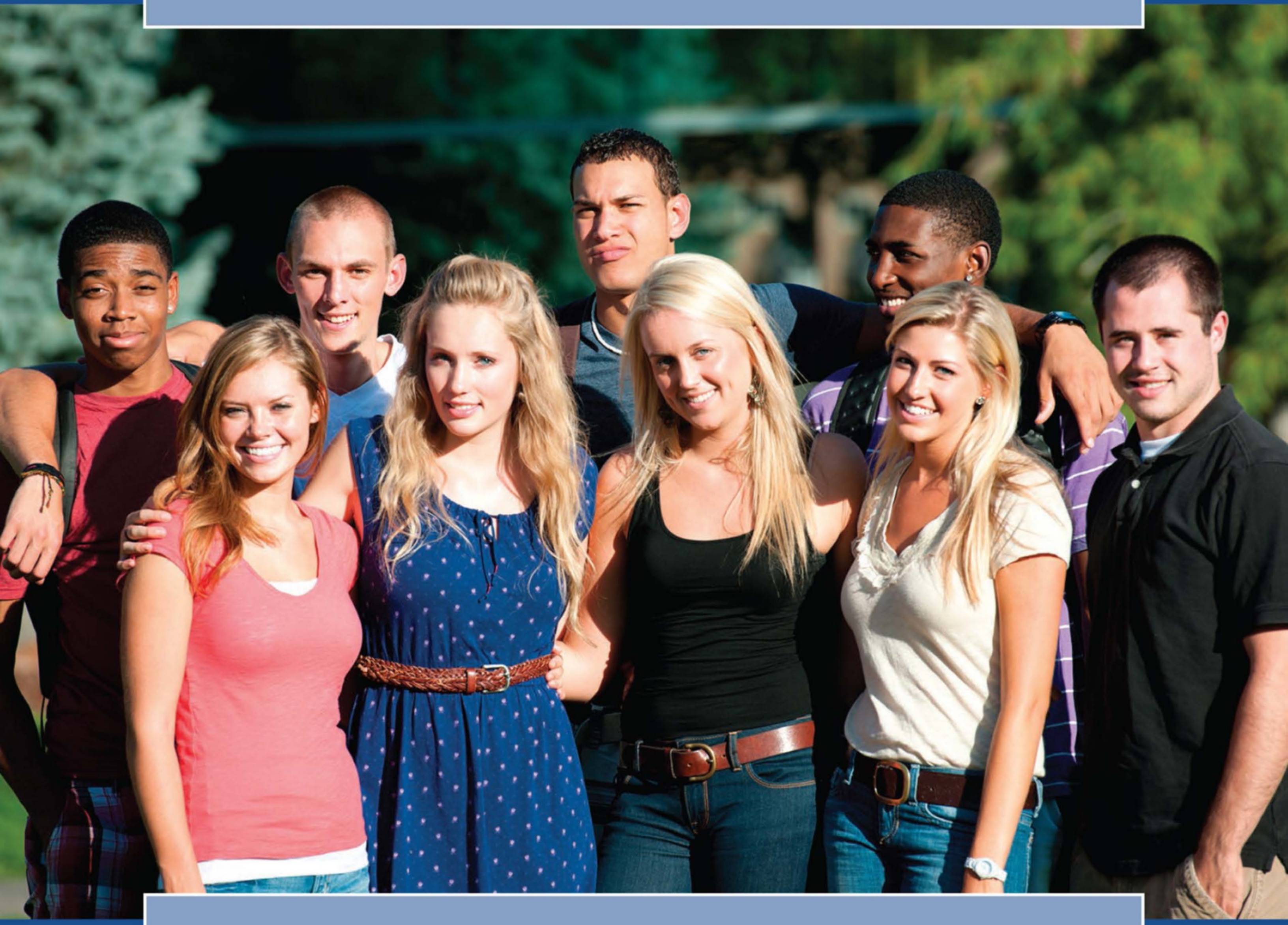
The Second Handbook of Research on the Psychology of Mathematics Education

The Journey Continues

Ángel Gutiérrez, Gilah C. Leder and
Paolo Boero (Eds.)



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The Second Handbook of Research on the Psychology of Mathematics Education

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The Journey Continues

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To the young researchers, throughout the world,
who are the future of mathematics education research
and of the PME community

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FOREWORD

Another Decade of PME Research

PME stands for Psychology of Mathematics Education; it is a society whose members form the International Group for the Psychology of Mathematics Education (IGPME). It holds an international conference annually, hosted by PME members in a diversity of countries around the world. From its beginnings in 1976, Psychology has underpinned the mathematics education research reported at PME conferences. For example, studies of students' learning of mathematical topics (e.g., ratio, algebra, calculus, geometry…) and recognition of the difficulties certain topics present to students. However, it has long been understood that the P (for Psychology) embraces a range of human sciences, such as (for example) Philosophy, Sociology, Anthropology and Semiotics, which have become central to the research in Mathematics Education of quite a few PME members. While the central themes of research continue to be the learning and teaching of mathematics, and cognitive studies root PME firmly within the Psychological domain, we have seen focuses on other themes like constructivism, socio-cultural theories, linguistics, equity and social justice, affect, and on the professional lives of teachers emerging over the years.

In all cases, research published in PME proceedings has gone through a critical review process. Reviewing is undertaken by PME members who have attended several PME conferences and had their own research reports in previous proceedings. PME is open to researchers in mathematics education throughout the world, and reviewers reflect the cultural and geographic diversity of PME itself. The review process requires that papers accepted for publication focus clearly on aspects of mathematics education and satisfy a set of criteria with demands on theory, methodology, reporting of results and discussion of implications and impact of the research. It is indicative of the quality of PME research reports and conference proceedings that PME papers are respected alongside those in high quality research journals.

In 2006, a PME handbook celebrating 30 years of PME was produced as a milestone for the PME community (1976–2006). This handbook synthesised PME research over the 30-year period and demonstrated the developing themes mentioned above through its chapters. Authors were chosen to acknowledge the scientific quality of their research and their active contributions to PME conferences over the years and, as a whole, to celebrate the diversity of PME culturally and geographically.

We now present a new handbook celebrating another decade of PME research – 40 years. The 40th PME conference is held in Szeged, Hungary. This is particularly fitting since Hungary was the birthplace of George Polya, who has been a great inspiration to PME members and students of mathematics widely over the years. His seminal book “How to Solve It” has influenced the doing of mathematics through problem-solving and the use of heuristics of problem solving. In fact, the title of Polya’s book is taken as the title for PME 40, and as a theme for the conference. The choice of guest speaker, Alan Schoenfeld, reflects the theme: Alan having been one of the pioneers in mathematical problem solving building on Polya’s work.

The editors of this new handbook were invited by the PME International Committee to produce a volume celebrating the most recent decade of PME. Their work in producing the handbook started with a survey of research published in PME proceedings since 2006, the recognition of key themes in this work and invitation to authors to study and provide a synthesis of each of the themes. Their introduction provides details of this process and a rationale for the themes chosen, showing a diversity from focuses within mathematics itself towards key aspects of the learning and teaching of mathematics and its relations to society and culture. Themes include numbers, algebra, geometry, functions and calculus, proof and argumentation, problem solving, mathematical modelling, language, the use of digital technology, curriculum and assessment, teachers’ knowledge and professional development, affect, and the socio-cultural-political axis in understanding mathematics education.

PME as a society is alive and well. Recent conferences in Taipei, Kiel, Vancouver, and Hobart have been extremely well attended. In recent years, PME has introduced a special day for early career researchers (the ERD) before each annual conference. These days have also been well attended, and participants have then attended the main conference. This means that PME is actively encouraging a new generation of researchers with every conference. In the coming years we plan to have conferences in Singapore (2017) and Sweden (2018) and are in conversation with other nations for planning conferences after this.

I recommend this handbook to all researchers in Mathematics Education. You will find here a strong taste of the research in PME, a synthesis of recent research and indications for future research directions. My thanks go to the editors and all authors and reviewers for their contribution to this important work.

Barbara Jaworski
President of PME
On behalf of the PME International Committee (IC)
February 2016

INTRODUCTION

A handbook compiling the research produced by the PME Group from its very beginning until 2005 was published in 2006 to celebrate the first 30 years of existence of the PME Group. During the last ten years the activities of the PME Group have undoubtedly grown and diversified. From inspection of the more recent conference presentations and Proceedings it is readily apparent that research areas have continued to evolve. It thus seems an appropriate moment to release a new handbook which captures both the new directions that have emerged as well as providing a rich overview of areas which continue having a sustained record of explorations by the PME community. The second PME handbook is published to celebrate the 40 years of activity of the PME Group. It focuses primarily on the research activities over the last ten years (2006–2015) and can be seen as a ready sequel to the first PME handbook, which covered the period 1976–2005. The proximity of the timing of the 2015 PME conference and a critical deadline to ensure the timely publication of this handbook has led to a slightly lighter review in several chapters of papers included in the 2015 Proceedings.

To test our impression that changes have occurred, since 2005, in the research interests of the PME Group, we analyzed the indexes of the Proceedings, identified the presentations related to the various research topics, and compared the different tallies. As for the first handbook, the editors' most sensitive decision was to use this analysis to select the topics for the chapters of the new handbook. The result was a list of fourteen chapters which cover the core and most relevant parts of the activity of the PME Group during the last ten years.

By comparing the indexes of the two handbooks, the main differences which have resulted from the evolution of researchers' interests can readily be seen. The emergence of new research directions was already becoming evident during the first years of this century. Since many of these explorations were still in an initial phase of development, it was deemed premature to include them in the first handbook. However, some of these research areas (like language or modelling) have increased in momentum and relevance, and now certainly warrant inclusion in the second handbook. Main "traditional" research areas (like algebra, arithmetic, geometry, and calculus) as well as domains that were "new" ten years ago (socio-cultural, political, affectivity, ICTs, and teacher related issues) have retained a dominant presence in the Proceedings. Accordingly they form part of the second handbook – although, in some cases, with reduced coverage.

Once we had decided which topics to include in the handbook, our second important task was to identify authors who could take on the responsibility for writing the chapters. We believed that it was important to select researchers who had not contributed a chapter to the first handbook and considered that effective coverage of each field would be enhanced by the selection of a team of at least two authors, located, for most chapters, in different parts of the world. Finally, to achieve optimum continuity between the two handbooks, we asked, whenever possible, for an author from the first handbook to act as a reviewer of the pertinent chapter in the second handbook. Each chapter, it should be noted, was constructively reviewed by at least two PME members, with the selection of reviewers based on their expertise in the relevant area. For this group, too, we aimed at geographic diversity. In summary, the 31 authors in this handbook came from 15 countries (or 17 countries if we count their place of birth); the 27 reviewers from 13 different countries. The combination of the scientific quality of authors and reviewers, and their wide geographical distribution, have given voice to diverse approaches, perspectives and delivered a meaningful document of relevance to both mature and emerging researchers.

The handbook chapters are organized into the same four sections used in the first handbook. The first group of chapters correspond to topics related to mathematics content areas: algebra, arithmetic, geometry (including measurement and visualization), and calculus. The second section, the main one in terms of page volume, is devoted to transverse topics: proof, ICTs, language, curriculum and assessment, concept learning, problem solving, and modelling. The third group of chapters comprises those focused on social, cultural, political or affective aspects of teaching and learning of mathematics. Finally, the last but equally important section consists of a chapter devoted to pre- and in-service teachers' activity.

In closing, we acknowledge the efforts of all those committed to the continuing growth and evolution of the PME Group, the persistent search for new knowledge aimed at fostering teachers', students', and society's understanding and appreciation of mathematics and its productive application in their personal and professional lives. Our thanks are also extended to the members of the International Committee who serve the Group and take care of scientific, organizational, and administrative matters that require attention if the health of the PME Group is to be preserved, to the local organizers of the annual PME conferences, who selflessly strive to provide PME members with the best possible environment to celebrate the yearly meeting, and, above all, to the PME members with their shared aim of achieving a better mathematical education for all members of society.

Ángel Gutiérrez
Gilah C. Leder
Paolo Boero

REVIEWERS OF THE CHAPTERS IN THIS VOLUME

The editors and authors thank the following people for their help in the review process of this volume, by reviewing one of the chapters.

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Barbara Jaworski (UK)	2013–2016

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The following persons acted as Chairs of the Local Organizing Committees of the PME International Conferences:

No.	*Year*	*Chair*	*Place*
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PME 4	1980	Robert Karplus	Berkeley (USA)
PME 5	1981	Claude Comiti	Grenoble (France)
PME 6	1982	Alfred Vermandel	Antwerpen (Belgium)
PME 7	1983	Rina Hershkowitz	Shoresh (Israel)
PME 8	1984	Beth Southwell	Sidney (Australia)
PME 9	1985	Leen Streefland	Noordwijkerhout (The Netherlands)
PME 10	1986	Leone Burton and Celia Hoyles	London (UK)
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PME 13	1989	Gérard Vergnaud	Paris (France)
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PME 23	1999	Orit Zaslavsky	Haifa (Israel)
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PME 29	2005	Helen L. Chick	Melbourne (Australia)
PME 30	2006	Jarmila Novotná	Prague (Czech Republic)
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PME 32	2008	Olimpia Figueras	Mexico D.F. (Mexico)
PME 33	2009	Marianna Tzekaki	Thessaloniki (Greece)
PME 34	2010	Márcia Maria Fusaro Pinto	Belo Horizonte (Brazil)
PME 35	2011	Behiye Ubuz	Ankara (Turkey)
PME 36	2012	Tai-Yih Tso	Taipei (Taiwan)
PME 37	2013	Aiso Heinze	Kiel (Germany)
PME 38	2014	Cynthia Nicol and Peter Liljedahl	Vancouver (Canada)
PME 39	2015	Kim Beswick	Hobart (Australia)
PME 36	2016	Csaba Csíkos	Szeged (Hungary)

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Due to the large number of citations in the chapters of this handbook to papers in PME Proceedings, and to conform with space limitations, the editors decided that a shortened format would be used for references to the PME Proceedings. To acknowledge the editorship of the PME Proceedings, a full listing is provided below:

No.	*Year*	*Editors*
PME 1	1977	There were no proceedings published.
PME 2	1978	E. Cohors-Fresenborg & I. Wachsmuth
PME 3	1979	D. Tall
PME 4	1980	R. Karplus
PME 5	1981	Equipe de Recherche Pédagogique
PME 6	1982	A. Vermandel
PME 7	1983	R. Hershkowitz
PME 8	1984	B. Southwell, R. Eyland, M. Cooper, J. Conroy, & K. Collis
PME 9	1985	L. Streefland
PME 10	1986	Univ. of London Institute of Education
PME 11	1987	J. C. Bergeron, N. Herscovics, & C. Kieran
PME 12	1988	A. Borbás
PME 13	1989	G. Vergnaud, J. Rogalski, & M. Artigue
PME 14	1990	G. Booker, P. Coob, & T. Navarro de Mendicuti
PME 15	1991	F. Furinghetti
PME 16	1992	W. Geslin & K. Graham
PME 17	1993	I. Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin
PME 18	1994	J. P. da Ponte & J. F. Matos
PME 19	1995	L. Meira & D. Carraher
PME 20	1996	L. Puig & A. Gutiérrez
PME 21	1997	E. Pehkonen
PME 22	1998	A. Olivier & K. Newstead

EDITORS OF THE PROCEEDINGS OF THE PME INTERNATIONAL CONFERENCES

No.	*Year*	*Editors*
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PME 24	2000	T. Nakahara & M. Koyama
PME 25	2001	M. van den Heuvel-Panhuizen
PME 26	2002	A. D. Cockburn & E. Nardi
PME 27	2003	N. A. Pateman, B. J. Dougherty, & J. T. Zilliox
PME 28	2004	M. J. Høines & A. B. Fuglestad
PME 29	2005	H. L. Chick & J. L. Vincent
PME 30	2006	J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková
PME 31	2007	J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo
PME 32	2008	O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda
PME 33	2009	M. Tzekaki, M. Kaldrimidou, & H. Sakoni
PME 34	2010	M. M. F. Pinto & T. F. Kawasaki
PME 35	2011	B. Ubuz
PME 36	2012	T.-Y. Tso
PME 37	2013	A. M. Lindmeier & A. Heinze
PME 38	2014	P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan
PME 39	2015	K. Beswick, T. Muir, & J. Wells
PME 36	2016	C. Csíkos, A. Rausch, & J. Szitányi

PME INTERNATIONAL GROUP CONFERENCES

The Proceedings of the PME International Conferences are available to the PME members at the PME web page http://www.igpme.org/ (except for PME4, PME5, PME6, PME8, and PME10). Many PME proceedings can be freely retrieved from the ERIC web page http://www.eric.ed.gov/. The table below indicates the ERIC ED numbers for all the Proceedings of the PME International Conferences stored at ERIC.

No.	*Year*	*Place*	*ERIC number*
2	1978	Osnabrück, Germany	ED226945 (not available)
3	1979	Warwick, United Kingdom	ED226956 (not available)
4	1980	Berkeley, California, USA	ED250186 (not available)
5	1981	Grenoble, France	ED225809 (not available)
6	1982	Antwerp, Belgium	ED226943 (not available)
7	1983	Shoresh, Israel	ED241295 (not available)
8	1984	Sydney, Australia	ED306127 (not available)
9	1985	Noordwijkerhout, The Netherlands	ED411130, ED411131
10	1986	London, United Kingdom	ED287715 (not available)
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128, ED411129
13	1989	Paris, France	ED411140, ED411141, ED411142
14	1990	Oaxtepex, Mexico	ED411137, ED411138, ED411139
15	1991	Assisi, Italy	ED413162, ED413l63, ED413164
16	1992	Durham, New Hampshire, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134, ED411135, ED411136
20	1996	Valencia, Spain	ED453070, ED453071, ED453072, ED453073, ED453074
21	1997	Lahti, Finland	ED416082, ED416083, ED4l6084, ED416085

No.	*Year*	*Place*	*ERIC number*
22	1998	Stellenbosch, South Africa	ED427969, ED427970, ED427971, ED427972
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301, ED452302, ED452303, ED452304
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Honolulu, Hawaii, USA	ED500857, ED500859, ED500858, ED500860
28	2004	Bergen, Norway	ED489178, ED489632, ED489538, ED489597
29	2005	Melbourne, Australia	ED496845, ED496859, ED496848, ED496851
30	2006	Prague, Czech Republic	ED496931, ED496932, ED496933, ED496934, ED496939
31	2007	Seoul, Korea	ED499419, ED499417, ED499416, ED499418

PME NORTH AMERICAN CHAPTER (PME-NA) CONFERENCES

Many Proceedings of the PME-NA Conferences can be freely retrieved from the ERIC web page http://www.eric.ed.gov/ The table below indicates the ERIC ED numbers for all the Proceedings of the PME-NA Conferences published separately from the Proceedings of PME International Conferences stored at ERIC.

No.	*Year*	*Place*	*ERIC number*
3	1981	Minneapolis, Minnesota, USA	ED223449
4	1982	Athens, Georgia, USA	ED226957 (not available)
5	1983	Montreal, Quebec, Canada	ED289688
6	1984	Madison, Wisconsin, USA	ED253432 (not available)
7	1985	Columbus, Ohio, USA	ED411127
8	1986	East Lansing, Michigan, USA	ED301443 (not available)
10	1988	Dekalb, Illinois, USA	ED411126
11	1989	New Brunswick, New Jersey, USA	ED411132, ED411133
13	1991	Blacksburg, Virginia, USA	ED352274
15	1993	Pacific Grove, Califomia, USA	ED372917
16	1994	Baton Rouge, Louisiana, USA	ED383533, ED383534
17	1995	Columbus, Ohio, USA	ED389534
18	1996	Panama City, Florida, USA	ED400178
19	1997	Bloomington-Normal, Illinois, USA	ED420494, ED420495
20	1998	Raleigh, North Carolina, USA	ED430775, ED430776
21	1999	Cuernavaca, Morelos, Mexico	ED433998
22	2000	Tucson, Arizona, USA	ED446945
23	2001	Snowbird, Utah, USA	ED476613
24	2002	Athens, Georgia, USA	ED471747

PART 1

COGNITIVE ASPECTS OF LEARNING AND TEACHING CONTENT AREAS

FERNANDO HITT AND ALEJANDRO S. GONZÁLEZ-MARTÍN

1. GENERALIZATION, COVARIATION, FUNCTIONS, AND CALCULUS

1. INTRODUCTION

In this chapter we review the main contributions of PME to research on the topics of functions and calculus, identifying ongoing trends as well as newly emerging issues and approaches. The first part of this chapter (Section 2) refers mainly to the chapter on *advanced mathematical thinking* (Harel, Selden, & Selden, 2006) in the last *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006), which is where most of the PME research results concerning functions and calculus appear. However, the field has evolved, and a considerable part of this chapter is devoted to identifying issues we consider to be some of the major approaches and research topics that have emerged in recent years.

Our chapter examines research that has been carried out on the topics of functions and calculus. At first glance, this would appear to be a 'condensed' area of research, however the reality is very different from what we first imagined. Research on the teaching and learning of functions extends to the early grades, and *Early algebra* researchers advocate encouraging algebraic thinking beginning in primary school, using a functional approach. This led us to consider some PME papers that look at this *Early algebra* perspective, allowing us to explore the origins of functional thinking in primary school by examining the main contributions of PME to the perspective of the functional approach. Furthermore, the topic of functions is taught beginning in secondary school, leading to the introduction of calculus and its study at the university level. In the last ten years, research conducted at the university level has also evolved both in terms of approaches and research topics (Artigue, Batanero, & Kent, 2007; Nardi, Biza, González-Martín, Gueudet, & Winsløw, 2014; Rasmussen, Marrongelle, & Borba, 2014), adapting to the characteristics of a rather different and varied educational level. As a consequence, when we were asked to write a chapter about 'Functions and Calculus,' we were actually confronted with a wide range of topics and even some contradictions. For instance, there is a long tradition of research on teacher training for primary and secondary education, but scarce research focused on the university level; there is abundant research on teaching practices in primary and secondary education, but a major gap appears at the tertiary level; other discrepancies are covered later in this chapter. It would

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 3–38.

be impossible to parse this diversity of research and multitude of approaches from primary school to the university level, and in this chapter we instead identify those works that, in our view, have propelled knowledge on the topic in the last ten years. Our choices are, of course, influenced by our own experience as researchers and the account presented here reflects our personal vision.

Concerning the theoretical approaches used to conduct research on the topics of functions and calculus, ten years ago, Harel et al. (2006) highlighted the fact that theoretical frameworks used in studies on *advanced mathematical thinking* (AMT) were largely cognitive. This phenomenon was not restricted to the PME community, but was a popular trend in research on advanced levels, which for a long time concentrated on "identifying cognitive processes underlying the learning of mathematics at advanced levels, investigating the relationships of these processes with respect to those at play at more elementary levels, and understanding students' difficulties with advanced mathematical concepts" (Artigue et al., 2007, p. 1011). While this situation has changed (more quickly in research on the primary and secondary levels than the tertiary level), the PME proceedings still publish a number of papers that follow some of these cognitive approaches, such as the *concept image – concept definition* approach or the use of representations. Although PME and PME-NA have played an important role in the consolidation of AMT, research in the last years has been critical of this approach and some of its implicit ideas, as Artigue et al. (2007) summarize. These critiques may have been the reason certain terms appear less frequently (or, at least, are being used more judiciously). For example, the Congress of European Research in Mathematics Education (CERME) had an *Advanced mathematical thinking* working group until its sixth edition (2009), but shifted to the *University mathematics education* working group in 2011 (Nardi, González-Martín, Gueudet, Iannone, & Winsløw, 2011), with some of the AMT content being redistributed to other groups. Consequently, we will not refer to AMT in this chapter and will instead refer to specific content related to functions and calculus and, of course, to the theoretical approaches pertaining to problems of teaching and learning this content.

As mentioned above, until ten years ago, researchers were using mainly cognitive approaches, especially in studying higher levels of education. Since then, some of these approaches have undergone developments that have opened up wider perspectives. For instance, research focusing on the cognitive (for example, articulation among representations, Duval, 1999) has evolved to other types of research connected to sociocultural processes, where communication in the classroom is the key ingredient (Mariotti, 2012; Radford, 2003, 2009). This last type of research has been at the origin of some interesting task-design activities (e.g. Prusak, Hershkowitz, & Schwarz, 2013). This is just one example, and later in this chapter we will discuss other instances of theoretical developments that have emerged in the last ten years. Another major evolution in the field has seen different approaches being used in a coordinated way; this has sparked an interest within the

PME community in comparing theories (e.g. Boero et al., 2002; Presmeg, 2006a) and in examining the networking of theories, as evidenced at the 2010 (Bikner-Ahsbahs et al., 2010), and 2014 (Clark-Wilson et al., 2014) Research Forums.

It is worth noting that PME and PME-NA communities have been extremely prolific in developing research related to the topics of functions and calculus. It would be therefore impossible to present a comprehensive summary, and in this chapter we address what we consider to be the most important advances, alongside research conducted outside the PME community. Although we will mainly refer to papers published in PME proceedings during the last ten years, references to the evolution of research that has appeared in journals are inevitable, and some of the papers presented during PME conferences have led to, or come from, papers published in international journals.

This chapter is divided into five main sections (not including this introduction). Section 2 synthetizes the main PME contributions developed using cognitive approaches, in most cases following pre-existing theoretical perspectives (although we also include some new approaches). The abundant literature on functions in different school grades led us to create a whole section summarizing the main results on this topic, which are discussed in Section 3, with a particular focus on the transition from mental to semiotic representations. In Section 4 we examine some attempts to expand past purely cognitive approaches, drawing mainly on socio-cultural or institutional perspectives and on the networking of theories. Section 5 addresses topics of research that have received more attention in recent years, compared to the previous handbook. Finally, in the last section, we reflect on the main contributions of PME research with respect to functions and calculus, and explore avenues for future research.

2. PME CONTRIBUTIONS USING COGNITIVE APPROACHES

2.1. Concept Image and Concept Definition

One important change becomes apparent when comparing the PME production of the last ten years with that of previous decades: the marked decline in the number of papers following the *concept image – concept definition* approach. This may be due to the shift in research towards favoring semiotic representations and issues that are more social and cultural rather than solely cognitive, as discussed in the introduction. Another possible reason is the trend towards using other cognitive approaches, as we will discuss later in this section.

The term *concept image* was introduced to define “the total cognitive structure that is associated with [a given] concept, which includes all mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). On the other hand, *concept definition* describes “a form of words used to specify that concept” (p. 152). This approach has been useful to show, for instance, that a learner can hold a contradictory concept image and concept definition. Building on the gap between the concept definition and the concept image, as well as the relationships between intuitive and formal knowledge (as considered by Fischbein, 1999), Kidron and Picard (2006) constructed an activity based on the discrete-continuous interplay to help university students understand the notion of limit in the definition of the derivative. Using a model from the field of dynamical systems – the logistic equation – as well as Euler's method to see the effect of replacing $\lim_{\Delta x \to 0} \Delta x / \Delta y$ by $\Delta x / \Delta y$, their data reveal the existence of the 'treasured intuition' that "*gradual causes have gradual effects* and that *small changes in a cause should produce small changes in its effect*" (p. 443, emphasis in the original). Although their activities helped some students overcome treasured intuitions and enrich their concept image, the treasured intuitions seemed to be quite persistent. They also pointed at the link between different representations as a reason for students' success at overcoming the gap. We will return to the importance of representations in Sections 3.1 and 3.3. This work was pursued by Kidron (2009), using Fischbein's notion of mental model (Fischbein, 2001), to show that "the role of intuitive structures does not come to an end when analytical (formal) forms of thinking become possible" (p. 314), again demonstrating the tenacity of certain treasured intuitions. In particular, Kidron's work shows that tacit models can coexist with logical reasoning, even in advanced students, and underlines the fact that some pictorial models may emerge when facing abstract questions, again reinforcing the importance of using different representations.

Although use of this approach in PME has dropped off significantly in recent years, it is not a static approach, as Artigue et al. (2007) clearly show. For instance, Bingolbali and Monaghan (2008) proposed a reinterpretation of the *concept image* construct, criticizing the fact that most studies using it adopt a purely cognitive approach. Their proposition – in connection with Bingolbali, Monaghan and Roper, 2006, discussed in Section 5.5 – takes into account the learning context, particularly in undergraduate studies. They demonstrate how department affiliations can have an impact on students' development of concept images (their work focuses on derivatives), which are influenced by teaching practices and departmental perspectives. Although their work adds an institutional component to the *concept image – concept definition* approach (we discuss the advantages of institutional approaches in Section 4.1), the impact of their work on PME research is unclear. However, it is worth noting that ten years ago, Harel et al. (2006) posed the question, "is a learner's current concept image a consequence of a specific teaching approach or is it an unavoidable construct due to the structure and limitation of the human brain, mind, culture, and social interaction?" (p. 163). Although institutional approaches (see Section 4.1) seem to provide answers to this question, the recent reinterpretation of the *concept image* construct also appears to respond to it, and it will be interesting to see the new directions this approach might take in light of this development.

2.2. The Duality between Process and Object

Like the *concept image – concept definition* approach, the APOS (Action, Process, Object, Scheme) framework has been less present in PME in the last 10 years. The main tenets of this approach, derived from Piaget's ideas about reflective abstraction, are clearly presented by Artigue et al. (2007), so we will only cite an example concerning infinity using an innovative perspective and developed over a number of years.

Mamolo (2014) used APOS theory as a lens through which to interpret her participants' struggle with various questions and paradoxes concerning infinity, in looking at the differences between *potential infinity* and *actual infinity* – both quite present in many notions and procedures of calculus; the former seen as a process in which every moment in time is finite, but also goes on forever, and the latter seen as a completed entity that envelops that which was previously potential (Fischbein, 2001). These two notions can be associated with the process and object conceptions of infinity, respectively, the latter being the encapsulation of the former. In this perspective, "through encapsulation, the infinity becomes cognitively attainable" (Dubinsky, Weller, McDonald, & Brown, 2005, p. 346) and can be conceived as an object, a complete entity which can be acted upon. However, recent studies have shown that in some cases, de-encapsulating infinity back to a process seems to be a useful strategy to cope with infinity (Brown, McDonald, & Weller, 2010).

Taking an innovative approach and studying paradox resolution, and building on previous work (Mamolo & Zazkis, 2008), Mamolo (2014) approached the participants' understanding of 'acting' on transfinite cardinal numbers via arithmetic operations, focusing particularly on struggles with the indeterminacy of transfinite subtractions. She used the Ping-Pong Ball Conundrum (P1, see Mamolo & Zazkis, 2008) with a variant (P2):

> P1 – Imagine and infinite set of ping-pong balls numbered 1, 2, 3…, and a very large barrel; you will embark on an experiment that will last for exactly 60 seconds. In the first 30s, you will place balls 1–10 into the barrel and then remove ball 1. In half the remaining time, you place balls 11–20 into the barrel, and remove ball 2. Next, in half the remaining time (and working more quickly), you place balls 21–30 into the barrel, and remove ball 3. You continue this task ad infinitum. At the end of the 60s, how many balls remain in the barrel? (p. 169)
>
> P2 – Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of this experiment, how many balls remain in the barrel? (p. 170)

Mamolo studied the work of two subjects, a high-achieving fourth year Mathematics major who had received formal instruction on comparing infinite sets, and a university lecturer who taught prospective teachers mathematics and didactics (including comparing cardinalities of infinite sets). While the first participant was able to cope with P2, the second one could not, shifting his attention from describing cardinalities of sets to enumerating their elements, and reasoning informally rather than deductively, in what is described as "attempts to make use of properties of a process of *infinitely many finite entities* rather that make use of properties of an object of *one infinite entity*" (p. 175, emphasis in the original). The behavior of the two participants led Mamolo (2014) to identify two ways of 'acting on infinity': (1) through the use of coordinating sets with their cardinalities and using bijections between sets, and, (2) through de-encapsulation of the object of infinite set to extend properties of finite cardinals to the transfinite case. Her results point to tensions between object, process, and de-encapsulation of an object that warrant further research, which could shed light on the uses of potential and actual infinity in questions dealing with calculus.

2.3. Embodied Cognition

The growing contribution to research of the theories of embodied cognition has already been acknowledged (e.g. Artigue et al., 2007), however, their use in advanced mathematics remains rare. Embodied cognition sees mathematical ideas as grounded in sensory-motor experience (Lakoff & Núñez, 2000) and considers the centrality of learners' gestures in grasping mathematical ideas. Two main conjectures of this approach are that mathematical abstractions grow to a large extent out of bodily activities (i.e. the latter are a part of conceptualizing processes), and that understanding and thinking are perceptuo-motor activities that are distributed across different areas of perception and motor action (Nemirovsky, 2003).

Before 2005, this approach was used in calculus in PME, for instance, by Maschietto (2004), who studied a key issue of the introduction of calculus – the global/local game – with the help of graphic-symbolic calculators. Her main hypothesis was that the zoom-controls of the calculator could support the production of gestures and metaphors that could help students shift from a global to a local point of view, this being seen as a major aspect of transition in calculus. To tackle this issue, she designed a didactical engineering[1] sensible to the principles of embodied cognition. Her paper explored the relationship between the physical features of the calculator (specifically, the different zooms) and the bodily activity involved. Her results seem to indicate that the exploration of several functions through the zooming process, aiming to shift between the local and the global, was supported by gestures and language, and that these remained in the students' repertoire even when the calculators were not available, which agrees with Hähkiöniemi's (2008) results concerning derivatives – except for the use of calculators – which we discuss in Section 3.3.

In 2005, Nemirovsky and Rasmussen (2005) – in connection with their work in Rasmussen, Nemirovsky, Olszervski, Dost and Johnson (2004) – also used this approach for an even more advanced mathematical topic: systems of differential equations. They constructed a physical tool called a 'water wheel,' based too on the premise of the "rich connections between kinesthetic activity and how people qualitatively understand and interpret graphs of motion" (p. 9). Interested in the notion of transfer, they explored how prior kinesthetic experiences with a physical tool (the water wheel) can provide students with resources that can be generalized to work with symbolic equations (in their case, systems of differential equations). Their results show how the students' interaction with the water wheel helped them develop sensitivity to the roles of different variables in a system of differential equations, as well as to the connections among them. One important implication of this work concerns the known dissociation of symbolic and graphical aspects of calculus concepts for students. Nemirovsky and Rasmussen argue that a bodily interpretation, or feeling, of the meaning of many topics in calculus could help students relate the results of calculations to the motion they describe, and hence their graphic representation.

More recently, Swidan and Yerushalmy (2013) also used this approach, taking into account elements of the objectification theory (which considers learning to be a process of becoming aware of the knowledge that exists in the culture, Radford, 2003), and the important role of accumulation in building the notion of integral (see Thompson & Silverman, 2008). In the three cases explored in this section, evidence shows a strong bodily connection with the tools used, becoming a 'bridge' between abstract mathematical concepts and students. However, the lack of studies on this approach at advanced levels calls for caution, and more research needs to be developed to better understand the knowledge that students build through this type of activity (Artigue et al., 2007, p. 1024). In any case, it seems that the mediations of the teacher in this kind of activity play a crucial role, particularly in helping students associate their developed knowledge with the targeted institutional forms of knowledge.

2.4. Other Approaches

Other approaches have been used in research on the topics of functions and calculus. For instance, the conceptual change approach, which postulates the necessity of going through intermediate states before gaining an understanding of mathematical notions. Using this approach, Vamvakoussi, Christou and Van Dooren (2010) showed the main difficulties students have apprehending the density property in different sets of numbers (rational, irrational and real numbers), and particularly the strong impact of the nature of interval end points (e.g. natural, decimal, or rational numbers) on students' answers related to the amount of numbers in between; these difficulties may interfere with the learning of notions such as convergence or the epsilon-delta definition of limit. Their results agree with those of Pehkonen, Hannula, Maijala and Soro (2006), who developed a longitudinal study of students in grades 5 to 8, mapping the development in the understanding of the density of rational numbers.[2] To explicitly tackle the idea of discreteness, they implemented the *rubber line* metaphor (Vamvakoussi, Katsigiannis, & Vosniadou, 2009), focusing on the 'no successor' aspect of density, to help students transition through intermediate states of understanding, and showing different degrees of success depending on the students' grade level.

Another approach used by PME researchers to investigate notions related to functions and calculus is the development of models on the understanding of a given notion. Roh (2010a, 2010b) and Roh and Lee (2011) established a framework for understanding the ε-N definition of limit and constructed activities aimed at helping students grasp this definition, and, in particular, understand the role and order of the quantifiers ε and N. Their second activity (Roh & Lee, 2011) exploits the use of ε-N strips, and their results seem to indicate they help students develop a better understanding of the ε-N definition. The use of strips is not new, and in the early 80s, Robert (1983) showed the potential of such approaches through the use of didactical engineering, and, in particular, the effect of classroom interaction in the construction of the notions of limit and convergence. It is important to note that, although Vamvakoussi, Roh and their colleagues built cognitive models, a big part of their research is based on social interaction among students. However, this element is not central to their research and their analyses focus on the individual. We explore other approaches in Sections 3.4 and 4, where going beyond the individual becomes an important factor.

Finally, it bears mentioning that some works presented at PME conferences have used the Abstraction in Context (AiC) approach (Schwarz, Dreyfus, & Hershkowitz, 2009). This approach investigates processes of constructing knowledge by taking into account the need for a new construct as part of the process of abstraction. This approach follows three stages: (1) identifying the need for a new construct; (2) the emergence of the new construct; and (3) the consolidation of the new construct. This approach has been used, for instance, to investigate the process of constructing a definition – in this case, inflection point, which is a problematic object in calculus – (Gilboa, Kidron, & Dreyfus, 2013). AiC confirms once again that students often appeal to their concept image and not to the concept definition of mathematical objects, and demonstrates that students must realize the need for a definition. Also, Kouropatov and Dreyfus (2013) used the concept of accumulation to introduce the Fundamental Theorem of Calculus (FTC) – following the work of Thompson and Silverman (2008) – and showed how AiC is useful for identifying actions and constructs to study processes of knowledge construction. Their results point to the relevance of the notion of accumulation to help students construct processes of integration, and indicate that understanding of the FTC can be achieved based on the notions of accumulation and rate of change. Although the role of the interviewer seems to have a strong effect on results (and one might question the stability of the knowledge constructed by the students in their experiment), the effectiveness of identifying different levels of actions calls for more research using this approach. It is also worth noting that even though the focus of this approach is mostly cognitive, it has recently been linked with more social approaches, as we discuss in Section 4.3 on the networking of theories.

3. SEMIOTIC REPRESENTATIONS IN THE CLASSROOM: THE DEVELOPMENT OF THE CONCEPT OF FUNCTION

3.1. Use of Representations, Patterns and Variation

Over the last ten years, a large number of PME researchers have focused on the idea of generalization related to variation in primary school; this allows the subsequent use of this same approach in secondary school, employing patterns as a tool. In this approach, children in primary and secondary school are asked to find a general rule for a given pattern and to produce a semiotic representation to explain their reasoning. Through these processes, pupils eventually develop a way to figure out the form of a general term related to the pattern (usually, from a visual, natural language or numerical point of view, although in some cases, particularly in secondary education, an algebraic expression is asked). Using this approach and favoring whole class discussion in primary school, Dooley (2009) proposes to analyse patterns not just focusing on a single variable, but rather on the functional relationship between variables, showing that generalization and justification are closely aligned. Making use of a whole class discussion, it is legitimate to ask whether every single pupil retains the knowledge constructed in class. In her study with 10-year-old children, Warren (2006) identified different types of performances regarding patterns (p. 380):

1. No response;
2. Nonsense response;
3. Quantification of the growing rule in symbols (e.g. +3);
4. Quantification using specific examples (e.g. $2 \times 3 + 2$, $3 \times 3 + 2$);
5. Correct symbolic relationship using unknowns (e.g. $3 \times ? + 2$).

These categories clearly reflect that not all children are able to immediately develop processes of generalization. Related to this, Wilkie (2015) analyzed 102 7th-graders' performances in generalization activities with patterns, finding that 18.6% of the population used correspondence, 14.7% gave a rule using letters and only 2.9% expressed their results as an equation. These research findings seem to indicate that several steps must be taken to help young students successfully perform this type of activity. This agrees, on the one hand, with Radford (2010, 2011) who posits that engaging in early algebra thinking is not immediate or spontaneous, and highlights the idea that while early algebra could be promoted, it must take into consideration specific pedagogical conditions. On the other hand, Wilkie's findings also agree with Trigueros and Ursini (2008), who indicate that several steps must be taken to acquire the notion of variable as *unknown,* as a *general number,* or as a *functional relationship*. As we discuss in the next section, these steps are a necessary precursor to acquiring the concept of function.

3.2. The Use of Covariation between Variables, Modelling and Task-Design

The late 90s and the first decade of this century saw a flurry of research that produced important results concerning the construction of the concepts of covariation and function. The PME and PME-NA groups on representations studied the processes of visualization and conversion between representations, both with students and pre-service teachers (see, for instance, the *Journal of Mathematical Behavior* special issue, edited by Goldin and Janvier, 1998). The construction of the concept of function appeared to be more complex than expected with respect to students (Janvier, 1998) as well as high school teachers (Hitt, 1998). Both cases highlighted the importance of building the concept of function through conversion processes among representations (Duval, 1995). The discussions continued in PME-NA, which led Carlson (2002) to present a more accurate approach showing the importance of the subconcept of covariation between variables as a prelude to the concept of function, which is concretized with the Five Mental Actions of the Covariation Framework (p. 65):

- MA1. Coordinating one variable with changes in the other;
- MA2. Coordinating the direction of change of one variable with changes in the other;
- MA3. Coordinating the amount of change;
- MA4. Coordinating the average rate of change;
- MA5. Coordinating the instantaneous rate of change of the function.

This framework underscored the importance of modelling in the construction of the concept of function. For instance, Thompson (2008) addressed a major problem in learning mathematics related to the notion of meaning (as a coherent conceptual approach), and exemplified this notion with three contents: trigonometry, linear and, exponential functions. With respect to linear functions, he explained the importance of meaning in associating the notions of linear functions to rate of change, proportionality, and average speed, showing the value of modelling (see also Thompson and Carlson, in press). Furthermore, Musgrave and Thompson (2014) explored teachers' mathematical meanings as influenced by function notation, finding that teachers read function notation by stressing only the content to the right of the equal sign while neglecting the importance of covariation between variables. The ability to shift from a variational to a covariational type of reasoning seems to be far from evident, and some efforts have been made to help students with this process. Here we can cite the recent contribution of Johnson (2015), who developed activities to promote this shift among 9th grade students, using a dynamic computer applet and a task-design approach. The use of dynamic computer representations seems to be an interesting approach to help students grasp covariational relations.

Over the last few years, modelling in the learning of mathematics has taken on greater importance from primary to university levels (e.g. Blum, Galbraith, Henn, & Niss, 2007), and as said above, its use can help students grasp content related to functions. This is the case of the work presented by González-Martín, Hitt and Morasse (2008), which assigns importance to representations and modelling processes using a task-design approach and collaborative learning in a sociocultural setting. Their work shows that secondary students' thinking processes during modelling activities can promote covariational thinking about variables, allowing the notions of independent and dependent variable to emerge naturally. In their study, they introduced the notion of spontaneous representations (non-institutional representations) constructed by the students to tackle modelling activities, showing that these representations act as an important 'bridge' between the students' first attempts at tackling the activity and the institutional representations expected by the teacher and the school system. The work on modelling can provide a suitable environment to facilitate the evolution of these spontaneous representations in a special socio-cultural setting, called ACODESA (see also Hitt & González-Martín, 2015).

3.3. Transition from Mental Images to a Focus on Semiotic Representations and Visualization as a Semiotic Process Related to Functions and Calculus

As we said in the previous section, the PME working group on representations (Goldin & Janvier, 1998) and further research published by PME-NA (see Hitt, 2002) revealed another side of the learning coin, contrasting with the *concept image – concept definition* approach. In this perspective, external representations of mathematical objects are fundamental, because they permit the apprehension of mathematical concepts. Conversion processes between different representations therefore play an essential role in the construction of mathematical concepts, articulation among registers of representations becoming an essential part of the learning process (Duval, 1995, 1999). It is worth noting that targeted external representations are usually connected to pre-existing institutional representations, such as those found in textbooks or on computer screens, and shared by a collectivity. All the papers in this section clearly illustrate that this theoretical approach distances itself from the *concept image – concept definition* approach. The main goal is to understand the difficulties students experience when doing a treatment in the same register of representations, when converting from one representation in one register to another representation in a different register, and, of course, to know more about their learning and understanding. One of the main characteristics of this approach is that it relates visualization to a cognitive activity that is intrinsically semiotic (Duval, 1999; Presmeg, 2006a). Although its first versions placed a clear emphasis on the cognitive aspects of learning – and, as a consequence, on the individual – this led to a noticeable shift from investigation focused on mental images (or related constructs) constructed by an individual, to investigation focused on the conversion processes between representations, and, finally, on how the individual in a learning process constructs a mathematical concept throughout this activity (Duval, 2006; Presmeg, 2006b, 2008). Under this theoretical approach a new era of research emerged, exposing the learning problems that materialize when converting from one representation to another.

We discuss two specific examples concerning derivatives. To contribute to the debate on the differences and complementarity between visualization and analytic thinking, Aspinwall, Haciomeroglu and Presmeg (2008) constructed an instrument to better understand the thinking of calculus students, particularly with respect to derivatives. In their work, which aligns with PME contributions on visualization (Presmeg, 2006b), mathematical visualization encompasses "processes of creating or changing visual mental images, a characterization that includes the construction and interpretation of graphs" (p. 98). The instrument they constructed predicts individuals' preferences for visual or analytic thinking, showing that successful students use a combination of visualization and analysis, and that verbal-descriptive thinking helps sustain the use of visual and analytic thinking. Moreover, their work shows that visual and analytical processes are mutually dependent during mathematical problem solving, and that the verbal-descriptive component acts as a necessary link, being one of the most useful modes of internal processing, supporting visual and analytic processes.

Finally, Hähkiöniemi's (2008) research used aspects of embodied cognition (see Section 2.3) to investigate the meaningfulness and durability of students' knowledge. This study is related to the promotion of an articulation between the definition of the derivative of real functions and graphical representations of the function and the tangent of the curve in one point. The formal definition of the derivative was not addressed; rather, the study investigated descriptions of five 12th grade students who were assigned tasks of conversion between the definition of the derivative and a graphical representation of the situation (qualitatively analyzing the rate of change of functions from graphs), one year after receiving instruction. Regarding the meaning of the derivative, all the students referred to the slope of the tangent, the rate of change, and the differentiation, giving embodied meaning to the derivative and using gestures to describe it. The author suggests "it seems that the graphical and embodied elements of the derivative were experientially real for the students and gave meaning to the abstract mathematical concept" (p. 116). The visualization of the tangent also seemed to be a helpful tool for the students, assisting with the durability of knowledge and leading in many cases to the use of gestures.

These two last examples illustrate the potential of research on visualization and representations to make connections with other ways of expressing mathematics that are usually neglected by traditional practices: language and gesture. We come back to the potential of these approaches in Section 6.3.

3.4. Sociocultural Approaches to Teaching and Learning Covariation between Variables and Functions

As we discuss in Section 4, the last few years have seen the emergence of some (and the consolidation of many) institutional and sociocultural approaches in mathematics education. These approaches have also figured in work on semiotics and representations. The PME community quickly realized the importance of representations for the teaching and learning of mathematics, evidenced by the "Semiotics" discussion group organized by Sáenz-Ludlow and Presmeg held from 2001 to 2004. The work carried out in this group bore fruit, leading to a special issue of *Educational Studies in Mathematics* in 2006 (Sáenz-Ludlow & Presmeg, 2006) and the book, "Semiotics in mathematics education" edited by Radford, Schubring and Seeger (2008). In the introduction to this book, the authors point out that the theoretical approach of semiotics attempts to understand "the mathematical processes of thinking, symbolizing and communicating" (p. vii), adding:

> But semiotics is more than a contemplative gesture: in contemporary semiotic perspectives the notions of culture and cultural praxis receive a new interpretation—interpretation which extends to history as well—making semiotics a form of practical understanding and social action (Thibault, 1991). This is why it does not come as a surprise that semiotics is increasingly considered as a powerful research field capable of shedding some light on what have traditionally been understood as self-contained domains of enquiry. (p. vii)

This perspective represented a break with the cognitive approach. Research studies did not focus exclusively on the individual, and communication processes were ascribed more power. The use of semiotics in the processes of signification (in a construction of the sign and concepts) has been in ascendancy as a theoretical approach in PME over the last ten years, and in these processes, communication is the main ingredient in a sociocultural setting, as illustrated by González-Martín et al. (2008), discussed in Section 3.2.

Communication and its combination with artefacts (considered broadly and not restricted to technology) are paramount in the theory of semiotic mediation (TSM). Mariotti (2012), in considering a collaborative setting where communication is a main element, argues that:

> The theoretical model of TSM offers a powerful frame for describing the use of an artefact in a teaching-learning context. Within this model the use of an artefact has a twofold nature: on the one hand it is directly used by the students as a means to accomplish a task; on the other hand it is indirectly used by the teacher as a means to achieve specific educational goals. (p. 36)

From this perspective, the planning of teaching activities where communication is important and the use of processes of co-construction of the sign, including artefacts, requires a careful task-design. This can allow teachers to reach their goals while giving students the opportunity to construct, in this process of signification, action schemas through which the artefact evolves into a tool. The conceptual and technological approach to learning mathematics was influenced by semiotics, communication, and the transformation of an artefact into a tool, leading to a better understanding of the technological approach to the teaching and learning of functions (e.g. Presmeg, 2008) and calculus (e.g. Lagrange & Artigue, 2009) in a technological environment, as we will see in the next subsection.

3.5. Semiotics and Technology, the Concept of Function and Modelling Processes

As there is a chapter in this *Handbook* concerning the use of technology, we will mention just a few works produced in the last ten years that reflect the rapid evolution of research on the problems of learning functions and calculus in a technological environment. At the end of the last century, advances in technology led many countries to adopt a high school syllabus that promoted the teaching of functions and calculus using three representations – numeric, graphic and algebraic (e.g. Schwarz, Dreyfus, & Brukheimer, 1990) – and some conversion activities appeared in calculus textbooks. However, conversion is not easy, even when using technology. Following the theory of reification (Sfard & Linchevski, 1994) for the case of functions, Campos, Guisti and Nogueira de Lima (2008) showed how secondary school teachers could not shift from the interiorization and condensation phases to the reification phase in a computational environment, when confronted with tasks about conversion among representations.

Such studies illustrated once again that conversion among representations (from a cognitive perspective) is not as easy as was previously thought, even using technology. A shift was made, not only with respect to the cognitive approach using technology, but also in studying the role of communication in the process of knowledge construction. This shift led to new studies on the processes of instrumentation and instrumentalisation when dealing with artefacts and semiotic mediation (Arzarello & Paola, 2008; Hegedus & Moreno-Armella, 2008; Mariotti, 2012), promoting a better understanding of the concepts of variation, covariation and function.

With regard to modelling processes, technology reinforced pupils' possibilities to reflect on covariation between variables and the construction of functions in a dynamic approach; for example, using MathWorlds (e.g. Rojano & Perrusquía, 2007) and video-clips (e.g. Naftaliev & Yerushalmy, 2009). These environments allow the use of different technological perspectives, thereby providing more opportunities to implement modelling processes than in the past (see also Arzarello, Robutti, & Carante, 2015).

As discussed in Section 3.2, modelling processes are gaining importance in research, and technology expands the possibilities for new approaches in the classroom. The connections between modelling and technology were addressed in PME39, during the plenary lecture given by English (2015). This lecture presented the STEM (Science, technology, engineering, and mathematics) project related to the unification of several scientific branches to tackle common goals; it also discussed the importance of a STEM perspective in education and how this project is changing syllabuses and curricula in countries such as the USA and Australia. The fact that curricula need to emphasize more data analysis from modelling processes and functional approaches was stressed, as well as variation and covariation processes; the use of technology may pave the way for major advances in these areas.

4. OTHER APPROACHES

As mentioned in the introduction, Harel et al. (2006), in their discussion on future research, noted the predominance of cognitive approaches in research, stating that "It would be enlightening to incorporate social and cultural constructs […] offered by PME scholars, into AMT studies" (p. 162). Section 3.4 outlined how these constructs have been exploited through the use of semiotic approaches. In fact, the emergence and consolidation of such approaches has now affected all levels of education, their impact at the tertiary level being more recent, as shown in a recent *Research in Mathematics Education* special issue in which Nardi, Biza et al. (2014) stated: "we see the emergence of institutional, sociocultural and discursive approaches to research in [University Mathematics Education] as a milestone" (p. 91). In this section, we discuss some of these approaches and how they have enriched our understanding of the processes of teaching and learning of functions and calculus.

4.1. Institutional Approaches

In this section, we consider the contribution of the Anthropological Theory of Didactics (ATD, Chevallard, 1999) to the study of processes related to the teaching and learning of calculus. The use of ATD has developed quickly in recent years and has shown its potential to deepen the study of processes of teaching and learning from an institutional point of view. Its use at the tertiary level has grown considerably (Winsløw, Barquero, De Vleeschouwer, & Hardi, 2014) and the number of CERME conference participants applying it at the university level is considerable (Nardi, Biza et al., 2014). Paradoxically, its presence in PME regarding the teaching and learning of functions and calculus is still scarce.

Like other approaches in this section, ATD puts forward the view that mathematical objects are not absolute but emerge from human practices. A fundamental notion is that of *institution*, which is broadly defined as a social organization that allows and imposes on its *subjects* (every person who occupies any of the possible positions within the institution), the development of *ways of doing and of thinking proper to itself* (Chevallard, 1989, pp. 213–214). Therefore, regarding mathematical objects, institutions develop sets of rules that define what it means to 'know' these objects, thus determining their *institutional relationship* with mathematical objects, i.e., the *ideal* relationship that their subjects should have regarding these objects. Subjects also have a *personal relationship* with any object, as a product of all the interactions they can have with these objects through contact with them as they are presented in different institutions. Institutional relationships have a strong effect on personal relationships, and the study of learning processes requires an examination of institutional practices.

This effect is illustrated in González-Martín's (2013, 2014) research. This study on how pre-university textbooks introduce infinite series of real numbers (González-Martín, Nardi & Biza, 2011), also examined teachers' practices, identifying several implicit *contract rules* that could potentially influence students in their learning:

- To solve given questions about series, the latter's definition is not necessary.
- Applications of series, inside or outside of mathematics, are not important.
- The notion of convergence can be reduced to the application of convergence criteria.
- To solve given questions about series, the use of visualization (or any visual representation of series) is not necessary.

The results of this study, which examined a group of 32 pre-university students studying under three different teachers, showed the effect of institutional organizations (in this case, through textbooks) and teachers' practices on their students' learning. If existing *praxeologies* take for granted that visualization is developed in a natural, spontaneous way, and if they address issues related to convergence and its meaning solely through the application of convergence criteria, it is no surprise that students do not develop any tools to tackle questions requiring the development of visual abilities, nor do they develop an interpretation of what convergence really is, calling instead on intuitions or using the potential infinity (see Section 2.2).

ATD has also been used in PME, as well as outside PME, to investigate the transition from secondary to tertiary studies, highlighting the impact of institutional choices (many of which are guided by societal choices) on how content is organized and what students can learn. For instance, Alves Dias, Artigue, Jahn and Campos (2010) investigated the kind of tasks associated with functions in the selective evaluations that serve as gateways to the tertiary level in Brazil and France. Their analysis shows that in these evaluations, functions belong to different *habitats*: algebra in Brazil, and analysis in France. The type of tasks that put functions into play in both contexts is considerably different, thereby leading students to develop different skills and, consequently, different *personal relationships* with functions. Given the presence of ATD on the international scene and its potential to illuminate the effects of institutional choices (at several levels) on students' learning, its negligible impact on the PME community is surprising. It is also worth noting that this theory is connected to the Theory of Didactical Situations and instrumental and documentational approaches, which have also had little impact in the PME community over the last ten years. However, these approaches and their combination with ATD have proven useful for studying teaching and learning phenomena concerning calculus, both at the secondary and the tertiary levels (see González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014; Gueudet, Buteau, Mesa, & Misfeldt, 2014; Winsløw et al., 2014).

4.2. Commognition

The commognitive framework, which emerged in recent years and consolidated with the publication of Sfard's book (2008),[3] appeared very early in PME. This framework stresses the close relationship between thinking and communicating, to the point that "Thinking is an individualized version of (interpersonal) communicating" (p. 81), and sees learning as a change in ways of communicating. It identifies four distinctive features of mathematical discourses, analyzing how they change over time: word use, visual mediators, routines, and narratives. To illustrate this principle of learning, which can be seen as a change in discourse, we mention the work of Kim, Sfard and Ferrini-Mundy (2005), who analyzed students' discourse concerning infinity and limits, going beyond other works focused on misconceptions and cognitive obstacles. They investigated two groups of students as they aged (Korean and US students), comparing the characteristics and evolution of their discourse concerning limits and infinity. Their results show that the fact the word *infinite* appears in the English language before it is used mathematically – which is not the case in Korean – seems to influence the way students define and refer to infinity; this appeared to lead the US students to take the object-like character of infinity for granted, and the researchers concluded that colloquial discourse effectively seems to have an impact on mathematical discourse.

The commognitive approach was also used by Güçler (2011) to analyze the historical development of limits, identifying junctures that resulted in changes in the discourse on limits, and which may also be critical for students' learning. She highlights that the dynamic view, which holds an underlying assumption of continuous motion, dominated mathematicians' discourse until the 18th century, and that it was not until Cauchy (1789–1857) that the notion of limit was objectified. Although he realized the necessity of a theory of limits and an explicit definition of the concept, his definition still called for the metaphor of continuous motion. Weierstrass (1815–1897) and Dedekind (1831–1916) replaced Cauchy's kinematic approach with an algebraic-arithmetic approach: the metaphor of continuous motion was replaced with the metaphor of discreteness. Objectification led to the elimination of dynamic motion; however, in spite of the precision that the current formal definition of limits provides, it "wipes out all the intuitive tools with which to make sense of the concept" (p. 470). Data coming from a study with students (see also Güçler, 2013) seems to indicate that these junctures are critical for students: even when writing expressions such as $\lim_{x \to 4} f(x) = 2$, students will say "it is approaching two", rather than "the limit is equal to two". This word choice is seen as an indicator of students only endorsing the narrative 'limit is a process' and not objectifying limits as a number at the end of their instruction (p. 447). The metaphor of continuous motion is also present in some students, indicating that they did not attend to the instructor's shifts in word use and metarules in the contexts of the informal and formal definition.

This approach provides a lens through which to examine aspects of interactions in detail, and has also been used to study phenomena related to the transition from school to university mathematics (e.g. Nardi, Ryve, Stadler, & Viirman, 2014). Recent developments have adapted it to study teachers' knowledge in terms of discourse (Cooper, 2014) and this approach offers great potential to study "'the macro-level' of historically established mathematical discourse, the meso-level of local discourse practices jointly established by the teacher and students […] and the micro-level of individual students' developing mathematical discourses" (Cobb, 2009, cited by Nardi, Ryve et al., 2014, p. 196).

4.3. Networking of Theories

The comparison of theories has been a subject of interest for the PME community. For instance, at the beginning of the century, Boero et al. (2002) worked on comparing theories of abstraction and, in the last ten years, Presmeg (2006a) compared two theoretical frameworks: that of Duval (1995, 1999) on semiosis and noesis (related to the articulation among registers of representations) and the semiotic means of objectification of Radford (2002, 2003).

The work of Prediger, Bikner-Ahsbahs and Arzarello (2008), and later of Bikner-Ahsbahs and Prediger (2009), which examined the importance of focusing on the networking of theories, may have spurred the PME scientific committees' promotion of research along these lines (with two Research Forums in 2010 and 2014, see the introduction of this chapter) in order to unite, differentiate and strengthen different theoretical frameworks. According to Clark-Wilson et al. (2014), the aim of networking theories is to unite, differentiate and strengthen different theoretical frameworks to better explain learning phenomena. This brings us to the discussion undertaken by the PME forum organized by Bikner-Ahsbahs et al. (2010), who identified some conditions for an efficient networking: "the underlying principles have to be 'near enough' and […] the empirical load of a concept plays a crucial role if integrating is the aim" (p. 146). Furthermore, they offered a broad analysis of different ways in which theories can be networked (Figure 1):

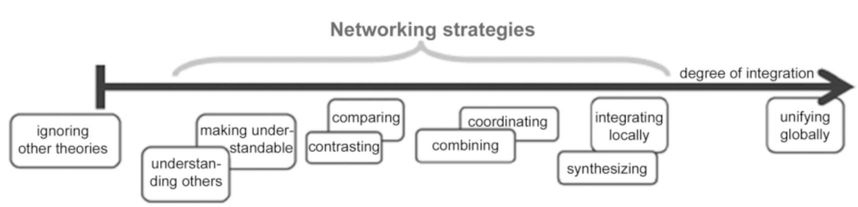

Figure 1. Networking strategies (Prediger et al., 2008, p. 170)

One of these ways is comparing and contrasting. Taking this into consideration, it is possible to make a critique of Presmeg's (2006a) discussion when comparing relationships amongst signs related to the theoretical approaches of Duval (1995, 1999) and Radford (2002, 2003). As explained in Section 3.3, Duval's approach mainly concerns institutional representations and the conversion processes among them that serve students in the construction of an articulation among representations related to a mathematical object. Meanwhile, according to Radford (see Section 3.4), in the process of objectification, elements are considered that are not necessarily institutional, but are an integrant part of the institution, such as culture, communication and representations. That is, the manipulation of objects, drawings, gestures, marks, and the use of linguistic categories, analogies, metaphors, etc., are key components of mathematical communication in the process of objectification. Presmeg's comparison analyzes excerpts from both theoretical approaches, albeit in a context related to institutional representations (concerning trigonometric functions, graphic representations, and processes of visualization). It is therefore not surprising that Duval's approach is seen as "paramount" (p. 32), but we can criticize the fact that the analyzed data are not 'near enough' Radford's theoretical approach, and the design of the investigation seems rather to have been conceived to be analyzed from a cognitive approach and not from a semiotic process of signification. This contrasts with the principles of the networking of theories, as highlighted by Bikner-Ahsbahs et al. (2010), in which research must be designed to allow the integration of different approaches, giving both theories the chance to interact (Figure 1).

One example of networking of theories, implying the use of technology, is provided by Clark-Wilson et al. (2014). They contrast theoretical approaches and constructs that have been frequently used to examine students' performances and have recently been applied to teachers – for instance, *instrumental genesis* and *instrumental orchestration*. New constructs emerged from this contrast to analyze students' errors or strategies when solving problems (not restricted to functions and calculus), such as *critical incidents*, *hiccups*, and the notions of *instrumental distance* and *double instrumental genesis*. A similar evolution took place with the notion of *Pedagogical Content Knowledge* (PCK, Shulman, 1986), which led to the emergence of the *Pedagogical Technology Knowledge* (PTK, Thomas & Hong, 2005); the latter focuses on mathematics and employs the theoretical base of instrumental genesis.

Finally, we mention the work of Kidron, Bikner-Ahsbahs, Cramer, Dreyfus and Gilboa (2010), who networked the Interest-Dense Situations (IDS) and the Abstraction in Context (AiC, see Section 2.4) approaches in activities concerning real numbers to assist in "uncovering blind spots of their methodologies" (p. 169): IDS considers social interactions as a basis for learning mathematics, and AiC develops tools to investigate the construction of learning, with social interaction as part of the context. Their paper shows how each theory helped uncover or refine elements of the data analysis performed by the team using the other theory, noting that, for instance, "the social interaction analysis offered by IDS reveals important cognitive aspects" (p. 175).

4.4. Mathematics Education, Psychology and Neurosciences

We end this section by looking at the increasing importance placed on interdisciplinary research between mathematics education and psychology and neurosciences at the last PME meeting (PME39), which leads to a questioning of “how different methodologies currently used in cognitive neuroscience afford, and constrain, research design and potential findings/implications for maths education” (Tzur & Leikin, 2015, p. 115). We name two works that illustrate this interdisciplinarity in recent research in mathematics education. Lithner (2015) presented a descriptive study to show how the brain works when solving tasks related to an algorithmic reasoning and when solving tasks related to creative mathematically founded reasoning; among the results reported, brain imaging seems to indicate that students learning by creative reasoning could use their mental resources more economically in different tasks. This work also relates to that of Waisman, Leikin and Leikin (2015), who used tasks related to functions and proposed to measure “mathematical ability” through the identification of brain activity. Their results also show that different mathematical abilities reflect in different ways on ERPs, and that these differences are dependent on the level of insight imbedded in the task solution. These works introduce new dimensions for research in mathematics education and in some cases seem to confirm or contradict some of the beliefs held about students’ learning. Although the presence of functions and calculus in the examples briefly presented here is peripheral, we believe that new areas can be explored to improve our understanding of how this content is learned.

5. TOPICS DESERVING SPECIAL ATTENTION

In this section we discuss topics that have been the focus of growing research in recent years, especially concerning functions and calculus. Space limitations have forced us to zero in on just a few topics. This obliges us not to discuss, for instance, studies that have been developed in the last few years on the secondary-tertiary transition, particularly with different approaches;[4] one example of these studies is given in Section 4.1.

5.1. Teachers’ Knowledge and Practice

The theoretical proposition of Shulman (1986), concerning the amalgam of specific content knowledge and teaching knowledge, gave origin to what we know as PCK and to a field of research on teachers’ knowledge. Research now focuses on preservice teachers to a greater extent than in the past, particularly regarding content related to functions and calculus. In this sense, research has identified that a strong PCK concerning a mathematical topic (inverse functions, in the case of Bayazit & Gray, 2006) does not guarantee adequate teaching. This result calls for further research to identify factors orientating PCK towards effective practices, avoiding procedural approaches (such as the ones identified by Lucus, 2006, concerning the composition of functions).

Developing Shulman's (1986) work for the teaching and learning of mathematics, Ball and Bass (2000) introduced a new dimension for teachers' knowledge, the *Mathematical Knowledge for Teaching* (MKT). Using this construct, Seago and Goldsmith (2006) worked with teachers possessing a conventional and 'compressed knowledge' of linear functions; they showed that some activities demanding conceptual and 'unpacked understanding' presented in a seminar can help teachers increase their MKT and develop their teaching abilities. However, the use of professional development materials does not always produce a significant impact on teachers' MKT, as Seago, Carroll, Hanson and Schneider (2014) discuss in the case of linear functions. The MKT construct has also been used to explore lower secondary mathematics teachers' abilities concerning mathematical language (Wang, Hsieh, & Schmidt, 2012). Results unveil difficulties teachers may have concerning competences related to thinking and reasoning about mathematical language, as well as difficulties they may have choosing teaching activities that could cultivate their students' competences related to mathematical language.

PME has also been interested in teachers' beliefs (see the chapter *Research on mathematics-related affect* in this Handbook). However, research on calculus teachers' belief systems is still scarce, and work in this area may make a useful contribution to an emerging area of calculus research (Rasmussen, Marrongelle et al., 2014, p. 512). In Section 5.3, we refer to some of these works concerning calculus in tertiary education. Regarding calculus in high school, Erens and Eichler (2014) were interested in the structure of belief systems, which characterize teachers' instructional planning. Working from the perspective of beliefs and goals, their work identifies some relations of coordination (for instance, presenting calculus as process-oriented and application-oriented) and subordination (for instance, presenting calculus as application-oriented as a means to facilitate students' motivation). However, their work also identifies contradictions that may be due to personal factors (for instance, although they hold a formalist view of calculus, some teachers do not activate it to avoid creating difficulties for students) or to constraints from external factors (for instance, although an instrumentalist view could be peripheral for a teacher, s/he could activate this goal in order to help students pass national exams). These results support the idea that beliefs and goals depend not only on individual factors (as we also illustrate in Sections 5.3 and 5.5 concerning tertiary education), and identify the existence of inconsistencies, calling for further research. Other inconsistencies, this time regarding the use of visualization, have been identified by Biza, Nardi and Zachariades (2008): some teachers can explicitly accept a visual argument, while at the same time claiming the need to support and verify algebraically for the same statement. The possibility of holding erroneous images, combined with a tendency to support visual reasoning in some cases, could lead to negative effects from the use of visualization.

5.2. Teachers and Task-Design

Task-design is not new, and already at the beginning of the twentieth century, psychology studies on intelligence had developed specific tasks suitable for these studies (Brownell, 1942). The evolution of different theoretical frameworks in mathematics education entailed the production of tasks associated with these frameworks, the coherence between the task and the theoretical approach being a fundamental element of research design. For instance, considering the notion of epistemological obstacle (Brousseau, 1997) related to the concept of function, and to raise awareness of the conception that "functions are continuous and expressed by a single algebraic expression", one task that has proven efficient consists of asking participants to construct two examples of one real variable function such that for all $x, f(f(x)) = 1$ (Hitt, 1994).

In reflecting on ways to improve task-design activities and the efficacy of these tasks, the 2014 Research Forum on mathematical tasks (Clark, Strømskag, Johnson, Bikner-Ahsbahs, & Gardner, 2014) was structured around four key questions:

1. What are the possible functions of a mathematical task in different instructional settings and how do these functions prescribe the nature of student task participation?
2. What contingencies affect the effectiveness of a mathematical task as a tool for promoting student higher order thinking skills?
3. How might we best theorize and research the learning processes and outcomes arising from the instructional use of any mathematical task or sequence of tasks from the perspective of the student?
4. What differences exist in the degree of agency accorded to students in the completion of different mathematical tasks and with what consequences? (p. 119–120).

These questions can provide a suitable basis for the discussion on the elaboration of efficient tasks, taking into account the mathematical content, the student and the teacher. Furthermore, a reflection on and transformation of the tasks used may have the potential to change teaching and learning approaches to functions and calculus, as highlighted, for instance, by English (2015).

Finally, regarding the use of technology, in a work related to functions and modelling processes and following the documentational approach, Psycharis and Kalogeria (2013) highlighted the existence of three factors that may hinder teachers' construction of tasks and materials: (1) Teachers' difficulties in developing their own teaching material, (2) Teachers' difficulties in learning the affordances of the software tools and integrating them into activities with added educational value (e.g. microworlds, scenarios, worksheets), (3) Teachers' knowledge, pedagogical conceptions and experiences regarding the everyday practice of teachers. These results highlight the need for additional research on task-design activities in technological environments (Clark-Wilson et al., 2014), particularly regarding functions, calculus and modelling.

5.3. Teachers, Teaching Practices and Their Effects in Tertiary Education

In the last PME *Handbook*, Harel et al. (2006) acknowledged the growing body of research on mathematicians' writing, problem solving, and proving (p. 160). Although they also mentioned the growing research on mathematicians' teaching practices, there was not enough space in their chapter to develop this point. More recently, with respect to research on the teaching and learning of calculus, Rasmussen, Marrongelle et al. (2014) identified work on teacher knowledge, beliefs, and practices as one of the most recent developments in the field. Furthermore, as we noted in the introduction, covering the topics of functions and calculus in a single chapter would lead us to encounter some gaps. In this section, we address an important one. PME researchers have focused a good deal on teachers' beliefs, as the last PME *Handbook* shows (Leder & Forgasz, 2006). However, although primary and secondary teachers' training, beliefs and practices have been a subject of research for many years, this is not the case for the tertiary level, and there is still little research on how lecturers actually teach at the university level (Speer, Smith, & Horvath, 2010; Weber, 2004). In this section, we give an overview of some important results obtained in PME regarding these issues.

Regarding undergraduate teaching practices relative to calculus content, we cite the works of Rowland (2009) and Petropoulou, Potari and Zachariades (2011), related to university teachers' training, beliefs, decisions, and practices, the former in connection to the Fundamental Theorem of Calculus and the latter in connection to general calculus content, with data from sequences. In both cases, the authors developed a single-case study concerning a particular subject: a lecturer with a background in mathematics and mathematics education. In both cases, different uses of examples are among the main strategies aimed at constructing mathematical meaning. These works indicate that the instructors' practices appear to be based on the professional knowledge they develop (or craft), their beliefs and vision concerning the nature of mathematics itself, the purposes of teaching and learning mathematics, and the ways in which mathematics is most effectively taught and learned, as well as their own experience. These results agree with those of Weber (2004), who noted that beliefs about mathematics as a research mathematician, and beliefs about students and teaching as an experienced mathematics lecturer, were the main influences on a lecturer's practice. We see here differences in research concerning primary and secondary teachers, who usually receive teacher training that influences their belief system. This shortcoming in the case of tertiary education may place higher importance on the need to hold discussions within teams of lecturers, as proposed by Rowland (2009). This is justified by the fact that if an instructor's beliefs are his/her own, and if other lecturers teach differently and do not articulate similar beliefs, it is unclear which version of 'being mathematical' students might construct. Rowland therefore recommends that the entire team of lecturers meet to discuss different ways of teaching, epistemological assumptions, students' role in lectures, and so on, in order to establish sociomathematical norms.

Having examined university teachers' training, beliefs and practices regarding calculus, it is worthwhile to question the effects of these elements on students. So far, experience and the literature suggest that there is much research to conduct, because Calculus courses prompt many students to change careers: this issue is addressed by Rasmussen and Ellis (2013) who sought to better characterize the profile of students who choose not to continue with Calculus and uncover the main reasons why students switch out of Calculus courses. Their data comes from an in-depth national survey with over 14,000 students responding to at least one of their instruments. One important result showed that 12.5% of STEM-intending students in their sample had planned to take Calculus II at the beginning of their Calculus I course, but decided not to do so upon completing the first course. This group, called switchers, displays a number of characteristics: the percentage of female switchers is significantly higher in comparison to males (20% and 11%), switcher rates differ significantly depending on career choice (engineers having the lowest rate), and, the mathematical background of switchers and students who go on to Calculus II was statistically similar at the start of their post-secondary education. This last result is quite significant and refutes a preconceived idea: their data indicate that students who abandon their STEM ambitions are not weaker when they enter university than those who continue on the STEM path. Regarding the reasons for changing majors, 31.4% of students in this situation acknowledged that their experience with Calculus I made them decide not to take Calculus II. Their study also points to teaching practices as influencing students' experiences and choices, and indeed, the researchers' subsequent paper (Rasmussen, Ellis, Zazkis, & Bressoud, 2014) shows that one characteristic of successful calculus programs is the existence of substantive graduate teaching assistant (GTA) training programs, varying from "a weeklong training prior to the semester together with follow up work during the semester to a semester course taken prior to teaching" (p. 37). This training of GTAs, who are seen as future lecturers, was the topic of a paper by Ellis (2014), who sees it as a way to make up for the lack of pedagogical or didactical training for university teachers. The need pointed out by Rowland to hold group discussions on practices, views and beliefs can be addressed somewhat preemptively through professional development programs. Ellis (2014, p. 13) underlines the important role played by mentoring with respect to K-12 professional development, and her results seem to show that mentoring augments the training of GTAs, suggesting "a relationship between GTA professional development and student success that needs to be further examined" (p. 14). Given the large number of Calculus students around the world, research concerning teachers' practices and training is needed, as "there is great need to better understand the factors that contribute to student decisions to stay in or to leave a STEM major" (Rasmussen, Marrongelle et al., 2014, p. 512).

5.4. Analysis of Textbooks

Sträßer (2009) addressed the important role of artefacts (such as textbooks, computers, tools, etc.) in the teaching and learning of mathematics, acknowledging that textbooks "have always played a major role in mathematics education" (p. 70). This being the case, it is surprising that research has not placed much focus on the analysis of textbooks until recently, and research concerning high school and university topics – including calculus – is still scarce.

Of all the topics included under the label 'Calculus', a wide variety have received little research attention, especially the most advanced topics. However, of the topics that students encounter first, continuity is one that has been studied the least by researchers, often appearing implicitly in studies on limits or functions. Taking this into account, Giraldo, González-Martín and Santos (2009) analyzed how continuity of single-valued real functions of one real variable is presented in undergraduate textbooks used in pre-service mathematics teachers' calculus courses. Their main results indicate that the notion of continuity is mostly introduced using the notion of limit, in many cases using intuitive images that call for the image of 'a curve drawn without removing the pencil from the paper.' This could have consequences for teachers' understanding of the notion of continuity, which has already been signaled as problematic (Hitt, 1994; Mastorides & Zachariades, 2004).

Regarding the concept of infinite series, Nardi, Biza and González-Martín (2009) analyzed a set of university textbooks used in the UK (and which are also used in many other countries). The analysis focused mainly on the use of visual representations, tasks, and examples to introduce series, finding that this concept is mostly introduced in a decontextualized way, with few graphical representations and even fewer applications and references to the concept's significance and relevance. The results agree with the analysis of a larger sample of pre-university textbooks used in Quebec (González-Martín, Nardi et al., 2011). In addition, the effects of these textbooks and their use by teachers on students' learning of series have been analyzed using ATD (see Section 4.1). Finally, regarding secondary education, González-Martín, Giraldo and Souto (2011, 2013), analyzed how real and irrational numbers are introduced by textbooks, using ATD. Their results revealed a similar situation, as well as a lack of justification for the need of these 'new' numbers. Moreover, although studies have identified difficulties in learning the topics addressed in this section, this research is often neglected, which raises the question of why research on calculus is not having a greater impact on practices and resources.

5.5. Calculus as Service Mathematics

One of the areas of tertiary mathematics research that have developed rather quickly in the last few years is the study of educational processes for audiences enrolled in faculties other than mathematics (Artigue et al., 2007). The landscape has changed a great deal: technology has altered the skills and knowledge required for many of these professions, the number of students enrolled in these faculties has dramatically increased, the background (particularly concerning mathematics) of students entering these faculties has changed, and societal expectations have also grown. The number of papers published in the last few years on the teaching and learning of mathematics as a service course has increased, and the field of engineering has attracted special attention. However, efforts are still needed to better understand the phenomena at play, and the claim made by Kent and Noss (2001, p. 395) 15 years ago seems to be still valid, namely that "The teaching of service mathematics remains relatively unexplored, and many of its fundamental assumptions (What is its purpose? What are the fundamental objects and relationships of study?) remain unexamined."

Earlier in this chapter (Section 2.1), we mentioned the work by Bingolbali and Monaghan (2008), indicating the differences in students' acquisition of the derivative according to their department of affiliation. This work is closely related to their paper presented at PME30 (Bingolbali et al., 2006) where they analyzed the views and practices held by lecturers teaching Calculus courses in different departments. This research collected data from six different lecturers with experience teaching mathematics or physics as a service subject. Their results indicate that lecturers behave in different ways according to their audience: they privilege different aspects of mathematics, place different questions on examinations, and use different textbooks. For instance, lecturers emphasized different aspects of topics based on the type of student: concerning derivatives, aspects related to rate of change were highlighted for engineering students, whereas aspects related to tangents were highlighted for mathematics students. The role and place given to proof also varied according to the audience. But not every decision is the result of personal choice, and the lecturers' perception of the department's priorities also seems to play an important role. The results of this research have at least two implications. First, the connections between this work and the results presented by Bingolbali and Monaghan (2008) seem to imply that students' learning is strongly conditioned by their lecturers' choices. This aligns with many results obtained using ATD, such as those already noted by González-Martín (2013, 2014) in Section 4.1. Secondly, lecturers' choices are influenced by the fact that while they each have their own background, they see themselves as members of an institution (department or faculty), although these institution-driven choices can sometimes conflict with their own background and views, as illustrated by Hernandes Gomes and González-Martín (2015).

This type of investigation calls for more research to better understand the interplay of elements in contexts such as the teaching of calculus to engineering students. For instance, the choice of textbooks – and resources in general – may have implications for students' learning, as well as the training of the lecturers, as acknowledged by research following the documentational approach (Gueudet et al., 2014; Gueudet & Trouche, 2009). Also, while some choices seem to be made to adapt lectures to a specific audience, the question remains whether these changes lead to a 'different' calculus course, or whether the same key characteristics endure. In this sense, Barquero, Bosch and Gascón (2011) identified a *dominant epistemology* in university teaching that has an impact on different mathematics teaching practices. They call this epistemology 'applicationism' and its main characteristics are: (1) mathematics is independent of other disciplines; (2) basic mathematical tools are common to all scientists; (3) the organization of mathematics content follows the logic of mathematical models instead of being built up by considering modelling problems that arise in different disciplines; (4) applications always come after basic mathematical training; (5) extra-mathematical systems could be taught without any reference to mathematical models (pp. 1940–1941). Whether these principles can be found (and to what extent) in the practices of lecturers who 'adapt' content to their students' profile also remains an open question for research.

6. FUTURE RESEARCH

We finish this chapter by examining issues that, in our opinion, warrant further research. Once more, we focus on just a few issues, although we are aware that several require more investigation (we have noted some of these in previous sections of this chapter).

One important issue already identified by researchers (Artigue, 2001; Rasmussen, Marrongelle et al., 2014), is the fact that while research in calculus has concentrated on a few topics (namely functions, limits, derivatives, and integrals), advanced topics remain relatively unexplored. For instance, the proportion of papers focusing on differential equations is quite small compared with functions and derivatives, and papers on multivariate calculus are few in number. Issues concerning transition in a broad sense (for instance, from high school to university, from calculus to analysis, from calculus to algebra, etc.) also deserve further research through a variety of lenses, especially institutional and sociocultural perspectives. And, as we noted in Section 5.5, there is a great need to investigate the relationships between calculus and client disciplines in terms of practices, what should be taught, and what students are learning, to cite just a few. In particular, "Post-secondary educational research has from this point of view a specific epistemological role to play in educational research thanks to its proximity with the professional world of mathematics. The increasing importance taken in post-secondary mathematics education by service courses faces us with the necessity of taking a wider perspective" (Artigue et al., 2007, p. 1044). This leads us, finally, to underline the importance of coordinating efforts to make various advances in research concerning functions and calculus available to the broader practitioner and policymaking communities. Furthermore, systematic research on teaching practices concerning calculus content, particularly at the tertiary level, is needed.

In the following paragraphs, we address other issues that warrant further research.

6.1. Networking of Theories

As noted in Section 4.3, for more than 15 years the PME community has been interested in the comparison and/or networking of theories, and theoretical advances are shown in this line under different perspectives. Bikner-Ahsbahs et al. (2010) offer some methodological elements to consider in this process of networking and they draw on research projects, such as TELMA and Re-Math, identifying the emergence of cross-experimentation methodology as a key element. As noted by Bikner-Ahsbahs and Prediger (2009), to overcome some of the limitations that arise from using only psychological approaches, the networking of theories appears to be a promising way of doing research. Furthermore, considering technology (and, in particular, its use by client courses), Rasmussen, Marrongelle, et al. (2014) also propose the networking of different theoretical perspectives and their respective findings as a promising way forward. Although PME has considered the networking of theories, more systematic research is needed. As we have highlighted throughout this chapter, many issues related to the teaching and learning of functions and calculus interact (teachers' training or the lack thereof, practices, beliefs, materials and resources, departments, etc.), and the networking of theories looks to be a promising way of taking into account several of these issues at the same time.

6.2. Task-Design

Rasmussen, Marrongelle et al. (2014) recently stated that "It is noteworthy that the research in calculus learning and teaching has not capitalized on advances in design research […] to further link theories of learning with theories of instructional design" (p. 509). Although not necessarily connected with calculus, task-design has been the focus of some interest recently, as indicated by the organization of topic study groups focusing on it at ICME conferences, resulting in an ICMI study on task-design (Watson & Ohtani, 2015). This interest can be explained, according to Clark-Wilson et al. (2014), by the great difficulty teachers face in building tasks and applying them in the classroom. One strategy for successful task-design consists of proposing sequences of enchained tasks covering broad mathematical topics (Artigue, 2002), preferably aiming at producing emergent models, necessary to symbolize and mathematize gradually (Gravemeijer, 2007). As discussed in Section 3.2, Hitt and González-Martín (2015) proposed ways (a method) to tackle these issues in the classroom at the secondary level: in a sociocultural approach, the construction of a sequence of activities promoting diversified thinking and the emergence of non-institutional representations. Combining individual work, teamwork and whole class debate, these tasks helped pupils co-construct the subconcept of covariation between variables, necessary to the construction of functions. This is just one example of task-design, but certainly more effort must be made to transfer results on students' learning of calculus into design research. At the undergraduate level, collaboration between mathematicians and mathematics education researchers seems to be a promising avenue for future research.

Finally, concerning technology, there is also a need (both for pre-university and university students) to relate theoretical work with computer activities, as well as the need to create sequences of activities around a given topic.

6.3. Semiotics

In looking at research developed over the last 10 years in the PME community concerning argumentative discourse, two types of components can be highlighted:

- A component that seeks to convince, to win the support of the other (called the seduce component by some authors).
- A component that aims to explain, based on reasoning.

These two components can be found in different works of the PME community: the first is seen more in research on collaborative learning when communicating mathematical ideas in primary and secondary school (more related to conjecturing and convincing in peer interaction), and the second is found mostly in research on university-level contexts, where proof is a requirement even if a previous conjecture has been made. Both components are always present in research on communication with others, but in the construction of mathematical thinking, instruction usually promotes the gradual diminution of the first component and an incremental increase in the second one. Some authors following a semiotic approach include gestures in the argumentative discourse – as we discussed in Section 3.3, – which is an interesting approach. As Sfard (2008, p. 94) puts it: "while defining thinking as individualized communication, I was careful to stress that all forms of communication need to be considered, not just verbal."

Many results related to semiotics do not adequately clarify the relationship between the process of resolution of the mathematical task (not exclusively with mathematical content) and the role the students or teachers assign to gestures to convince others. The importance of spoken language is not always highlighted either, as we discussed in Section 3.3. We believe that Sections 2.3 and 3.4 have emphasized the important role that gestures and signs (other than mathematical symbols) can play in the learning of mathematics, and certainly more research is needed to better understand how semiotics, in a sociocultural setting, can help in the understanding of learning processes concerning generalization, functions, calculus and modelling problems in context.

NOTES

[1] For a summarized overview of the Theory of Didactic Situations and the main principles of didactical engineering, particularly at the undergraduate level, see González-Martín, Bloch, Durand-Guerrier & Maschietto (2014).

[2] These authors also studied students' self-confidence, showing that students who gave better answers were also usually more confident in their answers, whereas this was not the case when ends were given in the form of decimal numbers (0.8 and 1.1 in their case). Issues surrounding affection and self-confidence are discussed in the chapter *Research on mathematics-related affect* in this Handbook.

[3] For a summarized overview of the main tenets of this framework, and some examples of its use at the undergraduate level, particularly in calculus, see Nardi, Ryve, Stadler & Viirman (2014).

[4] For instance, Rasmussen, Marrongelle et al. (2014) state that "the secondary vs. tertiary differences might be greater when viewed through a pedagogical or cultural lens, including institutional constraints and affordances. This is an interesting area of research" (p. 507).

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2. ON NUMBERS: CONCEPTS, OPERATIONS, AND STRUCTURE

All is number.
(The motto of the Pythagorean school)

Mathematics as well as mathematics education research has long progressed beyond the study of number. Nevertheless, numbers and understanding numbers by learners, continue to fascinate researchers and bring new insights about these fundamental notions of mathematics.

In progressing from one to infinity (and beyond) we follow our own curiosities as they led us through various domains of the research landscape. Highlighted throughout the research surveyed in this chapter is an emphasis on structures – be they number structures, structures for task design, or structures of mathematical thinking. We embark on our journey, with a few stops along the way (and with apologies to authors that were not included).

ON NATURAL NUMBERS: COUNTING, ORDERING, AND OPERATING

Some will tell you that three is the magic number. Others will tell you that there are eight days a week for love. Forty-two is the answer to everything; ninety-nine is the great one; six degrees of separation; and one ring to rule them all. Numbers! Naturally, we are drawn to them, and in this section we exemplify several themes that have attracted researchers and the PME community.

Following the "usual" trajectory of school learning, we start with a review of research on early number concepts, and we then devote several subsections to arithmetic operations with numbers, as those occupy a significant part of school curricula. Examining relationships amongst number concepts and operations lead us to structure, and in our last subsection we highlight learners' and teachers' attention to number structures found in elementary number theory.

Early Number Concepts and Number Sense

The pop culture references to numbers listed above are hardly mathematical, yet pop culture is one of the early contexts in which children learn to interpret and make sense of numbers. For young children, early number contexts tend to focus on

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counting, ordering, and comparing. For instance, Coles (2014) extended discussion on cardinality vs. ordinality of numbers in children's initial exposure to numbers, asking: which approach is primary in the development of number concepts? Based on neuro-science evidence that perceptions of ordinality and cardinality operate in different parts of the brain, Coles presented results from, and advocated for, an instructional approach that highlights ordinality. Such an approach links numbers with each other (rather than with collections of objects) and therefore supports the development of mathematical structure. Askew, Abdulhamid and Mathews (2014) highlighted the role of embodied cognition in learning early counting. Their focus on forward and backward counting implicitly emphasized ordinality of numbers and connected it to the embodied metaphors in teachers' gestures that support learning opportunities. Embodiment metaphors have also been linked to the development of mental computation, and we discuss this in a subsequent section.

The role of ordinality in children's early number learning may be linked to their development and use of different counting strategies, where the ability to associate numbers with each other can lend itself to more efficient approaches. Lampert and Tzur (2009) revisited children's transition from counting-all to counting-on and provided an explanation for what supports the transition from the former to the latter strategy. They noted that counting-on requires a coordination of two invariants – extending a count and keeping track of the second quantity. To that end, they suggested that the regress of children from counting-on to counting-all strategies "may be rooted in the stage at which a learner has formed each or both invariants, not mainly or entirely in the structure of a task" (p. 478). Strategies for composing numbers have also attracted research interest for early number concepts. Cayton and Brizuela (2008) investigated Grade 2 children's abilities to write multidigit numbers, which were presented to them either orally or by tokens in which different colours corresponded to different place values. The results demonstrated a large variety of notational strategies used in composing numbers, and that by the end of the second grade some children still had difficulty representing numbers, not yet grasping the concept of a number system.

As a learner develops, the need for an extended sense of number also develops. Similarities and differences between Israeli and Korean learners' number sense for whole numbers, fractions and decimals were discussed by Markovits and Pang (2007). They noted a cultural difference in learners' preferences regarding the use of exact calculations over what they described as "more of number sense considerations" (p. 247), such as benchmark comparisons or estimation. Pittalis, Pitta-Pantazi and Christou (2013) proposed a theoretical model that extends further the notion of number sense to include what they characterized as algebraic arithmetic. In particular, this includes recognition of number patterns and understanding of number equations. Pittalis et al. (2013) observed over a longitudinal study that the development of algebraic arithmetic may require a stabilization period after a growth period, yet it also had the largest latent slope compared to growth in early number sense and conventional arithmetic.

Number Line

Number line – as a metaphorical representation of numbers – was used in a several studies that focused on students' acquisition of number concepts. In schools, number line can be treated as a separate part of the curriculum, or as a model for teaching ordering numbers and number operations (Ernest, 1985). Bruno and Cabrera (2006) asserted that a number line is a "common representation for all number systems, and is a connecting thread in numerical knowledge" (p. 250). In their review of Spanish textbooks, Bruno and Cabrera noted that number lines are used mainly when a new number system (whole, integer, rational or real) is introduced. However, when introducing or presenting number operations, the use of number lines was infrequent.

Gray and Doritou (2008) described number line as "a sophisticated mathematical representation characterized as a metaphor of the number system" (p. 97). They noted that while number line is used as an essential aid in supporting learning and is a frequent pedagogical choice, it can present some conceptual difficulties. These difficulties are related to an ambiguous metaphorical association of conceptual number line with a physical and finite number track, which is used as a model to aid computation. In particular, PME researchers demonstrated that teachers associated number line with actions, used mostly in counting "jumps" when teaching addition and subtraction (see, for example, Heirdsfield & Lamb, 2006). In particular, teachers focused on perceptual, rather than conceptual, aspects of number line. Similarly, students in Grades 3 to 6 saw number line as a "line with numbers on it", stripped from its richness related to density and continuity. Consequently, the limited knowledge of the teacher did not promote the abstract conception of a number system among students.

In a follow up to their study reported in 2008, Doritou and Gray (2009) pointed out that in teaching (following the framework from teaching mathematics in England; DfEs, 2009) the number line is used mostly for demonstration, rather than for developing ideas. In observing teachers' use of a number line in instruction, Doritou and Gray noted, in resonance with their earlier work, that number line was seen as a tool supporting operations, rather than an aid in developing relational understanding of ordering numbers. The conceptual differences between number line and number track were not evident in instruction, as both can be used to represent "jumps". As such, children's understanding remained limited to whole numbers. The researchers considered this approach as contributing to children's focus on procedures and their failure to consider number line representations in a "relational" way. They suggested that association of a number line with number track can lead to significant difficulties when learning to work with fractions or larger numbers.

Diezmann and Lowrie (2007) investigated how the concept of number line (to which they refer as a "structured number line" to distinguish it from an "empty" number line, discussed below) develops over time. They noted considerable advantages of using number lines as a generic representational tool for a variety of concepts (mathematical variability) and as a possible representational tool among other tools (perceptual variability). They also revealed the considerable development of these related concepts among primary students over three years and noted significant gender differences in favour of boys. The former result is not surprising, and it was attributed to students' additional experience in schooling, rather than to particular instruction related to number lines. The latter result, while in accord with other studies that favour boys in tasks of spatial/visual representations (e.g., Battista, 1990; Maeda & Yoon, 2013), warrants further investigation.

Murphy (2008) discussed the idea of an "empty" number line (ENL) and its use in teaching mental calculations, contrasting the case of England and the Netherlands. She outlined the theoretical origins of the empty number line concept, which originated with the works of Soviet psychologists Vygotsky and Gal'perin and was further developed in the Freudenthal Institute, and which capitalized on the "abstractness" of the empty number line. However, likely due to space limitations, particular tasks associated with the ENL were not elaborated upon in this report. Particular tasks were mentioned in the two studies we describe in what follows.

Research by Gervasoni, Parish, Bevan, Croswell, Hadden, Livesey and Turkenburg (2011) has extended the seminal work of Siegler and Booth (2004) on the development of numerical estimation and showed that at times the "simplest" tasks in the experts' eyes can be most revealing. Working with 2-digit numbers and the notion of place value, these researchers presented students in Grades 2 and 3 with a number line segment, with marked locations of 0 and 100. A point was marked at half-distance between 0 and 100, and children were asked to identify what number "would go there". This novel task presented significant difficulty to children and helped in identifying students in need of additional experience with 2-digit numbers. Williamson (2013) also used empty number line (referring to it as "blank" number line) in tasks of positioning a number on the line and also estimating the value of a number positioned on the line. Randomized numbers were used across different ranges, with only end points labeled. Linear and logarithmic models were fitted to children's estimates. However, the report focused on strategies used by students in their estimations, rather than on the relative accuracy of estimated placements. It was noted though that the accuracy of children's estimates decreased on larger ranges.

The tasks used in Gervasoni et al. (2011) and Williamson (2013) are reminiscent of our experiences working with teachers, in which we present a number line segment with points 0 and 1,000,000 labeled, and request to place 100 on this number line. While all teachers place 100 "much closer" to 0 than to 1,000,000, the position they typically identify is actually closer to 10,000 rather than to 100. This task provokes an interesting conversation related to scaling and relative size of numbers and highlights the pedagogical power of the empty number line idea.

Number Operations and Computations

Operations with whole numbers comprise a significant part of any curriculum in the early grades. Researchers in Psychology attended to children's acquisition of number operations long before mathematics education emerged as a research field. Nevertheless, the focus on number operations continues to attract researchers, presenting nuanced theoretical approaches and instructional interventions.

Clarke, Clarke and Horne (2006) reported on a longitudinal study that looked at children's mental computational strategies, from their arrival to school at age 5, over 7 years. They presented "growth points" for arithmetic operations, which described the progression of a child's competency. While a steady progression over time was noted, a slower rate was observed in Grade 3. This was attributed, in part, to the introduction and emphasis on formal algorithms at that time. In line with Narode, Board and Davenport (1993), Clarke et al. (2006) advocated for a delay in presenting conventional algorithms to students and encouraged the development of personal computational strategies. In a related work, Gervasoni, Brandenburg, Turkenburg and Hadden (2009) explored the tensions faced by pupils and teachers when the teaching of arithmetic algorithms was delayed in favour of personal or informal computation strategies. Their study used empty number line as an intended catalyst for improving mental computation for students in Grades 3 and 4, and self-study and roundtable reflection as a methodology to challenge conventional approaches to teaching primary mathematics. Key themes included tensions amongst the usefulness of number lines for providing justifications and for developing computational reasoning, and the over-reliance and automaticity observed in pupils' use of algorithms (often without understanding) when presented with calculations. Not surprisingly, tensions also arose amongst community/parent/government expectations for pupils and the subsequent transitions from those strategies to more formal or abstract ones encountered in future mathematics classes. The researchers concluded that the teachers "were *caught in the middle* between research-based innovative practice and the tug of more conventional practice" (Gervasoni et al., 2009, p. 62, emphasis in original).

Notwithstanding these tensions, transitions from fostering informal or personal strategies to teaching more efficient and formal computational strategies have been met with some success via learners' use of metaphors. For instance, Murphy (2006, 2008) suggested that embodied metaphors – such as a "conceptual" number line – can aid with both fostering mental calculations, as well as transitioning from informal to formal mathematics. Murphy's use of empty number line differs from that of Heirdsfield and Lamb (2006) whose research explored number line as a "jumping" tool to aid mental computation. Through an embodied perspective, it was suggested that researchers could gain a better understanding of "how children's early reasoning develops into their first mathematical thinking beyond numerosities and… mental calculations" (Murphy, 2006, p. 223). Indeed, recent research has also begun to delve into the relationship between embodiment and more advanced mathematics such as linear algebra (e.g., Hannah, Stewart, & Thomas, 2014), periodicity and graphical representations (e.g., Bolite Frant, Quintaneiro, & Powell, 2014), and calculus (e.g., Swidan & Yerushalmy, 2013).

Number operations and computations have also attracted more in-depth research focusing on particular operations, the schemes and strategies which promote reasoning with these operations, and trends amongst learners of different ages, stages, and prior achievement, as we demonstrate below.

Addition-Subtraction

Ellemor-Collins and Wright (2008) argued that "facility in adding and subtracting without counting is a critical goal in early numeracy" (p. 439). Their report described instructional interventions that focused on learning addition and subtraction by low-achievers in Grades 3 and 4. They followed one student, 4th grader Robyn, in her successful progress from counting strategies to non-counting reasoning in addition and subtraction tasks for numbers up to 100. Gervasoni's (2006) study focused on children in Grades 1 and 2 identified as "vulnerable in number learning" and their addition strategies for one-digit numbers. She described children's strong preference for the "count-on" strategy and advocated for instructional intervention before this perceptual strategy becomes entrenched. Such a progression in abstraction was interpreted by Gilmore and Inglis (2008) through the lens of the process-to-object theories of conceptual development. The transition from process-based thinking (such as applying counting strategies for addition) to object-based thinking (non-counting reasoning) was described as requiring an ontological shift that allows the process to be viewed as something completed, upon which operations or transformations can be applied.

Torbeyns, Vanderveken, Verschaffel and Ghesquière (2006) analyzed the adaptive expertise of Grade 2 children of different abilities in solving addition and subtraction tasks with numbers up to 100. The instruction focused mainly on two strategies – referred to as jump and split. That is, in considering the addition of 45 and 21, for example, in the jump strategy one attends first to 45 and 20 and then to 1, while in the split strategy, the initial attention is on 40 and 20 and then on the 5 and 1. The researchers noted that only high-achievers adapted their strategy choice to particular features of the tasks, and they argued for instruction that supports children's adaptive expertise.

Learners' strategies for mental computations with addition and subtraction have been delineated in the research literature and used to both interpret understanding and foster it through task design. Csikos (2012) provided one such example in connection to Grade 4 students' 3-digit mental arithmetic. Csikos referred to Heinze, Marschick and Lipowsky (2009) whose framework for computational strategies proposed four types (stepwise, split, compensation, and indirect addition), which extend on the prior work of Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter and Fennema (1997) and Selter (2001). In the research conducted by Csikos, tasks which were designed to be easily solvable when using the compensation strategy (e.g., 176 + 135 = 180 + 135 – 4 or 176 + 135 = 171 + 140) were among the most challenging for students to complete correctly. It was noted that computational errors hindered otherwise appropriate uses of strategies and that students used a limited variety of strategies. In resonance with Torbeyns et al. (2006), Csikos advocated for a wide repertoire of strategies to be taught to provide a broader basis from which to develop learners' adaptive strategy use.

Multiplication-Division

Tzur, Johnson, McClintock, Xin, Si, Kenney, Woodward, Hord and Jin (2012) claimed that learning to reason multiplicatively requires a major conceptual shift and an abstracted notion of "coordinated counting". They explored the conceptual schemes children construct for multiplicative reasoning, and identified a framework for task design that deliberately builds with learners' existing schemes as a means of fostering more advanced knowledge. They discussed six schemes: multiplicative Double Counting (mDC), Same Unit Coordination (SUC), Unit Differentiation and Selection (UDS), Mixed-Unit Coordination (MUC), Quotitive Division (QD), and Partitive Division (PD). Each scheme provides a basis for the development of further arithmetic reasoning – e.g., UDS provides a basis for the distributive property of multiplication over addition and MUC is a precursor to partitioning a totality. Tzur, Xin, Si, Woodward and Jin (2009) and Woodward et al. (2009) discussed more on the MUC and mDC schemes in the context of task design for promoting progress and transitions in schemes. The role of reflection, via directing attention to structure and coordinated counting, and the importance of analyzing students' existing schemes were clearly emphasized. Such schemes may develop intuitively, as Bakker, van den Heuvel-Panhuizen and Robitzsch (2013) suggested, noting that young children demonstrated substantial knowledge of multiplicative reasoning before having been taught. Children's pre-instructional multiplicative knowledge included the ability to solve context problems as well as "bare number" problems of the form of doubling or with the verbal prompt of "times". The most challenging problems were ones that relied on a structural understanding of multiplicative situations, and these seemed to require formal instruction. Interestingly, their results suggested that multiplication and division may be considered equally difficult to a learner who has not received formal instruction on multiplicative reasoning. They speculated that simultaneous introduction of multiplication and division could benefit children's intuitive understanding of the connections between the two operations, and they called for further research to this end.

In most current curricula, multiplication precedes division, and the operation of division is known to present difficulty for students and elementary school teachers alike. As such, Leinonen and Pehkonen (2011) argued that division is an essential operation to consider in teacher education. They implemented an instructional strategy in which student teachers were encouraged to reflect on the principles of the long division algorithm and to write in their accounts about the reasoning involved. The results demonstrated the effectiveness of this strategy, in comparison to traditional teaching methods involving examination.

In addition to the mysteries of long division, division with remainders is not well understood, both as a decontextualized operation (e.g., Pehkonen & Kaasila, 2009), and as a model of situations (e.g., Spinillo & Lautret, 2006). Spinillo and Lautret (2006) acknowledged the difficulty students usually experience with the operation of division, in particular with situations involving division with remainder. Their instructional intervention, with students aged 8–11years from low income families, highlighted the notion of reminder in problems representing both quotitive and partitive division. They noted that understanding the role of remainder plays a significant role in learners' understanding of division and they suggested including division with remainder problems early in school curriculum. Pehkonen and Kaasila (2009), in investigating elementary school teachers' and upper secondary school students' understanding of division, asked to determine without any calculation, the result of 491 ÷ 6, when it was known that 498 ÷ 6 = 83. The failure of the majority of participants to produce a correct solution to this non-standard task was attributed to insufficient reasoning strategies.

Amato's (2011) report demonstrated student teachers' difficulty in representing various division situations (sharing, measuring and comparing) and focused on developing ways for helping them improve their understanding. In particular, it was noted that comparison in terms of ratio is not sufficiently present in (Portuguese) school and therefore was not sufficiently experienced by the participants. Involving teachers in performing children's activities, which in this study included translating from concrete materials to verbalization and discussion of division situations, appeared appropriate. The post-test conducted as part of the research indicated that the majority of participants improved their understanding with this approach.

The Role of Parentheses

A few studies have looked at students' understanding of numerical computations with and without the use of "useless" brackets as a means to gain further insight into students' arithmetic reasoning as well as to help scaffold transitions to algebraic thinking. Marchini and Papadopoulos (2011) used a series of simple addition and subtraction tasks – some with and some without "useless" brackets – with 292 primary school pupils. Their overall results supported the use of (useless) brackets for improving performance and for "understanding correctly the equality sign since this can be easily misunderstood as another operation sign" (p. 190). In particular, students in the control group had difficulty with questions such as $\square + 4 = 9$, whereas all students in the experimental group were able to correctly solve $(\square + 4) = 9$. This result is in line with Lincheski and Livneh's (1999) ESM study; Marchini and Papadopoulos suggested "useless" brackets may serve as a useful teaching tool for acquiring relational thinking. Gunnarsson, Hernell and Sonnerhed (2012) designed a similar study to test the suggestion of Lincheski and Livneh's (1999) that inserting brackets around the product in $a \pm b \times c$ would foster structure sense. In contrast to the previous studies, Gunnarsson et al. (2012) found that certain uses of useless brackets had a detrimental impact on student learning. In particular, they distinguished between two roles that brackets play in numerical computations – emphasizing and indicating precedence. With a sample size of 169 students aged 12 to 13, they found that emphasizing brackets impeded development from a left-to-right computation strategy to the use of precedence rules. They suggested that an implicit conflict between these two roles (emphasizing/indicating precedence) may create obstacles to learning the order of operations, and that a clearer articulation of bracket roles may offer a different outcome for student achievement.

Overall, as seen from the above examples, researchers around the world have experimented with various methods of helping learners acquire computational competence, with variable degrees of success. Insights into particular "stumbling stones" in learning may guide teachers and curriculum designers, and we expect future PME research will continue to explore this rich territory.

Number Theory

The theory of numbers, to which Gauss referred as "Higher Arithmetic", has found its place in PME research. Rephrasing the famous assertion of Gauss, that number theory is a "Queen of Mathematics" and the title of the Eric Temple Bell classic, "Mathematics: Queen and Servant of Science," Zazkis (2009) suggested that number theory was a queen and servant of mathematics education. The "queen" metaphor implied power, rather than perfunctory, while the "servant" metaphor implied utility, rather than lower societal status. Furthermore, by the servant metaphor it was articulated that researchers use number theory concepts and relations for investigating other topics in mathematics education, in particular transitions to algebra and proof.

Interest in number theory continued in the decade of interest for this chapter, primarily focusing on the "servant" role. The notions of proof and argumentation, and teachers' and undergraduate students' progress in acquiring and strengthening these concepts has been pursued in a variety of studies (e.g., Barkai, Tabach, Tirosh, Tsamir, & Dreyfus, 2009; Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2008; Gabel & Dreyfus, 2013; Kempen & Biehler, 2014; Toh, Leong, Toh, & Ho, 2014). These studies draw examples from elementary number theory.

As this review is concerned with numbers, within this set, we focus on the number theory examples used in these studies, rather than on their results. Participants in Barkai et al.'s (2009) study examined existential and universal statements related to the divisibility of numbers, such as "The sum of every five consecutive numbers is divisible by 5" or "There exists a sum of five consecutive numbers that is divisible by 5". In-service high school teachers were asked to examine the validity of the statements and provide either proofs or counterexamples. The study of Tsamir et al. (2008) focused on similar examples, where teachers were invited to suggest correct and incorrect justifications that their students might produce. Gabel and Dreyfus (2013) used the proof of the Euclidean algorithm to demonstrate the notion of "flow of a proof". Kempen and Biehler (2014) used the following task, "The sum 11 + 17 is an even number. Is this true for every sum of any two odd numbers? – Argue convincingly!" in their examination of argumentation used by student teachers. Toh, Leong, Toh and Ho (2014) discussed principles of task design, as well as conjecturing and proving strategies of student teachers, drawing on a particular number theory related task: "An L-Shaped number is one that can be written as a difference of two squares. Describe as completely as possible, which natural numbers are L-Shaped numbers."

Studies that focused on the use of examples also utilized the context of elementary number theory. For instance, "What can you tell about the divisors of two consecutive numbers?" was a particular task in the study of Morselli (2006). Levenson (2014) used the concept of parity in analyzing Grade 5 students' explanations and examples. Specifically, participants were asked to determine whether the numbers 14, 9, 284 and 0 were even or odd and explain their reasoning. Issues of parity were also of interest for researchers concerned with aspects of teachers' knowledge and its development. For example, Cooper (2014) analyzed mathematical discourse for teaching, using a lesson on parity of numbers that included definitions of even and odd numbers and parity result of operations. Also working with teachers, Zazkis and Chernoff (2006) introduced the notions of "pivotal example" and "bridging example" while discussing the understanding of prime numbers, divisibility and the fundamental theorem of arithmetic of one teacher, in her attempt to reduce the expression $\frac{13 \times 17}{19 \times 23}$. Tjoe (2014) investigated aesthetic predispositions of student teachers, using various approaches to find the greatest common factor and least common multiple.

The study of Zazkis and Zazkis (2013), as a noted exception, focused on student teachers' understanding of the structure of natural numbers, rather than using number theory as means to another end. A further development of this study is published in Zazkis and Zazkis (2014). This research used a method of script writing: participants were presented with two opposing views with respect to a mathematical claim, "Larger numbers have more factors", and were asked to continue the dialogue between two characters elaborating upon and arguing for or against the presented views. It was revealed that the majority of participants considered prime numbers as "exceptions" to the general view that larger numbers indeed have more factors. This view was supported by selected examples and revealed participants' articulated approaches for making and accepting arguments – touching once more upon the usefulness of number theory in its "servant role".

ON FRACTIONS AND (NON-INTEGRAL) RATIONAL NUMBERS

Fractions are hard. Fractions present difficulties to teachers and students alike. Over a hundred PME research reports in 2005–2014 supported this general claim, highlighting different aspects of understanding and operating with fractions.

One of the main difficulties highlighted in the research seems to relate to the fact that learners encounter fractions after they have established ideas and procedures for natural numbers. These early experiences may influence learners' expectations for working with fractions, a phenomenon referred to in the research as "natural number bias." Beginning with this phenomenon, we explore fraction interpretations, representations, models, and operations that have been brought into play by learners and teachers. Knowledge of the teacher, its relation to interpretations and models used by and for learners, and possible implications and issues for assessment are each considered in this section.

Natural Number Bias

Among repeatedly observed difficulties in operating with fractions is students' over-reliance on procedures and intuitions acquired when working with natural numbers. This was labeled in the research as a "natural numbers bias". For example, a person may naively conclude that 1/3 is smaller than 1/4 because $3 < 4$. In fact, such an inappropriate comparison was attributed to McDonald's fast food chain's failure to promote a third-pounder hamburger, as many customers did not understand why it was more expensive than the familiar and loved quarter-pounder.

While natural number bias was investigated mainly among elementary school children, recent PME reports from Belgium extended the research on this issue to populations more mathematically mature. Van Dooren, Van Hoof, Lijnen and Verschaffel (2012) studied the manifestation of natural number bias among secondary school students on tasks of fraction comparison. They employed a dual process perspective that differentiated between intuitive and analytic reasoning. Students responded much faster on so called "congruent items", that is, where the bias would lead to a correct solution (e.g., comparing 1/3 and 2/3), than to "incongruent items", for which the bias would lead to an error (e.g., comparing 1/5 and 1/9). However, the expectation that incongruent items will result in more errors was not confirmed. Similarly, Obersteiner, Van Hoof and Verschaffel (2013) studied expert mathematicians working on comparing fractions in order to get further evidence for the intuitive character of this bias. They found that even experts had a "trace" of natural numbers bias. This was concluded by observing that though experts responded to the tasks correctly, their response time was longer when fractions contained common numerators or denominators. This observation supported the claim that the source of the bias is in intuitive processes. In a related study, Van Hoof, Vandewalle and Van Dooren (2013) demonstrated that natural number bias persists not only in transition to fractions. They also found evidence of natural number bias in secondary school students working with algebraic expressions. That is, students exhibited more accuracy working with "congruent items", where the bias, if invoked, led to a correct conclusion (example: Is it always true that $9 + c > c$?) than with "incongruent items", for which the bias would have led to a wrong conclusion (example: Is it always true that $2 \times m > m$?).

Attending further to congruence and incongruence, Gomez, Jimenez, Bobadilla, Reyes and Dartnell (2014) explored the extent to which a natural number bias provides a useful account of the errors committed by students in Grades 5–7 in a fraction comparison questionnaire. About a quarter of the participants responded in a way considered as "extreme case of bias" – those students were 100% accurate on congruent items and erred on all incongruent items. Of note, this research distinguished between congruent items with common components (e.g., 4/9, 8/9), congruent items without common components, for which both larger numerator and denominator belong to a larger fraction (e.g., 5/7, 1/3), incongruent items with common components (e.g., 5/8, 5/17), and incongruent items without common components (e.g., 2/3, 5/17). The presence of common components made no difference for incongruent items, where in both cases the demonstrated success rate was about 40%. However, the presence of common components in congruent items resulted in a larger percentage (82% vs. 72%) of successful solutions. Interestingly, top students in this study (that is, students with overall better results) behaved in a way opposite to the predictions of the natural number bias for the case of items with no common components. This surprising finding was explained by the possibility of heuristic overgeneralization of the common remark made by teachers that the magnitude of a fraction grows if its denominator shrinks, and vice versa. This reasoning supports decision making when considering a single fraction, but leads to errors in fraction comparison.

In a study that looked at Grades 3 to 6 students' understanding of rational decimal numbers, Roche and Clarke (2006) found that ordering tasks provided considerable difficulty for students, even for those who were characterized as "experts" in decimal comparison tasks. A whole number bias resulted in inaccuracies when comparing the relative sizes of decimals in tasks that required ordering more than two decimal numbers. Roche and Clarke (2006) suggested that teaching strategies which confirm a whole number bias, such as adding strings of zeroes to equalise the lengths of two decimals before comparison may be detrimental to the development of conceptual understanding. They advocated for the frequent use of fractional language to describe decimals for purposes of fostering more appropriate understanding of the decimal numeration system.

A strong reliance on experience with natural numbers has also presented an obstacle in understanding the density of rational numbers (Vamvakoussi & Vosniadou, 2006; Vamvakoussi, Christou, & Van Dooren, 2010). Evidence of natural number bias – though not labeled as such in the studies on density – was found when students attempted to assign a successor to a rational number, or believed that there was a finite number of rational numbers in a given interval. These and similar mistakes were attributed to the property of discreteness of natural numbers that was misapplied in considering rational numbers. A further challenge in understanding the density of the rational and real numbers was linked to students' understanding of infinity (Pehkonen, Hannula, & Soro, 2006). While it was found that understanding of density improved with age, tasks which asked students from Grades 5–8 to identify how many numbers exist between 0.8 and 1.1 were consistently problematic for participants. In a similar work with an alternative angle, Kullberg, Watson and Mason (2009) analysed the responses of Grades 7 and 9 students through a covariational perspective. They suggested there is a two-way relationship, which develops over time, between decimal number understanding and number line representations, and that this relationship is necessary for correctly placing given numbers on a number line. Participants had difficulty ordering and positioning (spacing) decimal numbers such as 1.7, 1.71, 1.701, 1.7001, on an unscaled number line, which required an understanding of positional variation as well as variation in digits and numbers of decimal places.

Researchers have argued that conceptual change is required for successful transition from understanding natural to rational numbers. Such a conceptual change is not unlike that connected to the development of real number concepts. Merenluoto and Lehtinen (2006) conducted a large quantitative study with 17 and 18 year olds that found natural number bias hindered success in operating with real numbers. They suggested that students "need to tolerate the inevitable feeling of ambiguity" (p. 165) which accompanies letting go of familiar practices and procedures. Theory on conceptual change is also featured in Prediger's (2006) report, in which similarities and differences – referred to as "continuities and discontinuities" in the learning process – between natural numbers and fractions were examined. The mentioned "discontinuities" included the unique symbolic representation of natural numbers vs. the multiple representations for the same fractional value, the discreteness of natural numbers vs. the density of fractions, and the order-property of multiplication and division (e.g., "multiplication makes bigger"). Prediger proposed an integrated model that attempted to describe students' difficulties with discontinuities related to fractions as obstacles requiring a conceptual change to overcome.

Is it possible to overcome the natural number bias by means of pedagogical interventions? According to Gomez et al. (2014) this remains an open question. Recent research suggests that it may be impossible to overcome this bias in its totality.

Fractions and Teachers

Anna Sfard, in her plenary address at the International Congress of Mathematics Education in Copenhagen in 2004, noted that she was "pleased to find out that the last few years have been the era of the teacher as the almost uncontested focus of researchers' attention" (Sfard, 2004, p. 90). She also described the last two decades of the 20th century as "almost exclusively the era of the learner", and the several decades prior to that as the "era of the curriculum" (ibid.). In reviewing PME research for the past decade, it is evident that the era of the teacher is continuing and getting momentum. In reviewing over a hundred PME reports that focused on fractions and rational numbers, we note that about a third of these reports addressed various aspects of teachers' knowledge related to these concepts. There is a separate chapter in this handbook dedicated to research on teachers' knowledge and its development, however we highlight here several studies that addressed fraction-related concepts.

The majority of research reports from PME 2005–2015 have focused on aspects of pedagogical knowledge related to teaching fractions and interpreting students' work. For instance, Chick, Baker, Pham and Cheng (2006) proposed a framework for investigating teachers' pedagogical content knowledge (PCK) in the content of decimal fractions and confirmed its applicability. Focusing on particular aspects of fractions, PME researchers have found deficiencies in student teachers' knowledge of mathematics and pedagogy related to multiplication of fractions (Ho & Lai, 2012; Amato, 2009), division of fractions (Li & Smith, 2007), particular models and representations related to fraction operations (Izsak, 2006; Amato, 2006), fractional units (Lo & Grant, 2012), and changing referent wholes (Prediger & Schink, 2009).

Studies which have focused on teachers' knowledge of fractions have also presented ideas for how this knowledge can be strengthened. For example, PME researchers have reported on primary school teachers' misconceptions related to the order of decimals and fractions (Alatorre & Saiz, 2008) and student teachers' difficulty in attending to the place-value structure in decomposing decimals (Widjaja & Stacey, 2006). Ideas for how to strengthen knowledge in these domains included engaging teachers as leaners using realistic contexts that draw attention to structure (e.g., Peled, Meron, & Rota, 2007). While the teachers' personal knowledge was at times linked to their inability to interpret students' solutions, working with students or with student solutions was seen as an avenue for supporting teacher's knowledge, and in particular, for supporting various aspects of pedagogical content knowledge or knowledge of mathematics for teaching. A "learning study", as a particular model that combines students' and teachers' learning, was suggested as an example of teachers' professional development (Ling & Runesson, 2007). Learning study is similar to the Japanese Lesson Study "in which a group of teachers work collaboratively to explore and develop their teaching practice in a cyclic process of planning, observing and revising lessons" (p. 157). However, the goal of the learning study is to enhance students' learning, rather than to improve the lesson. A particular lesson of fraction addition was used as an example. Involvement in the learning study helped teachers re-evaluate their assumptions related to students' knowledge and re-evaluate their instructional emphases. The revised instruction included an emphasis on whole and part-whole relationships, rather than on the algorithms and operations.

Teachers' interpretation or evaluation of students' solutions of fraction tasks was featured in several studies (e.g., Alatorre, Mendiola, Moreno, Sáiz, & Torres, 2011; Ribeiro, Mellone, & Jakobsen, 2013; Callejo, Fernandez, & Marquez, 2013). The overall results pointed to teachers' difficulties in interpreting non-algorithmic solutions and judging their correctness for both contextualized (e.g., "What amount of chocolate would six children get if we divide the five bars equally among them?) and decontextualized (e.g., How many times does ⅓ fit in 2.5?) tasks involving multiplicative structures. Further, more procedural, rather than conceptual, errors were identified in students' solutions. In accord with other research, the task of working with students' solutions was seen as an important method in teacher education, fostering teachers' professional development, in particular their specialized content knowledge.

Naturally, as fractions are an integral part of any elementary school curriculum, these above mentioned studies focused on elementary school teachers. However, Zoitsakos, Zachariades and Sakonidis (2013) looked at secondary mathematics teachers understanding of the decimal expansion of rational numbers. In particular, they focused on a repeating period of 9, in the decimal expansion 0.3999… . They reported that despite a strong mathematics background, the majority of teachers considered this symbol as a process, rather than as a number. For readers interested in an extended discussion of repeating decimal expansion we recommend consulting Weller, Dubinsky and Arnon (2009, 2011, 2013).

Fraction Interpretations

According to Charalambous and Pitta-Pantazi (2005), "fractions are among the most complex mathematical concepts that children encounter in their years in primary education" (p. 233). The multifaceted fraction notion, which encompasses five interrelated sub-constructs contributes to this complexity. The sub-constructs are: part-whole, ratio, operator, quotient and measure (e.g., Kieren, 1980, 1988). A variety of PME research reports addressed these interrelated interpretations of fractions and considered how various interpretations are featured in children's problem solving.

Based on the model that presents a relationship between fractional constructs and operations, Charalambous and Pitta-Pantazi (2005) developed and implemented a test, carried out with over 600 students in Grades 5–6, and identified correlations between various components of the model. The data confirmed that the part-whole interpretation is a foundation for the other four 'subordinate' interpretations, and also indicated that all the interpretations contributed towards proficiency with fraction operations. In a follow up study, Charalambous (2007), in addition to developing a scale for measuring student understanding, examined the level of difficulty presented by different sub-constructs. Tasks related to part-whole recognition of fractions appeared less difficult for students, while tasks related to the measure and operator sub-constructs appeared most difficult. However, the differences in students' performance could have been attributed to the Cypriot curriculum in which students have more opportunity to practice part-whole related tasks. Clarke, Sukenik, Roche and Mitchell (2006), designed and used an assessment instrument, which attended to the five sub-constructs identified above, in individual interviews with Grade 6 students (n = 323). In resonance with the aforementioned studies, students performed much better on the part-whole tasks than on the tasks involving other interpretations of fractions. The weakest performance was demonstrated on the tasks that involved measure and division interpretations. This can be seen as a further support of the importance of the part-whole interpretation.

In contrast to these studies, the foundational positioning of the part-whole interpretation, which largely followed from the studies of Kieren (1980, 1988), was not supported by others (e.g., Freudenthal, 1983; Thompson & Sandanha, 2003), who argued for the importance of the inclusion of ratio interpretations in the early stages of fraction instruction. Following this latter view, Cortina and Zuniga (2008) suggested that ratio comparisons could be a viable starting point in introducing fractions, and an alternative to the "equal partitioning" traditional approach. This recommendation was based on the study of nine 11 year olds, reasoning about the relative capacity of cups – as the task appeared meaningful and supported quantitative reasoning about multiplicative relationships and about basic equivalencies.

Mamede and Nunes (2008) described a teaching experiment in which children were randomly assigned to groups and then introduced to fractions using quotient, part-whole or operator situations. The results of the test administered after the teaching experiment demonstrated that quotient situations were the most helpful for children in the tasks of equivalence and ordering of fractions. They noted that quotient situations were helpful for children in establishing connections between informal ideas and fractional representations of quantities, and could be used for introducing learners to fractions. In a similar vein, Naik and Subramaniam (2008) discussed the inadequacy of an exclusive emphasis on the part-whole interpretation, and demonstrated in their study with Grade 5 students the effectiveness of supplementary instruction that focused on measure and quotient interpretations of fractions. Particular gains were noted on tasks of comparing the relative size of fractions. However, the study of Mamede and Cardoso (2010) suggested that emphasizing one sub-construct over another was not the only indicator of differences amongst student understanding or performance, and they pointed instead to the centrality of context. In their study, students in Grade 6 performed better on equivalence and ordering tasks presented in quotient situations than in part-whole and operator situations, but they performed better on labeling tasks in part-whole and operator situations than in quotient situations. Their results suggested that "distinct situations affect differently students' understanding of fractions" (p. 262).

These studies combined emphasize the importance of exposing students to a variety of situations in their mathematics class, repeating the argument against almost exclusive reliance on part-whole notion. However, given the structure of the curriculum and teachers' dispositions, this recommendation appears easier to agree with than to implement.

Fraction Representations and Models

Researchers in mathematics education have acknowledged the importance of attending to representations of mathematical concepts in teaching and learning. Recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations, as well as the ability to move flexibly between different representations are essential for understanding the concept (Lesh, Post, & Behr, 1987). Several PME reports attended to a variety of models and representations in the domain of fractions.

Kyriakides (2006), for example, explored Grade 5 students' use of rectangular area models to represent fractions via a constructivist teaching experiment. He suggested that "the value of the rectangular area model as a tool for scaffolding the meaning of fractions lies in the multidimensional role area plays in human life" (p. 18). The findings demonstrated that vertical partitioning is central in children's perception of proper fractions. Deliyianni, Panaoura, Elia and Gagatsis (2008) attended to the addition of fractions represented in a variety of ways, such as number line, circle diagrams, rectangular diagrams, or symbols. They confirmed their hypothesis that flexibility with multiple representations influenced various dimensions of students' (in Grades 5 and 6) understanding of fraction addition. In a follow-up study, Deliyianni, Elia, Panaoura and Gagatsis (2009) focused on primary and secondary school students' understanding of decimal addition, and looked for similar trends with respect to the influence of different representations. In resonance with their previous work, they advocated for flexibility across representations, noting that different types of representations – including diagrammatic and symbolic – affected students' solving processes. Further, they noted an importance in making explicit connections amongst decimals and fractions for fostering structural sense of decimal number addition, and they suggested that flexibility across representations may help students transition from primary to secondary school mathematics.

While flexibility in attending to various representations is acknowledged in the above mentioned studies, representation of fractions were also attended to as relevant to particular operations. Kalogirou, Gagatsis, Michael and Deliyianni (2010) explored to what extent different types of fraction representations can help students overcome obstacles with fraction division as well as students' perceptions of the usefulness of different kinds of pictorial representations. The results indicated that students were more successful on tasks that were accompanied by a representational picture than ones without. The researchers reported significant differences in students' beliefs about the utility of various pictorial representations: decorative pictures, informational pictures, representational pictures, or organizational pictures. Unfortunately, this report was not accompanied by any pictorial representations, so we are left wondering what the various pictures could have been. Adding another dimension, Dreher, Kuntze and Winkel (2014) focused specifically on the conversions of representations of fractions. They administered a test for Grade 6 students that involved examining, performing and justifying conversions among representations. The results indicated greater success in examining conversions task, than in performing conversions, regardless of whether or not justifications were required.

Representations and context have also attracted research interest with regard to the teaching and learning of decimal number sense, arithmetic, and structural understanding. For instance, Bonotto (2006) noted that realistic modeling had a positive effect in fostering decimal number sense. Such modeling included cultivating particular socio-mathematical norms (such as pretending to be at a restaurant and splitting a bill) and using cultural artifacts that fostered decision-making and a grasp of the connections between numerical representations of decimals and their referent quantities (such as making choices from a menu). Further, while students who engaged with these experimental activities showed positive growth, students from the control group were found to have an increase in errors during the post-test. Peled, Meron and Rota (2007) also promoted the benefits of realistic modeling and contexts in fostering an understanding of decimal structure with Grade 3 students. Their approach included a long sequence of contextualized investigations that emphasized a re-invention of base ten groupings and place value, before introducing tasks which aimed at generalizing multiplicative structures. In particular, their multiplicative approach was based on viewing decimal numbers as special combinations of multiples and powers of 10. Peled, et al. found that students' activity was characterized by "deep thinking" and "good argumentation" (p. 71), and that the multiplicative approach enabled activation of prior learning for meaningful transfer. They also noted that while the teachers they worked with initially rejected such an instructional approach, they benefited from experiencing it as learners, which helped increase their appreciation for the techniques.

Deficits in fraction or decimal number understanding can have serious repercussion not only in school mathematics, but also in the workforce. Steinle and Pierce (2006) found fundamental misunderstandings of decimal numbers among student nurses could lead to calculation interpretation errors, even when correct procedural routines were followed. The nurses' self-professed problems with decimals included knowing how to treat repeating decimals – e.g., 3.77777 was viewed as the same as 3.7, with implications in dosage interpretation and administration – and ordering and comparing decimals – e.g., deciding whether or not a blood alcohol reading of 0.12 was over the legal driving limit of 0.05. While the nursing courses and supplementary learning material focused mainly on procedural understanding, Steinle and Pierce found that even minimal intervention with conceptually-focused teaching had a significant positive effect.

On Tests and Errors

Several researchers addressed the challenges of developing reliable tests for assessing students' understanding of fractions. For example, Nikolaou and Pitta-Pantazi (2013) developed a theoretical model of factors involved in understanding fractions and confirmed that the proposed factors determined hierarchical levels of understanding the fraction concept. At the low level, the factors included inductive reasoning and sense about the magnitude of fractions. At the middle level, the additional factors were argumentation and justification, connections with decimals and percentages, and ability to attend to and convert between different representations. At the high level, comprising the factors from the previous two levels, the additional factors were definitions and mathematical explanations, and ability to reflect on one's solution. The levels were found to be stable over time.

In a study on testing of fractions, decimals, and multiplicative reasoning, Hodgen, Kuchemann, Brown and Coe (2010) compared test results from 1976/7 and 2008/9 to examine changes in adolescents' achievement and difficulties. Their preliminary results indicated that while attainment in decimals was higher in 2008 than thirty years prior, the opposite was true for fractions, and in both cases a higher proportion in 2008 of very low performances was observed. The measures used included the items developed as part of the 1970s Concepts in Secondary Mathematics and Sciences (CSMS) study. This data was compared with the results of 3000 11–14 year olds' address of the same CSMS test items in 2008.

Acknowledging that developing reliable measures of understanding on any mathematical concept is difficult, Jones, Inglis, Gilmore and Hodgen (2013) presented a different approach, called Comparative Judgement (CJ). They argued that "The expense, lengthiness and difficulty of measuring conceptual understanding is a barrier to progress in mathematics education" (p. 113) and suggested that collective expertise of teachers and researchers could provide an alternative. In this approach, experienced educators make pairwise judgments on the quality of students' responses on a test. The idea is derived from a psychological principle that humans are better at comparing objects to one another, rather than comparing an object against a particular criterion. In this study eight mathematics educators assessed 25 student responses pairwise, each completing 50 judgment decisions. One particular item involved ordering a list of seven fractions (3/4, 3/8, 2/5, 8/10, 1/4, 1/25, 1/8) from smallest to largest and justifying the method. The researchers found strong inter-rater consistency and demonstrated the validity of the method as general mathematical achievement of students could predict the CJ parameters. They suggested further research should examine the extent to which CJ may offer a method that can be used routinely in different content domains. Further information on uses of CJ can be found in Jones and Inglis (2015).

The studies of Heemsoth and Heinze (2013, 2014) investigated whether or not instruction that involved discussion of errors supported students' performance. Their initial study demonstrated that reflection on errors – so called "negative knowledge" – was beneficial for advanced students only. The findings of the follow up study, which implemented two different error-handling strategies, demonstrated that students who reflected on the rationales behind erroneous solutions enhanced their knowledge more than students who reflected only on the corresponding correct solution. However, these results are limited to procedural knowledge, and further research was suggested to draw conclusions about conceptual knowledge.

THE INTEGRAL, THE IRRATIONAL, AND THE INFINITE

The (intended) pun in the titles of this and the previous section poke a little at the dominance of fractions, rational decimals, and their manipulations in mathematics education research. Non-integral in the mathematical sense (i.e., not of integer value), yet integral in a linguistic sense, the future of fractions and their representations in teaching, learning, and research will be interesting to watch. Eyes might look toward controversies around their necessary place in the curriculum (e.g., Mason, Taylor, Simmt, & Gourdeau, 2015) and new thinking around curricular design (e.g., Ministry of Education and Culture, Finland, 2015), in the meantime, we turn our attention to other "integral" mathematics.

In the following, our attention turns to integers, and the elements and relationships that are integral to integer understanding, yet offer challenges for learning and teaching. We then turn our attention toward pedagogical structures for developing ideas of irrational numbers, and to conceptual structures in reasoning with and about infinity.

Zero: More Than Nothing, Less Than Everything

Gallardo and Hernandez (2005, 2006) examined learners' conceptions of zero in the context of transitioning from arithmetic to algebra. From an advanced perspective, zero is recognized as an identity element in a group with additive structure (closure), such as the integers. For a novice learner, Gallardo and Hernandez (2006) observed that different meanings of zero can develop as different associations across contexts and (re)presentations are made. They offered a case study analysis of the conceptions of a high-achieving 12 year old pupil who was competent in "arithmetic and algebraic systems of signs" (p. 154). Five different meanings of zero were interpreted: the nil zero (valueless), the implicit zero (used in solving processes), the total zero (identity element), the arithmetic zero (result of arithmetic manipulations), and the algebraic zero (result of algebraic manipulations). Bofferding and Hoffman (2014) observed that strong conceptions about zero, and its place on the number line, could act as a hindrance in distinguishing between absolute value and ordered value of positive and negative integers. They suggested that teachers provide explicit instruction to help students from overgeneralizing the idea that zero is always the start of a number line, targeting instruction around the placement of zero relative to all integers and highlighting the structural symmetry of the number line.

On Negative Numbers

Number sense developed for natural numbers relates to, but is not sufficient for, number sense of negative integers (Kilhamn, 2009). Kilhamn (2009) suggested that number sense be an explicit part of teaching negative numbers, bearing in mind four interrelated components, which included: ability and intuition about natural numbers and arithmetic, ability to compare magnitudes and relative sizes, ability to recognize benchmark numbers and patterns (such as the symmetry of opposite numbers), and possessing knowledge of the effects of operations on integers. Appreciating the symmetric relation about zero of inverses in the group of integers may be seen as a precursor to "check and balance" type approaches to integer arithmetic. Koukkoufis and Williams (2006) analysed children's semiotic processes when applying the "compensation strategy" while playing dice games. They noted that in some contexts, children intuitively and comfortably used the compensation strategy, and that this intuition facilitated its symbolic generalization (Radford, 2003).

While Koukkoufis and Williams (2006) found children to have some success in reifying negative numbers through their experiences with realistic dice games, Chrysostomou and Mousoulides (2010) noted that student teachers relied on process-based explanations of negative numbers and lacked conceptual (and encapsulated) understandings. Their participants relied on rules and memorization, and could not justify comparison decisions or pedagogical models and representations. Chrysostomou and Mousoulides (2010) concluded that teachers' limited and un-encapsulated knowledge of negative numbers "created even more difficulties on their pedagogical content knowledge and prevented them from being able to realize what was actually needed for successfully teaching the negative numbers" (p. 272). Ekol (2010) probed students' and teachers' action-oriented thinking when learning negative numbers in a dynamic geometry environment. He observed that the use of a dynamic, and necessarily interactive, number line shifted attention away from object properties of the numbers to action-oriented thinking, which was found to be important for working with integer arithmetic and conceptual understanding. It is worth noting that while action- (or process-) based thinking was helpful in the digital context of interactive dynamic software, such thinking was deemed problematic when trying to provide a definition for negative numbers, or in placing decimals on a static number line (Chrysostomou & Mousoulides, 2010).

On the Seeming Irrationality of Irrational Numbers

What are these creatures, such as π, e, φ and fan favourite $\sqrt{2}$? How are these so-called numbers introduced and developed in school mathematics? And, what on earth are they for? These questions might be quite challenging for some to answer, particularly if they are relying on textbook explanations. Gonzalez-Martin, Giraldo and Machado Souto (2011) analysed a collection of textbooks commonly used across educational strands in Brazilian curriculum, and raised important criticisms about how irrational numbers were presented and not presented. The texts exclusively introduced irrational numbers via decimal representations and unjustified computational routines and algorithms. The lack of any explanation or argumentation to motivate the need for, or existence of, irrational numbers was highly problematic and to the detriment of structure and conceptual development of irrationals and of properties of the real number line, such as its density. Gonzalez-Martin et al. (2011) suggested that such textbook approaches ought to introduce these "new" kinds of numbers with a sense of their purpose and utility, such that the mathematical need for the construction of the field of real numbers is better motivated. They promoted the use and discussion of mathematics problems which cannot be solved by rational numbers, and as such may beget the concept of irrational numbers.

Similarly, Shinno (2007) suggested that a historical setting and determining the existence of a solution to equations such as $x^2 = 2$ could offer context for a meaningful rediscovery of irrational numbers. Shinno noted that several challenges exist when introducing irrational numbers, and he cautioned against an over-reliance on concrete representations or contexts. In particular, he suggested that to develop "incommensurability sense" a learner must appreciate the limitations of concrete representations and recognize a need for algebraic development and general forms. While Shinno's experiment made use of paper folding to draw attention to the existence of irrational numbers, the role of questioning whether or which numbers cannot be represented as a ratio of integers should be part and parcel to instructional approaches. Shinno and Iwasaki (2009) explored further the process of conceptual change in learners, and found that an understanding of incommensurability was essential for the ontological shift required in transitioning from ideas of rational to irrational number concepts. They observed lessons in a Grade 9, lower-tiered, mathematics class, which focused on square roots of numbers. They found that Euclid's algorithm and proof by contradiction served as a useful didactical aide in fostering a sense of incommensurability, in spite of the considerable challenges students can encounter with the formal logic of an indirect proof. More importantly, they reaffirmed the role of shifting from a 'realistic' knowledge of (natural) number relying on concrete existence, to an 'idealistic' knowledge of (real) number relying on abstracted reasoning. In analysing students' discursive processes when learning irrational number addition, Shinno (2013) emphasized the importance of reifying the signifiers $\sqrt{2} + \sqrt{3}$ and $5 + \sqrt{6}$ to represent one irrational number. Shinno suggested that the act of reifying was fostered through the use of questions similar to "if not, what yes?" (Koichu, 2008) which kept students focused and active, and which "implies the need for a new semantic space of the signifier" (p. 213) that can then be objectified through students' interactions with area models.

The representation $\sqrt{2} + \sqrt{3}$ can be interpreted as a procept (Gray & Tall, 1994), where the ambiguity between the process of adding two numbers, and the object or concept of the sum of two numbers, is visible in the signifier and linked to challenges in learner perceptions. The decimal representations of irrational numbers, particularly because of their infinite expansion and an associated tendency to view the expansion as a process but not a concept, can also be challenging for individuals' understanding of the magnitude of those numbers, and has implications for learners' understanding of infinity. Mamolo (2007) found that pre-Calculus students had a tendency to over-apply the term "infinite number" and in doing so, confounded the finite properties but infinite representation of π with the "infinite properties" but "finite representation" of points on a number line. An intuitive use of the notion of "measuring infinity" (Tall, 1980) and a lack of conceptual understanding of irrational numbers are some factors which contributed to challenges when moving beyond the reals to the surreal. Let us turn now to the surreal; off to infinity.

Making Space for Infinity

In a chapter dedicated to number, the place of infinity might be seen as dubious, yet we see a space for it. Not a number, but a number, we restrict our focus on the expansive topic to the notion of infinity as it emerges in Cantorian set theory – as transfinite numbers with their own idiosyncratic arithmetic structures. Whether or not one accepts infinity as a number, it is a concept that continues to attract attention and interest. Indeed, there has been no shortage of research within the PME community which has been dedicated to learners' understandings of infinity.

Tsamir and Tirosh (2006), who have written extensively on the topic, provided an overview of the first thirty years of PME research on *the psychological complexity, mathematical significance*, and *educational challenges* ("pme") of the learning and teaching of infinite sets. They highlighted series of tasks that have been used to uncover, explore, and foster student thinking around infinite divisibility, transfinite cardinal comparisons, and limits. Early findings unearthed the resilience of intuitive and contradictory approaches, which drew research attention toward teaching interventions and the effects of various presentations, and re-presentations, of infinite sets. With an eye on teaching, infinity in teacher education began to attract research attention, which had been previously focused on novice learners. The anomalous nature of infinity was no less challenging for teachers, despite more advanced mathematics backgrounds. Particular attention was paid toward teachers' awareness of the necessity for consistent and non-contradictory results when comparing sets and in determining by which criteria to make the comparison. Tsamir and Tirosh (2006) suggested that "when designing and teaching mathematics courses, attention should be given to the relations between formal and intuitive knowledge and to the conflicts which may arise in the mismatching of applications of these different types of knowledge" (p. 60).

The context of (set theoretic) infinity (infinities, actually) has continued to attract researchers interested in the interplay amongst intuitive and formal understandings, and in learners' strategies for coping with abstract mathematics. Singer and Voica (2009) identified four 'categories of structures' that children and adolescents used to reason about infinite set comparisons. They identified arithmetic structures, geometric structures, fractal-type structures, and density-type structures. These different structures were linked to other mathematical contexts, and if activated could help the understanding of some important concepts, such as graphical transformations, recursion, and division algorithms. This research supports the claim of Montes, Carrillo and Ribeiro (2014), who suggested that infinity is "intrinsic to school mathematics … [and] applicable to the day-to-day work of teaching" (p. 237). Teaching interventions, however, have had dubious influences on learner perceptions of infinity. For example, Narli, Delice and Narli (2009) confirmed prior research around the resilience of personal experiences and primary intuitions in young adolescents' conceptions of infinity. They noted the persuasiveness of non-mathematical contexts and emotional connections – infinity was likened to extreme feelings such as loneliness and freedom, and to notions of life and living. As one participant put it "if there was infinity plants would not die they would all still be alive" (p. 212). Similar considerations were observed in more mathematically sophisticated learners, whose consideration of paradoxes of infinity – such as Hilbert's Grand Hotel (HGH) and the Ping Pong Ball Conundrum (PPBC) – led to new insights on the tacit influences of philosophical or emotional orientations toward the concept (Mamolo & Zazkis, 2008a). Success in resolving such paradoxes seemed to hinge on an ability to separate belief from knowledge – that is, even participants who could accept the normative resolution of the PPBC felt compelled to identify that this acceptance was in spite of their instincts. Mamolo and Zazkis (2008a) provided one of the first empirical studies which used paradoxes as a research tool (rather than a teaching tool) to investigate and compare learners' understanding. Maes, Cornet, Verhoef and Hendrikse (2011) extended this work, using the PPBC with upper-year high school students, noting similar results and suggesting that the coordination of multiple infinite sets may have been too complex for the learning stages of their participants. More on the uses of paradoxes as windows to learners' understandings can be found in Dubinsky, Weller, Stenger and Vidakovic (2008), Mamolo and Zazkis (2008b) and Radu and Weber (2011).

The ambiguous nature of infinity as either (or both) a potential or an actual mathematical entity offers a fruitful ground for delving into the nuances of individuals' abstract reasoning processes. The tension between potential and actual infinities has been considered from the perspective of young learners (Singer & Voica, 2009), adolescents (Maes et al., 2011; Narli et al., 2009), student teachers and undergraduates (Mamolo & Zazkis, 2008a; Mamolo, 2014a). In their ESM papers, Dubinsky, Weller, McDonald and Brown (2005a, 2005b) applied the APOS Theory to correspond the notions of potential and actual infinity to process- and object-based conceptions, respectively. Through this perspective, the notion of actual infinity may evolve from an individual's encapsulation of potential infinity. In her consideration of learners' understanding of transfinite cardinal arithmetic, Mamolo (2014a) highlighted the nuances involved in reasoning about the cardinality and cardinal number of an infinite set. In particular, she showed that it is possible to conceptualize an infinite set as a completed object without conceiving of a transfinite cardinal number as one (where a transfinite cardinal number is that which represents the cardinality of the 'completed' infinite set). Further, it was suggested that de-encapsulation of an infinite set may be problematic for making sense of an encapsulated transfinite cardinal number when the properties of a process of infinitely many finite items is extrapolated to make sense of the properties of an object of one infinite entity. This and other studies focused on the genetic decomposition of infinity are discussed further in the Canadian Journal for Science, Mathematics and Technology Education; see for example, Mamolo (2014b) and Weller, Arnon and Dubinsky (2009, 2011, 2013).

Tsamir and Tirosh (2006) declared that the "value of examining the same issues from different viewpoints cannot be overstated" (p. 60). Indeed, in their own work, revisiting findings through a variety of theoretical lenses provided them with "a more extensive vocabulary to discuss the phenomena observed" (ibid). Theoretical approaches both informed, and were informed by, researches on learners' understanding of infinity. Kim, Sfard and Ferrini-Mundy (2005) investigated students' discourse on the topic of infinity, comparing and contrasting US and Korean participants. Clear differences amongst the discourses of US and Korean students were noted, and were "ascribed to the fact that only in English do the mathematical words infinity and infinite (as well as set) appear also in the colloquial discourse" (p. 207). Since colloquial discourse was the primary source for US students' conceptualization of infinity, the authors conjectured that this discourse may impact students' development of other aspects of their mathematical discourse, such as routines and endorsed narratives. This work was further developed in Kim, Ferrini-Mundy and Sfard's (2012) investigation on the language-dependent nature of mathematics learning, and has informed broader research on the influence of language in mathematics learning (e.g., Sfard, 2014).

AFFORDANCES OF TECHNOLOGY IN SUPPORTING THE STUDY OF NUMBER

Technological innovations support the development of new teaching and research tools and in such allow further insight into issues of affect and cognition of mathematics learners. These innovations range from computer-based interaction with tools and applets to advances in brain research.

For instance, virtual manipulatives have been implemented for fraction learning. Suh and Moyer-Packenman's (2007) study of 3rd graders found significant differences in student achievement in favour of virtual manipulatives vs. physical ones. Furthermore, Suh and Moyer-Packenman (2008) used virtual manipulatives to study the learning of fraction equivalence of students with special needs. The affordances of these manipulatives were discussed with particular attention to the connections between the visual and notational representations. The simultaneous change in multi-dimensional representations was considered advantageous, helping learners avoid cognitive overload, which may be present in a physical environment.

Olive (2011) used a dynamic number line and semiotic mediation to explore learners' understanding of the relative sizes and positions of fractions. He demonstrated the ways in which participants' reasoning advanced and how it was shaped by using this dynamic tool. Over the field of real numbers, arithmetic skills and number sense rely on a flexible understanding of place value, for which the affordances of digital technologies may provide helpful. Kortenkamp and Ladel (2013, 2014; also Ladel & Kortenkamp, 2013) looked at the effect of a digital, interactive place value chart on learners' growth of understanding. Their early qualitative analyses indicated that the use of the digital tool, while surprising to students at first, supported a conceptual understanding of place value and the ability to flexibly represent numbers in the place value chart. Their on-going quantitative analyses will investigate whether their interactive place value chart – which preserves the value of a number and allows for nonstandard partitions – will yield similar results consistently across a larger scale. Geiger, Dole and Goos (2011) noted the necessity of seamlessly integrating instructional models for numeracy learning – be they via digital tools or otherwise – such that the various representations and contexts for mathematics are understood. More to this end is discussed in the chapter dedicated to ICT in mathematics teaching and learning.

While the aforementioned technologies were useful in the study of number (as teaching tools), other technologies are being employed – and have been found similarly useful – in the study of "the study of number" (as research tools). For instance, Obersteiner, Moll, Beitlich, Cui, Schmidt, Khmelivska and Reiss (2014) used eye tracking to distinguish strategies used by mathematicians in comparing fractions. Cimen and Campbell (2012) used a wide spectrum of observational tools, including eye tracking, electrocardiography (EKG) and respiration rate data to compose learners' profiles as they attended to concepts of elementary number theory.

Results from neuroscience research have also entered the ongoing debates among mathematics education researchers, some of which we have discussed throughout this chapter – such as primality of ordinality vs. cardinality conceptions of number (Coles, 2014), or the connections between counting and measuring (Iannece, Mellone, & Tortora, 2009) – and are used to support researchers' views or to guide their investigations. An innovative study by Tzur and Depue (2014) used fMRI to study the brain activity of participants comparing fractions and whole numbers. Their research demonstrated that different brain regions are activated by comparison tasks in different numerical domains, and that constructivist-based interventions during which participants engaged in particular problem solving tasks impacted what brain regions were activated for the tasks. Tzur and Depue (2014) advocated for collaboration between educators and neuroscientists and argued that brain research can provide further insight into the conceptual frameworks in mathematics education that were to date developed through observational studies.

CONCLUDING COMMENT

On reflecting over this chapter, and in particular on the promising roles technologies may yet play in the discipline, we note that in the realm of mathematics, technological advances have enabled new discoveries about numbers, and continue to do so. For example, as of 2013, the largest known prime number is $2^{57,885,161} - 1$ and it has over 17 million digits. A Pi Calculator Applet can compute a million digits of π in a few seconds on a normal PC; in theory it can compute more than 10^{15} digits of π. In the realm of mathematics education, our curiosities turn toward the possibilities that new technologies may afford PME researchers. Will technological advances help researchers better understand how various numbers are processed in a human brain? Will (and in what ways could) they shape the next decades of PME research focused on (and beyond) number? We expect that the PME research handbook published in the third decade of the 21st century will shed light on these, and other, important questions.

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3. RESEARCH ON THE LEARNING AND TEACHING OF ALGEBRA

INTRODUCTION

For many years, the mathematics education community has investigated the difficulties students have with algebra. Different aspects of algebraic thinking, considered to be fundamental to overcome those difficulties, have been analyzed by researchers using a variety of theoretical frameworks. Different proposals to redress those difficulties, based on research results, have been suggested and their effectiveness has been investigated. Results of these research studies have signalled that in order to foster students' development of algebraic thinking it is necessary to help them: improve their sense and dynamic conception of variables (i.e., variables as changing entities); develop their capability to generalize and to express their generalization; become aware of the dynamic relation that exist between variables; and, identify algebraic structure (e.g., Kieran, 2006; Radford, 2008, 2011; Mason, Stephens, & Watson, 2009; Ursini & Trigueros, 2011; Cooper & Warren, 2008). Within the mathematical topics studied in high school, algebra plays an important role. Researchers have found that the higher level of mathematics courses students take in high school, the greater their chance of attending and graduating from college and finding better paid employment in the future (Carnevale & Desrochers, 2003).

Algebra still remains an area of interest for the PME community. There have been major shifts in the focus of this research since it first came to the attention of PME researchers (Kieran, 2006). The themes that have witnessed a growth in interest by the PME community over the last ten years are: algebraic thinking in the elementary years (early algebra); generalization; structure sense; advanced algebra; and, the use of technological tools to support the development of algebraic understanding of secondary school students. One of the major gains that early algebra research has seen over the past ten years is the inclusion of researchers whose previous focus of research has been predominantly 8th and 9th students (aged 13 to 14 years) (e.g., Becker & Rivera, 2005, 2008; Radford, 2006; Sabena, Radford, & Bardini, 2005). Recent findings have indicated that many of the difficulties young students experience with algebraic thinking mirror the difficulties students' exhibit as they begin formal algebra (e.g., Becker & Rivera, 2007; Warren, 2006). Thus research with 13–14 year-old students is not only informing research regarding upper secondary and tertiary students but also informing research involving elementary students. Additionally there has been an increase in the emphasis on theorizing

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related to the nature of algebraic thinking of 5–14 year old students (e.g., Radford, 2006, 2010; Rivera, 2013).

Algebraic thought rests on some basic pillars and fundamental among them are the two aspects: variables as general abstract entities that can be represented in different ways; and, the 'structure sense' of algebra. These two aspects are involved in any algebraic activity (generalization, equations, functions, etc.). Research conducted this last decade on the teaching and learning of algebra, in secondary and university level, has used a variety of theoretical frameworks to analyze the complexity of working with these two aspects, paying attention to the particularities of their specific instantiations. In particular, PME researchers have been concerned with them through the analysis of students' and teachers' work with equations, relation between variables, pattern recognition, and generalization. In addition, this last decade has witnessed a rapid development of technologies. Its influence on mathematics' learning and in particular, in algebra learning has demanded further research. The possibility to use technologies to develop students' algebraic knowledge and skills, and the obstacles involved in their use by teachers has strongly attracted the attention of researchers.

This report is organised into four sections: The development of algebraic thinking among students aged from 5–14 years; the development of algebraic thinking among upper secondary and tertiary students; the use of technology; and a final section which integrates these themes, identifies the gaps in these themes and presents possible directions for future research in the domain of the learning and teaching of algebra. The examples presented in these sections have been drawn from the research that has occurred over the ten-year period from 2005 to 2015, with a particular focus on PME conference proceedings.

THE DEVELOPMENT OF ALGEBRAIC THINKING AMONG STUDENTS AGED FROM 5–14 YEARS

Early algebraic thinking refers to thinking about algebra early and looking at number from a more structural perspective. Thus, its focus is on developing in students the awareness of the structure of patterns and the structure of arithmetic (Mason, Stephens, & Watson, 2009). Early algebraic reasoning entails encouraging young students to become naturally aware of generalizations in numerical and non-numerical contexts and expressing these generalizations using a variety of semiotic signs (Radford, 2006). Traditionally algebra has only been taught after students have acquired a substantial amount of arithmetic knowledge, with the assumption that arithmetic provides the grounding on which to build algebraic knowledge. But as evidenced by the current research conducted by PME researchers, many early secondary students (13–14 years) struggle with this domain, and most of these difficulties can be traced back to their prior experiences in arithmetic. Thus a research focus for the future is how (and should) arithmetic and algebraic thinking be intertwined with each supporting the other across the first 10 years of school.

Underpinning the research classified as early algebra is the perspective that for many students the meaning of algebra is derived from its numerical foundations. In particular the focus of research over this last ten year period has been on investigating the challenges of the well-documented discontinuities that arithmetic has created for students beginning to formally explore algebraic concepts (Kieran, 2006). For example, students' prior arithmetical use of letters in formulas and as labels can negatively impact on their understanding of the concept of a variable (e.g., Kuchemann, 1981; Clement, 1982); and, many students experience difficulties when (a) interpreting equations with several numerical terms and unknowns (e.g., Linchevski & Linveh, 1999), (b) manipulating algebraic expressions (e.g., Kirshner, 1989), and (c) articulating the structure of a pattern or relationship in ordinary language (Macgregor & Stacey, 1993). Traditionally early algebra has tended to be associated with the elementary years of schooling. Given that the recent findings have indicated that many of the difficulties young students experience mirror the difficulties students exhibit as they begin formal algebra, this section has been extended to include both of these groups of students (i.e., elementary school students – 5 to 12 year old – and students beginning formal algebra – 13 to 14 year old).

In all 62 PME full research reports informed this section of the review. No short communications were included in this analysis. However, if the PME research report was further elaborated in a journal article, book section or book this source was also utilized to inform this section of the review. This section reflects the types of research that have occurred, and the particular themes that researchers have investigated with regard students aged 5–14 years. Initially the papers were classified according to their focus and the methodology utilized to explore this focus. The number of students included in quantitative studies ranged from 50 to 1300. Table 1 presents the frequency of PME papers according to their focus and their data collection method.

Table 1. PME papers – Algebraic thinking 5–14 year old students

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	25	8	3		36
Teaching algebra	6	4	1		11
Teaching & learning	5	4	4	2	15
Total	36	16	8	2	62

Thus in the last ten years research has predominantly occurred in the area of student learning with a focus on the use of qualitative methods such as interviewing individual students or videoing small groups of students as they engage in algebraic tasks. Secondary to this interest is the concern with the act of teaching algebra, and the relationship between what is taught and what is learnt. A meta-analysis of these research papers revealed that the focus of PME algebra researchers investigating algebraic thinking amongst 5–14 year old students over this period were: noticing and representing pattern structure (35 papers); and, working with equations, expressions and variables – the influence of arithmetic thinking (27 papers). This analysis was driven by the understanding that research in early algebra in the elementary school arose from the many difficulties beginning high school students exhibited as they studied algebra, difficulties that emanated from 'operating from an arithmetic frame of reference' with a focus on calculating (Kieran, 2006, p. 25). Thus the fundamental purpose of early algebra research has focused on further investigating and theorising about redressing this issue (e.g., thinking about numeracy equalities as relational, symbolizing relationships between qualities, developing functional thinking). In addition, the categories reported in this section of the review are also utilised in the second section of the chapter, allowing for a more coherent analysis of the issues pertaining to learning algebra to occur. The next section presents a synopsis of the research findings related to these themes.

Noticing and Representing Pattern Structure

Pattern activities have been considered to be one of the main ways for introducing students to algebra (e.g., Ainley, Wilson, & Bills, 2003; Mason, 1996). From this perspective, algebra is about generalizing (Radford, 2006). Previous research has evidenced that visual approaches generated in tasks involving the generalization of geometric figures and numeric sequences can provide strong support for the development of algebraic expressions, variables, and the conceptual framework for functions (Healy & Hoyles, 1999). However, not all activities lead to algebraic thinking. For example, placing the emphasis on the construction of tables of values from pattern sequences can result in the development of closed-form formulas, formulas that students cannot relate to the actual physical situation from which the pattern and tables of values have been generated (e.g., Amit & Neria, 2008; Hino, 2011; Warren, 2005). This impacts on students' ability to identify the range of equivalent expressions that can be represented by the physical situation.

The patterns utilised in the 2005–2015 research encompassed both linear and quadratic functions that were represented as a string of visual figures or numbers. The activities students engaged in involved searching for the relationship between the discernable related units of the pattern (commonly called terms), and the terms' position in the pattern. These reflect the types of activities predominantly used in current curricula to introduce young adolescent students to the notion of a variable and equivalence.

Students noticing and representing the pattern structure. Fundamental to patterning activities is the search for mathematical regularities and structures. In this search, Rivera (2013) suggests that students are required to coordinate two abilities, their perceptual ability and their symbolic inferential ability. This coordination involves firstly noticing the commonalities in some given terms, and secondly forming a general concept by noticing the commonality to all terms (Radford, 2006; Rivera, 2013). Finally, students are required to construct and justify their inferred algebraic structure that explains a replicable regularity that could be conveyed as a formula (Rivera, 2013). At this stage the focus is no longer on the terms themselves but rather on the relations across and among them (Kaput, 1995).

While the 2005–2015 research involved the exploration of the notion of variables, the findings suggest that signs other than the conventional alphanumeric symbols of algebra can be used to express variables (Radford, 2010). Given this caveat, the findings of this research exhibit that algebraic thinking can appear in students at an early age (e.g., Radford, 2010; Anthony & Hunter, 2008; Rivera, 2011; Warren, Miller, & Cooper, 2011). Having students engage in *quasi-generalization* processing using *quasi-variables,* that is expressing generalizations in terms of specific large numbers as examples of 'any number' significantly assists students to noticing and representing pattern structure, and providing a generalization in language and other signs including alphanumeric symbols (Cooper & Warren, 2011).

Difficulties students experience in noticing pattern structure. Emerging from the findings of this current research is that while young students are capable of noticing pattern structure and engaging in pattern generalization, they exhibit many of the difficulties found in past research with older students. As revealed in the findings of this research: young students have difficulties moving from one representational system to another such as from the figures themselves to an algebraic form that conveys the relationships between the figures (Becker & Rivera, 2007); students tend to be answer driven as they search for pattern structure (Ma, 2007); they engage in single variational thinking or recursive thinking (Becker & Rivera, 2008; Warren, 2005); they fail to understand algebraic formula (Warren, 2006; Radford, 2006); and, they have difficulties expressing the structure in everyday language (Warren, 2005). In addition, initial representations of the pattern (e.g., pictorial, verbal and symbolic) can influence students' performance. This is particularly evident as 139 10 and 11 year-old students explored more complex patterns (Stalo, Elia, Gagatsis, Teoklitou, & Savva, 2006), with pictorial representations of patterns proving easier for students to predict terms in further positions and articulate the generalization.

Capabilities that assist students to notice structure. Adding to the research is a delineation of the types of capabilities that assist young students to reach generalizations. The ability to see the invariant relationship between the figural cues is paramount to success (Becker & Rivera, 2006; Stalo et al., 2006). The development of specific language that assists students to describe the pattern (e.g., position, ordinal language, rows) (Warren, Miller, & Cooper, 2011; Warren, 2006) and fluency with using variables (Becker & Rivera, 2006) help students to express and justify their generalization. In addition, Becker and Rivera (2006) found that students who had facility with both figural ability and variable fluency were more capable of noticing the structure, and developing and justifying generalizations. By contrast, students who fail to generalize tend to begin with numerical strategies (e.g., guess and check) as they search for generalizations and lack the flexibility to try other approaches (Becker & Rivera, 2005). This has implications for the types of instructional practices that occur in classroom contexts. It is suggested that instruction that includes verbal, figural and numerical representations of patterns, and emphasises the connections among these representations assists students to reach generalizations (Becker & Rivera, 2006). An ability to think multiplicatively has also been shown to assist students generalize figural representations of linear patterns (Rivera, 2013).

Theories pertaining to noticing structure and reaching generalisations. Results from Radford's longitudinal study of 120 8th grade (typically 13–14 year-olds) students over a three year period delineated three types of generalization that emerged from the exploration of figural pattern tasks: *factual; contextual;* and *symbolic* (Radford, 2006). The first structural layer is factual: 'it does not go beyond particular figures, like Figure 1000'. The generalization remains bound at the numerical level. Expressing a generalization as factual does not necessary mean that that is the extent of student's capability. It may simply be that this level can answer the question posed by others or the context in which algebra is needed (Lozano, 2008). The second layer is contextual; 'they are contextual in that they refer to contextual embodied objects' and use language such as *the* figure and the *next* figure. Finally, symbolic generalization involves expressing a generalization through alphanumeric symbols. The suggested criteria that can be used to assist teachers to distinguish these levels of early algebraic reasoning are: the presence of entities which have the character of generality; the type of language used; and, the treatment that is applied to these objects based on the application of structural properties (Aké, Godino, Gonzato, & Wilhelmi, 2013). The latter refers to how students express this generality. Aké et al. (2013) suggest that algebraic practice involves two crucial aspects, namely, being able to use literal symbols as a general expression and relate this expression to the visual context from which it is derived. In addition, with growing patterns gesturing between the variables (e.g., pattern term, pattern quantity) in conjunction with having iconic signs to represent both variables (e.g., counters for pattern term and cards for pattern quantity) helped 7–9 year old Indigenous students to identify the pattern structure (Miller & Warren, 2015).

Rivera (2013) from the results of his longitudinal study with 2nd grade to 7th grade students begins to provide insights into how these shifts in thinking occur, from the figural representation, to the factual, contextual and symbolic generalizations. He theorises that these shifts involve toing and froing between thoughts and pattern, and fundamental to this movement is the role of abduction and induction. At the initial stage, from a limited sample set, a generalization is abduced or inferred and a hypothesis constructed. Induction involves testing this hypothesis through intensive experimentation. As students obtain terms for larger positions (e.g., step 10 and step 100), they review their generalization and make necessary adjustments. Thus a combined abduction-induction process allows students to state their generalization. Rivera adds rigour to this process by suggesting the conditions required to be in place in order to help students and teachers evaluate their generalizations. These are: The generalization must 1) Be *Non-monotonic*: the generalization that offers the best explanation can still be shown to be false if additional or different assumptions are made. 2) Deal with the *cut off point:* the generalization can explain why the stated generalization that is based on a few examples hold for the whole population. 3) Allow for *vertical extrapolation*: the generalization must support conclusions for sequences of values beyond the values that are already known. 4) Accommodate *the eliminative dimension*: the generalisation has be chosen from several plausible ones and provides the best understanding of the pattern beyond what is superficially evident. These condition stem from the research of Psillos (1996) and Peirce (1960).

Reaching generalization from figural patterns is a complex and difficult process for many young students. From the results of interviews conducted with 19 7 year-old students, Rivera (2011) explored the use and implications of parallel distributing processing to begin to explain the differences between their ability to generalize the structure inherent in figural patterns. Underpinning this theory of cognition of learning is that ‘knowledge emerges and is stored in connections among neuron-like processing units with experiences and learning altering, strengthening and continuously making adjustments in connections among units’ (Rivera, 2013, p. 100). The complexity of the model reflects the complexity entailed in students thinking as they search for generalizations. One advantage of the model is that it begins to take into account the notion of context (and prior learning/connections) as we explain the differences between students’ capabilities. In addition, it has been shown that students activate and coordinate a number of different semiotic systems when exploring figural patterns. They engage in oral speech (utterances), draw figures, construct patterns, and use iconic gestures (e.g., Chen & Leung, 2012; Sabena, Radford, & Bardini, 2005). The specific role this synchronization of these systems plays in the objectification of knowledge, and in particular as students move through the three types of generalization needs further investigation.

The transition to noticing the structure of quadratic patterns. The difficulties that students exhibit when generalizing figural linear patterns appear to be compounded as they move into figural quadratic patterns. In an empirical study involving 50 talented students aged 12–14 years Amit and Neria (2008) found that 23 students used an additive strategy when finding successive terms in the pattern. These strategies encompassed drawing other terms in the pattern and counting or using tables and lists, and tended to result in the generation of recursive generalizations.

Aligning with the findings of research incorporating linear figural patterns, the 14(28%) students who were successful in reaching a global generalization used a visual-based approach (i.e., visualised the growth in the pattern). By contrast, Chua and Hoyles (2012, 2013, 2014) reported that 93(56%) of similar aged students investigating a similar quadratic figural pattern reached a correct global generalization. They conjectured that this is the result of students' prior experiences in algebra, specifically the teaching of number patterns followed by an introduction to the concept of a variable. This conjecture aligns with Sigley, Maher and Wilkison's (2013) results showing that being introduced to the technical language of algebra in conjunction with formal notation assisted the 11 year old student, who was the focus of their study, to correctly articulate his global generalization and link it to symbolic notation.

Concluding comments. The findings of the 2005–2015 research with regard to noticing and representing pattern structure are significant for four reasons. First, while noticing and representing the structure of patterns is a complex process, these findings show that young students are capable of engaging in pattern activities and expressing the pattern structure as generalizations. This practice involves two crucial aspects, namely, generating a general expression for the pattern and relating this expression to the visual context from which the pattern is derived. Initial representations of the pattern (e.g., tables of a values) and the visual cues inherent in the pattern can influence this practice. Additionally, the development of specific language to describe the pattern and an increase in fluency in using variables can help students to express their generalizations. Second, this research has produced a number of theoretical frameworks that will guide the research that occurs in the future. The two main dimensions further elaborated in this research pertain to (a) the levels students pass through as they notice pattern structures and express these structures in a symbolic form (*factual, contextual, symbolic*) and (b) the role combined abduction-induction processes play in helping students move through these levels. Both these dimensions not only assist teachers to evaluate what students know but also inform the types of instructional practices that occur within the classroom context. Third, this research has reaffirmed the finding that expressing generalizations in symbolic notation is not a necessary condition of thinking algebraically. Algebra can be 'practiced by resorting to other semiotic systems and signs' (Radford, 2006, p. 3). The findings also suggest that prior experiences with less complex patterns (e.g., number and linear patterns) influence students' ability to generalise the structure of more complex patterns (e.g., quadratic patterns). Fourth, this research has generated a number of directions for future research in this area: How do we help young students develop their mathematical language and visual capabilities as they progress through the elementary years? How do we help older students transfer this knowledge to more complex patterns including quadratic patterns? What representations and pattern sequences assist these processes?

Working with Variables, Expressions and Equations – Arithmetic Thinking

Although it is now well recognized that algebraic thinking in the early grades can occur without the need to use letter-symbolic algebra (Kieran, 2006), students' understanding of the structure of arithmetic and associated use of arithmetic symbols still impacts on their ability to effectively engage in letter-symbolic algebra. For example, past research has presented many examples of how adolescent students hold a persistent belief that the equal sign is a syntactic indicator for a place to put the answer. Additionally, many of the misconceptions of the meaning of a variable persist (e.g., Lim, 2007; Trigueros & Ursini, 2008). It is also recognized that this could be due to the types of activities that are occurring in the early grades. Thus in the last ten years attention has been drawn to this issue, particularly in terms of investigating students' ability to form and manipulate equations, and solve inequalities. The two types of equations utilised in the 2005–2015 research were equations containing only numeric symbols (arithmetic equations) (e.g., $5+ \ = +7$) and equations containing alpha-numeric symbols (algebraic equations) (e.g., $2x + 3 = y + x$).

Working with variables. The 2005–2015 research with regard to variables has evidenced that young students can engage in the concept of a variable without the use of letters. The inclusion of visual-gestural cues, such as the sign for 'secret' proved important for deaf students (Fernandes & Healy, 2014) understanding of a variable. Additionally, Khosroshani and Asghari (2013) showed that the notion of a *specular* number (a number that is *spe*cific to the user but is treated as a partic*ular* non-specific number) helped pre-schoolers to engage in algebraic thinking. It also needs to be acknowledged that the symbols and letters commonly used in representing variables have emerged from particular historical and cultural contexts. For Australian Indigenous students, allowing them to create symbols that are culturally appropriate and personal appears to be an effective way to introduce them to working with variables (Matthews, Cooper, & Baturo, 2007).

However, evidence also suggests that the use of letters and symbols does not necessarily mean that students understand the notion of a variable (Hewitt, 2014). In his study with 12–13 year old students, Hewitt (2014) found, through probing students' statement that letter/symbol could be 'any number', that many responses exhibited the 'natural number bias', interpreting a letter as a natural number. In addition, for some, the natural number value it could be was mitigated by the ease of calculation that could occur, for example, 'it can't be 572' or 'it is even'. Christou and Vosniadou (2009) suggested that a reorganisation of students' initial knowledge of number needs to produce a conceptual change so that students can recognize a variable as a symbol that can stand for any real number.

Interpreting, manipulating and generating expressions. The difficulties that many students experience in both arithmetic and algebra in interpreting and manipulating expressions involving more than one operation are well documented. Van Hoof, Vandewalle and Van Dooren (2013) claimed that secondary students in their second year (14 years of age) also interpreted a letter as a natural number rather than a rational number which led them to make consistent errors when determining whether expressions such as $2 \times m > m$ are always true. Additionally, students continued to exhibit the propensity to calculate all arithmetic expressions from left to right, a problem that Gunnarsson, Hernell and Soonerhed (2012) found in a large proportion of 169 students aged 12 to 13. However, intervention involving tasks where brackets were used to emphasise the precedence of the operations (e.g., $5 + (3 \times 2)$) did not result in a significant number of these students transferring from a left to right strategy to using precedence rules when computing arithmetic expressions without brackets (Gunnarsson et al., 2012).

How students manipulate and generate equivalent algebraic expressions is also guided by student's structural sense of arithmetic. Geraniou, Mavrikis, Hoyles and Noss (2011) showed that the use of pattern-based activities involving figural growing patterns helped 11 and 12 year-old students generate and justify equivalent expressions. They also identified three main categories that students used to justify the equivalence of their expressions: Structural Justification for Equivalence (focussing on structural aspects of the figural pattern with little reference to its symbolic rule); Symbolic Justification (focussing on both the symbolic rule and the figural pattern); and Empirical Justification (focussing solely on the numerical aspect of the rule).

Finally, Meyer (2014) conjectured that, when 12 year-old students manipulated expressions, they utilized two different processes: Giving relevance (relating the certain parts of an algebraic expression to each other, while neglecting other parts); and, Basic structure (recognizing the basic structure of the expression together with how it is represented symbolically). For example, in an expression like ab + ac + ab a student may *give relevance* to the two ab's while ignoring ac and reformulate the expression as 2ab + ac. By contrast, another student may recognise the basic structure of the expression (each term is a multiple of a) which might result in the transformation of the expression to a(b + c + b).

Understanding equality and inequality. The 2005–2015 research with regard to equality and inequality resulted in two broad findings. First, the types of representations students experience can influence their ability to form an equation and recognise the equivalence between the equations. For example, five year-old students successfully used the balance scales to model arithmetic problems in real world contexts as equations with more than one value on each side (Warren, 2007). Additionally, Carlo and Ioannis (2011) found that using brackets to encompass each side of an arithmetic equation (e.g., (5+) = (+7)) with 2nd and 3rd grade students helped them to 'see' the unity of different terms connected by a sign or operation, that is, the equivalence between the equations expressions.

Second, even if students are successful in representing and manipulating equations, this knowledge does not necessarily positively impact on their ability to understand and handle inequality (Verikios & Farmaki, 2006). The findings of this later research suggest that this is due to the fact that the manipulations required to represent and solve inequalities do not align with those used for equalities. Finally, teachers' perceptions of their students' ability with regard to the algebraic concept of an equation do not necessarily align with their capabilities (Alexandrou-Leonidou & Philippou, 2005). Students are often more capable than some teachers imagined.

The influence of teaching. Teaching arithmetic for algebraic purposes can have a positive impact on students' growth in mathematics. Pittalis, Pitta-Pantazi and Christou's (2014) empirical study of 204 6-year-old students showed that these students' growth in algebraic arithmetic (understanding of patterns, equations and functions) over an eight-month period had a direct effect on their growth in conventional arithmetic, and an indirect effect on their growth in elementary number sense. Teaching arithmetic for algebraic purposes can also assist at risk students to transfer their arithmetic knowledge to algebra contexts. Livneh and Linchevski (2007) in an empirical study with at risk 7th grade students showed that intervention focussing on developing an understanding of the structure of algebraic expressions in arithmetic contexts entailing arithmetic expressions prevents students from making structural mistakes in compatible algebraic expressions. The results of a post-test at the completion of one years teaching without intervention indicated that the students at risk were unable to meet the requirements of a basic algebra course by the end of their first year of algebra. In the second year of the study, 7th grade students (aged 12 years) at risk participated in a purposely designed intervention consisting of items, such as, "Is 75 – 25 + 25 equal or not equal to 75 – 50?" This was considered to be compatible with the algebraic task: "Is 16 – 4x + 3x equal to or not equal to 16 – 7x?" The results of the post-test at the end of this intervention evidenced that these students could successfully engage with compatible-algebra tasks (tasks that mirrored the arithmetic tasks utilized in the intervention). However, these students failed to show significant progress in algebra tasks that were not compatible with the numerical tasks used in the intervention.

If instruction is appropriate, young students can learn to understand powerful mathematics structures such as the backtracking (unwinding) principle and the balance principle (Cooper & Warren, 2008). In their five-year longitudinal study with 7 to 11 year-old students, Cooper and Warren (2008) showed that the combination of balance and number line models was powerful in assisting these young students to determine that change resulting from addition–subtraction requires the performance of the opposite change (subtraction-addition respectively of the same amount) if one wants to return the expression to its original state. This mathematical structure underpins solving equations using the backtracking (unwinding) principle and balance principle.

However, Eisenmann and Even (2008) found that the same teacher does not necessarily enact the same curriculum materials in a different classroom. The results of an intense study on a 7th grade teacher working in two different classrooms showed that the discipline problems in one of these classrooms resulted in them engaging in fewer global/meta level activities (e.g., activities involving generalizing, problems solving, proving and justifying; see Kieran, 2004), activities that are seen as at the very heart of algebraic thinking.

With regard to texts and the importance they place on developing students understanding of the structure of arithmetic, Demosthenous and Stylianides (2014) in their intensive study involving 2814 tasks from a series of 4th grade to 6th grade Cypriot texts discovered that only 10.7% of these tasks were algebra related and less than 12% of these attended to exploring the structure of arithmetic and generalizing its arithmetic relations. By contrast, in their study involving 60 10–11 year-old students, Slovin and Venenciano (2008) reported that the 19 students who had prior experience with their Measure Up Curriculum, a program built on the theoretical framework developed by Elkonin and Davydov (1966) and focussing on relationships among quantities and the use of literal symbols from the first grade, were more capable at working with variables.

Concluding comments. The findings of the 2005–2015 research with regard to working with variables, expressions and equations are significant for four reasons. First, young students can engage in the concept of a variable provided they are allowed to use signs and symbols that are culturally and developmentally appropriate (e.g., own invented signs, spectacular or secret numbers). Additionally, it seems that focussing on relationships amongst quantities (e.g., lengths and volumes) and using literal symbols from the first grade, helps these students later understanding of the concept of a variable. Second, students' lack of understanding of the structure of arithmetic and associated use of symbols persists in negatively impacting on their understanding of equations. While the types of representations that students experience at a young age can help them 'see' the equivalence between each side of an equation (e.g., balance scales to model equations, brackets to show the different expressions on each side of an equation), many students still exhibit a natural number bias when assigning meaning to letters and symbols. In addition, it seems that, as students grow older, the more resilient these misunderstandings become. Third, the knowledge students gain with regard to successfully representing and manipulating equations does not necessarily transfer to nor is applicable for inequality contexts. Fourth, teaching arithmetic for algebra purposes can positively impact directly on students' understanding of algebra and arithmetic, and indirectly on their growth in elementary number sense. In particular, intervention that focuses on developing an understanding of the structure of algebraic expressions in arithmetic contexts with at risk students who are beginning formal algebra study can assist them in successfully engaging in compatible algebraic tasks.

This research, and in particular the fourth point, has generated a number of directions for future research. Although the impact that students' understanding of the structure of arithmetic remains problematic with regard to their ability to transfer to formal algebra, there has been little research particularly relating to the arithmetic curricula and teaching interventions that can assist this transition to occur smoothly. The little that has transpired clearly shows that teaching arithmetic for algebra purposes can positively impact on both arithmetic and algebra.

THE DEVELOPMENT OF ALGEBRAIC THINKING IN STUDENTS 15-YEAR-OLD AND UP

In this section we provide a panoramic view of the research on algebraic thinking and on the learning and teaching of algebra in students 15-year-old and up, with special attention on that developed by researchers in the PME community in the last ten years. As in previous section, PME papers were classified according to their focus and the methodology employed. Table 2 presents the frequency of the PME papers according to their focus and their data collection method. As can be observed, the great majority focus on students' learning using a qualitative approach.

Table 2. PME papers – Algebraic thinking 15 year old students and up

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	21	4	1	3	29
Teaching algebra	2			1	3
Teaching & learning	2		4		6
Total	25	4	5	4	38

The research studies informing this section are grouped according to the topics mentioned in the introduction: Focus on variables; Generalization and skills for generalizing; Equations: Solutions and meanings; Related variables and functions; and Structure sense.

Focus on Variables

Algebra is the basis of all other fields of mathematics, and also of natural and social sciences and engineering. Solving problems in algebra involves abstraction and the capability to interpret and use symbols, together with the possibility to generalize, model different situations, and use rules to perform symbol manipulation.

Although it is true that the introduction of early algebraic thinking is not necessarily linked to the use of literals (Kieran, 2004) and, that "using letters does not amount to doing algebra" (Radford, 2006, p. 3), using and interpreting symbols to designate the objects of algebra is fundamental to the development of algebraic thinking and to the possibility of using them outside the classroom. At upper school levels this implies working with literal symbols to represent variables. Otherwise, the solution of problems would become troublesome. This is why this conceptual study area has been and continues to be of interest to researchers.

According to many studies, the development of algebraic thinking goes hand in hand with the development of the concept of variable, a multifaceted concept linked to the different facets of algebra (generalizing, problem solving, structure analysis, modelling, analysing related quantities). For more than forty years researchers have pointed out the many difficulties students encounter in grasping the essentials of the notion of variable and in working flexibly with its multiple uses at different levels of abstraction. When solving problems in school algebra the different uses of variable (unknown, general numbers, related variables, parameters) very often appear together, and the same symbols are used to represent them. Students are expected to grasp the essence of each use, work with each of them and shift fluently from one to the other as required by a specific task. Some researchers have insisted on the need to distinguish the different uses and aspects characterizing algebraic variables arguing that a more explicit distinction of the diverse meanings associated with the word 'variable' would help students make sense of the symbols used to represent them.

Research developed in the last thirty years has shown that each use of variable is linked to specific epistemological and didactical obstacles. It has been suggested as well that when algebra is taught taking only one specific use of variable as the starting point and central focus, the possibility of flexibly moving between its different uses and the richness derived from the relationships between them is lost or obscured and students' understanding of algebra remains limited (see Kieran, 2006). These difficulties continued to be studied in the last ten years and recent studies have focused on how students' can be helped to give meaning to variables. In spite of the important role played by parameters, these seem to have been neglected by algebra researchers. Since in almost any problem situation involving a variable its multifaceted character is present, a deep understanding of this concept becomes a source of richer comprehension of algebra and mathematics in general.

Several studies explored the appropriateness of different environments and approaches to promote the creation of meaning and better understanding of variables. Lim (2007), for example, created opportunities for students to attend to meaning and to use numbers as a platform to investigate algebraic expressions and structures. Through a case study this researcher illustrates the feasibility of helping 11th grade students improve their algebraic thinking, in particular, moving from manipulating symbols in a non-referential symbolic manner to reasoning with symbols in a goal-oriented manner, from association-based prediction to coordination-based prediction, and from impulsive anticipation to analytic anticipation. Wille (2008) signalled that the versatility of students' thinking about variable was enhanced when they experienced its different aspects. The possibilities of understanding the different uses of variable in parallel with each other, in real contexts and multi-representational environments was underlined by Tahir, Cavanagh and Mitchelmore (2009). They found that studying variable as a function in parallel with variable as a generalized number using multiple representations and real contexts helped students to come to a more complete meaning of the term 'variable'. Their results also showed a reduction in students' misconceptions and an improvement of their performance when using this approach. In a study about an effective strategy to help students develop ideas about the solution set of systems of linear equations, using modelling, Trigueros, Possani, Lozano and Sandoval (2009) suggested that students' strategies were strongly related to their flexibility in moving between the different uses of variable. Those students who showed proficiency when working with variables and those who developed this flexibility during the course employed richer strategies; were able to use them to model and work with different types of problems; and, were able to interpret different types of solution sets, including those containing free variables and restrictions (that is, variables in solution sets that can take any value in a given set of real numbers and that are associated to systems that have an infinity of solutions, for example t in $S =\{\mathbf{x} \in R^3/\ x = 2 - 3t,\ y = 2t,\ z = 1-t,\ t \in [0, 25]\}$), which have proven to be difficult for most studies, and to use them to analyze both real situations and models' solutions.

Little research has been conducted internationally using the same tools to explore students' capability to work with variables in order to establish similarities and differences in achievements and difficulties across different contexts and locations. The aim of these international studies is primarily to assist countries understand their own education systems by setting their strengths and weaknesses against the backdrop of those of other countries. This was the purpose of Alvarez, Gómez-Chacón and Ursini (2015) research. They analyzed 8th and 11th grade students' responses to a questionnaire testing their understanding of algebraic variables. The results provide evidences that might help the participating countries to revisit their curricula, focusing on their strengths and weaknesses and the support provided to students to develop the ability to think in algebraic terms.

Researchers in the last ten years have been less interested in students' use and understanding of parameters in spite of previous studies reporting that students have many difficulties when they encounter parameters in algebraic expressions (see Kieran, 2006). Bardini, Radford and Sabena (2005) investigated 11th graders' cognitive difficulties when working with parameters in the context of the generalization of patterns and showed how the semiotic problem of indeterminacy, a central element of the concepts of variable and parameter, reveals students' weak understanding of letters and algebraic formulas. Moreover, the many difficulties students of different school levels have with the interpretation, manipulation and symbolisation of parameters were documented by Ursini and Trigueros (2011). Students in this study conceived parameters as products of second order generalizations, that is, product of generalising first order general statements (for example, the equation $3x^2 + px + 7 = 0$ involves a family of quadratic equations). Their results also showed that students need a clear referent or statement that gives meaning to parameters to be able to work with them; otherwise, they perceive parameters as general numbers and have many difficulties handling parameters when they encounter them in any type of algebraic expression.

Generalization and Skills for Generalizing

Algebraic thinking is characterized by the capability to generalize and express generalization. Many researchers have stressed that generalization is one of the paths to algebra (e.g. Mason, Graham, & Johnston-Wilder, 2005). According to Carraher, Martinez and Schliemann (2008, p. 3):

> Mathematical generalization involves a claim that some property or technique holds for a large set of mathematical objects or conditions. The *scope* of the claim is always larger than the set of individually verified cases; typically, it involves an infinite number of cases (e.g., ‘‘for all integers’’). To understand how an assertion can be made about ‘‘all x’’ we need to consider the grounds on which the generalization is made. (p. 3)

Fundamental to the act of generalizing is the learner. A context that has gained the greatest attention by PME researchers in the last ten years is the patterning context. In this particular context, Radford (2006) stated that:

> Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some elements of the sequence S, being aware that this commonality applies to *all* terms of S and being able to use it to provide a direct *expression* of whatever term of S. (p. 5)

As delineated in the first section of this chapter, many studies have shown that both algebra beginners and more advanced students can deal successfully with particular cases of patterns, but have serious difficulties in generalizing and expressing the relationships in terms of algebraic language. Some of the reported dimensions that contribute to these difficulties for young 4th grade students are: the lack of spatial visualisation techniques (Warren, 2005); the lack of appropriate generalizing strategies (Moss & Beatty, 2006a, 2006b); and, difficulties in using algebraic language to express generality (Warren, 2005). Many of these difficulties have also been shown to exist in secondary students (Ursini, 2014; Alvarez, Gómez-Chacón, & Ursini, 2015). These results suggest that many students across all ages tend to lack the capability to reflect on their own actions and become conscious of them. They lack metacognitive abilities and the capability to use algebraic language as a tool to communicate mathematically.

In the PME tradition of algebra as generalization to develop and express mathematical proofs and modelling situations (Kieran, 2006), Boero and Morselli (2009) considered algebraic language as a system of signs and transformation rules useful to generalize arithmetic properties. An adaptation of Habermas' construct of rational behaviour is used by Boero and Morselli (2009) to describe and interpret some of the students' difficulties and mistakes, and to provide indications for the teaching of algebraic language. Their analysis showed that goal oriented reasoning and using verbal language are necessary in order to perform the actions needed in proving, modelling and problem solving.

Regarding teacher education programs, Hallagan, Rule and Carlson (2009) considered that strategies that involve inquiry, problem solving, and critical thinking helped pre-service teachers to focus on interpreting and making sense of the role of symbols involved in the generalization of patterns. In this same line, Radford (2006) has suggested that teachers and teacher educators should be aware of students' practices in order to distinguish algebraic generalization in students' activity from other forms of work with the general, which according to his previously mentioned definition, are not truly algebraic. He also warns that teachers must be equipped with knowledge to be able to distinguish different approaches to generalization.

Stressing the importance of focussing on teachers' explanations of students' responses to mathematical tasks, El Mouhayar and Jurdak (2015) explored teachers' arguments to explain students' responses to pattern generalisation tasks. They identified four different perspectives teachers assume (student lens, teacher lens, mixed teacher, and inability to explain students' responses) stressing that the pattern generalization types mediated teachers' perspectives.

Equations: Solution and Meaning

The PME community has devoted decades of research to the identification and the analysis of students' difficulties in interpreting and manipulating algebraic expressions. Attention has been paid to students' approaches to solve equations (particularly linear, less research has been carried out on students' solving of quadratic equations), systems of equations, and inequalities. Students' procedures and strategies to deal with such tasks have also been analyzed from different perspectives. Well-grounded teaching experiments have been designed, with and without the support of technology, to help students overcome their difficulties and to help them construct meaning for algebraic expressions, equations and solution procedures (see Kieran, 2006). In spite of these efforts, and probably due to different kinds of obstacles (from bureaucratic and political, to economic, socio-cultural and cognitive, from teachers' and parents' academic preparation, attitudes and beliefs, to curricular organization) students' difficulties are still present in most classrooms around the world and at different school levels. Research efforts have continued in the same lines and concerns about teaching equations, and the influence of students' difficulties with equations in university mathematics education have emerged (Borja-Tecuatl, Trigueros & Oktaç, 2013).

Efforts to deepen our understanding of the possible causes of students' difficulties and to look for ways to help students interpret equations and their solutions have continued during the last ten years. Nogueira de Lima and Tall (2006) and Nogueira de Lima (2007), for example, investigated Brazilian 14- to 16-year-old students' interpretation of the concept of equation and its solution set. Their findings show the influence of previous arithmetic studies, previous algebra experience and teachers' beliefs about algebra on students' conceptions. Based on their results they stress that teaching practices focussing on a single solution procedure limits the development of students' flexibility to give meaning to equations, to the solution set, and to their procedural knowledge. The question of meaning was also tackled by Caglayan and Olive (2008) who studied 8th grade students' use of a representational metaphor for writing and solving equations in one unknown. They reported that only one of 24 students was able to construct a "family of meanings" to make sense of equations and solutions, and to connect algebraic expressions to representational metaphors when negative quantities were involved. Based on the analysis of the knowledge used by a teacher and her 8th grade students when generating and evaluating equations to model word problems, Caglayan and Olive (2008) extended previous results on meta-representational competence and stressed the importance for teachers to develop the capacity to recognize and discuss students' criteria when choosing representations.

Students' cognitive tendencies when dealing with different tasks involving unknowns has continued to attract researchers' attention. Filloy, Rojano and Solares (2008) found that the cognitive tendencies identified when students operate with a single unknown reappeared when they were learning methods to solve two-unknown linear equations systems, showing that the reference to and the sense of the different representation of unknowns must be reconstructed when facing new types of problems.

The concern for students' understanding of equations and solution sets has led researchers to focus on university students' capabilities to use equations and inequalities, to manipulate them and to interpret solution sets. Studies have focused on the development of algebraic thinking when students encounter university mathematics topics. Researchers have found that algebra is central in the learning of advanced mathematics, but it can act as a "key and a lock" at the same time in understanding advanced concepts such as limits of functions (Alcock & Simpson, 2005). They insist that when students used mathematical notation as a way to apply mathematical procedures, without making sense of expressions or of the goal of manipulations, for example to solve inequalities, algebra can become a lock to their success. By contrast, for students who were able to interpret and use the different variables involved in definitions or inequalities, algebra acted as a key to their understanding of new concepts.

Attention was also paid to teachers and teachers' development in relation to their content knowledge and the way they teach equations. Findings suggest that their poor understanding of mathematical structures and their knowledge about students influence their teaching decisions. The impact of teachers' training programmes in their teaching practice and how innovative teaching strategies can support students' learning of algebra have been explored as well. Koirala (2005), for example, found that using mathmagic, a game in which students are invited to play with numbers ("think of a number", "add 10", "multiply it by 3", and so on), can help low-performing 14 years old students develop confidence and interest in learning basic algebraic concepts and enhance their understanding of variables, expressions and equations.

Related Variables and Functions

Function is a central concept in mathematics and has a significant role in mathematics education. Literature on mathematics education has paid in the past, and in the last decade as well, a lot of attention to this concept. Studies have used different theoretical perspectives and have reported the many difficulties students, pre-service and in-service teachers face in understanding and learning the meaning of function, such as, paths to understanding functions, process-object conceptions or the use of different representations (Elia, Panpura, Eracleus, & Gagatsis, 2007; Gerson, 2008; Bayazit, 2011). Specific tools based on results of diagnosis questionnaires have been used in the design of online applications that can help and give feedback to students, both in the case of algebraic manipulation and in drawing and interpreting the graphs of functions. These applications can also help teachers to design specific strategies to help students to better understand these central concepts.

Research has shown that students can understand the correspondence between numbers, independently of the representation used, but find a dynamic conception between numbers, which includes variation, difficult (Ursini & Trigueros, 2011). To help students make sense of variation, some researchers have proposed using a method where each operation is explicated so that students can reflect on the situation and "make tacit meanings explicit" (Thompson, 2008). Other researchers stress that using an appropriate methodology, such as games, modelling situations in context or using metaphors from everyday experiences to introduce functions to students can help them develop the notion of correspondence into that of covariance (Francisco & Hähkiöniemi, 2006; Dogan-Dunlap, 2007; Pierce, 2005). Teaching strategies based in knowledge theories, such as APOS theory, have also been successful in assisting at risk students make sense of the concept of function (Dubinsky & Wilson, 2013). Students participating in this study were able to develop their algebraic thinking from beginning with their own verbal explanations to ending with writing functions in symbolic forms.

For some researchers, the main difficulty with the understanding of functions is the fact that they present different facets depending on the representations used (Nyikahadzoyi, 2006; Adu-Gyamfi, Stiff, & Bossé, 2012). All these authors suggest that flexibly working with different representations, being able to understand the characteristics of functions in each representation, and relating different representations to each other are fundamental to the learning about the concept of a function.

Some researchers have indicated that attention to the role played by symbols and the use of models has proved to be successful in fostering students' capabilities to make sense of related variables. These researchers have focused on specific types of functions, for example, quadratic, exponential or trigonometric functions (Pierce, 2005; Francisco & Hähkiöniemi, 2006; Panasuk & Beyranevand, 2010). However, after all these years there has been no consensus on how to help students understand the concept of function, and on how to overcome the difficulties inherent in its learning. Nilsen (2015) focused on the way functions are introduced by teachers in both lower and upper secondary school, and the usual gap between formal explanations provided in textbooks. This author argued that the introduction of functions is done without explicitly considering mathematical aspects like the range and domain, or the uniqueness property, and that dependent and independent variables continue to prove problematic. He concluded that examples and explanations provided should underpin and support the mathematical properties of function.

All the previously referred-to studies have analyzed one variable functions. More recently, however, this field of research has expanded and two variable functions and parametric functions have been investigated. These studies have shown that generalization of knowledge from one variable functions to other types of functions is not straightforward and that difficulties students experience with one variable function transfer to other types of functions. Results demonstrate that each generalization involves particular obstacles that need to be studied in depth and that effort is needed in order for students to construct a formal and more inclusive definition of function (Trigueros & Martinez-Planell, 2010; Martinez-Planell & Trigueros, 2012; Stalvey, 2014; Weber & Thompson, 2014).

Function is clearly one of the most important concepts in mathematics. The teaching and learning of functions has received a lot of attention. For example, strategies to help overcome students' conceptual difficulties have been investigated, and activities that promote students' reflection and formally make aspects of functions explicit have been developed. However, more effort is required both in research and in making research results available to a wider educational community in order to help a majority of students learn more deeply about functions.

Structure Sense

Students at different school levels and university levels, all have difficulties in transferring what they learnt in the context in which they first met different concepts to other unfamiliar contexts (Hoch & Dreyfus, 2006). This research emphasises the importance of making sense of manipulations, generalization, functions, and different properties of algebraic expressions. Previous studies have already called attention to the importance of making sense of the structure in algebra (see Kieran, 2006). But the meaning of "structure sense" was and continues to be under debate.

Arcavi (1994), for example, referred to symbol sense as a complex feel for symbols. He included in it an appreciation of the power of symbols, the ability to manipulate and to interpret symbolic expressions, and have a sense of the different roles symbols can play in diverse contexts. Other researchers have emphasized identifying structure in expressions or equations; identifying visually repeating characteristics of expressions; and, exhibiting versatility of thought as key dimensions of structure sense (Kirshner & Awtry, 2004, Tall & Thomas, 1991). Hoch and Dreyfus (2007) gave a precise and pragmatic definition of structure sense based on their studies with high school students. They defined structure sense as: recognising a familiar structure in its simplest form; dealing with a compound term as a single entity and through appropriate substitutions; and, choosing appropriate manipulations to make best use of a structure. They also underlined the importance of the substitution principle in this definition. In accordance with these criteria and based on their observations, Novotná and her colleagues (Novotná, Stehlíková, & Hoch, 2006; Novotná & Hoch, 2008), suggested definitions linked to university algebra. They also suggested that there is a relationship between high school and university algebra structure sense, and viewed high school structure sense as sub-components of university structure sense components. In addition, they underlined the importance of structures as part of mathematics in general, and in the learning of algebra in particular. Studies on students' structure sense have played an important role in research on algebraic thinking in the last ten years. This interest has been reflected in the PME community's work where the importance and development of structure sense for successful performance in algebra has been stressed, for example identifying the structure of equations or inequalities.

In the last ten years research on advanced algebraic thinking has become an important area of interest for mathematics education researchers. Aiming at finding factors that could explain students' success in the solution of complex algebraic equations and inequalities, Trigueros and Ursini (2008) analyzed in depth the approach followed by 36 university students working with equations and high achieving university students working with inequalities (Ursini & Trigueros, 2009). Their results showed that understanding the different uses of variable, including parameters, together with other factors that can be associated to structure sense, in terms of Hoch and Dreyfus' definition are crucial for success. These results led these researchers to extend the definition of structure sense to include understanding of variable. They considered this understanding as critical to successfully solving problems and as a starting point to developing the capability to work with problems requiring advanced mathematics. Using their definition, Trigueros, Ursini and Escandón (2012) analyzed the responses of 270 Pre-Calculus university students to six complex algebraic problems in order to find possible relations among all the included categories and their role. Using Implicative Statistics they found that, although all the categories appeared to be strongly interrelated, two categories: distinguishing that a problem involves the analysis of different cases and, a correct use of definitions, played significant roles in students' ability to use all the aspects considered in the definition of structure sense. Moreover, a flexible use of variables and the capability to interpret parameters played a dominant role when implication relations were studied. These were tightly linked to students' ability to identify algebraic structures and use definitions correctly. Thus, there is an urgent the need to take these implications into account in the design of activities that aim to foster structure sense in students.

Systems of linear equations are a central topic in the transition from elementary Algebra to Linear Algebra. Students are known to have difficulties interpreting the solution set of systems of equations, particularly when systems have an infinite number of solutions. Research results on this topic signal that these difficulties are related to students' difficulties in interpreting a variable as a dynamic entity, that is as a "changing entity" which can take different values (Ursini & Trigueros, 2011); with the notion of set as an object; with the distinction of the process of solution and the notion of solution set; and with the interpretation of different representations of solution sets (Afamasaga-Fuata´I, 2006; Trigueros et al., 2007). All these difficulties can be related to the lack of structure sense in terms of the definitions described above.

Focusing on students' development of structure sense, Hoch and Dreyfus (2006) used a questionnaire to analyze 165 high achieving 10th grade students' performance. Their results showed that most of these students did not use a high level of structure sense when solving exercises requiring the use of algebraic techniques, and that those students who used structure sense made fewer mistakes.

Searching for factors that could explain the lack of progress in the development of algebraic proficiency of students from 8th to 12th grade, Van Stiphout, Drijvers and Gravemeijer (2011) considered structure sense as part of what Arcavi defined as symbol sense (see above) in order to analyze test items. The analysis revealed that most students were not able to deal flexibly with the mathematical structure of more complex expressions and equations that involve, for example, the use of subtraction technique. These researchers emphasised that this is an obstacle for attaining a higher level of conceptual understanding which requires a shift of thinking. Using the anthropological theory of didactics, Chevallard and Bosch (2012) propose a structural approach that considers algebra as a tool to model different intra and extra-mathematical situations. These authors consider such an approach as a way to overcome the difficulties that the learning of algebra presents nowadays. Other authors have used the same theory, together with the ontosemiotic approach, to offer a model that takes into account the structure of algebra and that can be used in the teaching of algebra at the secondary level and to design richer teaching sequences. The emergence of structure sense in classroom interactions at the secondary level was analyzed by Janßen and Bikner-Ahsbahs (2013). Focusing on linear equations and functions' structure they studied the development of structure sense. They searched for crucial moments of objectification claiming that when it happens the object is accessible in other situations and students can identify algebraic structures.

All these researchers agree that even students, who can display proficiency when working with elementary algebra problems, may have difficulties in applying the techniques to complex problems. They also concur on the need to teach explicitly the abilities included in structure sense. Some recommendations were: using brackets to help students to "see" algebraic structure and make evident the presence of a new expression that could be considered as an entity or a new variable; working with examples where analysis or classification of problems in terms of their structural properties is the goal of the activities; making the role of variable and changes of variable explicit in classroom discourse; asking how definitions and properties can be used; asking students for the goal of the activity instead of the solution; and, stressing the importance of validation. Such approaches would lead students to engage in greater levels of reflection and analysis related to algebraic situations.

Additionally, the development of structure sense would provide students with a stronger foundation on which more complex and abstract algebraic thinking can be developed. For instance, research conducted in the last ten years has shown that many students finish Linear Algebra courses with a limited understanding of the main ideas of this discipline. Analyzing possible causes of difficulties and looking for ways to help students develop a richer understanding of concepts of this discipline has been the main concern of these studies. Researchers have underlined that most students can cope with the manipulations involved in solving a variety of Linear Algebra exercises, but that they do not develop an understanding of the concepts involved in such manipulations and are unable to apply them to problems that need competencies that go beyond rote manipulation (Hannah, Stewart, & Thomas, 2014). Other researchers have suggested that Linear Algebra difficulties can also be related to students' conception that a great effort is needed in its learning (Martinez-Sierra, García, & Dolores-Flores, 2015). Some researchers have related students' difficulties in understanding particular Linear Algebra concepts to the lack of formal thinking (Britton & Henderson, 2009; Wawro, Sweeney, & Rabin, 2011), described in the definition of structure sense as based in a complex feel of symbols and a flexible use of variables (Arcavi, 1994; Ursini & Trigueros, 2009).

However, other studies have pointed out that the need to work jointly with several algorithms also generates obstacles for students (Hannah, Stewart, & Thomas, 2014). In a study focusing on students' understanding of linear independence and dependence, Stewart and Thomas (2006) found that working with the solution algorithm for homogeneous systems of equations, together with the interpretation of the obtained solution set in terms of these concepts was an obstacle for most students. They also concluded that interpreting free variables in the solution set seems to be key to understanding linear dependence and independence as well as their relation to other Linear Algebra concepts such as span and basis.

Results of studies related to specific concepts (Plaxo & Wawro, 2015; Stewart & Thomas, 2008) underline that difficulties in Linear Algebra are related mainly to lack of understanding of definitions, and the use of properties in the solution of exercises when working with modelling or application problems. In spite of difficulties associated with the learning of abstract concepts, several research studies have shown that using models to approach abstract concepts can help students to make sense of definitions and apply their learning in the solution of complex problems (Zandieh & Rasmussen, 2010).

Algebraic thinking in Abstract Algebra courses and its teaching also received some attention (Novotná, Stehlíková, & Hoch, 2006; Hare & Sinclair, 2015). Novotná et al. (2006) were interested in students' understanding of algebraic operations and their structure. They found that while some students are able to abstract specific properties of one or more mathematical objects to form the basis of the definition of new abstract objects, other students constructed abstract concepts through logical deduction from definitions. They concluded that few students were able to reason inside a new structure spontaneously and to find a structure's properties. On this basis they developed a model for teaching binary operations and their properties as a first step to develop students' structure sense. Hare and Sinclair (2015) used semiotic theory to analyse teachers' signs in an Abstract Algebra course. They found that the act of pointing is important in 'underlining' objects and relations among them.

It is interesting to observe that the development of structure sense, both from the perspective of high school algebra and from that of university algebra plays a fundamental role in university students making sense of algebraic structures and applying concepts related to different algebraic structures. It is of fundamental importance to do more research on this topic in order to help students to develop structure sense so that they can progress in their understanding of advanced mathematics topics and apply this structure sense to the solution of formal and real problems.

ABOUT THE USE OF TECHNOLOGY

The use of technology continues to be an interesting and important topic of research in the mathematics education community. As new software is developed these questions always arise: How do we use it in the classroom? What kind of tasks need to be designed within specific technological environments? What are the results of its use in terms of developing algebraic thinking and learning? Research in the last ten years evidences that technology has had a small but significantly positive impact on students' learning. However, this impact is dependent on teachers' use of the technology, the classroom interactions that occur (e.g., Ursini & Sacristan,

Table 3. PME papers – Use of technology in algebraic thinking

Focus	*Method*				
	Qualitative	*Quantitative*	*Mixed method*	*Theory*	*Total*
Student learning	14		1	3	18
Teaching algebra	3				3
Teaching & learning	3				3
Total	20	0	1	3	24

2006; Rakes, Valentine, McGatha, & Ronau, 2010; Guzman, Kieran, & Martinez, 2011) and, the tasks' design (Johnson, 2015). Table 3 gives a glimpse of PME papers related to technology use and clearly calls for more research in this area.

Already well-known technologies, such as CAS, continue to be studied particularly in terms of the potential of instruction design that includes their use and the development of teacher training programs. Researchers have found that high expectations on this tool can be considered as naive and that experiences show the complexities involved in their use in the classroom (Trouche & Drijvers, 2010). In spite of this complexity, it has been found that the possibilities offered by CAS can enrich and extend students' and teachers' view of algebra. CAS can for example foster the emergence of algebraic reasoning or be used to the development of novel tasks (Kieran & Guzmán, 2009; Heid, Thomas, & Zbiek, 2013; Kieran & Drijvers, 2006; Kieran & Saldanha, 2005).

PME researchers continue analyzing if and how the use of specific technologies help to improve communication and understanding in the classroom. Among others, Sacristán and Kieran (2006) analyzed a student's difficulties in understanding notation for general polynomials using CAS. They found that by making conjectures and trying them in TI-92 Plus calculator the student improved in the use of general notation and eventually could make sense of the ellipsis sign. The role of teachers in orchestrating class discussion and fostering attention of students to overcome their difficulties shows how some approaches to classroom communication can be useful for teachers (Kieran, Guzmán, Boileau, Tanguay, & Drijvers, 2008).

The impact of CAS in students' learning continues to play a role in research (Kieran & Damboise, 2007; Solares & Kieran, 2012). Researchers have demonstrated the impact of CAS in improving weak 10th grade algebra students in being able to do procedures and understand concepts, or help students' articulate different perspectives to understand equivalence of expressions. These studies found that CAS had a positive impact on the development of procedures and on concept development. In another study (Lim, 2007) technology and its graphing potential was used to teach transformation of functions and their graphs to secondary school students. The impact on students' learning was examined using APOS theory. They compared results of students who used CAS with those who didn't use it in their classroom. They found that the group that used CAS were able to apply transformations on mental objects they had constructed to construct new graphs, while the other students needed to calculate specific points to graph its transformations.

Comparing teachers' beliefs, how they are reflected in their use of CAS in secondary school algebra instruction, and how they shape CAS algebra tasks has also been studied (Kieran, Tanguay, & Solares, 2011; Özgün-Koca, 2011). Researchers concluded that technology can help to change teachers' views and to make sense of specific algebraic content by working with appropriate practical and theoretical rich experiences where they can reflect, and discuss to shape these experiences for their classroom needs.

Other technologies have been designed and tested in terms of their potential in the teaching and learning of algebra. The dynamic metaphors of change and dragging, together with the process of naming when working with spread-sheets, was studied by Wilson, Ainley and Bills (2005). Their results appeared to support the evolution of meaning and notation for variable. The ReMath European project was developed to investigate the role of representations of mathematical objects offered by different Digital Dynamic Artifacts (DDA) when used in educational contexts. The DDA that received more attention in terms of research reported at PME meetings was a microworld called Aplusix (Nicaud et al., 2006). Studies on its impact on students' learning were conducted in different settings. Maffei and Mariotti (2006) found that a specific tool of the software, called detached step, played an important role in making students conscious of their errors and in helping them to reflect on and overcome their difficulties. Exploring the potential of the feedback component of Aplusix when teachers used it during discussion in the classroom, Maffei, Sabena and Mariotti (2009) found that teachers' questioning played a pivotal role in developing a semiotic chain starting from the DDA's signs that led students to give mathematical meaning to algebraic expressions. The development of this chain of interpretations was not linear, but it helped to maintain students' interest in asking new questions and, at the same time, promoted interactions among them and with the teacher. As a result, a process of unfolding the meaning of the tool feedbacks signs was developed in the group. In a teaching experiment conducted by Maffei and Mariotti (2013) to investigate how the Graph representation provided by this DDA could become a tool of semiotic mediation, they found that the Graph representation tool helped students to carry out algebraic manipulation and allowed them to refer to the mathematical meaning of equivalence class of algebraic expressions. Based in semiotic considerations Chaachoua, Chiappini, Pedemonte, Croset and Robotti (2012) analysed and compared DDA possibilities in terms of their impact on students' learning.

In another study, Kynigos, Psycharis and Moustaki (2010) performed a design experiment to explore 17-year-old students' construction of meaning and the use of algebraic like equations. These researchers focused on students' engagement while they constructed and controlled animated models on MoPix and in their possibility to make sense of structural aspects of equations underlying the models' behaviour. Through the experiment, students edited ready-made algebraic like equations and constructed new ones to assign particular behaviours to objects. Results showed how students developed different degrees of structuring and shifted gradually from a view of equations as processes to a view of equations as objects. According to authors reification was not a one-way process of meaning making; it was a dynamic process of understanding supported by the use of the technology.

The effects of teaching algebraic solving of word problems using Hypergraph Based Problem Solver (HBPS) software were studied with 15–16 years old students (González Calero, Arnau, Puig, & Arevalillo–Herráez, 2013). Their strategies were analyzed. González et al. (2013) found the emergence of a tendency to construct equations where one of the variables appears on one side of the equation, without using all the available information, and how they tried to use it to calculate values of the related variable. They attributed this tendency to the interpretation of the equal sign in the equation as a comparison between quantities rather than as a sign to do something.

Naftaliev and Yerushalmy (2009) investigated innovative uses of technology in the domain of school algebra. They compared the contexts for mathematics learning created by printed diagrams vs. interactive diagrams and video clips vs. interactive animations, when presenting to students an activity describing a motion situation and another requiring the description of a linear function. The two activities included an interactive diagram. They argue that the process of concept construction occurred as a result of the students' decision to change the representation of the data in the activity to build a focused collection and to expand the given representations, or build new ones. The ways in which sketchy interactive diagrams were used by students transformed sketchy information into an important component of conceptual learning.

Concern about teachers' developmental process to integrate technology into their classroom practices led researchers to study how teachers use their knowledge of algebra, to teach different concepts using a Dynamic Software (GeoGebra). Studies have found that stressing interconnection between knowledge of teaching algebra and knowledge of teaching with technology is fundamental for success. Johnson, for example, analysed the transition from variation to covariation creating environments that involve non-temporal changing quantities. Her experience led her to underline the importance of activities that provide students with opportunities to attend to multiple varying quantities from the same measure space as well as the role of the teacher in promoting interaction between students who have a different conception of variation and covariation (Johnson, 2015).

Overall, the studies related with the use of technology in the teaching and learning of algebra in the last ten years have focused mainly on how technologies can help students to make sense of mathematical signs, calculations and results. Interesting results have been found on how teachers can use technologies in their classroom to promote communication and reflection. Research findings suggest that when used in an active and participative way, technology can help teachers in improving students learning. It is evident that technologies have evolved towards more dynamic and powerful designs that can be used both by teachers and students to conjecture, explore, try out and make sense of the meaning of algebraic expressions, to find solutions of equations and to work with related variables in a variety of interesting and motivating contexts. The dynamics of technology development continue to pose challenges to researchers and students, while new questions about their use in the classroom and the nature of the tasks to be designed to promote students' interaction and learning keep arising.

FINAL REFLECTIONS AND FUTURE DIRECTIONS

The study of how we teach and learn algebra has played a role in research within the PME community. However, some epistemological questions arise, as we need, as a community, to define what exactly we mean by algebra, algebraic thinking, algebraic reasoning, and algebraic problems. Researchers have worked so far as if those terms had the same meaning for all researchers. However, some positioning facing the difficulties in learning this subject make it clear that this is not the case. More effort to clarify these meanings is needed. This would not only add clarity to research results, but also point out to different research and teaching strategies tuned to possible different positions.

Even though algebraic thinking continues to be an important topic for the PME community of researchers, this review has evidenced that, while research in the domain of early algebra has increased and intensified, research pertaining to secondary and tertiary students has diminished as compared with previous decades. Juxtaposed against this trend is the continued difficulty that secondary and university students have in this domain. In spite of the progress we have made in our use of technology to teach algebra and our understanding of student difficulties, the problems that have been evidenced in past research still exist. Algebra remains an important domain of mathematics, and is fundamental for advanced mathematics learning. How can students learn university mathematics without a thorough understanding of algebra? Thus, not continuing research in this university and secondary sector would be a great mistake regarding the future of mathematics.

With respect to generalization, the majority of the research that has occurred has been in the early algebra area using qualitative methods with a focus on generalizing the structure of patterns. There is a call to reignite the focus on generalizing the structure of arithmetic, particularly in regard to its relation to the structure of algebra. This review evidences tentative findings that intervention focusing on the structure of arithmetic has positive pay offs for students successfully identifying the structure of algebra. However, this review shows, on the one hand, that many of the misconceptions that students have with regard to algebraic structure (e.g., the order of operations) are entrenched at an early age, and on the other hand, a lack of interest from researchers about the role that proofs play in generalization.

Thus important questions requiring further research are: When and how should this integration occur? When and how should we be engaging students in conversations and experiences that focus on arithmetic for algebraic purposes rather than arithmetic purely for arithmetic purposes? Should this occur somewhat simultaneously or consecutively? What is the role of algebraic proofs in this integration?

Research in the last decade has clearly put forward the importance of development of structure sense at the secondary and tertiary levels and its relation to the study of advanced algebra and advanced mathematics. Research efforts in this topic should be a priority in the years to come. Particularly, studies about intervention strategies, comparison and the influence of institutional constraints using sound methodological approaches need to be encouraged.

There is evidence throughout this review that how teachers teach has an impact on what students learn. Results of many studies also underline the importance of working both with students and teachers stressing the relevance of developing students' capability to generalize, to interpret symbols, and to express their thoughts and generalizations in correct mathematical (and algebraic) language, to fully understand functions, and to use modelling through all the school levels so the students can become acquainted with parameters and other elements of structure sense. Even though Kieran insisted ten years ago on the importance of doing more research about teachers' algebraic thinking development, this issue has not received enough attention in this decade. The knowledge that teachers have about mathematical structures and the beliefs they hold with regard to students' capability to learn algebra influences the decisions they make as they teach. In addition, it has been demonstrated in this review that coming to an understanding of algebraic concepts requires the use of a range of representations and modes of learning. Thus more research focusing on these topics is imperative to forwarding students' learning of algebra and new comparative studies may shed some light both in curricula and how it is enacted in different countries and school levels.

Finally, while research in the fields of use of technology in the classroom and its influence on students learning and on the use of algebraic language has been somewhat intense over the last 10 years, little research has occurred in the areas of generalization, cognition and students capability of understanding functions, developing structure sense and in how technology is used by teachers and students in real classroom contexts. We are thus calling for a balance of our knowledge across all these areas since it is important if our aim is to continue to enhance the development of students' algebraic thinking from elementary school up to the university.

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4. RESEARCH ON THE TEACHING AND LEARNING OF GEOMETRY

INTRODUCTION

The chapter provides a comprehensive review of recent research in geometry education, covering geometric and spatial thinking, geometric measurement, and visualization related to geometry, as well as encompassing theoretical developments and research into teaching and teacher development. Without doubt, the research of the *International Group for the Psychology of Mathematics Education* (PME) community in the field of geometry education has advanced since the first PME research handbook reviewed PME research over the 30 years from the inception of PME to 2005 (see Gutiérrez & Boero, 2006). In general, the emphasis of subsequent geometry education research has increasingly been on the use of technology (especially forms of dynamic geometry software) and how this impacts on geometry teaching and learners' geometrical thinking (especially on the teaching and learning of geometrical reasoning and proving), on teachers' geometric content knowledge, and on teacher development for geometry education. As such, studies examining the uses of forms of digital technology are addressed in every section of this chapter.

At same time, there has been continuing work related to spatial reasoning, geometric measurement, and visualization related to geometry. There has also been a continuing focus on the development of students' knowledge regarding understanding of geometric figures, definitions and inclusion relations, identification of shapes and language issues. In these studies, there are fewer examples of a furtherance of the Piagetian legacy, while use of the van Hiele model has continued alongside more recent developments in theory and methodology such as discursive, embodied, and eco-cultural perspectives (e.g. Ng, 2014; Owens, 2015). Thus, many research studies have focused on modes of understanding (visual, figural, conceptual), as well as on mental images and their manipulation, while employing new theoretical notions and methodologies.

The content of this chapter reflects the main emphases of research in geometry education as presented at PME conferences over the period 2005–2015. The synthesis is presented in the form of the following sections: spatial reasoning, geometric visualization, geometric measurement, geometric reasoning and proving, students' knowledge, teachers' knowledge and development, and teaching geometry and the design and use of geometric tasks.

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 109–149.

A NOTE ON REVIEW METHODOLOGY

There are a number of well-established methods for conducting a research review (Cooper, Hedges, & Valentine, 2009). While a literature review is a vital part of every research report, the purpose of this research synthesis is to make explicit some of the connections and relations between individual studies that otherwise may not be so visible. As such, constructing this review involved the purposeful selection, review, analysis and synthesis of research on geometry education that was presented at annual PME conferences over the period 2005–2015, inclusive. Where appropriate, connection is made to work presented at PME conferences prior to 2005, as is connection to work published in relevant journals and books. The content of each set of PME proceedings from 2005 to 2015 was digitally – searched, and also hand-searched, to create a database of research reports. Each research report was reviewed and analysed, and this set of analyses used to develop the synthesis presented in this chapter.

SPATIAL REASONING

> Spatial reasoning has always been a vital capacity for human action and thought, but has not always been identified or supported in schooling. (Whiteley, Sinclair, & Davis, 2015, p. 3)

Previously in the field of spatial reasoning, spatial capability was examined essentially for its relation to mathematical learning, connected to cultural and teaching factors as well as to imagery and strategies for geometric measurement of area and volume (Owens & Outhred, 2006). There were also some studies about spatial problem-solving strategies in relevant tasks (e.g. Oikonomou & Tzekaki, 2005). However, there was limited specific interest in this capability *per se*, its meaning and definition, its role in curricula, its development in school.

A link between spatial capability and geometric thinking was made during earlier PME research on the use of technology in approaching geometry, such as the use of *Logo* (e.g. Edwards, 1994). More systematic research increased when the learning of space acquired a particular value. As Sack, Vazquez and Moral (2010, p. 113) have argued, spatial reasoning is now seen as a vital component of learners' successful mathematical thinking and problem solving. More recently, Sinclair and Bruce (2014) led a compendium of reports on projects that have focussed on spatial reasoning for young learners. This mapped out "the terrain of established research on spatial reasoning" by examining "the actualities and possibilities of spatial reasoning in contemporary school mathematics" through offering "examples of classroom emphases and speculations on research needs that might help to bring a stronger spatial reasoning emphasis into school mathematics" (p. 173). Much of this work is expanded upon by Davis and the Spatial Reasoning Study Group (2015).

Studies of Students' Knowledge Related to Spatial Capabilities

Earlier studies investigated connections between spatial capability and geometric thinking. In their research, Xistouri and Pitta-Pantazi (2006) examined connections between spatial capabilities (mental rotation and perspective-taking) and geometrical thinking related to symmetry, while Kalogirou, Elia and Gagatsis (2013) investigated how visualization and mental rotation might be related to geometrical figure apprehension (perceptual and operative) as proposed by Duval (1999). Using data from relatively large-scale samples of primary and secondary school students, these studies showed significant relations between spatial capabilities and performance in symmetry, perspective-taking capability as well as geometrical figure apprehension. More specifically, the results of the first study indicated that perspective-taking capability is more related to symmetry performance than spatial rotation, being thus a predictor of students' performance in reflective symmetry, while data from the second showed that spatial capability is "positively related to geometry achievement and problem solving" (Kalogirou et al., 2013, p. 134). By examining the data from the sample of secondary school students, the authors suggested that it is likely that, as students get older and receive more advanced teaching in geometry, they tend to use figures not just as spatial representations but as "semiotic representations of geometric objects" (p. 135).

In a study of primary students on spatial visualization and spatial orientation with net tasks (matching net cubes to cubes) and model tasks (finding top views of models), Diezmann and Lowrie (2009) found that students mainly used matching or matching-and-eliminating strategies. The researchers' concluded that the students' difficulties in visualizing and explaining their thinking might be due to the lack of prior experience and under-developed mental imagery.

In investigating the development of spatial reasoning in pre-school children, Tzekaki and Ikonomou (2009) invited 30 children, aged 4.5 to 6.5 years old, to observe, one by one, two-dimensional Lego configurations and retain their characteristics in order to reconstruct them, either by watching or from memory. The analyses of the children's reconstructions demonstrated a continuous improvement of their spatial thinking and provided interesting information about the spatial characteristics that children at this age retain mentally when they attempt to copy a spatial situation. More specifically, such children easily retain information related to the number and shape of bricks, or to their left-right placement (corresponding to their own orientation), but they encounter difficulties in finding relative positions that demand combining spatial information.

In order to investigate young children's spatial strategies from kindergarten to primary age, Reinhold, Beutler and Merschmeyer-Brüwer (2014) video-recorded task-based one-to-one clinical interviews with 22 pre-schoolers (aged 5 to 7) as each child was presented with a series of four tasks that involved 'buildings' made of glued cubes and drawings of 'buildings' (shown in a 'cavalier' perspective).

Using Thurstone's (e.g. 1950) framework of distinguishing three major spatial capability factors (spatial relations, visualization, and spatial orientation) and using previous research on cube building (e.g. Battista & Clements, 1996), they reported on the nature of pre-schoolers' building strategies in relation to their capabilities of enumerating the number of cubes in a three-dimensional cube building. While Reinhold et al. found that while students' paying attention to intended structural elements (counting in rows or columns) does not guarantee an awareness of the structure of the 'building', they could gain insight into structural elements and could change "trial and error building strategies into orientation in structural elements" (p. 87).

More specific research by Panorkou and Pratt (2009, 2011) explored how individuals experience and think about dimension. In their first study, in which a phenomenographic approach was implemented, two pairs of 10 years old students and 10 teachers were interviewed with questions related to their dimensional thinking. The findings formed a characterization of this thinking in a variety of ways: dimension as action; as state (involving location); material dimension (involving measuring or conceptions based on vision or touch); abstract dimension; and dimension as prototype or hierarchy (with relationships between dimensions). Continuing their study Panorkou and Pratt (2011) designed tasks using *Google Sketchup* and conducted a number of extended task-based interviews with 10 year-old students. They observed the students expressing various "situated abstractions" such as "polygons can be 'flat' (in a 2-D space) or 'coming out' (in a 3D space)" and "polygons that look flat in 3D can be disconnected" or "twisted" (pp. 342–343). They concluded that "a key idea about dimension seems to be that it in some sense depicts the level of capacity of the space" (p. 343).

Studies by Diezmann and Lowrie (2008), and by Lowrie, Diezmann and Logan (2011), focused on primary students' knowledge of maps of localities. In the first study a GLIM (Graphical Languages in Mathematics) test was administrated to a sample of 378 4th grade students, plus 98 students were interviewed using 12 items from the test. The results revealed key difficulties including interpreting vocabulary incorrectly, attending to incorrect foci on maps, and overlooking critical information. In the later study, information is encoded in the form of fixed attributes (marks and symbols) in a particular spatial orientation. Lowrie et al. (2011) examined the performance on six map items of 583 students of 2nd and 3rd grades, from metropolitan and non-metropolitan locations. The results showed significant performance differences in favour of metropolitan students on two of six map tasks. In trying to explain the differences, they speculated that metropolitan students might be more likely to be exposed to coordinate map systems than students in non-metropolitan areas and that "the additional requirement for students to locate information besides what was provided in the direct instructions proved challenging for non-metropolitan students" (p. 149).

Summarizing, research in the field of spatial capabilities indicates a low development of skills related to spatial orientation, spatial relations and transformations, as well as understanding of dimensions and localities. However, spatial experiences such as reconstruction of spatial configurations or cube building are likely to support progress of spatial abilities. This kind of research is significant because, as noted above, spatial reasoning, more than being an important component of human action and thought, is known to be closely connected to geometric thinking and development of geometric knowledge.

Teaching Proposals Improving Spatial Reasoning

A range of studies has aimed at improving spatial reasoning for different ages. In earlier research, Owens (2005) examined how pre-service teachers were using substantive communication about space mathematics in primary schools. A qualitative analysis of observations in their classroom showed that, teachers, after taking a large number of example lessons, worked systematically with their students' knowledge attempting to extend it, by providing effective challenges and questions. In general, working with spatial tasks in the classroom, games, toys or relevant software improve significantly different aspects of spatial capabilities and spatial thinking.

More recently, Highfield, Mulligan and Hedberg (2008) studied the case of two children exploring a *Bee-bot* programmable toy, a tool that enabled them to engage in transformational geometry. These two children demonstrated relational thinking to plan, program and manipulate the toy through a complex pathway and developed interesting problem-solving strategies.

Experimenting with teaching approaches, Chino, Morozumi, Arai, Ogihara, Oguchi and Miyazaki (2007) proposed a spatial geometry curriculum utilizing 3-D dynamic geometry software in lower secondary grades. The results, coming after comparing experimental with control groups as well as results of the national survey of Japan, identified positive effects regarding the construction of spatial figures by moving a plane figure and the explanation the students gave for a 3-D figure represented in 2-D. Hegedus (2013) reported on a multi-modal interactive environment where young learners were able not only to "click-drag-deform mathematic objects on a screen as in traditional dynamic geometry" but also experience "force feedback related to mathematical properties through the same device" (p. 33). Psycharis (2006) reported on how 13 year-olds dynamically manipulated geometrical figures involving ratio and proportion tasks, while Samper, Camargo, Perry and Molina (2012) reported a case study of implication and abduction in dynamic geometry.

Both Moustaki and Kynigos (2011) and Ferrara and Mammana (2014) have researched the spatial capability of much older students. In their research, Moustaki and Kynigos (2011) looked for instances in which students' visualization, construction and mathematical reasoning processes might contribute to the enhancement of those capabilities. They developed a '3-D Modelling & Cutting' microworld and used it with some 12th grade engineering students specializing in Programming Computer Numerical Control (CNC) Machines. The analysis showed that the students initially perceived the figures and shapes represented in the 2-D drawing in a "purely iconic way instead of a mathematical one" (p. 262). With greater experience, the students came to realise that they had been 'misled' by the static 2-D drawing and needed to use 3-D geometrical objects to specify spatial relationships among the component's parts that would not differentiate as they changed viewpoints.

For Ferrara and Mammana (2014), the visual challenge involved in the approach of spatial geometry was the use of 'flat' diagrams for geometrical figures. Using the dynamic geometry software *Cabri 3D*, they introduced a definitional 'analogy' between quadrilaterals and tetrahedra for 'edges' and 'faces'. Undergraduate mathematics students tackled two main tasks; introducing the medians for quadrilaterals and tetrahedra, and conjecturing about the properties that hold in both cases. These tasks, say the researchers, pushed the students towards a search for similarities and differences, invariants and changes, between the two figures. In this way the learners managed to "see in space" (p. 59) through the affordances offered by the dynamic geometry software.

With elementary-age children (in Grade 3), Sack, Vazquez and Moral (2010) and Sack and Vazquez (2011) reported on using 3-D models, 2-D conventional and semiotic (abstract) representations, verbal descriptions of figures, and tasks using *Geocadabra* (Lecluse, 2005) software by which a multi-cube structure can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid coding. Working with different representations, the children had to calculate in multiple ways how many unit cubes were in relevant structures and connect the result to the sum of the numbers in the figures' top-view coding grid.

Summarizing the results of these studies, spatial tasks combining 2-D and 3-D geometric figures supported by relevant technological tools are likely to foster spatial-knowledge development and improve students' spatial reasoning, confirming, thus, the important role of technological environments in the development of spatial thinking.

GEOMETRICAL VISUALIZATION AND VISUAL THINKING

> Geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight. (Zeeman, quoted in Royal Society, 2001, p. 12)

In this section visualisation is taken to be the capacity to "represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz, 1990, p. 75) and attention is paid to visual intuition. For both, there is some inevitable overlap with spatial reasoning. As such, some research reported in the section on spatial reasoning may also appear in this section, and vice versa.

In the first PME handbook, Owens and Outhred (2006) covered a good deal of research on visualization alongside findings concerning the use of imagery in mathematics in general, and also in spatial processing and geometric thinking. In relation to this, Presmeg (2006) summarised issues in visualization by first clarifying terms relating to semiotics (such as signifier, registers, iconic, indexical, or symbolic signs) and then explicitly examining imagery (mental images) and externally-presented inscriptions involving visualization. Presmeg explained that "both visual imagery and inscriptions are sign vehicles that are instantiations of visualization in mathematics, insofar as they depict the spatial structure of a mathematical object" (p. 22).

Visual Cognition of Geometrical Objects

A number of research studies have focused on 'visual cognition', defining it as a mental process (perceiving, recognizing, retaining in memory, etc.) that refers to the way an individual acquires and processes visual information. Usefully, Kalogirou, Elia and Gagatsis (2013) pointed to differences between terms such as visual perception and visualization. They suggested that visual perception, while one of the most important factors affecting the capability to recognize plane shapes, only provides a "direct access to the shape and never gives a complete apprehension of it" (pp. 129–130). On the contrary, they argued, visualization is "based on the production of a semiotic representation of the concept and gives at once a complete apprehension of any organization of relations"; as such, visualization in mathematics "requires specific training in order to grasp directly the whole configuration of relations and to handle the figure as a geometrical object" (p. 130).

Widder, Berman and Koichu (2014) have been searching for "a better understanding of the visual obstacles' constituents, and the interaction between them" as that might be "the key to improve spatial geometry instruction" (p. 370). With data from testing high-attaining grade 12 students, their study confirmed "the existence of a prototype representing a cube" in that the overwhelming majority of the participants "drew the same normatively-positioned cube frequently used during spatial geometry instruction" (p. 375). While the prototypical use of normative drawings of cubes in spatial geometry instruction "may form a mental image meant to assist visualization", at the same time Widder et al. argued that this "may not allow enough flexibility, and therefore hinder identification and manipulation of a 3-D geometrical situation in un-normative sketches" (Widder et al., 2014, p. 375).

Relevant to students' visual cognition appears to be teachers' capability in visualization in geometry. For example, Markovits, Rosenfeld and Eylon (2006) investigated 25 teachers' performance in visual tasks along with their prior content knowledge and beliefs in the area of visual cognition. The results showed that the visual cognition of these teachers was limited, and their capabilities in visual estimation, free recall and graphical reproductions were close to those of 3rd grade students. Cohen (2008) examined pre-service and in-service teacher's knowledge of mental images and their beliefs about geometrical straight lines and planes. Their findings revealed conflicting teacher beliefs between formal knowledge and mental images as well as typical misconceptions about lines and planes.

Sack and Vazquez (2008), based on a spatial operation capacity model (SOC) conducted an after-school teaching experiment with two groups of 3rd and 4th grade students. The authors found that the student's performance on standardized test items that use verbal visualization terms (for example, top, side and front views) "may be compromised by unconventional language use rather than lack of visual cognition" (p. 224).

Haj-Yahya and Hershkowitz (2013) aimed at "linking visualization, students' construction of geometrical concepts and their definitions, and students' ability to prove" (p. 409). With data from testing grade 10 students, they found that many of them knew the formal definitions of the various quadrilaterals but did not make use of the definitions when faced with tasks using forms of visual representation of shapes. In many cases, say Haj-Yahya and Hershkowitz, "students know the formal definition but do not make use of it when faced with a visual task representation|" (p. 415).

Chumachemko, Shvarts and Budanov (2014) were also interested in the development of visual perception. Focusing on the Cartesian coordinate system and, in particular, the "transformations of perception that are needed to approach this mathematical visual model" (pp. 313–314), they compared the eye movements of participants at three levels of mathematics competence and they confirm their hypothesis: when detecting a point on the Cartesian plane "the better participants are educated, the shorter are their gaze paths, and the more the number of their fixations is reduced, and the durations of their tasks solving become shorter" (p. 316).

Overall, research agrees in the existence of limits to visual cognition and how there are visual obstacles in different recognition processes both for students and for teachers. In some cases, the visual aspect might even distract students from their mental or relevant theoretical knowledge, a finding that needs further investigation.

Visualization in Reasoning and Problem Solving

Introducing notions of 'linking visual active representations' (LVAR) and 'reflective visual reaction' (RVR), the aim of a teaching experiment by Patsiomitou and Koleza (2008) was to explore the role of these notions in a dynamic geometry software environment. With data from 14 secondary school students, the results showed that prior knowledge played a significant role in parallel with LVAR and RVR as a shift from visual to formal proof led students to formulate "if …then" propositions and to move "between two successive 'Linking Visual Active Representations' only by means of mental consideration, without returning to previous representations to reorganize his/her thoughts" (p. 94).

When undergraduate students are reading a 'worked proof', research by Lin, Wu and Sommers (2012) found that visualization corresponds to "needing to keep spatial representations in their working memory and to look between proof and figures" (p. 151). By studying the eye-tracking movements of undergraduate students as they read geometry proofs of different difficulty levels, the researchers found evidence that "visual reception and visualization occur simultaneously" (p. 152).

In their studies with pre-service teachers, Torregrosa and Quesada (2008, 2009) focused on what they call configural reasoning in which discursive and operative apprehensions (Duval, 1999) are coordinated in order to solve a problem or generate a proof. They found that visual predominance tends to inhibit the visualisation of the configuration sucg that configurative reasoning and the proving process are not always interrelated.

In the same context of solving geometrical problems, Pitta-Pantazi and Christou (2009) investigated whether individuals' cognitive styles, measured in terms of object imagery, spatial imagery and verbal capability, were related to their mathematical creativity. Some 96 pre-service teachers answered the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) and were examined in a mathematical creativity test for their capabilities in area, shape, pattern, problem solving and number. The results showed significant connections between spatial imagery and cognitive style, on the one hand, and mathematical fluency, flexibility and originality (as components of creativity) on the other, but no connections of object imagery and verbal cognitive capability to any dimension of creativity.

In their study Ramfull and Lowrie (2015) examined the connections between students' cognitive style, visualization and mathematics performance. They examined 807 6th graders from Singapore schools with three instruments: the C-OSIVQ questionnaire for measures of cognitive styles, the Paper Folding Test for spatial visualization and the Mathematics Processing Instrument for problem solving performance. The results align with previous studies by indicating significant correlations between cognitive styles (mainly spatial imagery information processing) and spatial visualization and problem solving abilities.

It is apparent from all aforementioned studies that visualization is indispensable in proving and problem solving. Visual aids support students' and teachers' thinking and both appear to improve their visual imagery for the needs of a solution or a proof. However, the visual representations or process they develop are not always effective in solving or proving relevant tasks, but there is still limited research related to the connection of visuality (as defined at the beginning of this section) to creative developments. Studies with digital technologies, such as DGEs, are providing more evidence and are offering new possibilities in the visualization of geometric objects.

Visualizing in Geometry and Use of Gestures

> Humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment (Roth, 2001, p. 9)

Research in geometry education has special interest in the role of gestures in mathematical communicating and thinking as an aspect of geometric visualization. In their Research Forum, Arzarello and Edwards (2005) examined gestures as a way of processing and communicating geometric ideas based on psychological, semiotic and psycholinguistics theoretical frameworks (Alibali, Kita, & Young, 2000; Bara & Tirassa, 1999; Peirce, 1955; Radford, 2003). Thus, they recorded the dynamic evolution in the use of gestures as pointed out by the social activity of the students in a geometric context and their discussion about solid shapes. They first analysed gestures and speech alongside written words and mathematical signs (c.f. Edwards, 2005). Later in the forum, Arzarello, Ferrara, Robutti and Paola (2005) extended this by examining relations between the use of gestures and the development of new 'perceivable signs'. They recorded the progression of students' solution during the construction of solids and examined the introduction of signs with gestures. At first, the students' gestures had an iconic function presenting the solid they were describing. Gradually they became 'indexes' (in the sense of Pierce) in the communicative attempt of transferring knowledge to others and finally they acquired a symbolic function; thus their relation developed in a piece of theoretical knowledge.

Maschietto and Bartolini Bussi (2005) approached the study of the construction of mathematical meanings in terms of development of semiotic systems (gestures, speech in oral and written form, drawings) in a Vygotskian framework with reference to cultural artefacts. In their paper they presented a teaching experiment related to perspective drawing with 4th-5th grade students. The authors described how they analysed "the appropriation of an element of the mathematical model of perspective drawing (visual pyramid) through the development of gestures, speech and drawings, starting from a concrete experience with a Dürer's glass to the interpretation of a new artefact as a concrete model of that mathematical object…" (p. 315). Analysis of the students' protocols highlighted the parallel development of different semiotic systems (gestures, speech in oral and written form, drawings) and their mutual complementary enrichment. Research by Sack, Vazquez and Moral (2010), mentioned earlier, also reveals the use of gestures by young students.

In their research, Ng and Sinclair (2013) studied children's use of gestures on spatial transformation tasks. They found that children used gestures "as multi-modal resources to communicate temporal relationships about spatial transformations" (p. 361). Subsequently, Ng (2014) reported on the interplay between language, gestures, dragging and diagrams in bilingual learners' mathematical communications, when students rely on "gestures and dragging as multimodal resources to communicate about dynamic aspects of calculus" (p. 289). For more on high school students engaged in perceptual, bodily, and imaginary experiences while discussing about calculus concepts in a dynamic geometry environment, see Ferrara and Ng (2014).

GEOMETRIC MEASUREMENT

> Measurement plays a central role in reasoning about all aspects of our spatial environment. (Battista, 2007, p. 891)

In their review of earlier PME research, Owens and Outhred (2006) depicted the complexities of measurement principles and their teaching. Here, subsequent research is reviewed – first on length, then on area, volume, and angle.

Length

> An understanding of linear measure is imperative, as it provides the basis for length, area, and volume. (Cullen & Barrett, 2010, p. 281)

As Watson, Jones and Pratt (2013, p. 76) confirm, research has shown that when children measure lengths they can end up "applying a poorly-understood procedure rather than focusing on the correspondence between the units on the ruler (which may be seen erroneously as a counting device) and the length being measured". What is more, research by McDonough (2010, p. 294) reports "confusion regarding unit name, length, and relationships" when the object being measured is longer than the ruler.

Given the different ways that measurement tasks can be presented, Cullen and Barrett (2010) compared the strategies used by young children (aged 4–5 years, and 7–8 years) when engaged in measurement tasks that were presented either using *Geometer's Sketchpad* (GSP) software or as paper-and-pencil. Noting that measurement strategies include the endpoint strategy (where the child refers either to the right or left endpoint as the length of the object) and the point-to-the-middle-of-an-interval strategy, the researchers found that "linking the intervals on a ruler to iterable discrete objects, or to virtual representations of those objects, were both successful ways to motivate students to use the effective 'point to midpoint' strategy" (p. 287). They concluded that interval-identifying strategies should be beneficial when teaching students to measure the length of an object with a ruler.

Beck, Eames, Cullen, Barrett, Clements and Sarama (2014) investigated whether grade 6 children's knowledge of measurement related to their capability to use double number lines when solving problems involving proportional reasoning. Using ideas of hierarchic interactionalism, Beck et al. defined a series of 'levels' – the first two of which are Length-Unit-Relater-and-Repeater (LURR) level, where children "measure by repeating, or iterating, a unit, and understand the relationship between the size and number of units", and the Consistent-Length-Measurer (CLM) level, whereby children "see length as a ratio comparison between a unit and an object" and "use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits" (p. 106). They found that children at the LURR level relied on iterative strategies, while children at the CLM level could "partition and correctly attend to units along one scale but not yet coordinate units along two scales simultaneously" (pp. 111–112).

Future research could build on what is already known about the foundational ideas of measurement such as identical units, iteration and zero-point.

Area

Given that area measurement is known to pose further challenges for learners (see Watson, Jones, & Pratt, 2013, p. 76), Gonulates and Males (2011) analysed US primary school mathematics textbooks and found little variety in the ways in which knowledge was expressed. The researchers concluded that the textbooks did not provide opportunities for students to engage with conceptual knowledge of area.

Whether primary-age children might benefit from being taught a curriculum that integrates 2-D geometry with area measurement, compared with a curriculum that stressed numerical calculation of area, was studied by Huang (2011). Huang's conclusion was that integrating area measurement instruction with numerical strategies and geometric materials seemed to be "a promising approach to promoting children's conceptual understanding of area measurement" as well as their capacity to "explain geometric reasoning with measurement when solving problems" (pp. 47–48).

The development of different components of students' knowledge about area measurement was investigated by Frade (2005). Frade found that students aged 11 to 12 showed a concept of area as a physical geographic space while by age 12–13 this had evolved to them being able to use "the rectangle area formula adequately" and having "the 'know how' to solve a number of problems" (p. 327).

Area concepts continue to appear in the mathematics curriculum through to university. Cabañas-Sánchez and Cantoral-Uriza (2010) focused on how first-year university mathematics students could transform convex and non-convex polygons so that area was conserved. In analysing the arguments presented by the students, the researchers found that the students used both 'parallelism' (area between parallel lines is conserved) and relevant formulae to calculate areas.

Future research might develop further promising ways of promoting children's conceptual understanding of area.

Volume

Turning to 3-D measures, Watson, Jones and Pratt (2013, p. 76) note that these introduce "even more complexity, not only by adding a third dimension and thus presenting a significant challenge for students' spatial sense, but also in the very nature of the entity being measured". As noted above in the section on spatial reasoning, in research on how 8–9 year-old children solve 3-D tasks using the software *Geocadabra* (Lecluse, 2005), Sack and Vazquez (2011) concluded that "coding of rectangular array structures fosters children's understanding of the volume formula in concert with their emerging multiplication skills" (p. 95). Huang (2012) was similarly interested in how children would benefit from a curriculum that integrates geometry with volume measurement, as compared to teaching that stresses numerical calculations and application of the formula. By designing different week-long teaching sequences for two 5th grade classes (pupils aged 10–11), Huang found that each approach "facilitated the children's acquisition of the idea of volume measurement" and their capability to "solve different types of problems embedded with volume measurement concepts" (p. 361).

While focusing on mass rather than volume, McDonough, Cheeseman and Ferguson (2012) developed a one-week teaching unit for 6–8 year olds. Through this they found that the children were capable of thinking constructively about the intricacies of mass measurement. In terms of comparing and ordering masses, they found that the children appeared to "draw on prior experiences and sometimes on visual cues, but with appearance-based comparison for mass not as likely a reliable strategy as it might be, say, for length" (p. 207).

These studies illustrate the continuing need for active research on the topic of volume, and for research on the related topics of mass and capacity.

Combinations of Measures

As well as studying individual measures, researchers have also conducted studies involving more than one measure. For example, Stephanou and Pitta-Pantazi (2006) analysed the answers that upper primary school students gave to area and perimeter tasks. They found that more than half of the students' answers were influenced "not so much by the specific context of a task (area or perimeter) or the presence of a diagram" but rather they were influenced "by the external features (change of one/both dimensions) of the task that trigger the intuitive rule 'if A then B, if not A then not B'" (p. 183). Huang (2010) also examined children's understanding of perimeter and area. The findings indicated that even where children (aged 8–9) had the computational capability to calculate perimeters, this did not necessary mean that they had complete comprehension of the meanings of multiplication and of the formula for area calculation.

Cullen, Miller, Barrett, Clements and Sarama (2011) compared three different unit-eliciting task structures for measurement comparison tasks. With a sample of children from grades 2–4, the researchers found that students were most successful with a task structure that asked "how much longer/bigger?" and were least successful with a task structure that asked "how many times longer/bigger?" (p. 249). What is more, in response to "how much longer/bigger?" the children tended to use an additive comparison while they tended to produce multiplicative comparisons in response to "how many times longer/bigger?" (ibid).

Research by Fernández and De Bock (2013, p. 297) focused on a frequently-investigated case of students' misuse of linearity; that of effect of an enlargement or reduction of a geometrical figure on its area or volume. Here, learners have the tendency to treat relations between length and area, or between length and volume, as linear instead of, respectively, quadratic and cubic – perhaps, the researchers suggest, because secondary school students struggle with the distinction between dimensionality and 'directionality' (an example of that latter being that while the perimeter of a square is one-dimensional, it has two 'directions' in the form of length and breadth). Analysis of the responses to a set of tasks by 13–14 year olds confirmed the preponderance of "linear" answers and also indicated that more than 20% of the students' answers were "directional" (ibid). The distinction between dimensionality and directionality was more a struggle for figures where the number of directions and dimensions coincided, such as when a square has two dimensions and also two directions.

Curry, Mitchelmore and Outhred (2006) surveyed 96 students of Grades 1–4 using tasks assessing understanding of the five measurement principles: the need for congruent units; the importance of using an appropriate unit; the need to use the same unit when comparing objects; the relationship between the unit and the measure; and the structure of the unit iteration. Their results showed that while some of these principles were found to be clearer to older children, a precise order of development was not evident. The researchers concluded that appropriate learning tasks could be ones that help focus students on "the reasons for using a fixed unit size, for not leaving gaps, for using multiplication in some contexts, for rejecting certain units and accepting others, and for the inverse principle" (p. 383).

Such suggestions can be compared to those of Owens and Kaleva (2008), who have studied the many differing indigenous communities of Papua New Guinea (PNG). In setting out to collect and analyse approaches to measurement for as many PNG language groups as possible, Owens and Kaleva generalise to say that PNG people "have a sense of area (tacit knowledge) developed through sleeping, gardening and house building in particular" and "are able to use this idea of area to make judgements such as the estimated amount of material needed for a house of a particular floor size"; likewise, PNG people "would visualise a garden by knowing its length" (p. 79). The researchers concluded "by making these points explicit, teachers can reduce the discontinuities in knowledge and hence build a firm basis for school mathematics" (ibid).

The issue of primary students' measurement estimates has been studied by Huang (2015) and by Ruwisch, Heid and Weiher (2015). Huang reported that good estimators tended to adopt multiple strategies and mental rulers more frequently than poor estimators, while Ruwisch and colleagues found that the children (and educators) that they studied gave better estimations for lengths than for capacities.

Angle

The measuring of angle is, according to Bryant (2009, p. 4), "another serious stumbling block for pupils". One problem, according to Bryant, is that turning 90 degrees (a 'dynamic' angle) appears very different to the corner of a book being 90 degrees (a 'static' angle). The study by Masuda (2009) confirms that learner difficulties range from grade 5 students having difficulty paying attention to an angle as one of the attributes of the shape (and distinguishing it from measuring a side of a shape) to grade 11 students being unclear about radians and degrees.

Kaur (2013) researched the ideas of elementary school children (aged 5–6) working on angle comparison using dynamic geometry software (DGS). Here, the children's gestures and motion played an important role in their decision-making on angle comparison tasks. In particular, the use of gestures, such as hands as the 'arms' of an angle, enabled the children to see the process of turning even in case of 'static' shapes. In this way, "embodied routines could be helpful in looking at dynamic thinking, especially in case of young children" (p. 151).

Dohrmann and Kuzle (2014) focused on the development from grade 5 to 10 of pupils' understanding of an angle of 1 degree. The results showed that many of the children's misconceptions were directly connected to the measuring tool, namely the set square, and to the way they tried to draw an angle of 1°. In the case of the set square, this tool was found to privilege a 'static', rather than 'dynamic', perspective on angle.

In shedding light on the meanings of angle in 3-D space held by 12-year-old students, Latsi and Kynigos (2011) used a specially-designed "Turtle Geometry with dynamic manipulation microworld" within a teaching experiment in which the children "addressed angle as a directed turn … in the context of noticing and understanding 3-D objects' spatial and geometrical properties" (p. 127). The researchers found that the students benefitted from experiencing "a vehicle of motion metaphor (e.g. flying the turtle)" (ibid). In this way the students came to use angle as "a spatial visualisation concept" (ibid).

In research by Tomaz and David (2011), the focus was on the definition of the bisector of an angle and measuring the angles formed by it. In the study, students aged 13–14 tackled the problem of finding the measure of an angle formed by the bisectors of two given adjacent angles. This "opened the possibilities to deepen their [the students'] understanding about the measure of angles" (p. 264). This illustrates, say the researchers, the "power of the visual representations for structuring and modifying the mathematical activity in the classroom" (p. 259).

While the difficulties that students encounter with the notion of angle are well known in the literature, these studies show how research is needed on fusing, rather than confusing, for learners the 'static' and 'dynamic' perspectives on angle.

GEOMETRICAL REASONING AND PROVING

> An important aspect of geometry is concerned with the development of deductive reasoning and proof. (Royal Society, 2001, p. 9)

Students' Developing Capabilities with Geometric Reasoning and Proving

Research continues to focus on the capabilities of students at different grade levels with geometric reasoning and proving. Investigating cognitive predictors of geometrical proof competence, Ufer, Heinze and Reiss (2008) proposed a model comprising three levels: basic calculations; one-step proofs; and multi-step proofs. With data from testing 341 students in grade 9, the research confirmed that while knowledge was an important predictor of geometric proof competence, other predictors were also significant. The authors concluded that "if a student does not understand the nature of mathematical proofs, or has no problem-solving strategies at hand, he or she will hardly be able to construct a proof in spite of the best geometric content knowledge" (p. 367). Such a conclusion was echoed by Yang, Lin and Wang (2007) in a study of students' capabilities when reading geometry proofs.

The issue of how geometrical proof competence is connected to the capability to define geometric concepts was studied by Silfverberg and Matsuo (2008). In data from testing 152 Japanese and 162 Finnish students at 6th and 8th grade on the definitions of quadrilaterals, the researchers found that in both countries the students' understanding of defining geometric concepts related to their "understanding of the class inclusion relations" (p. 263). In examining students' capabilities in making geometric generalizations, Yevdokimov (2008) found that the higher-attaining students could formulate generalized arguments. Antonini (2008) showed how students treated contradictions in geometric argumentations and proofs, indicating how proof by contradiction is not straightforward for learners. Ginat and Spiegel (2015) found an absence of the 'fluency' and 'flexibility' aspects of creativity in novices' geometry proofs.

Bieda (2011) reported on the aspects of proofs and non-proofs that were convincing to middle grade students. The analysis found that the students "valued the explanatory power of an argument when evaluating a proof for a true geometry statement that provided a diagram" (p. 153). In a study of the assumptions made by 10th grade students when proving geometric statements, Dvora and Dreyfus (2011) found that unjustified assumptions arose when students "misused theorems or assigned extraneous properties to geometric objects", and that unjustified assumptions were "made with the purpose of reaching a critical step in the proof" (p. 289).

Matos and Rodrigues (2011) investigated how the construction of geometric proof related to the social practice developed in the classroom, and, in particular, the role of geometric diagrams. The researchers concluded that diagrams played "an important role in the process of sharing and increasing the ownership of meaning of proof by highlighting the relevant properties" (p. 183). For an interesting analysis of geometric pictures, see Stenkvist (2012).

In proof problems involving 2-D representations of 3-D shapes, the diagram may not always help. For example, Jones, Fujita and Kunimune (2012) reported a study of lower secondary school pupils (aged 12–15) who tackled a 3-D geometry problem that used a particular diagram as a representation of the cube. The analysis showed how some of the students could "take the cube as an abstract geometrical object and reason about it beyond reference to the representation", while others needed to be offered "alternative representations to help them 'see' the proof" (p. 339). The influence of 3-D representations on students' level of 3-D geometrical thinking is reported by Kondo, Fujita, Kunimune and Jones (2013) and the follow-up paper by Kondo, Fujita, Kunimune, Jones and Kumakura (2014).

Attempting to deepen the ways in which visually-based geometric materials support students' generating of conjectures, Lin and Wu (2007) examined how 6th graders, still in the process of intuitive geometry, generated geometrical conjectures when geometrical conditions in diagrams were given. The analysis showed that students generated more related conjectures if they looked at one example, instead of two or three at the same time, and they generated more conjectures if the examples were conjunctive (that is, the example was the conjunction of the conditions given in the question with other conditions). Komatsu (2011) studied how grade 9 students generalized their conjecture through proving. After the students proved their conjecture and faced its counterexample, applying their proof to a boundary case between example and counterexample of their conjecture was found to be crucial.

Given that there can be a tension between the practical aspect of physically carrying out a geometrical construction and the theoretical aspect of constructing the related proof, Fujita, Jones and Kunimune (2010) studied the extent to which there might be 'cognitive unity' between students' geometrical constructions and their proving activities. The results suggested that while grade 9 students gained a much greater appreciation of how to use already-known facts to proceed with further investigations in geometry, the uniting of student conjecture production and proof construction was not automatic. As the authors concluded "further research is necessary to give a fuller answer to the matter of how, and to what extent, geometrical constructions encourage the uniting of student conjecture production and proof construction" (p. 15). In a follow-up report, the same authors reported two cases from grade 7 where the use of geometrical constructions enabled the students to shift "from relying on visual appearances or measurement to reasoning with properties of shapes" (Fujita, Kunimune, & Jones, 2014, p. 65).

A range of studies has examined students' proof and proving when using dynamic geometry software (DGS). Patsiomitou and Emvalotis (2010), for example, concluded from their study that "the dynamic manipulation of objects in the software led the students to construct the properties of figures" and this, in turn, helped the students classify the figures (see also, Patsiomitou, 2011). Baccaglini-Frank, Mariotti and Antonini, (2009) reported on different perceptions of invariants and generality of proof in dynamic geometry, while Baccaglini-Frank, Antonini, Leung and Mariotti (2011) reported on a study with upper secondary-age students (aged 16–18) that focused on constructing a proof by contradiction. The latter showed that "there can be a strong subjective element in the process of producing a geometrical proof (or a convincing argument) via the solver's conscious choices of construction and dragging in a DGS" (pp. 87–88). Olivero (2006) investigated the role of the DGS hide/show tool in the conjecturing and proving processes. While this facility offers students the possibility to focus on different elements during a geometric construction, the analysis confirmed that the visible elements on the screen guided the focus of the students and it was this that effected the construction of conjectures and the development of proofs. In a different approach, Leung and Or (2007) studied oral explanations and written proofs provided by secondary students working on construction tasks with DGS. The researchers concluded that writing up DGS proofs "may involve using mathematical symbols or expressions that transcend the usual semantic of a traditional mathematical symbolic representation (p. 183).

Fujita, Jones and Miyazaki (2011) and Miyazaki, Fujita and Jones (2014) reported on studies of a "web-based proof learning support environment" (p. 353) in which learners tackled geometrical congruency-based proof tasks by dragging sides, angles and triangles to cells of a flowchart-style proof while the web-based system automatically transfered figural to symbolic elements so that learners could concentrate on the logical and structural aspects of their proofs. From their research, the researchers argued that with this approach, alongside suitable guidance from the teacher on the structural aspects of a proof, students could "start bridging the gap in their logic and thereby begin to overcome circular arguments in mathematical proofs" (2011, p. 353).

Textbooks may, or may not, provide support for students' developing capabilities with geometric proving. Dolev and Even (2012) compared six 7th grade Israeli mathematics textbooks, examining the opportunities provided by the textbooks to justify and explain mathematical work about triangle properties. They found, compared with algebra, that all six textbooks included "considerably larger percentages of geometric tasks that required students to justify or explain their solutions" (p. 203). Miyakawa (2012) compared textbooks from France and Japan and found differences such as what gets called proof in the textbook, the form of proof used, and the functions of proof employed.

Given that definitions are integral to geometric proof, Okazaki (2013) found that for 5th grade pupils five situations should help: "(1) understanding the meaning of identifying geometric figures, (2) constructing examples from non-examples and justifying the constructions via comparisons, (3) recognizing equivalent combinations, (4) examining undetermined cases via counterexamples, and (5) conceiving figures as relations beyond the given actualities" (p. 409). Haj-Yahya, Hershkowitz and Dreyfus (2014) investigated 11th grade students' geometrical proofs through the lens of the students' definitions and found that the difficulties students had in understanding geometric definitions affected their understanding of the proving process and hence the capability to prove.

Several studies have examined the ways in which high-attaining students compose or construct a proof, or create a definition, and how this might help in understanding the proving approach of the students in general because the characteristics of their approaches are very close to the mathematical proving or defining processes. Examples include Lee (2005) and Song, Chong, Yim and Chang (2006) who examined the constituents of proving that high-attaining students produce, Ryu, Chong and Song (2007) who researched their spatial visualization of solid figures, Lee, Kim, Na, Han and Song (2007) who researched their use of utilize induction, analogy, and imagery, and Lee, Ko and Song (2007) who studied the ways they define geometric objects. These researchers concluded that teachers need to draw explicit attention to the value of informal proofs and that for students to develop their sense of geometrical reasoning there needs to be extensive experience of conjecturing and then verifying. Kim, Lee, Ko, Park and Park (2009) built on this work in a study of how high-attaining students can become aware of unjustified assumptions in geometric constructions.

Teaching Proposals Improving Students' Performance in Geometric Proving

In looking to help students, Cheng and colleagues examined strategies such as reading-and-colouring (Cheng & Lin, 2006), the use of coloured flashcards to support geometric argumentation (Cheng & Lin, 2007), and step-by-step reasoning in two-step geometry proofs (Cheng & Lin, 2008). With the reading-and-colouring teaching approach entailing students using colours to show known and unknown information in proving tasks, teaching experiments with 9th grade students found that the approach helped students to see the necessary information for proving a statement. As a way of supporting geometry proof reasoning in slower students, Cheng and Lin (2008) developed a step-by-step reasoning strategy and found that this teaching strategy improved the students' proving process. Research by Kuntze (2008) confirmed that writing about geometrical proving can foster "the competency of solving geometrical proof tasks" (p. 295).

Huang (2005) investigated how a sample of teachers in Hong Kong and Shanghai taught Pythagoras' theorem. The findings showed both similarities and differences in terms of the approach to the justification of the theorem. Although teachers in both places emphasized the justification of the theorem by various activities, the following differences were noticeable: Hong Kong teachers were what they called "visual verification-orientated" while Shanghai teachers were "mathematical-proof-orientated" (p. 166). Moreover, compared with Hong Kong teachers, Shanghai teachers made more effort to encourage students to speak about and construct their own proofs. Zaslavsky, Harel and Manaster (2006) also investigated the teaching of the Pythagoras theorem, in particular how examples were used and how this enabled analysis of teacher mathematical and pedagogical knowledge that may support or inhibit student learning.

The role of the teacher is known to be crucial to students' developing capabilities with geometric proving. Dimmel and Herbst (2014) found that teachers had different views of the appropriate level of detail in a student's geometrical proof. Focused on classroom interaction, Miyakawa and Herbst's (2007) study of classroom geometrical proving found differences between what they called "installing theorems" and "doing proofs": in the former, "details may be excluded, and a theorem may be established without proof" while when 'doing proofs' the conclusion "cannot be used until proved" (p. 288).

In the same direction, Fuglestad and Goodchild (2009) examined teachers' knowledge about proof and its necessity, concluding that some teachers do not appear certain about the nature and the necessity of a proof. Attempting to support teachers' understanding of geometric reasoning and proof, Bayazit and Jakubowski (2008) proposed constructions with compass and straightedge, while De Bock and Greer (2008) proposed to pre-service teachers a challenging task (in this case, finding and proving which rectangles with sides of integral length have equal area and perimeter). Lei, Tso and Lu (2012) examined how reading comprehension of geometry proof might be influenced by worked-out examples. With data from 85 grade 8 students who were novices at deductive proof in geometry, they found that lower-attaining students tended to overlook the overall logical structure of proof by only repeating the steps from worked-out examples and that these students failed to apply related knowledge in proving.

Brockmann-Behnsen and Rott (2014) reported on a long-term study conducted in four 8th grade classes. Two of these classes served as control groups, with the mathematics lessons of the other two classes frequently enriched by structured argumentation and the training in the use of heuristics. In the post-test, the treatment groups obtained significantly better results than the control groups (who had no special training in heuristics and argumentation strategies). While not a controlled trial, Fielding-Wells and Makar (2015) describe a teaching unit with a class of 10–11 year-olds which included the task "Can a pyramid have a scalene face?" (p. 297). Through their analysis the researchers identified several benefits of argumentation for the learners.

STUDENTS' GEOMETRIC KNOWLEDGE

Owens and Outhred (2006, p. 85) pointed to the impact of Piaget on earlier research on student's knowledge about geometric figures. In subsequent research, evidence of the legacy has been much less. In contrast, the van Hiele model (ibid, pp. 86–89) continues to feature. More recent studies have employed various frameworks, including figure apprehension according to Duval (1999), and the notion of figural concept by Fischbein (1993). In addition, use of more general frameworks includes Sfard's (2008) commognition approach, as well as notions of embodiment (Gibbs, 2006).

The Piagetian Legacy and Use of the van Hiele Model

Examples of continuation of the Piagetian legacy in PME research include the study by Cullen et al. (2011) who used the Piagetian idea of the importance of comparison in measurement and Maier and Benz (2014) who used Piagetian notions of drawing skills in investigating how children aged between 4 and 6 drew different kinds of triangles. Examples of the use of the van Hiele model include research by, for example, Wu and Ma (2005), Wu and Ma (2006), Wu, Ma, Hsieh and Li (2007). Such studies of elementary school students confirm the outcomes of previous research that students tend to judge geometric figures by their appearance, with the circle the easiest and quadrilaterals the more difficult.

More recently, Guven and Okumus (2011) tested the van Hiele levels of 8th grade Turkish students together with their classification preferences (hierarchical or partitional) about relationships between some quadrilateral pairs. They found that "most of the students were at van Hiele level 2 before starting their high school education and the students generally chose partitional classification" (p. 473). For Kospentaris and Spyrou (2009), after examining data on the van Hiele levels of secondary school students, it was because of geometry teaching methods that such students barely surpass level 1. Patsiomitou and Emvalotis (2010) used the van Hiele levels in a study of the development of students' geometrical thinking through a guided-reinvention process with DGS. They found that students "developed their geometrical thinking processes and applied skills, reaching a higher level of abstraction" (p. 39)

Apprehension of Geometric Figures

According to Duval's (1999) theoretical framework, there are four different ways to organize and process visual aspects in geometric figures: perceptual apprehension (recognizing figures); sequential apprehension (perceiving their different parts); discursive apprehension (on the basis of statements, definitions, descriptions); and operative apprehension (modifying a figure or some of its element). A study by Elia, Gagatsis, Deliyianni, Monoyiou and Michael (2009) of various aspects of figure modification confirmed students' tendency to apply part-whole modifications rather than modifications referring to the position or orientation of a figure. In later research (Deliyianni, Michael, Monoyiou, Gagatsis, & Elia, 2011), the researchers aimed at confirming a composite theoretical model concerning middle and high school students' geometrical figure understanding. More recently, Kalogirou, Elia and Gagatsis (2013) investigated how two major components of spatial capability, that of visualization and mental rotation, might be related to geometrical figure apprehension (perceptual and operative) as proposed by Duval (1999). Statistical analysis indicated a moderate though significant correlation between spatial capability and geometrical figure apprehension.

Sinclair and Kaur (2011) found that kindergarten children were able to "develop an understanding of symmetry that showed awareness of the properties of reflectional symmetry through the behaviour of dynamic images" (p. 193). For Sinclair, Moss and Jones (2010) the focus was children aged 5 to 7 trying to decide whether two lines on a DGS screen that they know continue (but cannot see all of the continuation) would intersect, or not. They report that, in tackling this question, the children engaged in "aspects of deductive argumentation" (p. 191). Kaur and Sinclair (2014) reported part of a longitudinal study of the development of young children's geometric thinking (aged 7–8). They found that "during the teacher-led explorations and discussions with dynamic sketches, children's routines moved from description of tool-based informal properties to formal properties" (p. 415), as well as from particular to more general discourse about what is a triangle.

Knowledge of Definitions and Inclusion Relations

A study by Ubuz (2006) of secondary school students' definitions of polygons and quadrilaterals, and the ways these figures are presented in the textbooks, found that "figures (in textbooks) often provide an instantiation of a definition, not a general and rigorous proof" so that the students "focus on figural understanding to produce conceptual understanding" (p. 347).

The understanding of the inclusion relations between quadrilaterals has been the focus of a number of studies (Guven & Okumus 2011; Okazaki, 2009; Okazaki & Fujita, 2007; Silfverberg & Matsuo, 2008). Such studies confirmed that students' difficulties in understanding the inclusion relations differ from grade to grade and can be related to tacit properties and significant prototype phenomena. In their study of how Japanese and Finnish students were able to apply class inclusion and disjunctive classification, Silfverberg and Matsuo (2008) found that about half of the students could identify the inclusion of squares into rectangles, and of rectangles into parallelograms. Okazaki and Fujita (2007), grounding their research on Hershkowitz's (1990) theoretical frame of prototype phenomenon, obtained data from Japanese 9th graders and from Scottish pre-service primary teachers. They found that for Japanese students the prototype phenomenon appeared "strongly in squares and rectangles" and that such prototype images and implicit properties were "obstacles for the correct understanding of the rectangle/parallelogram and square/rectangle relations" (p. 47), while even though the pre-service teachers had a "relatively flexible images of parallelograms" the strongest prototype phenomenon appeared with squares.

The image of angles in a parallelogram or a rectangle appears to be an obstacle in understanding inclusion properties, as shown in the study by Ozakaki (2009). The simple identification of geometric figures does not necessarily allow students to approach inclusion relations as they remain with the tacit properties that they have in mind.

Matsuo (2007) recorded the differences in students' understanding of geometric quadrilaterals. The results revealed four ordered states in understanding relations: not distinguishing between two geometric figures; identifying both figures respectively; distinguishing or identifying figures based on their differences or similarities; and understanding the inclusion relation. Serow (2006) examined the development of triangle property relationships using the SOLO taxonomy (Biggs & Collis, 1982). The analysis revealed differences in the ways students understood the relationships among properties. As notted earlier, Haj-Yahya and Hershkowitz (2013) found that, when definitional statements about quadrilaterals were given verbally to 10th graders without any visual support, more students were able to identify and explain the inclusion relationships.

Identification of 2-D and 3-D Shapes

A number of studies have investigated the identification of shapes such as triangles through different grades (e.g. Horne & Watson, 2008), as well as the type of criteria that students use to identify geometric figures more generally (e.g. Sophocleous, Kalogirou, & Gagatsis, 2009). Such studies have confirmed that students develop the concept of shapes through experiences both inside and outside school and from holistic visual approaches to properties recognition. Horne and Watson (2008) tested students across seven consecutive grades on a task related to identification of triangles. While they found an improvement across grades 1 to 4, most students' errors concerned the inclusion, rather than the exclusion, of triangles. Maier and Benz (2014) studied young children's ideas of triangles by analysing their drawings. They found that children aged 3–11 mainly drew isosceles triangles (although the researchers were not sure whether the children were attempting to draw equilateral triangles with limited drawing skills). Moreover, they found that prototypical presentations were dominant not only for the first drawn triangle but also as varying triangles because "most children varied their triangles through area size" (p. 160).

The study by Sophocleous, Kalogirou and Gagatsis (2009) compared the criteria of figure recognition with solutions that 5th and 6th grade students proposed in creativity tasks with overlapping figures. Their results indicated that the more critical attributes of shapes the students could recognize, the better they performed in creativity tasks.

More recently, Arai (2015) investigated how instructional tasks change the ways first graders identify geometric figures. A questionnaire with instructional tasks was administrated to three groups of 69 students. In the first group the students have to find the number of sides and vertices of triangles, to draw figures and read definition of triangles, while in the second group they have only to find the number of sides and vertices of triangles and draw figures, and in the third group students read definition of triangles. While most of the students "used visual reasoning to identify triangles, and were noticeable influenced by prototype examples", there were signs that they could change their reasoning "after engaging in instructional tasks" (p. 55).

A number of studies have investigated students' knowledge of 3-D shapes. Wu, Ma and Chen (2006) investigated students of different grades and found that higher grade students had more sophisticated representations of 3-D shapes. In a later study, Ma, Wu, Chen and Hsieh (2009) examined students' drawings of solid cuboids and compared their results to those given by Mitchelmore (1978) two decades earlier. This indicated an improved distribution of the stages compared with that presented by Mitchelmore.

Nevertheless, research by Pittalis, Mousoulides and Christou (2009) has underlined that students have many difficulties in representing, identifying, or interpreting. With data from 40 students from 5th to 9th grade, the researchers identified four levels of sophistication in the representations: no proper drawings; coordination of front and side views; proper conventions of 3-D drawings with some errors; proper drawings.

Hatterman (2008) observed 15 university students, trained in 2-D DGEs (Euklid-DynaGeo and Cabri 3D), while they worked in groups on Archimedes Geo3D and Cabri 3D. The results showed that experiences in 2D-environments appeared insufficient when students work in 3-D space. The students had problems in justifying simple facts in 3D-environments and benefitted from access to 3-D models to solve given tasks. In their study, Leung and Or (2009) investigated perspective dragging in Cabri 3D and found that this helped students to identify and reason about geometric properties of 3D objects.

Language Issues in the Development of Geometrical Thinking

In research on language issues in the development of geometrical thinking, Leung and Park (2009) found that common names in geometric and in everyday language both support and prevent students' understanding of figures and their properties because the terms direct students to fix their attention on some special characteristics that are not always consistent with the definition of the figures. More recently, Ng (2014), as noted above, studied the "interplay between language, gestures, dragging and diagrams" (p. 290) in 12th grade bilingual learners' mathematical communications about various aspects of Calculus through geometrical dynamic sketches using DGS. The findings suggested that "bilingual learners utilised a variety of resources, including language, gestures and visual mediators in their mathematical communication – with gestures taking on a prevalent role" (p. 295).

Summarizing, studies related to students' geometric knowledge keep attracting the interest of research on the teaching and learning of geometry, with older or newer approaches related to identification of 2-D or 3-D geometric figures. As a significant number of relevant studies have been accumulated in this field, a careful and systematic record of the findings and subsequent conclusions related to students' geometric knowledge might be imperative.

TEACHERS' GEOMETRIC KNOWLEDGE AND DEVELOPMENT

> Teaching geometry well involves [the teacher] knowing how to recognise interesting geometrical problems and theorems, appreciating the history and cultural context of geometry, and understanding the many and varied uses to which geometry is put. (Jones, 2002, p. 122)

Given that the nature and extent of teachers' knowledge affects the quality of their teaching (e.g. Ball & Bass, 2003), a number of studies have focused on examining pre-service and in-service teachers' knowledge of geometry – and on ways of developing this knowledge.

Geometric Knowledge of Teachers

Fujita and Jones (2006) reported on the geometric knowledge of Scottish pre-service primary teachers and the ways that these pre-service teachers defined and classified quadrilaterals. Based on the ideas of concept definition and concept image introduced by Tall and Vinner (1981), and of figural concept initiated by Fischbein (1993), Fujita and Jones (p. 130) distinguished what they called the individuals' "personal figural concept" (coming from personal experiences) from the "formal figural concept" (as defined in geometry). Almost 160 pre-service primary teachers in the first year of their studies were examined in questions related to quadrilateral properties, and 124 pre-service teachers in the third year of their studies were examined about quadrilateral relationships. Analysis of the first group's answers showed that there was a gap between figural concepts and definitions provided. Similarly, the analysis of the answers of the second group indicated a weak understanding of the hierarchical relationship of quadrilaterals.

For Tatsis and Moutsios-Rentzos (2013), their focus was the capability of pre-service primary school teachers to interpret and evaluate verbal information related to 2-D geometrical objects. The researchers found, in contrast with their conjecture, that the pre-service teachers mostly showed a stronger positive evaluation of the geometrical descriptions, followed by weaker positive evaluations of the topological descriptions. These, say the researchers, were accompanied by "relatively negative evaluations for everyday descriptions" (p. 270).

While the above studies focus on pre-service elementary teachers, Silfverberg and Joutsenlahti (2014) studied pre-service elementary and secondary teachers' notions of angles in a plane. They found that some of their respondents "interpreted an angle as a line consisting of two line segments, some consisting of two rays, and some as a region defined by these elements" (p. 190). What is more, interpretations differed as to "whether an angle continues outside the part shown in the drawing in the direction determined by the angle, or not" (ibid). Moore-Russo and Mudaly (2011) reported on a study of South African secondary school teachers' knowledge of gradient (or slope). Based on data from nine free-response test items completed by 251 practicing teachers pursing qualifications to teach grades 10–12 mathematics, their findings suggested that understanding of gradient of these teachers varied greatly, with many of the teachers lacking "even a basic understanding of this important concept" (p. 241).

In a similar vein, research by Son (2006) investigated pre-service primary and secondary teachers' conceptions of reflective symmetry and compared these with their teaching strategies. Based on the van Hiele model, the results showed that the pre-service teachers had a limited understanding of reflective symmetry and confused symmetry with rotation. Their deficiencies directed them to use procedural teaching approaches in their attempt to help students' understanding of symmetry and symmetrical constructions. Comparable results in the study of Van der Sandt (2005) showed that when secondary pre-service teachers did not adequately control the geometric subject matter, their deficiencies had implications in their classroom teaching. Paksu (2009) found that pre-service elementary teachers' self-efficacy in geometry was related to many factors such as their van Hiele geometric thinking level, their attitude towards geometry, and their attainment in geometry. Chiang and Stacey (2015) focused on in-service primary school teachers in Taiwan. In line with much existing research, they found the teachers lacked some basic geometric knowledge.

In a diagnostic test of the knowledge of both pre-service and in-service teachers about triangles, Alatorre and Saiz (2009) found both figural and conceptual misconceptions. These included the idea that the base of a triangle is necessarily horizontal (with the rest of the figure above it) and the height necessarily vertical and/or drawn from the highest point, the idea that triangles must necessarily be isosceles, that altitudes need to be internal, the idea that each triangle has only one base and one height, confusing the height with the median, the use of right-angled triangles terminology with non-right-angled ones, various misconceptions about the Pythagorean Theorem and its applications, and errors with the formula for the area of a triangle. Subsequently, Alatorre, Flores and Mendiola (2012) studied in-service primary teachers' reasoning and argumentation about triangle inequality. Their findings suggested that reasoning and argumentation "are not part of many [primary] teachers' professional practice" (p. 9).

Given the consistent findings of problems with teacher knowledge, more research could focus on how the geometrical knowledge of pre-service, and in-service, teachers could be improved. Existing PME research on this topic is addressed in the next section.

Teacher Development for Geometry Education

Various research studies have shown that the geometrical knowledge of pre-service, and in-service, teachers can be improved not only by the appropriate education (for instance, González & Guillén, 2008, proposed an *Initial Competence Model* for teachers for the teaching of geometric solids), but also by the use of technologies such as dynamic geometry environments (e.g. Haja, 2005). In the study by Haja, pre-service secondary teachers were studied for their problem-solving capabilities while they were undertaking geometrical constructions using DGS. The researcher applied a "knowledge-in-action design that expected them to "…apply their content knowledge to understand the given problem, construct the dynamic figures, make conjectures, verify the conjectures, and solve similar problems" (p. 82). Using open-ended tasks for which the pre-service teachers had to find a solution with the dynamic software, the evidence showed that they met the expectations of the knowledge in action design.

In a similar vein, Presmeg, Barrett and McCrone (2007) designed a course that included geometric constructions that the pre-service teachers could tackle both by using DGS and by traditional tools. These two different modes of representation of geometric concepts could, according to Duval's (1999) framework, support the pre-service teachers' constructions of generalized geometric knowledge. According to researchers' approach, the property of DGS sketches to stay together when the mouse moves points or lines, and the distinction between variant and invariant properties, were the two concepts that were more related to the development by the pre-service teachers of geometric generalizations. Moreover, collaborative discussions and sharing meanings were amongst the main factors for participants' accomplishments. Similarly, in a study conducted by Olvera, Guillén and Figueras (2008), the fostering of communities of practice of in-service primary teachers was found to improve their approaches in the teaching of solid geometry. Alqahtani and Powell (2015) studied teams of middle and high school in-service teachers during a semester-long professional development course in which the teachers participated in a collaborative online dynamic geometry environment. The researchers found that through this online dynamic geometry environment the teachers interacted to notice variances and invariances of objects and relations in geometrical figures and to solve open-ended geometry problems. For Morgan and Sack (2011) in their research with pre-service teachers, the van Hiele model remained "a useful framework to describe the evolving shape-building activities" (pp. 249–250).

Martignone's (2011) research provides examples of tasks for teachers involving artefacts (such as ruler and compasses) and how teachers can succeed in implementing such tasks in their classrooms. Lavy and Shriki (2012) studied how the skills of pre-service secondary school mathematics teachers in evaluating geometrical proofs could be improved through peer assessment of each other's proofs. The outcome was that the engagement of the pre-service teachers in peer assessment, both as assessors and as those being assessed, "contributed to the development of [their] assessment skills" (p. 41). By comparing the first and the second assessment tasks conducted by the pre-service teachers, the researchers found that the pre-service teachers developed their capabilities "to select a proper criteria list and assign a reasonable numerical weight to each criterion" (ibid).

Cirillo (2011) provides a case study of a beginning secondary school teacher working to improve the way of teaching geometrical proof and proving during their first three years of teaching. Given that the beginning teacher had a strong mathematics background, the study illustrated how content knowledge "is not necessarily sufficient preparation to teach proof" (p. 247). The study by Hähkiöniemi (2011) provides an account of how an experienced teacher was given the opportunity to try a pre-planned unit for high school students on approximating the area under a curve that was enriched with DGS-based tasks and how this raised the teachers' awareness of different teaching methods as well as the benefits and challenges of using such methods.

In general, the studies on teachers' geometric knowledge, and their pre-service and in-service education, indicate that attention needs to be given to how to build teachers' understanding of common 2-D and 3-D objects (e.g. triangles, quadrilaterals, or angles) with consequent implications for their teaching. In investigating approaches that improve teachers' geometrical education, relevant research confirms the effectiveness of general approaches (e.g. community in practice or peer assessment) but also the use of technological tools in geometric problem solving or proving.

TEACHING GEOMETRY AND GEOMETRIC TASKS

> Tasks shape the learners' experience of the subject and their understanding of the nature of mathematical activity. (Watson & Ohtani, 2015, p. 3)

Teaching Interventions

Of the various studies of geometry teaching, some entail genetic approaches involving historical, logical and epistemological, psychological and socio-cultural aspects (e.g. Safuanov, 2007) and some feature ethno-mathematical and humanist approaches valuing cultural and scientific heritage (e.g. Chorney, 2013; Gooya & Karamian, 2005), as well as the use of art work as a creative tool to approach geometric figures (Pakang & Kongtaln, 2007). On top of this, there have been studies related to the teachers' choices regarding the use of diagrams and examples (Zodik & Zaslavsky, 2007) and studies emphasizing algebraic approaches to solving geometrical problems (Dindyal, 2007). How students make sense of the 'figured world' of the geometry classroom was explored by Aaron (2008), while Ding and Jones (2006) investigated geometry teaching at the lower secondary school level in Shanghai, China.

Gal, Lin and Ying (2006) observed five different 9th grade classes aiming at investigating the factors and class characteristics that influenced students' low achievement. The findings suggested that the low achievers were provided with less learning opportunities. Similarly, Soares (2010) studied a 4th grade geometry class that was co-taught by two teachers with "different and complementary perspectives" (p. 201), one trying to encourage the students to solve challenging problems, and the other managing situations in which novel tasks are introduced. This combination of skills made for successful teaching. Both Hähkiöniemi (2011), as noted above, and

Hollebrands, Cayton and Boehm (2013) reported on the types of pivotal teaching moments, and related teacher actions, which can arise in a technology-intensive geometry classroom.

Geometric Tasks

For some studies, the design of geometric tasks was integral to the research. The research reported by Fujita, Jones and Kunimune (2010), Fujita, Jones and Miyazaki (2011), and Komatsu (2011), all relied on well-designed tasks. In Fujita, Jones and Kunimune (2010), the task was "how to construct the largest square within a given triangle ABC" (p. 12). The conclusion of the teaching experiment was that this task could be used to "encourage students' mathematical arguments, reasoning and proof" (p. 15). In Fujita, Jones and Miyazaki (2011), the tasks were integral to the design of a "web-based proof learning support environment" (p. 353). In the tasks, learners tackled proof problems by dragging sides, angles and triangles to cells of the flow-chart proof and the web-based system automatically transferred figural to symbolic elements so that the learners could concentrate on logical and structural aspects of proofs. The task included both ordinary proof problems such as prove the base angles of an isosceles triangles are equal (the researchers call these closed problems) and problems by which students construct different proofs by changing premises under certain given limitations (which the researchers called open problems). Each time the learners selected a next step in their flow-chart proof, the web-based system checked for any error via a database of possible next steps. If there was an error, the learners received feedback in accordance with the type of error.

The study by Komatsu (2011) utilised a task concerning a small triangle placed on top of a larger one and the change in length of two segments after rotation of one triangle around a common point. For the students, the task was deliberately ambiguous as they were unclear what the 'two segments' meant, but it was this ambiguity that made the task interesting as it resulted in the students. It was also the boundary case between example and counterexample that played a crucial role.

Aspinwall and Unal (2005) conducted a teaching experiment called geo-arithmetic with pre-service secondary mathematics teachers. Their results confirmed that implementing a variety of different representational systems helped the pre-service teachers to translate from one to another. Other studies have examined geometrical tasks involving toys, machines or other tools, the use of which appear to support problem-solving processes and advancements in understanding (e.g. the use of *Bee-bots* by Highfield, Mulligan, & Hedberg, 2008, mentioned above).

Using DGS, the dynamic manipulation of geometric objects by 'dragging' is commonly referred to as the 'drag mode' (Hölzl, 1996; Jones, 1996). This is when an object in an on-screen diagram is 'dragged', the diagram is modified yet all the geometric relations used in its construction are preserved. This function supports teaching tasks that provide different apprehensions to the viewing of geometric objects and support of dynamic representations that enrich internal thought of students (Xu & Tso, 2009). A range of studies continues to explore the affordances of dynamic geometry 'dragging' environments. For example, Chan (2012) studied a university mathematics teacher who, while an accomplished mathematician, was unfamiliar with DGS. Chan found that initially the mathematician considered the software "a computational tool for the system of Euclid's Elements" but while working on explorative tasks, the mathematician experienced "the powerfulness of dragging and developed a new understanding towards DGS" (p. 297). The affordance of dragging for geometrical problem solving was a feature of the research of Jacinto and Carreira (2013). Here, 14 year-olds tackling a problem involving a rectangular lawn and a triangular flowerbed used 'dragging' to check or verify their solution. Similarly, Leung and Or (2009), investigating perspective dragging in Cabri 3D, showed that this function helps students' identification and reasoning about geometric properties of 3D objects.

Certainly, dragging in 3D software presents some differences compared to the manipulation of 3D physical models. Hattermann (2008, 2010) focused on the drag-mode of the 3D digital environment and underlined its importance in explaining that it transforms the static figures of geometry to dynamic objects. However, the use of this function is not so apparent to students who need encouragement to implement it and appreciate its advantages. In their study with 13–14 year old students, Lee and Leung (2012) confirm that, while generating more examples is the central affordance of dragging, generating such examples becomes possible for the student "only when prompted" (p. 66). Building on this and related studies, Leung (2014) proposes four principles for task design in dynamic geometry, while Sollervall (2012) reports on the design of spatial coordination tasks that make use of mobile technologies.

In a different vein, Martignone and Antonini (2009) introduced pantographs for geometrical transformations. They presented a classification scheme efficient to analyse the interaction between a subject and the machine, and the processes involved. Subsequently, Martignone (2011) presented and discussed some examples of tasks for teachers that involved geometrical 'machines'; that is "reconstructions of tools belonging to the historical phenomenology of mathematics from ancient Greece to 20th century" (p. 193) such as curve drawers and pantographs. The teachers tackled tasks such as constructing an isosceles triangle and then later they adapted the tasks for their classroom.

Wu, Wong, Cheng and Lien (2006) designed a learning environment named *InduLab* that gave 4th grade students the possibility to discover the rules of triangle construction and thus approach the angle sum property. Lew and Yoon (2013) used a developing affordance of certain software to link geometry and algebra and reported how "constructing the solutions of quadratic equation offers an alternative approach that gives students an opportunity to connect algebra (quadratic equation) and geometry (construction)" (p. 255). Their study showed how understanding of the mathematics of geometric similarity connects quadratic equation with geometric construction.

Finally, Choy, Lee and Mizzi (2015) studied how textbooks support the teaching of the topic of gradient in Germany, Singapore, and South Korea. By examining textbooks in terms of "contextual (educational factors), content, and instructional variables" (p. 169), they concluded that the textbook 'signature' of each country is 'unique'.

Summarizing, several studies have, to date, focused explicitly on geometric task design and entailed the use of technology. As the teaching of geometry is a multidimensional challenge, there is scope for more research on geometry teaching and tasks.

CLOSING REMARKS

Research on spatial reasoning has analysed different components, including perspective taking, rotation and mental transformation. Findings emerging from these investigations, mainly from tests or task-based interviews, both on younger and older ages, have concerned students' and teachers' capabilities in spatial understanding and processing. These capabilities improve over the age-range, but some individuals still retain vague conceptions of dimensions or space, and thus face spatial situations (even maps) with strategies that tend to be rather non-elaborated. These shortcomings are attributed to the lack of appropriate education and are generally improved by teaching proposals, especially when appropriate tasks and technological tools are implemented.

In terms of geometrical visualisation and visual thinking, there is evidence that, even though the role of visual process is particularly important in the learning and teaching of space and geometry, the number of investigations related to the visualizing capabilities of either students or teachers, or proposals for teaching interventions, has been somewhat limited. One reason for this could be the greater range of studies conducted in earlier years. Nevertheless, there remains a considerable interest in investigating visual processes in geometrical proving and problem solving, as well as a special concern about the use of gesture as an aspect of visualization.

In contrast to the somewhat limited development of research on geometrical visualisation and visual thinking, research continues to search for ways to improve the learning and teaching of geometric measurement. Using tests and interviews to examine conceptions about measurement of length, area or volume both on young and older students, research indicates low achievement and confusions regarding different aspects such as units, partition or iteration. Appropriate teaching proposals and relevant activities appear to improve measurement understanding.

Research into the teaching and learning of geometrical reasoning and proving continues apace, spurred by the increasing availability and sophistication of computer software. Studies with tests or interviews, mainly on secondary students, are attempting to connect proving processes to other capabilities or social practices and to identify predictors of proving skill. There is a special research interest in teaching proposals or use of relevant software with encouraging results regarding students' development in argumentation, generalization and proving. However, these results are only parts of a wide field of investigation. Constituting an important component of mathematical activity, geometric reasoning and proving requires further research in several under-researched issues.

Studies of students' geometric knowledge continue to form a main thrust in research on the teaching and learning of geometry, mainly based on the van Hiele model, Duval's figure apprehension framework, or other approaches related to identification of 2-D or 3-D geometric figures. Such research focuses on many of the key geometric ideas in the curriculum, and attempt to find connections with other mathematical issues (like spatial reasoning, visualization, proving or use of language). A systemization of the results in this field might be needed.

Paralleling the studies of students' geometric knowledge are studies of teachers' geometric knowledge and studies of teacher development for geometry education, indicating important figural and conceptual misunderstandings. Based on the same frameworks as with students, researching teachers' knowledge across different geometric ideas mainly indicates low understanding of geometry subject matter. This fact raises the need for an improvement of teachers' education and attracts the interest of several studies with proposals including relevant tasks, geometric software or teaching approaches.

Another rich vein of research in geometry education is that focusing on the teaching of geometry and the design and use of classroom tasks, especially the use of technology. Even so, research with proposals for appropriate teaching tasks remains somewhat limited and would benefit from further systematic investigation.

Some topics of research are under-represented. For example, there seems limited research explicitly on the topics of congruency and similarity, and little on transformation geometry. Research on analytic/coordinate geometry is also limited, as is research on vector geometry. On the positive side, research in geometry education is embracing the use of more recent discursive, embodied and eco-cultural perspectives, and is also employing new methods such as eye-tracking.

As research develops further, the affordance of digital technologies is enriching approaches to geometric and spatial teaching and learning by providing new ways of apprehension and representation, new manipulation and processes, wider and deeper conceptual understanding and linking of different meanings and treatments.

In general, results concerning the better understanding of how space and geometry are comprehended by students but also related to the development of effective teaching approaches, give opportunities for an enhanced access to relevant concepts and procedures. Moreover, the improvement of teachers' geometrical knowledge as well as their awareness of appropriate teaching methods, including the use of digital technology, develops the overall image. As mentioned previously, throughout the research effort, the systematization of findings and methods continues to be of great importance.

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PART 2

COGNITIVE ASPECTS OF LEARNING AND TEACHING TRANSVERSE AREAS

LEONOR SANTOS AND JINFA CAI

5. CURRICULUM AND ASSESSMENT

INTRODUCTION

Curriculum and assessment have not been popular themes in PME conferences and proceedings. In the first Handbook of Research on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006), there was neither a chapter on curriculum nor on assessment. This is very understandable given that PME has traditionally been focused on the "*psychology of mathematics education.*" Therefore, most of the chapters in the first handbook involved cognitive aspects. In recent years, however, there has been an increased number of PME presentations and papers on both curriculum and assessment. This phenomenon shows the wider recognition of the importance of curriculum and assessment in studying the psychology of mathematics education. In fact, researchers have recognized curriculum and assessment as an integral part of research related to the psychology of mathematics education.

Searching the major publications in mathematics education, we found no chapter entitled "curriculum and assessment." Although curriculum and assessment are related, there seems to be a distance between the two. How, then, can we conceptualize a chapter on curriculum and assessment? We first conceptualize the chapter by viewing assessment as the link between the implemented and the attained curriculum. There is a broad acceptance of the conception of curriculum as having three levels: Intended curriculum, implemented curriculum, and attained curriculum (Husen, 1967). We have therefore tried to conceptualize the connections between curriculum and assessment in these three levels (Cai, 2014). After we present this conceptualization, the chapter will discuss curriculum and assessment in the following sections: Assessment as a way to analyse the intended curriculum, assessment as a means to implement the curriculum, and assessment as a validation of the attained curriculum. Because of the critical roles of teachers, we also include a section on teachers' knowledge of curriculum and assessment. In each section, we have tried to point out the methodological issues and possible future directions of research. We conclude the chapter by summarizing the major aspects that emerged from the analysis presented.

ASSESSMENT AS THE LINK BETWEEN IMPLEMENTED AND ATTAINED CURRICULUM

There is no consensus about the actual definition of curriculum. In this chapter, we use the term in a broad sense, following Cai and Howson (2013) from two

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 153–185.

perspectives. The first is that curriculum can be discussed from different levels. The International Association for the Evaluation of Educational Achievement (IEA)'s First International Mathematics Study (FIMS) (Husen, 1967) distinguished between three levels of curriculum (intended, implemented, and attained). The intended curriculum refers to the formal documents that set system-level expectations for the learning of mathematics. These usually include goals and expectations set for the educational system along with textbooks, official syllabi, and/or curriculum standards. The implemented curriculum refers to school and classroom processes for teaching and learning mathematics as interpreted and implemented by the teachers according to their experience and beliefs for particular classes. Thus, the implemented curriculum deals with the classroom level. The classroom is central to students' learning since students acquire much of their knowledge and form their attitudes from classroom instruction. Regardless of how well a curriculum is designed, it has little value outside of its implementation in classrooms. Finally, the attained curriculum refers to what is learned by students and is manifested in their achievement and attitudes. It exists at the level of the student, and deals with the aspects of the intended curriculum that are taught by teachers and actually learned by students.

The second is that curriculum can be used both as a product and a process. A curriculum is a product: A set of instructional guidelines and materials for students' acquisition of certain culturally-valued knowledge and skills. A curriculum can also be viewed as a process. In this sense the curriculum is not a physical thing, like textbooks, but rather the interaction of teachers, students and knowledge. In this view, teachers are an integral part of the curriculum constructed and enacted in classrooms (Cai & Howson, 2013). Thus, it is natural for us to discuss teachers' knowledge related to curriculum and assessment.

Figure 1 illustrates our conceptualization about curriculum and assessment. For the implemented curriculum, our focus will be on formative assessment, which we will discuss in the section on assessment as a means to implement the curriculum. To gain access to the attained curriculum, it is possible to use summative assessment of students' mathematics learning, taking into consideration the implemented curriculum. This will be considered in the section on assessment as validation of the attained curriculum. As both the implemented and attained curricula are built on the intended curriculum, we also discuss assessment as a way to analyse the intended curriculum. Finally, teachers' knowledge of curriculum and assessment could be about formative assessment or summative assessment, as well as the implemented or attained curriculum.

In developing this chapter, we mainly used the PME Proceedings from 2005 to 2015. Because the short oral and poster presentations have only one-page summaries in the PME proceedings, we faced a challenge in reviewing those studies. Thus, we have mainly relied on the PME research reports for this chapter. To help us discuss issues related to curriculum and instruction, we also refer to other relevant work whenever it is appropriate.

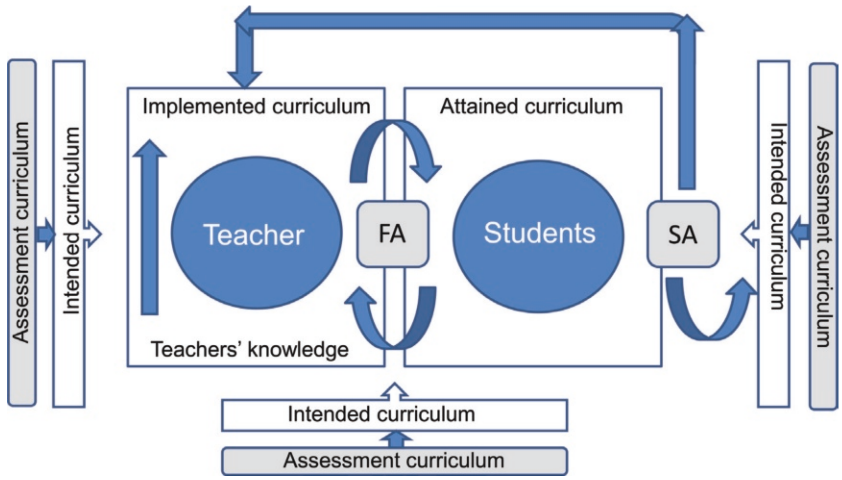

Figure 1. Relations between curriculum and assessment

ASSESSMENT AS A WAY TO ANALYSE THE INTENDED CURRICULUM

In the field of mathematics education, the analysis of intended curriculum has long been accepted as a line of scholarly inquiry (e.g., Cai & Cirillo, 2014; Fuson, Stigler, & Bartsch, 1988; Hamann & Ashcraft, 1986), and a number of researchers have published findings purely based on curriculum analysis in respected journals in the area. However, it is only in recent years that a few researchers have started to present studies based on their analysis of intended curriculum at PME conferences. Some researchers have directly analysed the intended curriculum for certain topics to examine the kinds of learning opportunities provided to students. For example, Jiang and Cai (2014) analysed problem-posing tasks in Chinese and U.S. elementary mathematics textbooks. In the past several decades, there have been efforts around the world to incorporate problem posing into school mathematics at different educational levels (e.g., Cai, Hwang, Jiang, & Silber, 2015; Singer, Ellerton, & Cai, 2015). If problem-posing activities are to play a more central role in classrooms, they must be more prominently represented in curricula. Similarly, if teachers are to engage students in problem posing in the classroom, they must have sources for problem-posing activities. The education reform both in China and the U.S. has recommended that problem-posing activities be included in mathematics curricula. By analysing problem-posing tasks in textbooks, Jiang and Cai (2014) observed how reform ideas were reflected in the mathematics curriculum, particularly in the increasing prominence of problem-posing tasks in textbooks. It would appear that curriculum reform has moved problem-posing tasks into greater prominence, but a great effort is still needed to make problem posing a reality in both curriculum and instruction. In fact, the analysis showed that even in the so-called reform textbooks, the proportion of problem-posing tasks was very small (about 3%). Moreover, most of the problem-posing tasks were embedded in the content strand of number and operations.

In longitudinally investigating the effect of middle school mathematical curricula on students' learning in the United States, Cai and his colleagues first analysed two types of middle school curricula in their LieCal Project (Cai, Nie, & Moyer, 2010; Cai, Moyer, & Wang, 2013a). The first type of middle school curriculum is called Connected Mathematics Project (CMP). The CMP curriculum is one of the Standards-based middle school curricula in the United States, designed to build students' understanding of important mathematics through explorations of real-world situations and problems. Students using the CMP curriculum are guided to investigate important mathematical ideas and develop robust ways of thinking as they try to make sense of and resolve problems based on real-world situations. Through exploring interesting mathematical situations, reflecting on solution methods, examining why the methods work, comparing methods, and relating methods to those used in previous situations, students build deep understanding of mathematical concepts and related procedures. The LieCal Project was designed to longitudinally investigate the ways and circumstances under which CMP and more traditional (non-CMP) middle school mathematics curricula could or could not enhance student learning in algebra as well as the characteristics of the curricula that led to student achievement gains. The project addressed a number of research questions, one of which is related to the intended curriculum: What are the differences between the intended treatment of algebra in the CMP curriculum and in the non-CMP curricula?

Cai and his colleagues (Cai et al., 2013a; Cai et al., 2010; Nie, Cai, & Moyer, 2009) found that the CMP curriculum takes a so-called functional approach to the introduction of algebraic concepts in the teaching of algebra, whereas the non-CMP curricula take a structural approach. The functional approach emphasizes the important ideas of change and variation in situations. It also emphasizes the representation of relationships between variables. In contrast, the structural approach avoids contextual problems in order to concentrate on developing the abilities to generalize, work abstractly with symbols, and follow procedures in a systematic way. In particular, CMP characterizes a variable as a quantity that changes or varies. In contrast, the non-CMP curricula define a variable as a symbol (or letter) used to represent a number. Variables are treated predominantly as placeholders and are used to represent unknowns in expressions and equations. In CMP, equations are a natural extension of the development of the concept of variable as a changeable quantity used to represent relationships. In the non-CMP curricula, the definition of a variable as a symbol develops naturally into the use of context-free equations with the emphasis on procedures for solving equations. To illustrate the functional approach in the CMP curriculum and the structural approach in the non-CMP curricula, all problems involving linear equations in the curricula were classified into three categories: (1) One equation with one variable, e.g., $2x + 3 = 5$; (2) One equation with two variables, e.g., $y = 6x + 7$; and (3) Two equations with two variables, e.g., the system of equations y = 2x + 1 and y = 8x + 9. In the CMP curriculum, over 90% of the problems involve one equation with two variables, but nearly 90% of the problems in the non-CMP curricula involve one equation and one variable. This finding confirms the functional approach to the introduction of algebraic concepts in the CMP curriculum, whereas the non-CMP curricula take a structural approach.

The analysis of intended curriculum as a line of scholarly inquiry is often the domain of researchers. However, Lee (2006) investigated how preservice teachers analysed mathematics textbooks in Korea. Thirty-four preservice teachers from Korea participated in the three-hour lecture and discussion for 15 weeks. These teachers were exposed to an existing framework for textbook analysis and, in particular, they were asked to evaluate two series of textbooks using the following seven categories from Kulm, Morris and Grier (2000): (1) Identifying Sense of Purpose, (2) Building on Student Ideas about Mathematics, (3) Engaging Students in Mathematics, (4) Developing Mathematical Ideas, (5) Promoting Student Thinking about Mathematics, (6) Assessing Student Progress in Mathematics, and (7) Enhancing the Mathematics Learning Environment. Teachers were asked to rate the textbooks High, Medium or Low in each category using scores of 3, 2, or 1, respectively. After the analysis, the preservice teachers were also asked to try reconstructing the textbooks based on the results of the evaluation and analysis. Lee (2006) found that the preservice teachers rated the textbooks quite low on the categories of 'Building on Student Ideas about Mathematics,' 'Engaging Students in Mathematics,' and 'Enhancing the Mathematics Learning Environment' (close to 1). As for 'Developing Mathematical Ideas' and 'Assessing Student Progress in Mathematics,' the preservice teachers rated the textbooks relatively high (about 2). Perhaps the most exciting finding from Lee (2006) was that participating in such textbook analysis is an effective way to learn and understand curriculum materials and then grow as an effective teacher. This type of curriculum analysis can play an important role in shifting from a passive curriculum user to an active curriculum developer. There have been similar findings from other studies suggesting that engaging curriculum analysis could increase teachers' curriculum knowledge (Ariav, 1991).

There has been little discussion about methodological issues when analysing the intended curriculum. Some researchers have recently begun to discuss methodological issues (Cai & Cirillo, 2014; Stylianides, 2014). For example, Cai and Cirillo (2014) raised several fundamental questions to consider when engaging in research on intended curriculum: (1) How many textbooks should we analyse? (2) Which textbook(s) should we analyse? (3) What text in the textbook(s) should we analyse? The exposition? The exercises? (4) How much of that text should we analyse? (5) How should we analyse it (i.e., what framework will we use to conduct our analysis)? and (6) What research questions should guide our analysis? For additional discussion about methodological issues with respect to the analysis of the intended curriculum, please see Lloyd, Cai and Tarr (in press).

In summary, the analysis of the intended curriculum has been widely accepted as a scholarly activity aimed at understanding students' learning opportunities. Guiding teachers to analyse intended curriculum can help them to be both curriculum users and curriculum developers. There is a need for the field of mathematics education to systematically consider the methodological issues with respect to the analysis of the intended curriculum.

ASSESSMENT AS A MEANS TO IMPLEMENT THE CURRICULUM

When we consider assessment strategies to enhance learning and consequently to attain the mathematics curriculum, three main aspects have to be considered: The definition and the nature of adequate settings that permit the successful deployment of such strategies, the characteristics of the strategies themselves, and finally the effectiveness of such strategies for mathematics learning.

Conditions of Assessment for Learning

As assessment for learning happens in the daily life of the classroom, some conditions have to be guaranteed to create an adequate environment for using an assessment that contributes to implementing the curriculum. Although we recognize the complexity of the educational environment, we highlight two particular conditions: (i) The way errors are considered by the teacher and the students and (ii) the knowledge of assessment criteria. The culture of error determines the students' confidence in sharing their reasoning and mathematical processes. The assessment criteria permit teachers to be conscious of what they do in their teaching practices and students to understand what is missing, so that they may close the gap between what they have done and what they are expected to do.

Culture of error. Errors made by students, revealed throughout the assessment process, are a familiar phenomenon to any mathematics teacher. They are present in the everyday life of the classroom and come through oral or written productions of students in learning situations. However, the way teachers and students face an error is decisive for the type of learning environment that is built in the mathematics classroom. From the perspective of assessment as a measure, the error is seen as an evil to eradicate and assumes an accountant function (Santos, 2002). Its causes are usually associated with the student. From the perspective of assessment for learning, error is seen as natural and inherent to the learning process (only one who is learning errs) and constitutes a fundamental source to access the different types of students' reasoning. The reasons for error will now be centered around the curriculum. The error is taken as an indicator of the degree of difficulty in the construction or ownership of a certain concept, or in the way it was approached and worked. It is an indicator of the need for educational intervention that requires adaptation. Taking these assumptions as a starting point, students' errors or misconceptions may be analysed using three perspectives: (i) As a curriculum support; (ii) the way teachers face and use the error in their teaching processes; and (iii) possible benefits for mathematics learning when students reflect on their own errors.

To use errors as a curriculum support, it is necessary to know the level of difficulty of the different mathematics topics for students' learning. Based on a sample of 8,829 children, Davis, Pampaka, Williams and Wo (2006) scaled the item difficulties of data-handling and statistics items for the UK national curriculum for ages 7 to 14 and identified errors on the same scale. They developed a five level hierarchy of item difficulty, using three dimensions: Cognitive load, arithmetic demand, and prototyping. For the data-handling problems used, "prototyping errors were most frequent at the lower end of the difficulty scale than those of cognitive demand" (p. 407), although the three dimensions considered for explaining the reasons for error were identified at all ability levels.

Chung (2007), focusing on the concept of time, identified 24 types of misconceptions in a national sample of 1,100 schools (9- to 12-year-old students) in Taiwan. "The more distance moved, the longer time spent" and "when the clock stops moving, the time stops as well" (p. 210) were the stronger misconceptions found in the study. Doig, Williams, Wo and Pampaka (2006) used a national representative sample involving a total of 14,000 students from schools in England and Wales to create a developmental map of students' understanding and skills about time. According to their report, although the students involved were exposed to a particular curriculum (that used in the UK), the map may be used as a diagnostic assessment tool, providing a description of students' development concerning the concept of time.

Knowing students' difficulties in the learning process of any mathematics topic is important to support teachers' work. However, several conditions are needed for teachers to use this knowledge profitably. First of all, it is necessary to pay attention to how teachers perceive error and how they deal with it in the mathematics classroom. Studying 137 German in-service secondary teachers, Kuntze (2009) concluded that teachers held a low behaviorist and a rather constructivist view about the role of errors in mathematics learning. Moreover, most of the teachers had a positive view of the importance of learning by errors.

In Taiwan, Leu and Wu (2005) developed a three-year case study of one teacher, Ms. Lin. As time passed, this teacher seemed to change her perception of the use of errors for learning – from the perception that error was an indicator of failure and the role of the teacher is to ignore it, to the recognition that discussing students' errors may help them to clarify and reflect. Nevertheless, this change was conditioned by extrinsic factors, such as time constraints and/or pressure for students' high score attainment.

Secondly, teachers need to skillfully use the errors in the classroom to implement the intended curriculum. Focusing on the subtraction algorithm, the division algorithm, fraction addition, and the relationship between area and perimeter, Chick and Baker (2005) aimed to identify which types of answers (procedural or conceptual) nine Australian teachers of grade 5 or 6 gave to their students. The authors concluded that the mathematical topic and the nature of the items might be related to the emphasis on procedural and/or conceptual explanations.

Finally, Heemsoth and Heinze (2014) highlighted the benefit that students may gain by reflecting on their own errors in fraction problems. Participants in the study included 174 German students in seventh and eighth grade. Heemsoth and Heinze (2014) provided evidence that procedural knowledge was enhanced when students reflected on the rationales behind their errors when compared with those who reflected only on the corresponding correct solutions.

The studies presented reinforce the importance of the way the error is faced, by students and teachers, as well as the need to have a deep understanding of the reasons underlying the students' errors. The perceptions and knowledge about errors allow teachers to orient their teaching, foster a deep understanding of the curriculum, and develop meaningful learning of mathematics.

Assessment criteria. Assessment criteria are statements that tell us what aspect of responses is important for students to solve mathematical problems, at a given time. They are lenses with which it is possible to analyze students' work. In assessment for learning, assessment criteria are elements of communication. The use of assessment criteria depends on the context and on the learning objectives stated in a certain moment. It is "a working tool, subject to improvements, adjustable" (Vial, 2012, p. 277).

As students better understand what is expected from them, they are predisposed to achieve greater learning. Pinto and Santos (2013) reviewed two Portuguese studies including two classes (first and sixth grade) working with problem solving and developing mathematical reasoning, respectively. The results showed that both group of students used and reshaped assessment criteria in a progressive way, being progressively able to improve their learning. These findings show that the assessment criteria can be an important resource for mathematics learning, despite a student's age.

The appropriation of assessment criteria by students is essential to the learning process (Black & Wiliam, 1998). Nevertheless, it is not easy for students to develop a deep understanding of the assessment criteria, as evidenced in a study carried out in Portugal by Santos and Gomes (2006) This large study involved a seventh grade class, and included the case of a student named Vanda. Data collection consisted of observations of 23 lessons, interviews with students, and document analysis of the students' written reports. Students were asked to write reports from mathematical tasks. At the beginning, Vanda is directed by an impression she creates from her previous assessment school experience, based on self-imposed standards. Self-imposed standards mediated her actions towards the criteria and the activity under development. From the teacher's continuous investment in students' appropriation of criteria, the relationship between assessment criteria and self-imposed standards changed, as reflected in Vanda's mathematical activity.

The appropriation of the assessment criteria by the students demands a committed investment by the teacher. It is not enough that teachers state the criteria, it is also necessary to negotiate with students and to work with them in a continuous way, allowing the ownership of the criteria by the students. However, those demands may constitute new teaching and assessment practices, which may create a new set of difficulties. But teachers' reflection, enhanced by a collaborative work setting, helps mathematics teachers to perform the role necessary to support the appropriation of the assessment criteria by the students (Semana & Santos, 2011).

Similar to what was found concerning the students (Santos & Gomes, 2006), in the beginning of their work with assessment criteria, teachers may also use previous mathematics values and discourse. In Greece, Klothou and Sakonidis (2009) investigated the pedagogical discourse adopted by eight primary teachers with respect to the nature of mathematics as well as the learning, teaching, and assessment processes. From the 2-hour-long interviews the researchers conducted with the teachers, the results concerning one of the teachers, Nikitas, evidenced that he "utilizes an unofficial discourse, adopting, however, the values of the traditional pedagogic discourse" (Klothou & Sakonidis, 2009, p. 357). In the same vein, Monoyiou, Xistouri and Philippou (2006) used semi-structured interviews of 16 teachers from Cyprus. Using students' arguments from 236 students' worksheets (fifth and sixth grades), they asked the teachers to mark the students' arguments on a scale from 0 to 5. Apparently, the teachers' appraisals were based on subjective criteria and differed greatly from one another. Similarly, Sakonidis and Klothou (2007) concluded that the teachers in their sample (553 primary teachers with different amounts of professional experience) used their own criteria, which in some cases did not allow them to offer diverse assessment judgments. Limited resources were used by teachers, namely related to their expectations of how mathematical knowledge should be communicated and their beliefs about the nature of mathematics (Sakonidis & Klothou, 2007; Klothou & Sakonidis, 2011).

Taking into account that the definition of assessment criteria may be a difficult task for teachers, based on the definition of a hypothetical learning trajectory (HTL), Siemon, Izard, Breed and Virgona (2006) identified and validated an integrated learning assessment framework for multiplicative thinking to help improve students' learning in years 4 to 8. A range of rich tasks, including at least two extended tasks, were applied to 3,200 students; teachers in the researched school scored the students' worksheets on the basis of the scoring rubrics provided. Finally, part of the previous group of students (1,500 students) was involved in an 18-month action research study that progressively explored a range of targeted teaching interventions aimed at scaffolding student learning in terms. Content analysis of items permitted "the identification of eight relatively discrete categories which described what students might be expected to be able to do if they scored within the corresponding band of item thresholds" (Siemon et al., 2006, p. 117).

In summary, clear specification and deep understanding of assessment criteria are challenging tasks for teachers and students. Self-imposed standards, that may include beliefs about the nature of mathematics and mathematics learning, constitute possible obstacles, which make an adequate and helpful use of assessment criteria a difficult task. The appropriation of the assessment criteria by the students is a progressive process, but it contributes to the students' knowledge of what is expected, namely in high quality writing (Santos & Semana, 2015).

Main Strategies of Assessment for Learning

Although it is possible to identify several strategies to develop assessment for learning practices in the mathematics classroom, we will highlight feedback and self-regulation. The first may be the responsibility of the teacher or the students. Although the second has to be developed by the student, the teacher nevertheless has a great responsibility to create favorable situations for its development.

Feedback. Black and Wiliam (1998), in their review of the literature on classroom formative assessment spanning the previous 15 years, showed that a large amount of research concerning feedback, mainly empirical studies, had already been developed. However, the effectiveness of feedback is a complex issue that depends on several factors. As Black and Wiliam concluded, it seems that "formative assessment is a static process of measuring the amount of knowledge currently possessed by the individual, and feeding this back to the individual in some way" (p. 52). Moreover, "it is the quality of feedback, not just the quantity of feedback that merits our closest attention" (Sadler, 1998, p. 84). These facts may explain the continuing interest in this topic.

There is no consensus on the definition of feedback. A broader definition is equivalent to information about the level of success of something. A second definition, with an explicit learning purpose, considers the information that allows one to identify the gap between what is already made and the reference level or the objective to be attained, and the attempt to change this gap in any way. A more restrictive definition is also used, in which something is only considered feedback when "it is used to alter the gap" (Sadler, 1989, p. 121). In other words, the first two definitions are centred on the type of feedback provided while the third one is centred on its effects. In this chapter we have taken the broader view of what constitutes feedback to permit us to include more information from the PME proceedings. The structure of this section will try to establish the relation between the effectiveness of feedback and: (i) The content of feedback; (ii) classroom factors; (iii) student dimensions; and (iv) mathematical context.

In the context of a Portuguese collaborative project, Santos and Pinto (2009) presented a meta-analysis of four studies developed along three school years by two mathematics teachers and four middle school classes with students aged from 12 to 14 years old. The results indicated that an interrogative or mixed (both interrogative and affirmative) feedback that is contextualized by the task tends to be clearer to the students, helping them to improve. The interrogative form, however, has one risk.

Instead of helping students reflect on and reorient their reasoning, it may lead them to answer the question directly. On these occasions, the purpose of the interrogative feedback is lost. Short feedback comments seem to be more effective than long ones, helping students to focus on certain specific aspects of the task. The results also suggested the existence of a possible relationship between the length of the feedback and the nature of the mathematical task. This relation was the focus of another study carried out in the same collaborative project (Dias & Santos, 2010). Following the methodological design of case studies, four students from eighth grade were studied. Considering four different types of mathematical tasks, two open tasks (investigations and bibliographical inquiries), and two focus tasks (problems and tests), the authors found patterns that seemed to confirm the existence of that relationship. In particular, the most challenging mathematical tasks led the teacher to write longer comments. Open tasks tended to originate feedback focused on the content, but for focused tasks, the feedback seemed to focus on encouragement, positive reinforcement, and underlying mathematical aspects. This study was part of a broader one that involved 50 students aged 13 years old and aimed to understand the effectiveness of teacher written feedback related to the nature of the mathematical task and to the work method developed (Dias & Santos, 2009). The results showed that feedback given for worksheets of tasks such as problems and exploratory or investigative tasks seemed to be more efficient than feedback given for tasks that appealed strictly to knowledge of mathematical concepts. Also, the feedback given for worksheets of tasks solved in small groups seemed to enhance learning in a more significant way when compared to that for tasks solved individually.

These results alert us to the complexity of the teaching and learning setting. Several other factors are present and may explain why students react in different ways to the same learning situation. Students' perceptions about feedback may include: (i) Feedback as a means to improve students' understanding, so marking right or wrong makes no sense; (ii) feedback as playing a role in building or breaking a learner's self-confidence; and (iii) feedback as a way to access the teacher's point of view (Bansilal, James, & Naidoo, 2009). When students' understanding of what is requested from the task is not what the teacher intended it to be, this may naturally constitute a new difficulty for the effectiveness of feedback. Santos and Pinto (2009), in the meta-analysis referenced previously, highlighted different attitudes of the students related to their mathematical achievement. The students with lower achievement in mathematics revealed more difficulties in understanding feedback when it related to or used mathematical concepts, or referred to more abstract ideas. Students with high achievement in mathematics, when not understanding the feedback, tended to orally question the teacher in an attempt to gain new feedback, thus creating a new learning opportunity. In contrast, students with lower achievement in mathematics did not question their teachers, thereby missing a new opportunity to learn.

When using artifacts, feedback may play a recursive role that moves from the modification of the artifact, to the construction of a new interpretation of a mathematics concept (e.g., fractions) or of solving the problem, returning to the modification of the artifact, and so on (Abtahi, 2014). The use by children of the feedback from a mathematical artifact depends on their initial idea about the task and the artifact. In the scope of a European project, Maffei, Sabena and Mariotti (2009) focused on the use of a Dynamic Digital Artefact, namely the feedback component (feedback-signs), in the learning of equivalence between algebraic expressions. Through continuous questioning, the teacher intended to facilitate the understanding of the meaning of the signs and the development of that understanding from a primary interpretation to a developed one. But, this evolution was not linear. The semiotic chain maintains a coexistence of both interpretations.

In summary, all the studies discussed concerned feedback provided by the teacher, although it is recognized that students' feedback given to their peers may have a lot of potentialities (e.g., Sadler, 2010). One transversal message that emerges from these studies is that the effectiveness of feedback is not guaranteed, whatever its quality. Looking at feedback as a communication process, the great challenge is to assure that feedback is a dialogical process and not an action of sending a message (Nicol, 2010). In a dialogical process, students receive external feedback and then adapt and integrate this information internally.

Self-regulation. Taking a progressive perspective on education, it is assumed that the construction of learning requires an active role by students throughout the construction process, including the assessment. In particular, when students are responsible for assessing their own learning process, with the main aim being to modify the current state of affairs and improve their achievement, we speak about the process as self-regulation. Self-regulation is an activity essential for effective learning (Allal, 2007).

The self-regulation process may consist of different components. There is, however, a certain consensus on the key role of cognition, metacognition, and emotions in this process. Panaoura and Panaoura (2006) developed a study in Cyprus that had as its main objective the exploration of the impact of processing efficiency and working memory on metacognitive processes with respect to mathematics, and to explore whether the interrelations between these processes tend to change with development. Participants in this study included 126 pupils who were 8 to 11 years old. The results provided evidence that "processing efficiency had a coordinator role on the growth of mathematical performance, while self-image, as a specific metacognitive ability, depended mainly on the previous working memory ability" (Panaoura & Panaoura, 2006, p. 319) and partially on recent mathematical performance.

In mathematics education, problem solving occupies a prominent place and it is therefore not surprising that the majority of the studies over the last decade of PME proceedings that approach self-regulation established relations with problem solving. That was the case for Marcou and Philippou (2005), who aimed to study the relationship between motivational beliefs and self-regulatory strategies in solving mathematical problems and in students' performance in problem solving. They considered self-efficacy beliefs, task-value beliefs, and goal orientations as characterizing motivational beliefs, and cognitive and metacognitive as components of self-regulatory strategies as well as volitional strategies. The study involved 219 Cypriot students from the fifth and sixth grades. They concluded that students who tend to use self-regulatory strategies while solving a mathematical problem are more likely to have increased motivational beliefs, and vice versa.

Aiming to better understand the relationship between self-regulation and problem solving, Marcou and Lerman (2006) proposed a model that combined the theory of Self-Regulated Learning (SRL), namely the Zimmerman model (e.g., Zimmerman, 2000), with Mathematical Problem Solving (MPS). To check and validate the mapping of their model, the authors conducted two studies. The first one was carried out in the UK and involved five students from fourth to sixth grade. Students were asked to solve at least two of the three problems posed, to write their work in detail, and to think aloud. Students chose to work in a group. Based on the results, an improvement of the initial version of the model was made. The second study involved Cypriot students and five teachers of year 4, 5, and 6. Teachers were asked to implement the theory and the model in their classrooms during three lessons within two months. Clinical interviews were added due to the difficulties the young children had in making their cognitive and metacognitive ability explicit. Although there were some very positive indications of the suitability of the model in the second study, due to the short period of time of the study, the authors concluded that it was not possible to make final conclusions concerning the impact and efficiency of the model.

Whatever model we use to access and relate students' problem solving and self-regulation capacities, these capacities have to be developed. The role of the teacher is crucial in this development process. In Germany, a study carried out by Collet and Bruder (2006) showed that the work with students by specially trained teachers integrating problem solving, self-regulation, or both leads to improvement in problem solving in all performance groups (low, medium, and high). Based on these results, the same authors conducted a subsequent study (Collet & Bruder, 2008) one year after the end of the intervention. The follow-up study confirmed the stability of the students' problem-solving capabilities.

From the studies described above, it seems possible to conclude that teachers' competences for teaching problem solving and promoting self-regulation can enhance students' problem-solving capacities. However, this may not be the case for changing students' self-regulation. Achmetli, Schukajlow and Krug (2014), also in Germany, studied the effects of prompting students to use multiple solution methods on students' self-regulation while solving real-world problems. No significant differences were found in self-regulation between students in the multiple solution methods condition and the one solution method condition when controlling for self-regulation on the pre-test. The authors explained that the absence of increase in students' self-regulation was due to the fact that students did not have the opportunity to make assumptions about missing information and to apply their assumptions to the task.

Using written reports, Semana and Santos (2010) carried out a study in Portugal that aimed to understand students' development of self-regulation capacity, supported by some assessment strategies that were the teacher's responsibility (the discussion of a report script, the investment over the students' appropriation of the assessment criteria, and feedback production). The results from two eighth grade students were presented. The students' self-regulation capacities evolved gradually but differentially, although both students tended not to identify areas for improvement in their work. Neither did the students outline intervention strategies in order to reduce or eliminate the differences between the current and the desired state of events.

Another context for enhancing self-regulation capacity, namely a reflexive portfolio, was studied in Portugal by Dias and Santos (2013), but with secondary students (11th grade). The results showed improvement in some metacognitive processes throughout the school year. Improvement in mathematical communication also emerged. Explicit work, continued over a sufficiently extended period, helped to change some of the students' behavioral habits, such as studying only on the eve of summative tests. The students considered that working with a portfolio had positive effects on the motivational dimension of self-regulation.

It is important to note that the students' consciousness of their own motivational beliefs is important due to the role that this component plays in the self-regulation process. It is also necessary to understand how students think about their own capacity to self-regulate their learning process. In the context of posing problems and its relation to the development of creativity, Shriki and Lavy (2014) carried out a study in Israel that was intended to examine the effect of students' self-assessment on their mathematical creativity and its development. Out of 190 students from six different regular upper-elementary schools (9th to 12th grades), two case studies were presented. The results showed that for students who possess an optimal mixture of resources from the outset (intellectual skills, thinking styles, personality, and motivation) self-assessment of creativity can be beneficial. Students that do not have these resources need some additional support, such as teacher feedback.

In summary, self-regulation is an important student capacity to enhance mathematical learning, in particular in problem solving. Nevertheless, it is highly demanding of students. It requires not only cognitive capacities, but also affective and emotional ones. For its development, a supportive environment has to be created by the teacher. In other words, fostering assessment for learning by allowing students to take part in self-assessment demands a major shift in student and teacher roles compared to those found in more traditional classrooms (Nortvedt & Santos, 2014).

Technological Classroom Environments and Assessment for Learning

Nowadays, we cannot ignore the important role of technology in any context of mathematics classrooms if our aim is to contribute to meaningful mathematical learning (NCTM, 2014). Leung (2013) calls for a greater understanding of how technology may improve mathematics teaching and learning compared to the curriculum of the past, stating that "curriculum and teaching and learning methods will need to be regularly reconceptualized to take advantage of the power of modern technology to improve mathematics education" (p. 523). This reconceptualization demands new ways to look for what it is really relevant to know in mathematics and which mathematics make sense to teach and, consequently, to rethink the assessment processes in the digital age.

The use of technology may offer a good opportunity to change all aspects of the assessment process (Stacey & Wiliam, 2013), such as the ways that assessment tasks are selected, created, and presented to the students and the ways with which to provide feedback. In Australia, Goodwin (2008) presented the development and implementation of an Early Digital Fraction Assessment (EDFA) that present students with open-ended tasks. Participating in the study were 40 male students (Kindergarten and grade 1) and eight case study students, consisting of four from each class. The results showed that the dynamic representations enabled by the EDFA elicited representations of common fractions and percentages, with apparent conceptual understanding.

With the objective of developing a formative assessment design for web project-based learning (WPBL) in elementary schools, Lin, Hung and Hsiao (2009) carried out a research study in Taiwan. The online device provided norms for individual accountability in the group and progress for each group, as well as assessment feedback with guidance for or models of how to conduct an operation within the project materials. The study took three years and involved 124 fifth grade students (62 in each group, with an experimental and a control group). The experimental group of students learned collaboratively for 15 months. A pre-test and post-test were used. According to the results, collective knowledge and creativity may be developed using formative assessment through a web-based PBL, even for elementary school students.

Broughton, Hernandez-Martinez and Robinson (2013) in the UK used computer-aided assessment (CAA) over ten years. The participants of the study were nine self-selected first-year undergraduates (four mathematicians, three aeronautical and automotive engineers, one materials engineer, and one sports technology engineer). Students were pleased because the feedback enabled them to demonstrate an improvement and, considering that the high marks in CAA indicate the attainment of learning goals, that they had learned the material. Moreover, "students had set superficial goals that did not indicate what had been learned, and perhaps this explains in part why more challenging goals were not set" (Broughton, Hernandez-Martinez, & Robinson, 2013, p. 119). The use of CAA encouraged interaction between peers, which developed more beneficial mutual support over time.

Finally, Roble (2014) studied how U.S. mathematics teachers use the Navigator system to formatively assess their students. She considered two high school mathematics teachers in the U.S., one from one an urban district and the other from a suburban district, who used technology. Several types of questions, such as check for understanding and recall of prior knowledge, were used as feedback. As they worked with connected classroom technology, teachers were able to seek feedback from all students, resulting in data-based instructional decision making.

We conclude that, despite the recognition of the role of technology in assessment for learning, we found only a few studies in the PME proceedings that considered the use of this resource in developing an assessment process that enhances learning. In the studies presented, the focus was different, from the opportunities contributing to students' mathematics learning to the way teachers used technology. Nevertheless, they evidence positive results, not ignoring some difficulties that did not fulfill the initial expectations. Further studies are needed, especially if we take into consideration that "technology resources is not yet evident among most teachers" (Leung, 2013, pp. 522–523) and even when teachers use digital tools it is not easy to integrate it with other elements of teaching and learning mathematics (Geiger, Dole, & Goos, 2011).

Assessment for Learning and Students' Mathematical Performance

Assessment is not an end in itself, but a means for learning and a means to attain the curriculum. Perhaps one of the most important results from the meta-analysis carried out by Black and Wiliam (1998, p. 61) is that "formative assessment does improve learning." But this key issue continues to be discussed and studied (e.g., Bennett, 2011; Kingston & Nash, 2011; McMillan, Venable, & Varier, 2013). Several studies previously presented in this part of the chapter made connections between assessment strategies and students' performance. Here, we will include very recent PME proceedings research, the objective of which was to study the effects of formative assessment strategies on students' mathematical performance.

In the Netherlands, Veldhuis and van den Heuvel-Panhuizen (2014) proposed studying the feasibility and effectiveness of classroom assessment techniques for mathematics in primary school. Two consecutive small-scale studies with 10 third grade teachers (four, then six) and a total of 214 students were conducted. The same method and assessment techniques, consisting of short activities of less than 10 minutes (Red/Green cards, Clouds, Hard or easy, Experiment, Find the error(s), and Find problems with the same result), were used in both studies in the context of the assessment of number sense, specifically when working with addition and subtraction. The first study focused on feasibility and sustainability issues. Subsequently, the second study aimed to study the effectiveness of the use of classroom assessment techniques. A pre-test and a post-test were used to measure students' mathematics achievement. The results provided evidence that in both studies students learned more when teachers made effective use of classroom assessment, compared with students from the national sample.

The effectiveness and sustainability of feedback was also studied in Turkey by Özdemir and Tekin (2011). Worksheets containing test questions about the subject and essay questions aiming to learn students' thoughts on the subject were given to students of the experimental group during and after the classes. After students completed the worksheets, the teacher collected them and gave feedback. Depending on the feedback, students had to resubmit their sheets. A pre-test and post-test were applied, as well as a retention test after eight weeks. A mathematical attitude scale was used to determine mathematical attitudes. According to the authors, the experimental group was more successful than the group of students who were not given feedback and, after eight weeks, the experimental group remembered the mathematical subject, while the other group had started to forget it.

In summary, the findings of these studies point in the same direction as the findings of Black and Wiliam (1998). In other words, the feasibility, effectiveness, and sustainability of mathematics classroom assessment for learning strategies may be possible.

A significant number of researchers have dedicated their effort to studying assessment as a means to attain the curriculum. Researchers from different countries have provided evidence supporting the use of assessment for learning as well as evidence of the positive relationship between assessment for learning and student performance. In the future, continued research efforts are needed to explore a deeper understanding of classroom assessments teachers and students may use to contribute to students' mathematics learning. In the last decade, the studies published in PME proceedings have mainly involved students or teachers in elementary or middle school levels. In the future, this line of research could be extended to the high school or college levels. That is, another direction for future research is to investigate the effect of formative assessments on high school and college students' learning.

ASSESSMENT AS VALIDATION OF THE ATTAINED CURRICULUM

Because of the critical role of curriculum in teachers' teaching and students' learning, it is important to empirically investigate the actual impact of curriculum on students' learning. In this section, we first summarize some findings from the LieCal Project (Cai et al., 2013a), mentioned in an earlier section because this is the only project reporting the effect of curriculum on students' learning in the PME proceedings. Then, we discuss a few methodological issues regarding this line of research. Finally, we discuss possible relationships between formative and summative assessment.

Findings from the LieCal Project

The LieCal Project had both a middle school component and a high school component (Cai, 2014; Cai et al., 2011; Moyer et al., 2011). The LieCal-Middle School Project investigated the differential effects of the Connected Mathematics Project (CMP) and more traditional mathematics curricula (non-CMP) on middle school students' learning of algebra. In the LieCal-Middle School component, more than 1,300 students (650 using CMP and 650 using non-CMP curricula) were followed as they progressed through grades 6–8 from 14 middle schools in an urban school district serving a diverse student population in the United States. In the LieCal-High School component, Cai and colleagues continued to follow about 1,000 of these students from 9th grade to 12th grade as they spread out to ten different high schools in the district. As was indicated in Cai (2014), the LieCal Project investigated not only the ways and circumstances under which these curricula could or could not enhance student learning in algebra, but also the characteristics of the curricula that led to student achievement gains. In the LieCal Project, a quasi-experimental design with statistical controls was used to examine longitudinally the relationship between students' learning and their curricular experiences. Student achievement in grades 6–8 was measured using the state test in mathematics, using multiple-choice items (assessing basic algebraic thinking skills) and open-ended tasks (assessing conceptual understanding and problem solving in algebra). On the open-ended tasks, the growth rate for CMP students over the three years was significantly greater than that for non-CMP students (Cai et al., 2011). Also used were problem-posing tasks that examined the impact of middle school curriculum on students' high school learning (Cai et al., 2013a; Cai et al., 2013b). Generally, when comparing the problem posing performance of the CMP students in each third to the non-CMP students in the same third, the CMP students performed as well or better than the non-CMP students in the same third. The evidence has shown that problem posing can be a feasible, reliable, and valid measure of the effect of middle-school curriculum on students' learning in high school.

Methodological Considerations for Assessing the Impact of Curriculum

We discuss three methodological issues related to assessing the impact of curriculum on students' learning: (1) Using various learning outcome measures; (2) using both classical and modern statistical methods; and (3) examining the curricular impact beyond the grade levels in which the curriculum was implemented.

Using various learning outcome measures. Even though various methods can be used to measure students' learning, the heart of measuring mathematical performance is the set of tasks on which students' learning is to be evaluated. It is desirable to use various types of assessment tasks, thereby measuring different facets of mathematical thinking. For example, different formats of assessment tasks (such as multiple-choice and open-ended tasks) may be used to measure students' learning. Multiple-choice tasks have many advantages, including the fact that more items can be administered within a given time period, and scoring responses can be done quickly and reliably. However, such items are difficult to use to infer students' cognitive processes from their responses. Thus, in addition to multiple-choice tasks, open-ended tasks may be used. For open-ended tasks, students are asked to produce answers as well as show their solution processes and provide justifications for their answers. In this way, the open-ended tasks provide a better window into the thinking and reasoning processes involved in students' mathematics learning. Of course, a disadvantage of open-ended tasks is that only a small number of these tasks can be administered within a given period of time. Also, grading students' responses is labor intensive. To help overcome the disadvantages of using open-ended tasks, a matrix sampling design of administering open-ended tasks to students is recommended. This can reduce both testing time and grading time while still obtaining a good overall estimate of students' learning of mathematics (Cai et al., 2011). In addition, as we noted above, problem-posing tasks can be a reliable and feasible measure of curricular effect on students' learning (Cai et al., 2013b).

Analyses of mathematical performance. The unit of analysis should be appropriately selected in analysing students' performance. In evaluating mathematics education programs at the student level, a reasonable number of students from different classes with different teachers in different schools or even different school districts are needed. To examine the impact of a curriculum on students' learning, pre- and post-tests are usually used and statistical analyses are conducted. In classical analysis using t-tests or Analysis of Covariance (ANCOVA), each student in the sample is treated as the same. This implies that students from different classes with different teachers in different schools are considered to have the same experience with the program. The reality is that students from different classes with different teachers in different schools are likely to have different experiences with the program. Thus, in evaluating mathematics education programs, researchers must select an appropriate unit of analysis, and take into account the degree to which a program is implemented as well as the relationship between the degree of implementation and students' achievement. Advanced analysis techniques, such as hierarchical linear modeling, can be used to examine the impact of a curriculum on student learning at the intended (system) and at the implemented (classroom) level simultaneously (Cai et al., 2011; Raudenbush & Bryk, 2002).

When assessing the attained curriculum, it is useful to know students' learning outcomes in terms of mean scores on various types of tasks. However, comparing the students' performances from different mathematical programs in terms of correctness on individual tasks is not particularly revealing unless the reviewers explore the thinking and methods that led students to their correct answers. For example, two students may receive the same mean score, but use very different solution strategies. Also, two students may receive the same mean score, but may make very different errors. Therefore, in evaluating students' learning outcomes, it is important to examine the cognitive aspects of their problem solving, such as solution strategies, mathematical misconceptions/errors, mathematical justifications, and representations. In fact, examining solution strategies can reveal qualitative aspects of students' mathematical thinking and reasoning, such as how they go about formulating goals and purposes in their learning and mathematical problem solving. Similarly, the examination of solution justifications and representations reveals the ways that students process a problem and express their mathematical ideas and thinking processes. In the LieCal Project, problem-solving strategies have been used as a measure of longitudinal curricular effects on student learning (Cai et al., 2014). Using assessment data, Cai et al. (2014) compared the CMP students' problem-solving performance and strategy usage on a multi-part open-ended problem to that of their non-CMP counterparts. When controlling for their sixth-grade state mathematics test performance, high school students who had used CMP in middle school had significantly higher scores. In addition, high school students who had used CMP appeared to have greater success algebraically abstracting the relationship in the task.

Examining the curricular effect beyond the grade levels. The LieCal Project is the only longitudinal study that has examined the effect of middle school curriculum beyond the middle school levels. Findings from research studies of the effectiveness of Problem-Based Learning (PBL) on the performance of medical students (Dochy, Segers, Van den Bossche, & Gijbels, 2003) showed that PBL students performed better than non-PBL (e.g., lecturing) students on clinical components in which conceptual understanding and problem solving ability were assessed. However, PBL and non-PBL students performed similarly on measures of factual knowledge. When these same medical students were assessed again 6 months or a few years later, it was found that the PBL students not only performed better than the non-PBL students on clinical components, but also on measures of factual knowledge (Vernon & Blake, 1993). This result may imply that the conceptual understanding and problem solving abilities learned in the context of Problem-Based Learning facilitate the retention and acquisition of factual knowledge over longer time intervals. The CMP curriculum can be characterized as a problem-based curriculum (Cai, 2014). Analogous to the results of research on the learning of medical students in the PBL research, it was found that CMP students outperformed non-CMP students on measures of conceptual understanding and problem solving during middle school. Also analogously, CMP and non-CMP students performed similarly on measures of computation and equation solving. Therefore, it is reasonable to hypothesize that the superior conceptual understanding and problem-solving abilities gained by CMP students in middle school may result in better performance on a delayed assessment of manipulation skills such as equation solving, in addition to better performance on tasks assessing conceptual understanding and problem solving (Cai et al., 2013a).

While the field of mathematics education has made advances in assessing the impact of curriculum on students' learning, recent research has shown the advantages of using multiple outcome measures to detect curriculum impact on student learning. In particular, it is important to use a combination of multiple-choice tasks and open-ended tasks for such measures. Both quantitative and qualitative analyses should be used to investigate curricular effect. In the quantitative analysis, both classical (e.g., ANCOVA) and modern statistical techniques (HLM) may be used. Finally, we must examine the curricular effect beyond the grade levels in which the curricula are used.

Relation between Formative and Summative Assessment

Researchers have shown that integrating formative assessment into the classroom can enhance learning. However, summative assessment is an imperative in many education systems. Therefore, it is essential to understand how these two types of assessments may be interrelated (Looney, 2011). Some researchers have started to address the issues of the relatedness of formative and summative assessments (Taras, 2005). This can be seen in Vial (2012, p. 353) when he speaks about the "ambiguity of the expression 'to do at the same time the formative and the certificate'" or when comparing the control functions (verification of conformity) with the accompaniment. He associates the two with oil and water, respectively, showing the impossibility of mixture. A less radical position, but also reserved, is presented by Shepard (2001) when she says that "the uniform nature of external assessment and their infrequency means that they will rarely ask the right questions at the right time to be an effective part of the ongoing learning process" (Shepard, 2001, p. 1080).

The very nature of the information provided by summative exams – quantitative scales, statistical measures, very general categories of mathematical topics such as "Geometry and Measurement" or "Algebra" – means that they make a very inefficient contribution if we want to intervene in terms of learning support (Foster & Noyce, 2004). The interpretation and consequent action need to be distinct in order to be suited to different purposes (Harlen, 2006). Thus, the integration of formative and summative assessment is coated with a high complexity and tension (Black & Wiliam, 2005; Price, Carroll, O' Donovan, & Rust, 2011). However, this complexity can be reduced when we consider only internal evaluation practices, that is, those that are the teacher's responsibility and not the responsibility of external experts. The information is thus collected and processed by the same person. It involves the teacher in planning the assessment and in defining the assessment criteria. This is the situation in the three articles from the PME proceedings that we analyse below. These three studies were situated in the primary, secondary, and pre-university education levels, respectively.

Pinto and Santos (2012), in Portugal, intended to understand a possible relationship between summative and formative assessment in a group of primary schools. Thirteen primary teachers with students from 6 to 10 years old applied mathematics written tests each trimester to all the students from the same grade of the group of schools. This assessment practice was intended to assess the students' achievement in mathematics and to contribute to the development of plans to support their learning. Since the information that the teachers got from students' tests was marked by a high level of generality, the teachers encountered difficulties in building supportive plans according to the specifications of each student. As the students progressed in age, the difficulties described moved from certain mathematical topics (numbers, operations) to mathematical capacities (problem solving, mathematical reasoning, and communication). These results indicate that these tests served a more summative purpose than as a way for teachers to collect useful information to regulate students' mathematical learning.

With a similar objective, Baroudi and Clarke (2009) studied ways in which the formative and summative goals of assessment could be achieved by using the same instruments in the context of an Australian secondary school for girls. The teachers used assessment worksheets throughout the instructional sequence. At the end of the unit, teachers predicted the marks that would be achieved by their students. They were most accurate about the group of students who achieved the lowest 25% of the test marks. "Formative assessment instruments complement, but do not replace, an end-of-unit summative test for the purpose of generating an accurate report of students' performance" (Baroudi & Clarke, 2009, p. 333).

The setting of these two previous studies included an attempt to change the usual assessment practices. This attempt may be also found in the study conducted by Povey and Angier (2006) in a higher education mathematics course in the UK. Different assessment tasks were developed in which students had space to explore and find out about mathematics, to try out different approaches to the subject, and to develop their own ideas. The part of the study presented involved two students, Geoff and Anna, who had previously failed university mathematics but went on to become effective mathematicians, achieving first class honors in their final mathematics assessments. Each student was interviewed, sometimes alone and sometimes together, and written reflections from one or both as well as email conversations with one or both were also considered in the data analysis. The authors concluded that it is possible to re-craft the demands on students in order to incorporate some opportunities for educative assessment (assessment as part of the learning process).

But changing assessment practices is demanding and requires the willingness to do it and the recognition of the importance of doing it. In Israel, Biton and Koichu (2009) aimed to characterize the process of creating conditions for adopting particular alternative assessment tools. Data was collected from interviews with the head teacher, the academic advisor, and 6 out of 10 mathematics teachers at the Centre for Pre-University Education. In addition, about 250 randomly chosen students' files were analyzed, representing about 600 files of students who had studied at this centre in the past and were then accepted to the Technion. Biton and Koichu (2009) found that statistical analyses were the most convincing arguments for the academic staff to implement alternative assessment. Moreover, as the level of prediction of the different exams is not the same, the use of alternative assessment tools will not decrease the predictive validity of overall grades.

A future direction for research is to investigate the alignment among various levels of curriculum. Thomas and Yoon (2014) have shown the challenge for a teacher to align the intended and implemented curricula, as has also been documented elsewhere (e.g., Lloyd et al., in press). The encouraging news from Thomas and Yoon (2014) is that it is possible to help teachers become aware of the curriculum alignment through professional development activities. It is also quite complex to align formative and summative assessment practices. Thus, one of the important directions for future research is to explore ways to align various levels of curriculum, as well as to align formative and summative assessments.

TEACHERS' KNOWLEDGE OF CURRICULUM AND ASSESSMENT

In the context of the mathematics classroom, the role of the teacher is essential. The success of assessments depends on the learning environment created by the teachers and their own role during the assessment process. But the quality and adequacy of these learning opportunities depend greatly on teachers' professional competence. In other words, teachers need to have deep curriculum and assessment knowledge to develop adequate assessment practices. In this section, we will discuss research about types of curriculum and assessment knowledge teachers need, as well as the ways to develop teachers' curriculum and assessment knowledge.

Teachers' Knowledge of Curriculum

Curriculum matters for teachers' teaching and students' learning (Cai, 2014). Teachers' knowledge of the intended curriculum facilitates their enactment of the curriculum in classrooms. Gilbert and Gilbert (2013) proposed the development of Educative Curriculum Materials (ECMs) to deepen teachers' knowledge of mathematics. According to Gilbert and Gilbert (2013), an ECM is not intended to script instruction, but rather is designed to help teachers learn and understand the intended curriculum. ECMs increase teachers' knowledge so that they can help students develop mathematical understanding through the construction of increasingly detailed relationships between concepts.

Teachers' curriculum knowledge can grow, and in fact, participation in curriculum-analysis workshops leads to changes in teachers' curriculum knowledge (Ariav, 1991). Ariav (1991) conducted a study based on Silberstein's three levels of teachers' curriculum knowledge (autonomous consumer, consumer-developer, and autonomous developer; cited in Ariav, 1991) that examined the effects of such curriculum-analysis workshops. The teacher is an autonomous developer who can plan, design, and develop an entire course of study, often in areas with no (or few) existing curriculum materials. Ariav (1991) discovered that at the beginning of the study, most of the participating teachers thought they were already at the level of autonomous consumers of curriculum materials, but in reality, they had not yet reached that highest level (autonomous). Participation of the curriculum-analysis workshops not only helped teachers to become aware of their level of curriculum knowledge, but also led to a deep understanding of the curriculum through curriculum analysis.

Teachers' Knowledge of Assessment

The professional knowledge necessary to develop assessment practices with success is currently a key, but problematic, issue. Several studies have indicated that teachers show a lack of declarative and procedural knowledge of assessment (e.g., Black & Wiliam, 1998; Clark, 2012). These findings are consistent with those presented by Clyatt (2014) that involved a group of 15 experienced U.S. mathematics teachers. Clyatt's study showed that a large majority of the teachers were not familiar with assessment terms. Moreover, the teachers did not establish any relation between teacher training education, understanding terms, and available assessment options.

In the last decade, some studies in the PME proceedings have had as their main objective the understanding of which kinds of knowledge teachers apply or need to apply when they are developing assessment activities. These studies have involved experienced teachers or preservice teachers from different school grades. For example, in Germany, Leuders and Leuders (2014) aimed to understand the diagnostic competencies of preservice teachers, viewed as the ability of teachers to accurately assess students' performance. From the short written statements made by the group of participants about primary students' solutions to open-ended tasks, the results pointed out that the quality of the diagnostic judgments seemed to be related to noticing specific features of students' solutions.

In Israel, Liora and Miriam (2012) examined 42 preservice and 25 novice elementary mathematics teachers' pedagogical knowledge in the area of assessment of student achievement. Two questionnaires were given to each group of participants: One related to declarative knowledge and another to actual knowledge. For the novice teachers, a declarative behavior questionnaire was also given to understand how they make use of the same terms used in declarative knowledge in the classroom. For most of the terms, a significant difference was found between the participants' declarative knowledge and its application. One year later, a similar study was conducted, also in Israel (Hoch & Amit, 2013). Similar questionnaires were used. No difference between the two groups of teachers was found regarding knowledge on both knowledge questionnaires. However, the results did show, again, a lack of knowledge, including basic concepts of assessment. Even when teachers know a term, they tend to not use it.

Taking PISA 2003 competencies as a benchmark for their study, Rubio, Font and Giménez (2010), from Spain, aimed to determine the initial competency level of preservice secondary teachers with respect to assessing the PISA mathematical competencies. Presenting one case study from a broader study, the authors concluded that this particular future teacher showed difficulties in determining the level of complexity of the competencies needed to solve a given problem, as well as difficulties assessing the competencies that can be inferred from the solution given to a problem. With the same framework, Lee and Na (2007) carried out a study in South Korea to understand the practical knowledge helpful in assessing students' mathematical power. Using the action research approach, the study was carried out by a researcher who was an elementary mathematics teacher, in three stages: Development of an assessment plan, with 15 open-constructed response problems and scoring guides; the application of these problems to students; and the analysis and reflection of the initial materials and the students' responses. After analysing the material and the students' responses, the teacher reported the improvement and development of practical knowledge for assessments.

When we consider teachers' professional knowledge, this can include teachers' attitudes. Krzywacki, Koistinen and Lavonen (2011), through their analysis of interviews with eight mathematics teachers in Finland who worked with a technological assessment tool, pointed out the importance of teachers' willingness to change their assessment practices.

In summary, one particular aspect that is important to notice from the few articles revisited from the PME proceedings is that most of them used concrete materials from students to study the teachers' assessment knowledge. This methodological option supports the importance of connecting teachers' knowledge with practices (Ponte & Chapman, 2006). Although these studies confirm what the large body of literature in the field already states about the need to develop teachers' assessment knowledge, they also call attention to the importance of attending to teachers' attitudes in any process of assessment practice innovation that is aligned with what mathematics learning actually means.

Teachers' Knowledge Development

There exist several strategies to develop teachers' professional knowledge in general and teachers' curriculum and assessment knowledge in particular. These strategies may be categorized into two types: Formal/institutional and informal. One of the formal ways to improve teachers' professional knowledge is through an institutional course or training. This strategy is not an end in itself, but rather is a way to achieve learning. Thus, the efficacy and sustainability of such approaches are essential considerations in studies related to professional courses. This is the case in the study of Koh and Chapman (2014) that investigated an intervention to improve the quality of Singaporean teachers' assessment literacy in mathematics teaching and learning. Guidelines for teachers to design authentic assessment tasks as well as criteria to characterize authentic intellectual quality were used. Eighteen grade 5 teachers from four schools participated (two schools in the experimental group, two schools in the control group). The data consisted of 116 assessment tasks designed by the teachers and 712 related pieces of students' work collected before and after the intervention. The experimental teachers increased their competence in designing assessment tasks that were of high authentic intellectual quality, focusing more than the control group teachers on students' mathematical understanding, thinking, problem solving, and connections. Olson, Slovin, Olson, Brandon and Yin (2010) compared a professional development model with formative assessment but without networked technology with formative assessment along with technology.

During a period of one year, the model without technology brought greater gains in assessment knowledge, but did not provide evidence for more positive attitudes toward using assessment.

In Germany, Besser and Leiss (2014) studied the possibility of developing tests and fostering teachers' pedagogical content knowledge concerning formative assessment in the context of dealing with modeling tasks in competency-oriented mathematics. Over a period of ten weeks, 27 mathematics teachers participated in teacher training. Teachers in experimental group A were trained in central ideas of formative assessment when dealing with modeling tasks in competency-oriented mathematics teaching. Teachers in experimental group B were trained in selected aspects of competency-oriented mathematics in general. A pre-test and a post-test on teachers' mathematical pedagogical content knowledge were applied to compare the two groups. The results showed that teachers' knowledge of students' learning was significantly higher if they were specifically trained within the topics being tested. In other words, special formative assessment knowledge when dealing with modeling tasks is more suitable for developing pedagogical content knowledge than general knowledge about competency-oriented mathematics.

Although the results from these studies of professional development have generally been positive, the PME proceedings also reflect other ways to develop teachers' assessment knowledge. There is, for instance, the case of reflection. Reflection is one strategy to develop knowledge that has been highlighted in the literature (e.g., Schön, 1991; Wood, 2001). There is also evidence that collaborative work that supports teachers' practice is another way to improve teachers' knowledge of assessment. Santos and Pinto (2010) accompanied one Portuguese mathematics teacher during three school years, focusing on her feedback practices. Although the teacher used feedback, the evolution of her feedback practices included the establishment of favorable moments for students' reflection, from noting fewer errors, to encouraging correction and varying the feedback's syntactic form.

Another possible strategy to develop teachers' assessment knowledge is to use particular instruments. For example, Clarke, Sukenik, Roche and Mitchell (2006) used task-based interviews in Australia as instruments to enhance teacher knowledge. Ten experienced primary teachers, after participating in a day's training on the use of the interview tasks, interviewed 323 grade 6 students (broadly representative of Victorian students) at the end of the school year. Previously, students had answered tasks focusing on rational numbers, prepared by the research team and piloted and refined in their own schools. In the results, the authors concluded that "the use of the interview provides teachers with considerable insights into student understanding, common misconceptions, and forms a basis for discussing the 'big ideas' of mathematics and curriculum implications of what they have observed" (p. 343). This instrument continued to be developed and studied, with a focus on its capacity to foster the improvement of teachers' knowledge, and in particular, knowledge about students (e.g., students' understanding, thinking, and reasoning), content knowledge, and pedagogical content knowledge (Clarke, 2013).

Tier-testing is another instrument particularly useful for diagnostic purposes (Haja & Clarke, 2009). Considering one Australian mathematics teacher and 12 of her seventh grade students, a qualitative and quantitative inquiry was used to answer the following research question: How does the teacher react to the assessment information, concerning proportional reasoning, from two-tier tasks? Although the teacher used tier-tasks frequently and was able to notice students' misconceptions, she seemed to have great difficulty adjusting her instruction according to the information she received. The peer-assessment environment may also be a promising setting for the teacher to get information about students' learning and difficulties. Biton (2013) used six 90-minute peer-assessment activities in two classes, with video and audio recorded, allowing the teacher to develop her knowledge about the students and improve her teaching in a grounded way.

In conclusion, research on teachers' curriculum and assessment knowledge is under development. Teachers' professional knowledge has been widely studied in the last few decades (e.g., Ball, Thames, & Phelps, 2008; Shulman, 1986). The frameworks developed by Ball et al. (2008) and Shulman (1986) have been adapted to study teachers' curriculum and assessment knowledge. Perhaps there is a need to develop specific frameworks to study teachers' curriculum and assessment knowledge. For example, what are the main components of knowledge in these two particular types of teachers' knowledge? What distinguishes curriculum and assessment knowledge from other types of professional knowledge? How can we measure teachers' curriculum and assessment knowledge?

In an era where mathematics curricula are changing in many countries and introducing new curricular orientations, it is imperative to improve teachers' curriculum knowledge and the assessment knowledge they will need to guarantee adequate assessment practices. In this section, we have reviewed several ways of improving teachers' curriculum and assessment knowledge. Although this knowledge is far from what is desired, it is possible to create favorable settings to change this situation. That promises to be another critical direction for future research.

CONCLUSIONS

In this chapter we have proposed a way to examine the relatedness of curriculum and assessment. Taking learning as the main purpose of education, and considering curriculum and assessment as processes, the establishment of this relation between curriculum and assessment allows us to look at teaching and learning in a more integrated way. We were conscious of possible difficulties that this conceptualization could bring to us, especially as the PME authors most likely did not consider this perspective in their research. We believe that our conceptualization of curriculum and assessment makes sense, is operational, and may be useful for future research in the area.

A great diversity of countries across different continents is represented in the research on curriculum and assessment that we have reviewed. Clearly, curriculum and assessment are on the agenda of diverse mathematics education research communities, all of which recognize their importance.

The methodological options presented in the great majority of the research considered in this chapter were in the category of empirical studies, whether they followed a quantitative or qualitative design. This fact is consistent with calls to develop a theory of assessment, particularly for formative assessment (Black & Wiliam, 2006; Santos, 2015). It should be indicated that there are only a few large-scale studies related to curriculum and assessment in the PME proceedings in the past decade.

The findings of the studies we reviewed were diverse and most of them confirmed the findings from other studies, a fact that makes for more robust research results. It is particularly interesting to note that some research studies conducted in different countries were very similar. The opportunity to have such a chapter in this volume allows us to highlight such similarities.

As we have indicated at the beginning of the chapter, the number of PME studies that have focused on curriculum or assessment or both is not high, when we consider the total number of such research studies in 10 years of PME proceedings. We hope that the interest in curriculum and assessment continues and develops in the PME research community. Curriculum and assessment constitute two main dimensions of teaching and learning mathematics. Without them we will not be able to understand and to contribute to mathematics learning, the fundamental aim of the work of any mathematics education researcher.

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6. DEVELOPMENTAL, SOCIOCULTURAL, SEMIOTIC, AND AFFECT APPROACHES TO THE STUDY OF CONCEPTS AND CONCEPTUAL DEVELOPMENT

THE STUDY OF CONCEPTUAL DEVELOPMENT: AN OVERVIEW

We live in a world rich and full of unique events and objects. The book you hold in your hands and read at this moment resembles another book placed on a bookshelf of your library. They both have different covers or different size or address different things, but both are "books". We could say that both are different categories of "books", such as "science fiction", "fantasy" or "many readable written sheets" etc. What is the difference between a magazine and a journal? Under what principles do we decide to classify a series of objects under the concept of "book"? What are the implications of ignoring the uniqueness of each book when we engage with this classification? Most of us would probably agree that without such classifications – or, concepts – we could hardly recall and process information, and communicate with others. It is not surprising then that how concepts arise and what role their emergence plays in the teaching and learning (of mathematics) is a major focus of (mathematics) education research. In order to contextualize and embed mathematics education work within the broader field of investigations into concepts and conceptual development, we review briefly some work from this broader field. We note that the approaches taken here are largely from the field of psychology. However, as our account progresses, it is gradually enriched by weaving in influences from other fields – and crucially the proportion of purely psychological works reviewed in the rest of the chapter reveals that contemporary developments within mathematics education feed from influences from a range of other disciplines, most notably sociology, anthropology and linguistics.

Conceptual development has been described either in terms of analyzing human ability to construct knowledge through representations of concepts, or by examining specific concepts that are crucial to human survival or critical to the very nature of knowledge constructed and used by humankind (Markman, 1999). Four approaches to describing and interpreting conceptual representations seem to dominate the psychological literature: representations in terms of necessary and sufficient features; probabilistic representations; representations through example; and, representations through causal relationships.

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 187–233.

The approach of conceptual representations in terms of necessary and sufficient features (Markman & Dietrich, 2000) is a "classical" approach: it suggests that two objects or events are put into the same category because, first, they are identical to each other in terms of certain features and, second, they are different from those that belong to other categories. All instances of a concept share common properties. Sharing these properties is a necessary and sufficient condition for an item to qualify as member of a category. It is worthwhile to note that although the classic approach to concepts has not proved fruitful for understanding everyday concepts – *inter alia* by seminal works in cognitive linguistics such as (Lakoff, 1987) that brought the study of categorizing as a fundamentally human capacity center-stage – it has much in common with the accepted approach to definition in logic and mathematics. In other words, even if humans do not use the 'categorical' approach in developing everyday concepts, they are capable of using it in a certain rule-governed domains, i.e., mathematics, science and law.

Another construal of conceptual representations views concepts as consisting not of necessary and sufficient features but of features that belong to the concept with a certain degree of probability. The approach of probabilistic representation refers to features that are associated with a concept, none of which, however, is a sufficient and necessary feature of the concept. Instead, each feature has some degree of correlation with the concept in the sense that an object that belongs to this category has a certain probability of possessing this feature.

With regard to the approach of probabilistic representations Rosch et al. (1976) proposed that humans mostly and with priority use concepts that belong to the so-called "basic" level. This is the core of the classification system and the level that maximizes similarities to and minimizes differences from members of other concepts. Consider, for example, the concept "chair": chairs can be made of metal, wood or plastic; they can be high-back or low-back; they can be with arms or without arms; etc. Yet are all recognized as chairs. In the terms of the probabilistic representations approach, deciding on whether an object belongs or not in a certain concept is based on a judgement that compares similarities and differences between families of objects.

The construal of conceptual representations based on specific examples of members of the category designated by the concept refers to the view that concepts are represented in terms of the specific and important examples of concept members, the exemplars. Central to this model is the notion that concepts are organized around family resemblances rather than features that are individually necessary and jointly sufficient for categorization. The exemplars for a category consist of the most common attribute values associated with the members of the category. A conceptual representation for a category like 'furniture' includes all the instances that belong to the category 'furniture' (Estes, 1994).

Finally, the approach of representation through causal relationships focuses on different aspects of concepts and refers to beliefs that shape a "simplistic" theory for the understanding and use of each concept. Kail's (1989) studies provide evidence that concepts develop as theories. Theories are bodies of knowledge about a particular domain or field of information. Such theories have been suggested to explain several psychological fields, for example, mental state attribution information from a mentalizing theory, visual perception information from a theory of how seen objects behave in relation to the observer etc. The paper of diSessa and Cobb (diSessa & Cobb, 2004) about the "ontological innovation and the role of theory in design experiments" distinguishes clearly different types and levels of theory.

The above mentioned four approaches have influenced several prominent psychologists, including Piaget (Inhelder & Piaget, 1964), Vygotsky (Vygotsky, 1962), and Bruner (Bruner, 1970), who have written extensively about conceptual development, often with a focus on learners. Research within mathematics education has been under the strong, even definitive influence of these works – PME itself was initially conceived as emerging out of this purely psychological tradition (see also Hershkovitz & Breen, 2006; Gutiérrez & Boero, 2006). Foreword and Introduction respectively to the first *Handbook*, in which the cognitive roots of PME research is traced also with reference to the Introduction by Fischbein (1990) to the earlier *Mathematics and Cognition* volume.

Following this description, we now turn to reviewing how PME studies of the last 11 years study conceptual development within mathematics. Given that this *Handbook* maintains the first one's focus on specific content and cross-content areas, our chapter focuses on the many and varied approaches that researchers in mathematics education, particularly those presenting at the 2005–2015 PME conferences, have been taking in their studies.

The studies reviewed here take a plethora of approaches and our review aims to do justice to this plethora. Our intention is to trace different trends in the scholarly work over the past 11 years and also ask: has scholarly interest changed over the past years, and, if so, how? Are, for example, some approaches being employed more than others? If so, how?

To do so, we review the studies presented at the PME conferences between 2005 and 2015 with a particular focus on the approach that these studies take in terms of conceptual development. The approaches we consider are developmental, sociocultural-discursive and semiotic (the next three sections), followed by a section on studies that examine conceptual development in relation to affect. We conclude with some observations on the patterns and shifts we have noticed since the publication of the first *Handbook* as well as highlights of what we see as potent ways forward. Our attempt is to include as many studies that used these approaches to study conceptual development as possible. However, because of the space limitations, in each topic that is discussed within each section we describe in greater detail those studies that we judged as more "representative" and we reference other studies (which are by no means lacking in quality or interest in comparison) more succinctly. Further, throughout, we note that the terms *concept* and *development* are treated differently by the different approaches reviewed here. We have therefore dedicated some space within each section to a potted history of each approach, in order to provide the necessary clarifications regarding use of terms – and we have done so in full awareness that such potted accounts can never be comprehensive or completely escape the risk of omission or oversimplification.

DEVELOPMENTAL APPROACHES TO CONCEPTUAL DEVELOPMENT

The concept of cognitive development has attracted considerable interest throughout the history of mathematics education research. This section refers to psychological research on conceptual development, which is based on the work of researchers in psychology who studied extensively in the field of mathematics education, as well as on the work of researchers in mathematics education who focused on conceptual development. Pegg and Tall (2002) proposed two kinds of theories of cognitive growth: global developmental theories and local theories of conceptual development. Global developmental theories suggest that cognitive development occurs in a series of global stages, each characterized by increasingly sophisticated and abstract levels of thought. Local theories of conceptual growth revolve around the transformation of processes into objects.

Regarding global developmental theories, the most well-known and influential theory that falls under this category is that of Piaget. Piaget's stage theory of cognitive development describes four distinct stages: sensori-motor, preoperational, concrete operational and formal operational (Piaget & Garcia, 1983). In addition, Bruner's (1966) constructivist theory describes how a learner's encounter with new knowledge follows a progression from enactive to iconic and then to symbolic representations.

In Piaget's (1972) seminal work three types of abstraction were defined: empirical, pseudo-empirical and reflective. Empirical abstraction gathers knowledge from the properties of objects, by acting on objects in the external world (Beth & Piaget, 1966). Pseudo-empirical abstraction mediates between empirical and reflective abstraction and "teases out properties that the actions of the subjects have introduced into objects" (Piaget, 1985, pp. 18–19). General coordination of actions leads to reflective abstraction, which is an absolutely internal process (Piaget, 1980). Further constructions can be built using existing structures and abstracting from them. In other words, an operation on such "entities" becomes in its turn an object of the theory. Thus, mathematical entities move from one level to another (Piaget, 1972).

Regarding local theories of conceptual growth, several have been proposed by influential theorists (Davis, 1984; Dubinsky, 1991; Gray, Pitta, Pinto, & Tall, 1999; Gray & Tall, 1994; Piaget, 1972; Sfard, 1991). Gray and Tall (1994), Sfard (1991) and Dubinsky (1991) all give emphasis on the way a process is condensed into an object. In addition, the process-object theories of Dubinsky (1991) and Sfard (1991) focused on formal concept development and more advanced mathematical thinking. Gray and Tall's procepts (1994) highlight the importance of mathematical symbolism to a larger extent than the other theories. Furthermore, while Sfard, at least in her writing at the time (1991), underscores that a process comes first and is followed by the object, Gray and Tall (1994) do not explicitly make reference to the ordering of process-and object-based thinking (Gilmore & Inglis, 2008).

The qualitative changes that occur when actions become objects of thought have been the focus of Davis (1984). Davis (1984) identified two kinds of procedures: a visually moderated sequence (VMS) and an integrated sequence. In a VMS, the learner carries out the procedure step by step, without being able to perceive the whole algorithm or patterns of the activity. Subsequently, as the procedure is practised, it becomes an object of scrutiny and analysis (pp. 29–30). In what is called an "integrated sequence", the learner conceives of the whole algorithm as a construction consisting of smaller component sequences.

Sfard (1991) claimed that concepts are developed through three successive stages: interiorization, condensation and reification. During the interiorization stage, a process is executed mentally on lower-level mathematical objects. The learner becomes progressively capable of performing these processes. At the second stage of concept development, which is called condensation, the learner becomes even more capable of consciously thinking of a particular process as a whole. '"Squeezing" lengthy sequences of operations into more manageable units' is a key feature of this stage (Sfard, 1991, p. 19). Reification is regarded as an ontological shift, since the learner suddenly becomes able to view a familiar concept through a different lens, as a 'fully-fledged object' (Sfard, 1991, p. 19). The construct is no longer dependent upon a process, as the student can recognize the conceptual category the construct falls under and attribute meaning to it. Therefore, although the first two stages involve quantitative changes, reification is characterized by a merely qualitative change, because 'a process solidifies into object, into a static structure' (Sfard, 1991, p. 20). More recently, Pitta-Pantazi, Christou and Zachariades (2013) identified these three stages (interiorization, condensation and reification) in 11th grade students' responses regarding the monotonicity of exponential functions (see also, Paschos & Farmaki, 2006).

Several research studies having in purpose to examine Piaget's theory of concept learning and development were carried out in the past decade. Recently, Paschos and Farmaki (2006) conducted a case study on a first year university student of mathematics focusing on the Piagetian theory of reflective abstraction for investigating the way in which the student acts in order to calculate the distance covered in a time. The findings indicate that the stages of knowledge, the concepts' images, and the mental mechanism and operations of the students are gradually revealed. Also, understanding this mechanism will allow teachers to decide and distinguish whether the students understand correctly the definition of the definite integral concept, and not just have an empirical perception of integration, by which they can act effectively only in a limited and particular framework (see also Simon, 2014).

Dubinsky (1991) formulated a similar theory on concept development, known as APOS theory, according to which a mathematical concept develops as one tries to transform existing physical or mental objects. The *acronym APOS* stands for Action-Process-Object-Schema. Weller, Arnon and Dubinsky (2009) argue that as an individual repeats an action (A), the action may be interiorized into a mental process (P). If one apprehends a process as an entirety, then the individual has encapsulated the process into an object (O). A coordination of these actions, processes and objects, is called a schema (S). A schema could help students '... understand, deal with, organise, or make sense out of a perceived problem situation' (Dubinsky, 1991, p. 102).

For example, Zuffi (2010) used APOS theory to analyse data from a qualitative study conducted in Brazil, with in-service mathematics teachers, in order to explore teachers' knowledge about functions. The aim of the study was to propose a spherical model for human beings' complex learning processes. Such a model considers elements from inner-individual aspects (like in Piagetian theory), but also how these aspects are affected by social-historical-cultural (as considered by Vygotsky), affective, or even extra sensorial – unconscious aspects (by psychoanalysis).

Gray and Tall (1994) relied on Sfard's (1991) and Dubinsky's (1991) theories, but extended them by giving emphasis to the role of mathematical symbols. They introduced the term of 'procept' to underline that a symbol flexibly and ambiguously represents both process and concept and acts as a pivot, switching from a process to a concept. Hence, procept is defined as 'the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object' (p. 121). They suggested that a major challenge in mathematics learning pertains to bridging the proceptual divide. Zoitsakos, Zachariades and Sakonidis (2013) examined secondary mathematics teachers' knowledge based on the notion of 'procept'. This study showed that, despite the subjects' strong mathematical background, notably few individuals appeared to hold an accurate understanding of a particular infinite decimal representation of a rational number, whereas a significant number see it only as a process.

According to Gray and Tall (1994), proceptual thinking refers to the ability to compress stages into procepts. Procepts are constructed usually through a procedure-process-procept sequence. Initially, the individual performs a procedure as a series of steps. Then, the procedure becomes more routine and carried out without paying attention to details, resulting in process. At this stage, a number of procedures yielding the same input-output are considered as the same process. Finally, the process is compressed into a procept, while the symbol is used to evoke either a process or a concept. They assumed that learners that think in a flexible manner are able to perceive symbols as a process and a concept. In contrast, less flexible thinkers are not aware of the dual role of symbols and rely on less flexible procedural methods (Gray & Tall, 1994). This theory was also enriched with research studies which tried to explore the relations and connections between students' concept development and their mental representations (Gray & Pitta, 1996; Gray, Pitta, & Tall, 1997; Pitta & Gray, 1996; Pitta & Gray, 1997; Pitta & Gray, 1999; Pitta-Pantazi, Gray, & Christou, 2002; Pitta-Pantazi, Gray, & Christou, 2004).

Apart from the procedure-process-procept sequence, Tall (2004, 2013) built a theory describing three distinct worlds of mathematics: the embodied, the symbolic-proceptual, and the formal. The embodied world begins with our perceptions and actions on the real world. The proceptual world is related to calculations in arithmetic and symbolic manipulation in algebra and calculus. The formal world is based on axioms, which are formulated to define specific mathematical structures. Tall, Gray, Bin Ali, Crowley, DeMarois, McGowen, Pinto, Pitta, Thomas and Yusof (2001) claimed that the three worlds focus on different qualities: The embodied world focuses on the objects and their properties, the proceptual world on processes represented by symbols, and the formal on properties and their relationships. Some studies investigated Tall's three worlds theory in certain mathematical concepts. For example, Pitta-Pantazi, Christou, and Zachariades (2013) explored this theory in the context of functions confirming Tall's theory. Likewise, Furinghetti, Morselli and Paola (2005) examined 15 year old students who explored a phenomenon of covariance, by using Cabri to draw geometric figures, measure, and sketch graphs. The findings of their study highlighted the path from the embodied stage to formal mathematics and that even the symbolic-proceptual world is hindered to students: the dynamic figure and the diagram may act as a burden in the effort to go on. A similar finding emerged in the study by Swidan and Yerushalmy (2009), on two 17 year old students.

Some research studies combined research frames mentioned above, such as the study of Stewart and Thomas (2007) which investigated the learning of the linear algebra concepts of linear combination, span, and subspace in a group of second year university students focusing on the implementation of APOS theory of Dubinsky (1991), particularly in the context of Tall's (2003) three worlds of mathematics. The results of their study confirmed that the students struggled to understand the concepts through mainly process conceptions, but embodied, visual ideas proved valuable for them. A year later, the same research team repeated their research, by teaching students a procedural way to find a basis for a subspace using matrix manipulation. They found that emphasis on matrix processes may not help students understand the concept, and embodied, visual ideas that could be valuable were usually lacking (Stewart & Thomas, 2008).

In addition, Hannah, Stewart and Thomas (2014) investigated linear algebra with second year undergraduates, who were taught fundamental linear algebra concepts with the use of the embodied, symbolic and formal dimensions. The results showed that student affect – the relationship of which with conceptual development we return to in the penultimate section of this chapter – was much more positive when concepts were first met in the embodied or symbolic worlds but there was little effect on the overall understanding. In general, by the end of the course student perspectives on formal aspects of mathematics, definitions, theorems and proofs, were much more positive than at the beginning. Verhoef and Tall (2011) focused on three upper level high school teachers development based on Tall's (2003) embodied and symbolic worlds. Their study revealed the significance of the complex reality of school practice in reference to the powerful claim of curriculum guidelines, study guides based on textbooks, and the attaining of high exam results.

What our review of studies with a developmental/cognitive take on conceptual development suggests is that these approaches – largely established in the 1980s and 1990s – continued to be deployed well into the 2000s, often with only minor amendments. A distinctive development – certainly inaugurated in the 1990s but shaping the field in a much more visible manner in more recent years – was the surge of approaches to conceptual development that address directly and systematically the aforementioned complex and multifaceted reality – the context – in which mathematical learning takes place. While some of the approaches mentioned in this section were to some extent revisited in the light of such considerations (e.g. Bingolbali & Monaghan, 2008), more visible were developments regarding studies with a sociocultural and/or discursive take. We showcase some of these developments in the section that follows with an overview of said studies.

SOCIOCULTURAL AND DISCURSIVE APPROACHES TO CONCEPTUAL DEVELOPMENT

In the last ten years or so sociocultural and discursive approaches to research in mathematics education have grown substantially. In this section we present a synthesis based on the 2005–2015 PME papers that deploy a sociocultural and/or discursive approach to conceptual development. Before we do so we remind the reader that what is meant by 'conceptual development' is meaning-making as defined and used in the participationist perspective of sociocultural researchers (according to which learning is an activity of participation determined by the situational, cultural and historical milieu in which it takes place)

A Sociocultural Take on Conceptual Development

In recent years, the field has opened up substantially to approaches that encompass strong consideration of the context in which learning occurs. We relay here from the observation Lerman (2006) offers in the first *Handbook* regarding this strong consideration (pp. 362–363): *identity* and a close, systematic study of *teaching practice* (both in terms of *teacher learning* and the influence on learners of various *teaching modalities*) are emerging as major foci for future work. The works we review in what follows – such as (Frade, Lerman, & Meira, 2014) and (NicMhuirí, 2014) – illustrate the accuracy of Lerman's observation, as well as Simon's (2015) call for work that elaborates further what Vygotsky (1978) labelled "internalization" of knowledge by individual learners.

Bradford and Brown (2005) deploy a poignant metaphor – paraphrasing the famous painting by Magritte, which depicts a smoker's pipe with the caption "Ceci n'est pas une pipe" – in order to remind us that what we see in a circle drawn on a piece of paper is not a circle but a drawing of a circle. They then discuss the syntactic filters and sociocultural factors that condition our perception of mathematical concepts.

Power within the group dynamics of a learning environment is amongst the most significant sociocultural factors that shape our engagement with mathematics (Barnes 2005). Barnes analysed the enactment of power during group discussions in high school mathematics. The class presented in the study was working on introductory calculus using a collaborative learning approach. In analysing a group discussion, Barnes first traces the flow of ideas, looking at when and by whom a new idea was introduced, and how others responded. She then divides the transcript of the conversation into what she terms "negotiative events" and examines how transitions from one event to the next come about. Her examination is driven by sociocultural constructs by Davydov (1995) and Lerman (2001) – and she concludes that the negotiations of mathematical meaning that she observed were driven by learners in a position of power not always determined by these learners' hitherto perceived mathematical capabilities.

Gorgorió and Planas (2005) start from the constructs 'cultural scripts' and 'social representations', and, on the basis of their empirical research, they revisit the construct of 'norms' (Cobb & Yackel, 1996) from a sociocultural perspective and propose that norms, both sociomathematical norms and norms of the mathematical practice, as cultural scripts influenced by social representations, mediate the learning of mathematics in multicultural classrooms. When taking into account the particular circumstances in which mediation occurs, they then claim, there is a need for a move from a cultural perspective to a broader sociocultural one.

Graves and Suurtamm (2006) bring together research into the fundamental question "What is mind?" that examines the relation between culture and cognition. This research, their account says, talks about the social mind, the discursive mind, mind as action, and the collective mind. Their research draws on complexity theory (Davis, 2004: Davis & Simmt, 2003; Maturana & Varela, 1987) and sociocultural theory (Bakhtin, 1986; Foreman, 2003; Vygotsky, 1986) to investigate how learning collectives evolve and are understood from the perspectives of multiple participants. They focus particularly on examples from their work with beginning teachers.

Hunter (2007) focuses on the challenges faced by teachers who aspire to develop discourse communities in which the students learn to construct and evaluate mathematical meaning – in her case, the focus is on the construction of mathematical arguments – collectively. Her paper examines the interactional strategies used by a teacher to constitute a classroom context in which the students participated in the discourse of collective argumentation. She reports the way the teacher used student explanations as the foundations for building justification and validation of reasoning. In a subsequent report from this study Hunter (2008) provides descriptions of the interactional strategies four teachers used which gradually scaffolded student use of more complex questions and prompts. Hunter reports the way the students appropriated the teachers' models of questions and prompts – and used them to engage in exploratory talk (Mercer, 2000) and to develop rich explanatory justification and generalisations. The theoretical framework of this study is derived from a sociocultural perspective. An analogous focus can be found in Tsai's (2007) study of classroom discourse, again from the perspective brought together by Paul Cobb and Erna Yackel (e.g. 1996).

Zahner, Moschkovich and Ball (2008) build on previous work on student interpretations of graphs and use a sociocultural perspective on mathematical reasoning to describe how four pairs of eighth-grade students interpreted horizontal segments on a distance versus time graph using a story about a bicycle trip.

Following the strand of work within mathematics education that gave us studies of ethnomathematics, everyday mathematics, situated cognition and workplace mathematics, Naresh and Presmeg (2008) focused on the nature of workplace mathematics, through the case study of a bus conductor in India. The general aim of their study is to develop a better understanding of the mathematics used in said workplace, both from the perspective of the workplace practitioner himself as well as that of the researchers. Likewise, Amit and Gurion (2015), showed that ethnomathematics can change students' attitudes to culture of their and the tribe's older generation (see also the Chinese research of mathematical effectiveness in Zhang & Seah, 2015).

Radford (2008) deals with students' transformation of meanings related to their understanding of Cartesian graphs in the context of a problem of relative motion. The investigation of the students' transformation of meanings is carried out in the course of a process that he terms objectification, i.e., a social process related to the manner in which students become progressively conversant, through personal deeds and interpretations, with the cultural logic of mathematical entities. Radford provides a multisemiotic analysis of the work done by one Grade 10 group of students and their teacher, and tracks the evolution of meanings through an intense activity mediated by multiple voices, gestures and mathematical signs. His study draws on a Vygotskian sociocultural perspective in which mathematical thinking is considered a cultural and historically constituted form of reflection and action, embedded in social praxis and mediated by language, interaction, signs and artefacts (Radford, 2006). Of particular interest here is Radford's treatment of meaning making as heteroglossic transformation. We return to Radford's seminal work also in the following section of this chapter (on semiotic and embodied approaches) as well as other works (e.g. Berger, 2015) that deploy a fusion of sociocultural and semiotic approaches to substantial effect.

Rivera and Becker (2008) draw on data obtained over three years to address how shared content and understanding of generalization emerged and evolved in a middle school class using a socially shared symbolic system in a community of minds framework.

Dooley (2009) also focuses on the emergence of generalization in the context of a primary classroom. Taking a sociocultural perspective, and treading beyond the common – and often superficial – pattern seeking type of activity that can be found in primary classrooms, Dooley describes a whole class discussion that shows that generalization and justification are closely aligned. She concludes from this that there is a need for teachers to press for justification and for students to attend to the functional relationship between variables rather than pattern finding in single variable data.

Goos (2012) sketches out two theoretical frameworks for understanding mathematical learning. One framework extends Valsiner's zone theory of child development (1987), and the other draws on Wenger's (1998) ideas about communities of practice. Her aim is to suggest how these two frameworks might help us understand the learning of others who have an interest in mathematics education, such as teacher educators and mathematicians.

Andersson and Seah (2012) present the mathematics learning story of a student named Sandra to demonstrate how a student's engagement changes with the learning contexts, via the identity narratives which are told with reference to different levels of contexts in and outside the mathematics classroom. Data were collected from a survey, interviews, spontaneous conversations, students' blogs and project logbooks. Changes in identity narratives and engagement appeared to be rooted in the relatively stable valuing of achievement, explanation, application and sharing. The extent to which Sandra's valuing was aligned with these facilitates our understanding of the complex interplay amongst context, valuing and agency. That is, sociocultural and personal valuing, and the extent to which these are aligned, promise to regulate and explain the role of learning contexts in student agency, including engagement and hence learning. In this vein Radford (2014) argues that, in joint labour, teaching and learning are *fused* into a single process: the process of teaching-learning—one for which Vygotsky used the Russian word *obuchenie*. Teachers and students therefore "are simultaneously teachers *and* students" (Freire, 2005, p. 76).

Clarke, Strømskag, Johnson, Bikner-Ahsbahs and Gardner (2014) refer to Rezat and Strässer (2012) who identify the students' mathematics-related activity as an example of the Vygotskian conception of an instrumental act, where the student's interaction with mathematics is mediated by artefacts, such as mathematical tasks. Most importantly, recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further recognition that (à la Vygotsky) the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity (i.e. student learning). Rezat and Strässer (2012) have re-conceptualized the familiar didactical triangle (teacher-student-mathematics) as a socio-didactical tetrahedron, where the vertices are teacher, student, mathematics and mediating artefacts. This reconception of didactical relationships recognizes that the connections represented by the sides of the original didactical triangle require mediation. The vehicles of this mediation are artefacts, which include everything from textbooks and IT tools to tasks and language. Use of the socio-didactical tetrahedron provides us with an important tool by which to give recognition to the mediational role of tasks in the teaching and learning of mathematics.

One virtue of the socio-didactical tetrahedron is that it facilitates the separate consideration of the triangles forming each face of the tetrahedron and the vertices of each of those triangles. Where Clarke et al. (2014) focus is on the task as mediating artefact and address the question of how the resultant socio-didactical tetrahedron might structure our consideration of research into the function of tasks in facilitating student learning and into the dynamic between student and task.

To paraphrase Rezat and Strässer (2012, p. 645): each of the triangular faces of the tetrahedron stands for a particular perspective on the role of tasks within mathematics education: the didactical role of the teacher is best described as an orchestrator of student mathematical activity as represented by the triangle teacher-task-student (Face A); the triangle student-task-mathematics represents the student's task-mediated activity of learning mathematics (Face B); the triangle teacher-task-mathematics depicts the teacher's task-mediated activity of representing mathematics in an instructional setting (Face C); the original didactical triangle constitutes the base of the model (i.e. student-teacher-mathematics) (Face D). The tetrahedral structure offers an important representation of the complexity of classroom teaching/learning that affords a level of detailed reflection on the didactical role of tasks. In utilizing this more complex conception of the instructional use of mathematical tasks, significant agency is accorded to each component (student, teacher, mathematics and task) in the determination of the actions and outcomes that find their nexus in the social situation for which the task provides the pretext.

Clarke et al.'s (p. 138) use of the socio-didactical tetrahedron puts forward a 'Vygotskian conception of the process of mathematical learning and the role of instructional tasks in facilitating this learning process'. They argue that recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further – and useful – recognition that the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity (i.e. student learning). We see this work as a demonstration of how researchers have been coming together to issues concerning conceptual development from several perspectives. Here these perspectives are: phenomenographic approaches capturing the reflexive connection between the teacher's use of tasks and the students' conceptions of those tasks and according significance to student intellectual agency; and, *didactique* (Brousseau, 1997) approaches highlighting the role of task in creating a milieu.

Douek (2014) takes a Vygotskian didactical perspective, conceiving teaching-learning as a dialectical construction of "scientific concepts" in relation to "everyday concepts". Among every day concepts here Douek includes spontaneous individual practices and scientific conceptualization is characterized by conscious management of concepts, their properties and related practices on a general level. This perspective implies the gradual construction of class references through cycles of individual production, negotiated and then synthetized with the teacher's guidance. For scientific conceptualization argumentation is a means and an aim, as it is involved in proving and conjecturing. Douek outlines direct implications of a Vygotskian perspective on classroom discourse particularly in relation to developing scientific conceptualization and argumentation. These are the five implications she lists:

> "(1) introduce students to cultural interest for finding reasons that have a theoretical relevance and/or can be shared as valid references; (2) make them aware that reasons can be various, not all based on the class references, and understand the relations between those and the specific references related to everyday concepts; (3) develop attention and concern for interaction and ability to adequately express one's views in a given socio-cultural context; (4) develop consciousness of one own's positioning and a critical attention to it; (5) establish mathematical references collectively—theoretical knowledge and practices—under teacher's guidance, built upon various sources, including students' contributions and cultural experience". (p. 210)

Adler and Ronda (2014) illustrate an analytic framework for teachers' mathematics discourse in instruction (MDI). MDI is built on three interacting components of a mathematics lesson: a sequence of examples and related tasks; accompanying talk; patterns of interaction. Together these illuminate what is made available to learn. MDI is grounded empirically in mathematics teaching practices in South Africa, and theoretically in socio-cultural theoretical resources. The framework is responsive to the goals of a particular research and professional development project with potential for wider use.

Alsalim (2014) refers to the social approach of research in mathematics education as promoting the notion that practice is not only a personal individual matter but situated in the social and cultural context. The Patterns of Participation (PoP) construct put forward here serves the purpose of exploring this situating, understanding the relationships between teachers' practice and social factors. In this sense PoP is a paradigmatic example of recent PME research which adopts participationism as a metaphor for learning as driven by the work of Vygotsky (1978), Lave and Wenger (1991), and Sfard (2008).

Anderson and Anderson (2014) note that research on mathematics found in 'everyday' interactions (e.g., Walkerdine, 1988) often relies on analysis of parent-child talk during studies of social interactions and/or literacy events more generally. In contrast, from the outset of the current study, parents were aware that mathematics was the focus of study and that each of them would determine the activities to be video-taped in their home. They report the types of activities six middle class mothers perceived as opportunities to engage their preschool child with mathematics. Analysis also included the patterns found within and across families. Overall, the mothers documented play-based events, many of which were common across four or more homes and entailed 'less conventional' mathematics. Parental styles of mathematical engagement are discussed. This study, as many of those we review in this section is informed by socio-historical theory (e.g. Vygotsky, 1978; Wertsch, 1998) and the notion that learning is social, as well as individual. Children learn to use the "cultural tools" such as mathematics of their community and culture inter-psychologically as they are guided and supported by parents and significant other people. As they practice using these "tools" and support is gradually withdrawn, children learn to use them intra-psychologically or independently.

Another typically Vygotskian construct, introduced by Barabash, Guberman and Mandler (2014), is *implementation ability* (IA), by which they mean the ability of a person to apply the recently acquired piece of mathematical knowledge, provided this piece is in his or her mathematical Zone of Proximal Development (ZPD) (Vygotsky, 1978). The implementation is expected to occur in the "neighborhood" of a learning issue in question – and, "the farther" within this "neighbourhood" a learner treads, "the better" (p. 95). In this vein Abtahi (2015) supports that the ZPD emerges as children participate in collective interactions with mathematical tools that involve the use of guidance provided by the physical properties of the tools in the process of solving problems.

The intensified focus on context – amply demonstrated in the samples of PME research presented in this section so far – has often implied a focus on communication and language. In other words, examination of learning in recent years has become increasingly discursive (Nardi, 2005).

Conceptual Development as a Discursive Shift

Much like the sociocultural approaches described in this section so far, discursive approaches to research in mathematics education are described as 'participationist' and are typically juxtaposed to those labelled 'acquisitionist' (Kieran, Forman, & Sfard, 2002) which, have often come into question on methodological and epistemological grounds (Sriraman & Nardi, 2013): clinical-experimental methods can be remote from where learning occurs; and, learning is not context invariant and universal. As Kieran et al. (2002) note, discursive perspectives aim to bridge individual and social dimensions of learning, espouse the sociocultural tenet that learning occurs in, and is co-constituted by, the situational, cultural and historical milieu – and emphasise human thinking as a type of communication. Within mathematics education discursive approaches have risen fast in recent years as Ryve's (2011) review of 108 papers amply demonstrates. One discursive approach that has come into full display and substantial deployment in recent years is Sfard's (2008) commognitive approach.

Roux and Adler (2012) indicate that first-year undergraduate student action is a complex interplay of the ways of talking about and looking at the mathematical objects, together with discursive, social and political ways of acting in the classroom. Additionally, Heyd-Metzuyanim's (2013) research included seventh grade students and the results of the study indicate that a conceptual division is made between mathematizing (talking about mathematical objects) and subjectifying (talking about the participants of the discourse). This division forms the basis of an operational set of discursive categorizations for "identifying" activity, enabling the extraction of identity narratives from spontaneous interactions in class (see also, Nachlieli, Heyd-Metzuyanim, & Tabach, 2013). Shinno (2013) suggests that the change of meta-discursive rule in a ninth grade classroom, which is an essential aspect of the reification of a new signifier, is inevitable for the transition from the template-driven use of signifier to the objectified use of symbol. Further Gholamazad (2007) examined pre-service elementary school teachers' methods of proving and demonstrated that proving through writing a dialogue can involve students in the process of creating a mathematical proof effectively (see also, Wille, 2011 on 11 years old students). This finding is in line with Knott, Olson and Currie's (2009) discursive approach to exploring the influence of an instructor's discursive practice on student learning (see also, Knott, 2010).

Bardelle (2013) – building on earlier work (2009) which investigated the role of graphs in the conceptualization of the derivative of real valued, differentiable functions and identified a lack of coordination of the semiotic systems involved in the representation of derivatives and in particular the occurrence of pragmatic aspects related to the use of graphs – in her study of Italian science undergraduates showed that "implicatures" occurring in everyday communication heavily affect the interpretation of a variety of sentences (see also, Sánchez & García, 2011; Morgan & Tang, 2012). In this vein, Ingram (2009) studied 13–15 year-old students and revealed that students engage in mathematics when their perceptions of what they want to achieve have not yet been realized and these perceptions are affected by students' views about mathematics, the context of the moment and the students' feelings about being able to do mathematics.

Several studies have deployed Sfard's theory in the context of conceptual development within geometry. Ng and Sinclair (2013) suggest that kindergarten student' use gestures as multi-modal resources to communicate temporal relationships about spatial transformations. In the same age range Sinclair and Kaur (2011) examine the effect of the use of dynamic geometry environments on children's thinking and offer evidence of how dynamic environments can facilitate the growth of more sophisticated mathematical discourse (see also, Kaur & Sinclair, 2014). Ng (2014) investigates secondary students' reliance on gestures and dragging as multimodal resources to communicate about dynamic aspects of calculus. We note here that these studies take a commognitive as well as semiotic perspective and we return to the latter in the following section of this chapter.

Hino and Koizumi (2014) explore how sixth-grade students' attention is brought to new mathematical content in whole-class interaction. They analyse the progression of social interaction in terms of how different foci were presented, problematized, or modified in the context of studying constancy of proportion. The results show that the children's vague attention to the constant number was questioned and made an object of examination. The children's attention was then carefully controlled by involving them in building new perspectives, which became the basis for meaning making about the constancy of proportion. Teaching actions that make this happen are also the focus here. In a tertiary level context, Jayakody (2014) studied first year university students' conflicts between different 'realizations' of the concept of 'continuous function'. Her study explains how these conflicting realizations have arisen from the inconsistent definitions presented in textbooks and other mathematical resources. This study also points to the need of elaborating further the notion of "commognitive conflict". An example of recent work that engages with this aim is the study by Kim and Kwon (2015) who conduct a commognitive analysis of students' progressive mathematisation (from situational to referential and then formal levels) when they resolve commognitive conflicts about conditional probability through storytelling.

In looking forward we see broadening the scope of the commognitive approach – as well as more broadly the variations of the study of discourses that have risen in recent PME studies – and linking it with other sociocultural approaches as a task to be taken up from now onwards. Commognitive analyses generate 'mathematically rich accounts of data' (Sriraman & Nardi, 2013, p. 326) – see the Research Cases presented in (Nardi, Ryve, Stadler, & Viirman, 2014) as examples. In our view these examples merely scratch the surface of the potentialities within the commognitive framework. The framework reveals the many and varied facets of participation in discourse 'without requiring us to reduce the complexity of the social and semantic interrelationships' (Stahl, 2009, p. 4) that govern this participation. We agree with Stahl that 'Sfard has done us the great service of bringing the "linguistic turn" of twentieth century philosophy (notably Wittgenstein) into twenty-first century learning science' (p. 5).

So far the bulk of commognitive analyses 'are confined to brief dyadic interchanges or even utterances by one student' (p. 7) – or lecturer, or a combination of both. The unit of commognitive analysis can certainly 'be scaled up' (p. 7) and can encompass the consideration of 'physical environment, history, culture, social institutions, power relationships, motivational influences and collective rememberings' (p. 7). What Jablonka and Bergsten (2010) describe as the "social brand" of approaches to research in mathematics education – such as Communities of Practice and Inquiry and Activity Theory – identify these as 'the web of agency' (p. 7) and the elaborate, mathematically intricate detail of the commognitive framework (Yackel, 2009) – its expansive and precise 'grammar' (Sriraman, 2009) – can assuredly strengthen our systematic inquiry into this web.

Coming back to the observation by Lerman (2006) on the rising importance of studies of identity, we see this discursive approach as having much to offer in this respect. As Sfard (2008) notes, there is a need for a 'commognitive operationalization of the notion of identity and the systematic study of the processes of subjectification' (p. 293). We agree with her and with the numerous reviewers of her work (Walker-Johnson, 2009; Lemke, 2009; Felton & Nathan, 2009; Cobb, 2009; Wing, 2011) who make a similar point. As Cobb (2009) notes, the commognitive framework has the capacity to attend to 'the macro-level of historically established mathematical discourse, the meso-level of local discourse practices jointly established by the teacher and students […] and the micro-level of individual students' developing mathematical discourses' (p. 207). For example, her construct of learning-teaching agreement can be seen as a variant of Brousseau's *didactic contract*, or 'systems of reciprocal obligations' (Sfard, 2008, p. 283). Sfard's model of teaching is also 'broadly compatible with Lave and Wenger's (1991) notion of legitimate peripheral participation' (Cobb, 2009, p. 209). We also agree with the now widespread appreciation of 'the essential pragmatism of Sfard's approach' (Wing, 2011, p. 364) and, are optimistic about the growth of its capacity in the light of the PME works reviewed here. The fact that it is becoming more broadly deployed in studies of mathematical reasoning in distinctly different and under-studied social and cultural contexts – Morris' (2014) South Pacific context of study is an example – adds evidence to this promise.

A very pronounced emphasis in the papers we review in this section is on communication and on our ways of mediating meaning. It is far from surprising then that a major focus of works within PME in the last 10 years has been on the semiotics of mathematical meaning making, including gestures. We now turn to a review of the approaches taken in these studies.

SEMIOTIC AND EMBODIED APPROACHES TO CONCEPTUAL DEVELOPMENT

In this section we focus on the contribution of semiotics to concept learning and development as it is reflected in the PME proceedings of this decade. Mathematics is a human activity that is carried out by means of signs since mathematical objects cannot be accessed and apprehended directly through the senses (Duval, 2007; Presmeg, 2006). This means that mathematics and its teaching and learning is an inherently semiotic activity. Mathematical signs mediate two processes: the development of a mathematical concept in the individual and that individual's interaction with the already codified and socially sanctioned mathematical world. In this way, the individual's mathematical knowledge is both cognitively and socially constituted (Radford, 2000; Berger, 2005). In line with this view, Ernest (2006) suggests that semiotics can offer a synthesis of the cognitive nature of mathematical activity and also of its social aspects. The seminal work of Radford and Arzarello on semiotics, which will be discussed more extensively below, has shown that semiotics is a powerful tool which can help us understand how people think, symbolize and communicate in mathematics (e.g., Arzarello, 2006; Radford, Schubring, & Seeger, 2008). Peirce (1931, 1958) defines sign, or representamen, as "something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent, its object. It stands for that object, not in all respects, but in reference to a sort of idea."

In the learning and teaching of mathematical concepts a wide range of semiotic resources are activated, including written numbers and equations, drawings of graphs and geometrical figures, bodily movements, gestures and various representations in computer environments (Sabena, 2008; Radford, 2009a). Radford (2003, 2008) claims that mathematical activity includes a set of semiotic resources (mediation), named as semiotic means of objectification, that direct individual's intentional activates (reflexivity).

Numerous PME studies in the last ten years investigated students' and teachers' conceptual development through a semiotic lens focusing on one or more of this array of signs. Considering the specificities of the nature, use and influence of different types of signs in mathematical activity, we aim to gain a better understanding of the role the semiotic resources play in the teaching, learning and development of mathematical concepts, by reviewing two research themes (or directions) based on the type of mathematics-related representations these studies have as a focus: a) semiotic representations and b) embodied actions with a focus on gestures.

a) The Role of Semiotic Representations in the Development of Mathematical Concepts

Initially, it is important to define the term "semiotic approach". Mathematical objects are not accessible by themselves but they require representations in suitable registers: "the only way to have access to them and deal with them is using signs and semiotic representations" (Duval, 2006, p. 107). In agreement with Duval's theory, the development of mathematical understanding is relative to the use of different semiotic representations of the mathematical objects being studied in order for the learner to distinguish any mathematical object from its representation. Thus the term "semiotic representations" in this section refers to mental productions or externally presented inscriptions in oral or written/visual form, or in computer environments (Duval, 2006; see also Pino-Fan, Guzmán, Duval, & Font, 2015).

According to Duval's theory (Duval, 1995, 2009) which is used in numerous studies (e.g. Andrài & Santi, 2011; González-Martín, Giraldo, & Souto, 2011; Tatsis & Moutsios-Rentzos, 2013), mathematical thinking and learning seems as a coordination of semiotic systems according to the following operations: treatment (transforming a representation into another one within the same semiotic system), and conversion (transforming a representation into another one, in another semiotic system).

Many studies were conducted based on a Peircean model (e.g. Arzarello, Ferrara, Paola, Robutti, & Sabena, 2005; Presmeg, 2006). However in terms of Ernest's (2005) semiotic processes, numerous researchers (e.g. Berger, 2008) argued that mathematics consists of three components: sets of signs which may be written, or uttered, or encoded electronically.

Framed within the Theory of Semiotics, several studies (e.g. Martignone, 2011; Olive, 2011; Mariotti, 2012; Chan, 2012; Maffei & Mariotti, 2013; Samper, Camargo, Molina, & Perry, 2013; Swidan & Yerushalmy, 2013) adopt the term of *semiotic mediation* and stressed that in a semiotic framework the teacher's main roles are the following: to develop suitable tasks; to make the condition for polyphony, eliciting the polysemic property of the artefact; to guide the transformation of situated "texts" (signs) into mathematical "texts". In this way the teacher mediates mathematical meanings, through the artefact as a tool of semiotic mediation (Bartolini Bussi, 2009).

The use of *multiple representations* and their interconnections in mathematics education have long been acknowledged (see Benke, 2006; Leung & Wang, 2006; Lin, 2008; Dreher, Winkel, & Kuntze, 2012). The National Council of Teachers of Mathematics (NCTM) reinforces this belief: "Different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses" (2000, p. 360). Barmby, Harries, Higgins and Suggate (2007) argue that extensive links between external representations imply a more developed network of internal representations (see also Shvarts & Chumachenko, 2011). Warner (2005) add that connecting contexts and linking multiple representations for the same idea, is associated to better understanding of the mathematical context (see Triantafillou & Potari, 2009; and also, Yamada, 2012).

Numerous studies explore the role of multiple representations in the development of different mathematical concepts, showing that linking between multiple semiotic representations can lead to multiple mathematical and real-world meanings (Amato, 2005; Bardini & Stacey, 2006; Triantafillou & Potari, 2008; Van Dooren, De Bock, & Verschaffel, 2012a). Furthermore, research on teachers' views about the use of multiple representations showed that there is a common lack of awareness regarding reasons for emphasizing multiple representations and their interrelations (Amit & Fried, 2005; Dreher, Kuntze, & Lerman, 2012) and that the choice of representations could be influenced by teachers' lack of subject knowledge and limitations in teachers' use of representations (Barmby & Milinkovic, 2011). What is more, Kertil, Delice and Aydin (2009) revealed that pre-service teachers had difficulties in the transition between different modes of representations and therefore in modeling process. Stylianou (2008) suggested that teachers should be explicitly aware of the purposes of a representation they use and, further, should make the students explicitly aware of this (see also, Izsák, Caglayan, & Olive, 2009; Henriques, 2010; Milinković, 2012).

In this vein, a line of research provided empirical evidence for the tenability of a theoretically-driven framework of *representational flexibility*. The study of Delice and Korkmaz (2009) showed that students had difficulties in writing down as representations the sets they described orally with mathematical language, thus representational flexibility was deficient. Gagatsis and Monoyiou (2011) confirmed a model for the structure of the understanding of the concept of functions related to multiple representational flexibility and problem solving ability and investigated its stability across pre-service teachers from two countries (Cyprus and Italy). Nevertheless, Presmeg and Nenduradu (2005) found that fluency of conversion among representational registers was not a sufficient criterion for inferring a robust, relational grasp of the mathematical concepts involved (see also, Verhoef & Broekman, 2005). Rossi Becker and Rivera (2005) showed that students who used pragmatic generalization employed both numerical and figural strategies and were representationally fluent; that is, they saw sequences of numbers as consisting of both properties and relationships (similar results are proposed by McNab, 2006;

Borba, Barreto, & Azevedo, 2012). Mulligan, Mitchelmore and Prescott (2005) found that children's perception and representation of mathematical structure generalized across a range of mathematical content domains and contexts. They added that representations over time became more complex with configurations and characters of the child's earlier 'system' used inappropriately. Relying on the developmental stages of Mitchelmore, Wu, Ma and Chen (2006) elementary students with higher Van Hiele levels of geometric thinking were classified into higher stages of the representation of five simple regular space figures (see also, a pattern research by Michael, Elia, Gagatsis, Theoklitou, & Savva, 2006).

Representational flexibility was examined also in the concepts of fraction and decimal number addition (Deliyianni, Gagatsis, Elia, & Panaoura, 2015; see also, Deliyianni, Panaoura, Elia, & Gagatsis, 2008; Deliyianni, Elia, Panaoura, & Gagatsis, 2009; Panaoura, Gagatsis, Deliyianni, & Elia, 2009). This type of flexibility was defined as the ability to handle within-representation transformations (intra-representation flexibility), that is, treatment, and between-representation transformations (inter-representation flexibility), that is recognition and conversion, of the same mathematical object. This triad of representational transformation competences (recognition, treatments, conversion) is of great significance in the learning and development of any mathematical concept. Taking a developmental perspective, Deliyianni and Gagatsis (2013) used dynamic modelling to show that there is a strong interrelation between representational flexibility and problem solving in fraction addition. Further research studies were developed examining *fraction's representations* and mathematical concept development (e.g. Amato, 2005; Gabriel & Content, 2009; Deliyianni & Gagatsis, 2013; Deliyianni, Gagatsis, Elia, & Panaoura, 2015). What emerges from such studies is that the multi-representational presentation of the fraction addition process activated interconnected systems of coding information (Suh & Moyer-Packenham, 2007). Thus translating between the fraction representations and using mental models that appear to include a variety of interpretations of a fraction led students to a better understanding of the concept (Morrow-Leong, 2014). Furthermore, particular kinds of pictorial representations were found to help students overcome the epistemological obstacle related to fraction division (Kalogirou, Gagatsis, Michael, & Deliyianni, 2010).

Izsák (2006) explored teachers' knowledge when using linear and area representations to teach fraction arithmetic for the first time. The researcher suggested that the role of representation in teaching mathematics needs to be used in ways that promote sense making. Moreover, Charalambous (2007) indicated that capitalizing on the affordances that several representational models offer, such as the number line, teachers might better scaffold students' construction of the concept of fractions. However, Dreher and Kuntze (2015) found that most teachers focused only on a few core aspects of fractions and suggested interrelations with more global views.

In a report on an analysis of the treatment of addition and subtraction of fractions in elementary mathematics textbooks used in Cyprus, Ireland, and Taiwan it was found that the Taiwanese textbooks employ a greater variety of representations compared to the Cypriot and Irish textbooks; they use a greater variety of representations (linear, area, volumetric, and discrete sets representations), which is considered critical for supporting students' understanding of rational numbers (Delaney, Charalambous, Hsu, & Mesa, 2007). What is more, Berger (2015) pointed out that in a typical textbook the usage of signs with mathematical meaning combined with everyday concepts is a necessary ingredient of learning.

The connection between using representations and *problem solving* in different mathematical content strands was the object of investigation by various PME studies. Besides representational flexibility, an additional core component of students' representational thinking of the mathematical notions of fraction and decimal number addition, is problem-solving ability (Deliyianni et al., 2015). This is also the case for the understanding of the concept of functions. Confirmatory factor analysis in the study of Gagatsis and Monoyiou (2011) revealed that multiple representational flexibility and problem solving ability are both dimensions of the conceptual understanding of functions, with this structure remaining invariant across Cyprus and Italy (see also, Monoyiou & Gagatsis, 2010).

Research conducted by Sevimli and Delice (2011) indicated that teacher trainees lacked sufficient levels of representation awareness in the process of solving definite integral problems. Similar results were shown in Dindyal's (2005) research with high school students using four different forms of representations (verbal, numerical, graphical, and symbolic) during problem solving which include algebraic thinking in geometry. In geometry it seems that visual elements have an effect on the students' geometrical concept formation (see also, Haj Yahya & Hershkowitz, 2013). Additionally, Bayik and Argun (2011), support that students internal and external representations in the problem solving process of geometrical reasoning can reveal students' understanding and difficulties about geometrical concepts. Students were able to separate each form of representation but they had difficulties switching from one form of representation to another and making links between parallel representations of the same concept (see also Leuders, Bruder, Wirtz, & Bayrhuber, 2009). In other words, as Guidoni, Iannece and Tortora (2005) emphasized, the most essential in problem solving and teaching mathematics is not in a standard hierarchy of multi-representations but is in a continuous shifting from one cognitive dimension to another in a mutual progressive enhancement (see also, Higgins, 2005). Furthermore, as Benko and Maher (2006) concluded, students built meaningful representations while working together on strands of problem tasks and specifically they used their representations to justify their solutions in order to make sense of the problems for themselves and to convince others (see also, San Diego, Aczel, Hodgson, & Scanlon, 2006). It is essential to note that, as Zodik and Zaslavsky (2007) argue, the specific choices of representation and diagram a teacher makes could reflect a pedagogical goal, such as maintaining the need to rely on the givens of the problem and not on the particularities of a specific visual representation (similar findings were detected in the study of Cayton & Brizuela's, 2008).

Research studies on the role of semiotic representations in mathematics learning were conducted also for young children (preschoolers and first grades of primary school). Selva, Da Rocha Falcão and Nunes (2005) provided empirical evidence for the importance of supplying diverse symbolic representations in order to support concept development in mathematics addition problems. Elia and Gagatsis (2006) explored the effects of two experimental programs on the development of arithmetic problem solving (APS) ability by 6–9 year-old pupils. The programs stimulated flexible interpretation and use of a plurality of semiotic representations in the context of APS with emphasis on a particular mode: informational picture or number line. An a priori model was validated for all the pupils, suggesting that different modes of representation of the problems significantly influence APS performance, irrespective of the kind of instruction they had received. What is more, Zigdon-Mark and Tirosh (2006) found in their research that most children, both kindergarten children and first graders, regarded quantity as an essential characteristic of number representation (see also, Papandreou, 2009). Another study about kindergarten children by Kafoussi (2006) investigated the capabilities of kindergarten children to read visual representations of data in a problem situation. The results showed that more difficulties seemed to occur in the process of counting in the two-dimensional block graph as well as in the reading of the cyclic diagram. Droujkova (2005) focused on children of four to seven years of age working with table representations. Using tables in qualitative, additive and multiplicative worlds, children developed algebraic and multiplicative ideas such as covariation, binary operation, distribution, or commutativity.

A significant body of PME studies focused on the *number line* and the contribution of its use to learning in different mathematical content areas. Van den Heuvel-Panhuizen (2003) found that the form of a double number line "can function on different levels of understanding, and that it can keep pace with the long-term learning process that students have to pass through" (p. 30). In the same vein, Kuchemann, Hodgen and Brown (2011) argued that an understanding of the double number line model is important for helping students make a shift in understanding multiplication as scaling. Beck, Eames, Cullen, Barrett, Clements and Sarama (2014) suggested that there exists a link between children's level of conceptual and procedural knowledge for length measurement and their ways of using the double number line representation when solving problems involving proportional reasoning.

However, Bruno and Cabrera (2006) in a study on textbooks demonstrated that the number line is mainly used when new number systems (whole, integer, rational or real numbers) are introduced, while it is less often connected with basic operations. Shiakalli and Gagatsis (2006) explored the difficulties that arise in the conversion from one mode of representation of the concepts of addition and subtraction to another, emphasizing number line representation. The results of the study showed that different types of conversions among representations of the same mathematical content were approached in a distinct way by students, indicating the existence of the phenomenon of compartmentalization, i.e., deficiency in the coordination of at least two modes of representation of a concept.

Several research studies explored *linear algebra (function)* and the representations connected to them, as representational flexibility is a core component of developing understanding of the concepts in this mathematical topic. Amado, Carreira, Nobre and Ponte (2010) showed the key role of the representations that are used to solve a problem and how they are critical in the development of a sustainable process for informal learning of formal methods of solving systems of linear equations (see also, Dewolf, Van Dooren, Hermens, & Verschaffel, 2013). What is more, Greenes, Chang and Ben-Chaim (2007) suggested that in order to enable students to build deep and meaningful understanding of the key concepts of linearity, it is recommended that teachers use the spiral method and devote much more time to teaching and systematically reviewing concepts of slope, y-intercept and the connection between the algebraic and graphical representation of a line (see also Rojano & Perrusquía, 2007; Caglayan & Olive, 2008; Schmitz & Eichler, 2013).

Stewart and Thomas (2007) confirmed that some students struggle with basic linear algebra concepts such as linear combination, span and subspace. Using embodied symbolic and formal experiences the researchers showed that students could obtain a better understanding of linear algebra concepts. Furthermore, Van Dooren, De Bock and Verschaffel, (2012b) investigated students' conceptual understanding of linear functions and discovered that the most difficult representational connection was the one between a formula and a graph, and vice versa. Kaldrimidou, Moroglou and Tzekaki (2008) also found that students' conceptions depend on the function's mode of representation. Triantafillou, Spiliotopoulou and Potari (2013) revealed that the graphical representation of a function, conceptions and procedures used are related and when a function is represented numerically or algebraically, conceptions and procedures are not related (see also Kaldrimidou & Moroglou, 2009).

During teaching linear relations Huang and Cai (2007) found that a U.S. teacher tried to treat all four representations (graphic, symbolic, verbal and numerical) equally and develop them simultaneously through different activities, while a Chinese teacher paid more attention to developing symbolic and graphic representations by treating numerical and tabular representations as tools for developing other representations (see also González-Martín, Giraldo, & Souto, 2011 about Brazilian public textbooks). However, teachers in Indigenous Australian and multi-cultural schools appeared to rely heavily upon a literacy approach to mathematics instruction, rather than a focus upon using rich mathematical representations to model concepts (McDonald, Warren, & de Vries, 2011).

The use of *dynamic environments* is also closely related to representations. There is a great number of publications regarding the use of such environments by teachers and students or relating these environments with representational flexibility. Different researchers have examined the *role of a dynamic environment of representation (ICT impact)* in the development of different mathematical concepts. Many researchers studied students' behaviour when using different representations of functions when they work with dynamic software (Canavarro & Gafanhoto, 2012; Lowrie & Diezmann, 2005; Ku & Aspinwall, 2007; Beatty, Brune, & McPherson, 2011; Fernandes & Healy, 2014). Numerous studies also found regarding the use of *dynamic environments in problem solving* (Maher & Gjone, 2006; Santana, 2008; González-Martín, Hitt, & Morasse, 2008), indicating their positive effect on the students' understanding of problems. We invite the reader to see a detailed review of studies of the role dynamic environments elsewhere in this handbook.

Although the discussion of semiotic representations in mathematics learning and development originated earlier than the past ten years, our review here suggests that it still receives much attention and it will probably remain the focus of a major body of research in the next years, because mathematical information is conceptualized and communicated through semiotic representations, which are written in books, students' and teachers' notes, or on computer screens. In a broader sense, embodied actions, including gestures, constitute also an important component of the semiotic approach to mathematics learning, which has received much attention in studies carried out under newer paradigms, as it will be discussed in what follows.

b) The Role of Gestures in the Development of Mathematical Concepts

A semiotic approach refers to representations which include not only written symbols, language or graphs, but also body movements, gestures and other types of signs (Radford, 2009a). By the term of "gestures" we mean the movements of the arms and hands that are produced in effortful cognitive activity, such as reasoning or problem solving (McNeill, 1992). McNeill (2005) proposes a dimensional framework for gestures, in which every gesture has a specific loading across the following four dimensions: deixis, iconicity, metaphoricity and temporal highlighting. Gestures can serve as a representational tool of various mathematical ideas through which children can get a deeper level of consciousness of their meaning. As McNeill (2005, p. 56) points out, to make a gesture "is to iconically materialize a meaning in actional and spatial form".

Godino, Font and Batanero (2009) put forward what they call an integrative theoretical system for mathematics education: an onto-semiotic approach to mathematical knowledge and instruction. Their perspective aims to bring together institutional (sociocultural) and personal (psychological) viewpoints.

There is an established and increasing focus in mathematics education research concerned with gestures in mathematical situations and classrooms contexts. Numerous studies have been developed to explore the effect of gestures in the teaching and learning process in mathematics. Several research studies have occurred about the four dimensions of gestures. The gestures used in the study of Arzarello, Thomas, Coballis, Hamm, Iwabuchi, Lim, Phillips and Wilson (2009) appear iconic, but they can become metaphoric gestures in nature when presented in a mathematical context. Additionally, the case study of Yoon, Thomas and Dreyfus

(2009) showed that a virtual space gives an iconic gesture a metaphoric quality by conferring onto it a set of mathematical properties that enable it to be considered not simply an iconic shape, but a mathematical object in a mathematical space.

Healy and Fernandes (2008) examined the role of gestures for blind learners showing that iconic gestures may actually be more important to the blind people. Highfield and Mulligan (2009) observed 19 grade one children while programming a simple robotic toy (Beebot or Pro-bot) to solve a spatial mapping task. They classified most children's gestures as 'deictic' and they were re-coded into four subcategories (discrete pointing, hand sliding, hand stepping and pointing with eyes or head). Similar findings were deduced by Ferrara and Nemirovsky (2005).

Gestures are considered as important components of the communication system, providing a tool to convey information (McNeill, 1992). Gesture, is a major component of the semiotic systems that teachers use in mathematics teaching (Arzarello, 2006; Radford, Edwards, & Arzarello, 2009; Williams, 2005). Specifically, gesture can be regarded as a semiotic resource that teachers use in building and improving mathematical notions (e.g., Bikner-Ahsbahs, Dreyfus, Kidron, Arzarello, Radford, Artigue, & Sabena, 2010; Robutti, Edwards, & Ferrara, 2012). Arzarello, Thomas, Coballis, Hamm, Iwabuchi, Lim, Phillips and Wilson (2009) make a clear recommendation from their research that gestures could be usefully employed to assist in the teaching of mathematical subject matter. Certainly this study confirmed that teachers (and students) are able to understand the semantic meaning of the gestures they observe, and hence they are in a position to respond to them. Radford's (2003, 2008) cultural-semiotic approach considers cognition as a reflexive mediated activity, of which a specific aspect is objectification. According to Radford cognition is a shared practice (activity) that involves the individual as a whole (both mind and body), as part of his socio-cultural contexts. Following the semiotic means that mediate activity, mathematical objects are layered in level of generality which Radford (2003) described as: a factual generalization, when the objectification of the general scheme is characterized by a perceptual/sensorimotor semiosis; a contextual generalization when the general scheme is objectified by more abstract semiotic means that, however, originated in/from the spatial and temporal context; and a symbolic generalization when the general scheme is objectified by symbolic language that is differed by a spatial-temporal dimension. The learner lives a desubjectification of meaning, namely a conflict with his spatial-temporal and sensorimotor experience.

Also, Radford (2009b) used another concept named semiotic node in order to analyze the students' use of speech, gestures and actions in their meaning making processes. Specifically semiotic node is an attempt to theorize the relations between the semiotic systems in knowledge objectification. A semiotic node is a part of the students' semiotic activity where action and diverse signs (e.g. gesture, word, formula) interplay in order to achieve knowledge objectification. Since knowledge objectification is a process of becoming aware of specified conceptual states of affairs, semiotic nodes are related to the developmental progress of becoming conscious of something. They are associated with layers of objectification, which are mentioned above.

Gestures, glances, drawings, and other extra-linguistic modes of expression do not satisfy all the properties of such definitions, but they seem to be an integral aspect of the semiotic activities that could be observed in the classroom. To take into account all these phenomena within a semiotic perspective, an enlarged notion of semiotic system must be used, the semiotic bundle (Arzarello, 2006). The semiotic bundle includes all the signs that are produced simultaneously, by a student or a group of students who interact in order to solve a problem and/or discuss a mathematical question. This mechanism that teachers use in relation to the development of knowledge using the semiotic resources is called semiotic game. Specifically, the semiotic game takes place when teacher harmonizes with the semiotic resources produced by the students and then guides the development of knowledge according to these resources (Arzarello, 2006). In particular, the semiotic bundle does not only give attention to the signs at a certain moment (synchronic analysis), but also their evolution over time (diachronic analysis), in a dynamic way. With the diachronic analysis of the semiotic bundle, we can consider signs produced at different (close or far) times, transformed into other signs. The semiotic bundle thus allows us to analyze the multimodal semiotic activity of the subjects in a holistic manner, showing the dynamic evolution of signs over time. A multimodal approach includes ‘‘the range of cognitive, physical, and perceptual resources that people utilize when working with mathematical ideas’’ (Radford et al., 2009, p. 91; also see Elia, Evangelou, & Hadjittoouli, 2014). What is more, Warren, Miller and Cooper (2011) investigated how six grade 1 students grasp and express generalisations using Piagetian clinical interviews. The findings of their study suggested that the use of gestures (both by students and interviewers), self-talk (by students), and concrete acting out, assisted students to reach generalisations and to begin to express these generalisations in everyday language. Furthermore, as students gained more awareness of the structure of functions, their use of gestures and self-talk tended to decrease (see also Miller & Warren, 2015).

Arzarello and Sabena (2011) adopt another term in order to specify the way that students' processes are managed and guided according to intertwined modalities of control, namely semiotic and theoretic control. The researchers speak of semiotic control "when the decisions concern mainly the selection and implementation of semiotic resources" (p. 191) and of theoretic control when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it. For example, a semiotic control is necessary to choose a suitable semiotic representation for solving a task (e.g., an algebraic formula vs a Cartesian graph), while a theoretic control intervenes when a subject decides to use a theorem of calculus or of Euclidean geometry for supporting an argument (ibid.).

Edwards (2005, 2008) proposes that the framework of embodied cognition, and the tools of cognitive linguistics and gesture analysis can help us discover the ways that both novices and more experienced students build and conceptualize mathematical ideas. In her research the question of how gestures evoke meaning is addressed within the context of two studies, one involving prospective elementary school teachers discussing fractions, and the other involving doctoral students in mathematics talking about and carrying out proofs. In both situations, gestures and their accompanying language are analyzed in terms of conceptual mappings from more basic conceptual spaces.

As was mentioned above, Arzarello (2006) proposes the concept of semiotic bundle that comprises of different types of semiotic resources that are used in mathematical activities. Sabena, Yoon, Arzarello, Dreyfus, Paola and Thomas (2009) attempt deepened the analysis of gestures within the semiotic bundle model by introducing a new construct: the virtual space of gestures. A virtual space of gestures is a space that is created by subjects through a set of gestures and the meanings associated with them. From a cognitive point of view, the virtual space endows the gesture space with a palpable structure.

Gestures may convey the same information as speech (Arzarello & Edwards, 2005), thus reinforcing the speech meaning. It is important to point out that at PME29 (2005), a research forum was established (vol. 1, pp. 123–154) coordinated by Arzarello and Edwards, in order to investigate the role of gesture in the construction of mathematical meaning. The research forum included seven articles in which several dimensions of the subject were discussed. Specifically, Arzarello, Ferrara, Robutti, Paola and Sabena (2005) presented a case involving geometric visualization to illustrate a new theoretical framework for analyzing gesture and speech in mathematics learning environments.

Edwards (2005) in her article emphasized that the original narrative-based classification of gestures of McNeill should be adjusted for gestures used in mathematical discourse. Moving on, the research team of Arzarello, Ferrara, Robutti and Paola (2005) conducted a study about the genesis of signs by gestures, which are used as a communicating tool of the mathematical thinking during an activity of 3D geometry problem. The results showed that starting gestures have an iconic meaning in that their shape looks like their referents (the geometric solids they express), but they become indexes (in the sense of Peirce) in the communicative attempt of transferring knowledge to the others. The indexical gestures acquire a symbolic function later, when they are used as existing objects of a virtual geometric world and in relation with the genuine geometric objects (p. 79).

Recent research has shown that speech and gesture are two facets of the same cognitive linguistic reality. Nevertheless, gestures' cognitive potential can be analyzed and understood only in the context of their interaction with other modalities and principally with language. As McNeill noted, "Speech and gesture are elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thought—global-synthetic and instantaneous imagery with linear segmented temporally extended verbalization" (McNeill, 1992, p. 35). This view suggests not only that, gestures should be examined in association with other modes of representation in our attempts to understand mathematical thinking, but also that the contribution of gesture to mathematical understanding, which almost always requires both analytic thinking and imagery, is distinct from the role of other modalities. Ferrara and Savioli (2009) concluded that gestures and words are well coordinated with each other, and they are pertinent to mathematical understanding. Similar to these findings are those of Simensen, Fuglestad and Vos (2014) who proposed that the use of artefacts, gestures and speech are intertwined with thinking in the meaning making process (see also Bartolini Bussi & Maschietto, 2005).

What is more, Sabena, Radford and Bardini (2005) in their article about synchronizing gestures, words and actions in pattern generalizations, presented the dynamics between gestures and speech. Their results showed the occurrence of gesture-speech match (with gesture and speech containing congruent information) and mismatch (with gesture and speech containing different information) and the critical role of gestures in the objectification of knowledge. Similar research studies have been carried out by Radford (2005), Edwards, Robutti and Bolite Frant (2006) and Edwards (2006).

When gestures and speech contain different information, gestures may provide information that is conflicting to the content of speech, or may supplement speech by providing additional information. As shown by the study of Radford, Miranda and Guzman (2008) students used gestures to communicate distances in a meaningful relational way. Thus, gestures helped them extend their way of seeing and interpreting a graph. Teachers' gestures, according to Bjuland, Cestari and Borgersen (2008), help students to make the connection between the semiotic representations, figure and diagram. Sinclair and Gol Tabaghi (2009) concluded that the mathematicians use gestures and metaphors to express their thinking about concepts. Additionally, linguistic and nonlinguistic expressions comprise a dynamic component of thinking (see also Kaur, 2013).

Last but not least, it must be said that producing gestures facilitates speakers to explore possible ways of organizing and packaging spatial information in speech. Findings from the research study of Ng and Sinclair (2013) suggested that children use gestures as multi-modal resources to communicate temporal relationships about spatial transformations. Likewise, Ng (2014) showed that the students relied on gestures and dragging as multimodal resources to communicate about dynamic aspects of calculus. He also suggested that language, gestures, and diagrams serve complementary functions in mathematical communications.

THE STUDY OF AFFECT IN RELATION TO CONCEPTUAL DEVELOPMENT

The concept development of any mathematical concept is related with the affective development and it seems that mathematical activity is marked out by a strong interaction between cognitive and emotional aspects. One of the major issues in the recent research on affect is the understanding of the interaction between affect and cognition. Researchers agree that apart from knowledge of the subject and its teaching, teachers' beliefs and attitudes towards mathematics, its teaching and learning play a predominant role in their instructional approaches (see also Donovan, 2015). Social cognitive theory contends that human behaviour, the environment and personal factors such as cognition, emotion and motivation, operate reciprocally on one another (Chiu & Klassen, 2010). Students' engagement is a significant issue in mathematics classrooms and affects achievement levels (Skilling, Bobis, & Martin, 2015). Schukajlow (2015) considers emotional factors essential for students' learning and supported that mathematical enjoyment, boredom and interest are interrelated.

The interrelations of the affective domain and cognitive domain were underlined by Goldin, Rösken and Törner (2009) by using the example of beliefs which do not exist in isolation; they are attached to objects and serve both affective and cognitive functions. DeBellis and Goldin (2006) interpreted affect as a representational system parallel to the cognitive system. As Goldin (2002) suggested, the affective domain is a complex structural system consisting of four main dimensions, emotions, values, attitudes and beliefs, while concepts such as motivation, feelings, conceptions, interest belong also in the field of affect. The significant correlation between the development of those components with the declarative and conceptual understanding and construction of any mathematical concept is underlined by the research interest of the last decades as expressed by the thematic working groups on Affect at the ERME conferences, the numerous presentations at the conferences of PME and the special issues of scientific journals such as the Educational Studies in Mathematics 63(2) (October 2006). Leder and Grootenboer (2005) reported a predominance of belief studies, a diminishing number of studies on attitudes and a few on values and emotions (e.g., Pierce, Chick, & Wander, 2015).

The affective factors of mathematical achievement have to be taken into account in order to understand all levels of performance (Roth, 2008) in mathematics in general and problem solving in particular. It is accepted that one's behaviour when confronted with a task is determined by her/his beliefs and personal theories rather than the respective knowledge of the specifics of the task. Beliefs are a multifaceted construct which can be described as one's subjective opinions about the world. Teachers have beliefs about themselves as teachers and learners of mathematics, about the nature of the mathematical knowledge and about the factors that affect the learning of mathematics. A part of teachers' beliefs are their self-efficacy beliefs about their ability to affect student learning through the planning of mathematical instruction. Teachers with high efficacy beliefs are expected to be more willing to adopt innovations and more successful in scheduling inquiry-based teaching. Further, self-efficacy theory predicts that students work harder on a learning task when they judge themselves as capable (Mayer, 1998; Baron, 2015).

The research forum at the 28th conference of PME, in 2004, discussed the theoretical frameworks of affect in mathematics education such as (i) a system of representation and communication, (ii) by a dynamic viewpoint in the functioning of self-system processes and (iii) under a socio-constructivist perspective. The literature review of the studies on the affective domain which had been presented at PME conferences and had been published at the respective proceedings indicated that the works of the last years can be divided into five main categories which are presented below:

- The first category consists of studies on the pre-service teachers' beliefs (e.g. Beswick & Callingham, 2014), their beliefs in relation to the practice (Bayazit & Aksoy, 2011; Beswick, 2015), their beliefs about the use of computer systems at the teaching of mathematics (Özgün-Koca, 2011; Berube et al., 2010) and teachers' epistemological beliefs in mathematics (Gattermann, Halverscheid, & Wittwer, 2012; Beswick, 2009; Bardini, 2015). It seems that the relationship with mathematics of prospective teachers is often built on negative experiences with mathematics as students. The key issue is to empower their desire for "redeeming" themselves from negative past experiences in order to become good mathematics teachers (Di Martino, Coppola, Mollo, Pacelli, & Sabena, 2013). Teachers' beliefs influence how a classroom in organized and what mathematics will be emphasized and valued (Kuhnke-Lerch et al., 2010).
- The second group of studies consists of teachers' belief systems on specific mathematical concepts such as calculus (e.g. Erens & Eichler, 2014), the place of proof in school mathematics (Iscimen, 2011), their conceptions of inequalities (Halmaghi, 2010), the beliefs and difficulties in dealing with rational numbers (Marceness & Frade, 2010), their ontological beliefs and their impact on teaching geometry (Girnat, 2009). One of the major shifts in thinking in relation to teaching and learning of mathematics the last years is related with the different views about the nature of mathematics. At the same time many studies examined the teachers' didactical beliefs about errors in classroom (Rahat & Tsamir, 2009) and mistakes (Kuntze, 2009). Rach, Ufer and Heinze (2012) investigated the effects on students' attitudes towards errors as learning opportunities for reconstructing the mathematical knowledge.
- A significant body of research suggested that negative attitudes are a major factor limiting the development of mathematical skills, knowledge and confidence (Palandri & Sparrow, 2009; Larkin, 2015; Martínez-Sierra, Socorro, González, & Dolores-Flores, 2015). The same category includes the students' perceptions on learning mathematics in general (Perger, 2009) or on specific mathematical concepts such as their perceptions of invariants and generality of proof (Baccaglini-Frank, Mariotti, & Antonini, 2009).
- The interrelations of students' cognitive behaviour with their affective or meta-affective behaviour are more obvious in the case of the problem solving where students have to self-monitor the engaged cognitive processes. The concept of meta-affect was defined by Goldin (2002) as the affect about affect, the affect about and within cognition, the monitoring of affect and the affect as monitoring. There are studies about students' affective self-regulatory monitoring during solving mathematical problems (Cimen & Campbell, 2012) and studies about their self-representation on activating processes, such as the study on students'

self-beliefs about using representations while solving geometrical problems (Panaoura, Deliyianni, Gagatsis, & Elia, 2011).

- Finally the interrelations of students' mathematical achievements with their goal orientation, attitudes and beliefs (Hannula & Laakso, 2011), constitute another group of studies on the affective domain. The study of Panaoura, Gagatsis, Deliyianni and Elia (2009) investigated the structure of students' beliefs about the use of different types of representation and their respective self-efficacy beliefs in relation to their cognitive performance on the concepts of fractions and decimals.

A great emphasis on the research about beliefs and attitudes was on the methodological issues concerning the measuring tools which are used. Significant were the ethical issues surrounding its highly personal nature and the interference of the way in which individuals behave. There are studies which underlined the use of qualitative approach such as the use of interview (Horne, 2009) or observations (Kuntze, 2009). Olson, Olson, and Okazaki (2008) used a detailed analysis of videotapes in order to provide interpretation of the notion of an affective structure. Additionally in a few cases the use of case study as a method of investigating the domain in depth was suggested. Meagher and Brantlinger (2008) examined the mathematics instructional practices and beliefs of novice mathematics teachers by using observation, interview and survey data. The emphasis of the discussion was on the limitations of using either quantitative or qualitative instruments (Bernack et al., 2011). Wang, Odell and Schwille (2008) explored the developmental processes about the teaching conceptions of practice teachers in mathematics by using the case study method, including classroom observations and pre and post-lesson interviews. Halverscheid and Rolka (2007) used pre-service teachers' pictures and texts in order to examine their mathematical beliefs, while Melo and Pinto (2007) used a written paper on a movie script about mathematics which was worked out by students in order to describe a case study's beliefs, feelings and attitudes. Lewis (2012) used an interpretative phenomenological analysis of the data concerning one student. A discussion took place on the criticism about the lack of validity of the questionnaires which are used during quantitative methods (Barmby & Bolden, 2014), the difficulties in interpreting the qualitative data of narrative processes and the advantages of using mixed methods (Chang & Wu, 2007; Asnis, 2013).

A branch of research presented at PME conferences focused on the role of affect in the social context of the classroom and mainly in relation to different cultural environments. Studies presented the results from cross-cultural comparison of mathematics teachers' beliefs, such as the work of Hannula et al. (2013) which compared Estonian, Latvian and Finnish mathematics teachers' self-reported constructivist teaching practices. This study was a part of a Research Forum which focused on affect and contrasted cognitive with sociocultural approaches to studying affect. Also, Varas et al. (2012) analysed the drawings made by third grade students from Chile and Finland when asked to draw their math classes and results indicated that there were many differences in these countries (Uegatani & Koyama, 2015).

While reviewing the research reports of the last years the high age range of the studies on the affective domain is obvious as well as the different levels of education which are covered. There are studies about pre-school teachers' perceptions (Yalcin & Ocal, 2009), elementary teachers' knowledge in mathematics (Li, Huang, & Tang, 2008), mathematical beliefs and behaviours of high school students (Francisco, 2008) and teachers' and students' perceptions of their mathematics classroom environments (Beswick, 2008). In all ages it seems that there are connections and relationships among affect and mathematical learning, while affect play a decisive role in the progress of cognitive development, the construction and the reconstruction of any mathematical concept. For example, Marcou and Lerman (2006) studied self-efficacy of primary school children and found that "students spend more time to read, analyze and understand the text of the problem and on verification processes in order to review and correct their work." (p. 142).

PATTERNS, SHIFTS AND WAYS FORWARD

Concepts and concept development in mathematics is in itself a multifarious and complex theme and is related to a diversity of factors. Thus, studies that aim to develop a better understanding of concepts and concept development need to use a broad and multidimensional framework, including major approaches similar to those involved in the present chapter: developmental, sociocultural/discursive, and semiotic. The most intensive and extensive developments seem to use the sociocultural/discursive and the semiotic/embodied approaches: the former with numerous works focusing on the context in which conceptual development occurs and the discursive activity occurring in this context; and, the latter with numerous works on semiotic representations, often enriched by a strong emphasis on embodied actions and gestures.

Our review reveals that the various approaches to the study of concept development are interrelated and complementary rather than in contrast to one another, as they focus on, and give insight into, different facets of concept development (e.g., semiotic representations, gestures, tasks as tools, social interactions) which are closely connected to each other in mathematics teaching and learning. Evidence for this is provided by the presence of the work of the same researchers in more than one section referring to different approaches (e.g. see Radford). However, it is to be noted that most of the PME studies focus on a specific research approach to study concepts and concept development. This seems to apply not only for PME but more generally, as according to Nemirovsky (2005), for example, some semiotics-oriented studies tend to rely on the formal analysis of semiotics representations whereas others bring to prominence the socio-cultural context of use.

In PME the number of publications on semiotic representations, which are extended by other approaches, such as embodied actions and gestures, is extensively growing. Further the social approach in mathematics education research is a major focus of studies in PME which has grown substantially in the past ten years. An important issue in studying concept development is whether the mathematics education community (including the PME community) is ready to accept new theoretical frameworks related to other fields of research, such as (neuro) cognitive science. In light of the above considerations, we propose that much further (probably collaborative) work is needed to reach a more holistic, integrated and inclusive line of research which could improve and deepen our knowledge of concept development in mathematics.

Finally, a major observation based on our review is the high degree of specialisation and specificity in PME studies in the sense that the papers that grapple with grand themes such as conceptual development are rarely found in the proceedings. Is this tendency a sign of maturity in the discipline of mathematics education, or is this the result of the small size of PME papers, or of the way reviewing works with the proposed texts? Is this observation applicable to papers published, say, in ESM and other leading journals too? Given this high degree of specialisation and specificity, will a chapter on *conceptual development* be present (or even necessary) in the *Third PME Handbook* of 2026?

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7. DIGITAL TECHNOLOGY IN MATHEMATICS TEACHING AND LEARNING

A Decade Focused on Theorising and Teaching

INTRODUCTION

Over the past decade, a wide variety of PME papers have focused on the use of digital technologies in mathematics education. The 2006 PME Handbook separated research on the use of digital technologies across three main chapters (geometry, arithmetic and algebra, and proof); however, this 2016 PME Handbook unites the topic areas together. Rather than categorise the papers by these topics, we have chosen to organize this chapter in terms of four main sections: technology innovation, theorizing, technology broadening and teaching. In this introduction, we explain our choice for this structuring and link it, where possible, to recommendations and predictions made in the 2006 Handbook.

In relation to the first section, we were initially struck by the relative lack of innovative technologies in the PME papers written over the past decade. We thus decided to consult outside sources to get a better sense of the major areas of innovation in technology in education, which pointed to three major types of advancements: (1) the move from comprehensive software packages to small, expressiveness, web applets; (2) the move towards mobile, touch-based interface; and (3) the changes in social norms (web and social technology). Within each type of innovation, we examine relevant research in the mathematics education research, where possible, and identify PME papers that relate to these innovations. Some of these innovations were anticipated in the 2006 Handbook, such as the touch-based interfaces (Confrey & Kazak, 2006) and the change in social technology (Lerman, 2006), including the development of online communication (which have yet to gain much attention in PME papers over the past decade).

While innovation was surprisingly limited, there was significant work on developing and refining theory, which is our second section. In their chapter on digital technology in the teaching and learning of geometry, Laborde et al. (2006) argue that, “research on the use of technology in geometry teaching needs more contributors” (p. 296). This has happened over the past decade, and in addition to developing existing theories, several new theoretical perspectives have emerged. We dedicate a central part of our review to these developments, highlighting how they are a consequence of: (1) developments of new technologies that are used in

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 235–274.

learning and teaching; (2) increases in use of technology and broadening of their implementation.

The third and largest section provides an overview of all the research papers related to the use of technology. Most of these involve well-known digital technologies such as dynamic geometry environments (DGEs), CAS, Graphing Calculators and Spreadsheets. A smaller but growing number of papers focus on smaller-scale environments that are constrained to the teaching and learning of particular concepts. In the previous Handbook, Ferrara et al. (2006) had observed a shift from larger, more comprehensive environments to smaller, more specialized ones. We conjecture that the acceleration of this process is due to two reasons: (1) Web 2.0 technology better support smaller, focused and linked learning objects, often evolving by different users and relatively easy to produce and use; and (2) the increasing need for and acceptance by the school system of digital technology across all levels and ages (often without much change to the curriculum), which requires modest, more easily integrated pieces.

Finally, in the fourth section, we focus on the teaching of mathematics using digital technology. Technology had opened up new challenges for teaching, not only in terms of their knowledge and beliefs, but also in terms of the complexities of integrating different kinds of resources. Several chapters in the previous Handbook underline the importance of the teacher and the need for more research on how teachers might become more effective at integrating digital technology and how their practices might change as a result. For example, Ferrara et al. (2006) highlight the need to focus on the substantial challenges of teaching with digital technology: "…re-awakens us to the complexity underlying the teaching and learning of mathematics" (p. 268).

Decade of Technology Innovation

The internationally recognized NMC Horizon Project (available at: http://k12.wiki.nmc.org), established in 2002 identifies on a yearly basis emerging technologies likely to have a large impact over the coming five years in education around the globe. More recently, the NMC project has published Horizon Reports (HR) examining technologies for their potential impact on and use in teaching and learning. Each year, the HR picks six major features of technology related to education, policy, inquiry and broader aspects of learning in the new society that are likely to have impact during the next five years. These trends are sorted into three time-related categories—fast moving trends that will realize their impact in the next one to two years, and two categories of slower trends that will realize their impact within three to five or more years. The reports also identify critical challenges that schools face, especially those that are likely to continue to affect education over the coming five-year time period.

A comprehensive review of the last decade suggests trends and related challenges that we grouped under three categories: (1) Mobile related technologies (any device that its mobility suggests ubiquitous world of computing including phones and gaming, sensors and GPS based applications); (2) Tangible Smart technology (augmented reality, internet of things, wearable computing) linking objects with online content; and (3) Personal-Web and social technologies (user created content, collaboration, social networking, personalization of learning and of teaching and learning analytics). Most of the examples in this category are not yet present in the PME work of the last decade.

Clearly there is a gap between the marked trends and the ability of the school-systems to change. Further, the research community (including PME papers) is often the product of a later stage of implementation of innovation. However, a review of the work done by the mathematics education community (some outside of PME) suggests some connections between these overall trends. We start by looking at these studies, which are pointing towards current challenges that schools and teachers are already facing and near future challenges that research could address.

Mobile Personal Devices in Mathematics Education

Ubiquitous computing for teaching and learning was best instantiated during the last decade by the capacities of new mobile phones and the networks to which they belong. The most apparent dimension is the unique capabilities of mobile devices to serve as tools for social participation in mathematical practices (Roschelle et al., 2007). However, research on mobile devices started some time ago, with the introduction of calculators.

Roschelle et al. (ibid.) reviewed three successful implementations of handheld devices in mathematics education: graphing calculators, classroom response systems and probeware, which have also produced valuable improvements in school learning. The success of graphing calculators has been documented in studies since the 1980's (see also section 3 for work carried out in the last decade). Classroom response systems are participatory and include feedback tools that with teacher mediation can increase the students' engagement in learning, enhance classroom communication between teacher and students. Hegedus and Kaput (2003) suggest that such systems can dramatically change students' engagement with core mathematics. The *SimCalc* Classroom Connectivity Project built on the opportunity to connect students within a classroom so that they may respond in real-time to a teacher's queries and have their "responses" instantly (and often anonymously) collected and posted to a public display, where they become the focus of classroom discussion. Probeware refers to the use of sensors with associated software incorporating instant multiple linked representations and immediate feedback to the data collected with the sensor.

From calculators to mobile phones. Many of the functions mentioned above, that were implemented and studied with calculators, became available to innovative work with mobile phones throughout the past decade. They mark an additional step towards the challenge of ubiquitous learning, learning in and out of the formal setting of the classroom and the change of the school-system infrastructure. An example of implementation that was designed to serve wide spectrum of activities in the mathematics classroom is the Math4Mobile system (Yerushalmy & Weizman, 2006). It consists of specialized applications (as graphing calculators, geometry, algebra and calculus applications) that use the connectivity affordances of the phone. Connectivity means also the inclusion of a voting system that offers immediate presentation of the group and personal collected data. This work follows investigation into uses of handheld devices for classroom collaborative learning that was built on Roschelle and Pea's (2002) *Wireless Internet Learning Devices* (WILDs) framework principles.

Capturing and communicating with images. The use of the mobile phone camera was a major component of the settings studied by Gadanidis, Borba, Hughes and Scucuglia (2010) and by Yerushalmy and Botzer (2011). In both studies, users captured phenomena with the video camera and shared it online. Yerushalmy and Botzer asked pre-service teachers to record phenomena and mathematize it into a rate of change model as part of a calculus unit that they were designing with mobile phones applications. Gadanidis et al. asked 7th graders to write dramatic scripts to communicate about their mathematics learning with a wider community. The scripts were written in small groups, performed and recorded using mobile phones, and shared online.

Mobile phones also feature location awareness and context-aware applications that can offer learning opportunities relevant to mathematics education. Spikol and Eliasson (2010) report on the findings from a mobile geometry project for middle school students designed together with teachers that consisted of outdoor and indoor activities. Sollervall (2012) reports positive results of an intervention lesson using GPS (Global Positioning System) with 12-year-old students, who were asked to coordinate themselves physically in an outdoor activity. The work was continued in the mathematics classroom and involved mobile software applications specifically developed to support spatial coordination activity.

Teaching with mobiles. A few researchers have started to study changes in teachers' practices in regard to the use of mobile technology. Sinclair and Wideman (2009) studied the effects of linked hand-held technology in early secondary mathematics. They developed a set of criteria specific to TI-Navigator, to evaluate teachers' implementation experiences and found that changes in teachers' practices in regard to sharing, checking, and modelling are related to the key affordances of TI-Navigator: (1) the provision of two-way communication between teacher and students, which enables sharing, and checking, and (2) the provision of a shared visual display, which facilitates investigation of mathematical models. Daher and Baya'a (2011) studied various aspects of learning and teaching secondary mathematics. They used activity-theory to analyze in-service and preservice teachers' attitudes and decisions regarding mathematics teaching and learning with mobile phones. Yerushalmy and Botzer (2011) designed tasks to support new types of practices such as guided explorations of real-life phenomena, collaborative group discussions and personal experience. They demonstrate why shifting attention from static, inert representations to dynamic personal constructions on learners' personal devices, and sharing these constructions among learners creates new challenges for teachers, such as having to deal immediately with a large amount of data.

Tangible Designs for Mathematical Experiences

The cognitive significance of our bodies and senses has become a growing interest in mathematics education in the embodied nature of mathematical thinking. It has been a feature of research on technology at least since Papert (1980) coined the term 'body syntonicity'. Some digital technologies seem to provide new types of bodily engagement with mathematics, primarily through visual, kinetic and tangible modes. For example, the exploration of motion graphs using the Microcomputer-Based Laboratory (MBL) proved to be a powerful tool for improving students' understanding of concepts in kinematics and for elaborating their graphing skills. Using MBL, one and two-dimensional motion is represented through graphs which reflect kinematic interactions with real objects connected to computers, as well as onscreen and haptic manipulations (e.g., Robutti, 2005; Botzer & Yerushalmy, 2008).

Working within this tradition, Ferrara (2006) considers the activity of high-school students who worked in groups to track a 3-D uniform circular motion. Ferrara used a device called 3D Motion Visualizer to display in real time the trajectory of motion on a computer screen. The task was to draw the 2-D temporal representations of motion along the three directions (width, height, depth). The analysis of speech and gestures shows how understanding the concept of motion in mathematical terms grows out of a complex dynamics between recollections (remembering) and expectations (imagining). Similarly, Ferrara and Savioli (2011) studied elementary school students throughout a teaching experiment. The students had used a motion sensor to work with graphical representations of position versus time. The students were asked to compare two graphs, choosing two cartoon characters, animals and vehicles, as subjects of possible corresponding motions. The authors' analysis revealed the students' thinking strategies, as well as key characterizations of graphs as models of motion.

Embodied experimentations of young students with technology is repeatedly acknowledged as providing unique opportunities for children to learn through action and reflection and for researchers to observe and get a new glimpse into students' conceptualization. One example of the work can be found in Abrahamson and Lindgren's (2014) Mathematical Imagery Trainer, which is designed to help students learn about proportion through a qualitative approach. Another example is Hegedus (2013), who investigates how young children make sense of 3-D mathematical objects in a multimodal environment involving a haptic, force-feedback device. A third example is Avraamidou and Monaghan (2009), who describe non-conventional abstraction carried by an 11 year-old child with the mediation of the Sim 2 simulation video-game.

Finally, Highfield, Mulligan and Hedberg (2008) describe learning through exploration with a Bee-bot programmable toy. They report on two case studies of young students (aged five and eight years), engaging in transformational geometry. The authors argue that in planning, programming and manipulating the toy through complex pathways, the children developed problem-solving strategies and relational thinking.

New sensual experiences: The touch-technology. Touch technology offers new sense to sensual experiences in mathematics with technology permitting both touch and multi-touch interactions. Touch-screen devices can enable intuitive interface suitable for whole-class and individual interactions and for young learners, allowing them to use their fingers and gestures to explore mathematics ideas and express mathematical understandings. Multitouch technology is capable of detecting multiple on-screen touch-locations simultaneously, which may also provide opportunities for multi-user interactions. Recent mathematics education research has started to explore both the use of personal and mobile multi-touch systems.

Arzarello, Bairral and Dane (2014) investigate the cognitive singularities involved in touchscreen manipulation that can be observed with respect to geometrical thinking. They identify and illustrate different types of touchscreen manipulation during the process of solving high-school problems using the Geometric Constructor tablet software. Soldano, Arzarello and Robutti (2015) explore game strategies to improve secondary students' geometric thinking with a multi-touch system.

Sinclair and Pimm (2014) explore the richer sense of finger gnosis (literally, knowledge of one's fingers) with respect to three- and four-year-olds' interactions with a novel iPad App (*TouchCounts*). The study focuses on direct and tactile engagement and examines the children's responses to an "inverse subitising" task. They report on a striking shift from index finger incrementation to deployment of several fingers all-at-once (in a cardinal touch gesture) to achieve a given target number that is then spoken by the iPad.

Also in a multi-touch environment, Ng (2014) provides a detailed analysis of the mathematical communication involving a pair of high school calculus students who are English language learners. She looked at the word-use, gestures and dragging actions in the student-pair communication about calculus concepts when paper-based static and then touchscreen dynamic diagrams. Of particular importance is Ng's notion of "dragsturing", which is the combined act of dragging and gesturing that seems to have both communicative and epistemic functions, and that points to a fundamentally new kind of mathematical practice in touch-based environment.

Outside of PME, research on learning with touch technology is expanding and we anticipate this to be an area of significant future growth. In particular, the use of touch technology seems to be promising for students with disabilities, as evidenced in Thompson Avant and Heller's (2011) research on the use of *TouchMath* with students with physical disabilities.

Social Technology: Learning and Teaching on the Web

Social computing, social technology, social media and social networking are terms used for describing computer technology to facilitate socializing, collaborating and working in groups. Social media has now proliferated to the point where it spans all ages and demographics. For educational institutions, social media enables two-way dialogues between students, educators, and the institution that are less formal than with other media. As social networks continue to flourish, educators are using them as professional communities of practice, as learning communities, and as a platform to share interesting activities and stories about topics students are studying in class. The 2014 Horizon Report states that understanding how social media can be leveraged for social learning is a key skill for teachers, and that teacher training programs are increasingly being expected to include this skill.

In his study of on-line, distance education course for teachers, Borba (2005) assumes that knowledge is generated by collectives of humans-with-media, and that different technologies modify the nature of the knowledge generated. He studied the transformation of mathematics in on-line courses, highlighting the highly written form of mathematics that occurred as the teachers exchanged solutions through chat. Borba and Zulatto (2006) report data that illustrate how teachers can collaborate online in order to learn how to use geometry software in teaching activities. They studied a virtual environment that allows construction to be carried out collectively, even if the participants are not sharing a classroom. Goos and Bennison's (2006) interest in online communities of practice brought them to set up pre-service teachers and beginning teachers communities focused on learning to teach secondary mathematics. Bulletin board discussions as well as email messages were archived and analyzed. The authors examined messages posted that span the transition from pre-service to beginning teaching. The participants expanded, transformed and maintained the community during the pre-service program and after graduation. The study shows that the emergent design of the community contributed to its sustainability in allowing the pre-service teachers to define their own professional goals and values.

Learning in an online community was also part of the Besamusca and Drijvers' (2013) study, which followed 8th grade teachers learning to implement an algebra system in the mathematics classroom. A community of practice was set up to study and evaluate the influence of the community on teachers' professional development. However, evaluation of the enterprise shows that the teachers' development was not optimally supported by the community. Teachers lost interest in writing their blogs and expressed a relatively low opinion of the added value of the blogs.

These results contradict those from other studies with a similar setup. As extended and more frequent use of online teachers' professional development communities emerge, we suggest that it is important to analyze the differences that may cause different outcomes and should be addressed by near future studies. One reason might be the change of the technological tools supporting collaborative work and learning. Social technology is changing daily and has been dramatically changed since the beginning of the decade. Today's web users are also creating content in different representations, uploading photographs, audio and video to the cloud to record and share their experiences, including their teaching and resources (e.g. http://betterlesson.com/ and http://www.seasamath.net). Producing, commenting and classifying these media has become just as important as the more passive tasks (that dominate the web 1.0 actions in the previous decade) of searching, reading, watching and listening. Open tools and sites make it easy to share (such as GoogleDocs). In addition to interacting with the content, social media makes it easy to interact with friends and institutions that produced the content. The other regards the participants' motivation for engaging in an online community. What makes a social technology phenomenon interesting? What is likely to make it long lasting? Is it the way it facilitates an almost spontaneous development of communities who share similar interests?

We expect that the study of communities of learning and construction of knowledge will soon expand to include the communication channels that became the natural and spontaneous way for social networking such as Facebook. A glimpse of such research can be seen in Biton, Hershkovitz and Hoch (2014), who studied new opportunities of Facebook for interactions between teachers and students preparing for the matriculation test.

THEORISING

Artigue and Cerulli (2008) describe eight different theoretical approaches in research on technology: theory of didactical situations, anthropological theory of didactics, activity theory, instrumental approach, theory of semiotic mediation, social semiotics, socio-constructivism and constructionism. These theories are not all specific to the use of digital technologies. Indeed, Drijvers, Kieran and Mariotti (2009) provide an historical overview of theoretical perspectives they consider relevant to integrating technology into mathematics education. They then focus on *Instrumentation Theory* and *Semiotic Mediation,* but make "a plea for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives" (p. 89). While these two theoretical perspectives attend explicitly to the use of digital tools, there have been several other theoretical perspectives that have been used in the PME proceedings over the last decade. In this section, we have chosen to focus on two main categories: those that attend explicitly to digital tools and those that do not. Within each category, there is great variety in terms of epistemological and ontological assumptions. Further, some researchers have chosen to combine different theoretical perspectives.

With respect to the second type, we highlight the need for a combination of new theoretical perspectives and new research methodologies (that do not rely only on questionnaires) that concern the affective dimension of students' and teachers' use of digital technology.

Theoretical Perspectives that do not Directly Involve Digital Tools

As evident in the list proffered by Artigue and Cerulli, there are several theories that are used in mathematics education that do not specifically involve the use of tools. For example, Furinghetti, Morselli and Paola (2005) use Bruner's enactive, iconic and symbolic modes to analyse the interaction modalities used by students using the DGE *Cabri*. This approach has some similarity to theories of embodied cognition and also semiotic mediation, both of which have been developed as mathematics-specific theories, as we will describe later.

A few reports also offer theoretical approaches that enable a more critical analysis. One example has been the work by Lerman and Zevenberger. In Lerman and Zevenberger (2006), they examine the equitable practices of ICT use in diverse contexts. They use the notion of productive pedagogy to do so, drawing on Bernstein's (2000) notions of *classifications* and *framing* to investigate controls on communication. In Lerman and Zevenberger (2007), they focus on the way in which the use of IWB's can shut down students' thinking, despite the widespread rhetoric to the contrary. Palmér and Ebbelind (2013) also use Bernstein's notions of classification and framing to study the possibilities to learn for preschool children using iPads. They find that applications with weak framing (i.e., they are interactive, instead of being exercise-focused) prompt free dialogues between the teacher and the student, and the focus of the dialogue often became mathematical, irrespective of whether the classification is strong (containing mostly mathematical elements) or not. The notions of framing and classification are specific to the situative perspective that highlights the social context in which an activity takes place. Several other researchers have classified technology using different approaches, such as Sedig and Sumner (2006), who classify them in terms of the forms of interaction they enable individual users.

Theories of embodied cognition arise in certain studies, particularly in relation to the use of gestures and in studies where the digital technology enables mathematical objects to be in motion. The studies by Ferrara (2006) as well as Ferrara and Savioli (2011), described in the previous section, use the "sensuous cognition" approach of Radford (to examine the role of gestures in the students' thinking. The latter paper focuses on more phenomenological accounts of perceptuo-motor-sensory activities and their relation to what the authors call "imaginary" activity, which they take to be constitutive of mathematical thinking. Radford's approach is also used by Highfield and Mulligan (2009), who study children's use of gestures and diagrams as they work on a problem-solving task involving robotic toys. These researchers use McNeill's

(2005) classification of gestures and highlight the role of deictic gestures, which were especially helpful in the problem solving process.

Within the research focused on teachers' use of digital technologies, many PME authors draw on participationist approaches, especially Wenger's (1997) work on communities of practice. The Goos and Bennison (2006) paper described in the previous section used it to examine how pre-service and beginning teachers of secondary mathematics can develop and sustain communities of practices. Fuglestad (2007) uses it to study the teachers' competence with ICT in mathematics developed through workshops. Camargo, Samper, Perry, Molina and Echeverry (2009) use it to analyse the way that teachers use dragging to make conjectures.

Participationist approaches also get used to analyse students' learning with digital technologies. Sfard's (2008) commognitive approach, which adopts the point of view that thinking is communicating, and that changes in mathematical discourse (which can be linguistic or non-linguistic in nature) correspond to changes in thinking. She characterises discourse in terms of word use, routines, visual mediators and endorsed narratives. Sinclair, Moss and Jones (2010) thus analyse the changes in geometric discourse of k-2 children using a DGE to explore the concept of parallel and intersecting lines. They highlight the pivotal role played by the children's gestures, which arose out of the dynamic diagrams shown on the overhead projector, in enabling students to produce two effective routines for determining whether or not two lines will intersect. The authors underscore the need to complement Sfard's approach with one that accords sufficient attention to children's gestures. Sinclair and Kaur (2011) conduct a similar analysis, but this time in the context of young children's thinking about reflectional symmetry in a DGE. In particular, they compare the children's discourse between the discrete and continuous contexts, where the former triggered a processual discourse while the latter led to a more structural discourse. Kaur (2013) combines Sfard's approach with an added focus on the role of gestures in thinking and learning to study the emergence of kindergarten children's dynamic thinking in angle comparison tasks. Kaur and Sinclair (2014) also use Sfard's approach to examine grades 2/3 children's thinking about scalene, isosceles and equilateral triangles in a DGE. They focus on dragging as a visual mediator for helping the children develop more mathematical routines for comparing triangles, and even helping some children think in terms of inclusive definitions of triangles.

Berger (2011) also draws on Sfard's commognitive framework to analyse a pair of in-service teachers' activities when using a DGE-based task involving polynomial and rational functions. She finds that the students did not exploit the graphs produced by the DGE as visual mediators, perhaps because it is not consistent with existing endorsed narratives around what graphs should look like. They also insisted in drawing the sketches by hand. Sfard's characterisation helped provide insight into the key aspects of their mathematical thinking, including their expectations about mathematical activity.

Combining Sfard's approach with Moschkovich's sociocultural view of bilingual learners, Ng (2014) examines the interplay between language, gestures, dragging and diagrams in bilingual secondary students communication about calculus concepts presented in a multitouch DGE. She describes how students' dragging actions also function as communicative acts as they are working in pairs, which increases their communicative resources, perhaps making them less dependent on language.

Mason's (2008) theory on shifts in awareness was used in Gol Tabaghi and Sinclair (2010) to investigate one undergraduate student's learning of eigen theory in the DGE *Sketchpad*. This theory permitted the authors to focus explicitly on how the students attended to the symbolic, geometric and dynamic aspects of eigenvectors and eigenvalues, and how the increased geometric awareness emphasised the invariant collinearity of infinitely many eigenvectors for a given eigenvalue, the relative importance of the eigenvector (which is usually found after the eigenvalue in algebraic approaches), and the possibility of having more than one eigenvector. The students' dragging strategies provided an important way of analysing shifts in attention. The link between dragging and attention was also used in Lee and Leung's (2012) paper, as a way of analysing how students understand geometric properties in a DGE. The authors drew on Marton and Booth's (1997) variation theory, which conceptualises learning in terms of improved discernment. They argue that the action of dragging helped draw students' attention to relevant geometric properties. Leung (2014) also draws on variation theory, but as a theoretical approach for task design.

Mason's (2002) Discipline of Noticing is used as a theoretical framework in Hewitt's (2010) analysis of grade 5 students' learning of about equation solving using *Grid Algebra*. This framework acknowledges that the viewing of data depends on what is stressed and ignored, which depends on the viewer's experience and interests. Hewitt also uses the pedagogical framework of the arbitrary/necessary divide (Hewitt, 1999), which shaped the design of the software as well as the teaching activities. Hewitt identifies the themes of subordination and fading as relevant to the students' learning of formal notation, which ended up helping students solve equations as well.

Theoretical Perspectives that Directly Involve Digital Tools

In addition to the theories of semiotic mediation and instrumental genesis (to which we dedicate the two next subsections), some researchers have drawn on Borba and Villarreal's (2005) construct of humans-with-media, which advances a theoretical assumption that humans and media are deeply interrelated, and that the use of media changes the way students and teacher think, as well as the mathematics. This is a perspective that had previously been articulated by earlier theorists such as Papert as well as Noss and Hoyles. Like constructionism, it lays out some epistemological and/or ontological assumptions about learning, but does not provide specific constructs or methods that can be used in the analysis of data. For example, Borba

(2005) asserts that the internet modifies the interactions and knowledge production of students in distance learning courses, but this is a function of his theoretical perspective rather than a result of the analysis of the empirical data he provides. Borba and Zulatto (2006) examine the way in which collective work with a DGE in an online teacher education context emerges. This perspective is also adopted by Jacinto and Carreira (2013) in their study of students' DGE-based problem solving in out-of-school mathematics competition context. Finally, Flores, Escudero and Aguilar (2014) use the humans-with-media approach to reflect on the possible transformations that online environments produce in the production of research knowledge within the area of online mathematics teacher education. They argue that *researchers-with-online environments* can access remote data in a less intrusive way, that data collection and processing is also facilitated, and that new theoretical tools are adapted and created

Focus on affordances and instrumentation. A few papers draw explicitly on Gibson's (1977) notion of affordances, which has become a key construct in the literature. For example, Brown (2005) studies how students and teachers perceive and enact graphing calculators. Brown (2006) focuses on the importance of helping students become aware of affordances. Finally, Brown (2014) investigates the variety of ways in which students use and perceive the same affordances.

This approach has several points in common with the instrumental approach, which also grows from a non-educational context. While Gibson draws attention to the potential tools might have, and hence to the importance of tool design, instrumental genesis is interested in the complex process through which an artefact (a physical tool) transforms into an instrument (a psychological tool), which includes scripts for how to use the artefact—which are called utilization schemes (see Laborde et al. (2006) for more information).

Some studies are conceptually informed by the instrumental approach, while not using it as a methodological tool. For example, the instrumental approach is mentioned by Stewart and Thomas (2005) as a way of recognising the process of instrumental genesis involved in learning to use a CAS, but their study focuses on university students' attitudes towards and perceptions of CAS use, as determined by questionnaires. Wilson, Ainley and Bills (2005) do not mention the instrumental approach directly, but draw heavily on Haspekian's (2003) study of the students' use of spreadsheets, which does explicitly use the instrumental approach. Wilson, Ainley and Bills also use the instrumental approach as a tool for analysing data in that they examine the way particular aspects of spreadsheets (the variable cell and naming columns) mediate students' meanings of variable.

Other authors use the instrumental approach explicitly. Prior work in the field has described schemes of utilisation, especially in relation to the use of DGEs. This work elaborated different dragging modes (Arzarello, Olivero, Paola, & Robutti, 2002) and measuring modes (Olivero & Robutti, 2007). Similarly, Hatterman (2008) identifies unique drag modes of students using a 3-D DGE and extends this work in

Hatterman (2010). Patsiomitou (2011) considers the notion of "theoretical dragging", which she links to Toulmin's theory of argumentation. The author claims that, in contrast to experimental dragging, theoretical dragging can provide a non-linguistic warrant that leads to "dynamic propositions" in a DGE. Minh (2012) investigates the process of instrumental genesis in the learning of functions within a geometrical and symbolic CAS environment and finds that this process takes a long time for students. Roorda, Vos, Drijvers and Goedhart (2014) examine the instrumentation schemes of one student using a graphing calculator to study the derivative concept, showing how that student develops symbolic understanding over time.

The instrumental approach allows researchers to analyse a student's use of a digital artefact, but it does not say much about the student's mathematical understanding of a concept that she explores using the artefact. This may explain why the instrumental approach if often used in combination with another theoretical approach, and usually one that addresses epistemological concerns. For example, a strong tradition of French research combines the instrumental approach with the Anthropological Theory of Didactics (ATD) (Chevallard, 1999), which proposes a general epistemological model of mathematical knowledge that is based on human activity, and thus inscribed within social institutions. Its main theoretical tool is that of praxeology, which is structured in two levels: praxis includes the *tasks* to be solved as well as the *techniques* available to solve them; and, logos includes the *technology* (the discourse that describes, explains and justifies the techniques used) and the *theory*, which is the formal argument that justifies the technology. Since the tasks and techniques that arise in the use of digital technologies can be quite distinct from those in paper-and-pencil environments, researchers can study how they affect the students' mathematical knowledge, that is, their description, explanation and justification of techniques. With an explicit focus on teachers, Lagrange (2011) uses ATD to analyse the ways that teachers take up innovative tools such as *Casyopée*, which introduce new techniques that compete with existing ones, thus requiring a reconsideration of praxeologies. Based on video recordings of meeting with teachers and of experimentations, Lagrange analyses how current praxeologies made it difficult for teachers to integrate *Casyopée*, especially those who were "mid-adopters".

Kieran and Saldanha (2005) examine the way that CAS-based techniques, which are different from pencil-and-paper ones, act as a bridge between task and theory. They further argue that this bridging process requires that students explicitly interpret their work with the CAS, which allows them to justify the use of the discourse on the techniques. Similarly, with respect to students' use of CAS, Drijvers et al. (2006) show how the use of CAS-based techniques give rise to a justification of the discourse on the techniques. Kieran and Damboise (2007) also combine the instrumental and anthropological approaches, but this time as a way to show how CAS "played three roles that were instrumental in increasing students' motivation and confidence: generator of exact answers, verifier of students' written work, and instigator of classroom discussion" (p. 105). ATD is the central theoretical framing for the study of Solares and Kieran (2012), which analyses the epistemic and pragmatic values arising from one student's interactions with a CAS while learning about the equivalence of rational expressions.

Psycharis (2006) combines the instrumental approach with Noss and Hoyles' (1996) theory of situated abstraction, which also offers an epistemological counterpart to the instrumental approach. This theory seeks to describe how mathematical knowledge can be conceived as being both situated and abstract—that is, it is seen as arising out of particular contexts (how and where it is learned), yet it maintains certain mathematical invariants. Psycharis aims to study the dynamic manipulation schemes used by students in geometrical constructions and argues that the latter theory enables the examination of the specific way that mathematical knowledge develops. Panorkou and Pratt (2011) also use situated abstraction to describe how one pair of 10-year old student used *Google SketchUp* to develop a mathematical meaning of dimension. While the authors attend explicitly to the tools that students use during the two tasks, much like the instrumental approach would warrant, the analysis rests on identifying the situated abstractions that the students express verbally as they work. Lee, Cho and Lee (2012) use situated abstraction as a theoretical lens to study the mediation of embodied symbols in a combinatorial microworld.

Doorman, Boon, Drijvers, van Gusbergen, Gravemeijer and Reed (2009) combine the instrumental approach with "form-function-shift", which permits an analysis of the interplay between tool use and conceptual development. They thus analyse grade eight students' acquisition of the mathematical concept of function through a teaching experiment method, showing how a "form-function-shift" occurred within the instrumental genesis process.

A different approach, by Lozano, Sandoval and Trigueros (2006), combines instrumentation with enactivism to student the mathematics learning of students using the primary school programme *Enciclomedia*. Enactivism can account for a broader set of phenomena than can instrumentation, including the social and affective dimensions of the teaching and learning situation.

The instrumental approach has developed over time, especially as researchers have turned their attention to the role of teachers in a technology-based classroom, to focus on the construct of instrumental orchestration (Trouche, 2004), which relates to the way a teacher, for example, externally steers a student's instrumental genesis. Orchestration is used in Kieran, Guzmán, Boileau, Tanguay and Drijvers' (2008) study of whole-class discussion involving a CAS activity as a way of analysing the way the teacher steers the discussion through, for example, "notational re-voicing". A similar study by Morera and Fortuny (2012) uses instrumental orchestration to analyse the whole class discussion involving a DGE. Their goal was to identify rich situations in terms of their potential contributions to student's mathematical learning. They assert that the students' progress centrally involves communicative nonverbal actions, and not just the verbal interactions that make up the list of types of instrumental orchestration. This suggests a need to complement this kind of study with theoretical approaches that can account for the importance of nonverbal communicative actions.

Drijvers, Doorman, Boon, van Gisbergen and Reed (2009) study the types of orchestrations of three grade 8 teachers using an applet related to the concept of function. This framework allows them to identify six ICT-specific orchestrations and follow up interviews indicate that the teachers' preferences for the types of orchestrations relate to their view on the teaching and learning of mathematics. Patsiomitou and Emvalotis (2010) draw on the insights of both instrumental genesis and instrumental orchestration to design a classroom intervention with secondary students in Greece using DGE. They show that the students' reasoning skills (in relation to van Hiele levels) improved and explain this by analysing both the instrumentation and orchestration processes.

Documentational genesis is another evolution of the instrumental theory that focuses on how teachers' work is developed with and on resources in a dialectic process where design and enactment are intertwined (Gueudet & Trouche, 2009). Psycharis and Kalogeria (2013) use this approach to study a prospective teachers' documentational work in a technology-enhanced mathematics classroom. They identify the various factors involved in the teacher's development of two new documents (a scenario and a worksheet), which include his limited understanding of the affordances of the software tool.

The theory of instrumental genesis focus on the relationship between tools and users. A different approach is offered in Chorney (2013), who argues for a material, post-human perspective in which the tool, the student and also the mathematics are not simply interacting as individual nodes in a graph. Instead, they form an assemblage in which, for example, the student and the tool can be seen as one ontological unit. He illustrates this by examining a DGE-student-circle assemblage as it changes over time.

Focus on mediation. Semiotic approaches are visible throughout the decade and, for the most part, increasingly become identified in terms of the theory of semiotic mediation (TSM), proposed by Bartolini-Bussi and Mariotti (2008). One exception can be found in the work of Berger (2008), who uses a semiotic approach, but one that is more specifically situated within the work of Peirce (1998). In this work, mathematics is seen as a semiotic system consisting of three components: a set of signs, a set of rules for sign production and a set of relationships between signs and their meanings. This socio-historical approach has much in common with Radford's (2012), which also attends to the embodied and affective dimensions of knowing. Indeed, Swidan and Yerushalmy's (2009) broad theoretical framing draws on Vygotsky's (1978) socio-cultural theory of learning, which accords a significant role to the mediation of external artefacts that are transformed into internally oriented tools. These authors focus also on theories of embodiment, through Berger (2004) and Radford et al. (2005), and use Radford's (2003) categories of attention, awareness and objectification to examine how students made sense of the accumulation function in two different applications involving dynamically and visually linked representations of functions. Their case study of two 17-year old students enabled them to identify four different meanings for the accumulation function, which the authors argue were interiorized in the course of their operations in the microworld. Swidan and Yerushalmy (2013) use the same theoretical tools, but with an increased emphasis on how bodily activities are involved in the conceptualisation process and on multimodality, to analyse the way a pair of 17-year old students embody the convergence of the Reimann accumulation function in the *Calculus UnLimited* application.

A few papers draw on activity theory, which can be seen as an extension of Vygotsky's work, as developed by Leont'ev, and which takes as its central unit of analysis activity, rather than the individual or even the group. Triantafillou and Potari (2006) used an ethnographic approach that draw on activity theory to study the uses of technology in the work place. A similar approach, informed by Vygotsky and Radford, was taken by Hassan, Fernandes and Healy (2014) to analyse the algebraic expressions of deaf students using the *Mathsticks* microworld. The authors show how the students were able to connect the visuo-gestural and dynamic digital representations, and highlight the importance of developing a shared sign for the concept of variable.

Early uses of TSM focused on how particular actions or presentations in a digital technology might enable students to develop associated mathematical meanings, which are presumably specific to that technology. For example, Keisoglou and Kynigos (2006) examine the mediation process of students working in trigonometry microworld; they specifically examine how students mediate representational registers. Similarly, Kynigos and Gavrilis (2006) focus on mediation in a microworld in which students construct sinusoidal periodic covariation. Arzarello and Paola (2008) combined TSM with the instrumental approach to study how students' use of a graphing calculator, in addition to paper-and-pencil methods, affects the way they choose the independent variable. Olive (2011) also combines these two theories in his analysis of a one student makes sense of fractions using a DGE dynamic number lines. Chan (2012) uses the construct from TSM of the double semiotic link, relates the artefact with both the DGE-based task and the mathematical knowledge, of a mathematician working on tasks in Euclidean geometry, such as Haruki's Cevian Theorem. The experience with dragging was seen as modifying his semiotic link between the DGE and mathematical knowledge.

Focusing on the role of feedback in *Aplusix*, Maffei, Sabena and Mariotti (2009) use TSM to perform an a priori analysis of the potentialities of the software. They also show, through excerpts of classroom discussion, how the semiotic process was triggered by the teacher's interventions. Maffei and Mariotti (2010) extend this work to analyse the way tree representations in *Aplusix* make structural and procedural aspects of algebraic expressions emerge in a teaching experiment involving two ninth grade Italian classes. Maffei and Mariotti (2013) also use TSM to examine the way in which the graph in *Aplusix* becomes a mathematics tool for students learning about algebraic equivalence.

The two papers described above highlight how TSM can be used not only to study the learning process, but also the teaching process. Indeed, the notion of a *didactical cycle* is mentioned explicitly in the latter paper, which provides explicit guidance for how teachers can guide the process of semiotic mediation both in terms of the use of discussions and in the design of tasks. Related to this, Arzarello and Paola (2007) use the notion of a "semiotic game" to describe how a teacher and her students navigate the process of semiotic mediation in a multimodal environment that involves graphing calculator signs as well as gestures, speech and visual representations. This research draws explicitly on the TSM, while also bringing in aspects of embodied cognition in order to privilege the role of gestures in mediation of meaning.

A Comment on Theories and Methodologies

Not all the papers we considered make explicit use of a theoretical perspective. These papers fall into two main types:

- Quantitative studies: examples include Lin and Chin's (2005) comparison of two grade five students' calculator-assisted learning of number sense in Taiwan; Pegg, Graham and Bellert (2005) study the effect of a computer-based system focusing on basic skills; Kimihno, Tatsuo, Hitoshi, Fumihiro, Yuichi and Mikio (2007) study of lower secondary students in Japan who were taught using a 3-D DGE-based curriculum involving the use of 3-D dynamic geometry; Hosein, Aczel, Clow and Richardson (2008) comparison of students' exploration using black-box, glass-box and open-box software; Beatty, Bruce and McPherson (2011) show that the use of CLIPS (computer-based environment designed to focus student's attention) improved increased pre-test scores. Ma, Xin, Tzur, Yang, Park, Liu and Ding (2014) evaluated the effects of an intelligent tutor system designed to support multiplicative reasoning on students with learning disabilities. Other related examples relevant to this category are discussed elsewhere in this chapter.

Studies of beliefs and/or preferences about the use of digital technologies: examples not discussed elsewhere in this chapter include Gkolia and Jervis (2006) on students' attitudes to the integration of technology in mathematics; Özgün-Koca (2011) on prospective teachers' views on the use of CAS.

With respect to the papers that do draw on theories, there has been significant development over the past decade, which suggests that the field of mathematics education related to digital technology has certainly matured; it has evolved from being an "experimentation niche" and has become an established domain of research that now carries a more solid message for the future. That being said, there remain important lacunae. We would like to highlight two of them. The first concerns the need to better integrate theories relating to tool use with well-known more general and established theories. The second is the relative lack of parallel evolution in terms of methods, as can be seen in the fact that it was the explicit focus of only one PME paper over the past decade. Hosein, James, Clow and Richardson (2007) propose the method of remote observations for analysing learners' mathematical activity with spreadsheets, a method that is especially well suited to on-line learning environments and that capitalises on current screen and voice capture technology. Elsewhere, Yerushalmy has used known linguistic theories of text analysis and extended them to analysis of interactive text (including dynamic diagrams); this work has informed the design and implementation on interactive text. Finally, given the increased use of neuroscientific methods in mathematics education research, such as fMRI and EEG, there may also be opportunities to employ such methods in the particular context of digital technology. We will return these two lacunas in the conclusion.

BROADENING THE PHENOMENON OF INTEREST

In their chapter on projecting trends in digital technology in mathematics education, Sinclair and Jackiw (2005) describe three ways of technology evolution in mathematics education. While the first wave focused almost exclusively on learners' interactions with technology, the second wave consisted of technologies such as spreadsheets, graphing calculators, computer algebra systems and dynamic geometry environments, all of which are more transparently related to the school mathematics curriculum. The PME papers over the past decade involve mostly on second wave digital technologies, but these vary in terms of both their mathematical expressivity and their curriculum specificity. In particular, we can see the following evolution in second wave digital technology: (1) *open digital technologies* do not contain embedded tasks, though these can be developed alongside the technology (*Logo*, *Cabri*, *Sketchpad*, CAS), and are thus open to a wide range of potential actions and uses; (2) *task embedded digital technologies* contain embedded tasks, which direct the actions and uses to more specific purposes; (3) *evaluative technologies* provide feedback on students' responses and actions. This evolution shows a growing increase in the accessibility of technology to the context of the mathematical classroom, moving towards an increased integration of the various practices of teaching, such as proposing problems and assessing student learning. In the following sections, we describe the PME research over the last decade in terms of these four categories.

Open Digital Technologies

The digital technologies in this section are often considered to be microworlds, in Papert's (1980) sense of the term (see Healy & Kynigos, 2010). We consider in this section papers relating to the use of DGEs (such as *Cabri*, *Sketchpad*, *Cinderella*, *Geometric Supposer*, and not applets in which a student can simply drag the vertex of a triangle). We then examine papers focussing on CAS, spreadsheets and graphing calculators.

Given the fact that DGEs are relevant not only for geometry, but also other topic that make use of geometric models, it may not be surprising that the research on the use of DGEs has broadened extensively. In her chapter, Mariotti (2006) focuses on the research involving DGEs in the context of the proving process. Some of the themes she discussed were also evident over the past decade. For example, Benítez Mojica and Santos-Trigo (2006) study the role of the DGE in high school students' formulating and verifying of conjectures; Rodríguez and Gutiérrez (2006) find that the use of DGE helps high school students identify conjectures, but does not help them find deductive proofs. Samper, Camargo, Perry and Molina (2011) focus on how the use of DGEs by university students supports their abductive inferences, which they claim to be an important component of the process of justification.

Shifting away from prior concerns, but still focussed on the processes of experimenting and justifying, Leung and Or (2007) consider the diachronic nature of DGE objects and its effect on the way a secondary student engages in oral explanation and written proof, arguing that a more diachronic discourse may help bridge the empirical-theoretical gap that is much discussed in the literature. Also with an interest in the importance of invariance in the use of DGEs, Baccaglini-Frank, Mariotti and Antonini (2009) provide a framework that describes the different invariants and show that students' interpretations of these invariants play a stronger role in their process of discovery than in the generality of proofs.

The emerging importance of curriculum materials that support the use of DGEs can be seen in Patsiomitou and Emvalotis (2010), who show how secondary school students developed their van Hiele levels as well as their geometric reasoning skills through the use of a DGE, and place particular emphasis on the role of the *Sketchpad*-based curriculum materials that were designed to promote conceptual change. Focusing more deeply on student's instrumental genesis form the previous study, Patsiomitou (2011) identifies two obstacles that students face when they construct dynamic diagrams, and also show how students' theoretical dragging can provide non linguistic *warrants*—in Toulmin's sense—for the construction of a "dynamic proposition," which is empowered in a dynamic geometry environment. The latter finding is an attempt to establish dynamic geometry as a context for geometric investigation that is strongly related to but also distinct from traditional Euclidean approaches, a goal that was first articulated by Goldenberg and Cuoco (1998), but that has not been much pursued, perhaps because of the pressure to use and evaluate DGEs within the context of current school curricula.

New themes in the research on the use of DGEs include a wider application in terms of strands of the curriculum and grade levels. For example, the following papers focus on the use of mathematical processes that are not specific to the proving process: Haja (2005) and Haug (2010) investigates the use of DGEs in problem solving, as do Jacinto and Carreira (2013), who study on an out-of-school problem solving context; Presmeg, Barrett and McCrone (2007) focus on the fostering of generalisation and the development of student norms; Sinclair and Crespo (2006) show that pre-service teachers who explore a DGE-based mathematical situation engage in better problem posing than when they are given paper-and-pencil contexts. Several papers have focused on specific types of reasoning that might be promoted or encouraged by the use of DGEs: Papageorgiou, Monoyiou and Pitta-Pantazi (2006) study how the use of DGE helped students overcome the intuitive rule "more A – more B" in the context of finding the sum of the angles of a triangle and a quadrilateral; Baccaglini-Frank, Antonini, Leung and Mariotti (2011) investigate how the use of DGE can help students understand proofs by contradiction; Samper, Perry, Camargo, Molina and Echeverry (2010) focus on students' conditional thinking.

Because DGEs offer visual, kinetic, graphical and symbolic forms of expression, researchers have also investigated how these different forms are used and combined by students. For example, Patsiomitou and Koleza (2008) study the use of linked representations in a DGE to develop students' geometric thinking; Ofri and Tabach (2013) inquire into the effect of the multiple representations for a dyad of eight grade students exploring a problem situation related to functions; Ferrara and Ng (2014) investigate the multimodal affordances of a multitouch DGE that involves not only visual and symbolic representations, but also gestural interactions of students exploring calculus concepts; Similarly, Ng (2014) investigates high school students investigating calculus concepts using a multitouch DGE. The theme of how teachers and students talk about and share mathematics can be seen in the papers by Berger (2011), who focuses on discourse with DGE classroom, and Morera and Fortuny (2011), who investigate group discussion following the use of DGE.

A particularly important theme examines the way students interpret the dynamic images found in DGEs. While researchers have pointed to the power of continuously changing shapes (like the triangle whose vertex is dragged to form a whole family of triangles), it is still not clear how exactly students made sense of these images (as many example, or as an invariance) and what effect this has on their thinking (see Battista, 2008). Talmon and Yerushalmy (2006) examine students' conceptualization of dragging studying the tension between the appeared figure, the figure-image and the additional knowledge that the software designers computed, and determines the behaviour of the dragged figures. Olivero (2006) studies the use of showing/hiding constructions as a way of helping students see geometrically significant features of a construction. Patsiomitu and Emvalotis (2009) propose the LVAR model of visualisation to study how students' use of a DGE affects their spatial reasoning.

Beyond the traditional focus on high school geometry, the use of DGEs has been studied in a growing variety of contexts. Gol Tabaghi and Sinclair (2010) described sketches that were designed to help undergraduate students learn about eigenvectors and eigenvalues. Lew and Yoon (2013) focus on quadratic equations; Ferrara and Mammana (2014) make use of a 3-D DGE to examine how students make sense of moving back and forth between 2-D to 3-D shapes. At the elementary school level, Olive (2011) describes upper primary school children working with fractions on a dynamic number line. Other papers already mentioned include: Sinclair, Moss and Jones (2010) on grade 1 children explored intersecting and parallel lines; Sinclair and Kaur (2011) on grades 2/3 children's thinking about symmetry; Kaur (2013) on kindergarten children's thinking about angle; Kaur and Sinclair (2014) on grades 2–3 children learn about triangle identification. Finally, Papadopoulos and Dagdilelis (2009) report on grades 5 and 6 students using a DGE in combination with other digital technologies including *MSPaint* and *GeoComputer* (an electronic geoboard) to explore the concept of area. The authors found that the 36 students using the digital technologies developed more strategies for estimating the area of irregular shapes than the 62 students working in a paper-and-pencil environment. Huang (2012) compared the results of fifth grade students using an enriched curriculum that integrated *Cabri-3D* to teaching volume measurement with students using only physical manipulatives and found that the former group performed better on a follow-up test.

Researchers have also continued to investigate the use of other technologies such as symbolic manipulators (in the form of CAS), spreadsheets (focusing on manipulations of numbers) and graphing algebra tools (graphing calculators or applications emphasizing multiple representations of expressions in algebra or calculus). Kieran and Saldanha (2005) study secondary students' reasoning about equivalent algebraic expressions and found that getting them to interpret their own work with the CAS was beneficial, Drijvers et al. (2006) argue that confronting CAS with paper-and-pencil algebra can be powerful in students' developing techniques that can lead to theoretical thinking. Yiasoumis, Mattheou and Christou (2009) show that 11th grade students improved their performance on analytic tasks related to the concept of limit after using a CAS-like applet. Sacristan and Kieran (2006) investigate student conjecturing with a CAS, describing how one high school student used CAS to test a conjecture regarding the general formula for x^n+1. Based on an analysis of a pair of university students working on quadratic approximations, Berger (2008) argues that CAS can facilitate the learning of mathematics because it functions as a tool for semiotic activity. Sevimli and Delice (2013) find that CAS helps students develop their concept image of definite integrals more than in a paper-and-pencil environment. In 2014, Sevimli and Delice conducted a comparative study and found that students in CAS-integrated group had less misconceptions related to "negative area" in the definite integral.

Other research elaborating on the use of well-known digital technologies includes spreadsheets: Ainley, Bills and Wilson (2005) focus on the design of "purposeful" tasks using spreadsheets and find that the secondary students' perception of the purpose of the task affected the way in which the technology was used by the students, which in turn affects the way the tool becomes *transparent* (in the sense of Lave & Wenger, 1991) for the students; Son and Lew (2006) describe the way in which six grade 10 students working in pairs used a spreadsheet to make hypotheses about a complex real-world problem, which would not be easy to solve in a paper-and-pencil environment, and the process in which they justify their discoveries—the teacher was found to play an important role in this process of justifying; Calder (2009) examines the visual tensions when mathematical tasks encountered in a spreadsheet; Challis, Jarvis, Lacicza and Monaghan (2011) found that spreadsheets were the digital technology that fit best with the objectives of the activities of undergraduate staff and students.

Another type of digital technology that continued to receive attention during the past decade is the graphing calculator: Pierce's (2005) study the influence of three secondary teachers, who used a particular textbook designed to teach linear functions through the use of graphing calculators, on students' algebraic understanding; Lew and So (2008) investigated two high school students' problem solving with a graphing calculator and found that the digital technologies help them make empirical and deductive justifications; Getenet and Beswisk (2014) study pre-service teachers' use of a graphing calculator to understand and describe the properties of logarithmic functions as the bases vary; Swidan and Yerushalmy (2013) study the personal meanings for the definite integral that arise from high school students' use of *Calculus Unlimited* (CUL), which is a dynamic and multi-semiotic graphing environment. Instead of focusing on a particular mathematical topic, Hegedus, Dalton, Cambridge and David (2006) describe the new patterns of participation (how students interact amongst themselves and with the teacher) that emerge out of students' use of *Simcalc* in a graphic calculators' networked classroom.

In a few studies, a combination of digital technologies was involved. For example, Fuglestad (2005) studied students' choice of software tools on given problems and found that only one half of them were able to choose the most appropriate tool for the problem; Jungwirth (2006) investigate a classroom in which multiple digital technologies were used, including CAS, spreadsheets and DGE, and found that when these digital technologies were used, "doing" dominated in the classroom, whereas "done" dominated the classroom when no digital technologies were used; Hong and Thomas (2013) studied undergraduate students' difficulties with a local or interval perspective of functions in a classroom where the lecturer used DGE, CAS as well as graphing software. Also focusing on an undergraduate context in which multiple digital technologies are used, Oates, Sheryn and Thomas (2014) study the design of a digital technology integrated undergraduate course in view of identifying its strengths and weaknesses. Finally, Pinkernell and Bruder (2013) focus on whether the use of digital technology in general (especially CAS and graphing calculators) in the classroom hinders the mastery of basic concepts and argue that it does not as long as teachers use methods that also involve mental mathematics exercises as well as repetition.

Despite the recent resurgence of programming in education, there have been few studies in the PME proceedings. Baccaglini-Frank, Antonini, Robotti and Santi (2014) report on a students' use of *Mac-Trace*, which is a drag-and-drop Logo-based programming environment. Cho, Song and Kim (2007) report on the design of an integrated DGS and Logo microworld that enables the creation of planar curves. At the undergraduate level, Buteau, Muller and Marshall (2014) report on a study conducted with students who were involved in a year-long course in which they designed, programmed and used microworlds to learn and do mathematics. They found that through these activities the students developed the fifteen theoretically identified competencies (such as "to programme mathematics" and "to visualise mathematics") and that choosing their own topics was key to the development of these competencies.

Several PME papers have focused on the use of MaLT, which is a 3-D programming environment that also enables dynamic manipulation. For example, Kynigos, Psycharis and Latsi (2009) used a microworld developed in MaLT with 20 grade seven students to explore their meanings of the concept of angle. They argue that the tools in the microworld enabled the students progressively coordinates the different facets of the concept in 3-D space. Continuing this work, Latsi and Kynigos (2011) studied 12 year-old students' novel meaning of angle as a result of changing virtual perspectives of their constructions. Also using MaLT, Moustaki and Kynigos (2011) studied engineering students' activities on three phases of tasks and showed how their success was related to the opportunities for visualising and spatial reasoning afforded by the microworld design. Also focusing on both 3-D and programming, Markopoulos, Kynigos, Alexopoulou and Koukliou (2009) used the *Cruislet* microworld, which uses GIS (geomgraphical information systems) with a Logo programming language and enables learners to navigate 3-D geometrical spaces. The authors describe how 12 grade 10 students constructed meanings related to vectors, coordinates and functional relationships.

Several new microworld environments that are not 3-D have also been developed and researched over the past decade. Geraniou, Mavrikis, Kahn, Hoyles and Noss (2009) describe the development of a microworld called *eXpressor* for expressing generality as well as insights from trials with students. Their iterative design highlighted five main elements of the microworld: (1) the need to provide a rationale for generality; (2) offering model construction and analysis simultaneously; (3) scaffolding the transition from number to variables; (4) attending to specific cases while also gaining awareness of the general; and (5) reflecting on expressions that have been derived from the model construction.

Also with a focus on algebra, Hewitt (2010) describes the *Grid Algebra* environment, which is designed to help students learn to solve equations and develop fluency with formal algebraic notation. He shows how a carefully sequenced set of activities helped a class of 9–10 year-old students gain confidence in the reading and writing of notation, despite their young age. With a focus on algebra, but at a higher level, Psycharis, Moustaki and Kynigos (2009) studied 17 year-old students' use of the *MoPiX* environment, which enables students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. They describe the students' development of a structural conception of the notion of equations.

In the context of inferential statistics, Prodromou (2013) describes a study conducted with 14 to 15 year-old students using the dynamic statistics software *Tinkerplots*. The author describes four phases of informal inferential reasoning in statistical modelling and simulation process and shows how the software helped foster the development of the students' logic of inferences over these phases.

Task-Embedded Digital Technologies

As evidenced by the recent ICMI Study on Task Design, this is an area of growing importance. An upcoming volume focused on task design and technology (Leung & Baccaglini-Frank, to appear) should provide a state-of-the-art overview of theoretical and empirical developments in this area. In the PME proceedings over the past decade, a small number of papers have focused explicitly on task design. One was Leung (2014), who considered task design with a DGE, with a specific focus on how tasks can be designed to help students notice and appreciate invariance. Yerushalmy and Shubash (2009) examine the unique features of tasks constructed using interactive diagrams to examine the learning of 15 year-old students working on two algebra and functions activities (one with which they were familiar (quadratic function) and the other not (difference function)). In the computational environment tasks come with examples that are either semi-random or can be generated by the user. The researchers found that random, non-structured sequences of examples were more difficult to acquire at first, but that the determined students could use them to leverage conceptual understanding.

Salle (2014) investigates the learning process of two sixth-grade students using animated worked-out examples in the domain of fractions. Using some quantitative data, the author argues that the worked-out examples are effective for pair learning and that the self-explanation prompts given to the students played an important role in their learning, as exemplified by a qualitative analysis. Naftaliev and Yerushalmy (2009) investigate whether and how tasks designed upon printed diagrams vs. interactive diagrams, or upon video clips vs. interactive animations, create different contexts for mathematics problem solving. Concentrating on high-school tasks in functions and algebra the researchers explore the ways in which problem solvers use sketchy interactive diagrams designed to encourage the problem solver to transform sketchy information into conceptual learning.

Jones and Pratt (2005) used the *Visual Fractions* microworld, in which users can create, manipulate, destroy and connect fractions (as well as operations, relations regions and Boolean flags) on the screen, to investigate the meanings for the equal sign that two thirteen year-old girls constructed. They found that the microworld, along with the tasks they were given, enabled the girls to create the meaning of equivalent for the equal sign. Lavy (2006) used a microworld that was created within the Logo-based Microworld Project Builder environment. She found that the 14 students who solved the problem in a traditional environment had the same percentage of success as the 78 students who saw the visualisation of the problem in the microworld, but that the visualisation distracted the latter group's attention away from the proper handling of the solution. Lee, Cho and Lee (2012) study the use of a combinatorial microworld. They find that the embodied symbols help students perform better on most tasks when comparing pre- and post-tests.

Suh and Moyer-Packenham (2007) compare the use of physical and virtual manipulatives in 36 grade 3 students' learning of fractions and found that students using the virtual manipulative performed showed statistically significant positive result. Using Dual Code theory, they also showed that the students' using the virtual manipulatives used more pictorial and numeric representations in explaining their work and the ones that did not used more algorithmic approaches.

Finally, Fesakis and Kafoussi (2009) provide a good transition between this section and the next in that it examines the effect of two combinatorics microworlds (using the *Scratch* programming environment) on 30 kindergarten children's activities, where one of the microworlds is evaluative in that is provides scaffolding and feedback on students' mistakes and the other is not. They found no significant difference in children's performance on either microworld, nor on children's performance with physical manipulatives, but note that the children using the evaluative microworld did not make use of the scaffolding.

Evaluative Digital Technologies

Evaluative feedback has long been a feature of computer-based learning, but has been restricted to providing right/wrong, single-input feedback on solutions to exercises or problems. Such evaluative feedback does not fit well with current interactive and collaborative technologies, which support multimodal forms of interaction, and not just symbolic ones. Over the past decade, a few PME papers have studied novel approaches of evaluative feedback (for the student as well as for the teacher).

There were several papers focused on *Aplusix*, which is a learning environment for algebra that includes embedded tasks as well as evaluative feedback Chaachoua, Bittar and Nicaud (2005) describe the work they carried out in trying to identify students' conceptions of linear equations in order to provide diagnostic information that could eventually be used to automatically produce suitable tasks for students. They tested their model with 342 Brazilian students aged 13–16 year-old and found that it cohered well with the teachers' opinions of the students' algebraic understanding. Maffei and Mariotti (2006) conducted a teaching experiment with three 9th grade classes in Italy and found that the use of *Aplusix* was effective not only in terms of decreasing the number of errors that students made, but mainly at improving students' attitudes towards their errors. Another diagnostic model was used by Gonzalez-Calero, Arnau, Puig and Arevalillo-Herráez (2013). They investigate two pairs of 15–16 year-old students' performances on an intelligent tutoring system (*Hypergraph Basic Problem Solver*) focused on solving arithmetic-algebraic word problems.

Other evaluative approaches aim to provide students formative feedback. Wu, Wong, Cheng and Lien (2006) described a computer-assisted learning environment called *InduLab* in which students learn how to perform inductive tasks (such as discovering properties of the angles of a triangle). It provides several tools for helping the learner to induce target properties, including a data table. Fujita, Jones and Miyazaki (2011) describe a web-based flowchart platform designed to help secondary school students overcome circular arguments in mathematics. They claim that with the support of the teacher on the structural aspects of a proof, the feedback from the platform helped learners begin to overcome circular arguments. Kelly, Heffernan, Heffernan, Goldman, Pellegrino and Solfer-Goldstein (2014) conducted a randomised-controlled trial of 63 seventh graders in which 30 students used a web-based homework system that provided immediate feedback as well as detailed item reports to teachers. They found that the students using the web-based homework outperformed the others in the post-test that was given, which the authors explain in part by the fact that these students could attempt each homework question multiple times.

Using activity theory, and drawing on the assessment literature, Broughton, Hernandez-Martinez and Robinson (2013) investigate the used of computer-aided assessment (CAA), which provides evaluative feedback on students' solutions. They found that such evaluative might be effective for low-level goals, but that it does not inspire students to continue the learning cycle and explore new learning goals.

Critical Review of the Three Types of Technology Broadening

Especially within the first type of technology broadening (*open digital technologies* and *task embedded digital technologies*), there has been a strong presence of research at the elementary school level. Several of the examples in the first category involved digital technologies that were initially designed for use at the high school level, such as DGEs. Others arose principally from changes in digital technology hardware, such as the development of programmable toys and of touchscreen devices. In contrast, all of the examples in the third type of technology broadening (*evaluative digital technologies*) focused on the secondary school of undergraduate level. This may be related to the very different assessment practices in the later grades.

Almost all the digital technologies that were studied were mathematically specific. There are a growing number of digital technologies, however, especially games, that are also being brought into the mathematics classroom. In addition to the Sim 2 paper that we discussed earlier, another PME paper related to non-mathematical games was that of Lowrie, Jorgenson and Logan (2012), who investigate gender differences in game that primary school children like to play. They found that girls preferred to play games that require logic and problem solving and boys preferred games that contain maps—these gender differences were more pronounced in non-metropolitan locations.

TEACHING WITH TECHNOLOGY

Compared with research on student learning with technology, research on the teacher has not been as well developed. This can be seen by the nascent theory development in this area that began with the descriptive framework of TPACK, which includes technology knowledge as additional to pedagogical and content knowledge that teachers may use in their teaching practices. More recently, as evidenced in Clark-Wilson, Robutti and Sinclair (2014), new theories that provide a more analytic lens on the role of the teacher in teaching with digital technology, stemming largely from the theories of instrumental genesis and instrumental orchestration, but also in Ruthven's (2014) framework for analysing the expertise that underpins successful integration of digital technology.

In the PME papers over the past decade, researchers have examined both the knowledge that teachers might need to teach with technology and the way teachers learn to use digital technologies in the classroom, both in pre-service and inservice contexts. There have also been studies that highlight some of the productive practices that teachers engage in when teaching with digital technologies.

Inservice and Pre-Service Teacher Knowledge as it Relates to Digital Technologies

Several papers investigated teachers' "pedagogical content knowledge" (PCK) as it relates to particular topics or technologies, often pointing to their deficiencies. For example, Akkoç, Bingolbali and Ozmantar (2008) point to the difficulties that a pre-service teacher has with the derivative at a point. The authors assert that the teacher's limited content knowledge, combined with her lack of experience in using graphing calculators, prevented her from appreciating how the graphing calculator could be used to teach the concept.

Bretscher (2012) presents a case study of a secondary school teacher using a spreadsheet to teach linear sequences. She summarises the different advantages that the teacher noticed (such as feedback to the student), but also highlights certain potentially problematic aspects of the teacher's actions, such as her choice to not point out the differences between standard notation and spreadsheet notation. In addition, Bretscher argues for a reframing of TPACK, writing that it "may be better understood as a transformation and deepening of existing mathematical knowledge rather than as a new category of knowledge representing the integration of technology, pedagogical and mathematical knowledge" (p. 83). In a similar vein, Rocha (2013) offers an alternate framework for investigating the teaching mathematics with technology, which she calls *Knowledge for Teaching Mathematics with Technology* (KTMT), and which she argues provides a more dynamic model of teacher growth than does TPACK.

Teachers' Views of Digital Technologies

In this subsection we consider papers concerned with teachers' perceptions of digital technologies in terms of how they viewed their affordances or potential use, as well as attitudes about the use of digital technology in teaching mathematics.

Ball and Stacey (2005) investigated four teachers' initial use of CAS. They describe the issues that the teachers identified at the end of the school year, such as the lack of intermediate steps available when CAS is used and the pressure of external examinations. Ball and Stacey (2006) focus on one teacher's early use of CAS to show how he moved from a functional to a pedagogical use of CAS and how he came to appreciate the pedagogical possibilities of CAS. Pierce and Stacey (2009) studied the classroom implementation of TI-Nspire in classrooms where lesson study was used, which gave them high quality insight into the pedagogical affordances of the digital technology. They focus especially on the availability of multiple representations.

Berger (2012) shows how two inservice teachers differed in the way they used a DGE; one used it as a tool with which to make sense and the other as a tool with which to explore various aspects of given functions. Lagrange (2011) shows how teachers differ in the way they take up innovative software. Tan and Forgasz (2006) compare teachers' view of the use of graphing calculators in Singapore and Australia, showing that the latter teachers were more enthusiastic about their use, perhaps because they are mandatory in the grade 12 examinations in Victoria, Australia. Kuntze and Dreher (2013) found that the 39 pre-service and 65 inservice teachers in Germany differed in terms of the use of digital technology, with the former being very optimistic and the latter showing very little use.

Taking a more longitudinal approach, Thomas' (2006) research spanned ten years and involved over 300 secondary New Zealand teachers' attitudes towards the use of digital technology. He found that while there are many more computers available in schools, access remains a key obstacle for teachers. He also found a change in terms of the kind of software teachers used, away from content-specific programs and towards generic software—especially the spreadsheet. Finally, he argues that a teacher's attitude is a key factor in whether or not teachers integrate the use of digital technology in the mathematics classroom.

Teacher Learning and Professional Development

There have been several studies on the use of digital technologies in both pre-service and inservice education. In terms of the former, researchers have examined pre-service teachers in on-line environments (Borba & Zulatto, 2006; Goos & Bennison, 2006; Flores, Escudero, & Sánchez Aguilar, 2014) and in communities of practice (Amado & Carreira, 2006). In terms of more specific topics Ozmantar et al. (2011) studied the development of pre-service teachers in linking multiple representations; Sinclair and Crespo (2006) investigated the use of technology to help pre-service elementary teachers pose better mathematical problems; and, Okumus and Thrasher (2014) focused on pre-service elementary teachers' use of the dynamic statistics software, *Tinkerplots*.

In terms of inservice teachers, researchers have investigated teachers' professional development in relation to the use of digital technology within a community of practice (Fuglestad, 2007; Besamusca & Drijvers, 2013) as well as teacher learning through virtual simulations (Meletiou-Mavrotheris & Mavrou, 2013). Researchers have also studied inservice teachers' problem solving capacities (Haja, 2005) as well as how they use dragging as an organizer for conjecturing in a DGE (Camargo, Samper, Perry, Molina, & Echeverry, 2009). Psycharis and Kalogeria (2013) undertook a case study of the documentational work of one inservice teacher preparing a lesson for his colleagues. The authors argue that the two new documents that the teacher developed were constrained by three factors; (1) the inherent difficulty of developing teaching materials; (2) the difficulties in knowing the affordances of the microworld that had been chosen; and (3) the teacher's knowledge and experience,

Teachers' Practices Involving Digital Technology

Instead of trying to change teachers' understanding or use of digital technology, several studies have been concerned with teachers' current practices. These studies usually aim to highlight the complexities of teaching with technology as well as the novel practices that teaching with technology entail. For example, Jungwirth (2006) examines how teachers are using a variety of different digital technologies (including CAS, spreadsheets and DGEs) in their practices and how this differs from their off-line practice. Lerman and Zevengergen (2007) examine current teachers' use of interactive whiteboards, highlighting how they differ from their anticipated use. Sinclair and Wideman (2009) study how secondary teachers use TI-navigator, and show how changes in their practice (in relation to criteria that the authors developed for effective use) were related to the teachers' conception of mathematical teaching.

Kieran, Guzman, Boileau, Tanguay and Drijvers (2008) examine teachers' orchestration in a whole class CAS discussion, highlighting the specific moves that the teacher makes in facilitating this discussion. Arzarello and Paola (2007) highlight the role of the teacher in the semiotic game of the classroom, that is, in using the different semiotic mediators that emerge from interactions with digital technology. Hollebrands, Cayton and Boehm (2013) highlight the pivotal teaching moments in secondary classrooms involving the use of DGE (such as technology confusion and incorrect technology use), describing how teachers respond to these moments and what impact it might have on the students.

In the future, the novel approaches, both theoretical and methodological, of research such as Herbst and Chazan (2015) would seem worth pursuing in the context of studying teacher practices. Their approach involves using digital animations of classroom scenarios designed to breach certain norms of teacher practice—these norms then become topics of discussion and reflection, as well as sites of insight into teachers' practical rationality.

CONCLUSION

In this final section we highlight some overall issues and point to some productive directions for future research. Following the structure of our chapter, we begin with the issue of innovation, which we found to be less well-represented than we had anticipated. There are several reasons why this might be the case, one being that it can be difficult to be successful in publishing PME papers that involve the use of innovative technology because the theoretical implications of the innovation are not yet well understood—and yet a solid theoretical framework is a strong requirements for PME papers. It may also be that innovative technologies tend to require detailed explanations, which leave little room for the inclusion of other sections of the typical PME paper. Finally, while innovative technologies would be described and critiqued within a PME paper, the mathematics education literature does not yet have a strong tradition of technology design (see Schoenfeld, 2008). Indeed, only one paper focused exclusively on the very important and theory-laden choices involved in designing new digital technologies. One way of increasing research on innovative technology may be to connect more strongly with some of the research in technology education, where the focus on design is more developed.

An issue related to design is the fact that there are an increasing number of new digital technologies that frequently get called by the same name (such as DGEs) but that can differ substantially. Even small differences can have an important effect on teaching and learning (see Mackrell (2011) for a discussion of the small by significant differences between *Cabri*, *Sketchpad, Cinderella* and *Geogebra* and well as the earlier Talmon and Yerushalmy (2004) on the different understanding of dragging that students develop when using *Cabri*, *Sketchpad* or *Supposer*). Researchers need to pay attention to these differences and, perhaps, to find more helpful ways—especially for teachers—of communicating their results.

In terms of theory, we have noticed a tendency for researchers to combine two or more theoretical perspectives in order to adequately account for their research contexts. Sometimes general theories of learning must be combined with theories that provide more of a focus on the use of tools and their role in teaching and learning. We see a need to better articulate theories of learning with theories of tool use, which is currently done, for the most part, by combining approaches. Also, perhaps more importantly, we see a need to find ways of incorporating affective dimension, which is one of the long-standing and oft-mentioned motivations for using digital technologies in the first place.

Connected to innovation, but also to theorizing, is the issue of methodology. Again, there was only one paper in the past decade that focused specifically on new methodological approaches to studying the use of digital technology in mathematics education. This was an area highlighted by Lerman (2006) as being an area of future potent research. With the increased use of digital tools used to collect and analyse data, and the ever-increasing sophistication of theory, we hope to see more methodological innovation in the future.

Given the changes we have seen over the past decade, we anticipate a continued growth in the development of small, tangible software that are geared to niche "markets" such as students with disabilities and homeschoolers and mixed classroom. While in its infancy now, we also anticipate a burgeoning of design and research on e-textbooks as well as on new forms of digital assessment. This will have an impact on future research on teachers using technology in the classroom, in part because there will be changing demands on their expertise: while they may not have to learn how and when to use powerful, comprehensive packages, they will have a new challenge of choosing and integrating a variety of different tools. When these tools become integrated in the form of e-textbooks, there will be new research questions that arise about the role of the teacher in the digital classroom.

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8. LANGUAGE IN MATHEMATICS EDUCATION RESEARCH

INTRODUCTION

In the past few decades, language has become an active focus of investigation in educational research, including research in mathematics education. Such a focus is a symptom of a relatively recent paradigmatic shift whose chief characteristics are a new understanding of the student and an increasing awareness of the complexities of learning contexts, such as, notably, the complexities arising from cultural and linguistic diversity. This paradigmatic shift appeared as the field attempted to move away from the two main models that emerged and evolved during the educational progressive reform of the early 20th century (see also Lerman, 2006).

The first of the two main models was the "transmissive model." With its intellectual origins in behaviourism, this model was promoted by bureaucratic pedagogues who focused on implementing mass education to efficiently address the demands of industrial and business production (Tyack, 1974). Two of the contemporary heritages of this model are a methodical and detailed curriculum and the obsession with systematic "objective" assessments. The second main model was the "child-centered" educational model. Intellectually rooted in a romantic pedagogy, this model focused on the child's interests and intellectual potential. "Progressivism," as this model came to be known, promoted the idea that "knowledge is … [a] personal acquisition, obtained by learning from experience" (Darling & Nordenbo, 2002, p. 298) and meant "promoting discovery and self-directed learning by the student through active engagement" (Labaree, 2005, p. 277).

Although language is not absent from these models, it does not appear as a central research problem. And when it does appear, it is generally related to problems surrounding the investigation of students' conceptualizations. Language is considered as a kind of *window* to see indirectly what is happening in the student's mind as, for instance, in Piaget's conservation tasks. When students' conceptualizations are perceived to be incorrect, language is often then seen as an obstacle or barrier to the effective communication of the desired knowledge or structures. Language, however, is clearly more than a window or an obstacle; language, talk, text and the production and interpretation of symbols are integral to the creation of learning, teaching and assessment, particularly in mathematics. In Piaget's conservation tasks, for example, language is not simply a neutral conduit for conveying thoughts between experimenter and subject; the tasks are constituted through linguistic

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 275–313.

processes. For language to move to the forefront as an educational research problem on its own, it was necessary to move beyond the conception of humans as Cartesian problem solvers promoted by progressive models. This move, from which emerges the idea of *homo communicans* and that opens up new spaces in which to conceive of the student in new terms, was not merely accidental. It responded to fundamental changes of a social, cultural, historical, and economic nature. As Paul Kelly puts it in his book *Multiculturalism Reconsidered*,

> With the retreat of European empires […] and, much more significantly, with the collapse of the old European empires following the Second World War, there has been a transformation of that earlier colonialist legacy […]. European states—especially the old colonial powers such as Britain, France, Holland, Belgium and, to a lesser extent, Spain and Portugal—became multicultural states as a consequence of colonial retreat […] In the British case, the retreat from empire began a process by which immigration from former colonies transformed the country into a multiethnic and multiracial society. (Kelly, 2002, p. 2)

The result is that today "All modern states face the *problems* of multiculturalism even if they are far from endorsing multiculturalism as a policy agenda or official ideology" (Kelly, 2002, p. 1; emphasis in the original). Although multiculturalism was a predominant feature of life in Ancient Greece and Rome, contemporary multiculturalism with its central interest in language is truly new. As Gress (1999) points out, "The [ancient] Greeks never learned foreign languages" (p. 565). He goes on to say that

> For the Greeks of the archaic and classical eras—from Homer to Alexander—encounters with the other were encounters with the marvelous or the dangerous. They took place in the framework of an evolving anthropology of curiosity and difference, accompanied and complemented from Herodotus on by an overarching dichotomy of Greek versus barbarian. (Gress, pp. 562–563)

To understand contemporary multiculturalism's interest in language we should add that the paradigmatic shift alluded to above has also been entangled with changes in new forms of production and colonization brought forward by global capitalism where "money, technology, people and goods […] move with increasing ease across national boundaries" (Hardt & Negri, 2000, p. xii). These new global forms of production have been accompanied by a variety of unprecedented kinds of virtual interaction and communication. Within the new global context of production and modes of human interaction, individuals from other cultural formations have ended up acquiring a central place—an ontological one, in fact—in the manner in which individuals have come to understand themselves. However, "the appearance of the *Other*," as we may term it, has not been a neutral experience. It has brought with it new questions about identity, power, ethnicity, multiculturalism, multilingualism, etc.

Perhaps the contemporary global context of production in which we live is leading us to experience a somewhat similar historical phenomenon as the one the 16th century Spaniards experienced when they confronted the multitude of communities of what is now called the American continent. That is, when they discovered a substantially different *other* and, along with it, they also discovered that gods, customs, morals, language, and worldviews may have a different order than the one they grew up with and knew. In his book *The Conquest of America*, the Bakhtinian specialist Todorov (1984) points out the strong need that Christopher Columbus felt to rename all things. For Columbus, language was an instrument through which things were possessed and individuals subjugated. Talking about the first island he found in his travels, Columbus said, "I gave [to the first island] the name of *San Salvador*, in homage to His Heavenly Majesty who has wondrously given us all this. The Indians call this island Guanahani" (Todorov, 1984, p. 27). And he went on to tell the King the names he had given to the other islands. Todorov comments:

> Hence Columbus knows perfectly well that these islands already have names, natural ones in a sense (but in another acceptation of the term); others' words interest him very little, however, and he seeks to rename places in terms of the rank they occupy in his discovery, to give them the *right* names. (Todorov, 1984, p. 27; emphasis in the original)

Naming things—which Columbus did through notarial acts written ceremoniously in front of the perplexed natives—provided him with a means to possess things and people. The difference between us and Columbus and the conquerors is that we are asking questions about power and culture within an array of new sensibilities. How, in our contemporary multicultural settings—in a culturally diverse classroom, for instance—could language not be an instrument of subjugation and possession? We will come back to this question later. For the time being, let us summarize the previous comments by noting that the invention of *homo communicans*—that is, the constitutive insight that what humans are is deeply entangled with, and rooted in, the individuals' historical and cultural communicative relationships with others—has not been embedded in epistemic questions only (e.g., how we name things, how we know things) but also in questions of alterity, power, identity, culture, and politics.

In this chapter, we review PME research on language from the past 10 years and offer a critical appraisal of this work. To begin, in the next section we set out an overview of the relevant research, looking at both the major themes that have appeared, as well as the different theoretical approaches to language that have been deployed. In the remaining sections, we discuss three themes in more depth: the role of language in mathematics conceptualization; cultural dimensions, such as the role of language in mediating between the individual and society, and, in particular, questions of power and authority in mathematics education; and language diversity in learning and teaching mathematics.

OVERVIEW

In the first PME Handbook, there was no chapter explicitly devoted to language as a focus of research. Questions of language are most salient in Lerman's (2006) chapter on socio-cultural research and in Gates's (2006) chapter on equity and social justice. Lerman's (2006) categories of socio-cultural research, for example, include:

- Cultural psychology, including work based on Vygotsky, activity theory, situated cognition, communities of practice, social interactions, semiotic mediation.
- Ethnomathematics.
- Sociology, sociology of education, poststructuralism, hermeneutics, critical theory.
- Discourse, to include psychoanalytic perspectives, social linguistics, semiotics. (p. 351)

It is apparent even from these brief characterizations that language is pretty central both explicitly (e.g., social interactions, discourse, semiotics) and more implicitly (e.g., as a key aspect of both Vygotskian and poststructuralist theory). Meanwhile, Gates (2006) includes a brief discussion of "Language, discourse and critical consciousness" as part of a section on the third decade of PME. In this section, he highlights contributions on language, the politics of discourse, and critical studies, with most emphasis on the issue of multilingual classrooms.

It seems, then, that in the first 30 years of PME, questions of language can best be described as an emerging theme: both Lerman and Gates highlight their absence in the early days of PME and their increasing presence in the third decade. In the past 10 years of PME, however, there are more than 150 research reports, contributions to research forums and plenary lectures devoted to language-related topics. Given the linguistic turn described in our introduction, it is perhaps no surprise that PME research has attended to the kinds of questions we have mentioned.

For this chapter, then, we have compiled a corpus of contributions to PME conferences from 2005 to 2014, consisting of research reports, plenary presentations and research forums. Research reports are, of course, the primary form of contribution to PME and most reflect the work of the members. Plenary presentations represent substantial contributions that discuss specific topics in more depth. Research forums offer multiple perspectives on a given topic and, although individual contributions can be somewhat brief, the overall contribution of a research forum can be substantial. We will refer to all three forms of contribution as 'papers'. We did not include short oral presentations, posters, discussion groups or working groups, since these activities are only represented by brief, single-page reports that lack important detail.

The corpus consists of papers that explicitly address language issues, or for which language is a relevant feature. Papers that explicitly address language, for example, include contributions on multilingual mathematics classrooms, the role of mathematics classroom interaction in learning or teaching mathematics, or the nature of mathematical discourse. In some papers, some aspect of language is identified as a factor within a broader research focus, such as the role of classroom discussion within a paper focused on teaching for equity in mathematics outcomes. In total, the corpus consists of 153 papers, for which we conducted two classificatory analyses. The first analysis looked at the substantive focus of each paper. The second analysis looked at the theoretical framework used in each paper. In the rest of this section, we summarize the outcomes of these analyses, in order to situate the thematic discussions which make up the rest of the chapter.

For the first analysis, a general emergent classification, conducted with the help of NVivo software, examined research topics within the corpus. This analysis highlighted four main conceptual categories . Of course, conceptual categories may overlap. Table 1 provides the main conceptual categories along with their corresponding common core:

Table 1. Main categories and their core

Conceptual category	*Core*
Cultural dimensions	Focus on the relationship between individual and society; language, mathematics, and culture; cultural discursive routines; and multilingualism.
Language and conceptualization	focus on language and conceptualization; language in collective participation and in embodiment; representations and symbol use, and Vygotskian semiotics.
Mathematics as discourse and mathematics discourse	Focus on mathematics discourse or mathematics as discourse; the investigation of students' discourse and teachers' discourse.
Theoretical approaches to language	Focus on theoretical approaches to language; problems of hermeneutics, the theoretical relationship between language and thinking, and the role of language in the construction of knowledge.

Figure 1 shows the distribution of papers according to these categories. In the NVivo software terminology, a "source" corresponds to a document made up of excerpts coming from papers of a same PME conference. An excerpt (or "unit of sense," comprising usually one or more paragraphs from a paper conveying the general focus and meaning(s) of the paper) is called a "reference." The ongoing analysis of references gives rise to conceptual categories (called "nodes" and "sub-nodes") that NVivo displays in the form of a "tree" (see Figure 1). Thus, within the general category (or "node") "Language and conceptualization" is a sub-node called

"Representation and symbol use." From Figure 1 we see that, from the pool of the 153 surveyed papers, 13 references fall under "representation and symbol use" and that the 13 references come from 7 PME proceedings ("sources"). The topic that has the biggest number of references is "ideology, power, agency, and gender." It contains 28 references coming from the 10 PME surveyed proceedings. The NVivo distribution of nodes provides us with a possible view of the research landscape on language in mathematics education research.

Name	Sources	References
Cultural Dimensions	0	0
Cultural discursive routines	3	4
Ideology, Power, Agency, Gende...	10	28
Individual -Society Relationship	4	5
Language, Math and Culture	5	12
Multilingualism	1	1
Effect on T&L, Supporting T&...	7	10
Language-Math proficency	5	7
Task Design	1	2
Values and expectations	1	1
Language and conceptualization	6	20
Collective participation & Intera...	9	13
Embodiment	8	13
Language as a thinking tool	1	1
Math and everyday languages	8	20
Representations and symbol use	7	13
Vygotskian Semiotics	5	9
Math as discourse and math disco...	5	16
Students' discourse	2	2
teachers' discourse (effect, prof...	7	11
Theoretical	1	1
Critical Discourse	1	1
Hermeneutics	1	1
Language and Constr Knwlge	1	1
Thinking and Language	2	2

Figure 1. NVivo tree showing main nodes and sub-nodes, as well as sources and references in nodes and sub-nodes

Although this categorisation has guided our work in this chapter, we do not discuss every category or subcategory, preferring to restrict ourselves to areas in which the field has developed the most.

For our second analysis, we attempted to identify the principal theoretical orientation for each paper. This process was not always straightforward; some papers had a rather general theoretical basis involving references to a variety of ideas and authors, while a few papers had no identifiable theoretical framework at all. Nonetheless, the majority of papers referred to one or two key sets of ideas as the basis for the research they reported, in some cases fairly briefly as part of a literature review, in other cases more elaborately. Some papers, of course, were entirely devoted to theoretical considerations. We further grouped the theories into higher-order categories, although distinctions between the different groupings are not necessarily especially clear. Any approach with fewer than five instances was recorded as 'other'. The results are summarised in Table 2.

Table 2. Theoretical orientations in PME papers on language topics from 2005–2014

Theoretical orientation	
Sociocultural	48
Discourse analysis	22
Sociopolitical	11
Informal/everyday language	9
Teachers' practice	9
Constructivism	7
Embodied cognition	5
Other	23
Total	134

The most striking observation arising from this fairly crude analysis is the prevalence of sociocultural theory as the basis for much PME research on language in mathematics education. This finding is particularly striking given Lerman's (2006) charting of the then recent rise of sociocultural perspectives across all PME research reports, not just those focusing on language. This work falls largely within the Vygotskian tradition, in which language is understood as a tool, and as mediating between subject and object in the production of mathematical meaning (e.g., Berger, 2005).

In more recent years, Sfard's development of Vygotskian theory in particular has formed the basis for numerous PME papers. Sfard (e.g., 2008) argues that mathematical thinking is an individual form of mathematical communication, reflecting Vygotsky's claim that development occurs first intermentally and then intramentally. Sfard's approach develops this idea in terms of participation in mathematical discourse as forming the basis for individual mathematical cognition, with learning conceptualised as change in discourse. Sfard has subsequently proposed a categorisation of mathematical discourse into four aspects: endorsed narratives, routines, word use and visual mediators. This work has informed almost 20 research contributions at recent PME meetings, including work on dynamic geometry environments (Sinclair & Kaur, 2011; Berger, 2011; Ng, 2014), fractions learning (Wille, 2011), the concept of limit (Güçler, 2011), and the concept of square root (Shinno, 2013). Sfard's work on identity in mathematics has also informed a number of contributions (e.g., Nachlieli, Heyd-Metzuyanim, & Tabach, 2013).

Several other interpretations of sociocultural theory have been proposed and used in the past 10 years. Radford has an approach that draws on semiotics, embodied cognition and dialectical materialism but which is fundamentally rooted in Vygotskian theory (Radford et al., 2005; Radford, Miranda, & Guzmán, 2008; Radford, 2011, 2014). Others have drawn on Gee's (discursive) theory of cultural models (Setati, 2006; Kleanthous & Williams, 2010); activity theory (e.g., Ohtani, 2007); and communities of practice (e.g., Hunter, 2008). Finally, some papers draw on Bakhtinian concepts, often in combination with Vygotskian theory (Mesa & Chang, 2008; Radford, Miranda, & Guzmán, 2008; Williams & Ryan, 2014) although not in all cases (e.g., Barwell, 2013).

The second most frequent theoretical orientation groups together various forms of discourse analysis. This category includes: papers drawing on positioning theory, such as Herbel-Eisenmann and Wagner's (2005) analysis of textbooks, Sakonidis and Klothou's (2007) analysis of students' written work, or Skog and Andersson's (2013) investigation of pre-service teachers' discourse; papers drawing on discursive psychology, such as Barwell's (2007, 2008) analyses of how mathematical thinking is constructed in the discourse of mathematicians and of mathematics education researchers; papers drawing on Halliday's systemic functional linguistics and his notion of mathematical register, including Leung and Or's (2007) study of students' explanations, Herbel-Eisenmann, Wagner and Cortes's (2008) analysis of lexical bundles, and Gol Tabaghi and Sinclair's (2011) study of pre-service teachers' diagramming practices.

The socio-political orientation covers contributions that mainly draw on sociological theories, including Fairclough's critical discourse analysis (e.g., Thornton & Reynolds, 2006; Le Roux & Adler, 2012; Le Roux, 2014), Goffman's participation frameworks (Hegedus et al., 2006), and Bernstein's theory of framing and pedagogical practice (Knipping & Reid, 2013). The total shown for socio-political papers is likely to be somewhat understated, since several other papers, particularly listed under discourse analysis or socio-cultural theory suggest at least socio-political leanings, even if the theoretical framework is not explicitly socio-political in nature (we comment more on this issue later in the chapter). This kind of orientation is relatively recent, following the changes to the PME constitution around ten years ago, which allowed research to address topics in addition to psychology for the first time.

The remaining categories are less represented and sometimes less well theorised. Several papers were based on a general theoretical distinction between everyday or informal language and mathematical language (e.g., Amit & Jan, 2006; García-Alonso & García-Cruz, 2007; Bardelle, 2010). Another group of papers focused on teachers' practices (Chen & Chang, 2012), or teachers' knowledge or understanding in relation to their teaching (Adler & Ronda, 2014), or look at the orchestration or conceptualization of mathematics classroom discussion (e.g., Kahn et al., 2008; Morera & Fortuny, 2012; Wang, Hsieh, & Schmidt, 2012). A handful of papers were based on the theoretical notion of embodied cognition (e.g., Bjuland, Cestari, & Borgersen, 2008; Edwards, 2010; Warren, Miller, & Cooper, 2011).

Finally, 'other' incorporates a wide variety of theoretical orientations to language that occurred relatively rarely. Some notable examples include Lunney Borden's (2009) use of decolonizing methodologies; Heinze et al.'s (2009) use of Cummins' theories of bilingual education to investigate the performance of language minority students in Germany; and Shinno's (2013) analysis of semiotic chaining.

The common thread that runs through the majority of theoretical frameworks adopted in PME research on language in the past 10 years is the idea that language is central to the processes of mathematical thinking, learning and teaching and, as such, is the link between the individual and the social. In this work, language is neither the means of *transmission* of mathematical knowledge, nor the learner's means of expression of their individually constructed schemas. Rather, it is through language that both learners and teachers are historically and culturally constituted *as* learners and teachers of mathematics. As we shall discuss in the remaining sections of the chapter, the predominant theoretical orientations necessitate, often implicitly, or at least, often without being fully developed, a central place for otherness, often termed alterity. In the next sections, we look in more depth at three main thematic foci for PME research on language in the past 10 years: ways of conceptualizing language and mathematics; cultural dimensions of language and mathematics; and language diversity in mathematics education.

WAYS OF CONCEPTUALIZING LANGUAGE AND MATHEMATICS

In this section, we discuss PME research that examines language in collective participation and in embodiment, representations and symbol use, and Vygotskian semiotics. We focus, in particular, on the role that is ascribed to language in the students' and teachers' mathematics conceptualization. Although there seems to be an agreement that "Language is an important tool in the construction of mathematical knowledge" (García-Alonso & García-Cruz, 2007, p. 258), we still need to understand how mathematics education researchers conceive of the relationship between language and conceptualization.

Natural Language and Mathematical Language

Several papers in our corpus deal with the problem of the relationship between natural language and mathematical language. Various terms of have been used for natural language, including 'informal language' and 'colloquial language'. Some of these papers stress the *influence* of natural language on the students' understanding of mathematical concepts. For instance, Fernández Plaza, Ruiz Hidalgo, and Rico Romero (2012) show that the students' mathematical concept of limit of a function at a point is influenced by colloquial uses of terms such as "to approach," "to tend toward," "to reach," and "to exceed" (2012, p. 235).

In a study dealing with the concept of monotonicity, Bardelle (2010) refers to the students' frequent "misuse of mathematical language" (p. 183) and the students' lack of awareness that mathematical terms have a specific scientific meaning:

> [The] interviews show that Matteo and Filippo understand the concept of monotonicity of a function but they cannot answer correctly because they do not realize that the term 'increasing' is a scientific one and hence it has just one well determined meaning. Matteo and Filippo give their own interpretation of the term. (Bardelle, 2010, p. 181)

In another investigation, Bardelle (2013) shows also the influence of natural language on the mathematical understanding of universal statements (e.g., "Not all A is B"): "the interpretation of verbal statements in a mathematical setting may happen to be based on everyday context and not on a mathematical one" (p. 71).

Expanding on Bardelle's work, Ye and Czarnocha (2012) carried out an investigation that "confirms, in a spectacular fashion, the impact of natural language on the mathematical understanding of negation by identifying, during the student interview, a source of misconception initiated from incorrect French/English translation" (Ye & Czarnocha, 2012, p. 235).

It is, therefore, clear that there is an influence of natural language on students' mathematical conceptualizations and that one of the problems is that students do not seem to be aware of the fact that the meanings of natural language do not necessarily coincide with those of mathematical language. Drawing on the work of Shuard and Rothery (1984), García-Alonso and García-Cruz (2007) suggest a distinction between "(1) those terms which have the same meaning in both [everyday and mathematical] contexts; (2) those terms whose meaning changes from one context [to] the other; and (3) those terms which are only seen in a mathematical context" (p. 258). Bearing this typology in mind, they carried out an investigation of four popular textbooks among high school teachers, and analyzed the meaning of 27 terms pertaining to statistical inference in everyday use as well as in the mathematical context (e.g., "population," "sample," and "confidence level"). They concluded that, often, "definitions that appear in the textbooks do not correspond to their mathematical meaning but instead to the one in their everyday use" (p. 263). The problem is thus not only the students' but also the textbook authors'.

The co-occurrence of mathematics and everyday language in the classroom, not only in its oral dimension but also in its written one, has led some researchers to investigate the impact of natural language on the understanding and performance of students (see, for example, Ilany & Margolin, 2008). Bergqvist (2009, p. 146) noted that "In order to read texts in mathematics it is necessary to be able to recognise which category words belong to in order to be able to interpret them correctly." Bergqvist endeavoured "to identify PISA mathematics items for which student performance is influenced by reading ability" (Bergqvist, 2009, p. 145).

Let us try to pose the problem in a more general manner. To do so, let λ_1 and λ_2 be two semiotic systems (a contemporary natural language and a contemporary language of "mathematics," respectively). To a semiotic system λ we can associate the "concepts" or "ideas"' i that individuals express, convey, and manifest with and through λ. Thus, i_1 is the system of ideas associated to with λ_1 and i_2 is the system of ideas associated with λ_2. With a few notable exceptions (e.g., Baber & Dahl, 2005; Lunney Borden, 2009; Edmonds-Wathen, 2014), PME language researchers seem to be, to an important extent, asking questions not about the relationship between λ and i, but about the *influence* of λ_1 in i_2 (Bergqvist, 2009; Fernández Plaza, Ruiz Hidalgo, & Rico Romero, 2012) or the *interference* of λ_1 in λ_2 and i_2 (Bardelle, 2010, 2013; Ye & Czarnocha, 2012).

For Makar and Canada (2005), the problem revolves around the pedagogical use teachers can make of the students' use of λ_1 and i_1 in moving towards λ_2 and i_2 Their research is about the concept of variation with prospective teachers. In a task from a post-interview, the prospective teachers were showed "weights for 35 different muffins bought from the same bakery, and asked what subjects thought their own (36th) muffin might weigh. The set of data for the 35 muffin weights were shown in both a boxplot and a histogram" (Makar & Canada, 2005, p. 276). Makar and Canada note that the subjects resort to terms of natural language to convey ideas of distribution (e.g., "bulk of this data," "concentration of data," data "really clustered," or, as in other interviews, "scattered" or "bunched" data, when the interviewed subjects referred to data presented in dot plots). They conclude by saying that the:

> informal use of language needs to be given a greater emphasis in research on statistical reasoning […] There are several reasons for this. For one, teachers need to learn to recognize and value informal language about concepts of variation and spread to better attend to the ways in which their students use this same language. Secondly, although the teachers in this study are using informal language, the concepts they are discussing are far from simplistic and need to be acknowledged and valued as statistical concepts. Thirdly, the scaffolding afforded by using more informal terms, ones that have meaning for the students may then help to redirect students away from a procedural understanding of statistics and towards a stronger conceptual understanding of variation and distribution. (2005, pp. 279–280)

At the practical level, PME mathematics education researchers seem to recognize that natural language may be both a source of interference and a support in the development of mathematical language and ideas. Even "vague language" may prove to be important: "vague language fosters construction of new mathematical ideas" (Dooley, 2011, p. 287; see also Tatsis & Rowland, 2006).

Previous studies have focused on the identification of linguistic functions to which students resort to express mathematical ideas *in natural language*. For instance, in research about pattern generalization of figural sequences, Radford (2000) identified two such functions, termed *deictic and generative action functions of language.* Radford focused on students' sentences like "OK. Alright, look. You . . . one has to add (*pointing to a figure on the paper*) . . . you *always* add 1 to the *bottom*, right?" He argued that the deictic function and the generative action function of language were at the root of the students' mathematical generalization. Through terms like "top" and "bottom," the deictic function of language provides students with the possibility to *notice* and *refer* to key parts of a *perceptual* term in order to imagine *non-perceptual objects* and their mathematical properties. The argument is that perception is somehow oriented by the meaning of deictic linguistic terms, suggesting thereby potential manners by which to look at, and attend to, objects in our environment. The "generative action function" refers to

> the linguistic mechanisms expressing an action whose particularity is that of being repeatedly undertaken in thought. In this case, the adverb 'always' provides the generative action function with its repetitive character, supplying it with the conceptual dimension required in the generalizing task. The relevance of generative action functions can be acknowledged by noticing that, in our example, generality is objectified as the *potential action* that can be reiteratively accomplished. (Radford, 2000, p. 248)

In other words, in λ, the adverb "always" plays a similar role as the universal quantifier ∀ plays in λ_2.

Consogno, Gazzolo, and Boero (2006) identified an additional linguistic function, which they termed the Semantic-Transformational Function (STF) of natural language. It refers to

> the construct that accounts for some advances of [the students'] conjecturing and proving process. The student produces a written text with an intention he/she is aware of; then he/she reads what he/she has produced. His/her interpretation (suggested by *key expressions* of the written text) can result in a *linguistic expansion* and in a *transformation of the content* of the text that allow advances in the conjecturing and proving process. (pp. 353–354; emphasis in the original)

The Relationship between λ_1 and λ_2

Naturally, the fact that students can start thinking mathematically within λ_1 (the semiotic system of a natural language) does not mean that λ_2 (the semiotic system of a contemporary language of "mathematics") can be dismissed. And reciprocally: it would be a mistake to think that a mathematical activity within λ_2 is independent of λ_1: there is a limit to what can be mathematically expressible within λ_1. Natural languages have not been created to calculate and to carry out relatively complex computations. Nor have they been created to investigate theoretical properties of Banach spaces or abstract topologies, for instance. The standard contemporary mathematical language to which students are exposed in school mathematics has acquired, since the Renaissance, an operational dimension it never had before. There was a rupture indeed in the conception of language in the Renaissance that led to the development of two different paths. On the one hand, there was a humanist trend that sought to remove from language the barbaric dimensions of scholastic Latin and other previous linguistic formations. The humanistic trend ended up in a research program whose goal was a simplification and purification of language, the identification of the various parts of discourse, a systematic approach to grammar, and a general theory of the structures of thought (Cassirer, 1963). Grammar, "was taken to provide access to the bases of thought itself" (Reiss, 1997, p. 23). On the other hand, the Renaissance witnessed the emergence of a new scientific language epitomized in the works of Galileo and the abacist mathematicians. The chief characteristic of this language was to reason in an operational manner.

Although both conceptions of language in the Renaissance take different directions, they each rest on a formidable cultural abstraction. On the one hand, there is a progressive development of the idea of a general grammar that in its reasonability, that is, in its appeal to a supposedly general and universal reason, applies to any particular language. On the other hand, there is a search for an efficient language where unknowns, variables, and parameters, and their operations can be carried out regardless of the reference—a minimalist language in which the subject vanishes.

The extinction of the subject is one of the most impressive accomplishments of the contemporary mathematical semiotic system. Such a semiotic system, that endlessly keeps scaring students and sometimes teachers as well is voiceless. Yet it cannot work alone. As Vergnaud notes, "No diagram, no non-linguistic symbolism, no algebra can fulfill its function without a linguistic accompaniment, even if it remains internal or inner only" (2001, p. 14). In short, even in its most developed form, λ_2 depends on λ_1: "Natural language is a metalanguage of all symbolisms" (p. 14). Natural language and the language of mathematics play different roles. With their own specificities, each one of them provides individuals with access to different layers of mathematical consciousness. They provide individuals with different forms of expressiveness and aesthetic experience.

How has this relationship been understood by PME researchers interested in language issues? As we have already noted, the predominant theoretical perspective used in PME language research draws on sociocultural theory, and for the most part the relationship between colloquial and mathematical discourse is framed by ideas from this theoretical tradition.

For Sfard (2010), the route to the development of mathematical language is through changes to colloquial language:

> If mathematics is a discourse, then learning mathematics means changing forms of communication. The change may occur in any of the characteristics with the help of which one can tell one discourse from another: words and their use, visual mediators and the ways they are operated upon, routine ways of doing things, and the narratives that are being constructed and labelled as "true" or "correct". Since uses of words and mediators create a tightly knit web of connections, we should probably consider this system in its entirety, even when interested in only some of its elements. In research on learning any mathematical concept, therefore, nothing less than the whole discourse of which the given concept is a part would suffice as a unit of analysis. (p. 218)

Sfard's approach construes individual learning in terms of change in individual communication, including thinking, which she considers to be communication with oneself. Her approach has been adopted and developed by many contributors to PME over the past 10 years.

For example, Sánchez and García (2011) examined the think-aloud responses of 14 pre-service primary school teachers to a set of nine questions about the properties and definitions of regular quadrilaterals. Sánchez and García analyzed the students' responses by looking for moments of 'commognitive conflict' (using a portmanteau word coined by Sfard to underline the fusion of communication and cognition in her theory). According to the theory, moments of commognitive conflict will arise due to the differential use of language in colloquial and mathematical discourse. Sánchez and García were able to show that such moments did arise for the participants in their study, and related them to the 'confrontation' of mathematical and socio-mathematical norms. For example, one such confrontation was:

> between the [Mathematical Norm] related with defining expressed in the criterion of minimality and the [Socio-Mathematical Norm] 'everything you see in a figure that goes with the presentation of a task has to necessarily indicate something'. It leads students to incorporate descriptive features/aspects, coming from the task presentation, in some of their responses that are neither necessary nor relevant (for example, length of the side). (p. 110)

This position appears to be based on a couple of important assumptions: first, that there is a clear separation or dichotomy between colloquial and mathematical language; and second, that teachers can make use of students' colloquial language to bridge to mathematical language and meaning.

Barwell (2013), however, argued that the relationship between colloquial and mathematical discourse (for which he used the terms 'informal' and 'formal' language) must be seen as dialogic. In particular, he argued that the implicitly linear sense of development from informal to formal mathematical language is problematic. Referring to data from a class of 10–12-year-olds, he concludes:

> A dialogic perspective on formal and informal language in mathematics classrooms highlights a relationship between formal and informal that is not uni-directional. Rather than steady progress from informal to formal, these students work at both. The teacher, too, must make skilful use of varying degrees of formality. Of course, students need to learn formal mathematical language as part of learning mathematics, but this does not mean that informal language disappears; nor is it simply a scaffold to reach more formal language. Both are necessary; they will always be in tension. (p. 79)

Embodiment

In truth, the situation is more complex than insinuated above. As research on embodiment suggests, in the classroom processes of conceptualization, students and teachers resort to more than colloquial and mathematical languages. They resort to gestures, body posture, kinaesthetic actions, artefacts, and signs in general. Instead of being epiphenomenally surplus to teaching and learning, these embodied and material resources are an important part of classroom activity. As Warren, Miller, and Cooper (2011) report, "the use of gestures (both by students and interviewers), self-talk (by students), and concrete acting out, assisted students to reach generalisations and to begin to express these generalisations in everyday language" (p. 329).

The proper cognitive and epistemological understanding of embodiment and material culture has been the object of an active line of research in PME. At the theoretical level, Edwards, Rasmussen, Robutti, and Frant (2005) led a working session in PME 29 to discuss the role of conceptual metaphor and conceptual blends, and language and gestures in the construction of mathematical ideas and in teaching, learning, and thinking. In the same PME conference, Arzarello and Edwards (2005) organized a Research Forum on "Gesture and the Construction of Mathematical Meaning." The Research Forum led to a Special Issue in Educational Studies in Mathematics (Edwards, Radford, & Arzarello, 2009) where the need of a "multimodal approach" is argued:

> Crucial to the production of knowledge is the individual's experience in the act of knowing and the fact that this experience is mediated by one's own body. However, this return of the body to epistemology and cognition does not amount to a disguised form of empiricism. Conceptual ideas are not merely the impression that material things make on us, as Hume (1991) and other 18th century empiricists once claimed. The return of the body is rather the awareness that, in our acts of knowing, different sensorial modalities—tactile, perceptual, kinesthetic, etc.—become integral parts of our cognitive processes. This is what is termed here the multimodal nature of cognition. (Radford, Edwards, & Arzarello, 2009, p. 92)

A great deal of research on multimodality has revolved around the understanding of the relationship between gestures and language in the students' conceptualizations (e.g., Askew, Abdulhamid, & Mathews, 2014; Edwards, 2010, 2011; Edwards, Bolite Frant, & Radford, 2010; Edwards, Bolite Frant, Robutti, & Radford, 2009; Hegedus, Dalton, Cambridge, & Davis, 2006; Ng, 2014; Radford, 2011; Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005; Robutti, Edwards, & Ferrara, 2012). Arzarello and his collaborators have investigated the role of gestures in the evolution of students' mathematical signs. Thus, in Arzarello, Bazzini, Ferrara, Robutti, Sabena and Villa (2006), the authors investigate "the genesis of written signs starting from specific gestures, progressively shared within the group." They suggest that gestures have various functions: "understanding the situation, looking for patterns or rules, anticipating and accompanying productions of written representations, drawings and symbols necessary to solve the problem" (p. 73).

There has also been an interest in understanding the role of the teacher's gestures on the students' gestures and conceptualization. For instance, in their PME 32 paper, Bjuland, Cestari, and Borgersen (2008) asked the following research question: "What kinds of communicative strategies does an experienced teacher use in her dialogues with pupils, introducing a task that involves moving between different semiotic representations?" (p. 185) They found that: "The [teacher's] gestures make the connection between the semiotic representations, figure and diagram" (p. 185). In the same PME conference, Radford, Miranda, and Guzmán (2008) dealt with a similar problem, cast in terms of the role of multimodality in the classroom evolution of meanings. Following the idea of conceiving of gestures as signs that constitute a genuine semiotic system on its own (Radford, 2002), Radford, Demers, Guzmán, and Cerulli (2003) suggest seeing gestures as embodying different views, voices, and meanings, much like words in natural language. Their analysis shows how, in a very subtle way, the students' gestures come to echo, with their own *intonation*, the teacher's gestures. The echoing of the teacher's gestures and the personal intonation that students bring forward opens up possibilities to *generalize* previous gestures. Within this context, gestures in particular, and multimodality in general, are conceived of as *polyphonic*, and the joint teacher-students classroom transformation of meanings appears as *heteroglossic*:

> Borrowing a term from M. M. Bakhtin, we want to call the transformative process undergone by the students' meanings as *heteroglossic*, in that heteroglossia, as we intend the term here, refers to a locus where differing views and forces first collide, but under the auspices of one or more voices (the teacher's or those of other students'), they momentarily become resolved at a new cultural-conceptual level, awaiting nonetheless new forms of divergence and resistance. (Radford, Miranda, & Guzmán, pp. 167–168)

In general terms, we can reformulate the question of language and conceptualization as follows. Instead of a relationship between two semiotic systems (natural and mathematical languages) and their corresponding (interrelated) conceptualizations alluded to in the previous section, conceptualization emerges in activities underpinned by a range of perceptual, tactile, kinesthetic, and other sensorial multimodal channels in dialectical interaction with semiotic systems (natural languages, mathematical languages, gestures, diagrams, etc.).

The systemic understanding of such interaction and the political forces that underpin the evolving relationships require more research that may complete the substantial number of PME investigations dealing with representations and symbol use (e.g., Misailidou, 2007; Verhoef & Broekman, 2005; Walter & Johnson, 2007), language and conceptualization (Armstrong, 2014; Baber & Dahl, 2005; Mellone, Verschaffel, & Van Dooren, 2014; Mesa & Chang, 2008; Meyer, 2014; Planas & Civil, 2010; Ruwisch & Neumann, 2014; Viirman, 2011) or classroom discourse (e.g., Asnis, 2013; Berger, 2005; Gholamazad, 2007; Le Roux, 2014; Sfard, 2010). Such a systematic understanding could also benefit from the interesting question of the role of society and culture in conceptualizations in natural and mathematical languages (e.g., Lunney Borden, 2009; Clarke & Mesiti, 2010; Clarke, Xu, & Wan, 2010; Edmonds-Wathen, 2010; Morgan & Tang, 2012).

CULTURAL DIMENSIONS OF LANGUAGE AND MATHEMATICS: AUTHORITY, POWER, AND COLLECTIVE DISCOURSE

In this section, we bring together the question of language as it appears in discussions where the focus is on ideology, power, agency, and gender, including the relationship between the individual and society; the question of language, mathematics, and culture; and cultural discursive routines. The topic of language diversity is addressed in the next section.

There is a growing sensitivity in PME research about the manner in which language embeds, conveys, perpetuates, and shapes ideological stances and social relations, like power. There is also a growing sensitivity in understanding the often subtle mechanisms through which language affords or constrains agency, and structures views about gender. Although the questions about ideology, power, agency, and gender are not necessarily related to multilingualism, it is in multilingual contexts that they often become more salient.

As mentioned previously, in our count, discussions about ideology, power, agency, and gender appear centrally in 28 papers. One of the main concerns is the manner in which students position themselves and also how they come to be positioned by current classroom practices, discourses, and texts (Herbel-Eisenmann & Wagner, 2005; Esmonde, Wagner, & Moschkovich, 2009; Moschkovich, Gerofsky, & Esmonde, 2010; Skog & Andersson, 2013). Another important concern is to describe and understand inclusive discursive practices and practices that exclude or marginalize students (e.g., Hunter, 2013; Hunter, Civi1, Herbel-Eisenmann, & Wagner 2014; Moschkovich, Gerofsky, & Esmonde, 2010). This "political/ideological" line of inquiry rests on a broad conceptualization of language that goes beyond the investigation of the relationship between language and the development of mathematical understanding to focus on "how language in the mathematics classroom illustrates power relationships" (Thornton & Reynolds, 2006, p. 273).

Power relations can appear in the manner in which communication happens in the classroom (e.g., Adler, 2012; Brown, 2011, Civil, 2012; Chapman, 2009; Hussain, Threlfall, & Monaghan, 2011; Radford, 2014; Wagner, 2014), but also in more subtle ways, as for instance in how teachers assess their students' achievements (Sakonidis & Klothou, 2007), how authority is asserted through lexicological choices (Herbel-Eisenmann, Wagner, & Cortes, 2008), or in how students' activity is constrained by recourse to the passive voice and nominalisations (Morgan & Tang, 2012). Behind the "political/ideological" line of inquiry is, of course, a conception of teachers and students that—at the most general level—rests on beliefs about the relationship between the individual and society, and about the nature of power and authority. As two theoreticians of power in classrooms noted a few years ago, "different understandings and practices of authority have been shaped for over a century by conflicting ideological belief systems" (Pace & Hemmings, 2007, p. 10). How, then, do language-minded mathematics education researchers publishing in PME proceedings tackle the question of power and authority? The answer is both difficult and easy.

The answer is difficult in the sense that in the PME language papers dealing with power there is rarely any specific theorization of the meaning of power and authority. A relatively elaborated instance appears in Herbel-Eisenmann, Wagner, and Cortes's (2008) paper, where the authors refer to Pace and Hemmings (2007), who define authority as "a social relationship in which some people are granted the legitimacy to lead and others agree to follow" (p. 6; emphasis in the original). Pace and Hemmings's definition—inspired by Max Weber's work and more precisely by Mary Haywood Metz (1978)—highlights an asymmetrical relation between the manner in which individuals act towards each other, and the social distinction between those who are granted legitimacy to lead and those who are expected to follow. The definition, however, is too abstract. Authority is eradicated from its context. Furthermore, the only explanation that is given for the existence and practice of authority is that authority serves to maintain a "*moral order*" (2007, p. 6; emphasis in the original) which, to make things worse, is equated with "shared purposes, values, and norms intended to hold individuals together and guide the proper way to realize institutional goals" (2007, p. 6). This definition of authority turns out to be very rationalist, simplifies the idea of moral order as something transparent and politically neutral, and portrays individuals as merely consenting and negotiating agents.

At the same time, the question about how language-minded mathematics education researchers publishing in PME proceedings tackle the question of power and authority has a relatively easy answer. It is easy in the sense that through the papers we see that power and authority are thematized along the lines of a *reaction* to transmissive teaching. Let us explain.

In transmissive teaching, the teacher appears as the holder of authority and the students as those who follow the authority of the teacher. The implicit conception of authority and power of transmissive teaching takes as its starting point the idea that the cultural mission of the teacher is to ensure that knowledge, values, and norms are properly passed on to the students. Likewise, the cultural mission of the student is to receive or appropriate this knowledge, values and norms. "In this view," Henry Giroux notes, "authority is frequently associated with unprincipled authoritarianism" (Giroux, 1986, p. 25).

The remedy against the affliction of authority is usually found in the students' freedom and autonomy. Freedom and autonomy—the two chief Western categories that have defined the idea of the human subject since the emergence of manufacturing capitalism in the 16th century (Beaud, 2004; Kaufmann, 2004; Radford, 2012)—are considered to provide the basis for students' escape from authority, and the central condition for students' emancipation and authentic learning.

This story is not new—and this is something on which we would like to insist, as it is only by understanding the educational story behind authority and its antithetical position, i.e., freedom and autonomy, that we believe we may be able to go beyond the predicaments in which the political/ideological research on language seems to be immersed today. Authority on the one hand, and freedom and autonomy on the other, were the axes around which the proponents of the two main models of the 20th century pedagogical reform mentioned in the introduction envisioned and organized their corresponding pedagogical programs. In the case of the transmissive model, authority provided the hierarchical relationship between teachers and students that was required to put in motion a specific form of knowledge production and reception. In the case of the progressive model, authority appeared as something to be overcome through the nurturing of the student's freedom and autonomy (see, e.g., Neill, 1992). In searching to promote the student's freedom and autonomy, progressive educators built their pedagogy through a dichotomy between teachers and students. This dichotomy offered the conceptual and methodological basis for their pedagogical action.

We should not jump to the conclusion that this is past history. The two main pedagogical programs of early 20th century educational reform have not disappeared. On the contrary: both have evolved under the influence of new societal and historical demands. The progressive model has moved from a discourse entrenched in the student to a discourse about students. However, the move from the singular to the plural, that is, the move from a child-centered pedagogy to a children-centered one, where collective discourses are emphasised, does not amount to a change of view of the learner. The move, as we shall see, is cosmetic, not ontological. More profound changes are noticeable in the transmissive program. In its search for efficiency and alignment with neo-liberal global capitalism's forms of material production, the transmissive program has undergone a profound refinement. It has developed sophisticated technologies of control to monitor students' achievement (e.g., through regional, national, and international tests) and the teachers' implementation of a technical, prescriptive curriculum. Ironically, the curriculum of the transmissive model is not shy about advocating for students' engagement in their learning. One of the best examples is the Ontario mathematics curriculum. Yet, in practice, students' engagement remains more often than not a purely rhetorical move. We do not need to go far to find other examples. Referring to the American educational context, the historian of education, David Labaree, argues that today “It is hard to find anyone in an American education school who does not talk the talk and espouse the principles of the progressive creed” (Labaree, 2005, p. 277). However, as Labaree notes, “We talk progressive but we rarely teach that way. In short, traditional methods of teaching and learning are in control of American education” (Labaree, 2005, p. 278). And referring to the endless war between progressives and bureaucratic, efficientist, transmissive pedagogues, he concludes that “The pedagogical progressives lost” (Labaree, 2005, p. 278; see also Kantor, 2001).

The lost war of the progressive model is a recurrent theme in many PME papers, even if the theme is not formulated explicitly in this way. Brown (2011), for instance, having in mind not only the UK context in which he works, but also the contemporary educational context at large, complains that teachers find themselves working under governmental demands that seek to promote prescriptive curricula that favour some social groups. “Specifically,” Brown (2011) notes, teachers “work to curriculums that mark out the field of mathematics in particular ways that favour certain priorities or groups of people” (p. 190), confining students and teachers to the sphere of cultural reproductive agents. Wagner (2014) makes a similar point: “I consider it unfortunate that mathematics classroom practices tend toward closed dialogue in which children are not invited to see the possibility of multiple approaches and possibilities” (p. 63). And he did not miss the opportunity to complain about the lack of autonomy with which students are left in traditional transmissive classrooms: “Teachers too frequently fail to raise the possibility of students’ autonomy” (p. 63).

It is, however, in empirical papers that the reaction to the traditional transmissive model is most salient. It is there that the question of students’ participation (or the lack thereof) comes to the fore (e.g., Høines & Lode, 2006; Hunter, 2007; Hodge, Zhao, Visnovska, & Cobb, 2007).

These empirical papers also show a great concern for understanding the role that teachers may play in promoting students’ dialogical participation in collective discussions (e.g., Hunter, 2008; Mesa & Chang, 2008; Chapman, 2009; Gilbert & Gilbert, 2011; Sánchez & García, 2011; Morera & Fortuny, 2012; Toscano, Sánchez, & García, 2013; Adler & Ronda, 2014; Cavanna, 2014; Hung & Leung, 2012; Thornton & Reynolds, 2006). For instance, Thornton and Reynolds (2006) investigate the extent to which Grade 8 Australian students have opportunities to express themselves and submit ideas to the classroom. A closer look at the analysis shows that the students’ opportunities for participation are still carried out against the background of the teacher-students dichotomy championed by the progressive reformers. Thornton and Reynolds (2006) contrast Noemi’s classroom—that is, the classroom they investigated—to many of the TIMSS 1999 video classrooms, which “featured reproductive discourse, with the apparent goal of students being to guess what was in the teacher’s mind” (p. 275). They remark: “In Noemi’s classroom students see themselves as active participants in learning, who have power over both the mathematics and the discursive practices of the classroom" (p. 277). They go on to say: "Power is located with students" (p. 277). With power on the side of the students, the teacher's authority has finally vanished.

Chapman (2009) offers us a similar view. As in the case of Thornton and Reynolds (2006), she poses the problem against the backdrop of the war between traditional mathematics classrooms and reformed classrooms. In a clear and succinct way, she summarizes how discourse is conceptualized in current reform mathematics education perspectives: "Discourse, as promoted in current reform perspectives of mathematics education, is not about classroom talk intended to convey exact meaning from teacher to student; instead, it is about communication that actively engages students" (Chapman, 2009, p. 297). Of course, there is nothing wrong with this. As Giroux notes, "student experience is the stuff of culture, agency, and self-production and must play a definitive role in any emancipatory curriculum" (1986, p. 36). To see the teacher-students dichotomy appear we have to consider the following part of the citation that we highlight in italics: "…instead, it is about communication that actively engages students *in a way that allows them to construct new meanings and understandings of mathematics for themselves*" (Chapman, 2009, p. 297; our emphasis). The second part of the citation tells us who is in control of the means of classroom knowledge production. It reveals that the conception of classroom discourse is still based on the teacher-students dichotomy. It is the students who, through their engagement in classroom communication, have to understand mathematics *for* themselves. This is what empowerment seems to be about.

Lee (2006) also stresses the need for students to take control of the means of classroom knowledge production. She pleads for an approach that engages students in classroom discourse and that is oriented towards helping them express and explain their ideas, so that "They take *ownership* of *their* ideas and become able to *control* and use them" (Lee, 2006, pp. 7–8; our emphasis).

In sum, contemporary progressive (or reform) views of mathematics classroom interaction revolve around the old progressive idea of students' participation. Although this is certainly a commendable idea, we see that students' participation is understood against the backdrop of a dichotomy between teachers and students. This dichotomy, the progressive pedagogues feel, is required in order to guarantee the overcoming of the teacher's authority. At the epistemological level, the dichotomy serves to define a specific form of knowledge production in the mathematics classroom, which is based on the idea that students have to gain control over, and ownership of, knowledge and its mechanisms of production.

How does the teacher understand the operating dichotomy that promises to set the students free from authority? Noemi, the teacher in Thornton and Reynolds's (2006) investigation, says:

> My aim in my Mathematics classroom is for students to regard Mathematics as an art which belongs to them, a means of regarding and interpreting the world,

> a tool for manipulating their understandings, and a language with which they can share their understandings. My students' aim is to have fun and to feel in control. My role is primarily that of observer, recorder, instigator of activities, occasional prompter and resource for students to access. Most importantly, I provide the stimulus for learning what students need, while most of the direct teaching is done by the students themselves, generally through open discussion. (Thornton & Reynolds, 2006, p. 278)

As we can see, the teacher conceptualizes herself as a *resource*, providing the students with occasional stimuli. In other PME papers, the teacher appears as a "facilitator" (Chapman, 2009, p. 298) or "guide" (Hodge, Zhao, Visnovska, & Cobb, 2007, p. 42) of the subjective expression of the students. There is a generalized patriarchal view of the teacher, who is reduced to playing a shepherding role—teachers appear as scaffolders, observers, and room-makers-for-students-to-think-and-act. They are there to promote student achievement and established forms of academic success. But the progressive model does more than that: most importantly, it provides teachers with technologies of subjectification to conceive of themselves as shepherds and facilitators.

We can try to go further and ask the question about how the teacher conceptualizes the students. The previous cited passage provides us with some interesting elements with which to answer the question. Understanding knowledge—mathematics, in this case—as something that can be *possessed*, the teacher conceives of the students as potential possessors. The teacher wants the students to regard mathematics as something that "belongs to them" (Thornton & Reynolds, 2006, p. 278).

Let us notice that this stance is not typical of teachers like Noemi. As we have seen, researchers also expect the students to understand mathematics *for* themselves; they are expected to take *ownership* of *their* ideas. The same goes for the Theory of Didactical situations, where teachers are advised not to show the students the answer. As Brousseau notes, if the teacher shows the student how to solve the problem, the student "does not make it her own" (Brousseau, 1997, p. 42). Since how to solve the problem is not "her own," in this line of thinking the student cannot be said to have achieved a genuine mathematical understanding.

In brief, the progressive (reform) model and the theories and pedagogies it has inspired tend to look at the students through the lenses of the students-as-private-owners paradigm. That is, the students are conceived of as subjects of a specific form of "knowledge production that equates doing and belonging: what belongs to the students is what they do by themselves. What they do not do by themselves does not belong to them" (Radford, 2014, p. 5; rephrased). Within this context, understanding is featured as the epistemic equivalent of belonging: Understanding is the product of the students' own cogitations and deeds. The students' understanding is the product of their own labor—not the teachers'. How indeed—the question runs—could students understand something that they did not themselves produce?

In the same way as we labor in society to acquire and possess things, students labor in the classroom to possess/understand knowledge. Hussain, Threlfall and Monaghan (2011) attempt to introduce a new approach to mathematics teaching and learning: "This paper introduces an approach to mathematics teaching and learning which we feel transcends the usual teacher-centered versus student-centered dichotomy by integrating two kinds of mathematics classroom discourse, the authoritative and the dialogic" (2011, p. 1). The solution that they envision is based on a partition of authority—sometimes authority rests with the students, sometimes it rests with the teacher. They continue:

> It is proposed that mathematics teaching and learning should engage students in dialogic communicative approaches to empower them to articulate their ideas and to take more responsibility, but that in order to enable students to build mathematics competences effectively it is also proposed that the teacher should at times involve periods of authoritative discourse on topics prompted by the dialogic discourse. (Hussain et al., 2011, p. 1)

The question of authority is again posed against the background of the opposition of teacher and students. The solution exists in the alternation of authority, a compromise between the two camps at war—the transmissive (traditional) and the progressive (reform) camps.

In his PME 38 plenary talk, Radford (2014) suggested a dialectical approach that puts at the center the idea of teaching and learning as a single process in which teachers and students work together—an idea captured in the term joint labour:

In joint labour teaching and learning are fused into a single process: the process of teaching-learning—one for which Vygotsky used the Russian word obuchenie. In this sense, teachers and students "are simultaneously teachers and students" (Freire, 2005, p. 76). They are simultaneously teachers and students, but not because both are learning (Roth & Radford, 2011). They are, of course. However, the real reason is because teachers and students are labouring together to produce knowledge. (pp. 10–11)

Here knowledge is neither something that teachers possess and pass on to the students (the transmissive model) nor something that students acquire through their own personal deeds (the progressive model). Knowledge is not something to possess; like music, it is a kind of evolving space to attend ("fréquenter" as Guillemette, 2015, p. 76 says), visit, and enjoy. More precisely, knowledge is a diverse cultural-historical set of potentialities that, through the teacher-students' joint labour, enables actions, imaginations, interpretations and new understandings.

This perspective moves away from the conception of the teacher as a shepherd discussed previously:

> regardless of how much the teacher knows about [mathematics], she cannot set [mathematical] knowledge in motion by herself. She needs the students—very

> much like the conductor of an orchestra, who may know Shostakovich's 10th Symphony from the first note to the last, needs the orchestra: it is only out of joint labour that Shostakovich's 10th can be produced or brought forward and made an object of consciousness and aesthetic experience. (Radford, 2014, p. 11)

Although teachers and students do not play the same role, they work together. They need each other. "Teachers and students are in the same boat, producing knowledge and learning together. In their joint labour, they sweat, suffer, and find gratification and fulfillment with each other" (Radford, 2014, p. 19).

LANGUAGE DIVERSITY IN MATHEMATICS EDUCATION

The perceived increase in language diversity in contemporary classrooms must, for education, be one of the most salient legacies of colonialism and globalization. There are two aspects to this legacy. First, the increasing movements of people around the world, initially as a result of colonial policies, more recently as a result of globalization, mean that classrooms now rarely fit the presumed ideal in which all students speak one and the same language. In developed countries, these circumstances have often come as something of a shock, leading to concepts like 'superdiversity' (Vertovec, 2007; see Barwell, 2016, for a more extended discussion in the context of mathematics education) as societies and, in particular, education systems struggle to come to terms with the presence of multiple languages and cultural backgrounds. The second aspect of the legacy of colonialism and globalization, however, is that the Eurocentric view of 'normal' societies as unilingual, with one language unifying one nation, is finally itself being overturned. A plurilingual view of society is no surprise to the 'rest' of the world, where living with multiple languages is the norm. Much as the peoples of the Americas must have been surprised to learn that they were 'Indians', so the 'discovery' of language diversity implies a complex and problematic relationship with otherness.

A focus on language diversity, including topics such as mathematics learning in multilingual classrooms, in bilingual education programs or of immigrant second language learners have featured at PME for some time. Indeed, in his paper at PME29, the first year of our current survey, Barwell (2005) reviewed research reports with a focus on language diversity from the previous 10 years. He identified 13 research reports in that period, indicating a good level of interest in topic of growing prominence. In our current survey, we have identified 21 papers addressing this topic, suggesting a degree of growth in work in this area. These papers cover a range of national contexts (Australia, Canada, Catalonia-Spain, Germany, Malaysia, New Zealand, Philippines, South Africa, Tonga, USA) and sociolinguistic settings, including bilingual classrooms, indigenous learners, immigrant learners, and multilingual societies. This work addresses several interrelated topics.

Several contributions examine aspects of students' mathematics learning in the context of language diversity, looking at how their mathematical understanding is linked to practices like code-switching (e.g., Manu, 2005; Planas, Iranzo, & Setati, 2009; Planas & Civil, 2010) and the challenges of word problems given in an 'imported' language (Verzosa & Mulligan, 2012). There has also been work seeking to understand students' perspectives on learning mathematics in a language other than their home language (Setati, 2006), and the perspectives of 'local' students on practices designed to support immigrant learners in their mathematics classes (Planas & Civil, 2008).

Another strand of research continues the search for a link between language proficiency and mathematics achievement. Some of the early work on this topic was reported in PME in earlier decades (e.g. Clarkson, 1996; Clarkson & Dawe, 1997). Recent papers include two quantitative studies conducted in Germany (Heinze et al., 2009; Prediger et al., 2013), as well as Essien and Setati's (2007) investigation of the effects on mathematics scores of an intervention designed to improve a group of South African students' proficiency in English.

Several researchers have reported their work with teachers to develop more effective tasks or teaching methods (Poirier, 2006; Nkambule, Setati, & Duma, 2010; Hunter, 2013) and Civil (2008) has also reported on similar work with parents. Lim and Ellerton (2009) reported teachers' views as part of their examination of changes to language policies in Malaysia.

Finally, three papers have examined the relationship between grammatical structures of indigenous languages and the related affordances for mathematical thinking and learning (Lunney Borden, 2009; Edmonds-Wathen, 2010, 2014).

This work reflects the kinds of tensions arising in mathematics classrooms in contexts of language diversity discussed by Barwell (2012a, 2012b, 2014), including tensions between home and school languages, between formal and informal mathematical language, and between language for learning and language for getting on in the world. Barwell draws on Bakhtin (1981) to theorize these tensions as reflecting an inherent tension in language. Bakhtin uses the metaphor of centripetal and centrifugal forces to conceptualize the nature of language both as diverse and constantly new and different (called *heteroglossia*), and as striving to reflect an ideal of purity and perfection (known as *unitary language*). Hence, most of the papers mentioned above subscribe to an idea of mathematical language as a stable, unified register or discourse, when it can instead be seen as multiple, diverse and unstable.

Bakhtin's understanding of language is based on relationality and, in particular, dialogue. Thus heteroglossia is not simply the presence of difference, but rather the relations and interactions between these differences. For Bakhtin, these interactions are dialogic in nature, meaning that they involve more than one perspective at once. Dialogue arises between languages, discourses, utterances or voices and is what make meaning possible. Fundamental to this view of language is the role of alterity. Difference requires otherness but, as we have seen, difference is also the source of an unavoidable tension within language. Whenever students must learn mathematics in a second language, or a language they do not use at home, they are learning mathematics with an Other's language (Barwell, 2013). And language, in Bakhtin's theory, is not just language, it is ideology—a worldview. Thus, learning mathematics in another language, or in multiple languages, is not just a question of getting through the language to the mathematics that lies beneath; rather, each language, or a particular variety of language or languages, offers a different mathematics (Edmonds-Wathen, 2014). A key question for the work reviewed in this section, then, is: How do PME researchers interested in language diversity deal with the fundamental issue of otherness in their research?

There are, inevitably, a variety of responses to this question apparent in the different papers. In some cases, the learner is the Other. For example, in Heinze et al.'s (2009) carefully designed quantitative study conducted in Germany, the goal was to understand the relationship between the language proficiency of immigrant students and performance in mathematics, such as in a high-stakes mathematics test. The assumption is (reflecting, we presume, the national policy context in Germany) that many immigrant students do not speak good German and should learn to do so in order to succeed in mathematics. Heinze et al. found some links between proficiency in German and mathematical performance. The students' proficiency in their home language, was not evaluated, however, despite much research showing that home language proficiency can also be an important factor in school success (e.g. Cummins, 2000). Immigrant students are characterized in terms of 'foreignness'—they are either migrants, or their parents are migrants, or they speak a foreign language at home (Heinze et al., 2009). (The study also found no difference between migrants and non-migrants on basic arithmetic performance.)

The othering of immigrants is also apparent in Planas and Civil's (2008) paper. They worked with a secondary school mathematics teacher who was implementing 'reform' teaching practices, which included problem-solving and collaborative group work. Planas and Civil report on interviews with some of the 'local' students, which reveal how they see immigrant students as language learners rather than mathematics learners:

> Helena [high achiever]: They put us in small groups and they say that this way we will learn more mathematics, but the real reason is that they do it so that those from outside get a chance to practice our language. I don't think this is right because I think that these decisions should be based on the mathematics. (Planas & Civil, 2008, p. 125)

Moreover, while there was interest from local students in the alternative mathematical methods displayed by the immigrant students, the prevailing view was that the immigrant students should learn 'our' methods.

It seems that the key basis for the construction of immigrant students as other is the perception that they do not speak the classroom language 'correctly' or are not proficient or simply speak differently. Khisty (2006) discusses this issue in some depth, in the context of Spanish/English bilingual students in the USA (not necessarily immigrants). She proposes a sociocultural view of learning in which learning mathematics is understood as socialization into the language of the mathematics community. She uses this perspective to look for explanations for underachievement:

> Academic discourse competence in this broader sense is acquired through active participation in the community that uses that discourse, and through interactions with a more capable other (Vygotsky, 1986). The lack of discourse competence suggests academic failings. […] Without the academic discourse or language, students are systematically excluded or marginalized from classroom curricula and activities. (Khisty, 2006, p. 436)

She also argues that the “denigration” (p. 437) of students’ home language amounts to an additional form of alienation from school and from mathematics and “silences students’ voice” (p. 437). Khisty’s argument is one of the more carefully developed positions apparent in the papers in this section. Nevertheless, it is not without some underlying tensions, at least when viewed from the perspective of Bakhtin’s theory. In particular, it is based on a view of mathematics and mathematical discourse as something students should learn. The nature of mathematical discourse is not itself questioned; students should learn it and will benefit from it. It appears that the students simply need to learn mathematical discourse, and hence the educational problem is to create suitable conditions (reflecting the progressive view of education). In fact, learning mathematical language also means learning a particular worldview; it means becoming a particular kind of person and could thus be seen as a kind of colonization of the mind. This tension is an example of the problem of moving beyond both transmissive and progressive approaches to teaching and learning mathematics.

An alternative approach to alterity is to assume from the start that language has a political dimension. Setati (2006), for example, assumes that “The political nature of language is not only evident at the macro-level of structures but also at the micro-level of classroom interactions. Language can be used to exclude or include people in conversations and decision-making processes” (p. 98). In her interviews with five South African students about the language they preferred to use to learn mathematics, three preferred English and two did not express a preference. The students all spoke four or five different languages. For Setati, a preference for English can be related to the political role of English; the students saw English as an international language and therefore as a “route to success” (p. 99) and in some cases preferred it even when they acknowledged that they would understand mathematics better if it were taught in one of their home languages.

Civil (2008) in her work with Mexican-American parents also sees language as political. Her study reveals how the language policies in South-West USA which enforce a strong preference for English in schooling serve to marginalize the parents in her study. They are less able to attend class when their children are young, or to support their children in mathematics. They also noticed that their children were often grouped with other learners of English, so reducing their chance to interact with English-speakers, and they often studied mathematics they had previously learned in Mexico.

In both these studies, then, it is language itself that is seen as the Other. In Setati's study, English, although widely used in education in South Africa, is seen as the language of 'international' and of 'social goods.' In order to succeed, the students felt that they must learn this language, even to the detriment of their understanding of mathematics. In Civil's paper, the parents report how the use of English (in a particular way), positions them as different, and so as less capable. In both studies, English is colonizing students of mathematics and, as a result, may marginalize and alienate them. Indeed, in the case of the students in South Africa, they may be alienated from the very languages they speak at home. Again, then, there is a tension, between the many ways students have of talking about mathematics, including the different languages they may know (*mathematical heteroglossia*), and the educational ideal of a single language of instruction for mathematics.

A third approach to alterity is to attempt to understand the Other better. Three papers reported studies focused on analyzing the linguistic structure of other languages, particularly indigenous languages in Canada and Australia. Edmonds-Wathen's work (2010, 2014) draws on the concept of linguistic relativity, which assumes that the structures of language influence ways of thinking. For Edmonds-Wathen, this principle applies to mathematics. In the first of her papers, she reports on her work in a remote community in the Northern Territory, Australia, in which mathematics is taught in an indigenous language called Iwaidja. She sets out how spatial language in English is structured very differently from in Iwaidja (Edmonds-Wathen, 2010). In the second paper, she looks at the structures relating to number in various languages around the world to show how presumed universal features of mathematics are actually culturally and linguistically specific.

In her paper, Lunney Borden (2009) describes some of her work with Mi'kmaw schools in Nova Scotia, Canada. Her experiences illustrate the alienating effects of an English-language perspective on mathematics. For example, she describes how the English concept of 'middle' is not easily translatable into Mi'kmaw, so that a student asked in English to show the middle of something may appear not to understand the mathematical notion, when in fact it is language that is most relevant. Edmonds-Wathen characterizes well the deeper issue at stake in all three papers:

> It is difficult to avoid a deficit perspective in a discussion of people not using numbers because Western culture and mathematics education values quantification so highly. Nevertheless, it also does learners a disservice if their prior learning and conceptual development is not taken into account by mathematics educators. This is particularly relevant for remote Indigenous

> Australian children who enter a compulsory school system that is largely designed and taught by English-speaking non-Indigenous people who learnt their own number words from their parents within their own cultural milieu. (Edmonds-Wathen, 2014, p. 437)

Much as Columbus named the new world in his own image, as part of the process of conquest and appropriation, so mathematics has also been named by Eurocentric thinkers. Recognizing that this naming can itself be a form of colonization, however, makes it possible to consider alternative positions. In Edmonds-Wathen's and Lunney Borden's work, the Other is relative; the 'English-speaking non-Indigenous people' are others to Iwaidja people or Mi'kmaw people and vice versa. The Other is no longer singular, identified with the oppressed, the marginalized, the alienated; there is a relation—Bakhtin's ideas would suggest a dialogic relation—through which each constructs the other, although of course this relation is not necessarily equal (see the section on cultural dimensions).

Given the complex relationship between language, mathematics teaching and learning, and alterity, what (again) can teachers do? And how, for that matter, can researchers conduct their research in a way that does not marginalize and alienate (if this is even possible)?

For some, the answer to both these questions can be found in the concepts of voice and dialogue. Khisty (2006) explicitly draws attention to the role of student voice in supporting mathematics learning and concludes by raising questions about how teachers position themselves in relation to students' home languages:

> Do teachers and others understand and appropriately consider the political implications of which language is used and how? Do they view it as a learning resource or as something that does not have a place in mathematics classrooms, that should be ignored? Do they genuinely value the home language, do they recognize that differential status among students, including language status, is detrimental to students' learning, and do they seek ways to equalize language status? Do they seek ways to validate what students' have to say even when they do not speak the dominant language of instruction? (p. 438)

Khisty's questions point towards approaches to teaching that involve dialogue between languages, as well as between the voices of students, the teacher and mathematics. Nkambule, Setati and Duma (2010), for example, working in a South African classroom of 46 Grade 11 students analyzed what happened when the teacher used dual language versions of mathematics problems. All of the students and the teacher spoke multiple languages and were grouped according to the main language they used at home. The mathematics problems were presented in English and one of isiZulu, isiXhosa, Sepedi or Sesotho. Nkambule et al.'s analysis shows how the use of multiple languages supported the students to invoke 'horizontal mathematization'; that is, to make links between the mathematics in problems and their own experiences of similar situations. The study sets out a teaching strategy that values students' home languages, as well as their interpretations of the mathematics problems.

Poirier (2006), in a contribution to a research forum, describes her contribution to mathematics curriculum development with a school board in Nunavut, the Canadian province with a majority Inuit population. She recognizes the dangers of the situation:

> If we want to re-examine the Inuit curriculum and develop learning activities adapted to the Inuit culture, the researcher who is not a member of that community can not do that alone. The risk of developing activities that will not be suitable, or well-adapted, is too great. (p. 110)

She describes how, to mitigate these risks, she worked collaboratively with a team of four Inuit teachers and three Inuit teacher trainers. Her approach is highly dialogic, with the team exploring Eurocentric and Inuit mathematical concepts and ways of thinking, each in relation to the other. She reports the comments of a member of the school board:

> This research proposal is also a unique project in the history of KSB research specifically addressing curriculum questions in a minority, bicultural, and bilingual situation. As described in your paper, the dual phenomena with two cultures in contact in a learning environment, and in a school setting using the subject of math, is like an unexplored expedition to a foreign area of the universe of learning. (Betsy Annahatak, Curriculum development department, Kativik School Board, September, 2002). (Poirier, 2006, p. 112)

These remarks suggest that a degree of dialogue was established, although there remains an underlying sense of tension arising from the dominant nature of Eurocentric mathematics and European languages.

Lunney Borden (2009) has perhaps gone furthest towards a fully dialogic approach. Having taught for many years in Mi'kmaw schools, she drew on decolonizing methodologies, engaging in discussions with Mi'kmaw elders to develop an acceptable approach to her research. An important aspect of decolonizing methodologies is questioning the way research itself—frequently a colonizing activity—is conducted. The outcomes of her research, then, not only challenge Eurocentric notions of mathematics, but challenge Eurocentric approaches to research.

Mathematics education is still mostly conceived of in terms of unquestioned forms of alterity. What is transmissive education, if it not a form of colonization of the mind? Perhaps less obviously, progressive education can be seen in the same light: the imposition of a particular view of students, teachers and mathematics. The starting point for the development of a more dialogic approach is the awareness of the value of the Other, and an acceptance of heteroglossia as a normal state of affairs. This position suggests the need not just for a more effective approach to teaching mathematics in the context of language diversity, but also the need for a more ethical approach.

CONCLUSION

In this chapter, we have surveyed research on language published in PME conference proceedings from 2005 to 2014. We have discussed some of the main trends, such as language and conceptualization, questions surrounding authority and power, and language diversity. In this conclusion we ask the question: What is missing in current research on language?

What is missing, we think, is the constitution of a language of critique that may help us move from the two models of the early 20th century educational reform that continue to inform educational practice today. We have lived for more than a century pulled by a transmissive conception of education and a children-centered notion of education that, in the end, has been engulfed by schooling tailored to respond to the needs of contemporary capitalist forms of production. It is against the backdrop of the century-long struggle of these two models of educational reform that an important line of research on language has been moving for some time towards questions of power, authority, student participation, and equity. These questions have often been dealt with along the lines of a neo-liberal "redistributive" pedagogy. That is, a pedagogy that seeks to re-order the structures of knowledge and power in order to ensure "equal opportunities for all to learn through accessing both the mathematics curriculum and qualified teachers" or "equality of mathematical achievement outcomes across student groups" (Hunter, 2013, p. 97).

Although commendable on several counts, this pedagogy falls short of questioning the societal forces that produce inequalities and oppression. It fails to question, for instance, the mathematics curriculum, its political and economical orientation, and the kind of subjectivities it favors. While this critique has been made by Walkerdine (1988) and Giroux (1989) some 30 years ago (and developed in more recent work by, for example, Appelbaum, 2012; Valero, 2007; Walshaw, 2014), it is not well developed in PME research on language (or in PME research in general). Yet, it is within a redistributive pedagogy that questions of power or language diversity are often formulated in the PME proceedings: they are often formulated as the search for pedagogical actions that capitalize on minority group languages to lead the members of these groups to dominant mathematics. Language diversity becomes a tool to attain, maintain, and affirm Western mathematics. What is missing here, we suggest, is a critical language that could help us understand that the tensions between languages, or between forms of language, are not simply the source of pedagogical or ontological challenges: they are political, through and through. Such a critical language should help us transcend the shortcomings of redistributive pedagogy and to go beyond the conception of knowledge as something politically neutral to be possessed, the conception of students as private owners and teachers as technical implementers of a prescribed curriculum (shepherds, scaffolders, observers, instigators, helpers, etc.). As one of the reviewers put it, “so-called reform classrooms risk to privilege privileged students again.” Instead of conceiving of teachers as curriculum technologists whose role is to promote conventional forms of academic success, we argue for a conception of teachers as intellectual practitioners who critically problematize the knowledge and values that they and the students bring to, and co-produce in, the classroom. We argue for a conception of teachers as critical agents who acknowledge the fact that classrooms are first of all places of conflict and resistance and that it is out of conflict and resistance that subjectivities are formed and transformed, the teachers’ included. Such an approach would connect the research in our first theme (on language and mathematical conceptualization) with research in our second and third themes (on language, power, authority and language diversity).

What remains to be done to address the challenges we have highlighted in PME research on language in mathematics education, we think, is the elaboration of a new emancipatory conception of knowledge, authority and power. To do so, we need to start working from a non-substantialist perspective. That is, we need to think of knowledge, authority, and power not as “things” that people have or lack. We might be better off thinking of authority and power as rather a set of fluid and always moving relations that are enacted as individuals engage in human life. Authority and power are at the heart of the social practices of the division of labor and the tensions that result from the manner in which persons, groups of persons, and communities envision, define, and pursue their individual-societal purposes and truths. It is through human practices that authority and power are produced (not in situ, but historically). In turn, authority and power come to shape, embrace, and orient these practices, thereby making it possible that “certain forms of subjectivity, certain object domains, certain types of knowledge come into being” (Foucault, 2000, p. 4).

What is also missing in PME research on language and discourse, then, is a vision of teachers and students where authority is not an authoritarian relationship but rather a communal social and cultural construction “that expresses a democratic conception of collective life, one that is embodied in an ethic of solidarity, social transformation, and an imaginative vision of citizenship” (Giroux, 1986, pp. 22–23). Power and authority should rather serve as methodological lenses to critically reflect on the school values that we promote, nurture, and convey, as well as the kinds of rationalities and ways of knowing that we privilege. By looking at power and authority in this way, we may become reflectively able to notice those that we exclude, allowing us to envision more encompassing inclusive and just courses of action. Such a conception of authority and power may also allow us to rethink the positions, stances, and ideologies we come to embrace and promote in the school and beyond. We need to rethink the forms of classroom knowledge production and the forms of human collaboration that could be consonant with an emancipatory critical pedagogical agenda.

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9. PROOF AND ARGUMENTATION IN MATHEMATICS EDUCATION RESEARCH

INTRODUCTION

In the chapter on proof in the previous PME Research Handbook, Mariotti (2006) observed that there had seemed to be "a general consensus on the fact that the development of a sense of proof constitutes an important objective of mathematics education" and also "a general trend towards including the theme of proof in the curriculum" (p. 173). A decade later, Mariotti's observations are equally, if not more, applicable: there is currently a widespread agreement among mathematics educators on the significance of proof in students' learning of mathematics, with a number of educational policy documents or curriculum frameworks in different countries calling for an important place for proof in all students' mathematical experiences and as early as the elementary school (see, e.g., the U.S. *Common Core State Standards for School Mathematics* (CCSSI, 2010) and the most recent *National Mathematics Curriculum* in England (Department for Education, 2013)).

It has been suggested that mathematics education research has influenced or even pressured curriculum authors into giving proof a place in the mathematics curricula of different countries (Hoyles, 1997; Mariotti, 2006). There are several arguments for the importance of proof in students' mathematical experiences from the beginning of their education. These arguments have been elaborated in various publications including PME reports (e.g., Stylianides & Stylianides, 2006; Yackel & Hanna, 2003), so we will not repeat them here. Yet, there is a big difference between recommending or accepting the idea that proof should have an important place in school mathematics and realizing this recommendation for all students.

During the past few decades mathematics education research has cast light on many different issues related to proof, thus generating useful knowledge with implications for teaching practice. However, there are still many open debates in the field and important research questions remaining to be addressed. Over the past decade in particular there has been an upsurge of research activity related to the teaching and learning of proof, including many articles in all major journals in the field, a number of books or edited volumes (e.g., Stylianides, 2016; Stylianou, Blanton, & Knuth, 2010), and an international study conference on proof that was conducted under the auspices of the International Commission on Mathematical Induction (ICMI) (Hanna & de Villiers, 2012).

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 315–351.

The PME community is, and continues to be, a main contributor to debates and research advances in the area of proof, with a plethora of relevant reports published in the PME proceedings following the period covered by Mariotti's (2006) review. Our aim in this chapter is to review and reflect on major research advances of the PME community in the area of proof, based primarily on the PME proceedings during the period 2005–2015. A review of PME research on argumentation and proof focusing on post-elementary education and covering the period 2010–2014 can be found in Sommerhoff, Ufer and Kollar (2015). More comprehensive reviews of the state of the art in the field as a whole can be found in Harel and Sowder (2007), and in Stylianides, Stylianides and Weber (2016).

In what follows, we explain our decision to widen the scope of this review by considering issues related to argumentation and proof rather than just proof, and we discuss the meanings of these two key terms. We describe also the methodology we followed in the review and how the rest of the chapter is organized.

Argumentation and Proof

The concepts of *argumentation* and *proof* have been discussed in detail by Mariotti (2006, pp. 181–184) who also presented part of the debate about whether the relationship between the two concepts can be more productively viewed as a possible rupture (e.g., Duval, 1989) or as a possible continuity (e.g., Boero, Garuti, & Mariotti, 1996). No matter which position one takes in this debate, with which the PME community has engaged from its early stages as illustrated by the previous two references, the following points stand: (1) argumentation and proof are closely related, and (2) considering both argumentation and proof helps draw attention to a wider range of important processes related to proving than when considering them separately. Indeed, these two points, together with the increased attention that argumentation and proof have received at PME conferences over the years and elsewhere (e.g., Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012), have guided our decision to address in this chapter issues related to both argumentation and proof.

There seems to be a fairly shared understanding among researchers about the meaning of *argumentation*, a term which is generally used to describe the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false (e.g., Boero, Garuti, & Mariotti, 1996; Duval, 1989; Krummheuer, 1995). Thus argumentation focuses on the epistemic value of a given statement and can embody a link between the process of *ascertaining* (i.e., the process employed by an individual to remove his or her own doubts about the truth or falsity of a statement) and the process of *persuading* (i.e., the process employed by an individual or a group to remove the doubts of others about the truth or falsity of a statement) (Harel & Sowder, 2007). Argumentation is often situated in the context of a broader mathematical activity which has been described using different terms (e.g., proving or reasoning-and-proving) and can involve the following: exploration of examples or particular cases, generation or refinement of conjectures, and production of arguments for these conjectures that may not necessarily qualify as proofs or support the development of proofs (e.g., Buchbinder & Zaslavsky, 2009; Komatsu, 2011; Lockwood, Ellis, Dogan, Williams, & Knuth, 2012; Morselli, 2006; Stylianides, 2008; Zaslavsky, 2014; Zazkis, Liljedahl, & Chernoff, 2008).

In contrast to the rather consistent meaning attributed to argumentation in the field, the meaning of *proof* has been subject to debate among researchers at PME conferences and elsewhere (e.g., Balacheff, 2002; Reid, 2005; Stylianides, 2007; Weber, 2014). Some of these researchers have reviewed definitions of proof used in different research studies thus illustrating the multiplicity of perspectives in the field, while others proposed specific definitions of proof and discussed their affordances or domains of application in mathematics education research. Different definitions may be better suited to serve different research purposes; this implies that it may be neither possible nor desirable for all researchers to adopt a common definition. Yet, it is important that researchers specify their perspective on proof so as to facilitate understanding of their claims or findings (Balacheff, 2002; Reid, 2005; Stylianides, 2007).

Accepting the importance for such specificity, we describe our perspective on proof, without however suggesting that this is better than alternative perspectives. We begin with Mariotti's (2006) observation that "the crucial point that has emerged from different research contributions [in the field of mathematics education] concerns the need for proof to be acceptable from a *mathematical* point of view but also to make sense for *students*" (p. 198; italics added). Following up on this observation, we define *proof* in the context of a classroom community as a mathematical argument for the truth or falsity of a mathematical statement that meets both of the following criteria, where criterion 1 reflects a mathematical consideration and criterion 2 a student consideration (Stylianides, 2007):

- *Criterion 1*: An argument qualifying as a proof should use true statements, valid modes of reasoning, and appropriate modes of representation, where the terms "true," "valid," and "appropriate" are meant to be understood with reference to what is typically agreed upon nowadays in the field of mathematics, in the context of specific mathematical theories.
- *Criterion 2*: An argument qualifying as a proof should use statements, modes of reasoning, and modes of representation that are accepted by, known to, or within the conceptual reach of students in a given classroom community.

While not comprehensive, this definition is sufficiently "elastic" to allow description of proof across different levels of education, which is a pressing issue given the (positive) trend towards making proof part of the mathematics curriculum beginning from the elementary school grade levels. Also, the definition integrates different perspectives on proof discussed in the literature. These include, for example, the view of proof as a logical deductive chain of reasoning linking premises with conclusions in the context of a mathematical theory (e.g., Healy & Hoyles, 2001; Knuth, 2002; Mariotti, 2000), as well as others that highlight the cognitive or social aspects of proof thus viewing proof as an argument that establishes the truth or falsity of a statement for a person or a community (Harel & Sowder, 2007) or as an argument that is accepted by a community at a given time (Balacheff, 1988).

Methodology for the Review and Chapter Organization

As we noted earlier, our aim in this chapter is to review and reflect on major research advances of the PME community in the area of proof, based primarily on the PME proceedings during the period 2005–2015. We used the following two complementary and partly overlapping approaches to identify which reports to include in our review. The term "report" refers in this chapter to any published piece in the proceedings with length more than 1 page (such pieces could be labelled in the proceedings as plenary papers, plenary panels, research reports, research forums, discussion groups, or working sessions).[1]

- *Approach 1*: We included all of the reports with any of the keywords "proof/proving" and "argument/argumentation" in their titles or abstracts, though we did filter out few reports where the use of these terms was incidental (as, e.g., in the phrase "In this paper we make the *argument* that…").[2]
- *Approach 2*: We included all of the reports that were listed under the domain "Proof, proving and argumentation" in the section typically called "Index of Presentations by Research Domain" and found in volume 1 of the proceedings.[3] The particular domain under which a report is listed is specified by the authors of the report at the point of submission.

Approach 1 offered a rather objective way of identifying relevant reports, while Approach 2 gave voice to authors themselves to indicate whether they considered their reports to be primarily about argumentation and proof. Interestingly, a considerable number of reports were identified by only one of the two approaches; this emphasizes the complementarity of the approaches and helps justify our choice to consider in the review the union of their returns.

Over 160 reports qualified for inclusion in the review, with more than 80% of them being categorized into the following three general themes according to the approximate ratio 2:2:1. The bulk of this chapter is a discussion of reports under these three themes.

- *Theme 1*: Research on student conceptions and learning;
- *Theme 2*: Classroom-based research; and
- *Theme 3*: Research on teacher knowledge and development.

The reports that did not fit under any of these themes addressed a large variety of topics that defied broader grouping. Yet, one topic received relatively more attention than others, and so we comment on it briefly. This concerned the place or treatment of concepts related to argumentation and proof in curricular resources, notably mathematics textbooks. The studies on this topic were typically empirical and took the form of comparative analyses of textbooks in different countries (e.g., Miyakawa, 2012) or of different textbooks in the same country (e.g., Dolev & Even, 2012). Their findings showed a multiplicity of approaches to the place and treatment of argumentation and proof in textbooks not only across but also within countries. In educational contexts where teachers rely rather heavily on textbooks for their everyday planning and teaching, these findings raise a concern about the presumably large variation in the learning opportunities offered to students in different classes, even within the same country, depending on which textbook their teachers follow.[4] A recently published journal special issue on the place and treatment of concepts related to argumentation and proof in mathematics textbooks (Stylianides, 2014) offered a forum for more reports of empirical findings in this area and for discussion of methodological issues surrounding textbook analyses.

In the sections that follow, we discuss separately the three themes. In each section, we begin with a general description of the theme and any sub-themes within it, we continue with review of the reports belonging to the theme, and we conclude with a reflection on the state of PME research within the particular theme, including possible directions for future research.

THEME 1: RESEARCH ON STUDENT CONCEPTIONS AND LEARNING

General Description of Theme 1

In this section, we review the PME reports that focused primarily on issues of learners' conceptions when engaging in argumentation and proof. Although the set of studies reviewed for this theme included some research on mathematicians' strategies and conceptions, the vast majority of studies focused on students' conceptions and learning in secondary and undergraduate grade levels. There were a smaller number of studies focused on middle school students and even fewer studies with elementary school students. The reports reviewed go beyond reporting that students at particular levels have difficulty engaging in proof; collectively, they reveal aspects of learners' perspectives on proof and proving and add nuance to our understanding of learners' conceptions when engaging in proving.

Among the reports we reviewed, there were three related sub-themes:

- Students' conceptions of proof and the proof process;
- Experts and novices' use of examples in argumentation and proving; and
- Knowledge, tasks, and tools that promote success in generating proof.

In what follows, we review separately reports belonging to each sub-theme. We conclude with a reflection on the state of PME research within Theme 1.

Review of PME Reports Belonging to Theme 1

Students' conceptions of proof and the proof process. A relatively large number of reports discussed results of studies to learn more about students' conceptions when generating or validating proofs. A feature of many of these studies is that they approached their research from what one could call an actor-oriented perspective (Lobato, 2003), where students' conceptions are considered without reference to, or grounding with, expert or normative conceptions. The importance of capturing students' perspectives is underscored in Knapp and Weber (2006), whose study of Advanced Calculus students' proving led to a reconceptualization of Weber's (2001) construct of strategic knowledge for proof. In Weber's original framing of the construct, strategic knowledge for proof was the body of strategies, heuristics, and techniques known and utilized by the prover with the implicit goal of using this knowledge to attain a proof by any means. Knapp and Weber reframed the construct to be strategies, heuristics, and techniques that are used to attain students' goals of proving, as students' goals may range beyond simply attaining a proof.

Fried and Amit (2006), reporting on a sub-study of the large-scale, comparative *Learners' Perspective Study* (Clarke, 2001), investigated eighth graders' perspectives on proof. They argued that students' positioning as mathematical authorities influences their confidence in determining whether their written argument is a proof. They highlighted an important point from their findings: positioning students as distinct from mathematical authorities can perpetuate students' beliefs that there is a definitive notion of "proof" and that they do not have agency to construct their own conceptions of proof. They further challenged mathematics educators to help students "see that their continual debate, defining, and self definition is a normal state of affairs in mathematics" (p. 119).

Another study by Kunimune, Kumakura, Jones and Fujita (2009) of lower-secondary students (eighth and ninth graders) contributes to the existing body of literature on students' understandings of proof and generality (e.g., Chazan, 2000; Ellis, 2007). Their sample consisted of approximately 400 students, who responded to written survey items that aimed to assess their conceptions of algebraic proof. The researchers found that students who were consistently successful in producing algebraic proofs did not necessarily recognize the generality of their proofs. An inability to recognize the generality of a proof may suggest that students do not conceive of proof as a means for establishing truth. Coupled with the work of Fried and Amit (2006) discussed above, these findings suggest further that abilities to produce a proof do not need to go hand-in-hand with understanding a proof's specific role in the discipline of mathematics. Bieda's (2011) work illustrates similar complexity to students' conceptions of argumentation and proof as shown in the studies of Fried and Amit, and Kunimune et al.; Bieda showed that middle grades students' conceptions of what makes an argument convincing for showing the truth of a given statement involves both explanations of why a statement is always true and specific instantiations of the true statement to illustrate the phenomenon to the reader. Additional information regarding the findings of this study can be found in Bieda and Lepak (2014).

Some additional studies focused on advanced students and mathematicians' processes for generating proof. Edwards (2008, 2010) employed embodied cognition perspectives to better understand expert (advanced doctoral students and mathematicians) proof processes. Edwards found her participants evoked a conceptual metaphor of "proof is a journey" as they thought aloud when doing a proof. Wilkerson (2008) also found that embodiments (a term used by the author to describe examples, constructions, and prototypes) emerged in think alouds when mathematicians were interpreting a proof relating to new content to make sense of. This suggests that examples may play a similar role for novices and possibly relates to the extensive work on students' use of examples in argumentation and proving reviewed earlier.

Alcock and Weber (2005) illustrated, with two case studies selected from the interviews of 11 undergraduates, that learners might take either a referential or a syntactic approach to attempting proof. In the referential approach, the prover uses particular or generalized instantiations of the statement to guide formal inferences. Those who attempted the proof syntactically tended to stick with manipulating formally stated facts without the use of examples to guide their process. From their analysis of the case studies, Alcock and Weber discovered that students who took a more referential approach had a more difficult time producing formal proof from their intuitions. However, those who approached proof more syntactically tended to generate proofs without a good sense of the meaning of those proofs. Students with more syntactic attempts also tended to describe general decision rules about when to use particular proof techniques.

Zazkis, Weber and Mejia-Ramos (2014) investigated the kind of thinking that is needed to move from referential attempts at proof to a successful proof product. They identified three kinds of action employed by students as they worked on formal proof from informal arguments: *syntactifying, re-warranting*, and *elaborating*. In brief, syntactifying involves taking statements from a more informal register and transforming them into a form suitable for a formal proof; re-warranting involves identifying a deductive reason for a step from an informal, non-deductive justification; and elaborating involves adding detail to the formal proof to clarify ideas from the informal argument. From analyzing interviews with 73 undergraduates, they concluded that students who employed all three activities were highly likely to successfully produce a proof. Only in 14% of cases where a student did not use all three activities was a proof produced.

The studies reviewed in this sub-theme add richness and nuance to our understanding of students' conceptions of proof and the proving process. Although many of the studies illustrate that learners may hold conceptions that impede successful performance in producing a convincing argument or a proof, several also illustrate that students can have sophisticated conceptions of proof while their abilities to produce proof may not be as robust. Additionally, several of the studies contribute ways of describing and naming students' conceptions and actions during the proving process that can build shared understanding within the field as well as generate further research.

Experts and novices' use of examples in argumentation and proving. The reports we reviewed in this sub-theme generally aimed to identify how example use can be a productive, generative part of the argumentation and proving process. This focus is an evolution from the body of existing literature documenting students' over-reliance upon empirical evidence when proving, reflecting a general interest about the pedagogical value of examples in mathematics. The research reported here speaks to the practices of example use across a range of participants engaged in argumentation and proof; half of the papers focused on example use by middle school students (Chrysostomou & Christou, 2013; Ellis, Lockwood, Dogan, Williams, & Knuth, 2013; Lin & Wu, 2007) and high school students (Buchbinder & Zaslavsky, 2013), whereas the other half examined practices with example use by undergraduates (Morselli, 2006; Watson, Sandefur, Mason, & Stylianides, 2013) as well as by professional mathematicians (Antonini, 2006; Ellis et al., 2013). As such, this collection of reports represents a significant body of literature to inform the field about the nature of students' (and mathematicians') example use and supports further inquiry into this common practice of proving.

Several of the papers in this sub-theme focused on identifying types of example use that emerged in students' or mathematicians' argumentation and, in some cases, identifying which types led to desirable outcomes (e.g., a proof). The researchers tended to use qualitative approaches with small numbers of participants to gather rich data about example use. One of the studies with a larger sample was that of Morselli (2006), who conducted interviews with 47 university students and found that participants' argumentative processes could be classified into four profiles: (1) work exclusively through algebraic manipulation; (2) short explorations with examples and shift to algebraic proof; (3) extended explorations with examples leading to reasoning about the conjecture; (4) unfocused explorations with examples. She identified, in particular, that participants exhibiting argumentation habits categorized into the fourth profile were less successful than other students. This suggests that exploration with examples can be very productive for proving as long as the exploration is focused and purposeful.

Similarly, Lin and Wu (2007) found that the type of examples students investigate influences successful conjecturing. They posed conjecturing tasks in interviews with sixth grade students, where students were asked to conjecture what other geometric invariants would exist under the conditions shown in the given examples (see Figure 1). Figure 1 includes one example considered to be *typical*, meaning that the example showed the typical representation in textbooks. A second example, of the *conjunctive* type, satisfied all of the given conditions as well as other conditions. The *extreme* example, the third example, was one that satisfied all of the givens, but also contained some boundary features such as very large or small angle measures. The researchers randomized the order in which the examples were organized for each interview. During the interview, the researchers noted the number of conjectures generated as the participants considered each example, as well as noted each conjecture. A key finding was that conjectures students generated while analyzing extreme conjectures were fewer in number and more likely to be incorrect than if they conjectured from observations of conjunctive or typical examples. Otten, Gilbertson, Males and Clark (2014) raised questions about the influence of the typical example accompanying claims to be proven in geometry textbooks, which they called a case of *general with particular instantiation.* The findings of Lin and Wu suggest that, as Otten et al. hypothesized, the features of given examples influence the kinds of generalizations that students make. In cases where students are asked to reason from examples to prove a given conjecture, it may support students' argumentation process if a range of examples are provided for their review or if students are encouraged to generate their own examples so that they can determine which features are variant under the given conditions.

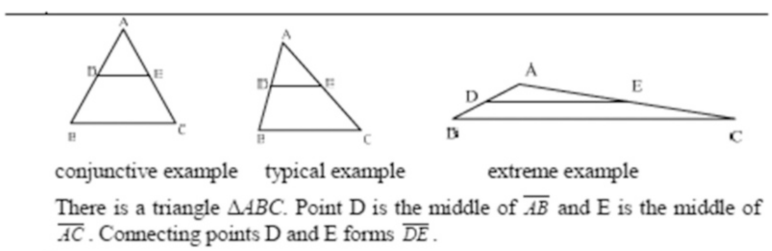

Figure 1. A conjunctive, a typical, and an extreme example (derived from Lin & Wu, 2007, p. 211)

Ellis, Lockwood, Dogan, Williams and Knuth (2013) compared the example choice and use of middle school students with that of expert provers (mathematicians). Their analysis focused exclusively on the practices of successful provers within each participant group, and characterized example choice and use of successful provers. Their findings suggest that successful provers insightfully navigate a range of examples. Experts (mathematicians) tend to reflect upon the utility of a particular example before choosing it and exhibit more metacognitive awareness of the utility of examples in the proving process.

The work of Buchbinder and Zaslavsky (2013) illustrates that students' notions about the role of empirical evidence in determining the validity of a claim are complex and nuanced. They conducted dyad interviews with seven pairs of ninth and tenth grade students, and provided them arguments to discuss. While students generally recognized the limitations of arguments that over-rely upon empirical evidence, they also tended to believe a statement to be true "unless proven otherwise"; that is, students had particular difficulty accepting a claim to be false when it was initially shown to be true with confirming examples but then a counterexample was found that disproved the claim.

In summary, the studies reviewed within this sub-theme showed the important role examples play in the proving practice of experts and suggest ways to support novices in using examples strategically to support the generation of conjectures and proof. Further, these studies suggest more work is needed to better understand learners' beliefs about the use of examples in the argumentation and proving process as well as the efficacy of interventions aimed at developing learners' sophistication in using examples.

Knowledge, tasks, and tools that promote success in generating proof. The third, and final, subcategory of reports within Theme 1 discussed aspects of students' knowledge that relate with their success in generating proofs, and tasks and tools that can promote students' competencies in generating proofs. Further discussion of tasks and tools for teaching argumentation and proof can be found in Theme 2, which focuses more on classroom-based studies discussing teachers' actions and interactions between teachers and students. The studies reviewed in this theme address questions about students' thinking and experiences during the learning process.

One question that emerged in many of the reports in this subcategory could be stated as, "What is the knowledge students need to be able to generate mathematical proof?" Hsu (2010) investigated the relationship between students' performance on geometrical calculation items, such as those to find missing measures, and geometrical proof generation questions. Calculation and proof generation items were paired so that they referred to the figures. Similar in scope to the well-cited Healy and Hoyles (2000) study, Hsu surveyed over 900 eighth- and ninth-grade Taiwanese students and found, unsurprisingly, that ninth graders performed better overall on both types of items. However, of note was that students did better on geometrical calculation items *after* completing a related geometry proof task (where the order of tasks was systematically varied). The order in which the participants received the calculation item or the proof generation item did not matter for performance on the geometry proof task. These findings suggest that generating proof may lead to a deeper understanding of the content.

Ufer, Heinze and Reiss (2008) reported on a study of knowledge needed for doing geometry proof, particularly the correlations among key predictors, identified through a review of the literature, and geometry proof performance. They collected survey responses from over 300 students who were enrolled in the highest track within the German secondary school system (Gymnasium). The survey consisted of proof generation items of various difficulty levels, where difficulty was based on the number of steps in the proof, and questions assessing knowledge of basic facts and problem solving skill. Using a linear regression model, the researchers showed that the three cognitive predictors (declarative basic knowledge, procedural basic knowledge, and problem solving skill) all significantly predicted geometry proof performance. While declarative knowledge was found to be the most significant predictor, problem solving skill was the least. The authors cautioned against an interpretation of these findings that there is little relationship between students' problem solving skill and their ability to generate proofs. Indeed, some studies have yielded valuable information about the process of generating proof by considering proof construction as a problem-solving task (see Weber, 2005). Rather, the authors highlighted that proving involves processes such as associative thinking that may not be a part of problem solving and suggested that the strength of a student's declarative knowledge likely determines her or his success in producing a proof. A later paper by the same authors (Ufer et al., 2009) provides additional information about how students' performance in producing proof is enhanced by the quality of their geometrical knowledge, particularly the availability of *perceptual chunks,* defined as mental associations between prototypical figures and mathematical concepts.

Some other studies discussed the importance of students' declarative knowledge of definitions for better performance in generating proofs. Dickerson and Pitman's (2012) work shows that knowledge of definitions and ability to use definitions in proof is challenging even for advanced students. Of the five undergraduates interviewed, none were able to make a clear distinction between a mathematical theorem and a mathematical definition and many generated arguments solely from their concept images (Tall & Vinner, 1981) rather than the concept definitions. Haj Yahya, Hershkowitz and Dreyfus (2014) found similarly problematic findings regarding high school students' understanding of geometric concept definitions.

Several studies in this sub-theme discussed aspects of students' thinking and proofs after the implementation of tasks or activities designed to improve students' proof competencies. Lin's (2005) plenary talk on his work studying the effects of refuting and the colouring strategy on students' skill to engage in proof was frequently cited in the PME papers reviewed for this theme. Lin found from his sample of Taiwanese ninth graders that the colouring strategy, where students would use colored pens to highlight given diagrams or draw information from the given statements, promoted better performance in completing proofs. However, he cautioned that these findings suggest that the colouring strategy diverts students' attention to extraneous features of the diagram or irrelevant information from the givens. Theme 2 provides further discussion of studies on the colouring strategy.

There were a few other studies of note describing particular tasks that promoted meaningful engagement in argumentation and proof. Mamona-Downs (2009) showed positive effects in undergraduates' ability to articulate their reasoning and refine their arguments after reading and interpreting selected work from peers. Beitlich, Obsteiner and Reiss (2015) discussed how secondary students read heuristic worked examples, a special type of textual pedagogical example where an imaginary peer reveals their thinking, including missteps, while completing a proof. In particular, this contribution used eye-tracking data to study how students attended to multiple representations in the argument. Their results showed that students spent the most time reviewing pictorial representations, and subsequently showed to have better comprehension of what was presented in pictures than in the symbolic or verbal parts of the argument. Brockmann-Behnsen and Rott (2014) investigated the effects of a structured training given to two classes of approximately 30 eighth-grade students in a German secondary school compared with two similarly-sized classes of their peers who did not receive the training. The training consisted of educating students about the structure of argumentation, such as developing the ability to distinguish claims from evidence, and on problem-solving heuristics, such as working backwards. Using pre-post test design, the treatment students performed significantly better on geometry proof tasks than their counterparts in the control classes. Finally, the work of Cramer (2014) suggests that logical games may foster equitable participation in argumentation, based on her analysis of students' participation in classroom argumentation using Habermas' theory of communicative action.

In addition to investigating the influence of tasks on proof performance, a range of studies focused on learners' interactions with various tools during proving tasks. Several papers discussed the potential of Dynamic Geometry Environments (DGEs) for supporting students' investigation of geometric conjectures and moving from informal argumentation to proof. Notably, Baccaglini-Frank, Antonini, Leung and Mariotti (2011) refined Leung and Lopez-Real's (2002) notion of *pseudo object* through observations of high school students' work with a DGE, where they defined pseudo object as "a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory" (p. 83). The pseudo object emerged through the actions of construction and dragging, mediating students' reasoning within the DGE and their theoretical knowledge of Euclidean geometry.

Rodriguez and Gutiérrez (2006) studied undergraduate mathematics students' use of a DGE while proving, particularly how these students used a DGE to produce deductive arguments as solutions of geometry proof problems. They sought to find differences between students' performance when proving without tools and when using a DGE. The authors found that a DGE helps students to identify and empirically check conjectures, but it does not provide an advantage over paper-and-pencil when students bridge informal arguments to proof.

Antonini and Martignone (2011) investigated novice and experts' argumentation when explaining the pantograph machine (Figure 2). Designed with pedagogical aims, this machine consists of "two leads fixed in two plotter points of an articulated system composed by some rigid rods and some pivots" (p. 41) and performs geometric transformations. They interviewed three pre-service teachers, two university students, and one early career mathematician as they attempted to identify and justify the transformation performed by the machine. The mathematician, unlike the other participants, only referenced features of the physical drawing performed by the machine if he had difficulty identifying the law embodied by the machine's structure. Similar to DGEs, the learning environment afforded by the pantograph exposes the nature of students' declarative knowledge.

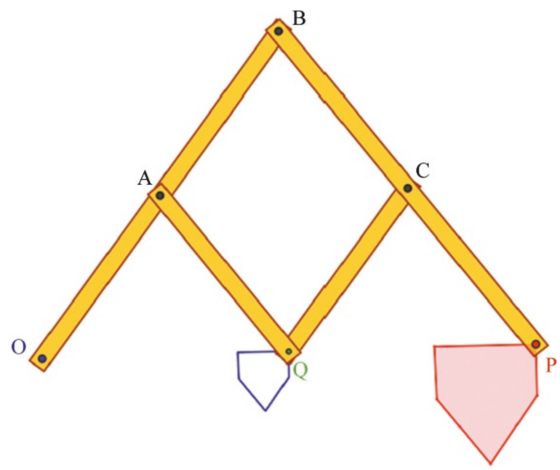

Figure 2. Scheiner's pantograph and its products (derived from Antonini & Martignone, 2011, p. 42)

Reflection on the State of PME Research within Theme 1

Taken together, the papers reviewed for this theme added many important insights to our understanding of students' knowledge, skills, and beliefs for argumentation and proof, especially with regard to students' understanding and use of examples in argumentation and proof. The findings from these studies come from research conducted with a range of methodological approaches. However, our review noted two interesting features of the set of studies that should be taken into account when considering research that builds upon these studies.

The first feature was the predominance of research on secondary students' argumentation and proof within the domain of geometry: of the studies we reviewed for this theme, about 40% focused solely on argumentation and proof in secondary school geometry. The second feature was the focus on participants at the upper end of the novice/expert continuum: another about 35% of the studies we reviewed involved undergraduate students, advanced doctoral students, or mathematicians. A minority of papers reviewed addressed issues of learning argumentation and proof in school settings other than secondary school and in domains other than Euclidean geometry. If argumentation is a practice that complements, or can complement, learning throughout the school mathematics curriculum, as many curriculum standards documents suggest (CCSSI, 2010; NCTM, 2000), more attention should be paid to students' understanding of argumentation and proof in domains beyond geometry.

Many of the studies presented at PME over the past decade illustrate how learning to prove, even in upper level mathematics, continues to be a persistent challenge for students. However, continued research is needed because engaging in argumentation is a key activity for developing a foundation of mathematical knowledge that can be applied to learning more mathematics and to doing mathematics in a range of contexts (CCSSI, 2010). There is an emerging interest in research on student thinking in a variety of activities related to argumentation beyond producing an argument or proof, such as pattern generalization (Lockwood, Ellis, & Knuth, 2013) or proof comprehension (Samkoff & Weber, 2015; Weber, 2015). The promise of this research is, in part, to gain insight to students' conceptions in a range of proof-related activities and investigate relationships between competencies in, for example, comprehending a proof and producing a proof. Moreover, this research may suggest a variety of activities that can be incorporated throughout experiences in K-16 mathematics that address students' development in adopting argumentation as a habit of mind.

THEME 2: CLASSROOM-BASED RESEARCH

General Description of Theme 2

In this section, we review those PME reports that dealt with the role and status of argumentation and proof in the classroom. In comparison to Theme 1, the classroom context played out more prominently in the reports under Theme 2.

Some of these reports aimed at understanding the role and status of argumentation and proof in ordinary classrooms (with a focus on students and/or teachers), while others presented and discussed teaching experiments where specific strategies and/or tasks were developed and used in order to improve the teaching and learning of argumentation and proof. Accordingly, we organize our review of PME reports belonging to Theme 2 under the following sub-themes:

- Students' processes of argumentation and proof;
- Teachers' ways of dealing with argumentation and proof in the classroom; and
- Interventions aimed at improving the teaching and learning of argumentation and proof.

The last part of this section will be devoted to a reflection on the state of PME research within Theme 2.

Like in Theme 1, the most represented educational level among the identified reports was secondary school, especially its upper part (grades 8–13). Also, the most prevalent mathematical domain was again geometry, followed by arithmetic and elementary number theory. Only few studies referred to other mathematical domains such as probability.

Review of PME Reports Belonging to Theme 2

Students' processes of argumentation and proof. As outlined in the Introduction, researchers in mathematics education agree that argumentation and proof are closely related (e.g., Durand-Guerrier et al., 2012). Thus, considering together students' argumentative and proving processes is highly relevant.

A series of PME reports focused primarily on students' argumentation processes during classroom activities. Among them, Douek (2006) studied the evolution from everyday to scientific concepts (from a Vygotskyan perspective) in the context of elementary school mathematics. Douek highlighted the key role played by argumentative activities in fostering concept development: "[A] continuous development of argumentative skills allows to nourish the backing of mathematical reasoning on other (and easier to master) kinds of reasoning. As a result, argumentation can effectively move the everyday concepts/scientific concepts under the teacher's guide" (p. 456). This study linked argumentation and concept development, thus showing that argumentation is not only a crucial step towards proof, but it has also an educational value in itself. In a similar way, Stoyanova Kennedy (2006) examined conceptual change as it took place through argumentation in a fifth grade class working on the theme of finite and infinite sets as a community of mathematical inquiry. The author presented and discussed the key phases of the activity, starting from the orientation phase, where spontaneous conceptions about finite and infinite sets emerged, to the building phase, where students collaborated to verbalize some solutions, to the conflict phase, where contradictions emerged, to the synthesis phase, where a resolution was found that gave birth to a new conceptual formation. More recently, Fielding-Wells and Makar (2015) analyzed class situations of epistemic argumentation, defined as a discourse that seeks the truth through critical reasoning and justification. They argued that epistemic argumentation is a fruitful activity that may have the following effects: supporting access to the cognitive and metacognitive processes that are typical of expert performance, supporting the development of communicative competencies, supporting the achievement of mathematical literacy, supporting the enculturation into the practices of mathematics, and supporting rational reasoning.

Another series of PME reports focused on the crucial link between argumentation and proof. These studies relied primarily on the idea of *cognitive unity* (Boero, Garuti, & Lemut, 2007), defined as the continuity between the processes of conjecture production and proof construction. Martinez and Li (2010) focused on the conjecturing process of grade 9–10 students in the domain of arithmetic. They defined *conjecturing* as a complex process "that involves the production of several mathematical statements; from which, one of the conjectures emerges as a conjecture to prove; and, through which a person comes to believe the likely truth-value of the conjecture to prove" (p. 269). The authors observed students producing conjectures during classroom activities, noting that, before reaching the conjecture to prove, students explored the problem and tested examples and/or counterexamples. Such a process led them to dismiss or accept conjectures and also to find out mathematical relations that would be employed in the subsequent proving phase. This is well aligned with the hypothesis of cognitive unity. The authors emphasized that the conjecturing phase is a complex and rich process and advocated the diffusion of conjecturing and proving activities in the US, where the curriculum has traditionally centred primarily on proof production and appreciation. Fujita, Jones and Kunimune

(2010) investigated the extent to which complex geometrical construction tasks may foster a cognitive unity between conjecturing and proving. Their data derived from classroom experiments carried out in Japanese lower secondary schools. They analyzed excerpts from group discussions using Toulmin's (1958) model for argumentation. They found that, when students were asked to produce conjectures and prove them, cognitive unity was not automatic. Their study also suggested the importance of designing teaching sequences where students, sharing their mathematical arguments with peers, gradually develop their "appreciation of how to use already known facts to proceed with further investigation" (p. 15). This links the study also to the last sub-theme, which concerns intervention studies.

The aforementioned studies all acknowledge the importance of students' interactions during the argumentation and proving processes. Matos and Rodrigues (2011) even more explicitly focused on proving in the classroom as a form of social practice. The authors adopted the *social theory of learning* perspective, according to which mathematics is a situated and social phenomenon. Consequently, the construct of *community of practice* was central to their work. The authors analyzed excerpts from group interactions in a grade 9 teaching experiment on geometry, focusing on the use of diagrams. Their analysis showed that, although a group may be seen as a community of practice where students share the same concern for the task and develop a shared practice, the members of the group can behave differently in terms of participation depending on their level of mathematical competence. The analysis showed further that the converse can also happen: "all the members of the team increased their ownership of meaning in different degrees depending of the degree of participation" (p. 183). The analysis highlighted also the complexity of the proving process in groupwork and the key role of the teacher in suggesting the use of the diagram as a powerful means for "sharing and increasing the ownership of meaning of proof" (p. 183). The crucial role of the teacher in fostering students' engagement with argumentation and proof has been discussed in many PME reports, some of which we review next.

Teachers' ways of dealing with argumentation and proof in the classroom. Scholars agree that teachers should set up proper actions so as to arouse students' need for proof and proving (e.g., Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki-Landman, 2012). Indeed, teachers face many challenges when dealing with proof in the classroom (e.g., Lin, Yang, Lo, Tsamir, Tirosh, & Stylianides, 2012): For example, teachers must establish suitable socio-mathematical norms, choose or design appropriate tasks and manage them in the proper way so as to foster understanding, and guide the students towards deductive thinking without turning proving into a "ritual" activity. Teachers must also be able to establish a proving culture in the classroom. One key point is the way proof is introduced in the classroom and what the role and purpose of such a treatment is: Furinghetti and Morselli (2011) distinguished between teaching proofs and teaching by proof, with the aim of proof in the second case being to promote understanding.

Several PME reports addressed the role of the teacher in fostering students' engagement with argumentation and proof. Most studies focused on teachers' interactions with students. Huang (2005) compared Hong Kong and Shanghai lessons on the Pythagorean theorem using video recordings of lessons. The study showed that Hong Kong teachers tended to value visual verification, while Shanghai teachers were keen to present a deductive argument that met the standard of proof. In terms of interacting with students and involving students in the proving process, Shanghai teachers made more efforts to involve students into proof construction. These findings may be interpreted in terms of the influence of the Confucian and British cultural traditions on the educational settings of Shanghai and Hong Kong, respectively. This study highlighted the role of cultural issues in examinations of the teaching of proof, brought to the fore the crucial dialectics between visualization and deductive reasoning, and took into account not only the status of proof in the classroom, but also the way teachers promote students' involvement with proof.

Several reports (e.g., Azmon, Hershkowitz, & Schwarz, 2011; Rigo, Rojano, & Pluvinage, 2008; Schwarz, Hershkowitz, & Azmon, 2006) addressed the role of the teacher when interacting with students, under the theoretical assumptions that social interaction must have a key role in mathematics instruction and that argumentation may foster concept development. Schwarz et al. (2006) identified recurrent patterns of interaction between two teachers and their students dealing with probability concepts in eighth grade. One of the teachers (whom they called teacher A) played a mediating role, while the other (teacher B) called for short and quick answers, with no provision for argumentation and in a sort of Socratic dialogue: "[W]ith teacher A, students feel obligated to support claims by explaining; they are used to crystallize ideas by reaching agreement and negotiating mathematical meanings; with teacher B, students are committed to tune to the teacher's questions and to adopt her explanations as theirs" (p. 71). In a further study by the same group of researchers, Azmon et al. (2011) conducted a quantitative analysis to explore the relationship between teacher-students patterns of interaction in the same two classes and individual students' subsequent argumentative processes. They found that students' explanations in the two classes differed not in terms of correctness, but in terms of richness, with richer explanations offered in the class of teacher A. An interpretation of these findings is that in teacher A's class there were socio-mathematical norms concerning students' responsibility for elaboration on their explanations and engagement in knowledge construction. The study highlights the importance of the mediating role of the teacher and suggests the significance of educating teachers so that they can efficiently manage discussions in their classrooms.

The mediating role of the teacher was discussed also by Cusi and Malara (2009), who studied grade 9–10 students' conscious use of algebraic language through teaching experiments on proof in elementary number theory. Drawing on the idea of *cognitive apprenticeship* (Collins, Brown, & Newman, 1989), the authors affirmed that the teachers should serve as a "role model," thus fostering students' development of those skills that are crucial for proving. The research was carried out in two steps:

in the first, the authors analyzed the teachers' interventions when leading students to prove, highlighting positive and negative behaviours; in the second, the authors drew, from their previous analysis, a characterization of the theoretical construct of *teacher as a role model*. Some characteristics of the teacher as a role model are the following: (a) stimulating students' attitude of research and acting as an integral part of the class in the research work; (b) acting as a practical/strategic guide and as a reflective guide in identifying effective practical/strategic models during class activities; (c) maintaining a balance between semantic and syntactic aspects of algebraic language; (d) acting as an "activator" of interpretative processes and anticipating thoughts; (e) acting both as an "activator" of reflective attitudes and as an "activator" of meta-cognitive actions. This is a useful characterization that may also have implications for teacher education.

Other studies addressed the role of guide played by the teacher. Ubuz, Dinçer and Bulbul (2012, 2013, 2014) presented a series of research reports on the structure of argumentation during teacher-students interactions. Their research was conducted in undergraduate mathematics courses and data analysis was performed using Toulmin's model for argumentation. Their findings suggested that the teacher plays a crucial role, providing guide-backing (approving warrants, backing or intermediate conclusions given by students) and guide-redirecting (proposing examples or suggestions when the students get stuck or do not start the argumentation from a good point).

Finally, Rodríguez and Rigo (2015) studied the emergence of a culture of rationality in the classroom, adopting an ethnographic approach and employing Toulmin's model as an interpretative lens. They defined the *culture of rationality* as being made up of standards of sustentation (those arguments that a given community employs to sustain mathematical facts and those recurring practices that are used in a given community) and trajectories of participation and distribution of responsibilities (the succession of interventions of the class actors in the argumentation process). The authors examined episodes from classroom teaching, highlighting the nature of arguments (always backed by mathematical considerations) and the recurrent trajectories of participation (dialectic exchanges between the teacher and the students). This study contributes to the description of a culture of rationality in the classroom and, in the authors' own words, highlights the role of the teacher in promoting the development of such a culture: "how the teacher negotiates her own rationality practices—an objective that, by way of dialogical exchange, involves the students by means of constant questions, not only about what but also about why—and how this enculturates her students in that rationality" (p. 93).

Interventions aimed at improving the teaching and learning of argumentation and proof. Setting up interventions (i.e., planning task sequences and devising learning strategies) for the teaching and learning of proof is a crucial theme of research in the teaching and learning of argumentation and proof, as evidenced by the recent ICMI Studies 19 "Proof and proving in mathematics education" and 22

"Task design in mathematics education." Lin, Yang, Lee, Tabach and Stylianides (2012) discussed principles for task design for conjecturing, proving, and the transition between conjecture and proof. Regarding conjecturing, the authors highlighted the importance of providing students with an opportunity to engage in observation, construction, and reflection. Regarding proving, the authors pointed out the importance of promoting the expression of arguments using different modes of argument representation (verbal arguments, symbolic notations, etc.), asking students to create and share their own proofs and to evaluate proofs produced by the teacher. Furthermore, regarding the transition from conjecture to proof, the authors suggested that the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317).

Within this general strand of research, a number of PME reports dealt with interventions aimed at promoting students' approach to argumentation and proof. These studies are important, as they bring to the fore a third main element of classroom-based research besides students and teachers, namely tasks. Also, these studies help illustrate the link between theoretical and applied research by examining how theoretical ideas can be turned into proposals for classroom implementation.

An example of the shift from theoretical considerations to classroom implementation is found in a collection of reports including a research forum (Boero, 2006; Boero, Douek, Morselli, & Pedemonte, 2010; Boero & Morselli, 2009; Boero & Planas, 2014) concerning the possible adaptation of Habermas' (1998) construct of *rational behaviour* to study different aspects of proving and other mathematical activities. The construct of rational behaviour deals with the complexity of discursive practices in the intersection of three kinds of rationality: epistemic (relating to the development of knowledge and questions about the validity of judgments), teleological (relating to strategic choices and corresponding actions to achieve a set goal), and communicative (relating to the reflective use of language oriented toward understanding). The work of Boero and his colleagues illustrates how an important theoretical construct from outside mathematics education can be conveniently interpreted and flexibly adapted to offer, in combination with other constructs, a new and promising perspective into the study of discursive practices related to proving. An interesting aspect is the fact that this construct, integrated with other theoretical tools such as Toulmin's model for argumentation, may provide a comprehensive frame that allows one to (1) better analyze students' proving processes and (2) plan and carry out innovative classroom interventions. Within the integrated model proposed by Boero et al. (2010), two levels of argumentation are outlined: the meta-level, concerning the awareness of the constraints related to the three components of rational behaviour in proving, and the level concerning the proof content. Thus, students' enculturation into the world of theorems is a long-term process where the teacher must create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological, and communicative requirements of proving.

Another example of theoretical considerations that inform task design is offered in a series of research reports by Cheng and Lin (2006, 2007, 2008). The authors proposed and tested a learning strategy, called "reading and coloring," aimed at helping students to take into account all of the necessary information to develop a proof. The strategy derives from theoretical considerations about the cognitive processes underlying multi-step proof production and is explicitly addressed to low achievers in proving. The authors emphasized that the strategy should be cultivated in line with the following two design principles: the strategy must provide an operative tool to students, and the strategy must be in continuity with the teacher's regular teaching approach. The strategy, tested in geometry grade 9 courses, was found to be efficient in reducing memory workload when organizing several steps into a proof sequence. The strategy is not efficient when colours may cause visual disturbance and for those students who have difficulty in devising intermediate hypothetical conditions. For those students who do not perform hypothetical bridging thinking, Cheng and Lin (2008) proposed a different learning strategy called "step-by-step unrolled reasoning."

Another strand of research focused on the development of tasks to foster students' approach to argumentation and proof, with an emphasis on the process of proving and meta-level knowledge about proof (e.g., Heinze, Reiss, & Groß, 2006; Kuntze, 2008; Miyazaki, Fujita, & Jones, 2014). Heinze et al. (2006) proposed worked-out examples as a tool for helping students to learn argumentation and proof. Drawing on Boero's (1999) description of the phases of the proving process and on Schoenfeld's (1983) idea of teaching heuristic methods in problem solving, the authors set up a learning environment based on heuristic worked-out examples. The study was carried out in grade 8. The sample comprised of 243 German students, who were divided into an experimental group (150 students) and a control group (93 students) according to their performance on a pre-test on reasoning and proving and a questionnaire about their interest towards mathematics. The control group received regular instruction on proving, while the experimental group followed a learning path that guided exploration and more reproductive phases. Heuristic worked-out examples were embedded into stories: students could follow the proving process of hypothetical characters, accompanied by meta-level comments and explanations. Moreover, students were involved in self-explanation activities by working with short texts including blanks. In the words of the authors, "heuristic worked-out examples provide scaffolding and might on the other hand encourage students to perform their own mathematical activities" (p. 279). The findings showed that the learning path based on heuristic worked-out examples is particularly efficient for low-achievers.

The work by Heinze et al. focused on proving as a process and on the idea of offering students some element of meta-level knowledge about proof. In the same vein, Kuntze (2008) set up a learning environment, called the "topic study method," where students were asked to write texts on different aspects of the proving process so as to foster their proof-related meta-knowledge. For example, students were asked to evaluate argumentations of hypothetical characters containing mistakes, or to comment on mathematicians' quotations about proof. In the first part of the study, 232 grade 8 participants were split into two groups – one following the topic study method and the other following the heuristic examples method (Heinze et al., 2006). The findings showed that the two methods are comparable in efficiency. In the second part of the study, 153 university students were assigned to different groups: 24 received no specific training on proof, 22 solved geometry tasks without proving, 18 worked with the topic study method, and 89 worked with heuristic worked-out examples. Students who worked with the topic study method scored significantly better than the first two control groups and comparably with those who followed the worked-out examples method. Kuntze (2008) found that the topic study method might improve students' proof-related meta-knowledge. The study also opened up for reflection a possible correlation between meta-knowledge and proof competence.

Miyazaki et al. (2014) addressed the issue of setting up efficient introductory lessons to proof. In order to help students appreciate the structure of a proof, they proposed to combine two pedagogical ideas: flow-chart proofs (showing the "story line" of the proof) and open problems (see Figure 3). Proof construction was an open problem in the sense that students could "construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion" (p. 228). The use of flow-chart proving was tested in grade 8, in a teaching experiment of nine lessons: during the first four lessons students constructed flow-chart proofs in open problem tasks; during two lessons they constructed a proof by reference to a flow chart in a closed problem task; during three lessons they refined proofs by placing them into flow-chart proof format in a closed problem situation. The findings showed that such an approach might foster an understanding of proof. More precisely, the flow chart proof helped students identify necessary conditions and combine them to reach conclusions.

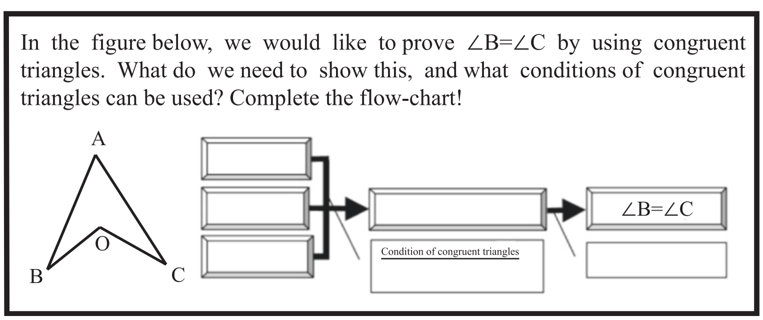

Figure 3. An example of flow-chart proving in an "open problem" situation (derived from Miyazaki et al., 2014, p. 227)

Reflection on the State of PME Research within Theme 2

The review of reports belonging to Theme 2 reveals some common trends and a consensus among scholars on some key issues concerning the teaching and learning of argumentation and proof. The first issue is the deep interconnection between argumentation and proof and the benefits of addressing argumentative activities. This is also linked to the growing consensus about the importance of social interaction when doing mathematics (e.g., Schwarz, Dreyfus, & Hershkowitz, 2009). Another key issue refers to the importance of providing students with opportunities to appreciate the process of proving, and not only proof as a final product. To this end, proving is conceptualized as a special case of problem solving (e.g., Weber, 2005), thus suggesting the importance of meta-knowledge (Boero et al., 2010; Kuntze, 2008) and heuristics (Heinze et al., 2006). Further research should study the role of heuristics and meta-knowledge and ways of promoting them. It should also aim to address the link between long-term argumentative activities and consequent proving competencies.

Other issues emerging from our review in this section concern task design. The research reports we reviewed proposed and tested tasks or task sequences aimed at improving students' approach to argumentation and proof, with a focus on proving as a process. These reports paid special attention to the theoretical considerations that guided or underpinned task design. The field would benefit from more research that would use theoretical ideas to design practical tools for use in the classroom and in the service or particular learning goals in different areas including argumentation and proof (Stylianides & Stylianides, 2013).

THEME 3: RESEARCH ON TEACHER KNOWLEDGE AND DEVELOPMENT

General Description of Theme 3

The PME reports falling in this theme addressed a range of issues pertaining to the nature of teachers' knowledge or the development of teachers' knowledge, with a relative balance in focus between preservice and inservice teachers and between the elementary and secondary school levels. The issues addressed could be categorized into the following sub-themes:

- Teachers' knowledge of argumentation and proof (nature and development);
- Teachers' knowledge of teaching argumentation and proof (nature and development); and
- Teachers' beliefs related to argumentation and proof.

In what follows, we review separately reports belonging to each of these three sub-themes. The first sub-theme received by far the most attention in the PME proceedings we reviewed, and this is reflected in the space we have devoted to it. We conclude with a reflection on the state of PME research within Theme 3.

Review of PME Reports Belonging to Theme 3

Teachers' knowledge of argumentation and proof (nature and development). The reports in this sub-theme have predominantly examined the nature of preservice elementary teachers' mathematical knowledge about different aspects of argumentation and proof (e.g., Zazkis & Zazkis, 2013), though there are also examples of studies with inservice teachers including secondary mathematics teachers (e.g., Gabel & Dreyfus, 2013; Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2008). Also, there are few studies that specifically aimed to develop teachers' knowledge about argumentation and proof (e.g., Gholamazad, 2007; Reichersdorfer et al., 2012). Most of the studies were conducted in mathematics courses in teacher education or professional development programs, with data collection being directly linked to or forming part of research participants' coursework, supplemented in few cases with individual interviews.

Overall, the contributions made by the reports in this sub-theme fall in one or more of the following three categories: (1) Empirical findings about the nature of teachers' mathematical knowledge about argumentation and proof (What do teachers know?); (2) Empirical findings about the effectiveness of interventions designed to enhance teachers' mathematical knowledge about argumentation and proof (How can teachers' knowledge be developed?); and (3) Theoretical or methodological contributions to research on the nature or development of teachers' mathematical knowledge about argumentation and proof. We present few examples of reports to illustrate these contributions.

The report of Zazkis and Zazkis (2013) is an example of a report that made a contribution within categories (1) and (3). The research was conducted in a mathematics course for preservice elementary teachers, with the data comprising the written responses of 24 preservice teachers to a task in an elective course assignment. The task presented a scenario in which two characters (presumably students) had opposing views about the truth or falsity of a mathematical generalization, which was actually false but this piece of information was not revealed to solvers. The research participants were asked, first, to imagine and write a dialogue in which the two characters attempted to convince each other of their viewpoint, and, second, to comment on their dialogues thus distinguishing between the argumentation attributed to the characters and the argumentation that participants themselves considered appropriate. Only one third of the participants indicated clearly that the generalization was false, with many considering the generalization as "not totally wrong" or "only partly correct." Also, many participants created characters that were not convinced by a single counterexample and found certain counterexamples more convincing than others. While with these dialogues participants demonstrated good understanding of the different forms that students' argumentation might take, participants did not clarify in their commentaries whether they themselves considered these forms of argumentation mathematically appropriate, which raises concern about their understanding of the power of a single counterexample to refute a generalization. A similar concern derived from the findings of other PME reports with preservice elementary teachers (e.g., Zeybek & Galindo, 2014).

To examine preservice teachers' mathematical knowledge, Zazkis and Zazkis (2013) used a task that was both *mathematical*, as it asked solvers to comment on their written dialogues thus demonstrating their own mathematical knowledge about argumentation and proof, and *connected to teaching*, as it put solvers in a situation of imagining and articulating different arguments students might offer or consider to be convincing for the particular generalization. This task is an exemplar of a special category of tasks that Stylianides and Stylianides (2006) called "teaching-related mathematics tasks" and defined as follows: "These are *mathematics* tasks that are *connected to teaching*, and have a dual purpose: (1) to foster [or assess] teacher learning of mathematics that is important for teaching, and (2) to help teachers see how this mathematics relates to the work of teaching" (p. 205).[5] In their report, whose contribution is mainly theoretical and thus illustrative of category (3), Stylianides and Stylianides (2006) argued that teaching-related mathematics tasks might serve as a means to promote or assess inservice or preservice teachers' knowledge of mathematics, including argumentation and proof, by taking seriously the idea that these are adults who are, or are specifically preparing to become, teachers of mathematics. Interestingly, many reports that addressed aspects of teachers' mathematical knowledge about argumentation and proof did not offer a compelling argument about why and how these aspects are, or could be, essential for mathematics teaching.

While the majority of reports that made a contribution within category (1) involved (preservice) elementary teachers and identified weaknesses in their mathematical knowledge about argumentation and proof, reports that examined (preservice or inservice) secondary mathematics teachers' knowledge also identified weaknesses (e.g., Gabel & Dreyfus, 2013; Tsamir et al., 2008). For example, in a study with 50 inservice secondary mathematics teachers Tsamir et al. (2008) found the following: Although all research participants correctly proved or refuted six given statements using different predicates and quantifiers and they also correctly recognized the validity of given symbolically-presented proofs for each of those statements, only about half of them identified as invalid a symbolically-presented argument that was not general.

Few reports made a contribution within category (2), which concerns empirical findings about the effectiveness of interventions to enhance teachers' knowledge. Gholamazad (2007) engaged preservice elementary teachers in writing down dialogues between two imaginary characters: "EXPLORER, the one who tries to prove the proposition, and WHYer, the one who asks all the possible questions related to the process of proof" (p. 266). Analysis of the created dialogues showed that such an activity was efficient in leading preservice teachers to "explain *why* and *how* to do instead of just doing" (p. 271). The study of Reichersdorfer et al. (2012), which involved 119 preservice secondary mathematics teachers, also made a contribution within category (2). Using an experimental research design with pre- and post-tests and random allocation of participants in four intervention groups, the researchers examined the effect on participants' argumentation skills of two collaborative learning settings (one with and another without a collaboration script) and two instructional approaches (one based on heuristic worked examples and another based on authentic problems). Key findings included the following: there was no significant difference between the effects of the two collaborative settings; the instructional approach based on heuristic worked examples was more effective for the development of a certain kind of argumentation skills that the researchers characterized as "low level" (e.g., schematic argumentation skills based on a routine application of simple rules); the instructional approach based on authentic problems was more effective for the development of a different kind of argumentation skills that the researchers characterized as "high level" (e.g., evaluating and proving or refuting conjectures). This research cast some light on the complex network of factors that might determine the efficacy of an intervention aiming to enhance teachers' (and possibly other individuals') argumentation skills.

Teachers' knowledge of teaching argumentation and proof (nature and development). The reports in this sub-theme collectively examined various aspects of preservice or inservice teachers' knowledge of teaching argumentation and proof, with attention paid to both the nature and the development of that knowledge. Some aspects of knowledge of mathematics teaching that were addressed by the reports related to knowledge of students, which we define broadly as knowledge of how students learn or understand argumentation and proof (including knowledge of common student conceptions or misconceptions); other aspects related to knowledge of pedagogical practices, which we also define broadly as knowledge of how to support or assess students' learning or understanding of argumentation and proof.[6]

Overall, the reports in this sub-theme make the point that, while teachers' knowledge of teaching argumentation and proof has weaknesses (some of them having their roots in limitations of teachers' mathematical knowledge about argumentation and proof), improvement of this knowledge is possible. Such an improvement can be purposefully engineered in the context of teacher education or professional development courses, or it can happen more naturalistically in the context of teachers' own professional practice. We present few reports to exemplify aspects of this general point.

Monoyiou, Xistouri and Philippou (2006) examined the nature of inservice elementary teachers' knowledge of pedagogical practices, with a focus on teachers' assessment of different kinds of student arguments. Specifically, they conducted semi-structured interviews with 16 teachers who were asked to mark on a given scale different kinds of student arguments for three mathematical generalizations. A key finding was that most teachers gave high marks to empirical arguments, which were generally marked at least as highly as valid arguments.

The report of Barkai, Tabach, Tirosh, Tsamir, and Dreyfus (2009) is an example of a study on the development of teachers' knowledge of students. The research was conducted in a professional development course for inservice secondary mathematics teachers, which aimed to enhance participants' knowledge of mathematics and mathematics teaching related to argumentation and proof. Barkai et al. compared participants' responses before and after the course to the part of a questionnaire that asked them to suggest as many valid and invalid arguments they thought their students would offer for six given statements using different predicates and quantifiers. At the end of the course participants' suggestions of valid and invalid student arguments had increased both in number (by about 50%) and variety. Specifically, participants could offer more valid arguments presented verbally, more invalid arguments based on numerical examples, and more invalid arguments with a repertoire of symbolic lapses. These findings imply that the participants improved their ability to anticipate valid and invalid student arguments expressed with different modes of representation.

Barkai et al. (2009) purposefully engineered the development of teachers' knowledge in a professional development course, which may be viewed as an intervention of long duration. Cirillo (2011) studied in a more naturalistic way the development of a teacher's knowledge of teaching argumentation and proof. Specifically, Cirillo documented the classroom experiences of a beginning secondary mathematics teacher, with strong mathematical background, across his first three years of teaching proof in a geometry class of 15–16-year-olds. With this longitudinal interpretive case study Cirillo cast some light on the challenges faced by beginning teachers in learning to teach proof (even when their mathematical knowledge is not a problem) and on the rather long journey that individual teachers might have to persevere through in order to independently develop their pedagogical practices.

Teachers' beliefs related to argumentation and proof. The reports in this sub-theme have focused predominantly on secondary mathematics teachers (mostly inservice) and have examined teachers' beliefs about the place or purposes of argumentation and proof (or related concepts) in school mathematics (e.g., Chua, Hoyles, & Loh, 2010; Dickerson & Doerr, 2008; Iscimen, 2011), including teachers' views about pedagogical practices related to proof (Dimmel & Herbst, 2014; Miyakawa & Herbst, 2007). We review the findings or broader methodological contributions of some reports in this sub-theme.

The reports of Chua et al. (2010) and Dickerson and Doerr (2008) both focused on the beliefs of inservice secondary mathematics teachers. The first was a questionnaire-based study with 29 teachers who took a 9-hour workshop on pattern generalization and were asked to write their thoughts about the purposes of written justification in pattern generalizations; the second was an interview-based study with 17 teachers concerning their beliefs about the purposes of proof in school mathematics. A key finding of Chua et al. (2010) was that, while almost 60% of the teachers in their study viewed the purpose of a justification to be *explanation*, only one teacher mentioned *conviction*, which is generally recognized to be a core purpose of proof. Chua et al. interpreted this finding with reference to the following distinction between "justification" and "proof" and to the typical form that justification takes in students' work with pattern generalizations:

> A proof is a form of justification given to establish the validity of the rule, but not all justifications are proofs… [In pattern generalization] the explanations provided by learners for justifying how they derive the rule are far less formal than what is expected of in a formal proof. (p. 279)

Dickerson and Doerr's (2008) study focused on proof, and so their findings are not directly comparable to those of Chual et al. A key finding of Dickerson and Doerr was that some teachers believed a major purpose of proof in school mathematics was to develop students' thinking skills and that proofs that deviate from the normative form may undermine this purpose.

Iscimen (2011) examined the development of teachers' beliefs about the place of proof in school mathematics during a geometry course for preservice middle school teachers. Although the course did not focus on proof per se, it did offer plenty of opportunities for participants to engage with proof. Iscimen's findings were based on case studies of six participants who started the course with varying knowledge and beliefs about proof. Over the duration of the course participants started to appreciate the value of proof and its explanatory power for themselves as teachers. Yet, they questioned the value of proof for their students and their students' ability to engage with proof. Similar disappointing findings concerning preservice secondary mathematics teachers' beliefs about the place of proof in school mathematics were reported by Hallman-Thrasher and Connor (2014), though the participants in their study were teacher candidates with STEM backgrounds and thus not typical preservice secondary mathematics teachers who usually are mathematics majors.

Dimmel and Herbst (2014) examined inservice secondary mathematics teachers' views about the appropriate level of detail in a proof being scrutinized during a lesson. The researchers used a novel methodological approach to elicit teachers' views that involved use of comics-based, animated representations of lessons in an experimentally controlled way. Findings, derived from application of the methodological approach with a sample of 34 teachers, showed that teachers held different views about the appropriate level of detail in a proof depending on the kind of statements used in a proof. For example, teachers reacted unfavourably to lesson episodes that showed a teacher asking for explicit justification of statements that were tacitly warranted by a diagram, whereas they favoured asking for explicit justification of statements that were tacitly entailed by definitions. In an earlier study that used again representations of lessons to elicit teachers' views about normative practices in instruction, Miyakawa and Herbst (2007) found that secondary mathematics teachers did not always consider that a proof was the best way to convince students about the truth of a theorem. Rather, teachers valued spending time on other kinds of arguments (including empirical) so as to raise students' epistemic value of the theorem. These findings may help explain some of the teacher pedagogical choices and assessments of student work that we reviewed earlier (e.g., Monoyiou et al., 2006): teachers may privilege the kinds of arguments that raise students' epistemic value of the theorem rather than uphold the norms of the discipline. The findings offer further some insight into the possible sources of student misconceptions related to the power of a proof to establish conclusively the truth of a theorem (e.g., Fischbein & Kedem, 1982) and illustrate the tension that may exist between argumentation and proof (e.g., Duval, 1989).

Reflection on the State of PME Research within Theme 3

The focus of PME research on the nature of teachers' mathematical knowledge about argumentation and proof, with an emphasis on limitations of that knowledge and with less attention being paid to ways of developing that knowledge, reflects the state of research on teachers' mathematical knowledge more broadly (e.g., Ponte & Chapman, 2008). It also reflects a general trend in mathematics education research whereby a disproportionally larger number of studies have identified problems of instruction (limitations of teachers' mathematical knowledge being a case in point) than those studies that have aimed to offer solutions to some of these problems (Stylianides & Stylianides, 2013). More research is thus needed on the development of teachers' mathematical knowledge about argumentation and proof, with the designed interventions taking explicitly into account the idea that effective mathematics teaching requires teachers not only to have good mathematical knowledge but also to be able to use flexibly that knowledge as they support students' learning (e.g., Ball, Lubienski, & Mewborn, 2001).

Of course good mathematical knowledge is in itself insufficient for effective teaching (e.g., Kilpatrick et al., 2001), and so a coordinated approach to improving the teaching of argumentation and proof would have to consider also other teacher-related factors, notably, teachers' knowledge of teaching argumentation and proof and teachers' beliefs about the place or purposes of these concepts in school mathematics. Indeed, some PME reports reviewed earlier and reports published elsewhere have shown that beginning teachers with good mathematical knowledge still face serious challenges in trying to teach argumentation and proof (e.g., Cirillo, 2011; Stylianides, Stylianides, & Shilling-Traina, 2013). They have shown further that teachers tend to have negative beliefs about the appropriateness of proof for their students or about their students' ability to engage with proof (e.g., Iscimen, 2011; Knuth, 2002), as well as beliefs that may foster proof-related misconceptions among students (e.g., Miyakawa & Herbst, 2007). Teacher education and professional development programs have a key role to play in preparing or supporting teachers to teach argumentation and proof. Teachers themselves can also view their teaching practice as a context for ongoing inquiry and development (e.g., Ponte & Chapman, 2008), provided of course that they believe in the importance of argumentation and proof for their students' learning.

CONCLUSION

What this review affords us is a chance to consider the collective approach that has been taken to address the persistent challenge of improving students' experiences with argumentation and proof in school mathematics, and to reflect on how the intellectual resources of the field are being used to address this challenge. In the conclusion of the previous review of PME reports published during 1976–2005 concerning mathematics education research on proof, Mariotti (2006) cautioned against investigations into the teaching and learning of proof being divorced from the reality of the classroom. It is significant, then, that one of the major themes that has emerged from our review of PME reports published during 2005–15 features classroom-based research. However, while we (as a field) have gained more knowledge over the past decade about the nature of argumentation and proof in the mathematics classroom, the findings of PME and other relevant reports show that the typical school experience of students and the treatment of argumentation and proof in textbooks (e.g., Stylianides, 2014) continue to fall short of what is needed to achieve the intent of educational policy documents or curriculum frameworks (e.g., CCSSI, 2010; Department for Education, 2013).

The reports of research on students' conceptions of argumentation and proof and of classroom-based research reviewed in Themes 1 and 2, respectively, focused more on post-elementary school students, while the reports of research on teachers' knowledge and development reviewed in Theme 3 focused more on elementary teachers (notably preservice). What might this variation in focus between the reports belonging to the three themes imply for the status of proof in school mathematics or for researchers' priorities/assumptions regarding that status? The scarcity of classroom-based studies at the elementary school level may be due, for example, to the fact that argumentation activity in most elementary classrooms is sparse. Yet, the same reason could be offered as a justification for the need of more classroom-based research at the elementary school level that would aim to elevate the status of argumentation and proof in elementary classrooms (e.g., Yackel & Hanna, 2003; Stylianides, 2016).

While the classroom-based research reviewed in Theme 2 generally took a wider view of argumentation and proof, exploring what might be involved in helping students appreciate the process of proving and argumentative activity, the research on teachers' knowledge and development reviewed in Theme 3 focused primarily on teachers' understanding of proof as the final product of an argumentative activity. This is problematic because the teachers' role in teaching argumentation and proof is multifaceted and not restricted only to judgements of whether different arguments meet the standard of proof (e.g., Herbst, 2006).

The research on students' conceptions reviewed in Theme 1 and the research on teachers' knowledge of argumentation and proof reviewed in Theme 3 both placed more emphasis on documenting difficulties that students or teachers have with argumentation and proof rather than on developing effective ways to address some of these difficulties. This observation is in agreement with the state of the art in the field as a whole (Stylianides et al., 2016). Design-based research might offer a promising approach to respond to the need for developing effective ways to address students' and teachers' difficulties with argumentation and proof (Stylianides & Stylianides, 2013). Since at the heart of design-based research is to iteratively design and investigate classroom-based interventions (e.g., Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), the methodology provides an opportunity to engineer situations that address questions about students' and teachers' conceptions of argumentation and proof in the "messiness" of real classrooms at the school and teacher education levels, respectively.

Finally, we consider what could be new themes in research about argumentation and proof in school mathematics given the premium put on argumentation and proof in different curriculum frameworks. One theme might focus on ensuring students' equitable access to learning opportunities related to argumentation and proof. Given the complex social dynamics at play inside mathematics classrooms (e.g., Chazan, 2000) and the difficulties teachers face in managing classroom dialogue which is at the core of meaningful learning in mathematics (e.g., Stein, Engle, Smith, & Hughes, 2008), it is important that the field investigates ways of empowering teachers to support all of their students to meaningfully participate in argumentation and proof in the mathematics classroom. Another theme might concern productive ways for assessing students' capacities to not only engage in producing proof, but also to engage in processes that are "on the road" to proof. Research such as the studies reviewed in Theme 1 on students' use of examples or studies published elsewhere (e.g., Zaslavsky, 2014; Zazkis et al., 2008) might support further work into how teachers can identify students' approaches to example use and then act upon their assessments. In addition, as educational policy documents have featured specific standards regarding knowledge for argumentation and proof (c.f., CCSSI, 2010), the research community may need to offer a response about meaningful ways to practically assess students' knowledge and understanding of these mathematical practices.

NOTES

1. Our decision to restrict this review to reports longer than 1 page implies that we did not consider any short orals or poster presentations.
2. The search function was not available in the 2010 Proceedings and we were thus unable to search for our keywords in the abstracts of that year. Instead, we looked for keywords in the titles of the 2010 Proceedings.
3. We were unable to apply Approach 2 for the 2008, 2014, and 2015 Proceedings (which did not include the specific index) and for the 2010 Proceedings (which did not include page numbers or volumes where the relevant reports could be located).
4. Of course, this is not to say that different classes that are taught by the same teacher who uses the same textbook will necessarily receive the same learning experiences (see, e.g., Even, 2008).

[5] The notion of "teaching-related mathematics tasks" was further developed and elaborated in Stylianides and Stylianides (2014) under the slightly modified term "pedagogy-related mathematics tasks."

[6] Our definitions of these two kinds of teachers' knowledge of mathematics teaching draw on the respective definitions of similar kinds of teacher knowledge discussed in Kilpatrick, Swafford, and Findell (2001, pp. 370–372).

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10. RECENT ADVANCES IN RESEARCH ON PROBLEM SOLVING AND PROBLEM POSING

INTRODUCTION

The goal of this chapter is to describe recent advances in mathematical problem solving, as they were represented in research reports from the annual international conferences for the Psychology of Mathematics Education. Delimiting the scope of this chapter was a challenge. If we believe Halmos (1980), that the heart of mathematics is problem solving, and deVault (1981), that "doing mathematics is problem solving" (p. 40), then mathematics education could sensibly consist entirely of the psychological and didactical study of problem solving. Indeed, if we interpreted every investigation that involved individuals engaging in non-routine mathematical tasks or exploring their perceptions of such tasks as an investigation in mathematical problem solving, then the large majority of PME research reports would be classified this way. In this book chapter, we limited our search to PME research reports that focused on the processes of solving or posing problems, including what might be learned from engaging in these activities and how such activities could be implemented in the classroom. Even with this more restrictive search, we found over 200 research reports in the last decade satisfying this criterion. Consequently, we do not aim to provide a comprehensive review of all the articles written on problem solving. Rather, we instead discuss what we as a field have learned in several areas that we, the authors, subjectively judged to be important. Specifically, we chose to organize our chapter around the following themes: (i) problem solving and problem posing as a research tool; (ii) studies on problem solving including problem solving processes and strategies and problem solving competences and expertise in problem solving; (iii) the activity of problem posing; and (iv) teaching problem solving and posing.

The current PME Handbook builds on the previous Handbook of Research on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006) that surveyed the first thirty years of research presented at the PME conferences, ending in 2006. The aim of the current volume is to describe advances in mathematics education research that have occurred since the previous volume a decade ago. However, the previous volume did not contain a chapter on problem solving. As it would clearly be infeasible to extend our literature review to PME's inception (1976), we chose to reference several volumes that represented the state of the field around 2004,

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 353–382.

including Cai, Mamona-Downs and Weber (2005), and begin our review of PME Proceedings in 2004.

State of the Field Around 2004

In this chapter, we distinguish between four broad traditions of problem solving based research: Problem solving as a research tool; problem solving as an object of study; problem posing as an object of study; and problem solving as a didactical tool. Here we clarify what we mean by those terms and discuss the state of the field within these research traditions circa 2004.

By *problem solving as a research tool*, we are referring to studies in which students or teachers are faced with a problem solving task but the researcher is not interested in their problem solving behavior on the task *per se*. Rather, the researcher uses participants' work on the problem solving task as an opportunity to investigate some other construct, such as their understanding of the concepts involved in the task or their beliefs about mathematics as a discipline. We do not give extended focus on this topic in this chapter, as significant studies in this regard will most likely be covered in other chapters of this volume, although we do discuss studies that we found to be particularly noteworthy.

By *problem solving as an object of study*, we are referring to research on what occurs in the processes of problem solving (as opposed to whether students learn content from problem solving or researchers can learn about students' understanding by assigning them a problem solving task). In particular, how do students solve mathematical problems and how can these skills be taught? Schoenfeld (1992) provided an exemplary literature review clarifying the research knowledge base of mathematics education in the early 1990s, including a discussion of what are widely accepted as the four components of mathematical problem solving: *resources* which refer to the mathematical contents and understandings that the problem solver has access to, *heuristics* which refer to techniques or rules of thumb that will not necessarily lead to a solution but are likely to provide the problem solver with an insight that is useful for solving a problem, *metacognition* which refers to the problem solver's propensity to understand the problem and form a plan before engaging in computations and monitoring his or her progress as they work on the problem, and *beliefs* which refer to what the problem solver thinks about the nature of mathematics and how this influences his or her mathematical reasoning. We refer the reader to Schoenfeld (1992) for a more comprehensive review. Since that time, other researchers have elaborated Schoenfeld's problem solving model by considering issues such as individuals' affective states, orientations, and values, as well as how the components of Schoenfeld's model interact during the problem solving process. Perhaps the most notable improvement is the more refined problem solving model presented by Carlson and Bloom (2005) and Goldin's focus on how the role of emotions and affect guide the problem solving process (deBellis & Goldin, 2006; Goldin, 2000).

By *problem posing as an object of study*, we are acknowledging that problem posing is recognized as an important activity in its own right. Problem posing not only has a dialectic relationship with problem solving (i.e., in solving a large problem, we often pose and solve smaller ones, and in posing a problem, we often consider problems that would be challenging to solve), but problem posing is sometimes used as an activity to advance students' understanding and reasoning in its own right.

By *problem solving as a didactical tool*, we are referring to research where problem solving is not the goal of instruction, but rather a *means* to obtain some other end, such as conceptual understanding. Schroeder and Lester (1989) originally defined this as *teaching via problem solving*, referring to this as an important but (at the time) overlooked area of problem solving research. Since that publication, many instructional interventions use problem solving as a tool to elicit students' understandings, introduce conflict, motivate new concepts, or invite students to invent or define new concepts. (See, for instance, Rasmussen and Marrongelle's (2006) discussion of how problems can advance these classroom goals). Indeed, the use of problem solving to teach new mathematics is a central part of many contemporary instructional methodologies and theories, including the constructivist teaching experiment (Simon, 1995) and Realistic Mathematics Education (RME) (Gravemeijer, 1994).

Critical Research Shifts in Problem Solving Research

When Schoenfeld (1992) published his review of problem solving research in the early 1990s, much of problem solving research involved observing how students applied (or failed to apply) domain-general heuristics to solve non-routine problems in domains such as algebra, geometry, and number theory. In 2005, Cai, Mamona-Downs and Weber (2005) published an edited volume on current research in problem solving that illustrated shifts in the topics and methods of mathematical problem solving research. Among the trends noted in this volume were calls to consider practical issues the teacher must face when integrating problems in mathematics classrooms (Leikin & Kawass, 2005; Silver et al., 2005), how problem solving and problem posing interact (Cifarelli & Cai, 2005), problem solving in domain specific areas such as proving (Weber, 2005), and expanding the scope of problem solving tasks to domains such as exceptionally challenging problems (Grugnetti & Jaquet, 2005).

In this subsection, we discuss how problem solving has continued to evolve since Cai, Mamona-Downs and Weber's (2005) edited volume, both in terms of what is being investigated and in the methodologies used to investigate it. Here we identify several trends and cite representative PME research reports that are emblematic of these changes. For the sake of brevity, we do not discuss these articles in any depth. Rather our aim here is to give the reader a sense of the broader types of research related to problem solving that are being conducted. We cite the related articles to provide the reader with direction in case the reader is interested in studying these trends further.

There are more recent studies on the practical difficulties that the teacher faces when implementing problem solving activities in his or her classroom. Researchers have continued to venture out of laboratory settings and toward examining the complexity of implementing problem solving in classroom practice. Some researchers are concerned with the mathematical and pedagogical knowledge that teachers need to implement problems effectively in their classrooms (e.g., Barabash, Guberman, & Mandler, 2014; Levav-Waynberg & Leikin, 2006), with Chapman (2012) and Foster, Wake and Swan (2014) proposing practice-based models of the mathematical knowledge that is necessary for teaching problem solving effectively. Other researchers have analyzed the complexity and practical difficulties involved in orchestrating problem solving lessons in a manner that is consistent with reform-oriented documents (dePaepe, De Corte, & Verschaffel, 2006; Fritzlar, 2004). Zodik and Zaslavsky (2004) have analyzed effective dispositions and actions that a tutor modeled to promote effective problem solving with his students.

Some research on problem solving tends to focus on domain-specific, rather than domain-general, problem solving processes. Some researchers on problem solving have focused on identifying strategies and disposition for solving certain types of problems, rather than identifying heuristics that might be useful across a wide range of mathematical content (e.g., look at simple cases). Domains that were investigated included measurement estimation (Hurang, 2004), geometric area (Mamona-Downs, 2006), algebraic functions (Mousolides & Gagatsis, 2004), problem solving with dynamic geometry software (Haug, 2010; Leikin, 2004), proof reading (Tay et al., 2014), proof writing (Torregrosa & Quesada, 2008; Weber, 2004), and unconventional Fermi problems (Albarracín & Gorgorió, 2012).

There is a wider range of methodologies to investigate problem solving.Most reports published in the volume of Cai, Mamona-Downs and Weber (2005) investigated students' problem solving using verbal protocol analysis or video-recording students' or teachers' work as they solved problems in situ. In a desire to increase the number of participants in their research, some researchers have attempted to infer students' problem solving processes from the written records of their work (e.g., Assmus, Forster, & Fritzlar, 2014; Mousolides & Gagatsis, 2004), with the researchers sometimes using modified worksheets to make students' reasoning and metacognition more transparent (Leppaho, 2008; Quek, Toh, Leong, & Ho, 2014). Others have adopted tools from semiotics (Arzarello, 2006) or cognitive science, such as brain imaging (Leikin, Waisman, Shaul, & Leikin, 2012, 2014) and eye-tracking (Obersteiner et al., 2014, Andrà, Arzarello, Ferrara, Holmqvist, Lindström, Robutti, & Sabena, 2011), to investigate problem solving processes.

The goals of problem solving have become more expansive. In Schoenfeld (1992) and Cai, Mamona-Downs and Weber (2005), the didactical goals associated with problem solving generally consisted of improving students' abilities to solve problems or to learn some mathematical content. However, in the research reports that we have reviewed, we found problem solving also used for fostering creativity (Amit & Gelat, 2012; Vale et al., 2012), valuing and validating graphical reasoning (Gonzales-Martin & Camacho, 2004), promoting equity and diversity (Powell, 2004), increasing students' agency (Radu, Tozzi, & Weber, 2006), eliciting students' social values (Shimada & Baba, 2012), and improving their values and identity (Park, 2014).

Technology has influenced the problems that we pose and the strategies that students use to understand these problems. We discuss this trend in more detail later in this section. This wide array of research aims and methods will be present throughout our chapter. In the concluding section, we discuss the strengths and weaknesses of the heterogeneous approaches to researching problems solving in mathematics education.

Characterizing Mathematical Problems and Problem Solving

Starting with Polya's (1945) *How to solve it*, many researchers focused on questions such as, "what is a mathematical problem?", "how can problems be characterized?", and "how can problems be classified?". To Polya (1945), solving a problem was associated with overcoming some difficulties to deduce something that was previously unknown to the problem solver. Schoenfeld (1985) provided a similar description, defining a problem as a task in which the solver did not possess an algorithm to complete. Hošpesová and Novotná (2009) noted that there was not a shared definition as to what constitutes a word problem. They followed Verschaffel, Greer and De Corte (2000) in treating word problems as "verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement". Hence to Hošpesová and Novotná, a word problem could be solved by a sequence of correctly ordered operations. On the contrary Son (2005) proposed a different definition of problem, identifying a problem as a task or problem that does not have accompanying solutions or answers presented. Hence, Son's description is broader than Hošpesová and Novotná's as it might include traditional exercises.

More recently, mathematics educators have explored the link between problem solving and mathematical modeling (e.g., English & Watters, 2005; Mousolides & English, 2008; Peled & Bassan-Cincinatus, 2005) and problem solving and problem posing (Cifarelli & Cai, 2006), often including modeling eliciting activities and problem posing as problem solving tasks. This expansion of problem solving extends some previous descriptions of problem solving. With problem posing, one is not deducing what is unknown but is instead deciding on what might be valuable or interesting to deduce.

With mathematical modeling, one is not merely applying operations but often making assumptions that specify what operations make sense and what the quantities in the operations should be (e.g., Schukajlow & Krug, 2012). According to English and Watters (2005), "a modeling problem is a realistically complex situation where students engage in mathematical thinking (beyond that of the traditional school problem) and generate conceptual tools needed for some purpose. Modeling problems foster and reveal children's mathematical thinking thus enabling teachers to capitalize on the insights gained into their children's mathematical developments" (ibid, v. 2, p. 302). Other researchers have considered solving non-standard 'problematic' problems (Dewolf, Van Dooren, Hermens, & Verschaffel, 2013), addressing real-life situations (Bonotto, 2009) that require the solver to incorporate real world knowledge not explicitly stated in the problem task.

In our review of the research reports of PME, we found most studies treated the constructs of problem and problem solving as commonly accepted, often implicitly adopting the similar conceptualizations of Polya (1945), Schoenfeld (1985) and Carlson and Bloom (2005). Given the expansion of the constructs of problem and problem solving in recent mathematics education research, specifying what one means by problem and problem solving would add clarity to the field.

Changes in Problems Design and Capacity Associated with Uses of Technology in School Mathematics

In recent years, there has been a rapid advance in the quality of educational technology in mathematics and the access that students have to this technology. We first discuss how technology has changed the types of problems that we can give to students and then describe how the presence of technology changes the way that students solve problems.

Han and Chang (2007) explored problem solving with a computer algebra system (CAS) with eight secondary students from Korea. These students were observed as they completed three types of problem solving tasks: modeling in algebra, finding patterns in integration, and optimizing the surface area or volume of a particular solid. Using CAS advanced the level of complexity of the tasks that students were able to solve. Moreover, the students showed great interest in solving more complicated modeling problems with the CAS. Kieran and Guzman (2009) and Guzmán, Kieran and Martínez (2011) also explored how a CAS can expand the types of problems that students can address. They presented one teacher with a task where he used CAS to simplify algebraic expressions of the form $x^n - 1$ for many natural numbers n then used these results to form conjectures and write a proof. By engaging in this activity, the teacher came to appreciate how CAS can enable more sophisticated activities for students, including those that involve conjecturing and proving.

Leikin (2004) developed a framework on the quality of a mathematical problem based on several seminal works in problem solving (Polya, 1981; Schoenfeld, 1985; Charles & Lester, 1982) using four conditions (1) having *motivation* to find a solution; (2) having *no readily available procedures* for finding a solution; (3) making an *attempt* and persists to reach a solution; (4) the task or a situation has several solving approaches. Leikin stressed that these criteria are relative and subjective; for instance, a task that is cognitively demanding for one person may be trivial for another. She argued that use of a dynamic geometry environment had the potential to raise the quality of mathematical tasks with all four criteria. Lavy and Shriki (2007) explored problem posing supported by a dynamic geometry environment as a means for developing mathematical knowledge of prospective teachers. They found that these problem posing tasks in this environment led prospective teachers to examine definitions and attributes of objects, form connections between objects, and evaluate the validity of their arguments. Nonetheless, the prospective teachers were reluctant to pose difficult problems in which they could not find a conventional proof for their solution. This illustrates how a dynamic geometry environment expands the problems that prospective teachers will pose while the demands of formal proof may narrow them.

With respect to how technology may influence the problem solving process, Naftaliev and Yerushalmy (2009) explored how the use of printed diagrams or interactive diagrams create different contexts for mathematics learning. The interactive diagrams made it possible to address a given graph as a sketch that reveals the "big picture". The process of concept construction was based on the students' decision to change the representation of the data, build a focused collection of data, expand existing or build new representations. By exploring two students' solution processes to an algebra problem, Lew and So (2008) documented how graphing technology influenced students' problem solving processes. In particular, the technology allowed the students to evaluate the effects of introducing assumptions, explore the graphical consequences of changing assumptions, and test their conjectures using graphs. The authors argued that such exploration would simply be infeasible using only paper and pencil.

Jacinto and Carreira (2013) conducted a study in which they investigated problem solving activity within the context of a web-based beyond-school competition. By exploring four different solutions from students who employed GeoGebra, a dynamic geometry software, the authors found different levels of mastery of the mathematical and technological content. This illustrates that the benefits that technology can bestow on the problem solving process depends on the background of the individual student.

PROBLEM SOLVING AND PROBLEM POSING AS A RESEARCH TOOL

As noted in the introduction to this chapter, problem solving is commonly used as a research tool for exploring different aspects of mathematical reasoning, learning and teaching. Here are some representative examples. Some researchers have used problem solving tasks to infer students' understanding of mathematical concepts. Barmby, Harries, Higgins and Suggate (2007) illustrated how mathematical understanding can be assessed by seeing what connections students form between symbolic procedures and informal problem solving situations. Barmby et al. (2007) equated mathematical understanding with participants' ability to cope successfully with 'open' problem solving situations where what is required is not obvious. Deliyianni, Elia, Panaoura and Gagatsis (2009) used Confirmatory Factor Analyses on a large sample of 1701 primary and secondary school students of students working on problems concerning decimal numbers to infer their understanding of decimals. They performed a similar analysis to infer students' understandings of fractions.

Similarly, some researchers have used problem solving tasks to infer students' general mathematical abilities. Ryu, Chong and Song (2007) analyzed mathematically gifted students' spatial visualization ability with regard to three-dimensional figures; the research team asked these students to mentally rotate these figures given their two dimensional representations and perform other tasks that required mental visualization. Park, Ko, Lee and Lee (2011) analyzed the analogies that mathematically gifted students would create in statistics by using problem posing as a research tool. The gifted students were asked to construct similar problems in statistics.

Other researchers have used problem solving to uncover conceptual gaps in students' understanding. In a study by Gervasoni et al. (2011), the researchers utilized problem solving to identify and explain difficulties in mathematics of children who were judged as having a strong understanding of two digit numbers based on traditional measures. By giving these students a specially designed problem solving task, Gervasoni et al. demonstrated that these students could not identify 50 on a number line. Still other researchers have used the problems that teachers pose in their classrooms as a means to infer their goals of instruction. Hiraoka and Yoshida-Miyauchi (2007) proposed a framework for creating or analyzing Japanese lessons from the viewpoint of mathematical activities, illustrating their framework by analyzing a specific lesson on fractions. These authors illustrated that although few problems were posed in the lesson that they explored, the purpose of these problems was to advance students' understanding of the mathematical content.

Finally, some researchers have used problem solving for the purposes of cross-cultural comparisons. Yuan and Presmeg (2010) presented high achieving secondary students from China and the United States with problem posing tasks as a lens for investigating different levels of creativity between these two groups of students. Yuan and Presmeg found that while United States students were better able to articulate their problems, the Chinese students were less likely to produce trivial problems and more likely to produce problems requiring a creative solution. In a study by Smith, Gerretson, Olkun, Akkurt and Dogbey (2009), the researchers examined whether combining causal (how events cause changes in the situation described in the problem) and outcome related elements (that can be computed based on the story described in the problem) with mathematical content improves mathematical word problem solving performance. The researchers found significant differences among the formats (schema and situation models), as well as between genders and countries.

Among the studies presented in the PME volumes the authors used different criteria for the selection of problems used in their studies. The criteria were clearly related to the purposes of having students engage in the problem, the mathematical content of the activity, and the types of participants of the research. Among these criteria, the authors used the complexity of the problem as associated with problem familiarity or similarity to another problem and the complexity of the problem solution (Jiang & Cai, 2015).

In summary, the majority of the PME research reports that we reviewed involved individuals solving or posing problems in a variety of ways. However, in many of these cases, the problem solving and problem posing tasks were used as a research tool for addressing issues aside from problem solving. These issues include the types of students reasoning in different branches of mathematics, differences in mathematical reasoning of a particular type (e.g., critical or creative reasoning, analogical or inductive reasoning, generalization and visualization) of different groups of participants, exploring mathematical difficulties associated with concrete mathematical concepts, properties and procedures. This makes sense. If a researcher wants to analyze a student's understanding or reasoning, then the researcher will probably avoid giving that student a task that he or she could use a standard algorithm to complete, as the execution of an algorithm reveals little about that student's cognition. It is usually more productive to give students a challenging non-routine task.

STUDIES ON PROBLEM SOLVING

In this section, we discuss research on strategies and approaches for solving problems and then competencies and expertise associated with solving mathematical problems.

Strategies and Approaches for Solving Problems

A number of researchers have tried to advance Schoenfeld's (1985, 1992) problem solving model by identifying specific strategies that students use to solve problems. For instance, some researchers have illustrated how students used their experiences solving previous problems to inform their work on future problems. Uptegrove and Maher (2004) described how five students first asked students to consider the number of 4-tall towers made with two colors as well as the number of such towers that have exactly one color being used *r* times. Later, these students were asked to count the number of pizzas made with *r* toppings when there were four toppings to choose from. In the latter case, the students identified similarities between the pizza task, the tower task, and Pascal's triangle, and used these similarities to develop a more efficient way to solve the problem. Powell (2006) described how students formed similar relationships between the Pizza and the Towers problem with a structurally isomorphic Taxicab problem (i.e., how many ways are there on a city block grid map to get from one specific point to another specific point). As a point of contrast, Karp (2006) studied the problem solving of 117 secondary students who were taught mathematics in a similar manner. Karp found that when these students solved structurally similar absolute value problems, for the most part they saw no relationship between the two problems. Uptegrove and Maher's (2004) and Powell's (2006) research occurred in a special longitudinal study in which students could revisit the same task and were given unlimited time to work on individual tasks, suggesting that experiences such as those may be useful for students recognizing similarities between structurally isomorphic tasks. Based on observing 89 primary school students solving a pair of isomorphic tasks, Assmus, Forster and Fritzlar (2014) developed a theoretical model of how students can recognize analogous tasks consistent with the account above. In particular, in this model, the role of reflecting on one's solution to a problem (often led by prompting from an interviewer) was integral in students' construction of analogies- the extended time and revisiting in Uptegrove and Maher's (2004) and Powell's (2006) research environments may have encouraged the reflection that Assmus, Forster and Fritzlar considered to be important.

A second area of investigation is the role that social processes play in solving problems. In his analysis of a group of secondary students successfully solving a problem, Powell (2006) documented that no single student's contributions would be sufficient to obtain a solution leading him to argue that the social cognition generated in the group was needed to solve the problem. Tatsis and Koleza (2004) examined the critical role that social processes played in students developing an understanding of the elements of a problem solving task and documented the roles that different students played in negotiating that meaning. In studying 40 fifth grade students solving a problem in pairs, Lange (2012) focused on cooperation acts that he found to be specific to problem solving, such as presenting an incomplete solution to one's work for another student to build on as well as evaluating and finding errors in a partner's work. What these studies have in common is that the processes that students used working collaboratively transcended individuals' efforts in problem solving and accounted for the benefits of collaboration on problem solving tasks.

A third area of investigation has been documenting the complexity of trying to model the problem solving process. Czocher (2014) observed that most frameworks of how students solve model-eliciting problems were cyclic and iterative. However, in her analysis of four university engineering students completing modeling-eliciting problems, she found that students' problem solving processes were neither cyclic nor iterative. Consequently, Czocher called for the development of a more fine-grained model that can account for the often messy behavior of students completing modeling tasks. Other researchers have documented how students' self-regulation (Marcou & Lerman, 2006) and confidence (de Hoyos, Gray, & Simpson, 2004) play a critical role in their decision-making processes, even though such factors are not currently included in some existing models of students' problem solving.

A fourth area of investigation concerns identifying domain specific reasoning strategies. Na, Han, Lee and Song (2007) explored mathematically gifted 6th grade (age 12) students' problem solving approaches on conditional probability, a topic which they had not yet studied. The questions were of the form, "find the probability of an event B given that A happened". Na and colleagues classified students' solutions strategies into three approaches: (i) finding P(B) and ignoring whether A occurred, (ii) finding the probability of A and B both occurring, or (iii) finding P(B|A) and attempting to provide a valid reason for this. The researchers found that the way the question was framed influenced the strategies that were used. Pehkonen and Kaasila (2007) focused on the understanding and reasoning in a non-standard division task. The following non-standard division problem was used: "We know, that 498/6 = 83. How could you conclude from this relationship (without using long division algorithm) what is 491/6?" When analyzing students' problem solving strategies, the researchers concluded that division seems not to be fully understood: only one fifth of participants produced a completely correct solution and the central reason for mistakes was insufficient reasoning strategies. In a similar study, Panoutsos, Karantzis and Markopoulos (2009) analyzed 16 sixth graders strategies for solving real life problems involving ratio. Students' reasoning appeared not to follow a predetermined path, but varied their approaches depending on the structure and qualitative features of the task. In some cases, students reverted to incorrect reasoning strategies such as addition. These results complement the findings of an earlier study by Van Doreen, De Bock, Vleugas and Verschaffel (2008). In their study, these researchers demonstrated that students would be less likely to make naïve mistakes about operator choice if they were asked to classify the problem prior to solving it. This research team had other investigations into how wording and context could alter problem solving performance. Van Dooren, De Bock, Janssens and Verschaffel (2005) showed that students' problem solving behavior strongly improves when the non-linear problem is embedded in a meaningful, authentic performance task whereas experience does not improve students' performance when the non-linear problem is offered as a word problem. Dewolf, Van Dooren, Hermens and Verschaffel (2013) asked whether students attend to and profit from representational illustrations of non-standard mathematical word problems. These non-standard problems were originally presented in Verschaffel et al.'s (1994) study to upper primary school children. The current study analyzed how illustrations that represent the problematic situation helped higher education students to visualize the problem situation and solve it more realistically. The researchers found no effect of the illustrations on the realistic nature of their solutions.

Rott (2011) explored the problem solving processes of fifth graders solving a particular task called the Beverage Coasters task. The Beverage Coasters task was difficult for these students as it involved covering area, a concept that students had not yet discussed in their curriculum. An analysis of the problem solving processes of 32 students working in pairs on a geometry task was performed using Schoenfeld's (1985) model for the analysis of problem solving strategies. Only one of those students showed behavior that was coded as *planning* and none of the students attempted to *verify* his or her solution. Only two pairs had *full access* and found correct solutions. Rott found a significant correlation between the students' problem solving behavior and their success. Papandreou (2009) investigated preschoolers' semiotic activity during their solving arithmetic problem concerning the addition of seven quantities. Based on the analysis of 117 preschoolers' drawings in combination with their verbal descriptions, Papandreou produced 16 "notation types" distributed in four main categories: letters or words, pictograms, arbitrary non-conventional symbols and numerals. The researcher recommended integration of drawing in preschoolers' mathematics education. Fesakis and Kafoussi (2009) examined whether and how computer microworlds and manipulatives could influence kindergarten children's capabilities in combinatorial problems. They found that children could solve combinatorial problems without applying any systematic strategy. The microworld did not seem to have a significant impact on their problem solving strategies, although the children participating in the experiment did use the microworlds and manipulatives to make sense of their solutions and showed significant decrease in errors.

A final category of studies concerns the types of processes that students might engage in to improve their problem solving performance. Kolovou, van den Heuvel-Panhuizen and Elia (2007) performed a quasi-experimental study on how primary school students solved non-routine puzzle-like word problems. The study showed that the online learning environment and writing down the solution procedure had a significant positive effect on students' problem solving performance. Jacinto and Carreira (2013) analyzed students' problem solving activity within the context of a web-based beyond-school competition devise for GeoGebra supported approaches: Using GeoGebra to obtain, interpret, validate, and explore a solution. These approaches are consistent with Polya's (1945) problem solving stages of implementing a plan and looking back. The results of this study were consistent with previous research that the dragging capabilities of dynamic geometry software can facilitate understanding a problem and planning a solution path (Healy & Hoyles, 2002; Hölzl, 1996), illustrating how technology can change the way that students reason about problems.

Problem Solving Competencies and Expertise

Problem solving expertise is usually considered by studying "experts" – that is, individuals or groups of individuals who exhibit superior performance in problem solving. This population often includes mathematicians (e.g., Schoenfeld, 1985) but can also include students that were "excelling in mathematics and generally gifted students" (Leikin, Waisman, Shaul, & Leikin, 2012, 2014). Before proceeding further, it is important to note that both what constitutes an "expert mathematician" and a "mathematically gifted student" are usually based on vague, ill-defined, and not universally shared categories by researchers (deFranco, 1996; Leikin et al., 2012, 2014). Previous research has demonstrated that the expertise of highly successful problem solvers includes the ability to focus attention on appropriate features of problems, having a more robust representation and mental imagery of mathematical concepts and situations, having more images, and having the capacity to strategically switch efficiently and effectively between different images (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994). The strategic use of representation systems and choice of solution methods is often based on the problem solver's experience and ability to classify problems. The expert problem solver "knows into which category the problem should be placed and knows which moves are most appropriate, given that particular type of problem" (Sweller, Mawer, & Ward, 1983, p. 640). Experts also have the ability to monitor their progress, marshal heuristics when they do not know how to proceed, and hold productive beliefs about mathematics (Schoenfeld, 1985, 1992). In this section, we report further work in this area as well as researchers who wish to extend the notion of problem solving expertise to include other constructs.

Some researchers have attempted to focus on components of expertise that students need to be taught and compare those that will naturally emerge if students are put in the right environments. For instance, Rott (2012) examined 64 fifth grade students solving problems and found they spontaneously used heuristics such as drawing diagrams and working backwards and that these heuristics played an important role in some students' solutions. He argued that these findings imply that, as a field, we do not have a good sense of what young students are capable of doing. Mousolides and English (2008) and Doyle (2006) arrived at a similar conclusion when they gave model-eliciting tasks to fourth and fifth grade students. These students were capable of using strategies such as organizing their data, building representations, and posing productive problems, despite not having extensive experience in these areas. These findings all suggest that even young children may have problem solving capacities, but special tasks are required to elicit them.

Andrà et al. (2011) used an eye-tracking methodology to examine how students with different levels of mathematical expertise read different mathematical representations- formulas, graphs and words. They used eye tracking to highlight the ongoing process of making sense of mathematical representations during problem solving. The study indicated quantitative and qualitative differences between the novice and expert group. Waisman, Leikin and Leikin (2015), used event related potentials (ERP) methodology, the brain activation associated with mathematical problem solving. They focused on the adolescents who differed in the combination of expertise in school problem solving and general giftedness. The researchers presented all participants learning-based and insight-based problems. They defined insight-based problems as ones that have a relatively simple solution which is difficult to discover until a solver changes his or her way of thinking (Weisberg, 2015), which solutions are usually based on the understanding of the problem structure and on intuition, and thus are unexpected and surprising. Patterns of individuals' brain activation associated with solving learning-based and insight-based problems were explored by measuring the strength of electrical potentials (measured with ERP procedure) and their scalp distribution. Waisman, Leikin and Leikin demonstrated that patterns of brain activation of experts who are generally gifted differed from those of experts who were not identified as generally gifted. They also demonstrated that these differences are task-dependent and differ for learning-based and insight-based problems.

Researchers who study problem solving have sought to expand the attributes of what makes an expert problem solver. In the introduction to the PME plenary panel, Gravemeijer (2007) argued for a holistic way of understanding expertise in problem solving. He claimed that the qualities of a successful (expert) students should not only be comprised of being able to produce solutions to problems, but also include notions such as: perseverance of effort, a willingness to engage with challenging problems, an ability to transfer knowledge to new contexts and novel problem solving situations, a broad awareness of mathematics around us, creativity and flexibility in thinking, and an ability to explain their thinking. Success in solving a variety of novel problems accompanied by a willingness to do so can be considered a critical feature of students who are developing their problem solving expertise.

To add to the list of expert problem solving competencies, Lee, Kim, Na, Han and Song (2007) examined how mathematically gifted students utilize induction, analogy and imagery. The subjects of this research were 6th and 8th graders receiving education in an academy for the gifted attached to a university. They demonstrated that induction, analogy and imagery played important roles in making mathematical discoveries by the study participants. Rott (2013) was curious if it was reasonable to say that students could have expertise at solving problems when they were as young as twelve years of age. Rott compared the problem solving strategies of fifth and sixth graders who were adept at solving problems with those who were not. His study demonstrated that the young expert problem solvers showed superior performance in all of the tasks that they completed as well as higher "mental flexibility" in executing the processes, which seemed to distinguish them from the novices without this expertise. Based on these findings, Rott (2013) suggested that the abilities to use inductive evidence, form analogies, and exhibit mental flexibility should all be considered as components of expertise, especially when analyzing the capabilities of children.

Another characteristic that can be considered as related to problem solving expertise is mathematical creativity. Amit and Gilat (2012) highlighted the importance of creativity, showing it was a critical component of fifth graders solving a modeling eliciting problem. Leikin and Lev (2007) introduced multiple solution tasks as a lens for observing mathematical creativity. In this perspective, expert solution spaces were defined as the most complete sets of solution strategies known at the moment. Leikin and Lev developed a scoring scheme for the evaluation of creativity based on the number of different problem solving strategies used by a student as well as rating how uncommon the students' solution sets were. Using their framework, the authors distinguished between generally gifted, students who were proficient in problem solving, and typical students. They also found that presenting students with unconventional tasks provided a better lens into students' creativity. Similarly, Amit and Gilat (2013) also analyzed the process of mathematical modeling through a creativity lens. The participants were mathematically talented primary school students who were members of "Kidumatica" math club. The researchers found that students' creative skills as manifested in the diversity of their significant mathematical ideas and the variety of approaches led them to create, invent and discover significant conceptual tools. Leu, Lo and Luo (2015) assessed the mathematical creativity of preservice Taiwanese teachers. Participants' performance on three cognitive dimensions of divergent production: fluency, flexibility, originality, as well as on overall mathematical creativity for each item were also discussed.

While the standard components of traditional expertise—the flexible use of robust representation systems, the application of a wide range of metacognition, and the ability to monitor or self-regulate one's work—continue to influence how researchers analyze problem solving research, other constructs, such as persistence and creativity, also are recognized as important. We concur that there should be more research in these emerging areas.

STUDIES ON PROBLEM POSING

Currently problem posing is recognized as an important mathematical activity and there has been a concerted effort to make problem posing play a more prominent role in mathematics instruction (Singer, Ellerton, & Cai, 2013, 2015). However, in a curricula study of Chinese and United States textbooks, Jiang and Cai (2014) argued that that problem posing plays a minor role in both countries, with few problem posing tasks appearing even in reform-oriented textbooks. In this subsection, we discuss recent trends in research on problem posing.

In 2011, Singer, Ellerton, Cai and Leung organized a PME research forum on problem posing in mathematics learning and teaching. One aim of this research forum was to discuss critical perspectives on problem posing and to provide critical organization and structure to problem posing research. A contribution towards this goal was Kontorovich and Koichu's (2009) proposed comprehensive framework of mathematical problem posing. Their framework consists of four facets: resources, problem posing heuristics, aptness, and social context in which problem posing occurs. The notions of resources and heuristics largely were direct problem posing correlates of Schoenfeld's (1985) problem solving framework. The notion of aptness considered whether the posed problem met the purposes or intention of the problem posing exercise. The notion of social context analogizes loosely with Schoenfeld's (1992) notion of mathematical practices, in that what counts as a suitable or good problem would depend on the knowledge of the audience and the social norms of the community receiving it. Kontorovich and

Koichu's framework is notable in that it suggests directions for problem posing research in each of these areas.

Some researchers have examined the role that problem posing has played in problem solving. For instance, Armstrong (2014) studied four groups of three middle school children solving a particularly challenging mathematical problem. She found that each group frequently posed problems to facilitate their investigations, with the four groups collectively posing 62 problems. The problem posing activities helped students clarify the task they were solving, make sense of and make use of the hypotheses of their investigations, and form and test plans for how their investigations could proceed. These results are consistent with a study by Cifarelli and Cai (2006), in which these researchers observed one pair of students solving two open-ended problem solving tasks in which the students were posed with a situation and asked to develop mathematical relationships. The students in their study posed problems to develop and test the mathematical relationships that they generated. Bonotto (2009) reported similar findings when she asked students to solve problems that required the incorporation of real world knowledge. Both Armstrong (2014) and Cifarelli and Cai (2006) noted the diversity of problems that the students posed, cautioning researchers not to pose a single model for how problems should be posed, and both remarked on how the social roles inherent in collaborative problem solving encouraged problem posing.

A second trend concerns the cognitive processes used by teachers in posing mathematical problems for didactical purposes. Harel, Koichu and Manaster (2006) explored how 24 middle school teachers could design a story problem whose solution can be found by dividing 4/5 by 2/3. They found that 60% of their sample was able to produce some answerable questions but only 20% were able to successfully complete Harel et al.'s task. One strategy used by both successful and unsuccessful teachers was that of utilizing reference points: focusing on an easier task (e.g., posing a simpler computation that would yield 4/5 as a solution) and then attempting to transform this into a problem that met the criteria of the task. Pelczer, Voica and Gamboa (2008) asked 18 first year university students to pose related problems about sequences (e.g., $a_n = 2n$ is a sequence) increasing in difficulty. Their results paralleled the findings of Harel et al. (2006). They found students largely used a selection-transformation-pose strategy in which they generated a problem and then attempted to transform the problem to obtain more difficult problems. While this strategy worked well for some students, it produced uninteresting problems for others. Jiang and Cai (2015) investigated the impact of sample questions on the sixth grade students' mathematical problem posing. They evaluated students' posed problems according to their solvability, similarity to sample problems and their complexity level. This research is notable in that it calls for and introduces a framework to evaluate the utility of posed problems.

A third category of studies explores the benefits that students can gain from engaging in problem posing. Yuan and Presmeg (2010) found a correlation between advanced high school students' problem posing abilities and their score on a creativity test, supporting the claim that there is a relationship between problem posing and creativity. Powell (2004) hypothesized that having students from marginalized groups participate in activities such as problem posing could increase their agency. Radu, Tozzi and Weber (2006) found support for Powell's claim. They analyzed an after-school program of mathematical problem solving for middle school students, focusing on how the teacher-researchers' actions expanded students' participation. They found that encouraging students to pose problems led them to take ownership of the mathematical investigations and increased students' understanding. Bragg and Nicol (2008) found that engaging preservice teachers in problem posing tasks led them to adopt more desirable beliefs and attitudes toward teaching mathematics and Bonotto (2006) illustrated the potential that problem posing has for increasing content understanding.

TEACHING PROBLEM SOLVING AND PROBLEM POSING

We break this section into four parts. First we report the influence that teachers' perceptions of problem solving have on instruction. Second, we describe the effects that interventions had on students' problem solving performance. Third, we discuss various efforts to educate teachers to use problem solving effectively in their classroom. Finally, we discuss how problem solving is sometimes used as a tool to teach new mathematical concepts.

Teachers' Perceptions of Problem Solving and its Influence on Their Teaching

The ways in which teachers use mathematical problems in the classroom are central in evaluating the quality of mathematics teaching. Henningsen and Stein (1997) stressed the importance and the difficulty of teaching high-level mathematical problems with high cognitive demand. However, it is not sufficient for mathematics educators simply to supply practicing teachers with rich mathematical problems. Which problems teachers choose to use and how they implement these problems are necessarily influenced by their own understanding of the mathematics involved, their pedagogical goals, and their beliefs about mathematics, teaching, and the capabilities of their students.

In this section, we first analyze the knowledge and beliefs of teachers and how this impacts on their use of problem solving activities in their classrooms. Shajahan (2005) examined the problem solving competency of prospective teachers in a dynamic geometry environment. He found that teachers' subject matter played a significant role in how they interpreted, solved, and appreciated the problem. Sullivan, Clarke, Clarke and O'Shea (2009) investigated the relationship between the way three teachers in Australia implemented tasks in fifth and sixth grade classrooms in order to explore the links between tasks, teacher actions, and student learning. They found that the three teachers implemented the tasks in different ways. Two teachers modified the task to reduce its complexity and encouraged a single solution method for students, while the other encouraged multiple solutions. Sullivan et al. suggested that it was the latter teacher's increased mathematical confidence in her ability that led her to allow more room for exploration in the classroom. These findings are broadly consistent with Chapman (2012) and Foster, Wake and Swan's (2014) claims that teaching problem solving requires deep mathematical knowledge on the part of the teachers and more work on what specific knowledge is required is needed.

Chapman (2011) investigated prospective teachers' ways of making sense of mathematical problem posing [PP] and the impact of posing various types of problems on their learning. Chapman highlighted five perspectives of problem posing and nine categories of PP tasks important to support teachers' development of proficiency in problem posing knowledge for teaching. Klinshtern, Koichu and Berman (2013) explored how mathematics teachers posed problems for students in their teaching. The majority of the teachers identified themselves as problem posers as they taught. This challenged previous research that teachers did not pose problems as they taught. Klinshtern et al. also looked at strategies that teachers used to pose problems, such as elaborating upon an existing problem, combining ideas, and shifting contexts.

A second group of studies explored teachers' perceptions and evaluations of problems, as well as their suitability for teaching. Applebaum and Leikin (2007) examined teachers' conceptions of mathematical challenge in school mathematics. They found that the teachers possessed a broad conception of mathematical challenge, including the relative and contextual nature of challenge based on the intended audience for the problem. The teachers, however, were not always convinced of the possibility of incorporating challenging mathematics in everyday classrooms. Jacobson, Singletary and De Araujo (2011) explored United States secondary teachers' conceptions of what constituted an integrated curriculum. The study found that the teachers needed to engage in fairly sophisticated problem solving behavior to recognize the mathematical connections between the problems in that curriculum. Koichu, Katz and Berman (2007) and Sinclair and Crespo (2006) examined what prospective teachers thought was a beautiful problem. Both research teams found that prospective teachers expressed a wide range of conceptions of what constituted a beautiful problem. Interestingly, Koichu, Katz and Berman reported that the correlations between problem difficulty, beauty, and challenge were very small for all the problems, suggesting that to prospective teachers, these constructs are relatively independent of one another. (For an interesting comparison, Inglis and Aberdein (2015) found that mathematicians appraised qualities of mathematical proofs, such as simplicity, utility, and aesthetics, as relatively independent constructs).

The Effect of Instruction on Students' Problem Solving Performance

Two PME research reports presented large-scale studies that summarized the curricular impacts of students' problem solving ability. Cai and his colleagues

(2014) compared the performance of high school students in the United States who were taught using the *Connected Mathematics Program* (CMP), a curriculum rich with problem solving tasks, with those who were taught in a traditional fashion. The 11th and 12th grade students who used the CMP performed better on multi-step problems than those who did not, demonstrating that the CMP had a long-term effect on students' abilities to solve problems. Eade and Dickinson (2006) described the effects of implementing a Realistic Mathematics Education (RME) based curricula in English middle schools. Eade and Dickinson found students' problem solving abilities improved after receiving the RME curriculum after one year, but they also identified practical issues that teachers faced while implementing it, such as having vocabulary and methods to measure student improvement, a reluctance of students not to rely on symbolic manipulation, and a lack of support to help students who struggle in learning this new curriculum.

Two other smaller scale studies explored the effect of instruction on improving students' problem solving abilities. Rubio and Del Valle (2004) proposed a sequence of steps that students can apply when reading algebraic word problems, finding that students' performance on such problems improved as they internalized and applied these steps. The results of a study in one classroom with 49 ninth grade students by Lee and Yang (2013) suggested that instruction in cognitive and metacognitive strategies had a statistically significant effect on students' problem solving performance in probability.

Teacher Education and Problem Solving

Collet, Bruder and Komorek (2007) analyzed the effects of a teacher development program that sought to connect the learning of mathematical problem solving with self-regulation. These researchers found that their instruction influenced teachers' in-class behavior a year later, enabling them to use problem solving in a more productive manner. Boero, Guala and Morselli (2013) argued that in mathematics teaching, teachers may see different mathematical domains as compartmentalized with no connections between them. These dispositions may prevent teachers from discussing problem solving in a productive manner or encouraging strategies that draw on connections between mathematical domains. In order to help teachers see connections between mathematical domains, teacher education must encompass suitable tasks that allow the problem solver to incorporate connections between mathematical domains. The authors suggested analytic geometry as a possible starting point, as this seems to be one area in which teachers do see connections (in this case between algebra and geometry).

Chapman (2005) described a teaching experiment in which she attempted to help preservice teachers construct pedagogical knowledge of problem solving. The participants were asked to interact with peers in cooperative social settings. They were asked to find similarities among the given problems without solving them, analyze the problem in terms of where students might reach impasses and what insights they might generate, and solve an assigned problem. They were then asked to reflect on their experiences to select a problem appropriate for a secondary school student. The initial knowledge of the participants indicated that most of them made sense of problems in terms of the traditional, routine problems they had previously experienced. The participants understood the problem solving process in a way consistent with the traditional classroom way of dealing with such problems. Chapman (2013) also evaluated the effectiveness of teacher education directed at developing mathematical problem solving knowledge for teaching. The study demonstrated the importance of the awareness of cognitive aspects involved in the problem solving process, as well as of perceiving problem solving as a way of thinking in facilitating teachers' knowledge of problem solving.

Nicol, Bragg and Nejad (2013) examined the types of problems that preservice teachers create, what they noticed and attended to, and the challenges they experience when designing mathematical problems within the context of a teacher education course. The researchers attempted to help preservice teachers by giving them criteria for a "good" problem that included: the problem requires more than remembering a fact of reproducing a skill, students learn mathematical content by doing the task and teachers learn about students' reasoning from the students' attempts, and there are several acceptable answers (taken from Sullivan & Lilburn, 2004). Nicol, Bragg and Nejad suggested that the preservice teachers found it difficult to design open-ended problems.

Problem Solving as a Didactical Tool

Many researchers, particularly those using modeling activities or the perspective of Realistic Mathematics Education (RME), use problems or investigations to help students learn and understand mathematical content. In this subsection, we review successful interventions in this regard. Shabhari and Peled (2012) described a realistic modeling approach using theoretical ideas from RME for teaching the concept of percent. They implemented this instruction in a seventh grade classroom where students were asked to work on a sequence of activities designed to invoke the concept of percent; these activities included having students price individual items so that they summed to a desired amount. Through analysis of classroom transcripts, the researchers documented how students' understanding of percent grew as they participated in these activities. These seventh graders also did better on a posttest on percent than a control group of seventh graders who received traditional instruction using a conventional textbook unit on percent.

Dougherty and Slovin (2004) attempted to help elementary students solve word problems. They taught these students how to represent the situations in word problems using generalized diagrams that represented the relationships between the parts and the whole in the word problems that they were solving. Dougherty and Slovin illustrated how working through problems led students to internalize this technique of solving problems, eventually developing representation systems similar to those commonly used to solve algebra problems.

Bonotto (2006) conducted a quasi-experiment in which she compared a traditional and an innovative way to teach decimals to students. Her innovative instruction was based on the premise that students should solve and pose realistic problems about decimals related to their real world situations and critically interpret their solutions to the problems they solved. The instruction was successful. The two classes receiving her innovative instruction outperformed two control groups on an assessment concerning decimals, both in items on practical realistic problems (e.g., problems concerning the Euro) and in terms of pure abstract mathematics (e.g., order these three decimal values). Ilany, Keret and Ben-Chaim (2004) developed a model of teaching preservice teachers the concepts of ratio and proportion using authentic investigative activities. They implemented their instruction with 11 preservice teachers, asking them to engage in investigations that elicited proportional reasoning. The preservice teachers showed robust improvement on a proportional reasoning assessment and also demonstrated more productive attitudes related to the importance of teaching students about proportional reasoning.

Connolly and Nicol (2015) illustrated how problem solving could be used to teach financial literacy to middle school students. The instructors designed activities based on students' prior knowledge and interests. The students were especially motivated during the financial problem solving role-play exercises, confirming the expectations of the instructors. This research found considerable variation between the financial lessons different students were taught at home and also in the student's experience and ability to work with financial math concepts. This highlights how instruction drawing from students' prior experiences may yield heterogeneous responses based on their diversity of experiences.

Delikanlis (2009) explored the use of historical problems in problem solving activities with secondary school students. These problems were chosen to incorporate historical aspects of the mathematical development in support of a method of teaching mathematics that comes closer to a humanity education and to engage students in a challenging problem solving activity. Papageorgiou (2009) conducted a study aimed at proposing and assessing an instructional intervention that integrates both inductive reasoning problem solving and the development of mathematical concepts. The examined teaching program included problem solving activities directed at developing students' abilities in solving problems that require the use of inductive reasoning. The findings revealed a significant improvement in students' use of inductive reasoning for problem solving for the group of students who received the training. Maher (2011) suggested supporting the development of mathematical thinking through problem solving and reasoning. Her findings were based on the results of longitudinal and cross-sectional studies that followed the mathematical thinking and reasoning of cohort groups of students who were thoughtfully engaged in doing mathematics in and out of classrooms. Aspects of her learning environment that she found critical to her success were having students be the arbiter of the correctness of solutions, encouraging sense making, and allowing students to spend extended time on the problem solving tasks in the study (see Powell, 2004, 2006 who reported on data from the same longitudinal study for a further description of this specialized problem solving environment).

Abtahi (2015) focused her attention on the zone of proximal development (ZPD) and the affordances of mathematical tools. Abtahi found that the physical properties and perceived affordances of mathematical tools act as mediators between children's physical actions and their mathematical problem solving and that this is recursively related to their perception of the tool and of the task. These findings also suggest that the ZPD may expand as children participate in collective interactions with mathematical tools that involve the use of guidance provided by the physical properties of the tools in the process of solving problems.

SUMMARY AND DIRECTIONS FOR FUTURE RESEARCH

In this concluding section, we would like to propose areas of problem solving research that we think can be particularly fruitful given our review of the literature. First, we believe that there is a bifurcation of research. Some researchers focus on problems that are clearly defined and have unambiguous solutions that can be obtained by the application of valid calculations and deductions. However, especially recently, many other researchers have explored open-ended and sometimes ill-defined problems. The latter problems often involve the solver to incorporate his or her real world understanding, make reasonable assumptions, and provide answers whose correctness depends on the real world understanding and the assumptions that they invoke. Modeling eliciting activities are a prominent example of these types of problems. Key research questions include: Can we identify the similarities and differences between the processes and dispositions needed to address these problems effectively? Are there different learning goals associated with these problems than with traditional well-defined problems? Are conventional problems better at establishing deductive connections between pure mathematical concepts and are open-ended problems better at helping students see mathematics as relevant to their lives? At a minimum, we implore researchers to be clear about which kinds of problems they are discussing in their research reports. If not, there is the risk of building an incoherent and inconsistent body of research.

Second, we believe more research is needed on the relationship between technology and problem solving. In this chapter, we have documented how technology can expand the kinds of problems that we can reasonably give students and influence the kinds of solutions that they obtain. We think it would be useful to address the following: For which problem solving processes using technology are there analogous processes that can be used without technology? Some processes, such as dragging in dynamic geometry environments (as discussed in Jacinto & Carreira, 2013) or exploring families of functions (as discussed in Lew & So, 2008), do not appear to have analogs in traditional problem solving with paper and pencil. What other problem solving strategies afforded by technology are unique to that environment? Given that technological competence affects one's use of technology (Jacinto & Carreira, 2013), should technical competence with educational technology be included in our models of problem solving expertise? And if so, how should the technological competence related to problem solving be taught to students? How can technological environments be used in developing problem solving expertise? Can technology in mathematics education support model eliciting activities?

Third, problem posing is continuing to emerge as an important frontier on problem solving research. Further investigations into the relationship between problem solving and problem posing are needed. Following Kontorovich and Koichu's (2009) model for problem posing, we can ask: What are the heuristics for posing good problems? Given that a key component of problem posing is recognizing aptness, more research on what experts think is an apt problem is needed. Do experts even agree on what a good problem is? Further, given the importance of problem posing in instruction (e.g., Klinshtern, Koichu, & Berman, 2013), how do we as researchers decide what is a good problem? How do we as researchers go beyond judging problems as trivial or interesting (e.g., Yuan & Presmeg, 2010) and develop robust assessment tools for evaluating problems in a more sophisticated manner? As Kontorovich and Koichu (2009) argued, we are only beginning to consider how social factors affect problem choice. Given the diversity of behavior that has been observed in expert mathematicians' practice (e.g., deFranco, 1998; Weber, Inglis, & Mejia-Ramos, 2014), do all mathematicians utilize the same competencies when posing problems? Do all gifted children?

Fourth, recent research has seen an expansion of what counts as problem solving expertise (e.g., Gravemeijer, 2007) and what constitutes an expert problem solver (e.g., Leikin et al., 2012, 2014). In particular, some researchers have highlighted the importance of creativity in problem solving and problem posing (Amit & Gelat, 2012; Leikin & Lev, 2007; Yuan & Presmeg, 2010). This invites the following questions: What are the specific processes by which creativity enables problem solving and posing? To what extent can mathematical creativity be taught to students and what are effective ways to teach creativity?

Fifth, we have found relatively few international comparison studies in the literature, but those that do exist find differences in the ways that problem solving is taught (Jiang & Cai, 2014) and in the problem solving abilities of students (Yuan & Presmeg, 2010) across different cultures. Given these differences, it is important to resist the temptation to aggregate our findings about students' behavior and the effectiveness of instructional interventions on studies that occurred in different countries until more international comparisons are conducted.

Finally, while research has documented the immense complexity of preparing teachers to implement problem solving tasks in the classroom, we are only beginning as a field to design instruction to help students do this. Hence, like many areas of mathematics education, there exists a large gap between theory and practice in problem solving research. Studies such as Collet, Bruder and Komorek (2007), who investigated the actions of teachers as a result of their teacher development program, are difficult, but necessary if our research is to improve the mathematical achievement of our future students.

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11. REFLECTIONS ON PROGRESS IN MATHEMATICAL MODELLING RESEARCH

INTRODUCTION

The terms, *models* and *modelling*, have been used variously in the literature, including in reference to solving word problems, conducting mathematical simulations, generating representations of problem situations (including constructing explanations of natural phenomena), creating cognitive representations while solving a particular problem, and engaging in a bidirectional process of translating between a real-world situation and mathematics (e.g., Cai, Cirillo, Pelesko, Borromeo Ferri, Borba, Geiger, Stillman, English, Wake, Kaiser, & Kwon, 2014; Doerr & Tripp, 1999; English & Halford, 1995; Gravemeijer, 1999; Greer, 1997; Lesh & Doerr, 2003; Romberg, Carpenter, & Dremock, 2005).

A frequently made distinction in the literature when discussing modelling is that between *modelling* and *application*. Stillman (2012, p. 903) describes this distinction as follows:

> With applications the direction (mathematics > reality) is the focus. "Where can I use this particular piece of mathematical knowledge?" The model is already learnt and built. With mathematical modelling the reverse direction (reality > mathematics) becomes the focus. "Where can I find some mathematics to help me with this problem? The model has to be built through idealising, specifying and mathematising the real world situation.

Related to this distinction between applications and modelling are the two main strands in which modelling is promoted in educational settings: *modelling as content* and *modelling as vehicle*. According to Julie and Mudaly (2007), "*[m]athematical modelling as content* entails the construction of mathematical models of natural and social phenomena without the prescription that certain mathematical concepts, procedures or the like should be the outcome of the model-building process" (p. 504, italics added), meaning that modelling is considered to be a learning goal in itself. *Modelling as a vehicle* on the other hand takes a more methodological and instrumental view of modelling in that modelling is used to achieve other curricular objectives, as for example expressed by Ilany and Margolin (2008) "that development

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 383–413.

of modelling skills is one of the important aims of mathematics curriculum, and serves as a pedagogical central tool" (p. 210).

Although the notions of mathematical models and modelling are understood and used in many various ways, educational research in this area typically uses or develops some general description of the process of mathematical modelling (Kaiser, Blomhøj, & Sriraman, 2006). One of the three prevailing general descriptions in the literature summarises in a schematically and idealised way how the modelling process connects the extra-mathematical world (domain) and the mathematical world (domain) (Blum, Galbraith, & Niss, 2007) through the so called *modelling cycles.* One of these modelling cycles featured prominently in the literature is the one found in the work of Blum and his colleagues (e.g., Blum & Leiss, 2007). Their perspective describes a modelling cycle that incorporates the processes of "constructing, simplifying/structuring, working mathematically, interpreting, validating, and exposing" (p. 25). However, modelling cycles reported in the literature can be quite diverse and highlight different aspects of the modelling process depending on the purpose and focus of the particular research (Borromeo Ferri, 2006; Haines & Crouch, 2010).

A further prevailing general description of modelling refers to modelling competence, modelling competency or modelling competencies (Blomhøj & Jensen, 2007; Chapter 3.3 in Blum et al., 2007; Maass, 2006). Modelling competency is often directly or indirectly defined by drawing on or referring to a view of modelling expressed in terms of a modelling cycle. Blomhøj and Jensen (2003), for example, refer to a modelling cycle in terms of a "mathematical modelling process", which they define as follows: "[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context" (p. 126). Also Maass' (2006) definition refers to a "modelling process", drawing on the work by Blum and Kaiser (1997) and listing a number of sub-competencies to specify modelling competencies: "Modelling competencies include skills and abilities to perform modelling processes appropriately and goal-oriented as well as the willingness to put these into action" (p. 117).

The last of the popular notions of models and modelling we noted is the models and modelling perspective typically addressed through model-eliciting activities (MEAs; Lesh & Doerr, 2003). Here, a model is a "system of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system" (Doerr & English, 2003, p. 112). From this perspective, modelling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products generated often include complex artefacts or conceptual tools that are needed for some purpose, meaning-making, or to accomplish some goal (Lesh & Zawojewski, 2007). When students work on modelling problems they engage in activities that iteratively develop and refine their models through multiple cycles of descriptions, interpretations, conjectures and explanations, often in interactions with other students (Lesh & Doerr, 2003).

Efforts have been made to clarify and differentiate various approaches to mathematical models and modelling (Blomhøj, 2008; Kaiser & Sriraman, 2008; Kaiser, Sriraman, Blomhøj, & Garcia, 2007). According to Kaiser and Sriraman (2006), different perspectives on, and approaches to, mathematical modelling can be classified as realistic or applied modelling; contextual modelling; educational modelling (with either a didactical or conceptual focus); socio-critical modelling; epistemological or theoretical modelling; or cognitive modelling. For a detailed discussion of these perspectives, the interested reader can see the studies by Kaiser and Sriraman (2006), by Kaiser, Blomhøj and Sriraman (2006), by Sriraman, Kaiser and Blomhøj (2006), and by Kaiser, Sriraman, Blomhøj and García (2007). Indeed, these different aspects and interpretations of models and modelling highlight the point raised at the 2014 PME Forum on the topic (Cai et al., 2014), namely, that researchers have come to the agreement that there is no unified perspective on what counts as a modelling activity, reflecting Lesh and Fennewald's (2013) reference to a conceptual confusion.

Aim of the Chapter

The aim of this chapter is to provide an overview of the research presented in the PME proceedings during the period 2005–2015 focusing on different aspects and issues involved in the teaching and learning of, and through, mathematical modelling. The chapter is intended to be both a review in the sense that it summarises the research presented in the proceedings, and to provide a road map for follow-up reading about this line of research reported by the PME community. In addition, the chapter attempts to highlight some challenges and possibilities for future research in the field.

Criteria for Selection

In reviewing the PME conference proceedings from 2005 to 2015 we identified in total 37 papers to consider (see Table 1 for the distribution of the selected papers across the proceedings). We selected the papers using word searches on key words and phrases such as model, modelling, application, realistic, real-world, real-life, etc. In addition, we examined the titles and read the abstracts, again paying attention to key words and phrases, and also skimmed those papers that appeared borderline with respect to modelling. We excluded papers that were neither Research Reports nor part of a Research Forum, that is, we did not consider those papers that were presented as Short Orals, as Posters, or as part of Discussion or Working Groups. Papers that appeared to address models and modelling from the perspective of mathematising a real-world situation were reviewed. Papers that used a different notion of modelling such as in reference to a framework or to a pedagogical guide (e.g., models of teacher education, curriculum models, or developing models of students' mastery of various topics) were also excluded.

Table 1. Distribution of papers included in the review

Proceeding	*Number of papers*
PME 39	2
PME 38/PME-NA 36	4
PME 37	7
PME 36	4
PME 35	1
PME 34	3
PME 33	6
PME 32/PME-NA 30	3
PME 31	1
PME 30	4
PME 29	2

After substantial analysis of the focus and content of the PME papers of the past decade, four broad categories became apparent for structuring the chapter, namely: *perspectives on models and modelling*, *curricular and instructional approaches in fostering modelling competence*, *the inclusion of generic processes, and approaches to models and modelling in teacher education.* Of course, there are other possible ways in which we might have structured this chapter including the perspectives on modelling we noted at the beginning of this chapter (e.g., modelling as vehicle and modelling as content). However, the categories we decided upon appeared the most salient, despite several papers appearing in more than one of these broad categories.

PERSPECTIVES ON MODELS AND MODELLING ACROSS THE PERIOD 2005–2015

First, it should be noted that some of the reviewed papers do not explicitly state the view of modelling adopted in the research and often key notions are not or implicitly defined. Hence, identifying the actual perspectives taken in the reported research was at times challenging. Nevertheless, as might be expected we found considerable diversity in perspectives on models and modelling used during this period.

Models and Modelling with a Focus on Model-Eliciting Activities

In their review of the range of modelling perspectives in the 2014 Proceedings, Geiger and Kaiser noted the enduring popularity of the models and modelling perspective advanced by Lesh and his colleagues many years ago (e.g., Lesh & Doerr, 2003; Lesh & English, 2005). Lesh and English (2005) addressed *Trends in the evolution of models and modelling perspectives on mathematical learning and problem solving* (p. 192) as part of a Research Forum on Theories of Mathematics Education.

A good deal of the research within the models and modelling perspective found in the PME proceedings during the period 2005–2015 use *model-eliciting activities* (MEAs), which are activities that purposefully put students in meaningful situations where they are confronted with a need to develop or recall a model (c.f. Freudenthal, 1983). MEAs also aim to make students' previous experiences and models visible to themselves, their peers and teachers, as well as making their models explicitly articulated objects that can be reflected upon and discussed (Lesh, Hoover, Hole, Kelly, & Post, 2000).

Studies in the 2005 Proceedings that adopted the models and modelling perspective (with the use of MEAs) include those of Schorr and Amit as well as English and Watters. Typically these researchers presented in-depth analyses of students' mathematical learning as they worked an MEA in small group situations. This learning involved the students in generating the mathematical ideas from the problem itself rather than being supplied with the core ideas by the teacher or textbook.

The study by Mousoulides, Pittalis and Christou (2006) presented twenty 11-year-olds working with an MEA aimed at assisting students to develop and use an understanding of the concept of average. By applying interpretative techniques to analyse students' responses from various sources, the authors concluded that students with no prior experience in modelling activities could effectively apply their informal mathematical knowledge to solve a modelling problem. The social interactions within groups enhanced the learning process and the generation of mathematical knowledge.

In a somewhat similar study, Mousoulides and English (2008) investigated the mathematical developments of ten-year-old students in Cyprus and Australia as they worked a complex MEA focusing on interpreting and dealing with multiple sets of data. The research revealed that the students in both countries adopted similar approaches to create models to solve the problem, regardless of their different cultural and educational backgrounds as well as being inexperienced in modelling. The students progressed through a number of modelling cycles in working the problem, with each cycle exhibiting successively more sophisticated thinking, namely: from initially focusing only on subsets of information, to applying mathematical operations in dealing with the data sets, and finally, to identifying trends and relationships.

Another example of research presented in the PME Proceedings drawing on the models and modelling perspective is the work by Amit and Gilat, who co-authored a collection of papers (Gilat & Amit, 2012; Gilat & Amit, 2013; Amit & Gilat, 2013; Gilat & Amit, 2014) that used MEAs to investigate different aspects of creativity and the development of creativity in gifted students. The authors used qualitative methods to analyse students' work on MEAs in terms of their creative thinking abilities. Their analysis involved categorizing students' modelling abilities in a schematic diagram of appropriateness, mathematical resourcefulness (fluency, flexibility, elaboration), and inventiveness or originality (Amit & Gilat, 2013; Gilat & Amit, 2014). Taken together, Amit and Gilat's research presented in the PME proceedings reveals that student engagement in modelling processes encouraged students to creatively utilise their fluent thinking skills as well as their skills in elaborating, refining, and generalising.

Modelling Cycles

Graphical representations of modelling as an iterative cyclic process connecting the disjoint "world of mathematics" and "the real world" (outside of mathematics) at least date back to Pollak's (1979) discussion about "*The interaction between mathematics and other school subjects"*. In terms of research into the teaching and learning of mathematical modelling this conceptualisation of modelling has proven productive (c.f. Blum, Galbraith, & Niss, 2007). Two of the current frequently used references to this perspective on modelling, and found in the reviewed PME proceedings, are those of Blum and colleagues (e.g., Blum & Leiss, 2007) and Galbraith and Stillman (2006).

Huang (2012) adopted the modelling cycle proposed by Galbraith and Stillman (2006) as a research framework in a teaching experiment on the integration of modelling in the calculus courses of university engineering students. Specifically, the author investigated the mathematical modelling processes and mathematical competency of first-year engineering students as they engaged in an optimisation problem aimed at reducing transportation costs. Using this framework, the author presented the results of the analysis as transitions between different phases in the modelling cycle during the students' problem solving activity, namely, the transitions of *real-world situations – real-world models – mathematical models – mathematical solutions – real-world meaning of solution*. The author concluded that the introduction of the modelling activity in the calculus course provided students with opportunities to learn mathematics in a new and different way. Students' engagement in the activity further made it possible to identify some of the students' mathematical shortcomings, something that would not be possible in a more ordinary teaching setting.

In the same Proceedings, Schukajlow and Krug (2012) investigated the multiple solutions and modelling approaches of 9th-grade students while solving complex problems. These researchers viewed modelling as engaging students in simplifying complex situations through mathematising and working mathematically. In this study, the cyclic conceptualisation of modelling was a key structuring factor for the designed learning environment in which the study was conducted.

Grigoraş and Halverscheid (2008) used a cyclic view of modelling to investigate how 11 to 13 year-olds moved back and forth between the real and mathematical domains as they worked on the classical *travelling salesman problem.* The research focused on identifying the links established by the students between the two domains, what mathematical tool the students used, and the patterns of students' arguments and working procedures. Although "[i]t was often not clear whether the students are in the mathematical world or in the rest of the world" (p. 111), the authors found that the students established links between a graph-based representation in the real world (a map of Germany) and mathematics by extracting basic and important concepts and aspects of graph theory.

Modelling Competency

Two papers were identified as applying the notion of modelling competency as an analytical framework, namely, the research of Huang's (2012) reviewed in the previous section and that of Brand's (2014). Brand applied a holistic and an atomistic approach (c.f. Blomhøj & Jensen, 2003) to foster students' mathematical modelling competencies. Using a pre- post- and follow-up-test design, students' modelling competency was measured in a test designed to capture students' *overall modelling competency* as well as three sub-competencies of mathematical modelling (*simplifying/ mathematising*, *working mathematically*, and *interpreting/ validating*). The results showed that although both approaches foster students' modelling competency and sub-competencies, they each have their strengths and weaknesses. Although the analysis revealed that "[a] general superiority of one approach could not be stated" (p. 191), the data indicated that the holistic approach is more effective for students with weaker performance in mathematics.

Realistic Mathematics Education

Three of the reviewed papers applied the mathematics domain-specific instructional theory of Realistic Mathematics Education (RME) developed in the Netherlands (Freudenthal, 1983). Central to the RME perspective are rich, meaningful, and imaginable situations that provide the learner with a context functioning as a hotbed for the development of mathematical procedures, tools, and concepts (van den Heuvel-Panhuizen & Drijvers, 2014). Working from this perspective, Bonotto (2009) used genuine advertisement leaflets to study the relationships between school mathematics and mathematics incorporated in real-life situations. Studying fifth-grade students' (10-year-olds) engagement in modelling and problem posing, Bonotto concluded that, due to the rich variation of students' experiences from outside school, the posed problems created and formulated by the students showed remarkable originality and involved both different and complex aspects. In addition, when the students subsequently solved and discussed the posed problem, they shifted to problem critiquing, that is, "the children attempted to criticize and make suggestions or correct the problems created by their classmates or the results obtained" (p. 198).

Doorman, Boon, Drijvers, van Gisbergen, Gravemaijer and Reed, H. (2009), also based their research within the RME tradition, using both qualitative and quantitative analysis of two teaching experiments to investigate grade 8 students' acquisition of the mathematical concept of function. The study focused on the potential of the integration of computer tools in instructional sequences on the learning of function and the simultaneous tool acquisition and concept development. The authors' analysis of students' engagement in an activity centred on comparing two cell-phone offers, showed that form-function-shifts (c.f. Saxe, 2002) could be detected in the intertwined processes of acquisition of the computer tool and learning about functions. According to Doorman and colleagues (2009), the results indicate that instructional sequences using multimedia environments support the co-emergence of external representations and mathematical concepts; this emergence appeared to be independent of the computer tools used in the design of the sequence.

The study by Van Stiphout, Drijvers and Gravemeijer (2013) also drew on the RME perspective, and more specifically on the notion of *emergent modelling*, which is an instructional design heuristic of the RME theory (c.f. Gravemeijer, 1999). Van Stiphout and colleagues investigated to what extent the mathematics textbooks in the Netherlands support students' development of conceptual understanding in algebra. The analysis of two textbook series covering grades 7–10 revealed two distinct didactical tracks within the books: one track following the RME approach and one track with a more traditional approach introducing new concepts as ready-made mathematics. In addition, the authors also concluded that the textbooks featured a considerable number of activities focusing on the exploration of contextual problems. On the other hand, there was little emphasis on activities that promote emergent modelling in terms of supporting students in developing more formal mathematical relations and concepts.

Mathematical and Cognitive Perspectives

The 2014 PME Forum included a discussion on *mathematical perspectives* and *cognitive perspectives* on models and modelling. A mathematical perspective was referred to as asking fundamental ontological and epistemological questions about the nature of modelling, especially the relationships between the real world and the world of mathematics. Examples of different approaches to addressing issues from these perspectives are presented. However, the following three core questions were cited as of special importance from mathematical perspectives on models and modelling:

- If we view mathematical modelling as a bidirectional process of translating between the real-world and mathematics, what are its essential features?
- Which of those essential features differentiate mathematical modelling from problem solving in school mathematics?
- From the viewpoint of a practitioner of mathematical modelling, what are the essential competencies and habits of mind that must be developed in students to allow them to become competent mathematical modellers? (Cai et al., 2014, p. 146).

An example of modelling from this mathematical perspective includes Pelesko's (2014) reference to "the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviours of that aspect of reality being modelled" (p. 150). A "good mathematical model" was thus considered to be "both an instrument, like a microscope or a telescope, allowing us to see things previously hidden, and a predictive tool allowing us to understand what we will see next" (p. 150).

Connected to the mathematical perspective, the cognitive perspective focuses on the particular cognitive processes and abilities that come into play and facilitate the transforming of a real-world problem or situation from outside of mathematics into a mathematical counterpart. The 2014 Forum presenters were invited to address the following aspects regarding the cognitive perspective, although not all of these issues were examined, at least not in substantive detail:

- What are factors that have an impact on students' formulation of researchable questions in modelling situations?
- If we view mathematical modelling as ill-structured problem solving, how does one convert an ill-structured problem into a well-structured problem with specified research questions?
- What are cognitive differences between expert modellers and novice modellers? (p. 147).

In addressing some of these questions, Borromeo Ferri and English (2014) examined the overall modelling process in terms of whether it is cyclic or linear, a rather frequent debate in the literature. They came to the conclusion that modelling is of a cyclic nature rather than rigidly linear, as is borne out in numerous studies (e.g., English, 2010; English & Mousoulides, 2015). Drawing on Kaiser and Sriraman's (2006) classification of modelling in terms of five core perspectives, Borromeo Ferri and English (2014) proposed cognitive modelling as a sixth core perspective in addition to: "realistic or applied modelling", "contextual modelling" that is akin to the MEA approach, "educational modelling, socio-critical modelling, and epistemological modelling" (p. 154).

Other Perspectives on Models and Modelling

In addition to papers using and drawing on the perspectives on models and modelling discussed above and in the introduction to this chapter, our review of PME Proceedings revealed a number of other perspectives and approaches adopted in research focusing on aspects involving modelling.

A couple of the reviewed papers discussed modelling from what can be described as a French didactical tradition. For example, in their theoretical work, Bosch, García, Gascón and Ruiz Higueras (2006) described the evolution of the research domain "modelling and applications" (p. 209), using the Anthropological Theory of Didactics (ATD) as an analytical tool. Emerging from Brousseau's theory of didactic situations (e.g. Brousseau, 1997), the theory of didactic transposition (Chevallard, 1991), and anthropological considerations, ATD seeks to provide a unitary theory of didactic phenomena centred around the "conception of knowledge as a practice and a discourse on practice together – that is, as a praxeology – along with a pragmatist epistemology which gives a prominent place to praxis" (Chevallard & Sensevy, 2014, p. 38, italics added).

Bosch and colleagues' analysis presented the developments of research in mathematical modelling, with an attempt to link the different aspects of mathematical modelling with the various levels of codetermination provided by the ATD (e.g., thematic level, discipline level). The result is a proposed reformulation of the modelling processes from the point of view of ATD. Based on their analysis and the main tenets of ATD the authors claimed that all mathematical activity is a modelling activity by itself, and that modelling is not restricted to one aspect of mathematics. As a consequence, Bosch and her colleagues further proposed to reformulate the modelling processes as general processes of reconstruction and integration of praxeologies of increasing complexity.

Another interesting perspective on the difficulties and challenges secondary students' might encounter when engaged in modelling was provided by Boero and Morselli (2009). The authors focused on the use of algebraic language in modelling and proving, and employed an adaptation of Habermas' (2003) construct of rational behaviour to describe and analyse various groups of students' modelling behaviour. By interpreting students' modelling activity in terms of epistemic rationality (consisting of modelling requirements), systemic requirements, teleological rationality and communicative rationality, they argued that the Habermasian-based framework provides teachers and researchers with a set of indicators to guide educational design and classroom choices when students are engaged in modelling and proving using algebra as a tool.

CURRICULAR AND INSTRUCTIONAL APPROACHES TO DEVELOPING MODELS AND MODELLING

Returning to the 2014 Forum, a number of issues related to both curricular and instructional approaches were raised for attention, although not all were addressed. In introducing the Forum, Cai empathised that in devoting an appropriate amount of time to mathematical modelling tasks, teachers need to consider, among others, which aspects of a task to emphasize, ways to organize students' work, how to cater for students of different levels of expertise, and how to support students without thinking for them.

With respect to the curricular approaches, the following questions were highlighted for consideration:

- Looking within existing mathematics textbooks, are there activities specifically geared toward mathematical modelling?
- Is it possible or even desirable to identify a core curriculum in mathematical modelling within the general mathematical curriculum?
- In Common Core State Standards for Mathematics (CCSSM) in the United States, mathematical modelling is not a separate conceptual category. Instead, it is a theme that cuts across all conceptual categories. Given this orientation, how might mathematical modelling be integrated into textbooks throughout the curriculum? (2010, p. 147).

Issues pertaining to instructional approaches included:

- What does classroom instruction look like when students are engaged in mathematical modelling activities and what inquiry-based pedagogies have emerged?
- What mathematical-modelling tasks have been used in classrooms, and what are the factors that have an impact on the implementation of those tasks in classrooms? (p. 148).

Answering the above questions has been an increasingly complex endeavour and remains a challenge. Since the beginning of the 1990s modelling was becoming increasingly more emphasised in documents governing the mathematics curricula worldwide (Blum & Niss, 1991), a trend that has been sustained to this day (Blum et al., 2007). Blum and Niss (1991) proposed the following six possible modes for implementing modelling in the mathematics curriculum: *the separation approach*; *the two-compartment approach*; *the island approach*; *the mixing approach*; *the mathematics curriculum integrated approach*; and *the interdisciplinary integrated approach*.

Related to these six modes of implementation, Blomhøj and Jensen (2003) discussed how to best support students in developing their mathematical modelling competence by juxtaposing the “two extreme positions” (pp. 128–129) of the holistic approach and the atomic approach. The essence of the holistic approach is that modelling is best learned by engaging with complex problems that demand the students work through the whole modelling process. In the atomic approach on the other hand, the students first practise the different sub-processes involved in modelling and analyse pre-existing models mathematically, before they engage in more complex modelling activities. The notions of the holistic and atomic approaches are also elaborated and used in a comparative study by Brand (2014), who found no overall significant differences between groups of students being subjected to teaching according to the two approaches in terms of performances measured on modelling sub-competencies. However, as noted earlier, depending on the level of students’ performance in mathematics generally, Brand’s results indicated that the choice of approach could have differences in terms of students’ learning outcome.

As evident from the previous section, a range of approaches to developing students’ modelling abilities was evident across the PME Proceedings in the period reviewed. On the one hand, the studies report on the possibilities and potential in activities and instructional sequences for students to learn modelling as a goal in itself, as well as a vehicle to learn specific mathematical content. On the other hand, the research also highlights some of the issues involved in the teaching and learning of, and through, modelling from both teachers’ and students’ perspectives. It is also apparent that the age group targeted has been predominantly the secondary years of schooling and beyond, with limited studies addressing younger students. Looking across the school levels in the PME Proceedings, there seems to be a gradual shift from more exploratory studies on the potential and possibilities of modelling (modelling as a goal in it self) at the lower levels to studies more focused on learning mathematical content (modelling as a vehicle) at the higher levels. We now briefly showcase the modelling research focusing on primary years, secondary years, and the tertiary level respectively.

Primary Years

One paper focusing on modelling in the primary classroom is the study by Mousoulides and English (2008), reviewed briefly earlier. The activity (an MEA) was purposefully designed so that the students needed to interpret and find relationships in and between, given tables of data. This task was structured to encourage students to explore the notion of rate and proportional reasoning, and to present and illustrate their results in visual and written forms. The classroom teacher gave no formal instruction or direct inputs to the students during their working of the activity. As mentioned previously, the findings revealed similar approaches to tackling and solving the problem by the students in the two countries. Specifically, the students were observed to progress through multiple modelling cycle iterations, taking into account more information in doing so: initially focusing on a subset of information leading to contradictory results, to later beginning to use mathematical operations consistently on all multiple datasets, to finally identifying trends and relationships in the data. The authors also noted that the students spontaneously began engaging in self-evaluation by "constantly questioning the validity of their solutions, and wondering about the representativeness of their models" (p. 428), which supported the students to develop and advance their models in a productive way.

Mousoulides, Pittalis and Christou (2006) used a sequence of two MEAs in a class of twenty 11 year-olds with no previous experiences in solving problems in a mathematical modelling context. The two MEAs were similar in the sense that both activities were intended to provide the students with opportunities to develop models for solving problems by using statistical reasoning and to explore and organize data. No formal instruction was provided to the students as they engaged in the MEAs. The activities were implemented in the following stages (c.f. Lesh et al., 2000): students read an article or text with the purpose of letting the students familiarize themselves with the context of the MEA; a whole class discussion of the reading and readiness questions followed; students worked in groups of three or four on the MEAs; they shared and compared their work with the rest of the class; students returned to their groups to revise and refine their models; and a whole-class discussion on the key mathematical ideas and processes developed in the activity concluded the learning experiences.

In comparing the students' work on the two activities the authors noted that many of the students were able to identify some structural elements of the problem in the first activity in such a way that they could transfer, modify, and apply their models to the second activity. The authors also stressed that "[a]n important conclusion of the present study is that the participating students were able to work successfully with mathematical modelling problems when presented as meaningful, real-world case studies" (p. 207). The activities and their implementation supported students in applying their prior and informal knowledge in approaching and analysing the problem, preserving students' freedom and autonomy.

Related to the above studies, Bonotto (2009) also illustrated the potential for learning in modelling activities at the primary level, as we reviewed previously. Bonotto argued that taking an approach that explicitly draws on students' experiences originating from outside school will motivate and create a genuine interest in the students. Her findings revealed that fifth- grade students could formulate and solve more original and realistic problems than normally would be found in mathematics textbooks. As in the study by Mousoulides and English (2008), the students in Bonotto's research also spontaneously started to engage in critiquing their own and their peers' work as they tried to solve the problems created in the class.

Secondary Years

Working with 12–16 year-olds in an exploratory study, Albarracín and Gorgorió (2012) used mathematical modelling and Polya's (1945) four steps to introduce Fermi problems with a focus on problem comprehension and the application of problem-solving strategies. The six problems used in the study, referred to as "inconceivable magnitude estimation problems" (p. 13), required students to estimate the value of a substantially large real magnitude that was well beyond the range of their regular daily experience. The problems all had or were presented in a real-world context. Three examples of the problems used are: *How many tickets could we sell for a (sold-out) concert in the school schoolyard? How many people are there in a demonstration? How many SMS messages do Catalans send to each other in one day?* (p. 14). The students wrote individual explanations about which steps they would follow to solve the problems, and were explicitly instructed not to make any calculations. The analysis of the procedures the students suggested revealed a number of different categories of strategies in solving the problems. Approximately half of the total number of the procedures suggested by the students were considered to be productive in solving the problems. The authors concluded that because these strategies displayed aspects of modelling, this type of problem could be a useful tool for introducing modelling processes into the classroom, supporting the argument made by Ärlebäck (2009).

Doorman and his colleagues (2009) applied a modelling framework (RME) to design an instructional sequence in a multimedia environment aiming to support the co-emergence of external representations and mathematical concepts in students' learning of functions. The authors based their design on one of the principles of the RME perspective, namely, "[o]pen questions cast in realistic problem situations offer students opportunities to develop tentative situation-specific external representations. These representations give rise to (informal) models and new mathematical goals emerge" (p. 449). The authors found that the multimedia environment indeed did support the students in developing more formal conceptualisations of functions from their initial more intuitive understandings. This result illustrates what in RME is referred to as emergent modelling (c.f. Gravemeijer, 1999).

Adopting a refinement of Chevallard's (1989) notion of mathematical modelling, Martinez and Brizuela (2009) conducted a case study analysis of three grade 9/10 students. Chevallard (1989) originally described mathematical modelling as comprising three stages: (1) Identification of variables and parameters; (2) Establishing relationships among variables and parameters; and (3) Working the model to establish new relationships (Martinez & Brizuela, 2009, p. 113). According to Martinez and Brizuela, the non-linearity of modelling processes and the complexity of students' work occur at all different stages of modelling. Their findings led the authors to a refinement of Chevallard's original three stages by adding the stages of Interpretation of the problem and Production of competing hypotheses, to the mathematical modelling process.

Tertiary Level

Studies at the tertiary level include Huang's (2012) study, reviewed previously. The study revealed the difficulties the university students experienced in transitioning between different modes of mathematical representations and the classifications of variables and parameters as known or unknown, implicit or explicit, and as independent or dependent variables. This difficulty to transition between modes of representations is in line with what Kertil, Delice and Aydin (2009) concluded from their study of pre-service teachers engaged in modelling.

Trigueros, Possani, Lozano and Sandoval (2009) constructed a research framework (which Lester, 2005, would refer to as a construction scaffold) that incorporated a 3UV model (c.f. Trigueros & Ursini, 2003) to study how the use of models influenced the development of university students' understanding of linear equations. This framework enabled them to simultaneously focus on the development of students' conceptual tools in decision-making situations when engaged in a modelling problem, and the different dimensions and roles played by the students' use and understanding of (algebraic) variables. The authors highlighted the motivating aspect of modelling and how students' understandings of, and abilities to use, variables became more explicit and visible during their modelling activities, compared to working with more traditional activities.

In another study Trigueros and Lozano (2010) discussed the effectiveness of using modelling as a teaching strategy to develop university students' ideas about linear dependence and linear independence. Using the APOS theory (Dubinsky & McDonald, 2002), the authors analysed how students' schemas evolved during their work with modelling activities. The study was conducted within a course on Linear Algebra with 35 students in Mathematics, Engineering and Economics programs, in Mexico. The results showed that students' engagement in modelling had a positive effect in the evolution of their linear dependence and independence schemas, as it provided a means for students to associate concrete meanings with abstract mathematical concepts. Hence, the authors concluded that the use of a modelling approach as a didactic strategy is useful to guide students' acquisition of new concepts and modelling techniques.

Czocher (2014) studied the work of four engineering majors enrolled in a course on differential equations. The author addressed the cyclic nature of modelling and questioned whether mathematical modelling is a "regular, quasiperiodic process" (p. 353). Czocher transformed the modelling diagram of Blum and Leiss (2007) into stages of model construction and transitions among stages and produced MAD-diagrams (c.f. Ärlebäck, 2009). The results showed that "the mathematical thinking involved in mathematical model construction is not sequential nor [sic] quasi periodic" (p. 359). Czocher thus concluded that the findings should lead to a revision of the conceptualisation of modelling as cyclic in nature as manifested in, for example, Blum and Leiss (2007).

Modelling and Project Based Learning

Some of the papers reviewed used modelling in the context of project work. For example, in the study of the effects of interdisciplinary project work on students' perception of mathematics, Dawn, Stillman and Stacey (2007) used a pre-post design and analysed the quantitative data from 409 Singaporean students aged 12–14, who participated in a 12–16 week interdisciplinary project. The project engaged students in *designing an environmental friendly building* and incorporated learning from mathematics, science and geography. Using measures from an *Interconnectedness of Mathematics Scale* and a *Belief & Efforts in making Connections Scale*, the authors found a small improvement in students' appreciation of interconnectedness after participating in the project work. It was concluded that "[s]tudents after the project work were somewhat more likely to appreciate mutual reinforcement of learning among mathematics and other subjects" (p. 191). However, the analysis showed no positive effects of interdisciplinary project work on students' willingness and initiative to actually make an effort to connect mathematics and other subjects.

Using the theoretical lens of *Activity Theory*, Araujo, Santos and Silva (2010) studied groups of students engaged in defining, developing and working on a mathematical modelling project. As part of their assessment in a one-semester university course on functions, derivatives and integral calculus, students worked in groups to develop mathematical modelling projects. The projects commenced with negotiating themes to investigate, followed by agreeing on a topic and questions to address, and then engaging in modelling to resolve the questions formulated. The paper presented data from one group of students gathered during the development phase of the projects, and through interviews that took place three years after the end of the course. By adopting the theoretical constructs of motives and objects of the activity from Activity Theory, the authors concluded that students had different motives in the activity The authors maintained that it is important to explicitly track the object of an activity since it is a key element in promoting *expansive transformation* (the process in Activity Theory describing learning) in the activity.

More recently, Hernandez-Martinez and Harth (2015) applied activity theory analysis to group work in mathematical modelling within an undergraduate engineering course. With a focus on the group's social interactions, the authors concluded that a major factor in students' mathematical learning during collaborative work is the quality of peer interactions, which is dependent on the students' communication and interpersonal skills.

Returning to project work, Villarreal, Esteley, Mina and Smith (2010) reported on the developments of three experienced in-service secondary mathematics teachers from the same school, while designing a mathematical modelling project for their students. Specifically, defining project work in line with Blomhøj and Kjeldsen (2006), the aim of the study was to explore how the teachers understood, created and implemented mathematical modelling projects in their classrooms and the decisions they made during the design of the projects. Based on their results, the authors highlighted the time-demanding nature of designing a long-term mathematical modelling project. Among the difficulties the teachers faced during modelling project design, was the need to control and foresee possible student difficulties, the latter being evident at every decision they made during the design of the projects.

The authors also identified a 'paradox' in the work of the teachers: while on the one hand they wanted to stimulate their students, on the other hand, they provided strict guidelines in their proposed projects and included very few opportunities for their students to be creative.

Use of Technology

Researchers interested in the teaching and learning of, and through, mathematical modelling have tried to acknowledge both the potential as well as the challenges technology's integration offers. Indeed, a number of issues regarding the place and use of technology in modelling were raised already by Blum and Niss (1991), although with reference mainly to desktop computers and computer hardware. More recently, however, one concrete example is provided by Siller and Greefrath (2010) who extended the work of Blum and Leiss (2007) on modelling cycles, by introducing the concept of *modelling in mathematics with technology*. In doing so, they added a "third world of technology" to the existing two worlds of "reality" and "mathematics". This third world inhabited, for example, by computer algebra systems (CAS), spreadsheet and dynamic geometry software (DGS), represents the "'world' where problems are solved through the help of technology" (p. 2137). A more general theoretical conceptualisation integrating perspectives of mathematical modelling and technology (in a wide sense) is discussed by Williams and Goos (2013). In their chapter, the authors argue that mathematical modelling should be "conceived as adding "theoretical thinking" to real, practical problem-solving activity" (p. 566). They use that conceptualisation to situate modelling and technology within a neo-Vygotskyan perspective, while they suggest that "mathematics inevitably […] appears alongside and even fused with, technologies in the solution of problems". Williams and Goos (2013) also refer to the powerful contemporary technologies "which expand the language of mathematics, and allow learners wider scope for theoretical thinking and modelling in practice" (p. 566).

In reviewing the PME Proceedings with regard to research on using technologies in the teaching and learning of modelling, modelling was predominately used as a vehicle for learning specific mathematical content. The technological tools typically supported the modelling process by providing easy access to modes of manipulations and modes of representations, and hence technological tools could be thought of as a vehicle for (doing) modelling. One example is the study by Lagrange and Artigue (2009), who developed a grid for designing and then analysing the "potentialities, as a tool for functional modelling" of Casyopée (p. 467). Casyopée, a technological tool developed in the ReMath research project connects in a meaningful and curriculum related way a symbolic and a dynamic geometry window, allowing better "connecting and integrating theoretical frames in technology enhanced mathematics learning" (p. 466). The authors report on an experiment with two eleventh-grade classes, who worked on a teaching and learning scenario. The scenario included sessions on the capabilities of the environment's symbolic window and quadratic functions, on students' knowledge of geometrical situations, and on students' work with the software to activate their algebraic knowledge for solving an optimization problem. The authors concluded that although general theories of mathematics education are influential, it is necessary to develop "local frames, like the grid", in "piloting in a precise way the design of artefacts or the use of these" (p. 471) in solving modelling problems.

Son and Lew (2006) showed how tenth-grade students worked in a spreadsheet environment to model the effects on the concentration of chlorine in a swimming pool with different initial concentrations of chlorine and different sized daily-added doses of refills. The students were reported to successfully use pencil and paper to derive the basic algebraic recurrence formulas, but that these relationships conveyed little meaning and understanding for the students. However, by using a spreadsheet environment to investigate the derived formulas to produce tables and graphs, the authors claim the students were able to more accurately understand the meaning of a model of how the chlorine concentration varied in the different scenarios. As well as concluding that with respect to time, the concentration of chlorine did not depend on the initial amount of chlorine in the swimming pool, but that the long time behaviour only depended on the daily amount of chlorine added, the students were also able to use their generated tables and graphs to justify this claim.

GENERIC PROCESSES SUCH AS METACOGNITION AND AFFECT

Also featured in the PME papers was a focus on generic processes, with metacognition and affective issues being prominent.

Cognitive and Affective Issues

As noted in the introduction to this chapter, modelling is considered to involve many higher-level skills. Dahl (2009) illustrated this feature in her analysis of the eight mathematical competencies developed in the Danish KOM project (e.g., Niss & Jensen, 2002). Based on the five distinct levels of understanding of the SOLO Taxonomy, Dahl defined a *competence progression* and used this on the stated intended learning outcomes in curricula documents ranging from compulsory to tertiary level. In terms of this framework, the modelling competence and sub-competencies describing what it means to do mathematical modelling are all categorized as level 4 or level 5. Related to these are affective dimensions of mathematics and modelling, dimensions that are often not easy to change. This issue is illustrated in the results by Dawn, Stillman, and Stacey (2007) on the effects of interdisciplinary project work on students' perceptions of mathematics. Their results indicated that engaging in a 12–16 week interdisciplinary project did not have a positive effect on students' beliefs about mathematics and its relationship to other subjects, nor on students' willingness to make an effort to connect mathematics to other subjects.

Schukajlow and Krug (2012) focused on both cognitive and affective issues in the context of modelling in a series of papers presented in the reviewed PME proceedings. At PME 36, the researchers considered the essence of a modelling activity to be "simplifying a complex situation that is presented in the task, mathematizing and working mathematically to reach a mathematical result" (p. 60). They completed a quasi-experimental study that investigated 138 German 9th graders' values, self-regulation, and self-efficacy expectations before and after a five lesson intervention. The intervention focused on teaching advocating single or multiple solutions to modelling tasks. The students were divided into two groups that were subjected to two different teaching scripts centred around working with modelling problems. One script emphasised multiple solutions and different results as a consequence of students' making different estimates of missing data combined with explicitly prompting them to find more than one solution. In the second script, the emphasis on multiple solutions was lacking. Five-point Likert scales were used to measure students' self-perceptions: self-regulation (6 items); self-efficacy (4 items); and value (3 items). Although the five lesson intervention improved students' self-regulation, self-efficacy and values in both groups, the analysis showed no significant statistical differences between the two groups in the students' values, their self-regulation, or their self-efficacy. However, the authors found significant statistical differences between the numbers of solutions the students developed in the respective groups. It was concluded that encouraging students to find multiple solutions while solving modelling problems is a worthwhile strategy, which has a positive impact on their self-image and values.

The very same research setting was also used in the paper by Schukajlow and Krug presented at PME 37 (Schukajlow & Krug, 2013). This time the authors aimed to explore whether treating multiple solutions while solving modelling problems results in more frequent planning and monitoring activities, and whether the development of multiple solutions has a positive influence on students' planning and monitoring activities. By using a questionnaire on planning and monitoring developed by Rakoczy, Buff and Lipowsky (2005), the authors concluded that there is a positive influence of treating and developing multiple solutions on students' planning and monitoring activities.

In the second study by Krug and Schukajlow-Wasjutinski in the 2013 PME proceedings (Krug & Schukajlow-Wasjutinski, 2013), the focus was on examining students' interest in working with tasks connected to everyday situations. The participants in the study were 9th and 10th grade students from Germany, who were randomly assigned to two experimental groups. Both groups were assigned 12 problems: four modelling problems; four word problems; and four intra-mathematical (pure mathematical, without any connection to reality) problems. Students in experimental group 1 first solved problems and then reported on their task-specific interest regarding these problems, while students in the second group first reported on their task-specific interest, and then solved the tasks that were used. The authors concluded, in contrary to previous research findings (see for instance

Schukajlow, Leiss, Pekrun, Blum, Müller, & Messner, 2012), that students have lower interest in modelling problems than in the other two types of problems. A suggested reason for this finding is that students do not regularly solve modelling problems in the mathematics class, and may be unsure of their ability to solve such problems. Comparing the measures of the two groups, the analysis also showed that task-specific interest across all types of problems decreases if measured after the tasks have been worked on.

Creativity

The interplay between mathematical modelling and creativity is a theme that has been explored in a series of co-authored papers by Gilat and Amit (Gilat & Amit, 2012; Gilat & Amit, 2013; Amit & Gilat, 2013; Gilat & Amit, 2014). In a case study investigating the potential of real-life modelling activities in simulating students' creativity, Gilat and Amit (2012) analysed the work of two mathematics high achieving students (aged 10 and 13) as they individually engaged in two sequences of modelling activities. The data collected and analysed consisted of researchers' notes of students' working of the modelling activities, recordings of the presentations of the students' working together with the discussion that followed, and recorded post-interviews with the students. The students were found to iteratively refine their thinking and models through multiple cycles; results were presented in terms of cognitive- and affective characteristics involved in the students' modelling processes. Cognitive characteristics identified were flexibility, combination, and analogy whereas the affective characteristics were motivation and interest, self-efficacy and perspective, and metacognition and self-reflection. The authors concluded that "[t]he findings clearly show some cognitive and affective characteristics that could establish the foundations for creative process development methodology using MEAs" (p. 272). It was suggested that working with non-routine modelling problems, like MEAs, can simulate students' creativity in solving problems using mathematics.

Along the same lines, Amit and Gilat (2013) analysed the creative thinking of 85 mathematically gifted students (5th–7th grade) in terms of fluency, flexibility, originality, and elaboration. The results revealed that student engagement in the modelling process encouraged the students to utilise their fluent thinking skills, as well as to stimulate students' elaboration skills, including refinement and generalization.

To investigate the more long-term effects on students' creativity of engaging in modelling activities, Gilat and Amit (2013) conducted an experiment involving 71 school students (5th–7th grade) who were members of a mathematics club for gifted students. The participants were divided into an experimental/intervention group (47 students) and a control group (24 students). Students in the intervention group worked with modelling activities for a period of 9 months, while students in both groups worked with other mathematics and creativity enrichment activities.

By using the Torrance Test of Creative Thinking (TTCT), the authors adopted a pre- post- test design to examine the possible impact of the students' engagement in the modelling activities. The analysis revealed that students in the experimental group scored significantly higher on the creativity test than students in the control group, although both improved their scores. Further, the results also indicated that the girls in the experimental group were more creative than the boys, although these differences were not tested for statistical significance.

The paper by Gilat and Amit (2014) presented at the joint conferences of PME 38 and PME-NA 36 is a continuation of their paper from PME 37 that aimed to further reveal the cognitive abilities exhibited by students engaged in a creative modelling activity. The paper elaborates rather extensively on the methodology of the study as well as on the different phases in the qualitative analysis undertaken. An intervention study with 71 gifted students (5th–7th grade) participating in 75 minute weekly sessions in a mathematics club, including four workshops based on MEAs, was implemented. The data collected included videotapes, classroom observation, and modelling products. The iterative and interpretative analysis of the data yielded three core categories and a number of subcategories describing the cognitive abilities that the students applied and activated as they engaged in the modelling activities. The categories identified were mathematical appropriateness (consisting of the three subcategories of knowledge, documentation, and utility), mathematical resourcefulness (with the subcategories involving fluency, flexibility, and elaboration), and inventiveness or originality. The authors claimed that the findings may provide both a better theoretical and practical understanding of the larger concept of mathematical creativity.

Parental Engagement

Another somewhat different affective dimension was studied by Mousoulides (2014) in his paper Using Modeling-Based Learning as a Facilitator of Parental Engagement in Mathematics: The Role of Parents' Beliefs. Mousoulides reported on the findings from a large study on "connecting mathematics and science to the world of work by promoting mathematical modelling as an inquiry based approach" (p. 263). He focused on: (a) parents' beliefs about inquiry-based mathematical modelling and parental engagement, and (b) the impact of a modelling-based learning environment on enhancing parental engagement. Mousoulides presented the results from semi-structured interviews with 19 parents from one elementary school. He found that parents hold strong positive beliefs on engagement in their children's learning, an appreciation of the modelling approach for bridging school mathematics and home, and a strong willingness to collaborate with teachers when they integrate modelling in their teaching. With regard to the research on parental engagement in school (mathematics), the author claimed that there is a need for researchers to expand their definitions of such engagement towards a dimension related to the inclusion of modelling and inquiry based initiatives.

MODELS AND MODELLING IN TEACHER EDUCATION

Returning to the 2014 PME Forum, further issues for consideration were raised with respect to teacher education, namely:

- Are there programs worldwide which successfully support pre-service and in-service teachers to teach mathematical modelling, and what are the features of these successful programs?
- What level of familiarity with disciplines other than mathematics is it necessary for pre-service and in-service teachers to have in order to successfully teach mathematical modelling? (p. 148).

Within the studies of teacher education examined, there were few that addressed interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modelling directly. As the Forum noted, it is well documented that teachers have difficulties with mathematical modelling given that teaching becomes "more open and less predictable" when students engage in such activities (p. 148). The Forum concluded that initial and in-service teacher training, as well as curriculum documents, has tended not to provide opportunities for making mathematical modelling an integral component of students' learning.

In providing some initial thoughts on teacher education, Stillman and Kwon (2014, p. 165) reviewed related research including studies of interdisciplinary or extra-mathematical knowledge requirements for teaching modelling. Few such studies addressed these requirements directly, although several did allude to the necessity for teachers, including those in elementary classrooms, to have the background knowledge to develop their students' modelling abilities. Other findings have suggested that pre-service teachers tend to isolate modelling from the real-world situation in focus, or activated their real-world knowledge and attempted to incorporate this into their modelling (Widjaja, 2013).

In another study in the 2014 Proceedings, Barabash, Guberman, and Mandler addressed the depth of knowledge required by primary school teachers in learning modelling. Focusing on interdisciplinary perspectives of expert mathematics teachers in primary school, the researchers investigated the question: Is "deep mathematical knowledge of primary school mathematics a necessary basis for the understanding of the concept of mathematical model?" (p. 90). They concluded that successful teachers of modelling demonstrated "higher levels of mathematical insight" and that a deeper understanding of the required mathematical knowledge is needed (p. 94).

RECOMMENDATIONS FOR ADVANCING THE FIELD

The aim of this chapter has been to provide an overview of the research presented in the PME proceedings during the period 2005–2015, focusing on issues pertaining to the teaching and learning of, and through, mathematical modelling. Based on the research reported in the PME community, we now close this chapter by suggesting a few challenges and possibilities for the research field in the near future.

Broadening Models and Modelling within Multidisciplinary Contexts

Reflecting critically on the 14 research questions posed in the 2014 PME Forum on mathematical modelling, it could be concluded that some advances have been made during the last past decade by the research field as a whole. There remain, however, many questions in need of further attention. It is beyond the scope of this chapter to identify more than a few of these; indeed, questions regarding models and modelling evolve as the world changes. Furthermore, issues for further attention depend in large part on the meaning and conceptualisation of mathematical models and mathematical modelling. Some questions worthy of consideration include: Do we need to rethink how we conceive of models and modelling, especially given their increased prevalence in today's world? How are they interpreted and applied in addressing local and global issues? One only has to reflect on the increased applications of modelling in efforts to understand and predict core national and global events. Indeed, modelling is an essential tool in so many aspects of our lives: predicting the national impact of economic downturns, forecasting growth in business and industry, and anticipating the consequences of dam spillage during heavy rains, are only a few examples.

The increased global prevalence of models and modelling was highlighted in a recent chapter on "Problem Solving in a 21st-Century Mathematics Curriculum" by English and Gainsburg (2016). They emphasised the importance of developing students' appreciation of how models are represented mathematically and technologically in many fields, including engineering, finance, manufacturing, and agriculture. Given that so many aspects of modern life have been mathematised using modelling, students need to be aware of, and understand how, such mathematisation shapes their daily lives.

In helping students become more mathematically aware, English and Gainsburg recommended selecting contexts that approximate real-world situations and foster students' appreciation of learning through classroom problem solving. This recommendation supports one issue we consider in need of further research, namely, the increased use of interdisciplinary contexts. Such contexts would appear especially important with the increased focus on STEM education, in particular, from an integrative perspective (e.g., US STEM Taskforce Report, 2014; Vasquez, 2014). The Report adopts the view that STEM education is far more than a "convenient integration" of its four disciplines, rather, it encompasses "real-world, problem-based learning" that links the disciplines "through cohesive and active teaching and learning approaches" (p. 9). The Report argues that the disciplines "cannot and should not be taught in isolation, just as they do not exist in isolation in the real world or the workforce" (p. 9). Mousoulides and English (2011) expressed similar sentiments: "How we might assist students in better understanding how their mathematics and science learning in school relates to the solving of real problems outside the classroom and how we might broaden students' problem-solving experiences to promote creative and flexible use of mathematical ideas in interdisciplinary contexts?" (p. 25).

One example of how this challenge might be met is through the selection of appropriate engineering contexts. Engineering and engineering practice provide an ideal basis for creating problems that are meaningful to students and enable them to see how their school learning of mathematics, science, and technology relates to real-world problems (e.g., Barrett, Moran, & Woods, 2014; English, 2015a; English & Mousoulides, 2015; Moore, Stohlmann, Wang, Tank, Glancy, & Roehrig, 2014). Further research is needed that explores how interdisciplinary contexts can foster students' appreciation and learning of models and modelling, including ways in which modelling is increasingly evolving in many domains.

In addressing more interdisciplinary contexts, consideration also needs to be given to modelling with younger learners. Traditionally, modelling has not featured prominently in the elementary grades despite research showing young learners' capabilities in this area (e.g., English, 2015b; Lehrer & Schauble, 2000, 2002, 2012; Mousoulides & English, 2011). The elementary school years are ideal for implementing interdisciplinary modelling problems, given the rich contexts of numerous cross-curricular themes and investigative topics. For example, students can develop models for improving the water quality of their local creek, for designing the reconstruction of damaged bridges, for sourcing water during shortages, and for assessing the impact of cyclones in selecting suitable new coastal resort site (e.g., English, 2009, 2015b; English & Mousoulides, 2015). Lehrer and Schauble's work (e.g., 2005; 2015) provides further rich examples of how younger learners can deal with modelling problems in interdisciplinary contexts with a particular focus on science and mathematics.

Greater and More Effective Use of Technologies

A number of advances in technology are available today, many of which are highly relevant for modelling and its applications. Research has shown how technology can play a pivotal role in supporting and promoting mathematical modelling (e.g., Gravemeijer, Lehrer, Van Oers, & Verschaffel, 2013; Hamilton, Lesh, Lester, & Brilleslyper, 2008; Lagrange & Hoyles, 2009; Mousoulides, 2013; Mousoulides, Christou, & Sriraman, 2008). Yet there is still limited emphasis on the affordances of technology among the PME contributions (e.g., Son & Lew, 2006; Doorman et al., 2009). The question remains why technology's infusion in modelling is still limited, even when research claims that computer-based learning environments can contribute to developing students' modelling competences (Mousoulides, 2013).

In his discussion document on the 'ICMI Study 14: Applications and modelling in mathematics education', Blum (2002) stated a significant question, which still remains unanswered: "How should technology be used at different educational levels to effectively develop students' modelling abilities and to enrich the students' experience of open-ended mathematical situations in applications and modelling?" (Blum, 2002, p. 167).

In trying to provide some guidelines for answering aspects of Blum's (2002) question, it might be useful to think of the availability of contemporary tools, including a broader definition of technologies, not just software. At the software level, recent advances in the use of dynamic geometry environments (e.g., *Sketchpad*, *Geogebra*, *Cabri*), programming tools for younger students (e.g., *Scratch*), simulation and microworld environments (e.g., *Simcalc*), and spreadsheets open new venues for rethinking the teaching and learning of modelling, especially when considering that a number of these environments (e.g., *Simcalc, dynamic geometry environments*) are rich enough, to be considered as modelling activities by themselves. At the hardware level, advances in virtual and augmented reality, 3D representations, data loggers (virtual labs), tablets and mobile devices, make it easier to rethink the use of technology in the teaching and learning of mathematical modelling.

The above technology can facilitate students' modelling in numerous ways. For example, computer programming environments for younger students, dynamic geometry, and computer algebra systems can be used to simulate and model a real-world problem, with simulations examining the various parameters in a model. Further, using data logging equipment provides an opportunity to collect data to validate models, especially when students deal with problems that involve real-life data. Virtual and augmented reality environments can be used to engage students in complex mathematical modelling situations, thereby facilitating greater access to abstract and quite difficult mathematical ideas and processes.

Of course, using technology in the teaching and learning of mathematics can be a complex process in itself (e.g., Artigue, 2002; Lagrange & Artigue, 2009; Lagrange & Hoyles, 2009; Vandebrouck, Monaghan, & Lagrange, 2013). It is not a simple matter of making available computers and software to school children and teachers, rather, we should be careful not to lose sight of the mathematical modelling processes per se.

To better examine the complexity of integrating technology into classrooms there is a strong need for longitudinal studies with mathematics teachers and students who use technology in mathematical modelling. There is also the need to design studies that use mixed research methods to examine how modelling processes are influenced by the use of technological tools (e.g., Lagrange & Hoyles, 2009; Mousoulides, Sriraman, & Lesh, 2008). Studies can also address whether some modelling processes cannot be developed without technology (Blum, 2002), as well as investigations that explore the variations in students' developed models when using (or not using) tools like augmented reality or microworlds/ programming environments.

As this chapter has attempted to show, the PME research community has been active in the field of models and modelling over the past decade, opening up new avenues for further work. We have highlighted just a few areas of the many that warrant increased attention from the international community. With models and modelling becoming more pervasive throughout societies as they grapple with increasingly diverse and pressing global challenges, the importance of models and modelling as a core curriculum component cannot be underestimated.

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PART 3

SOCIAL ASPECTS OF LEARNING AND TEACHING MATHEMATICS

PETER LILJEDAHL AND MARKKU S. HANNULA

12. RESEARCH ON MATHEMATICS-RELATED AFFECT

Examining the Structures of Affect and Taking the Social Turn

INTRODUCTION

In the first Handbook of Research on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006) Gilah Leder and Helen Forgasz began the chapter on Affect and Mathematics Education with an overview of research on affect outside and before PME. They noted that, although affect has often been claimed to be somewhat forgotten, there has been continuous interest in this field since the 1920's. Looking at this history they identified three phases of research – the first of which was focussed on observable facets of affect, measurement of attitude, and its relation to behaviour. The second phase shifted the focus on the role of cognition in the structure and shaping of attitudes. And the last phase has been studying the structure and function of affective systems. These same phases were reflected also in PME research on mathematics related affect. They also identified that there were four repeated themes in PME affect research: measurement of affective factors, descriptive studies, comparisons of affective and cognitive variables, and a small number of theoretical papers. They also observed some decline of PME research on affect over the period 1996–2005.

In this chapter, we review the more recent development of affect research in PME proceedings from 2005–2015. First, we discuss how relevant articles were selected for inclusion in this chapter. For the years where author indicated domains of research were included we used these to identify relevant papers. For the remaining proceedings we used careful reading of article titles and keyword searches of the electronic proceedings. After closer reading and deliberation we made the final selection for inclusion or exclusion of papers. For example, we excluded papers that focussed on student conceptions from a cognitive perspective. The result was a total of 188 Research Reports and papers from two Research Fora spanning the period 2005–2015.

From these 188 papers we attempted to extract information regarding a) the theoretical frameworks that were used, b) the findings regarding some classical questions of affect research, c) any new research questions or areas of interest that may have emerged, and d) the methodological approaches used to study mathematics related affect. More specifically, for each paper we identified the key

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 417–446.

concepts and theoretical framework used, the methodological approach and research design, the number of research subjects and their grade level, and the key affect related conclusions. Not all papers had a clearly explicated theoretical frame and the research design was sometimes difficult to determine. These difficulties may be partly due to the constraints posed by PME format.

From this analysis a number of themes emerged pertaining to both classic questions and new interests in affect research. In what follows, we present the results of this analysis in the form of these themes.

IN SEARCH OF A SHARED THEORETICAL FRAMEWORK

In the research of mathematics related affect, the lack of a shared terminology has been raised repeatedly (e.g., Furinghetti & Pehkonen, 2002; Hannula, 2011, 2012; McLeod, 1992). This same issue was an important element in the PME 2004 research forum on affect, where Gerald Goldin (2004) said:

> We do not now have a precise, shared language for describing the affective domain, within a theoretical framework that permits its systematic study. (p. 109).

In their chapter in the first handbook, Leder and Forgasz (2006) also discussed the issue of concepts and their definitions. Drawing from research within and outside mathematics education, Leder and Forgasz listed concepts used for researching affect and presented several definitions. Specifically, they presented McLeod's (1992) classification of mathematics related affect research into domains of beliefs, attitudes and emotions and Goldin's (2002) definition for emotions, attitudes, beliefs and values, ethics and morals as the four subdomains of affect. They also discussed the variety of definitions given for attitude.

Over the ten years of PME that are the focus of this review, research on mathematics-related affect has continued to use these concepts frequently and often either explicit or implicit reference was made to McLeod's or Goldin's frameworks. While some papers address the issues of defining these constructs more specifically, more thorough theoretical discussions have often taken place outside of PME publications. We make reference to these publications, when the echoes of these discussions can be heard in PME papers. In what follows we organize our discussion of definitions around McLeod's (1992) categories of beliefs, attitudes, and emotions, adding to this the category of motivation as an important category (Hannula, 2011, 2012).

Resolving the Conflict between Attitudes and Beliefs

According to Hannula (2011, 2012), one of the key problems in the terminology of mathematics related affect is the incompatibility of the two most frequently used concepts – attitudes and beliefs. While McLeod (1992) identified beliefs and attitudes as two main domains of affect, definitions of attitude typically see beliefs as an aspect of attitude.

In their 2010 journal article Di Martino and Zan (2010), in an effort to organize the different definitions for attitude in mathematics education, identified three types of definitions: (1) attitude as positive or negative degree of affect, (2) emotions and beliefs as two components of attitude, and (3) attitude as consisting of cognitive (beliefs), affective (emotions), and conative (behaviour) dimensions (see also Zan & Di Martino, 2009). This tripartitive attitude framework has been used in PME articles (e.g., Barmby & Bolden, 2014) and there is strong empirical support for the framework. For example, Pampaka and Wo (2014) used a Rasch model and data from multiple surveys with total of 17,448 respondents to identify the dimensions of attitude. Their results confirmed that attitude could be decomposed into the affective, conative, and cognitive components. How it is used in the literature is, however, not always clear – is it being used as an umbrella term or as a narrower concept? When attitude is used as an umbrella term, beliefs are one subcategory of it. However, when the term attitude is used in the narrow sense, attitudes and beliefs are two separate categories of affect.

Critically on Beliefs

Influential frameworks for research on mathematics-related beliefs are Bandura's (1977) self-efficacy (e.g., Chang & Wu, 2012, 2014; Panaoura, Gagatsis, Deliyianni, & Elia, 2009) and Ernest's (1989) view of mathematics (e.g., Beswick & Callingham, 2014; Erens & Eichler, 2013, 2014; Halverscheid & Rolka, 2007). However, these frameworks seem to be fairly fixed and we found no attempt to develop these frameworks further. We found two theoretical papers that elaborated the concept of teacher's beliefs in more detail. Both of these were critical of the notion of inconsistency between teachers' beliefs and practices.

O'Donovan (2015) proposed a simple definition for beliefs as "what people think (or hope) are true (or probably true)" (p. 307), which—as he acknowledges—is not a new definition for beliefs. What is more interesting is his discussion of Fives and Buehl (2012) taxonomy on how beliefs influence teacher action in three different ways:

> [as] interpretive *filters* – a layer of unconscious assumptions or habits that implicitly filter experiences and perception; *frames* within which problems are dealt with – this is the level at which ideological beliefs define or frame problems and situations that arise; and, action *guides* which motivate teachers to act. (pp. 307–308)

He then discussed the apparent discrepancy between teacher beliefs and teacher behaviour, pointing out that beliefs "need not be connected in a rigorously logical and coherent system either" (p. 309). Rather, he supports Thagard's (2000) analogy of "beliefs being like rafts floating at sea forming mutually supportive clusters, as opposed to being arranged hierarchically" (p. 309). He also discussed how behaviour need not be wholly driven by beliefs. If behaviour is merely influenced by beliefs, it allows for other internal or external non-belief factors to also influence actions, providing an explanation allowing consistency of teacher beliefs.

Skott (2010), posed an even more fundamental criticism towards the traditional belief research. His main criticism was targeted at the explanatory model provided for the discrepancies between teachers' beliefs and practices. Skott saw belief research as assuming (teacher) beliefs to be objectified (Sfard, 2008), and having an impact on their practice, and that this assumption leads to blaming teachers for being inconsistent and not enacting their espoused beliefs. He recommended belief research to "make a social turn" and focus on teacher patterns of participation and to redefine beliefs as "value-laden, reified patterns of participation" (p. 198). We will return to the issue of social turn later.

Expanding Emotions

One of the recent developments of research on emotions is to study several different emotions in one study. The Leder and Forgasz (2006) review did not find reason to discuss variation in emotions beyond the positive – negative dimension. However, we found an increasing interest in the multidimensionality of emotions in research presented in the last 10 years of PME. One influential source for these elaborations has been Pekrun's (e.g., Pekrun, Elliot, & Maier, 2006) framework of academic emotions (e.g., Heinze & Frenzel, 2010; Schukajlow, 2015). Further, Pesonen and Hannula (2014) traced these roots of multidimensional view of emotions further back to the literature of basic emotions (e.g., Ekman & Friesen, 1971). Another approach to multidimensionality of emotions has been Goldin, Epstein and Schorr's (2007) framework of archetypal affective structures (Alston et al., 2008).

Heinze and Frenzel's (2010) study was exceptional also because of their focus on the relationships between trait and state type emotions. They found trait mathematics enjoyment to correlate with state mathematics enjoyment in mathematical and educational contexts and trait mathematics anxiety to correlate with state mathematics anxiety in mathematical context. However, state mathematical anxiety had no correlation with other mathematics anxiety variables.

Multiple Motivations

In McLeod's (1992) framework motivation was perceived as a somewhat hidden subdomain of beliefs. This may explain why motivation was not, for a long time, a popular concept in mathematics education. Recently it seems to have gained more popularity in our field. Motivation research is conceptually diverse (Murphy & Alexander, 2000), and the diversity is reflected also in PME.

In our review, we found four main directions of research on motivation. First, there was research (primarily by Cypriot researchers) based on achievement goal theory, which typically measures students' inclination to focus on performance goals or mastery goals (Athanasiou & Philippou, 2006, 2009; Hannula & Laakso, 2011; Marcou & Philippou, 2005; Mousoulides & Philippou, 2005; Pantziara & Philippou, 2006, 2007, 2009; Porras, 2012). We found also research that looked at student needs and goals more qualitatively. For example Erens and Eichler (2014) identified a multiple-layered hierarchical system of goals explaining teacher choices when teaching calculus, while Naresh and Presmeg (2008) analysed goals behind a bus conductor's practice and motivation and Zazkis and Nejad (2014) analysed teachers' perceptions of students' intellectual needs. There was a number of studies which applied Schoenfeld's (1998, 2010) framework for "goal oriented decision making" (Hannah, Stewart, & Thomas, 2013; Paterson, Thomas, & Taylor, 2011; Thomas & Yoon, 2011). The third important approach to motivation was to explore mathematical values (Bishop, 2001). Many of these papers came from the Third Wave project (Andersson & Österling, 2013; Seah, 2007, 2011, 2013; Seah & Ho, 2009; Seah, Zhang, Barkatsas, Law, & Leu, 2014), but not all (Frade & Machado, 2008; Lin, Wang, Chin, & Chang, 2006; Wang & Chin, 2007). The last significant category we found were studies that conceptualized motivation as interest (Krug & Schukajlow-Wasjutinski, 2013; Olson, Slavin, Olson, Brandon, & Yin, 2010; Rach & Heinze, 2011; Schukajlow 2015). Moreover, there was one study that used the extrinsic–intrinsic distinction in their analysis (Zeybek & Galindo, 2011).

THE SOCIAL TURN IN AFFECT RESEARCH

The 'social turn' in mathematics education (Lerman, 2000) highlighted the need to address mathematics-related affect through social theories. Hannula noted that "emotions are, by their very nature, linked closely both to the biological human body and to social systems" (2012, p. 155) and used this as an argument to acknowledge three different levels for theorizing about affect: the embodied, the psychological, and the social. In our review, we found an increased use of social theories in PME research reports and research fora.

Identity

One often appearing concept used in research on mathematics related affect over the last ten years has been identity. Many of the studies focused on identity construction and how it is influenced by peer interaction (Kotsopoulos, 2009), specific mathematical tools (Chorney, 2011) and practices in actual classroom life (Heyd-Metzuyanim, 2013). This social construction of identity was highlighted in studies also where ethnicity is at play (e.g., Mulat & Arcavi, 2009; Gorgorió & Prat, 2013)

Several papers have also discussed the bridging between the social identity theories and earlier theories of beliefs and attitudes. For example, Ingram (2009) used a grounded theory approach to recognize students' identifying stories in a study where she followed the same students for several years. She concluded that students engage in mathematics when their perceptions of what they want to achieve has not yet been realized. These perceptions are affected by the students' views about mathematics, the context of the moment and the students' feelings about being able to do mathematics. In other words, students' beliefs about mathematics and self are important factors in their identity formation.

The bridging between the individual affect and the social identity was explicitly discussed in the 2010 Research Forum *Identity and Affect in the Context of Teachers' Professional Development* (Frade, Rösken, & Hannula, 2010). The forum discussed a spectrum of theoretical and methodological perspectives, including the structured nature of affect and beliefs (Goldin, Rösken, & Törner, 2010), Lacan's model of subjectivity and new teacher identity (Brown, 2010), the social nature of affect and a pragmatist perspective on identity formation (Frade & Meira, 2010), present and ideal teacher identity (Krzywacki & Hannula, 2010), and local and global affective structures (Gómez-Chacón, 2010). The Research Forum suggested the different theoretical frameworks to be complementary to each other rather than contradictory. Several contributions emphasized the inherently dynamic nature of identity. Krzywacki and Hannula (2010) wrote about positions available in the social context and a continuous negotiation between how one perceives oneself and the positions that are available. Based on a pragmatist perspective and on Vygotsky's ideas, Frade and Meira (2010) provided a social theory that emphasizes discourse and semiotic spaces. Their perspective strongly rejects a dichotomy between "individual realms" and "social fields", and they see both identity and affect as a combination of contingency and circumstance of social intercourse of the individual's activity and life. Moreover, they share Gee's perspective of people having multiple only temporally stable identities connected to "their performances in society" (Gee, 2000, p. 99 cited in Frade & Meira, 2010, p. 263).

Aside from the aforementioned research forum, Österholm (2009) approached the bridging of individual and social differently. He took two theoretical constructs, one individual (epistemological beliefs) and another social (communication) and analysed these systematically trying to create a coherent theoretical foundation for the study of relations between the two constructs. He concluded that it seems possible to join the theories of epistemological resources and discursive psychology, and presented some necessary additions and clarifications: (1) to include a model of the structure and utilization of mental representations, (2) that mental representations are primarily seen as describing the memory of prior experiences, and (3) that the utilization of prior experiences is seen as a central aspect of the contextualization of discourses. This suggested unification of theories was therefore seen as a good starting point for a continued development of theory and for future empirical studies.

Norms

Another approach to the social nature of affect, has been through the sociomathematical norms, which were originally introduced by Yackel and Cobb (1996). For example, Toscano, Sánchez and García (2013) identified five socio-didactic mathematical norms. Three of them were in some way related to the mathematical content and its learning and the other two are related to teachers' role. Gilbert and Gilbert (2011) reported about developing effective sociomathematical norms in classrooms to support mathematical discourse and found that teachers who engaged students in the culturally relevant oral tradition of "talk-story" were better able to initiate and sustain a level of discourse that extended student learning. Liljedahl and Allan's (2013) study can also be seen as an exploration of classroom norms. They used Fenstermacher's (1986) concept of "studenting" to address those student behaviours that are not related to intentional learning. They described how grade 10 students were finding ways to game the norms the teacher was trying to establish in the class.

We also noted that researchers who use activity theory (e.g., Engeström, 1987) pay attention to affective factors, such as norms, but they often treat affect as a peripheral factor in their framework (e.g., Andrà & Santi, 2013; Goodchild, 2013; Tomaz 2013). On the other hand, Asnis (2013) successfully combined activity theory and identity theory to study teachers' professional identity from a sociopolitical perspective.

Other Approaches

In addition to the social turn, Hannula (2011, 2012) identified the potential to research affect as a physiological phenomenon using embodied theories. So far, this has been rare. Andrá (2010) explored the non-linguistic modes of communication, such as body posture, prosody, and gesturing. From an explicitly embodied perspective, Cimen and Campbell (2012) measured psychophysiological data including electrocardiography (EKG) and respiration rate, which allowed them to observe the levels of relaxation of their research subject.

PME researchers have also introduced other frameworks for research on mathematics-related affect that do not fit within any of the above. For example, Ng and Andersin (2011) used the concept teacher empathy, and specifically mathematical empathy. They concluded that prospective teachers are empathetic in nature, and that they are able to relate affectively to struggling students. Yet, they rarely are able to relate cognitively.

Williams (2010, 2011, 2013) used Seligman's (Seligman, Reivich, Jaycox, Gillham, & Kidman, 1995) concept of optimism to explain the role of positive affect and its construction in mathematical activities. Lewis (2011, 2012, and 2014) has used *reversal theory* as a theoretical framework in his studies on disaffection. The *reversal theory* perspective has allowed him to discuss the complexities of motivation of disaffected students: they are "highly motivated, but their motivational needs are frustrated and unsatisfied by their experiences of school mathematics" (Lewis, 2011, p. 137). Both optimism and reversal theory address resilience, which is an important concept that gives some explanation why some students can better endure disappointments and difficulties in mathematics than some others.

Structure of Affect: Components and their Relations

A specific and often revisited issue of the theoretical framework of affect is the structure of affect. Originally Green (1971) suggested that beliefs would be organized in clusters around specific situations and contexts, more or less isolated from each other. This discussion has continued within PME, and there were also interesting empirical studies on the structure of mathematics-related affect.

Goldin et al. (2010) discussed the "structures of affect, motivation, and beliefs" summarizing their perspective as follows:

> mathematical beliefs (a) have structure, (b) belong to structured systems of beliefs, and (c) are embedded in complex affective structures that are important to understanding students' and teachers' motivations and behavioural patterns. Those affective structures form an essential part of a teacher's identity whether in terms of a social belonging or of a personal development. (p. 253)

Hannula (2011) claimed that empirical research on mathematics-related beliefs indicates an overall pattern, where positive (or negative) beliefs are related to each other and to positive (or negative) emotions and positive (or negative) motivation. Zan and Di Martino (2009) found such connections in their study that used a very open approach to student affect. They collected 1600 essays from 1st to 13th grade students with a title "me and mathematics". In their analysis they observed that a negative emotional disposition was always linked to either a negative belief about mathematics (instrumental) or a negative self-belief (low perceived competence). There have also been several quantitative studies in PME confirming the relatedness of beliefs, emotional disposition, and motivation (e.g., Hannula, Kaasila, Laine, & Pehkonen, 2005; Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014). In addition to identifying such correlations between students' motivation, emotions and self-beliefs Hannula and Laakso (2011) observed that the structure of beliefs was more coherent (i.e., scale reliabilities and correlations between variables were higher) in grade 8 than in grade 4.

Where are We with Theories?

Although the PME format limits opportunities for theoretical discussions we have seen a positive development with increased sensitivity to terminological variation and more careful elaboration of concepts. The structural properties of affect have been taken into account and considered jointly rather than separately. There has also been a clear social turn, where not only the social influence is accepted, but researchers have also started using the theoretical tools that allow careful study of the social context. Moreover, the relationship between the individual level and the social level has been elaborated to some extent.

ROLE OF AFFECT IN MATHEMATICAL PROBLEM POSING AND PROBLEM SOLVING

Because problem solving is such an affective experience there has long been an interest in the relationship between problem solving and affective elements. Over the last 10 years this relationship has been explored through research reports looking at specific affective characteristics, relationships between multiple affective characteristics, and student and teacher belief change.

Four research reports looked at the relationship between problem solving and a single affective characteristic. Van Harpen and Presmeg (2011) compared US and Chinese student attitudes towards problem posing. They found that, despite neither group having any exposure to problem posing, the two groups of students had very different attitudes, with the US students focusing more on context and Chinese students more on mathematics. Seok and Choi-Koh (2015) looked at mathematics anxiety form a neuroscience perspective. They found that high mathematics anxiety students had longer reaction times and larger amplitude than low math anxiety students when involved in problem solving activities. Interestingly, they also found that both high and low anxiety students have larger amplitude in graph-to-algebra problem solving tasks than algebra-to-graph tasks. Williams (2013) found a strong association between students' willingness to explore challenging mathematics problems and the ontogenesis of their confidence. Pesonen and Hannula (2014) looked at common emotional states during a solitary GeoGebra problem solving session. They found the most common states to be: neutral (40% of time), sad (34% of time), happy (15% of time) and angry (8% of time).

Three research reports looked at relationships between multiple affective variables and problem solving. Interestingly, all of these came from Cyprus. Panaoura et al. (2009) found that "multiple-representation flexibility, ability on solving problems with various modes of representation, beliefs about use of representations and self-efficacy beliefs about using them constructed an integrated model with strong interrelations in different educational levels" (p. 273). Michael, Panaoura, Gagatsis and Kalogirou (2010) found, in working with geometric shapes, that there were both differences and similarities among primary and secondary school students' self-concept beliefs. Marcou and Philippou (2005) found a significant relationship between motivational beliefs and self-regulated learning, as well as between self-efficacy, intrinsic goal orientation, and performance in mathematical problem solving.

Looking only at the qualities of problems, Koichu, Katz and Berman (2007) found that undergraduates could judge a problem as beautiful if it was affiliated with mathematics associated with a high level of aesthetic value, it looked new, and its solution was accessible but included elements of surprise.

THE RELATIONSHIP BETWEEN AFFECT AND ACHIEVEMENT

The positive correlation between affective variables and achievement has been well documented in the mathematics education literature (e.g., McLeod, 1992). In the time since the last handbook we found PME articles that seem to add some new details to this relationship. This literature has been focused primarily on the specific relationships between motivation and achievement and self-efficacy and achievement.

With respect to the motivation-achievement dyad Pantziara and Philippou (2009) used Achievement Goal Theory (Elliot & Church, 1997) to try to identify exogenous and endogenous factors that influence students' mathematical performance. Within their sample of 15 teachers and 321 students they found that the dominant exogenous factor influencing student performance was teacher practice, while the dominant endogenous factor was student motivation. Ufer (2015) found that specific motives play an important role in the activation of learning activities that go along with study success. In particular,

> students from the more application-oriented financial mathematics programme reported stronger motives related to professional perspectives and application, while students in the regular mathematics programme agreed more to motives of engaging in mathematics problems and mathematics as a science. (p. 269)

Looking more at how to motivate students, Koirala (2005) in his mixed method research of 23 high-school freshman, found that the use of mathmagic motivated students to learn basic algebraic concepts and that "their engagement in mathmagic activities enhanced their understanding of variables and expressions" (p. 215).

With respect to the relationship between self-efficacy and achievement, Mousoulides and Philippou (2005) found that self-efficacy was a strong predictor of mathematics achievement among pre-service teachers. Nuancing this Hannula, Bofah, Tuohilampi and Metsämuuronen (2014) also confirmed a strong relationship between self-efficacy and achievement, but also that this effect is reciprocal and more strongly from achievement to self-efficacy. Moreover, the study found a weaker unidirectional effect from achievement to liking mathematics.

Aside from the role of motivation and self-efficacy, Eleftherios and Theodosios (2007), in looking at survey data from 1645 grade 10, 11, 12 students, found that love of mathematics is the factor which correlates most positively with mathematical performance and ability. Looking from the direction of low achievement, Heinze (2005) found that particularly low achieving students are not aware of the learning opportunities afforded by mistake-handling activities.

GENDER AND AFFECT

In the Leder and Forgasz (2006) review, they noticed that, unlike other research on affect, research on affect and gender "has had a recognized and discernible impact on the development and delivery of mathematics instruction" (p. 412). Perhaps the consensus on the issue has been the reason that rather few studies on affect in 2005–2015 have focused on gender. Yet, some studies have provided additional details. Chiu's (2009) analysis of TIMSS data confirmed the old results that affective, cognitive, and social factors influence gender difference in mathematics achievement, but a new observation was that the influence is different for measurement and algebra. Hannula (2009) used generalized linear models (GLM) to identify whether the variation of different beliefs was primarily individual, between classes or between genders, concluding that student confidence in mathematics and perceived difficulty of mathematics were primarily gendered.

A qualitative study of students' narratives found a gender influence on students' identity construction, but no difference was found between single-sex and coeducational schools (Simpson & Che, 2015). Gender differences in mathematical self-efficacy are well documented across many countries. Other gender differences may be more culturally specific. Gattermann, Haverscheid and Wittwer (2012) found that in addition to self-concept there are gender differences also in German students' epistemological beliefs and Eleftherios and Theodosios (2007) reported that Greek girls believed more than Greek boys in a procedural learning approach. Leder (2009) conducted a survey of 90 Australian adults, who had been high achievers in mathematics (e.g., participants in mathematics Olympiads), to understand why so few of them had chosen mathematics-related careers.

Forgasz, Leder and Tan (2013) explored the perceptions of the general public in nine countries regarding gender differences in mathematics through a survey distributed via social media while Gómez-Chacón, Leder and Forgasz (2014) approached 393 pedestrians. Both studies found relatively little evidence of gender stereotyping. However, when found, the traditional male stereotype prevailed.

HOW AFFECTIVE TRAITS DEVELOP

Affective traits do not exist *ex nihlo*. They are developed over time. Within the review of the PME literature this fact was explored both within the context of students and teachers.

Change of Students Affect

The overall declining trend of student affect was well documented early (e.g., McLeod, 1992). PME research over the last ten years, provided some additional detail to these observations. For example, students' beliefs and attitudes have been found to be independent from their social status (Eleftherios & Theodosios, 2007).

Many studies confirmed a decline over the transition from primary to secondary school (Michael et al., 2010). More specifically, Athanasiou and Philippou (2006, 2009) observed that student perception regarding their classroom culture also became more negative over this transition and Horne (2009) reported that student perception of their learning environment also becomes more traditional as they go to upper secondary level. Tuohilampi et al. (2014) provided additional detail, when they reported a general decline of enjoyment of mathematics during primary school years, whereas self-efficacy declined during lower secondary school years.

The outcome of this development – a negative affect – may become so fixated that it is impossible to change later (Melo & Pinto, 2007). Students who had failed in traditional classrooms felt ‘cheated’ when an outreach intervention was based on a constructivist approach when they wanted procedurally based activities instead (Ewing, Baturo, Cooper, Duus, & Moore, 2007). On the other hand, Lewis’s (2012) case study illustrated the motivational and emotional complexity of a disaffected students’ relationship to mathematics, where both positive and negative emotions weave in and out of their experience.

Some studies also reported cases of a more positive change in student affect. One effective way to promote more positive affect is through problem solving (Shimada & Baba, 2015; Williams, 2010) or modelling tasks (Schukajlow & Krug, 2012) – possibly in a collaborative setting (Park, 2014). Also the instructional practices suggested by achievement goal theory (Pantziara & Philippou, 2007) or self-regulation and self-efficacy theories (Lavy & Zarfin, 2012) have been successful ways to promote a positive affective disposition.

Student beliefs are obviously influenced by the learning context and the teacher. Some research suggested that the development of students’ affective disposition is influenced by classroom ‘microculture’ (Lewis, 2014). Hannula (2009) identified that perceived teacher quality and enjoyment of mathematics were largely influenced by the classroom factor rather than individual difference or gender, and these beliefs were more positive in classes where the teacher had progressive teaching beliefs. Other studies showed that an error-tolerant classroom culture (Rach, Ufer, & Heinze, 2012), mathematics teachers’ efficacy beliefs (Chang & Wu 2014), and teachers’ values (Frade & Machado, 2008) had a significant influence on student affect. Regarding teachers’ values Seah’s (2007) results suggested that rather than values *per se*, the alignment of teacher and student teaching-related values was beneficial. In a similar tone, Hernandez-Martinez (2008) showed that when students and lecturers engaged in common practices within their institutions, they co-constructed their identities and this in turn shaped the practices in which they participate.

A few studies had specifically looked at the role of the teacher in cases of underprivileged ethnic groups. Several studies showed how integrating cultural elements into mathematics classrooms impacted student affective disposition (Amit & Abu Qouder, 2015; Hunter, 2013; Kidman, Cooper, & Sandhu, 2013; Kidman, Grant, & Cooper, 2013). In such classrooms the teacher played a significant role not only in the mathematical meaning-making but also in the reconstruction of identity (Gorgorió & Prat, 2013). What Lerman (2012) reminded us, was that the teacher's role of a significant other in their identity construction was not limited to students of underprivileged ethnicity.

It has been long accepted that student affect is influenced not only by their personal experiences in school, but also by their other social environment (e.g., McLeod, 1992). One significant influence are parents, and this influence may last until adult age (Hannula, Kaasila, Pehkonen, & Laine, 2007). Chang and Wu's (2012) study found that the socioeconomic background of the family and parenting styles were critical elements in the development of fifth-graders' mathematics self-efficacy. Some studies have reported successful interventions targeted at parents (Mousoulides, 2014). A comparative study between China and Australia indicated that Chinese students perceived parental influences to be stronger than students in Australia (Cao, Forgasz, & Bishop, 2005).

Change of Teacher Affect

A specific subset of how affective traits develop was a collection of research on how such traits change among teachers. With respect to preservice teachers, Rolka, Rösken and Liljedahl (2006) found that the problem solving environment within a preservice methods class had impact on the recasting of these preservice teachers' beliefs about what mathematics is, and what it means to teach and learn mathematics. Bragg and Nicol (2008) found that involving preservice teachers in the design of open-ended problems similarly shifted "the ways they viewed mathematics and how it might be taught" (p. 201). More recently, Thanheiser, Philipp and Fasteen (2014) found that having preservice teachers find, modify, or develop tasks to use with elementary school students was both exciting and motivating to them as they saw relevance between the activity and their future career as teachers.

Erens and Eichler (2013) found that a shift from self-referred reflection to reflection of the classroom practice impacted preservice teacher's perception of authority. That same year, Di Martino, Coppola, Mollo, Pacelli and Sabena (2013) found that future primary teachers have strong negative emotions towards mathematics, but they also have a desire to redeem these negative emotions. Bjerke, Eriksen, Rodal, Smestad and Solomon (2013) looked at the tensions experienced by preservice teachers—specifically between their prior experiences and beliefs and what they were learning at university. Finally, Kuntze and Dreher (2013) looked at affect related to computer use and found that it was possible to improve views towards this in both preservice and inservice teachers.

With respect to changes in inservice teachers' affect there was a large variety of change and affective factors without any convergence of focus. For example, Shy, Tsai and Chiou (2009) found that teachers were becoming more and more willing to work with gifted students. Meanwhile, Olson et al. (2010), testing alternative professional development programs, found that a focus on formative assessment actually had a greater effect on teacher change than a professional development program that focused more on technology. Looking at teacher motivation, Zeybek and Galindo (2011) found that "professional development activities form complex interrelationships with teacher motivation consisting of intrinsic and extrinsic motivators" (p. 361). After participating in professional development built around communities of practice, Besamusca and Drijvers (2013) saw that teachers became more confident in their use of ICT. That same year, Goodchild (2013) found that "development is a slow extrapolation of practice based on shared ideas that, when supported by experience, are meaningful in the imagination of teachers" (p. 369). In the same study, he also found that teachers are dismissive of novel approaches suggested by mathematics education researchers. Based on the results of their study, Kidman, Grant and Cooper (2013) suggested that, when working within an indigenous context, a climate of collaboration among teachers was imperative to maximizing change. Kuntze and Dreher (2013) confirmed that it was possible to improve teachers' views around computer use through professional development courses. And Oksanen and Hannula (2013), in a comparison of data from 2012 with similar data from 25 years earlier, found changes in mathematics teachers' beliefs about teaching.

In addition to the aforementioned research reports, there were also relevant results presented in the 2007 research forum Researching Change in Early Careers Teachers (Hannula & Sullivan, 2007). For example, Sullivan (2007), in his introduction at the research forum, commented on the assumption that changes in practice relate to changes in beliefs, and it may be that changes in practice precede changes in orientation (Guskey, 1986). Consistent with this position, Zaslavsky and Linchevski (2007) were anticipating changes in teachers' beliefs as they implemented an innovative program. Quite aside from this, Hannula, Liljedahl, Kaasila and Rösken (2007) explored therapeutic approaches to helping students cope with and change their negative mathematical affect. The results from their aggregated independent work showed that it was possible to change this negative affect through: (1) narrative rehabilitation, (2) bibliotherapy, (3) reflective writing, and (4) drawing schematic pictures. Finally, in his summary of the research forum, Hannula (2007) commented that the studies in the forum had been exploratory in nature—that there had been no clear hypothesises that could have been tested. Instead, the aim of the research had been to describe and understand the process of change. According to Hanula (2007), this was indicative of the low level of our understanding about teacher change to that point.

TEACHER BELIEFS AND PRACTICE (INSERVICE)

In a decade where teacher knowledge was extensively researched it is not surprising that there was also a complementary focus on teacher beliefs. In all, there were 24 research reports presented on this topic in the ten years from 2005 to 2015, the majority of which were focused on the beliefs of inservice teachers in general and the relationship between beliefs and practice or beliefs about teaching in particular. We remind the reader of the critical views about research on teacher beliefs discussed earlier.

With respect to beliefs and practice, Yates (2006) looked at the relationships between teachers' beliefs, practices, and experiences with curriculum reform in their elementary grades. She found that reform experiences, age, qualifications, and length of mathematics teaching experience did not significantly affect their teaching practices. Surprisingly, she also found that the participants' beliefs about the mathematics were unrelated to their beliefs about the teaching and learning of mathematics. However, the participants did differ in their beliefs and this was correlated to some of their child-centred practices. Finally, teachers holding strong views about the beauty of mathematics and those that scored well on constructivism used manipulatives more often, used worksheets less often, and used tests less frequently. Likewise, Hannula, Pipere, Lepik and Kislenko (2013), found that teachers' beliefs and cultural context explained 15% of the variation in self-reported constructivist teaching practices. They also found that the school micro-culture did not provide any significant additional effect. Beswick (2008), on the other hand, found that the teachers in her sample had very realistic views of the extent to which their classrooms conformed to constructivist principles. Zachariades, Nardi and Biza (2013), found that the use of multi-step tasks with inservice teachers gave them insights into their pedagogical perceptions and intentions, as well as epistemological beliefs, and revealed discrepancies between the teachers' stated beliefs and intended practice. Finally, Liljedahl, Andrà, Di Martino and Rouleau (2015), looking at the tensions teachers experience, including the tensions between their beliefs about practice, found that teachers do not simply manage these opposing forces, but also work at, and seek help in, resolving them.

Shifting now to teachers' beliefs about teaching, Anderson and Bobis (2005) found that, although their participants seemed to be well-aware of what the reform-based movement recommends regarding the teaching and learning of mathematics, some common responses were indicative of more traditional views of mathematics. For example, comments such as "I like my students to master basic mathematical operations before they tackle complex problems", "a lot of things in maths must simply be accepted as true and remembered", "if students use calculators they don't master the basic maths skills they need to know" came from teachers who simultaneously were able to talk about the reform-based movements. Staying with this tension, Beswick (2009) looked at a case study of one mathematics teacher as an example of how some teachers can hold differing beliefs about the nature of mathematics, viewing it either as a discipline or as a school subject. Meanwhile, Zazkis and Nejad (2014) used script writing to look at teachers' ideas about teaching. These scripts demonstrated their views of teaching as well as the traditional views they may be facing in their practice. Somewhat as an aside, Alexandrou-Leonidou and Philippou (2005) found that 5th and 6th grade teachers could only partially predict students understanding and reasoning in arithmetic and algebraic equations in different representation formats. They found that, contrary to the teachers` perceptions, the students could manage word equations and story problems more easily than they could handle tasks represented by pictures and diagrams.

Aside from the research focusing on the relationship between beliefs and practice there were also three papers that looked more closely at the beliefs themselves. For example, Diamond (2014) found five categories of beliefs about transfer of learning in her participants, thereby both extending current conceptualizations of transfer and identifying new beliefs regarding students' transfer of learning. In the same year, Frade, Lerman and Meira (2014) found that "teachers' affective positionings towards others emerge from shared social scenarios, manifested in response/reaction to such scenarios, and reflect their attempts to redescribe themselves in the eyes of others" (p. 105). Finally, Kuntze and Reiss (2005) found that cognitive constructivist and direct-transmission views of teaching and learning impacted the situated beliefs of teachers, as seen during their interpretation of videotaped classroom situations.

TEACHER BELIEFS AND PRACTICE (PRESERVICE)

The broad interest in teachers' beliefs was not restricted to inservice teachers, with eight papers focusing on the beliefs of preservice teachers, some of which focused specifically on teachers' beliefs about some aspect of mathematics education. For example, Bayazit and Aksoy (2011) found that the 22 teachers in their sample believed that analogies would contribute to student learning. Rahat and Tsamir (2009) looked specifically at high school teachers' beliefs about errors in the mathematics classroom and found that the mathematical quality of a classroom depends on a teacher's didactical beliefs about how to address errors. More recently, Hallman-Thrasher and Connor (2014) looked at prospective secondary teachers with a STEM background and the views that they held about how mathematics should be taught. They concluded that the teachers' views of what should be taught and how it should be taught was affected by their content experience.

Other papers focused on the relationship between prospective teachers' beliefs and some other dimension of mathematics education. Gomez and Conner (2014) found that preservice teachers' "affective responses while learning mathematics were a strong influence in the prospective teachers' evolving professional identities" (p. 177). Goos (2005), also looking at identity (as users of technology), found that Valsiner's (1997) zone theory was useful in understanding the relationship between teachers' pedagogical beliefs, the teaching repertoire offered by their pre-service course, and their practicum and initial professional experiences" (p. 49). Meanwhile, Flores and Carrillo (2014) looked at the relationship between a specific preservice teacher's conceptions of mathematics teaching and learning and her specialized knowledge. Their findings indicate that this connection is visible in her intentions for a lesson. Beswick (2015), also focusing on a single pre-service teacher, found that his choices relied heavily on his beliefs. Beswick used this result to argue that this blurs the boundary between beliefs and pedagogical content knowledge.

Finally, Di Martino and Sabena (2010) argued that we need to be cautious when discussing inconsistencies between beliefs and practice and that such inconsistencies may be external to a teacher with the true inconsistency being between the researcher's and teacher's assigning of words to meanings. Their research found this to be true with respect to inconsistent meanings assigned to the expressions "word problem" and "problem solving", especially with respect to what makes good word problems.

NEW IDEAS

When doing this review, we started from our educated preconception of what are the important topics in the research area of mathematics related affect. As such, we found what we were looking for. In addition, however, we found things that we had not anticipated. For example, there was a large number of studies focusing on affect among tertiary level mathematics students (Andrà, 2010; Baldino & Cabral, 2005; Bjerke et al., 2013; Ufer, 2015) and specifically on the transition to the university (Di Martino & Morselli, 2006; Di Martino & Maracci, 2009; Rach & Heinze, 2011). And we found studies on tertiary level teachers' affect (Hannah et al., 2013; Lerman, 2012; Paterson et al., 2011).

Another emerging area of research was on the affect and identity of indigenous and immigrant students. For these groups there are specific issues that influence their affect and identity, for example their cultural needs (e.g., Cooper, Baturo, Warren, Catholic, & Grant, 2006; Howard & Perry, 2005; Tomaz, 2013) and paying attention to power (Baldino & Cabral, 2005; Skog & Andersson, 2013; Tomaz, 2013).

RESEARCH METHODS

Leder and Forgasz (2006) listed in their review common measures of attitude and beliefs. They identified Likert-scale questionnaires, interviews and observations as typical methods and a "near even split of qualitative, quantitative, and mixed methods" (p. 415). They also observed a trend towards more qualitative research. Over the period 2005–2015 Likert-scale questionnaires, interviews and observations were still typical methods. Almost half of the studies were qualitative, one quarter were quantitative, one fifth were mixed, with the remaining papers being theoretical. It should be noted that a handful of studies (Bragg & Nicol, 2008; Bruce, Flynn, Ross, & Moss, 2011; Hunter, 2013) used a design research method.

The size of studies was rather evenly distributed into five categories (Table 1). There was quite a large number of very small studies with one to five participants and these were typically qualitative case studies. When we take into account the clustered nature of typical data collected from intact classrooms, the statistical power is typically sufficient to make solid conclusions only in studies that have several hundred participants.

Table 1. Frequency table of studies according to their size

Number of subjects in the study	*Number of studies*	*Frequency (%)*
1…5	34	21.25
6…20	29	18.13
21…100	41	25.63
101…500	33	20.63
500…	23	14.38
Total	160	100

While qualitative studies and case studies in general are valuable especially for exploring new ideas and for generating hypotheses, the progress of science needs also rigorous testing of hypotheses and the rejection of those hypotheses that do not survive such tests. It was somewhat worrying that the weight of the studies lay so strongly on small-scale and descriptive studies and experiments or quasi-experiments were quite rare in PME. One possible reason is that it is much more difficult to confirm solid results in an experiment than it is to observe something interesting in a qualitative case study. For example, Olson, Slavin, Olson, Brandon and Yin (2010) had to conclude that while participants in two different professional development programs did have some differences in knowledge gains, their experiments did not show any influence on teacher affect.

In a series of quasi-experimental studies on mathematical modelling the first results (Schukajlow & Krug, 2012) indicated that a learning environment combining directive instruction and group work has a positive influence on students' self-regulation, self-efficacy and values. Moreover, while the treatment of multiple solutions guided the majority of students to develop multiple solutions and increased their self-regulation, it had no effect on students' self-efficacy and values. More detailed experiments confirmed that intervention increased the number of solutions, but no hypothesized influence on affect was confirmed. A specific task processing intervention only decreased students' task specific interest (Krug & Schukajlow-Wasjutinski, 2013) and no differential effects on three alternative treatments on students' self-regulation could be observed (Achmetli, Schukajlow, & Krug, 2014). Also, the previously mentioned study reporting positive effects of the error-tolerant classroom culture on affect was a quasi-experimental intervention study (Rach et al., 2012).

Increased computing power of tabletop computers has made path analysis and structural equation modelling (SEM) accessible to all researchers. The methodology requires quite large data sets, which may explain why these are not popular among

PME research reports on affect. Pantziara and Philippou (2006) collected data from 302 grade six Cypriot students and their SEM analysis showed that fear of failure had a direct effect on students' achievement and to their interest in mathematics, and an indirect effect on both variables via mastery and performance goals. Moreover, mastery goals were predicted by self-efficacy and performance goals were predicted by both, fear of failure and self-efficacy. And lastly, performance-approach goals facilitated interest in mathematics but proved to have a negative effect on students' achievement in mathematics.

A Finnish research team reported results of a longitudinal, nationally representative data (grades 3, 6, 9; 5161 students). The sample size and the longitudinal design allowed simultaneous testing for the direction of causality between achievement and self-efficacy and achievement and enjoyment (Hannula et al., 2014).

In quantitative surveys, Likert-scales dominate the field. The few alternative approaches included semantic differential (Bernack, Holzäpfel, Leuders, & Renkl, 2011) and selecting items from a list in a preference order (Löfström, Hannula, & Poom-Valickis, 2010). An interesting new quantitative approach has been to make discourse analysis quantitative (e.g., Asnis, 2013; Nachlieli, Heyd-Metzuyanim, & Tabach, 2013). Asnis (2013) analyzed an impressive text corpora of 17,724,172 words from different sources, representing different communities: policy documents, newspaper articles, various education-related websites, and interviews with mathematics teachers. His tentative conclusion was that Israeli mathematics teachers' "identity discourse [was] influenced most strongly by the governmental discourse, and also, at least to some degree, by conceptual-ideological discourse of educational establishment" (p. 47).

On the qualitative side, interviews and observations are the norm. Within this line of research, interesting new approaches that seem to suit the area of affect well, have been narrative inquiry (Coppola, Di Martino, Pacelli, & Sabena, 2015; Simpson & Che, 2015) and narrative reporting (Frade et al., 2014; Liljedahl et al., 2015). As an alternative, Rolka and colleagues used students' drawings together with verbal expressions to study student beliefs about statistics (Bulmer & Rolka, 2005) and mathematics (Halverscheid & Rolka, 2006, 2007). Student drawings were used as the data also in a study by Varas, Pehkonen, Ahtee and Martinez (2012). We also found a 9-month ethnographic study involving Aboriginal students, their parents, Aboriginal educators, and non-Aboriginal teachers living in a remote Australian rural community in (Howard & Perry, 2005).

We also noted that among our data set a large variety of different mixed designs were used. However, the space allows us only to give two examples. Galligan (2005) study is an example of a mixed study with numerous data sets: eight lesson observations; pre and post-lesson semi structured interviews of teachers; teacher values questionnaire; teacher marking of student written work; student questionnaires; questionnaire given to teachers and moderators on their perceptions of students answers; other documents as necessary (Education Department approval forms; contracts, etc.); and two informal interviews with administrators of the program. The other example is Vollstedt's (2011) study, where stimulated recall interview data was analysed with a *grounded theory* approach to identify meanings given for mathematics and then a quantitative cluster analysis was performed to identify which meanings tended to appear together.

A number of studies used methodological triangulation, which can either confirm the validity of a new research instrument (Bernack et al., 2011; Löfström et al., 2010; Nachlieli et al., 2013) or allow a critical discussion of shortcomings of an instrument (Andersson & Österling, 2013).

New Directions

Leder and Forgasz (2006) expected physiological measures and new technologies to become more common in PME after 2005. We saw some changes happening towards this direction, but the studies were mostly just examples of possibilities rather than established methodological approaches. With regards to physiological measures, Cimen and Campbell's (2012) study gave an example of what is possible in this realm. They used a wide spectrum of observational methods ranging from audiovisual, keyboard and screen capture, eye-tracking, and self-report data, to psychophysiological data including electrocardiography (EKG) and respiration rate data. Also Seok and Choi-Koh (2015) found a correlation between the anxiety scale results and EEG measurements.

A much more common use of technology has been to use it for recording audiovisual data. Especially with regards to research on affect, it is now rare to make observational studies without the support of video-recordings. Screen recordings (Chorney, 2011; Pesonen & Hannula, 2014) or screen captures (Hähkiöniemi, 2011) can be used to record student's use of dynamic geometry software. Moreover, students can use smartphones to take photographs to record their perspective of significant learning events (Seah, 2011) or their learning can be recorded digitally (e.g., Andersson & Seah, 2012; Larkin & Jorgensen, 2015). New software promises also relief from laborious coding of audiovisual data. Pesonen and Hannula (2014) report successful use of software to automatically recognize student emotions from video data.

One unexpected use of technology, has been recruiting respondents through social media (Forgasz et al., 2013). It allows an easy access to international population, but, as the authors discussed, there is the danger of the sample being skewed with respect to socioeconomic background and age.

One additional example of the novel use of technology, was the simulated classroom simSchool as research context (Meletiou-Mavrotheris & Mavrou, 2013). SimSchool simulates student behaviour using the *Big Five* model of personality (McCrae & Costa, 1996) and interpersonal circumplex theory (Kiesler, 1983). Using these theories each virtual student was programed with a unique personality leading some students to get frustrated (if a task was too difficult) or bored (if a task was too easy). Pre-service and novice teachers could use this simulation as a safe environment for experimenting with teaching techniques, seemingly increasing their perceived instructional self-efficacy.

Another new direction in methodology was to use different projective techniques (e.g., Lewis, 2012). Instead of asking directly about a participant's experiences or affect, they could be asked to redesign a lesson (Olson & Kirtley, 2005) or to imagine conversations with a colleague, a school principal, or a concerned parent (Zazkis & Nejad, 2014), or the researchers could examine the metaphors teachers (Oksanen, Portaankorva-Koivisto, & Hannula, 2014) or students (Taing, Bobis, Way, & Anderson, 2015) produced.

CONCLUSIONS

Reviews like this provide us as an opportunity to reflect on our progress. What have we accomplished as a research community? Where should we be heading next?

The review of the research reported at PME over the last ten years indicates a development in the theories of mathematics-related affect. The research tradition on attitudes, beliefs and emotions is reaching a level of sophistication suggested in McLeod's (1992) article: affect has been examined as a system that interacts with learning and social context. The social turn, suggested by Lerman (2000), has taken place: research on identity has expanded rapidly, research on classroom norms (Yackel & Cobb, 1996) has emerged and there is some indication that activity theory (Engeström, 1987) is beginning to address emotions more explicitly. There is also some indication of new approaches – at least new within PME – of examining affect using theories that perceived affect as a psychophysiological phenomenon.

Such expansion of theoretical approaches may be fruitful, but only if the different approaches are able to communicate with each other. This requires that each researcher clearly explicates their theoretical approach and clearly defines the concepts they use – in a language that is transparent to researchers from other paradigms. One example is the ongoing discussion for alternative ways to explain the apparent discontinuities in teacher beliefs and between their beliefs and practices. This area will most likely benefit from a dialogue between the more traditional belief research and more recent social theories looking at the same question.

The research in and outside PME has accumulated solid evidence on the reciprocal relationship between student affect and achievement as well as about gender differences in mathematical self-efficacy. It was also well documented that student affect tends to decline during school, but that interventions can change this development. There is little need for work that simply repeats the same results. Future work in these areas needs to focus on rigorous testing of well formulated hypotheses.

Specifically important for a community like PME is to explore the cultural variation in mathematics related affect. For example, are the well-established findings – such as the ones mentioned above – truly universal, or are these findings limited to the most thoroughly researched industrialized countries.

In this review, we identified two recently expanded research areas of mathematics-related affect. Research on mathematics-related affect on tertiary level education is not a completely new area, but it seems to be expanding, especially research on mathematics teachers at tertiary level has been rare in the past. An even more prominently expanding area of mathematics education research is to study the affect of indigenous and immigrant populations. Although research on minorities can be seen as less relevant than research on large populations, such studies are important for testing how universal our findings are and deeply contextualised qualitative studies are also likely to produce new understandings of the underlying relations between affect and contextual factors.

Methodologically, we have seen a shift towards qualitative studies over the last ten years, cases studies being quite popular. This may be related to the social turn in affect research. As new theoretical approaches are used, the first stage of research is usually qualitative, as researchers need to generate an understanding of the phenomena before generating testable hypotheses. There are some examples of quantifying discourse analysis (Asnis, 2013; Heyd-Metzuyanim, 2013) and we expect larger scale quantitative studies employing social theories to increase their number in the future. A similar stage of numerous case studies may be expected also for studies using psychophysiological measures of affect.

While the general trend has been towards qualitative, the quantitative studies seem to be developing in their complexity. New software allows more complex analysis, which is a welcome development in a field where typical data is multilevel (individual and classroom level) and several variables need to be accounted for. One thing that seems to endure over time, is the popularity of Likert-type items for surveys.

Ten years ago, Leder and Forgasz (2006) forecasted an increase in studies using physiological measures and new technologies. Although such increase has been hardly visible in PME to date, we make the same prediction. Instruments for observing physiological measures have become cheaper and less intrusive. While cheaper technology makes such new data collection accessible for more researchers, the true potential for expansion comes from software that makes physiological data much faster to process and easier to interpret. Automatic analysis of facial expressions (Pesonen & Hannula, 2014) and large text corpora (Asnis, 2013) are only the beginning.

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NÚRIA PLANAS AND PAOLA VALERO

13. TRACING THE SOCIO-CULTURAL-POLITICAL AXIS IN UNDERSTANDING MATHEMATICS EDUCATION

INTRODUCTION

Research is always carried out from a standpoint, an epistemological stance that shapes the ontological assumptions about what is being researched, even though the researchers might be unaware of or unconcerned about what it is. Despite apparently being evident, this assertion needs to be revisited when reviewing the insertion of socio-cultural approaches to mathematics thinking, learning and education in the last 10 years of research in PME. The way in which theoretical tools and frameworks from other areas of study have been appropriated into mathematics education and how they have been transformed for the purpose of studying it are important issues in carrying out a research review. In this review we pay attention to the stances and assumptions that, together, articulate a logic about what it means to understand mathematics education as social and cultural. Connected to this, an initial observation, from which we derive the structure of this review, is the very same meaning that the category *socio-cultural* seems to have in the research community of PME – and more broadly in the international research in mathematics education. Our present investigation asks three leading questions:

1. What are the meanings that the authors who explicitly frame their research work as socio-cultural have given to this category?
2. Which are the identifiable directions and specific lines of concern in this body of research?
3. How have these meanings, directions and lines been enlarged and transformed in the span of 10 years?

It is our contention that there has been a growing adoption of socio-cultural frameworks in mathematics education research, and that such an expansion has led to important developments in the field, within and outside PME. This is not new; it continues along the path outlined by Gates (2006), among other authors, in the previous decades. We also argue, however, that there has been a move towards the configuration of a *socio-cultural-political axis*. While the progressive expansion of *socio-cultural* frameworks is not new, the clear featuring of the *political* framework

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is undeniable. Thus, the configuration of such an axis is particular to the last decade. From the very beginnings of PME, there has been discussion about the socio-cultural aspects of mathematics education research: what defines a socio-cultural approach, what constitutes a socio-cultural trend in the history of PME… This discussion became evident in the first *Handbook of Research on the Psychology of Mathematics Education*, in the chapters by Confrey and Kazak (2006), Gates (2006) and Lerman (2006) and their review of 'Social aspects of learning and teaching mathematics' for the period 1977–2005. The intersecting perspectives adopted in these chapters respectively addressed: constructivism, the understandings of the social, equity issues and access, and socio-cultural research. To a certain extent these perspectives could be identified separately in research; however, things have changed a great deal since then. In the decade 2006–2015, the adoption of a series of related theories allowed researchers to address not only the micro-constitution of mathematical thinking, but also its macro-configuration within larger societal fields. Consequently, it is no longer possible or at least not feasible to demarcate the boundaries between what the social, cultural and political embrace, although particular theoretical tools may emphasize one aspect more than others. This expansion to incorporate the political contributes to realising the ambitions of a research field that provides deep understandings of the complexity of mathematics education in contemporary societies.

Institutionally, the engagement of PME with an emerging socio-cultural-political axis has had several key moments. One such moment can be located at PME31, where the theme of the conference was 'School Mathematics for Humanity Education'. This took place just three years after PME28, where the theme had been 'Inclusion and Diversity'. In his plenary in PME31, Breen (2007) emphasized the fact that "individuals do not operate outside of a context – the social and political are ever-present in our teaching" (p. 76). Breen went on as follows:

> Thinking about PME, one might argue that PME conferences have always been held with the express purpose of annually celebrating the light … We each have our own template of what that light looks like and how it should be explored, and we judge each other's contributions against this template in our search for certainty. [D]evelopments indicate a welcome willingness on PME members' part to look beyond the light of mathematics education and embrace the shadow as an integral part of our field. (pp. 76–77)

The metaphor of integrating light and shadow represents well the challenges that socio-cultural-political research poses to the field. This metaphor points to the importance of how decisions and choices are made on what to research, why, with whom and for what purposes. As researchers we have the power, privilege and responsibility to illuminate the complexity of mathematics teaching and learning both towards the details of children's thinking processes in meaning mediation, and towards the broader significance of mathematics and mathematics education in contemporary societies. On tracing the paths of the socio-cultural-political axis in the last 10 years of research in PME, we will map out how these decisions and choices have been made and how they have constituted lines of thought about mathematics thinking, teaching and learning.

We have organised the chapter in three main sections. In 'Expanding views: what does the socio-cultural-political axis mean?' we look at recent work that has contributed to an expansion of the socio-cultural views of mathematics teaching and learning, embracing the political views. From there we go to 'Mapping the socio-cultural-political axis: what is it like?', where we provide an overview of some clusters of topics and findings in PME research. In the last section, 'Moving the field forward: what is next?', we discuss strengths, challenges, gaps, in addition to future directions in PME research and open this up to mathematics education research as a whole. Before we discuss the main sections, we will explain how we proceeded methodologically to conduct the research review.

REVIEWING RESEARCH: HOW DID WE PROCEED?

Reviewing academic literature also follows a logic. In our case, the logic adopted was framed by our knowledge of existing socio-cultural research in the field and its expansion in certain directions. Our three leading questions allowed us to identify the lines of concern, connections between those lines and who represents them, as well as the resonances between the perspectives expressed in particular papers. All this work around the state-of-the-art followed the stages of *selecting* (literature), *organising* (connections) and *analysing* (novelties). These stages were planned to be inductively accomplished, with the elucidation of connections and novelties being highly iterative in nature. We assumed that for any area of study to encourage the emergence of new ideas and trends, connections and advances between what has come to be known in the most recent past are necessary.

The *selecting stage* consisted of choosing the set of papers for the literature review. In this first stage, we drew on the whole of Plenaries, Panels, Research Fora and Research Reports (RRs), and identified and counted the papers that explicitly declared a theoretical perspective identifiable as socio-cultural for 2006–2015. Plenaries, Panels and Research Fora were read almost in totality in order to decide whether they added significant new debate to socio-cultural research. In the case of Research Fora, where a number of traditions are usually represented by a collection of short papers, we searched for evidence of such debate in at least one of the papers. Some more work was needed for the study of RRs. Six sets of PME Proceedings from this period include in their first volume an index of the authors of RRs within a system of research domains. This index helped us to trace the collection of RRs with socio-cultural approaches that were presented that year. For the Proceedings without an index of this kind, a selection of candidates from among all RRs came after reading titles, abstracts, introductions and references. We thus addressed the issue of *how much*, that is to say the relative weight of the socio-cultural approaches with respect to the total number of papers year by year. Later in this chapter we will provide the more generally obtained quantitative data. It is important to note that, although many authors refer to socio-cultural theories in some form, we only considered work with explicit statements of justification for the relevance of these theories in the investigation reported. This option reflects our idea of the research that can genuinely be seen as socio-cultural in its orientation because social and cultural principles are declared in substantial ways.

From our reading of research papers, the observation that a number of articles were conceptually close to the socio-political emerged strongly. In a second stage of the review, the nuances in the interpretation of the socio-cultural and political were identified. This allowed us to follow the traces of authors and their work presented in the entire material, together with the connections with other authors. In this strategy of mapping the networks of relationships and authors, it was possible to identify the traces of major directions and lines of concern that delineate the socio-cultural-political axis. Far from a rigid view of a structure, directions and lines of concern were explored as an interconnected system of ways in which socio-cultural-political PME researchers study and make sense of mathematics education and mathematics education research. This *organising stage* served not only to examine related insights in the sample of papers; it also guided the analysis of salient topics and issues for the purposes of delineating new paths of present and future socio-cultural-political research. In this respect, the analysis of new paths and emerging topics and issues in current research reveals our dynamic interpretation of the directions and lines of concern.

A third stage was to see, through the lines of concern, which topics and issues were addressed, and which new insights with respect to former research were provided in the papers. This *analysing stage* was planned to detect some of the newly integrated ideas in the context of PME that could be taken to the next period of follow-up socio-cultural-political research. We privileged the detection of topics or themes instead of or complementary to the detection of methodologies and methods in the narrow sense of techniques. In relation to this issue it can easily be found that a wide range of empirical papers show a qualitative analysis of qualitative data, commonly based on the development of small-scale qualitative studies. From among these, many draw on specific methodological orientations with their own technical language such as grounded theory, discourse analysis, narrative analysis, ethnography, interpretivism or phenomenology. To detect major topics and issues, we looked for relationships in the socio-cultural papers selected from one set of Proceedings as a first step to guide the search in another set of Proceedings. All in all, we encountered a number of emerging topics and issues which indicate theoretical links among several papers and authors concerning the diversity of lines of concern identified. In this stage, therefore, the approach was centred on detecting topics and issues that somehow play a role in unifying and extending the socio-cultural-political lines of concern through the introduction of pioneering conceptualisations in the context of contemporary PME work.

On using this type of logic to review literature we were looking for alignments and recurrences in the theoretical perspectives and findings in the papers, thus permitting the depth, breadth and progress of socio-cultural-political PME research from 2006–2015 to emerge to the surface. The intention was to map the field and not to give a detailed account of each single paper. Therefore, our review cannot be exhaustive on referring to each paper, nor can it be complete on detecting all the newly integrated topics and issues in and across the socio-cultural-political lines of concern. However, we hope that, on the one hand, authors can see their work represented in our mapping of the field and, on the other, the lines of concern, topics and issues examined provide a sufficiently rich state-of-the-art.

EXPANDING VIEWS: WHAT DOES THE SOCIO-CULTURAL-POLITICAL AXIS MEAN?

Twenty years ago, Lerman (1996) pointed to a central distinction in the study of mathematics education. The conceptualisation of the relationship between the individual and the social is the core difference between Piagetian inspired studies of mathematics education and socio-cultural studies. Lerman defined the latter as research involving "frameworks which build on the notion that the individual's cognition originates in social interactions … and therefore the role of culture, motives, values, and social and discursive practices are central, not secondary" (p. 4). The focus on the classroom *context* and how it influences teaching and learning was an entry point for theories that went beyond the (social) constructivist assumptions on the effects of external factors – including other people – on individual cognition. In the second half of the 1990s, a series of conceptualisations from other fields of study (see Bartolini Bussi, 1998, for elaboration on this) had been incorporated into mathematics education in an attempt to give an account of the "individual in context". Vygotsky's cultural historical psychology was an important ground for further interpretation in the form of Cultural-Historical Activity Theory (CHAT) (Engeström & Middleton, 1996), together with the work of Lave (1988) on situated cognition, and the works of Lave and Wenger (1991) and Wenger (1998) on communities of practice. Less known but still in the same area were alternative discursive psychology approaches, in particular the work of Walkerdine (1988, 1998). Sociological and political theories of education (Bernstein, 1990; Bourdieu & Passeron, 1977) had not been so broadly adopted in PME, even though they had started to provide a frame for dealing with problems emerging from contexts of mathematics education (Appelbaum, 1995; Mellin-Olsen, 1987; Skovsmose, 1994).

In the decade between 1996 and 2005, there is varied research on mathematics thinking, teaching and learning that could be identified as socio-cultural and which was part of PME, as reported in Lerman (2006). In the period 2006–2015, attention to mathematics and school mathematics as social, cultural and political gained recognition as a principle for a large number of PME researchers (i.e. the researchers who participate and present their work at PME conferences). Gates (2006) situates the origins of such recognition in the preparation of PME29 in 2005, when the International Committee decided to broaden the domains of research through the inclusion of the category 'Equity, diversity and inclusion' for participants to submit their work. Moreover, during the General Assembly of that conference, a proposal was approved to remove from the PME constitution the preference to consult psychology as the fundamental field of scholarship for the PME community. Our analysis indicates that this opening up has enlarged PME research by adopting various integrated interpretations of the social, cultural and political in mathematics education. This trend has persisted over the last decade, articulating what we call the socio-cultural-political axis.

For further characterisation of the socio-cultural-political axis in this section, we first discuss the identifiable, constitutive directions and lines of concern within PME in relation to the expansion movement. The description of lines of concern serves as an initial survey of some of the work and authors that have contributed to socio-cultural research in forms that did not exist or at least were quite rare ten years ago. We then elaborate on the newer cultural-historical and socio-political trends in PME, addressing particularly some of the authors whose works have been crucial in grounding the socio-cultural-political axis. We finish this section by relating the newest orientations in PME to research in the broader international field of mathematics education (i.e. the research that has not been presented at PME conferences and reported in the PME proceedings, although some PME researchers may strongly draw their PME research from it).

The Micro-Macro Constitution of the Socio-Cultural-Political in PME

In the early nineties, the interest in understanding individual mathematical thinking in context was the beginning of how some of the approaches in PME research, which originate in the work of Vygotsky, would later result in the demise of the dichotomy between the individual and the social. More than 20 years later, a refined language to engage in such an endeavour has been achieved. A basic theoretical distinction has been constructed between saying that individual mathematical thinking is influenced by interaction with others, and saying that there is no thinking – mathematical or of any other kind – outside the relationship between the self and the other. The "social" is not simply a matter of the "influence" of "the other" on the "person" – as if these were entities with a recognizable separate existence. The inseparability of the individual – the I or the self – from the other – one and many, now and in a past that is constantly present – in the production of the material and symbolic world through practice is a grounding premise to think about humans, their life and activities.

The issue of the inseparability of the individual from the other is a basic assumption of Vygotsky's cultural historical psychology rooted in Marx's historical materialism. Opposing Western European rationalism, which places the defining element of humanity in thinking understood as inner, mental activity, historical materialism breaks with the idea of the individual as a monad and proposes a configuration of three elements – *people in activity*, *artefacts and products of* *activity*, and *systems of meaning* – as the inseparable unit to think about the social world. This view has a number of implications for the notion of "individual in context" as a cultural-historical phenomenon rather than socio-cognitive or social psychological. Radford (2008a, 2008b) presents a delicate elaboration of the difference between the paradigms involved – socio-cognitive/social psychological and cultural-historical, – together with a discussion of how the long-established, two-way relationship between the individual and the social has raised the issue of the inseparability of the two.

Ideas about people in activity, artefacts and products of activity, and systems of meaning are tantamount to the newer cultural-historical and socio-political trends in the field, and are displayed in the papers that, in PME, can be mapped in relation to the socio-cultural-political axis. These are ideas that, expressed in diverse forms and with different emphases, have been present in many of the theories that mathematics education researchers have drawn upon in the study of mathematics thinking, learning and teaching in context. For example, Lave (1988, pp. 178–179) refers to three levels in the analysis of human cognition in social practice: the level of the *lived experience*, where people in activity, and activity and settings, are the constitutive elements of thinking-in-doing in everyday life; the level of the *semiotic systems*, with the structures they entail in a *constitutive order* of meaning; and the level of the *dialectic relationship between the lived experience and the constitutive order* in the generation of sense, meaning and thinking.

Our main point here is that, in 10 years of research outside and inside mathematics education, and inside and outside PME, more nuanced and rich languages to study thinking and education in mathematics outside of rationalist and socio-cognitive paradigms are now available. The original issue of understanding the "activity of the individual mind in context" from a social standpoint has been progressively unpacked and given precision by means of two major directions and several interrelated lines of concern. The two directions concerning the socio-cultural-political, which will be called "micro" and "macro" throughout the chapter, are complementary in that they dialectically connect local and systemic forces in contexts of mathematics education and mathematics education research.

In the direction of the micro-details of knowledge and meaning-making in cultural configurations, two lines have become evident in PME:

- The line of the micro-genetic analysis of semiotics, to which groups of scholars contribute, together and overlapping but also with some nuanced distinctions in their approaches to classroom activity in mathematics teaching and learning. Here we can mention the work of groups such as Radford and collaborators (e.g. Radford, Miranda, & Guzmán, 2008); and Arzarello and collaborators (e.g. Arzarello & Paola, 2007).
- The line of the micro-analysis of classroom discourse, in which a variety of research methods and central concepts coexist. Here we find the later work of

Sfard and her progressive turns toward participation and commognition, which have been researched by various scholars (e.g. Heyd-Metzuyanim, 2013); the work of Wagner, Herbel-Eisenmann and collaborators (e.g. Herbel-Eisenmann, Wagner, & Cortes, 2008); and that of Morgan and collaborators (e.g. Morgan & Tang, 2012).

As developed in the context of PME since 2006, the lines concerning semiotics and classroom discourse share important similarities in the light of attention paid to some primary notions. In particular, the more traditional assumption that people communicate as individuals has been replaced in both cases with frameworks around the idea of people communicating in activity across contexts of various kinds and through a variety of artefacts that are historically and socially realised.

A second direction studies the macro-details of the connections between the different participants in mathematics education and how they relate to each other in institutional arrangements in classrooms, schools, and outside schools. In this direction, work that adopts theoretical tools to study power becomes more evident. The study by Wagner, Herbel-Eisenmann and Cortes (2008) that we referred to above, for instance, is also about power. In the "macro" direction, however, power is taken to mean a decisive feature of broader social and political structures, while in Wagner et al. (2008) the decisive feature to be researched is the classroom discourse from the perspective of micro-level actions.

Again, this second direction has been expressed through different not mutually exclusive lines of study and intellectual traditions in recent PME research:

- The line of identities and identity-construction along different combined dimensions such as language, age, socio-economic status, immigrant background, race, ethnicity, gender, etc. Here we find the work of Barwell (2013) on language and language users, and the work of Lerman (2012) on socio-economic status and working-class students.
- The line of communities of practice in contexts of research such as teacher education, professional development or out-of-school mathematics. Here we find the work of Jaworski and Goodchild (2006) on mathematics teachers' professional learning communities, and the work of Bose and Subramaniam (2011) on children knowledge-building communities.
- The line of ex/inclusion of particular groups of students from access to and full participation in school mathematics. Here we find the work on the creation of teaching and learning opportunities of Planas and Civil (2015) with bilingual immigrant children in urban contexts, and the work of Hunter and Anthony (2014) with Māori and Pāsifika students.
- The line of society and the politics of mathematics education and mathematics education research. Here we find the work of Walshaw and Anthony (2006) on the power of discourse and hegemonic discourses of power, and the work of Setati (2006) on the critical role of language ideologies in institutions of mathematics teaching and learning.

The identification of these two distinct – but related – directions in PME work allows us to organise the diversity of theoretical and empirical frameworks in use for the understanding of mathematics education as social, cultural and political. The logic of this organisation is represented in Figure 1. As researchers choose to focus on the micro-dimensions of the constitution of mathematical thinking and learning, cultural-historical approaches with an emphasis on semiotics or on classroom discourse are productive ways of researching. Complementarily, as researchers choose to direct their gaze towards the constitution of thinking, learning and education in relation to the broader systems of signification that articulate the practices of mathematics education in society, a socio-political trend would offer ways of linking mathematics education practices to broader macro-issues and dimensions. In that case the study of identities, communities of practice, processes of ex/inclusion, and the linkage between society, politics and mathematics education become productive. This logic of moves towards the micro-details, and the macro-tendencies provides a different way of thinking about the field of the socio-cultural-political as complementary analytical moves, rather than boundary crossing between discrete, not connected categories and problems.

We will now further elaborate on the micro-macro constitution of the socio-cultural-political axis by referring to some of the PME works that have made important contributions to the grounding of theoretical tools and analyses with emphasis on one of the levels – micro or macro, – but with explicit mention of the two of them. We will map the newly arrived PME cultural-historical and socio-political research. This body of research is critically confronting some of the taken-for-granted relationships between interaction and knowledge in ways that may be seen as one of the features of contemporary PME work.

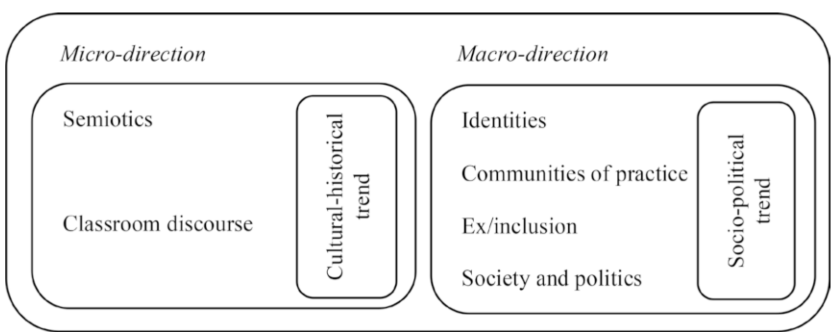

Figure 1. Representation of elements of the socio-cultural-political axis

Newer Cultural-Historical and Socio-Political Trends in PME

The micro-macro constitution of the socio-cultural-political axis has been possible through the number of works that, for the last decade of PME research, have included neighbouring disciplines such as sociology, history, philosophy, anthropology, political science and linguistics. The study and inclusion of related disciplines have allowed an expansion of what is meant by “the individual’s cognition being originated in social interactions” (Lerman, 1996, p. 4) in terms of the inseparability of the individual from the other. In line with the issue of inseparability, what is new in contemporary socio-cultural PME research is the dual emphasis on *how* mathematics learning occurs (i.e. the nature of interaction) and *what* mathematics learning is (i.e. the nature of knowledge and learning). Such an emphasis has led to changes in earlier socio-cultural interpretations of cognition as socially *occurring* and individually *being*. In the last decade, the interpretations of cognition in which the social is only considered in part of the “story” – learning is socially shaped but the products of learning shape the individual rather than the social – have been largely problematised. A move has taken place from viewing the social as only part of the “story” to conceptualising cognition as unified processes of cultural, social, historical and political induction into communities of thinking and practice. This newer conception of cognition as socially *occurring* and socially *being* appears in works of the micro and macro directions.

In order to explain more carefully the expansion of PME socio-cultural research and the micro-macro constitution of the socio-cultural-political axis, for this part of the chapter we have chosen influential work within two newly-established PME trends: the cultural-historical and the socio-political. Our analysis of the PME literature over the period 2006–2015 mostly relates the emergence of the cultural-historical trend to works on the micro level of semiotics and classroom discourse, and that of the socio-political trend to works on the macro level of identities, communities of practice, ex/inclusion processes, society and politics (see Figure 1). We indicate some of the authors who have been contributors to the advancement of either cultural-historical or socio-political ideas in recent PME research.

The rise and development of cultural-historical orientations in the context of PME can be illustrated through the research presented by Luis Radford and collaborators. Earlier in this chapter, we related Radford to the body of socio-cultural-political research in the direction of the study of knowledge and meaning-making at the micro level of specific cultural configurations, and particularly to the line of semiotic analyses of practices in the mathematical culture of the mathematics classroom. This is certainly an important part of the work by Radford inside and outside of PME. Nevertheless, his theoretical work on possibilities and boundaries of cultural-historical orientations in mathematics education research has also served to reinforce the complementary character of the two directions, micro and macro, in the constitution of the socio-cultural-political. The problem is not only to examine the configuration of local practices but also to consider broader cultural and historical forces – including the intellectual traditions in the field – intervening in the configuration, development and study of such practices.

In his plenary at PME38, Radford (2014) observed a cultural and historical concept of the individual, and reflected on the production of mathematical knowledge as a historically-based social process. In that plenary, Radford highlighted the decisive role given to culture, history and society in the understanding and formation of knowledge and thinking: knowledge is always articulated and enunciated in social, cultural, historical and political conditions, and these are the conditions that structure how it is materially accessible in culture. Some years before, in his commentary for a PME Research Forum, Radford (2010) reflected on the cultural and historical nature of theory and knowledge in mathematics education research:

> Theories in mathematics education reflect and refract implicit specific national-cultural "world views". They are unavoidably immersed in those symbolic systems of cultural significations... of the symbolic structures of society structures from where (implicitly or explicitly) our theories draw their views of what constitutes a good student, a good teacher, and, of course, a good researcher. (pp. 169–170)

When reviewing PME research with a focus on *what* is mathematical knowledge and learning from the perspective of its historical construction, we encountered the term *objectification* very frequently over the last decade in socio-cultural research on meaning-making in the mathematics classroom. This is how Radford calls the social process involved in becoming progressively aware of the cultural logic of mathematical entities and knowledge. Radford and collaborators have taken many initiatives to illustrate how the cultural theory of knowledge objectification, as a strong socio-cultural and historical framework to think about the ontology and epistemology of mathematics teaching and learning (Radford, 2008b), works and what it adds to the understanding of mathematics teaching, learning and thinking processes. For example, Radford, Bardini and Sabena (2006) show how the objectification of generalisation in mathematical tasks requires the coordination of eye, word and gesture in a rhythm. The latter provides ways for students to become gradually aware of the moves involved in generalising. Radford, Miranda and Guzmán (2008) examine how students recreate cultural and historical mathematical meanings around the objectification of Cartesian graphs in school mathematics. Radford (2011) reports another investigation in which young children interact with different manifestations of school knowledge constructed around early algebraic thinking by means of material artefacts and gestures.

A related but not identical research agenda is the socio-cultural path taken by some Italian groups interested in a semiotic analysis of mathematics teaching and learning embedded in the idea of classroom activity. With a variety of topics of concern, the semiotic analysis undertaken in mathematics classrooms shows how learning unfolds in the interaction of individuals and groups in a space of action, production and communication – the APC space (e.g. Arzarello, Bazzini, Ferrara, Robutti, Sabena, & Villa, 2006). The APC space is built up in the mathematics classroom as a dynamic system, where the different components – body, physical world and cultural environment – are integrated into a unity that reports on the embodied and cultural nature of the mathematical concepts. Further elaborating on

Vygotsky's activity theory, the issue of mediation in mathematics learning through specific social classroom practices and cultural frameworks of rationality is taken up in Arzarello and Paola (2007).

The study of the expansion movement and of the micro-macro constitution of the socio-cultural-political axis also points to the rise of socio-political orientations in PME. Here, the dynamics of power in mathematics education and mathematics education research is examined in relation to a variety of notions such as identity, positioning, disposition and agency, often borrowed from neighbouring disciplines and placed in relation to issues of social class, gender, age, race, ethnicity or language. In her introduction to the PME38 Panel (Phakeng, Halai, Valero, Wagner, & Walshaw, 2014), Phakeng (formerly Setati) traced the rise of the socio-political trend back to the intellectual tradition of critical mathematics education:

> In his paper entitled, 'Critical mathematics education for the future', Skovsmose (2004) argues that while mathematics education can empower, it can also suppress, and while it can mean inclusion, it can also mean exclusion and discrimination. Mathematics education, Skovsmose explains, does not contain any strong 'spine', because it can collapse into forms of dictatorship and support the most problematic features of any social development, or it can contribute to the creation of a critical citizenship and support democratic ideals. (p. 56)

The rise and development of socio-political orientations in the context of PME can be illustrated through the research presented, often in collaboration, by Lerman and Jorgensen (formerly Zevenbergen). Lerman and Jorgensen have contributed to the body of socio-cultural-political research in the direction of the study of connections between participants in mathematics education at the macro level of institutional arrangements in classrooms, school and out-of-school, and in respect of the lines of identities and ex/inclusion of groups of students. However, their investigations are especially valuable because they have contributed to widening, reinforcing and connecting the various lines of concern in the configuration of the macro direction of socio-cultural-political research.

In the introduction to the Research Forum entitled 'Sociological frameworks in mathematics education research' in PME33, Lerman (2009) indicated the importance of drawing on the work of authors such as Bernstein, Bourdieu or Foucault in order to "discuss how such [sociological] frameworks can shape our research questions and methodologies and form a basis for change in mathematics education" (p. 217). These frameworks are expected to help to examine *what* is mathematical knowledge and learning from the perspective of its political construction in ways that can complement the role and use of the philosophical frameworks previously used and revisited by Skovsmose (1994). More generally, in a number of PME collaborations, Jorgensen and Lerman (e.g. 2006, 2007) drew on sociology to understand the (educational) role of mathematics and knowledge in society, and the problematic nature of (school) mathematical knowledge. In these works, the investigation of a pure historical subject, whose ideal cannot be empirically found, is replaced with the investigation of concrete dynamic subjects engaged in attempts to reconstruct meanings and relations of power in culture, in the spaces of mathematics classrooms.

Even if classrooms have remained the privileged site for PME socio-cultural-political research over the last decade – although with efforts to include non-prototypical classrooms, – some of the socio-political research traces the political in relationships among people and institutions beyond the classroom. Walls (2006), for instance, addresses the impact of standardised tests on students' learning and their identities as mathematics learners. In a primary school community, she investigated the accounts of children, parents, teachers and managers on their perception of Year 5 Aspects of Numeracy Test in Queensland, Australia. Using a Foucauldian framework, Walls traced how testing was not just a way of detecting what children know mathematically, but rather a way of comparing and differentiating them, creating hierarchies and, thus, in/excluding some. The test was seen "as a major event in school and home life in its perceived authority to tell the truth, that is, to objectively measure and rank each child. In this perception, school management and teacher behaviour were modified, pupil identity reworked, and relationships within families adjusted" (Walls, 2006, p. 359). This type of research reveals that the different technologies of mathematics education – including testing – affect children and their identities, but also many of the participants in the network of practices of mathematics education.

Another feature of the newer PME socio-political trend is the view of systems of meanings and activity as changeable by the people involved in them. When focusing on the social and political conditions that constitute school mathematics, and on how such conditions are partially made accessible to marginalised groups in the dominant culture, a number of studies have emphasized the transformative dimension of power by conceptualising ideas of change, resistance and agency. In Jorgensen (2015), mathematics teaching and learning are seen in terms of progressive access to dominant worldviews and induction into mainstream cultural systems in ways that leave room for alternative realities. Jorgensen explores two rural schools and shows teachers who move to the very limits of the mainstream culture to modify interpretations of mathematics and professional learning. While the stories of successes and failures provided by the teachers' activity in these schools remind us of the lights and shadows in Breen (2007), the appreciation of power as "positive" refreshes the principles of the critical theory pioneer project by Skovsmose (1994) and outlines the consideration of Foucault (1980).

So far we have characterised the socio-cultural-political axis in the context of PME by (1) pointing to two main identifiable directions – with respect to the emphasis on either micro or macro levels of analysis; (2) describing a number of researched lines of concern in relation to these directions; and (3) illustrating two influential theoretical orientations which cross over lines of concern and, by doing so, contribute to the conceptualisation of the socio-cultural in newer forms, namely the cultural-historical and the socio-political (see Figure 1). The ways in which we have characterised the socio-cultural-political axis offer an idea of its richness and complexity. This is a complexity that goes beyond describing different dimensions, combinations between them and zones of intersection, as the geometrical image of an axis might suggest. We are documenting constructions in development, which depend on the dynamics of theories and groups of researchers continually contributing new analyses and interpretations to the field.

Some Evidence of the Socio-Cultural-Political Axis Outside PME

We have discussed the emergence and vitality of the newer cultural-historical and socio-political trends in PME research, as well as their transversal role in the configuration of the directions and lines of concern of the socio-cultural-political axis. However, it is fair to say that socio-cultural-political research is not a PME creation, at least not solely. The traditional perspectives for which mathematics education research is primarily linked to the search for didactical responses to technical problems, faced by students in their learning and by teachers in their teaching, currently coexist close to the newest perspectives that address problems in the field which are of a political or ideological nature.

When looking at the whole field of mathematics education research, cultural-historical and socio-political trends are identifiable along with renewed interpretations of the relationship between the individual and the social in ways that demand a variety of theories which originated in other fields. Several of the international initiatives on how to move the field forward through the adoption and recontextualisation of theories from other fields have been driven by researchers who are active in the PME community. In their report of the ICME Survey Team on ‘The notions and roles of theory in mathematics education research’, Assude, Boero, Herbst, Lerman and Radford (2008) indicate the potential and some of the benefits of using theories of an “external type” rooted in sociology or anthropology. Another initiative in pushing the integration of disciplines for the development of theories and perspectives in mathematics education research can be found in the section ‘Social, political and cultural dimensions in mathematics education’ of the *Third International Handbook of Mathematics Education*, in the chapter by Jablonka, Walshaw and Wagner (2013). These authors provide a critical overview of how diverse social, political and cultural theories are allowing us to widen our contemporary perspectives of mathematics education and mathematics education research. This chapter actually addresses the extent to which notions from literary theory, discourse analysis, social linguistics, sociology, positioning theory and postmodern approaches were also present in the PME proceedings from 2007 to 2010.

Additional evidence of the constitution of the socio-cultural-political axis outside of PME comes from initiatives on how to move the field forward through theorizing work. In this respect, the theorizing work in a collaboration between Roth and Radford (e.g. 2011) has had an enormous impact in that it has contributed to frame CHAT perspectives (Engeström & Middleton, 1996) in mathematics education research.

Through the conceptualisation of mathematics thinking and learning as dynamic systems of meanings and activity, these authors have challenged the psychologically-based distinctions between the individual and the social. Their work is simultaneous with the theoretical revision of the affordances and hindrances of social and cultural psychology in mathematics teacher education research. In this respect, the third and the fourth volumes of *The International Handbook of Mathematics Teacher Education*, edited respectively by Krainer and Wood (2008) and Jaworski and Wood (2008), illustrate the international opening up of a significant part of mathematics teacher education researchers to cultural-historical and socio-political perspectives, and more generally the theoretical interrogation of dominant ideologies at work in teacher education research designs. In her introductory chapter of the fourth volume, Jaworski (2008) refers to this opening up to theorizing within newer terrains from outside psychology:

> [T]here has been a shift. One obvious difference is that constructivism has moved from a largely cognitive, psychological focus to take into account social contextual and institutional factors... In parallel, socio-cultural theories, rooted in the work of Vygotsky and followers have become better known and understood in mathematics education, with a challenge, implicit or explicit, to constructivism… and social, cultural, political and policy issues have become more evident in the mathematics education literature... [P]erspectives of teacher educators have moved into more social frames... with recognition also of the wider influences of system and society. (p. 4)

Particularly in relation to socio-political perspectives, Valero and Zevenbergen (2004a) challenged the socio-cultural-psychological approaches by offering sociologically-oriented alternative ways to theorize mathematics education and mathematics education research. At the time of writing, the collaborative project of that book – which was linked to their participation in PME activities – remained unique in many senses. It meant bringing together authors from different parts of the world, socio-cultural traditions and emerging critical political perspectives, sharing the challenge of rethinking mathematics education research in relation to the political, economic and social conditions of schooling and other institutions with a role in the regulation of practices on the micro level. The book chapters provided a number of early responses to why mathematics educators and mathematics education researchers should care about power, equity, social justice and critical pedagogy. In their introduction, Valero and Zevenbergen (2004b) anticipated the need for the revision of the socio-cultural-psychological dominant orientations in socio-cultural research toward the inclusion of the political:

> [I]t is possible to identify another trend strongly rooted in sociology, critical theory and the politics of education. This trend stands on the assumption that mathematics education is, in essence, a *social* and *political practice*. This practice is social because it is historically constituted in complex systems of

> action and meaning... This practice is political because the exercise of power, both in it and through it, is one of its paramount features. (p. 2)

Gutiérrez (2013) outlined the enormous potential for the field, the socio-political turn having been initiated in the ways of thinking about mathematics teaching, learning and education, but also in the critical ways of thinking about research and research methodologies. As in Valero and Zevenbergen (2004b), Gutiérrez distinguishes between socio-cultural approaches to mathematics education research, with an emphasis on understanding psychological processes from a social base (in which the social is only a part in the "story"), and socio-political approaches, with an emphasis on understanding social and political processes on their own. She argues that, although socio-cultural-political issues still remain under-researched in the field, the dominance of socio-cultural-psychological orientations in socio-cultural research is being reduced. There are now more authors who are bringing power and mathematics (education) together, and placing issues of equity to the fore in their investigations. These investigations pose crucial challenges to the whole field, which is however still operating under poorly articulated approaches and agendas surrounding issues of equity, and threatened by tacit deficit-based beliefs that not *all* students and groups can learn.

In a similar way to the refinement of theoretical language to study the cultural constitution of mathematics education practices, the theoretical language to grasp the political in mathematics education has also become more nuanced. Nowadays, the adoption and recontextualisation of a variety of tools to study power have been brought to the field of mathematics education. Besides the interest in thinking about pedagogies that help improve the achievement of students from socially minoritised groups who have not succeeded in mathematics, the field is producing more solid analyses of why and how mathematics education practices, in a broad network inside and outside the school, operate inclusion/exclusion and differentiation of groups and learners. Examples of this type of work can be found in recent special issues of international journals, such as *Educational Studies in Mathematics*, on social theory in mathematics education (Morgan, 2014a), or *ZDM* on socio-economic influences on mathematical achievement: what is visible and what is neglected (Valero & Meaney, 2014).

MAPPING THE SOCIO-CULTURAL-POLITICAL AXIS: WHAT IS IT LIKE?

Overall, the absolute and relative frequencies of the socio-cultural RRs indicate a stable high representation of this domain for the last decade of PME. While in Lerman (2006), the result was that "the number of Research Reports classified as socio-cultural has grown substantially from 1990 onwards" (p. 353), in 2016 it can be said that the domain has become firm and consolidated, and it represents between a quarter and a third of the total RRs, by year and across years. To gain a more complete picture, it can be added that we also identified about a total of thirty

Plenaries, Panels and Research Fora as contributing to the socio-cultural-political axis in any of the identified directions, lines of concern and/or newer trends (see Figure 1) for the period 2006–2015.

The reality created by the numbers above points to the *extent*, in intensity and volume, of socio-cultural-political research. But these numbers are poor data in terms of understanding *how* the socio-cultural-political axis has been framed in PME over 10 years. In the previous section, we presented the existence of two major related directions, a diversity of lines of concern and two newer trends with a role in the rise and development of the socio-cultural-political axis inside and outside PME. We argued that these directions, lines of concern and newer trends cross over different topics of study in the field, groups of researchers and intellectual traditions around the world. Below, we map some of the topics newly addressed within the socio-cultural-political axis. These topics cross over more than one line of concern (i.e. semiotics, classroom discourse, identities, communities of practice, ex/inclusion and society and politics), and provide a window to PME contemporary socio-cultural-political research.

Knowledge Creation and Knowledge Use

The question of what mathematics is in/for teaching/learning is addressed through a series of topics of knowledge creation, knowledge use, frameworks and field development, which are present in practically all lines of concern through a number of papers. Although all papers are theory-building in some form, we refer here to those that search for evidence and arguments mostly in theory.

An example of this type, in the micro-direction of meaning mediation in classroom settings, is Hershkowitz, Tabach, Rasmussen and Dreyfus (2014). These authors expand the idea of knowledge *agent* to knowledge *agency* by considering an empirical bottom-up approach. The study combines two approaches – Abstraction in Content and the Documenting Collective Activity – to place knowledge and its mechanisms at the core of research on classroom discourse and the teaching and learning processes involved in it. Some more papers of this type can be found in the Research Forum in PME37 by Tabach, Nachlieli, Heyd-Metzuyanim, Morgan, Tang and Sfard (2013) on the development of "strong" discursive research. Different issues around the theory of commognition (Sfard, 2008) are addressed to make the argument that mathematics is a discursive activity and that mathematical objects result from the ways of communicating about them. In this framework, mathematics knowledge is conceptualised as the development of mathematical ways of using discourse, and mathematics learning as participation in a certain discourse. In particular, this means that mathematics knowledge is a kind of discourse, and consequently discourse is a topic of research in its own right, not a window to something else. This standpoint raises questions concerning how participation in one discourse is subject to participation in other discourses. Drawing on a socio-cultural-political agenda, we see the relative status of different discourses and the access to them by different participants situated as strong lines of reasoning for commognitivist research at present and in the years to come.

Another example of work about knowledge creation and knowledge use, with emphasis on the movement toward the socio-political in classroom discourse research, is the Plenary by Morgan (2009) in PME33. The question of *what* is mathematical knowledge and learning is here linked to the questions of *what* is language and *what* is discourse. Morgan addresses the need to connect the various perspectives in mathematics education research for the construction of powerful theoretical tools that can help gain insight into ways of integrating social structures and individual processes. She refers to the analytical challenges posed by the integrated study of the social and the individual as follows:

> My concern with social inequalities precludes adoption of a perspective that denies or ignores the influences of social relationships and structures on individual experience and achievement. My personal search for theory has thus been shaped by a need to understand how individual and social may be connected... Theories of learning and activity based in the Vygotskian tradition offer powerful ways of understanding such connections... However, because many of my questions seek to address the uneven distribution of knowledge and educational success, I intend to focus here on the contributions of sociolinguistic, discursive and sociological theory to my way of understanding. (p. 51)

In her research, Morgan draws on a variety of theories such as linguistics and social semiotics, critical discourse analysis and social theory. The articulation of these three “toolboxes” allows her to unfold an analytical strategy in which any classroom data are understood through knowledge of the immediate context of the practice and knowledge of the broader socio-cultural context shared by participants in this practice. Within this framework, classroom interaction is investigated along with more general social practices and larger structural arrangements of education that frame the particular contexts of mathematics education. Morgan’s work constitutes a perspective that binds the micro direction of classroom discourse and semiotics sensitive analysis to the macro political structuring, that is, “how mathematics education functions in society for individuals and for various social groups” (Morgan, 2014b, p. 130). As in the case of commognition, where the combination of communication and cognition in the very same name of the theory grasps the unity of the individual and the social, Morgan’s social semiotics assumes the socio-cultural-political principle of inseparability.

Some more papers centred on theoretical issues of knowledge creation and knowledge use, now located in the direction of the politics of mathematics education and mathematics education research, can be found in the Panel by Phakeng et al. (2014) in PME38. The hypothetical case of building a new school project in South Africa intending to provide mathematics education to improve the living conditions of children in a poor marginal area is used as a basis for a discussion of how mathematics education research, with its many discourses, relates to concrete social contexts and makes theory work for action. One of the foci of the papers in that panel is the ways in which discourses about what counts as an adequate organisation of mathematics education in schools are constituted by the social, political, material and cultural conditions of the schools, communities and countries. A second focus is the approach to theory and theorizing as the preceding necessary stage for performance and political action toward more democratic dynamics in the case of the imagined South African school. Similarly to Jorgensen (2015), theory and theorizing are viewed as transformative tools for the development of alternative more equitable worlds.

Community Work and Participant Development

The conceptualisation of theory as action, present in many papers regarding knowledge creation and knowledge use, is linked to another prominent topic in socio-cultural-political PME research: community work and participant development. As further interpretations of Lave and Wenger (1991) and Wenger (1998), there are a number of papers that move their focus towards communities of practice, identity, ex/inclusion and the politics of mathematics education. Some of these papers engage with community work, community development and action for social change, both in school and out-of-school contexts.

Mathematics education research has expanded to include a variety of cultural, historical and political considerations that have led some researchers to become engaged in intense community work with groups of students, teachers and families. This topic of research has been addressed by Civil and Planas with data from very different political contexts but similar methods and findings. These authors have presented a number of RRs of their work together and with their own teams in Arizona and Catalonia about the mathematics education of minority groups of students in the relationship between schools, families and other groups of students (e.g. Planas & Civil, 2008). In the RR by Phakeng, Bose and Planas (2015), a case is made concerning research on the relationship between educational policies and ideologies of mathematical achievement of language minority groups in contexts of poverty, with direct participation and political action by the team of researchers. An example of this type is the work with families and after-school programmes conducted by Civil (e.g. 2008, 2012). With a focus on the role that language plays in the mathematics classroom placement of some of the children, in her Plenary in PME36 Civil (2012) reports issues of parental engagement in mathematics education for Mexican-American working-class communities of the US. She reflects on how to establish bridges between home and school for mathematics learning and teaching with attention to the diversity of social practices, institutional discourses and out-of-school identities that students meet and struggle with:

> Largely due to the restrictive language policy affecting the schools where my work was located, I became interested in the interplay between language and mathematics, particularly for students whose home language is different from the language of schooling... The tension between in-school and out-of-school mathematics often goes hand in hand with what forms of mathematics are more valued... I see three elements at play as I reflect on opportunity to learn in the context of non-dominant communities: the nature of the mathematics problem; the language(s) involved; and the valorization of knowledge. (p. 45)

An increasing number of PME papers over the last decade have addressed issues of community work and participant development, while bringing out the tensions and possibilities of "bridging" communities and discourses during the research process. In Cooper, Baturo, Duus and Moore (2008), a researcher-teacher collaboration for the teaching of mathematics in vocational education for indigenous blocklayers in Australia is analysed. Thinking about the relationship between three main actors – researchers, an experienced blocklaying teacher, and indigenous, blocklaying students – in the context of vocational education, the concept of communities of practice (Wenger, 1998) is fruitful. In this research, the notion of community does not only include the series of practices of learning in school, but also extends beyond the school as practices of blocklaying happen outside vocational education and are important for the Torres Strait Communities to which students belong. The analysis of mathematics learning in such a context demonstrates that learning extends to community service, and in this way in-school and out-school practices are connected in a sense of community. In this broad sense of community, also including the participation of the researchers, new initiatives that would allow students to make sense of mathematics in blocklaying could emerge.

In Planas, Iranzo and Setati (2009), the analysis of classroom events with bilingual students in Catalonia is part of a community project with mathematics teachers in schools with a high percentage of working-class students learning the language of instruction. In the project, classroom events are examined and selected for inclusion in mathematics teacher education programmes to be conducted by the researchers under the principle that the use of the students' languages has positive effects on the increase in mathematical participation and learning. Developmental work with teachers is also present in the shared design, implementation and evaluation of the pedagogic practices and tasks in a sequence of lessons for the exploration of mathematical participation, in line with issues of authority and power, and the ways in which these are encoded in classroom discourse. Planas and Civil (2015) report the learning of some of the teachers who were engaged in that project for several years, and who gained awareness of the extent to which issues of authority and power are pervasive in their lessons. On the basis of these teachers' accounts about the criticality of the language practices in their classrooms, developmental work is conceived as a site where practices and identities can be modified in ways that allow us to imagine new forms of mathematical participation for all students, regardless of their dominant languages. This is therefore another example of study where theorizing is expected to promote change and identity work by students, teachers, and also researchers.

Teachers, Student Teachers and Teaching

With respect to communities and developmental work, a large number of papers address teachers and student teachers in studies that have been conducted in different parts of the world. Johnsen Høines and Lode (2006) examine the initial education of mathematics teachers in Norway and the use of post-teaching, collaborative, subject-based discussions as part of conversations in teaching practicum as a powerful learning setting. Student teachers and their tutors were invited to collaborate with the researchers. The intention of the collaboration was to explore the qualities of the conversations after a period of students' practicum to learn about the different aspects of the practice of a mathematics teacher. Inspired by Alrø and Skovsmose's (2002) proposal of learning as dialogue, they found that the institutionally dominant *evaluative discourse* in teacher education can be challenged by an ongoing *investigative dialogue*. The former is the type of discourse to which student teachers are often subjected, due to the fact that many conversations on their activity when they meet students during practicum are of an evaluative nature. The collaboration in the setting introduced by the researchers and among the researchers, tutors and student teachers, opens up for conversations that explore possibilities for what the student teachers could have done in the practicum. The emergence of a new possible conversation challenges the power of institutionally framed traditions, and offers a new rich collective learning opportunity not only for student teachers, but for tutors and researchers alike.

Adler and Ronda (2014) also bring to the fore the complexities of stimulating dialogue and participation in either a mathematics lesson or a teacher education context in South Africa. In their report, they explain a framework for describing teachers' mathematics discourse in instruction to be used later in teachers' professional development. This framework provides for a responsive and responsible description of the teacher as a professional who is in the process of challenging pedagogies and increasing her or his professional knowledge by paying attention to the opportunities for dialogue and interaction with students in the classroom, and to how these opportunities can be provoked and supported by the use of particular types of mathematics discourse in instruction. Under the influence of the literature on teaching dilemmas (Adler, 2002), Adler and Ronda elaborate on teacher development, classroom discourse and learner participation as the core elements of their teaching and teacher education framework.

Another example of a paper, this time on the knowledge and preparation of the mathematics teacher educators, is Jaworski and Goodchild (2006). These authors present findings from a developmental research project, again in Norway, that seeks to create knowledge and improve practice in mathematics learning and teaching by developing inquiry communities between teachers in schools and mathematics teacher educators in a university setting. The powerful idea of inquiry communities in the collaborative work between teachers and teacher educators has the broader potential to inform about how learning opportunities can be created in the mathematics classroom. In her PME36 Plenary, Goos (2012) similarly reflects on the idea of community of inquiry as a resourceful way of understanding the creation of learning opportunities in a variety of settings in mathematics education. In particular, Goos traces out certain questions for possible future research trajectories in mathematics teacher education that consider connections between different communities of inquiry and their cultures:

> Calls for improvements to mathematics education are implicitly based on the assumption that well prepared mathematics teacher educators are available who can foster change in teachers' practices... The ethical, social, political and intellectual challenges inherent in bringing about this type of change are well known. However, much less is known about the professional preparation of the mathematics educators who undertake these tasks, or about how they continue to learn throughout their careers... Creating opportunities to learn across interdisciplinary boundaries may lead to new understanding of how to integrate the mathematical and pedagogical expertise of community members to enrich mathematics education. (p. 80)

Digital Technologies and Pedagogies

The ways in which teachers and students use technologies (commonly called ICT) as tools for mathematics teaching and learning have been examined from a variety of perspectives and theories in the last two decades of PME research. There is, however, an emerging topic in the examined PME papers over the last decade regarding the beginning of socio-cultural-political research concerning the study of the role, use and effect of digital technologies in mathematics teaching and learning within schools across a range of sites and socio-cultural backgrounds. Some of these papers provide micro-level findings of the complex communication and meaning-mediation processes involved in the production and interpretation of signs (e.g. gestures, drawings, natural talk, mathematical register) when working with technology. While ICT tools are interpreted as crystallizers of historical forms of thinking available in contemporary societies, the analyses are focused rather on how these tools mediate students' mathematical thinking and learning, as well as particular pedagogies of mathematics teaching. Dynamic geometry packages are, for instance, analysed as elements of culturally produced forms of thinking, doing and teaching mathematics in educational settings.

An example of this type of work on the micro level is the investigation by Geiger, Dole and Goos (2011) in Australia on the integration of digital technologies, such as electronic calculators, into classroom practice for numeracy teaching. This research adopts a critical framework for the understanding of numeracy in the connection between mathematical knowledge, dispositions and cultural tools, for the purposes of their use and relevance in dimensions of life such as the personal/social, work and citizenship. ICT as part of numeracy allows the well-documented gap to be bridged between, on the one hand, school knowledge and learning and, on the other, out-of-school knowledge and action. In this respect, ICTs are studied and used as mediation tools not only for learning but also for social action in relation to differential access and outcomes. The influence of a Bourdieuian approach is clear in how Geiger and colleagues relate differential access and outcomes to the various structuring practices that serve to recognise and validate particular dispositions and skills within schools and classrooms.

Also under a socio-cultural approach but now completed with a Bernsteinian analysis for an investigation on a macro level, Lerman and Zevenbergen (2006, 2007) examine how the digital divide affects students, families, educational institutions and classroom pedagogies. In their 2007 RR, these authors present a study of the ways in which teachers use interactive whiteboards in their classrooms in the curricular context of the Australian New Basics. Despite the potential of using the tool to enhance student learning, the analysis of a number of lessons showed a restricted approach in its use, for quick lesson introduction preceding whole class teaching. Interviews with teachers indicated that the approach observed was based on assumptions about students, mathematics learning and technologies. These teachers failed to recognise serious questions in terms of equity concerning the experiences and access to computers and ICT programs in the home. In the 2006 RR, the Bernsteinian framework for the same three-year research project is introduced to reflect on the potential role of technologies to support numeracy learning for all students with a focus on disadvantaged learners in particular. The notions of visible and invisible pedagogies, together with those of recognition and realisation rules, are considered in relation to "the digital divide" between children from middle-class and working-class homes. It is concluded that newer forms of pedagogy based on ICT innovation and the related approved pedagogic interactions need to be made visible in the schooling contexts of mathematics education:

> Research shows that working within a progressive paradigm, that is, where the pedagogy is invisible, but mitigating the weak framing through strengthening some of the features of the pedagogy can make a substantial difference to the success of disadvantaged students... We conjecture that, without explicit awareness by teachers of the implications of different forms of pedagogy on different social groups the aims of the New Basics in terms of more equitable outcomes are not likely to be met. (p. 55)

Out-of-School and Workplace Mathematics

From our presentation of some clusters of topics in contemporary socio-cultural-political PME research, it would certainly be inaccurate to infer that these clusters were not present to some extent in earlier times. Even though we may deal with areas of study that have persisted since earlier time spans, our purpose in this chapter is to show what is new from the perspective of what is being added to the micro and macro directions identified in PME socio-cultural-political research. In this respect, a further topic in different papers over the last decade is the investigation of mathematics out of school, and particularly of mathematics cultures other than school mathematics in the workplace. When looking at what persists, we find a first type of studies that primarily give continuity to the socio-cultural studies of children's out-of-school mathematics and adults' mathematics in the workplace under situated cognition approaches that were already common in PME in the nineties. When looking at what is new, we find a second type of studies that expand long-established approaches by including cultural, social and political considerations, along with issues about valorisation/status of knowledge in the sense indicated by Civil (2012). We have only identified a few studies of the latter type in our literature review, but we see it as the beginning of an opening of this topic to the socio-cultural-political axis.

In two related papers, Bose and Subramaniam (2011) and Subramaniam (2012) report a study on children's everyday mathematical knowledge associated with participation in work activities. In India, it is common for children in low socio-economic positions to undertake different kinds of work. Conversations with school children age 10–12 living in a slum in Mumbai showed how arithmetic strategies based on the values of the currency that they operate with allow children to complete certain complex calculations. Their knowledge of measurement units used in packages and products that they manage are also present in their calculation strategies. The research confirms what has been found in previous studies of the transition between school and out-of-school mathematical practices: while people show a quite sophisticated contextualised capacity for dealing with qualities and measurements, these do not necessarily transfer into the realm of formalised school performance. The issue remains of how teachers and educators can bridge this gap to open learning opportunities up to these groups of students. Together with the presentation of a variety of arithmetic strategies, an additional point is made: different strategies may be valued differently in mathematics classrooms depending on whose knowledge is being represented by them and whose participation may be favoured. In this way, the topic of out-of-school mathematics versus in-school mathematics in previous decades of PME work moves in a direction that studies power and issues of valorisation of knowledge more explicitly. As posed by Subramaniam (2012), the focus moves toward critical issues surrounding the relationship of school learning to knowledge accessed outside the school:

> Optimism about knowledge acquired by children outside school, especially mathematical knowledge, being a potential springboard for learning school mathematics is evident even in the early writings on ‘out-of-school’ mathematical knowledge... However, despite many studies exploring the contours of such knowledge and its settings, its integration with the school mathematics curriculum remains limited. (p. 107)

Regarding the study of adult workplace mathematics, in the last decade some papers have addressed mathematical activity in the workplace from a micro perspective where mathematics is embedded in the work context and is mediated through tools. While, in some of the studies examined on children’s out-of-school mathematics, the socio-political trend is visible, in the majority of studies on workplace mathematics the cultural-historical trend through a variety of interpretations of CHAT perspectives is common. CHAT perspectives are useful frameworks to analyse mathematics learning to become a professional within specific institutional settings that call for new forms of practice, knowledge and resources in the development of professional agency. Triantafillou and Potari (2006), for instance, report an ethnographic study in Greece with groups of technicians, some of them with vocational qualifications and others with an academic background. The detection of mathematical strategies in the activities of all the groups (e.g. locating a fault in an underground wire-pair; installing and programming a telecommunication network; working in an Earth satellite station) is discussed in terms of what these results tell mathematics education research about differences in the valorisation of knowledge at school and in the workplace.

Social Views, Discourses and Values

Issues concerning valorisation of knowledge and knowledge users are strongly connected with the type of study that addresses mathematics in society and the framing of social views, discourses and values about mathematics and mathematics education. All these issues are highly related to the research domain on affect. Within the socio-cultural-political axis, however, this cluster of topics appears in papers mostly informed by discursive and sociological perspectives. Together with the papers that critically unpack official pedagogic discourses (e.g. Lerman & Zevenbergen, 2006, 2007), we find a few papers that address the role and use of “unofficial” media discourses in the wider cultural field. An example is the paper by Evans, Tsatsaroni and Czarnecka (2009). Here, the increasing use of mathematical images in distinct advertisements of nine English newspapers and the reproduction of certain public images of mathematics in the suggested messages are examined. These authors argue that such images in the media intertwine with pedagogical discourses of mathematics since both, as interconnected cultural productions, regulate people’s construction of identity and subjectivity. Their results raise questions as to how different mathematical images are used in the media addressing audiences with different social classes:

> We note the emergence of a trend – supported by our evidence – whereby mathematics equations or formulae are recruited as global communication technologies of subjectivity, shaping desire especially for those strata of the middle classes that are the most promising clients in the global consumers' market. This emerging strategy might undercut the use of maths as a critical discourse for citizens. (pp. 23–24)

This type of investigations opens up clearly for the discussion of mathematics and broader (un)official political and economic discussion in society. It also raises questions concerning the extent to which mathematics education can and should currently provide a space for critique in society, particularly regarding how certain cultural productions (e.g. advertisements) work to cancel out initiatives designed to improve the level of mathematical knowledge in the general population.

In relation to social views and discourses entering the field of mathematics education, there is another group of papers focused on how social views, power relationships and discourses enter the culture of the mathematics classroom in the form of values and actions of valuing. Seah (2013) refers to the category of mathematical values as those linked to the convictions that have been emphasized in the tradition of Western mathematics. Rationalism, control and progress are some of the values emerging from the development of a large-scale study with teachers from a variety of cultural settings and backgrounds, who were asked to respond to what they find important in mathematics education. This study brought together research teams from eleven regions across the world, such as China, Hong Kong, Japan, Singapore and Sweden. Rather than interpreting the values identified as individual qualities of the teachers, Seah proposes an analysis based on values as the internalisation of dominant cultural ways of viewing the world (of mathematics education), with an effect on the enactment of specific dispositions to teaching and learning (mathematics). Similarly to what is claimed by Evans et al. (2009), Seah alerts us about the risks of accepting without critique the ways in which mathematics and practices and participants in mathematics education are valued in contemporary societies.

MOVING THE FIELD FORWARD: WHAT IS NEXT?

In a review chapter of this kind, we cannot consider that the review has been finished just because we have come to the end. The review of literature could have gone on and on, allowing more evidence of important discussions in contemporary PME socio-cultural-political research to be added. We are aware that important clusters of topics such as multilingual mathematics teaching and learning have not been directly addressed. At this point, however, we have already made our main arguments: PME socio-cultural-political research over the last decade has matured to distinguish a variety of related approaches to conceive of mathematics education and mathematics education research as social, cultural and political. While a majority of prior PME socio-cultural research supported the idea that cognition is a socially-originated individual process, and therefore focused on socio-cultural-psychological orientations, the domain has taken newer socio-cultural-political directions in support of the inseparability of the individual and the social. The study of works in these newer directions points to the constitution of a socio-cultural-political axis in the field inside and outside PME.

As an area of study develops, examining what kind of research and how it has been recently conducted is fundamental in order to be able to think about future work, and also to point to some of the gaps and priorities to be further researched. Our examination of papers has allowed us to identify some lines of concern, topics and issues in socio-cultural-political PME research for which there is an increasing amount of evidence, either theoretical or empirical. One of the findings from the analysis of contemporary socio-cultural-political PME work is that most of the studies reported are empirically oriented, and among these a majority are classroom-based. These studies privilege analyses of data on students and teachers in their classroom environments in order to explore their processes of interaction and engagement with school mathematics. Here, the meanings of classroom interaction and discourse have gone through multiple rounds of refinement and interpretation in the last decade of PME for the development of knowledge about mathematical identities and many other related topics. What we want to outline is the fact that a majority of the classroom-based studies reported were conducted at the school level up to the students' age of 16 years, and practically none at other levels or sites of mathematics education practice, such as pre-school education, higher education or adult education. It could be argued that research at these other levels and sites is the focus of other regular meetings of the field – such as the study group on Adults Learning Mathematics (ALM), or the Congress of Ethnomathematics. In any case, the lack of studies at these levels and sites constitutes a current gap in PME socio-cultural-political research that needs to be filled. The institutional circumstances intervening on the different levels of mathematics education may lead to differences in the type of processes involved in classroom interaction, discourse, institutional framings, processes of in/exclusion and also the forming of mathematics in these contexts.

Excluding the papers which draw on data from research in classroom contexts, we have also seen that papers which report studies in out-of-school and/or vocational contexts are rare. This was not the case in the nineties, when several PME papers were regularly presented on this topic (see, e.g., the plenary by Schliemann, 1995, at PME19). Together with the exploration of the causes involved in the progressive misrepresentation of this topic, a planning needs to be undertaken to fill this research gap. Actions towards the construction of a more extended scientific community with its researchers in connection with other scientific communities and researchers may be essential. The development of PME socio-cultural-political work depends greatly on the expansion and enrichment of research conducted in out-of-school contexts, for which strategic collaboration with other mathematics education researchers from communities entirely focused on this type of work would be very beneficial. There is a significant amount of out-of-school mathematics education research outside the context of PME, making important, foundational contributions to the field. In particular, international groups such as Mathematics Education and Society (MES) conduct regular conferences that have become a forum for research on the social, cultural, ethical and political dimensions of mathematics education. Out-of-school and non-classroom-based research on situations of poverty and inequity has an important presence in these conferences.

Yet another finding that deserves consideration concerns issues of geographic representation in relation to the location of the authors and participants in the studies examined. The production of socio-cultural-political PME papers has been concentrated in about a dozen countries over the last decade. Taking into account that what lies at the core of socio-cultural-political mathematics education research is the need to address the uneven distribution of knowledge and success, it is significant to note the uneven distribution of geographical representation and the silence coming from the low "ranked" countries. A majority of the studies reported refer to participants in countries like Australia, Canada, South Africa, United Kingdom and the US, while fewer papers report studies including participants in regions such as East and Middle East Asia, Eastern and Southern Europe, and Central and South America. When paying attention to some of these regions, it is not always the case that socio-cultural-political research is misrepresented there. In Southern Europe, for instance, a number of researchers from Greece, Portugal and Spain are developing influential work in the domain, but they present their studies in conferences other than PME. All in all, the development of the socio-cultural-political axis requires an increased representation of regions for a better understanding of the many social and political challenges faced by participants in mathematics education across different contexts worldwide. The inclusion of more diverse settings would certainly result in stronger conceptualisations of culturally-grounded notions and theories.

Based on our review we find that, although several of the socio-cultural-political PME papers examined address important methodological questions, only some of them primarily consider these questions as a major topic of discussion and overtly elaborate on the need for, development and evaluation of particular analytical approaches. Thus, there remain a larger number of unexplored questions with respect to the possibilities, limitations and suitability of the variety of research methodologies that different authors use in their studies. One aspect of the reflexivity of research is researchers' awareness of their own participation in the reproduction of particular cultural and political relations concerning mathematics education. This topic, raised many years ago in PME (Valero & Vithal, 1998), is taken up by Baturo et al. (2008) when reflecting on research collaboration and methodologies that strive for de-colonial knowledge relations when studying mathematics education in/with indigenous communities. Alongside the strength of rigorous small-scale qualitative research, another feature of PME socio-cultural-political research is that most qualitative research on mathematics classroom discourse typically focuses on a few episodes, and rarely provides quantitatively larger evidence (e.g. Herbel-Eisenmann, Wagner, & Cortes, 2008). Moreover, the adoption of long-term methodologies and comparative cross-cultural studies with a socio-cultural-political orientation has remained minimal. All these approaches and kinds of evidence are necessary to further advance the domain in the field.

In the previous section, we discussed the interrelated topics of knowledge creation and knowledge use to support their presence in practically all lines of concern in socio-cultural-political PME research through a number of papers. However, even in relation to these topics, our review revealed that conceptual theoretically-based papers are less frequent than empirical papers. Few theoretical discussions on the emerging trends in research in PME form part of the proceedings. One of the exceptions is Brown (2009), who discusses Radford's concepts of culture and subjectivity in his theory of knowledge objectification, from the point of view of what psychoanalytical frameworks to theorize learning may offer the field. This type of paper is quantitatively rare in comparison to papers centred on the analysis of empirical data, rather than on the discussion of the theoretical construction that precedes the identification of a particular construct. This finding has several implications for development of the field as a whole, and for the ways in which we are building the socio-cultural-political terrain. Due to the empirical tradition in mathematics education research, and particularly in PME, it is not surprising that the model of theory building for the development of the socio-cultural-political axis draws mostly on the accumulation of data and data analysis as an argument for the discussion of theory. There is, however, a substantive assumption in this way of building theory: it presupposes that socio-cultural-political phenomena can be directly observed or linked to something that can be directly observable. Cultural-historical and socio-political approaches problematise those meanings of observable based on the search for external measures of constructs. Further elaboration on what can be designed to be observable, along with what kinds of observation matter and why, is still required.

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PART 4

PROFESSIONAL ASPECTS OF TEACHING MATHEMATICS

FOU-LAI LIN AND TIM ROWLAND

14. PRE-SERVICE AND IN-SERVICE MATHEMATICS TEACHERS' KNOWLEDGE AND PROFESSIONAL DEVELOPMENT

INTRODUCTION

As Adler, Ball, Krainer, Lin and Novotna (2005) remarked in their landmark ICME survey report, research into mathematics teacher education was rather sparse until the mid-1990s. From its roots in mathematics and psychology – witness the name of the sponsor organisation of this handbook – the output of researchers in mathematics (or 'mathematical') education had previously been more directed, and often in an anecdotal way, towards learners, curricula, purposes and innovative instruction (Kilpatrick, 1992). The shift of attention towards – or at least, to include – teachers roughly coincided with the advent in the late 1980s of the 'social turn' (Lerman, 2000) and growing attention to professional communities, and to the lives and roles of the actors within those communities. Ponte and Chapman (2006: see p. 462) confirm this chronology in the significant case of PME activity. By the mid-1990s, Tom Cooney, one of the leading researchers in the field, was able to comment: "Although it has been 30 years coming, it appears that the field of mathematics education is poised to seriously consider teacher education as a legitimate field of inquiry" (1994, p. 626). A clear and visible sign of this emergent interest in teachers, and mathematics teacher education in particular, as objects of research, is the 1999 book *Mathematics Teacher Education*, edited by Barbara Jaworski, Terry Wood, and Sandy Dawson. The book was an outcome of activity in a PME Working Group on in-service teacher education between 1990 and 1994. Cooney's claim was further vindicated when he himself was appointed founding editor of a respected journal devoted to the topic: the *Journal of Mathematics Teacher Education*, first published in 1998. One decade later, the coming-of-age of this field of research was evidenced in the four-volume *International Handbook of Mathematics Teacher Education*, with Series Editor Terry Wood.

Considering the scope of this present chapter, it is notable that teacher *knowledge* as such is given only passing attention in the aforementioned PME-rooted book (Jaworski et al., 1999), although some authors make reference to Shulman's identification of the need for a kind of mathematical knowledge beyond confident (or even profound) knowledge of mathematics per se, and the index lists seven references to Shulman's 'pedagogical content knowledge'. The (relatively late)

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 483–520.

emergence of mathematics teacher knowledge as a research field will be discussed further later in this chapter, but it is indicative that 'teacher knowledge' was not listed as a PME research domain[1] in its own right until PME28 in 2004.

Our First Steps: Scoping the Task

Our commission was to write a critical overview of PME research into pre-service and in-service mathematics teachers' knowledge and teaching development, in the years since the publication of the first *Handbook* (Gutiérrez & Boero, 2006). This earlier work included two chapters in a section entitled 'Professional Aspects of Teaching Mathematics': between them, the authors of these chapters surveyed PME research into mathematics teachers' beliefs, knowledge, learning and classroom practices. Our brief is less comprehensive than theirs, and (accordingly) we have just one chapter in which to tell our story.

A team of researcher-colleagues of the first author undertook a content-appraisal of PME Proceedings between 2006 (Prague) and 2014[2] (Vancouver) inclusive, searching on keywords such as *teacher knowledge, teacher belief, teacher education, educator education, professional development, professional growth*. The relevance of the various PME outputs was then confirmed, or otherwise, by a rapid inspection of each paper. The same team then entered key features of each paper, such as student education-phase, teacher career-phase, methodology, sample size (where relevant) and relevant keywords. This search identified 975 candidate outputs to be considered in our survey, and the need to reduce this number significantly was apparent, but it is worth noting here that about two-thirds of these studies concerned in-service teachers. The first blunt instrument to be applied in the reduction process was to restrict reading to four types of papers, namely: Research Reports, Plenary Presentations, Plenary Panels and Research Fora. This was not because we believed that the 'best' research was reported in these outputs, but because the single page made available to authors of other presentations and group activities, such as Short Oral, Poster Presentation, Discussion Groups and Working Sessions, restricts the detail that it is possible to report in a written account (as opposed to the 'live' presentation). Around 530 papers remained, and our next decision was a tighter focus on our brief – teacher knowledge and teaching development – as indicated in the title of our chapter. With this in mind, we eliminated those papers whose focus was on teacher beliefs or teacher practices, unless they also engaged significantly with teacher knowledge and/or teachers' professional development. This was a difficult but pragmatic choice, and does not deny the complex interaction between all these elements in the effort to understand teachers and teaching. Even then, we were left with 130 papers on teachers' professional development and 220 on teacher knowledge, with the intention of reducing both to about 50 papers, in line with the number of papers cited in the Ponte and Chapman (2006).

In the case of the 220 papers on teacher knowledge, this final reduction (to 53) was achieved with the assistance of several colleagues of the second author with relevant expertise, mostly in the UK and Norway, who were able to evaluate each of the outputs in detail against criteria such as the methodological thoroughness and theoretical coherence of the paper, and its relation to established work in the field. In the case of the 130 papers on teacher professional development, the team of the first author prioritised those on the learning of in-service and mathematics teacher educators (MTEs), reflecting a growing interest in MTE learning and professional development. One Plenary Address, two Plenary Panels, and four Research Fora relevant to this focus were retained, together with seven Research Reports related to MTEs' learning. Then, according to the main research questions of the remaining papers, we divided these papers into theoretical reports, teachers' learning outcomes, and learning processes through several rounds of group discussions. Finally, 42 representative papers on teacher professional development are cited in this chapter. The 95 papers which then underpin our survey are included in the reference list for this chapter.

MATHEMATICS TEACHER KNOWLEDGE

As we noted in the introduction to this chapter, the investigation of mathematics teacher *knowledge* is a relative latecomer to the field of mathematics teacher [education] research. The contents pages of early issues of the *Journal of Mathematics Teacher Education* bear out this observation, as does the Editor's retrospective on the first volume in particular (Cooney, 1998) which makes no specific reference to mathematics teacher knowledge. By contrast, and as a rough estimate, something like a third or more of all articles published in *JMTE* in recent years have addressed mathematics teacher knowledge, focusing on aspects such as knowledge to teach particular topics or content domains, the use of particular resources or technologies to develop teacher knowledge, the impact of a particular teacher education program, and efforts to theorise the nature of mathematics teacher knowledge itself; and this trend has been paralleled in the Proceedings of PME Conferences.

Before proceeding to survey the PME outputs identified for close attention, we note that papers with an exclusive focus on either elementary or secondary schooling each accounted for about one third of the 220 papers on mathematics teacher knowledge, with the remaining third mainly unspecified or mixed. The kindergarten/pre-school phase and tertiary education were under-represented by comparison, with three and four papers respectively. While tertiary teaching was the focus of several papers identified in the initial keyword search, few took tertiary teacher knowledge as their principal theme.

These 220 papers with a focus on mathematics teacher knowledge exhibited a geographical bias with a Euro-North American axis. Specifically, the first authors of almost a half were institutionally-located in Europe[3] or the Middle East, and about a quarter in the USA or Canada, with about 10% in each of Australasia and the Far East, 5% in South America, and only one paper originating in Africa. This distribution would account for the dominant voice in the current discourse around mathematics teacher knowledge, within but also beyond PME, in which the influence of Lee Shulman and knowledge-categories (Shulman, 1987) is very powerful. This remark is not at all a criticism of that particular ‘take’ on mathematics teacher knowledge, but there is the possibility that the deluge of Shulman-influenced papers drowns the particular wisdom and insight to be gained from alternative cultural perspectives on the topic (see e.g., Lee, Huang, & Shin, 2008).

Organisation of the Survey on Mathematics Teacher Knowledge

Scrutiny of the papers targeted for detailed attention revealed a great many characteristics, orientations and topics. In order to organise the survey in a coherent and manageable fashion, the following account of PME research on mathematics teacher knowledge is organised into four main sections, namely: theories of mathematics teacher knowledge; elaboration of mainstream theory; growth of mathematics teacher knowledge; and aspects of mathematics teacher knowledge (in particular the choice and use of representations and examples, teacher noticing and attention to ‘big ideas’, and teaching with technology). The distribution of space is indicative of the prevalence of these issues in the 53 ‘representative’ papers.

Theories of Mathematics Teacher Knowledge

The seminal work of Lee Shulman and his colleagues in the 1980s underpins most of the frameworks currently in use for conceptualising mathematics teacher knowledge. Shulman’s tripartite conception of teachers’ knowledge of the *content* that they teach includes not only knowledge of *subject matter*, but also *pedagogical* content knowledge, as well as knowledge of *curriculum*. Subject matter knowledge (SMK) refers to the “amount and organization of the knowledge *per se* in the mind of the teacher” (Shulman, 1986, p. 9); and pedagogical content knowledge (PCK) consists of “ways of representing the subject which makes it comprehensible to others…[it] also includes an understanding of what makes the learning of specific topics easy or difficult …” (Shulman, 1986, p. 9). In addition to his taxonomy of *kinds* of teacher knowledge, Shulman (1986) also draws out three *forms* of such knowledge, *viz.* ‘propositional’, ‘case’, and ‘strategic’.

A Research Forum at PME33 brought together teams of proponents of three prevalent post-Shulman theories of teacher knowledge, each being articulated first around 2003, together with two commentators (Ball, Charalambous, Thames, & Lewis, 2009b; Ball et al., 2009a; Rowland & Turner, 2009; Davis & Renert, 2009a; Even, 2009; Neubrand, 2009). The first of these theories, *Mathematical Knowledge for Teaching,* (Ball, Thames, & Phelps, 2008; Ball et al., 2009c) refines and re-configures the three kinds of content knowledge – subject-matter, pedagogical and curricular – identified by Shulman (1986). This (MKT) framework, developed by the group at Michigan University, has already been adopted (or adapted) by numerous researchers as a theoretical framework for their own enquiries, and it would be reasonable to describe MKT as the dominant theoretical framework in current research in the field. In the MKT deconstruction of Shulman, SMK is separated into 'common content knowledge' (CCK), 'specialized content knowledge' (SCK) and 'horizon content knowledge' (HCK). CCK is essentially '*learners'* mathematics', applicable in a range of everyday and professional contexts demanding the ability to calculate and to solve mathematics problems. SCK, on the other hand, is knowledge of mathematics content that mathematics *teachers* need in their work, but others do not. On the other hand, they suggest that knowing about typical errors in advance, thereby enabling them to be anticipated, is a type of *pedagogical* content knowledge which they call 'knowledge of content and students' (KCS). In MKT, *horizon* content knowledge includes knowing what mathematical experiences precede those in a given grade-level, and what will follow in the next, and subsequent, grades.

The second theory, the *Knowledge Quartet* (KQ), similarly underpinned by Shulman's work, arose from observation, codification and classification of teachers' actions in the classroom, specifically those that could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. The KQ identifies three *categories of situations* in which teachers' mathematics-related ('foundation') knowledge is revealed in the classroom: these categories, or dimensions, of the KQ are named 'transformation', 'connection' and 'contingency' (Rowland, Huckstep, & Thwaites, 2005). The first two of these dimensions are evidenced in the ways that the teacher represents and exemplifies the mathematics in focus, and how they sequence material to smooth the path of learning; the third dimension, contingency, attends to how the teacher's knowledge is mobilised as they 'think on their feet' in response to unanticipated events in the course of instruction.

A third approach to understanding mathematics teacher knowledge, *mathematics for teaching* (Davis, 2010), takes a more critical stance towards the legacy of Shulman's theoretical framework, in that the latter suggests (though not necessarily) a cognitive, individual perspective on an entity (teacher knowledge) which is only meaningful in social contexts. (This critique resonates with e.g., Hodgen, 2011; Proulx, 2010). *Mathematics for teaching* is rooted in complexity theory and approaches its enquiries through 'concept studies', a group setting for the collaborative sharing, exploration and enhancement of teachers' knowledge, explained and exemplified in an account (Davis & Renert, 2009b; Davis, 2010) of the collective unravelling by such a group of the concept 'multiplication', in monthly meetings over a two-year period. Concept study enquiry into mathematics-for-teaching begins from the stance that the professional knowledge in focus is mostly tacit, and most profitably viewed in terms of participation, and as an 'active disposition' than an 'in the head' asset.

The task assigned to the three research teams at the PME33 Research Forum was to present a reading of two 10-minute video segments through the lens of their particular theory of teacher knowledge. The three analyses (Ball et al., 2009b; Rowland & Turner, 2009; Davis & Renert, 2009a) are not incompatible, and at times they coincide (e.g., in attention to selecting and sequencing examples) but their emphases are, as would be expected, very different. In her commentary, Even (2009,

p. 148) asks: "Are the different perspectives compatible? Do they complement each other?" In a recent article, Rowland, Turner and Thwaites (2014) have answered the second question in the affirmative with respect to MKT and the KQ, arguing that "In the Knowledge Quartet, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other" (p. 320). Turner (2010) supports this claim in her PME34 paper, a KQ-based analysis of longitudinal records of one teacher's approach to teaching addition, by explicit reference to MKT concepts such as SCK and KCS.

In their PME34 paper, Proulx and Bednarz (2010) adopt the situated view of MTK, having already illustrated in Proulx and Bednarz (2009) that such knowledge is embedded in their classroom practice. They present findings from a program for inservice teachers with some features in common with Davis' concept study. The authors report that their approach leads to new mathematical comprehensions, and that some beliefs concerning 'mathematical norms' are being brought to the surface, and challenged.

Although the organization of teacher knowledge into categories of one kind or another might be convenient to try to capture and articulate distinctions between knowledge-types, the boundaries between different categories is usually fuzzy, promoting from the outset disputes about what exactly characterizes different knowledge-types, and indeed whether the supposed boundaries exist at all. In their extensive review, Depaepe, Verschaffel and Kelchtermans (2013) summarise this debate and the attempts to resolve it. In her PME38 plenary address on the professional knowledge of (prospective) mathematics teachers, Gabriele Kaiser (Kaiser, Blömeke, Busse, Döhrmann, & König, 2014) raised once again the 'paradigmatic differences' in conceptualisations of mathematics teacher knowledge (and PCK in particular) as 'in the head' in one view, and 'situated' in another, as captured by Depaepe et al. (2013, p. 22):

> Advocates of a cognitive perspective on PCK believe it can be measured independently from the classroom context in which it is used, most often through a test. […] Adherents of a situated perspective on PCK, on the contrary, typically assume that investigating PCK only makes sense within the context in which it is enacted. Therefore, they often rely on classroom observations ….

Reporting findings from the 16-country *Teacher Education and Development Study* (TEDS-M), and its follow-up TEDS-FU, Kaiser (2014) proposed that these two studies suggest a way that the cognitive and situated conceptions of PCK can be integrated. The TEDS-M theoretical framework of teachers' 'professional competencies' begins from Shulman-type categories, but also includes an affective dimension, and extends (like the Knowledge Quartet) to include beliefs related to mathematics and mathematics teaching and learning, as well as metacognitive factors. Using instruments developed (or adapted) for the purpose, these aspects of prospective teachers' professional competencies were surveyed. Various kinds of comparison of competencies, and some ranking, between countries or groups of countries are reported, and some of these findings are related to cultural characteristics of the countries in question (Hofstede, 1986). For example, it was found that prospective teachers from more collectivistic-oriented countries hold more static views about mathematics (as a theory and a set of rules), whereas prospective teachers from individualistic countries are more associated with a dynamic view (as a process). The follow-up TEDS-FU study investigated how mathematics teachers' professional knowledge develops as they begin their teaching careers, and how this professional knowledge can be investigated in a more performance-oriented way. Practice-oriented, situated indicators of teacher expertise such as 'perception accuracy' (related to 'noticing' – see later), knowledge-based reasoning, and rapid identification of errors in the classroom, were annexed to the existing TEDS-M theoretical framework. These were assessed using web-based instruments requiring participants' responses to items related to short teaching sequences viewed online. The researchers found, inter alia, that the ability to notice classroom situations adequately, and to reason appropriately, are strongly related to both aspects of disciplinary knowledge (both mathematical and pedagogical). On the other hand, the ability to recognise student errors depends more strongly on content knowledge than on pedagogical knowledge.

While the studies reported by Kaiser do indeed integrate both *in vitro* and practice-based approaches to evaluating mathematics teachers' professional knowledge, they both reflect a view that such knowledge can be evaluated – 'measured', in fact – on the basis of teachers' individual responses, out of the classroom, to suitable test/questionnaire items. The 'paradigmatic differences' in conceptualisations of mathematics teacher knowledge, and how teachers are best supported and enabled to grow professionally, remain intact. We return to the issue of knowledge growth later.

Elaboration of Mainstream Theory

Papers presented at PME include a number of proposals for the elaboration, or modification, of extant theories of mathematics teacher knowledge, as outlined in the previous section. While such studies usually add to acronym-overload in the field, some draw attention to gaps or conflicts in the mainstream teacher knowledge discourse. Both Chapman (2012) and Foster, Wake and Swan (2014) take up a critique that Shulman's framework and its derivatives focus on knowledge of mathematical concepts at the expense of problem solving proficiency. Chapman proposes a four-part framework of 'mathematical problem-solving knowledge for teaching' (MPSKT), namely knowledge: of problems; of problem solving; of instructional approaches; and of students as problem solvers. In a study with 11 practising secondary school teachers, it was found that the participants held different (up to six) different conceptions in relation to each of the four PS dimensions.

Foster et al. (2014) propose a more conservative adaptation of Ball et al.'s (2008) MKT framework, in which each occurrence of 'content' is replaced by 'concepts and processes' (thus e.g., 'knowledge of concepts and processes and teaching'). They then report a case study of two problem-solving lessons taught in the context of a lesson study-based professional development program. Their analysis of lesson observations and post-lesson discussions leads them to offer observed aspects of the three PCK-components of the MKT model from a process perspective. In a somewhat similar adaptation, or application, of the MKT framework to pedagogical knowledge of technology, Getenet, Beswick, and Callingham (2015) propose a mathematics – specific version of the TPACK framework (Mishra & Koehler, 2006: see later in this chapter) named Specialised Technological and Mathematics Pedagogical Knowledge (STAMP), which somehow blends the two frameworks to take advantage of the affordances of both.

Cooper (2014) proposes a radical re-versioning of MKT by locating the Michigan theory within Sfard's (2008) commognitive epistemological framework, which views thinking as a form of communication. In this commognitive embedding of MKT, each of the MKT components (CCK, etc.) becomes (or is viewed as) a discourse, and the theory as a whole is renamed *Mathematical Discourse for Teaching* (acronym: MDT). A significant theoretical distinction in Cooper's data analysis is that between discourses (and meta-discourses) concerning mathematics and those concerning pedagogy, each of which has its own keywords, mediators, routines and narratives (with reference to Sfard's characteristic features). He proceeds to an analysis of a PD session on the notion of parity, arguing that two kinds of 'knowing' (about parity) can be discerned in the data, corresponding to the two discourses.

Features of the MKT theory that have attracted considerable attention from researchers are the Common/Specialized content knowledge distinction (CCK/SCK) and, to a lesser expend, horizon content knowledge (HCK). One approach to the CCK/SCK distinction question is theoretical argument. Another, less common approach is to design test items purporting to activate/assess one or either of these constructs, but not the other as far as possible. Of course, the construction of such items will initially draw upon theoretical conceptualisations of the two constructs in the first instance, and eventually *define* them when used as instruments to measure those constructs. Drageset (2009) presented findings from a Norwegian study investigating "the existence of SCK and CCK as two separate constructs" (p. 475) as regards Norwegian primary and lower-secondary teachers. Twenty-seven test items (derived from the Michigan *Learning Mathematics for Teaching* item bank) were administered to 356 teachers; 10 of the items were deemed to test SCK, the others CCK. A rather brief statement of correlation analysis of the test responses concludes that the two constructs are "connected, but still sufficiently different empirically to indicate that there are two different constructs" (p. 479). We note that Michigan-based Schilling (2007) had found that "sometimes SCK shows up as a separate factor in factor analyses and sometimes it does not" (p. 106). The debate continues.

In a paper pre-dating MKT (Ball et al., 2008) and Depaepe et al.'s (2013) PCK survey, Chick, Baker, Pham and Cheng (2006) proposed a literature-based three-part framework for PCK based on the interaction between pedagogical and 'pure' content knowledge (CCK, perhaps). The components are labelled 'Clearly PCK', 'Content Knowledge in a Pedagogical Context' and 'Pedagogical Knowledge in a Content Context'. Several components of each dimension are identified and listed. The adequacy of the framework is tested empirically by reference to questionnaire and interview data concerning decimals from 14 upper-primary teachers. The authors conclude that the framework was adequate, with some redundant components in the case of their sample.

Another group of papers apply and elucidate aspects of the Knowledge Quartet theory of mathematics knowledge in teaching. Petrou (2008) uses the framework in an investigation of Cypriot pre-service teachers' knowledge in relation to their classroom practice. Her case study analysis raises for attention issues in connection with one pre-service teacher's lesson on fractions, in particular concerning representations of fractions and fraction-related division structures. Whereas other PME researchers cite the KQ in their theoretical framework, the most detailed elaborations of KQ-theory are by Turner (2008, 2009, 2011) and Rowland (2010, 2011). The *Contingency* dimension of the KQ – associated with teachers' responses to unplanned and unanticipated events in their mathematics classrooms – receives particular attention in these papers. Rowland (2010) highlights the potential for teacher learning presented by contingent events, especially in post-lesson reflection-on-action (Schön, 1983), and within teacher education programs. He exemplifies this potential with an incident in which a trainee teacher is surprised by a Grade 2 student's division of a rectangle into quarters. Turner (2009) takes up the same developmental theme regarding contingency, with reference to a longitudinal study in which beginning teachers learned to analyse their own teaching using the KQ as a tool. Drawing on an international resource of KQ-analyses of mathematics teaching at elementary and secondary levels, Rowland (2011) presents a taxonomy of 'triggers' of contingent events, the main components being students' responses to questions and tasks, teachers' in-the-moment insights, and the use of pedagogical tools, including technology. Finally, Turner (2008) draws out social, community-of-practice factors in the development of mathematics teaching in early-career teachers, and the interaction of such factors with the kind of critical reflection supported by the KQ.

Growth of Mathematics Teacher Knowledge

Several PME papers address the growth of mathematics teacher knowledge and how it comes about, approaching the issue from a number of directions. Three such papers evaluate the effect of pre-service education, of teaching experience, and of a particular PD program. Blömeke and Kaiser (2008) reported findings from an international study (MT21, a precursor of TEDS-M) of the efficacy of pre-service mathematics teacher education. Participants were 849 German student teachers in three cohorts representing the beginning, middle and end of teacher education (over 5–7 years), who took a situation-based assessment of their knowledge of mathematics, of mathematics pedagogy, and of general pedagogy. Findings confirmed significant knowledge growth between the first and third cohorts, although much less so in general pedagogy than in the two mathematics-specific domains. Blömeke and Kaiser raise the caveat that quasi-longitudinal designs make assumptions about cohort comparability.

In their paper, Doerr and Lerman (2009) address the growth of knowledge for teaching mathematics as a consequence of experience of teaching. Lesson observations and interviews with one teacher participant (Cassie) in a four-year longitudinal study support the claim that the role of commonplace pedagogical routines (‘local strategies’ such as a particular rubric to support students’ mathematical writing) shifted from procedural tools to conceptual *principles* for instruction. Doerr and Lerman point to the vital role of interactions between the project teachers, and between them and the researchers. The roles of reflection and teacher-community participation once again emerge as crucial.

In the third paper, Seago, Carroll, Hanson and Schneider (2014) examine the impact of a topic-specific two-year PD program (*Learning and Teaching Linear Functions* – LTLF) on teachers’ understanding and teaching of linear functions. An experimental design involved 63 teachers and 1645 students in California. Multiple instruments, including questionnaires, observations and tests, were used to assess relevant teacher knowledge, teacher practice and student knowledge before and after the intervention. The ‘impact analyses’ found modest short-term (only) improvements in the intervention teachers’ knowledge for teaching mathematics, but student-related aspects of their teaching were enhanced, relative to the control group.

Verhoef and Tall (2011) report research in the Netherlands on lesson study as an approach to mathematics teacher learning. Three upper-secondary teachers took part in two lesson study cycles on ‘derivative’ over one school year. Questionnaires administered at the beginning and end of the year probed beliefs about educational goals, teaching methods, and associations with the derivative concept. An exit interview elicited views about students’ understanding. It was found that the potential benefits of Lesson Study were undermined by other ‘controlling’ influences such as curricula, ingrained habits, textbooks and student examination preparation. The study seems to confirm the need for caution in transplanting LS to western cultural contexts.

Gilbert and Gilbert (2009) take up the theme of teachers’ “systemic, intentional analysis of their own practice” (p. 76) within a Professional Learning Community (PLC) as an effective means of transforming practice. They report findings from a project in which high school teachers worked together on GAMUT[4] tasks designed to highlight the mathematics that teachers use in their teaching. The paper shows how the tasks are sequenced and ‘layered’ so that each part has potential to deepen and extend participants' mathematical thinking in relation to earlier parts. The same authors develop the notion of school-based PLCs in their PME37 paper (Gilbert & Gilbert, 2013), in which they describe the development within a PLC of Educative Curriculum Materials (ECMs) envisaged as guides for teachers to support teacher learning and lesson planning. Taken together, the two papers illustrate the value of collaborative work on carefully-designed tasks, and of PLC networks, as a means of developing mathematics teacher knowledge in organic and sustainable ways.

In their introduction to the PME35 Research Forum on the use of tasks in mathematics teacher education, Sullivan and Zaslavsky (2011) offer a useful taxonomy of such tasks,[5] making the broad distinction between those that resemble tasks that could be used with school students and those that are peculiar to teacher education (such as analysis of videos of teaching). In a contribution to that Research Forum, Chazan, Herbst, Sela and Hollenbeck (2011) articulate a rather novel approach to the representation of classroom practice in which animations are used to present classroom scenarios for consideration and critique by (in this instance) pre-service teachers. The animation in question concerned a student's unexpected (and correct) approach to solving a particular linear equation. It typifies both a contingent situation (c.f. the Knowledge Quartet) and a provocation of specialised content knowledge (c.f. Mathematical Knowledge for Teaching) with rich learning potential for the PSTs.

Noh and Kang (2007) also explored the contribution of ECMs to the development of mathematics teacher knowledge, but in their case the ECMs were published 'reform-oriented' curriculum materials developed for use with school students, specifically the NSF-funded curriculum *Contemporary Mathematics in Context* (CMIC). Twelve high school teachers participated in individual, task-based interviews with the researchers, using CMIC problems on rate of change. It was found that although many of the participants held a procedural view of derivative, most demonstrated ability to move between different representations of rate of change – a strong feature of CMIC. Although the specific findings reported are appropriately tentative, a social and distributed view of MTK (and its development) necessarily assigns significance to the role of ECMs in professional settings.

The teaching of proof has exercised PME researchers over the years (see Stylianides, Bieda and Morselli, Chapter 9, this volume) but rather less attention has been given to the related teacher knowledge. Drawing on observation and interview data from a three-year case study, Cirillo (2011) presents a beginning teacher (Matt) with a strong mathematics background. Initially, Matt doubted that it was possible to teach proof, but by the third year he likened himself to a 'sherpa' who had climbed the 'mountain' (proof) many times in the past, and who now accompanies his students on the same journey. Cirillo emphasises that Matt's secure subject matter knowledge was not sufficient to enable him to teach his students how to prove, calling for studies of teachers with proven success at doing so.

Whereas reflecting on teachers and teaching practice is now a commonplace means of achieving growth in professional knowledge, the value of studying and understanding learners can be another, lately neglected, means to the same end. In his PME Plenary address, Doug Clarke (2013) pointed to a resurgence of interest in Piaget's clinical interview (e.g., Ginsburg, 1997) in Australia and New Zealand, "as a professional tool *for teachers of mathematics*" (Clarke, 2013, p. 19; emphasis in the original). Clarke reports that regular use of a research-based, one-to-one interview by teachers with their students has contributed to growth of their subject matter knowledge (SCK and HCK in particular) and pedagogical content knowledge, in particular knowledge of students' mathematical understanding, thinking and reasoning.

The *growth* of MTK clearly falls within the remit of the second major focus of this chapter – teachers' professional development – and some of the approaches addressed in this section will be revisited later in this chapter, with the focus directed more towards the development of teaching practice.

Aspects of Mathematics Teacher Knowledge

Apart from the elaboration of theory and attention to the growth of professional knowledge, PME outputs on MTK in the decade under consideration have clustered around particular themes, three of which we review below: namely, teachers' choice and use of representations and examples; teacher noticing and attention to 'big ideas'; and teaching with technology.

Choice and use of representations and examples. In his exposition of the concept of PCK, Shulman (1986, p. 9) referred to "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations", and so the selection and use of representations and examples for pedagogical purposes has been central to the notion of PCK from the outset. These two aspects of PCK are distinguishing components of the Transformation dimension of the Knowledge Quartet (Rowland et al., 2005), and very visible in the exposition of specialised content knowledge in the Mathematical Knowledge for Teaching framework (Ball et al., 2008; p. 400). Although particular PME papers tend to focus on just one of these aspects of mathematics teachers' knowledge and practice, the two are intimately connected. For example, Turner and Rowland (2007) describe how a teacher's last-minute switch from symbolic (numerals) to spatial (100-square) representation of two-digit numbers in a lesson on subtraction caused her prepared examples to 'misfire' in her 'how-to' explanation.

Multiple representations and cross-national comparisons feature in many of the papers on *representations*. Drawing on video classroom data from the Learners Perspective Study, Huang and Cai (2007) report analysis of the representations used by teachers in high-performing schools in Shanghai, China and California, USA, in 10 lessons (each) on linear functions. Huang and Cai (2007) cite NCTM (2001) in stating that teachers' selection of pedagogical representations reflects their knowledge and beliefs about mathematics teaching and learning. It was found that the US teacher drew on multiple representations in most lessons, frequently more than three (e.g., graphic, symbolic, tabular). By contrast, the Chinese teacher normally used only one or two types of representation, with verbal and numerical representations predominating. The US teacher used numerical representations least. The authors suggest that these differences may explain a separate finding regarding Chinese/US students' preference for abstract/concrete representations in problem solving. The US teacher's practice is consistent with the notion inherent in the paper of Koh and Kang (2007) discussed earlier, that the promotion of multiple representations by ECMs such as CMIC is beneficial for both students and teachers.

This cultural difference might explain the claim of Dreher, Kuntzer and Lerman (2012) that "fostering students' competencies in dealing with multiple representations should be a central goal" (p. 212). In another 'inter-cultural' study, British and German pre-service teachers (PSTs) answered a questionnaire in which they rated various items about multiple representations in the teaching of fractions. They detected a difference, deemed to be cultural, in that the British PSTs favoured use of multiple representations, irrespective of their mathematical appropriateness, in the interest of providing for students' individual learning differences. The German PSTs, by contrast, had concerns that multiple representations could confuse students.

Investigating teachers' knowledge to discriminate between different representations to achieve particular learning goals, papers by Barmby and Milinković (2011) and Milinković (2012) explore British and Serbian PSTs' choice between several alternative representations (such as sets, number line, area and arrays) to represent different entities and relations, for students of different ages. Their responses were indicative of the participants' SMK and PCK, but also the stress placed on particular representations in the two countries. Deher and Kuntze (2015), and also Way, Bobis and Anderson (2015), conclude that knowledge about representations, and how to use them in assessing and developing conceptual understanding (of fractions, in these papers) should be an explicit focus in mathematics teacher education.

Concerning the *choice and use of examples* in mathematics teaching, two rather different, extended contributions stand out as 'state of the art' reviews at the beginning and end of the PME decade under consideration. The first, Bills, Mason, Watson and Zaslavsky (2006), is the paper associated with a Research Forum on 'exemplification' at PME30, co-authored by several leading researchers in the field. The scope of the paper includes different meanings of 'example'; a historical survey of the pedagogical use of examples; theoretical perspectives, including the notion of 'personal example space' (Watson & Mason, 2005); teachers' selection and use of examples, with reference in particular to the work of Orit Zaslavsky and her collaborators; the learner's perspective, the role of examples in concept formation and problem solving, including non-examples, counter-examples and generic examples; research perspectives, including instructional design and theory building; and pointers for further research. The notion of generic example (otherwise called a prototype or paradigm) as a provocation to concept formation and reasoning is recurrent throughout the paper, and was subsequently the focus of a Working Session on Generic Proving (Leron & Zaslavsky, 2009) at PME33. In recent years, the notion of 'variation' in pedagogical exemplification has entered more fully into the discourse of instruction, following psychologist Ference Marton's perception that we learn from discerning variation, and what varies in our experience influences what we learn. The provision of examples must therefore take into account the 'dimensions of variation' (Marton & Booth, 1997; Watson & Mason, 2005) inherent in the objects of attention. Western thinking on this notion is also being linked to the practice of *bianshi* ('variation') within Chinese pedagogical practice (Gu, Marton, & Huang, 2004).

At the PME30 meeting, Zaslavsky, Harel and Manaster (2006) also contributed a paper on a secondary teacher's treatment of examples in a lesson on the theorem of Pythagoras as indicative of teacher knowledge, citing work by Zaslavsky and Peled some ten years earlier on teachers generating examples. At PME33, Sinitsky, Ilany and Guberman (2009) reported on pre-service teachers' ability to generalise and explain from fractions-examples.

Drawing on her sustained research into the topic, Orit Zaslavsky gave a PME34 plenary address on mathematical thinking with and through examples (Zaslavsky, 2014). The paper is organised around consideration of three inter-related settings – spontaneous example-use, evoked example-production, and provisioning of examples – with reference to the body of Zaslavsky's work investigating them. Students' (and especially teachers') spontaneously-generated examples can be problematic (Rowland, Thwaites, & Huckstep, 2003) but they can also be productive – Zaslavsky cites the student who wrote $5 + 6 + 7 + 8 + 9$ to exemplify a rule for summation of 5 consecutive integers, and then represented the sum as $(7 – 2) + (7 – 1) + 7 + (7 + 1) + (7 + 2)$, thereby providing insight into the rule. The second setting illustrates an expanding comprehension of the concept 'periodic function' at a PD workshop resulting from the provocation formula 'Give an example of…, and another one…, and now another one, different from the previous ones…'. The design of teacher-provided examples relates to aspects of teachers' content and pedagogical knowledge, and needs to take into account what the learner is likely to (and subsequently does, or does not) "see" in the example(s). The key didactic consideration here is 'transparency' and genericity.

Teacher noticing and attention to 'big ideas'. The notion of Horizon Content Knowledge (HCK) made explicit in the MKT framework (and c.f. Shulman's (1986) 'vertical curriculum knowledge') includes a synoptic perspective on mathematics enabling the teacher to look beyond the subject-matter immediately in focus to see the 'big picture'. Kuntze et al. (2011) report two studies from an EU-funded project related, respectively, to assessing and developing German PSTs' (elementary and secondary: N=117) knowledge of Big Ideas. The paper lists key characteristics of (the researchers' perception of) Big Ideas, such as potential to support conceptual understanding and meta-knowledge of the nature of mathematics. Their projects focused on three such Big Ideas (e.g., 'argumentation and proof'). Although the first study identified weak access to content linked to these Big Ideas, the second found that related professional knowledge and awareness of Big Ideas can be built up in professional development courses. Nicol, Bragg and Nejad (2013) report a Canadian study in which six elementary PSTs were asked to adapt a task on reasoning with fractions in order to make it more accessible, or more challenging. Their analysis of the PST's proposals indicates that none takes into account the big mathematical ideas in the original problem, specifically the relationship of a fraction to the 'whole'. These authors frame their finding in the context of teachers' noticing and attention (Mason, 2002) when considering/preparing tasks for the classroom. Papers by Pang (2011) and Vondrová and Žalská (2013) take up this 'noticing' theme, with regard to Czech PSTs' analysis of videotaped mathematics lessons. In a previous study, Vondrová and Žalská (2012) had found that PSTs pay little attention to 'mathematics-specific phenomena' (MSPs) when observing a full mathematics lesson. In this one, six short video clips were shown, so that a greater 'density' of MSPs were present in the material viewed, but the PSTs' ability to notice them was not significantly improved. Rather, their attention was mainly guided by generic motivational concerns. The authors ask the telling question: would practising teachers be more likely to notice the MSPs? Pang's (2011) paper reports very similar findings with PSTs in Korea, although it does note some improvement in sensitivity to mathematics-specific aspects of what they observe later in a case-based teacher preparation course in which such classroom events were regularly analysed and discussed.

Teaching with technology. Despite the very significant presence of digital technology in mathematics education research and practice, little progress has been made to date in integrating pedagogical knowledge of technology into frameworks for mathematics teacher knowledge, or in conceptualising mathematical knowledge of technology-for-teaching. In her PME36 paper, Bretscher (2012) turns to the TPACK framework (Mishra & Koehler, 2006) as a candidate to achieve this integration, and presents an analysis of the use of a PowerPoint presentation, an interactive whiteboard and a spreadsheet by one of three case study teachers in a lesson on nth terms of sequences. She concludes that TPACK is a useful tool for the purpose of including consideration of technology factors in the analysis of mathematics teacher knowledge, but that "the central TPACK construct may be better understood, not as a new category of knowledge …. but rather as a transformation and deepening of existing mathematical knowledge for teaching using technology" (p. 89). Ruthven (2014, p. 380) has subsequently suggested that TPACK "provides a rather coarse-grained tool for conceptualising and analysing teacher knowledge; one that generally needs to be supplemented by other systems of ideas to accomplish analysis to the depth required for effective professional development and improvement".

Two studies by Kuntze and Dreher (2013) used questionnaires to investigate the PCK of 39 PSTs in relation to computer use in mathematics teaching, and how it can develop in pre-service education; and also the views of 65 practising teachers about such computer use, and the extent to which they used it themselves for various purposes. The questionnaires were in part informed by a distributed view of teacher knowledge, to include relevant 'tools', and a framework of Martin (2012) of pedagogical functions of educational technologies (viz. 'connection', 'translation', 'off-loading', and 'monitoring'), which has some potential to enhance the existing technology-free theoretical frameworks for MKT. Findings from the study indicate that the PSTs were moderately optimistic about computer use at the beginning of their course, but that their lack of technology-related PCK rendered them unable, on the whole, to be specific about actual applications. At the end of the course, 25 PSTs who had chosen a computer-related unit showed significant gains in technology-related PCK and positive attitudes, whereas the remainder did not show such gains. As for the practising teachers, it was found that, on the whole, they lacked optimism, experience and PCK in relation to computer use. Clearly comparison between the pre-service and in-service cohorts is problematic, but the first study offers some hope that PCK for technology use is learnable.

Mathematics teachers' professional learning in relation to technology use is taken up in a wide-ranging Research Forum paper by Clark-Wilson et al. (2014) which introduces (with several examples) a number of theoretical frameworks, at different levels of generality, underpinned by the theory of *Meta-Didactical Transposition*, a model for the analysis of teacher education which was itself the focus of a PME37 Research Forum (Aldon et al., 2013). Although teacher knowledge is not foregrounded in the paper, examples of teachers' learning/practice trajectories "provide insight into how the particular features and functionalities of the different digital mathematical tools impact upon teachers' motivation and confidence to integrate them into classroom teaching involving mathematical digital technologies" (p. 102). These 'cases' also illustrate the use of different theories including TPACK (see above) and also Pedagogical Technology Knowledge (PTK) (Thomas & Hong, 2005). In contrast to TPACK, PTK relates specifically to mathematics teacher knowledge, and incorporates the understanding of the principles and techniques that enable teachers to design and manage instruction likely to promote *mathematical* learning with technology.

TEACHER PROFESSIONAL DEVELOPMENT

We turn now to mathematics teacher professional development. Llinares and Krainer (2006) concluded that programs aiming to promote teachers' learning addressed their awareness of mathematical process and content, and of children's mathematical thinking. Llinares and Krainer also identified the factors which promote or hinder teachers' learning as: structure of teachers' learning; mathematical tasks used in teachers' learning; support network; engagement in extended conversation about teaching and learning mathematics; time spent; and action research on teachers' beliefs and practice.

In considering PME research in the decade since Llinares and Krainer's review, we focus on the professional development of in-service mathematics teachers, hereinafter teacher professional development (TPD), in relation to the "knowledge construction or the incremental refinement of practice or both" (Clarke, 2009, p. 85). This part of our review emphasises the *refinement of teacher practice, especially practice influenced by teachers' newly constructed knowledge*. The review of PME studies specific to TPD includes 130 papers with this geographical distribution: about one third of the first authors were institutionally-located in North America, about a quarter each in Europe (6% in UK) and Asia, about 12% in Australasia, 5% in South America, and 2% in Africa. It is worth noting that roughly half of the 130 papers come from English-speaking countries such as America, Canada, UK, and Australia.

Organisation of the Review on Teacher Professional Development (TPD)

Three theories of teacher knowledge were elaborated in the previous section, but theories of TPD are still in process of development. The following sections address, in turn: theoretical perspectives on TPD; *description* of TPD; *interpretation* of TPD; and *prediction* of TPD. We also review PME research on *mathematics teacher educators'* education – an emergent TPD-related theme.

Theoretical Perspectives of Teacher Professional Development (TPD)

In the plenary panel at the PME33, Clarke (2009) summarised the mainstream theoretical perspectives in mathematics teacher education and addressed issues related to the bridge between research and practice via mediation of different theoretical perspectives. Perspectives on mathematics teacher education can generally be described as either *researching TPD from the cognitive or the socio-cultural perspective*; *viewing theory as a static entity or an evolving process*; or *the opposing or complementary nature of theories.* These three perspectives structure the following review of PME studies specific to TPD during 2006–2014.

Researching TPD from the cognitive or the socio-cultural perspective. Ponte (2009) claimed that cognitive theories have been the dominant view in teacher education. For example, Tzur (2007), from a cognitive perspective, asserted that TPD is "progress from intuitive to formal ways of thinking about teaching" (p. 143). He further pointed out that the learning progression does not only refer to behavioural changes but also to a paradigm shift in teachers' thinking from know-what to know-how. Muñoz-Catalán, Climent and Carillo (2009) attempted to make an analogy between student learning and teacher learning about teaching. They adopted the hierarchical stages of interiorisation, condensation and reification from Sfard (1991) to elaborate TPD. Muñoz-Catalán et al. showed that teachers are more inclined to take students' learning difficulties into account and adapt teaching plans that can meet students' need during the condensation stage.

While recognising this dominant cognitive perspective, Ponte (2009) pointed out the emergence of theories that emphasise social processes and how they influence TPD (e.g., social interactions between participants, communities of practice, and activity structures involving participants). Llinares and Krainer (2006) also claimed that investigations of TPD increasingly consider social and organisational aspects. Ponte and Chapman (2006) further elaborated sociocultural theory, based on the work of Vygotsky, which has become prominent in the PME community and has evolved as one of the more productive lines of work regarding teachers' practices. The review of PME studies specific to TPD between 2006 and 2014 also reveals the increasing interest in socio-cultural perspectives. For example, Ohtani (2009) adopted activity theory, one of the socio-cultural theories commonly used to interpret TPD, to argue that Japanese Lesson Study could be a successful approach. Likewise, Jaworski and Goodchild (2006) used activity theory as the framework with which to analyse issues and tensions with respect to the essence of TPD occurring within an inquiry learning community.

Viewing conception of theory as static entity vs. evolving process. In the panel, Clarke (2009) applied the definition proposed by Niss (2007, p. 1308) to elaborate the static conception of theory as "an organised network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain, or a class of domains, consisting of objects, processes, situations and phenomena"; The TEDS-M study (Kaiser et al., 2014) was judged to be conducted under the static perspective which mainly evaluated the content and pedagogical content knowledge, and the learning opportunities, of practising teachers. By contrast, due to the demands of new situations and research purposes, Clarke (2009) proposed that "theories need to be fluid and evolving" (pp. 87–88), e.g., in the context of online distance courses for teachers (Borba & Zulatto, 2006).

The opposing or complementary nature of theories. Different theoretical perspectives on teacher professional development, like those of teacher knowledge, need not be in opposition. Thus Clarke (2009) viewed "alternative theories as potentially complementary rather than necessarily opposed" (p. 91). One example of complementary alternative theories could be seen in the Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002), using four domains of teachers' professional growth identified by Guskey (1986). Clarke and Hollingsworth (2002, p. 947) identified "the specific mechanisms by which change in one domain is associated with change in another. The interconnected, non-linear structure of the model enabled the identification of particular 'change sequences' and 'growth networks', giving recognition to the idiosyncratic and individual nature of teacher professional growth".

In this review we find that theories are, for the most part, introduced by the researchers, and that theories of mathematics learning have their analogies in theoretical perspectives on mathematics teachers' learning. If there were be a comprehensive theory of TPD, relevant to multiple contexts and situations, what would be the necessary functions of such a theory? Our review suggests that they would be: to *describe* what teachers have learned; to *interpret* how teachers learn to refine their practice; and to *predict* TPD (that is *to design, evaluate, and research* it). We now consider each of these three functions in turn.

Description of Teacher Professional Development (TPD)

For the most part, PME research which described the refinement of teacher practice also interpreted the processes of refinement. Here we will emphasise the following four foci: *expert teachers*; *beginning teachers*; *inquiry-based teaching*; and *raising teachers' awareness of students' thinking*.

Expert teachers. The meaning of 'expert' is interpreted in different ways because different aspects of teachers' expertise are valued within different cultures and societies (e.g., Berliner, 2001). A PME36 Research Forum on Conceptualizing and Developing Expertise in Mathematics Instruction focused on teacher expertise and its development (Li & Kaiser, 2012), in which Ponte (2012) and Lin (2012) portrayed their perspectives about expert teachers implementation of the new curriculum in Portugal and their long-term participation in teacher professional development programs in Taiwan, respectively. They both considered expert teachers to be being those who employed student-centred teaching, including selecting tasks and conducting classroom discussions before and during teaching. About the selection of tasks, Ponte (2012) claimed the expert teacher "is able to select and perhaps adjust suitable tasks, …, involving students actively in mathematical work, stimulating them to develop their own strategies, concepts, and representations" (p. 126), effectively an elaboration of Lin's (2012) "designing and using tasks that support rich mathematics thinking" (p. 133). About conducting classroom discussions, Ponte (2012) indicated that expert teachers should "conduct classroom discussions that create opportunities for negotiation of meaning, development of mathematical reasoning, and institutionalization of new knowledge" (p.127); consistent with Lin's (2012) "purposely selecting and sequencing students' solutions for whole class discussion; critically questioning and using students' errors or misconceptions for discussion; responding to students' questions adequately" (p. 133).

Lin (2012) further proposed that the expert teacher would create and allocate creative assignments after lessons. Likewise, also in the Forum, Leikin (2012) claimed that "expertise in mathematics instruction is characterised by fluency, flexibility, originality and elaboration" (p. 143): she referred to creative teaching;

giving concrete empirical examples to elaborate these concepts, and to support her claim. It is notable that, in different contexts and from diverse viewpoints, Ponte, Lin and Leikin all proposed similar characteristics of expert teachers. Nevertheless, whether each society has a coherent definition of the expert teacher is still under investigation. The emphasis has shifted to whether there is only one view of what an expert teacher is within one society. Pang (2012) argued that even in the same society, the description of an expert teacher might differ depending on one's role; e.g., Korean mathematics educators usually considered expertise in mathematics instruction from the perspective of "mathematics-specific analysis ability", whereas educators in general in Korea considered expert teachers from the perspective of a "specific case-based pedagogy" (p. 136). This seems to be an important issue when developing expertise in mathematics instruction within one society in the future.

Beginning teachers. At a PME31 Research Forum entitled Researching Change in Early Career Teachers, Hannula and Sullivan (2007) focused on ways in which teacher educators might facilitate effective change in beginning teachers. It was proposed (p. 151) that beginning teachers might be in need of change if they:

1. Have fixed views of the nature of mathematics and limitations in relevant mathematics discipline knowledge;
2. Have anxieties about mathematical knowledge and teaching that can be potentially constraining and even disabling;
3. Are unfamiliar with desired pedagogies and curriculum, not having experienced these as school students themselves; and
4. See learning to teach as a short-term, once-only event as distinct from a career-long process.

Point 1 is similar to Leikin's (2006) intuitive thinking about mathematics teaching. The unfamiliarity with pedagogy and curriculum in Point 3 is the opposite of the "fluency" proposed by Leikin (2012), and the lack of experience as a student seems to be contrary to "awareness of children's mathematical thinking" claimed by Llinares and Krainer (2006). Furthermore, Points 1 and 3 are related to the concept of teacher efficacy. Chang and Wu (2007) studied 64 beginning elementary teachers' sense of efficacy related to mathematics teaching, finding that those who had majored in mathematics or science showed greater efficacy. Hannula, Liljedahl, Kaasila and Rösken (2007, p. 154) summarised the therapies aiming to reduce the mathematics anxiety of pre-service teachers into four types: narrative rehabilitation; bibliotherapy; reflective writing; and drawing schematic pictures. Whether these strategies could also be adopted for beginning (and more experienced) teachers is an interesting issue for future research. Additionally, it should be noted that point 4 deals with societal-based issues, which vary between countries; the correlation between beginning teachers' willingness to refine their teaching and their stance in relation to point 4 is also worth future investigation.

In addition, to enhance teacher efficacy and reduce mathematics anxiety, the Research Forum concluded: "One fruitful approach is to engage innovative mathematics teachers as experts or facilitators (teacher-researchers) for new projects" (p. 175); examples could be seen in Wang and Chin (2007) who investigated the ways mentors intervene in the mathematics teaching of practice teachers, and the principles and underlying values for their interventions.

Inquiry-based teaching. Inquiry-based teaching is widely promoted in mathematics education around the world, e.g., in European countries, implementation of inquiry-based learning in day-to-day teaching has been reported by Maass, Artigue, Doorman, Krainer and Ruthven (2013). Teachers' competence with inquiry-based teaching is often identified as a key indicator of expertise in mathematics instruction. For instance, the view of what makes an expert teacher in Portugal was portrayed by Ponte (2012) as a teacher who is able to select, and perhaps adjust, suitable tasks, especially exploratory tasks, that involve students actively in mathematics work, stimulating them to develop their own strategies, concepts and representations. These are inquiry-based learning tasks. Chapman (2010) maintains that "Inquiry, as a basis of teaching, is being associated with notions of learner-focused, question driven, investigation/research, communication, reflection, and collaboration" (pp. 361–362). Chapman reported the experience of a group of elementary teachers in "researching" how to adopt inquiry-based teaching in their classrooms. They developed an inquiry-teaching model, guided by their mentor to plan lessons for different grades. As a result, Chapman claimed, the teachers gained a deeper and more meaningful understanding of: inquiry-oriented teaching; questioning techniques that guide and enrich student thinking; posing thought provoking questions to motivate students to discuss and understand mathematics at a deeper level; and instructional strategies that allow students to assume ownership of their knowledge and knowledge construction.

Chin at al. (2006) reported a collaborative action research study on implementing inquiry-based instruction in an eighth grade mathematics class. An experienced teacher and a trainee teacher together carried out the action research, supported by an educator. After two-semesters, the trainee teacher gained a deeper understanding of the complex role of a mathematics teacher and had more confidence to conduct inquiry-based teaching on his own. The experienced teacher had also developed from being a novice at inquiry-based instruction to a confident teacher with the intention of communicating the teaching strategy to his peers.

Raising teachers' awareness of students' thinking. Studies focused on the intervention of using students' thinking as the basis of professional development are still ongoing, and some examples are cited here.

Regarding the role of students' mistakes in teachers' learning process, Heinze and Reiss (2007) investigated the effects of teacher training on teachers' ability to handle mistakes and assist students' learning of reasoning and proof in geometry. They conducted a quasi-experimental study and showed that students in the experimental group performed significantly better in the post-test.

Proulx and Bednarz (2009) invited teachers to explore the following fraction division task:

$$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$

Is this procedure adequate/correct? Does it always work? How?

A variety of resources, mathematical, didactical and pedagogical, were used by teachers when making sense of this mathematical situation. Some approached the “same” situation from different perspectives, some came at it from different perspectives at different times, and some employed ways that implicitly had a double nature (e.g., mathematical and didactical). All those points of entry appear to play a role.

Goldsmith, Doerr and Lewis (2009) reviewed over 100 studies on teachers' learning to challenge the issue: How do practising mathematics teachers continue to improve their teaching over time? They illuminated the “black box” of teacher learning by exploring teachers' changing attention to and use of student thinking.

Interpretation of Teacher Professional Development (TPD)

In order to interpret how in-service mathematics teachers learn to refine their practices, three contexts in which they learn were identified in PME research: *learning via teaching*, *via researching*, and *via participating in a learning community*. We consider these in turn.

Teachers' learning via teaching. A Research Forum at PME 31 (Leikin & Zazkis, 2007) considered how teachers might learn through teaching. The main sources of teachers' learning through teaching is their interaction with students, use of learning materials (such as textbooks and teachers' guides), communication with colleagues and attending workshops. By giving opportunities for students to initiate interactions and by managing lessons according to students' ideas, teachers also make opportunities for their own learning (p. 124). However, this way of learning is not always made explicit to teachers. Simon (2007) proposed that teachers' current understanding imposes limits on what teachers can learn from their teaching. Tzur (2007) conceptualised such learning in terms of a change in anticipation. That is, whenever teachers direct their activities towards certain goals, such as correcting student mistakes, predicting student responses, providing students with experiences that differ from one's own school experiences, resolving disagreements and/or one's cognitive conflicts, satisfying school's requirement to use software, improving one's own mathematics, etc., they essentially learn through noticing unanticipated ways in which others (e.g., one's students or peers) react to plans the teacher executes (p. 144). Such reactions may become prompts for the teachers' reflection on pedagogical/mathematical activity-effect relationships. That is, the teachers continually consider the extent to which their goal-directed teaching fosters certain effects, effects in the sense of inferred student/peer understanding. Whenever teachers noticed and revisited student/peer unanticipated actions, this prompted further reflection, hence they were learning. These three constructs: anticipation, reflection, and noticing, can powerfully explain the complex mechanisms, contexts, and stages in teacher change via teaching.

Teachers' learning via researching. Research is one of the best methods for teachers to learn how to refine their teaching practice. The focal issues of research in the process of teachers' learning can be collectively summarised from the discussion of the Research Forum at PME34 (Santos-Wagner & Chapman, 2010): (1) the goal is to develop teachers' reflective, analytical and critical thinking, (2) the helpful tools for collecting data from teachers are reflection, noticing and biographical writing, (3) the stimulus to autonomous teacher disposition in relation to mathematics pedagogy, and (4) making use of teachers' classroom practice and learning experience to help them to gain knowledge. In Llinares and Krainer's (2006) review, they suggested that "in the future, we need more of these research-oriented stories, putting an emphasis on explaining phenomena by using empirical evidence as well as theoretical consideration. Action research by teachers (…) and corresponding action research by teacher educators (…), and we need identified efforts in the future (p. 451)." Therefore, teachers' learning from research has been emphasised for its theoretical and practical importance in TPD.

The power of research for TPD is what "practice and theory can offer through learning processes of engaging teachers in *research projects*" (Santos-Wagner & Chapman, 2010, p. 354). In the context of research, teachers can be learners or researchers, depending on the goals they set. Traditionally, there are two methodological approaches to teachers' learning through the process of research in PD; one is *participating in a research project* and the other one is *conducting action research* (including *design tasks* for classroom practice) connected with the teacher's practice. These approaches are equally important in providing opportunities for learning with both theory and practice, although the actions taken in each research project might differ. Through engagement *in* a research project, teachers can quickly receive theoretical support from fellow researchers and follow the arrangement of the research design to learn. The majority of research in TPD can be categorised in this domain. However, with the other approach, conducting action research, teachers have to invest considerable effort in the process of linking theory and practice in order to improve their classroom practice (e.g., Serrazina, 2010) or to refine their teaching (e.g., Chapman, 2010). Generally, action research can function in a mentoring structure where both participants, i.e., mentors and early career teachers, learn together (e.g., Chin, Lin, Ko, Chien, & Tuan, 2006), or in a cooperative community where all participating teachers learn from each other in a systematic arrangement. For example, the experienced teachers learn to design tasks for enhancing their expertise in hierarchical stages to improve their design, take the experiences of their peer colleagues, and then develop further in participating in a design-based TPD program and conducting their own research (see Lin, Chen, Hsu, Yang, & Wu, 2013). This can also be found in lesson study (LS) group learning in which group member-teachers apply LS cycles to continuously refine their lesson (e.g., Robinson & Leikin, 2009). These studies of action research have one characteristic in common, in that they are all trying to stimulate innovation in teachers' professional expertise.

Teachers' learning via participating in learning community. Generally speaking, theories used to interpret TPD in learning communities are oriented to social and cultural perspectives. Review of PME papers with respect to TPD via participation in a community can be categorised into four main research topics: inquiry community, Lesson Study, design-based community, and online learning community.

The notion of *inquiry community* brings together characteristics of "being together" and "exploring" for triggering professional development. Fundamentally, the inquiry community involves an activity system where teachers are able to ask questions and seek answers to discover more about the teaching and learning of mathematics (Jaworski & Goodchild, 2006). To this end, Jaworski and Goodchild (2006) suggested that activity theory based on the work of Vygotsky can well articulate TPD in an inquiry community. They argued that activity theory offers a unit of analysis and the possibility of exploring the mediating elements and dialectical relationship between different tiers of participants and interactions with their environments.

Lesson Study entails a professional community where in-service teachers study lessons in depth on a school basis (Fernandez & Yoshida, 2009). Pang (2015) studied five in-service teachers and argued that lesson study motivates teachers to analyse the strengths and weakness of teaching approaches implemented in one class and to come up with alternatives. In principle, teachers should volunteer to participate in a lesson study community. However, in reality, Krainer (2011) argued that the participation can be regarded as quasi-required because socio-cultural commitment or pressure from principals plays a role in influencing the participation in such professional communities. Thus, Krainer (2011) concluded that culturally-situated theories such as cultural-historical activity theory, anthropological theory of didactics, and community of practice theory become promising theories that can be used to elaborate teacher learning in such professional environments. The use of those theories brings additional lenses in exploring and interpreting new aspects in the Lesson Study community.

Design-based community highlights design as an intervention approach by which teachers are involved in creating instructional tasks for student learning of mathematics. Design-based community does not only highlight learning through participating in practice as in Lesson Study, but also the facilitation of TPD by bridging theory and practice so that teachers have a picture of how theoretical ideas can be incorporated into their teaching. Thus, Lin et al. (2012; 2013) adopted a three-layer structure comprising grand theory, intermediate framework and a design tool (Gravemeijer, 1994; Ruthven, Laborde, Leach, & Tiberghien, 2009) for the design of professional programs, where the intermediate framework and design tool serve to coordinate and contextualise the theoretical insights from grand theory. Being task designers, teachers have opportunities to explore curriculum materials and student learning in detail, as well as to incorporate professional development materials into their designs, all of which become important sources for improving their teaching. The theory adopted for the investigation of TPD by Lin et al. (2012, 2013) is also aligned with situated learning theory, by which teachers development through interaction with others can be identified.

Online environments for teacher professional development have been seen as important for their potential benefits in responding to teachers' needs during the last decade, as compared with face-to-face professional development. It is generally thought that these online TPD programs can provide learning opportunities for teachers at their convenience, and when they are needed (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009). Flores, Escudero and Aguilar (2014) use the term 'online mathematics teacher education [OMTE]' in their literature review of this emerging research area. They found that the main issues investigated include *interactions* among teachers in online settings, and teachers' professional development (*growth*). The *theoretical approaches* employed in this area are partly extrapolations of tools designed originally for face-to-face settings, such as the concept of *community of practice* and *mathematical knowledge for teaching*. Some theoretical approaches are specifically designed for online settings, such as the concept of *humans-with-media* (Borba & Zulatto, 2006) and the instructional model of *online asynchronous collaboration*.

The data collected for supporting teachers' development in online environments include their *written productions*, *teaching materials*, and *mathematical productions* through graphing software, platform resources such as online forums, chat rooms and questionnaires, and digital recording artefacts. Lastly, the transformations of researchers in online environments can be summarised with reference to three aspects. One is that the way they *access data* is less intrusive than the methods used in a face-to-face setting, e.g., observations. Secondly, the efficiency of *data collection and processing* is higher than in a face-to-face setting. Thirdly, online environments create the need for researchers *to create theoretical tools* adapted from face-to-face settings.

Prediction of Teacher Professional Development and Processes—Design, Evaluation, and Research

Our survey found only a few papers which pinpoint the apparatus that can predict TPD. From a socially situated learning perspective, Hsu, Lin, Chen and Yang (2012) proposed a coordination mechanism defined as the ability to innovate for teaching by transforming and coordinating sources of information observed and experienced in different learning environments. Based on this definition, Hsu et al. identified two kinds of coordination, namely coordination as making connections between others and personal ideas in a superficial way; and coordination as integrating sources of information into the creation of novelty. Similarly, Boesen et al. (2014) identified two kinds of interpretation of information in respect of their teaching. One is assimilation, which refers to the ways that teachers interpret information that is in line with their preference. The second is adaptation, which highlights the coordination of information into the learning process.

Lin et al. (2013) further grounded their study on the analysis of teachers' intention to design tasks and evaluate them in alignment with the goals prescribed by professional development programs. Based on a case study, the analysis reveals three stages of teacher growth: self-expression; combining other ideas into personal design; and investigating the essences of mathematics learning. These can be used to evaluate and forecast teacher learning in design-based professional settings.

Although studies on TPD have a predictive orientation, the field in teacher education still lacks fundamental and comprehensive theories that can articulate and predict TPD outcomes appropriately across different professional settings. As suggested by Ponte (2009), this requires new theories about teacher education that can be used to design, evaluate and research processes of teacher education and development. We emphasise that design here does not only refer to planning and arrangement of professional programs by teacher educators, but also to an intervention approach of designing instructional tasks through which teachers have opportunities to improve their practice for better student learning of mathematics.

Various PME papers attempt to conceptualise TPD in terms of elements that can better explain, interpret and predict TPD. Sztajn, Campbell, and Yoon (2009) suggested that TPD should be designed, evaluated, and researched on the basis of four elements: goal, contexts, theories and structure. Goal involves the shared version of mathematics teaching and learning, understanding of mathematics knowledge for teaching, and equity and sense of self as a mathematics teacher. Contexts for TPD include curricular, participant background, teacher engagement in decision-making processes related to the intervention, participation attitudes (e.g., compulsory or voluntary), and the role of accountability in the community. With respect to theory, both teacher growth and instruction are involved. When structuring an intervention, there needs to be consideration of content and format to ensure how opportunities for learning are best organised and presented. Sztajn, et al. argued that conceptualisation contributes to a more careful examination of the fundamental aspects of TPD.

The prediction function also permits design, sustenance and evaluation of professional development programs on a large scale. Marrongelle, Sztajn and Smith (2013) made eight recommendations for the arrangement of large-scale, system-level implementation of professional development programs. They are

(1) to emphasise substance so that teachers have opportunities to engage in practising new content; (2) to enable teachers to create and adapt professional materials; (3) to design professional development programs utilising effective ways to organise learning experiences for mathematics teachers; (4) to build programs which provide a continuous and coherent set of experiences over an extended period of time; (5) to prepare and employ knowledgeable professional development facilitators; (6) to tailor to key role groups (e.g., department chairs, instructional leaders, school administrators and superintendents), ensuring that all understand the new content and practices; (7) to educate all stakeholders such as parents, politicians, school boards and so on; (8) to assess professional development programs continuously. These recommendations would ensure the successful implementation of high-quality professional development programs.

Research on Mathematics Teacher Educators' Education (MTEE)

Llinares and Krainer (2006) identified characteristics underlying research on mathematics teacher educators:

> Mathematics teacher educators' growth is viewed as a learning-through-teaching process supported by reflective practice – growth through practice – and the use of theoretical references generated in the reflection on professional development of mathematics teachers to think and offer explanation on mathematics teacher educators' growth. (p. 447)

They make reference to Zaslavsky and Leikin's (1999, 2004) three-layer action/reflection model, working contexts which allow different levels of autonomy in the development of mathematics teachers and teacher educators (Krainer, 1999), and Tzur's (1999, 2001) four-focus model for MTE development. Consideration of PME studies during the last decade points to what and how mathematics teacher educators learn. Ten research reports, one plenary address and one plenary panel paper were related to mathematics teacher educators (though three of these papers discussed MTEEs' views or dispositions and are not included in this review). Two discussion groups and one working session on mathematics teacher educators' knowledge were held in 2012, 2013, and 2014, respectively, reflecting increasing interest in research on mathematics teacher educators' knowledge.

Six of the nine papers in focus were classified as aiming to reveal or characterise mathematics educators' learning outcomes (what-oriented-questions), the others as aiming to explore or comment on mathematics educators' learning processes (how-oriented-questions). Concerning mathematics educators' learning outcomes, educative power and disposition of mathematics educators are identified as another two categories in addition to knowledge. There were two papers related to mathematics educators' knowledge. One concerned mentors, whose content knowledge, pedagogical knowledge and knowledge of students' cognition were tested as part of their learning outcome (Lin, 2007). The other paper reviewed the main issues investigated in online mathematics teacher education. Two categories were identified as a focus on analysing interactions among teachers, and a focus on teachers' professional development in online settings (Flores, Escudero, & Aguilar, 2014). These two issues can be treated as what mathematics teacher educators should know, and thus be classified as research on mathematics educators' knowledge. However, mathematics educators' knowledge did not extend to mathematical knowledge for educating in PME papers. Thus, Beswick and Chapman (2013) initiated a discussion of mathematics teacher educators' knowledge in 2013, followed by a working session in 2014 (Beswick, Goos, & Chapman, 2014).

Two papers investigated what can be learned from mathematics teacher educators' design, implementation, reflection and revision of their instruction, while one paper investigated mentors' approaches to intervening in the mathematics teaching of trainee teachers. Mathematics teacher educators' or mentors' approaches include metacognitive awareness and discussion (Kalogeria & Kynigos, 2009), documentational work motivated by fieldwork activities (Psycharis & Kalogeria, 2013), and interventions in (trainee) teachers' teaching (Wang & Chin, 2007).

In the remaining four papers concerned with mathematics educators' learning process, three categories of learning process were identified: understanding mathematics education research and practice; cooperatively solving pedagogical and educative problems; and participating in mathematics education research and practice.

Concerning the first category, Rhodes (2009) examined MTE's 'disequilibrium' while observing, analysing, and discussing a mathematics content class for preservice teachers. He found that participants who experienced disequilibrium were analytical in their thoughts and struggled to reconcile their own teaching experiences with their observations. Thus, experiencing disequilibrium is a promising approach to educating MTEs.

Two papers address cooperatively solving pedagogical and educative problems. Reflecting on mathematics education research and its interrelation with mathematics teachers, Krainer (2011) concluded that researchers cannot transmit knowledge directly to practitioners, and proposed viewing researchers as stakeholders in practice and teachers as stakeholders in research as a way to increase the further development of both parties through collaboration. From this point of view, it appears that teachers and teacher educators can mutually support each other to solve pedagogical (how to teach) and educative (how to learn to teach) mathematics problems.

The other paper (Erbilgin & Fernandez, 2011) focused on how one university supervisor (mathematics teacher educator) supported mathematics teachers (mentors) to solve an educative problem, that is, how to mentor student teachers. They found that a program based on educative supervision developed the supervisory knowledge of the mentor and changed the mentor's style of supervisory practice. This study demonstrated how an educative problem can be solved through researchers as stakeholders in practice and teachers as stakeholders in research.

The fourth paper (Liljedahl, Williams, Borba, Krzywacki, & Gebremichael, 2013) discussed the education of young mathematics education researchers, proposing that mentorship is required for them to develop a professional identify as scholars in their field. The issues related to mentoring young researchers beyond supervision were discussed. In particular, Liljedahl proposed that "there is room for, and need of, more explicit and active mentorship of our young researchers within our organization" (pp. 1–90). This implies that the learning of young researchers is viewed as participation in an academic community. That is, we shift our focus away from the individual acting on the world and onto the individual acting in the world (Lave & Wenger, 1991), so young researchers may move from peripheral to full participation in the (PME) community.

FINAL REFLECTIONS

We conclude with some thoughts arising from our survey of PME research on mathematics teacher knowledge and professional development in the decade since the previous overview.

Concerning mathematics teacher knowledge. Interest in mathematics teacher knowledge both within and beyond PME shows no sign of abating at the present time. In our survey we considered PME research concerning: theories of mathematics teacher knowledge; elaboration of mainstream theory; growth of mathematics teacher knowledge; and three particular aspects of mathematics teacher knowledge. Through Research Fora, Plenary Presentations and particular Research reports, mainstream theories of MTK have been thoroughly promulgated, elaborated and exemplified. There is scope for more effort to look for common ground, or complementarity, in the available theories, and PME is an ideal potential forum for doing so in an interactive and collegial context. Fundamental 'paradigmatic differences' between individual/cognitive and situated/social perspectives on MTK remain unresolved, and are perhaps unresolvable (in the sense of reflecting different world views). Most (but not all) theories of MTK naturally follow the lead of Shulman in identifying categories – of kinds of knowledge, or of situations in which it is manifested. The recent trend towards attempting to delineate the boundaries between such categories is interesting, even if potentially futile, but the interdependence of different aspects of knowledge also merits further study.

The theoretical understanding of MTK is intimately linked to designing efforts to promote its growth, and the papers reviewed present several fruitful approaches, of which structured reflection in a (teacher) learning community seems to be especially powerful. Indeed these are characteristics of the lesson study approach to the improvement of teaching and teacher knowledge; we can expect, and welcome, further investigation of the transfer of lesson study to diverse cultural, curricular and praxis contexts. Likewise, a distributed notion of MTK would recognise the crucial contribution of Educative Curriculum Materials to a common-wealth of professional knowledge, and more work can be expected towards theorising and investigating the role of ECMs as a component of MTK, and a stimulus for its development.

Concerning teachers' professional development. This review raises two substantial issues concerning the design, evaluation and investigation of TPD.

The first issue concerns the development of more fundamental and comprehensive theories to better describe, interpret and predict TPD in professional settings. Those studies orienting to culturally and socially situated learning perspectives attempt to articulate TPD in terms of becoming a member of a certain community in which they gradually learn the ability to communicate and act according to its particular norms (Cobb, 1992; Cobb, Yackel, & Wood, 1992; Yackel & Cobb, 1996). However, such studies might be limited in terms of elaborating TPD across different professional settings with different teacher backgrounds and populations. By contrast, studies on TPD orienting to the cognitive and psychological perspective do not consider how teachers appropriate sources of information, and how others such as teacher-colleagues or students play a role in influencing TPD. Developing fundamental and comprehensive theories that embrace both social and cognitive perspectives for better elaborating TPD becomes the emergent issue in teacher education research, along with the identification of fundamental and comprehensive theories to underpin the arrangement and implementation and evaluation of large-scale professional development programs across different mathematics content, teacher attributes and cultural characteristics.

The second issue is about how teachers can learn effectively. Teachers' learning via teaching, researching, and participating in learning communities have been reviewed in this chapter. Teachers' current understanding imposes limits on what teachers can learn from their teaching (Simon, 2007). A design-based community which integrates research and participation in a learning community can better facilitate teachers' learning. The studies of Lin et al. (2012, 2013) point to several requirements for developing such a design-based community: First, to develop a way to link research and practice perspectives in the program. As discussed earlier in this review, Lin et al. suggest a three-layer structure including grand theory, intermediate framework and a design tool (Ruthven, Laborde, Leach, & Tiberghien, 2009) for the design of professional programs. Secondly, to engage teachers in designing instructional tasks and to detect their pedagogical challenges, formulate instructional strategies to overcome these challenges, and then to test whether the strategies are useful or not in interaction with classroom students. In order to facilitate teachers to design tasks, Lin et al. (2102) propose three starting points: student misconceptions, standard 'results' in school mathematics, and engaging with student conjectures, each of which allows teachers to create tasks more easily. Thirdly, to develop strategies for enabling teachers to incorporate theoretical ideas into their design of instructional tasks. The adaptation of the above three considerations could be taken into account for future research on designing TPD in various contexts.

Concerning mathematics teacher educators' education. In general, papers relating to teacher educators' learning paid more attention to learning outcomes than to their learning processes. From our review, 'mathematical knowledge for educating' – the knowledge of mathematics teacher learning or principles of designing educative tasks – has not been well structured. Ideas about investigating mathematics teacher educators' competencies originated in research on mathematics teachers. Nonetheless, mathematics teacher educators' goals, resources and orientations are different from those of mathematics teachers (Schoenfeld, 2011), in addition to their action, reflection, autonomy and networking. Moreover, mathematics educators' power in communicating with teachers and reasoning for solving educative problems and connecting research and practice are less investigated (Yang Hsu, Lin, Chen, & Cheng, 2015). As for mathematics educators' disposition, affective factors are seldom considered.

The meanings and goals of research and practice are different for mathematics teachers and teacher educators, but there can be synergy between them (Krainer, 2011), and there is potential in developing mutually-supportive communities involving both groups.

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NOTES

[1] A published list of research domains, or categories, enables authors to indicate the substantive focus of their research report submissions, and reviewers to indicate their substantive expertise. These domains are reviewed from time to time by the PME International Committee.

[2] A similar search of the 2015 PME proceedings (identifying 27 additional papers for scrutiny) was undertaken after submission of the first draft of this survey, and is reflected in its content.

[3] Of the 530 papers remaining after the one-page contributions had been eliminated (as described), none were from France, and so a distinctive 'didactique' perspective is necessarily absent from this survey.

[4] Guides for Accessing Mathematical Understanding for Teaching.
[5] See also Zaslavsky and Sullivan (2011).

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