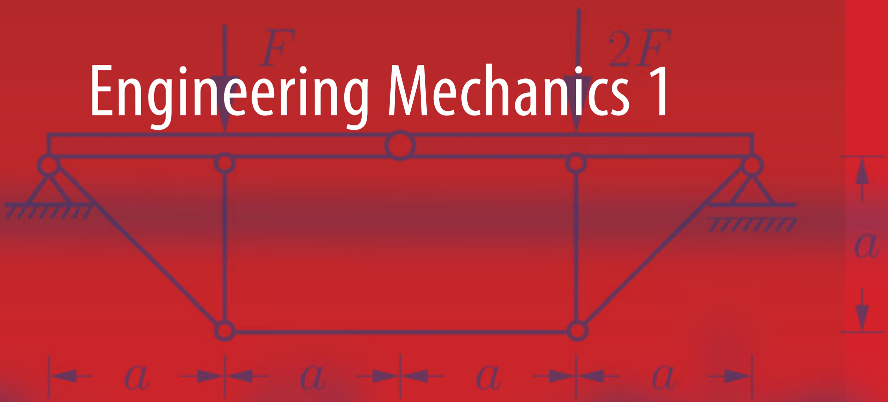
Dietmar Gross · Wolfgang Ehlers
Peter Wriggers · Jörg Schröder
Ralf Müller

Statics – Formulas and Problems

Engineering Mechanics 1



Springer

Statics – Formulas and Problems

Dietmar Gross · Wolfgang Ehlers
Peter Wriggers · Jörg Schröder
Ralf Müller

Statics – Formulas and Problems

Engineering Mechanics 1

Springer

Dietmar Gross
Division of Solid Mechanics
TU Darmstadt
Darmstadt
Germany

Jörg Schröder
Institute of Mechanics
Universität Duisburg-Essen
Essen
Germany

Wolfgang Ehlers
Institute of Applied Mechanics
Universität Stuttgart
Stuttgart
Germany

Ralf Müller
Institute of Applied Mechanics
TU Kaiserslautern
Kaiserslautern
Germany

Peter Wriggers
Institute of Continuum Mechanics
Leibniz Universität Hannover
Hannover
Germany

ISBN 978-3-662-53853-1 ISBN 978-3-662-53854-8 (eBook)
DOI 10.1007/978-3-662-53854-8

Library of Congress Control Number: 2016956635

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer-Verlag GmbH Germany
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface

This collection of problems results from the demand of students for supplementary problems and support in the preparation for examinations. With the present collection "Engineering Mechanics 1 - Formulas and Problems, Statics" we provide more additional exercise material. The subject "Statics" is commonly taught in the basic course of Engineering Mechanics classes at universities.

The problems analyzed within these courses use equilibrium conditions and the principle of virtual work to analyze static problems and to compute reaction forces and stress resultants. These concepts are the basic of many structural analyses of components used in civil and mechanical engineering.

We would like to make the reader aware that pure reading and trying to comprehend the presented solutions will not provide a deeper understanding of mechanics. Neither does it improve the problem solving skills. Using this collection wisely, one has to try to solve the problems independently. The proposed solution should only be considered when experiencing major problems in solving an exercise.

Obviously this collection cannot substitute a full-scale textbook. If not familiar with the formulae, explanations, or technical terms the reader has to consider his or her course material or additional textbooks on mechanics of materials. An incomplete list is provided on page IX.

Darmstadt, Stuttgart, Hannover,
Essen and Kaiserslautern, Summer 2016

D. Gross
W. Ehlers
P. Wriggers
J. Schröder
R. Müller

Table of Contents

Literature

Textbooks

Gross, D., Hauger, W., Schröder, J., Wall, W., Rajapakse, N., Engineering Mechanics 1: Statics, 2nd edition, Springer 2013

Hibbeler, R. C., Engineering Mechanics, Statics, Prentice Hall, 2012

Beer, F., Johnston, E.R., Mazurek, D., Vector Mechanics for Engineers: Statics, 11th edition, McGraw-Hill Education 2015

Beer, F., Johnston, E.R., Eisenberg, E., Cornwell, P., Mazurek, D., Vector Mechanics for Engineers: Statics and Dynamics, 11th edition, McGraw-Hill Education 2015

Collection of Problems

Hibbeler, R. C., Engineering Mechanics, Statics, Prentice Hall, 2012

Schaum's Outlines of Engineering Mechanics: Statics, 6th Edition, McGraw-Hill Education 2010

Beer, F., Johnston, E.R., Mazurek, D., Vector Mechanics for Engineers: Statics, 11th edition, McGraw-Hill Education 2015

Notation

For the solutions of the problems we used the following symbols:

$\uparrow$: Short notation for *sum of all forces in direction of the arrow equal to zero.*

$\overset{\curvearrowleft}{A}$: Short notation for *sum of all moments with respect to reference point A (with predetermined direction of rotation) equal to zero.*

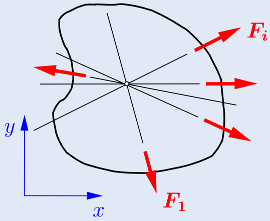
$\rightsquigarrow$ Short notation for *from this follows that.*

Chapter 1

Equilibrium

1

Forces with a common point of application in a plane



A system of forces with a common point of application can be replaced by a *statically equivalent force*

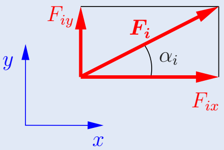
$$\boldsymbol{R} = \sum \boldsymbol{F_i}\,.$$

The system is in *equilibrium*, if

$$\sum \boldsymbol{F_i} = \boldsymbol{0}$$

or in cartesian components

$$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,.$$

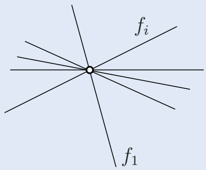


Here we used the notation

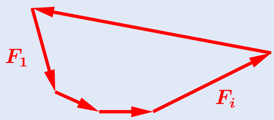
$$\begin{aligned}
\boldsymbol{F_i} &= F_{ix}\boldsymbol{e}_x + F_{iy}\boldsymbol{e}_y\,,\\
F_{ix} &= F_i \cos\alpha_i\,,\\
F_{iy} &= F_i \sin\alpha_i\,,\\
|\boldsymbol{F}_i| = F_i &= \sqrt{F_{ix}^2 + F_{iy}^2}\,.
\end{aligned}$$

In a *graphical solution*, the equilibrium condition is expressed by a *closed force polygon*.

lines of action



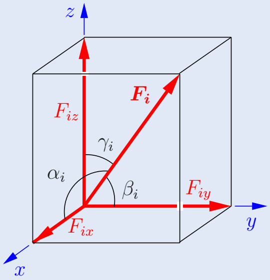
force polygon



Forces with a common point of application in space

Equilibrium exists, if the resultant $\boldsymbol{R} = \sum \boldsymbol{F_i}$ vanishes, i.e. if $\sum \boldsymbol{F_i} = \boldsymbol{0}$ or in cartesian components

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum F_{iz} = 0\,.$$



Here, the following notation is used

$$\boldsymbol{F_i} = F_{ix}\boldsymbol{e}_x + F_{iy}\boldsymbol{e}_y + F_{iz}\boldsymbol{e}_z \ ,$$

$$F_{ix} = F_i \cos\alpha_i \ ,$$

$$F_{iy} = F_i \cos\beta_i \ ,$$

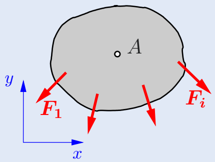
$$F_{iz} = F_i \cos\gamma_i \ ,$$

$$\cos^2\alpha_i + \cos^2\beta_i + \cos^2\gamma_i = 1 \ ,$$

$$|\boldsymbol{F}_i| = F_i = \sqrt{F_{ix}^2 + F_{iy}^2 + F_{iz}^2} \ .$$

General systems of forces in a plane

A general system of forces can be replaced by a resultant $\boldsymbol{R} = \sum \boldsymbol{F_i}$ and a resulting moment $M_R^{(A)}$ with respect to an arbitrary reference point A. Equilibrium exists, if

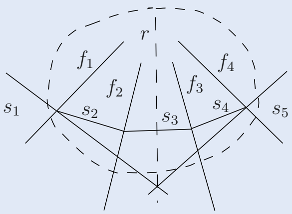


$$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum M_i^{(A)} = 0\,.$$

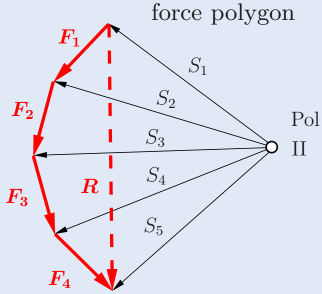
Instead of using the two force conditions, two alternative moment conditions with different reference points (e.g. B and C) may be applied. Here the points A, B and C *must not* lie on a *straight* line.

Graphical solutions for the resultant force are obtained with the help of the link polygon and the force polygon.

link polygon in layout diagram



force polygon



- The link lines s_i are parallel to the lines S_i in the force polygon.
- The line of action r of the resultant $\boldsymbol{R}$ (amplitude and direction follow from the force polygon) goes through the intersection of the outer link lines s_1 and s_5 of the link polygon.
- Equilibrium exists, if the link polygon and the polygon of forces are closed.

General systems of forces in space

Equilibrium exists, if the resultant of forces

$$\boldsymbol{R} = \sum \boldsymbol{F_i}$$

and the resulting moment

$$\boldsymbol{M_R^{(A)}} = \sum \boldsymbol{r_i} \times \boldsymbol{F_i}$$

with respect to an arbitrary reference point A vanish:

$$\sum \boldsymbol{F_i} = \boldsymbol{0}\,, \qquad \sum \boldsymbol{M_i^{(A)}} = \boldsymbol{0}$$

or in component form

$$\begin{aligned} &\sum F_{ix} = 0\,, &&\sum F_{iy} = 0\,, &&\sum F_{iz} = 0\,, \\ &\sum M_{ix}^{(A)} = 0\,, &&\sum M_{iy}^{(A)} = 0\,, &&\sum M_{iz}^{(A)} = 0 \end{aligned}$$

with

$$M_{ix}^{(A)} = y_i F_{iz} - z_i F_{iy}\,, \quad M_{iy}^{(A)} = z_i F_{ix} - x_i F_{iz}\,, \quad M_{iz}^{(A)} = x_i F_{iy} - y_i F_{ix}\,.$$

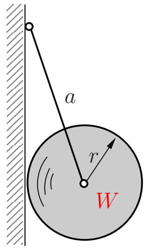
Here, x_i, y_i and z_i denote the components of the position vector $\boldsymbol{r_i}$, pointing from the reference point A to an arbitrary point on the action line of the force $\boldsymbol{F_i}$ (e.g., pointing to the point of application).

Remark: As in the plane case, it is possible to replace the force equilibrium conditions by additional moment equilibrium conditions with respect to suitable axes.

Problem 1.1 A sphere with the weight W is suspended by a wire at a smooth wall. The wire is fixed at the ball center. **P1.1**

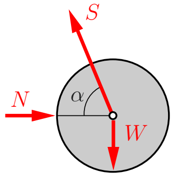
Determine the force S in the wire.

Given: $a = 60$ cm, $r = 20$ cm.



Solution **a)** *analytical:* All forces acting on the ball are made visible in the free-body diagram. Therefore, we cut the wire and separate the sphere from the wall. The force S in the wire is acting in the direction of the wire, the contact force N is acting perpendicular to the smooth wall, and the external force W points in vertical direction. The three forces are concurrent forces.

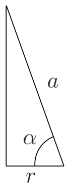
The equilibrium conditions are



$$\rightarrow: \quad N - S\cos\alpha = 0\,,$$

$$\uparrow: \quad S\sin\alpha - G = 0\,.$$

Solving for S and N yields



$$S = \frac{G}{\sin\alpha}\,,$$

$$N = S\cos\alpha = G\cot\alpha\,.$$

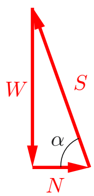
The angle α follows from the system geometry:

$$\cos\alpha = \frac{r}{a} = \frac{20}{60} = \frac{1}{3} \qquad \text{and} \qquad \sin\alpha = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{1}{3}\sqrt{8}\,.$$

This yields

$$\underline{\underline{S}} = \frac{3}{\sqrt{8}}\,G \approx \underline{\underline{1.06\,G}}\,.$$

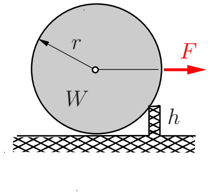
b) *graphical:* We draw a closed force polygon consisting of the known force W (*magnitude* and *action line* are given) and the two forces S and N, with given action lines. We obtain from the image



$$\underline{\underline{S = \frac{G}{\sin\alpha}}}\,, \qquad N = G\cot\alpha\,.$$

P1.2 **Problem 1.2** A smooth circular cylinder (weight W, radius r) touches an obstacle (height h), as depicted in the Figure.

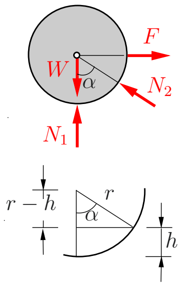
Determine the magnitude of the required force F to roll the cylinder over the obstacle.



Solution **a)** *analytical:* We isolate the cylinder from the base and the obstacle. In the free-body diagram, we see the four concurrent forces F, W, N_1 and N_2 acting at the cylinder. The equilibrium conditions are

$$\leftarrow: \quad N_2 \sin\alpha - F = 0\,,$$

$$\uparrow: \quad N_1 + N_2\cos\alpha - W = 0$$



with the angle α following from the system geometry:

$$\cos\alpha = \frac{r-h}{r}\,.$$

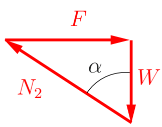
The two equilibrium conditions contain the three unknowns

$$N_1\,, \quad N_2 \quad \text{and} \quad F\,.$$

The force that initiates the cylinder to roll over the obstacle, also causes the cylinder to lift-off from the base. Then, the contact force N_1 vanishes:

$$N_1 = 0 \quad \rightsquigarrow \quad N_2 = \frac{W}{\cos\alpha} \quad \rightsquigarrow \quad \underline{\underline{F}} = N_2 \sin\alpha = \underline{\underline{W\tan\alpha}}\,.$$

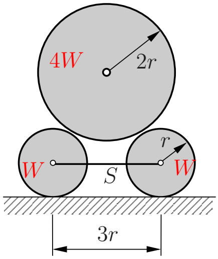
b) *graphical:* If $N_1 = 0$, we can draw a closed force polygon consisting of the known force W (*magnitude* and *action line* are given) and the two forces N_2 and F with given action lines. We read from the image



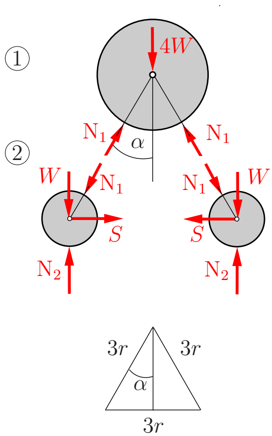
$$N_2 = \frac{W}{\cos\alpha}\,, \qquad \underline{\underline{F = W\tan\alpha}}\,.$$

Problem 1.3 A large cylinder (weight $4W$, radius $2r$) lies on top of two small cylinders, each having weight W and radius r. The small cylinders are connected by a wire S (length $3r$). All surfaces are smooth. **P1.3**

Determine all contact forces and the magnitude of force S in the wire.



Solution We isolate the three cylinders and introduce the contact forces in the free-body diagram. The forces acting at each cylinder are concurrent forces. Due to the symmetry of the problem we have only one equilibrium condition at the large cylinder and two equilibrium conditions for one of the small cylinders. These are three equations for the three unknown forces N_1, N_2 und S:



$$\text{①}\ \uparrow: \quad 2N_1 \cos\alpha - 4W = 0\,,$$

$$\text{②}\ \rightarrow: \quad S - N_1 \sin\alpha = 0\,,$$

$$\uparrow: \quad N_2 - N_1 \cos\alpha - W = 0\,.$$

The angle α follows from the geometry of the problem:

$$\sin\alpha = \frac{3r/2}{3r} = \frac{1}{2} \quad \rightsquigarrow \quad \alpha = 30^\circ$$

$$\rightsquigarrow \quad \cos\alpha = \frac{\sqrt{3}}{2}\,, \quad \tan\alpha = \frac{\sqrt{3}}{3}\,.$$

Solving for the forces yields

$$\underline{\underline{N_1 = \frac{2W}{\cos\alpha} = \frac{4\sqrt{3}}{3}W}}\,, \quad \underline{\underline{S = 2W\tan\alpha = \frac{2\sqrt{3}}{3}W}}\,, \quad \underline{\underline{N_2}} = 2W + W = \underline{\underline{3W}}\,.$$

Remark: The contact force N_2 could have been determined from the equilibrium condition for the complete system:

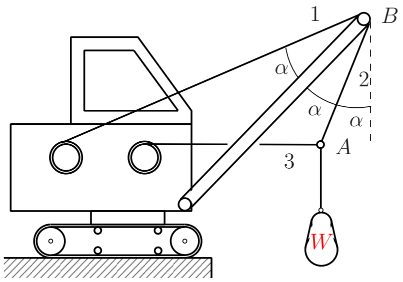
$$\uparrow: \quad 2N_2 - 2W - 4W = 0 \quad \rightsquigarrow \quad \underline{\underline{N_2 = 3W}}\,.$$

P1.4

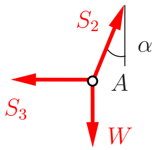
Problem 1.4 An excavator has been converted to a demolition machine.

Determine the forces in the cables 1, 2 and 3 as well as in the jib due to the weight W.

Remark: The jib only transfers a force in the direction of its axis (strut).



Solution We isolate the points A and B. The equilibrium conditions for point A yield

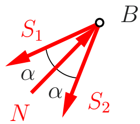


$$\left.\begin{aligned} \uparrow:&\quad S_2\cos\alpha - W = 0\,,\\ \rightarrow:&\quad S_2\sin\alpha - S_3 = 0\,,\end{aligned}\right\}\quad \begin{aligned} S_2 &= \frac{W}{\cos\alpha}\,,\\ S_3 &= W\tan\alpha\end{aligned}$$

and for point B (N is the force in the jib):

$$\rightarrow:\quad -S_2\sin\alpha + N\sin 2\alpha - S_1\sin 3\alpha = 0\,,$$

$$\uparrow:\quad -S_2\cos\alpha + N\cos 2\alpha - S_1\cos 3\alpha = 0\,.$$



Alternatively, we obtain for point B with a *clever* choice of the coordinate directions

$$\nearrow:\quad N - S_2\cos\alpha - S_1\cos\alpha = 0\,,$$

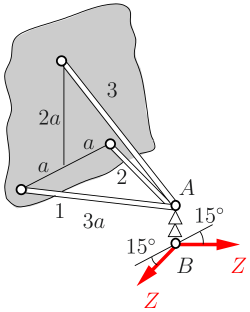
$$\nwarrow:\quad S_1\sin\alpha - S_2\sin\alpha = 0\,.$$

Thus, from the $2\times 2 = 4$ equilibrium conditions, we obtain the following results for the four unkowns S_1, S_2, S_3, N:

$$\underline{\underline{S_1 = S_2 = \frac{W}{\cos\alpha}}}\,,\qquad \underline{\underline{S_3 = W\tan\alpha}}\,,\qquad \underline{\underline{N}} = 2S_2\cos\alpha = \underline{\underline{2W}}\,.$$

Problem 1.5 A high-voltage power line is attached to an insulator which is held by three bars. The tensile force Z in the sagging power line at the insulator is to 1000 N. **P1.5**

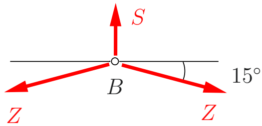
Determine the magnitude of the three forces in the three bars.



Solution Equilibrium at the insulator yields (plane subproblem):

$$\uparrow: \quad S - 2Z\sin 15^\circ = 0\,,$$

$$\rightsquigarrow \quad S = 2Z\sin 15^\circ = 517\,\text{N}\,.$$

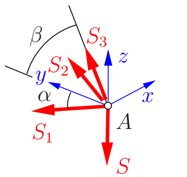


With the now known force S, the 3 forces in the bars result from the 3 equilibrium conditions at point A:

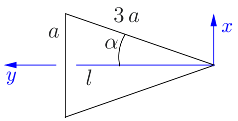
$$\sum F_x = 0: \qquad S_2\sin\alpha - S_1\sin\alpha = 0\,,$$

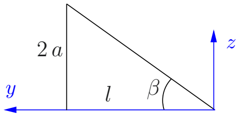
$$\sum F_y = 0: \quad S_1\cos\alpha + S_2\cos\alpha + S_3\cos\beta = 0\,,$$

$$\sum F_z = 0: \qquad S_3\sin\beta - S = 0\,.$$



The used auxiliary angles α and β follows from the geometry:





$$\sin\alpha = \frac{a}{3a} = \frac{1}{3} \rightsquigarrow \alpha = 19.5^\circ, \quad \tan\beta = \frac{2a}{l} = \frac{2}{\sqrt{3^2-1^2}} \rightsquigarrow \beta = 35.3^\circ.$$

Thus, we obtain the results

$$\underline{\underline{S_3}} = \frac{S}{\sin\beta} = \underline{\underline{1.73\ S}} = \underline{\underline{895\ \text{N}}}\,,$$

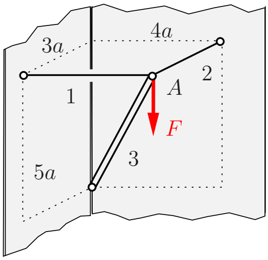
$$\underline{\underline{S_1 = S_2}} = -S_3\frac{\cos\beta}{2\cos\alpha} = -\frac{S}{2\tan\beta\cos\alpha} = \underline{\underline{-0,75\ S}} = \underline{\underline{-388\ \text{N}}}\,.$$

Remark: Due to symmetry conditions (geometry and load), it holds $S_2 = S_1$.

P1.6

Problem 1.6 The system consists of bar 3 which is held by two horizontal wires 1 and 2 and which is loaded by a force F.

Calculate the forces in the bar and the wires.

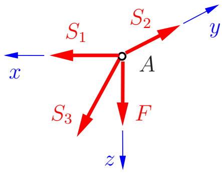


Solution We isolate point A by passing imaginary sections through the bar and the wires. The internal forces are visualized in the free-body diagram, they are assumed to be tensile forces. A suitable coordinate system, with the base vectors pointing into the direction of the wires and the force F, facilitates the calculation. The resulting forces lead to

$$\sum F_x = 0 : \quad S_1 + S_{3x} = 0\,,$$

$$\sum F_y = 0 : \quad S_2 + S_{3y} = 0\,,$$

$$\sum F_z = 0 : \quad S_{3z} + F = 0\,.$$



The components of S_3 are related to the geometrical lengths (L = length of bar 3)

$$\frac{S_{3x}}{S_3} = \frac{4a}{L}\,, \qquad \frac{S_{3y}}{S_3} = \frac{3a}{L}\,, \qquad \frac{S_{3z}}{S_3} = \frac{5a}{L}$$

or

$$S_{3x} : S_{3y} : S_{3z} = 4 : 3 : 5\,.$$

Substitution into the equilibrium conditions yields

$$S_{3z} = -F\,, \qquad \underline{\underline{S_2}} = -S_{3y} = -\frac{3}{5}\,S_{3z} = \underline{\underline{\frac{3}{5}\,F}}\,,$$

$$\underline{\underline{S_1}} = -S_{3x} = -\frac{4}{5}\,S_{3z} = \underline{\underline{\frac{4}{5}\,F}}\,,$$

$$\underline{\underline{S_3}} = S_{3z}\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + 1^2} = \underline{\underline{-\sqrt{2}\,F}}\,.$$

Remark: The negative sign of S_3 indicate a pressure force in bar 3.

Alternative approach: We can also solve the problem by directly starting from the equilibrium conditions in vectorial form:

$$\boldsymbol{S_1} + \boldsymbol{S_2} + \boldsymbol{S_3} + \boldsymbol{F} = \boldsymbol{0}\,.$$

Each force is expressed by its directional vector (unit vector) and its magnitude. For example, the directional vector for S_3 is given by

$$\boldsymbol{e_3} = \frac{1}{\sqrt{4^2+3^2+5^2}}\begin{pmatrix}4\\3\\5\end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix}4\\3\\5\end{pmatrix}\,.$$

Thus, we obtain for the forces

$$\boldsymbol{S_1} = S_1\,\boldsymbol{e_1} = S_1\begin{pmatrix}1\\0\\0\end{pmatrix}\,, \qquad \boldsymbol{S_2} = S_2\,\boldsymbol{e_2} = S_2\begin{pmatrix}0\\1\\0\end{pmatrix}\,,$$

$$\boldsymbol{S_3} = S_3\,\boldsymbol{e_3} = S_3\,\frac{1}{5\sqrt{2}}\begin{pmatrix}4\\3\\5\end{pmatrix}\,, \qquad \boldsymbol{F} = F\,\boldsymbol{e_F} = F\begin{pmatrix}0\\0\\1\end{pmatrix}\,,$$

and the equilibrium conditions now read

$$S_1\begin{pmatrix}1\\0\\0\end{pmatrix} + S_2\begin{pmatrix}0\\1\\0\end{pmatrix} + S_3\,\frac{1}{5\sqrt{2}}\begin{pmatrix}4\\3\\5\end{pmatrix} + F\begin{pmatrix}0\\0\\1\end{pmatrix} = \boldsymbol{0}\,.$$

Hence, we obtain for the components

$$S_1 + \frac{4}{5\sqrt{2}}\,S_3 = 0\,,$$

$$S_2 + \frac{3}{5\sqrt{2}}\,S_3 = 0\,,$$

$$\frac{5}{5\sqrt{2}}\,S_3 + F = 0\,,$$

which from the forces in the bar and wires result as

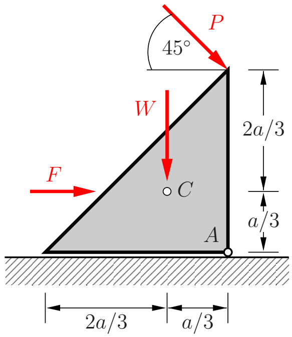
$$\underline{\underline{S_3 = -\sqrt{2}\,F}}\,, \qquad \underline{\underline{S_2 = \frac{3}{5}\,F}}\,, \qquad \underline{\underline{S_1 = \frac{4}{5}\,F}}\,.$$

P1.7 **Problem 1.7** An isosceles triangluar body is loaded by the forces F, P and the weight W.

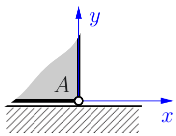
The forces acting on the body shall first be replaced by a resultant force and a resultant moment at point A (reduction to referencepoint A).

Determine the magnitude of the force F, that the resultant moment around point A vanishes, such that the body cannot tilt.

Given: $W = 6$ kN, $P = \sqrt{2}\,$kN, $a = 1\,$m.



Solution We solve the problem in vectorial form and introduce a coordinate system with its origin point A. The resultant force $\boldsymbol{R}$ is obtained from the sum of the individual forces:



$$\boldsymbol{W} = -W\,\boldsymbol{e}_y\,, \quad \boldsymbol{F} = F\,\boldsymbol{e}_x\,, \quad \boldsymbol{P} = \frac{\sqrt{2}}{2}P(\boldsymbol{e}_x - \boldsymbol{e}_y)\,,$$

$$\boldsymbol{R} = \boldsymbol{W} + \boldsymbol{F} + \boldsymbol{P} = (F + \frac{\sqrt{2}}{2}P)\,\boldsymbol{e}_x - (W + \frac{\sqrt{2}}{2}P)\,\boldsymbol{e}_y\,.$$

The resultant moment $\boldsymbol{M}_A$ is calculated with the lever arms

$$\boldsymbol{r}_{AW} = -\frac{a}{3}\,\boldsymbol{e}_x + \frac{a}{3}\,\boldsymbol{e}_y\,, \quad \boldsymbol{r}_{AF} = -\frac{2\,a}{3}\,\boldsymbol{e}_x + \frac{a}{3}\,\boldsymbol{e}_y\,, \quad \boldsymbol{r}_{AP} = a\,\boldsymbol{e}_y$$

of each force with respect to A as

$$\boldsymbol{M}_A = \boldsymbol{r}_{AW} \times \boldsymbol{W} + \boldsymbol{r}_{AF} \times \boldsymbol{F} + \boldsymbol{r}_{AP} \times \boldsymbol{P} = \frac{W\,a}{3}\,\boldsymbol{e}_z - \frac{F\,a}{3}\,\boldsymbol{e}_z - \frac{\sqrt{2}\,P\,a}{2}\,\boldsymbol{e}_z\,.$$

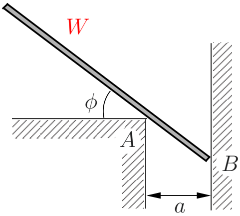
With the given values for G, P and a, the magnitude of the force $\boldsymbol{F}$ can be chosen such that the moment $\boldsymbol{M}_A$ vanishes. It follows from the condition $\boldsymbol{M}_A \overset{!}{=} \boldsymbol{0}$:

$$\frac{W\,a}{3} - \frac{F\,a}{3} - \frac{\sqrt{2}\,P\,a}{2} = 0 \quad \rightsquigarrow \quad \underline{\underline{F}} = W - \frac{3\sqrt{2}\,P}{2} = 6\,\text{kN} - 3\,\text{kN} = \underline{\underline{3\,\text{kN}}}.$$

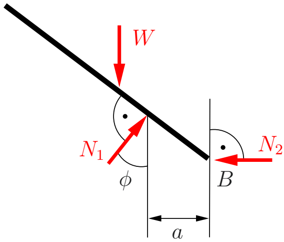
Remark: In the two-dimensional case, the resultant moment only has a z-component. This component can be more easily calculated from the sum of the individual moments with respect to A (pay attention to positive sense of rotation!) than by evaluating the cross product: $M_z^{(A)} = (a/3)G - (a/3)F - a(\sqrt{2}/2)P$.

Problem 1.8 A uniform beam (weight W, length $4a$) rests upon the corner A and the smooth wall at B. **P1.8**

Calculate the angle ϕ for which the beam is in equilibrium.



Solution We isolate the beam and sketch the free-body diagram. From the condition *„smooth"*, it follows that the unkown forces N_1 and N_2 are perpendicular to the respective contact plane. Thus, the equilibrium conditions read



$$\rightarrow: \quad N_1 \sin\phi - N_2 = 0\,,$$

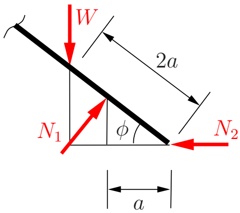
$$\uparrow: \quad N_1 \cos\phi - W = 0\,,$$

$$\overset{\curvearrowright}{B}: \quad \frac{a}{\cos\phi} N_1 - 2a\cos\phi\, W = 0\,.$$

They can be used to determine the three unknowns N_1, N_2 and ϕ. The solution for ϕ is obtained by substitution equation 2 into equation 3:

$$\frac{a\,W}{\cos^2\phi} - 2a\cos\phi\, W = 0 \quad \leadsto \quad \underline{\underline{\cos^3\phi = \frac{1}{2}}}\,.$$

The solution can be found more easily with the aid of the statement: *„Three forces are in equilibrium if their lines of action pass through* **one** *point and the according force polygon is closed"*. Thus, it follows from the geometry:



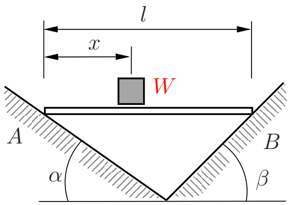
$$2a\cos\phi = \frac{a/\cos\phi}{\cos\phi}\,,$$

$$\leadsto \quad \underline{\underline{\cos^3\phi = \frac{1}{2}}}\,.$$

P1.9

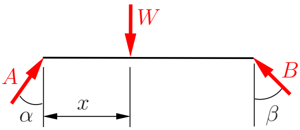
Problem 1.9 A beam of length l and negligible weight is placed horizontally between two smooth inclined planes. A block with the weight W rests upon the beam.

At what distance x the block must be placed in order to obtain equilibrium? Determine the reaction forces?



Solution a) *analytical:* We sketch a free-body diagram and formulate the equilibrium conditions:

$$\uparrow: \quad A\cos\alpha + B\cos\beta - W = 0\,,$$
$$\rightarrow: \quad A\sin\alpha - B\sin\beta = 0\,,$$
$$\overset{\curvearrowleft}{A}: \quad x\,W - l\,B\cos\beta = 0\,.$$



It follows therefrom

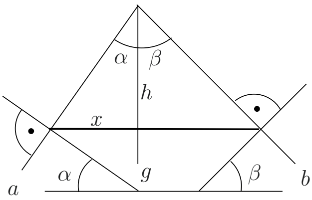
$$\underline{\underline{A = W\,\frac{\sin\beta}{\sin(\alpha+\beta)}}}\,, \qquad \underline{\underline{B = W\,\frac{\sin\alpha}{\sin(\alpha+\beta)}}}\,,$$

$$\underline{\underline{x}} = l\,\frac{\sin\alpha\cos\beta}{\sin(\alpha+\beta)} = \underline{\underline{\frac{l}{1+(\tan\beta/\tan\alpha)}}}\,.$$

b) *graphical:* Three forces are in equilibrium if their lines of action pass through one point and their force polygon is closed. Thus, the line of action w of W follows directly from the intersection of the lines of action a and b of the reaction forces A and B. We can see from the sketch:

$$\left.\begin{aligned} h\tan\alpha + h\tan\beta &= l \\ h\tan\alpha &= x \end{aligned}\right\}$$

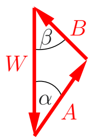
$$\rightsquigarrow \quad \underline{\underline{x = \frac{l}{1+\tan\beta/\tan\alpha}}}\,.$$



The reaction forces (e.g. force A) follow from the force triangle (sine rule):

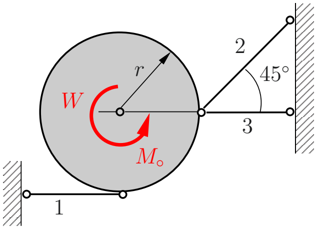
$$\frac{A}{\sin\beta} = \frac{W}{\sin\left[\pi-(\alpha+\beta)\right]}\,,$$

$$\rightsquigarrow \quad \underline{\underline{A = W\,\frac{\sin\beta}{\sin(\alpha+\beta)}}}\,.$$



Problem 1.10 A homogeneous cylinder (weight W, radius r) is held by three struts and loaded by an external moment $M_\circ$. **P1.10**

Determine the forces in the struts. For what magnitude of $M_\circ$ the force in strut 1 is zero?

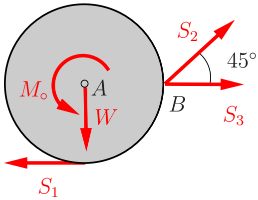


Solution We isolate the cylinder and sketch the free-body diagram. The equilibrium conditions lead to

$$\rightarrow: \quad \frac{\sqrt{2}}{2} S_2 + S_3 - S_1 = 0\,,$$

$$\uparrow: \quad \frac{\sqrt{2}}{2} S_2 - W = 0\,,$$

$$\overset{\curvearrowleft}{A}: \quad r\,\frac{\sqrt{2}}{2}\, S_2 - r\, S_1 + M_\circ = 0\,.$$



Therefrom we obtain

$$\underline{\underline{S_1 = \frac{M_\circ}{r} + W}}\,, \qquad \underline{\underline{S_2 = \sqrt{2}\, W}}\,, \qquad \underline{\underline{S_3 = \frac{M_\circ}{r}}}\,.$$

The required moment follows from setting S_1 to zero:

$$S_1 = 0 \quad \rightsquigarrow \quad \underline{\underline{M_\circ = -rW}}\,.$$

Remarks:

- Instead of point A, it is more convenient to use point B as the reference point for the moment equilibrium condition. In this we have only one unknown:

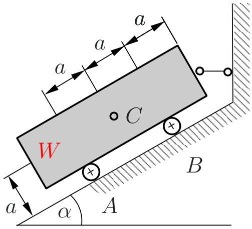
$$\overset{\curvearrowleft}{B}: \quad rW - rS_1 + M_\circ = 0 \quad \rightsquigarrow \quad S_1 = \frac{M_\circ}{r} + W\,.$$

- All forces in the bars are tensile forces.
- The force S_2 in bar 2 is independent of $M_\circ$.
- The moment $M_\circ$ is in equilibrium due to the forces S_1 and S_3 in the bars 1 and 3.

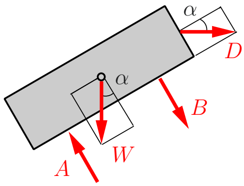
P1.11

Problem 1.11 A vehicle of weight $W = 10\,\text{kN}$ and known mass center C, stands on an inclined, smooth surface ($\alpha = 30^\circ$) and is held by a horizontal rope.

Calculate the compressive forces on the wheels.



Solution We cut the rope, isolate the vehicle from the plane and sketch the free-body diagram.



We use the equilibrium condition of forces in the direction of the tilted plane and two moment equilibrium conditions with inclined to points A and B. For the latter ones, we decompose the forces G and D into their components in direction of the plane and perpendicular to it. It follows

$$\nearrow: \quad D\cos\alpha - W\sin\alpha = 0\,,$$

$$\overset{\curvearrowleft}{A}: \quad 2aB + aW\sin\alpha - aW\cos\alpha - aD\cos\alpha - 3aD\sin\alpha = 0\,,$$

$$\overset{\curvearrowleft}{B}: \quad -2aA + aW\sin\alpha + aW\cos\alpha - aD\cos\alpha - aD\sin\alpha = 0\,.$$

Thus, we obtain

$$D = W\tan\alpha = \frac{W}{\sqrt{3}} = 5,77\,\text{kN}\,,$$

$$\underline{\underline{B}} = \frac{W}{2}(\cos\alpha - \sin\alpha) + \frac{D}{2}(\cos\alpha + 3\sin\alpha) = \frac{\sqrt{3}}{2}W = \underline{\underline{8.66\,\text{kN}}}\,,$$

$$\underline{\underline{A}} = \frac{W}{2}(\sin\alpha + \cos\alpha) - \frac{D}{2}(\cos\alpha + \sin\alpha) = \frac{W}{2\sqrt{3}} = \underline{\underline{2.89\,\text{kN}}}\,.$$

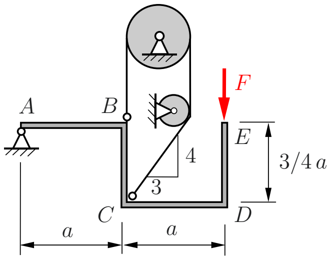
We check the result with an additional equilibrium condition:

$$\nwarrow: \quad A + B - W\cos\alpha - D\sin\alpha = 0$$

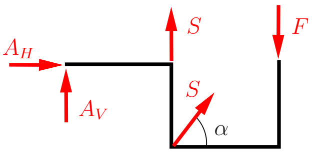
$$\leadsto \qquad \frac{W}{2\sqrt{3}} + W\frac{\sqrt{3}}{2} - W\frac{\sqrt{3}}{2} - \frac{W}{2\sqrt{3}} = 0\,.$$

Problem 1.12 A frame A to E is pin supported in A and held by a rope at B and C, which is passed over two pulleys without friction. **P1.12**

Determine the force in the rope for a given load F. The dead load of the frame can be neglected.



Solution We separate the system and consider, that the forces in the rope at both sides of the pulleys are equal (hence, the radius of the pulley does not enter the solution!):



So that the frame is in equilibrium, the equations

$$\uparrow: \quad A_V + S + S\sin\alpha - F = 0\,,$$

$$\rightarrow: \quad A_H + S\cos\alpha = 0\,,$$

$$\overset{\frown}{A}: \quad 2aF - aS - a(S\sin\alpha) - \frac{3}{4}a(S\cos\alpha) = 0$$

must hold. Together with

$$\cos\alpha = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}\,, \qquad \sin\alpha = \frac{4}{5}$$

it follows

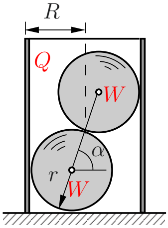
$$\underline{\underline{S = \frac{8}{9}F}}\,, \qquad A_H = -\frac{8}{15}F\,, \qquad A_V = -\frac{3}{5}F\,.$$

We calculate the moment equilibrium with respect to C to check the result:

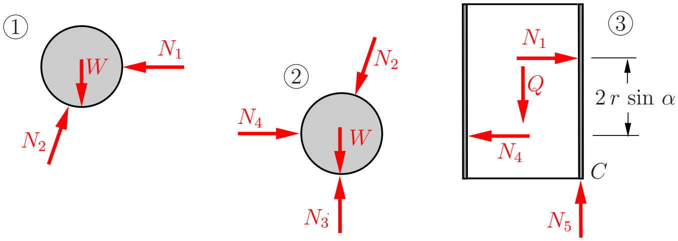
$$\overset{\frown}{C}: \quad aA_V + \frac{3}{4}aA_H + aF = 0 \quad \leadsto \quad -\frac{3}{5}aF - \frac{3}{4}a\frac{8}{15}F + aF = 0\,.$$

P1.13 **Problem 1.13** Two smooth spheres (each of weight W and radius r) are stacked inside a narrow circular tube (weight Q, radius R), which stands perpendicular to the ground ($r = \frac{3}{4}R$).

Determine the required weight W such that the tube does not tilt.



Solution We separate the spheres and the tube and sketch the free-body diagrams for the limiting case in which tilting just occurs. Then, the tube is only supported in point C by force N_5. (If the tube does not tilt, the contact force is distributed along the complete circumference of the tube.)



The equilibrium conditions for the spheres and the tube yield:

$$
\begin{aligned}
&① \uparrow: \quad N_2 \sin\alpha - W = 0\,, &&② \uparrow: \quad N_3 - N_2 \sin\alpha - W = 0\,,\\
&\leftarrow: \quad N_1 - N_2 \cos\alpha = 0\,, &&\leftarrow: \quad N_2 \cos\alpha - N_4 = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
③ \leftarrow: \quad & N_4 - N_1 = 0\,, \qquad \uparrow: \quad N_5 - Q = 0\,,\\
\overset{\curvearrowright}{C}: \quad & (r + 2r\sin\alpha)N_1 - rN_4 - RQ = 0\,.
\end{aligned}
$$

It follows therefrom

$$
N_1 = N_4 = \frac{W}{\tan\alpha}, \qquad N_2 = \frac{W}{\sin\alpha}, \qquad N_3 = 2W, \qquad Q = N_5 = \frac{3}{2}\,W\cos\alpha.
$$

With the geometric relationship

$$
\cos\alpha = (R - r)/r = 1/3,
$$

we obtain the limit weight for which tilting occurs:

$$
Q_{\text{tilting}} = W/2\,.
$$

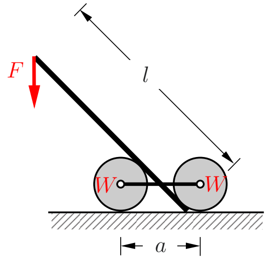
Such that the tube does not tilt, the condition

$$
\underline{\underline{Q > Q_{\text{tilting}} = W/2}}
$$

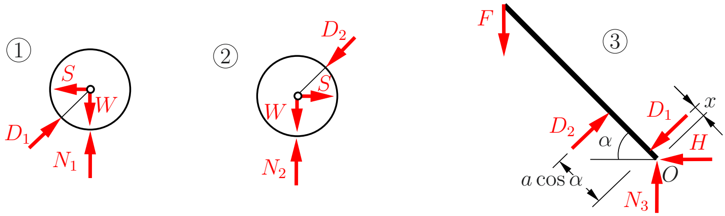
must be fullfilled.

Problem 1.14 Two smooth drums (weight W, radius r) are connected by a stiff rope of length a. The force F is applied using a lever of length l. **P1.14**

Determine the forces between the drums and the ground.



Solution We isolate the drums and the lever:



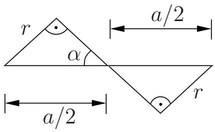
For the three separate systems, $2\times 2+1\times 3=7$ equations are available for the determination of the 7 unknowns (D_1, D_2, N_1, N_2, N_3, H, S):

$$
\begin{aligned}
&(1)\ \rightarrow: \quad S - D_1\sin\alpha = 0\,, \qquad \uparrow: \quad N_1 - W + D_1\cos\alpha = 0\,,\\
&(2)\ \rightarrow: \quad D_2\sin\alpha - S = 0\,, \qquad \uparrow: \quad N_2 - W - D_2\cos\alpha = 0\,,\\
&(3)\ \rightarrow: \quad H + D_1\sin\alpha - D_2\sin\alpha = 0\,,\\
&\qquad\ \uparrow: \quad N_3 - D_1\cos\alpha + D_2\cos\alpha - F = 0\,,\\
&\qquad\ \overset{\curvearrowright}{O}: \quad l\cos\alpha\ F - (a\cos\alpha + x)D_2 + xD_1 = 0\,.
\end{aligned}
$$

The angle α is obtained from the geometry:

$$\sin\alpha = \frac{r}{a/2}\,,$$

$$\cos\alpha = \sqrt{1-4(r/a)^2}\,.$$

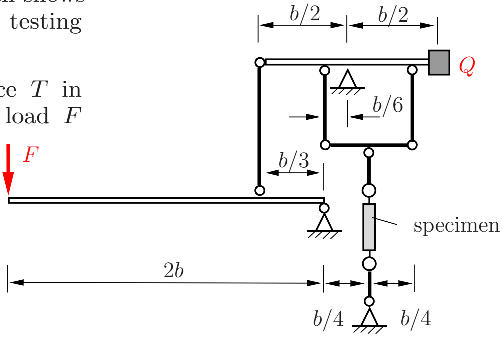


Summation of equations 1 and 3 yields $D_1 = D_2$. Thus, we obtain $H = 0$ and $N_3 = F$ and from equation 7, the unknown distance x drops out. Solving the equations yields

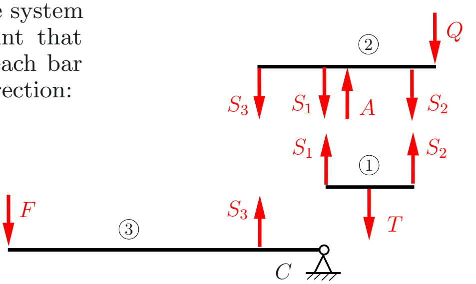
$$\underline{\underline{N_1 = W - F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}}}\,, \qquad \underline{\underline{N_2 = W + F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}}}\,.$$

P1.15 **Problem 1.15** The sketch shows the principle of a material testing machine.

Determine the tensile force T in the specimen for a given load F and weight Q.



Solution We separate the system where we take into account that the forces at the ends of each bar are equal with opposite direction:



$$\textcircled{1} \qquad S_1 = S_2\,, \qquad \text{(symmetry or moment equilibrium)}$$

$$\uparrow: \quad S_1 + S_2 = T\,,$$

$$\textcircled{2} \quad \overset{\curvearrowright}{A}: \quad \frac{b}{2}\,Q + \left(\frac{b}{2} - \frac{b}{6}\right) S_2 - \frac{b}{6}\,S_1 - \frac{b}{2}\,S_3 = 0 \quad \leadsto \quad S_1 = 3S_3 - 3Q\,,$$

$$\textcircled{3} \quad \overset{\curvearrowright}{C}: \quad \frac{b}{3}\,S_3 - 2\,b\,F = 0 \quad \leadsto \quad S_3 = 6F\,.$$

Thus, we obtain

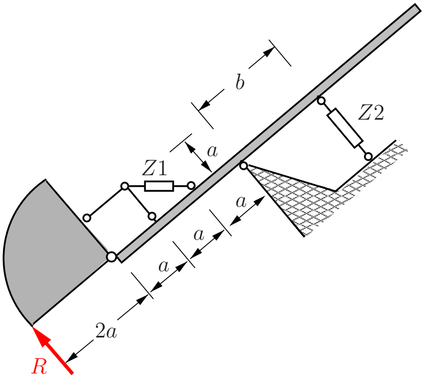
$$\underline{\underline{T}} = S_1 + S_2 = 6S_3 - 6Q = \underline{\underline{36F - 6Q}}\,.$$

Remarks:

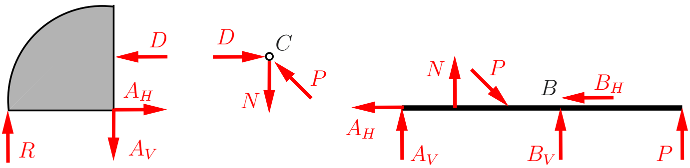
- By choosing suitable reference points for the moments, the support reactions A and C do not enter the equations.
- The load Q is used as counterweight to the weight of the levers and bars, which are neglected here.
- The magnitude of the force transferred to the specimen by the lever mechanicsm is 36 times the magnitude of the load F.

Problem 1.16 A hydraulic excavator arm shall be designed such that it exerts a force R at the cutting edge. **P1.16**

Determine the lever arm b of the cylinder $Z2$ such that it operates with the same pressure force as cylinder $Z1$.



Solution We isolate the system and sketch a free-body diagram. Therein, we a priori presume the same pressure force P in both cylinders.



Then, the equilibrium equations read for the shovel

$$\overset{\curvearrowright}{A}: \quad 2aR - aD = 0 \quad \leadsto \quad D = 2R\,,$$

$$\rightarrow: \quad A_H - D = 0 \quad \leadsto \quad A_H = 2R\,,$$

$$\uparrow: \quad R - A_V = 0 \quad \leadsto \quad A_V = R$$

and for point C

$$\rightarrow: \quad D - P\cos 45^\circ = 0 \quad \leadsto \quad \underline{\underline{P}} = D\sqrt{2} = \underline{\underline{2\sqrt{2}R}}\,,$$

$$\uparrow: \quad P\sin 45^\circ - N = 0 \quad \leadsto \quad N = 2R$$

as well as the moment equilibrium for the excavator arm

$$\overset{\curvearrowright}{B}: \quad 3\,aA_V + 2\,aN - a\,P\cos 45^\circ - b\,P = 0\,.$$

Solving the equations leads to the length of the lever arm:

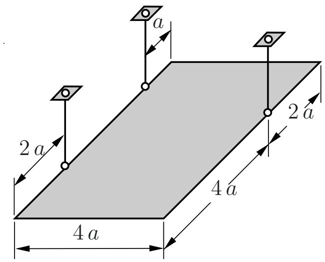
$$\underline{\underline{b = \frac{5}{4}\sqrt{2}\;a\,.}}$$

Remark: The support reactions B_V and B_H can be determined from the *equilibrium of forces* at the excavator arm.

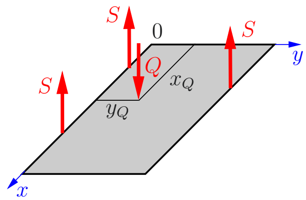
P1.17 **Problem 1.17** A rectangular plate of negligible weight is suspended by three vertical wires.

a) Assume that the plate is subjected to a concentrated vertical force Q. Determine the location of the point of application of Q so that the forces in the wires are equal.

b) Calculate the forces in the wires if the plate is subjected to a vertical constant area load p.



Solution a) We introduce a coordinate system. The unknown coordinates of the point of application of the force Q are denoted by x_Q and y_Q. If the forces in the wires are equal, the equilibrium conditions (parallel forces) are

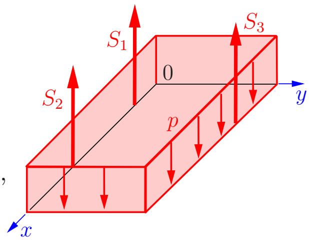


$$\sum F_z = 0 : \quad 3S - Q = 0 ,$$
$$\sum M_x^{(0)} = 0 : \quad 4\,aS - y_Q\,Q = 0 ,$$
$$\sum M_y^{(0)} = 0 : \quad -4\,aS - a\,S - 2aS + x_Q\,Q = 0 .$$

This yields

$$S = \frac{Q}{3} , \qquad \underline{\underline{y_Q = \frac{4}{3}a}} , \qquad \underline{\underline{x_Q = \frac{7}{3}a}} .$$

b) Now the plate is subjected to a constant area load p which can be replaced by a constant resultant force $F = 4 \cdot 6\,a^2 p = 24\,pa^2$. The forces in the wires are denoted by S_1, S_2 and S_3. The equilibrium conditions



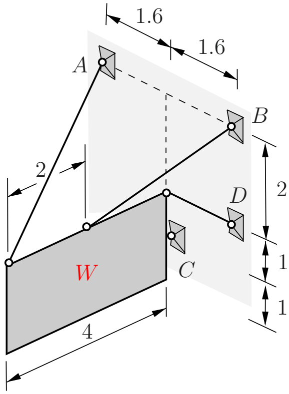
$$\sum F_z = 0 : \quad S_1 + S_2 + S_3 - 24\,pa^2 = 0 ,$$
$$\sum M_x^{(0)} = 0 : \quad 2\,a\,24\,pa^2 - 4\,aS_3 = 0 ,$$
$$\sum M_y^{(0)} = 0 : \quad 3\,a\,24\,pa^2 - 4\,aS_2 - aS_1 - 2\,aS_3 = 0$$

now lead to

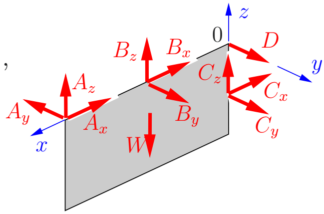
$$\underline{\underline{S_3 = 12\,pa^2}} , \qquad \underline{\underline{S_1 = 0}} , \qquad \underline{\underline{S_2 = 12\,pa^2}} .$$

Problem 1.18 A rectangular traffic sign of weight W is attached to a wall via two wires in A and B. It is held perpendicular to the wall by a joint in point C and a bar in D. All lengths are given in meters (m). **P1.18**

Determine the forces at the joint, in the wires and the bar.



Solution We isolate the traffic sign and sketch a free-body diagram with the components of all forces. Thus, the six spatial equilibrium conditions yield:



$$\sum F_x = 0 : \quad -A_x - B_x - C_x = 0\,,$$

$$\sum F_y = 0 : \quad -A_y + B_y + C_y + D = 0\,,$$

$$\sum F_z = 0 : \quad A_z + B_z + C_z - W = 0\,,$$

$$\sum M_x^{(0)} = 0 : \quad 1\,C_y = 0\,,$$

$$\sum M_y^{(0)} = 0 : \quad -4A_z - 2B_z + 2\,W + 1\,C_x = 0\,,$$

$$\sum M_z^{(0)} = 0 : \quad -4\,A_y + 2\,B_y = 0\,.$$

This provides six equations for 10 unknowns. Another $2 \times 2 = 4$ equations follow from the decomposition of the wire forces A and B into components (the components are related to eachother according to their respective lengths!):

$$\frac{A_x}{A_y} = \frac{4}{1.6}\,, \qquad \frac{A_x}{A_z} = \frac{4}{2}\,, \qquad \frac{B_x}{B_y} = \frac{2}{1.6}\,, \qquad \frac{B_x}{B_z} = \frac{2}{2}\,.$$

by solving for the forces, we obtain:

$$\underline{\underline{A_x = B_x = \frac{W}{3}}}\,, \quad \underline{\underline{C_x = -\frac{2}{3}\,W}}\,, \quad \underline{\underline{A_y = \frac{2}{15}\,W}}\,, \quad \underline{\underline{B_y = \frac{4}{15}\,W}}\,,$$

$$\underline{\underline{C_y = 0}}\,, \quad \underline{\underline{A_z = \frac{W}{6}}}\,, \quad \underline{\underline{B_z = \frac{W}{3}}}\,, \quad \underline{\underline{C_z = \frac{W}{2}}}\,, \quad \underline{\underline{D = -\frac{2}{15}\,W}}\,.$$

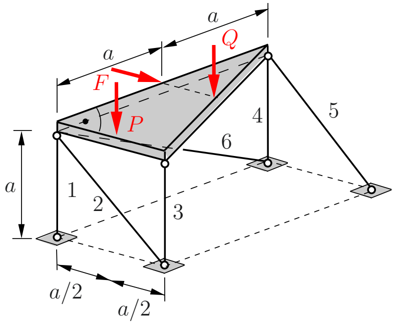
With the components of A and B, the forces in the ropes result as:

$$\underline{\underline{S_A = 0.4\,W}}\,, \quad \underline{\underline{S_B = 0.34\,W}}\,.$$

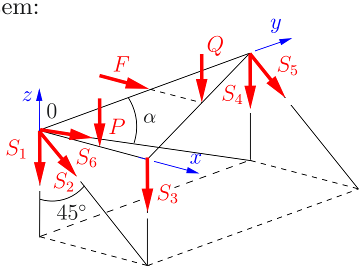
P1.19

Problem 1.19 A right-angled triangle (weight negligible) is supported by six bars. It is subjected to the forces F, Q and P.

Calculate the forces in the bars.



Solution First, we sketch the free-body diagram and choose a coordinate system:



Then we write down the equilibrium conditions (since the geometry of the problem is very simple, we do not resort to the vector formalism):

$$\sum F_x = 0 \ : \qquad \frac{\sqrt{2}}{2}\, S_2 + \frac{\sqrt{2}}{2}\, S_5 + F = 0\,,$$

$$\sum F_y = 0 \ : \qquad S_6 \cos\alpha = 0\,,$$

$$\sum F_z = 0 \ : \qquad -S_1 - \frac{\sqrt{2}}{2}\, S_2 - S_3 - S_6 \sin\alpha - S_4 - \frac{\sqrt{2}}{2}\, S_5 - Q - P = 0\,,$$

$$\sum M_x^{(0)} = 0 \ : \qquad -2aS_4 - 2a\,\frac{\sqrt{2}}{2}\, S_5 - a\,Q = 0\,,$$

$$\sum M_y^{(0)} = 0 \ : \qquad a\,S_3 + \frac{a}{2}\,Q + \frac{a}{2}\,P = 0\,,$$

$$\sum M_z^{(0)} = 0 \ : \qquad -2a\,\frac{\sqrt{2}}{2}\, S_5 - aF = 0\,.$$

Solving this system of equations for the forces in the bars yields

$$\underline{\underline{S_1 = \frac{F-P}{2}}}\,, \qquad \underline{\underline{S_2 = -\frac{\sqrt{2}}{2}\,F}}\,, \qquad \underline{\underline{S_3 = -\frac{1}{2}(Q+P)}}\,,$$

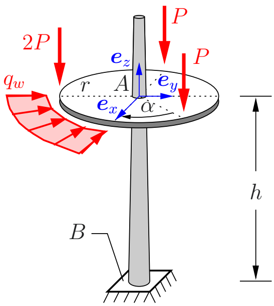
$$\underline{\underline{S_4 = \frac{1}{2}\,(F-Q)}}\,, \qquad \underline{\underline{S_5 = -\frac{\sqrt{2}}{2}\,F}}\,, \qquad \underline{\underline{S_6 = 0}}\,.$$

Problem 1.20 On the platform of a television tower, the shown external forces act due to the attached constructions and wind loads. **P1.20**

First, the external forces shall be replaced by a resultant force and a resultant moment with respect to the support point A of the platform.

Subsequently, the moment at the bottom B of the tower shall be determined with the aid of an offset moment.

Given: $\alpha = 45°$.



Solution In order to determine the resultant force and resultant moment with respect to point A, we need the forces and the respective lever arms. For the vertical single forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$, it follows with the shown basis system:

$$\text{forces:} \qquad \boldsymbol{F}_1 = -2\,P\,\boldsymbol{e}_z\,, \quad \boldsymbol{F}_2 = -P\,\boldsymbol{e}_z\,, \quad \boldsymbol{F}_3 = -P\,\boldsymbol{e}_z\,,$$

$$\text{lever arms:} \quad \boldsymbol{r}_{AF_1} = -r\,\boldsymbol{e}_y\,, \; \boldsymbol{r}_{AF_2} = -r\,\boldsymbol{e}_x\,, \; \boldsymbol{r}_{AF_3} = \frac{\sqrt{2}}{2}\,r\,(\boldsymbol{e}_x + \boldsymbol{e}_y)\,.$$

Since the wind load q_w acts in radial direction at each position, it does not induce a moment with respect to point A. For the resultant wind load it follows

$$\boldsymbol{F}_w = \frac{\sqrt{2}}{2}\,\frac{\pi}{2}\,r\,q_w\,(-\boldsymbol{e}_x + \boldsymbol{e}_y)\,, \qquad \boldsymbol{r}_{AF_w} = \boldsymbol{0}\,.$$

The overall resultant is given by

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_w = \frac{\sqrt{2}}{2}\,\frac{\pi}{2}\,r\,q_w\,(-\boldsymbol{e}_x + \boldsymbol{e}_y) - 4\,P\,\boldsymbol{e}_z\,,$$

and the resultant moment with respect to A is obtained as

$$\boldsymbol{M}^{(A)} = \sum_{i=1}^{3} \boldsymbol{r}_{AF_i} \times \boldsymbol{F}_i = P\,r\,(2 - \frac{\sqrt{2}}{2})\,\boldsymbol{e}_x + P\,r\,(\frac{\sqrt{2}}{2} - 1)\,\boldsymbol{e}_y\,.$$

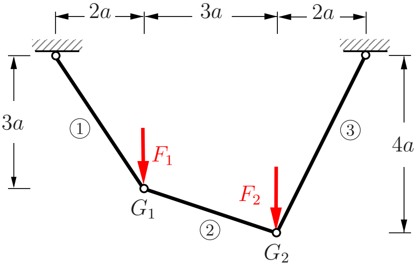
Now, in order to determine the moment with respect to point B, the *offset moment* $\boldsymbol{M}_V = \boldsymbol{r}_{BA} \times \boldsymbol{R}$ needs to be added to $\boldsymbol{M}^{(A)}$. It is calculated with the lever arm $\boldsymbol{r}_{BA} = h\,\boldsymbol{e}_z$ as

$$\boldsymbol{M}_V = \boldsymbol{r}_{BA} \times \boldsymbol{R} = \frac{\sqrt{2}}{2}\,\frac{\pi}{2}\,r\,q_w\,h\,(-\boldsymbol{e}_x - \boldsymbol{e}_y)\,.$$

Hence, the moment at the bottom of the tower is given by

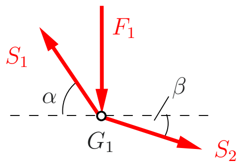
$$\boldsymbol{M}^{(B)} = \boldsymbol{M}^{(A)} + \boldsymbol{M}_V\,.$$

P1.21 **Problem 1.21** A system of three pin connected bars takes the depicted equilibrium position under the given loading with the forces F_1 and F_2.



a) Determine the required ratio of F_1/F_2 for this case.
b) Calculate the forces S_1, S_2 and S_3 in the bars?

Solution We isolate the joints G_1 and G_2 and obtain two central systems of forces, for which we have two equilibrium conditions each. From the free-body diagram of G_1 we obtain



$$\rightarrow: \quad -S_1 \cos\alpha + S_2 \cos\beta = 0\,,$$

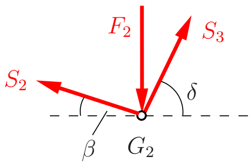
$$\uparrow: \quad S_1 \sin\alpha - S_2 \sin\beta - F_1 = 0\,.$$

This results in the relations

$$S_1 = S_2 \frac{\cos\beta}{\cos\alpha}\,,$$

$$S_2 \cos\beta \frac{\sin\alpha}{\cos\alpha} - S_2 \sin\beta - F_1 = 0 \quad \rightsquigarrow \quad S_2 = \frac{F_1}{\cos\beta\tan\alpha - \sin\beta}\,.$$

From the equilibrium conditions at G_2



$$\rightarrow: \quad -S_2 \cos\beta + S_3 \cos\delta = 0\,,$$

$$\uparrow: \quad S_2 \sin\beta + S_3 \sin\delta - F_2 = 0\,,$$

it follows

$$S_3 = S_2 \frac{\cos\beta}{\cos\delta}\,,$$

$$S_2 \sin\beta + S_2 \cos\beta \frac{\sin\delta}{\cos\delta} - F_2 = 0 \quad \rightsquigarrow \quad S_2 = \frac{F_2}{\sin\beta + \cos\beta\tan\delta}\,.$$

Thus, we have two solutions for S_2, one given as a function of the force F_1 and one as a function of F_2. The ratio of the forces F_1/F_2 results from equating the two solutions:

$$\frac{F_1}{\cos\beta\tan\alpha - \sin\beta} = \frac{F_2}{\sin\beta + \cos\beta\tan\delta}\,,$$

$$\leadsto \quad \underline{\underline{\frac{F_1}{F_2}}} = \frac{\cos\beta\tan\alpha - \sin\beta}{\sin\beta + \cos\beta\tan\delta} = \underline{\underline{\frac{\tan\alpha - \tan\beta}{\tan\beta + \tan\delta}}}\,.$$

From the geometry of the equilibrium position, we obtain for the angles:

$$\begin{aligned}
\tan\alpha &= \frac{3a}{2a} = \frac{3}{2}, \quad & \sin\alpha &= \frac{3}{\sqrt{13}}, \quad & \cos\alpha &= \frac{2}{\sqrt{13}}\,,\\
\tan\beta &= \frac{a}{3a} = \frac{1}{3}, \quad & \sin\beta &= \frac{1}{\sqrt{10}}, \quad & \cos\beta &= \frac{3}{\sqrt{10}}\,,\\
\tan\delta &= \frac{4a}{2a} = 2, \quad & \sin\delta &= \frac{2}{\sqrt{5}}, \quad & \cos\delta &= \frac{1}{\sqrt{5}}\,.
\end{aligned}$$

This results in

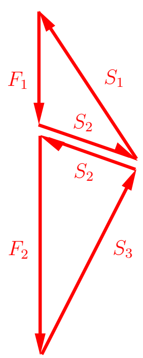
$$\underline{\underline{\frac{F_1}{F_2}}} = \frac{\frac{3}{2} - \frac{1}{3}}{\frac{1}{3} - 2} = \frac{9-2}{2+12} = \underline{\underline{\frac{1}{2}}}\,.$$

For the forces in the bars, it follows

$$\underline{\underline{S_2}} = \frac{F_1}{\frac{3}{\sqrt{10}}\frac{3}{2} - \frac{1}{\sqrt{10}}} = \frac{2\sqrt{10}}{7}F_1 = \underline{\underline{0.903\,F_1}}\,,$$

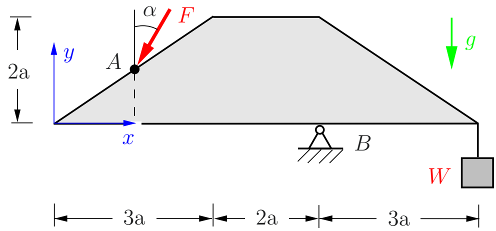
$$\underline{\underline{S_1}} = \frac{3}{\sqrt{10}}\frac{\sqrt{13}}{2}S_2 = \underline{\underline{1.545\,F_1}}\,, \qquad \underline{\underline{S_3}} = \frac{3}{\sqrt{10}}\frac{\sqrt{5}}{1}S_2 = \underline{\underline{1.916\,F_1}}\,.$$

The solution of the problem can also be obtained graphically. First we sketch the closed triangle of forces with F_1, S_1 and S_2 for G_1 with the given angles. Then we sketch the according triangle of forces for G_2 with the same force scaling (the magnitude of S_2 has to be equal and the direction of rotation must be opposite). From both triangles of forces we obtain by inspection the ratio of the magnitudes of the forces F_1 and F_2:



$$\underline{\underline{\frac{F_1}{F_2} \approx 0.5\,.}}$$

P1.22 **Problem 1.22** The given trapezoidal plate (weight is negligible) is loaded by a weight W and a force F such that it is in equilibrium.

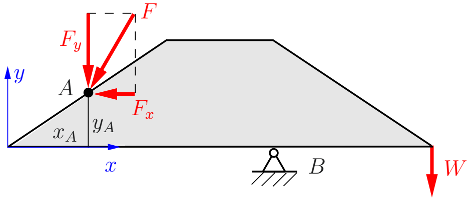


Calculate the coordinates of the loading point A at the edge of the plate.

Given: $W, F = \frac{3}{4}W, \alpha = 30°$

Solution We decompose the force F into its components

$$\begin{aligned} F_x &= F\sin\alpha = \frac{3}{4}W\sin 30° \\ &= \frac{3}{4}W\frac{1}{2} = \frac{3}{8}W, \\ F_y &= F\cos\alpha = \frac{3}{4}W\cos 30° \\ &= \frac{3}{4}W\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}W. \end{aligned}$$



Thus, the moment equilibrium condition with respect to B reads:

$$\overset{\curvearrowleft}{B}: \quad y_A F_x + (5a - x_A)F_y - 3aW = 0.$$

With

$$\frac{y_A}{x_A} = \frac{2a}{3a} \quad \leadsto \quad y_A = \frac{2}{3}x_A,$$

and the components of the force it follows

$$\frac{2}{3}x_A\frac{3}{8}G + (5a - x_A)\frac{3\sqrt{3}}{8}G - 3aW = 0.$$

After solving the equations, we obtain

$$\begin{aligned} (\frac{1}{4} - \frac{3\sqrt{3}}{8})x_A &= (3 - \frac{15\sqrt{3}}{8})a \\ \leadsto \quad \underline{\underline{x_A}} &= \frac{24 - 15\sqrt{3}}{2 - 3\sqrt{3}}a = \underline{\underline{0.619a}}. \end{aligned}$$

Hence, the y-coordinate is calculated as

$$\underline{\underline{y_A}} = \frac{2}{3}x_A = \underline{\underline{0.413a}}.$$

Chapter 2

2

Center of Gravity, Center of Mass, Centroids

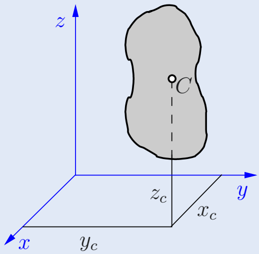
Centroid of a Volume

The coordinates of the Centroid of Volume of a body with volume V are given by

$$x_c = \frac{\int x \, \mathrm{d}V}{\int \mathrm{d}V}\,,$$

$$y_c = \frac{\int y \, \mathrm{d}V}{\int \mathrm{d}V}\,,$$

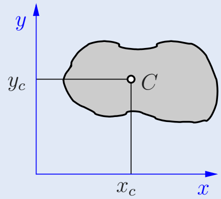
$$z_c = \frac{\int z \, \mathrm{d}V}{\int \mathrm{d}V}\,.$$



Centroid of an Area

$$x_c = \frac{\int x \, \mathrm{d}A}{\int \mathrm{d}A}\,,$$

$$y_c = \frac{\int y \, \mathrm{d}A}{\int \mathrm{d}A}\,.$$

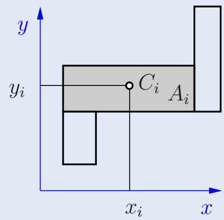


Here, $\int x \, \mathrm{d}A = C_y$ and $\int y \, \mathrm{d}A = C_x$ denote the *first moments of the area* with respect to the y- and x-axis, respectively.

For *composite* areas, where the coordinates (x_i, y_i) of the centroids C_i of the individual subareas A_i are known, we have

$$x_S = \frac{\sum x_i A_i}{\sum A_i}\,,$$

$$y_S = \frac{\sum y_i A_i}{\sum A_i}\,.$$



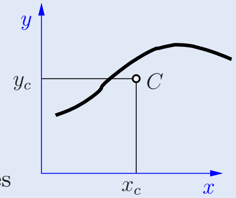
Remarks:

- When analyzing areas (volumes) with holes, it can be expedient to work with "negative" subareas (subvolumes).
- If the area (volume) has an axis of symmetry, the centroid of the area (volume) lies on this axis.

Centroid of a Line

$$x_c = \frac{\int x \, \mathrm{d}s}{\int \mathrm{d}s}\,,$$

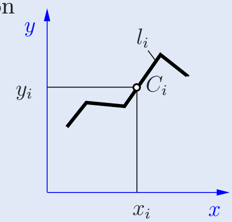
$$y_c = \frac{\int y \, \mathrm{d}s}{\int \mathrm{d}s}\,.$$



If a line is composed of several sublines of length l_i with the associated coordinates x_i, y_i of its centroids, the location of the centroid follows from

$$x_c = \frac{\sum x_i l_i}{\sum l_i}\,,$$

$$y_c = \frac{\sum y_i l_i}{\sum l_i}\,.$$



Center of Mass

The coordinates of the center of mass of a body with density $\rho(x, y, z)$ are given by

$$x_c = \frac{\int x\rho \mathrm{d}V}{\int \rho \, \mathrm{d}V}\,, \qquad y_c = \frac{\int y\rho \mathrm{d}V}{\int \rho \, \mathrm{d}V}\,, \qquad z_c = \frac{\int z\rho \mathrm{d}V}{\int \rho \, \mathrm{d}V}\,.$$

Consists a body of several subbodies V_i with (constant) densities ρ_i and associated known coordinates x_i, y_i, z_i, of the centroids of the subvolumes then it holds

$$x_c = \frac{\sum x_i \rho_i V_i}{\sum \rho_i V_i}\,, \qquad y_c = \frac{\sum y_i \rho_i V_i}{\sum \rho_i V_i}\,, \qquad z_c = \frac{\sum z_i \rho_i V_i}{\sum \rho_i V_i}\,.$$

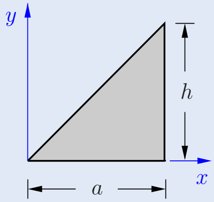
Remark:

For a *homogeneous* body ($\rho =$ const), the center of mass and the centroid of the volume coincide.

Location of Centroids

Areas

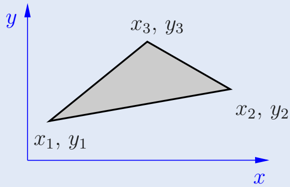
triangle



$x_c = \frac{2}{3}\, a$

$y_c = \frac{1}{3}\, h$

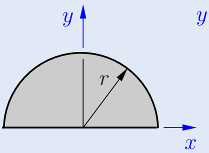
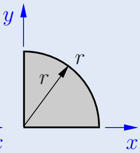
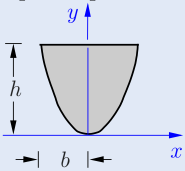
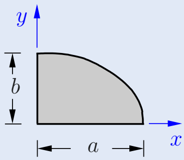
$A = \frac{1}{2}\, ah$



$x_c = \frac{1}{3}(x_1 + x_2 + x_3)$

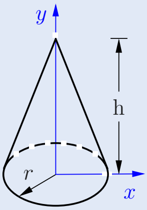
$y_c = \frac{1}{3}(y_1 + y_2 + y_3)$

$$A = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

semicircle	quater circle	quadr. parabola	quater ellipse
			
$x_c = 0$	$= \frac{4}{3\pi}\, r$	$= 0$	$= \frac{4}{3\pi}\, a$
$y_c = \frac{4}{3\pi}\, r$	$= \frac{4}{3\pi}\, r$	$= \frac{3}{5}\, h$	$= \frac{4}{3\pi}\, b$
$A = \frac{\pi}{2}\, r^2$	$= \frac{\pi}{4}\, r^2$	$= \frac{4}{3}\, bh$	$= \frac{\pi}{4}\, ab$

Volumes

cone

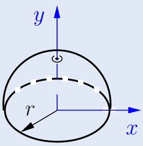


$x_c = 0$

$y_c = \frac{1}{4}\, h$

$V = \frac{1}{3}\, \pi r^2 h$

hemisphere



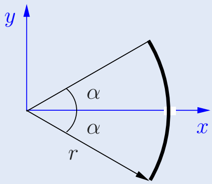
$x_c = 0$

$y_c = \frac{3}{8}\, r$

$V = \frac{2}{3}\, \pi r^3$

Line

circular arc



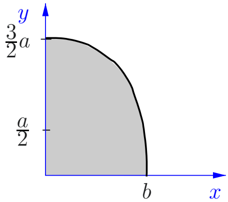
$x_c = \frac{\sin\alpha}{\alpha}\, r$

$y_c = 0$

$l = 2\,\alpha\, r$

Problem 2.1 The depicted area is bounded by the coordinate axes and the quadratic parabola with its apex at $x = 0$. **P2.1**

Determine the coordinates of the centroid.



Solution The equation of the parabola is given by

$$y = -\alpha\, x^2 + \beta\,.$$

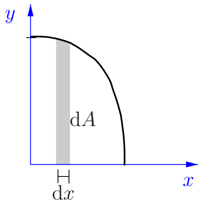
The constants α and β follow with the aid of the points $x_0 = 0$, $y_0 = 3a/2$ and $x_1 = b$, $y_1 = a/2$ as $\beta = 3a/2$ and $\alpha = a/b^2$. Thus, the equation of the can be written as

$$y = -a\left(\frac{x}{b}\right)^2 + \frac{3a}{2}\,.$$

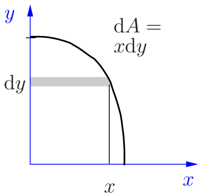
With the infinitesimal area $\mathrm{d}A = y\,\mathrm{d}x$, it follows

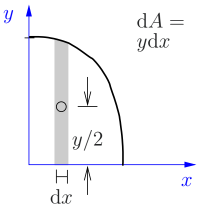
$$\underline{\underline{x_C}} = \frac{\int x\,\mathrm{d}A}{\int \mathrm{d}A} = \frac{\int x\,y\,\mathrm{d}x}{\int y\,\mathrm{d}x}$$

$$= \frac{\int\limits_0^b x\left[-a\left(\frac{x}{b}\right)^2 + \frac{3a}{2}\right]\mathrm{d}x}{\int\limits_0^b \left[-a\left(\frac{x}{b}\right)^2 + \frac{3a}{2}\right]\mathrm{d}x} = \frac{\frac{1}{2}\,ab^2}{\frac{7}{6}\,ab} = \underline{\underline{\frac{3}{7}\,b}}\,.$$



In order to determine the y-coordinate, we choose for simplicity again the infinitesimal area element $\mathrm{d}A = y\,\mathrm{d}x$ instead of $\mathrm{d}A = x\,\mathrm{d}y$,

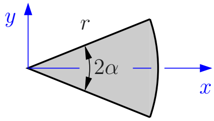




because we have already used it above. Now, we have to take into account that its centroid is located at the height y/2. Hence, we obtain

$$\underline{\underline{y_C}} = \frac{\int \frac{y}{2}\,y\,\mathrm{d}x}{\frac{7}{6}\,ab} = \frac{6}{14\,ab}\int\limits_0^b \left(a^2\,\frac{x^4}{b^4} + \frac{3a^2}{b^2}\,x^2 + \frac{9a^2}{4}\right)\mathrm{d}x = \underline{\underline{\frac{87}{140}\,a}}\,.$$

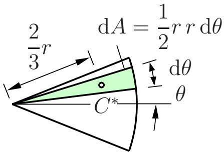
P2.2 **Problem 2.2** Locate the centroid of the depicted circular sector with the opening angle 2α.



Solution Due to symmetry reasons, we obtain $y_C = 0$. In order to determine x_C we use the infinitesimal sector of the circle (= triangle) and integrate over the angle θ

$$\underline{\underline{x_C}} = \frac{\int\limits_{-\alpha}^{\alpha} \left(\frac{2}{3}\, r\cos\theta\right) \frac{1}{2}\, r\, r\,\mathrm{d}\theta}{\int\limits_{-\alpha}^{\alpha} \frac{1}{2}\, r\, r\,\mathrm{d}\theta} = \frac{r^3\, 2\sin\alpha}{3\, r^2 \alpha}$$

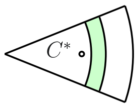
$$\underline{\underline{= \frac{2}{3}\,\frac{\sin\alpha}{\alpha}\, r\,.}}$$



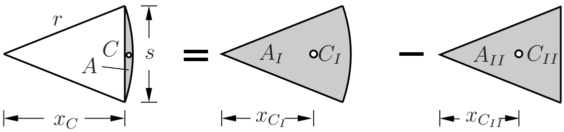
In the limit case of a semicircular area ($\alpha = \pi/2$), the centroid is located

$$\underline{\underline{x_C = \frac{4}{3\pi}\, r\,.}}$$

Remark: Alternatively, the determination of the centroid may be done by the decomposition of the area into circular rings and integration over x. In this case the centroid C^* of the circular rings has to be known or determined a priori.

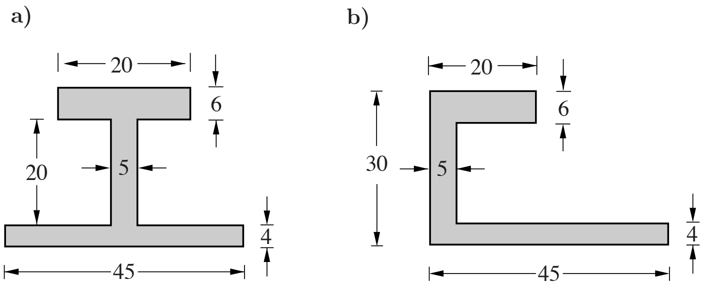


We may determine the centroid of a circular segment with the aid of the above calculations and by subtraction:



$$\underline{\underline{x_C}} = \frac{x_{C_I} A_I - x_{C_{II}} A_{II}}{A_I - A_{II}} = \frac{\dfrac{2\sin\alpha}{3\,\alpha}\, r\, r^2\alpha - \dfrac{1}{2}\, s\, r\cos\alpha\, \dfrac{2}{3}\, r\cos\alpha}{r^2\alpha - \dfrac{1}{2}\, s\, r\cos\alpha} = \underline{\underline{\frac{s^3}{12A}}}\,.$$

Problem 2.3 Locate the centroids of the depicted profiles. The measurements are given in mm. **P2.3**

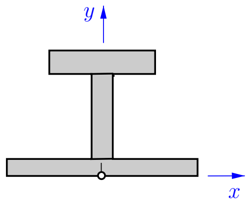


Solution a) The coordinate system is placed, such that the y-axis coincides with the symmetry axis of the system. Therefore, we know $x_C = 0$. In order to determine y_C, the system is decomposed into three rectangles with known centroids and it follows

$$\underline{\underline{y_C}} = \frac{\sum y_i A_i}{\sum A_i}$$

$$= \frac{2\,(4\cdot 45) + 14(5\cdot 20) + 27\,(6\cdot 20)}{4\cdot 45 + 5\cdot 20 + 6\cdot 20}$$

$$= \frac{5000}{400} = \underline{\underline{12.5\ \mathrm{mm}}}\,.$$



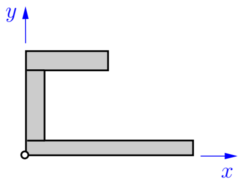
b) The origin of the coordinate system is placed in the lower left corner. Decomposition of the system into rectangles leads to

$$\underline{\underline{x_C}} = \frac{22.5\,(4\cdot 45) + 2.5\,(5\cdot 20) + 10\,(6\cdot 20)}{4\cdot 45 + 5\cdot 20 + 6\cdot 20}$$

$$= \frac{5500}{400} = \underline{\underline{13.75\ \mathrm{mm}}}\,,$$

$$\underline{\underline{y_C}} = \frac{2\,(4\cdot 45) + 14\,(5\cdot 20) + 27\,(6\cdot 20)}{400}$$

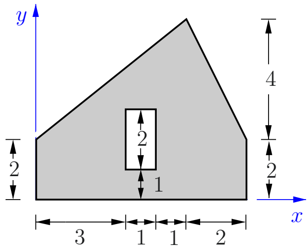
$$= \underline{\underline{12.5\ \mathrm{mm}}}\,.$$



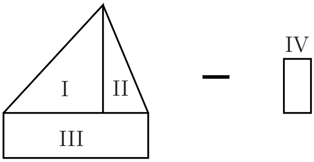
Remark: Note that a displacement of the system in the x-direction does not change the y-coordinate of the centroid.

P2.4

Problem 2.4 Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in cm.



Solution First we decompose the system into two triangles (I,II) and one rectangle (III), from which we subtract the rectangular cutout (IV). The centroids are known for each subsystem.



The calculation is conveniently done by using a table.

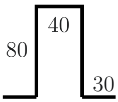
Sub-system i	A_i [cm^2]	x_i [cm]	$x_i A_i$ [cm^3]	y_i [cm]	$y_i A_i$ [cm^3]
I	10	$\frac{10}{3}$	$\frac{100}{3}$	$\frac{10}{3}$	$\frac{100}{3}$
II	4	$\frac{17}{3}$	$\frac{68}{3}$	$\frac{10}{3}$	$\frac{40}{3}$
III	14	$\frac{7}{2}$	49	1	14
IV	-2	$\frac{7}{2}$	-7	2	-4
	$A = \sum A_i = 26$		$\sum x_i A_i = 98$		$\sum y_i A_i = \frac{170}{3}$

Thus, we obtain

$$\underline{\underline{x_C}} = \frac{\sum x_i A_i}{A} = \frac{98}{26} = \underline{\underline{\frac{49}{13}\ \text{cm}}}\,, \qquad \underline{\underline{y_C}} = \frac{\sum y_i A_i}{A} = \frac{170/3}{26} = \underline{\underline{\frac{85}{39}\ \text{cm}}}\,.$$

Problem 2.5 A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. **P2.5**

Locate the centroid.



Solution We choose coordinate axes, such that y is the symmetry axis. Then, due to symmetry reasons, we can identify $x_C = 0$. The y-coordinate of the centroid follows generally by decomposition as

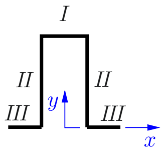
$$y_C = \frac{\sum y_i\, l_i}{\sum l_i}\,.$$

Three alternative solutions will be shown. The total length of the wire is

$$l = \sum l_i = 2 \cdot 30 + 2 \cdot 80 + 40 = 260 \text{ mm}\,.$$

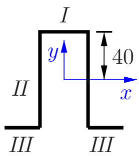
a)

$$\underline{\underline{y_C}} = \frac{1}{260}(\underbrace{80 \cdot 40}_{I} + \underbrace{2 \cdot 40 \cdot 80}_{II})$$
$$= \frac{9600}{260} = \underline{\underline{36.92 \text{ mm}}}\,.$$



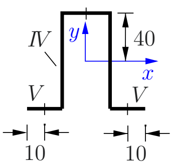
b)

$$\underline{\underline{y_C}} = \frac{1}{260}(\underbrace{40 \cdot 40}_{I} - \underbrace{2 \cdot 40 \cdot 30}_{III})$$
$$= \underline{\underline{-3.08 \text{ mm}}}\,.$$



c) We choose a specific subsystem IV such that its centroid coincides with the origin of the coordinate system:

$$\underline{\underline{y_C}} = \frac{1}{260}\left[\underbrace{2 \cdot (-40) \cdot 10}_{V}\right] = \underline{\underline{-3.08 \text{ mm}}}\,.$$

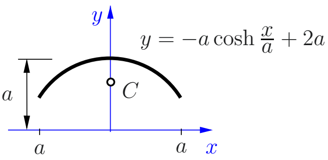


The advantage of alternative **c)** is, that only the first moment of subsystem V has to be taken into account.

P2.6

Problem 2.6 A thin wire is bent to a hyperbolic function.

Locate the centroid.



Solution The centroid is located on the y-axis due to the symmetry of the system ($x_C = 0$). We obtain the infinitesimal arc length $\mathrm{d}s$ with aid of the derivative $y' = -\sinh\frac{x}{a}$ as

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1+(y')^2}\mathrm{d}x = \sqrt{1+\sinh^2\frac{x}{a}}\mathrm{d}x = \cosh\frac{x}{a}\,\mathrm{d}x\,.$$

The total arc length follows by integration:

$$s = \int \mathrm{d}s = \int_{-a}^{+a} \cosh\frac{x}{a}\,\mathrm{d}x = 2\,a\sinh 1\,.$$

The first moment of the line with respect to the x-axis is given by

$$S_x = \int y\,\mathrm{d}s = a^2\left(4\,\sinh 1 - \frac{1}{2}\sinh 2 - 1\right)\,.$$

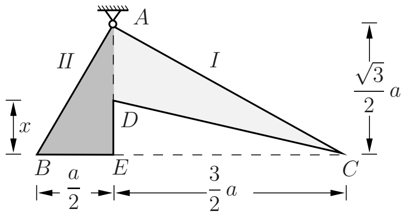
Hence, the centroid is located at

$$\underline{\underline{y_C}} = \frac{\int y\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{4\,a^2\sinh 1 - \frac{1}{2}\,a^2\sinh 2 - a}{2\,a\sinh 1} = \underline{\underline{0.803\,a}}\,.$$

P2.7

Problem 2.7 From the triangular-shaped metal sheet ABC, the triangle CDE has been cut out. The system is pin supported in A.

Determine x such that $\overline{BC}$ adjusts horizontal.

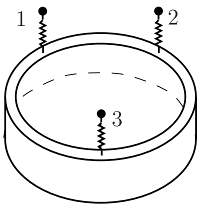


Solution The system is in the required position, if the centroid is located vertically below A. Consequently, the first moments of the triangular-shaped subsystems ADC and ABE have to be equal with respect to the point A:

$$\underbrace{\frac{1}{2}\left(\frac{\sqrt{3}}{2}\,a - x\right)\frac{3}{2}a}_{\text{area } ADC}\ \underbrace{\frac{1}{3}\,\frac{3}{2}\,a}_{\text{distance}} = \underbrace{\frac{1}{2}\,\frac{a}{2}\,\frac{\sqrt{3}}{2}\,a}_{\text{area } ABE}\ \underbrace{\frac{1}{3}\,\frac{a}{2}}_{\text{distance}} \quad \leadsto \quad \underline{\underline{x = \frac{4}{9}\sqrt{3}\,a\,.}}$$

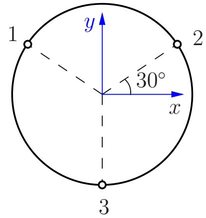
Problem 2.8 A piece of a pipe of weight W is fixed by three spring scales as depicted. The spring scales are equally distributed along the edge of the pipe. They measure the following forces:
$F_1 = 0.334\,W$, $F_2 = 0.331\,W$,
$F_3 = 0.335\,W$.

P2.8



Now an additional weight shall be attached to the pipe in order to shift the centroid of the total system into the center of the pipe (=static balancing). Determine the location and the magnitude of the additional weight.

Solution We know, due to the different measured forces, that the system is not balanced. Thus, the gravity center C (=location of the resulting weight) is not located in the middle of the ring, but coincides with the location of the resultant of the spring forces. Therefore, in a first step, we determine the location of the center of these forces. This can be done by the equilibrium of moments about the x- and y-axis:



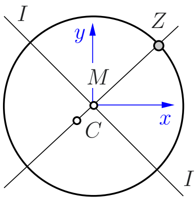
$$y_C\,W = r\sin 30^\circ(0.334\,W + 0.331\,W) - r\,0.335\,W\,,$$

$$\rightsquigarrow \quad y_C = -0.0025\,r\,,$$

$$x_C\,W = r\cos 30^\circ(0.331\,W - 0.334\,W)\,,$$

$$\rightsquigarrow \quad x_C = -0.0026\,r\,.$$

In order to recalibrate the gravity center into the center M of the ring, the additional required weight Z has to be applied on the intersection point of the ring and the line $\overline{CM}$. The weight of Z can be determined from the equilibrium of the moments about the perpendicular axis I:

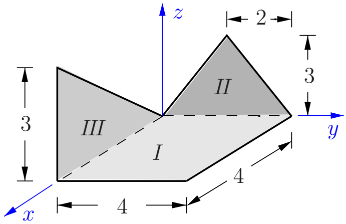


$$r\,Z = \overline{CM}\,W \quad \rightsquigarrow \quad r\,Z = \sqrt{x_C^2 + y_C^2}\,W$$

$$\rightsquigarrow \quad \underline{\underline{Z}} = \sqrt{(0.0025)^2 + (0.0026)^2}\,W = \underline{\underline{0.0036\,W}}\,.$$

P2.9 **Problem 2.9** A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in cm).

Locate the center of gravity.



Solution The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from

$$x_C = \frac{\sum \rho_i x_i V_i}{\sum \rho_i V_i}, \quad y_C = \frac{\sum \rho_i y_i V_i}{\sum \rho_i V_i}, \quad z_C = \frac{\sum \rho_i z_i V_i}{\sum \rho_i V_i}.$$

Since the thickness and the density of the sheet is constant, these terms cancel out and we obtain

$$x_C = \frac{\sum x_i A_i}{\sum A_i}, \quad y_C = \frac{\sum y_i A_i}{\sum A_i}, \quad z_C = \frac{\sum z_i V_i}{\sum A_i}.$$

The total area is

$$A = \sum A_i = 4 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 3 + \frac{1}{2} \cdot 4 \cdot 3 = 28 \text{ cm}^2.$$

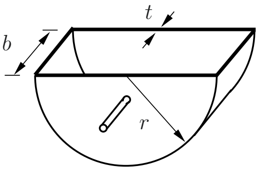
Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_{II} = 0$, $y_{III} = 0$, $z_I = 0$. Thus, we obtain

$$\underline{\underline{x_C}} = \frac{x_I A_I + x_{III} A_{III}}{A} = \frac{2 \cdot 16 + (\frac{2}{3} \cdot 4)\, 6}{28} = \underline{\underline{1.71 \text{ cm}}},$$

$$\underline{\underline{y_C}} = \frac{y_I A_I + y_{II} A_{II}}{A} = \frac{2 \cdot 16 + 2 \cdot 6}{28} = \underline{\underline{1.57 \text{ cm}}},$$

$$\underline{\underline{z_C}} = \frac{z_{II} A_{II} + z_{III} A_{III}}{A} = \frac{(\frac{1}{3} \cdot 3)\, 6 + (\frac{1}{3} \cdot 3)\, 6}{28} = \underline{\underline{0.43 \text{ cm}}}.$$

Problem 2.10 A semi-circular bucket is produced from a steel sheet with the thickness t and density ρ_S. **P2.10**



a) Determine the required distance of the bearing pivots to the upper edge, such that it is easy to turn the empty bucket around the pivots.

b) Consider a steel bucket which is filled with material of the density ρ_M. How does this change the required distance of the pivots?

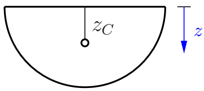
Given: $b = r$, $t = r/100$, $\rho_M = \rho_S/3$

Solution The bucket tilts easiest by positioning the pivots in the axis of the center of mass.

a) In case of an empty bucket (=homogeneous body), the center of mass coincides with the center of volume. Since the sheet thickness is constant, it cancels out. With the centroids of the subareas

semi circle $\quad z_1 = \dfrac{4\,r}{3\,\pi}$

semi circular arc $\quad z_2 = \dfrac{2r}{\pi}$



we obtain

$$\underline{\underline{z_{C_E}}} = \frac{z_1 A_1 + z_2 A_2}{A_1 + A_2} = \frac{\dfrac{4r}{3\pi}\,2\,\dfrac{\pi r^2}{2} + \dfrac{2r}{\pi}\pi r b}{2\,\dfrac{\pi r^2}{2} + \pi r b} = \underline{\underline{\frac{4\,r + 6\,b}{3\,\pi(r+b)}\,r}}\,.$$

b) In case of the filled bucket, we obtain with the mass of the steel bucket $m_S = \pi\left(r^2 + rb\right) t\rho_S$ and the mass of the filling $m_M = \frac{1}{2}\pi r^2 b\,\rho_M$ the location of the mass center as

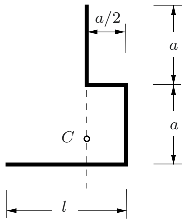
$$\underline{\underline{z_{C_F}}} = \frac{z_{C_E} m_S + \dfrac{4r}{3\pi} m_M}{m_S + m_M} = \underline{\underline{\frac{4\,(2r+3b)\,t\,\rho_S + 4\,rb\,\rho_M}{3\pi\left[2\,(r+b)\,t\,\rho_S + r\,b\,\rho_M\right]}\,r}}\,.$$

Using the given data $b = r$, $t = r/100$, $\rho_M = \rho_S/3$, it follows

$$z_{C_E} = \frac{10}{3\pi\cdot 2}\,r = 0.53\,r\,, \qquad z_{C_F} = \frac{4\cdot 5\,\dfrac{1}{100} + 4\cdot\dfrac{1}{3}}{3\pi\left(4\cdot\dfrac{1}{100} + \dfrac{1}{3}\right)}\,r = 0.44\,r\,.$$

Remark: Since the mass of the filling is much bigger than the mass of the bucket, we find the common center of mass close to the center of mass of the filling: $z_{C_M} = 4r/(3\pi) = 0.424\,r$.

P2.11 **Problem 2.11** The depicted stirrer consists of a homogenous wire that rotates about the sketched vertical axis.
Determine the length l, such that the center of mass C is located on the rotation axis.

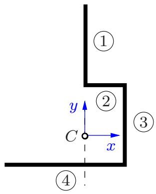


Solution Using the given coordinate system and decomposing the stirrer into four subparts, we obtain the center of mass from

$$x_C = \frac{\sum x_i\, l_i}{\sum l_i}\,.$$

For convenience, we use a table.

i	l_i	x_i	$x_i\, l_i$
1	a	0	0
2	$\frac{a}{2}$	$\frac{a}{4}$	$\frac{a^2}{8}$
3	a	$\frac{a}{2}$	$\frac{a^2}{2}$
4	l	$\frac{a}{2} - \frac{l}{2}$	$\frac{al}{2} - \frac{l^2}{2}$
$\sum$	$\frac{5a}{2} + l$	–	$\frac{5a^2}{8} + \frac{al}{2} - \frac{l^2}{2}$



The centroid shall lie on the rotation axis. Therefore, from the condition $x_C = 0$, follows the quadratic equation

$$\sum x_i\, l_i = \frac{5a^2}{8} + \frac{al}{2} - \frac{l^2}{2} = 0 \quad \rightsquigarrow \quad l^2 - al - \frac{5a^2}{4} = 0\,.$$

It has two solutions

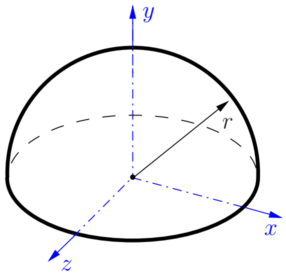
$$l_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + \frac{5a^2}{4}} = \frac{a}{2} \pm \frac{\sqrt{6}}{2} a\,,$$

from which only the positive one is physically reasonable:

$$\underline{\underline{l = \frac{a}{2}\,(1 + \sqrt{6})\,.}}$$

Problem 2.12 Determine the location of the centroid for the depicted surface of a hemisphere with the radius r.

P2.12

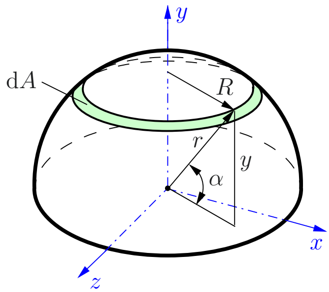


Solution We choose the coordinate system, such that the y-axis coincides with the symmetry axis. Therefore, we know:

$$\underline{\underline{x_C = 0}}, \quad \underline{\underline{z_C = 0}}.$$

T he remaining coordinate y_C, follows from

$$y_C = \frac{\int y \, \mathrm{d}A}{\int \mathrm{d}A}.$$



As infinitesimal area element, we choose the circular ring with the width $r\,\mathrm{d}\alpha$ and the circumference $2\pi\,R$ as our infinitesimal area element:

$$\mathrm{d}A = 2\pi\,R\,r\,\mathrm{d}\alpha.$$

Using $R = r\,\cos\alpha$ and $y = r\,\sin\alpha$, it follows

$$\mathrm{d}A = 2\pi\,r^2 \cos\alpha\,\mathrm{d}\alpha.$$

Now, we can determine the surface area as

$$A = \int \mathrm{d}A = 2\pi\,r^2 \int_{\alpha=0}^{\pi/2} \cos\alpha\,\mathrm{d}\alpha = 2\pi\,r^2\,\sin\alpha\,\Big|_0^{\pi/2} = 2\pi\,r^2$$

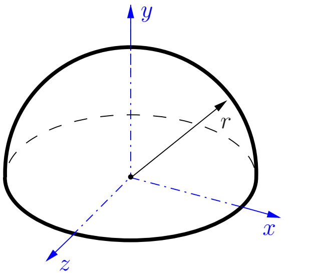
and the first moment of the area as

$$\int y\,\mathrm{d}A = 2\pi\,r^3 \int_{\alpha=0}^{\pi/2} \sin\alpha\,\underbrace{\cos\alpha\,\mathrm{d}\alpha}_{\mathrm{d}\sin\alpha} = 2\pi\,r^3 \frac{1}{2}\sin^2\alpha\,\Big|_0^{\pi/2} = \pi\,r^3.$$

Thus, the location of the centroid results as

$$\underline{\underline{y_C}} = \frac{1}{A}\int y\,\mathrm{d}A = \underline{\underline{\frac{r}{2}}}.$$

P2.13 **Problem 2.13** Determine the center of the volume for the depicted hemisphere of radius r.

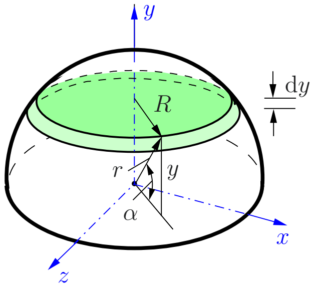


Solution Due to the axisymmetric geometry, we know

$$\underline{\underline{x_c = 0}}, \quad \underline{\underline{z_c = 0}}.$$

The remaining coordinate is determined from

$$y_C = \frac{\int y \, \mathrm{d}V}{\int \mathrm{d}V}.$$



As infinitesimal volume element we select the circular disk with radius R and thickness $\mathrm{d}y$:

$$\mathrm{d}V = R^2 \, \pi \, \mathrm{d}y \, .$$

By parametrization of the radius R and coordinate y

$$R = r \, \cos\alpha \, , \quad y = r \, \sin\alpha \quad \leadsto \quad \mathrm{d}y = r \, \cos\alpha \, \mathrm{d}\alpha \, ,$$

the volume of the hemisphere follows as

$$V = \int \mathrm{d}V = \int_{\alpha=0}^{\pi/2} \pi \, r^3 \, \cos^3\alpha \, \mathrm{d}\alpha = \int_{\alpha=0}^{\pi/2} \pi \, r^3 \, (1 - \sin^2\alpha) \underbrace{\cos\alpha \, \mathrm{d}\alpha}_{\mathrm{d}\sin\alpha}$$

$$= \pi \, r^3 \left(\sin\alpha - \frac{\sin^3\alpha}{3} \right) \Bigg|_0^{\pi/2} = \frac{2}{3} \, \pi \, r^3 \, .$$

With the the first moment of the area as

$$\int y \, \mathrm{d}V = \pi \, r^4 \int_{\alpha=0}^{\pi/2} \cos^3\alpha \underbrace{\sin\alpha \, \mathrm{d}\alpha}_{-\mathrm{d}\cos\alpha} = -\frac{\pi \, r^4}{4} \cos^4\alpha \, \Bigg|_0^{\pi/2} = \frac{\pi \, r^4}{4} \, ,$$

the center of the volume is determined as

$$\underline{\underline{y_C}} = \frac{1}{V} \int y \, \mathrm{d}V = \frac{\pi r^4}{4} \, \frac{3}{2\pi \, r^3} = \underline{\underline{\frac{3}{8} \, r}} \, .$$

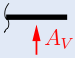
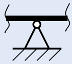
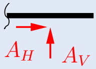
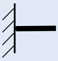
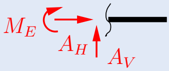
Chapter 3

Support Reactions

3

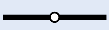
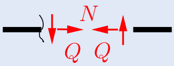
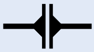
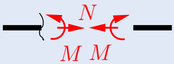
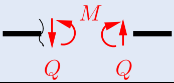
Plane Structures

In coplanar systems, we have 3 equilibrium conditions. Thus, we have maximum 3 support reactions in a statically and kinematically determinate structure. We distinguish the following supports:

Technical term	Symbol	Support Reactions
simple support		 A_V
hinged support		 A_H A_V
clamped support		 M_E A_H A_V

Note: At a *free end*, no *force* and no *moment* are acting.

Between 2 parts of a structure, the following connecting members can be present:

Technical term	Symbol	Transfered Re-actions
hinge		 N Q Q
parallel motion		 N M M
sliding sleeve		 M Q Q
hinged support		 N

With the degrees of freedom f, the number of support reactions r, the number of joint reactions v, and the number parts n, the following relation holds:

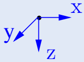
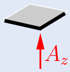
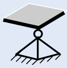
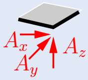
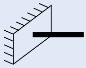
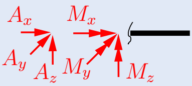
$$f = 3n - (r + v)\,.$$

Note:

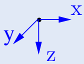
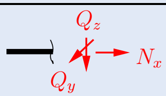
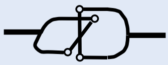
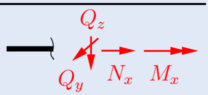
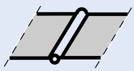
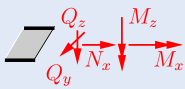
$$f \begin{cases} > 0: & f\text{-times movable,} \\ = 0: & \text{statically determinate (necessary condition),} \\ < 0: & f\text{-times statically indeterminate.} \end{cases}$$

Spatial Structures

In spatial systems, we have 6 equilibrium conditions. Thus, we have 6 support reactions in a statically and kinematically determinate structure. We distinguish the following supports:

Technical term	Symbol	Support Reactions 
simple support		
hinged support		
clamped support		

Between 2 parts of a structure, the following connecting members can be present:

Technical term	Symbol	Transfered Reactions 
hinge		
Cardan joint (U-joint)		
(door) hinge		

With the degrees of freedom f, the number of support reactions r, the number of joint reactions v, and the number parts n, the following relation holds:

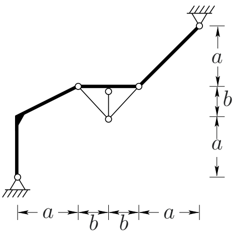
$$f = 6\,n - (r + v)\,.$$

Note:

$$f \begin{cases} > 0: & f\text{-times movable,} \\ = 0: & \text{statically determinate (only necessary condition),} \\ < 0: & f\text{-times statically indeterminate.} \end{cases}$$

P3.1 **Problem 3.1** Verify if the following structures are statically determinate and serviceable (=kinematically determinate). Note that kinematic determinacy is independent from static determinacy. For movable systems ($f > 0$), kinematic determinacy may be excluded a priori, but in case of static determinacy ($f = 0$) or indeterminacy ($f < 0$), the kinematic determinacy has to be investigated separately.

a)

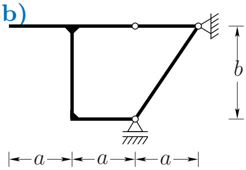


Solution We have $n = 3$ (bodies), $r = 4$ support reactions and $v = 2 \cdot 2 + 3 = 7$ (2 hinges and 3 bars), we obtain

$$f = 3 \cdot 3 - (4 + 7) = -2\,.$$

Therefore, the system is statically indeterminate. That it is also kinematically indeterminate can be recognized by considering the middle beam between the 2 hinges together with the 3 bars as a single rigid body. The we obtain with $n = 3$, $r = 4$ and $v = 2 \cdot 2$ $f = 1$. Therefore, the system obtains 1 kinematic degree of freedom and is movable, i.e. not serviceable.

b)

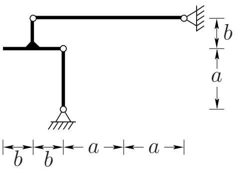


Solution With $n = 3$ (bodies), $r = 3$ support reactions and $v = 6$ (3 joints), it follows

$$f = 3 \cdot 3 - (3 + 6) = 0\,.$$

This system is statically determined and serviceable. Here an immobile three-hinged is attached on an also immobile beam.

c)

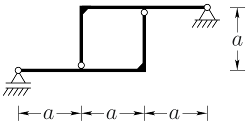


Solution The system consists of $n = 3$ beams/frames, contains $r = 4$ support reactions and $v = 4$ joint reactions (2 hinges). Thus we obtain

$$f = 3 \cdot 3 - (4 + 4) = 1\,.$$

The system is movable and therefore not serviceable.

d)

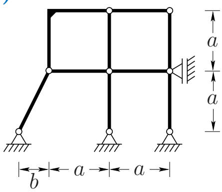


Solution With $n = 2$ frame parts, $r = 3$ support reactions and $v = 4$ joint reactions (2 joints), we obtain

$$f = 3 \cdot 2 - (3 + 4) = -1\,.$$

The parts of the frame are joined immovable, such that the system can be considered as a single rigid, immovable body. This system is statically determinate supported. These kinds of systems are externally statically determinate and internally statically indeterminate.

e)

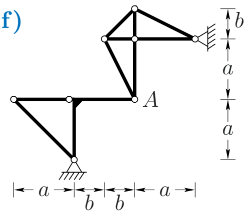


Solution Here we have $n = 9$, $r = 7$ and $v = 20$ (note: each additional beam connected at a joint grings 2 additional joint reaction). We obtain

$$f = 3 \cdot 9 - (7 + 20) = 0\,.$$

This system is statically determinate and serviceable. The lower right vertical beam is fixed, by its supports. To the left side of that beam, two immobile three-hinged frames are attached. This system is extended by two additional three-hinged frames on the top.

f)



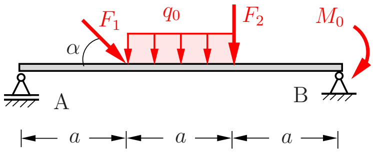
Solution In this case we have

$$f = 3 \cdot 10 - (4 + 26) = 0\,.$$

Although the depicted system is statically determined, it is not serviceable, because an infinitesimal movement is possible. Note that both supports and the joint A, which connects both subsystems, are on a straight line. This results in a very “soft” construction which is not serviceable.

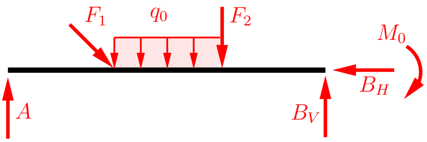
P3.2

Problem 3.2 Determine the support reactions for the depicted system.



Given: $F_1 = 2\,\text{kN}$, $F_2 = 3\,\text{kN}$,
$a = 1\,\text{m}$, $M_0 = 4\,\text{kNm}$,
$q_0 = 5\,\text{kN/m}$, $\alpha = 45°$.

Solution The beam is statically and kinematically determinate. We free the beam from its supports and make the reaction forces visible in the free-body diagram:



We write down the equilibrium conditions

$$\stackrel{\curvearrowleft}{A}: \quad 3a\,B_V - M_0 - 2a\,F_2 - \frac{3}{2}a\,(q_0 a) - a\,F_1 \sin\alpha = 0\,,$$

$$\stackrel{\curvearrowleft}{B}: \quad -3a\,A + 2a\,F_1 \sin\alpha + \tfrac{3}{2}a\,(q_0 a) + a\,F_2 - M_0 = 0\,,$$

$$\rightarrow: \quad F_1 \cos\alpha - B_H = 0\,.$$

They lead to

$$\underline{\underline{B_V}} = \frac{4 + 6 + \dfrac{3}{2}\cdot 5 + 2\cdot\dfrac{1}{2}\sqrt{2}}{3} = \underline{\underline{6.30\ \text{kN}}}\,,$$

$$\underline{\underline{A}} = \frac{2\cdot 2\cdot\dfrac{1}{2}\sqrt{2} + \dfrac{3}{2}\cdot 5 + 3 - 4}{3} = \underline{\underline{3.11\ \text{kN}}}\,,$$

$$\underline{\underline{B_H}} = 2\cdot\frac{1}{2}\sqrt{2} = \underline{\underline{1.41\ \text{kN}}}\,.$$

As a check, we use the force equilibrium in the vertical direction:

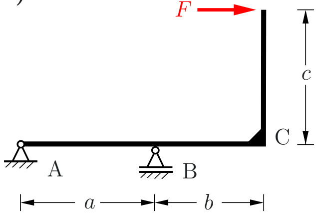
$$\uparrow: \quad A + B_V - F_1 \sin\alpha - q_0\,a - F_2 = 0\,,$$

$$\leadsto \quad 3.11 + 6.30 - 2\cdot 0.71 - 5 - 3 = 0\,.$$

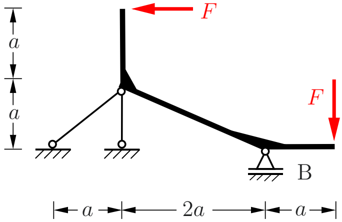
Remark: Note that the support reactions are given with an accuracy of only two digits after the decimal point. Therefore, this equation is not satisfied exactly.

Problem 3.3 Determine the support reactions for the depicted systems **P3.3**

a)



b)



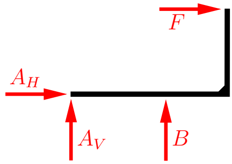
Solution The free-body diagrams are used to formulate the equilibrium conditions from which the support reactions are determined. Additionally, we check the obtained results by an additional equilibrium condition.

a)

$$\overset{\curvearrowleft}{A}: \quad a\,B - c\,F = 0 \quad \rightsquigarrow \quad \underline{\underline{B = \frac{c}{a}\,F}}\,,$$

$$\overset{\curvearrowleft}{B}: \quad -a\,A_V - c\,F = 0 \quad \rightsquigarrow \quad \underline{\underline{A_V = -\frac{c}{a}\,F}}\,,$$

$$\rightarrow: \quad A_H + F = 0 \quad \rightsquigarrow \quad \underline{\underline{A_H = -F}}\,.$$



Check:

$$\overset{\curvearrowleft}{C}: \quad -(a+b)\,A_V - b\,B - c\,F = 0$$

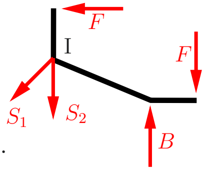
$$\rightsquigarrow \quad (c + \frac{b}{a}\,c)\,F - b\,\frac{c}{a}\,F - c\,F = 0\,.$$

b)

$$\overset{\curvearrowleft}{I}: \quad 2a\,B + a\,F - 3a\,F = 0 \quad \rightsquigarrow \quad \underline{\underline{B = F}}\,,$$

$$\rightarrow: \quad -F - S_1 \cos 45^\circ = 0 \quad \rightsquigarrow \quad \underline{\underline{S_1 = -\sqrt{2}F}}\,,$$

$$\uparrow: \quad B - F - S_2 - S_1 \sin 45^\circ = 0 \quad \rightsquigarrow \quad \underline{\underline{S_2 = F}}\,.$$



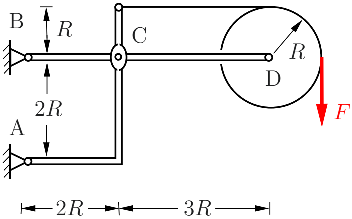
Check:

$$\overset{\curvearrowleft}{B}: \quad 2a\,S_2 + a\,S_1 \cos 45^\circ + 2a\,S_1 \sin 45^\circ + 2a\,F - a\,F = 0$$

$$\rightsquigarrow \quad 2a\,F - a\,F - 2a\,F + 2a\,F - a\,F = 0\,.$$

P3.4

Problem 3.4 Determine the support reactions of the depicted system. Neglect the friction of the pulley.



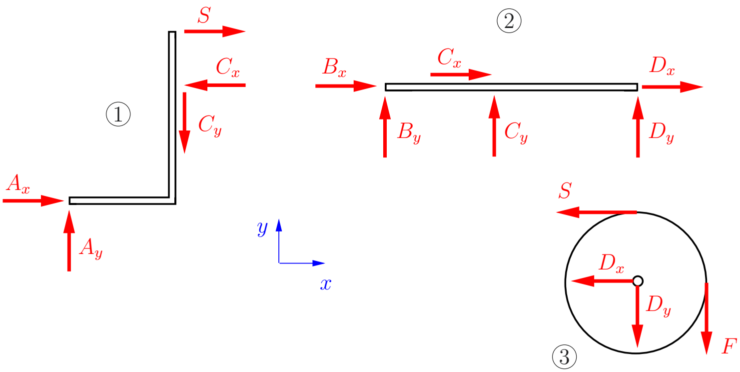
Solution First, we verify that the structure is statically determinate. The system consists of

$r = 4$	support reactions (2 force components at A and 2 components at B),
$n = 3$	bodies,
$v = 5$	transferred joint reactions (2 reactions at C, 2 reactions at D and the force in the rope).

Thus, the condition

$$f = \underbrace{3 \cdot 3}_{3\,n} - (\underbrace{4}_{r} + \underbrace{5}_{v}) = 0$$

is satisfied. We isolate the three bodies and obtain the sketched free-body diagrams.



The, the equilibrium equations read for the pulley 3

$$\overset{\curvearrowleft}{D}: \quad R\,S = R\,F \quad \rightsquigarrow \quad S = F\,,$$

$$\uparrow: \quad D_y = -F\,,$$

$$\rightarrow: \quad D_x = -F\,,$$

for the angled beam (1)

$$
\begin{aligned}
\overset{\curvearrowleft}{A}: \quad & 2R\,C_x - 2R\,C_y - 3R\,S = 0\,,\\
\uparrow: \quad & A_y = C_y\,,\\
\rightarrow \quad & A_x = C_x - S
\end{aligned}
$$

and for the beam (2) (using the results from the pulley)

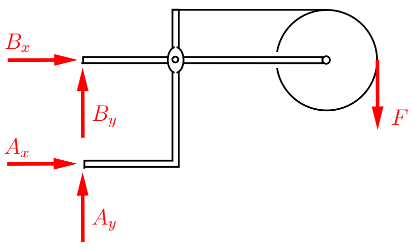
$$
\begin{aligned}
\overset{\curvearrowleft}{D}: \quad & -5R\,B_y - 3R\,C_y = 0\,,\\
\uparrow: \quad & B_y + C_y - F = 0\,,\\
\rightarrow: \quad & B_x + C_x - F = 0\,.
\end{aligned}
$$

The four support reactions and the two joint reactions at C can be calculated from the last six equations:

$$
\underline{\underline{B_y = -\frac{3}{2}\,F}}\,, \qquad C_y = \underline{\underline{A_y = \frac{5}{2}\,F}}\,,
$$

$$
C_x = 4\,F\,, \qquad \underline{\underline{B_x = -3\,F}}\,, \qquad \underline{\underline{A_x = 3\,F}}\,.
$$

Note that the support reactions in the horizontal direction can also be determined from the equilibrium condition for the complete system:



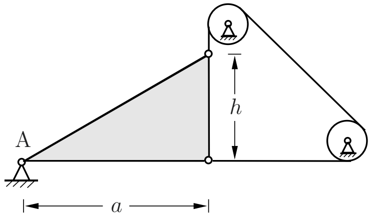
$$
\begin{aligned}
\overset{\curvearrowleft}{A}: \quad & 6R\,F + 2R\,B_x = 0 \quad \rightsquigarrow \quad B_x = -3F\,,\\
\rightarrow: \quad & A_x + B_x = 0 \quad \rightsquigarrow \quad A_x = 3F\,.
\end{aligned}
$$

In order to find A_y and B_y, we have to cut through the structure anyway.

P3.5

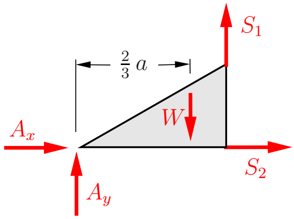
Problem 3.5

A homogenous triangular-shaped plate (specific weight per unit thickness ρg) is hold in the depicted position.



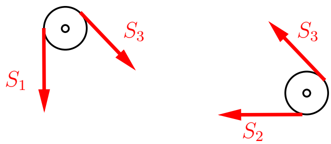
Determine the force in the rope and the support reactions. Neglect the the friction of the pulleys.

Solution Isolating the triangular plate



and sketching the free-body, we recognize the four unknown forces A_x, A_y, S_1, S_2. Using the equilibrium of the rolls, we obtain

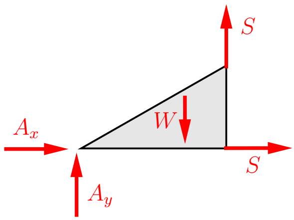
$$\left.\begin{aligned} S_3 &= S_1 \\ S_3 &= S_2 \end{aligned}\right\} \quad \rightsquigarrow \quad S_1 = S_2\,.$$



With that the number of unknowns is reduced to 3, since $S_1 = S_2 = S$. The resulting weight

$$W = \frac{1}{2}\,ah\rho g$$

acts in the centroid at distance $\frac{2}{3}\,a$ from support A. Thus, we obtain the equilibrium equations



$$\overset{\curvearrowleft}{A}: \quad \frac{2}{3}a\,W - a\,S = 0\,,$$

$$\uparrow: \quad A_y - W + S = 0\,,$$

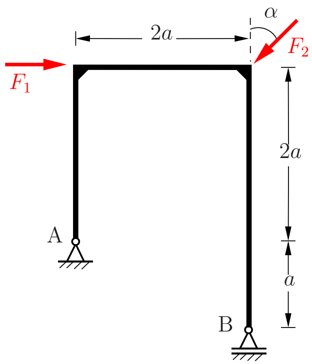
$$\rightarrow: \quad A_x + S = 0$$

and the demanded forces result as

$$\underline{\underline{S}} = \frac{2}{3}\,W = \underline{\underline{\frac{1}{3}\,ah\rho g}}\,, \qquad \underline{\underline{A_y}} = \frac{1}{3}\,W = \underline{\underline{\frac{1}{6}\,ah\rho g}}\,, \qquad \underline{\underline{A_x = -\frac{1}{3}\,ah\rho g}}\,.$$

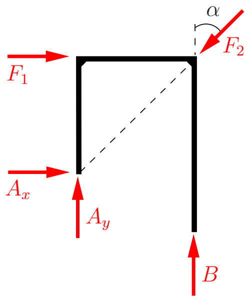
Problem 3.6 Determine the support reactions for the depicted frame.

P3.6



Given: $F_1 = 2000$ N,
$F_2 = 3000\sqrt{2}$ N,
$\alpha = 45°$,
$a = 5$ m.

Solution The free-body diagram shows that the line of action of F_2 passes through the support A. Thus, the equilibrium condition of the moments with respect to A yields



$$\overset{\curvearrowleft}{A}: \quad 2a\,B - 2a\,F_1 = 0 \quad \rightsquigarrow \quad B = F_1\,.$$

Additionally, we obtain from the equilibrium of forces

$$\uparrow: \quad A_y + B - F_2 \cos\alpha = 0 \quad \rightsquigarrow \quad A_y = F_2 \cos\alpha - F_1\,,$$

$$\rightarrow: \quad A_x + F_1 - F_2 \sin\alpha = 0 \quad \rightsquigarrow \quad A_x = F_2 \sin\alpha - F_1\,.$$

Inserting the numerical values, yields

$$\underline{\underline{A_x}} = 3000\sqrt{2}\,\frac{1}{2}\,\sqrt{2} - 2000 = \underline{\underline{1000\text{ N}}}\,,$$

$$\underline{\underline{A_y}} = 3000\sqrt{2}\,\frac{1}{2}\,\sqrt{2} - 2000 = \underline{\underline{1000\text{ N}}}\,,$$

$$\underline{\underline{B}} = \underline{\underline{2000\text{ N}}}\,.$$

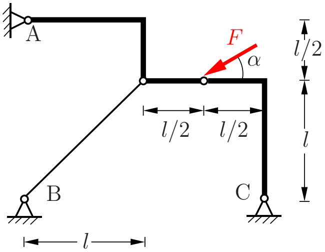
Check:

$$\overset{\curvearrowleft}{B}: \quad 3a\,F_2 \sin\alpha - 3a\,F_1 - a\,A_x - 2a\,A_y = 0$$

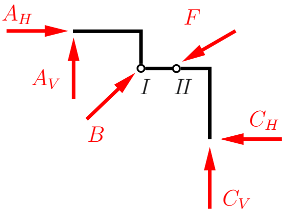
$$\rightsquigarrow \quad 15 \cdot 3000\,\sqrt{2}\,\frac{1}{2}\,\sqrt{2} - 15 \cdot 2000 - 5 \cdot 1000 - 10 \cdot 1000 = 0\,.$$

P3.7

Problem 3.7 Determine the support reactions for the depicted frame.
Given: $\alpha = 30°$



Solution The sketch of the free-body diagram shows the 5 unknown support reactions: 2 force components each in A and in C, and the force B in the bar (here assumed to be a compressive force).



First, we write down the equilibrium conditions for the complete system (the hinges are assumed to be frozen):

$$\stackrel{\curvearrowleft}{A}: \quad l\,B\frac{1}{2}\sqrt{2} + \frac{l}{2}\,B\frac{1}{2}\sqrt{2} - \frac{3}{2}l\,\frac{1}{2}\,F - \frac{1}{2}l\,\frac{\sqrt{3}}{2}\,F$$
$$-\frac{3}{2}l\,C_H + 2l\,C_V = 0\,,$$
$$\uparrow: \quad B\,\frac{1}{2}\sqrt{2} + C_V + A_V - F\,\frac{1}{2} = 0\,,$$
$$\rightarrow: \quad A_H + B\,\frac{1}{2}\sqrt{2} - C_H - F\,\frac{\sqrt{3}}{2} = 0\,,$$

Then, we use moment equilibrium conditions for the part to the left of hinge I and the part to the right of hinge II, respectively:

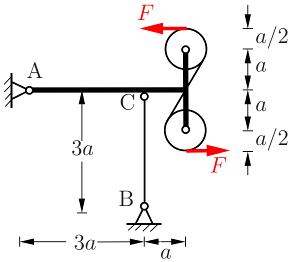
$$\stackrel{\curvearrowleft}{I}: \quad -l\,A_V - \frac{l}{2}\,A_H = 0\,, \qquad \stackrel{\curvearrowleft}{II}: \quad \frac{l}{2}\,C_V - l\,C_H = 0\,.$$

Solving these 5 equations for the 5 unknowns, yields

$$\underline{\underline{A_H = 0.7440\,F}}\,, \quad \underline{\underline{A_V = -0.372\,F}}\,, \quad \underline{\underline{B = 0.5261\,F}}\,,$$

$$\underline{\underline{C_H = \frac{F}{4}}}\,, \quad \underline{\underline{C_V = \frac{F}{2}}}\,.$$

Problem 3.8 The sketched system can be used to determine the force F in the rope, if a suitable measuring device is attached to the vertical bar $\overline{BC}$. **P3.8**



Determine under the assumption of a frictionless rope

a) the support reaction in A and B,

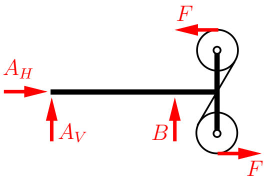
b) the support reaction of the rolls.

Solution **a)** The part $\overline{BC}$ is a hinged column. The 3 support reactions follow from the equilibrium equations

$$\rightarrow: \quad A_H = 0\,,$$

$$\uparrow: \quad A_V + B = 0\,,$$

$$\overset{\curvearrowleft}{A}: \quad 3aB + 3aF = 0,$$

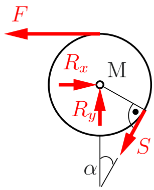


as

$$\underline{\underline{B = -F}}\,, \quad \underline{\underline{A_V = F}}\,, \quad \underline{\underline{A_H = 0}}\,.$$

b) We isolate on of the rolls and introduce the support reactions R_x and R_y. From the given geometry follows the auxiliary angle α:

$$\sin\alpha = \frac{a/2}{a} = \frac{1}{2} \quad \rightsquigarrow \quad \alpha = 30^\circ\,.$$



Thus, the equilibrium equations yield finally:

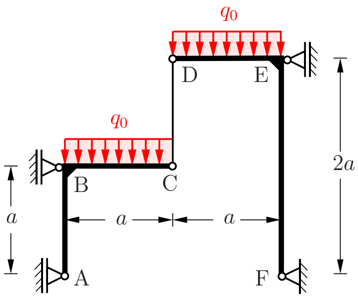
$$\overset{\curvearrowleft}{M}: \quad \frac{a}{2}F - \frac{a}{2}S = 0 \qquad \rightsquigarrow \quad \underline{\underline{S = F}}\,,$$

$$\uparrow: \quad R_y - S\cos\alpha = 0 \qquad \rightsquigarrow \quad \underline{\underline{R_y = \frac{1}{2}\sqrt{3}\,F}}\,,$$

$$\rightarrow: \quad R_x - S\sin\alpha - F = 0 \quad \rightsquigarrow \quad \underline{\underline{R_x = \frac{3}{2}\,F}}\,.$$

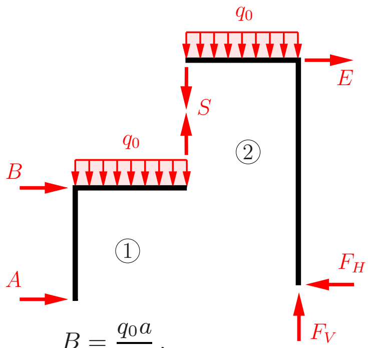
P3.9

Problem 3.9 Determine all support reactions for the depicted structure.



Solution The subsystems $\overline{ABC}$ and $\overline{DEF}$ are connected by the hinged column $\overline{CD}$. With $n = 2$, $v = 1$ and $r = 3 \cdot 1 + 1 \cdot 2 = 5$ we obtain $f = 3 \cdot 2 - (5 + 1) = 0$. Thus the necessary condition for statical determinacy is fulfilled.

We separate the system and sketch the free-body diagram. Therewith the equilibrium conditions can be formulated:



Equilibrium for subsystem 1:

$$\left.\begin{aligned}
\rightarrow:&\quad A + B = 0\\
\uparrow:&\quad S = q_0 a\\
\overset{\curvearrowleft}{A}:&\quad a\,S - \frac{q_0 a^2}{2} - a\,B = 0
\end{aligned}\right\} \quad \leadsto \quad \begin{aligned}
&\underline{\underline{B = \frac{q_0 a}{2}}}\,,\\
&\underline{\underline{A = -\frac{q_0 a}{2}}}\,.
\end{aligned}$$

Equilibrium for subsystem 2:

$$\left.\begin{aligned}
\overset{\curvearrowleft}{F}:&\quad a\,S + \frac{q_0 a^2}{2} - 2aE = 0\\
\uparrow:&\quad F_V - S - q_0 a = 0\\
\rightarrow:&\quad E - F_H = 0
\end{aligned}\right\} \quad \leadsto \quad \begin{aligned}
&\underline{\underline{F_V = 2 q_0 a}}\,,\\
&\underline{\underline{E = \frac{3}{4} q_0 a}}\,,\\
&\underline{\underline{F_H = \frac{3}{4} q_0 a}}\,.
\end{aligned}$$

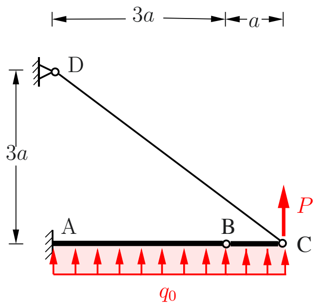
Test: Equilibrium of the moments for the complete system

$$\overset{\curvearrowleft}{D}: \quad 2aA + aB + \frac{q_0 a^2}{2} - \frac{q_0 a^2}{2} - 2aF_H + aF_V = 0$$

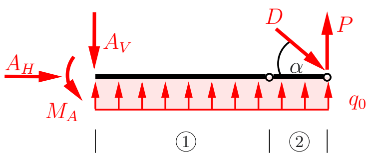
$$\leadsto \quad -q_0 a^2 + \frac{q_0 a^2}{2} - \frac{3}{2} q_0 a^2 + 2 q_0 a^2 = 0\,.$$

Problem 3.10 Determine the support reaction forces and moment for the sketched system.

P3.10



Solution The free-body diagram shows all forces acting on the system (the bar $\overline{CD}$ acts like a hinged column).
Thus the equilibrium conditions for the complete system and the subsystem (2), respectively, can be formulated



Complete system:

$$\uparrow: \quad -D\sin\alpha - A_V + P + q_0 4a = 0\,,$$

$$\rightarrow: \quad A_H + D\cos\alpha = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad -M_A + 4\,aD\sin\alpha - 2\,aq_0 4\,a - 4aP = 0\,,$$

Subsystem (2):

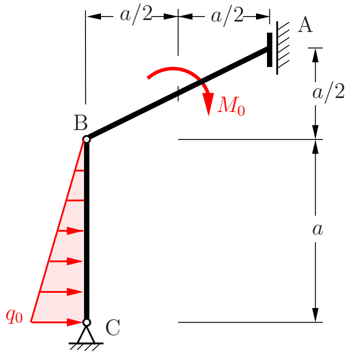
$$\overset{\curvearrowright}{B}: \quad a\,D\sin\alpha - Pa - \frac{1}{2}\,aq_0 a = 0\,.$$

Solving the 4 equations for the 4 unknown forces yields with $\sin\alpha = 3/5$ and $\cos\alpha = 4/5$ the support reactions

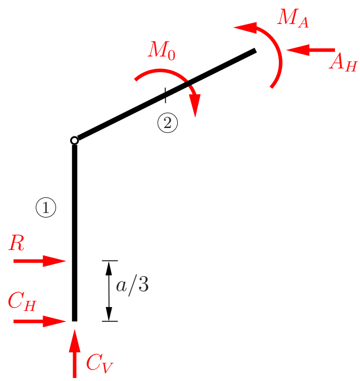
$$\underline{\underline{D = \frac{5}{3}P + \frac{5}{6}q_0 a}}, \quad \underline{\underline{A_V = \frac{7}{2}q_0 a}}, \quad \underline{\underline{A_H = -\frac{4}{3}P - \frac{2}{3}q_0 a}}, \quad \underline{\underline{M_A = -6q_0 a^2}}.$$

P3.11

Problem 3.11 A triangular- The part $\overline{BC}$ of the structure is loaded by a triangular-shaped line force. In addition a single moment M_0 is exerted to the part $\overline{AB}$. Determine the support reaction in A and C.



Solution The free-body diagram shows all forces and moments acting on the complete system. Here the line force has been replaced by its resultant R. Thus, the equilibrium conditions for the complete system and subsystem (2) are given as follows:



Complete system:

$$\uparrow: \quad C_V = 0\,,$$

$$\rightarrow: \quad -A_H + C_H + R = 0\,,$$

$$\overset{\curvearrowleft}{C}: \quad M_A - M_0 + \frac{3}{2}\,aA_H - \frac{1}{3}\,aR = 0\,,$$

Part (2):

$$\overset{\curvearrowleft}{B}: \quad -a\,C_H - \frac{2}{3}\,aR = 0\,.$$

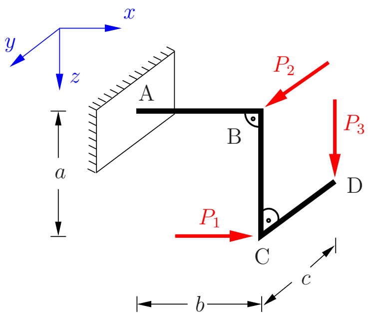
Therefrom, with $R = \frac{1}{2}\,q_0\,a$ the reactions forces as

$$\underline{\underline{C_H = -\frac{1}{3}\,q_0 a}}\,, \quad \underline{\underline{C_V = 0}}\,, \quad \underline{\underline{A_H = \frac{1}{6}\,q_0 a}}\,, \quad \underline{\underline{M_A = M_0 - \frac{1}{12}\,q_0 a^2}}.$$

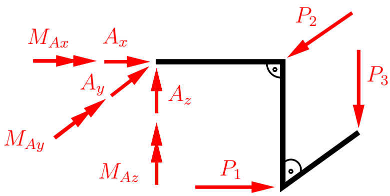
Problem 3.12 The shown structure is loaded at points B, C and D by the single forces P_1, P_2 and P_3. Each line of action is parallel to one of the coordinate axes.

P3.12

Determine the support reaction at point A.



Solution We recognize from the free-body diagram three components of the force and three components of the moment acting at point A. Thus, the equilibrium conditions for the forces and the moments yield:



$$\sum F_x = 0 \ : \quad \underline{\underline{A_x = -P_1}}\,,$$

$$\sum F_y = 0 \ : \quad \underline{\underline{A_y = P_2}}\,,$$

$$\sum F_z = 0 \ : \quad \underline{\underline{A_z = P_3}}\,,$$

$$\sum M_x^{(A)} = 0 \ : \quad \underline{\underline{M_{Ax} = c\,P_3}}\,,$$

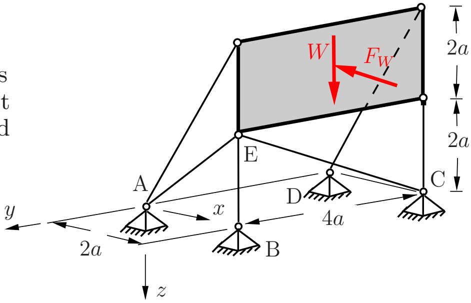
$$\sum M_y^{(A)} = 0 \ : \quad \underline{\underline{M_{Ay} = a\,P_1 - b\,P_3}}\,,$$

$$\sum M_z^{(A)} = 0 \ : \quad \underline{\underline{M_{Az} = b\,P_2}}\,.$$

P3.13

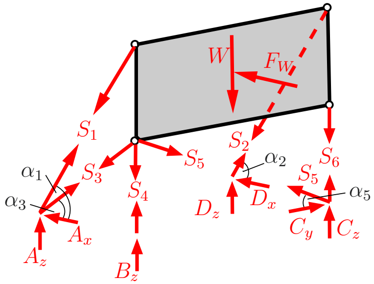
Problem 3.13 A signpoast is fixed by bars as depicted. It is loaded by its weight W and the resulting wind load F_W.

Determine the support.



Solution We obtain from the geometry the following angles

$$\cos\alpha_1 = \frac{1}{\sqrt{5}}\,, \qquad \cos\alpha_2 = \frac{1}{\sqrt{5}}\,,$$
$$\cos\alpha_3 = \frac{1}{\sqrt{2}}\,, \qquad \cos\alpha_5 = \frac{2}{\sqrt{5}}\,.$$



The equilibrium equations are given by:

$$\sum F_y = 0: \; -S_5\cos\alpha_5 = 0 \quad \leadsto \quad S_5 = 0\,,$$

$$\sum M_z^{(B)} = 0: \; -S_2\cos\alpha_2\, 4a - F_W\, 2a = 0 \quad \leadsto \quad S_2 = -\tfrac{1}{2}\sqrt{5}\,F_W\,,$$

$$\sum M_x^{(E)} = 0: \; -W\,2a - S_6\,4a - S_2\,\sin\alpha_2\,4a = 0$$
$$\leadsto \quad S_6 = -\tfrac{1}{2}W + F_W\,,$$

$$\sum M_y^{(E)} = 0: \; +S_1\cos\alpha_1\,2a + S_2\cos\alpha_2\,2a + F_W\,a = 0 \quad \leadsto \quad S_1 = 0\,,$$

$$\sum F_x = 0: \; -S_1\cos\alpha_1 - S_3\cos\alpha_3 - S_2\cos\alpha_2 - F_W = 0$$
$$\leadsto \quad S_3 = -\tfrac{1}{2}\sqrt{2}\,F_W\,,$$

$$\sum F_z = 0: \; +W + S_4 + S_6 + S_2\sin\alpha_2 + S_5\sin\alpha_5$$
$$+S_1\sin\alpha_1 + S_3\sin\alpha_3 = 0 \quad \leadsto \quad S_4 = -\tfrac{1}{2}W + \tfrac{1}{2}F_W\,.$$

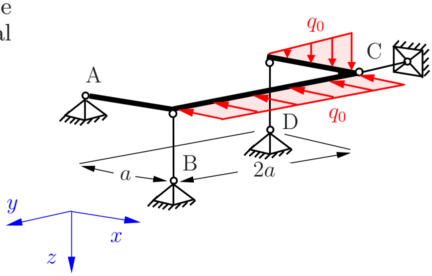
Thus, the reaction forces result as

$$\underline{\underline{A_x = -\tfrac{1}{2}\,F_W}}\,, \qquad \underline{\underline{D_x = -\tfrac{1}{2}\,F_W}}\,,$$

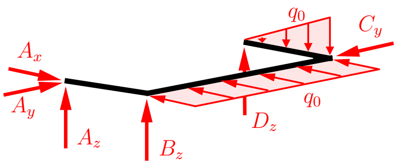
$$\underline{\underline{A_z = \tfrac{1}{2}\,F_W}} \qquad \underline{\underline{B_z = \tfrac{1}{2}\,W - \tfrac{1}{2}\,F_W}}\,, \quad \underline{\underline{C_z = \tfrac{1}{2}\,W - F_W}}\,, \quad \underline{\underline{D_z = F_W}}\,,$$

All other components of the support forces are zero.

Problem 3.14 Determine the support reactions forces for the depicted three-dimensional structure. P3.14



Solution We isolate the system and make visible all reaction forces and acting external forces in the free-body diagram. Since the supports at B, C and D are hinged columns, their resulting forces point into the direction of the hinged column.



From the the 3 equilibrium conditions for the forces and 3 for the moments, we obtain the following results for the 6 unknown support reactions. Here it is advantageous to pay attention to a suitable choice of the reference points for the moments.

$$\sum F_x = 0 : \quad A_x - 2q_0\, a = 0 \quad \leadsto \quad \underline{\underline{A_x = 2q_0\, a}}\,,$$

$$\sum M_x^{(A)} = 0 : \quad +D_z 2a - q_0\, \tfrac{1}{1}\, a\, 2a = 0 \quad \leadsto \quad \underline{\underline{D_z = \frac{q_0\, a}{2}}}\,,$$

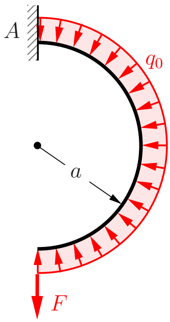
$$\sum M_y^{(A)} = 0 : \quad +B_z\, a - q_0\, \frac{a}{2}\, \frac{2a}{3} = 0 \quad \leadsto \quad \underline{\underline{B_z = \frac{q_0\, a}{3}}}\,,$$

$$\sum M_z^{(A)} = 0 : \quad C_y a\, - 2q_0\, a\, a = 0 \quad \leadsto \quad \underline{\underline{C_y = 2q_0\, a}}\,,$$

$$\sum F_y = 0 : \quad -A_y + C_y = 0 \quad \leadsto \quad \underline{\underline{A_y = 2q_0\, a}}\,,$$

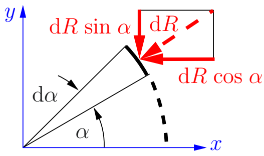
$$\sum F_z = 0 : \quad -A_z - B_z - D_z + q_0\, \frac{1}{2}\, a = 0 \quad \leadsto \quad \underline{\underline{A_z = -q_0\, \frac{a}{3}}}\,.$$

P3.15 **Problem 3.15** A semicircular arc with the radius a is loaded by a radial line load q_0 and a vertical force F.



Determine the support reactions.

Solution We replace the radial line load q_0 by its resultant R. For this purpose, we introduce a coordinate system and determine the force on an infinitesimal piece of the arc with an opening angle $\mathrm{d}\alpha$. The infinitesimal resultant in radial direction is



$$\mathrm{d}R = q_0\, a\, \mathrm{d}\alpha\,.$$

The components of the resultant (positive in positive coordinate direction) are

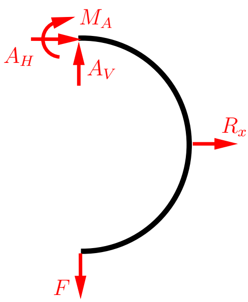
$$\mathrm{d}R_x = -\mathrm{d}R\,\cos\alpha\,, \quad \mathrm{d}R_y = -\mathrm{d}R\,\sin\alpha\,.$$

Integration over the semicircular arc yields

$$R_x = -\int_{-\pi/2}^{\pi/2} q_0\, a\, \cos\alpha\, \mathrm{d}\alpha = -q_0\, a\, \sin\alpha\Big|_{-\pi/2}^{\pi/2} = -2\, q_0\, a\,,$$

$$R_y = -\int_{-\pi/2}^{\pi/2} q_0\, a\, \sin\alpha\, \mathrm{d}\alpha = \quad q_0\, a\, \cos\alpha\Big|_{-\pi/2}^{\pi/2} = 0\,.$$

The three support reactions follow from the equilibrium conditions



$$\rightarrow: \quad A_H + R_x = 0\,,$$

$$\uparrow: \quad A_V - F = 0\,,$$

$$\stackrel{\curvearrowleft}{A}: \quad -M_A + R_x\, a = 0$$

as

$$\underline{\underline{A_H = 2\, q_0\, a}}\,, \quad \underline{\underline{A_V = F}}\,, \quad \underline{\underline{M_A = -2\, q_0\, a^2}}\,.$$

Chapter 4

Trusses

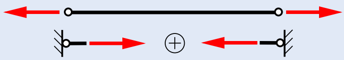
4

Assumptions for an ideal truss:

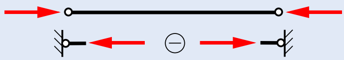
- All slender members of the truss are straight.
- The slender members are connected by frictionless pins.
- External forces are applied at the pins only.

Plane truss: All truss members and forces are in the same plane.

Rule of sign:



tension member



compression member

Computation of statical determinacy:

$$\boxed{f = 2j - (m + r)} \qquad \text{plane truss,}$$

$$\boxed{f = 3j - (m + r)} \qquad \text{spatial truss,}$$

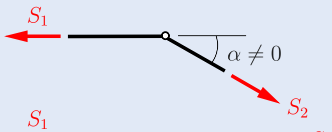
with

f = number of degrees of freedom, $\quad j$ = number of joints,
m = number of members, $\quad r$ = number of support reactions.

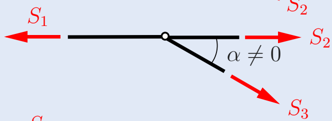
Note:

$$f \begin{cases} > 0 & f\text{-times movable}\,, \\ = 0 & \text{statically determinate (only necessary condition)}\,, \\ < 0 & f\text{-times statically indeterminate.}\,. \end{cases}$$

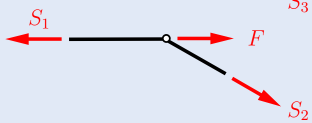
Zero-force members are members with vanishing internal forces. For plane trusses the following applies:



$$S_1 = S_2 = 0$$



$$S_3 = 0$$



$$S_2 = 0$$

Methods for the determination of internal forces:

Method of Joints

is usefull, when *all* internal forces have to be determined.

a) Analytical approach

- Applying the equilibrium conditions to the free-body diagram of each joint of the truss. Solution of the system of equations yields the internal forces and the support reactions.
- A large number of joints yields a large system of equations.

b) Graphical approach for plane trusses: Cremona diagram

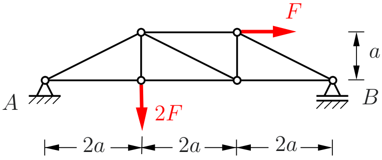
1. Determination of the support reactions.
2. Define direction of calculation: counter-clockwise $\curvearrowleft$ or clockwise $\curvearrowright$.
3. Draw a closed force polygon consisting of the external forces and the support reactions in your defined direction of calculation. (Choose proper scale for the forces!).
4. Enumerate trusses and identify zero-force members.
5. Starting at a joint with only *two* unknown internal forces, draw for every joint a force polygon. The hierarchy to be maintained for the internal forces is the one defined by the direction of calculation.
6. The direction of the forces at the joint have to be transfered in the free-body diagram in order to detect if we have a tension or a compression member.
7. The last force polygon is used for the verification of the calculation.
8. Summarize internal forces (including its sign) in a table.

Method of Sections

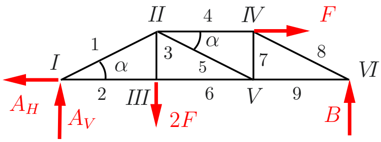
according to Ritter, can be applied to plane (spatial) trusses, if several internal forces *only* are of interest.

1. Determination of the support reactions.
2. The truss is divided by a cut into two parts. The cut has to be made in such a way that it goes through three members that do not built a system of coplanar (concurrent) forces.
3. The equilibrium conditions applied to the individual parts of the truss yield the internal forces of the members divided by cutting.

P4.1 **Problem 4.1** For the given truss, the forces in the bars shall be determined.



Solution We sketch a free-body diagram and number the nodes and the bars. The support reactions result from the equilibrium conditions of the entire system:



$$\overset{\curvearrowright}{A}: \quad 4aF + aF - 6aB = 0 \quad \rightsquigarrow \quad \underline{\underline{B = \frac{5}{6}F}}\,,$$

$$\overset{\curvearrowright}{B}: \quad 6a\,A_V - 4a\,2F + a\,F = 0 \quad \rightsquigarrow \quad \underline{\underline{A_V = \frac{7}{6}F}}\,,$$

$$\rightarrow: \quad -A_H + F = 0 \quad \rightsquigarrow \quad \underline{\underline{A_H = F}}\,.$$

The forces in the bars can be determined from the nodal equilibrium conditions. Using

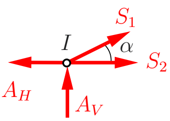
$$\sin\alpha = \frac{1}{\sqrt{5}}\,, \qquad \cos\alpha = \frac{2}{\sqrt{5}},$$

it follows

$$I \quad \uparrow: \quad A_V + S_1\frac{1}{\sqrt{5}} = 0\,,$$

$$\rightarrow: \quad S_2 + S_1\frac{2}{\sqrt{5}} - A_H = 0\,,$$

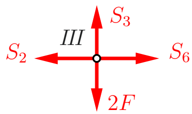
$$\rightsquigarrow \quad \underline{\underline{S_1 = -\frac{7\sqrt{5}}{6}F}}\,, \qquad \underline{\underline{S_2 = \frac{10}{3}F}}\,,$$



$$III \quad \rightarrow: \quad S_6 - S_2 = 0\,,$$

$$\uparrow: \quad S_3 - 2F = 0\,,$$

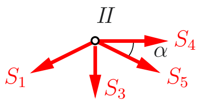
$$\rightsquigarrow \quad \underline{\underline{S_6 = \frac{10}{3}F}}\,, \qquad \underline{\underline{S_3 = 2F}}\,.$$



$$II \quad \downarrow: \quad S_1 \frac{1}{\sqrt{5}} + S_5 \frac{1}{\sqrt{5}} + S_3 = 0\,,$$

$$\rightarrow: \quad -S_1 \frac{2}{\sqrt{5}} + S_5 \frac{2}{\sqrt{5}} + S_4 = 0\,,$$

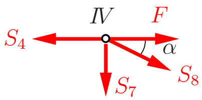
$$\rightsquigarrow \quad \underline{\underline{S_5 = -\frac{5\sqrt{5}}{6}\,F}}\,, \quad \underline{\underline{S_4 = -\frac{2}{3}\,F}}\,.$$



$$IV \quad \rightarrow: \quad -S_4 + F + S_8 \frac{2}{\sqrt{5}} = 0\,,$$

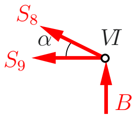
$$\downarrow: \quad S_7 + S_8 \frac{1}{\sqrt{5}} = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_8 = -\frac{5\sqrt{5}}{6}\,F}}\,, \quad \underline{\underline{S_7 = \frac{5}{6}\,F}}\,.$$



$$VI \quad \leftarrow: \quad S_9 + S_8 \frac{2}{\sqrt{5}} = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_9 = \frac{5}{3}F}}\,.$$

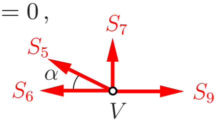


We check the second equilibrium condition at node *VI* as well as both nodal equilibrium conditions at node *V*:

$$VI \quad \uparrow: \quad S_8 \frac{1}{\sqrt{5}} + B = -\frac{5}{6}\,F + \frac{5}{6}F = 0\,,$$

$$V \quad \rightarrow: \quad S_9 - S_5 \frac{2}{\sqrt{5}} - S_6 = \frac{5}{3}\,F + \frac{5}{3}\,F - \frac{10}{3}\,F = 0\,,$$

$$\uparrow: \quad S_7 + S_5 \frac{1}{\sqrt{5}} = \frac{5}{6}\,F - \frac{5}{6}\,F = 0\,.$$



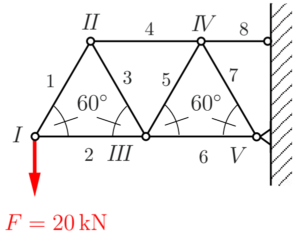
The results of the forces in the bars are summarized in the following table:

i	1	2	3	4	5	6	7	8	9
S_i/F	-2.61	3.33	2	-0.67	-1.86	3.33	0.83	-1.86	1.67

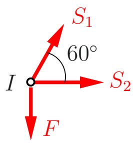
The largest forces occur in bar 2 and 6.

P4.2

Problem 4.2 For the given truss, all bar forces have to be determined.



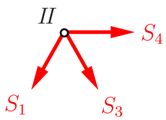
Solution Here, it is not necessary to determine the support reactions first. The forces in the bars can be obtained by formulating the equilibrium conditions for all nodes, starting with the loaded node I:



$$I \quad \uparrow: \quad S_1 \sin 60^\circ - F = 0\,,$$

$$\rightarrow: \quad S_2 + S_1 \cos 60^\circ = 0\,,$$

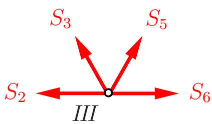
$$\rightsquigarrow \quad \underline{\underline{S_1}} = \frac{2}{\sqrt{3}} F = \underline{\underline{23.1 \text{ kN}}}\,, \quad \underline{\underline{S_2}} = -\frac{1}{2} S_1 = \underline{\underline{-11.6 \text{ kN}}}\,.$$



$$II \quad \downarrow: \quad S_1 \sin 60^\circ + S_3 \sin 60^\circ = 0\,,$$

$$\rightarrow: \quad S_4 - S_1 \cos 60^\circ + S_3 \cos 60^\circ = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_3}} = -S_1 = \underline{\underline{-23.1 \text{ kN}}}\,, \quad \underline{\underline{S_4}} = S_1 = \underline{\underline{23.1 \text{ kN}}}\,.$$



$$III \quad \uparrow: \quad (S_3 + S_5) \sin 60^\circ = 0\,,$$

$$\rightarrow: \quad -S_2 + (S_5 - S_3) \cos 60^\circ + S_6 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_5}} = -S_3 = \underline{\underline{23.1 \text{ kN}}}\,, \quad \underline{\underline{S_6}} = \underline{\underline{-34.7 \text{ kN}}}\,.$$

$$IV \quad \downarrow: \quad S_5 \sin 60^\circ + S_7 \sin 60^\circ = 0\,,$$

$$\rightarrow: \quad -S_4 + (S_7 - S_5)\cos 60^\circ + S_8 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_7}} = -S_5 = \underline{\underline{-23.1\text{ kN}}}\,, \qquad S_8 = \underline{\underline{46.2\text{ kN}}}\,.$$

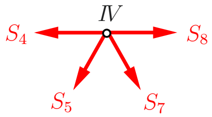


Table of bar forces:

i	1	2	3	4	5	6	7	8
S_i/kN	23.1	-11.6	-23.1	23.1	23.1	-34.7	-23.1	46.2

To check the results, we determine the forces in the bars 6, 7 and 8 using a RITTER-cut:

$$\overset{\curvearrowleft}{IV}: \quad \frac{3}{2}\,aF + a\,\sin 60^\circ\, S_6 = 0\,,$$

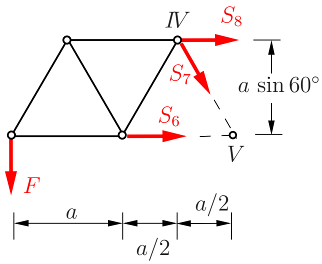
$$\leadsto \quad \underline{\underline{S_6 = -34.7\text{ kN}}}\,,$$

$$\overset{\curvearrowleft}{V}: \quad 2a\,F - a\,\sin 60^\circ\, S_8 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_8 = 46.2\text{ kN}}}\,,$$

$$\downarrow: \quad F + S_7 \cos 30^\circ = 0\,,$$

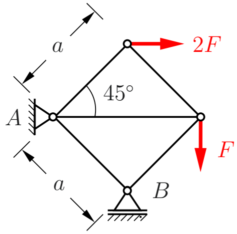
$$\leadsto \quad \underline{\underline{S_7 = -23.1\text{ kN}}}\,.$$



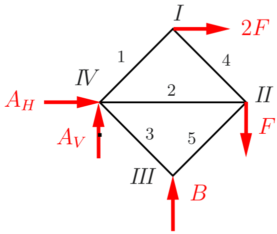
Remark: For cantilever trusses, the forces in the bars can be determined without previous calculation of the support reactions.

P4.3

Problem 4.3 For the given truss, the bar forces have to be determined with the Method of Joints.



Solution The reaction forces result from the equilibrium conditions for the entire system



$$\rightarrow: \quad A_H + 2F = 0\,,$$

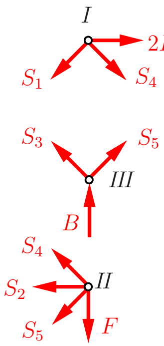
$$\uparrow: \quad A_V + B - F = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad \sqrt{2}\,aF + \frac{\sqrt{2}}{2}\,a2F - \frac{\sqrt{2}}{2}\,aB = 0$$

to

$$\underline{\underline{A_V = -3F}}\,, \quad \underline{\underline{A_H = -2F}}\,, \quad \underline{\underline{B = 4F}}\,.$$

Equilibrium at nodes I, III and II yields:



$$I \;\swarrow: \; S_1 - 2F\frac{\sqrt{2}}{2} = 0 \;\rightsquigarrow\; \underline{\underline{S_1 = \sqrt{2}\,F}}\,,$$

$$\searrow: \; S_4 + 2F\frac{\sqrt{2}}{2} = 0 \;\rightsquigarrow\; \underline{\underline{S_4 = -\sqrt{2}\,F}}\,,$$

$$III \;\nwarrow: \; S_3 + B\frac{\sqrt{2}}{2} = 0 \;\rightsquigarrow\; \underline{\underline{S_3 = -2\sqrt{2}\,F}}\,,$$

$$\nearrow: \; S_5 + B\frac{\sqrt{2}}{2} = 0 \;\rightsquigarrow\; \underline{\underline{S_5 = -2\sqrt{2}\,F}}\,,$$

$$II \;\leftarrow: \; S_2 + \frac{\sqrt{2}}{2}S_4 + \frac{\sqrt{2}}{2}S_5 = 0\,,$$

$$\underline{\underline{S_2 = 3F}}\,.$$

To check, we make sure that the equilibrium conditions at node IV are fulfilled:

$$IV \;\rightarrow: \; A_H + \frac{\sqrt{2}}{2}S_1 + S_2 + \frac{\sqrt{2}}{2}S_3 = -2F + F + 3F - 2F = 0\,,$$

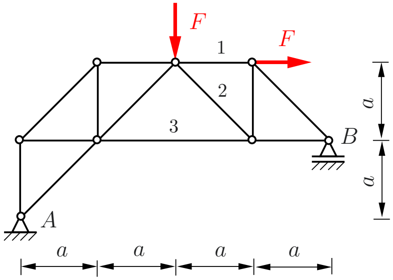
$$\uparrow: \; A_V + \frac{\sqrt{2}}{2}S_1 - \frac{\sqrt{2}}{2}S_3 = -3F + F + 2F = 0\,.$$

Table:

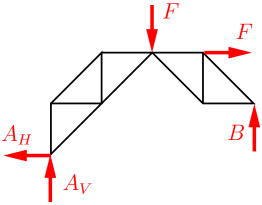
i	1	2	3	4	5
S_i	$\sqrt{2}F$	$3F$	$-2\sqrt{2}F$	$-\sqrt{2}F$	$-2\sqrt{2}F$

Problem 4.4 For the given truss, the reaction forces and the bar forces S_1, S_2 and S_3 have to be determined.

P4.4



Solution The reaction forces result from the equilibrium conditions of the complete system:



$$\rightarrow: \quad F - A_H = 0\,,$$

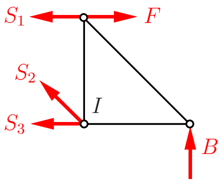
$$\uparrow: \quad A_V + B - F = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad 2aF + 2aF - 4aB = 0\,.$$

Therefrom, we obtain

$$\underline{\underline{A_V = 0}}\,, \quad \underline{\underline{B = F}}\,, \quad \underline{\underline{A_H = F}}\,.$$

The bar forces of interest follow from the equilibrium conditions of the sub system. For simplification purposes, we use the right part of the system:



$$\uparrow: \quad \frac{\sqrt{2}}{2} S_2 + B = 0\,,$$

$$\leadsto \quad \underline{\underline{S_2 = -\frac{2}{\sqrt{2}} F}}\,,$$

$$\overset{\curvearrowright}{I}: \quad a\,F - a\,S_1 - a\,B = 0\,,$$

$$\leadsto \quad \underline{\underline{S_1 = 0}}\,,$$

$$\leftarrow: \quad S_3 + S_1 + \frac{\sqrt{2}}{2} S_2 - F = 0\,,$$

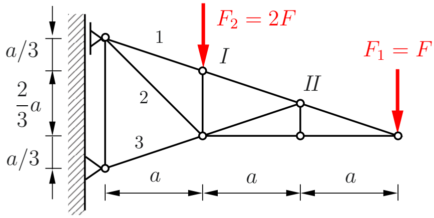
$$\leadsto \quad \underline{\underline{S_3 = 2F}}\,.$$

We check the equilibrium conditions in vertical directions for the left sub system:

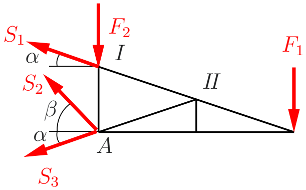
$$\uparrow: \quad A_V - F - \frac{\sqrt{2}}{2} S_2 = 0 - F + F = 0\,.$$

P4.5 **Problem 4.5** How large are the bar forces S_1, S_2 and S_3 for the given system?

How do they change, when the load F_2 is applied on node II?



Solution The equilibrium conditions for the separated system follow with the help of angle α and β:



$$\leftarrow: \quad S_1 \cos\alpha + S_2 \cos\beta + S_3 \cos\alpha = 0\,,$$

$$\uparrow: \quad S_1 \sin\alpha + S_2 \sin\beta - S_3 \sin\alpha - F_1 - F_2 = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad 2aF_1 - \frac{2}{3}aS_1 \cos\alpha = 0\,.$$

With

$$\sin\alpha = \frac{1}{\sqrt{10}}\,, \quad \cos\alpha = \frac{3}{\sqrt{10}}\,, \quad \sin\beta = \cos\beta = \frac{\sqrt{2}}{2}\,,$$

it follows

$$\underline{\underline{S_1 = \sqrt{10}F = 3.16\ F}}\,, \quad \underline{\underline{S_2 = \frac{3\sqrt{2}}{4}F = 1.06\ F}}\,,$$

$$\underline{\underline{S_3 = -\frac{5\sqrt{10}}{4}F = -3.95\ F}}\,.$$

If load F_2 is moved to node II, only the moment equilibrium condition changes:

$$\overset{\curvearrowright}{A}: \quad 2aF_1 + aF_2 - \frac{2}{3}aS_1 \cos\alpha = 0\,.$$

Thus, the bar forces result as

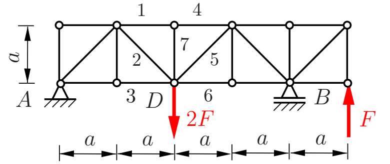
$$\underline{\underline{S_1 = 2\sqrt{10}F = 6.32\ F}}\,, \quad \underline{\underline{S_2 = -\frac{3\sqrt{2}}{4}F = -1.06\ F}}\,,$$

$$\underline{\underline{S_3 = -\frac{7\sqrt{10}}{4}F = -5.53\ F}}\,.$$

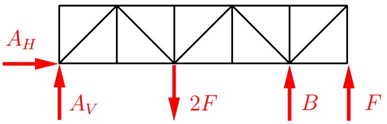
Remark: With the larger moment, S_1 and S_3 become larger and the tension bar changes into a compression bar.

Problem 4.6 For the given truss, the forces in the bars 1 through 7 shall be determined.

P4.6



Solution The reaction forces follow from the equilibrium conditions of the complete system:



$$\overset{\curvearrowright}{A}: \quad 2a\,2F - 4a\,B - 5a\,F = 0 \quad \rightsquigarrow \quad \underline{\underline{B = -\frac{1}{4}F}}\,,$$

$$\uparrow: \quad A_V + B - 2F + F = 0 \quad \rightsquigarrow \quad \underline{\underline{A_V = \frac{5}{4}F}}\,,$$

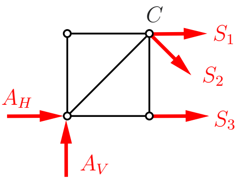
$$\rightarrow: \quad \underline{\underline{A_H = 0}}\,.$$

The bar forces 1 to 3 can be calculated from the sub system:

$$\overset{\curvearrowright}{C}: \quad a\,A_V - a\,A_H - a\,S_3 = 0\,,$$

$$\uparrow: \quad A_V - \frac{\sqrt{2}}{2}S_2 = 0\,,$$

$$\rightarrow: \quad A_H + S_1 + S_3 + \frac{\sqrt{2}}{2}S_2 = 0\,,$$

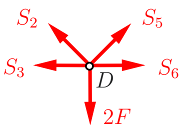


$$\rightsquigarrow \underline{\underline{S_3 = \frac{5}{4}F}}\,, \quad \underline{\underline{S_2 = \frac{5\sqrt{2}}{4}F}}\,, \quad \underline{\underline{S_1 = -\frac{5}{2}F}}\,.$$

Bar 7 is an unloaded bar: $S_7 = 0$. Furthermore, $S_4 = S_1$ holds. Equilibrium at node D finally yields

$$\uparrow: \quad \frac{\sqrt{2}}{2}S_2 + \frac{\sqrt{2}}{2}S_5 - 2F = 0\,,$$

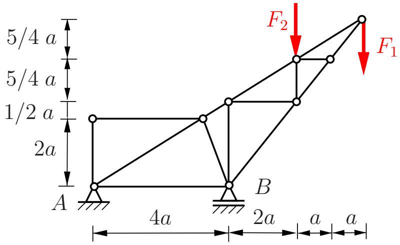
$$\rightarrow: \quad \frac{\sqrt{2}}{2}S_5 - \frac{\sqrt{2}}{2}S_2 + S_6 - S_3 = 0\,,$$



$$\rightsquigarrow \quad \underline{\underline{S_5 = \frac{3\sqrt{2}}{4}F}}\,, \quad \underline{\underline{S_6 = \frac{7}{4}F}}\,.$$

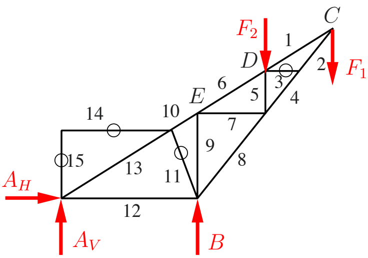
P4.7

Problem 4.7 How large are the reaction forces and bar forces in the given jib?



Given: $F_1 = 20$ kN ,
$F_2 = 10$ kN ,
$a = 1$ m .

Solution From the equilibrium conditions for the complete system,



$$\rightarrow: \quad A_H = 0\,,$$

$$\uparrow: \quad A_V + B - F_2 - F_1 = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad 6a\,F_2 + 8a\,F_1 - 4a\,B = 0\,,$$

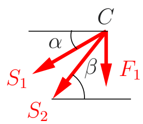
the reaction forces follow as

$$\underline{\underline{A_H = 0}}\,, \quad \underline{\underline{A_V = -25\text{ kN}}}\,, \quad \underline{\underline{B = 55\text{ kN}}}\,.$$

The bars 3, 11, 14 and 15 are unloaded bars. Therefore, it holds

$$\underline{\underline{S_2 = S_4}} \quad \text{and} \quad \underline{\underline{S_{10} = S_{13}}}\,.$$

Equilibrium at node C,



$$\leftarrow: \quad S_1 \cos\alpha + S_2 \cos\beta = 0\,,$$

$$\downarrow: \quad F_1 + S_1 \sin\alpha + S_2 \sin\beta = 0\,,$$

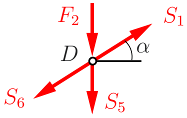
yields with

$$\sin\alpha = \frac{5}{\sqrt{89}}\,, \quad \cos\alpha = \frac{8}{\sqrt{89}}\,,$$
$$\sin\beta = \frac{5}{\sqrt{41}}\,, \quad \cos\beta = \frac{4}{\sqrt{41}}$$

the bar forces

$$\underline{\underline{S_1 = \frac{\sqrt{89}}{5} F_1 = 37.7\text{ kN}}}\,, \quad \underline{\underline{S_2 = -\frac{2}{5}\sqrt{41}\,F_1 = -51.2\text{ kN}}}\,.$$

Equilibrium at node D:

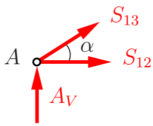


$$\rightarrow: \quad S_1 \cos\alpha - S_6 \cos\alpha = 0\,,$$

$$\uparrow: \quad S_1 \sin\alpha - S_6 \sin\alpha - F_2 - S_5 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_6 = S_1}}\,, \quad \underline{\underline{S_5 = -F_2 = -10\ \text{kN}}}\,.$$

Equilibrium at node A:

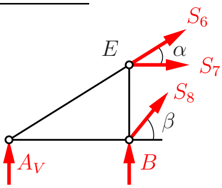


$$\uparrow: \quad A_V + S_{13} \sin\alpha = 0\,,$$

$$\rightarrow: \quad S_{12} + S_{13} \cos\alpha = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_{13} = 5\sqrt{89}\ \text{kN} = 47.2\ \text{kN}}}\,, \quad \underline{\underline{S_{12} = -40\ \text{kN}}}\,.$$

Cutting through bars 6, 7 and 8:

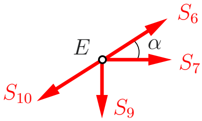


$$\overset{\curvearrowright}{E}: \quad 4a\, A_V - \frac{5}{2}\, a\, S_8 \cos\beta = 0\,,$$

$$\rightarrow: \quad S_7 + S_6 \cos\alpha + S_8 \cos\beta = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_8 = -10\sqrt{41} = -64\ \text{kN}}}\,, \quad \underline{\underline{S_7 = 8\ \text{kN}}}\,.$$

Finally equilibrium in vertical direction at node E yields



$$\uparrow: \quad S_6 \sin\alpha - S_{10} \sin\alpha - S_9 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_9 = -5\ \text{kN}}}\,.$$

Equilibrium in horizontal direction at node E can be used for checking

$$\rightarrow: \quad S_7 + S_6 \cos\alpha - S_{10} \cos\alpha = 8 + \frac{\sqrt{89}}{5} 20 \frac{8}{\sqrt{89}} - 5\sqrt{89}\frac{8}{\sqrt{89}}$$
$$= 8 + 32 - 40 = 0\,.$$

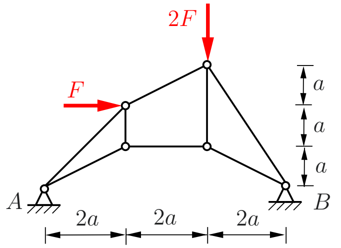
Table of bar forces:

i	1	2	3	4	5	6	7	8	9
S_i/kN	37.7	-51.2	0	-51.2	-10	37.7	8	-64	-5

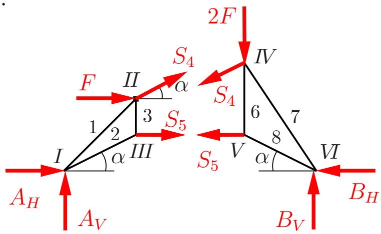
i	10	11	12	13	14	15
S_i/kN	47.2	0	-40	47.2	0	0

P4.8

Problem 4.8 For the given truss, the reaction forces and bar forces have to be determined.



Solution The truss has $j = 6$ joints, $m = 8$ bars and $r = 4$ reaction forces. The condition for statical determinism, $f = 2j - (m + r) = 12 - (8 + 4) = 0$, is therefore fulfilled.



The *four* reaction forces cannot be determined from the equilibrium conditions of the complete system solely. We therefore separate the system with a cut through two bars. Then $2 \times 3 = 6$ equilibrium conditions are available for determining the four reaction forces and two bar forces S_4 and S_5.
From the equilibrium conditions of the complete system,

$$\uparrow: \quad A_V + B_V - 2F = 0\,,$$

$$\rightarrow: \quad A_H - B_H + F = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad 2aF + 4a\,2F - 6aB_V = 0\,,$$

and for the right sub system,

$$\uparrow: \quad B_V - 2F - S_4 \sin\alpha = 0\,,$$

$$\leftarrow: \quad S_4 \cos\alpha + S_5 + B_H = 0\,,$$

$$\overset{\curvearrowright}{IV}: \quad 2aS_5 + 3aB_H - 2aB_V = 0\,,$$

we obtain with $\sin\alpha = 1/\sqrt{5}$ and $\cos\alpha = 2/\sqrt{5}$ the results

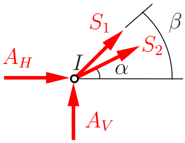
$$\underline{\underline{A_V = \frac{1}{3}\,F}}\,, \quad \underline{\underline{B_V = \frac{5}{3}\,F}}\,, \quad \underline{\underline{A_H = F}}\,, \quad \underline{\underline{B_H = 2\,F}}\,,$$

$$\underline{\underline{S_4 = -\frac{\sqrt{5}}{3}F}}\,, \quad \underline{\underline{S_5 = -\frac{4}{3}\,F}}\,.$$

The remaining bar forces can be obtained from the Method of Joints. With use of

$$\sin\beta = \cos\beta = 1/\sqrt{2}\,, \quad \sin\gamma = 3/\sqrt{13}\,, \quad \cos\gamma = 2/\sqrt{13}\,,$$

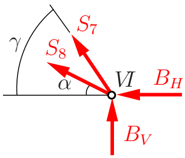
one obtains



$$I \rightarrow: \quad A_H + S_2\cos\alpha + S_1\cos\beta = 0\,,$$

$$\uparrow: \quad A_V + S_2\sin\alpha + S_1\sin\beta = 0\,,$$

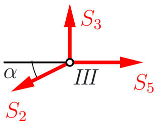
$$\leadsto \quad \underline{\underline{S_1 = \frac{\sqrt{2}}{3}F = 0.47\,F}}\,, \quad \underline{\underline{S_2 = -\frac{2}{3}\sqrt{5}\,F = -1.49\,F}}\,.$$



$$VI \leftarrow: \quad B_H + S_8\cos\alpha + S_7\cos\gamma = 0\,,$$

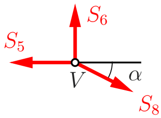
$$\uparrow: \quad B_V + S_8\sin\alpha + S_7\sin\gamma = 0\,,$$

$$\leadsto \quad \underline{\underline{S_7 = -\frac{\sqrt{13}}{3}F = -1.20\,F}}, \quad \underline{\underline{S_8 = -\frac{2}{3}\sqrt{5}\,F = -1.49\,F}}.$$



$$III \uparrow: \quad S_3 - S_2\sin\alpha = 0\,,$$

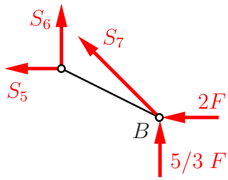
$$\leadsto \quad \underline{\underline{S_3 = -\frac{2}{3}\,F = -0.67\,F}}\,.$$



$$V \uparrow: \quad S_6 - S_8\sin\alpha = 0\,,$$

$$\leadsto \quad \underline{\underline{S_6 = -\frac{2}{3}F = -0.67\,F}}\,.$$

The Ritter Method of Sections can only be used in this exercise, if the reaction forces are already known. Exemplarily, one obtains with a cut through the bars 5, 6 and 7:



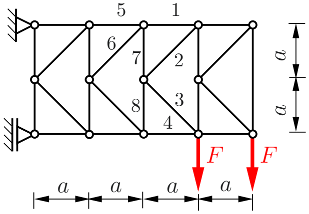
$$\leftarrow: \quad S_5 + S_7\cos\gamma + 2F = 0\,,$$

$$\uparrow: \quad S_6 + \frac{5}{3}\,F + S_7\sin\gamma = 0\,,$$

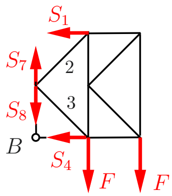
$$\overset{\curvearrowright}{B}: \quad 2a\,S_6 - a\,S_5 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_5 = -\frac{4}{3}\,F}}\,, \quad \underline{\underline{S_6 = -\frac{2}{3}\,F}}\,, \quad \underline{\underline{S_7 = -\frac{\sqrt{13}}{3}\,F}}\,.$$

P4.9 **Problem 4.9** For the given truss, the bar forces S_1 through S_7 shall be determined.



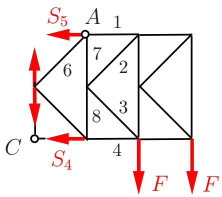
Solution At first, the bar forces S_1 and S_5 are calculated with the help of suitable cuts. Therefore, by exception, we cut through *four* bars such that three forces pass through the same point. The fourth force then follows from the moment balance at this point (for a cut through 1, 4, 7 and 8, this is point B):



$$\overset{\curvearrowright}{B}: \quad 2aF + aF - 2aS_1 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_1 = \frac{3}{2}\,F}}\,.$$

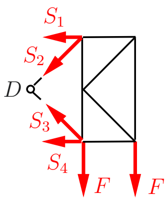
Analogously, from momentum in C follows



$$\overset{\curvearrowright}{C}: \quad 3aF + 2aF - 2aS_5 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_5 = \frac{5}{2}\,F}}\,.$$

The cut through 1, 2, 3 and 4 yields



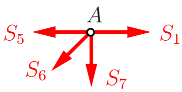
$$\overset{\curvearrowright}{D}: \quad 2a\,F + a\,F - a\,S_1 + a\,S_4 = 0\,,$$

$$\uparrow: \quad \frac{\sqrt{2}}{2}S_3 - \frac{\sqrt{2}}{2}S_2 - 2F = 0\,,$$

$$\leftarrow: \quad S_1 + S_4 + \frac{\sqrt{2}}{2}S_2 + \frac{\sqrt{2}}{2}S_3 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_4 = -\frac{3}{2}F}}\,, \quad \underline{\underline{S_3 = \sqrt{2}\,F}}\,, \quad \underline{\underline{S_2 = -\sqrt{2}\,F}}\,.$$

From equilibrium at node A, S_6 and S_7 can be calculated:

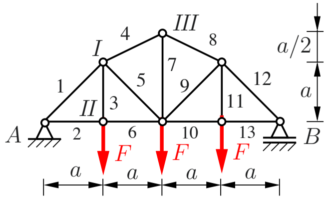


$$\rightarrow: \quad S_1 - S_5 - \frac{\sqrt{2}}{2}S_6 = 0\,,$$

$$\downarrow: \quad S_7 + \frac{\sqrt{2}}{2}S_6 = 0\,,$$

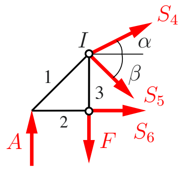
$$\leadsto \quad \underline{\underline{S_6 = -\sqrt{2}F}}\,, \quad \underline{\underline{S_7 = F}}\,.$$

Problem 4.10 Determine the bar forces for the given truss. **P4.10**



Solution The truss is symmetrically constructed and loaded. Thus, it holds that $S_4 = S_8$, $S_5 = S_9$, $S_1 = S_{12}$ etc. The vertical reaction forces in A and B follow as $A = B = 3F/2$.

Equilibrium at the cut system,



$$\overset{\curvearrowright}{I}: \quad aA - aS_6 = 0\,,$$

$$\uparrow: \quad A - F + S_4 \sin\alpha - S_5 \sin\beta = 0\,,$$

$$\rightarrow: \quad S_6 + S_4 \cos\alpha + S_5 \cos\beta = 0\,,$$

yields with $\sin\alpha = 1/\sqrt{5}$, $\cos\alpha = 2/\sqrt{5}$, $\sin\beta = \cos\beta = 1/\sqrt{2}$ the forces in the bars:

$$\underline{\underline{S_6 = A = \frac{3}{2}F}}\,, \quad \underline{\underline{S_4 = -\frac{2}{3}\sqrt{5}\,F}}\,, \quad \underline{\underline{S_5 = -\frac{\sqrt{2}}{6}F}}\,.$$

The remaining bar forces can be calculated with the Method of Joints:

$$III \;\downarrow: \quad S_7 + 2S_4 \sin\alpha = 0\,,$$

$$\leadsto \quad \underline{\underline{S_7 = \frac{4}{3}F}}\,,$$

$$II \rightarrow: \quad \underline{\underline{S_2 = S_6 = \frac{3}{2}F}}\,,$$

$$\uparrow: \quad \underline{\underline{S_3 = F}}\,,$$

$$A \;\uparrow: \quad A + S_1 \sin\beta = 0\,,$$

$$\leadsto \quad \underline{\underline{S_1 = -\frac{3}{2}\sqrt{2}\,F}}\,.$$

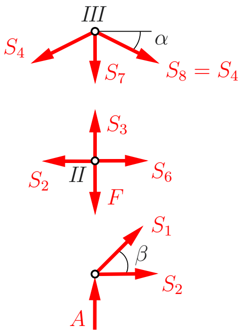


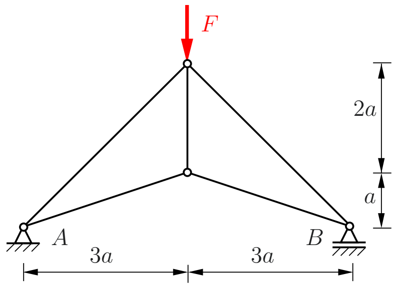
Table of bar forces:

i	1	2	3	4	5	6	7
S_i/F	$-3\sqrt{2}/2$	$3/2$	1	$-2\sqrt{5}/3$	$-\sqrt{2}/6$	$3/2$	$4/3$

Remark: The largest cut force according to its volume occurs at bar 1.

P4.11 **Problem 4.11** Determine the bar forces in the shown roof girder with the help of a CREMONA diagram.

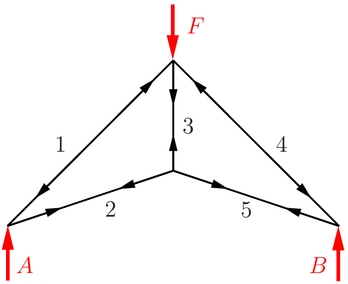
Given: $F = 10$ kN.



Solution Only vertical reaction forces occur in A and B:

$$A = B = \frac{1}{2}\,F = 5\text{ kN}\,.$$

We sketch the known forces into the free body diagram and add the sense of direction of each bar force at the individual nodes according to the CREMONA diagram.



CREMONA diagram

scale: 2 kN

sense of rotation:

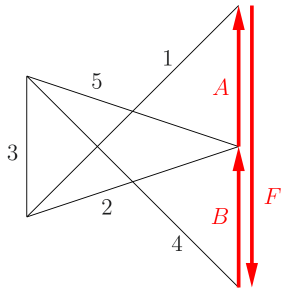
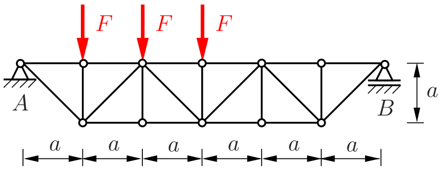


Table of bar forces:

i	1	2	3	4	5
S_i/kN	-10.6	7.9	5.0	-10.6	7.9

Remark: Because of symmetry, it holds that $S_1 = S_4$ and $S_2 = S_5$.

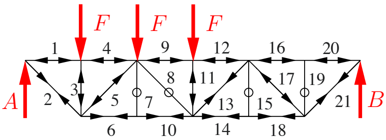
Problem 4.12 All bar forces have to be determined graphically. **P4.12**



Solution From the equilibrium conditions for the complete system, the vertical reaction forces result as

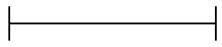
$$\underline{\underline{A = 2F}}\,, \quad \underline{\underline{B = F}}\,.$$

free body sketch:

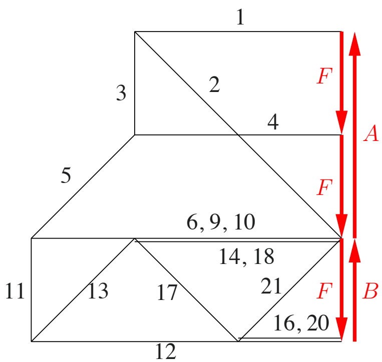


The bars 7, 15 and 19 are found to be unloaded bars. From the CREMONA diagram we additionally obtain bar 8 as an unloaded bar.

CREMONA diagram

scale: 

sense of rotation: 



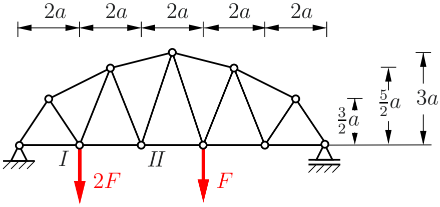
bar forces:

i	1	2	3	4	5	6	7	8	9	10	11
S_i/F	-2	$2\sqrt{2}$	-1	-2	$-\sqrt{2}$	3	0	0	-3	3	-1

i	12	13	14	15	16	17	18	19	20	21
S_i/F	-3	$\sqrt{2}$	2	0	-1	$-\sqrt{2}$	2	0	-1	$\sqrt{2}$

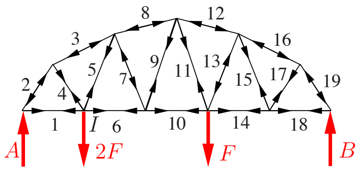
P4.13 **Problem 4.13** Determine the bar forces for the given truss.

How do the forces change, if the load $2F$ is moved from node I to node II?



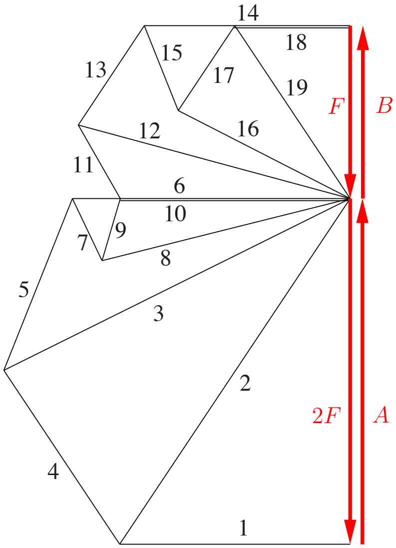
Solution In the shown truss, the reaction forces result from the equilibrium conditions as

$$A = 2F\,, \quad B = F\,.$$



The bar forces can be determined with the help of the Cremona diagram.

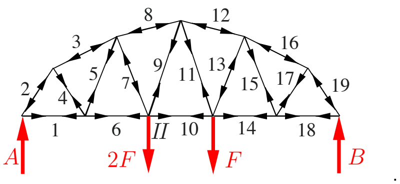
scale: F sense of rotation:



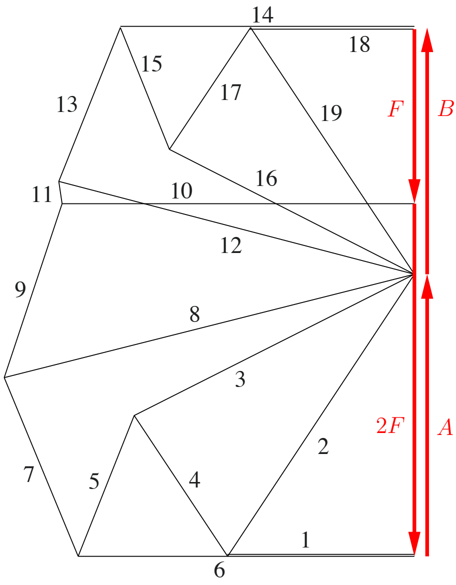
i	S_i/F
1	1.33
2	-2.39
3	-2.22
4	1.21
5	1.07
6	1.58
7	-0.38
8	-1.48
9	0.37
10	1.33
11	0.38
12	-1.48
13	0.69
14	1.19
15	-0.54
16	-1.10
17	0.59
18	0.67
19	-1.17

If the force $2F$ acts on node II, the reaction forces result as

$$A = 1.6F\,, \quad B = 1.4F\,.$$



With the same scale and sense of rotation, we obtain the following CREMONA diagram:

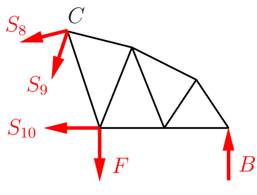


i	S_i/F
1	1.06
2	-1.92
3	-1.78
4	0.96
5	-0.86
6	1.91
7	1.09
8	-2.38
9	1.04
10	2.00
11	0.12
12	-2.08
13	0.94
14	1.67
15	-0.72
16	-1.57
17	0.83
18	0.93
19	-1.67

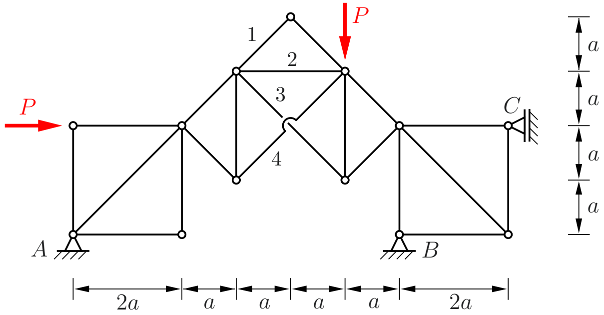
To check the result, the bar forces can be determined with the Method of Sections analytically. One obtains for S_{10}

$$\overset{\curvearrowright}{C}: \quad 3a\,S_{10} + a\,F - 5a\,B = 0$$

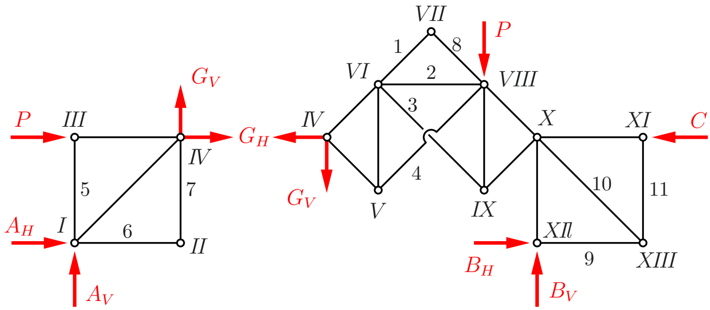
$$\leadsto \quad S_{10} = \frac{6}{3}\,F = 2F\,.$$



P4.14 **Problem 4.14** Determine the unloaded bars, all reaction forces and the bar forces S_1, S_2, S_3 and S_4 for the shown truss.



Solution The unloaded bars can be determined by examining each node.



Considering the equilibrium at nodes *II*, *III*, *VII* and *XI* shows that bars 6, 7, 5, 1, 8 and 11 are unloaded: $S_6 = S_7 = S_5 = S_1 = S_8 = S_{11} = 0$. Since $S_{11} = 0$, bars 9 and 10 can be identified as unloaded bars by examining node *XIII*, hence $S_9 = S_{10} = 0$. It follows that $\underline{\underline{B_H = 0}}$.

For the determination of reaction forces, we form the moment equilibrium at the left part of the system at node *IV* and for the complete system at node *X*:

left part of the system:

$$\overset{\curvearrowright}{IV}: \quad 2a\,A_V - 2a\,A_H = 0 \quad \rightsquigarrow \quad A_V = A_H\,,$$

complete system:

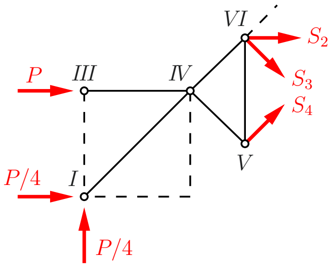
$$\overset{\curvearrowright}{X}: \quad -a\,P + 6a\,A_V - 2a\,A_H = 0 \quad \rightsquigarrow \quad \underline{\underline{A_V = A_H = \frac{P}{4}}}\,.$$

The remaining reaction forces can be computed by considering the equilibrium of forces in horizontal and vertical direction at the complete system:

$$\rightarrow: \quad P + A_H - C = 0 \quad \rightsquigarrow \quad \underline{\underline{C = \frac{5}{4}\,P}}\,,$$

$$\uparrow: \quad A_V - P - B_V = 0 \quad \rightsquigarrow \quad \underline{\underline{B_V = \frac{3}{4}\,P}}\,.$$

Using the Method of Sections, the remaining bar forces S_2, S_3 and S_4 can be computed:



$$\overset{\curvearrowright}{VI}: \quad -a\,P - 3a\,\frac{P}{4} + 3a\,\frac{P}{4} - 2a\,\frac{\sqrt{2}}{2}\,S_4 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_4 = -\frac{\sqrt{2}\,P}{2}}}\,,$$

$$\uparrow: \quad \frac{P}{4} + \frac{\sqrt{2}}{2}\,S_4 - \frac{\sqrt{2}}{2}\,S_3 = 0\,,$$

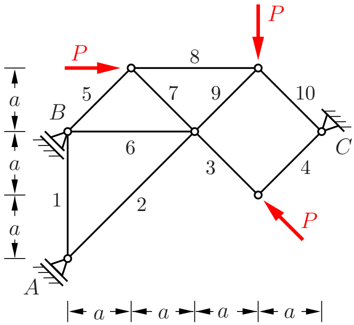
$$\rightsquigarrow \quad \underline{\underline{S_3 = -\frac{\sqrt{2}\,P}{4}}}\,,$$

$$\rightarrow: \quad P + \frac{P}{4} + S_2 + \frac{\sqrt{2}}{2}\,S_3 + \frac{\sqrt{2}}{2}\,S_4 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_2 = -\frac{P}{2}}}\,.$$

P4.15

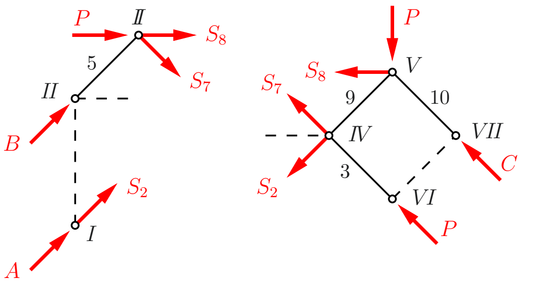
Problem 4.15 Determine the number of degrees of freedom and the unloaded bars for the given truss. Then, compute the remaining forces in the bars.



Solution The truss consists of $j = 7$ joints, $m = 10$ bars and $r = 4$ reaction forces. The number of degrees of freedom is consequently $f = 2\,j - (m + r) = 2 \cdot 7 - (10 + 4) = 0$. Thus, the system is statically determined.

By applying the rules for finding unloaded bars, we see that the bar forces S_1 and S_4 are zero. With $S_1 = 0$, S_6 also has to be equal to zero.

With the Method of Sections, (cutting through bars 2, 7 and 8) we divide the complete system into a left and a right subsystem. The respective free body sketches have the following form:



Considering equilibrium of forces of the complete system the reaction forces in point C follows as

$$\nwarrow: \quad C + P - \frac{\sqrt{2}}{2}\,P - \frac{\sqrt{2}}{2}\,P = 0 \quad \leadsto \quad \underline{\underline{C = (\sqrt{2} - 1)\,P}}\,.$$

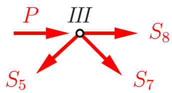
The equilibrium conditions at the right sub system yield

$$\overset{\curvearrowright}{IV}: \quad -a\,S_8 + a\,P - \sqrt{2}\,a\,C = 0 \quad \rightsquigarrow \quad \underline{\underline{S_8 = (\sqrt{2}-1)\,P}}\,,$$

$$\overset{\curvearrowright}{V}: \quad \sqrt{2}\,a\,S_7 + \sqrt{2}\,a\,P = 0 \quad \rightsquigarrow \quad \underline{\underline{S_7 = -P}}\,,$$

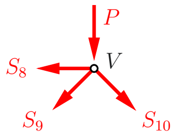
$$\nearrow: \quad -S_2 - \frac{\sqrt{2}}{2}\,S_8 - \frac{\sqrt{2}}{2}\,P = 0 \quad \rightsquigarrow \quad \underline{\underline{S_2 = -P}}\,.$$

The remaining forces in the bars can be determined with the Method of Joints:



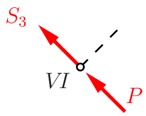
$$III \nearrow: \quad -S_5 + \frac{\sqrt{2}}{2}\,P + \frac{\sqrt{2}}{2}\,S_8 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_5 = P}}\,,$$



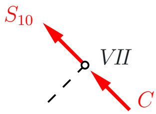
$$V \nearrow: \quad -S_9 - \frac{\sqrt{2}}{2}\,P - \frac{\sqrt{2}}{2}\,S_8 = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_9 = -P}}\,,$$



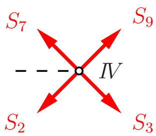
$$VI \nwarrow: \quad S_3 + P = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_3 = -P}}\,,$$



$$VII \nwarrow: \quad S_{10} + C = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{S_{10} = (1-\sqrt{2})\,P}}\,.$$



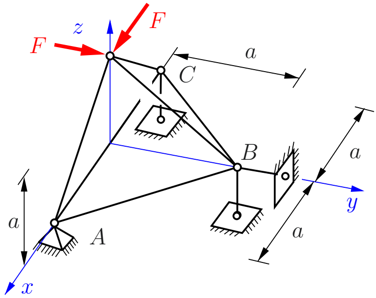
We check the results by considering the equilibrium conditions at node IV:

$$IV \nwarrow: \quad S_7 - S_3 = -P + P = 0\,, \quad \surd$$

$$\nearrow: \quad S_9 - S_2 = -P + P = 0\,. \quad \surd$$

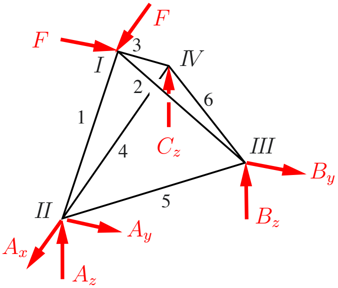
P4.16

Problem 4.16 Determine the reaction forces and bar forces for the given spatial truss.



Solution The truss consists of $j = 4$ joints, $m = 6$ bars and $r = 6$ reaction forces. Subsequently, the necessary condition for statical determination is fulfilled:

$$
\begin{aligned}
f &= 3j - (m + r) \\
&= 12 - (6 + 6) = 0\,.
\end{aligned}
$$



From the equilibrium conditions at the complete system,

$$\sum F_x = 0 \;:\quad A_x + F = 0\,,$$

$$\sum F_y = 0 \;:\quad A_y + B_y + F = 0\,,$$

$$\sum F_z = 0 \;:\quad A_z + B_z + C_z = 0\,,$$

$$\sum M_x = 0 \;:\quad a\,F - a\,B_z = 0\,,$$

$$\sum M_y = 0 \;:\quad a\,F + a\,C_z - a\,A_z = 0\,,$$

$$\sum M_z = 0 \;:\quad a\,A_y = 0\,,$$

the reaction forces follow as

$$\underline{\underline{A_x = -F}}\,,\quad \underline{\underline{A_y = 0}}\,,\quad \underline{\underline{A_z = 0}}\,,$$

$$\underline{\underline{B_y = -F}}\,,\quad \underline{\underline{B_z = F}}\,,\quad \underline{\underline{C_z = -F}}\,.$$

The forces in the bars can be obtained using the equilibrium conditions at the nodes. Under consideration that all bars except of bar 4 are tilted at 45^o to the respective coordinate axes, we obtain at node I and II:

$$I \quad \sum F_x = 0 \ : \quad \frac{1}{\sqrt{2}}S_1 - \frac{1}{\sqrt{2}}S_3 + F = 0\,,$$

$$\sum F_y = 0 \ : \quad \frac{1}{\sqrt{2}}S_2 + F = 0\,,$$

$$\sum F_z = 0 \ : \quad -\frac{1}{\sqrt{2}}S_1 - \frac{1}{\sqrt{2}}S_2 - \frac{1}{\sqrt{2}}S_3 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_1 = 0}}\,, \quad \underline{\underline{S_2 = -\sqrt{2}\,F}}\,, \quad \underline{\underline{S_3 = \sqrt{2}\,F}}\,.$$

$$II \quad \sum F_x = 0 \ : \quad A_x - \frac{1}{\sqrt{2}}S_1 - S_4 - \frac{1}{\sqrt{2}}S_5 = 0\,,$$

$$\sum F_y = 0 \ : \quad A_y + \frac{1}{\sqrt{2}}S_5 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_4 = -F}}\,, \quad \underline{\underline{S_5 = 0}}\,.$$

$$III \quad \sum F_y = 0 \ : \quad B_y - \frac{1}{\sqrt{2}}S_6 - \frac{1}{\sqrt{2}}S_2 - \frac{1}{\sqrt{2}}S_5 = 0\,,$$

$$\leadsto \quad \underline{\underline{S_6 = 0}}\,.$$

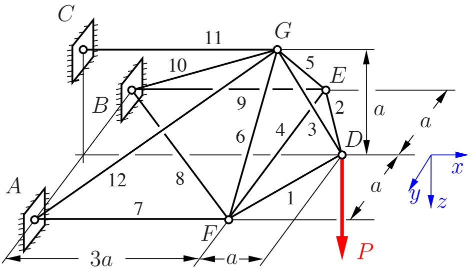
We check our results using the equilibrium at node IV:

$$\sum F_x = 0 \ : \quad \frac{1}{\sqrt{2}}S_6 + S_4 + \frac{1}{\sqrt{2}}S_3 = 0 \quad \leadsto \quad 0 - F + F = 0\,,$$

$$\sum F_y = 0 \ : \quad \frac{1}{\sqrt{2}}S_6 = 0\,,$$

$$\sum F_z = 0 \ : \quad C_z + \frac{1}{\sqrt{2}}S_3 = 0 \qquad \leadsto \quad -F + F = 0\,.$$

P4.17 **Problem 4.17** Determine the forces in the bars for the given spatial truss.



Solution The truss contains $j = 7$ joints, $m = 12$ bars and $r = 9$ reaction forces. Therefore it is statically determined:

$$f = 3j - (m + r) \quad \rightsquigarrow \quad f = 21 - (9 + 12) = 0\,.$$

We calculate the forces in the bars using the Method of Joints with the spatial equilibrium at the nodes:

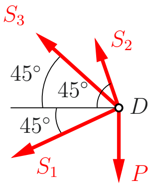
node D

$$\sum F_x = 0 \;: \quad -S_1 \cos 45^\circ - S_2 \cos 45^\circ - S_3 \cos 45^\circ = 0\,,$$

$$\sum F_y = 0 \;: \quad S_1 \sin 45^\circ - S_2 \sin 45^\circ = 0\,,$$

$$\sum F_z = 0 \;: \quad P - S_3 \sin 45^\circ = 0$$

$$\rightsquigarrow \quad \underline{\underline{S_3 = \sqrt{2}\,P}}\,, \quad \underline{\underline{S_1 = S_2 = -\frac{1}{2}\sqrt{2}\,P}}\,.$$



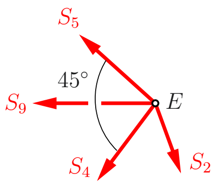
node E

$$\sum F_x = 0 \;: \quad -S_9 + S_2 \sin 45^\circ = 0\,,$$

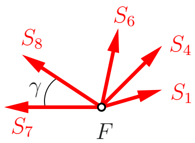
$$\sum F_y = 0 \;: \quad S_4 + S_5 \cos 45^\circ + S_2 \cos 45^\circ = 0\,,$$

$$\sum F_z = 0 \;: \quad S_5 \sin 45^\circ = 0$$

$$\rightsquigarrow \quad \underline{\underline{S_9 = -\frac{1}{2}\,P}}\,, \quad \underline{\underline{S_5 = 0}}\,, \quad \underline{\underline{S_4 = \frac{1}{2}\,P}}\,.$$



node F



$$\sum F_z = 0 \;:\quad S_6 \sin 45^\circ = 0\,,$$

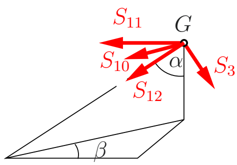
$$\sum F_x = 0 \;:\quad S_1 \sin 45^\circ - S_7 - S_8 \cos\gamma = 0\,,$$

$$\sum F_y = 0 \;:\quad -S_1 \cos 45^\circ - S_6 \cos 45^\circ - S_8 \sin\gamma - S_4 = 0$$

$$\rightsquigarrow \quad \underline{\underline{S_6 = 0}}\,, \quad \underline{\underline{S_7 = -\frac{1}{2}\,P}}\,, \quad \underline{\underline{S_8 = 0}}\,.$$

(Considering the symmetry of the loading, it can be found that $S_6 = S_5$, $S_7 = S_9$, $S_8 = 0$.)

node G



We introduce the angle α (between bar 12 and a vertical line through G) and β (between the projection of bar 12 onto the x-y-plane and the x-axis). From this, it follows

$$\cos\alpha = \frac{1}{\sqrt{11}}\,, \quad \sin\alpha = \frac{\sqrt{10}}{\sqrt{11}}\,, \quad \cos\beta = \frac{3}{\sqrt{10}}\,.$$

The equilibrium condition $\sum F_y = 0$ yields with $S_6 = S_5 = 0$ another conclusion about symmetry: $S_{10} = S_{12}$. The remaining conditions yield

$$\sum F_z = 0 \;:\quad S_3 \cos 45^\circ + 2\,S_{12} \cos\alpha = 0\,,$$

$$\sum F_x = 0 \;:\quad -S_{11} - 2\,S_{12} \sin\alpha \cos\beta + S_3 \sin 45^\circ = 0$$

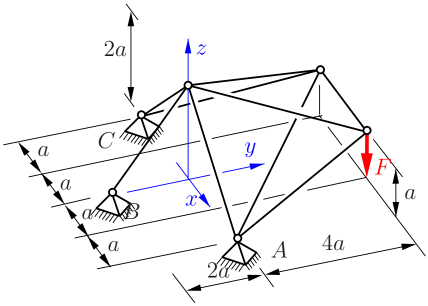
$$\rightsquigarrow \quad \underline{\underline{S_{10} = S_{12} = -\frac{\sqrt{11}}{2}\,P}}\,, \quad \underline{\underline{S_{11} = 4\,P}}\,.$$

We compute S_{11} from the equilibrium of the complete system in order to check our results. Therefore, we formulate the moment equilibrium condition with respect to a parallel to the y-axis through point A and B:

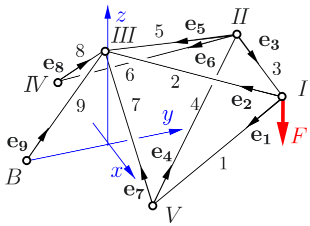
$$\sum M_y = 0 \;:\quad 4a\,P - a\,S_{11} = 0 \quad \rightsquigarrow \quad S_{11} = 4P\,.$$

P4.18 **Problem 4.18** The spatial truss is loaded by a force F.

Determine the forces in the bars.



Solution We consider the force acting in bar 9 (hinged column) as a reaction force. Now, the truss consists of $j = 5$ joints, $m = 8$ bars and $r = 1 + 2 \times 3 = 7$ reaction forces. Thus, the necessary condition for statical determination is fulfilled:



$$f = 3j - (m + r)$$
$$= 15 - (8 + 7) = 0\,.$$

We introduce the unit vectors $\mathbf{e_1}$ to $\mathbf{e_9}$ in order to express the directions of the bars and their respective components:

$$\mathbf{e}_1 = \frac{1}{\sqrt{18}}\begin{pmatrix}1\\-4\\-1\end{pmatrix}, \quad \mathbf{e}_2 = \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\-4\\1\end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$\mathbf{e}_4 = \frac{1}{\sqrt{26}}\begin{pmatrix}-3\\4\\1\end{pmatrix}, \quad \mathbf{e}_5 = \frac{1}{\sqrt{18}}\begin{pmatrix}1\\-4\\1\end{pmatrix}, \quad \mathbf{e}_6 = \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\-4\\-1\end{pmatrix},$$

$$\mathbf{e}_7 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \mathbf{e}_8 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \mathbf{e}_9 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

With the definition of all tensile forces to be positive, the equilibrium conditions at nodes I, II and III in vectorial (component) form read:

node I:

$$
\begin{aligned}
& S_1\mathbf{e}_1 + S_2\mathbf{e}_2 - S_3\mathbf{e}_3 - F\mathbf{e}_z = 0\,,\\
\rightsquigarrow\quad & \frac{1}{\sqrt{18}}\,S_1 - \frac{1}{\sqrt{18}}\,S_2 - S_3 = 0\,,\\
& -\frac{4}{\sqrt{18}}\,S_1 - \frac{4}{\sqrt{18}}\,S_2 = 0\,,\\
& -\frac{1}{\sqrt{18}}\,S_1 + \frac{1}{\sqrt{18}}\,S_2 - F = 0\,,
\end{aligned}
$$

$$
\rightsquigarrow\quad \underline{\underline{S_1 = -\frac{3}{2}\sqrt{2}F}}\,,\quad \underline{\underline{S_2 = \frac{3}{2}\sqrt{2}F}}\,,\quad \underline{\underline{S_3 = -F}}\,.
$$

node II:

$$
\begin{aligned}
& -S_4\mathbf{e}_4 + S_5\mathbf{e}_5 + S_6\mathbf{e}_6 + S_3\mathbf{e}_3 = 0\,,\\
\rightsquigarrow\quad & \frac{3}{\sqrt{26}}\,S_4 + \frac{1}{\sqrt{18}}\,S_5 - \frac{1}{\sqrt{18}}\,S_6 - F = 0\,,\\
& -\frac{4}{\sqrt{26}}\,S_4 - \frac{4}{\sqrt{18}}\,S_5 - \frac{4}{\sqrt{18}}\,S_6 = 0\,,\\
& -\frac{1}{\sqrt{26}}\,S_4 + \frac{1}{\sqrt{18}}S_5 - \frac{1}{\sqrt{18}}S_6 = 0\,,
\end{aligned}
$$

$$
\rightsquigarrow\quad \underline{\underline{S_4 = \frac{1}{4}\sqrt{26}F}}\,,\quad \underline{\underline{S_5 = 0}}\,,\quad \underline{\underline{S_6 = -\frac{3}{4}\sqrt{2}\,F}}\,.
$$

node III:

$$
\begin{aligned}
& -S_7\mathbf{e}_7 - S_8\mathbf{e}_8 - S_9\mathbf{e}_9 - S_2\mathbf{e}_2 - S_5\mathbf{e}_5 = 0\,,\\
\rightsquigarrow\quad & \frac{1}{\sqrt{2}}\,S_7 - \frac{1}{\sqrt{2}}\,S_8 + \frac{1}{\sqrt{18}}\,\frac{3}{2}\sqrt{2}\,F = 0\,,\\
& -\frac{1}{\sqrt{2}}\,S_9 + \frac{4}{\sqrt{18}}\,\frac{3}{2}\,\sqrt{2}\,F = 0\,,\\
& -\frac{1}{\sqrt{2}}\,S_7 - \frac{1}{\sqrt{2}}\,S_8 - \frac{1}{\sqrt{2}}\,S_9 - \frac{1}{\sqrt{18}}\,\frac{3}{2}\,\sqrt{2}\,F = 0\,,
\end{aligned}
$$

$$\rightsquigarrow \quad \underline{\underline{S_7 = -\frac{3}{2}\sqrt{2}\,F}}\,, \quad \underline{\underline{S_8 = -\sqrt{2}\,F}}\,, \quad \underline{\underline{S_9 = 2\sqrt{2}\,F}}\,.$$

The equilibrium condition at nodes *IV* and *V* as well as at bearing *B* can be used to determine the Cartesian components of the reaction forces:

node *IV*:

$$C_x\mathbf{e}_x + C_y\mathbf{e}_y + C_z\mathbf{e}_z + S_8\mathbf{e}_8 - S_6\mathbf{e}_6 = 0\,,$$

$$\rightsquigarrow \quad C_x - \frac{1}{\sqrt{2}}\sqrt{2}\,F - \frac{1}{\sqrt{18}}\,\frac{3}{4}\,\sqrt{2}\,F = 0\,,$$

$$C_y - \frac{4}{\sqrt{18}}\,\frac{3}{4}\,\sqrt{2}\,F = 0\,,$$

$$C_z - \frac{1}{\sqrt{2}}\,\sqrt{2}\,F - \frac{1}{\sqrt{18}}\,\frac{3}{4}\,\sqrt{2}\,F = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{C_x = \frac{5}{4}F}}\,, \quad \underline{\underline{C_y = F}}\,, \quad \underline{\underline{C_z = \frac{5}{4}\,F}}\,.$$

node *V*:

$$A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z - S_1\mathbf{e}_1 + S_4\mathbf{e}_4 + S_7\mathbf{e}_7 = 0\,,$$

$$\rightsquigarrow \quad A_x + \frac{1}{\sqrt{18}}\,\frac{3}{2}\,\sqrt{2}\,F - \frac{3}{\sqrt{26}}\,\frac{1}{4}\,\sqrt{26}\,F + \frac{1}{\sqrt{2}}\,\frac{3}{2}\,\sqrt{2}\,F = 0\,,$$

$$A_y - \frac{4}{\sqrt{18}}\,\frac{3}{2}\,\sqrt{2}\,F + \frac{4}{\sqrt{26}}\,\frac{1}{4}\,\sqrt{26}\,F = 0\,,$$

$$A_z - \frac{1}{\sqrt{18}}\,\frac{3}{2}\,\sqrt{2}\,F + \frac{1}{\sqrt{26}}\,\frac{1}{4}\,\sqrt{26}\,F - \frac{1}{\sqrt{2}}\,\frac{3}{2}\,\sqrt{2}\,F = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{A_x = -\frac{5}{4}F}}\,, \quad \underline{\underline{A_y = F}}\,, \quad \underline{\underline{A_z = \frac{7}{4}\,F}}\,.$$

bearing B:

$$\underline{\underline{B_x = 0}}\,, \quad \underline{\underline{B_y = B_z = -\frac{1}{2}\,\sqrt{2}\,S_9 = -2\,F}}\,.$$

Remark:

- The largest force acts in bar 9.
- The magnitude of the reaction forces is $A = \sqrt{90}F/4 = 2.37\ F$, $B = S_9 = 2\sqrt{2}\,F = 2.83\ F$ and $C = \sqrt{66}\,F/4 = 2.03\ F$.
- C lies in the plane which also contains S_6 and S_8.

Chapter 5

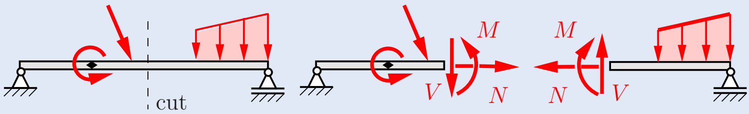
Beams, Frames, Arches

5

Stress Resultants

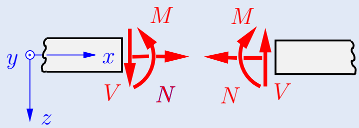
The stress resultants (normal force, shear force, bending moment) replace the internal stresses (forces per unit area) distributed across the cross-sectional area.

Plane systems



stress resultants:

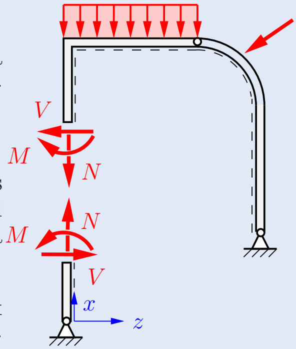
normal force	N,
shear force	V,
bending moment	M.



- Sign convention:
  Positive stress resultants at a positive face point in the positive directions of the coordinates.

- Coordinate system:
  x = axis of the beam (points right in case of a horizontal beam), z downwards for a horizontal beam.

- For frames, arches and complex structures we define the coordinate systems using *dashed lines* (lower side): x in direction of the dashed line, z away from the dashed line.

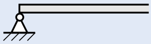
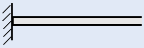
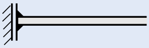
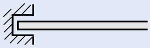


For **straight beams and straight portions of a frame,** the following differential relations between loading and stress resultants hold:

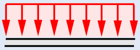
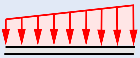
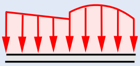
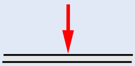
$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}x} = -q\,, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V} \quad \text{or} \quad \boxed{\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q\,.}$$

If we integrate the differential relations, the constants of integration have to be determined using the boundary conditions.

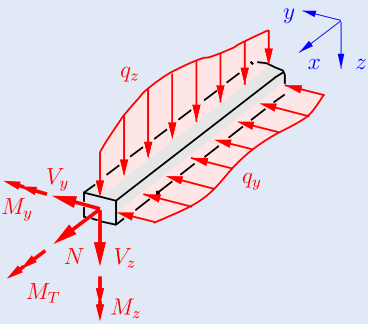
Boundary conditions:

hinged support		$(V \neq 0)$,	$M = 0$
free end		$V = 0$,	$M = 0$
clampend support		$(V \neq 0)$,	$(M \neq 0)$
parallel motion		$V = 0$,	$(M \neq 0)$
sliding sleeve		$(V \neq 0)$,	$(M \neq 0)$

Relation of V and M and loading:

loading		V diagram	M diagram
$q = 0$		const.	linear
$q =$ const.		linear	parabola
$q =$ linear		parabola	cubic parabola
jump of q		kink	continuous
concentrated force		jump	kink
external couple (moment)		continuous, no kink	jump

Spatial structures



Stress Resultants:

normal force	N,
shear forces	V_y, V_z,
bending moments	M_y, M_z,
torque	$M_x = M_T$.

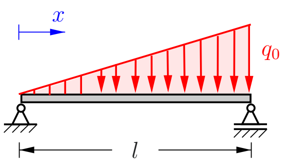
For **straight beams** the following differential relations between the loadings q_z, q_y and the shear forces V_y, V_z and the bending moments M_y, M_z hold:

$$\frac{\mathrm{d}V_z}{\mathrm{d}x} = -q_z\,, \qquad \frac{\mathrm{d}M_y}{\mathrm{d}x} = V_z\,,$$

$$\frac{\mathrm{d}V_y}{\mathrm{d}x} = -q_y\,, \qquad \frac{\mathrm{d}M_z}{\mathrm{d}x} = -V_y\,.$$

If we integrate the differential relations, the constants of integration have to be determined by using the boundary conditions.
The dependency of the shear forces and bending moment diagrams due to the characteristics of the loading are conferrable from the plane systems.

Problem 5.1 Determine the shear force and bending moment diagrams once for the depicted simply supported beam, carrying a linearly varying line load and additionally for a clamped support on the right and left side. **P5.1**



Solution *1. Simply supported beam.*

The linearly varying load, which can be described by

$$q(x) = q_0\,\frac{x}{l},$$

yields by integration

$$V(x) = -\int q(x)\mathrm{d}x = -q_0\,\frac{x^2}{2l} + C_1\,,$$

$$M(x) = \int V(x)\mathrm{d}x = -q_0\,\frac{x^3}{6l} + C_1 x + C_2\,.$$

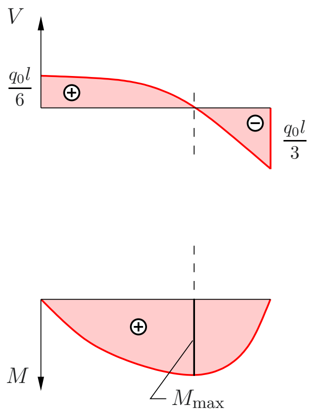
The constants follow from the support conditions:

$$M(0) = 0 \quad \rightsquigarrow \quad C_2 = 0\,,$$

$$M(l) = 0 \quad \rightsquigarrow \quad C_1 = \frac{q_0 l}{6}\,.$$

The shear force is obtained by

$$\underline{\underline{V(x) = \frac{q_0 l}{6}\left[1 - 3\frac{x^2}{l^2}\right]}}\,.$$



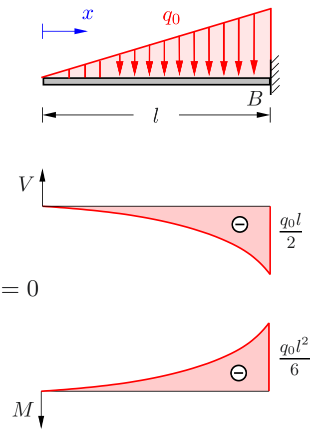
The values of $q_0 l/6$ and $q_0 l/3$ correspond to the support reactions. Due to the sign convention, a negative shear force indicates a force into the upwars direction. The bending moment diagram results in

$$\underline{\underline{M(x) = \frac{q_0 l x}{6}\left[1 - \frac{x^2}{l^2}\right]}}\,.$$

The maximum $M_{\max}$ appears at the root of the shear force: $V = 0$ for $\sqrt{3}\,l/3 = 0.577\,l$. It follows:

$$M_{\max} = q_0\,\frac{\sqrt{3}}{3}\,l^2\,\frac{1}{6}\big(1 - \frac{1}{3}\big) = \frac{\sqrt{3}}{27}\,q_0 l^2\,.$$

2. Beam, clamped on the right side



$$q(x) = q_0 \frac{x}{l}\,,$$

$$V(x) = -q_0 \frac{x^2}{2\,l} + C_1\,,$$

$$M(x) = -q_0 \frac{x^3}{6\,l} + C_1 x + C_2\,.$$

Due to the left boundary conditions

$$V(0) = 0 \leadsto C_1 = 0, \quad M(0) = 0 \leadsto C_2 = 0$$

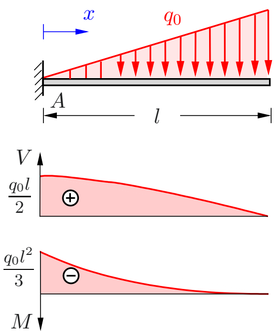
we obtain

$$\underline{\underline{V(x) = -\frac{q_0 x^2}{2l}}}\,, \qquad \underline{\underline{M(x) = -\frac{q_0 x^3}{6l}}}\,.$$

The result can be checked by determination of the moment at the right edge by the equilibrium conditions for the complete beam as

$$\uparrow: \quad B - \frac{1}{2} q_0 l = 0\,, \qquad \overset{\curvearrowleft}{B}: \quad M_B + \frac{l}{3}\,\frac{q_0 l}{2} = 0\,.$$

3. Beam, clamped on the left side



$$q(x) = q_0 \frac{x}{l}\,,$$

$$V(x) = -\frac{q_0 x^2}{2\,l} + C_1\,,$$

$$M(x) = -\frac{q_0 x^3}{6\,l} + C_1 x + C_2\,.$$

Due to the right boundary conditions

$$V(l) = 0 \leadsto C_1 = \frac{q_0 l}{2}\,,$$

$$M(l) = 0 \leadsto C_2 = \frac{q_0 l^2}{6} - C_1 l = -\frac{q_0 l^2}{3}\,,$$

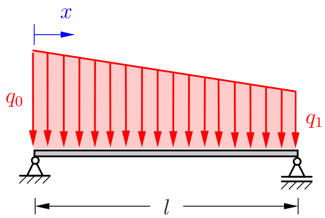
we obtain

$$\underline{\underline{V(x) = \frac{q_0 l}{2}\left[1 - \frac{x^2}{l^2}\right]}}\,, \qquad \underline{\underline{M(x) = -\frac{q_0 l^2}{6}\left[2 - 3\,\frac{x}{l} + \frac{x^3}{l^3}\right]}}\,.$$

To check the result, the clamping moment is obtained as

$$\overset{\curvearrowleft}{A}: \quad -M_A - \frac{2\,l}{3}\,\frac{q_0 l}{2} = 0 \quad \leadsto \quad M_A = -\frac{q_0 l^2}{3}\,.$$

Problem 5.2 A simply supported beam is loaded by a trapezoidal shaped load. Determine the location and the maximum of the bending moment for $q_0 = 2q_1$. **P5.2**



Solution The function of load is linear:

$$q(x) = a - b\,x\,.$$

We obtain due to the boundary conditions

$$q(0) = q_0 \quad \rightsquigarrow \quad a = q_0\,,$$

$$q(l) = q_1 \quad \rightsquigarrow \quad q_1 = a - b\,l \quad \rightsquigarrow \quad b = \frac{q_0 - q_1}{l}\,.$$

It follows

$$q(x) = q_0 - \frac{q_0 - q_1}{l}\,x\,.$$

This leads by integration to

$$V(x) = -q_0 x + \frac{q_0 - q_1}{l}\,\frac{x^2}{2} + C_1\,,$$

$$M(x) = -q_0\frac{x^2}{2} + \frac{q_0 - q_1}{l}\,\frac{x^3}{6} + C_1\,x + C_2\,.$$

Due to the boundary conditions the constants follow as

$$M(0) = 0 \quad \rightsquigarrow \quad C_2 = 0\,,$$

$$M(l) = 0 \quad \rightsquigarrow \quad C_1 = \frac{q_0 l}{2} - \frac{q_0 - q_1}{l}\,\frac{l^2}{6}\,.$$

The shearing force and the moment are obtained for $q_1 = \frac{1}{2}q_0$ as:

$$V(x) = \frac{q_0}{4\,l}x^2 - q_0\,x + \frac{5\,l\,q_0}{12}$$

$$M(x) = \frac{q_0}{12\,l}x^3 - \frac{q_0}{2}x^2 + \frac{5\,l\,q_0}{12}x\,.$$

Since $M' = V$, the maximal moment is found at the root of V:

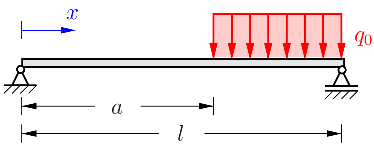
$$V = 0 \quad \rightsquigarrow \quad \underline{\underline{x^*}} = 2\,l \pm \sqrt{4\,l^2 - \frac{5}{3}\,l^2} = \underline{\underline{0.47\,l}}\,.$$

Inserting x^* into $M(x)$ yields

$$\underline{\underline{M_{max}}} = M(x^*) = \underline{\underline{0.09\;q_0 l^2}}\,.$$

P5.3

Problem 5.3 The depicted beam is partially loaded by q_0. Determine the shear force and bending moment diagrams.



Solution Due to the discontinuity of the loading we separate the integration into two parts:

$$
\begin{array}{ll}
0 \le x \le a: & a \le x \le l: \\
\qquad q = 0\,, & \qquad q = q_0\,, \\
\qquad V = C_1\,, & \qquad V = -q_0 x + C_3\,, \\
\qquad M = C_1 x + C_2\,, & \qquad M = -\frac{1}{2}\,q_0\,x^2 + C_3 x + C_4\,.
\end{array}
$$

The 4 integration constants follow from the boundary conditions and the transition condition at $x = a$:

$$M(0) = 0 \;\rightsquigarrow\; C_2 = 0\,, \qquad M(l) = 0 \;\rightsquigarrow\; -\frac{1}{2}\,q_0 l^2 + C_3 l + C_4 = 0.$$

Here V and M have to be continuous (no jumps, since no concentrated force and no single moment occur)

$$V(a^-) = V(a^+) \quad \rightsquigarrow \quad C_1 = -q_0 a + C_3\,,$$

$$M(a^-) = M(a^+) \quad \rightsquigarrow \quad C_1 a = -\frac{1}{2}\,q_0 a^2 + C_3 a + C_4\,.$$

It follows

$$C_1 = \frac{q_0 l}{2}\,\frac{(l-a)^2}{l^2}\,, \quad C_2 = 0\,, \quad C_3 = \frac{q_0 l}{2}\,\frac{l^2 + a^2}{l^2}\,, \quad C_4 = -\frac{q_0 a^2}{2}\,.$$

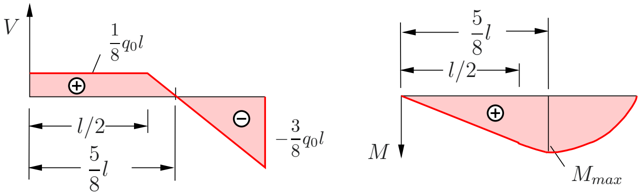
We obtain for the first part $0 \le x \le a$

$$\underline{\underline{V = \frac{q_0 l}{2}\,\frac{(l-a)^2}{l^2}}}\,, \qquad \underline{\underline{M = \frac{q_0 l^2}{2}\,\frac{(l-a)^2}{l^3}\,x}}$$

and for the second part $a \le x \le l$

$$\underline{\underline{V = \frac{q_0}{2}\left[\frac{(l-a)^2}{l} - 2\,(x-a)\right]}}\,, \qquad \underline{\underline{M = \frac{q_0}{2}\left[\frac{(l-a)^2}{l}\,x - (x-a)^2\right]}}.$$

For $a = l/2$, the shear-force and bending moment diagrams follow as



Remark:

- Instead of considering the parameter x for the complete length of the beam, we may also introduce separated parameters x_1, x_2
- For the special case of $a = 0$, the first part vanishes, such that we obtain for the shear force and the bending moment

$$V = \frac{1}{2}\, q_0\, (l - 2x)\,, \qquad M = \frac{1}{2}\, q_0\, (l\, x - x^2)\,.$$

Alternative: The diagrams can be determined by use of the FÖPPL symbol. Therefore we represent the discontinuous loading for the complete beam as

$$q = q_0 < x - a >^0 \quad \text{für} \quad 0 \le x \le l.$$

With application of the integration rules for the FÖPPL symbol, we obtain

$$V = -q_0 < x - a >^1 + C_1\,,$$
$$M = -\frac{q_0}{2} < x - a >^2 + C_1 x + C_2\,.$$

Due to the boundary conditions, this leads to (the transition conditions are directly fulfilled)

$$M(0) = 0 \leadsto C_2 = 0$$

$$M(l) = 0 \leadsto 0 = -\frac{q_0}{2}\,(l - a)^2 + C_1 l \quad \leadsto \quad C_1 = \frac{q_0}{2}\,\frac{(l - a)^2}{l}\,.$$

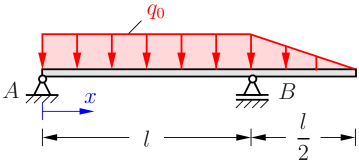
We obtain the solution for the entire beam as

$$\underline{\underline{V = \frac{q_0}{2}\left[\frac{(l - a)^2}{l} - 2 < x - a >^1\right]}}\,,$$

$$\underline{\underline{M = \frac{q_0}{2}\left[\frac{(l - a)^2 x}{l} - < x - a >^2\right]}}\,.$$

P5.4

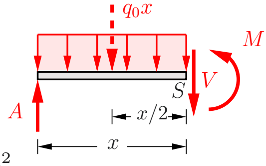
Problem 5.4 Determine the shear force and bending moment diagrams for the depicted beam.



Solution In a first step, the support reaction are determined (with A and B positive in the upwars direction):

$$A = \frac{11}{24}\, q_0 l\,, \qquad B = \frac{19}{24}\, q_0 l\,.$$

For the left subsection (between A and B) we obtain by the equilibrium conditions

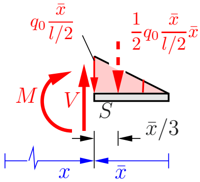


$$\uparrow: \quad A - q_0 x - V = 0\,,$$

$$\overset{\curvearrowleft}{S}: \quad -x\,A + \frac{x}{2}\,(q_0 x) + M = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{V = A - q_0\, x}}\,, \quad \underline{\underline{M = A\,x - \frac{q_0}{2}\,x^2}}$$

and for the right subsection, using an additional coordinate $\bar{x}$ starting at the free edge,

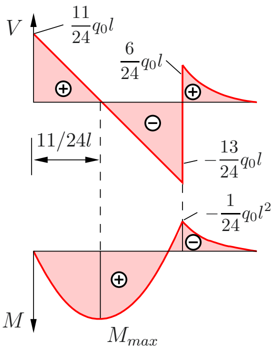


$$\uparrow: \quad -\frac{1}{2}\Big(q_0 \frac{\bar{x}}{l/2}\Big)\bar{x} + V = 0\,,$$

$$\overset{\curvearrowleft}{S}: \quad -\frac{\bar{x}}{3}\,\frac{1}{2}\Big(q_0 \frac{\bar{x}}{l/2}\Big)\bar{x} - M = 0\,,$$

$$\rightsquigarrow \quad \underline{\underline{V = \frac{q_0}{l}\,\bar{x}^2}}\,, \quad \underline{\underline{M = -\frac{q_0}{3l}\,\bar{x}^3}}\,.$$

Remarks:



- The value of the shear force reduces linearly from A to B. At B, it jumps by the magnitude of the reaction force. On the right section, it decays quadratically. At the free edge, the shear force is zero.
- At the free edge, we obtain $q = 0$. Due to $\mathrm{d}V/\mathrm{d}x = -q$, the slope of V is zero (horizontal tangent!) at this point.
- Due to the concentrated force, we obtain a kink in the moment diagram.

- M_{max} is located at $x = \frac{11}{24}l$ (since $V = 0$) and has a value of $M_{max} = \frac{1}{2}\left(\frac{11}{24}\right)^2 q_0 l^2 = 0.105\, q_0 l^2$.
- The slope of M at A is positive due to the positive V and $\mathrm{d}M/\mathrm{d}x = V$. At the free edge, the slope of M is zero, since $V = 0$.
- The bending moment at the support B is

$$M_B = -\frac{q_0}{3\,l}\,(l/2)^3 = -\frac{1}{24}\,q_0 l^2\,.$$

In an *alternative approach* the V and M diagrams are determined with help of the FÖPPL symbols. In this approach the supporting forces have not to be calculated a-priori. In a first step, the loading has to be constituted for the complete beam as the difference of the constant and linear line loadings:

$$q = q_0 - \frac{2q_0}{l} < x - l >^1$$

(the factor 2 is necessary, since q has to be reduced to zero over the length $l/2$). We obtain by integration

$$V = -q_0\,x + \frac{q_0}{l} < x - l >^2 + B < x - l >^0 + C_1$$

(the jump of the shear force due to the unknown supporting force B has to be considered by an additional FÖPPL bracket!)

$$M = -q_0\,\frac{x^2}{2} + \frac{q_0}{3\,l} < x - l >^3 + B < x - l >^1 + C_1 x + C_2\,.$$

The three unknowns C_1, C_2 and B can be solved, taking into account the three equations from the boundary conditions:

$$\begin{aligned} M(0) = 0 \quad &\leadsto \quad C_2 = 0\,, \\ V(\tfrac{3}{2}l) = 0 \quad &\leadsto \quad -\frac{3}{2}\,q_0 l + \frac{1}{4}\,q_0 l + B + C_1 = 0\,, \\ M(\tfrac{3}{2}l) = 0 \quad &\leadsto \quad -\frac{9}{8}\,q_0 l^2 + \frac{1}{24}\,q_0 l^2 + B\,\frac{l}{2} + \frac{3}{2}\,C_1 l = 0\,. \end{aligned}$$

We obtain

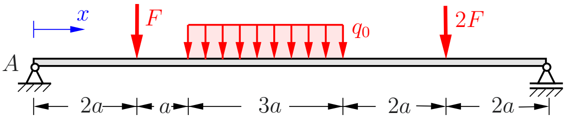
$$B = \frac{19}{24}\,q_0 l\,, \qquad C_1 = \frac{11}{24}\,q_0 l\,,$$

Remark: The constant C_1 is equal to the shear force at the support A and thus, the resulting reaction force.

P5.5

Problem 5.5 Determine the shear force and bending moment diagrams for the depicted multi-section beam. Calculate additionally the values at the intersections.

Use: $q_0 = F/a$.



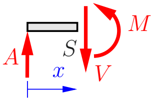
Solution In a first step, we compute the support reactions (positive in the upwards direction):

$$\overset{\curvearrowleft}{A}: \quad -2a\,F - 4,5a\,(3q_0 a) - 8a\,2F + 10a\,B = 0 \quad \rightsquigarrow \quad B = 3.15\,F\,,$$

$$\uparrow: \quad A + B - F - 3q_0 a - 2F = 0 \quad \rightsquigarrow \quad A = 2.85\,F\,.$$

With help of the free body diagrams, we obtain for the subsections:

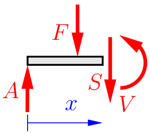
$0 < x < 2a$:



$$\uparrow: \quad V = A = 2.85\;F\,,$$

$$\overset{\curvearrowleft}{S}: \quad M = x\,A = 2.85\,Fx\,,$$

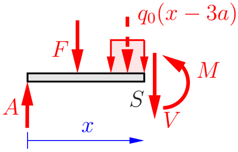
$2a < x < 3a$:



$$\uparrow: \quad V = A - F = 1.85\,F\,,$$

$$\overset{\curvearrowleft}{S}: \quad M = x\,A - (x-2a)F\,,$$

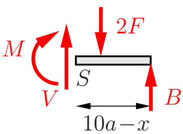
$3a < x < 6a$:



$$\uparrow: \quad V = 1.85\,F - q_0(x-3a)\,,$$

$$\overset{\curvearrowleft}{S}: \quad M = x\,A - (x-2a)F - \tfrac{1}{2}q_0(x-3a)^2\,,$$

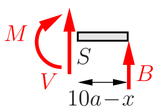
$6a < x < 8a$:



$$\uparrow: \quad V = -B + 2F = -1.15\,F\,,$$

$$\overset{\curvearrowright}{S}: \quad M = (10a-x)\,B - (8a-x)\,2F\,,$$

$8a < x < 10a$:



$$\uparrow: \quad V = -B = -3.15\,F\,,$$

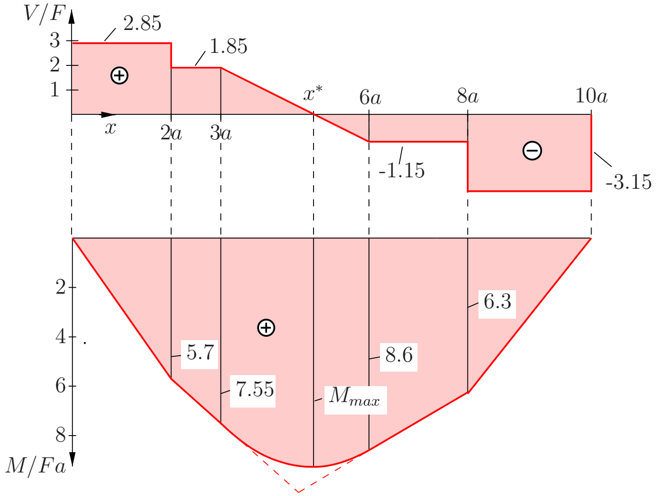
$$\overset{\curvearrowright}{S}: \quad M = (10a-x)\,B\,.$$

The maximal value of M can be found by the root of V ($M' = V$) in the 3^{rd} subsection ($3a < x < 6a$):

$$V = 1.85\,F - q_0(x-3a) = 0 \quad \rightsquigarrow \quad x^* = 1.85\,F/q_0 + 3a = 4.85\,a\,.$$

Thus we obtain

$$\underline{\underline{M_{max}}} = M(x^*) = 4.85\,a\,2.85\,F - 2.85\,a\,F - \frac{1}{2}\,q_0(1.85\,a)^2 = \underline{\underline{9.26\,Fa}}\,.$$



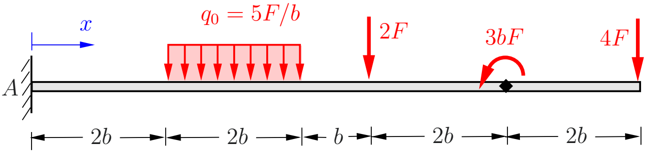
The V and M diagrams can also be determined using the Föppl symbol. Therefore, the discontinuities of $q(x)$ and $V(x)$ have to be taken into account:

$$\begin{aligned}
q &= q_0 < x-3a >^0 - q_0 < x-6a >^0\,,\\
V &= -q_0 < x-3a >^1 + q_0 < x-6a >^1 - F < x-2a >^0\\
&\quad -2F < x-8a >^0 + C_1\,,\\
M &= -\frac{1}{2}q_0 < x-3a >^2 + \frac{1}{2}q_0 < x-6a >^2 - F < x-2a >^1\\
&\quad -2F < x-8a >^1 + C_1 x + C_2\,.
\end{aligned}$$

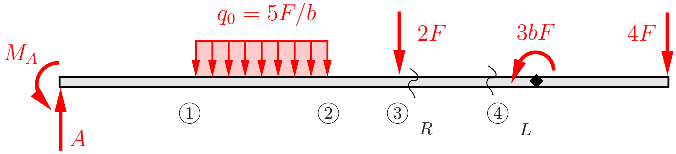
It follows from the boundary conditions

$$M(0) = 0 \rightsquigarrow C_2 = 0\,, \qquad M(10a) = 0 \rightsquigarrow C_1 = 2.85\,F\,.$$

P5.6 **Problem 5.6** Determine the shear force and bending moment diagrams for the depicted system.



Solution In a first step the support reactions are determined.



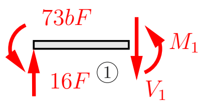
$$\uparrow: \quad A = 5\,\frac{F}{b}\cdot 2b + 2\,F + 4\,F = 16\,F\,,$$

$$\overset{\curvearrowleft}{A}: \quad M_A = 3b\,\big(5\,\frac{F}{b}\cdot 2b\big) + 5b\cdot 2\,F - 3b\,F + 9b\cdot 4F = 73\,b\,F\,.$$

For the calculation of V and M, we cut the beam at every discontinuity of the loading with respect to the internal forces and moments. With the equilibrium conditions, we can determine V and M with respect to the external loads and the internal forces and moments.

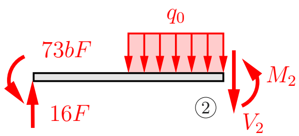
$$V_1 = 16\,F\,,$$

$$M_1 = 2b\cdot 16\,F - 73\,bF = -41\,bF\,,$$



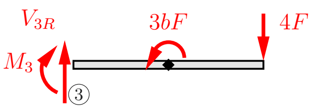
$$V_2 = 16\,F - 5\,\frac{F}{b}\cdot 2b = 6F\,,$$

$$M_2 = 4b\cdot 16F - 73\,bF - b\,\big(5\frac{F}{b}\cdot 2b\big)$$
$$= -19\,bF\,,$$



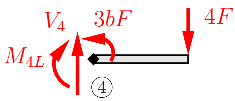
$$V_{3R} = 4F\,,$$

$$M_3 = 3b\,F - 4b\cdot 4F = -13\,bF\,,$$

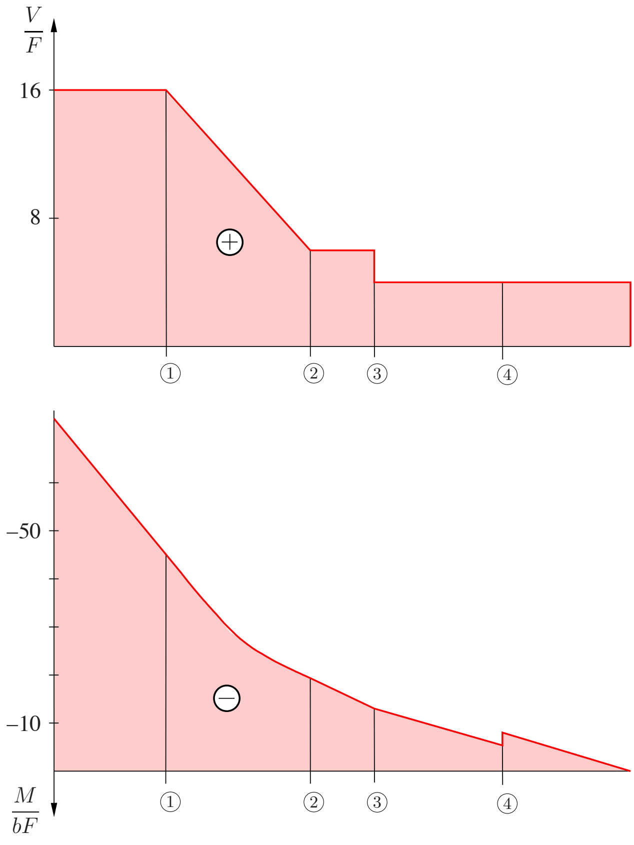


$$V_4 = 4F\,,$$

$$M_{4L} = 3b\,F - 2b\cdot 4F = -5\,bF\,.$$



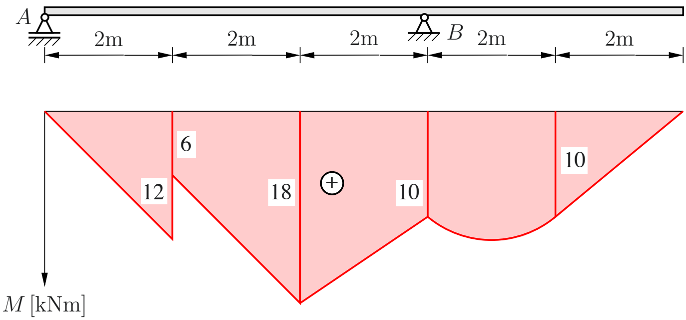
These results and the general relations between the external loadings and V and M (e.g. if $q = 0$, then $V =$ constant and $M =$ linear, see table at page 97) yield to shear force and bending moment diagrams:



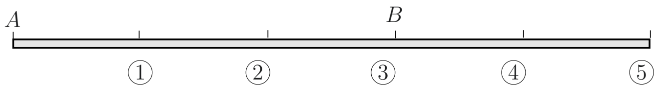
In the section between ① and ② of the bending moment diagram, the quadratic parabola has to pass tangentially into the straight line, since there is no concentrated force (concentrated forces lead to kinks in the bending moment diagram!).

P5.7

Problem 5.7 In this problem, the bending moment diagram of the system is known.
Determine the related loadings.



Solution We consider the highlighted points at the beam and the M-diagram in between them:



Due to the linear behavior, starting at point A with $M_1 = 12\,\mathrm{kNm} = 2\,\mathrm{m} \cdot A$, we obtain the reaction force as

$$\underline{\underline{A = 6\,\mathrm{kN}}}\,.$$

This is followed by a jump at the M diagram at point 1, such that we can identify a single moment at this point

$$\underline{\underline{M^* = 6\,\mathrm{kNm}}}$$

We can check the results, computing the M at 2 with A and M^*:

$$M_2 = 4\,\mathrm{m} \cdot 6\,\mathrm{kN} - 6\,\mathrm{kNm} = 18\,\mathrm{kNm}\,.$$

At point 2, we can identify the single force F due to the kink at the M-diagram. It can be calculated by

$$M_3 = 6\,\mathrm{m} \cdot 6\,\mathrm{kN} - 6\,\mathrm{kNm} - 2\,\mathrm{m} \cdot F = 10\,\mathrm{kNm} \quad \rightsquigarrow \quad \underline{\underline{F = 10\,\mathrm{kN}}}\,.$$

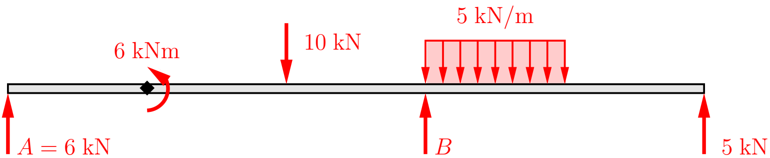
At the right edge, a concentrated force has to act into upwards direction, since a linear M diagram is considered. It is obtained by

$$M_4 = 2\,\mathrm{m} \cdot P = 10\,\mathrm{kNm} \quad \rightsquigarrow \quad \underline{\underline{P = 5\,\mathrm{kN}}}\,.$$

The quadratic parabola between point 3 and 4 is caused by a constant line load q_0. We obtain q_0 by the equilibrium of the right sub system as:

$$M_3 = 4\,\text{m} \cdot 5\,\text{kN} - 1\,\text{m} \cdot (q_0 \cdot 2\,\text{m}) = 10\,\text{kNm} \quad \rightsquigarrow \quad \underline{\underline{q_0 = 5\,\text{kN/m}}}\,.$$

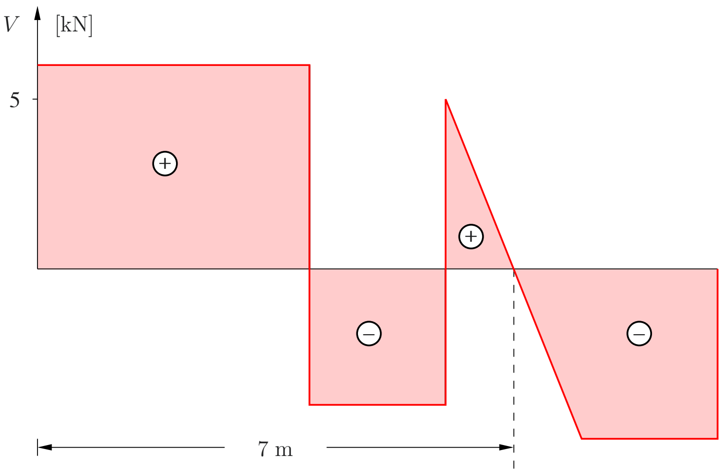
Now all external loadings have been determined and can be sketched as follows:



The omitted support reaction B follows from:

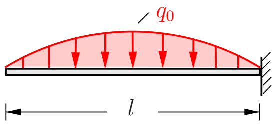
$$\uparrow: \quad B = 10 + 2 \cdot 5 - 5 - 6 = 9\,\text{kN}\,.$$

Finally, we are able to construct the shear-force diagram:



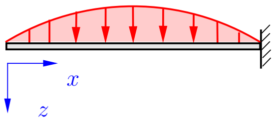
The (local) maximal bending moment can be found at the root of the shear-force diagram with $x = 7\,\text{m}$.

P5.8 **Problem 5.8** The depicted beam is loaded by a sinusoidal line load. Determine the bending moment diagram.



Solution We select the left edge of the beam as the origin of the coordinate system, since the shear force and the bending moment are zero at this point:

$$q(x) = q_0 \sin \frac{\pi x}{l} \,.$$



Integration leads to

$$V(x) = -\int q_0 \, \sin \frac{\pi x}{l} \mathrm{d}x = q_0 \, \frac{l}{\pi} \cos \frac{\pi x}{l} + C_1 \,,$$

$$M(x) = q_0 \left(\frac{l}{\pi}\right)^2 \sin \frac{\pi x}{l} + C_1 x + C_2 \,.$$

Using the boundary conditions, we obtain

$$V(0) = 0 \quad \rightsquigarrow \quad C_1 = -\frac{q_0 l}{\pi} \,,$$

$$M(0) = 0 \quad \rightsquigarrow \quad C_2 = 0 \,.$$

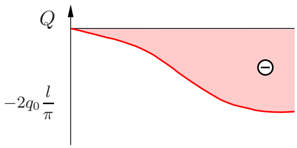
Therefore, the function of the shear force and the bending moment are

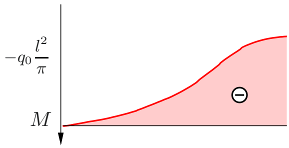
$$V(x) = \frac{q_0 l}{\pi} \left(\cos \frac{\pi x}{l} - 1\right) \,, \qquad M(x) = -\frac{q_0 l^2}{\pi^2} \left(\frac{\pi x}{l} - \sin \frac{\pi x}{l}\right) \,.$$

The maximal values of V and M are found at the clamped edge $x = l$:

$$V(l) = -\frac{2}{\pi} \, q_0 l \,, \qquad M(l) = -\frac{1}{\pi} \, q_0 l^2 \,.$$

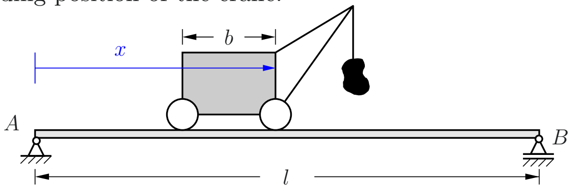
The sketch of the diagrams yield





Problem 5.9 A gantry crane of weight W moves across a bridge with the length l. The front axle of the crane carries $\frac{3}{4}W$, whereas the rear axle carries $\frac{1}{4}W$. The distance of the axles is $b = l/20$. **P5.9**

Determine the maximum value of the bending moment and the corresponding position of the crane.



Solution In a first step, the support reaction A (positive into the upwards direction) is calculated for an arbitrary distance x of the front axle:

$$\overset{\curvearrowright}{B}: \quad l\,A = (l-x)\frac{3}{4}W + (l-x+b)\frac{W}{4} \quad \leadsto \quad A = \left(\frac{81}{80} - \frac{x}{l}\right)W\,.$$

The maximal bending moment may occur at the rear (R) or at the front (F) axle. We obtain

$$M_R = (x-b)A = \left(x - \frac{l}{20}\right)\left(\frac{81}{80} - \frac{x}{l}\right)W\,,$$

$$M_F = x\,A - b\,\frac{W}{4} = x\left(\frac{81}{80} - \frac{x}{l}\right)W - \frac{l}{80}W\,.$$

The extreme values of the bending moment are obtained at the roots of the derivatives. Thus, for the front axle,

$$\frac{\mathrm{d}M_F}{\mathrm{d}x} = \frac{81}{80}W - 2\,\frac{x}{l}\,W = 0 \qquad \text{yields} \qquad \underline{\underline{x_1 = \frac{81}{160}\,l}}$$

and therefore

$$\underline{\underline{M_{Fmax} = \frac{6241}{25600}\,W\,l\,.}}$$

For the rear axle,

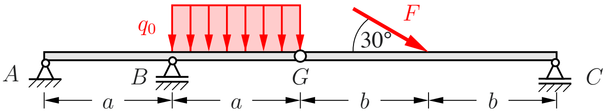
$$\frac{\mathrm{d}M_R}{\mathrm{d}x} = \frac{81}{80}W - 2\,\frac{x}{l}\,W + \frac{1}{20}W = 0 \qquad \text{yields} \qquad \underline{\underline{x_2 = \frac{85}{160}\,l}}$$

and therefore

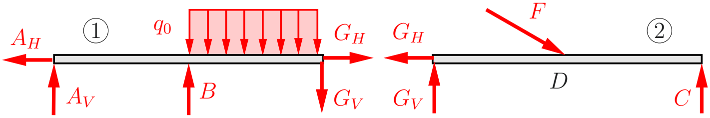
$$\underline{\underline{M_{Rmax} = \frac{5929}{25600}\,W\,l\,.}}$$

The highest value of the bending moment is obtained at x_1.

P5.10 **Problem 5.10** Determine the shear force, bending moment and axial force diagrams of the depicted hinged girder system.



Solution The free-body sketch is used to determine the support reactions:



From the equilibrium equations, we obtain

$$
\begin{aligned}
&① && &&②\\
&\rightarrow: & -A_H + G_H = 0\,, \qquad && \rightarrow: & -G_H + F\cos 30^\circ = 0\,,\\
&\uparrow: & A_V + B - q_0 a - G_V = 0\,, \qquad && \uparrow: & G_V + C - F\sin 30^\circ = 0\,,\\
&\overset{\curvearrowright}{G}: & 2a\,A_V + a\,B - \frac{a}{2}\,q_0 a = 0\,, \qquad && \overset{\curvearrowright}{G}: & b\,F\sin 30^\circ - 2b\,C = 0
\end{aligned}
$$

Applying $\sin 30^\circ = 1/2$ and $\cos 30^\circ = \sqrt{3}/2$ yields

$$
A_H = \frac{\sqrt{3}}{2}\,F\,, \qquad A_V = -\frac{q_0 a}{2} - \frac{F}{4}\,, \qquad B = \frac{3}{2}\,q_0 a + \frac{F}{2}\,,
$$

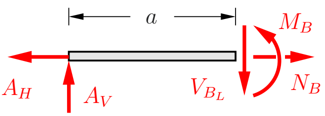
$$
C = \frac{F}{4}\,, \qquad G_V = \frac{F}{4}\,, \qquad G_H = \frac{\sqrt{3}}{2}\,F\,.
$$

The internal forces and moments are determined for the indicated points. At the points A, G and C, we have hinges, therefore the moment is zero. At the points B and D the shear force has a jump of the magnitude of the support forces, respectively, the vertical component of F ($F\sin 30^\circ = F/2$). At D, the axial force jumps with the magnitude of the horizontal component of F ($F\cos 30^\circ = \sqrt{3}F/2$). We obtain

$$
N_B = A_H\,,
$$

$$
V_{B_L} = A_V\,,
$$

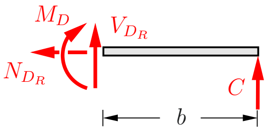
$$
M_B = a\,A_V\,,
$$



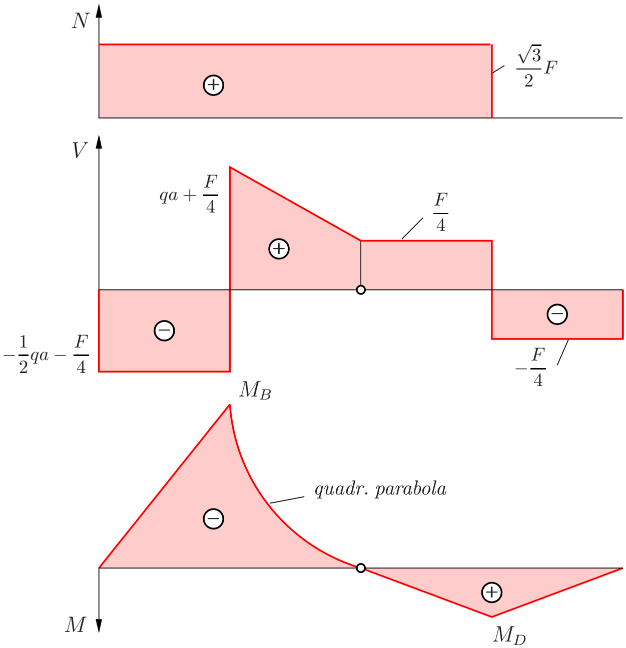
$$N_{D_R} = 0\,,$$

$$V_{D_R} = -C\,,$$

$$M_D = b\,C = b\,\frac{F}{4}\,.$$



Therefore, the diagrams can be sketched as



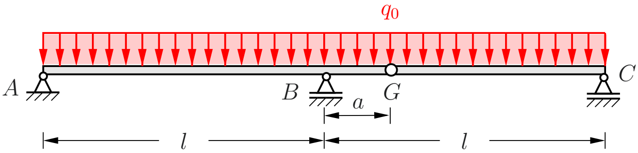
Remark:

- In the area $\overline{BG}$, the function of the bending moment is a quadratic parabola. From the V-function, it follows that the magnitude of the slope at B is higher than at point G.

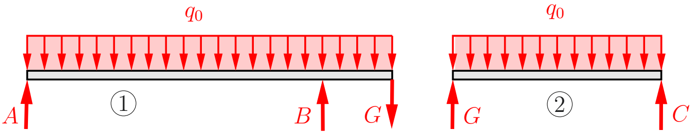
- The quadratic parabola has to merge into the linear function between G and D without a kink, since no additional load is located at the hinge.

P5.11 **Problem 5.11** Determine the shear force, bending moment and axial force diagrams of the depicted hinged girder system.

Calculate the distance a of the hinge G, such that the maximal bending moment is as small as possible.



Solution The support reactions and hinge forces are determined with the help of the free body sketch



and the equilibrium equations

$$① \uparrow: \quad A + B - G - q_0(l+a) = 0\,,$$

$$\overset{\curvearrowright}{G}: \quad (l+a)\,A + a\,B - \frac{q_0(l+a)^2}{2} = 0\,,$$

$$② \uparrow: \quad G + C - q_0(l-a) = 0\,,$$

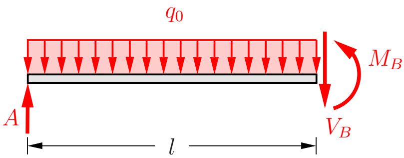
$$\overset{\curvearrowright}{G}: \quad \frac{q_0(l-a)^2}{2} - (l-a)\,C = 0.$$

Therefore, we obtain

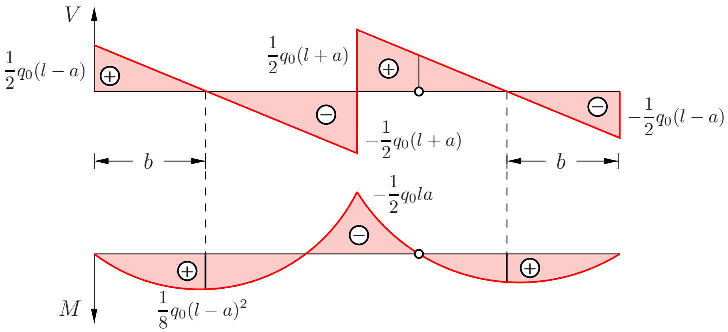
$$A = G = C = \frac{q_0(l-a)}{2}\,, \qquad B = q_0\,(l+a)\,.$$

The bending moment in B can be expressed as

$$M_B = l\,A - \frac{q_0 l^2}{2} = -\frac{1}{2}\,q_0 l\,a\,.$$



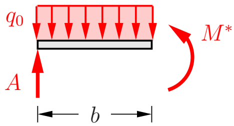
This leads to the depicted shear force and bending moment diagrams



Remark:

- The shear-force diagram is antisymmetric regarding B.
- The shear force in the midpoint between G and C, b=(l-a)/2, has to be zero. This is a consequence of the symmetric loading regarding the free-body sketch.
- The shear-force diagram indicates that the magnitude of the slope of M at A has to be smaller than at B
- The bending-moment diagram is symmetric regarding B.

The relative extreme values of M are located at the roots of V. They have a distance $b = (l-a)/2$ from A and from C. We obtain



$$M^* = b\,A - \frac{q_0 b^2}{2} = \frac{q_0}{8}(l-a)^2 .$$

In order to minimize the extremum, we use

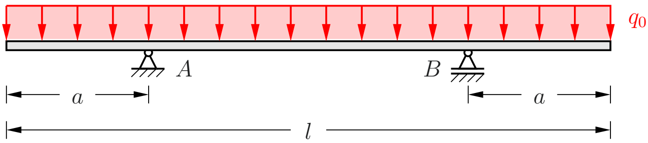
$$|M_B| = |M^*| .$$

Insertion yields the distance

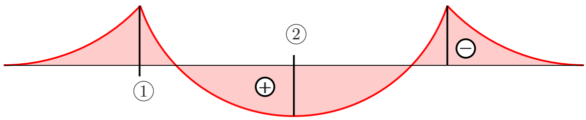
$$\frac{1}{2}\,q_0 l\,a = \frac{1}{8}\,q_0(l-a)^2 \quad \leadsto \quad \underline{\underline{a}} = (3-\sqrt{8})l = \underline{\underline{0.172\,l}} .$$

P5.12 **Problem 5.12** A beam, overlapping at both ends, is loaded by a uniform line load.

Find the overlapping length a for a given total length l, which minimizes the extreme value of the moment.



Solution The local extremal values of the moments are found at the supports and in the midpoint of the beam:



They can be determined as (with the support reactions, due to the symmetry, as $A = B = q_0 l/2$)

$$M_1 = -q_0 \frac{a^2}{2},$$

$$M_2 = -q_0 \frac{(l/2)^2}{2} + \frac{q_0 l}{2}\left(\frac{l}{2} - a\right).$$

The extreme value is minimal for the case $|M_1| = |M_2|$:

$$q_0 \frac{a^2}{2} = q_0 \frac{l}{2}\left(\frac{l}{2} - a\right) - q_0 \frac{l^2}{8}.$$

Thus, we obtain the overlapping length as

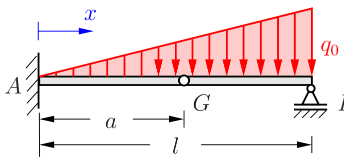
$$\underline{\underline{a = \frac{1}{2}\left(\sqrt{2} - 1\right)}}\, l = 0.207\, l$$

with the corresponding moment as

$$\underline{\underline{|M_{max}| = \frac{3 - 2\sqrt{2}}{8}\, q_0 l^2}} = 0.0214\, q_0 l^2.$$

Note that the magnitude of the maximal moment is only 17 % compared to the case with supports at the edge of the beams ($|M_{max}| = q_0 l^2/8$).

Problem 5.13 Determine the shear force and bending moment diagrams for the depicted system by integration. **P5.13**



Solution We obtain by integration of $q(x) = q_0 x/l$

$$V(x) = -q_0\,\frac{x^2}{2l} + C_1\,, \qquad M(x) = -q_0\,\frac{x^3}{6l} + C_1 x + C_2\,.$$

The constants C_1 and C_2 are determined by the boundary and transition conditions. We know that the moment is zero in G and B:

$$M(x=a) = 0\ :\quad -q_0\,\frac{a^3}{6l} + C_1\,a + C_2 = 0\,,$$

$$M(x=l) = 0\ :\quad -q_0\,\frac{l^2}{6} + C_1\,l + C_2 = 0\,.$$

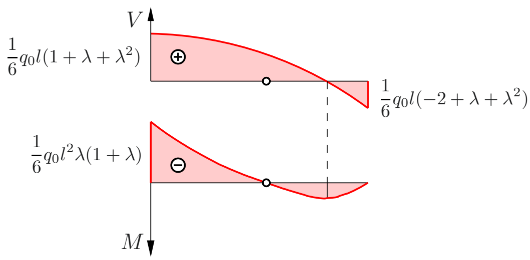
Introducing the abbreviation $\lambda = a/l$ leads to

$$C_1 = \frac{q_0 l}{6}\left(1+\lambda+\lambda^2\right)\,, \qquad C_2 = -\frac{q_0 l^2}{6}\lambda(1+\lambda).$$

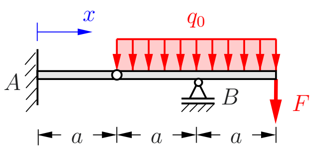
Thus, we obtain

$$\underline{\underline{V(x) = \frac{q_0 l}{6}\left[(1+\lambda+\lambda^2) - 3\left(\frac{x}{l}\right)^2\right]}}\,,$$

$$\underline{\underline{M(x) = -\frac{q_0 l^2}{6}\left[\lambda(1+\lambda) - (1+\lambda+\lambda^2)\left(\frac{x}{l}\right) + \left(\frac{x}{l}\right)^3\right]}}\,.$$



P5.14 **Problem 5.14** The depicted hinged girder system is loaded by a uniform line load q_0 and a concentrated force F .



Determine the shear force and bending moment diagrams.

Solution The line load can be depicted by use of the FÖPPL symbol as

$$q(x) = q_0 < x - a >^0 .$$

We obtain by integration

$$V(x) = -q_0 < x - a >^1 + B < x - 2a >^0 + C_1 ,$$

$$M(x) = -\frac{1}{2}\, q_0 < x - a >^2 + B < x - 2a >^1 + C_1 x + C_2 .$$

The unknown supporting reaction B causes a jump in the shear force diagram. This has to be considered in $V(x)$! The support reaction B and the constants C_1 and C_2 can be determined by the following conditions:

$$V(x = 3a) = F \quad \rightsquigarrow \quad -2q_0 a + B + C_1 = F ,$$

$$M(x = a) = 0 \quad \rightsquigarrow \quad C_1 a + C_2 = 0 ,$$

$$M(x = 3a) = 0 \quad \rightsquigarrow \quad -2\, q_0 a^2 + B\, a + 3a\, C_1 + C_2 = 0$$

This leads to

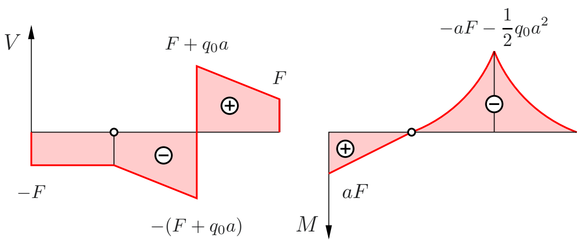
$$C_1 = -F , \qquad C_2 = a\, F , \qquad B = 2\, q_0 a + 2F .$$

Thus, we obtain for example in the points A and B

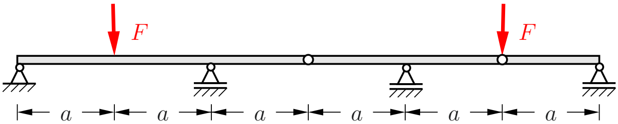
$$M_A = M(0) = C_2 = a\, F ,$$

$$M_B = M(2a) = -\frac{1}{2}\, q_0 a^2 + C_1 2a + C_2 = -\frac{1}{2} q_0 a^2 - aF .$$

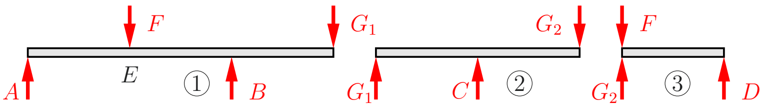
The diagrams are given as



Problem 5.15 Determine the shear force and bending moment diagrams for the depicted hinged girder system. **P5.15**



Solution In a first step, the support and hinge reactions are calculated. From the equilibrium equations



$$
\begin{aligned}
&① \uparrow: \quad A + B - F - G_1 = 0\,, && \overset{\curvearrowright}{A}: \quad a\,F - 2a\,B + 3a\,G_1 = 0\,,\\
&② \uparrow: \quad G_1 + C - G_2 = 0\,, && \overset{\curvearrowright}{C}: \quad a\,G_1 + a\,G_2 = 0\,,\\
&③ \uparrow: \quad G_2 + D - F = 0\,, && \overset{\curvearrowright}{D}: \quad a\,G_2 - a\,F = 0\,,
\end{aligned}
$$

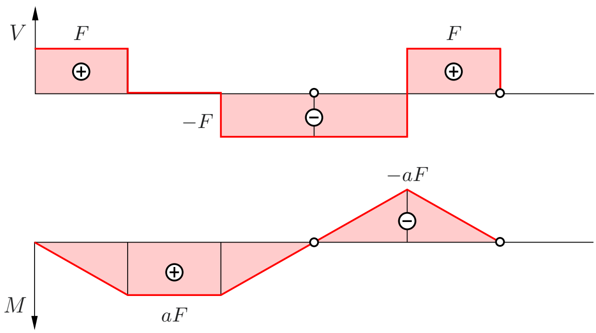
the support and hinge reactions are computed

$$
A = F\,, \quad B = -F\,, \quad C = 2F\,, \quad D = 0\,, \quad G_1 = -F\,, \quad G_2 = F\,.
$$

We obtain the bending moment at B, C and E as

$$
M_E = a\,F\,, \qquad M_B = 2\,a\,F - a\,F = a\,F\,, \qquad M_C = -a\,F\,.
$$

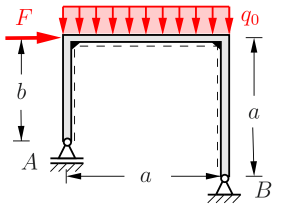
Thus, the diagrams are as follows:



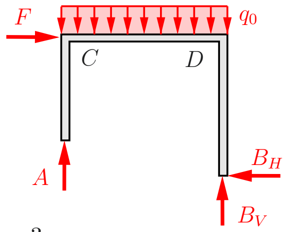
P5.16

Problem 5.16 The depicted frame is loaded by a force F and a uniform line load $q_0 = F/a$.

Determine the shear force and bending moment diagrams.



Solution The equilibrium equations



$$\uparrow: \quad A + B_V - q_0 a = 0\,,$$

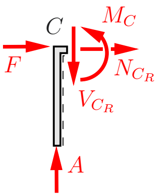
$$\rightarrow: \quad F - B_H = 0\,,$$

$$\overset{\curvearrowleft}{B}: \quad -a\,A + \tfrac{1}{2}\,q_0 a^2 - a\,F = 0$$

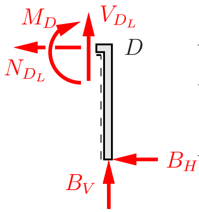
yield the support reactions

$$A = \frac{q_0 a}{2} - F = -\frac{F}{2}\,, \qquad B_V = \frac{q_0 a}{2} + F = \frac{3}{2}\,F\,, \qquad B_H = F\,.$$

We cut at the corners of the frame, directly right of C and directly left of D. The internal forces follow as

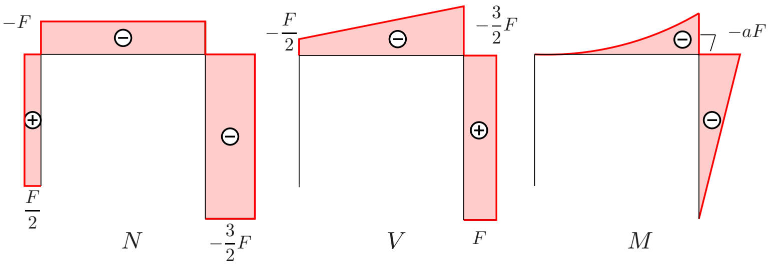


$$N_{C_R} = -F\,, \qquad V_{C_R} = -F/2\,, \qquad M_C = 0\,,$$

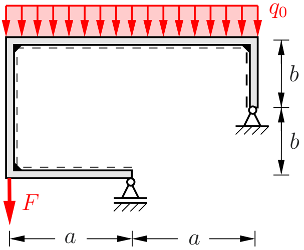


$$N_{D_L} = -F\,, \qquad V_{D_L} = -3F/2, \qquad M_D = -a\,F\,.$$

Due to the general coherences of the loadings and the internal forces, we obtain the depicted diagrams (Remark: the bending moments do not change at unloaded 90°-corners but the normal and shear forces are interchanged!):

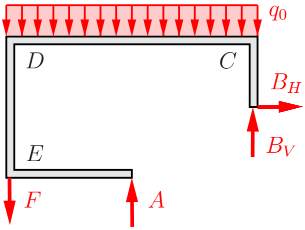


Problem 5.17 Determine the normal force, shear force and bending moment diagrams for the depicted frame. P5.17



Solution We obtain the support reactions with help of the equilibrium equations as

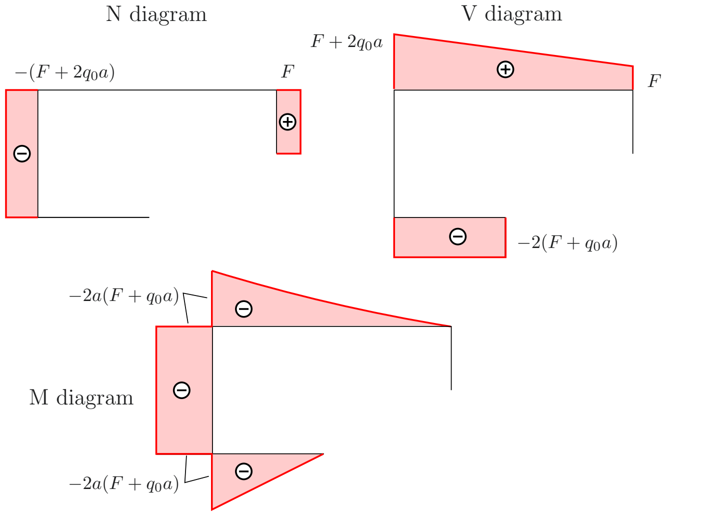
$$A = 2F+2q_0a, \quad B_V = -F, \quad B_H = 0.$$



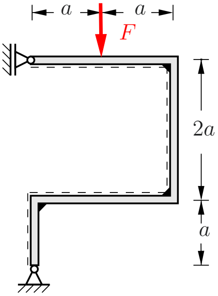
We obtain the bending moments at C, D and E as

$$M_C = 0\,, \qquad M_D = M_E = -a\,A = -2a(F+q_0a)\,.$$

Due to the relations between q, V and M together with N-V or V-N at the corners, we obtain following diagrams:

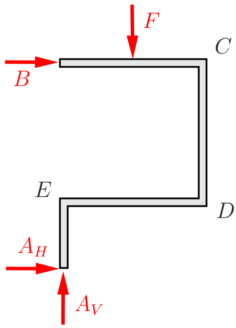


P5.18 **Problem 5.18** Determine the normal force, shear force and bending moment diagrams for the depicted frame.



Solution We obtain the support reactions with the help of the equilibrium equations as

$$A_V = F\,, \quad A_H = \frac{F}{3}\,, \quad B = -\frac{F}{3}\,.$$



We obtain the bending moments at the cornes C, D and E as

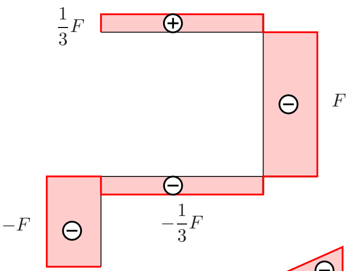
$$M_C = -a\,F\,,$$

$$M_D = 2a\,B - a\,F = -\frac{5}{3}\,a\,F\,,$$

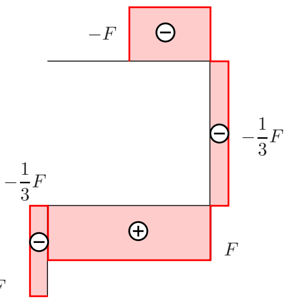
$$M_E = \frac{1}{3}\,a\,F\,.$$

Thus, the diagrams follow as:

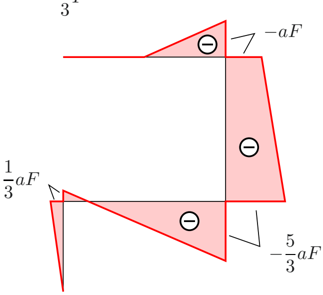
N diagram



V diagram

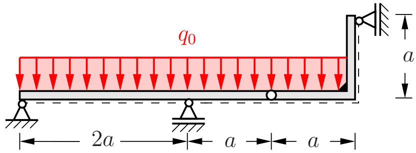


M diagram

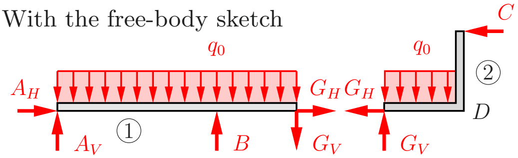


Problem 5.19 Determine the normal force, shear force and bending moment diagrams for the depicted hinged frame.

P5.19



Solution With the free-body sketch



the equilibrium equations for the sub systems are obtained as

$$
\begin{aligned}
&① \uparrow: \quad A_V + B - G_V - 3q_0 a = 0\,, && ② \uparrow: \quad G_V - q_0 a = 0\,,\\
&\rightarrow: \quad A_H + G_H = 0\,, && \rightarrow: \quad -G_H - C = 0\,,\\
&\overset{\curvearrowright}{A}: \quad -2a\,B + 3a\,G_V + \frac{9}{2}\,q_0 a^2 = 0\,, && \overset{\curvearrowright}{G}: \quad -a\,C + \frac{1}{2}\,q_0 a^2 = 0\,.
\end{aligned}
$$

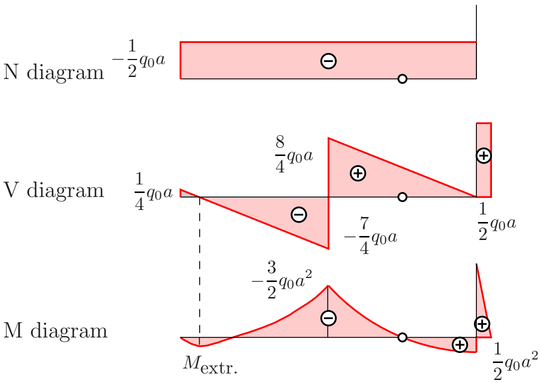
Thus, the support and hinge reactions follow by

$$
A_V = \frac{q_0 a}{4}\,, \quad A_H = \frac{q_0 a}{2}\,, \quad B = \frac{15}{4}\,q_0 a\,, \quad C = \frac{q_0 a}{2}\,,
$$
$$
G_H = -\frac{q_0 a}{2}\,, \quad G_V = q_0 a\,.
$$

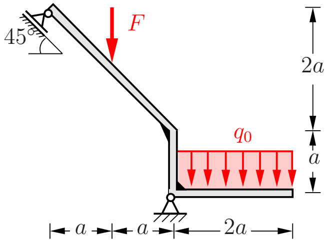
The bending moments result at B and D in

$$
M_B = 2a\,A_V - \frac{1}{2}\,(q_0\,2a)^2 = -\frac{3}{2}\,q_0 a^2\,, \qquad M_D = aC = \frac{1}{2}\,q_0 a^2\,.
$$

The diagrams can be sketched as

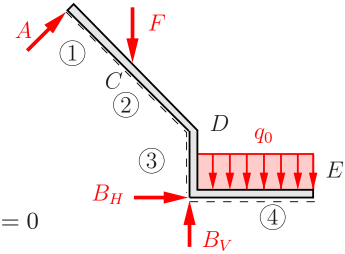


P5.20 **Problem 5.20** The depicted frame is loaded by a single force F and a uniform line load $q_0 = F/a$.



Determine the normal force, shear force and bending moment diagrams.

Solution We obtain by the equilibrium equations for the system



$$\rightarrow: \quad \frac{\sqrt{2}}{2} A + B_H = 0 ,$$

$$\uparrow: \quad \frac{\sqrt{2}}{2} A + B_V - F - 2q_0 a = 0 ,$$

$$\overset{\curvearrowright}{A}: \quad a\,F - 3a\,B_H - 2a\,B_V + 6q_0 a^2 = 0$$

the support reactions as

$$A = -\frac{\sqrt{2}}{5} F , \qquad B_V = \frac{16}{5} F , \qquad B_H = \frac{1}{5} F .$$

Considering the equilibrium in the sub systems, we obtain the normal and shear forces for ① up to ③ as

$$N_1 = 0 , \qquad V_1 = A = -\frac{\sqrt{2}}{5} F ,$$

$$N_2 = -\frac{\sqrt{2}}{2} F , \qquad V_2 = A - \frac{\sqrt{2}}{2} F = -\frac{7}{10}\sqrt{2} F ,$$

$$N_3 = \frac{\sqrt{2}}{2} A - F = -\frac{6}{5} F , \qquad V_3 = -\frac{\sqrt{2}}{2} A = \frac{F}{5} .$$

The bending moments result in the cutting points B, C and D in

$$M_B = -a\, 2q_0 a = -2aF ,$$

$$M_C = a\sqrt{2}\, A = -\frac{2}{5} aF ,$$

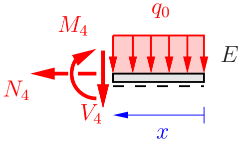
$$M_D = 2a\sqrt{2}\, A - aF = -\frac{9}{5} aF .$$

For subsection ④, we obtain

$$N_4 = 0\,,$$

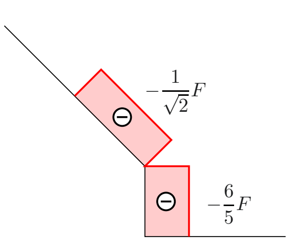
$$V_4 = -q_0 x = -F\,\frac{x}{a}\,,$$

$$M_4 = -\frac{1}{2}\,q_0 x^2 = -\frac{1}{2}\,a\,F\,\frac{x^2}{a^2}\,.$$

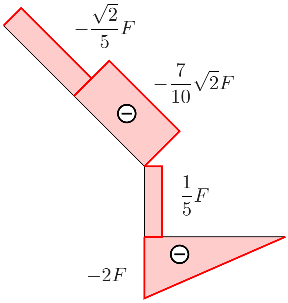


Remark: Since x runs from the right to the left, V_4 is positive downwards.
Thus, the diagrams follow as depicted. Jumps are obtained in N and V at the corners of the frame and at the point of attack of the single-forces; bending moments are constant over the angles.

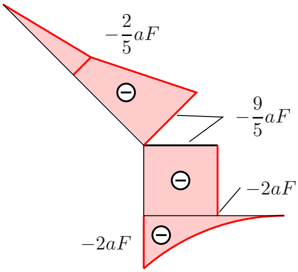
N diagram



V diagram



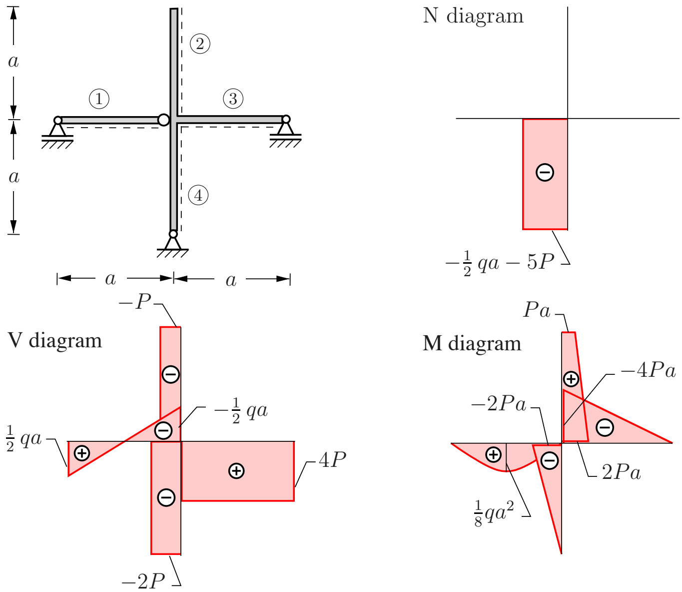
M diagram



P5.21

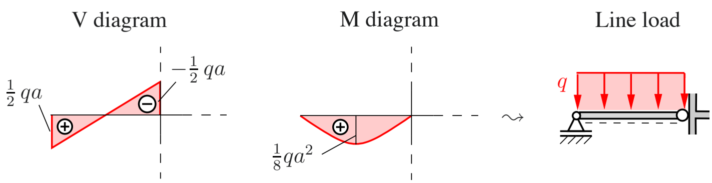
Problem 5.21 The composed structure is connected by a hinge. The diagrams of the normal force N, shear force V and bending moment M are also depicted.

Determine the corresponding loading.

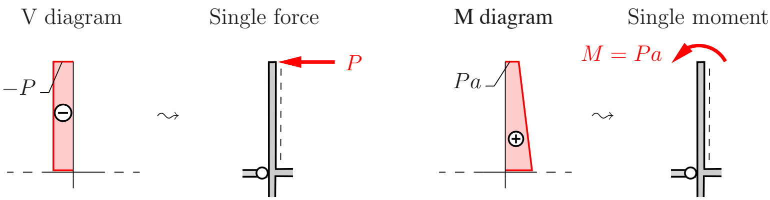


Solution The external loadings of the individual part can be reconstructed starting at the outer boundaries as follows. A subsequent consideration of the equilibrium equations at the middle point provides the potentially external loadings in this point.

We start the derivation with the left beam (1). We can conclude, due to the linear development of the shear-force with the boundary values $\pm q\,a/2$ and the parabola-shaped bending moment with the maximum of $qa^2/8$, that the subsystem has to be loaded by a uniform line load.



The additional sub systems exhibit a constant normal force as well as a constant shear force diagram. The bending moment diagrams are linear. Therefore, we can conclude that no external loadings occur within the subsystems (2), (3) and (4). Thus, additional external loadings can only appear at the free edge of subsystem (2) and in the middle point. Consideration of the boundary values of the shear-force and the bending moment leads to following loading:

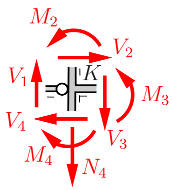


The equilibrium condition at the cut middle point yields the missing loading:

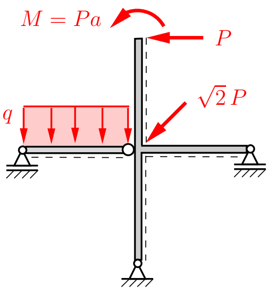
$$\overset{\curvearrowleft}{K}: \quad M_{MK} = M_2 + M_3 - M_4$$
$$= 2\,P\,a - 4\,P\,a - (-2\,P\,a) = 0$$
$$\rightsquigarrow \quad \text{no single moment,}$$

$$\rightarrow: \quad H_{MK} = V_2 - V_4 = -P - (-2\,P) = P$$
$$\rightsquigarrow \quad \text{horizontal single force,}$$

$$\uparrow: \quad V_{MK} = V_1 - V_3 - N_4$$
$$= -\tfrac{1}{2}qa - 4\,P + \tfrac{1}{2}qa + 5\,P = P$$
$$\rightsquigarrow \quad \text{vertical single force.}$$

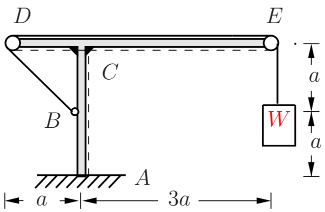


Thus, we obtain the external loading for the structure:



P5.22 **Problem 5.22** The simplified depicted crane supports a weight W with a rope. The rope is clamped in B and is guided by the frictionless pulleys D and E. The crane is additionally loaded by its self-weight q_0 (per length unit a).

Determine the N, V, and M diagrams for $W = q_0 a$

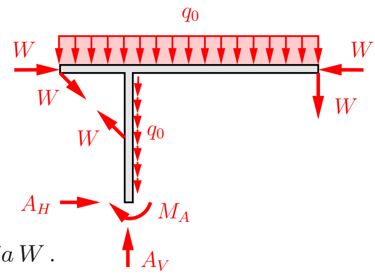


Solution The force in the rope is $S = W$. Thus, we obtain the free-body sketch and the equilibrium conditions as follows

$$\uparrow: \quad -W + A_V - 4\,q_0 a - 2 q_0 a = 0\,,$$

$$\rightarrow: \quad A_H = 0\,,$$

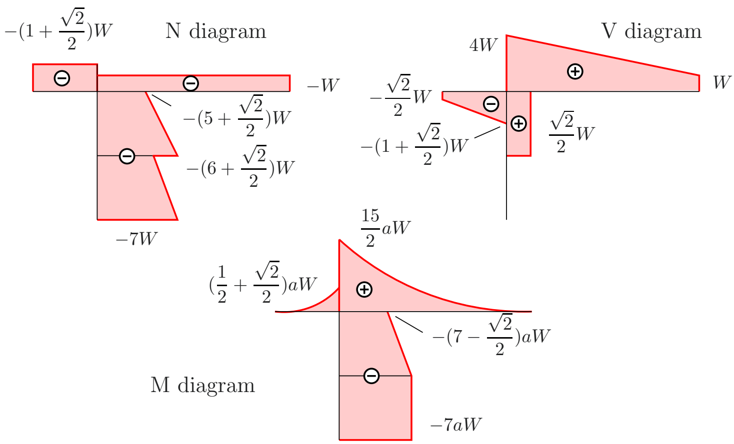
$$\overset{\curvearrowright}{A}: \quad M_A + 3a\,W + 4\,q_0 a^2 = 0.$$



The support reactions are

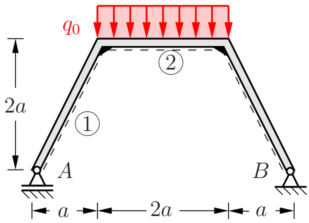
$$A_V = 7\,W\,, \quad A_H = 0\,, \quad M_A = -7a\,W\,.$$

Due to the general coherences of the loadings and the internal forces we obtain the depicted diagrams (Remark: Jumps in N and V are obtained, because of concentrated forces; at point C the sum of all moments has to vanish (Note direction of contributions!)):



Problem 5.23 Determine the diagrams of the internal forces and moments for the symmetric frame.

P5.23



Solution We obtain the support reactions, due to the symmetry, as

$$A = B = q_0 a\,.$$

The internal forces and moments in the sections ① and ② are calculated by the equilibrium equations of the sub systems. They yield with the help of $\cos\alpha = 1/\sqrt{5}$ and $\sin\alpha = 2/\sqrt{5}$

$$① \nearrow: \quad N_1 = -A\sin\alpha = -\tfrac{2}{\sqrt{5}}\,q_0 a\,,$$

$$\searrow: \quad V_1 = A\cos\alpha = \tfrac{1}{\sqrt{5}}\,q_0 a\,,$$

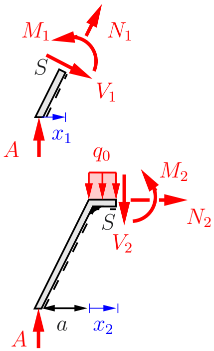
$$\overset{\curvearrowright}{S}: \quad M_1 = x_1 A = x_1 q_0 a\,,$$

$$② \rightarrow: \quad N_2 = 0\,,$$

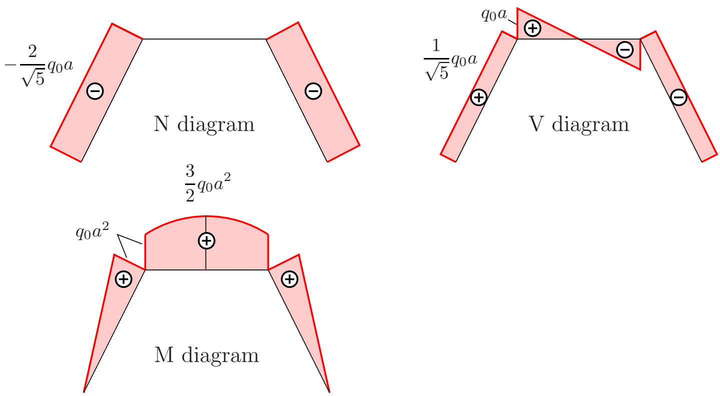
$$\uparrow: \quad V_2 = A - q_0 x_2 = q_0(a - x_2)\,,$$

$$\overset{\curvearrowright}{S}: \quad M_2 = (a + x_2)A - \tfrac{1}{2}q_0\,x_2^2$$

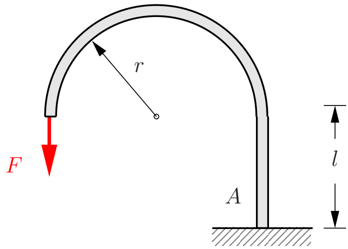
$$= q_0(a^2 + a x_2 - \tfrac{1}{2}x_2^2)\,.$$



Thus, the diagrams can be sketched.



P5.24 **Problem 5.24** Determine the normal-force, shear force and bending moment diagrams for the depicted arch.

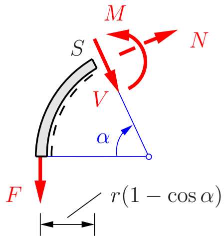


Solution In this problem, it is not necessary to determine the support reaction a priori. The equilibrium equations yield:

$$\nearrow: \quad N(\alpha) = F\cos\alpha\,,$$

$$\searrow: \quad V(\alpha) = -F\sin\alpha\,,$$

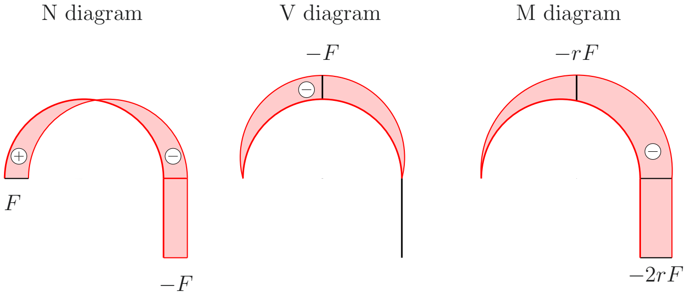
$$\overset{\curvearrowright}{S}: \quad M(\alpha) = -r\,F(1-\cos\alpha)\,.$$



We obtain for the vertical part

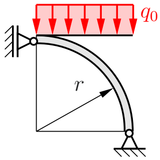
$$N = -F\,, \qquad V = 0\,, \qquad M = -2rF\,.$$

These values are related to the support reactions in A. Thus, the diagrams follow

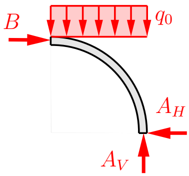


Remark: The internal forces and moments as well as the support reactions are independet of l.

Problem 5.25 The depicted arc is loaded by a constant line load. Determine the diagrams of the internal forces and the extreme values of N and M. **P5.25**

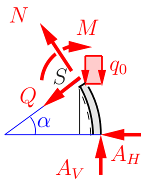


Solution We obtain the reaction forces by the equilibrium equations as



$$A_V = q_0 r\,, \qquad A_H = B = \frac{q_0 r}{2}\,.$$

The internal forces of the arc are determined as



$$\nearrow: \quad N(\alpha) = -[A_V - q_0 r(1-\cos\alpha)]\cos\alpha - A_H \sin\alpha$$
$$= -\tfrac{1}{2} q_0 r(2\cos^2\alpha + \sin\alpha)\,,$$

$$\searrow: \quad V(\alpha) = [A_V - q_0 r(1-\cos\alpha)]\sin\alpha - A_H\cos\alpha$$
$$= \tfrac{1}{2} q_0 r(2\cos\alpha\sin\alpha - \cos\alpha)\,,$$

$$\overset{\curvearrowright}{S}: \quad M(\alpha) = A_V r(1-\cos\alpha) - A_H r\sin\alpha - \tfrac{1}{2} q_0 r^2 (1-\cos\alpha)^2$$
$$= \tfrac{1}{2} q_0 r^2(1 - \sin\alpha - \cos^2\alpha)\,,$$

with extreme values of the bending moment and normal force as

$$\frac{\mathrm{d}M}{\mathrm{d}\alpha} = 0 \;:\; (-1 + 2\sin\alpha)\cos\alpha = 0\,,$$

$$\cos\alpha_1 = 0 \quad \rightsquigarrow \quad \alpha_1 = \pi/2 \quad \rightsquigarrow \quad \underline{\underline{M(\alpha_1) = 0}}\,,$$

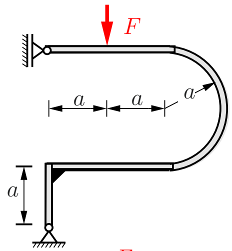
$$\sin\alpha_2 = 1/2 \quad \rightsquigarrow \quad \alpha_2 = \pi/6 \quad \rightsquigarrow \quad \underline{\underline{M(\alpha_2) = -\frac{q_0 r^2}{8}}}\,,$$

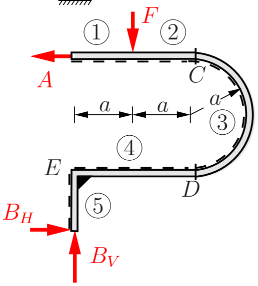
$$\frac{\mathrm{d}N}{\mathrm{d}\alpha} = 0 \;:\; (-4\sin\alpha + 1)\cos\alpha = 0\,,$$

$$\cos\alpha_3 = 0 \quad \rightsquigarrow \quad \alpha_3 = \pi/2 \quad \rightsquigarrow \quad \underline{\underline{N(\alpha_3) = -\frac{q_0 r}{2}}}\,,$$

$$\sin\alpha_4 = 1/4 \quad \rightsquigarrow \quad \cos^2\alpha_4 = \frac{15}{16} \quad \rightsquigarrow \underline{\underline{N(\alpha_4) = \frac{17}{16}\,q_0 r}}\,.$$

P5.26 **Problem 5.26** Determine the internal force diagrams for the depicted system.





Solution With the equilibrium equations

$$\uparrow: \quad B_V - F = 0\,,$$

$$\rightarrow: \quad B_H - A = 0\,,$$

$$\overset{\curvearrowright}{B}: \quad -3a\,A + a\,F = 0$$

the reactions forces follow

$$A = \frac{F}{3}\,, \quad B_V = F\,, \quad B_H = \frac{F}{3}\,.$$

Therefore, the normal and shear forces in the subsystems ①, ②, ④ and ⑤ are obtained as

$$N_1 = A = F/3\,, \qquad V_1 = 0\,,$$

$$N_2 = A = F/3\,, \qquad V_2 = -F\,,$$

$$N_4 = -B_H = -F/3\,, \quad V_4 = B_V = F\,,$$

$$N_5 = -B_V = -F\,, \qquad V_5 = -B_H = -F/3\,.$$

The bending moments are calculated at the loading points C, D and E to be

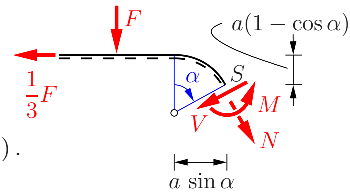
$$M_F = 0\,, \quad M_C = -a\,F\,, \quad M_D = -\frac{5}{3}\,a\,F\,, \quad M_E = \frac{a\,F}{3}\,.$$

Now, we obtain for the arc in ③

$$\swarrow: \quad V_3 = -\frac{F}{3}\,(\sin\alpha + 3\cos\alpha)\,,$$

$$\searrow: \quad N_3 = \frac{F}{3}\,(\cos\alpha - 3\sin\alpha)\,,$$

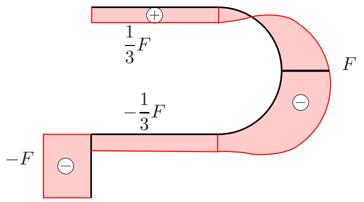
$$\overset{\curvearrowright}{S}: \quad M_3 = -\frac{aF}{3}\,(4 + 3\sin\alpha - \cos\alpha)\,.$$



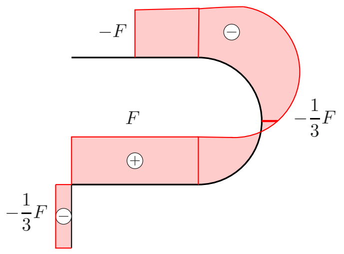
Some values of V_3, N_3 and M_3 are summarized in the following table.

α	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
V_3	$-F$	$-\dfrac{2\sqrt{2}}{3}\,F$	$-\dfrac{1}{3}\,F$	$\dfrac{\sqrt{2}}{3}\,F$	F
N_3	$\dfrac{1}{3}\,F$	$-\dfrac{\sqrt{2}}{3}\,F$	$-F$	$-\dfrac{2\sqrt{2}}{3}\,F$	$-\dfrac{1}{3}\,F$
M_3	$-aF$	$-\dfrac{1}{3}(4+\sqrt{2})aF$	$-\dfrac{7}{3}\,aF$	$-\dfrac{1}{3}(4+2\sqrt{2})\,aF$	$-\dfrac{5}{3}\,aF$

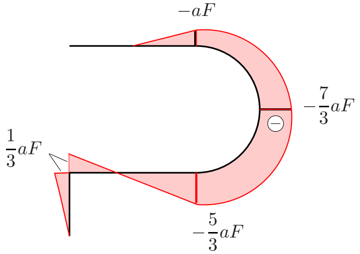
N diagram



V diagram

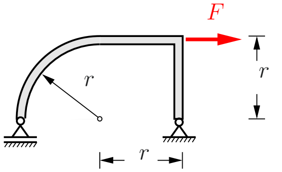


M diagram

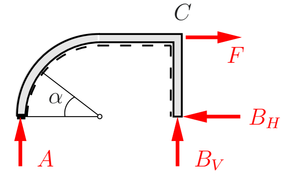


P5.27

Problem 5.27 Determine the internal force diagrams for the depicted system.



Solution The equilibrium conditions for the complete system lead to following reaction forces:

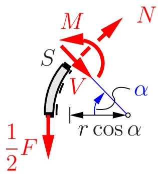


$$A=-\frac{F}{2}\,,\quad B_V=\frac{F}{2}\,,\quad B_H=F\,.$$

We obtain the bending moment at C considering equilibrium for the depicted sub system

$$M_C=-rB_H=-r\,F\,.$$

In the curved part the forces are

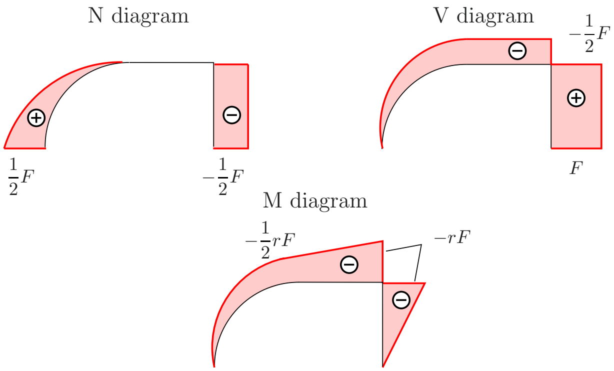


$$\nearrow:\quad N(\alpha)=\frac{F}{2}\cos\alpha\,,$$

$$\searrow:\quad V(\alpha)=-\frac{F}{2}\sin\alpha\,,$$

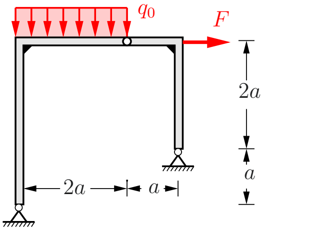
$$\overset{\curvearrowright}{S}:\quad M(\alpha)=-\frac{rF}{2}(1-\cos\alpha)\,.$$

Therefore, the diagrams can be depicted as

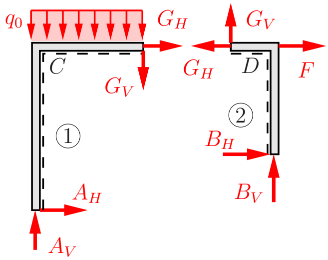


P5.28

Problem 5.28 Determine the normal force, shear force and bending moment diagrams for the depicted three-hinged frame.



Solution The equilibrium equations



$$
\begin{aligned}
① \uparrow: &\quad A_V - G_V - 2q_0 a = 0\,,\\
\rightarrow: &\quad A_H + G_H = 0\,,\\
\overset{\curvearrowright}{G}: &\quad 2aA_V - 2q_0 a^2 - 3aA_H = 0\,,\\
② \uparrow: &\quad B_V + G_V = 0\,,\\
\rightarrow: &\quad -G_H + B_H + F = 0\,,\\
\overset{\curvearrowright}{G}: &\quad -aB_V - 2aB_H = 0
\end{aligned}
$$

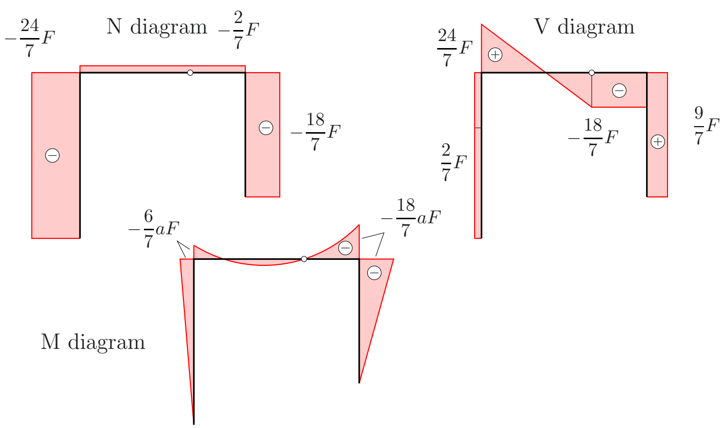
yield the reaction forces as

$$
A_V = \frac{24}{7}F, \quad B_V = -G_V = \frac{18}{7}F, \quad A_H = -G_H = \frac{2}{7}F, \quad B_H = -\frac{9}{7}F.
$$

The bending moments are calculated as

$$
M_C = -3aA_H = -\frac{6}{7}aF\,, \qquad M_D = 2aB_H = -\frac{18}{7}aF,
$$

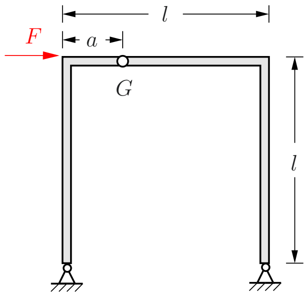
which leads to the following diagrams:



P5.29

Problem 5.29 Determine the position of the hinge G, such that the value of the maximal bending moment is minimal.

Depict the bending moment diagram for that case.



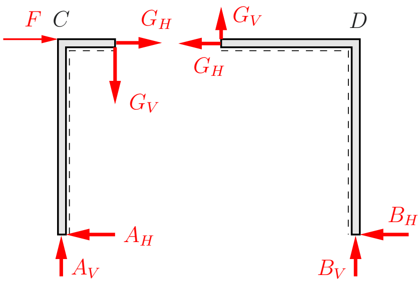
Solution We obtain from the equilibrium equations for the complete system and the right sub system

$$\uparrow: \quad A_V + B_V = 0\,,$$

$$\rightarrow: \quad F - A_H - B_H = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad F\,l - l\,B_V = 0\,,$$

$$\overset{\curvearrowright}{G}: \quad l\,B_H - (l-a)B_V = 0$$



yielding the reaction forces as

$$B_V = -A_V = F\,, \qquad B_H = F(1-\frac{a}{l})\,, \qquad A_H = F\frac{a}{l}\,.$$

At the points C and D the bending moment is obtained as

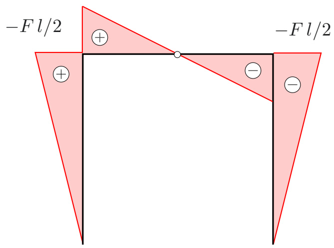
$$M_C = l\,A_H = F\,a\,, \qquad M_D = -l\,B_H = F\,a - F\,l\,.$$

In the entire system the bending moment is linear, therefore we find the minimum of the magnitude moment by equating

$$|M_C| = |M_D|\;: \quad a = l/2.$$

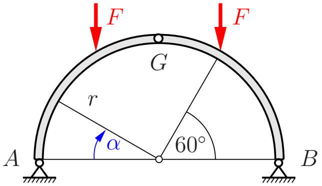
For that case the moments M_C, M_D and the bending moment diagram appear as

$$M_C = -M_D = F\,l/2\,.$$



M-Verlauf

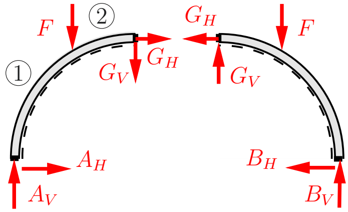
Problem 5.30 Consider the symmetrical loaded three hinged semi-circular arc. Determine the internal forces as a function of α. **P5.30**



Solution From the symmetry conditions, we obtain

$$A_V = B_V\,,\quad A_H = B_H\,,\quad G_V = 0\,.$$

The equilibrium equations for the left and the right sub systems yield



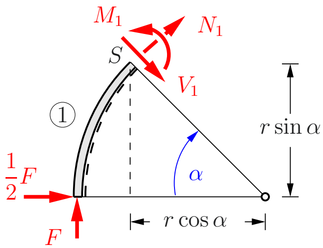
$$A_V = B_V = F\,,\quad A_H = B_H = -G_H = \frac{F}{2}\,.$$

The internal forces are obtained by the equilibrium conditions in the free-body sketches. They lead for sub system 1 between support A and the point of the loading $(0 \le \alpha < 60°)$ to

$$\nearrow:\quad N_1 = -F\cos\alpha - \tfrac{1}{2}F\sin\alpha = -F(\cos\alpha + \tfrac{1}{2}\sin\alpha)\,,$$

$$\searrow:\quad V_1 = F\sin\alpha - \tfrac{1}{2}F\cos\alpha = F(\sin\alpha - \tfrac{1}{2}\cos\alpha)\,,$$

$$\overset{\curvearrowleft}{S}:\quad M_1 = r\,F(1-\cos\alpha) - \tfrac{1}{2}r\,F\sin\alpha = r\,F(1-\cos\alpha - \tfrac{1}{2}\sin\alpha)\,.$$

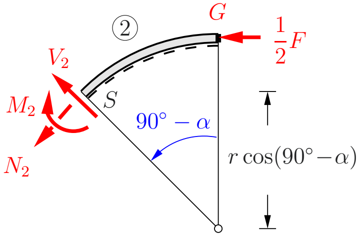


In sub system 2 between the point of loading and the hinge G, we obtain

$$\swarrow:\quad N_2 = -\tfrac{1}{2}F\cos(90° - \alpha) = -\tfrac{1}{2}F\sin\alpha\,,$$

$$\nwarrow:\quad V_2 = -\tfrac{1}{2}F\sin(90° - \alpha) = -\tfrac{1}{2}F\cos\alpha\,,$$

$$\overset{\curvearrowleft}{S}:\quad M_2 = \tfrac{1}{2}Fr(1-\sin\alpha)\,.$$

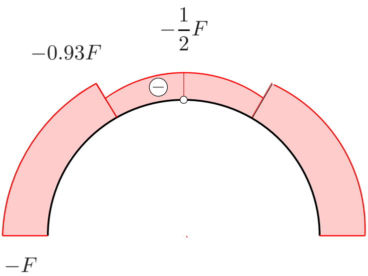


Some values of N, V and M are given in the following table.

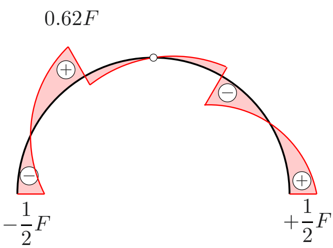
α	0	30°	45°	60°	90°
N_1	$-F$	$-1.12\ F$	$-1.06\ F$	$-0.93\ F$	
N_2				$-0.43\ F$	$-F/2$
V_1	$-F/2$	$-0.07\ F$	$+0.35\ F$	$+0.62\ F$	
V_2				$-0.25\ F$	0
M_1	0	$-0.12\ rF$	$-0.06\ rF$	$+0.07\ rF$	
M_2				$+0.07\ rF$	0

The internal force diagrams are depicted below. At the point of loading, we obtain a jump in the normal force of the magnitude $F\cos 60^\circ = F/2$ and in the shear force of $F\sin 60^\circ = 0.87\,F$. The bending moment experiences a kink at that point. Due to the symmetry of the structure and the loading, N and M are symmetric and V is antisymmetric.

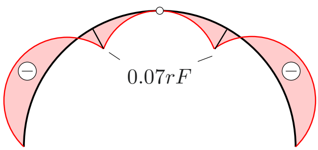
N diagram



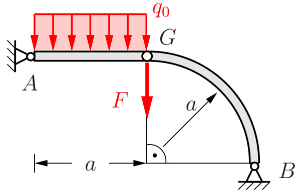
V diagram



M diagram



Problem 5.31 The three-hinged arc is loaded by a force F and a constant line load $q_0 a = 2F$. Determine the internal force diagrams. **P5.31**



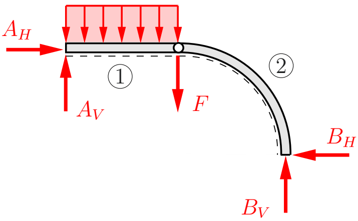
Solution Considering the equilibrium equations of the complete system, the sub systems of the arc ② and the beam ①

$$\uparrow: \quad A_V + B_V - F - q_0 a = 0\,,$$

$$\rightarrow: \quad A_H - B_H = 0\,,$$

$$② \quad \overset{\curvearrowleft}{G}: \quad a\,B_V - a\,B_H = 0\,,$$

$$① \quad \overset{\curvearrowleft}{G}: \quad -a\,A_V + \tfrac{1}{2} q_0 a^2 = 0\,,$$



yield the reaction forces as

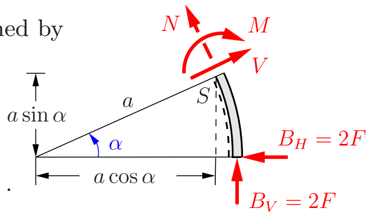
$$A_V = F\,, \qquad B_V = B_H = A_H = 2F\,.$$

The internal forces at the arc are obtained by

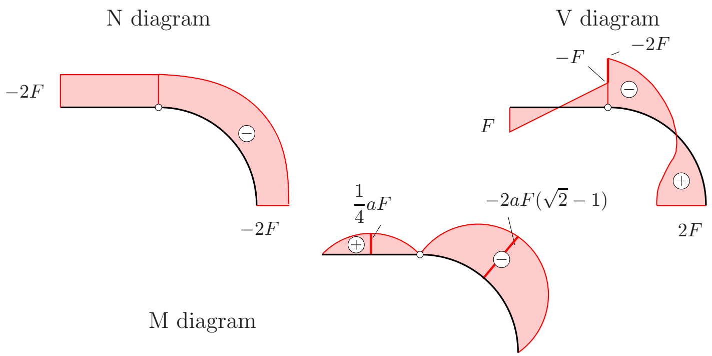
$$\nwarrow: \quad N(\alpha) = -2F(\cos\alpha + \sin\alpha)\,,$$

$$\nearrow: \quad V(\alpha) = 2F(\cos\alpha - \sin\alpha)\,,$$

$$\overset{\curvearrowleft}{S}: \quad M(\alpha) = 2Fa\,(1 - \cos\alpha - \sin\alpha)\,.$$

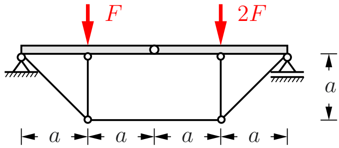


The diagrams are obtained as follows. Note that N and M are maximal in the middle of the arc.



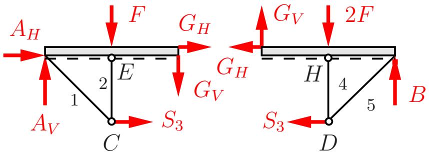
P5.32

Problem 5.32 Determine the internal forces for the depicted system.



Solution The reaction forces are obtained by the equilibrium equations of the complete system as

$$A_V = \frac{5}{4}\,F\,,$$
$$B = \frac{7}{4}\,F\,,$$
$$A_H = 0\,.$$

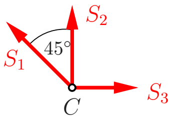


Consideration of the right sub system yields the joint-forces and S_3:

$$\uparrow: \quad G_V + B - 2F = 0 \quad \leadsto \quad G_V = F/4\,,$$
$$\overset{\curvearrowright}{G}: \quad a\,S_3 + 2a\,F - 2a\,B = 0 \quad \leadsto \quad \underline{\underline{S_3 = \frac{3}{2}\,F}}\,,$$
$$\rightarrow: \quad -G_H - S_3 = 0 \quad \leadsto \quad G_H = -\frac{3}{2}\,F\,.$$

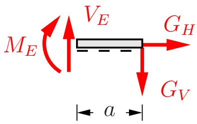
The remaining forces in the bars are obtained by the equilibrium condition at point C. Note that the situation is mirror-inverted at C and D ($S_1 = S_5$, $S_2 = S_4$):

$$\rightarrow: \quad -\frac{\sqrt{2}}{2}\,S_1 + S_3 = 0 \quad \leadsto \quad \underline{\underline{S_1 = S_5 = \frac{3}{2}\sqrt{2}\,F}}\,,$$
$$\uparrow: \quad S_2 + \frac{\sqrt{2}}{2} S_1 = 0 \quad \leadsto \quad \underline{\underline{S_2 = S_4 = -\frac{3}{2}\,F}}\,.$$



We obtain the bending moment at the point of loading E as

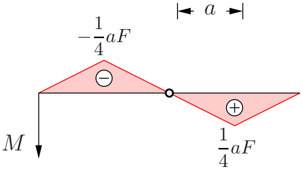
$$M_E = -a\,G_V = -\frac{a\,F}{4}\,.$$



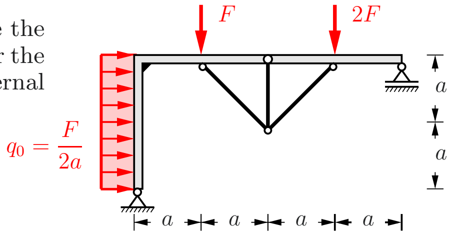
Analogously, the bending moment at H is

$$M_H = \frac{a\,F}{4}\,.$$

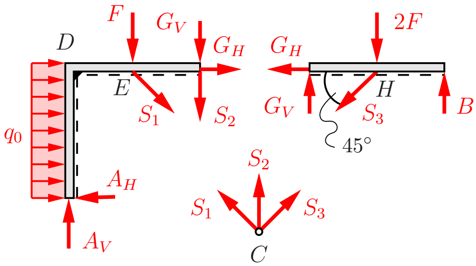
This yields the depicted bending moment diagram.



Problem 5.33 Determine the internal force diagrams for the beam and also the internal forces in the bars. **P5.33**



Solution The free-body sketch depicts the separated system. First of all the reaction forces are determined. Considering the equilibrium equations of the complete system



$$\rightarrow: \quad -A_H + 2q_0 a = 0\,,$$

$$\uparrow: \quad A_V - F - 2F + B = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad 2q_0 a^2 + a\,F + 6a\,F - 4a\,B = 0$$

leads with $q_0 = F/2a$ to

$$A_V = F\,, \qquad A_H = F\,, \qquad B = 2F\,.$$

The joint forces and the forces S_3 in the bar are determined in the right sub system. Equilibrium yields with $\sin 45° = \cos 45° = \sqrt{2}/2$

$$\uparrow: \quad G_V - 2F - \frac{\sqrt{2}}{2}\,S_3 + B = 0\,,$$

$$\rightarrow: \quad -G_H - \frac{\sqrt{2}}{2}\,S_3 = 0\,,$$

$$\overset{\curvearrowright}{G}: \quad 2a\,F + \frac{\sqrt{2}}{2}\,a\,S_3 - 2a\,B = 0\,.$$

Therefore, we obtain

$$\underline{\underline{S_3 = 2\sqrt{2}\,F}}\,, \qquad G_H = -2F\,, \qquad G_V = 2F\,.$$

The bar forces of S_1 and S_2 are obtained considering node C as

$$\rightarrow: \quad -\frac{\sqrt{2}}{2}\,S_1 + \frac{\sqrt{2}}{2}\,S_3 = 0 \qquad \rightsquigarrow \quad \underline{\underline{S_1 = S_3}}\,,$$

$$\uparrow: \quad \frac{\sqrt{2}}{2}\,S_1 + \frac{\sqrt{2}}{2}\,S_3 + S_2 = 0 \quad \rightsquigarrow \quad \underline{\underline{S_2}} = -\sqrt{2}\,S_3 = \underline{\underline{-4\ F}}\,.$$

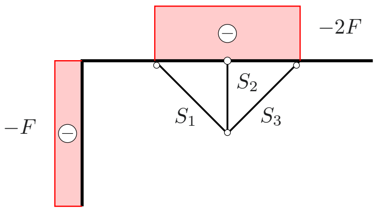
Finally, the bending moment diagram is obtained with help of the point values at D, E and H

$$M_D = 2a\,A_H - a\,(q_0 2a) = a\,F\,,$$

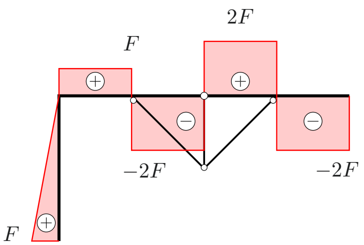
$$M_E = -a\,(G_V + S_2) = 2a\,F\,,$$

$$M_H = a\,B = 2a\,F\,.$$

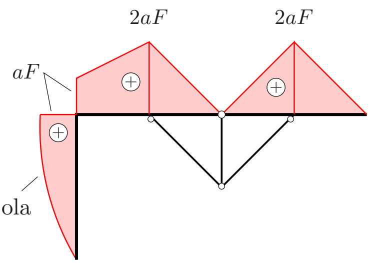
N diagram



V diagram

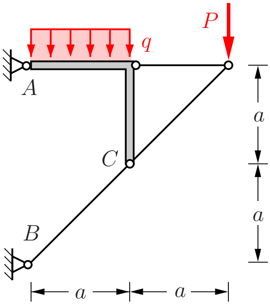


M diagram

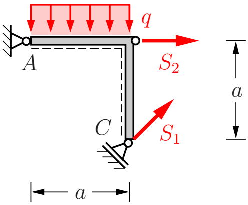


Problem 5.34 Determine the reaction forces in A and B and all internal force diagrams (normal, shear, and bending moment). Specify also all extreme values in the $A-C$ (angular frame). **P5.34**

Given: $P = q\,a/4$

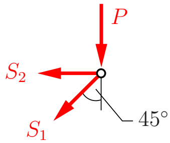


Solution In order to determine the reactions forces, we will simplify the structure. At first, the hinged column between B and C is replaced by a sliding support which is tilted by 45°. Additionally, the normal forces of the bars, where the load P acts, can be computed by equilibrium conditions at the hinge.



The internal bar forces are obtained by the equilibrium equations at the hinge as

$$\uparrow: \quad -\frac{\sqrt{2}}{2}\,S_1 - P = 0\,,$$
$$\rightarrow: \quad -S_2 - \frac{\sqrt{2}}{2}\,S_1 = 0\,,$$



which leads to

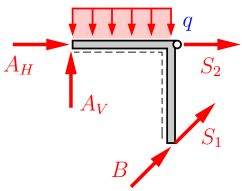
$$S_1 = -\sqrt{2}\,, \qquad S_2 = P\,.$$

The equilibrium equations for the simplified structure

$$\overset{\curvearrowleft}{A}: \quad \sqrt{2}\,a\,B + \sqrt{2}\,a\,S_1 - \frac{q\,a^2}{2} = 0\,,$$

$$\uparrow: \quad A_V + \frac{\sqrt{2}}{2}\,B + \frac{\sqrt{2}}{2}\,S_1 - q\,a = 0\,,$$

$$\rightarrow: \quad A_H + S_2 + \frac{\sqrt{2}}{2}\,B + \frac{\sqrt{2}}{2}\,S_1 = 0$$



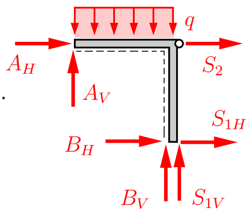
yield the reactions forces

$$\underline{\underline{B = \frac{\sqrt{2}}{2}\, q\, a}}\,, \qquad \underline{\underline{A_V = \frac{3}{4}\, q\, a}}\,, \qquad \underline{\underline{A_H = -\frac{q\, a}{2}}}\,.$$

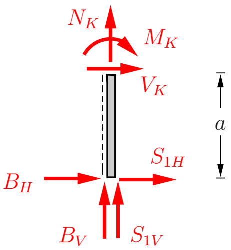
In order to sketch the internal force diagrams, it makes sense to disassemble the forces B and S_1 into its parts vertical and parallel to the angular frame. Then an easy assignment to the normal and shear forces is possible. In addition, M_K at the kink of the frame has to be determined:

$$B_H = \frac{\sqrt{2}}{2}\, B = \frac{q\, a}{2}\,, \qquad B_V = \frac{\sqrt{2}}{2}\, B = \frac{q\, a}{2}\,,$$

$$S_{1H} = \frac{\sqrt{2}}{2}\, S_1 = -\frac{q\, a}{4}\,, \quad S_{1V} = \frac{\sqrt{2}}{2}\, S_1 = -\frac{q\, a}{4}\,, \,.$$

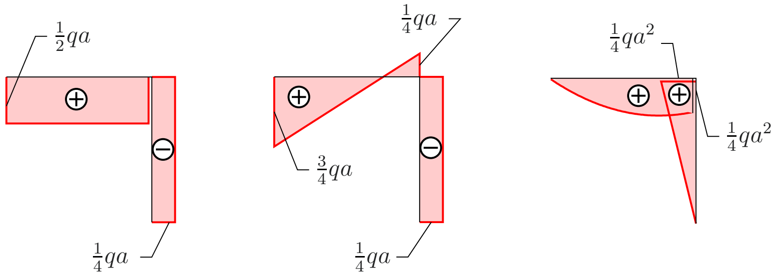


$$M_K = a\,(B_H + S_{1H}) = \frac{q\, a^2}{4}\,.$$



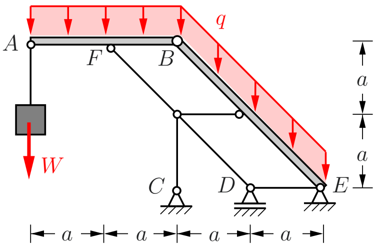
The internal force diagrams are obtained as

N diagram V diagram M diagram



Problem 5.35 Determine for the given structure all reaction forces as well as all extreme values of the internal forces (normal, shear force, and bending moments) and the sketch of the internal force diagrams for the beam $A-B$.

P5.35

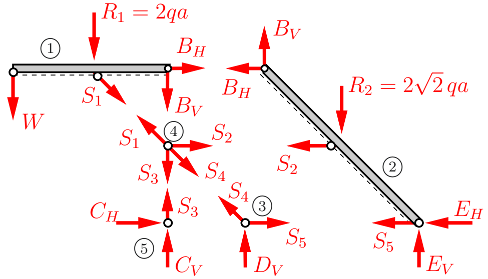


Solution With the help of the free-body sketch, we can determine the reaction forces as follows.

Consideration of the equilibrium of sub system ⑤ yields

$$\underline{\underline{C_H = 0\,.}}$$

Therefore, the horizontal reaction force E_H vanishes, since the system does not have any horizontal loading.



In order to calculate the remaining vertical reaction forces C_V, D_V, and E_V, we determine the internal forces of the sub system ①. A handy choice of the equilibrium equations leads directly to the respective forces:

$$\overset{\curvearrowright}{B}: \quad -2\,a\,W - a\,R_1 - \frac{\sqrt{2}}{2}\,S_1\,a = 0 \quad \rightsquigarrow \quad S_1 = -2\,\sqrt{2}\,(q\,a + W)\,,$$

$$\overset{\curvearrowright}{F}: \quad B_V\,a - W\,a = 0 \quad \rightsquigarrow \quad B_V = W\,,$$

$$\rightarrow: \quad \frac{\sqrt{2}}{2}\,S_1 + B_H = 0 \quad \rightsquigarrow \quad B_H = 2\,(q\,a + W)\,.$$

Equilibrium in vertical direction of sub system ② leads to

$$\uparrow: \quad E_V + B_V - R_2 = 0\,.$$

Thus, the reaction force E_V follow as

$$\underline{\underline{E_V = 2\,\sqrt{2}\,q\,a - W}}\,.$$

The remaining reaction forces C_V and D_V can be determined by

$$\overset{\curvearrowright}{D}: -a\,E_V + a\,C_V - 3\,a\,W - 2\,a\,R_2 = 0 \rightsquigarrow \underline{\underline{C_V = qa\,(4 + 2\sqrt{2}) - 2\,W}},$$

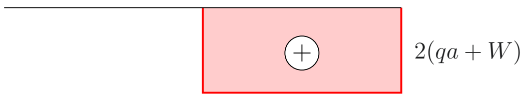
$$\uparrow: D_V + E_V + C_V - W - R_1 - R_2 = 0 \rightsquigarrow \underline{\underline{D_V = -q\,a\,(2\sqrt{2} + 2)}}.$$

The substructure $A - B$ can be idealized as a beam with two supports. The support reactions B_V, B_H and S_1 have already been determined. The bending moment in point F can be computed by

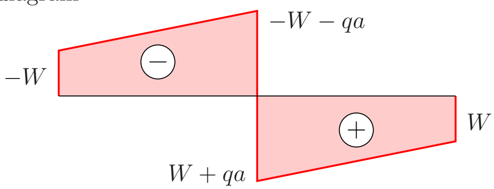
$$M_F = -W\,a - \frac{q\,a^2}{2}\,.$$

Thus, the internal force diagrams can be sketched as

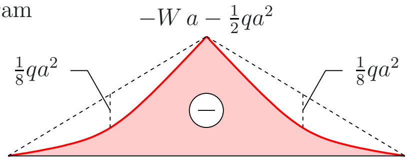
N diagram



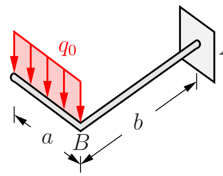
V diagram



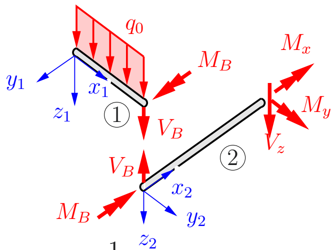
M diagram



Problem 5.36 Determine the internal forces for the depicted tilted beam. **P5.36**



Solution We introduce coordinate systems in both sub systems to determine the sign of the internal forces. We obtain for sub system ① by double integration of q_0 and with regard of the boundary conditions $V_z(0) = 0$, $M_y(0) = 0$



$$\underline{\underline{V_z = -q_0 x_1}}\,, \qquad \underline{\underline{M_y = -\frac{1}{2}\,q_0 x_1^2}}\,.$$

Thus, at the angle B, it follows

$$V_B = V_z(a) = -q_0 a\,, \qquad M_B = M_y(a) = -\frac{1}{2}\,q_0 a^2\,.$$

For the sub system ② the equilibrium conditions yield

$$\sum F_z = 0\ :\quad \underline{\underline{V_z}} = V_B = \underline{\underline{-q_0 a}}\,,$$

$$\sum M_x = 0\ :\quad \underline{\underline{M_x}} = -M_B = \underline{\underline{\frac{1}{2} q_0 a^2}}\,,$$

$$\sum M_y = 0\ :\quad \underline{\underline{M_y}} = x_2 V_B = \underline{\underline{-q_0 a x_2}}\,.$$

Remark:

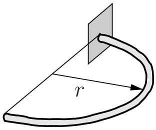
- The remaining internal forces are zero.
- The reaction forces at the clamping follow from the internal forces of sub system ② by

$$A = -V_z(b) = q_0 a,\ M_{Ax} = M_x(b) = \frac{q_0 a^2}{2},\ M_{Ay} = M_y(b) = -q_0 ab.$$

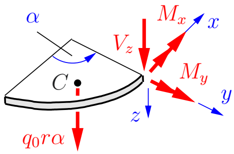
- The bending moment M_y in sub system ① changes at the angle B into the torsion moment M_x in sub system ② .

P5.37 **Problem 5.37** The clamped semicircular beam with radius r is loaded by its body force ($q_0 = \text{const}$).

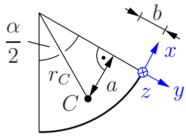
Determine the internal forces.



Solution The beam is cut at a arbitrary angle α, and a coordinate system is introduced, in order to determine the sign of the internal forces. Using the arc length $r\alpha$, the body force of the subsystem is equal to $q_0 r\alpha$. This can be idealized as a unified load in the centroid, which is located at



$$r_C = 2r\,\frac{\sin\alpha/2}{\alpha}$$



(compare chapter 2). The lever arm is computed by

$$a = r_C\,\sin(\alpha/2) = (2r/\alpha)\sin^2(\alpha/2) = (r/\alpha)(1-\cos\alpha)\,,$$

$$\begin{aligned} b &= r - r_C\,\cos(\alpha/2) = (r/\alpha)[\alpha - 2\sin(\alpha/2)\cos(\alpha/2)] \\ &= (r/\alpha)(\alpha - \sin\alpha)\,, \end{aligned}$$

which yields for the equilibrium equations

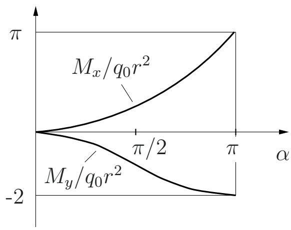
$$\sum F_z = 0\ :\quad \underline{\underline{V_z(\alpha) = -q_0 r\alpha}}\,,$$

$$\sum M_x = 0\ :\quad \underline{\underline{M_x(\alpha)}} = b(q_0 r\alpha) = \underline{\underline{q_0 r^2(\alpha - \sin\alpha)}}\,,$$

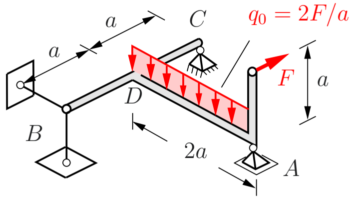
$$\sum M_y = 0\ :\quad \underline{\underline{M_y(\alpha)}} = -a(q_0 r\alpha) = \underline{\underline{-q_0 r^2(1-\cos\alpha)}}\,.$$

The remaining internal forces are zero. The reactions forces can be determined by the internal forces at $\alpha = \pi$.

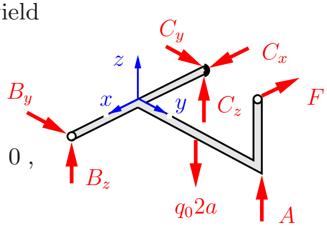
The diagrams of the bending moment M_y and the torque M_x are depicted.



Problem 5.38 Determine the internal force diagrams for the depicted structure. **P5.38**



Solution The equilibrium conditions yield



$$\sum F_x = 0 \ : \quad C_x - F = 0 \,,$$

$$\sum F_y = 0 \ : \quad B_y + C_y = 0 \,,$$

$$\sum F_z = 0 \ : \quad A + B_z + C_z - q_0 2a = 0 \,,$$

$$\sum M_x^{(D)} = 0 \ : \quad 2aA - a\,(q_0 2a) = 0 \,,$$

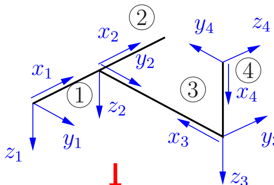
$$\sum M_y^{(D)} = 0 \ : \quad -aB_z + a\,C_z - aF = 0 \,,$$

$$\sum M_z^{(D)} = 0 \ : \quad aB_y - a\,C_y + 2aF = 0.$$

Thus, the reaction forces are determined with $q_0 = 2F/a$, as

$$A = 2F \,, \quad B_z = \frac{F}{2} \,, \quad B_y = -C_y = -F \,, \quad C_x = F \,, \quad C_z = \frac{3}{2}\,F \,.$$

For the following calculations, we subdivide the system into 4 parts and introduce individual coordinate systems. We obtain from the equilibrium equations:

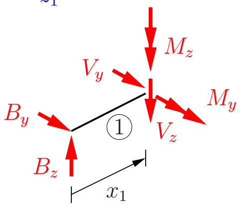


① $V_y = -B_y = F \,,$

$V_z = B_z = F/2 \,,$

$M_y = B_z x_1 = \frac{1}{2} F x_1 \,,$

$M_z = B_y x_1 = -F x_1 \,,$

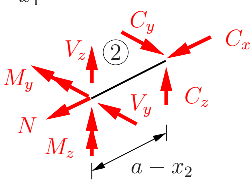


② $N = -C_x = -F \,,$

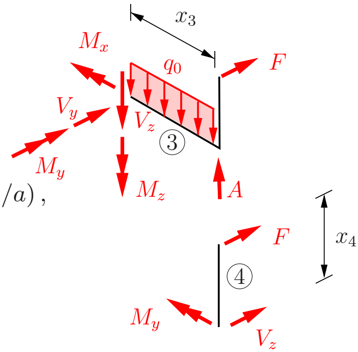
$V_y = +C_y = +F \,,$

$V_z = -C_z = -3F/2 \,,$

$M_y = +C_z(a - x_2) = +\frac{3}{2} F(a - x_2) \,,$



$$M_z = C_y(a - x_2) = F(a - x_2)\,,$$



③ $$V_y = -F\,,$$

$$V_z = A - q_0 x_3 = 2F(1 - x_3/a)\,,$$

$$M_x = -Fa\,,$$

$$M_y = A\,x_3 - \tfrac{1}{2} q_0 x_3^2 = F\,(2x_3 - x_3^2/a)\,,$$

$$M_z = F\,x_3\,,$$

④ $$V_z = -F\,,$$

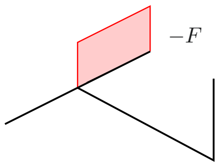
$$M_y = -F\,x_4\,.$$

The sketches of the internal forces are depicted below.

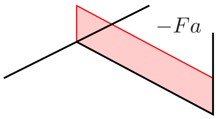
Remark:

- The bending moment at sub system ④ changes into the torque at sub system ③. The latter causes in the beam $\overline{BC}$ at point D a jump of the bending moment M_y.
- Analogously, the bending moment M_z of sub system ③ causes a jump of M_z of the beam $\overline{BC}$.

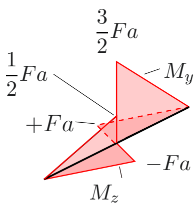
normal force

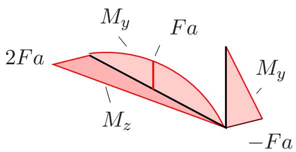


torque



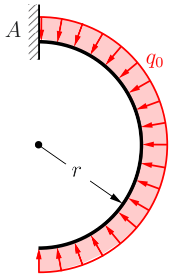
bending moment



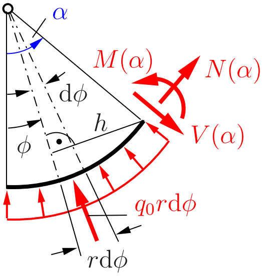


Problem 5.39 The depicted semi-circular arc of radius r is loaded by a constant radial line load q_0. **P5.39**

Determine normal force, shear force and bending moment.



Solution A cut of the arc for an arbitrary α frees internal forces. Considering an arc element of the length $r\mathrm{d}\phi$, we obtain an infinitesimal loading $q_0 r\mathrm{d}\phi$ in radial direction. Decomposition of the force in N and V direction and integration over the arc element leads to



$$\underline{\underline{N(\alpha)}} = -\int_0^\alpha q_0\, r\, \sin(\alpha-\phi)\, \mathrm{d}\phi$$

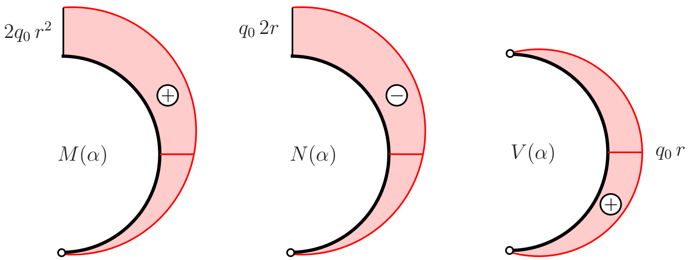
$$= \underline{\underline{-q_0\, r\, (1-\cos\alpha)}}\,,$$

$$\underline{\underline{V(\alpha)}} = -\int_0^\alpha q_0\, r\, \cos(\alpha-\phi)\, \mathrm{d}\phi = \underline{\underline{-q_0\, r\, \sin\alpha}}\,.$$

The infinitesimal moment $q_0 r\mathrm{d}\phi$ with respect to the cut yields $h\, q_0 r\mathrm{d}\phi$. We obtain from the equilibrium equation for the moments using the lever $h = r\sin(\alpha-\phi)$:

$$\underline{\underline{M(\alpha) =}} - \int_0^\alpha q_0\, r^2\, \sin(\alpha-\phi)\, \mathrm{d}\phi = \underline{\underline{-q_0\, r^2\, (1-\cos\alpha)}}\,.$$

Thus, the internal force diagrams are obtained as



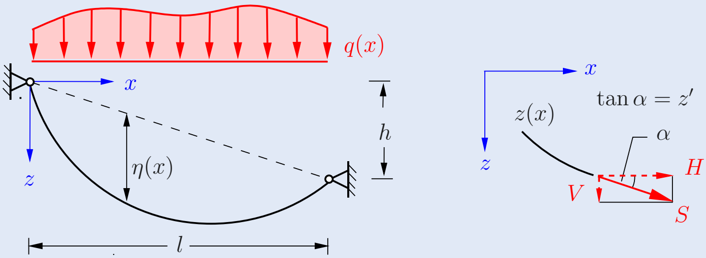
Chapter 6

Cables

6

1. Cables subjected to vertical loads

Loading $q(x)$ is a function of x.



The horizontal force H and the tension of the cable S are

$$H = \text{const}\,, \qquad S = H\sqrt{1+(z')^2}\,.$$

Integration of the differential equation

$$z'' = -\frac{1}{H}\,q(x)$$

leads to the *curve of the cable* $z(x)$ and the *sag* $\eta(x)$:

$$z(x) = -\frac{1}{H}\int_0^x\int_0^x q(\tilde{x})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{x} + C_1\,x + C_2\,, \qquad \eta(x) = z(x) - \frac{h}{l}\,x\,.$$

For a given H, the integration constants C_1, C_2 are determined by the geometrical boundary conditions $(z(0), z(l))$. For an unknown H, an additional constraint is needed. Possible constraints are:

1. maximum sag $\eta_{\max} = \eta^*$ given,
2. maximum tension $S_{\max} = S^*$ given,
3. length of cable $L = L^*$ given.

In the **special case** of a constant vertical loading $q(x) = q_0 = \text{const}$, we obtain for the curve of the cable and the sag

$$z(x) = \left(\frac{h}{l} + \frac{q_0 l}{2\,H}\right)x - \frac{q_0}{2\,H}\,x^2\,, \qquad \eta(x) = \frac{q_0}{2\,H}\,(l\,x - x^2)\,.$$

The horizontal force H is determined by

1. η^* given: $\quad H = \dfrac{q_0 l^2}{8\eta^*}$,

2. S^* given: $\quad S^* = H\sqrt{1 + \left(\dfrac{|h|}{l} + \dfrac{q_0 l}{2\,H}\right)^2}$,

3. L^* given: $\quad L^* = -\dfrac{H}{2\,q_0}\left[z'\sqrt{1+z'^{\,2}} + \operatorname{arsinh} z'\right]_{z'_1}^{z'_2}$

$$\text{with} \quad z'_1 = \frac{h}{l} + \frac{q_0\,l}{2\,H}\,, \qquad z'_2 = \frac{h}{l} - \frac{q_0\,l}{2\,H}\,.$$

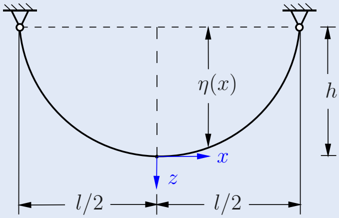
(in cases 2. and 3., H has to be determined using an implicit equation)

2. Cables subjected to self-weight

Loading $\bar{q}(s)$ is a function of the arc length s (weight per unit length of the cable). Here the relation $\mathrm{d}s = \sqrt{1+z'^2}\,\mathrm{d}x$ holds.

The horizontal force H and the tension of the cable S are

$$H = \text{const}\,, \qquad S = H\sqrt{1+(z')^2}\,.$$



The *curve of the cable* follows from

$$\boxed{z'' = -\frac{\bar{q}}{H}\sqrt{1+(z')^2}\,.}$$

For the special case

- weight is distributed constantly over cable length: $\bar{q} = \bar{q}_0 = \text{const}$
- both cable supports are at the same height

follows

cable curve: $\quad \boxed{z(x) = \dfrac{H}{\bar{q}_0}\left(1 - \cosh\dfrac{\bar{q}_0\,x}{H}\right),} \qquad$ (*catenary*)

sag: $\quad \eta(x) = z(x) + h\,, \qquad \rightsquigarrow \qquad h = \eta_{\max} = -z\left(\frac{l}{2}\right)$

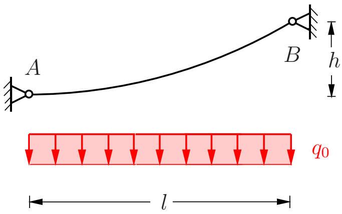
cable tension: $\quad S(x) = H\cosh\dfrac{\bar{q}_0\,x}{H}$,

cable length: $\quad L = \dfrac{2\,H}{\bar{q}_0}\sinh\dfrac{\bar{q}_0\,l}{2\,H}$.

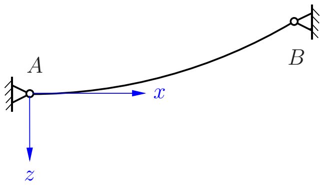
For given η^*, S^* or L^*, the horizontal force H has to be determined from a transcendental equation.

P6.1 **Problem 6.1** A supporting rope is loaded with a line load $q(x) = q_0$ between points A and B. The rope shall be stretched such that the slope of the rope at point A is zero.

Determine the horizontal tensile force and the maximum force in the rope.



Solution We place the coordinate system in point A. Integrating the differential equation of the rope curve twice yields



$$z''(x) = -\frac{q_0}{H}\,,$$

$$z'(x) = -\frac{q_0}{H}\,x + C_1\,,$$

$$z(x) = -\frac{q_0}{2\,H}\,x^2 + C_1 x + C_2\,.$$

The integration constants C_1, C_2 and the horizontal tensile force H follow from the boundary conditions:

$$z(0) = 0 \quad \rightsquigarrow \quad C_2 = 0\,,$$

$$z'(0) = 0 \quad \rightsquigarrow \quad C_1 = 0\,,$$

$$z(l) = -h \quad \rightsquigarrow \quad h = \frac{q_0}{2\,H}\,l^2 \quad \rightsquigarrow \quad \underline{\underline{H = \frac{q_0\,l^2}{2\,h}}}\,.$$

The force in the rope can be calculated by

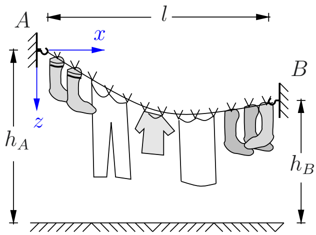
$$S = H\sqrt{1 + (z')^2}$$

$$= \frac{q_0\,l^2}{2\,h}\sqrt{1 + \left(\frac{2\,h\,x}{l^2}\right)^2}\,.$$

It obtains its largest value at $x = l$ (support point B):

$$\underline{\underline{S_{\max} = \frac{q_0\,l^2}{2\,h}\sqrt{1 + \left(\frac{2\,h}{l}\right)^2}}}\,.$$

Problem 6.2 A washing line is fixed at the end points A and B at the heights $h_A > h_B$ above the ground. The clothes on the line resemble approximately a constant line load $q(x) = q_0$. **P6.2**



Compute the maximal force in the line for a minimal distance of the line to the ground given by h^*.

Given: $h_A = 5\,a$, $h_B = 4\,a$, $h^* = 3\,a$, $l = 10\,a$.

Solution With the boundary condition $z(0) = 0$ and $z(l) = h_A - h_B = a$ we obtain the curve of the line, (comp. Exercise 6.1)

$$z(x) = \left(\frac{1}{10} + \frac{10\,q_0\,a}{2\,H}\right)x - \frac{q_0}{2\,H}\,x^2\,.$$

The yet unknown horizontal tensile force H can be determined from the minimal distance h^*. It occurs at the position x^* with $z' = 0$:

$$z'(x) = \frac{1}{10} + \frac{5\,q_0\,a}{H} - \frac{q_0}{H}\,x = 0\,.$$

It follows

$$x^* = \frac{1}{10}\frac{H}{q_0} + 5\,a \qquad \text{and} \qquad z_{\max} = z(x^*) = \frac{(H + 50\,q_0\,a)^2}{200\,q_0\,H}\,.$$

Inserting into $h^* = h_A - z_{\max}$ together with the given values yields the quadratic equation

$$H^2 - 300\,q_0\,a\,H + 2500\,(q_0\,a)^2 = 0$$

with the solution

$$H = (150 \pm 100\sqrt{2}\,)q_0\,a\,.$$

In case of the plus sign, the value of x^* is not between A and B. Thus, the solution for the minus sign holds

$$H = (150 - 100\sqrt{2}\,)q_0\,a \approx 8.579\,q_0\,a\,.$$

The maximal force in the line occurs at the position with the largest value of z', thus at the higher support of the line:

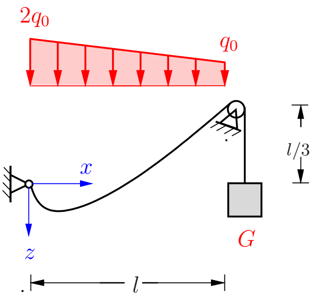
$$\underline{\underline{S_{\max}}} = S(0) = H\sqrt{1 + z'^2(0)} = \underline{\underline{10.388\,q_0\,a}}\,.$$

P6.3

Problem 6.3 A rope is stretched by a weight and loaded with a line load $q(x)$. The pulley at the right bearing is frictionless and its dimensions are negligible.

Determine the position and value of the maximal sag.

Given: $q_0 = \frac{1}{2}\sqrt{2}\, W/l$



Solution With the line load given by

$$q(x) = q_0\left(-\frac{x}{l} + 2\right)$$

integrating the differential equation of the curve of the rope twice yields

$$z''(x) = -\frac{q_0}{H}\left(-\frac{x}{l} + 2\right),$$

$$z'(x) = -\frac{q_0 l}{H}\left[-\frac{1}{2}\left(\frac{x}{l}\right)^2 + 2\,\frac{x}{l}\right] + C_1\,,$$

$$z(x) = -\frac{q_0 l^2}{H}\left[-\frac{1}{6}\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^2\right] + C_1 x + C_2\,.$$

It follows from the geometric boundary conditions

$$z(0) = 0 \quad \leadsto \quad C_2 = 0\,,$$

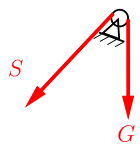
$$z(l) = -\frac{l}{3} \quad \leadsto \quad C_1 = -\frac{1}{3} + \frac{5}{6}\,\frac{q_0\, l}{H}\,.$$

Thus, the curve of the rope is given by

$$z(x) = -\frac{q_0\, l^2}{H}\left[-\frac{1}{6}\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^2 - \frac{5}{6}\left(\frac{x}{l}\right)\right] - \frac{1}{3}\,x\,.$$

The yet unknown horizontal tensile force H can be obtained from the given force in the rope at the position $x = l$. The condition

$$S(l) = W \qquad \text{bzw.} \qquad H\sqrt{1 + z'(l)^2} = W$$



yields together with

$$z'(l) = -\left(\frac{2}{3}\,\frac{q_0\, l}{H} + \frac{1}{3}\right) = -\frac{1}{3}\left(\sqrt{2}\,\frac{W}{H} + 1\right)$$

the quadratic equation

$$\left(\frac{W}{H}\right)^2 - \frac{2\sqrt{2}}{7}\left(\frac{W}{H}\right) - \frac{10}{7} = 0$$

with the solution

$$\frac{W}{H} = \begin{cases} -\dfrac{5}{7}\sqrt{2} < 0 & \text{(not feasible)}\,, \\ \sqrt{2}\,. \end{cases}$$

The horizontal tensile force follows as

$$H = \frac{1}{2}\sqrt{2}\,W = q_0\,l\,,$$

with the curve of the rope finally calculated as

$$z(x) = l\left[\frac{1}{6}\left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^2 + \frac{5}{6}\left(\frac{x}{l}\right)\right] - \frac{1}{3}\,x\,.$$

The curve of sagging yields with $h = -l/3$

$$\begin{aligned} \eta(x) &= z(x) - \frac{h}{l}\,x \\ &= l\left[\frac{1}{6}\left(\frac{x}{l}\right)^3 - \left(\frac{x}{l}\right)^2 + \frac{5}{6}\left(\frac{x}{l}\right)\right]\,. \end{aligned}$$

The maximum amount of sagging $\eta_{\max}$ follows from the condition

$$\eta' = 0 \qquad \rightsquigarrow \qquad \left(\frac{x}{l}\right)^2 - 4\left(\frac{x}{l}\right) + \frac{5}{3} = 0\,,$$

which leads to the solutions

$$\frac{x^*}{l} = 2 \pm \sqrt{7/3}.$$

The first solution $x^* = (2 + \sqrt{7/3}\,)\,l > l$ is geometrically not feasible, thus $\eta_{\max}$ occurs at the position

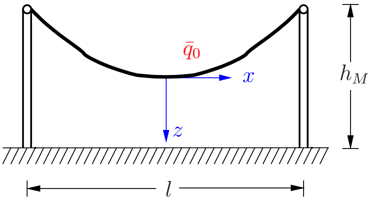
$$\underline{\underline{x^* = (2 - \sqrt{7/3}\,)\,l}}.$$

Inserting x^* yields the value of the maximal sag

$$\underline{\underline{\eta_{\max}}} = \eta(x^*) = \left(\frac{7}{9}\sqrt{\frac{7}{3}} - 1\right)\,l \approx \underline{\underline{0.188\,l}}\,.$$

P6.4

Problem 6.4 A rope is stretched across a street between two poles of the same height and loaded by its dead weight $\bar{q}_0$. At the footing of the poles a maximum momentum M_{max} can be absorbed.



Determine the maximal clearance height in the middle of the rope and the maximal force in the rope.

Given: $h_M = 20\,\text{m},\ l = 50\,\text{m},\ \bar{q}_0 = 10\,\text{N/m},\ M_{max} = 10\,\text{kNm}.$

Solution From the absorbable momentum at the footing of a pole, we can first compute the feasible maximal horizontal tensile force:

$$H = \frac{M_{max}}{h_M} = 500\,\text{N}\,.$$

The curve of the rope in the given coordinate system reads

$$z(x) = \frac{H}{\bar{q}_0}\left(1 - \cosh\frac{\bar{q}_0\,x}{H}\right)\,.$$

Thus the maximal sag is

$$\eta_{max} = -z(l/2) = -\frac{H}{\bar{q}_0}\left(1 - \cosh\frac{\bar{q}_0\,l}{2\,H}\right) = 6.381\,\text{m}\,.$$

The clearance height is obtained as

$$\underline{\underline{h_D}} = h_M - \eta_{max} = \underline{\underline{13.618\,\text{m}}}\,.$$

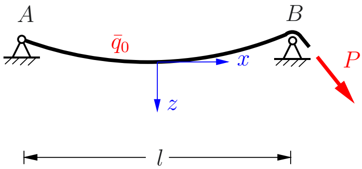
The maximum force in the rope occurs at the fixing points ($x = \pm\,l/2$):

$$\underline{\underline{S_{max}}} = H\,\cosh\frac{\bar{q}_0\,l}{2\,H} = 500\,\text{N}\cosh 0.5 = \underline{\underline{563.8\,\text{N}}}\,.$$

Remark: The length of the rope at the total weight follows as

$$L = \frac{2H}{\bar{q}_0}\sinh\frac{\bar{q}_0\,l}{2\,H} = 52.11\,\text{m}\,, \qquad G = L\,\bar{q}_0 = 521.1\,\text{N}\,.$$

Problem 6.5 A measuring tape (dead weight $\bar{q}_0$) is used to determine the distance between the points A and B. The actual distance is given by l. **P6.5**



Determine the force P which is necessary to stretch the measuring tape such that the measuring error is 0.5%. Determine the maximal sag of the tape.

Solution With the known measuring error of 0.5 %, the length L of the tape is calculated as

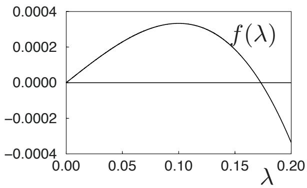
$$L = 1.005\, l = \frac{2\,H}{\bar{q}_0} \sinh \frac{\bar{q}_0\, l}{2\,H}\,.$$

Reformulation reveals the transcendental equation for the horizontal tension H,

$$1.005\, \frac{\bar{q}_0\, l}{2\,H} = \sinh \frac{\bar{q}_0\, l}{2\,H}\,,$$

or by substituting $\lambda = \dfrac{\bar{q}_0\, l}{2\,H}$

$$f(\lambda) = 1.005\,\lambda - \sinh\lambda = 0\,.$$



The solution is obtained from a graphical determination of $f = 0$, or, more precisely, through iteration using the Newton method:

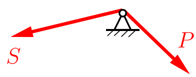
$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)} \qquad \text{with} \qquad f'(\lambda) = \frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} = 1.005 - \cosh\lambda.$$

Step	0 (starting value)	1	2	3	4
λ	0.2000	0.1777	0.1733	0.1731	0.1731

It follows

$$\lambda = 0.1731 \qquad \leadsto \qquad H = \frac{\bar{q}_0\, l}{2\,\lambda} = 2.889\,\bar{q}_0\, l\,.$$

The stretching force P and the maximal value of sag are calculated by

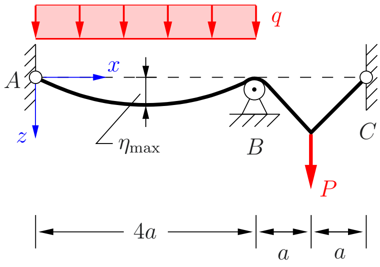


$$\underline{\underline{P}} = S(l/2) = H \cosh \frac{\bar{q}_0\, l}{2\,H} = H\, \cosh\lambda = \underline{\underline{2.932\,\bar{q}_0\, l}}\,.$$

$$\underline{\underline{\eta_{max}}} = -z(l/2) = -\frac{H}{\bar{q}_0}\left(1 - \cosh\frac{\bar{q}_0\, l}{2\,H}\right) = \underline{\underline{0.0434\, l}}\,.$$

P6.6

Problem 6.6 A rope is laid across a frictionless pulley (dead weight negligible) and has a total length $L = 7\,a$. It is loaded with a line load q and a single load P.



Determine P for a maximal sag $\eta_{\max} = a/10$ between the points A and B.

Solution We first determine the length of the rope in the region $A-B$ and $B-C$. It follows for region $A-B$

$$L^{AB} = -\frac{H}{2\,q}\left[z'\sqrt{1+z'^{\,2}} + \operatorname{arsinh} z'\right]_{z_1'}^{z_2'}$$

with the horizontal tension H and the limits z_1', z_2' given by

$$H = \frac{q\,l^2}{8\,\eta^*} = \frac{q\,(4\,a)^2}{8\,\frac{a}{10}} = 20\,qa, \quad z_1' = \frac{q\,l}{2\,H} = \frac{1}{10}, \quad z_2' = -\frac{q\,l}{2\,H} = -\frac{1}{10}\,.$$

Inserting yields $L^{AB} = 20\,a\left(\frac{1}{10}\sqrt{\frac{101}{100}} + \operatorname{arsinh}\left(\frac{1}{10}\right)\right) \approx 4.01\,a\,.$

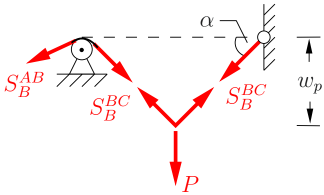
From the geometry, it follows for the length of the rope in region $B-C$:

$$L^{BC} = 2\sqrt{w_p^2 + a^2} \quad \text{or} \quad w_p^2 = \frac{1}{4}(L^{BC})^2 - a^2\,.$$

With $L = L^{AB} + L^{BC} = 7\,a$ and the known length L^{AB}, the maximal sag in region $B-C$ yields $w_p \approx 1.11\,a$. For the angle α, $\sin\alpha = (w_p/\sqrt{w_p^2 + a^2})$ holds. Consequently, S_B^{BC} can be determined from equilibrium in vertical direction at point P:

$$\uparrow: \quad 2\,S_B^{BC}\,\sin\alpha - P = 0 \quad \leadsto \quad S_B^{BC} = \frac{P}{2\,w_p}\sqrt{w_p^2 + a^2}\,.$$

With $S_B^{AB} = S_B^{BC}$ and



$$S_B^{AB} = H\sqrt{1 + \left(\frac{q_0\,l}{2\,H}\right)^2} \approx 20.10\,q\,a\,,$$

the magnitude of the force P yields

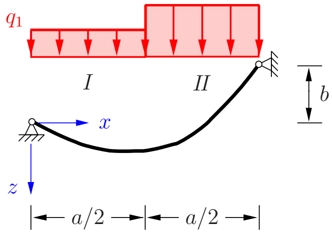
$$\underline{\underline{P}} \approx \frac{40.20\,q\,a \cdot 1.11\,a}{\sqrt{1.11^2\,a^2 + a^2}} = \underline{\underline{29.87\,q\,a}}\,.$$

Problem 6.7 A rope is loaded by two piecewise constant line loads q_1 and q_2.

P6.7

Determine the curve of the rope in the areas I and II for the known maximal rope force $S_{\max}$.

Given: $q_1 = 1\,\text{kN/m}$, $a = 20\,\text{m}$, $q_2 = 2\,\text{kN/m}$, $b = 4\,\text{m}$, $S_{\max} = 100\,\text{kN}$.



Solution Due to the jump in the load function at the position $x = a/2$, the rope is divided into two regions. By integrating the differential equation of the rope $z'' = -\frac{1}{H}\,q(x)$ twice, it follows for the rope curve in the regions:

$$\underline{\underline{z_I(x) = -\frac{1}{2H}\,q_1\,x^2 + C_1\,x + C_2}}\,, \qquad \underline{\underline{z_{II}(x) = -\frac{1}{2H}\,q_2\,x^2 + C_3\,x + C_4}}\,.$$

The 4 constants can be determined with the help of boundary and transition conditions:

$$\begin{aligned} &\text{boundary cond.:} \quad && z_I(0) = 0\,, && z_{II}(a) = -b\,, \\ &\text{transition cond.:} \quad && z_I(a/2) = z_{II}(a/2)\,, \quad && {z_I}'(a/2) = {z_{II}}'(a/2)\,. \end{aligned}$$

One obtains

$$\underline{\underline{C_1 = \frac{a}{8\,H}\,(3\,q_1 + q_2) - \frac{b}{a}}}\,, \qquad \underline{\underline{C_2 = 0}}\,,$$

$$\underline{\underline{C_3 = \frac{a}{8\,H}\,(-q_1 + 5\,q_2) - \frac{b}{a}}}\,, \qquad \underline{\underline{C_4 = \frac{a^2}{8\,H}\,(q_1 - q_2)}}\,.$$

In order to completely describe the rope curve, the unknown horizontal tension force H must be computed:

$$S_{\max} = H\,\sqrt{1 + ({z_{II}}'(a))^2} = 100\,\text{kN} \quad \text{with} \quad {z_{II}}'(a) = -\frac{35}{2\,H} - \frac{1}{5}\,.$$

The quadratic equation

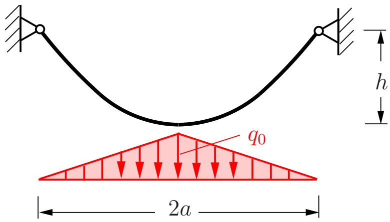
$$\frac{26}{25}\,H^2 + 7\,H - \frac{38775}{4} = 0$$

has the solution (H_2 is negative and thus physically not admissible!)

$$H_{1,2} = \frac{-7 \pm \sqrt{40375}}{\frac{52}{25}} \quad \rightsquigarrow \quad \underline{\underline{H = H_1 \approx 93.24\,\text{kN}}}\,.$$

P6.8

Problem 6.8 A rope hangs between two bearings at the same height and is loaded by a triangular line load with the maximal amplitude q_0 in the middle of the rope.



Determine the force in the rope at the lowest point of the rope and the shape of the rope curve.

Solution Choosing the coordinate system in the middle between the bearings, the load is given by

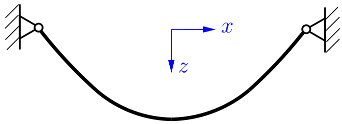
$$q(x) = q_0(1 - \frac{x}{a}) \,, \qquad 0 \le x \le a \,.$$

Integrating the differential equation of the rope curve twice yields

$$\frac{H}{q_0} z'' = \frac{x}{a} - 1 \,,$$

$$\frac{H}{q_0} z' = \frac{x^2}{2a} - x + C_1 \,,$$

$$\frac{H}{q_0} z = \frac{x^3}{6a} - \frac{x^2}{2} + C_1 x + C_2 \,.$$



We obtain the integration constants from the boundary conditions:

$$z'(0) = 0 \quad \rightsquigarrow \quad C_1 = 0 \,, \qquad z(a) = 0 \quad \rightsquigarrow \quad C_2 = \frac{a^2}{3} \,.$$

The curve of the rope then reads (symmetry!)

$$\underline{\underline{z = \frac{q_0}{H}\left(\frac{x^3}{6a} - \frac{x^2}{2} + \frac{a^2}{3}\right)}} \,, \qquad 0 \le x \le a \,.$$

The horizontal force H is calculated from the boundary condition

$$z(0) = h \quad \rightsquigarrow \quad h = \frac{q_0}{H}\frac{a^2}{3} \quad \rightsquigarrow \quad H = \frac{q_0\, a^2}{3h} \,.$$

Then the force in the rope at the lowest point follows as

$$S = H\sqrt{1 + (z')^2} \quad \text{mit} \quad z'(0) = 0 \quad \rightsquigarrow \quad \underline{\underline{S = H = \frac{q_0\, a^2}{3h}}} \,.$$

Chapter 7

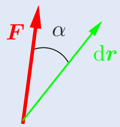
Work and Potential Energy

7

Work

A force $\boldsymbol{F}$ acting along an infinitesimally displacement $\mathrm{d}\boldsymbol{r}$ accomplishes the *work*

$$\mathrm{d}U = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = F\,\mathrm{d}r\,\cos\alpha\,.$$



Analogously, the work of a moment $\boldsymbol{M}$ along a rotation $\mathrm{d}\boldsymbol{\varphi}$ is

$$\mathrm{d}U = \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\varphi}\,.$$

In case that force and displacement vectors or moment and rotation vectors, respectively, are parallel to each other, the relations simplify to scalar expressions

$$\mathrm{d}U = F\,\mathrm{d}r \qquad \text{bzw.} \qquad \mathrm{d}U = M\,\mathrm{d}\varphi\,.$$

Principle of Virtual Work

Instead of the displacement $\mathrm{d}\boldsymbol{r}$, one introduces in statics a "virtual" displacement $\delta\boldsymbol{r}$. As a result, the *Principle of Virtual Work* can be formulated: A system of forces, which is in equilibrium, does not accomplish virtual work along a virtual displacement $\delta\boldsymbol{r}$:

$$\delta U = 0\,.$$

Virtual Displacements are:

1. imagined
2. infinitesimally small
3. compatible with the kinematic constraints of the system.

Remarks:

- In case that support reactions or internal forces have to be determined, it is necessary to use a free-body diagram and to introduce the support reactions or the internal forces as external loads.
- The symbol δ indicates the relation to the variational calculus.
- The work of a force along a finite path is given by

$$U = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,.$$

Stability of an Equilibrium Position

Conservative forces (weights, spring forces) can be derived from a potential $V = -U$ via

$$\delta V = -\delta U \, .$$

The equilibrium condition reads

$$\boxed{\delta V = 0 \, .}$$

The stability of the equilibrium position results from the sign of $\delta^2 V$:

$$\delta^2 V \begin{cases} > 0 & \text{stable,} \\ < 0 & \text{unstable.} \end{cases}$$

If V is given as a function of *one* spatial coordinate z, then

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}z}\delta z \, , \qquad \delta^2 V = \frac{1}{2}\,\frac{\mathrm{d}^2 V}{\mathrm{d}z^2}\,(\delta z)^2$$

apply. Therewith and given $\delta z \neq 0$, the following statements hold:

equilibrium condition $\boxed{\dfrac{\mathrm{d}V}{\mathrm{d}z} = 0 \, ,}$

stability $\boxed{\dfrac{\mathrm{d}^2 V}{\mathrm{d}z^2} \begin{cases} > 0 & \text{stable position,} \\ < 0 & \text{labile position.} \end{cases}}$

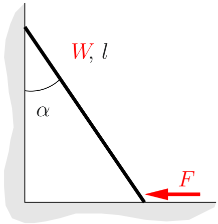
Remarks:

- In case of $\dfrac{\mathrm{d}^2 V}{\mathrm{d}z^2} = V'' = 0$, it is necessary to investigate higher derivatives.
- The equilibrium position is *indifferent*, if in addition to $V'' = 0$ also all higher derivatives are zero.
- The potential of a weight W is $V = W\,z$, if z is counted vertically *upwards* from the zero level.
- The potential of a spring tensioned by x (spring constant k) is $V = \frac{1}{2}k\,x^2$.
- The potential of a torsion spring tensioned by φ (torsion spring constant k_T) is $V = \frac{1}{2}k_T\,\varphi^2$.

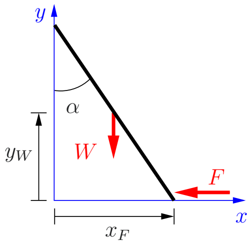
P7.1

Problem 7.1 A ladder (weight W, length l) leans on a smooth wall.

What is the magnitude of F to obtain an equilibrium state for the angle α?



Solution For the determination of equilibrium positions using the principle of virtual displacements, it is suitable to define the coordinates of the acting points of the forces. In the chosen coordinate system, they are given by x_F and y_W. Then, δx_F and δy_W are oriented against F and W, respectively. Therefore, the equilibrium position is found as



$$\delta U = -F\,\delta x_F - W\,\delta y_W = 0\,.$$

With

$$x_F = l\,\sin\alpha\,, \qquad y_W = \frac{l}{2}\,\cos\alpha\,,$$

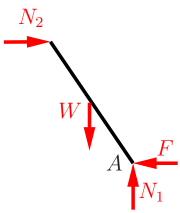
$$\delta x_F = l\,\cos\alpha\,\delta\alpha\,, \quad \delta y_W = -\frac{l}{2}\,\sin\alpha\,\delta\alpha\,,$$

it follows that

$$\delta U = -F\,l\,\cos\alpha\,\delta\alpha + \frac{1}{2}\,W\,l\sin\alpha\,\delta\alpha = 0$$

$$\leadsto \quad \underline{\underline{F = \frac{1}{2}\,W\,\tan\alpha\,.}}$$

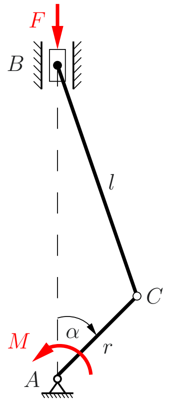
This result can be easily verified by formulating the equilibrium conditions for the acting forces:



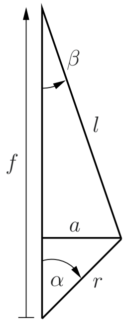
$$\left.\begin{aligned} \uparrow: &\quad N_1 - W = 0\,,\\ \rightarrow: &\quad N_2 - F = 0\,,\\ \overset{\curvearrowright}{A}: &\quad N_2\,l\,\cos\alpha - \frac{l}{2}\,W\,\sin\alpha = 0\,, \end{aligned}\right\} \quad \begin{aligned} &N_1 = W\,,\\ &N_2 = F\,,\\ &\underline{\underline{F = \frac{1}{2}\,W\tan\alpha\,.}} \end{aligned}$$

Problem 7.2 A crank $\overline{AC}$ is hinged in A and loosely bolted in C with the rod $\overline{BC}$. At the rod's end, the piston in B is subjected to the force F. Moreover, a moment M is acting on the crank. **P7.2**

Determine $M(\alpha)$ for equilibrium. Neglect the weight of the crank, the rod and the piston.



Solution The displacement f of the piston is introduced. Since F is acting in the opposite direction of δf just as M acts against the virtual rotation $\delta\alpha$, the equilibrium equation (principle of virtual work) reads



$$\delta U = -M\,\delta\alpha - F\,\delta f = 0\,.$$

From the sketch, one obtains

$$f = r\cos\alpha + l\cos\beta$$

$$\leadsto \quad \delta f = -r\sin\alpha\;\delta\alpha - l\sin\beta\;\delta\beta\,.$$

The angle β needs to be eliminated. Based on the sketch, it is

$$a = l\,\sin\beta = r\,\sin\alpha \quad \leadsto \quad \sin\beta = \frac{r}{l}\sin\alpha\,.$$

Differentiation yields

$$\cos\beta\;\delta\beta = \frac{r}{l}\cos\alpha\;\delta\alpha \quad \leadsto \quad \delta\beta = \frac{r}{l}\,\frac{\cos\alpha}{\cos\beta}\;\delta\alpha\,.$$

With $\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-(r/l)^2\sin^2\alpha}\,,$ one obtains

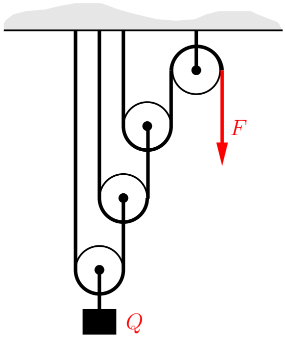
$$-M\,\delta\alpha + F\Big(r\,\sin\alpha\;\delta\alpha + l\,\frac{r}{l}\sin\alpha\;\frac{r}{l}\,\frac{\cos\alpha}{\sqrt{1-(r/l)^2\sin^2\alpha}}\;\delta\alpha\Big) = 0$$

or

$$\underline{\underline{M = Fr\;\sin\alpha\Big(1 + \frac{r\;\cos\alpha}{\sqrt{l^2 - r^2\sin^2\alpha}}\Big)}}\,.$$

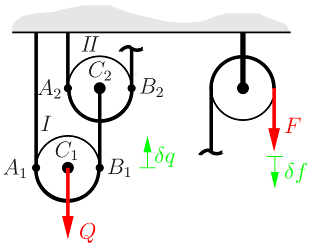
P7.3 **Problem 7.3** Determine the ratio of the load Q and the pulling force F at the example of a power pulley

a) in the sketched case (3 free pulleys)
b) in the general case (n free pulleys)?



Solution The load Q is fixed at C_1. Under a virtual displacement δq of Q, C_1 also rises with δq.

The pulley I rotates around A_1, since the point A_1 is fixed by a rope at the top. Therefore, B_1 and, thus, C_2 move by $2\,\delta q$.



Analogous considerations for the pulley II (A_2 is fixed) yield $4\,\delta q = 2^2 \delta q$ for the virtual displacement of B_2.

The pulley, fixed at the top, rotates around its center point. Therefore, the displacement δf of the pulling force F corresponds to the displacement of the point B_n at the final pulley.
Following the equilibrium condition

$$\delta U = -Q\,\delta q + F\,\delta f = 0\,,$$

it is found that

a) for 3 free pulleys with $\delta f = 2^3 \delta q = 8\,\delta q$

$$\underline{\underline{\frac{Q}{F} = 2^3 = 8\,.}}$$

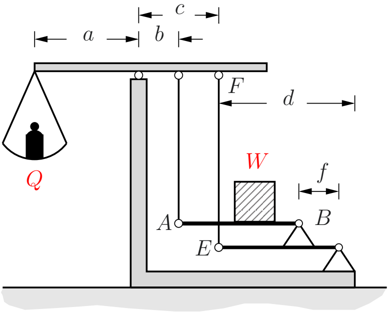
b) For n free pulleys one obtains with $\delta f = 2^n \delta q$

$$\underline{\underline{\frac{Q}{F} = 2^n\,.}}$$

Remark: This result explains the notion *power* pulley!

Problem 7.4 The depicted balance is constructed such that the amplitude of Q is independent of the position of the weight W on the load arm $\overline{AB}$.

Determine the ratio of the dimensions b, c, d and f for a given length a as well as the relation between Q and W.

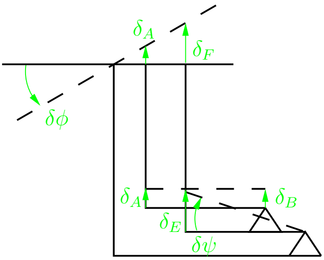


P7.4

Solution In order to fulfil the requirement, the load arm $\overline{AB}$ must stay horizontally. Thus,

$$\delta_A = \delta_B \,.$$

According to the sketch, a rotation of the upper beam with $\delta\phi$ yields



$$\delta_A = b\,\delta\phi\,, \qquad \delta_B = f\,\delta\psi\,.$$

Both angles are coupled via the displacement of the bar $\overline{EF}$:

$$\delta_F = c\,\delta\phi = d\,\delta\psi = \delta_E \quad \rightsquigarrow \quad \delta\psi = \frac{c}{d}\,\delta\phi\,.$$

Therefore, one obtains

$$\delta_A = b\,\delta\phi = f\,\frac{c}{d}\,\delta\phi = \delta_B \quad \rightsquigarrow \quad \underline{\underline{\frac{b}{c} = \frac{f}{d}}}\,.$$

Q results from the principle of virtual work via

$$Q\,\delta q - W\,\delta_A = 0$$

with

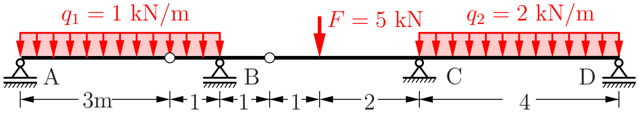
$$\delta q = a\,\delta\phi$$

to

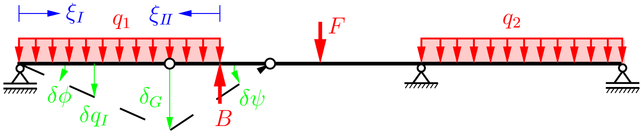
$$\underline{\underline{Q = \frac{b}{a}\,W\,.}}$$

P7.5

Problem 7.5 Determine the support reactions in A, B and D for the illustrated girder with two hinges by using the principle of virtual work.



Solution To obtain the reaction force in B, this support reaction is introduced as *external* force and the system is subjected to a kinematically admissible displacement.



A consideration of both parts of q_1 yields

$$\delta U = \int_0^3 q_1\,\delta q_I\,\mathrm{d}\xi_I + \int_0^1 q_1\,\delta q_{II}\,\mathrm{d}\xi_{II} - B\,\delta_B = 0\,.$$

With

$$\delta_B = 1\cdot\delta\psi\,,\quad \delta q_I = \xi_I\,\delta\phi\,,\quad \delta q_{II} = (1+\xi_{II})\delta\psi\,,$$

it follows that

$$B\,\delta\psi = q_1\,\frac{3^2}{2}\,\delta\phi + q_1\left(1+\frac{1^2}{2}\right)\delta\psi\,.$$

Including the geometrical relation at the hinge,

$$\delta_G = 3\,\delta\phi = 2\,\delta\psi \quad\rightsquigarrow\quad \delta\phi = \frac{2}{3}\,\delta\psi\,,$$

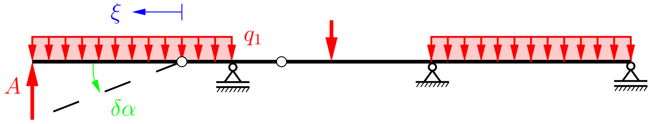
leads to

$$B = q_1\,\frac{3^2}{2}\,\frac{2}{3} + q_1\left(1+\frac{1}{2}\right) = 4.5\,q_1$$

or

$$\underline{\underline{B = 4.5\text{ kN}}}\,.$$

2) Using the free-body diagram, the support reaction in A results



from the virtual-work equation

$$\delta U = -A\,\delta_A + \int_0^3 q_1\,\delta q_1\,\mathrm{d}\xi = 0$$

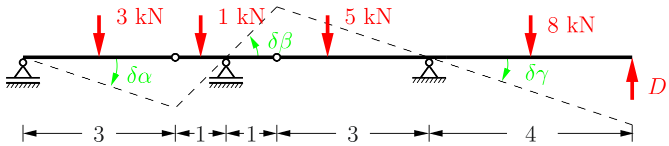
and the geometrical relations

$$\delta q_1 = \xi\,\delta\alpha\,, \quad \delta_A = 3\,\delta\alpha$$

leading to

$$-3\,A\,\delta\alpha + q_1\,\frac{3^2}{2}\,\delta\alpha = 0 \quad \rightsquigarrow \quad \underline{\underline{A}} = \frac{3}{2}\,q_1 = \underline{\underline{1.5\text{ kN}}}\,.$$

3) The free-body diagram for the support reaction D yields



The following geometrical relations hold:

$$\left.\begin{aligned} 3\,\delta\alpha &= 1\,\delta\beta \\ 3\,\delta\gamma &= 1\,\delta\beta \end{aligned}\right\} \quad \rightsquigarrow \quad \begin{aligned} \delta\alpha &= \delta\gamma\,, \\ \delta\beta &= 3\,\delta\gamma\,. \end{aligned}$$

For the application of the principle of virtual work, the distributed loads are replaced by their resultant forces. Then, one obtains

$$3\cdot 1.5\,\delta\alpha + 1\cdot 0.5\,\delta\beta - 5\cdot 2\,\delta\gamma + 8\cdot 2\,\delta\gamma - D\cdot 4\,\delta\gamma = 0\,,$$

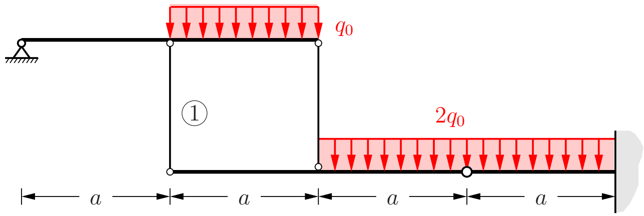
from which the support reaction yields

$$\underline{\underline{D}} = \frac{1}{4}\,(4.5 + 1.5 - 10 + 16) = \underline{\underline{3\text{ kN}}}\,.$$

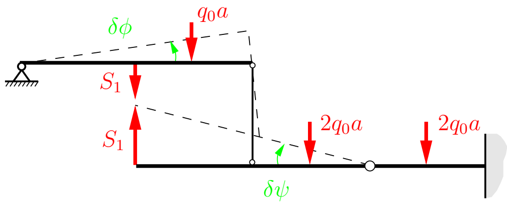
4) The support reaction in C is finally obtained by equilibrium in vertical direction:

$$\underline{\underline{C}} = F + q_1\cdot 4 + q_2\cdot 4 - A - B - D = \underline{\underline{8\text{ kN}}}\,.$$

P7.6 **Problem 7.6** For the illustrated system, composed of beams and rods, the force S_1 in the rod ① has to be determined.



Solution The resultant forces of the distributed loads are placed in the respective area centroids, and the system is subjected to a virtual displacement after rod ① is cut free.



The following geometric relation holds:

$$2a\,\delta\phi = a\,\delta\psi \quad \rightsquigarrow \quad \delta\psi = 2\,\delta\phi\,.$$

The principle of virtual work yields

$$\delta U = -\,q_0 a \cdot \frac{3}{2} a\,\delta\phi - S_1 \cdot a\,\delta\phi + S_1 \cdot 2\,a\,\delta\psi - 2\,q_0 a \cdot \frac{a}{2}\,\delta\psi = 0$$

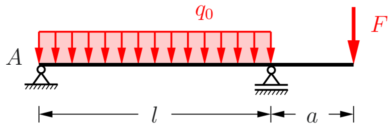
or

$$-\frac{3}{2}\,q_0 a^2 \delta\phi - S_1 a\,\delta\phi + 2\,a\,S_1 2\,\delta\phi - q_0 a^2 2\,\delta\phi = 0$$

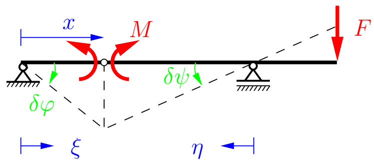
$$\rightsquigarrow \quad 3\,S_1 = \frac{7}{2}\,q_0 a \quad \rightsquigarrow \quad \underline{\underline{S_1 = \frac{7}{6}\,q_0 a\,.}}$$

Remark: The vertical distributed load at the lower beam can *not* be replaced by *one* resultant load in the hinge.

Problem 7.7 Determine for the depicted beam the function of the internal moment between the pins by using the principle of virtual work. **P7.7**



Solution In order to determine the internal moment M at an arbitrary position x by means of the principle of virtual work, one needs to introduce a hinge at x and has to let M acting as an external load on the adjacent parts of the beam. For a virtual displacement, it follows that



$$\delta U = -M\,\delta\varphi - M\,\delta\psi - F\,a\,\delta\psi + \int_0^x q_0\,(\xi\,\delta\varphi)\,\mathrm{d}\xi + \int_0^{l-x} q_0\,(\eta\,\delta\psi)\,\mathrm{d}\eta = 0.$$

With the geometric relation

$$x\,\delta\varphi = (l-x)\,\delta\psi \quad \rightsquigarrow \quad \delta\varphi = \frac{l-x}{x}\delta\psi\,,$$

one obtains

$$M\left(\frac{l-x}{x}+1\right)\delta\psi = \left[-F\,a + q_0\frac{x^2}{2}\frac{l-x}{x} + q_0\frac{(l-x)^2}{2}\right]\delta\psi\,.$$

After some rearrangements, the requested function for the internal moment is found as

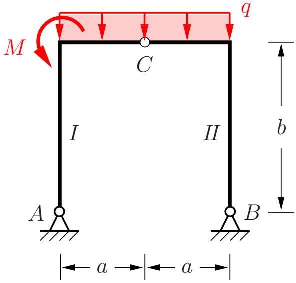
$$\underline{\underline{M\,(x) = \frac{x}{l}\left[-F\,a + \frac{q_0 l^2}{2}\left(1-\frac{x}{l}\right)\right]}}\,.$$

Obviously, proceeding from the support reaction $A = \frac{1}{2}\,q_0\,l - \frac{a}{l}\,F$ and the equilibrium equation $M = A\,x - \frac{1}{2}\,q_0\,x^2$, the same result for $M(x)$ is found.

P7.8

Problem 7.8 A frame is subjected to a distributed load q and a moment M.

Determine the horizontal support reaction in B by using the principle of virtual work.



Solution For the determination of the horizontal support reaction B_H, the horizontal support of the hinge B is released and B_H is applied as an external load. Then, the (now movable) system is subjected to a virtual displacement.
To obtain the correct displacement figure, the centers of rotation of the frame components I and II have to be found. For part I, the hinge in A acts as the center of rotation DP_I. For part II, one needs to find two possible motion directions of two points at II. The center of rotation can then be found at the intersection of two lines, which can be constructed perpendicular to the motion directions of these two points. In the present case, the motion possibilities of points C (linked to part I) and B (in horizontal direction) are known. This determines the center of rotation DP_{II}.

The virtual-work equation reads

$$\delta U = 2\,q\,a\,\delta u_q - M\,\delta\varphi - B_H\,\delta u_B\,.$$

With the geometrical relations

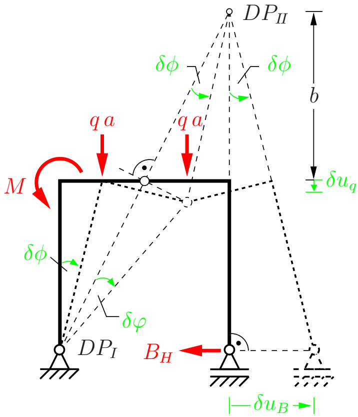
$$\delta u_q = \frac{a}{2}\,\delta\varphi\,, \qquad \delta u_B = 2\,b\,\delta\varphi\,,$$

it follows that

$$\delta U = \left(q\,a^2 - M - 2\,b\,B_H\right)\,\delta\varphi = 0\,.$$

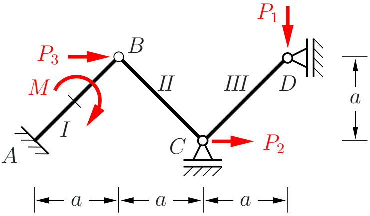
This yields the horizontal support reaction B_H:

$$\underline{\underline{B_H = \frac{q\,a^2 - M}{2\,b}}}\,.$$



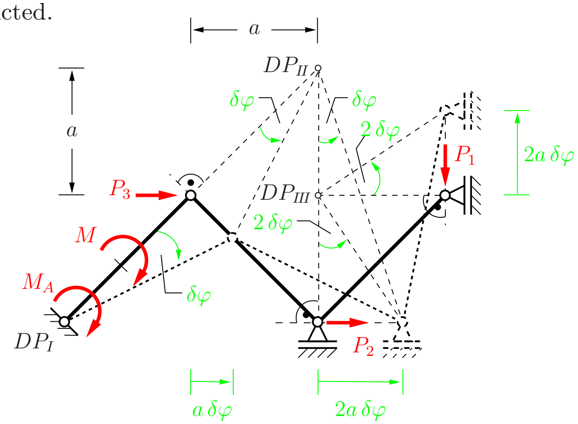
Problem 7.9 The illustrated system is loaded by the forces P_1, P_2, P_3 and the moment M.

Using the principle of virtual work, the value of the force P_1 should be determined such that the moment in A vanishes.



P7.9

Solution In order to obtain the moment in A, the clamped support is replaced by a hinge and the couple moment M_A is introduced as an external quantity. The now moveable system can be subjected to a virtual displacement. To evaluate the geometrical relations, the displaced system is constructed.



For the determination of the particular centers of rotation, one proceeds from beam I. Its center of rotation DP_I can be found at point A. Due to the obtained movement abilities of points B (endpoint of beam I), C (in horizontal direction) and D (in vertical direction), the centers of rotation of the beams II and III can be obtained according to the above sketch. Hence, all virtual displacements can be expressed in terms of $\delta\varphi$. The equilibrium condition is then given as

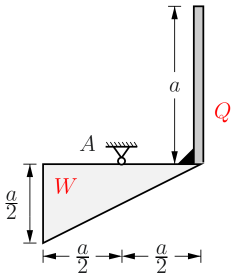
$$\delta U = M_A\,\delta\varphi + M\,\delta\varphi + P_3\,a\,\delta\varphi + P_2\,2\,a\,\delta\varphi - P_1\,2\,a\,\delta\varphi = 0\,.$$

Using the constraint $M_A = 0$, one obtains the required force P_1:

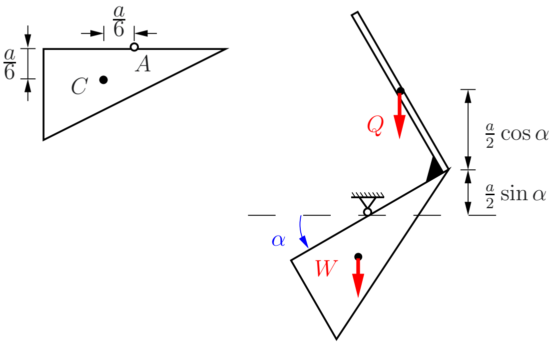
$$\underline{\underline{P_1 = \frac{M}{2\,a} + \frac{1}{2}\,P_3 + P_2\,.}}$$

P7.10 **Problem 7.10** A homogeneous bar with weight Q is connected with a triangular disc (weight W). Furthermore, the system is hinged in A.

Calculate all possible equilibrium positions and investigate the stability of theses equilibrium positions.



Solution First, the system is deflected by an arbitrary angle α, see sketch.



In consideration of the position of the center of gravity, the potential energy of the system comparing the non-deflected ($\alpha = 0$) with the deflected system yields

$$V = Q\left(\frac{a}{2}\sin\alpha + \frac{a}{2}\cos\alpha\right) + W\left(-\frac{a}{6}\sin\alpha - \frac{a}{6}\cos\alpha\right) .$$

Thus, one obtains the equilibrium equation

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}\alpha} &= Q\,\frac{a}{2}\,(\cos\alpha - \sin\alpha) - W\,\frac{a}{6}\,(\cos\alpha - \sin\alpha)\\ &= \frac{a}{2}\left(Q - \frac{W}{3}\right)(\cos\alpha - \sin\alpha) = 0\,.\end{aligned}$$

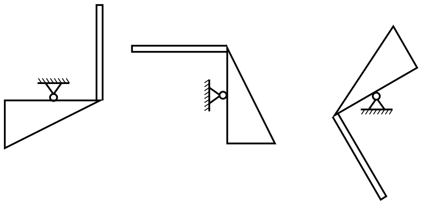
Following this, the possible equilibrium positions can be calculated:

$$1)\quad Q - \frac{W}{3} = 0 \qquad\qquad \rightsquigarrow \quad \underline{\underline{Q = \frac{W}{3}}}\,,$$

$$2)\quad \cos\alpha - \sin\alpha = 0 \quad\rightsquigarrow\quad \tan\alpha = 1 \quad\rightsquigarrow\quad \underline{\underline{\alpha_1 = \frac{1}{4}\pi}}\,,$$

$$\underline{\underline{\alpha_2 = \frac{5}{4}\pi}}\,.$$

For the investigation of the stability, the second derivative and, if necessary, higher derivatives of V are determined.
In the first case, the second and all higher derivatives of V are equal to zero. Following this, the equilibrium for this special weight ratio is neutral and thus equilibrium is possible in any position (see examples):



In the second case, one obtains

$$V'' = \frac{\mathrm{d}^2 V}{\mathrm{d}\alpha^2} = -\frac{a}{2}\left(Q - \frac{W}{3}\right)(\sin\alpha + \cos\alpha)\,.$$

Thus, the sign of this term depends on α and on the weight ratio. Finally, one obtains

a) $\alpha_1 = \dfrac{\pi}{4}$:

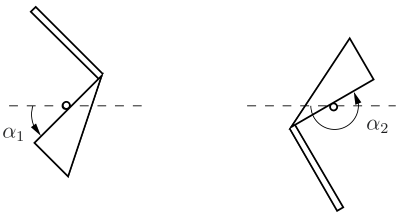
$$Q > \frac{W}{3} \quad \rightsquigarrow \quad V''(\alpha_1) < 0 \quad \rightsquigarrow \quad \text{unstable}\,,$$

$$Q < \frac{W}{3} \quad \rightsquigarrow \quad V''(\alpha_1) > 0 \quad \rightsquigarrow \quad \text{stable}\,,$$

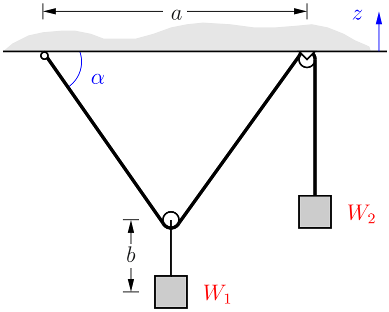
b) $\alpha_2 = \dfrac{5}{4}\pi$:

$$Q > \frac{W}{3} \quad \rightsquigarrow \quad V''(\alpha_2) > 0 \quad \rightsquigarrow \quad \text{stable}\,,$$

$$Q < \frac{W}{3} \quad \rightsquigarrow \quad V''(\alpha_2) < 0 \quad \rightsquigarrow \quad \text{unstable}\,.$$



P7.11 **Problem 7.11** For the illustrated system, the equilibrium position $\alpha = \alpha_0$ and the corresponding limit cases should be discussed. Assume that the rolling radii are negligibly small and that the length l of the rope is given.



Solution Weight forces are conservative forces and accordingly have a potential character. Using the coordinate z (directed vertically upwards), the position of the weights can be determined by the geometry:

$$z_1 = -b - \frac{a}{2}\tan\alpha\,,$$

$$z_2 = -\left(l - 2\,\frac{a}{2}\,\frac{1}{\cos\alpha}\right) = -\left(l - \frac{a}{\cos\alpha}\right).$$

Thus, the potential energy of the system can be formulated as

$$V = W_1 z_1 + W_2 z_2 = -W_1\left(b + \frac{a}{2}\tan\alpha\right) - W_2\left(l - \frac{a}{\cos\alpha}\right) = V(\alpha)\,.$$

The equilibrium position can be determined by the condition

$$\frac{dV}{d\alpha} = 0 \quad \rightsquigarrow \quad -W_1\,\frac{a}{2}\,\frac{1}{\cos^2\alpha} + W_2\,\frac{a\sin\alpha}{\cos^2\alpha} = 0$$

yielding

$$\underline{\underline{\sin\alpha_0 = \frac{1}{2}\,\frac{W_1}{W_2}}}\,.$$

Limit cases:

$W_1 > 2\,W_2 \quad \rightsquigarrow$ no possible equilibrium (because of $\sin\alpha_0 \leq 1$),

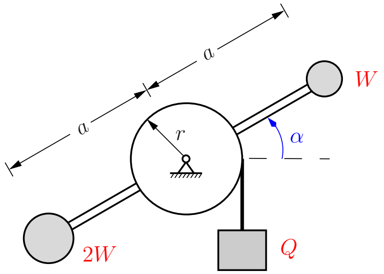
$W_1 = 2W_2 \quad \rightsquigarrow$ $\alpha_0 = \pi/2$, i.e. $a = 0$ for a finite length of the rope,

$W_1 = 0 \quad \rightsquigarrow$ $\alpha_0 = 0$.

Note: The length l of the rope and the distance b have no effect on the solution.

Problem 7.12 A disc (radius r) is hinged in its center and is connected to two weights via two rods (length a). A weight Q is attached to a rope which is wrapped around the disc. P7.12

Calculate all equilibrium positions and investigate the stability of these equilibrium positions.



Solution Since all active forces are weight forces, the system is conservative. Assuming an unwounded rope length l and specifying the zero level for $\alpha = 0$, the potential energy of the system for a deflection related to a certain angle α can be computed as follows

$$V = -2\,W\,a\,\sin\alpha + W\,a\,\sin\alpha - Q\,(l - r\alpha)$$

or

$$V = -W\,a\,\sin\alpha - Q\,(l - r\alpha) = V(\alpha)\,.$$

The equilibrium positions result from

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = 0\ :\quad -W\,a\,\cos\alpha + Q\,r = 0 \quad\leadsto\quad \underline{\underline{\cos\alpha = \frac{Q\,r}{W\,a}}}\,.$$

Due to the ambiguity of circular functions, two solutions exist:

$$\alpha_1 = \arccos\frac{Q\,r}{W\,a}\,,\qquad \alpha_2 = -\alpha_1\,.$$

The second derivative

$$V'' = \frac{\mathrm{d}^2 V}{\mathrm{d}\alpha^2} = W\,a\,\sin\alpha$$

specifies the stability at the equilibrium positions. One obtains

$$V''(\alpha_1) = W\,a\,\sin\alpha_1 > 0\,,$$

$$V''(\alpha_2) = -W\,a\,\sin\alpha_1 < 0\,,$$

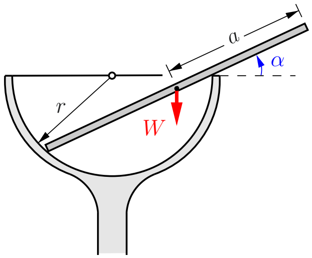
i.e. the state α_1 is stable and the state $\alpha_2 = -\alpha_1$ is unstable.

Note: Because of $\cos\alpha \leq 1$, these solutions only exist for $Q\,r < W\,a$. In the limit case $Q\,r = W\,a$ one obtains $\cos\alpha = 1$, i.e. the system is in equilibrium if the rods are horizontal.

P7.13

Problem 7.13 A straw (weight W, length $2a$) lies in a hemispherical glass with ideally smooth walls.

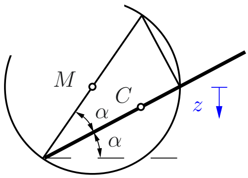
Calculate the equilibrium position $\alpha = \alpha_0$ and investigate its stability.



Solution Specifying the zero level to lie on the edge of the glass and using the coordinate z (directed vertically downward), the distance to the center of gravity of the straw can be determined as follows:

$$z = r\sin 2\alpha - a\sin\alpha\,.$$

Thus, the potential energy of the system can be calculated as



$$V(z) = -W\,z = -W\,(r\,\sin 2\alpha - a\,\sin\alpha)\,.$$

Using the equilibrium condition, one obtains

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\alpha} = W\,(-2r\,\cos 2\alpha + a\,\cos\alpha) = 0\,.$$

Furthermore, using $\cos 2\alpha = 2\cos^2\alpha - 1$, the equilibrium state follows

$$4\,r\,\cos^2\alpha - a\,\cos\alpha - 2\,r = 0$$

or

$$\underline{\underline{\cos\alpha_0 = \frac{a + \sqrt{a^2 + 32\,r^2}}{8\,r}}}\,,$$

where only angles $\alpha > 0$ are useful. With the second derivative

$$V'' = W\,(4\,r\sin 2\alpha - a\,\sin\alpha) = W\,(8\,r\,\cos\alpha - a)\,\sin\alpha$$

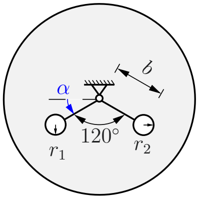
of the potential energy of the system and by inserting α_0, one obtains

$$V''(\alpha_0) = W\sqrt{a^2 + 32\,r^2}\,\sin\alpha_0\,.$$

The equilibrium is stable since this expression is positive for $0 < \alpha_0 < \pi/2$.

Problem 7.14 A disc (radius a) is hinged in its center and has two circular holes (radii r_1 and r_2) each at a distance of b from the hinge. **P7.14**

Calculate the equilibrium positions and investigate the stability at these equilibrium positions for $r_1 = \sqrt{2}\, r_2$.



Solution Since the disc without the holes is in equilibrium in any position, only the influence of the two holes needs to be taken into account. Therefore, the holes are considered as "negative" weights which must be "added" to the disc. Specifying the hinge as zero level, the potential energy of the system can be formulated as

$$V = W_1 b \sin\alpha + W_2 \sin(180^o - 120^o - \alpha) = V(\alpha)\,.$$

The states of equilibrium result from

$$V' = W_1 b \cos\alpha - W_2 b \cos(60^o - \alpha) = 0\,.$$

With the weights

$$W_1 = \pi r_1^2 \rho g = 2\pi r_2^2 \rho g\,, \qquad W_2 = \pi r_2^2 \rho g$$

one obtains

$$V' = \pi r_2^2 \rho g [2\cos\alpha - \cos(60^o - \alpha)] = 0$$

and thus

$$2\cos\alpha - \frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha = 0 \quad \leadsto \quad \tan\alpha = \sqrt{3}$$

$$\leadsto \quad \underline{\underline{\alpha_1 = 60^o}}\,, \qquad \underline{\underline{\alpha_2 = 240^o}}\,.$$

The second derivative

$$V'' = -W_1 b \sin\alpha - W_2 b \sin(60^o - \alpha)$$

specifies the stability at the equilibrium position yielding

$$\underline{\underline{V''(\alpha_1)}} = -W_1 b\,\frac{\sqrt{3}}{2} < 0 \quad \leadsto \quad \underline{\underline{\text{unstable}}}\,,$$

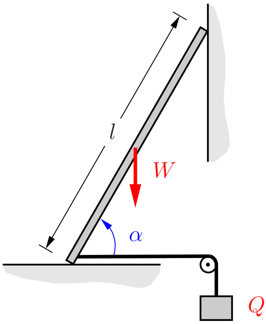
$$\underline{\underline{V''(\alpha_2)}} = +W_1 b\,\frac{\sqrt{3}}{2} > 0 \quad \leadsto \quad \underline{\underline{\text{stable}}}\,.$$

Note: Since the center of gravity of the perforated disc is located *above* the support for α_1 and *below* the support for α_2, the statement about the stability is well illustrated.

P7.15

Problem 7.15 A rod with weight W is leaned against a vertical, smooth wall. The lower end of the rod rests on the smooth ground and is supported by a rope (length L), which is loaded with a weight Q.

How heavy does the weight Q have to be for a given angle α to ensure that the system remains in position? Is this equilibrium position stable?



Solution Compared to the ground, the potential energy of the system is obtained by

$$V = W\,\frac{l}{2}\sin\alpha - Q\,(L - l\cos\alpha)\,.$$

The equilibrium equation

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = W\,\frac{l}{2}\cos\alpha - Ql\,\sin\alpha = 0$$

yields the required weight Q:

$$\underline{\underline{Q = W\,\frac{\cot\alpha}{2}}}\,.$$

From the second derivative

$$\frac{\mathrm{d}^2V}{\mathrm{d}\alpha^2} = -W\,\frac{l}{2}\sin\alpha - Q\,l\,\cos\alpha\,,$$

it follows by inserting the required weight Q:

$$\frac{\mathrm{d}^2V}{\mathrm{d}\alpha^2} = -W\,\frac{l}{2}\sin\alpha - W\,\frac{l}{2}\cot\alpha\cos\alpha = -\frac{W\,l}{2\sin\alpha}\,.$$

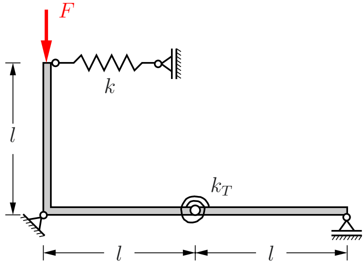
Thus, the equilibrium position is unstable for an angle

$$0 \le \alpha \le \frac{\pi}{2}.$$

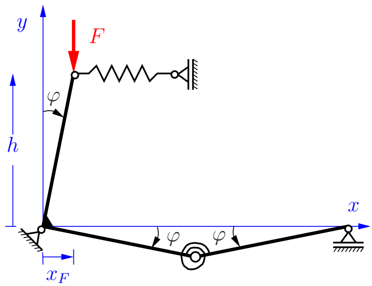
Note: The length L of the rope has no effect on the solution.

Problem 7.16 A system consisting of rigid, weightless beams, a spring (stiffness k) and a torsion spring (stiffness k_T) is in equilibrium in the depicted position. **P7.16**

Calculate the critical load F_{crit} for which the system becomes unstable.



Solution Deflecting the system to a certain angle φ, the potential energy of the system consists of the potentials V_F of the load, V_k of the spring and V_{k_T} of the torsion spring:



$$
\begin{aligned}
V &= V_F + V_k + V_{k_T} \\
&= F\,h + \frac{1}{2}\,k\,x_F^2 + \frac{1}{2}\,k_T\,(2\varphi)^2 \\
&= F\,l\,\cos\varphi + \frac{1}{2}\,k\,(l\,\sin\varphi)^2 \\
&\quad + \frac{1}{2}k_T\,(2\,\varphi)^2 = F\,l\,\cos\varphi + \frac{1}{2}\,k\,l^2\sin^2\varphi + 2\,k_T\,\varphi^2 \,.
\end{aligned}
$$

The equilibrium positions result from

$$
\frac{\mathrm{d}V}{\mathrm{d}\varphi} = -F\,l\,\sin\varphi + k\,l^2\,\sin\varphi\,\cos\varphi + 4\,k_T\,\varphi = 0\,.
$$

Besides the equilibrium position $\varphi = 0$, further equilibrium positions can be determined numerically from the transcendental equation, if required. Using the second derivative

$$
\frac{\mathrm{d}^2V}{\mathrm{d}\varphi^2} = -F\,l\,\cos\varphi + k\,l^2\,\cos 2\varphi + 4\,k_T
$$

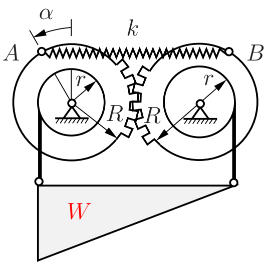
of the potential energy, one obtains for the position $\varphi = 0$

$$
\left.\frac{\mathrm{d}^2V}{\mathrm{d}\varphi^2}\right|_{\varphi=0} = -F\,l + k\,l^2 + 4\,k_T\,.
$$

If the second derivative is negative, this position becomes unstable. Following this, the critical load can be determined from $V'' = 0$ such that

$$
\underline{\underline{F_{\text{crit}} = k\,l + 4\,\frac{k_T}{l}\,.}}
$$

P7.17 **Problem 7.17** A homogeneous triangular plate with weight W hangs on two rope drums (radius r). The drums are connected with each other by gearwheels (radius R). A spring (stiffness k), unstretched for $\alpha = 0$, is connected in A and B.



Calculate the angle α for which the system is in equilibrium assuming $W\,r/k\,R^2 = 1$. Increasing $W\,r/k\,R^2$, what ratio limits the domain of equilibrium?

Solution Deflecting the system to a certain angle α, the spring is stretched by $x_k = 2R\sin\alpha$ and the triangular disc is moved downwards by $x_W = r\alpha$. Following this, the potential energy of the system can be determined as follows:

$$V = \frac{1}{2}\,k\,x_k^2 - W\,x_W = \frac{1}{2}\,k\,(2R)^2\sin^2\alpha - W\,r\,\alpha\,.$$

Using the equilibrium equation

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = \frac{1}{2}\,k\,(2R)^2\,2\sin\alpha\,\cos\alpha - W\,r = 0\,,$$

one obtains

$$\sin 2\alpha = \frac{W\,r}{2\,k\,R^2}\,.$$

The equilibrium positions can be determined inserting the given numerical values yielding

$$\sin 2\alpha = \frac{1}{2} \quad \rightsquigarrow \quad \underline{\underline{\alpha_1 = 15^\circ}}\,, \quad \underline{\underline{\alpha_2 = 75^\circ}}\,, \quad \underline{\underline{\alpha_3 = 195^\circ}}\,, \quad \underline{\underline{\alpha_4 = 255^\circ}}\,.$$

From the second derivative

$$V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\alpha^2} = 4\,k\,R^2\cos 2\alpha\,,$$

for stability at the equilibrium positions follows

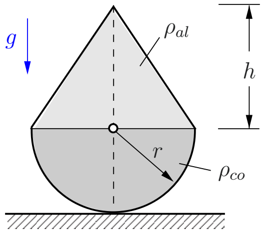
$$V''(\alpha_1) > 0 \quad \text{stable}\,, \qquad V''(\alpha_2) < 0 \quad \text{unstable}\,,$$

$$V''(\alpha_3) > 0 \quad \text{stable}\,, \qquad V''(\alpha_4) < 0 \quad \text{unstable}\,.$$

Because of $\sin 2\alpha \le 1$, equilibrium is only possible if $W\,r/k\,R^2 \le 2$.

Problem 7.18 An isosceles triangle with constant thickness t made of aluminium (density ρ_{al}) is attached to a semicircular disc with the same thickness made of copper (density ρ_{co}). P7.18

Determine the maximum height h of the triangle to ensure that the system returns to its initial position after a deflection of $|\varphi| \leq 90°$.

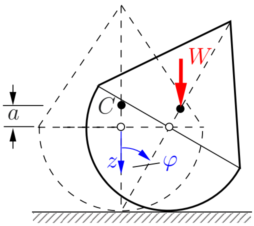


Solution Introducing a distance a between the overall center of gravity C and the center of the semicircle, the potential energy of the system for an arbitrary position $(-90° \leq \varphi \leq 90°)$ yields

$$V = -(W_{co} + W_{al})\, a\, (1 - \cos\varphi)\,,$$

where the weight of the semicircular plate can be calculated as

$$W_{co} = g\, m_{co}\,, \quad m_{co} = \rho_{co}\, t \frac{r^2 \pi}{2}\,.$$



The weight of the triangular disc is

$$W_{al} = g\, m_{al}\,, \quad m_{al} = \rho_{al} g\, t\, r\, h\,.$$

The equilibrium positions result from

$$\frac{\mathrm{d}V}{\mathrm{d}\varphi} = 0 \quad \rightsquigarrow \quad -(W_{co} + W_{al})\, a\, \sin\varphi = 0\,.$$

Following this, only $\varphi^* = 0$ provides an equilibrium position. This equilibrium state is stable for

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2}\right|_{\varphi^*} > 0 \quad \rightsquigarrow \quad -(W_{co} + W_{al})\, a > 0\,.$$

This condition is fulfilled, when $a < 0$, i.e. the overall center of gravity must be below the center of the semicircle:

$$z_C = \frac{m_{co}\, z_{co} + m_{al}\, z_{al}}{m_{co} + m_{al}} > 0 \quad \rightsquigarrow \quad m_{co}\, z_{co} > -m_{al}\, z_{al}\,.$$

Using the centers of gravity $z_{co} = \dfrac{4}{3\pi} r$ and $z_{al} = -\dfrac{h}{3}$, one obtains

$$\rho_{co}\, t \frac{r^2 \pi}{2} \frac{4}{3\pi} r > \rho_{al}\, t\, r\, h \frac{h}{3} \quad \rightsquigarrow \quad \underline{\underline{h < r\sqrt{2\frac{\rho_{co}}{\rho_{al}}}}}\,.$$

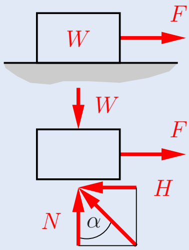
Chapter 8

Static and Kinetic Friction

8

Static Friction

Based on the surface roughness, a body remains in equilibrium until the static friction force H is smaller than the limit force H_0. H_0 is proportional to the normal force N.



$$\boxed{|H| < H_0\,, \qquad H_0 = \mu_0 N}$$

μ_0 = coefficient of static friction.

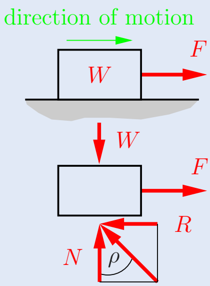
The static friction force is a *reaction force*. In a statically determinate system, it can be determined from the equilibrium conditions.

Static friction angle: For the direction of the resultant of N and H_0 (limiting friction), it follows that

$$\tan\varrho_0 = \mu_0 = \frac{H_0}{N}\,, \qquad \varrho_0 = \text{static friction angle}\,.$$

Kinetic Friction

As a result of surface roughness, a moving body is affected by the kinetic friction force R. The kinetic friction force is an *active force*. It is proportional to the normal force N (Coulomb's friction law):



$$\boxed{R = \mu N}$$

μ = coefficient of kinetic friction.

Kinetic friction angle: For the direction of the resultant of N and R, it follows that

$$\tan\varrho = \mu = \frac{R}{N}\,, \qquad \varrho = \text{kinetic friction angle}\,.$$

Problem Types:

1. Static friction: $H < \mu_0 N$
2. Limiting friction: $H = \mu_0 N$
3. Kinetic friction: $R = \mu N$

Remarks:

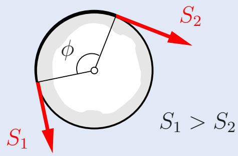
- The static friction force applies in the contact area of the bodies.
- The direction of the static friction force is opposite to the direction of relative motion (that would occur if it would not be hindered by static friction).
- The magnitude of the static friction force is independent of the contact area.
- In case of static friction, the resultant of N and H is located inside of the *static friction cone* with the opening angle ϱ_0 $(\alpha < \varrho_0)$.
- The coefficient of static friction is generally larger than the coefficient of kinetic friction.
- Coefficients of static and kinetic friction (approximate values) for dry materials:

material	μ_0	μ
steel on steel	0.15 - 0.5	0.1 - 0.4
steel on teflon	0.04	0.04
wood on wood	0.5	0.3
leather on metal	0.4	0.3
car tyres on streets	0.7 - 0.9	0.5 - 0.8

Static and Kinetic Belt Friction:

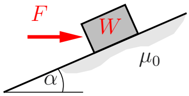
static belt friction: $S_1 \leq S_2 e^{\mu_0 \phi}$

kinetic belt friction: $S_1 = S_2 e^{\mu \phi}$

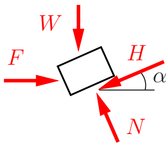


P8.1 **Problem 8.1** A block with weight W is resting on a rough inclined plane.

Specify the range of the external force F for which the block stays at rest.



Solution From the equilibrium conditions



$$\nearrow: \quad F\cos\alpha - W\sin\alpha - H = 0\,,$$

$$\nwarrow: \quad -F\sin\alpha - W\cos\alpha + N = 0\,,$$

the static friction force and the normal force can be determined:

$$H = F\cos\alpha - W\sin\alpha\,, \qquad N = F\sin\alpha + W\cos\alpha\,.$$

An *upward movement* is prevented if

$$H < \mu_0 N\,.$$

This condition is fulfilled, when

$$F < W\,\frac{\sin\alpha + \mu_0\cos\alpha}{\cos\alpha - \mu_0\sin\alpha}\,.$$

Using the addition theorems and inserting $\mu_0 = \tan\varrho_0$, one obtains

$$F < W\tan(\alpha + \varrho_0)\,.$$

To prevent a *downward movement*, the direction of H has to be reversed. Following this, the static friction condition can be formulated as

$$-H < \mu_0 N$$

yielding

$$F > W\,\frac{\sin\alpha - \mu_0\cos\alpha}{\cos\alpha + \mu_0\sin\alpha} \quad \rightsquigarrow \quad F > W\tan(\alpha - \varrho_0)\,.$$

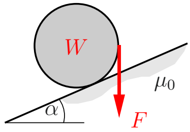
Thus, one obtains

$$\underline{\underline{\tan(\alpha - \varrho_0) < \frac{F}{W} < \tan(\alpha + \varrho_0)\,.}}$$

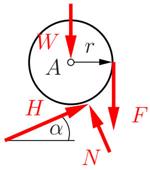
Note: The two static friction conditions can be summarized as follows: $|H| < \mu_0 N$.

Problem 8.2 A cylindrical roller with weight W is resting on an inclined plane (slope angle α). **P8.2**

Specify the external force F and the coefficient of static friction μ_0 such that the roller stays at rest.



Solution Using the equilibrium conditions



$$\nwarrow: \quad N - (W + F)\cos\alpha = 0\,,$$

$$\nearrow: \quad H - (W + F)\sin\alpha = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad Fr - Hr = 0$$

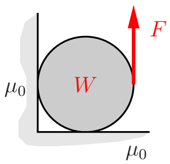
and the static friction condition

$$H < \mu_0 N\,,$$

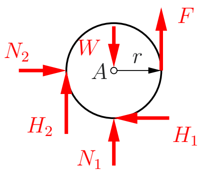
one obtains

$$\underline{\underline{F = W\,\frac{\sin\alpha}{1 - \sin\alpha}}}\,, \qquad \underline{\underline{\mu_0 > \tan\alpha}}\,.$$

Problem 8.3 How large does the force F have to be in order to ensure that the cylindrical roller with weight W is set in motion, assuming that the coefficient of static friction μ_0 is the same at both contact points? **P8.3**



Solution Using the equilibrium conditions



$$\rightarrow: \quad N_2 - H_1 = 0\,,$$

$$\uparrow: \quad N_1 + H_2 + F - W = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad H_1 r + H_2 r - Fr = 0$$

and the static friction conditions

$$H_1 = \mu_0 N_1\,, \qquad H_2 = \mu_0 N_2$$

one obtains

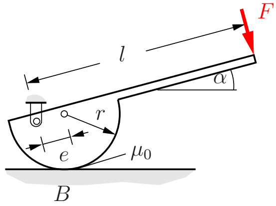
$$\underline{\underline{F = W\,\frac{\mu_0(1 + \mu_0)}{1 + \mu_0 + 2\mu_0^2}}}\,.$$

Remarks: - The system is statically indeterminate.
- The forces H_1 and H_2 are oriented in the opposite direction of the incipient motion.

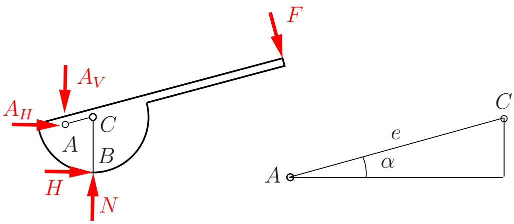
P8.4

Problem 8.4 An eccentric device with dimensions l and r is inclined by an angle α and is loaded with an applied force F.

Determine the required value of the eccentricity e such that the normal force N acts at the contact point B for a given coefficient of static friction μ_0.



Solution The free-body diagram of the system results in:



Using the equilibrium conditions

$$\rightarrow: \quad A_H + H + F\sin\alpha = 0\,,$$

$$\uparrow: \quad -A_V + N - F\cos\alpha = 0\,,$$

$$\overset{\curvearrowright}{C}: \quad F(l-e) - A_H e\sin\alpha - A_V e\cos\alpha - Hr = 0$$

and eliminating A_H and A_V, one obtains

$$H = \frac{F\,l - N\,e\,\cos\alpha}{r - e\,\sin\alpha}\,.$$

Furthermore, using the static friction condition

$$|H| < \mu_0 N\,,$$

it follows that

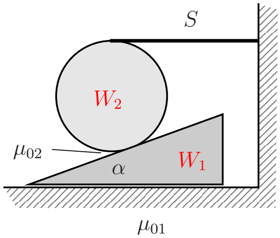
$$F\,l - N\,e\,\cos\alpha < \mu_0 N(r - e\,\sin\alpha)\,.$$

This yields the eccentricity e:

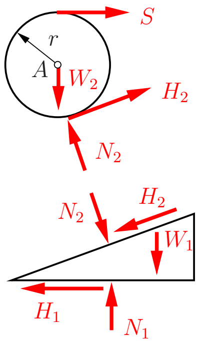
$$\underline{\underline{e > \frac{l\,\dfrac{F}{N} - \mu_0\,r}{\cos\alpha - \mu_0\sin\alpha}}}\,.$$

Problem 8.5 A wedge with weight W_1 and inclination angle α is resting on a horizontal plane. A cylindrical roller with weight W_2 lies on the wedge and is held by a rope S. **P8.5**

Determine the required values of the coefficients of static friction μ_{01} (between the wedge and the plane) and μ_{02} (between the roller and the wedge) in order to prevent slipping.



Solution From the equilibrium conditions for the roller



$$\rightarrow: \quad S + H_2 \cos\alpha - N_2 \sin\alpha = 0\,,$$

$$\uparrow: \quad -W_2 + H_2 \sin\alpha + N_2 \cos\alpha = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad S\,r - H_2\,r = 0$$

and the wedge

$$\uparrow: \quad -W_1 + N_1 - H_2 \sin\alpha - N_2 \cos\alpha = 0\,,$$

$$\rightarrow: \quad -H_1 - H_2 \cos\alpha + N_2 \sin\alpha = 0\,,$$

the forces at the contact points are determined as

$$N_2 = W_2\,, \qquad H_2 = W_2 \frac{\sin\alpha}{1+\cos\alpha}\,,$$

$$N_1 = W_1 + W_2\,, \qquad H_1 = W_2 \frac{\sin\alpha}{1+\cos\alpha}\,.$$

Using the static friction conditions

$$H_1 < \mu_{01} N_1\,, \qquad H_2 < \mu_{02} N_2\,,$$

one obtains the required values of the coefficients of static friction

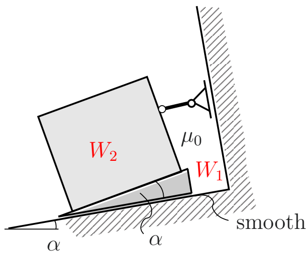
$$\underline{\underline{\mu_{01} > \frac{W_2 \sin\alpha}{(W_1+W_2)(1+\cos\alpha)}}}\,, \qquad \underline{\underline{\mu_{02} > \frac{\sin\alpha}{1+\cos\alpha}}}\,.$$

P8.6

Problem 8.6 A block with weight W_2 is resting on a *smooth* inclined plane and is held by a rope. A *rough* wedge is pushed between the block and the plane (coefficient of static friction μ_0).

a) Calculate the required rope force S and the normal force N_1 between the plane and the wedge.

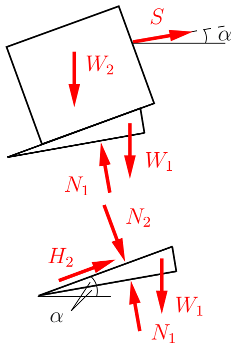
b) Determine the necessary value of the coefficient of static friction μ_0 such that the system stays at rest.



Solution **a)** The equilibrium conditions for the overall system yield

$$\nearrow: \quad \underline{\underline{S = (W_1 + W_2)\sin\alpha}}\,,$$

$$\nwarrow: \quad \underline{\underline{N_1 = (W_1 + W_2)\cos\alpha}}\,.$$



b) Using the equilibrium conditions for the wedge

$$\nearrow: \quad H_2 - W_1 \sin 2\alpha + N_1 \sin\alpha = 0\,,$$

$$\nwarrow: \quad -N_2 - W_1 \cos 2\alpha + N_1 \cos\alpha = 0$$

and inserting the normal force N_1, one obtains

$$H_2 = W_1 \sin 2\alpha - (W_1 + W_2)\sin\alpha\cos\alpha = \frac{1}{2}(W_1 - W_2)\sin 2\alpha\,,$$

$$N_2 = (W_1 + W_2)\cos^2\alpha - W_1\cos 2\alpha = \frac{1}{2}(W_1 + W_2) - \frac{1}{2}(W_1 - W_2)\cos 2\alpha.$$

Furthermore, using the static friction condition

$$|H_2| < \mu_0 N_2\,,$$

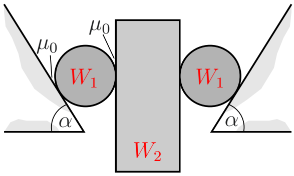
the necessary value of the coefficient of static friction follows as

$$\underline{\underline{\mu_0 > \frac{|W_1 - W_2|\sin 2\alpha}{W_1 + W_2 - (W_1 - W_2)\cos 2\alpha}}}\,.$$

Note: Depending on the values of W_1, W_2 and α, the wedge moves downwards or upwards in case of a violation of this condition.

Problem 8.7 A block (weight W_2) is clamped between two cylinders with the weight W_1 resting on inclined planes (slope angle α). All surfaces are rough (coefficient of static friction μ_0).

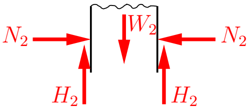
P8.7



Find the required value of W_2 in order to prevent slipping.

Solution Using the equilibrium conditions for the block

$$\uparrow: \quad 2H_2 - W_2 = 0$$

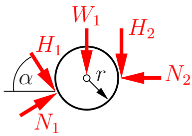


and for one cylinder

$$\uparrow: \quad N_1 \cos\alpha - H_2 - H_1 \sin\alpha - W_1 = 0\,,$$

$$\rightarrow: \quad N_1 \sin\alpha + H_1 \cos\alpha - N_2 = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad H_2 r - H_1 r = 0\,,$$



one obtains

$$H_1 = H_2 = \frac{W_2}{2}\,,$$

$$N_1 = \frac{W_2(1+\sin\alpha) + 2W_1}{2\cos\alpha}\,,$$

$$N_2 = \frac{W_2(1+\sin\alpha) + 2W_1 \sin\alpha}{2\cos\alpha}\,.$$

Furthermore, using the static friction conditions

$$H_1 < \mu_0 N_1\,, \qquad H_2 < \mu_0 N_2\,,$$

it follows that

$$W_2 < \frac{2\mu_0}{\cos\alpha - \mu_0(1+\sin\alpha)} W_1\,, \qquad W_2 < \frac{2\mu_0 \sin\alpha}{\cos\alpha - \mu_0(1+\sin\alpha)} W_1\,.$$

With $\sin\alpha \le 1$, one obtains

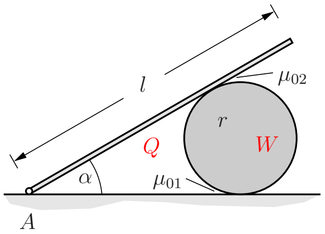
$$\underline{\underline{W_2 < \frac{2\mu_0 \sin\alpha}{\cos\alpha - \mu_0(1+\sin\alpha)} W_1\,.}}$$

Note: If $\mu_0 = \cos\alpha/(1+\sin\alpha)$, the right term is tending to infinity. Thus for $\mu_0 > \cos\alpha/(1+\sin\alpha)$, the system is *self-locking*.

P8.8

Problem 8.8 A rod (length l, weight Q) is supported in A and is leaned against a cylindrical roller (weight W, radius r) at an angle of α.

Determine the required values of the coefficients of static friction μ_{01} and μ_{02} in order to keep the system in equilibrium.



Solution Using the equilibrium conditions for the roller

$$\rightarrow: \quad -H_1 + N_2 \sin\alpha - H_2 \cos\alpha = 0\,,$$

$$\uparrow: \quad N_1 - W - N_2 \cos\alpha - H_2 \sin\alpha = 0\,,$$

$$\overset{\curvearrowright}{B}: \quad H_1 r - H_2 r = 0$$

and for the rod

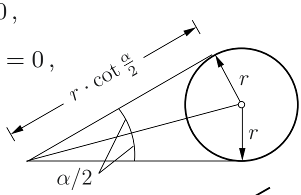
$$\overset{\curvearrowright}{A}: \quad Q\,\frac{l}{2}\cos\alpha - N_2\, r \cot\frac{\alpha}{2} = 0\,,$$

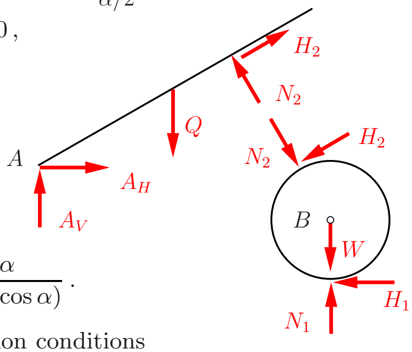
one obtains

$$N_1 = W + Q\,\frac{l}{2r}\,\frac{\cos\alpha}{\cot(\alpha/2)}\,,$$

$$N_2 = Q\,\frac{l}{2r}\,\frac{\cos\alpha}{\cot(\alpha/2)}\,,$$

$$H_1 = H_2 = Q\,\frac{l}{2r}\,\frac{\sin\alpha\cdot\cos\alpha}{\cot(\alpha/2)\,(1+\cos\alpha)}\,.$$





Furthermore, using the static friction conditions

$$H_1 < \mu_{01} N_1\,, \qquad H_2 < \mu_{02} N_2\,,$$

it follows with

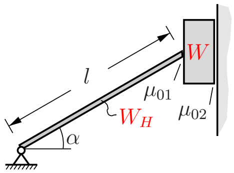
$$\cot\frac{\alpha}{2} = \frac{1+\cos\alpha}{\sin\alpha}$$

that

$$\underline{\underline{\mu_{01} > \frac{1}{\dfrac{W}{Q}\,\dfrac{2r}{l}\,\dfrac{\cot^2(\alpha/2)}{\cos\alpha} + \cot(\alpha/2)}}}\,, \qquad \underline{\underline{\mu_{02} > \tan(\alpha/2)}}\,.$$

Problem 8.9 A block is clamped between a rod with weight W_H and a wall. Block and wann have rough surfaces. The coefficients of static friction μ_{01} and μ_{02} at the two points of contact are given. **P8.9**

Determine the weight W of the block in order to prevent slipping.

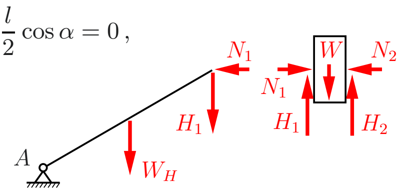


Solution Using the equilibrium conditions for the rod and for the block

$$\overset{\curvearrowright}{A}: \quad N_1 l \sin\alpha - H_1 l \cos\alpha - W_H \frac{l}{2}\cos\alpha = 0\,,$$

$$\uparrow: \quad H_1 + H_2 - W = 0\,,$$

$$\rightarrow: \quad N_1 - N_2 = 0\,,$$



the static friction conditions

$$H_1 < \mu_{01} N_1\,, \qquad H_2 < \mu_{02} N_2$$

and the assumption that $\mu_{01} < \tan\alpha$, one determines after eliminating H_1, H_2 and N_2 the two inequalities

$$N_1 < \frac{W_H}{2(\tan\alpha - \mu_{01})}\,, \qquad \frac{2W + W_H}{2(\tan\alpha + \mu_{02})} < N_1\,.$$

Thus, it follows that

$$\frac{2W + W_H}{2(\tan\alpha + \mu_{02})} < \frac{W_H}{2(\tan\alpha - \mu_{01})}$$

or

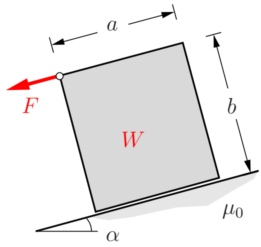
$$\underline{\underline{W < \frac{W_H}{2}\,\frac{\mu_{01} + \mu_{02}}{\tan\alpha - \mu_{01}}}}\,.$$

Remarks:

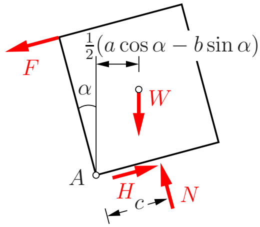
- If $\mu_{01} = \tan\alpha$, the denominator becomes zero. Following this, W can be increased as required. Generally, the system is *self-locking* for $\mu_{01} \geq \tan\alpha$ independently from W_H.
- The system is statically indeterminate. Thus, the forces H_i and N_i can not be determined.
- Assuming the limiting friction case with $H_1 = \mu_{01} N_1$ and $H_2 = \mu_{02} N_2$, the "<"-sign needs to be replaced by the "="-sign in the final result.

P8.10 **Problem 8.10** A homogeneous cuboid with weight W is resting on a rough inclined plane.

Determine the required values of the applied force F and the coefficient of static friction μ_0 such that the cuboid moves in form of slipping or tilting.



Solution From the equilibrium conditions



$$\nearrow: \quad H - F - W\sin\alpha = 0\,,$$

$$\nwarrow: \quad N - W\cos\alpha = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad \frac{W}{2}(a\cos\alpha - b\sin\alpha) - Fb - Nc = 0\,,$$

the forces in the contact surface and the position of N can be determined:

$$H = F + W\sin\alpha\,, \quad N = W\cos\alpha\,, \quad c = \frac{1}{2}(a - b\tan\alpha) - \frac{Fb}{W\cos\alpha}\,.$$

In order to cause *slipping*, the following must apply:

$$H = H_0 = \mu_0 N\,, \qquad c > 0\,.$$

This yields

$$\underline{\underline{F = W(\mu_0\cos\alpha - \sin\alpha)}}\,, \qquad \underline{\underline{\mu_0 < \frac{1}{2}(\frac{a}{b} + \tan\alpha)}}\,.$$

In order to cause *tilting* around the point A, the following must hold:

$$c = 0\,, \qquad H < \mu_0 N\,.$$

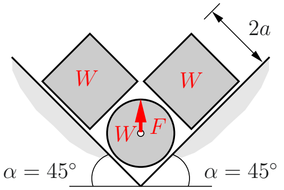
Thus, one obtains

$$\underline{\underline{F = W\,\frac{a\cos\alpha - b\sin\alpha}{2b}}}\,, \qquad \underline{\underline{\mu_0 > \frac{1}{2}(\frac{a}{b} + \tan\alpha)}}\,.$$

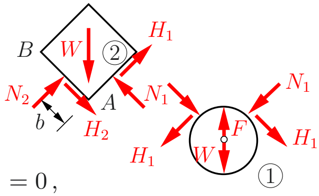
Hence tilting is only caused in case of a sufficiently rough plane.

Problem 8.11 Two cubes and a cylindrical roller each with weight W are resting between two inclined planes. The coefficient of static friction is μ_0 on all contact surfaces. **P8.11**

Determine the required force F needed to pull the roller upwards. What is the magnitude of μ_0 to prevent the cubes from tilting?



Solution Considering the symmetry and using $\sin\alpha = \cos\alpha = \sqrt{2}/2$, the equilibrium conditions can be determined as



$$① \uparrow: \quad F - W - 2\frac{\sqrt{2}}{2}N_1 - 2\frac{\sqrt{2}}{2}H_1 = 0\,,$$

$$② \rightarrow: \quad \frac{\sqrt{2}}{2}N_2 + \frac{\sqrt{2}}{2}H_2 + \frac{\sqrt{2}}{2}H_1 - \frac{\sqrt{2}}{2}N_1 = 0\,,$$

$$\uparrow: \quad \frac{\sqrt{2}}{2}N_2 - \frac{\sqrt{2}}{2}H_2 + \frac{\sqrt{2}}{2}H_1 + \frac{\sqrt{2}}{2}N_1 - W = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad N_2 b - N_1 a = 0\,.$$

In order to overcome the friction, the following must hold:

$$H_1 = \mu_0 N_1\,, \qquad H_2 = \mu_0 N_2\,.$$

Thus, one obtains from the first three equilibrium conditions

$$\underline{\underline{F = 2W\,\frac{1+\mu_0+\mu_0^2}{1+\mu_0^2}}}\,.$$

From the fourth equilibrium condition, it follows that

$$b = a\,\frac{1+\mu_0}{1-\mu_0}\,.$$

In order to prevent tilting around point B, the following must hold:

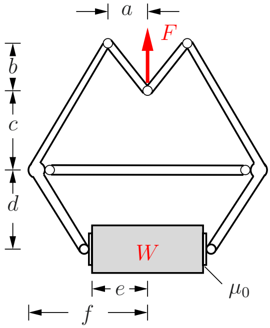
$$b < 2a\,.$$

This leads to

$$\frac{1+\mu_0}{1-\mu_0} < 2 \quad \rightsquigarrow \quad \underline{\underline{\mu_0 < \frac{1}{3}}}\,.$$

Note: Pulling the roller upwards, the contact forces between the roller and the inclined planes disappear. The orientation of the static friction forces must be plotted correctly (in the opposite direction of the beginning motion).

P8.12 **Problem 8.12** Determine the required value of the coefficient of static friction μ_0 such that the load W can be held by the stone pincers.



Solution Using the equilibrium conditions for the overall system

$$\uparrow: \quad F - W = 0\,,$$

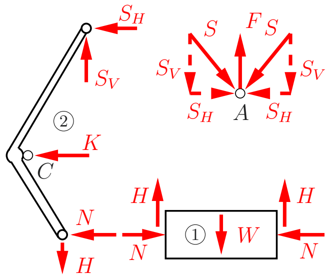
for the point A

$$\uparrow: \quad F - 2S_V = 0\,,$$

for the body ①

$$\uparrow: \quad 2H - W = 0$$

and for the body ②



$$\stackrel{\curvearrowright}{C}: \quad Nd + H(f-e) - S_V(f-a) - S_H(b+c) = 0\,,$$

one obtains with

$$\frac{S_H}{S_V} = \frac{a}{b}$$

for the forces H and N:

$$H = \frac{W}{2}\,, \qquad N = \frac{W}{2}\,\frac{ac+be}{bd}\,.$$

Furthermore, using the static friction condition

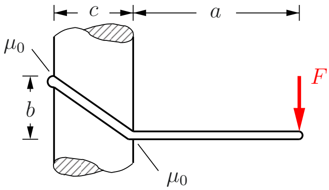
$$H < \mu_0 N\,,$$

it follows that

$$\underline{\underline{\mu_0 > \frac{bd}{ac+be}}}\,.$$

Problem 8.13 A climbing iron is clamped onto a pole and is subjected to the force F. **P8.13**

Determine the required value of μ_0 in order to prevent the climbing iron from slipping.

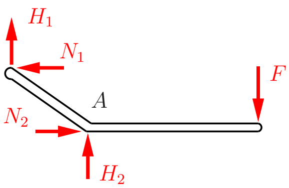


Solution Using the equilibrium conditions

$$\rightarrow: \quad N_2 - N_1 = 0\,,$$

$$\uparrow: \quad H_1 + H_2 - F = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad Fa + H_1 c - N_1 b = 0\,,$$



yields

$$N_2 = N_1\,, \qquad H_1 = N_1\frac{b}{c} - F\frac{a}{c}\,, \qquad H_2 = F(1+\frac{a}{c}) - N_1\frac{b}{c}\,.$$

From the static friction conditions

$$H_1 < \mu_0 N_1\,, \qquad H_2 < \mu_0 N_2\,,$$

one derives

$$N_1\frac{b - c\mu_0}{a} < F \quad \text{and} \quad F < N_1\frac{b + c\mu_0}{c+a}$$

or

$$\frac{b - c\mu_0}{a} < \frac{b + c\mu_0}{c+a}\,.$$

Solving the equation for μ_0, the required value of the coefficient of static friction follows

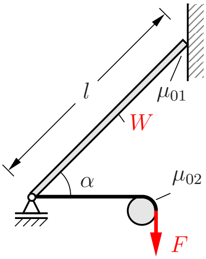
$$\underline{\underline{\mu_0 > \frac{b}{c+2a}}}\,.$$

Remarks:

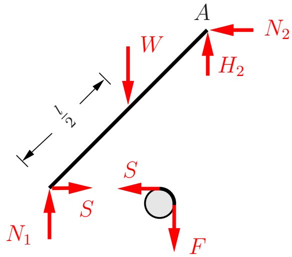
- As the system is statically indeterminate, the forces N_1, N_2, H_1 and H_2 cannot be determined.
- Alternatively, one can also solve the problem by discussing the limiting friction case. Then, the inequality is replaced by an equation and the determined value μ_0^* corresponds to the lower limit of the coefficient of static friction.

P8.14 **Problem 8.14** A rod, length l and weight W, is leaned against a rough wall at an angle of α. The lower end of the rod is supported by a rope which is wrapped around a rough pin.

Specify the range of the applied force F such that the system is in equilibrium.



Solution Using the equilibrium conditions



$$\rightarrow: \quad S - N_2 = 0\,,$$

$$\uparrow: \quad N_1 + H_2 - W = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad N_1 l\cos\alpha - S l\sin\alpha - W\,\frac{l}{2}\cos\alpha = 0\,,$$

yields

$$H_2 = \frac{W}{2} - S\tan\alpha\,, \qquad N_2 = S\,.$$

Furthermore, using the static friction condition

$$|H_2| < \mu_{01} N_2\,,$$

it follows that

$$\frac{W}{2} - S\,\tan\alpha < \mu_{01} S \qquad \text{or} \qquad -\frac{W}{2} + S\,\tan\alpha < \mu_{01} S$$

depending on the orientation of H_2. Therefore, one obtains

$$\frac{W}{2\,(\tan\alpha + \mu_{01})} \;<\; S \;<\; \frac{W}{2\,(\tan\alpha - \mu_{01})}\,.$$

Belt friction at the pin may occur if

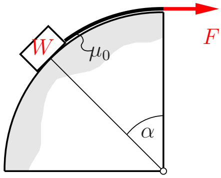
$$S\,\mathrm{e}^{-\mu_{02}\pi/2} \;<\; F \;<\; S\,\mathrm{e}^{+\mu_{02}\pi/2}\,.$$

Inserting the lower (upper) bound of S in the left (right) term, it follows that

$$\underline{\underline{\frac{\mathrm{e}^{-\mu_{02}\pi/2}}{2\,(\tan\alpha + \mu_{01})} \;<\; \frac{F}{W} \;<\; \frac{\mathrm{e}^{+\mu_{02}\pi/2}}{2\,(\tan\alpha - \mu_{01})}}}\,.$$

Problem 8.15 A block with weight W is supported by a rope. The coefficient of static friction μ_0 between the surface and the block or the rope is given. P8.15

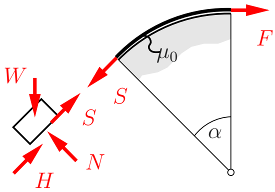
Specify the limits of the force F such that the block stays at rest.



Solution The equilibrium conditions

$$\nwarrow: \quad N - W\cos\alpha = 0\,,$$

$$\nearrow: \quad H + S - W\sin\alpha = 0\,,$$



lead to

$$N = W\cos\alpha\,, \qquad H = W\sin\alpha - S\,.$$

Furthermore, using the static friction condition

$$|H| < \mu_0 N\,,$$

yields

$$W(\sin\alpha - \mu_0\cos\alpha) \; < \; S \; < \; W(\sin\alpha + \mu_0\cos\alpha)\,.$$

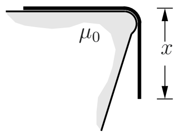
With the static friction condition for the rope

$$S\mathrm{e}^{-\mu_0\alpha} \; < \; F \; < \; S\mathrm{e}^{\mu_0\alpha}\,,$$

one obtains

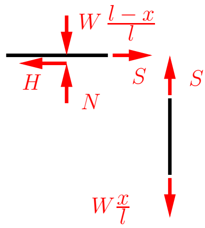
$$\underline{\underline{\mathrm{e}^{-\mu_0\alpha}(\sin\alpha - \mu_0\cos\alpha) \; < \; \frac{F}{W} \; < \; \mathrm{e}^{\mu_0\alpha}(\sin\alpha + \mu_0\cos\alpha)\,.}}$$

Problem 8.16 Determine the maximum overhang x of the heavy rope with total length l in order to prevent slipping. P8.16



Solution The equilibrium conditions lead to

$$N = W\,\frac{l-x}{l}\,, \qquad H = S = W\frac{x}{l}\,.$$

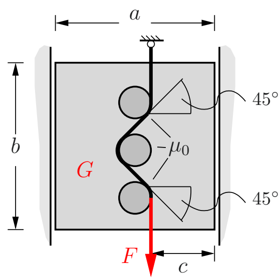


Inserting these relations in the static friction condition $H < \mu_0 N$, it follows that

$$\underline{\underline{\frac{x}{l} < \frac{\mu_0}{1+\mu_0}}}\,.$$

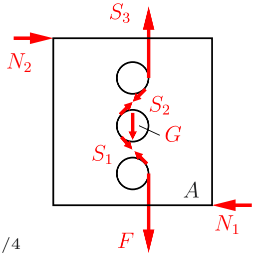
P8.17

Problem 8.17 A block with weight W can move vertically between two smooth walls. It is held by a rope which passes around three fixed rough pins.



Determine the minimum for the force F, such that the block does not slide. Furthermore, find the forces which are exerted from the walls onto the block.

Solution Using the equilibrium conditions



$$\uparrow: \quad S_3 - W - F = 0\,,$$

$$\rightarrow: \quad N_1 - N_2 = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad W\tfrac{1}{2}a + Fc - S_3 c - N_2 b = 0$$

and the static friction conditions

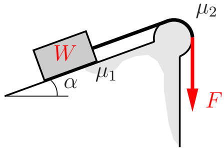
$$S_1 < F\mathrm{e}^{\mu_0\pi/4}, \quad S_2 < S_1\mathrm{e}^{\mu_0\pi/2}, \quad S_3 < S_2\mathrm{e}^{\mu_0\pi/4}\,,$$

one obtains

$$\underline{\underline{F > \frac{W}{\mathrm{e}^{\mu_0\pi}-1}}}\,, \qquad \underline{\underline{N_1 = N_2 = W\frac{a-2c}{2b}}}\,.$$

P8.18

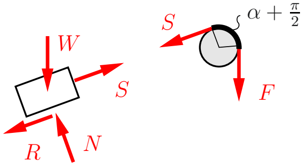
Problem 8.18 Determine the required value of the force F such that the block with weight W can be raised with uniform velocity assuming that the plane and the curved surface are rough.



Solution Using the equilibrium conditions

$$\nwarrow: \quad N - W\cos\alpha = 0\,,$$

$$\nearrow: \quad S - R - W\sin\alpha = 0$$



and the static friction conditions

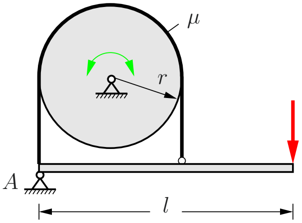
$$R = \mu_1 N\,, \qquad F = S\mathrm{e}^{\mu_2(\alpha+\pi/2)}\,,$$

one obtains

$$\underline{\underline{F = W\mathrm{e}^{\mu_2(\alpha+\pi/2)}(\sin\alpha + \mu_1\cos\alpha)}}\,.$$

Problem 8.19 A braking torque M_B is applied to a rotating shaft by a band break. Determine the magnitude of the applied force F for a given coefficient of kinetic friction μ, when the shaft is rotating
a) clockwise or
b) counterclockwise.

P8.19

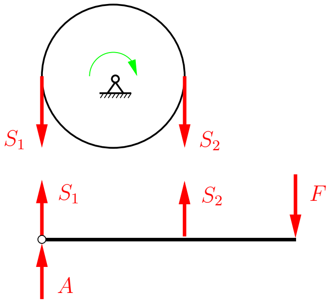


Solution Using the equilibrium condition for the lever

$$\overset{\curvearrowright}{A}: \quad -S_2\, 2r + F\, l = 0\,,$$

yields

$$S_2 = F\,\frac{l}{2r}\,.$$



For a clockwise rotation, the kinetic friction condition results in

$$S_1 = S_2\,\mathrm{e}^{\mu\pi}\,,$$

and the breaking moment can be calculated as

$$M_B = S_1\, r - S_2\, r = S_2\, r\,(\mathrm{e}^{\mu\pi} - 1)\ .$$

Further using the result of S_2, one obtains

$$\underline{\underline{F_R = \frac{2M_B}{l\,(\mathrm{e}^{\mu\pi} - 1)}}}\,.$$

For a counterclockwise rotation, the kinetic friction condition results in

$$S_2 = S_1\,\mathrm{e}^{\mu\pi}$$

and the breaking moment can be calculated as

$$M_B = S_2\, r - S_1\, r = S_2\, r\,(1 - \mathrm{e}^{-\mu\pi})\,.$$

Inserting the result of S_2, one obtains

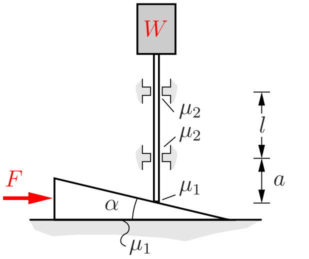
$$\underline{\underline{F_L = \frac{2M_B\,\mathrm{e}^{\mu\pi}}{l\,(\mathrm{e}^{\mu\pi} - 1)}}}\,.$$

Note: Due to $\mathrm{e}^{\mu\pi} > 1$, $F_L > F_R$ holds for the same value of M_B.

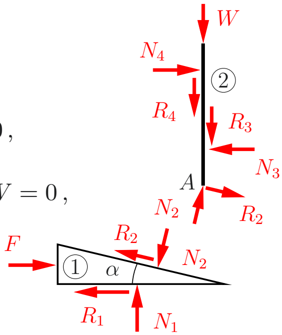
P8.20

Problem 8.20 A block with weight W should be raised with uniform velocity by pushing a weightless wedge forward.

Determine the required value of the force F assuming a coefficient of kinetic friction μ_1 in the contact areas of the wedge and a coefficient of kinetic friction μ_2 at the contact points of the rod.



Solution Using the equilibrium conditions for the wedge and the rod



$$① \rightarrow: \quad F - R_1 - R_2\cos\alpha - N_2\sin\alpha = 0\,,$$

$$\uparrow: \quad N_1 - N_2\cos\alpha + R_2\sin\alpha = 0\,,$$

$$② \rightarrow: \quad N_2\sin\alpha + R_2\cos\alpha - N_3 + N_4 = 0\,,$$

$$\uparrow: \quad N_2\cos\alpha - R_2\sin\alpha - R_3 - R_4 - W = 0\,,$$

$$\overset{\curvearrowright}{A}: \quad -N_3 a + N_4(l+a) = 0$$

and the kinetic friction conditions

$$R_1 = \mu_1 N_1\,, \qquad R_2 = \mu_1 N_2\,, \qquad R_3 = \mu_2 N_3\,, \qquad R_4 = \mu_2 N_4\,,$$

one obtains the required force by solving the system of equations for F, such that

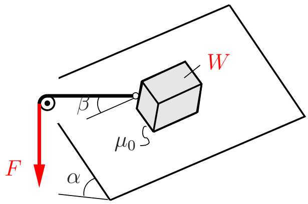
$$\underline{\underline{F = W\,\frac{\mu_1(\cos\alpha - \mu_1\sin\alpha) + (\sin\alpha + \mu_1\cos\alpha)}{(\cos\alpha - \mu_1\sin\alpha) - \mu_2\,\dfrac{l+2a}{l}\,(\sin\alpha + \mu_1\cos\alpha)}}}\,.$$

Remarks:

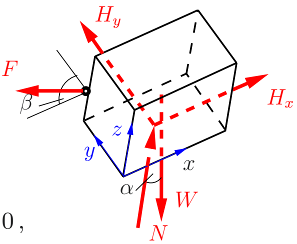
- The kinetic friction forces are oriented in the opposite direction of the motion.
- If the denominator tends towards zero ($F \rightarrow \infty$), the system is self-locking.

Problem 8.21 A body with weight W is resting on a rough inclined plane. A force F is applied to the body through a rope at an angle β (parallel to the inclined plane). **P8.21**

Specify the required value of the coefficient of static friction μ_0 such that the system stays at rest.



Solution At first, a suitable coordinate system is introduced. The free body diagram shows the acting forces, which have to fulfil the equilibrium conditions



$$\sum F_x = 0 \;:\quad H_x - F\cos\beta = 0\,,$$

$$\sum F_y = 0 \;:\quad H_y + F\sin\beta - W\sin\alpha = 0\,,$$

$$\sum F_z = 0 \;:\quad N - W\cos\alpha = 0\,.$$

Therein, H_x and H_y characterize the static friction force H. One finds for H and N

$$|H| = \sqrt{H_x^2 + H_y^2} = \sqrt{F^2 - 2FW\sin\alpha\sin\beta + W^2\sin^2\alpha}\,,$$

$$N = W\cos\alpha\,.$$

Inserting into the static friction condition

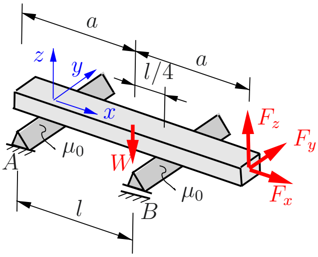
$$|H| < \mu_0 N \qquad \text{resp.} \qquad \mu_0 > \frac{|H|}{N}$$

yields the required value of μ_0:

$$\underline{\underline{\mu_0 > \frac{\sqrt{F^2 - 2FW\sin\alpha\sin\beta + W^2\sin^2\alpha}}{W\cos\alpha}}}\,.$$

P8.22

Problem 8.22 A rigid body (weight W) rests eccentrically on two rails and is subjected to forces at one end. The body can slide in B in x-direction.



Determine the maximum force for that the body stays at rest.

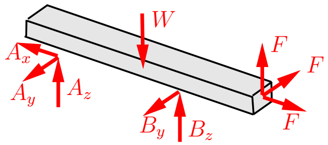
Given: $F_x = F_y = F_z = F$, $a = l$, $\mu_0 = 2/3$.

Solution The reaction forces at the supports can be determined by using the equilibrium conditions

$$A_x = F\,,$$

$$A_y = -\frac{3}{4}F\,, \qquad B_y = \frac{7}{4}F\,,$$

$$A_z = \frac{W}{4} + \frac{3}{4}F\,, \quad B_z = \frac{3}{4}W - \frac{7}{4}F\,.$$



The normal forces and the static friction forces at A and B read

$$N_A = A_z = \frac{W}{4} + \frac{3}{4}F\,, \qquad H_A = \sqrt{A_x^2 + A_y^2} = \frac{5}{4}F\,,$$

$$N_B = B_z = \frac{3}{4}W - \frac{7}{4}F\,, \qquad H_B = |B_y| = \frac{7}{4}F\,.$$

When a motion is considered at A, the limiting friction condition yields

$$H_A = \mu_0 N_A \quad \rightsquigarrow \quad F_1 = W\,\frac{\mu_0}{5 - 3\mu_0} = \frac{2}{9}\,W\,.$$

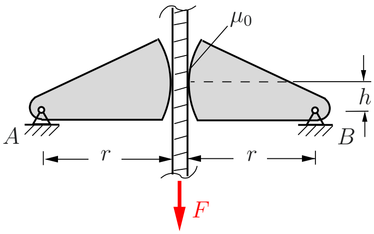
In case of a motion at B, one obtains

$$H_B = \mu_0 N_B \quad \rightsquigarrow \quad \underline{\underline{F_2}} = W\,\frac{3\mu_0}{7(1+\mu_0)} = \underline{\underline{\frac{6}{35}\,W}}\,.$$

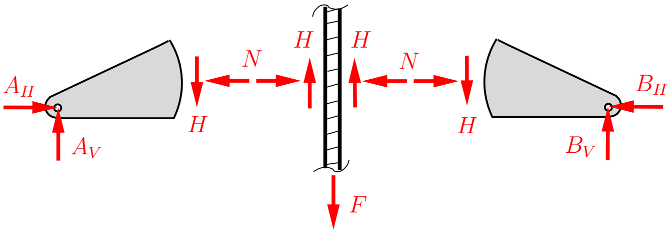
Since $F_1 > F_2$, the motion initiated at B in case F exceeds the critical force F_2.

Problem 8.23 A rope is clamped by two clamping jaws, which are fixed by two hinged supports in A and B. A force of magnitude F is applied to the rope. P8.23

Determine
a) the reaction forces in A and B, and
b) the coefficient of static friction μ_0, such that there is no slipping.



Solution a) Draw the free-body diagram:



The reaction forces can be determined by evaluating the equilibrium conditions for the subsystems.

$$\text{Rope} \uparrow: \quad 2H - F = 0 \quad \rightsquigarrow \quad \underline{\underline{H = \frac{F}{2}}}$$

The equilibrium conditions at the left clamping jaw yield

$$\overset{\curvearrowleft}{A}: \quad Nh - Hr = 0 \quad \rightsquigarrow \quad N = \frac{r}{h}H = \frac{r}{2h}F\,,$$

$$\uparrow: \quad A_V - H = 0 \quad \rightsquigarrow \quad \underline{\underline{A_V = \frac{1}{2}F}}\,,$$

$$\rightarrow: \quad A_H - N = 0 \quad \rightsquigarrow \quad \underline{\underline{A_H = \frac{r}{2h}F}}\,.$$

Since the overall system is symmetric, one finds $B_H = A_H$ and $B_V = A_V$.

b) As long as the condition $|H| < \mu_0 |N|$ is fulfilled, slipping does not occur:

$$\frac{F}{2} < \mu_0\, F\, \frac{r}{2h} \quad \rightsquigarrow \quad \underline{\underline{\mu_0 > \frac{h}{r}}}\,.$$

This result is valid for any F. Hence, the system is self-locking.

Chapter 9

9

Moments of Inertia

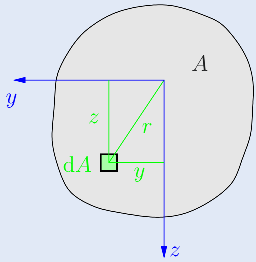
Moments of Inertia are used in beam theories (cf. Volume 2). Moments of second order of an area A, as for example of a cross sectional area of a beam, are defined as follows:

$$I_y = \int_A z^2 \mathrm{d}A$$

$$I_z = \int_A y^2 \mathrm{d}A$$

$$I_{yz} = I_{zy} = -\int_A yz \mathrm{d}A$$

$$I_p = I_y + I_z = \int_A r^2 \mathrm{d}A$$



I_y, I_z	:	rectangular moments of inertia with respect to the y- or z-axis, respe
I_{yz}	:	products of inertia (centrifugal moments),
I_p	:	polar moment of inertia.

Note: Moments of inertia depend on the position of the coordinate origin and the orientation of its axes

Radii of gyration: = "distance" r_g of the area A, from which by multiplication with A one recovers the moment of inertia:

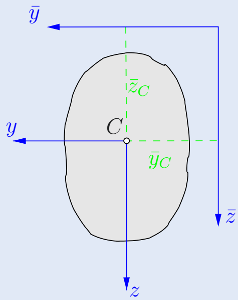
$$I_y = r_{gy}^2 \, A \,, \quad I_z = r_{gz}^2 \, A \,, \quad I_p = r_{gp}^2 \, A \,.$$

Parallel-Axis Theorem (Steiner's Theorem)

$$I_{\bar{y}} = I_y + \bar{z}_C^2 A \,,$$

$$I_{\bar{z}} = I_z + \bar{y}_C^2 A \,,$$

$$I_{\bar{y}\bar{z}} = I_{yz} - \bar{y}_C \bar{z}_C A \,.$$

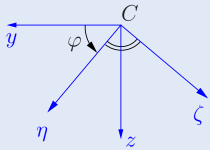


C = area centroid,

y, z = centroidal axes.

Rotation of the Coordinate System (transformation equations)

$$I_\eta = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2}\cos 2\varphi + I_{yz}\sin 2\varphi\,,$$
$$I_\zeta = \frac{I_y + I_z}{2} - \frac{I_y - I_z}{2}\cos 2\varphi - I_{yz}\sin 2\varphi\,,$$
$$I_{\eta\zeta} = -\frac{I_y - I_z}{2}\sin 2\varphi + I_{yz}\cos 2\varphi\,.$$



Principal Moments of Inertia: For each area, there exists a system of axes perpendicular to each other (*principal axes*), for which I_η and I_ζ assume extreme values (*principal moments of inertia*) and the products of inertia $I_{\eta\zeta}$ vanish.

Principal Moments of Inertia:

$$I_{1,2} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}\,.$$

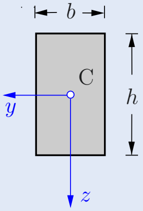
Directions of Principal Axes:

$$\tan 2\varphi^* = \frac{2I_{yz}}{I_y - I_z}\,.$$

Remarks:

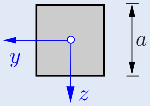
- In case of symmetric areas, the axis of symmetry and the axis perpendicular to the axis of symmetry are principal axes.
- Moments of inertia are coefficients of a tensor (*moment of inertia tensor*).
- Plotting the pairs $(I_\eta\,, I_{\eta\,\zeta})$ or $(I_\zeta\,, I_{\eta\,\zeta})$, respectively, for all possible angles in a coordinate system (abscissa = rectangular moment of inertia, ordinate = product of inertia), one obtains the *circle of inertia*. The construction of the circle of inertia is performed analogously to MOHR's *stress circle* (cf. Volume 2).
- The quantities $I_\eta + I_\zeta = I_p$ and $I_\eta\, I_\zeta - I_{\eta\zeta}^2$ are *invariants*, meaning that they are independent of the angle φ.

Rectangle



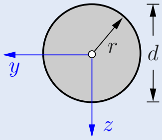
$$I_y = \frac{b\,h^3}{12}\,, \qquad r_{gy} = \frac{\sqrt{3}}{6}\,h\,,$$
$$I_z = \frac{h\,b^3}{12}\,, \qquad r_{gz} = \frac{\sqrt{3}}{6}b\,,$$
$$I_{yz} = 0\,,$$
$$I_p = I_y + I_z = \frac{b\,h}{12}(h^2 + b^2)\,.$$

Square



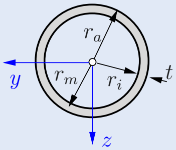
$$I_y = I_z = \frac{a^4}{12}\,, \qquad r_{gy} = r_{gz} = \frac{\sqrt{3}}{6}\,a\,,$$
$$I_p = \frac{a^4}{6}\,.$$

Circle



$$I_y = I_z = \frac{\pi\,r^4}{4} = \frac{\pi\,d^4}{64}\,, \qquad r_{gy} = r_{gz} = \frac{r}{2}\,,$$
$$I_p = \frac{\pi\,r^4}{2} = \frac{\pi\,d^4}{32}\,, \qquad r_{gp} = \frac{\sqrt{2}}{2}r\,.$$

(Thin-walled)
Circular Ring

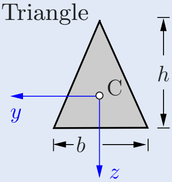


$$I_y = I_z = \frac{\pi}{4}(r_a^4 - r_i^4)\,, \quad r_{gy} = r_{gz} = \frac{1}{2}\sqrt{r_a^2 + r_i^2}\,,$$
$$I_p = 2\,I_y\,, \quad r_{gp} = \frac{\sqrt{2}}{2}\sqrt{r_a^2 + r_i^2},$$

with $t = r_a - r_i$ and $r_m = (r_a + r_i)/2$ follows for the thin-walled profile ($t \ll r_m$)

$$I_y = I_z \approx \pi r_m^3 t\,, r_{gy} = r_{gz} \approx \frac{\sqrt{2}}{2}\,r_m\,.$$

Isosceles
Triangle

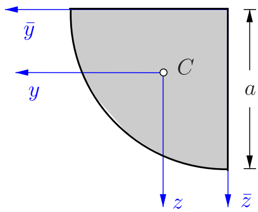


$$I_y = \frac{b\,h^3}{36}\,, \qquad r_{gy} = \frac{h}{3\sqrt{2}}\,,$$
$$I_z = \frac{h\,b^3}{48}\,, \qquad r_{gz} = \frac{b}{2\sqrt{6}}\,.$$

P9.1

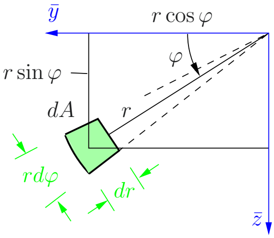
Problem 9.1 Determine for a quadrant with radius a:

a) $I_{\bar y}$, $I_{\bar z}$, $I_{\bar y \bar z}$
b) I_y, I_z, I_{yz} ($\{y,z\}$: centroidal axes)
c) the direction of the principal axes,
d) the principal moments of inertia.



Solution a) It is convenient to use polar coordinates. With the differential area element in polar coordinates

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\varphi\,,$$



the solution reads

$$\underline{\underline{I_{\bar z}}} = \int_A \bar y^2 \mathrm{d}A = \int_0^{\pi/2}\int_0^a (r^2\cos^2\varphi) r\mathrm{d}r\mathrm{d}\varphi$$

$$= \frac{r^4}{4}\Big|_0^a \left(\frac{\varphi}{2} + \frac{1}{4}\sin 2\varphi\right)\Big|_0^{\pi/2} = \underline{\underline{\frac{\pi\, a^4}{16}}}\,,$$

$$\underline{\underline{I_{\bar y}}} = \underline{\underline{I_{\bar z}}} \qquad \text{(symmetry !)}\,,$$

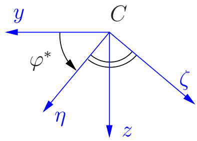
$$\underline{\underline{I_{\bar y \bar z}}} = -\int_0^{\pi/2}\int_0^a (r\cos\varphi)(r\sin\varphi)\, r\,\mathrm{d}r\,\mathrm{d}\varphi = -\frac{a^4}{4}\,\frac{1}{2} = \underline{\underline{-\frac{a^4}{8}}}\,.$$

b) Applying the parallel-axis theorem with $\bar y_C = \bar z_C = 4a/3\pi$ (cf. Center of Gravity, pp. 32), one finds

$$\underline{\underline{I_y = I_z}} = I_{\bar y} - \bar z_C^2 A = \frac{\pi\, a^4}{16} - \left(\frac{4\,a}{3\pi}\right)^2 \frac{\pi\, a^2}{4} = \underline{\underline{\left(\frac{\pi}{16} - \frac{4}{9\pi}\right) a^4}}\,,$$

$$\underline{\underline{I_{yz}}} = I_{\bar y \bar z} + \bar y_C \bar z_C A = \underline{\underline{\left(-\frac{1}{8} + \frac{4}{9\,\pi}\right) a^4}}\,.$$

c) Symmetry leads to



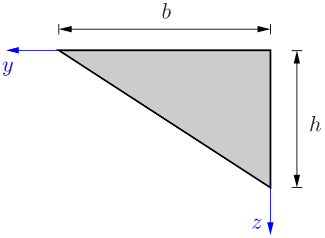
$$\underline{\underline{\varphi_1^* = \pi/4}} \quad \rightsquigarrow \quad \underline{\underline{\varphi_2^*}} = \varphi_1^* + \pi/2 = \underline{\underline{3\,\pi/4}}$$

d) With $I_y = I_z$ one can find the principal moments of inertia

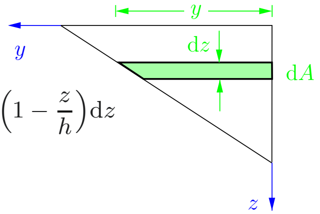
$$\underline{\underline{I_1}} = I_y + I_{yz} = \underline{\underline{\left(\frac{\pi}{16} - \frac{1}{8}\right) a^4}}\,,$$

$$\underline{\underline{I_2}} = I_y - I_{yz} = \underline{\underline{\left(\frac{\pi}{16} - \frac{8}{9\,\pi} + \frac{1}{8}\right) a^4}}\,.$$

P9.2 **Problem 9.2** Determine for a right-angled triangle the moments of inertia I_y, I_z, I_{yz}.



Solution The crucial step is the choice of a convenient area element. Three different possibilities are examined for I_y.
1st approach: Area element $\mathrm{d}A$ (width y, height $\mathrm{d}z$) with distance z to the y-axis.

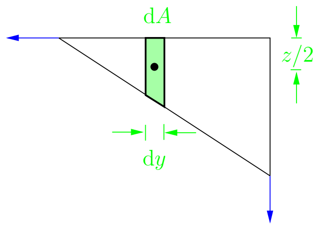


$$\mathrm{d}A = y\,\mathrm{d}z\,, \qquad y = b\left(1 - \frac{z}{h}\right),$$

$$\underline{\underline{I_y}} = \int z^2 \mathrm{d}A = \int z^2 (y\,\mathrm{d}z) = \int_0^h z^2 b\left(1 - \frac{z}{h}\right)\mathrm{d}z$$

$$= b\left(\frac{z^3}{3} - \frac{z^4}{4h}\right)\Bigg|_0^h = \underline{\underline{\frac{b\,h^3}{12}}}\,.$$

2nd approach: "Summation" (= Integration) of the moment of inertia of infinitesimal rectangles (height z, width $\mathrm{d}y$).



$$\mathrm{d}A = z\,\mathrm{d}y\,, \quad \mathrm{d}y = -\,\frac{b}{h}\,\mathrm{d}z\,.$$

Since the centroidal axis of the area element $\mathrm{d}A$ does not coincide with the y-axis, the parallel-axis theorem needs to be applied. With

$$\mathrm{d}I_y = \frac{\mathrm{d}y\;z^3}{12} + \left(\frac{z}{2}\right)^2 z\;\mathrm{d}y = \frac{1}{3}\,z^3\mathrm{d}y\,,$$

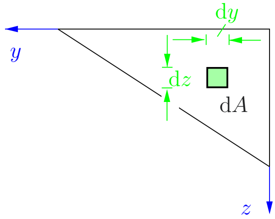
one obtains by integration

$$\underline{\underline{I_y}} = \int_0^b \mathrm{d}I_y = -\frac{b}{3\,h}\int_h^0 z^3\mathrm{d}z = -\frac{b}{3\,h}\;\frac{z^4}{4}\Bigg|_h^0 = \underline{\underline{\frac{b\,h^3}{12}}}$$

(as y is integrated from 0 to b, z needs to be integrated from h to 0!).

3rd approach: Consider the area element $\mathrm{d}A$ (width $\mathrm{d}y$, height $\mathrm{d}z$) with the distance z from the y-axis:

$$\mathrm{d}A = \mathrm{d}y\ \mathrm{d}z\,.$$



Integration yields

$$\underline{\underline{I_y}} = \int\int z^2\,\mathrm{d}y\ \mathrm{d}z$$

$$= \int\limits_0^b \left\{ \int\limits_0^{z(y)} z^2 \mathrm{d}z \right\} \mathrm{d}y$$

$$= \int\limits_0^b \left\{ \frac{z^3}{3}\bigg|_0^{h-\frac{h}{b}y} \right\} \mathrm{d}y = \frac{1}{3}\int\limits_0^b \left\{ h^3 - 3\frac{h^3}{b}y + 3\frac{h^3}{b^2}y^2 - \frac{h^3}{b^3}y^3 \right\} \mathrm{d}y$$

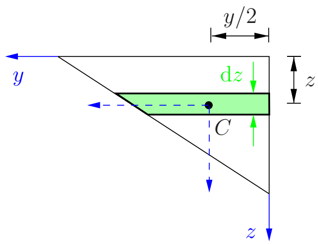
$$= \frac{1}{3}\left[h^3 b - \frac{3}{2}h^3 b + h^3 b - \frac{1}{4}h^3 b \right] = \underline{\underline{\frac{1}{12}bh^3}}\,.$$

Clearly, the *1st approach* is the simplest alternative, since the area element has a constant distance to the reference axis.

The moment of inertia I_z can be calculated according to the procedure for I_y by interchanging the sides h and b of the triangle:

$$\underline{\underline{I_z = \frac{hb^3}{12}}}\,.$$

The product of inertia is computed with the area element from the *1st approach*. Since the product of inertia vanishes with respect to the centroidal axes, the remaining part results from the parallel-axis theorem:



$$\underline{\underline{I_{yz}}} = -\int\int \frac{y}{2}\, z(y\ \mathrm{d}z)$$

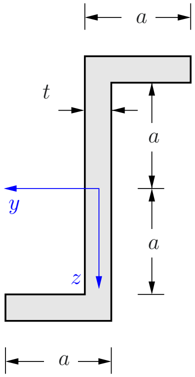
$$= -\int\limits_0^h \frac{1}{2}zb^2\left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right)\mathrm{d}z$$

$$= -\frac{1}{2}b^2\left(\frac{h^2}{2} - \frac{2h^2}{3} + \frac{h^2}{4}\right) = \underline{\underline{-\frac{b^2h^2}{24}}}\,.$$

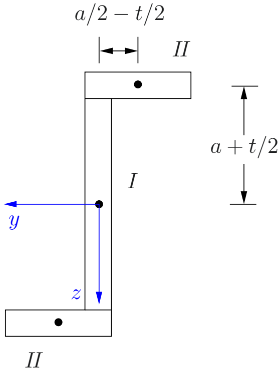
P9.3

Problem 9.3 Determine the direction of the principal axes and the principal moments of inertia for the displayed profile with constant thickness t.

Given: $a = 10$ cm, $t = 1$ cm.



Solution First, the moments of inertia are determined with respect to the y- and z-axis. Therefore, the profile is divided into three rectangles. According to the parallel-axis theorem, the moment of inertia for each rectangle is composed of the moment of inertia with respect to the respective centroidal axes and by considering the perpendicular distance of the rectangular area to the reference axes:



$$I_y = \frac{t(2a)^3}{12} + 2\left\{\frac{at^3}{12} + \left(a + \frac{t}{2}\right)^2 at\right\}$$

$$= 2873 \text{ cm}^4 ,$$

$$I_z = \frac{(2a)t^3}{12} + 2\left\{\frac{ta^3}{12} + \left(\frac{a}{2} - \frac{t}{2}\right)^2 at\right\}$$

$$= 573 \text{ cm}^4 .$$

The product of inertia for the partial areas vanishes with respect to their centroidal axes. Therfore, I_{yz} can be calculated by considering the parallel-axis theorem for partial areas II.

$$I_{yz} = -2\left[\left(a + \frac{t}{2}\right)\left(\frac{a}{2} - \frac{t}{2}\right) at\right] = -945 \text{ cm}^4 .$$

The direction of the principal axes can be determined with

$$\tan 2\varphi^* = \frac{2I_{yz}}{I_y - I_z} = \frac{2 \cdot -945}{2873 - 573} = -0.822$$

and results in

$$2\varphi^* = -39.4^\circ \quad \rightsquigarrow \quad \underline{\underline{\varphi_1^*}} = \underline{\underline{-19.7^\circ}}\,,$$

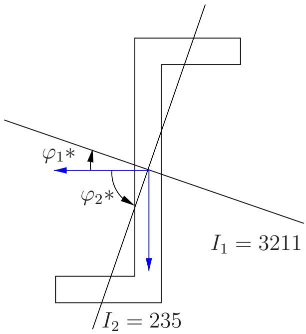
$$\underline{\underline{\varphi_2^*}} = \varphi_1^* + 90^\circ = \underline{\underline{70.3^\circ}}\,.$$

The principal moments of inertia follows as

$$I_{1,2} = \frac{2873 + 573}{2} \pm \sqrt{\left(\frac{2873 - 573}{2}\right)^2 + 945^2} = 1723 \pm 1488$$

$$\rightsquigarrow \quad \underline{\underline{I_1 = 3211\ \text{cm}^4}}\,, \qquad \underline{\underline{I_2 = 235\ \text{cm}^4}}\,.$$

The allocation of the principal axes to their respective principal moments of inertia can be derived by the transformation equations. In the present case, it is clear that the maximum principal moment of inertia I_1 belongs to the direction represented by φ_1^* , since the distance of the areas in that case is larger compared to the direction φ_2^*.



Remarks:

- The *invariants* can be verified for the numerical example:
  a) $I_y + I_z = I_1 + I_2 = 3446\ \text{cm}^4\,,$
  b) $I_y I_z - I_{yz}^2 = I_1 I_2 = 7.5 \cdot 10^5\ \text{cm}^8\,.$

- Terms of higher order can be neglected for a thin-walled profile ($t \ll a$), such that

$$I_y \simeq \frac{8}{3}ta^3 = 2667\ \text{cm}^4, \qquad I_z \simeq \frac{2}{3}ta^3 = 667\ \text{cm}^4,$$

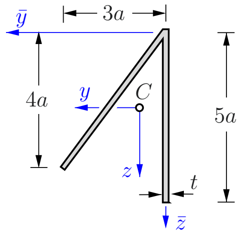
$$I_{yz} \simeq -ta^3 = -1000\ \text{cm}^4, \qquad \varphi^* \simeq -22.5^\circ\,,$$

$$I_1 \simeq 3080\ \text{cm}^4, \qquad I_2 \simeq 252\ \text{cm}^4.$$

However, these approximations lead to inaccurate results for the present numerical example, since the condition $t \ll a$ is not sufficiently fulfilled.

P9.4

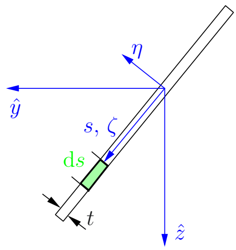
Problem 9.4 Determine for the thin-walled cross section ($t \ll a$) the direction of the principal axes and the principal moments of inertia with respect to the centroidal axes.



Solution Initially, the center of gravity is determined:

$$\bar{y}_C = \frac{\frac{3}{2}a\, 5at}{2 \cdot 5at} = \frac{3}{4}a\,, \quad \bar{z}_C = \frac{2a\, 5at + \frac{5}{2}a\, 5at}{2 \cdot 5at} = \frac{9}{4}a\,.$$

The moment of inertia for the oblique partial area with respect to its centroidal axes can be derived by introducing the coordinate s, whereby the following holds:



$$\mathrm{d}A = t\,\mathrm{d}s \quad \text{and} \quad s^2 = \hat{y}^2 + \hat{z}^2\,.$$

With the slope m of the corresponding part of the cross section, $\hat{z}$ can be expressed by $\hat{z} = m\,\hat{y}$, such that $\hat{y}$ and $\hat{z}$ can be expressed by s:

$$\hat{y}^2 = \frac{1}{1+m^2}\,s^2\,, \quad \hat{z}^2 = \frac{m^2}{1+m^2}\,s^2\,.$$

The moments of inertia result in

$$I_{\hat{y}} = \int \hat{z}^2\,\mathrm{d}A = \int_{-2.5a}^{2.5a} \frac{m^2}{1+m^2}\,s^2\,t\,\mathrm{d}s = \frac{m^2}{1+m^2}\,\frac{125}{12}\,a^3\,t\,,$$

$$I_{\hat{z}} = \int \hat{y}^2\,\mathrm{d}A = \int_{-2.5a}^{2.5a} \frac{1}{1+m^2}\,s^2\,t\,\mathrm{d}s = \frac{1}{1+m^2}\,\frac{125}{12}\,a^3\,t\,,$$

$$I_{\hat{y}\hat{z}} = -\int \hat{y}\hat{z}\,\mathrm{d}A = \int_{-2.5a}^{2.5a} \frac{m}{1+m^2}\,s^2\,t\,\mathrm{d}s = -\frac{m}{1+m^2}\,\frac{125}{12}\,a^3\,t\,.$$

For the given cross section, the slope is $m = \frac{4}{3}$, such that

$$I_{\hat{y}} = \frac{20}{3}\,a^3\,t\,, \qquad I_{\hat{z}} = \frac{15}{4}\,a^3\,t\,, \qquad I_{\hat{y}\hat{z}} = -5\,a^3\,t\,.$$

Another way to derive this result is to apply the transformation equations. For the given geometry, $I_{\hat{y}}$ can be determined with the moments of inertia in the rotated configuration: $I_\eta = (5a)^3\, t \,/\, 12, \quad I_\zeta = I_{\eta\zeta} = 0$ and the corresponding angle $\varphi = -\arctan\frac{3}{4} = -36.87°$. Thus,

$$\begin{aligned} I_{\hat{y}} &= \frac{I_\eta + I_\zeta}{2} + \frac{I_\eta - I_\zeta}{2}\cos 2\varphi + I_{\eta\zeta}\sin 2\varphi \\ &= \frac{1}{2}\,[1+\cos(-73.74°)]\,\frac{(5a)^3\,t}{12} = \frac{20}{3}\,a^3\,t\,. \end{aligned}$$

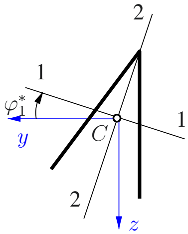
For the calculation of the moments of inertia for the complete cross section with respect to the centroidal axes, the parallel-axis theorem needs to be applied for each part of the cross section, such that

$$\begin{aligned} I_y &= \frac{20}{3}a^3\,t + 5at\,(\frac{9}{4}a - 2a)^2 + \frac{(5a)^3\,t}{12} + 5at\,(\frac{5}{2}a - \frac{9}{4}a)^2 = \frac{425}{24}\,a^3\,t, \\ I_z &= \frac{15}{4}a^3\,t + 5at\,(\frac{3}{2}a - \frac{3}{4}a)^2 + 0 + 5at\,(\frac{3}{4}a)^2 = \frac{225}{24}\,a^3\,t\,, \\ I_{yz} &= -5\,a^3\,t - 5at\,(\frac{3}{2}a - \frac{3}{4}a)(2\,a - \frac{9}{4}a) - 5at\,(-\frac{3}{4}a)(\frac{5}{2}a - \frac{9}{4}a) \\ &= -\frac{25}{8}\,a^3\,t\,. \end{aligned}$$

With these results, the directions of the principal axes are

$$\tan 2\,\varphi^* = \frac{2\,I_{yz}}{I_y - I_z} = \frac{-\dfrac{25}{4}\,a^3\,t}{\dfrac{425-225}{24}\,a^3\,t} = -\frac{3}{4} \quad\rightsquigarrow\quad \begin{aligned} &\underline{\underline{\varphi_1^* = -18.43°}}\,, \\ &\underline{\underline{\varphi_2^* = \quad 71.57°}}\,. \end{aligned}$$

One can derive the result also by considering the symmetry of the cross section with respect to the axis 2–2. The slope of the axis 2–2 is



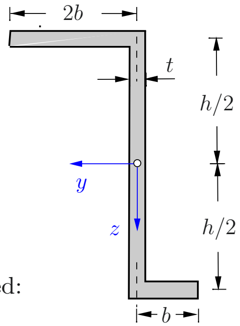
$$m_{2-2} = 3\,,$$

which leads to $\varphi_2^* = 71.57^o$.

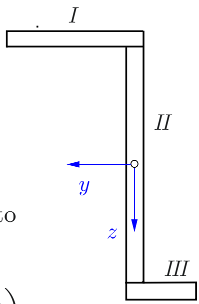
The principal moments of inertia are

$$\begin{aligned} I_{1,2} &= \left[\frac{425+225}{48} \pm \sqrt{\left(\frac{425-225}{48}\right)^2 + \left(\frac{25}{8}\right)^2}\,\right] a^3\,t \\ &= \left[\frac{325}{24} \pm \frac{125}{24}\right] a^3\,t \\ &\rightsquigarrow \quad \underline{\underline{I_1 = \frac{75}{4}\,a^3\,t}}\,, \qquad \underline{\underline{I_2 = \frac{25}{3}\,a^3\,t}}\,. \end{aligned}$$

P9.5 **Problem 9.5** Determine for the non-symmetric and thin-walled Z-profile (t = const, $t \ll h, b$) the rectangular moments of inertia I_y, I_z and the product of inertia I_{yz}.



Solution The area is decomposed into three rectangles. The parallel-axis theorem is applied:



$$I_y = \overbrace{\left(2\,b + \frac{t}{2}\right)\frac{t^3}{12} + t\left(2\,b + \frac{t}{2}\right)\left(\frac{h}{2}\right)^2}^{I}$$

$$+ \overbrace{\frac{t(h-t)^3}{12}}^{II}$$

$$+ \overbrace{\left(b + \frac{t}{2}\right)\frac{t^3}{12} + t\left(b + \frac{t}{2}\right)\left(\frac{h}{2}\right)^2}^{III} .$$

When $t \ll h, b$, the expression can be simplified to

$$\underline{\underline{I_y}} = \overbrace{2b\,t\,\frac{h^2}{4}}^{I} + \overbrace{t\,\frac{h^3}{12}}^{II} + \overbrace{b\,t\,\frac{h^2}{4}}^{III} = \underline{\underline{b\,th^2\left(\frac{3}{4} + \frac{1}{12}\,\frac{h}{b}\right)}} .$$

When terms of higher order are also neglected for I_z and I_{yz}, the following expressions can be derived

$$\underline{\underline{I_z}} = \overbrace{\left[\frac{t(2b)^3}{12} + (2bt)b^2\right]}^{I} + \overbrace{\left[\frac{tb^3}{12} + b\,t\left(\frac{b}{2}\right)^2\right]}^{III} = \underline{\underline{3\,t\,b^3}} ,$$

$$\underline{\underline{I_{zy}}} = -\overbrace{\left[b\left(-\frac{h}{2}\right)2\,b\,t\right]}^{I} + \overbrace{\left[\left(-\frac{b}{2}\right)\frac{h}{2}\,b\,t\right]}^{III} = \underline{\underline{\frac{5}{4}\,t\,b^2h}} .$$

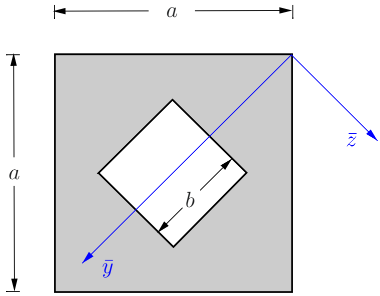
Remarks:

- The origin of the $\{y,z\}$-coordinate system does not coincide with the center of gravity.
- The product of inertia I_{yz} vanishes for the partial areas with respect to their centroidal axes. The above results can be derived by using the parallel-axis theorem only.

P9.6

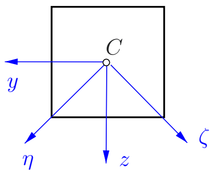
Problem 9.6 The ratio of the moments of inertia for the shaded area with respect to the axes $\bar{y}$ and $\bar{z}$ is 1:5.

Determine the length b of the small square.



Solution The moments of inertia of a square (length a) with respect to the centroidal axes are

$$I_y = I_z = \frac{a^4}{12}\,, \qquad I_{yz} = 0$$



and for rotated axes η, ζ after applying the transformation equations

$$I_\eta = I_\zeta = \frac{a^4}{12}$$

(the centroidal and principal axes coincide for a square!). Therefore, one finds for the given area

$$I_{\bar{y}} = \frac{a^4}{12} - \frac{b^4}{12} = \frac{1}{12}\,(a^2+b^2)(a^2-b^2)\,.$$

With the parallel-axis theorem, it follows

$$I_{\bar{z}} = \frac{1}{12}\,(a^4-b^4) + \left(\frac{\sqrt{2}}{2}a\right)^2 (a^2-b^2)\,.$$

When the requirement

$$\frac{I_{\bar{z}}}{I_{\bar{y}}} = 5$$

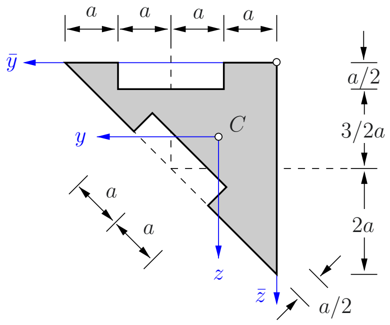
is taken into account, length b follows:

$$5 = \frac{\frac{1}{12}(a^4-b^4) + \frac{a^2}{2}(a^2-b^2)}{\frac{1}{12}(a^4-b^4)} = 1 + \frac{6}{1+\left(\frac{b}{a}\right)^2}$$

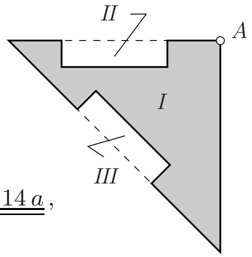
$$\leadsto \quad 1+\left(\frac{b}{a}\right)^2 = \frac{6}{4} \quad \leadsto \quad \left(\frac{b}{a}\right)^2 = \frac{1}{2} \quad \leadsto \quad \underline{\underline{b = \frac{1}{2}\sqrt{2}\,a\,.}}$$

P9.7 **Problem 9.7** Determine for the shaded area

a) the location $\bar{y}_C$, $\bar{z}_C$ of the center of gravity,
b) the moments of inertia with respect to the centroidal axes y, z.



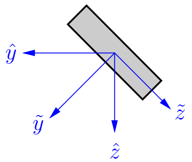
Solution **a)** The calculation of the center of gravity is carried out by considering the center of gravity of the partial areas:



$$\underline{\underline{\bar{y}_C}} = \frac{\left(\frac{4}{3}a\right) 8\,a^2 - (2\,a)\,a^2 - (2\,a - \frac{1}{4\sqrt{2}}\,a)a^2}{8\,a^2 - a^2 - a^2} \approx \underline{\underline{1.14\,a}}\,,$$

$$\underline{\underline{\bar{z}_C}} = \frac{\left(\frac{4}{3}a\right) 8\,a^2 - \left(\frac{1}{4}\,a\right) a^2 - (2\,a - \frac{1}{4\sqrt{2}}\,a)\,a^2}{6\,a^2} \approx \underline{\underline{1.43\,a}}\,.$$

b) The moments of inertia of the partial area *III* with respect to the local coordinate system $\{\tilde{y}, \tilde{z}\}$ has already been derived in previous examples. These can be transformed into the $\{\hat{y},\hat{z}\}$-coordiante system by a 45° rotation:



$$I_{\tilde{y}} = \frac{(2a)^3\ \frac{1}{2}a}{12} = \frac{a^4}{3}\,, \quad I_{\tilde{z}} = \frac{\left(\frac{1}{2}a\right)^3 2\,a}{12} = \frac{a^4}{48}\,, \quad I_{\tilde{y}\tilde{z}} = 0\,.$$

$$\leadsto \quad I_{\hat{y}} = I_{\hat{z}} = \frac{1}{2}\,(I_{\tilde{y}} + I_{\tilde{z}}) = \frac{17}{96}\,a^4, \quad I_{\hat{y}\hat{z}} = \frac{1}{2}\,(I_{\tilde{y}} - I_{\tilde{z}}) = \frac{5}{32}\,a^4.$$

As a next step, the moments of inertia are calculated with respect to the $\{\bar{y}, \bar{z}\}$-coordinate system in A by considering *II* and *III* as negative partial areas:

$$I_{\bar{y}} = \frac{(4\,a)^4}{12} - \left(\frac{a^4}{48} + \frac{a^4}{16}\right) - \left(\frac{17}{96}\,a^4 + (2\,a - \frac{a}{4\sqrt{2}})^2 a^2\right) \approx 17.75\,a^4\,,$$

$$I_{\bar{z}} = \frac{(4\,a)^4}{12} - \left(\frac{a^4}{3} + 4\,a^4\right) - \left(\frac{17}{96}\,a^4 + (2\,a - \frac{a}{4\sqrt{2}})^2 a^2\right) \approx 13.50\,a^4\,,$$

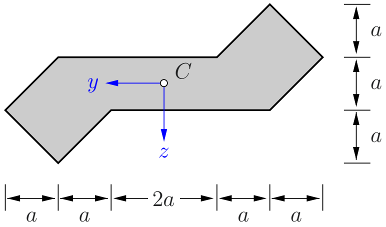
$$I_{\bar{y}\bar{z}} = -\frac{(4\,a)^4}{24} - \left(0 - \frac{a^4}{2}\right) - \left(\frac{5}{32}\,a^4 - (2\,a - \frac{a}{4\sqrt{2}})^2 a^2\right) \approx -7.00\,a^4\,.$$

Applying the parallel-axis thereom yields the moments of inertia with respect to the centroidal axes:

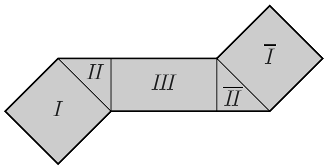
$$
\begin{aligned}
\underline{\underline{I_y}} &= I_{\bar{y}} - \bar{z}_C^2\, A &&\approx 17.75\, a^4 - (1.43\, a)^2\ 6\, a^2 &&\approx \underline{\underline{5,48\, a^4}}\,,\\
\underline{\underline{I_z}} &= I_{\bar{z}} - \bar{y}_C^2\, A &&\approx 13.50\, a^4 - (1.14\, a)^2\ 6\, a^2 &&\approx \underline{\underline{5.70\, a^4}}\,,\\
\underline{\underline{I_{yz}}} &= I_{\bar{y}\bar{z}} + \bar{z}_C\, \bar{y}_C\, A &&\approx -7.00\, a^4 + (1.43\, a)\ (1.14\, a)\ 6\, a^2 &&\approx \underline{\underline{2.78\, a^4}}\,.
\end{aligned}
$$

P9.8 **Problem 9.8** Determine for the shaded area:

a) I_y, I_z, I_{yz},
b) the direction of the principal axes,
c) the principal moments of inertia.



Solution **a)** The area is devided into five partial areas. Since the area is point-symmetric with respect to C, the values of the moments of inertia of the partial areas I and $\bar{I}$ are the same as is for II and $\overline{II}$.



$$\underline{\underline{I_y}} = 2\overbrace{\left[\frac{a^4}{3}+\left(\frac{a}{2}\right)^2 2\,a^2\right]}^{I \text{ and } \bar{I}} + 2\overbrace{\left[\frac{a^4}{36}+\left(-\frac{a}{6}\right)^2\frac{a^2}{2}\right]}^{II \text{ and } \overline{II}} + \overbrace{\frac{2\,a^4}{12}}^{III} = \underline{\underline{\frac{23}{12}\,a^4}},$$

$$\underline{\underline{I_z}} = 2\overbrace{\left[\frac{a^4}{3}+(2\,a)^2\;2\,a^2\right]}^{I \text{ and } \bar{I}} + 2\overbrace{\left[\frac{a^4}{36}+\left(\frac{4\,a}{3}\right)^2\frac{a^2}{2}\right]}^{II \text{ and } \overline{II}} + \overbrace{\frac{a\,(2\,a)^3}{12}}^{III}$$

$$= \underline{\underline{\frac{115}{6}\,a^4}},$$

$$\underline{\underline{I_{yz}}} = \overbrace{2\left[0 - (2\,a)\,\left(\frac{a}{2}\right)\,2\,a^2\right]}^{I \text{ and } \overline{I}} + \overbrace{2\left[\frac{a^4}{72} - \left(\frac{4\,a}{3}\right)\left(-\frac{a}{6}\right)\frac{a^2}{2}\right]}^{II \text{ and } \overline{II}} + \overbrace{\ 0\ }^{III}$$
$$= \underline{\underline{-\frac{15}{4}\,a^4}}\,.$$

b) With these results, the directions of the principal axes can be found as

$$\tan 2\,\varphi^* = \frac{2\,I_{yz}}{I_y - I_z} = \frac{90}{207} \quad \leadsto \quad \underline{\underline{\varphi_1^* = 11.75^\circ}}\,, \quad \underline{\underline{\varphi_2^* = 101.75^\circ}}\,.$$

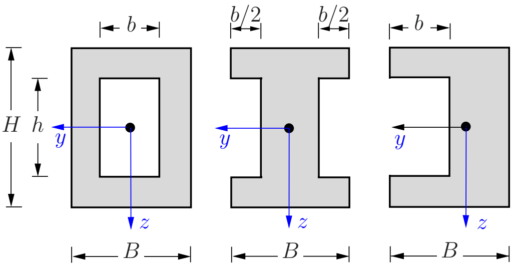
c) One obtains for the principal moments of inertia

$$I_{1,\,2} = \left[\frac{23+230}{24} \pm \sqrt{\left(\frac{23-230}{24}\right)^2 + \left(-\frac{15}{4}\right)^2}\,\right] a^4$$
$$= \left[\frac{253}{24} \pm \sqrt{\frac{5661}{64}}\right] a^4$$
$$\leadsto \quad \underline{\underline{I_1 \approx 19.95\,a^4}}\,, \quad \underline{\underline{I_2 \approx 1.14\,a^4}}\,.$$

Remark: The match of the principal axes to their respective principal moments of inertia can be derived by the transformation equations. In the present case, it is clear that the smaller principal moment of inertia I_2 corresponds to the direction with angle φ_1^*. The areas have in average a much smaller distance to the axis with direction φ_1^* than to the axis with direction φ_2^*.

P9.9

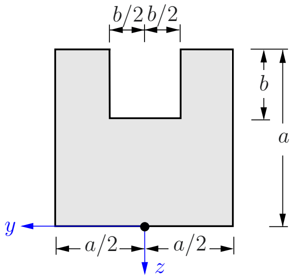
Problem 9.9 Determine for the three cross sections the moments of inertia about the y-axis.



Solution The respective cross sections are decomposed into a shaded area and a white area. The moment of inertia of the cross sections can be derived by considering the moment of inertia for the rectangle $B \times H$ and subtracting the moment of inertia for the white area $b \times h$. Since the z-coordinate of the center of gravity is the same for all of these white areas, the solution is identical for all cross sections:

$$I_y = \frac{B\,H^3}{12} - \frac{b\,h^3}{12}\,.$$

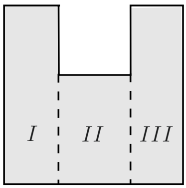
Problem 9.10 Determine the moments of inertia with respect to the $\{y,z\}$-coordinate system. **P9.10**



Solution Since the cross section is symmetric with respect to the z-axis, the product of inertia vanishes: $I_{yz} = 0$.
There are two possibilities to calculate the remaining moments of inertia I_y and I_z. The first alternative is shown for I_y: The cross section is devided into three rectangles. According to the parallel-axis theorem, the moment of inertia I_y for the cross section reads

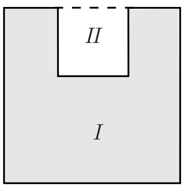
$$I_y = 2\left[\frac{(a-b)}{2}\frac{a^3}{12} + \frac{(a-b)}{2}\,a\left(\frac{a}{2}\right)^2\right] + \frac{b(a-b)^3}{12} + b(a-b)\left(\frac{a-b}{2}\right)^2,$$

$$\underline{\underline{I_y = (a-b)\,\frac{a^3}{3} + b\,\frac{(a-b)^3}{3}}}\,.$$



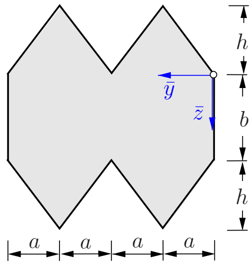
The other possibility to determine the moments of inertia is shown for I_z: the cross section is devided into two areas, the shaded area and the white area. Applying this procedure yields

$$\underline{\underline{I_z}} = \frac{a\,a^3}{12} - \frac{b\,b^3}{12} = \underline{\underline{\frac{a^4 - b^4}{12}}}\,.$$

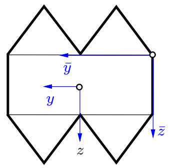


P9.11

Problem 9.11 Determine the moments of inertia of the cross section with respect to the $\{\bar{y}, \bar{z}\}$-coordinate system.



Solution It is convenient to start with the calculation of the moments of inertia with respect to the principal axes y and z. Therefore, the cross section is devided into four isosceles triangles and one rectangle. For each triangle, the moment of inertia about the y-axis reads



$$\frac{2a\,h^3}{36} + ah\,\left(\frac{h}{3} + \frac{b}{2}\right)^2 .$$

The principal moment of inertia I_y is the sum of the moments of inertia of the triangles and the rectangle:

$$I_y = 4\left(\frac{2a\,h^3}{36} + ah\,\left(\frac{h}{3} + \frac{b}{2}\right)^2\right) + \frac{4a\,b^3}{12} .$$

With the cross sectional area $A = 4a(h+b)$ and the parallel-axis theorem $I_{\bar{y}} = I_y + (b/2)^2 A$, one obtains

$$\underline{\underline{I_{\bar{y}} = 4\left(\frac{2a\,h^3}{36} + \left(\frac{h}{3} + \frac{b}{2}\right)^2 ah\right) + \frac{4a\,b^3}{12} + 4a(h+b)\left(\frac{b}{2}\right)^2}} .$$

Analogously, one obtains

$$I_z = 4\left(\frac{h\,(2a)^3}{48} + ah\,a^2\right) + \frac{b\,(4a)^3}{12} .$$

Considering the parallel-axis theorem again yields

$$\underline{\underline{I_{\bar{z}}}} = I_z + (2a)^2 A = \underline{\underline{\frac{14}{3}a\,h^3 + \frac{16}{3}b\,a^3 + 4a(h+b)\,(2a)^2}} .$$

For $I_{\bar{y}\bar{z}}$, one finds with $I_{yz} = 0$

$$\underline{\underline{I_{\bar{y}\bar{z}}}} = I_{yz} - \frac{b}{2}2aA = \underline{\underline{-4a^2 b\,(h+b)}} .$$