ATLANTIS PRESS

Atlantis Studies in Probability and Statistics
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Ordered Random Variables: Theory and Applications

Atlantis Studies in Probability and Statistics

Volume 9

Series editor

Chris P. Tsokos, Tampa, USA

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Atlantis Press
29, avenue Laumière
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ISSN 1879-6893 ISSN 1879-6907 (electronic)
Atlantis Studies in Probability and Statistics
ISBN 978-94-6239-224-3 ISBN 978-94-6239-225-0 (eBook)
DOI 10.2991/978-94-6239-225-0

Library of Congress Control Number: 2016951690

Printed on acid-free paper

To my Father, Mother and my Wife Saman

Muhammad Qaiser Shahbaz

To my Father, Mother and my Wife, Masuda

Mohammad Ahsanullah

To my Father, Mother and my Husband Shahbaz

Saman Hanif Shahbaz

To my parents

Bander Al-Zahrani

Preface

Ordered random variables have attracted several researchers due to their applicability in many areas, like extreme values. These variables occur as a natural choice when dealing with extremes like floods, earthquakes, etc. The use of ordered random variables also appears as a natural choice when dealing with records. In this book we have discussed various models of ordered random variables with both theoretical and application points of view. The introductory chapter of the book provides a brief overview of various models which are available to model the ordered data.

In Chap. 2 we have discussed, in detail, the oldest model of ordered data, namely order statistics. We have given the distribution theory of order statistics when sample is available from some distribution function F(x). Some popular results regarding the properties of order statistics have been discussed in this chapter. This chapter also provides a brief about reversed order statistics which is a mirror image of order statistics. We have also discussed recurrence relations for moments of order statistics for various distributions in this chapter.

Chapter 3 of the book is dedicated to another important model of ordered variables, known as record values introduced by Chandler (1952). Record values naturally appear when dealing with records. This chapter discusses in detail the model when we are dealing with larger records and is known as upper record values. The chapter contains distribution theory for this model alongside some other important results. The chapter also presents recurrence relations for moments of record values for some popular probability distributions.

Kamps (1995) introduced a unified model for ordered variables, known as generalized order statistics (GOS). This model contains several models of ordered data as a special case. In Chap. 4, we have discussed, in detail, this unified model of ordered data. This chapter provides a brief about distribution theory of GOS and its special cases. The chapter also contains some important properties of the model, like Markov chains property and recurrence relations for moments of GOS for some selected distributions.

In Chap. 5 the model of reversed order random variables known as dual generalized order statistics (DGOS) is discussed. The model was introduced by Burkschat et al. (2003) as a unified model to study the properties of variables arranged in decreasing order. The model contains reversed order statistics and lower record values as a special case. We have given some important distributional properties for the model in Chap. 5. We have also discussed recurrence relations for moments of DGOS when sample is available from some distribution F(x). The chapter also provides relationship between GOS and DGOS.

Ordered random variables have found tremendous applications in many areas such as estimation and concomitants. Chapter 6 of the book presents some popular uses of ordered random variables. The chapter presents use of ordered random variables in maximum likelihood and Bayesian estimation.

Chapters 7 and 8 of the book present some popular results about probability distributions which are based on ordered random variables. In Chap. 7 we have discussed some important results regarding the characterization of probability distributions based on ordered random variables. We have discussed characterizations of probability distributions based on order statistics, record values, and generalized order statistics. Chapter 8 contains some important results which connect ordered random variables with extreme value distribution. We have discussed the domains of attractions for several random variables for various types of extreme values distributions.

Finally, we would like to thank our colleagues and friends for their support and encouragement during compilations of this book. We would like to thank Prof. Chris Tsokos for valuable suggestions which help in improving the quality of the book. Authors 1 and 3 would like to thank Prof. Muhammad Hanif and Prof. Valary Nevzorov for healthy comments during compilation of this book. Authors 1, 3, and 4 would also like to thank Statistics Department, King Abdulaziz University, for providing excellent support during compilation of the book. Author 2 would like to thank Rider University for their excellent facilities which helped in completing the book.

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Chapter 1
Introduction

1.1 Introduction

Ordered Random Variables arise in several areas of life. We can see the application of ordered random variables in our daily life for example we might be interested in arranging prices of commodities or we may be interested in arranging list of students with respect to their CGPA in final examination. Another use of ordered random variables can be seen in games where; for example; record time in completing a 100 m race is recorded. Several such examples can be listed where ordered random variables are playing their role. The ordered random variables has recently attracted attention of statisticians although their use in statistics is as old as the subject. Some simple statistical measures which are based upon the concept of ordered random variables are the range, the median, the percentiles etc. The ordered random variables are based upon different models depending upon how the ordering is being done. In the following we will briefly discuss some popular models of ordered random variables which will be studied in more details in the coming chapters.

1.2 Models of Ordered Random Variables

Some popular models of ordered random variables are discussed in the following.

1.2.1 Order Statistics

Order Statistics is perhaps the oldest model for ordered random variables. Order Statistics naturally arise in life whenever observations in a sample are arranged in increasing order of magnitude. The order statistics are formally defined as under.

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_1

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from some distribution function $F(x)$. The observations arranged in increasing order of magnitude $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ are called order statistics for a sample of size n. The joint distribution of all order statistics is given in David and Nagaraja (2003) as

$$f_{1,2,\ldots,n:n}(x_1, x_2, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i). \tag{1.1}$$

Order statistics are very useful in studying distribution of maximum, minimum, median etc. for specific probability distributions. We will study order statistics in much detail in Chap. 2. The rth order statistics $X_{r:n}$ can be viewed as life length of $(n - r + 1)$–out–of–n system.

1.2.2 Order Statistics with Non-integral Sample Size

The study of order statistics is based upon size of the available sample and conventionally that sample size is a positive integer. The model of order statistics is easily extended to the case of fractional sample size to give rise to fractional order statistics as defined by Stigler (1977). The conventional order statistics appear as a special case of fractional order statistics. The distribution function of rth fractional Order Statistics based upon the parent distribution $F(x)$ is given as

$$F_{r:\alpha}(x) = \frac{1}{B(\alpha, r)} \int_0^{F(x)} t^{r-1} (1 - t)^{\alpha - r} \, dt. \tag{1.2}$$

If $\alpha = n$ (integer) then we have simple order statistics. The fractional order statistics do not have significant practical applications but they do provide basis for a general class of distributions introduced by Eugene, Lee and Famoye (2002).

1.2.3 Sequential Order Statistics

If we consider simple order statistics as life length of components then we can interpret them as random variables where the probability distribution of components remains same irrespective of the failures. In certain situations the probability distribution of the components changes after each failure and hence such components can not be modeled by using simple order statistics. Sequential Order Statistics provide a method to model the components with different underlying probability distributions after each failure. The Sequential Order Statistics are defined as follow.

Let $\left\{Y_j^{(i)}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n-i+1\right\}$ be independent random variables so that $\left\{Y_j^{(i)}; j = 1, 2, \ldots, n-i+1\right\}$ is distributed as F_i and $F_1, \ldots, F_n$ are strictly increasing. Moreover let $X_j^{(1)} = Y_j^{(1)}$, $X_*^{(1)} = \min\left\{X_1^{(1)}, \ldots, X_n^{(1)}\right\}$ and for $2 \le i \le n$ define

$$X_j^{(i)} = F_i^{-1}\left[F_i\left(Y_j^{(i)}\right)\left\{1 - F\left(X_*^{(i-1)}\right)\right\} + F_i\left(X_*^{(i-1)}\right)\right];$$
$$X_*^{(i)} = \min\left\{X_j^{(i)}, 1 \le j \le n-i+1\right\}$$

then the random variables $X_*^{(1)}, \ldots, X_*^{(n)}$ are called Sequential Order Statistics. The joint density of first r Sequential Order Statistics; $X_*^{(1)}, \ldots, X_*^{(r)}$; is given by Kamps (1995b) as

$$f_{1,2,\ldots,r:n}\left(x_1, x_2, \ldots, x_r\right) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left\{\frac{1 - F_i\left(x_i\right)}{1 - F_i\left(x_{i-1}\right)}\right\}^{n-i} \times \frac{f_i\left(x_i\right)}{1 - F_i\left(x_{i-1}\right)}. \tag{1.3}$$

The Sequential Order Statistics reduces to simple order statistics if all $F_i(x)$ are same.

1.2.4 Record Values

The Record values has emerged as an important model for ordered random variables. The record values appear naturally in real life where one is interested in successive extreme values. For example we might be interested in Olympic record or records in World Cricket Cup. When we are interested in successive maximum observations then records are known as *Upper Records* and when one is interested in successive minimum observations then records are known as *Lower Records*. Chandler (1952) presented the idea of records in context with monitoring of extreme weather conditions. Formally, the record time and upper record values are defined as follows.

Let $\{X_n; n = 1, 2, \ldots\}$ be a sequence of iid random variables with a continuous distribution function F. The random variables

$$L(1) = 1$$
$$L(n+1) = \min\left\{j > L(n); X_j > X_{U(n)}\right\}; n \in \mathbb{N}$$

are called the record time and $X_{U(n)}$ is called Upper Record Values. The joint density of first n upper record values is

$$f_{X_{U(1)},\ldots,X_{U(n)}}(x_1,\ldots,x_r)=\left\{\prod_{i=1}^{n-1}\frac{f(x_i)}{1-F(x_i)}\right\}f(x_n). \tag{1.4}$$

Record values have wide spread applications in reliability theory. We will discuss the upper record values in Chap. 3 and lower record values in Chap. 5.

1.2.5 k–Record Values

The upper record values provide information about largest observation in a sequence of records. Often we are interested in knowing about specific record number. The k–Record values provide basis for studying distributional behavior of such observations. The k–Record values are formally defined by Dziubdziela and Kopocinski (1976) as below.

Let $\{X_n; n=1,2,\ldots\}$ be a sequence of iid random variables with a continuous distribution function F and let k be a positive integer. The random variables $U_K(n)$ defined as $U_K(1)=1$ and

$$U_K(n+1)=\min\left\{r>U_K(n): X_{r:r+k-1}>X_{U_K(n),U_K(n)+k-1}\right\}; n\in\mathbb{N}.$$

where $X_{r:r+k-1}$ is rth order statistics based on a sample of size $r+k-1$; are called the record time and $X_{U_K(n),U_K(n)+k-1}$ is called nth $k-$record values. The joint density of n k–records is

$$f_{U_K(1),\ldots,U_K(n)}(x_1,\ldots,x_r)=k^n\left\{\prod_{i=1}^{n-1}\frac{f(x_i)}{1-F(x_i)}\right\} \times[1-F(x_n)]^{k-1}f(x_n). \tag{1.5}$$

The simple upper record values appear as special case of $k-$record values for $k=1$. The $k-$record values are discussed in Chap. 3.

1.2.6 Pfeifer's Record Values

The upper record values and $k-$upper record values are based upon the assumption that the sequence of random variables $\{X_n; n=1,2,\ldots\}$ have same distributions F. Often this assumption is very unrealistic to be kept intact and a general record model is needed. Pfeifer (1979) proposed a general model for record values when observations in a sequence are independent but are not identically distributed. The Pfeifer's record values are defined as below.

Let $\left\{X_j^{(n)}; n, j \in \mathbb{N}\right\}$ be a double sequence of independent random variables defined on some probability space with

$$P\left(X_j^{(n)}\right) = P\left(X_1^{(n)}\right); n, j \in \mathbb{N}$$

Define the inter record times as

$$\Delta_1 = 1$$
$$\Delta_{n+1} = \min\left\{j \in \mathbb{N}; X_j^{(n+1)} > X_{\Delta_n}^{(n)}\right\}; n \in \mathbb{N}.$$

In this case the random variables $X_{\Delta_n}^{(n)}$ are called Pfeifer's record values.

The joint density function of n Pfeifer's record values is given as

$$f_{\Delta_1^{(1)},\ldots,\Delta_n^{(n)}}(x_1, \ldots, x_r) = \left\{\prod_{i=1}^{n-1} \frac{f_i(x_i)}{1 - F_{i+1}(x_i)}\right\} f_n(x_n); \tag{1.6}$$

where F_i is distribution function of the sequence until occurrence of ith record. If all random variables in the sequence are identically distributed then Pfeifer's record values transformed to simple upper records.

1.2.7 k_n–Records from Non-identical Distributions

The Pfeifer's record values and k–record values can be combined together to give rise to k_n–records from Non-identical distributions. Formally, the k_n–records from Non-identical distributions are defined as below.

Let $\left\{X_j^{(n)}; n, j \in \mathbb{N}\right\}$ be a double sequence of independent random variables defined on some probability space with $P\left(X_j^{(n)}\right) = P\left(X_1^{(n)}\right); n, j \in \mathbb{N}$ and let

$$X_j^{(n)} \sim X_1^{(n)} \sim F_n; n, j \in \mathbb{N}$$

Also let $(k_n; n \in \mathbb{N})$ be a sequence of positive integers. Define inter record times as

$$\Delta_1 = 1;$$
$$\Delta_{n+1} = \min\left\{j \in \mathbb{N}; X_{j:j+k_{n+1}-1}^{(n+1)} > X_{\Delta_n,\Delta_n+k_n-1}^{(n)}\right\}; n \in \mathbb{N};$$

where $X_{j:j+k_{n+1}-1}$ is jth order statistics based on a sample of size $j + k_{n+1} - 1$; then the random variables

$$X_{\Delta_n,\Delta_n+k_n-1}^{(n)} = X_{\Delta_n,k_n}^{(n)}$$

are called k_n–records from Non-identical distributions.

The joint density function of r k_n–records from Non-identical distributions is given as

$$f_{\Delta^{(1)}_{1,k_1},\Delta^{(2)}_{2,k_2},\ldots,\Delta^{(r)}_{r,k_r}}(x_1, x_2, \ldots, x_r) = \left(\prod_{j=1}^{r} k_j\right) \prod_{i=1}^{r} \left[\left\{ \frac{1 - F_i(x_i)}{1 - F_i(x_{i-1})} \right\}^{k_i - 1} \times \left\{ \frac{f_i(x_i)}{1 - F_{i+1}(x_i)} \right\} \right]; \tag{1.7}$$

where F_i is distribution function of the sequence until occurrence of ith record. If $k_n = 1$ for all $n \in \mathbb{N}$ then k_n–records from Non-identical distributions reduces to Pfeifer's record values.

Chapter 2
Order Statistics

2.1 Introduction

Order Statistics naturally appear in real life whenever we need to arrange observations in ascending order; say for example prices arranged from smallest to largest, scores scored by a player in last ten innings from smallest to largest and so on. The study of order statistics needs special considerations due to their natural dependence. The study of order statistics has attracted many statistician in the past. Formerly, order statistics are defined in the following.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $F(x)$ and so all X_i are i.i.d. random variables having same distribution $F(x)$. The arranged sample $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ is called the *Ordered Sample* and the rth observation in the ordered sample; denoted as $X_{r:n}$ or $X_{(r)}$; is called the rth *Order Statistics*. The realized ordered sample is written as $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$. The distribution of rth order statistics and joint distribution of rth and sth order statistics are given below.

2.2 Joint Distribution of Order Statistics

The joint distribution of all order statistics plays an important role in deriving several special distributions of individual and group of order statistics. The joint distribution of all order statistics is easily derived from the marginal distributions of available random variables. We know that if we have a random sample of size n from a distribution function $F(x)$ as $X_1, X_2, \ldots, X_n$ then the joint distribution of all sample observations is

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i);$$

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_2

where $f(x_i)$ is the density function of X_i. Now since all possible ordered permutations of $X_1, X_2, \ldots, X_n$ can be done in $n!$ ways, therefore the joint density function of all order statistics is readily written as

$$f(x_{1:n}, x_{2:n}, \ldots, x_{n:n}) = n! \prod_{i=1}^{n} f(x_i). \tag{2.1}$$

The joint density function of all order statistics given in (2.1) is very useful in deriving the marginal density function of a single and group of order statistics.

The joint density function of all order statistics is useful in deriving the distribution of a set of order statistics. Specifically, the joint distribution of r order statistics; $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{r:n}$; is derived as below

$$\begin{aligned}
f_{1,\ldots r:n}(x_1, \ldots, x_r) &= \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} f(x_{1:n}, x_{2:n}, \ldots, x_{n:n}) \\
&\quad \times dx_{r+1} \cdots dx_n \\
&= \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} n! \prod_{i=1}^{n} f(x_i) dx_{r+1} \cdots dx_n \\
&= n! \prod_{i=1}^{r} f(x_i) \int_{x_r}^{\infty} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} \prod_{i=r+1}^{n} f(x_i) \\
&\quad \times dx_{r+1} \cdots dx_n \\
&= \frac{n!}{(n-r)!} \left[\prod_{i=1}^{r} f(x_i) \right] [1 - F(x_r)]^{n-r}.
\end{aligned} \tag{2.2}$$

Expression (2.2) can be used to obtain the joint marginal distribution of any specific number of order statistics.

The distribution of a single order statistics and joint distribution of two order statistics has found many applications in diverse areas of life. In the following we present the marginal distribution of a single order statistics.

2.3 Marginal Distribution of a Single Order Statistics

The marginal distribution of rth order statistics $X_{r:n}$ can be obtained in different ways. The distribution can be obtained by first obtaining the distribution function of $X_{r:n}$ and then that distribution function can be used to obtain the density function of $X_{r:n}$ as given in Arnold, Balakrishnan and Nagaraja (2008) and David and Nagaraja (2003). We obtain the distribution function of $X_{r:n}$ by first obtaining distribution function of $X_{n:n}$; the largest observation; and $X_{1:n}$; the smallest observation.

The distribution function of $X_{n:n}$ is denoted as $F_{n:n}(x)$ and is given as

$$\begin{aligned} F_{n:n}(x) &= P\{X_{n:n} \leq x\} \\ &= P\{all\ X_i \leq x\} = F^n(x). \end{aligned} \tag{2.3}$$

Again the distribution function of $X_{1:n}$; denoted as $F_{1:n}(x)$; is

$$\begin{aligned} F_{n:n}(x) &= P\{X_{1:n} \leq x\} = 1 - P\{X_{1:n} > x\} \\ &= 1 - P\{all\ X_i > x\} = 1 - [1 - F(x)]^n. \end{aligned} \tag{2.4}$$

Now the distribution function of $X_{r:n}$; the rth order statistics; is denoted as $F_{r:n}(x)$ and is given as

$$\begin{aligned} F_{r:n}(x) &= P\{X_{r:n} \leq x\} \\ &= P\{atleast\ r\ of\ X_i\ are\, less\, than\, or\, equal\, to\, x\} \\ &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}. \end{aligned} \tag{2.5}$$

Now using the relation

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} dt;$$

the distribution function of $X_{r:n}$ is given as

$$\begin{aligned} F_{r:n}(x) &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i} \\ &= \int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} dt \\ &= \int_0^{F(x)} \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} t^{r-1}(1-t)^{n-r} dt \\ &= \frac{1}{B(r, n-r+1)} \int_0^{F(x)} t^{r-1}(1-t)^{n-r} dt \\ &= I_{F(x)}(r, n-r+1); \end{aligned} \tag{2.6}$$

where $I_x(a, b)$ is incomplete Beta Function ratio. From (2.6) we see that the distribution function of $X_{r:n}$ resembles with the distributions proposed by Eugene, Lee and Famoye (2002). Expression (2.6) is valid either if sample has been drawn from a discrete distribution. An alternative form for the distribution function of $X_{r:n}$ is given as

$$\begin{aligned}F_{r:n}(x) &= \sum_{i=r}^{n}\binom{n}{i}F^{i}(x)[1-F(x)]^{n-i}\\ &= \sum_{i=r}^{n}\binom{n}{i}F^{i}(x)\sum_{k=0}^{n-i}(-1)^{k}\binom{n-i}{k}F^{k}(x)\\ &= \sum_{i=r}^{n}\sum_{k=0}^{n-i}(-1)^{k}\binom{n}{i}\binom{n-i}{k}F^{i+k}(x). \qquad (2.7)\end{aligned}$$

Assuming that $X_i's$ are absolutely continuous, the density function of $X_{r:n}$; denoted by $f_{r:n}(x)$; is easily obtained from (2.6) as below

$$\begin{aligned}f_{r:n}(x) &= \frac{d}{dx}F_{r:n}(x)\\ &= \frac{d}{dx}\left[\frac{1}{B(r,n-r+1)}\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt\right]\\ &= \frac{1}{B(r,n-r+1)}\left[\frac{d}{dx}\left\{\int_0^{F(x)} t^{r-1}(1-t)^{n-r}dt\right\}\right]\\ &= \frac{1}{B(r,n-r+1)}f(x)F^{r-1}(x)[1-F(x)]^{n-r}\\ &= \frac{n!}{(r-1)!(n-r)!}f(x)F^{r-1}(x)[1-F(x)]^{n-r}. \qquad (2.8)\end{aligned}$$

The density function of $X_{1:n}$ and $X_{n:n}$ can be immediately written from (2.8) as

$$f_{1:n}(x) = nf(x)[1-F(x)]^{n-1}$$

and

$$f_{n:n}(x) = nf(x)F^{n-1}(x).$$

The distribution of $X_{r:n}$ can also be derived by using the multinominal distribution as under:

Recall that probability mass function of multinomial distribution is

$$P(Y_1=y_1,Y_2=y_2,\ldots,Y_k=y_k) = \frac{n!}{y_1!y_2!\ldots y_k!}p_1^{y_1}p_2^{y_2}\ldots p_k^{y_k};$$

and can be used to compute the probabilities of joint occurrence of events. Now the place of $x_{r;n}$ in ordered sample can be given as

$$\underbrace{x_{1:n}\le x_{2:n}\le\ldots\le x_{r-1:n}}_{\substack{r-1\ observations\\ Event\ 1}}\le\underbrace{x_{r:n}}_{Event\ 2}\le\underbrace{x_{r+1:n}\le\ldots\le x_{n:n}}_{\substack{n-r\ observations\\ Event\ 3}}$$

In the above probability of occurrence of *Event 1* is $F(x)$, that of *Event 2* is $f(x)$ and probability of *Event 3* is $[1 - F(x)]$. Hence the joint occurrence of above three events; which is equal to density of $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x) F^{r-1}(x)[1 - F(x)]^{n-r};$$

which is (2.8). When the density $f(x)$ is symmetrical about μ then the distributions of rth and $(n - r + 1)th$ order statistics are related by relation

$$f_{r:n}(\mu + x) = f_{n-r+1:n}(\mu - x).$$

Above relation is very useful in moment relations of order statistics.

When the sample has been drawn from a discrete distribution with distribution function $F(x)$ then the density of $X_{r:n}$ can be obtained as below

$$\begin{aligned} f_{r:n}(x) &= F_{r:n}(x) - F_{r:n}(x-1) \\ &= I_{F(x)}(r, n-r+1) - I_{F(x-1)}(r, n-r+1) \\ &= P\big\{F(x-1) < T_{r,n-r+1} < F(x)\big\} \\ &= \frac{1}{B(r, n-r+1)} \int_{F(x-1)}^{F(x)} u^{r-1}(1-u)^{n-r} du. \end{aligned} \tag{2.9}$$

Expression (2.9) is the probability mass function of $X_{r:n}$ when sample is available from a discrete distribution. The probability mass function of $X_{r:n}$ can also be written in binomial sum as under

$$\begin{aligned} f_{r:n}(x) &= F_{r:n}(x) - F_{r:n}(x-1) \\ &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i} \\ &\quad - \sum_{i=r}^{n} \binom{n}{i} F^i(x-1)[1 - F(x-1)]^{n-i} \\ &= \sum_{i=r}^{n} \binom{n}{i} \big\{F^i(x)[1 - F(x)]^{n-i} \\ &\quad - F^i(x-1)[1 - F(x-1)]^{n-i}\big\}. \end{aligned} \tag{2.10}$$

We now obtain the joint distribution of two ordered observations, namely $X_{r:n}$ and $X_{s:n}$ for $r \leq s$; in the following.

2.4 Joint Distribution of Two Order Statistics

Suppose we have random sample of size n from $F(x)$ and observations are arranged as $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. The joint distribution function of $X_{r:n}$ and $X_{s:n}$ for $r \leq s$ is given by Arnold et al. (2008) as

$$\begin{aligned}
F_{r,s:n}(x_r, x_s) &= P(X_{r:n} \leq x_r, X_{s:n} \leq x_s) \\
&= P(atleast\ r\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_r \\
&\quad and\ atleast\ s\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_s) \\
&= \sum_{j=s}^{n} \sum_{i=r}^{s} P(Exactly\ r\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_r \\
&\quad and\ exactly\ s\ of\ X_i\ are\ less\ than\ or\ equal\ to\ x_s) \\
&= \sum_{j=s}^{n} \sum_{i=r}^{s} \frac{n!}{i!(j-i)!(n-j)!} F^i(x_r) \\
&\quad \times [F(x_s) - F(x_r)]^{j-i}[1 - F(x_s)]^{n-j}.
\end{aligned}$$

Now using the relation

$$\begin{aligned}
&\sum_{j=s}^{n} \sum_{i=r}^{s} \frac{n!}{i!(j-i)!(n-j)!} p_1^i (p_2 - p_1)^{j-i} (1 - p_2)^{n-j} \\
&= \int_0^{p_1} \int_{t_1}^{p_2} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} t_1^{r-1} (t_2 - t_1)^{s-r-1} (1 - t_2)^{n-s} dt_2 dt_1;
\end{aligned}$$

we can write the joint distribution function of two order statistics as

$$\begin{aligned}
F_{r,s:n}(x_r, x_s) &= \int_0^{F(x_r)} \int_{t_1}^{F(x_s)} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\
&\quad \times t_1^{r-1} (t_2 - t_1)^{s-r-1} (1 - t_2)^{n-s} dt_2 dt_1 \qquad (2.11) \\
&; -\infty < x_r < x_s < \infty;
\end{aligned}$$

which is incomplete bivariate beta function ratio. Expression (2.11) holds for both discrete and continuous random variables. When $F(x)$ is absolutely continuous then density function of $X_{r:n}$ and $X_{s:n}$ can be obtained from (2.11) and is given as

$$\begin{aligned}
f_{r,s:n}(x_r, x_s) &= \frac{d^2}{dx_r dx_s} F_{r,s:n}(x_r, x_s) \\
&= \frac{d^2}{dx_r dx_s} \Big[\int_0^{F(x_r)} \int_{t_1}^{F(x_s)} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\
&\quad t_1^{r-1} (t_2 - t_1)^{s-r-1} (1 - t_2)^{n-s} dt_2 dt_1 \Big]
\end{aligned}$$

or

$$f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \times \frac{d^2}{dx_r dx_s}\left[\int_0^{F(x_r)} \int_{t_1}^{F(x_s)} t_1^{r-1}(t_2-t_1)^{s-r-1}(1-t_2)^{n-s} dt_2 dt_1\right]$$
$$= \frac{1}{B(r, s-r, n-s+1)} f(x_r) f(x_s) F^{r-1}(x_r) \times \left[F(x_s) - F(x_r)\right]^{s-r-1} [1-F(x_s)]^{n-s}$$

or

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r) f(x_s) F^{r-1}(x_r)[F(x_s) - F(x_r)]^{s-r-1} \times [1-F(x_s)]^{n-s}, \; -\infty < x_r < x_s < \infty, \tag{2.12}$$

where $C_{r,s,n} = [B((r, s-r, n-s+1))]^{-1} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

The joint probability mass function $P\{X_{r:n} = x; X_{s:n} = y\}$ of $X_{r:n}$ and $X_{s:n}$ can be obtained by using the fact that

$$f_{r,s:n}(x, y) = F_{r,s:n}(x, y) - F_{r,s:n}(x-1, y) - F_{r,s:n}(x, y-1) + F_{r,s:n}(x-1, y-1)$$
$$= P\{F(x-1) < T_r \le F(x), F(y-1) < T_s \le F(y)\}$$
$$= C_{r,s,n} \int\int_B v^{r-1}(w-v)^{s-r-1}(1-w)^{n-s} dv dw \; ; \tag{2.13}$$

where integration is over the region

$$\{(v, w) : v \le w, F(x-1) \le v \le F(x), F(y-1) \le w \le F(y)\}\cdot$$

Consider the joint distribution of two order statistics as

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r) f(x_s) F^{r-1}(x_r)\left[F(x_s) - F(x_r)\right]^{s-r-1} \times [1-F(x_s)]^{n-s};$$

where $C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\cdot$ Using $r = 1$ and $s = n$ the joint density of smallest and largest observation is readily written as

$$f_{1,n:n}(x_1, x_n) = n(n-1) f(x_1) f(x_n)[F(x_n) - F(x_1)]^{n-2}. \tag{2.14}$$

Further, for $s = r + 1$ the joint distribution of two contiguous order statistics is

$$f_{r,r+1:n}(x_r, x_{r+1}) = \frac{n!}{(r-1)!(n-r-1)!} f(x_r) f(x_{r+1}) F^{r-1}(x_r) \times \left[1 - F(x_{r+1})\right]^{n-r-1}. \tag{2.15}$$

Analogously, the joint distribution of any k order statistics $X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n}$; for $x_1 \le x_2 \le \cdots \le x_k$; is

$$f_{r_1,r_2,\ldots,r_k:n}(x_1, x_2, \cdots, x_k) = n! \prod_{j=0}^{k} \left\{ \frac{\left[F(x_{r+1}) - F(x_r)\right]^{r_{j+1}-r_j-1}}{\left(r_{j+1} - r_j - 1\right)!} \right\} \times \left\{ \prod_{j=1}^{k} f\left(x_j\right) \right\}. \tag{2.16}$$

where $x_0 = -\infty, x_{n+1} = +\infty, r_0 = 0$ and $r_{n+1} = n + 1$. Expression (2.16) can be used to obtain joint distribution of any number of ordered observations.

Example 2.1 A random sample is drawn from Uniform distribution over the interval [0, 1]. Obtain distribution of rth order statistics and joint distribution of two order statistics.

Solution: The density and distribution function of $U(0, 1)$ are

$$f(u) = 1; \;\; F(u) = u.$$

The distribution of rth order statistics is

$$f_{r:n}(u) = \frac{1}{B(r, n-r+1)} f(u) F^{r-1}(u)[1 - F(u)]^{n-r} = \frac{1}{B(r, n-r+1)} u^{r-1}(1-u)^{n-r};$$

which is a Beta random variable with parameters r and $n - r + 1$. Again the joint distribution of two order statistics is

$$f_{r,s:n}(u_r, u_s) = \frac{1}{B(r, s-r, n-s+1)} f(u_r) f(u_s) F^{r-1}(u_r) \times \left[F(u_s) - F(u_r)\right]^{s-r-1} [1 - F(u_s)]^{n-s} = \frac{1}{B(r, s-r, n-s+1)} u_r^{r-1}(u_s - u_r)^{s-r-1}(1-u_s)^{n-s}.$$

The joint distribution of largest and smallest observation is immediately written as

$$f_{r,s:n}(u_r, u_s) = n(n-1)(u_n - u_1)^{n-2}.$$

Example 2.2 A random sample of size n is drawn from the standard power function distribution with density

$$f(x) = vx^{v-1}; 0 < x < 1, v > 0.$$

Obtain the distribution of rth order statistics and joint distribution of rth and sth statistics.

Solution: For given distribution we have

$$F(x) = \int_0^x f(t)dt = \int_0^x vt^{v-1}dt = x^v; 0 < x < 1.$$

Now distribution of rth order statistics is

$$\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r, n-r+1)} f(x) F^{r-1}(x)[1 - F(x)]^{n-r} \\ &= \frac{1}{B(r, n-r+1)} vx^{rv-1}(1 - x^v)^{n-r}. \end{aligned}$$

The distribution function of rth order statistics is readily written as

$$\begin{aligned} F_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \int_0^{F(x)} t^{r-1}(1-t)^{n-r} dt \\ &= \frac{1}{B(r, n-r+1)} \int_0^{x^v} t^{r-1}(1-t)^{n-r} dt \\ &= \sum_{i=r}^{n} \binom{n}{i} x^{iv}(1 - x^v)^{n-i}. \end{aligned}$$

The joint distribution of $X_{r:n}$ and $X_{s:n}$ is

$$\begin{aligned} f_{r,s:n}(x_r, x_s) &= \frac{1}{B(r, s-r, n-s+1)} f(x_r) f(x_s) F^{r-1}(x_r) \\ &\quad \times \left[F(x_s) - F(x_r) \right]^{s-r-1} [1 - F(x_s)]^{n-s} \\ &= C_{r,s,n} v^2 x_r^{rv-1} x_s^{v-1} \left(x_s^v - x_r^v\right)^{s-r-1} \left(1 - x_s^v\right)^{n-s}. \end{aligned}$$

where $C_{r,s,n} = [B(r, s-r, n-s+1)]^{-1} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

2.5 Distribution of Range and Other Measures

Suppose a random sample of size n is available from $F(x)$ and let $X_{r:n}$ be the rth order statistics. Further let $X_{s:n}$ be sth order statistics with $r < s$. The joint density function of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} f(x_r) f(x_s) F^{r-1}(x_r)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \times [1 - F(x_s)]^{n-s}.$$

Using above we can obtain the density of $W_{rs} = X_{s:n} - X_{r:n}$ by making the transformation $w_{rs} = x_s - x_r$. The joint density of w_{rs} and x_r in this case is

$$f_{W_{rs}}(w_{rs}) = C_{r,s,n} f(x_r) f(x_r + w_{rs}) F^{r-1}(x_r) \times [F(x_r + w_{rs}) - F(x_r)]^{s-r-1}[1 - F(x_r + w_{rs})]^{n-s}.$$

The marginal density of w_{rs} is

$$f_{W_{rs}}(w_{rs}) = C_{r,s,n} \int_{-\infty}^{\infty} f(x_r) f(x_r + w_{rs}) F^{r-1}(x_r) \times [F(x_r + w_{rs}) - F(x_r)]^{s-r-1}[1 - F(x_r + w_{rs})]^{n-s} dx_r.$$

When $r = 1$ and $s = n$ then above result provide the density function of *Range* (w) in a sample of size n and is given as

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} f(x_r) f(x_r + w)[F(x_r + w) - F(x_r)]^{n-2} dx_r. \quad (2.17)$$

The distribution function of sample range can be easily obtained from (2.17) and is

$$\begin{aligned} F_W(w) &= n \int_{-\infty}^{\infty} f(x_r) \int_{0}^{w} (n-1) f\big(x_r + w^{/}\big) \\ &\quad \times \big[F\big(x_r + w^{/}\big) - F(x)\big]^{n-2} dw^{/} dx_r \\ &= n \int_{-\infty}^{\infty} f(x_r) \big[F\big(x_r + w^{/}\big) - F(x_r)\big]^{n-1} \Big|_{w^{/}=0}^{w^{/}=w} dx_r \\ &= n \int_{-\infty}^{\infty} f(x_r)[F(x_r + w) - F(x_r)]^{n-1} dx_r. \end{aligned} \quad (2.18)$$

Again suppose that number of observations in sample are even; say $n = 2m$; then we know that the sample median is

$$\widetilde{X} = \frac{1}{2}\big[X_{m:n} + X_{m+1:n}\big].$$

The distribution of median can be obtained by using joint distribution of two contiguous order statistics and is given as

$$f_{m,m+1:n}(x_m, x_{m+1}) = C^{/}_{m,n} f(x_m) f(x_{m+1}) F^{m-1}(x_m)\big[1 - F(x_{m+1})\big]^{m-1};$$

where $C^{/}_{m,n} = \frac{n!}{[(m-1)!]^2}$. Now making the transformation $\widetilde{x} = \frac{1}{2}\big[x_m + x_{m+1}\big]$ and $y = x_m$ the jaccobian of transformation is 2 and hence the joint density of $\widetilde{x}$ and y is

$$f_{\widetilde{X}Y}(\widetilde{x}, y) = 2C^{/}_{m,n} f(y) f(2\widetilde{x} - y) F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1}.$$

The marginal density of sample median is, therefore

$$f_{\widetilde{X}}(\widetilde{x}) = 2C^{/}_{m,n} \int_{-\infty}^{\widetilde{x}} f(y) f(2\widetilde{x} - y) F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1} dy \qquad (2.19)$$

The density of sample median for an odd sample size; say $n = 2m + 1$; is simply the density of mth order statistics for a sample of size $2m + 1$.

Example 2.3 Obtain the density function of sample range for a sample of size n from uniform distribution with density

$$f(x) = 1; 0 < x < 1.$$

Solution: The distribution of sample range for a sample of size n from distribution $F(x)$ is

$$f_W(w) = n(n-1) \int_{-\infty}^{\infty} f(x_r) f(x_r + w)[F(x_r + w) - F(x_r)]^{n-2} dx_r.$$

Now for uniform distribution we have

$$f(x) = 1;\; F(x) = x.$$

So

$$f(x_r + w) = 1;\; F(x_r + w) = (x_r + w),$$

hence the density function of range is

$$\begin{aligned} f_W(w) &= n(n-1) \int_0^{1-w} [(x_r + w) - x_r]^{n-2} dx_r \\ &= n(n-1) \int_0^{1-w} w^{n-2} dx_r \\ &= n(n-1) w^{n-2}(1 - w); 0 < w < 1. \end{aligned}$$

Example 2.4 Obtain the density function of range in a sample of size 3 from exponential distribution with density

$$f(x) = e^{-x}; x > 0.$$

Solution: The distribution of sample range for a sample of size n from distribution $F(x)$ is

$$f_W(w) = n(n-1)\int_{-\infty}^{\infty} f(x_r)f(x_r+w)[F(x_r+w) - F(x_r)]^{n-2}dx_r.$$

which for $n = 3$ becomes

$$f_W(w) = 6\int_{-\infty}^{\infty} f(x_r)f(x_r+w)[F(x_r+w) - F(x_r)]dx_r.$$

Now for exponential distribution we have

$$f(x) = e^{-x} \text{ and } F(x) = 1 - e^{-x},$$

hence

$$f(x_r) = e^{-x_r} \text{ and } F(x_r) = 1 - e^{-x_r},$$
$$f(w + x_r) = e^{-(w+x_r)} \text{ and } F(w + x_r) = 1 - e^{-(w+x_r)}.$$

Using these in above expression we have

$$\begin{aligned} f_W(w) &= 6\int_0^{\infty} e^{-x_r}e^{-(w+x_r)}\left[e^{-x_r} - e^{-(w+x_r)}\right]dx_r \\ &= 6e^{-w}\left(1 - e^{-w}\right)\int_0^{\infty} e^{-3x_r}dx_r \\ &= 2e^{-w}\left(1 - e^{-w}\right); w > 0, \end{aligned}$$

as required density function of range.

Example 2.5 Obtain the density function of median in a sample of size $n = 2m$ from exponential distribution with density function

$$f(x) = e^{-x}; x > 0.$$

Solution: The distribution of sample median for a sample of size $n = 2m$ from distribution $F(x)$ is

$$f_{\widetilde{X}}(\widetilde{x}) = 2C^{/}_{m,n}\int_{-\infty}^{\widetilde{x}} f(y)f(2\widetilde{x} - y)F^{m-1}(y)[1 - F(2\widetilde{x} - y)]^{m-1}dy;$$

where $C^{/}_{m,n} = \frac{n!}{[(m-1)!]^2}$. For the given distribution we have

$$f(y) = e^{-y};\ F(y) = 1 - e^{-y}$$
$$f(2\widetilde{x} - y) = e^{-(2\widetilde{x}-y)};\ F(2\widetilde{x} - y) = 1 - e^{-(2\widetilde{x}-y)}.$$

Substituting these values in above equation we have

$$\begin{aligned}
f_{\widetilde{X}}(\widetilde{x}) &= 2C^{/}_{m,n} \int_0^{\widetilde{x}} e^{-y} e^{-(2\widetilde{x}-y)} \left(1 - e^{-y}\right)^m \\
&\quad \times \left[e^{-(2\widetilde{x}-y)}\right]^{m-1} dy \\
&= 2C^{/}_{m,n} e^{-2\widetilde{x}} e^{-2(m-1)\widetilde{x}} \int_0^{\widetilde{x}} e^{-2y} \left(1 - e^{-y}\right)^m \\
&\quad \times e^{(m-1)y} dy \\
&= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \int_0^{\widetilde{x}} e^{-(3-m)y} \left(1 - e^{-y}\right)^m dy.
\end{aligned}$$

Now expanding $\left(1 - e^{-y}\right)^m$ we have

$$\begin{aligned}
f_{\widetilde{X}}(\widetilde{x}) &= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \int_0^{\widetilde{x}} e^{-(3-m)y} \sum_{h=0}^{m} (-1)^h \binom{m}{h} e^{-hy} dy \\
&= 2C^{/}_{m,n} e^{-2m\widetilde{x}} \sum_{h=0}^{m} (-1)^h \binom{m}{h} \int_0^{\widetilde{x}} e^{-[(3-m)+h]y} dy \\
&= \frac{2C^{/}_{m,n}}{(3-m)+h} \sum_{h=0}^{m} (-1)^h \binom{m}{h} e^{-2m\widetilde{x}} \left(1 - e^{-[(3-m)+h]\widetilde{x}}\right),
\end{aligned}$$

as required density of median.

2.6 Conditional Distributions of Order Statistics

The conditional distribution plays very important role in studying behavior of a random variable when information about some other variable(s) is available. The study of conditional distributions is easily extended in the case of order statistics. The conditional distributions of order statistics provide certain additional information about them and we present these conditional distributions in the following theorems as discussed in Arnold et al. (2008).

We know that when we have a bivariate distribution; say $f(x, y)$; of two random variables X and Y, then the conditional distribution of random variable Y given X is given as

$$f(y|x) = \frac{f(x, y)}{f_2(x)};$$

where $f_2(x)$ is the marginal distribution of X. Analogously, the conditional distributions in case of order statistics can be easily defined; say for example the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is defined as

$$f(x_s|x_r) = \frac{f_{r,s:n}(x_r, x_s)}{f_{r:n}(x_r)},$$

where $f_{r,s:n}(x_r, x_s)$ is joint distribution of $X_{r:n}$ and $X_{s:n}$ and $f_{r:n}(x)$ is marginal distribution of $X_{r:n}$. The conditional distributions of order statistics are discussed in the following.

Theorem 2.1 *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$; for $r < s$; is same as the distribution of $(s - r)th$ order statistics from a sample of size $(n - r)$ from a distribution $F(x)$ which is truncated on the left at x_r.*

Proof The marginal distribution of $X_{r:n}$ and the joint distribution of $X_{r:n}$ and $X_{s:n}$ are given in (2.8) and (2.12) as

$$f_{r:n}(x_r) = \frac{n!}{(r-1)!(n-r)!} f(x_r)\Big[F(x)\Big]^{r-1}\Big[1 - F(x_r)\Big]^{n-r};$$

and

$$f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r) f(x_s)\Big[F(x_r)\Big]^{r-1} \times \Big[F(x_s) - F(x_r)\Big]^{s-r-1}[1 - F(x_s)]^{n-s}.$$

Now the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$\begin{aligned} f(x_s|x_r) &= \frac{f_{r,s:n}(x_r, x_s)}{f_{r:n}(x_r)} \\ &= \Bigg\{\frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r) f(x_s)\Big[F(x_r)\Big]^{r-1} \\ &\quad \times [F(x_s) - F(x_r)]^{s-r-1}[1 - F(x_s)]^{n-s}\Bigg\}/ \\ &\quad \frac{n!}{(r-1)!(n-r)!} f(x_r)\Big[F(x_r)\Big]^{r-1}[1 - F(x_r)]^{n-r} \end{aligned}$$

or

$$
\begin{aligned}
f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!} f(x_s)[1-F(x_s)]^{n-s} \\
&\quad \times \frac{[F(x_s)-F(x_r)]^{s-r-1}}{[1-F(x_r)]^{n-r}} \\
&= \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{f(x_s)}{1-F(x_r)} \\
&\quad \times \left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1} \left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s}. \qquad (2.20)
\end{aligned}
$$

Noting that $\frac{f(x_s)}{\{1-F(x_r)\}}$ and $\frac{\{F(x_s)-F(x_r)\}}{\{1-F(x_r)\}}$ are respectively the density and distribution function of a random variable whose distribution is truncated at left of x_r completes the proof.

Theorem 2.2 *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$; for $r < s$; is same as the distribution of rth order statistics from a sample of size $(s-1)$ from a distribution $F(x)$ which is truncated on the right at x_s.*

Proof The marginal distribution of $X_{r:n}$ and the joint distribution of $X_{r:n}$ and $X_{s:n}$ are given in (2.8) and (2.12). Now the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$ is

$$
\begin{aligned}
f(x_r|x_s) &= \frac{f_{r,s:n}(x_r,x_s)}{f_{s:n}(x_s)} \\
&= \Bigg\{\frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_r)f(x_s)\Big[F(x_r)\Big]^{r-1} \\
&\quad \times [F(x_s)-F(x_r)]^{s-r-1}[1-F(x_s)]^{n-s}\Bigg\}/ \\
&\quad \frac{n!}{(s-1)!(n-s)!} f(x_s)[F(x_s)]^{s-1}[1-F(x_s)]^{n-s}
\end{aligned}
$$

or

$$
\begin{aligned}
f(x_r|x_s) &= \frac{(s-1)!}{(r-1)!(s-r-1)!} f(x_r)\Big[F(x_r)\Big]^{r-1} \\
&\quad \times \Big[F(x_s)-F(x_r)\Big]^{s-r-1} \frac{1}{[F(x_s)]^{s-1}} \\
&= \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{f(x_r)}{F(x_s)} \left[\frac{F(x_r)}{F(x_s)}\right]^{r-1} \\
&\quad \times \left[1-\frac{F(x_r)}{F(x_s)}\right]^{s-r-1}. \qquad (2.21)
\end{aligned}
$$

Proof immediately follows by noting that $f(x_r)/F(x_s)$ and $F(x_r)/F(x_s)$ are respectively the density and distribution function of a random variable whose distribution is truncated at right of x_s.

Theorem 2.3 *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$; for $r < s < t$; is same as the distribution of $(s-r)th$ order statistics for a sample of size $(t-r-1)$ from a distribution $F(x)$ which is doubly truncated on the left at x_r and on the right at x_t.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$\begin{aligned} f_{r,s,t:n}(x_r, x_s, x_t) = {} & \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r) f(x_s) f(x_t) \\ & \times \Big[F(x_r)\Big]^{r-1} \Big[F(x_s) - F(x_r)\Big]^{s-r-1} \Big[F(x_t) - F(x_s)\Big]^{t-s-1} \\ & \times \Big[1 - F(x_t)\Big]^{n-t}. \end{aligned}$$

Also the joint distribution of $X_{r:n}$ and $X_{t:n}$ is

$$\begin{aligned} f_{r,t:n}(x_r, x_t) = {} & \frac{n!}{(r-1)!(t-r-1)!(n-t)!} f(x_r) f(x_t) \Big[F(x_r)\Big]^{r-1} \\ & \times [F(x_t) - F(x_r)]^{t-r-1} [1 - F(x_t)]^{n-t}. \end{aligned}$$

Now the conditional distribution of $X_{s:n}$ given $X_{r:n}$ and $X_{t:n}$ is

$$\begin{aligned} f(x_s|x_r, x_t) = {} & \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{r,t:n}(x_r, x_t)} \\ = {} & \Big\{ \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} \\ & \times f(x_r) f(x_s) f(x_t) \Big[F(x_r)\Big]^{r-1} \Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\ & \times [F(x_t) - F(x_s)]^{t-s-1} [1 - F(x_t)]^{n-t} \Big\} / \\ & \Big\{ \frac{n!}{(r-1)!(t-r-1)!(n-t)!} f(x_r) f(x_t) [F(x_r)]^{r-1} \\ & \times [F(x_t) - F(x_r)]^{t-r-1} [1 - F(x_t)]^{n-t} \Big\} \end{aligned}$$

or

$$
\begin{aligned}
f(x_s|x_r, x_t) &= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)!} f(x_s)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \times \Big[F(x_t) - F(x_s)\Big]^{t-s-1} \frac{1}{\Big[F(x_t) - F(x_r)\Big]^{t-r-1}} \\
&= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)!} \frac{f(x_s)}{F(x_t) - F(x_r)} \\
&\quad \times \left[\frac{F(x_s) - F(x_r)}{F(x_t) - F(x_r)}\right]^{s-r-1} \times \left[\frac{F(x_t) - F(x_s)}{F(x_t) - F(x_r)}\right]^{t-s-1}. \qquad (2.22)
\end{aligned}
$$

Proof immediately follows by noting that $\frac{f(x_s)}{F(x_t)-F(x_r)}$ and $\frac{F(x_s)-F(x_r)}{F(x_t)-F(x_r)}$ are respectively the density and distribution function of a random variable whose distribution is doubly truncated from left at x_r and at the right of x_s.

Theorem 2.4 *Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{r:n}$ and $X_{s:n}$ given $X_{t:n} = x_t$; for $r < s < t$; is same as the joint distribution of rth and sth order statistics for a sample of size $(t-1)$ from a distribution $F(x)$ which is truncated on the right at x_t.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$
\begin{aligned}
f_{r,s,t:n}(x_r, x_s, x_t) &= \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r)f(x_s)f(x_t) \\
&\quad \times \Big[F(x_r)\Big]^{r-1}\Big[F(x_s) - F(x_r)\Big]^{s-r-1}\Big[F(x_t) - F(x_s)\Big]^{t-s-1} \\
&\quad \times [1 - F(x_t)]^{n-t}.
\end{aligned}
$$

Also the marginal distribution of $X_{t:n}$ is

$$
f_{t:n}(x_t) = \frac{n!}{(t-1)!(n-t)!} f(x_t)[F(x_t)]^{t-1}[1 - F(x_t)]^{n-t}.
$$

Now the conditional distribution of $X_{r:n}$ and $X_{s:n}$ given $X_{t:n}$ is

$$
\begin{aligned}
f(x_r, x_s|x_t) &= \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{t:n}(x_t)} \\
&= \left\{\frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!}\right. \\
&\quad \times f(x_r)f(x_s)f(x_t)[F(x_r)]^{r-1}\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \times [F(x_t) - F(x_s)]^{t-s-1}[1 - F(x_t)]^{n-t}\Big\}/ \\
&\quad \left\{\frac{n!}{(t-1)!(n-t)!} f(x_t)[F(x_t)]^{t-1}\right. \\
&\quad \times [1 - F(x_t)]^{n-t}\Big\}
\end{aligned}
$$

or

$$
\begin{aligned}
f(x_r, x_s|x_t) &= \frac{(t-1)!}{(r-1)!(s-r-1)!(t-s-1)!} f(x_r) f(x_s) [F(x_r)]^{r-1} \\
&\quad \times \left[F(x_s) - F(x_r) \right]^{s-r-1} \frac{\left[F(x_t) - F(x_s) \right]^{t-s-1}}{[F(x_t)]^{t-1}} \\
&= \frac{(t-1)!}{(r-1)!(s-r-1)!(t-s-1)!} \frac{f(x_r)}{F(x_t)} \frac{f(x_s)}{F(x_t)} \\
&\quad \times \left[\frac{F(x_r)}{F(x_t)} \right]^{r-1} \left[\frac{F(x_s)}{F(x_t)} - \frac{F(x_r)}{F(x_t)} \right]^{s-r-1} \\
&\quad \times \left[1 - \frac{F(x_s)}{F(x_t)} \right]^{t-s-1}. \qquad (2.23)
\end{aligned}
$$

The proof is complete by noting that $f(x_r)/F(x_t)$ and $F(x_r)/F(x_t)$ are respectively the density and distribution function of a random variable whose distribution is truncated at the right at x_t.

Theorem 2.5 *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be order statistics for a sample of size n from an absolutely continuous distribution $F(x)$ then the conditional distribution of $X_{s:n}$ and $X_{t:n}$ given $X_{r:n} = x_r$; for $r < s < t$; is same as the joint distribution of $(s-r)th$ and $(t-r)th$ order statistics for a sample of size $(n-r)$ from a distribution $F(x)$ which is truncated on the left at x_r.*

Proof The joint distribution of $X_{r:n}$, $X_{s:n}$ and $X_{t:n}$ is obtained from (2.16) as

$$
\begin{aligned}
f_{r,s,t:n}(x_r, x_s, x_t) &= \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} f(x_r) f(x_s) f(x_t) \\
&\quad \times [F(x_r)]^{r-1} \left[F(x_s) - F(x_r) \right]^{s-r-1} \left[F(x_t) - F(x_s) \right]^{t-s-1} \\
&\quad \times [1 - F(x_t)]^{n-t}.
\end{aligned}
$$

Also the marginal distribution of $X_{r:n}$ is

$$
f_{r:n}(x_r) = \frac{n!}{(r-1)!(n-r)!} f(x_r) [F(x_r)]^{r-1} [1 - F(x_r)]^{n-r}.
$$

Now the conditional distribution of $X_{s:n}$ and $X_{t:n}$ given $X_{r:n}$ is

$$
\begin{aligned}
f(x_s, x_t|x_r) &= \frac{f_{r,s,t:n}(x_r, x_s, x_t)}{f_{r:n}(x_t)} \\
&= \Bigg\{ \frac{n!}{(r-1)!(s-r-1)!(t-s-1)!(n-t)!} \\
&\quad \times f(x_r) f(x_s) f(x_t) [F(x_r)]^{r-1} \left[F(x_s) - F(x_r) \right]^{s-r-1}
\end{aligned}
$$

$$\times [F(x_t) - F(x_s)]^{t-s-1}[1 - F(x_t)]^{n-t}\Big\}/$$
$$\left\{\frac{n!}{(r-1)!(n-r)!} f(x_r)[F(x_r)]^{r-1} \times [1 - F(x_r)]^{n-r}\right\}$$

or

$$\begin{aligned}
f(x_s, x_t|x_r) &= \frac{(n-r)!}{(s-r-1)!(t-s-1)!(n-t)!} f(x_s) f(x_t) \\
&\quad \times \Big[F(x_s) - F(x_r)\Big]^{s-r-1} [F(x_t) - F(x_s)]^{t-s-1} \\
&\quad \times [1 - F(x_t)]^{n-t} \frac{1}{[1 - F(x_r)]^{n-r}} \\
&= \frac{(n-r)!}{(s-r-1)!(t-s-1)!(n-t)!} \frac{f(x_s)}{1 - F(x_r)} \\
&\quad \times \frac{f(x_t)}{1 - F(x_r)} \left[\frac{F(x_s) - F(x_r)}{1 - F(x_r)}\right]^{s-r-1} \\
&\quad \times \left[\frac{F(x_t) - F(x_s)}{1 - F(x_r)}\right]^{t-s-1} \left[\frac{1 - F(x_t)}{1 - F(x_r)}\right]^{n-t}. \qquad (2.24)
\end{aligned}$$

The proof is complete by noting that $\frac{f(x_s)}{1-F(x_r)}$ and $\frac{F(x_s)-F(x_r)}{1-F(x_r)}$ are respectively the density and distribution function of a random variable whose distribution is truncated at the left of x_r.

Theorems 2.1 to 2.5 provide some interesting results about the conditional distributions of order statistics from a distribution $F(x)$. From all these theorems we can see that the conditional distributions in order statistics are simply the marginal and joint distributions of corresponding order statistics obtained from the truncated parent distribution and appropriately modified sample size.

Example 2.6 A random sample of size n is drawn from the Weibull distribution with density function

$$f(x) = \alpha x^{\alpha-1} \exp(-x^{\alpha}); x, \alpha > 0.$$

Obtain the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$.

Solution: The conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is given as

$$\begin{aligned}
f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{f(x_s)}{1 - F(x_r)} \\
&\quad \times \left[\frac{F(x_s) - F(x_r)}{1 - F(x_r)}\right]^{s-r-1} \left[\frac{1 - F(x_s)}{1 - F(x_r)}\right]^{n-s}.
\end{aligned}$$

Also the conditional distribution of $X_{r:n}$ given $X_{s:n} = x_s$ is

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{f(x_r)}{F(x_s)}\left[\frac{F(x_r)}{F(x_s)}\right]^{r-1} \times \left[1-\frac{F(x_r)}{F(x_s)}\right]^{s-r-1}.$$

Now for the given distribution we have

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha t^{\alpha-1}\exp(-t^\alpha)dt = 1-\exp(-x^\alpha);\ x,\alpha > 0.$$

So, the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$f(x_s|x_r) = \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{\alpha x_s^{\alpha-1}e^{-x_s^\alpha}}{e^{-x_r^\alpha}} \times \left(\frac{e^{-x_r^\alpha}-e^{-x_s^\alpha}}{e^{-x_r^\alpha}}\right)^{s-r-1}\left(\frac{e^{-x_s^\alpha}}{e^{-x_r^\alpha}}\right)^{n-s}.$$

$$= \frac{(n-r)!\alpha x_s^{\alpha-1}e^{-(n-s+1)x_s^\alpha}}{(s-r-1)!(n-s)!}\frac{\left(e^{-x_r^\alpha}-e^{-x_s^\alpha}\right)^{s-r-1}}{e^{-(n-r)x_r^\alpha}}.$$

Again, the conditional distribution of $X_{r:n}$ given $X_{s:n}$ is

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{\alpha x_r^{\alpha-1}e^{-x_r^\alpha}}{1-e^{-x_s^\alpha}}\left[\frac{1-e^{-x_r^\alpha}}{1-e^{-x_s^\alpha}}\right]^{r-1} \times \left[1-\frac{1-e^{-x_r^\alpha}}{1-e^{-x_s^\alpha}}\right]^{s-r-1},$$

or

$$f(x_r|x_s) = \frac{(s-1)!}{(r-1)!(s-r-1)!}\frac{\alpha x_r^{\alpha-1}e^{-x_r^\alpha}}{\left(1-e^{-x_s^\alpha}\right)^{s-1}} \times \left(1-e^{-x_r^\alpha}\right)^{r-1}\left(e^{-x_r^\alpha}-e^{-x_s^\alpha}\right)^{s-r-1},$$

Above conditional distributions can also be derived from the parent truncated distribution.

2.7 Order Statistics as Markov Chain

In the previous section we have presented the conditional distributions of order statistics. The conditional distributions of order statistics enable us to study their additional behavior. One of the popular property which is based upon the conditional distributions of order statistics is that the order statistics follows the Markov chain. We prove this property of order statistics in the following.

We know that a sequence of random variables $X_1, X_2, \ldots, X_r, X_s$ has Markov chain property if the conditional distribution of X_s given $X_1 = x_1, X_2 = x_2, \ldots,$ $X_r = x_r$ is same as the conditional distribution of X_s given $X_r = x_r$, that is if

$$f(x_s|X_1 = x_1, \ldots, X_r = x_r) = f(x_s|X_r = x_r).$$

Now to show that the order statistics follow the Markov chain we need to show that the conditional distribution of sth order statistics given the information of rth order statistics is same as the conditional distribution of sth order statistics given the joint information of first r order statistics. From Theorem 2.1 we know that the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ is

$$\begin{aligned} f(x_s|x_r) &= \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{f(x_s)}{1-F(x_r)} \\ &\quad \times \left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s}. \end{aligned} \tag{2.25}$$

Also, the conditional distribution of of $X_{s:n}$ given $X_{1:n} = x_1, X_{2:n} = x_2, \ldots,$ $X_{r:n} = x_r$ is

$$f(x_s|x_1, \ldots x_r) = \frac{f_{1,2,\ldots,r,s:n}(x_1, \ldots, x_r, x_s)}{f_{1,2,\ldots r:n}(x_1, \ldots, x_r)}.$$

The joint distribution of first r order statistics is given in (2.26) as

$$f_{1,\ldots r:n}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right][1-F(x_r)]^{n-r}. \tag{2.26}$$

Now the joint distribution of $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ and $X_{s:n}$ is obtained as

$$\begin{aligned} f_{1,\ldots r,s:n}(x_1, \ldots, x_r, x_s) &= \int_{x_s}^{\infty} \cdots \int_{x_s}^{x_{s+3}} \int_{x_s}^{x_{s+2}} \int_{x_r}^{x_s} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} \\ &\quad \times f_{1,\ldots n:n}(x_1, \ldots, x_n) \\ &\quad \times dx_{r+1} \cdots dx_{s-1} d_{s+1} \ldots dx_n \\ &= \int_{x_s}^{\infty} \cdots \int_{x_s}^{x_{s+3}} \int_{x_s}^{x_{s+2}} \int_{x_r}^{x_s} \cdots \int_{x_r}^{x_{r+3}} \int_{x_r}^{x_{r+2}} \\ &\quad \times n! \prod_{i=1}^{n} f(x_i) dx_{r+1} \cdots dx_n \end{aligned}$$

or

$$f_{1,\ldots r,s:n}(x_1,\ldots,x_r,x_s) = n!\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)$$

$$\times \left\{\int_{x_r}^{x_s}\cdots\int_{x_r}^{x_{r+3}}\int_{x_r}^{x_{r+2}}\prod_{i=r+1}^{s-1} f(x_i)dx_{r+1}\ldots dx_{s-1}\right\}$$

$$\times \left\{\int_{x_s}^{\infty}\cdots\int_{x_s}^{x_{s+3}}\int_{x_s}^{x_{s+2}}\prod_{i=s+1}^{n} f(x_i)dx_{s+1}\ldots dx_{n}\right\}$$

or

$$f_{1,\ldots r,s:n}(x_1,\ldots,x_r,x_s) = \frac{n!}{(s-r-1)!(n-s)!}\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)$$

$$\times\left[F(x_s)-F(x_r)\right]^{s-r-1}[1-F(x_s)]^{n-s}.$$

Hence the conditional distribution of $X_{s:n}$ given $X_{1:n}=x_1, X_{2:n}=x_2,\ldots,X_{r:n}=x_r$ is

$$f(x_s|x_1,\ldots x_r) = \left\{\frac{n!}{(s-r-1)!(n-s)!}\left[\prod_{i=1}^{r} f(x_i)\right] f(x_s)\right.$$

$$\left.\left[F(x_s)-F(x_r)\right]^{s-r-1}[1-F(x_s)]^{n-s}\right\}/$$

$$\left\{\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right][1-F(x_r)]^{n-r}\right\}$$

or

$$f(x_s|x_1,\ldots x_r) = \frac{(n-r)!}{(s-r-1)!(n-s)!}f(x_s)$$

$$\times\left[F(x_s)-F(x_r)\right]^{s-r-1}\frac{[1-F(x_s)]^{n-s}}{[1-F(x_r)]^{n-r}}$$

$$= \frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{f(x_s)}{1-F(x_r)}$$

$$\times\left[\frac{F(x_s)-F(x_r)}{1-F(x_r)}\right]^{s-r-1}\left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-s};$$

which is (2.25). Hence order statistics from a distribution $F(x)$ form a Markov chain. The transition probabilities of order statistics are easily computed from conditional distributions. We know that the transition probability is computed as

$$P(X_{r+1:n} \geq y | X_{r:n} = x) = \int_y^{\infty} f(x_{r+1} | x_r = x) dx_{r+1}.$$

Now the conditional distribution of $X_{r+1:n}$ given $X_{r:n} = x_r$ is obtained from (2.25) by using $s = r + 1$ as

$$\begin{aligned} f(x_{r+1}|x_r) &= \frac{(n-r)!}{(n-r-1)!} \frac{f(x_{r+1})}{1-F(x_r)} \left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-r-1} \\ &= (n-r)\frac{f(x_{r+1})}{1-F(x_r)} \left[\frac{1-F(x_s)}{1-F(x_r)}\right]^{n-r-1}. \end{aligned}$$

Hence the transition probability is

$$\begin{aligned} P(X_{r+1:n} \geq y | X_{r:n} = x) &= \int_y^{\infty} f(x_{r+1}|x_r = x) dx_{r+1} \\ &= \int_y^{\infty} (n-r) \left[\frac{1-F(x_s)}{1-F(x)}\right]^{n-r-1} \\ &\times \frac{f(x_{r+1})}{1-F(x)} dx_{r+1} \\ &= \frac{n-r}{\{1-F(x)\}^{n-r}} \int_y^{\infty} \{1-F(x_{r+1})\}^{n-r-1} \\ &\times f(x_{r+1}) dx_{r+1} \end{aligned}$$

or

$$\begin{aligned} P(X_{r+1} \geq y | X_r = x) &= \frac{n-r}{\{1-F(x)\}^{n-r}} \times \left.\frac{-\{1-F(x_{r+1})\}^{n-r}}{n-r}\right|_y^{\infty} \\ &= \left[\frac{1-F(y)}{1-F(x)}\right]^{n-r}. \end{aligned}$$

We can readily see that the transition probabilities depends upon value of n and r.

2.8 Moments of Order Statistics

The probability distribution of order statistics is like conventional probability distribution and hence the moments from these distributions can be computed in usual way. Specifically, the pth raw moment of rth order statistics is computed as

$$\begin{aligned} \mu_{r:n}^p &= \int_{-\infty}^{\infty} x_r f_{r:n}(x) dx \\ &= \frac{1}{B(r, n-r+1)} \int_{-\infty}^{\infty} x_r^p f(x_r) F^{r-1}(x_r) [1-F(x_r)]^{n-r} dx. \quad (2.27) \end{aligned}$$

Using the probability integral transform property of order statistics, the moments can also be written as

$$\mu_{r:n}^{p} = \frac{1}{B(r, n-r+1)} \int_{0}^{1} \{F^{-1}(u_r)\}^p u_r^{r-1} (1-u_r)^{n-r} du; \tag{2.28}$$

where u_r is rth Uniform Order Statistics. The mean and variance of $X_{r:n}$ can be computed from (2.27) or (2.28). Further, the joint pth and qth order moments of two order statistics; $X_{r:n} = x_1$ and $X_{s:n} = x_2$; are computed as

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\
&= C_{r,s:n} \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f(x_1) f(x_2) F^{r-1}(x_1) \\
&\quad \times \left[F(x_2) - F(x_1) \right]^{s-r-1} [1 - F(x_2)]^{n-s} dx_1 dx_2.
\end{aligned} \tag{2.29}$$

Using probability integral transform, we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n} \int_{0}^{1} \int_{0}^{v} \{F^{-1}(u)\}^p \{F^{-1}(v)\}^q u^{r-1} (v-u)^{s-r-1} \\
&\quad \times (1-v)^{n-s} du dv.
\end{aligned} \tag{2.30}$$

The pth and qth joint central moments are given as

$$\begin{aligned}
\sigma_{r,s:n}^{p,q} &= E\left[(X_{r:n} - \mu_{r:n})^p (X_{s:n} - \mu_{s:n})^q\right] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} (x_1 - \mu_{r:n})^p (x_2 - \mu_{s:n})^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\
&= C_{r,s:n} \int_{0}^{1} \int_{0}^{v} \left[F^{-1}(u) - \mu_{r:n}\right]^p \left[F^{-1}(v) - \mu_{s:n}\right]^q \\
&\quad \times u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} du dv.
\end{aligned} \tag{2.31}$$

Specifically, for $p = q = 1$; the quantity $\sigma_{r,s:n}$ is called covariance between $X_{r:n}$ and $X_{s:n}$. Also if $F(x)$ is symmetrical; say about 0; then following relations holds

$$\mu_{r:n}^{p} = (-1)^p \mu_{n-r+1:n}^{p}$$

and

$$\mu_{r,s:n}^{p,q} = (-1)^{p+q} \mu_{n-s+1,n-r+1:n}^{p,q}.$$

Further, the moments of linear combinations of order statistics can be easily obtained.

The conditional distributions of order statistics provide basis for computation of conditional moments of order statistics. Specifically, the pth conditional moment of sth order statistics; $X_{s:n}$; given $X_{r:n} = x_r$ is obtained by using the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ as

$$\begin{aligned}
\mu^p_{s|r:n} &= E\left(X^p_{s:n}|x_r\right) = \int_{x_r}^{\infty} x_s^p f(x_s|x_r)dx_s \\
&= \frac{(n-r)!}{(s-r-1)!(n-s)![1-F(x_r)]^{n-r}} \\
&\quad \times \int_{x_r}^{\infty} x_s^p f(x_s)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \times [1-F(x_s)]^{n-s} dx_s. \qquad (2.32)
\end{aligned}$$

The conditional mean and variance can be obtained from (2.32). Further, the pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$ is computed from the conditional distribution of $X_{s:n}$ given $X_{r:n} = x_r$ and $X_{t:n} = x_t$ as

$$\begin{aligned}
\mu^p_{s|r,t:n} &= E\left(X^p_{s:n}|x_r, x_t\right) = \int_{x_r}^{x_t} x_s^p f(x_s|x_r, x_t)dx_s \\
&= \frac{(t-r-1)!}{(s-r-1)!(t-s-1)![F(x_t)-F(x_r)]^{t-r-1}} \\
&\quad \times \int_{x_r}^{\infty} x_s^p f(x_s)\Big[F(x_s) - F(x_r)\Big]^{s-r-1} \\
&\quad \times [F(x_t) - F(x_s)]^{t-s-1} dx_s. \qquad (2.33)
\end{aligned}$$

In similar way the joint conditional moments of order statistics can be defined by using corresponding conditional distribution.

Example 2.7 A random sample has been obtained from the density

$$f(x) = \upsilon x^{\upsilon-1}; 0 < x < 1; \upsilon > 0.$$

Obtain expression for pth moment of rth order statistics and that for joint pth and qth moment of rth and sth order statistics. Hence or otherwise obtain mean and variance of $X_{r:n}$ and covariance between $X_{r:n}$ and $X_{s:n}$.

Solution: The distribution of rth order statistics is given as

$$f_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} f(x_r)F^{r-1}(x_r)[1-F(x_r)]^{n-r}.$$

For given distribution we have $F(x) = x^{\upsilon}$ and hence the density of $X_{r:n}$ is

$$f_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} \upsilon x_r^{r\upsilon-1}\left(1 - x_r^{\upsilon}\right)^{n-r}.$$

Now we have

$$\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x_r^p f_{r:n}(x_r)dx_r \\
&= \frac{1}{B(r,n-r+1)} \int_0^1 x_r^p v x_r^{rv-1}\left(1-x_r^v\right)^{n-r} dx_r \\
&= \frac{v}{B(r,n-r+1)} \int_0^1 x_r^{p+rv-1}\left(1-x_r^v\right)^{n-r} dx_r \\
&= \frac{v}{B(r,n-r+1)} \int_0^1 x_r^{p+rv-1}\left(1-x_r^v\right)^{n-r} dx_r \\
&= \frac{v}{B(r,n-r+1)} \int_0^1 x_r^{p+v(r-1)} x_r^{v-1}\left(1-x_r^v\right)^{n-r} dx_r
\end{aligned}$$

Making the transformation $x_r^v = y$ we have $v x_r^{v-1} dx_r = dy$, hence

$$\begin{aligned}
\mu_{r:n}^{p} &= \frac{1}{B(r,n-r+1)} \int_0^1 y^{\frac{p}{v}+r-1}(1-y)^{n-r} dx_r \\
&= \frac{B\left(r+\frac{p}{v}, n-r+1\right)}{B(r,n-r+1)} = \frac{\Gamma(n+1)\Gamma\left(\frac{p}{v}+r\right)}{\Gamma(r)\Gamma\left(n+\frac{p}{v}+1\right)}.
\end{aligned}$$

The mean is readily written as

$$\mu_{r:n} = \frac{\Gamma(n+1)\Gamma\left(\frac{1}{v}+r\right)}{\Gamma(r)\Gamma\left(n+\frac{1}{v}+1\right)}.$$

Again the joint distribution of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s:n}(x_r, x_s) = C_{r,s,n} v^2 x_r^{rv-1} x_s^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1}\left(1-x_s^v\right)^{n-s}.$$

The product moments are therefore

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^p X_{s:n}^q\right) = \int_0^1 \int_0^{x_s} x_r^p x_s^q f_{r,s:n}(x_r, x_s) dx_r dx_s \\
&= C_{r,s:n} v^2 \int_0^1 \int_0^{x_s} x_r^p x_s^q x_r^{rv-1} x_s^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1} \\
&\quad \times \left(1-x_s^v\right)^{n-s} dx_r dx_s \\
&= C_{r,s:n} v \int_0^1 x_s^q x_s^{v-1}\left(1-x_s^v\right)^{n-s} \\
&\quad \times \left\{ v \int_0^{x_s} x_r^{p+v(r-1)} x_r^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1} dx_r \right\} dx_s \\
&= C_{r,s:n} v \int_0^1 x_s^q x_s^{v-1}\left(1-x_s^v\right)^{n-s} \{v I(x_r)\} dx_s
\end{aligned}$$

Now consider

$$\begin{aligned} vI(x_r) &= v\int_0^{x_s} x_r^{p+v(r-1)} x_r^{v-1}\left(x_s^v - x_r^v\right)^{s-r-1} dx_r \\ &= v x_s^{v(s-r-1)} \int_0^{x_s} x_r^{p+v(r-1)} x_r^{v-1}\left(1 - \frac{x_r^v}{x_s^v}\right)^{s-r-1} dx_r. \end{aligned}$$

Making the transformation $y = x_r^v / x_s^v$ we have

$$\begin{aligned} vI(x_r) &= x_s^{(s-r-1)} \int_0^1 y^{p/v+(r-1)}(1-y)^{s-r-1} dy \\ &= x_s^{p+v(s-2)} B\left(\frac{p}{v} + r, s - r\right) \\ &= x_s^{p+v(s-2)} \frac{\Gamma\left(r + \frac{p}{v}\right)\Gamma(s-r)}{\Gamma\left(s + \frac{p}{v}\right)}. \end{aligned}$$

Using above value of $vI(x_r)$ in above equation we have

$$\begin{aligned} \mu_{r,s:n}^{p,q} &= C_{r,s:n} \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} v \int_0^1 x_s^{p+q+v(s-1)} x_s^{v-1}\left(1 - x_s^v\right)^{n-s} \\ &= C_{r,s:n} \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)} \end{aligned}$$

Now using the value of $C_{r,s:n}$ we have

$$\begin{aligned} \mu_{r,s:n}^{p,q} &= \frac{1}{B(r, s-r, n-s+1)} \times \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \\ &\quad \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(s-r)\Gamma(n-s+1)} \times \frac{\Gamma(s-r)\Gamma\left(r + \frac{p}{v}\right)}{\Gamma\left(s + \frac{p}{v}\right)} \\ &\quad \times \frac{(n-s)\Gamma(n-s)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma\left(\frac{p+q}{v} + n + 1\right)} \\ &= \frac{\Gamma(n+1)\Gamma\left(r + \frac{p}{v}\right)\Gamma\left(\frac{p+q}{v} + s\right)}{\Gamma(r)\Gamma\left(s + \frac{p}{v}\right)\Gamma\left(\frac{p+q}{v} + n + 1\right)}; r < s. \end{aligned}$$

The covariance can be easily obtained from above.

Example 2.8 Find pth moment of rth order statistics for standard exponential distribution.

Solution: The density and distribution function of standard exponential distribution are

$$f(x) = e^{-x} \text{ and } F(x) = 1 - e^{-x}.$$

The pth moment of rth order statistics is

$$\begin{aligned}
\mu_{r:n}^p &= E\left(X_{r:n}^p\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx \\
&= C_{r:n} \int_0^{\infty} x^p e^{-x} \left(1 - e^{-x}\right)^{r-1} \left(e^{-x}\right)^{n-r} dx \\
&= \sum_{j=0}^{r-1} (-1)^j C_{r:n} \binom{r-1}{j} \int_0^{\infty} x^p e^{-x(n-r+j+1)} dx \\
&= \sum_{j=0}^{r-1} (-1)^j \frac{n!}{j!(r-1-j)!(n-r)!} \frac{\Gamma(p+1)}{(n-r+j+1)^{p+1}}.
\end{aligned}$$

The mean and variance can be obtained from above.

Example 2.9 A random sample of size n is drawn from the Weibull distribution with density

$$f(x) = \alpha x^{\alpha-1} \exp(-x^{\alpha});\ x, \alpha > 0.$$

Obtain the expression for pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$.

Solution: The pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ is given in (2.34) as

$$\begin{aligned}
\mu_{s|r:n}^p &= E\left(X_{s:n}^p | x_r\right) = \int_{x_r}^{\infty} x_s^p f(x_s|x_r) dx_s \\
&= \frac{(n-r)!}{(s-r-1)!(n-s)![1-F(x_r)]^{n-r}} \int_{x_r}^{\infty} x_s^p f(x_s) \\
&\quad \times \left[F(x_s) - F(x_r)\right]^{s-r-1} [1 - F(x_s)]^{n-s} dx_s.
\end{aligned} \tag{2.34}$$

Now for given distribution we have

$$f(x) = \alpha x^{\alpha-1} \exp(-x^{\alpha})$$

and

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha t^{\alpha-1} \exp(-t^{\alpha}) dt = 1 - e^{-x^{\alpha}}.$$

Hence the pth conditional moment of $X_{r:s}$ given $X_{r:n} = x_r$ is

$$\mu^p_{s|r:n} = \frac{K_{r,s:n}}{(e^{-x_r^\alpha})^{n-r}} \int_{x_r}^{\infty} x_s^p \alpha x_s^{\alpha-1} e^{-x_s^\alpha} \left(e^{-x_r^\alpha} - e^{-x_s^\alpha}\right)^{s-r-1} \times \left(e^{-x_s^\alpha}\right)^{n-s} dx_s$$

$$= \frac{\alpha K_{r,s:n}}{e^{-(n-r)x_r^\alpha}} \int_{x_r}^{\infty} x_s^{p+\alpha-1} e^{-(n-s+1)x_s^\alpha} \sum_{j=0}^{s-r-1} (-1)^j \times \binom{s-r-1}{j} e^{-(s-r-j-1)x_r^\alpha} e^{-jx_s^\alpha} dx_s;$$

or

$$\mu^p_{s|r:n} = \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \int_{x_r}^{\infty} x_s^{p+\alpha-1} e^{-(n-s+j+1)x_s^\alpha} dx_s$$

Making the transformation $(n - s + j + 1)x_s^\alpha = y$ we have

$$\mu^p_{s|r:n} = \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \int_{(n-s+j+1)x_r^\alpha}^{\infty} \frac{y^{p/\alpha}}{(n-s+j+1)^{p/\alpha+1}} e^{-y} dy$$

$$= \alpha K_{r,s:n} \sum_{j=0}^{s-r-1} \frac{1}{e^{-(n-s+j+1)x_r^\alpha}} (-1)^j \binom{s-r-1}{j} \times \Gamma_{(n-s+j+1)x_r^\alpha}\left(\frac{p}{\alpha}+1\right);$$

where $K_{r,s:n} = \frac{(n-r)!}{(s-r-1)!(n-s)!}$. The conditional mean and variance can be obtained from above expression. Using $\alpha = 1$ in above expression we can obtain the expression for pth conditional moment of $X_{s:n}$ given $X_{r:n} = x_r$ for Exponential distribution.

2.9 Recurrence Relations and Identities for Moments of Order Statistics

In previous section we have discussed about single, product and conditional moments of order statistics in detail. The moments of order statistics posses certain additional characteristics in that they are related with each other in certain way. In this section we will give some relationships which exist among moments of order statistics. The relationships among moments of order statistics enable us to compute certain moments of higher order statistics on the basis of lower order statistics and/or on the basis of lower order moments. Some of the relationships which exist among moments of order statistics are distribution specific and some of the relationships are free from the underlying parent distribution of the sample. In the following we discuss some of the popular relationships which exist among moments of order statistics. We present both type of relationships; that is distribution free relationships and distribution specific relationships.

2.9.1 Distribution Free Relations Among Moments of Order Statistics

The moments of order statistics have several interesting relationships and identities which hold irrespective of the parent distribution. Some relations hold among moments of order statistics and several relations connect moments of order statistics with moments of the distribution from where sample has been drawn. In the following we first present some relationships that hold among single and product moments of order statistics and latter we will give some identities which hold between moments of order statistics and population moments.

We first give an interesting relationship which hold between distribution of rth order statistics and other single ordered observations and a relationship that connect joint distribution of two rth and sth order statistics with joint distribution of other ordered observations in the following.

We know that the distribution of rth order statistics is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}$$

or

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r-1}[1-F(x)] \\ &= \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{(n-1)-r} \\ &\quad - \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{(r+1)-1}[1-F(x)]^{n-(r+1)} \end{aligned}$$

or

$$\begin{aligned}f_{r:n}(x) &= \frac{n}{n-r}\frac{(n-1)!}{(r-1)!(n-r-1)!}f(x)[F(x)]^{r-1}[1-F(x)]^{(n-1)-r}\\&\quad-\frac{r}{n-r}\frac{n!}{r!(n-r-1)!}f(x)[F(x)]^{(r+1)-1}[1-F(x)]^{n-(r+1)}\\&= \frac{n}{n-r}f_{r:n-1}(x)-\frac{r}{n-r}f_{r+1:n}(x).\end{aligned}$$

So we have following relationship for distribution of rth order statistics and other ordered observations

$$(n-r)f_{r:n}(x) = nf_{r:n-1}(x) - rf_{r+1:n}(x). \tag{2.35}$$

In similar way we can show that

$$\begin{aligned}(n-s)f_{r,s:n}(x_1,x_2) &= nf_{r,s:n-1}(x_1,x_2) - rf_{r+1,s+1:n}(x_1,x_2)\\&\quad-(s-r)f_{r,s+1:n}(x_1,x_2).\end{aligned} \tag{2.36}$$

The relationships given in (2.35) and (2.36) enable us to derive two important relationships which hold between single and product moments of order statistics. The relationships are given below.

We know that the pth moment of rth order statistics is

$$\mu^p_{r:n} = E\left(X^p_{r:n}\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx.$$

Using the relationship given in (2.35) we have

$$\begin{aligned}(n-r)\mu^p_{r:n} &= \int_{-\infty}^{\infty} x^p\{nf_{r:n-1}(x) - rf_{r+1:n}\}dx\\&= n\int_{-\infty}^{\infty} x^p f_{r:n-1}(x)dx - r\int_{-\infty}^{\infty} x^p f_{r+1:n}(x)dx\\&= n\mu^p_{r:n-1} - r\mu^p_{r+1:n}\end{aligned}$$

or

$$r\mu^p_{r+1:n} = n\mu^p_{r:n-1} - (n-r)\mu^p_{r:n}. \tag{2.37}$$

The relation (2.37) was derived by Cole (1951) and is equally valid for expectation of function of order statistics; that is

$$rE\left[\{g(X_{r+1:n})\}^p\right] = nE\left[\{g(X_{r:n-1})\}^p\right] - (n-r)E\left[\{g(X_{r:n})\}^p\right];$$

also holds. Again consider the product moments of order statistics as

$$\mu_{r,s:n}^{p,q} = E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s:n}(x_1, x_2)dx_1dx_2.$$

Using the relationship (2.36) we have

$$\mu_{r,s:n}^{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q \left\{ \frac{n}{n-s} f_{r,s:n-1}(x_1, x_2) - \frac{r}{n-s} \right. \\ \left. \times f_{r+1,s+1:n}(x_1, x_2) - \frac{(s-r)}{n-s} f_{r,s+1:n}(x_1, x_2) \right\} dx_1dx_2$$

or

$$\begin{aligned}(n-s)\mu_{r,s:n}^{p,q} = n &\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s:n-1}(x_1, x_2)dx_1dx_2 \\ &-r \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r+1,s+1:n}(x_1, x_2)dx_1dx_2 \\ &-(s-r) \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r,s+1:n}(x_1, x_2)dx_1dx_2\end{aligned}$$

or

$$(n-s)\mu_{r,s:n}^{p,q} = n\mu_{r,s:n-1}^{p,q} - r\mu_{r+1,s+1:n}^{p,q} - (s-r)\mu_{r,s+1:n}^{p,q}$$

or

$$r\mu_{r+1,s+1:n}^{p,q} = n\mu_{r,s:n-1}^{p,q} - (n-s)\mu_{r,s:n}^{p,q} - (s-r)\mu_{r,s+1:n}^{p,q}. \tag{2.38}$$

which is due to Govindarajulu (1963). Again the relationship (2.38) is equally valid for function of product moments of order statistics. The relationship (2.37) further provide us following interesting relationship for n even

$$\frac{1}{2}\left(\mu_{\frac{n}{2}+1:n}^{p} + \mu_{\frac{n}{2}:n}^{p}\right) = n\mu_{\frac{n}{2}:n-1}^{p}; \tag{2.39}$$

which can be easily proved by using $r = \frac{n}{2}$ in (2.37). The relationship (2.39) also provide following interesting result for symmetric parent distribution

$$\begin{aligned}\mu_{\frac{n}{2}:n-1}^{p} &= \mu_{\frac{n}{2}:n}^{p} \; for\ p\ even \\ &= 0\ for\ p\ odd.\end{aligned}$$

The relationships given in (2.37) and (2.38) enable us to compute moments of order statistics recursively. We have certain additional relationships available for moments of order statistics which are based upon the sum of moments of lower order statistics. We present those relationships, which are due to Srikantan (1962); in the following theorems.

Theorem 2.6 *For any parent distribution following relation holds between moments of order statistics*

$$\mu_{r:n}^{p} = \sum_{i=r}^{n}(-1)^{i-r}\binom{i-1}{r-1}\binom{n}{i}\mu_{i:i}^{p};\ r = 1, 2, \ldots, n-1. \tag{2.40}$$

Proof We have

$$\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x^{p} f_{r:n}(x)dx \\
&= \int_{0}^{1} \left\{F^{-1}(u)\right\}^{p} f_{r:n}(u)du \\
&= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r-1}(1-u)^{n-r}du
\end{aligned}$$

Now expanding $(1-u)^{n-r}$ using binomial expansion we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r-1}\sum_{j=0}^{n-r}(-1)^{i}\binom{n-r}{j}u^{j}du \\
&= \sum_{j=0}^{n-r}(-1)^{j}\binom{n-r}{j}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(u)\right\}^{p} u^{r+j-1}du \\
&= \sum_{j=0}^{n-r}(-1)^{j}\frac{(n-r)!}{j!(n-r-j)!}\frac{n!(r+j)}{(r-1)!(n-r)!(r+j)}\mu_{r+j:r+j}^{p}.
\end{aligned}$$

Now writing $r+j=i$ we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \sum_{i=r}^{n}(-1)^{i-r}\frac{(n-r)!}{i(i-r)!(n-r-i+r)!}\frac{n!}{(r-1)!(n-r)!}\mu_{i:i}^{p} \\
&= \sum_{i=r}^{n}(-1)^{i-r}\frac{(i-1)!n!}{i(i-1)!(i-r)!(n-i)!(r-1)!}\mu_{i:i}^{p} \\
&= \sum_{i=r}^{n}(-1)^{i-r}\binom{i-1}{r-1}\binom{n}{i}\mu_{i:i}^{p}
\end{aligned}$$

as required.

Theorem 2.7 *Following relationship holds between moments of order statistics*

$$\mu_{r:n}^{p} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{1:i}^{p}; \, r = 2, 3, \ldots, n. \tag{2.41}$$

Proof Consider the expression for pth moment of rth order statistics as

$$\begin{aligned}
\mu_{r:n}^{p} &= E\left(X_{r:n}^{p}\right) = \int_{-\infty}^{\infty} x^{p} f_{r:n}(x)dx \\
&= \int_{0}^{1} \left\{F^{-1}(u)\right\}^{p} f_{r:n}(u)du \\
&= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^{p} u^{r-1}(1-u)^{n-r} du.
\end{aligned}$$

Writing u^{r-1} as $\{1-(1-u)\}^{r-1}$ and expanding binomially in power series of $(1-u)$ we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^{p} \sum_{j=0}^{r-1} (-1)^{j} \\
&\quad \times \binom{r-1}{j} (1-u)^{j} (1-u)^{n-r} du \\
&= \sum_{j=0}^{r-1} (-1)^{j} \frac{(r-1)!}{j!(r-1-j)!} \frac{n!}{(r-1)!(n-r)!} \\
&\quad \times \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^{p} (1-u)^{n-r+j} du
\end{aligned}$$

Now using $j = r + i - n - 1$ or $i = j - r + n + 1$ we have

$$\begin{aligned}
\mu_{r:n}^{p} &= \sum_{i=n-r+1}^{n} (-1)^{r+i-n-1} \frac{(r-1)!}{(r+i-n-1)!(n-i)!} \frac{n!}{r!(n-r)!} \\
&\quad \times r \int_{-\infty}^{\infty} \left\{F^{-1}(u)\right\}^{p} (1-u)^{i-1} du \\
&= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{1:i}^{p};
\end{aligned}$$

as required.

Theorem 2.8 *Following relationship holds between moments of order statistics for $r, s = 1, 2, \ldots, n$ and $r < s$*

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1} \sum_{j=n-s+i+1}^{n} (-1)^{j+n-r-s+1} \binom{i-1}{r-1} \binom{j-i-1}{n-s} \binom{n}{j} \mu_{i,i+1:j}^{p,q} \quad (2.42)$$

Proof Consider the expression for product moments of order statistics as

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\
&= C_{r,s:n} \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f(x_1) f(x_2) F^{r-1}(x_1) \\
&\quad \times \left[F(x_2) - F(x_1) \right]^{s-r-1} [1 - F(x_2)]^{n-s} dx_1 dx_2.
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} = C_{r,s:n} &\int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r-1} (v-u)^{s-r-1} \\
&\times (1-v)^{n-s} du dv.
\end{aligned}$$

Now expanding $(v-u)^{s-r-1}$ in power series we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n} \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r-1} \\
&\quad \times \sum_{h=0}^{s-r-1} (-1)^h \binom{s-r-1}{h} v^{s-r-1-h} u^h (1-v)^{n-s} du dv \\
&= \sum_{h=0}^{s-r-1} (-1)^h \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \frac{(s-r-1)!}{h!(s-r-h-1)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r+h-1} v^{s-r-h-1} (1-v)^{n-s} du dv \\
&= \sum_{h=0}^{s-r-1} (-1)^h \frac{n!}{h!(r-1)!(s-r-h-1)!(n-s)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{r+h-1} v^{s-r-h-1} (1-v)^{n-s} du dv.
\end{aligned}$$

Now using $i = h + r$ we have:

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \sum_{i=r}^{s-1} (-1)^{i-r} \frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!} \\
&\quad \times \int_0^1 \int_0^v \left\{F^{-1}(u)\right\}^p \left\{F^{-1}(v)\right\}^q u^{i-1} v^{s-i-1} (1-v)^{n-s} du dv.
\end{aligned}$$

Now writing v^{s-i-1} as $\{1-(1-v)\}^{s-i-1}$ and expanding in power series we have

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}(-1)^{i-r}\frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!}$$
$$\times \int_0^1\int_0^v \{F^{-1}(u)\}^p\{F^{-1}(v)\}^q u^{i-1}$$
$$\times \sum_{m=0}^{s-i-1}(-1)^m\binom{s-i-1}{m}(1-v)^m(1-v)^{n-s}dudv$$

or

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}(-1)^{i-r+m}\frac{n!}{(i-r)!(r-1)!(s-i-1)!(n-s)!}$$
$$\times \frac{(s-i-1)!}{m!(s-i-1-m)!}\int_0^1\int_0^v \{F^{-1}(u)\}^p\{F^{-1}(v)\}^q$$
$$\times u^{i-1}(1-v)^{n-s+m}dudv.$$
$$= \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}\frac{(-1)^{i-r+m}n!}{(i-r)!(r-1)!m!(s-i-1-m)!(n-s)!}$$
$$\times \int_0^1\int_0^v \{F^{-1}(u)\}^p\{F^{-1}(v)\}^q u^{i-1}(1-v)^{n-s+m}dudv$$

or

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{m=0}^{s-i-1}\frac{(-1)^{i-r+m}n!(i-1)!(n-s+m)!}{(i-r)!(r-1)!m!(s-i-1-m)!(n-s)!}$$
$$\times (n-s+i+m+1)! \times \ \mu_{i,i+1:n-s+m+i+1}.$$

Now using $j = n-s+i+1+m$ and rearranging the terms we have

$$\mu_{r,s:n}^{p,q} = \sum_{i=r}^{s-1}\sum_{j=n-s+i+1}^{n}(-1)^{j+n-r-s+1}\binom{i-1}{r-1}\binom{j-i-1}{n-s}\binom{n}{j}\mu_{i,i+1:j}^{p,q};$$

as required.

Proceeding in the way of Theorem 2.8, we also have following relationships between product moments of order statistics.

$$\mu_{r,s:n}^{p,q} = \sum_{i=s-r}^{s-1}\sum_{j=n-s+i+1}^{n}(-1)^{n-j-r+1}\binom{i-1}{s-r-1}\binom{j-i-1}{n-s}\binom{n}{j}\mu_{1,i+1:j}^{p,q}; \tag{2.43}$$

and

$$\mu_{r,s:n}^{p,q} = \sum_{i=s-r}^{n-r} \sum_{j=r+i}^{n} (-1)^{s+j} \binom{i-1}{s-r-1} \binom{j-i-1}{r-1} \binom{n}{j} \mu_{j-i,j:j}^{p,q}. \tag{2.44}$$

The relationships given in (2.43) and (2.44) link product moments of order statistics with those based upon the product moments of order statistics based upon smaller sample sizes and on lower order product moments.

The relationships given in equations (2.40) to (2.44) are useful in computing single and product moments of a specific order statistics as a sum of moments of lower order statistics. We have some additional interesting identities which relates sum of moments of a specific order statistics with sum of moments of lower order statistics. We also have some interesting identities which relate moments of order statistics with population moments and are free from any sort of distributional assumptions. We present these identities in the following section.

2.9.2 Some Identities for Moments of Order Statistics

The moments of order statistics posses certain simple identities. These identities are based upon following very basic formulae

$$\sum_{r=1}^{n} X_{r:n}^{p} = \sum_{r=1}^{n} X_{r}^{p};\ p \geq 1 \tag{2.45}$$

and

$$\sum_{r=1}^{n} \sum_{s=1}^{n} X_{r:n}^{p} X_{s:n}^{q} = \sum_{r=1}^{n} \sum_{s=1}^{n} X_{r}^{p} X_{s}^{q};\ p,q \geq 1. \tag{2.46}$$

Now if all X_r have same distribution $F(x)$ with $E\left(X_r^p\right) = \mu_p$, variance σ^2 and $E\left(X_r^p X_s^q\right) = \mu_{p,q}$ then we have following interesting identities.

Taking expectation on (2.45) we

$$E\left(\sum_{r=1}^{n} X_{r:n}^{p}\right) = E\left(\sum_{r=1}^{n} X_{r}^{p}\right)$$

or

$$\sum_{r=1}^{n} E\left(X_{r:n}^{p}\right) = \sum_{r=1}^{n} E\left(X_{r}^{p}\right)$$

or

$$\sum_{r=1}^{n} \mu_{r:n}^{p} = \sum_{r=1}^{n} \mu_{p} = n\mu_{p}. \tag{2.47}$$

In particular

$$\sum_{r=1}^{n} \mu_{r:n} = n\mu. \tag{2.48}$$

Again taking the expectation of (2.46) we have

$$E\left(\sum_{r=1}^{n}\sum_{s=1}^{n} X_{r:n}^{p} X_{s:n}^{q}\right) = E\left(\sum_{r=1}^{n}\sum_{s=1}^{n} X_{r}^{p} X_{s}^{q}\right)$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} E\left(X_{r:n}^{p} X_{s:n}^{q}\right) = \sum_{r=1}^{n}\sum_{s=1}^{n} E\left(X_{r}^{p} X_{s}^{q}\right)$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = \sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{p,q}.$$

Since X_r and X_s have same distribution therefore above relation can be written as

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = \sum_{r=1}^{n} \mu_{p+q} + \sum_{r=1}^{n}\sum_{s\neq r=1}^{n} \mu_{p}\mu_{q}$$

or

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n}^{p,q} = n\mu_{p+q} + n(n-1)\mu_{p}\mu_{q}. \tag{2.49}$$

In particular

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n} = n\mu_2 + n(n-1)\mu^2 = n\sigma^2 + n^2\mu^2; \tag{2.50}$$

and as a result we have

$$\begin{aligned}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mu_{r,s:n} &= \frac{1}{2}\left\{\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_{r,s:n} - \sum_{r=1}^{n} \mu_{r:n}^{2}\right\} \\ &= \frac{1}{2}\left\{n\mu_2 + n(n-1)\mu^2 - n\mu_2\right\} \\ &= \frac{n(n-1)}{2}\mu^2 = \binom{n}{2}\mu^2. \end{aligned} \tag{2.51}$$

We also have following identity for $n = 2$

$$\mu_{1,2:2}^{p,q} + \mu_{1,2:2}^{q,p} = 2\mu_{p}\mu_{q}; \tag{2.52}$$

which for $p = q = 1$ reduces to

$$\mu_{1,2:2} = \mu^2.$$

Also from (2.48) and (2.50) we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n} = \sum_{r=1}^{n}\sum_{s=1}^{n}\mu_{r,s:n} - \left(\sum_{r=1}^{n}\mu_{r:n}\right)\left(\sum_{s=1}^{n}\mu_{s:n}\right)$$
$$= n\sigma^2 + n^2\mu^2 - n^2\mu^2 = n\sigma^2. \quad (2.53)$$

Above identities are very useful in checking the accuracy of single and product moments of order statistics by their comparison with the population moments.

The single moments of order statistics have an additional identity which relates the sum of single moments in terms of sum of moments of lower order statistics. These identities are given in the following theorem.

Theorem 2.9 *The single moments of order statistics satisfies following identities*

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{1}{r}\mu_{1:r} \quad (2.54)$$

and

$$\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{1}{r}\mu_{r:r}. \quad (2.55)$$

Proof Consider the expression for single moments of order statistics as

$$\mu_{r:n}^{p} = \int_{-\infty}^{\infty} x_r^p f_{r:n}(x_r)dx_r$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p f_{r:n}(u)du$$
$$= \frac{n!}{(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du.$$

Applying summation over both sides we have

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^{p} = \sum_{r=1}^{n}\frac{n!}{r(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p$$
$$\times u^{r-1}(1-u)^{n-r}du$$
$$= \sum_{r=1}^{n}\binom{n}{r}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{u}\left\{\sum_{r=1}^{n}\binom{n}{r}u^r(1-u)^{n-r}\right\}du$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{u}\left\{1-(1-u)^n\right\}du.$$

Now using the identity

$$1-(1-u)^n = u\sum_{r=1}^{n}(1-u)^{r-1};$$

we have

$$\sum_{r=1}^{n}\frac{1}{r}\mu_{r:n}^p = \int_0^1 \left\{F^{-1}(u)\right\}^p \sum_{r=1}^{n}(1-u)^{r-1}du$$
$$= \sum_{r=1}^{n}\frac{1}{r}r\int_0^1 \left\{F^{-1}(u)\right\}^p(1-u)^{r-1}du$$
$$= \sum_{r=1}^{n}\frac{1}{r}\mu_{1:r}^p;$$

which is (2.54).

For second identity again consider the expression for single moments as

$$\mu_{r:n}^p = \frac{n!}{(r-1)!(n-r)!}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

or

$$\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^p = \sum_{r=1}^{n}\frac{n!}{(r-1)!(n-r+1)!}$$
$$\times\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$
$$= \sum_{r=1}^{n}\binom{n}{r-1}\int_0^1 \left\{F^{-1}(u)\right\}^p u^{r-1}(1-u)^{n-r}du$$

or

$$\sum_{r=1}^{n}\frac{\mu_{r:n}^p}{n-r+1} = \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{1-u}\left\{\sum_{r=1}^{n-1}\binom{n}{r}u^r(1-u)^{n-r}\right\}du$$
$$= \int_0^1 \left\{F^{-1}(u)\right\}^p \frac{1}{1-u}\left(1-u^n\right)du.$$

Now using the identity

$$1-u^n=(1-u)\sum_{r=1}^{n}u^{r-1};$$

we have

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{n-r+1}\mu_{r:n}^{p}&=\int_0^1\left\{F^{-1}(u)\right\}^p\sum_{r=1}^{n}u^{r-1}du\\&=\sum_{r=1}^{n}\frac{1}{r}r\int_0^1\left\{F^{-1}(u)\right\}^p u^{r-1}du\\&=\sum_{r=1}^{n}\frac{1}{r}\mu_{r:r}^{p};\end{aligned}$$

which is (2.55). Hence the theorem.

Example 2.10 Prove that in a random sample from a continuous distribution with cdf $F(x)$ following relation holds:

$$E[X_{s:n}F(X_{r:n})]=\frac{r}{n+1}\mu_{s+1:n+1}.$$

Solution: We have:

$$\begin{aligned}f_{r,s:n}(x_1,x_2)&=C_{r,s:n}f(x_1)f(x_2)[F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\\&\quad\times[1-F(x_2)]^{n-s}.\end{aligned}$$

Now

$$\begin{aligned}E[X_{s:n}F(X_{r:n})]&=C_{r,s:n}\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}\{x_2F(x_1)\}f(x_1)f(x_2)[F(x_1)]^{r-1}\\&\quad\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_1dx_2\\&=C_{r,s:n}\int_0^1\int_0^v F^{-1}(v)u^r(v-u)^{s-r-1}(1-v)^{n-s}dudv\\&=C_{r,s:n}\int_0^1 F^{-1}(v)(1-v)^{n-s}\left[\int_0^v u^r(v-u)^{s-r-1}du\right]dv\\&=C_{r,s:n}\int_0^1 F^{-1}(v)(1-v)^{n-s}(I)dv;\end{aligned}\tag{2.56}$$

where $I=\int_0^v u^r(v-u)^{s-r-1}du$. Now consider

$$\begin{aligned} I &= \int_0^v u^r (v-u)^{s-r-1} du \\ &= v^{s-r-1} \int_0^v u^r \left(1 - \frac{u}{v}\right)^{s-r-1} du. \end{aligned}$$

Making the transformation $\frac{u}{v} = w$ we have

$$\begin{aligned} I &= v^s \int_0^1 w^r (1-w)^{s-r-1} dw \\ &= v^s B(r+1, s-r) = v^s \frac{\Gamma(r+1)\Gamma(s-r)}{\Gamma(s+1)} \\ &= v^s \frac{r!(s-r-1)!}{s!}. \end{aligned}$$

Using above value in (2.56) we have

$$\begin{aligned} E[X_{s:n} F(X_{r:n})] &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\ &\quad \times \int_0^1 F^{-1}(v)(1-v)^{n-s} v^s \frac{r!(s-r-1)!}{s!} dv \\ &= \frac{r(r-1)!n!}{(r-1)!s!(n-s)!} \int_0^1 F^{-1}(v) v^s (1-v)^{n-s} v^s \\ &= \frac{r}{n+1} \frac{(n+1)!}{s!(n-s)!} \int_0^1 F^{-1}(v) v^{s+1-1} (1-v)^{n-s} v^s \\ &= \frac{r}{n+1} \mu_{s+1:n+1}; \end{aligned}$$

as required.

2.9.3 Distribution Specific Relationships for Moments of Order Statistics

In previous section we have given some useful relations which exist among moments of order statistics irrespective of the parent distribution. There exist certain other relationships among moments of order statistics which are based upon the parent distribution from where sample has been drawn. In this section we will give some recurrence relations for single and product moments of order statistics which are limited to parent probability distribution. We first give two useful results in following theorem which can be used to derive the recurrence relations for single and product moments of order statistics for certain special distributions.

Theorem 2.10 *Following relations hold for single and product moments of order statistics from a distribution $F(x)$*

$$\mu^p_{r:n} - \mu^p_{r-1:n} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx. \tag{2.57}$$

and

$$\mu^{p,q}_{r,s:n} - \mu^{p,q}_{r,s-1:n} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \times [1-F(x_2)]^{n-s+1}dx_2dx_1. \tag{2.58}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Proof We know that the pth moment of rth order statistics is

$$\begin{aligned}\mu^p_{r:n} &= E\left(X^p_{r:n}\right) = \int_{-\infty}^{\infty} x^p f_{r:n}(x)dx \\ &= \int_{-\infty}^{\infty} x^p \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx.\end{aligned}$$

Integrating above equation by parts taking $f(x)[1-F(x)]^{n-r}$ as function for integration we have

$$\mu^p_{r:n} = \frac{n!}{(r-1)!(n-r)!}\Bigg[-x^p[F(x)]^{r-1}\frac{\{1-F(x)\}^{n-r+1}}{n-r+1}\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left\{px^{p-1}[F(x)]^{r-1} + (r-1)x^p[F(x)]^{r-2}f(x)\right\} \times \frac{-[1-F(x)]^{n-r+1}}{n-r+1}dx\Bigg]$$

or

$$\mu^p_{r:n} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}[1-F(x)]^{n-r+1} \times [F(x)]dx + \frac{(r-1)}{n-r+1}\frac{n!}{(r-1)!(n-r)!}$$

$$
\begin{aligned}
&\times \int_{-\infty}^{\infty} x^p f(x)[1-F(x)]^{n-r+1}[F(x)]^{r-2}dx \\
&= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}[F(x)]^{r-1} \\
&\times [1-F(x)]^{n-r+1}dx + \frac{n!}{(r-2)!(n-r+1)!}\int_{-\infty}^{\infty} x^p f(x) \\
&\times [F(x)]^{r-2}[1-F(x)]^{n-(r-1)}dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mu_{r-1:n}^p = &\frac{n!}{(r-2)!(n-r+1)!}\int_{-\infty}^{\infty} x^p f(x)[F(x)]^{r-2} \\
&\times [1-F(x)]^{n-(r-1)}dx,
\end{aligned}
$$

hence above equation can be written as

$$
\begin{aligned}
\mu_{r:n}^p = &\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}[F(x)]^{r-1} \\
&\times [1-F(x)]^{n-r+1}dx + \mu_{r-1:n}^p
\end{aligned}
$$

or

$$
\begin{aligned}
\mu_{r:n}^p - \mu_{r-1:n}^p = &\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \\
&\times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx,
\end{aligned}
$$

which is (2.57).

To prove the second result we consider the expression for product moments of order statistics as

$$
\begin{aligned}
\mu_{r,s:n}^{p,q} &= E\left(X_{r:n}^p X_{s:n}^q\right) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{r,s:n}(x_1,x_2)dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1} \\
&\quad \times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty} x_1^p f(x_1)[F(x_1)]^{r-1}I(x_2)dx_1, \qquad (2.59)
\end{aligned}
$$

where

$$
I(x_2) = \int_{x_1}^{\infty} x_2^q f(x_2)\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2.
$$

Integrating above integral by parts using $f(x_2)\{1-F(x_2)\}^{n-s}$ for integration we have

$$\begin{aligned}
I(x_2) &= -x_2^q\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\frac{\{1-F(x_2)\}^{n-s+1}}{n-s+1}\Bigg|_{x_1}^{\infty} \\
&\quad +\frac{1}{n-s+1}\int_{x_1}^{\infty}\Bigg[qx_2^{q-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \\
&\quad +(s-r-1)x_2^q\{F(x_2)-F(x_1)\}^{s-r-2}f(x_2)\Bigg] \\
&\quad \times\{1-F(x_2)\}^{n-s+1}dx_2 \\
&= \frac{q}{n-s+1}\int_{x_1}^{\infty}x_2^{q-1}[F(x_2)-F(x_1)]^{s-r-1} \\
&\quad \times\{1-F(x_2)\}^{n-s+1}dx_2+\frac{s-r-1}{n-s+1}\int_{x_1}^{\infty}x_2^q f(x_2) \\
&\quad \times[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2. \qquad (2.60)
\end{aligned}$$

Now using the value of $I(x_2)$ from (2.60) in (2.59) we have

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= C_{r,s:n}\int_{-\infty}^{\infty}x_1^p f(x_1)[F(x_1)]^{r-1}\Bigg[\frac{q}{n-s+1} \\
&\quad \times\int_{x_1}^{\infty}x_2^{q-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}\{1-F(x_2)\}^{n-s+1}dx_2 \\
&\quad +\frac{s-r-1}{n-s+1}\int_{x_1}^{\infty}x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-2} \\
&\quad \times[1-F(x_2)]^{n-s+1}dx_2\Bigg]dx_1
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}f(x_1)[F(x_1)]^{r-1} \\
&\quad \times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1 \\
&\quad +\frac{s-r-1}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1} \\
&\quad \times[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2dx_1
\end{aligned}$$

or

$$\begin{aligned}
\mu_{r,s:n}^{p,q} &= \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}f(x_1)[F(x_1)]^{r-1} \\
&\quad \times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1
\end{aligned}$$

$$+C_{r,s-1:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2dx_1$$

or

$$\mu_{r,s:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1$$
$$+\mu_{r,s-1:n}^{p,q}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)$$
$$\times [F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1}$$
$$[1-F(x_2)]^{n-s+1}dx_2dx_1,$$

which is (2.58) and hence the theorem.

The results given in Theorem 2.10 are very useful in deriving distribution specific recurrence relations for single and product moments of order statistics. In the following subsections we have discussed recurrence relations between single and product moments of order statistics for certain distributions.

2.9.4 Exponential Distribution

The Exponential distribution has been the area of study in order statistics by many researchers. The density and distribution function of this distribution are

$$f(x) = \alpha \exp(-\alpha x);\ x, \alpha > 0$$

and

$$F(x) = 1 - \exp(-\alpha x).$$

We can readily see that the density and distribution function for exponential distribution are related through the equation

$$f(x) = \alpha[1 - F(x)]. \tag{2.61}$$

The recurrence relation for single and product moments of order statistics for exponential distribution are derived Joshi (1978, 1982) by using (2.57), (2.58) and (2.61) as below.

For recurrence relation between single moments of order statistics consider relation (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r}[1-F(x)]dx.$$

Now using (2.61) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha(n-r+1)}\mu_{r:n}^{p-1}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\alpha(n-r+1)}\mu_{r:n}^{p-1}; \tag{2.62}$$

as given in Balakrishnan and Rao (1998). The recurrence relation (2.62) provide following relation as a special case for $r = 1$

$$\mu_{1:n}^{p} = \frac{p}{\alpha n}\mu_{1:n}^{p-1}.$$

The recurrence relation for product moments of order statistics for exponential distribution is readily obtained by using (2.58) as

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2)-F(x_1)\Big]^{s-r-1} \times [1-F(x_2)]^{n-s+1}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \times [1 - F(x_2)]^{n-s} [1 - F(x_2)] dx_2 dx_1.$$

Now using (2.61) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha(n-s+1)} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) f(x_2) \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \times [1 - F(x_2)]^{n-s} dx_2 dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha(n-s+1)} \mu_{r,s:n}^{p,q-1}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{\alpha(n-s+1)} \mu_{r,s:n}^{p,q-1}. \tag{2.63}$$

Using $s = r+1$ in (2.63) we have following recurrence relation for product moments of two contiguous order statistics from exponential distribution

$$\mu_{r,r+1:n}^{p,q} = \mu_{r:n}^{p+q} + \frac{q}{\alpha(n-r)} \mu_{r,r+1:n}^{p,q-1}. \tag{2.64}$$

Certain other relations can be derived from (2.63) and (2.64). Some more recurrence relations for single and product moments of order statistics from exponential distribution can be found in Joshi (1982).

2.9.5 The Weibull Distribution

The Weibull distribution has wide spread applications in almost all the areas of life. The density and distribution function for a Weibull random variable are

$$f(x) = \beta \alpha^{\beta} x^{\beta-1} \exp\big[-(\alpha x)^{\beta}\big];\ x, \alpha, \beta > 0$$

and

$$F(x) = 1 - \exp\big[-(\alpha x)^{\beta}\big].$$

We can see that the density and distribution function are related as

$$f(x) = \beta\alpha^{\beta}x^{\beta-1}[1 - F(x)]. \tag{2.65}$$

The recurrence relations for single and product moments of order statistics for Weibull distribution are derived below.

The recurrence relation for single moments is given in (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r}[1-F(x)]dx.$$

Using (2.65) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}x^{1-\beta} \times f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-\beta} \times f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\alpha^{\beta}\beta(n-r+1)}\mu_{r:n}^{p-\beta}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\alpha^{\beta}\beta(n-r+1)}\mu_{r:n}^{p-\beta}. \tag{2.66}$$

We can see that (2.66) reduces to (2.62) for $\beta = 1$ as it should be. Using $\beta = 2$ in (2.66) we obtain the recurrence relation for single moments of order statistics from Rayleigh distribution as

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{2\alpha^{2}(n-r+1)}\mu_{r:n}^{p-2}. \tag{2.67}$$

Also for $r = 1$ we have following recurrence relations for single moments of first order statistics from Weibull distribution

$$\mu_{1:n}^{p} = \frac{p}{n\alpha^{\beta}\beta}\mu_{1:n}^{p-\beta}. \tag{2.68}$$

We now present the recurrence relations for product moments of order statistics for Weibull distribution. We have relation (2.58) as

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s+1}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-1}f(x_1) \times [F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s}[1 - F(x_2)]dx_2dx_1.$$

Using (2.65) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha^{\beta}\beta(n-s+1)}C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^{p}x_2^{q-\beta}f(x_1) \times f(x_2)[F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \times [1 - F(x_2)]^{n-s}dx_2dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{\alpha^{\beta}\beta(n-s+1)}\mu_{r,s:n}^{p,q-\beta}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{\alpha^{\beta}\beta(n-s+1)}\mu_{r,s:n}^{p,q-\beta}. \tag{2.69}$$

We can immediately see that (2.69) reduces to (2.63) for $\beta = 1$. Using $\beta = 2$ in (2.69) we have following recurrence relation for product moments of order statistics for Rayleigh distribution

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{q}{2\alpha^2(n-s+1)}\mu_{r,s:n}^{p,q-2}. \tag{2.70}$$

Also for $s = r+1$ the recurrence relation for product moments of two contiguous order for Weibull distribution is

$$\mu_{r,r+1:n}^{p,q} = \mu_{r:n}^{p+q} + \frac{q}{\alpha^\beta \beta(n-r)}\mu_{r,r+1:n}^{p,q-\beta}. \tag{2.71}$$

The relationships (2.67) and (2.69) can be used to derive recurrence relations for mean, variances and covariances of order statistics from Weibull distribution.

2.9.6 The Logistic Distribution

The Logistic distribution has density and distribution function as

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}; -\infty < x < \infty$$

and

$$F(x) = \frac{1}{1+e^{-x}}.$$

We can readily see that the density and distribution function are related as

$$f(x) = F(x)[1-F(x)]. \tag{2.72}$$

Shah (1966, 1970) used the representation (2.72) to derive the recurrence relations for single and product moments of order statistics from the Logistic distribution. These relations are given below.

For single moments consider (2.57) as

$$\mu_{r:n}^p - \mu_{r-1:n}^p = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

or

$$\begin{aligned}\mu_{r:n}^p - \mu_{r-1:n}^p &= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}F(x) \\ &\quad \times [F(x)]^{(r-1)-1}[1-F(x)]^{n-r}[1-F(x)]dx. \\ &= \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} \\ &\quad \times F(x)[1-F(x)][F(x)]^{(r-1)-1}[1-F(x)]^{n-r}dx.\end{aligned}$$

Now using (2.72) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} f(x) \\ \times [F(x)]^{(r-1)-1} [1 - F(x)]^{n-r} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n(n-1)!}{(r-1)(r-2)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} \\ \times f(x)[F(x)]^{(r-1)-1} [1 - F(x)]^{n-r} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{np}{(r-1)(n-r+1)} \mu_{r-1:n-1}^{p-1}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{np}{(r-1)(n-r+1)} \mu_{r-1:n-1}^{p-1}; \tag{2.73}$$

as the recurrence relation for single moments of order statistics from Logistic distribution.

We now present the recurrence relation for first order product moments of order statistics from Logistic distribution. For this consider

$$\begin{aligned} \mu_{r:n} = E\left(X_{r:n} X_{s:n}^{0}\right) &= \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1 x_2^0 f_{r,s:n}(x_1, x_2) dx_2 dx_1 \\ &= C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1 f(x_1) f(x_2) [F(x_1)]^{r-1} \\ &\quad \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2 dx_1 \\ &= C_{r,s:n} \int_{-\infty}^{\infty} x_1 f(x_1) [F(x_1)]^{r-1} I(x_2) dx_1; \end{aligned} \tag{2.74}$$

where

$$I(x_2) = \int_{x_1}^{\infty} f(x_2) \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2.$$

Using (2.72) in above equation we have

$$I(x_2) = \int_{x_1}^{\infty} F(x_2)\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1}dx_2$$
$$= \int_{x_1}^{\infty}[1 - \{1 - F(x_2)\}]\Big[F(x_2) - F(x_1)\Big]^{s-r-1}$$
$$\times [1 - F(x_2)]^{n-s+1}dx_2$$

or

$$I(x_2) = \int_{x_1}^{\infty}\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1}dx_2$$
$$- \int_{x_1}^{\infty}[F(x_2) - F(x_1)]^{s-r-1}[1 - F(x_2)]^{n-s+2}dx_2$$
$$= I_1(x_2) - I_2(x_2);$$

Using above equation in (2.74) we have

$$\mu_{r:n} = C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1} I_1(x_2)dx_1$$
$$-C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1} I_2(x_2)dx_1. \tag{2.75}$$

Now consider

$$I_1(x_2) = \int_{x_1}^{\infty}\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1}dx_2.$$

Integrating by parts, taking dx_2 as integration and rest of the function for differentiation we have

$$I_1(x_2) = \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1 - F(x_2)]^{n-s+1}x_2\Big|_{x_1}^{\infty}$$
$$- \int_{x_1}^{\infty} x_2\Big[\Big\{(s - r - 1)[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1 - F(x_2)]^{n-s+1} f(x_2)\Big\} - \{(n - s + 1) f(x_2)$$
$$[F(x_2) - F(x_1)]^{s-r-1}[1 - F(x_2)]^{n-s}\Big\}\Big]dx_2$$
$$= (n - s + 1)\int_{x_1}^{\infty} x_2 f(x_2)\Big[F(x_2) - F(x_1)\Big]^{s-r-1}$$
$$\times [1 - F(x_2)]^{n-s}dx_2 - (s - r - 1)\int_{x_1}^{\infty} x_2 f(x_2)$$
$$\times [F(x_2) - F(x_1)]^{s-r-2}[1 - F(x_2)]^{n-s+1}dx_2. \tag{2.76}$$

Similarly

$$I_2(x_2) = (n-s+2)\int_{x_1}^{\infty} x_2 f(x_2)\Big[F(x_2)-F(x_1)\Big]^{s-r-1}$$
$$\times [1-F(x_2)]^{n-s+1}dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2 f(x_2)$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+2}dx_2. \tag{2.77}$$

Now using (2.76) and (2.77) in (2.75) we have

$$\mu_{r:n} = (n-s+1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_2)f(x_1)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2$$
$$-(n-s+2)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times\Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2dx_1$$
$$+(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_2)f(x_1)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}dx_2.$$

Now rearranging the terms we have

$$\mu_{r,s:n+1} = \frac{n+1}{n-s+2}\Big[\mu_{r,s:n} - \mu_{r,s-1:n} - \frac{n-s+2}{n+1}\mu_{r,s-1:n+1}$$
$$-\frac{1}{n-s+2}\mu_{r:n}\Big]. \tag{2.78}$$

Using $s = r+1$ in above equation the relation for product moments of two contiguous order statistics from Logistic distribution turned out to be

$$\mu_{r,r+1:n+1} = \frac{n+1}{n-r+1}\Big[\mu_{r,r+1:n} - \frac{r}{n+1}\mu^2_{r+1:n+1} - \frac{1}{n-r}\mu_{r:n}\Big]. \tag{2.79}$$

Certain other relations can be derived from (2.78) and (2.79).

2.9.7 The Inverse Weibull Distribution

The Inverse Weibull distribution is another useful distribution which has several applications in almost all the areas of life. The density and distribution function for an Inverse Weibull random variable are

$$f(x) = \frac{\beta}{x^{\beta+1}} \exp\left(-\frac{1}{x^{\beta}}\right); x, \beta > 0$$

and

$$F(x) = \exp\left(-\frac{1}{x^{\beta}}\right).$$

The density and distribution function of Inverse Weibull distribution are related as

$$f(x) = \frac{x^{\beta+1}}{\beta} F(x). \tag{2.80}$$

The recurrence relations for single and product moments of order statistics for Inverse Weibull distribution are derived below.

The recurrence relation for single moments is given in (2.57) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} \times [F(x)]^{r-1}[1-F(x)]^{n-r+1} dx.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} F(x) \times [F(x)]^{r-2}[1-F(x)]^{n-r+1} dx.$$

Using (2.80) in above equation we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(n-r+1)} \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^{p-1} x^{\beta+1} \times f(x)[F(x)]^{(r-1)-1}[1-F(x)]^{n-(r-1)} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(r-1)} \frac{n!}{(r-2)!(n-r+1)!} \int_{-\infty}^{\infty} x^{p+\beta} \times f(x)[F(x)]^{(r-1)-1}[1-F(x)]^{n-(r-1)} dx$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{\beta(r-1)} \mu_{r-1:n}^{p+\beta}$$

or

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{\beta(r-1)} \mu_{r-1:n}^{p+\beta}. \tag{2.81}$$

as a recurrence relation for single moments of order statistics from Inverse Weibull distribution. Using $\beta = 2$ in (2.81) we have following recurrence relation for single moments of order statistics from Inverse Rayleigh distribution

$$\mu_{r:n}^{p} = \mu_{r-1:n}^{p} + \frac{p}{2(r-1)} \mu_{r-1:n}^{p+2}. \tag{2.82}$$

We now present the recurrence relations for product moments of order statistics for Inverse Weibull distribution. We have relation (2.58) as

$$\begin{aligned} \mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} & x_1^p x_2^{q-1} f(x_1) \\ & \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \\ & \times [1 - F(x_2)]^{n-s+1} dx_2 dx_1. \end{aligned}$$

or

$$\begin{aligned} \mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} & x_1^p x_2^{q-1} f(x_1) \\ & \times [F(x_1)]^{r-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} \\ & \times [1 - F(x_2)]^{n-s} [1 - F(x_2)] dx_2 dx_1. \end{aligned}$$

or

$$\begin{aligned} \mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = & \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} \\ & \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2 dx_1 \\ & - \frac{q C_{r,s:n}}{n-s+1} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} F(x_2) \\ & \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1 - F(x_2)]^{n-s} dx_2 dx_1. \end{aligned}$$

Now using (2.80) in above equation we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1 - \frac{q C_{r,s:n}}{\beta(n-s+1)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+\beta} f(x_1) f(x_2) [F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1.$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q}{n-s+1} C_{r,s:n} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1) [F(x_1)]^{r-1} \times \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2 dx_1 - \frac{q}{\beta(n-s+1)} \mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{q C_{r,s:n}}{n-s+1} \int_{-\infty}^{\infty} x_1^p f(x_1) [F(x_1)]^{r-1} \times I(x_2) dx_1 - \frac{q}{\beta(n-s+1)} \mu_{r,s:n}^{p,q+\beta}; \tag{2.83}$$

where

$$I(x_2) = \int_{x_1}^{\infty} x_2^{q-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1-F(x_2)]^{n-s} dx_2.$$

Now integrating above equation by parts using x_2^{q-1} as function for integration we have

$$I(x_2) = \Big[F(x_2) - F(x_1) \Big]^{s-r-1} [1-F(x_2)]^{n-s} \frac{x_2^q}{q} \Bigg|_{x_1}^{\infty} - (s-r-1) \int_{x_1}^{\infty} \Big\{ [F(x_2) - F(x_1)]^{s-r-2} [1-F(x_2)]^{n-s} f(x_2) - (n-s) [1-F(x_2)]^{n-s-1} \Big[F(x_2) - F(x_1) \Big]^{s-r-1} f(x_2) \Big\} \frac{x_2^q}{q} dx_2$$

or

$$I(x_2) = \frac{n-s}{q}\int_{x_1}^{\infty} x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-1}$$
$$\times [1-F(x_2)]^{n-s-1}dx_2 - \frac{s-r-1}{q}\int_{x_1}^{\infty} x_2^q$$
$$\times f(x_2)[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s}dx_2.$$

Using above in (2.83) we have

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{qC_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty} x_1^p f(x_1)[F(x_1)]^{r-1}$$
$$\times \left\{\frac{n-s}{q}\int_{x_1}^{\infty} x_2^q f(x_2)[F(x_2)-F(x_1)]^{s-r-1}\right.$$
$$\times [1-F(x_2)]^{n-s-1}dx_2 - \frac{s-r-1}{q}\int_{x_1}^{\infty} x_2^q f(x_2)$$
$$\left.\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s}dx_2\right\}dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{(n-s)C_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{(n-1)-s}dx_2dx_1$$
$$-\frac{(s-r-1)C_{r,s:n}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{(s-1)-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{nC_{r,s:n-1}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{(n-1)-s}dx_2dx_1$$
$$-\frac{nC_{r,s-1:n-1}}{n-s+1}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{(s-1)-r-1}[1-F(x_2)]^{n-s}dx_2dx_1$$
$$-\frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} - \mu_{r,s-1:n}^{p,q} = \frac{n}{n-s+1}\mu_{r,s:n-1}^{p,q} - \frac{n}{n-s+1}\mu_{r,s-1:n-1}^{p,q} - \frac{q}{\beta(n-s+1)}\mu_{r,s:n}^{p,q+\beta}$$

or

$$\mu_{r,s:n}^{p,q} = \mu_{r,s-1:n}^{p,q} + \frac{1}{n-s+1}\left[n\mu_{r,s:n-1}^{p,q} - n\mu_{r,s-1:n-1}^{p,q} - \frac{q}{\beta}\mu_{r,s:n}^{p,q+\beta}\right]. \quad (2.84)$$

The recurrence relations for product moments of order statistics for Inverse Exponential and Inverse Rayleigh distribution can be easily obtained from (2.84) by using $\beta = 1$ and $\beta = 2$ respectively. Some other references on recurrence relations for moments of order statistics include Al-Zahrani and Ali (2014), Al-Zahrani et al. (2015), Balakrishnan et al. (2015a, b).

Example 2.11 Show that for standard exponential distribution having density $f(x) = e^{-x}$; following relation holds for moments of order statistics:

$$\mu_{r:n}^{p} = \mu_{r-1:n-1}^{p} + \frac{p}{n}\mu_{r:n}^{p-1}; \quad 2 \le r \le n.$$

Solution: We have $(p-1)th$ moment of rth order statistics as:

$$\begin{aligned}\mu_{r:n}^{p-1} &= E\left(X_{r:n}^{p-1}\right) = \int_{-\infty}^{\infty} x^{p-1} f_{r:n}(x)dx \\ &= \int_{-\infty}^{\infty} x^{p-1}\frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1} \\ &\quad \times [1-F(x)]^{n-r}dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1} f(x)[F(x)]^{r-1} \\ &\quad \times [1-F(x)]^{n-r}dx.\end{aligned}$$

Now for standard exponential distribution we have $f(x) = 1 - F(x)$ so we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!}\int_{0}^{\infty} x^{p-1}[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

Now integrating by parts taking x^{p-1} for integration and rest of the function for differentiation we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!}\Bigg[[F(x)]^{r-1}[1-F(x)]^{n-r+1}\frac{x^p}{p}\bigg|_0^\infty$$
$$- \int_0^\infty \Big\{(r-1)[F(x)]^{r-2}[1-F(x)]^{n-r+1} f(x)$$
$$-(n-r+1)[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)\Big\}\frac{x^p}{p}dx\Bigg]$$
$$= \frac{n!}{(r-1)!(n-r)!p}\Bigg[(n-r+1)\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$
$$-(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r+1}dx\Bigg].$$

Now splitting the first integral we have:

$$\mu_{r:n}^{p-1} = \frac{n!}{(r-1)!(n-r)!p}\Bigg[n\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$
$$-(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$
$$-(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r+1}dx\Bigg]$$
$$= \frac{n!}{(r-1)!(n-r)!p}\Bigg[n\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx$$
$$-(r-1)\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r}dx\Bigg]$$
$$= \frac{n}{p}\Bigg[\frac{n!}{(r-1)!(n-r)!}\int_0^\infty x^p f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}dx\Bigg]$$
$$-\frac{(r-1)}{p}\Bigg[\frac{n!}{(r-1)!(n-r)!}\int_0^\infty x^p f(x)[F(x)]^{r-2}[1-F(x)]^{n-r}dx\Bigg]$$

or

$$\mu_{r:n}^{p-1} = \frac{n}{p}\mu_{r:n}^{p} - \frac{n}{p}\mu_{r-1:n-1}^{p}$$
$$\text{or } \mu_{r:n}^{p} = \mu_{r-1:n-1}^{p} + \frac{p}{n}\mu_{r:n}^{p-1};$$

as required. Further, for $r = 1$ we have a special relation as:

$$\mu_{1:n}^{p} = \frac{p}{n}\mu_{1:n}^{p-1}.$$

Example 2.12 Show that for standard exponential distribution having density $f(x) = e^{-x}$; following relations holds for joint moments of order statistics:

$$\mu_{r:r+1:n} = \mu_{r:n}^2 + \frac{1}{n-r}\mu_{r:n}; \quad 1 \le r \le n-1$$

$$\text{and } \mu_{r,s:n} = \mu_{r,s-1:n} + \frac{1}{n-s-1}\mu_{r:n}; \; 1 \le r < s \le n, s-r \ge 2.$$

Solution: We first obtain the first relation. Consider:

$$\begin{aligned}
\mu_{r:n} = E\left(X_{r:n}X_{s:n}^0\right) &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 x_2^0 f_{r,s:n}(x_1,x_2)dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_2}^{\infty} x_1 f(x_1)f(x_2)[F(x_1)]^{r-1} \\
&\quad \times \Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}dx_2dx_1;
\end{aligned}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. Now for standard exponential distribution we have $f(x) = 1 - F(x)$ and hence writing $f(x_2) = 1 - F(x_2)$ above equation can be written as:

$$\begin{aligned}
\mu_{r:n} &= C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}\Big[F(x_2) - F(x_1)\Big]^{s-r-1} \\
&\quad \times \Big[1 - F(x_2)\Big]^{n-s+1} dx_2dx_1 \\
&= C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}I(x_2)dx_1; \qquad (2.85)
\end{aligned}$$

where

$$I(x_2) = \int_{x_1}^{\infty}\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s+1}dx_2.$$

Now integrating above by parts, using dx_2 for integration and rest as differentiation we have

$$\begin{aligned}
I(x_2) = \Big[F(x_2) - F(x_1)\Big]^{s-r-1}&[1-F(x_2)]^{n-s+1}x_2\Big|_{x_1}^{\infty} \\
&- \int_{x_1}^{\infty}\Big[\Big\{(s-r-1)[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1}f(x_2)\Big\} \\
&\Big\{-(n-s+1)\Big[F(x_2) - F(x_1)\Big]^{s-r-1}[1-F(x_2)]^{n-s}f(x_2)\Big\}\Big]x_2dx_2
\end{aligned}$$

or

$$I(x_2) = \left[F(x_2) - F(x_1)\right]^{s-r-1}[1 - F(x_2)]^{n-s+1}x_2\Big|_{x_1}^{\infty}$$
$$+(n-s+1)\int_{x_1}^{\infty} x_2\left[F(x_2) - F(x_1)\right]^{s-r-1}[1-F(x_2)]^{n-s}$$
$$\times f(x_2)dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1-F(x_2)]^{n-s+1}f(x_2)dx_2. \quad (2.86)$$

Now for $s = r + 1$ in (2.86) we have

$$I(x_2) = (n-r)\int_{x_1}^{\infty} x_2[1-F(x_2)]^{n-r-1}f(x_2)dx_2$$
$$-x_1[1-F(x_1)]^{n-r}. \quad (2.87)$$

Now using (2.87) in (2.85); with $C_{r,s:n} = \frac{n!}{(r-1)!(n-r-1)!}$; we have:

$$\mu_{r:n} = C_{r,s:n}\int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}\Big[(n-r)\int_{x_1}^{\infty} x_2[1-F(x_2)]^{n-r-1}$$
$$\times f(x_2)dx_2 - x_1[1-F(x_1)]^{n-r}\Big]dx_1$$
$$= \frac{n!}{(r-1)!(n-r-1)!}(n-r)\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)$$
$$\times [F(x_1)]^{r-1}[1-F(x_2)]^{n-r-1}dx_2dx_1$$
$$-\frac{n!}{(r-1)!(n-r-1)!}\int_{-\infty}^{\infty} x_1^2 f(x_1)[F(x_1)]^{r-1}$$
$$\times [1-F(x_1)]^{n-r}dx_1$$

or

$$\mu_{r:n} = (n-r)\mu_{r,r+1:n} - (n-r)\mu_{r:n}^2$$

or

$$\mu_{r,r+1:n} = \mu_{r:n}^2 + \frac{1}{n-r}\mu_{r:n};$$

as required. Again for $s - r \geq 2$ we have, from (2.86):

$$I(x_2) = (n-s+1)\int_{x_1}^{\infty} x_2\left[F(x_2) - F(x_1)\right]^{s-r-1}[1-F(x_2)]^{n-s}$$
$$\times f(x_2)dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2[F(x_2) - F(x_1)]^{s-r-2}$$
$$\times [1-F(x_2)]^{n-s+1}f(x_2)dx_2. \quad (2.88)$$

Now using (2.88) in (2.85) we have:

$$\mu_{r:n} = C_{r,s:n} \int_{-\infty}^{\infty} x_1 f(x_1)[F(x_1)]^{r-1}$$
$$\Bigg\{(n-s+1)\int_{x_1}^{\infty} x_2 \Big[F(x_2) - F(x_1)\Big]^{s-r-1}$$
$$\times [1-F(x_2)]^{n-s} f(x_2)dx_2 - (s-r-1)\int_{x_1}^{\infty} x_2$$
$$[F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1} f(x_2)dx_2\Big\}dx_1$$

or

$$\mu_{r:n} = (n-s+1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times \Big[F(x_2)-F(x_1)\Big]^{s-r-1} [1-F(x_2)]^{n-s}dx_2dx_1$$
$$-(s-r-1)C_{r,s:n}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1x_2 f(x_1)f(x_2)[F(x_1)]^{r-1}$$
$$\times [F(x_2)-F(x_1)]^{s-r-2}[1-F(x_2)]^{n-s+1} f(x_2)dx_2dx_1$$

or

$$\mu_{r:n} = (n-s+1)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n}$$

or

$$\mu_{r,s:n} = \mu_{r,s-1:n} + \frac{1}{n-s+1}\mu_{r:n};$$

as required.

2.10 Relations for Moments of Order Statistics for Special Class of Distributions

In previous section we have discussed recurrence relations for single and product moments of order statistics for certain distributions. We have seen that the recurrence relations for single moments of order statistics can be derived from (2.89) as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty} x^{p-1}$$
$$\times [F(x)]^{r-1}[1-F(x)]^{n-r+1}dx. \tag{2.89}$$

Using probability integral transform above relation can be written as

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1} \times \left\{F^{-1}(t)\right\}^{/} t^{r-1}(1-t)^{n-r+1}dt. \tag{2.90}$$

We now present a recurrence relation for single moments of order statistics for class of distributions having special structure of $\left\{F^{-1}(t)\right\}^{/}$. The relation is given in the following.

Theorem: For the class of distributions defined as

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{p_1}(1-t)^{q-p_1-1} \text{ on } (0,1);$$

the following relation holds for single moments of order statistics

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = pC(r,n,p_1,q)\mu_{r+p_1:n+q}^{p-1}; \tag{2.91}$$

where

$$C(r,n,p_1,q) = \mu_{r:n} - \mu_{r-1:n} = \frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}.$$

Proof We have from (2.90)

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1} \times \left\{F^{-1}(t)\right\}^{/} t^{r-1}(1-t)^{n-r+1}dt. \tag{2.92}$$

Now using the representation for $\left\{F^{-1}(t)\right\}^{/}$ we have

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1} \times \frac{1}{d}t^{p_1}(1-t)^{q-p_1-1}t^{r-1}(1-t)^{n-r+1}dt.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = \frac{1}{d}\frac{p}{n-r+1}\frac{n!}{(r-1)!(n-r)!}\int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1} \times t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt.$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = p\frac{1}{d}\frac{n!}{(r-1)!(n-r+1)!}\frac{(r+p_1)\binom{n+q}{r+p_1}}{(r+p_1)\binom{n+q}{r+p_1}} \times \int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = p\frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}(r+p_1)\binom{n+q}{r+p_1} \times \int_{-\infty}^{\infty}\left\{F^{-1}(t)\right\}^{p-1}t^{r+p_1-1}(1-t)^{(n+q)-(r+p_1)+1}dt$$

or

$$\mu_{r:n}^{p} - \mu_{r-1:n}^{p} = pC(r,n,p_1,q)\mu_{r+p_1:n-q}^{p-1};$$

as required. Using $p = 1$ we can readily see that

$$C(r,n,p_1,q) = \mu_{r:n} - \mu_{r-1:n} = \frac{1}{d}\frac{\binom{n}{r-1}}{(r+p_1)\binom{n+q}{r+p_1}}.$$

The class of distribution defined as

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{p_1}(1-t)^{q-p_1-1};$$

give rise to several special distributions for suitable choices of p_1 and q. Some of these special cases are given below.

1. For $p_1 = 0$ and $q = 0$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log\left(\frac{1}{1-t}\right)$$

or

$$t = F(x) = 1 - e^{-dx};$$

that is the Exponential Distribution.

2. For $p_1 = 0$ and $q \neq 0$ we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{q-1}$$

or

$$F^{-1}(t) = -\frac{1}{dq}(1-t)^q$$

or

$$t = F(x) = 1 - (dqx)^{1/q};$$

which for $q < 0$ provides Pareto distribution and for $q > 0$ provides Pearson type I distribution.

3. For $p_1 \neq -1$ and $q = p + 1$ we have

$$\left\{F^{-1}(t)\right\}' = \frac{1}{d}t^{q-1}$$

or

$$F^{-1}(t) = \frac{1}{dq}t^q$$

or

$$t = F(x) = (dqx)^{1/q};$$

which for $q > 0$ provides Power function distribution.

4. For $p = -1$ and $q = 0$ we have

$$\left\{F^{-1}(t)\right\}' = \frac{1}{d}t^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log(t)$$

or

$$t = F(x) = e^{dx};$$

which is reflected Exponential distribution.

5. For $p = -1$ and $q = -1$

$$\left\{F^{-1}(t)\right\}' = \frac{1}{d}t^{-1}(1-t)^{-1}$$

or

$$F^{-1}(t) = \frac{1}{d}\log\left(\frac{t}{1-t}\right)$$

or

$$t = F(x) = \frac{1}{1+e^{-dx}};$$

which is the Logistic distribution.

We can obtain other distributions for various choices of p_1 and q. The recurrence relations for these special cases can be directly obtained from (2.91) by using the corresponding values.

2.11 Reversed Order Statistics

Reversed Order Statistics appear frequently when data is arranged in descending order of magnitude, say for example marks of students arranged from highest to lowest or population of cities; in million; arranged in decreasing order. The distribution theory of such variables can be studied in the context of *Reversed Order Statistics* which appear as a special case of Dual Generalized Order Statistics; discussed in Chap. 5. The reversed order statistics and their distribution are defined in the following.

Let $x_1,\ x_2,\ \ldots, x_n$ be a random sample from a distribution $F(x)$ and suppose that the sample is arranged in descending order as $x_1 \geq x_2 \geq \cdots \geq x_n$ then this descendingly ordered sample constitute the reversed order statistics. The joint distribution of n reversed order statistics is same as the joint distribution of ordinary order statistics. The joint marginal distribution of r reversed order statistics is given as

$$f_{1(re),\ldots,r(re):n}(x_1,\ldots,x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right]\{F(x_r)\}^{n-r}. \tag{2.93}$$

Further, the marginal distribution of rth reversed order statistics and joint marginal distribution of rth and sth reversed order statistics; for $r < s$; are easily written from (2.93) as

$$f_{r(re):n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1}. \tag{2.94}$$

and

$$f_{r(re),s(re):n}(x_1,x_2) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_1)f(x_2)\{1-F(x_1)\}^{r-1} \times [F(x_1)-F(x_2)]^{s-r-1}\{F(x_2)\}^{n-s}. \tag{2.95}$$

We can readily see that the distribution of rth reversed order statistics from distribution $F(x)$ is same as the the distribution of $(n-r+1)$th ordinary order statistics from the distribution $F(x)$.

Example 2.13 A random sample is available from the density

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha});\ x,\alpha > 0$$

Obtain the marginal density function of rth reversed order statistics and joint density function of rth and sth reversed order statistics for this distribution.

Solution: The density function of rth reversed order statistics and joint density function of rth and sth reversed order statistics are given in (2.94) and (2.95) as

$$f_{r(re):n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1}.$$

and

$$f_{r(re),s(re):n}(x_1,x_2) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} f(x_1)f(x_2)\{1-F(x_1)\}^{r-1} \times [F(x_1)-F(x_2)]^{s-r-1}\{F(x_2)\}^{n-s}.$$

Now we have

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha})$$

So

$$F(x) = \int_0^x f(t)dt = \int_0^x \alpha t^{\alpha-1}\exp(-t^{\alpha})dt = 1-\exp(-x^{\alpha}).$$

The density function of rth reversed order statistics is therefore

$$\begin{aligned} f_{r(re):n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1-F(x)\}^{r-1} \\ &= \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\exp(-x^{\alpha}) \\ &\quad \times \{1-\exp(-x^{\alpha})\}^{n-r}\{\exp(-x^{\alpha})\}^{r-1} \\ &= \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\exp(-rx^{\alpha}) \\ &\quad \times \sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\exp(-jx^{\alpha}) \end{aligned}$$

or

$$f_{r(re):n}(x) = \frac{n!}{(r-1)!(n-r)!}\alpha x^{\alpha-1}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j} \times \exp\left\{-x^{a}(r+j)\right\}.$$

Again the joint density of rth and sth reversed order statistics is

$$\begin{aligned} f_{r(re),s(re):n}(x_1,x_2) = {} & C_{r,s:n}\alpha x_1^{\alpha-1}\exp\left(-x_1^{\alpha}\right)\alpha x_2^{\alpha-1}\exp\left(-x_2^{\alpha}\right) \\ & \times \left\{\exp\left(-x_1^{\alpha}\right)\right\}^{r-1}\left[\exp\left(-x_1^{\alpha}\right)-\exp\left(-x_2^{\alpha}\right)\right]^{s-r-1} \\ & \times \left\{1-\exp\left(-x_2^{\alpha}\right)\right\}^{n-s}, \end{aligned}$$

or

$$\begin{aligned} f_{r(re),s(re):n}(x_1,x_2) = {} & C_{r,s:n}\alpha^2 x_1^{\alpha-1}x_2^{\alpha-1}\exp\left(-rx_1^{\alpha}\right)\exp\left(-x_2^{\alpha}\right) \\ & \times \sum_{j=0}^{s-r-1}(-1)^j\binom{s-r-1}{j}\exp\left\{-x_1^{\alpha}(s-r-j-1)\right\} \\ & \times \exp\left(-jx_2^{\alpha}\right)\sum_{k=0}^{n-s}(-1)^k\binom{n-s}{k}\exp\left(-kx_2^{\alpha}\right). \end{aligned}$$

or

$$\begin{aligned} f_{r(re),s(re):n}(x_1,x_2) = {} & C_{r,s:n}\alpha^2 x_1^{\alpha-1}x_2^{\alpha-1}\sum_{k=0}^{n-s}\sum_{j=0}^{s-r-1}(-1)^{j+k}\binom{n-s}{k} \\ & \times \binom{s-r-1}{j}\exp\left\{-x_1^{\alpha}(s-j-1)\right\} \\ & \times \exp\left\{-x_2^{\alpha}(j+k+1)\right\}, \end{aligned}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Chapter 3
Record Values

3.1 Introduction

Several situations arise where one is interested in studying the behavior of observation(s) which exceed already established maximum. For example one might be interested to know the new highest stock price of a product or new highest runs score in an innings of a cricket match. These are the examples where we are talking about the *Upper Records*. Upper Records naturally arises in daily life whenever we want to replace an already recorded maximum observation whit one which exceeds that value.

The upper record values have widespread applications in several areas of life and have attracted number of statisticians to study the distributional behavior of upper record values when underlying phenomenon follow some specific distribution. The concept of upper record values was first introduced by Chandler (1952) and are formerly defined as below.

Suppose that we have a sequence of independently and identically distributed random variables $X_1, X_2, \ldots$ having the distribution function $F(x)$. Suppose $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We call X_j is an Upper Record Value of the sequence $\{X_n, n \geq 1\}$ if $Y_j > Y_{j-1}$. From this definition it is clear that X_1 is an upper record value. We also associate the indices to each record value with which they occur. These indices are called the record time $\{U(n)\}, n > 0$ where

$$U(n) = \min\left\{j | j > U(n-1), X_j > X_{U(n-1)}, n > 1\right\}.$$

We can readily see that $U(1) = 1$. We will denote the upper record values by $X_{U(n)}$. Many authors have characterized various probability distributions by using the upper record values. Some notable references are Ahsanullah (1978, 1979, 1986, 1991a, 1991b), Ahsanullah and Holand (1984), Ahsanullah and Shakil (2011), Balakrishnan,

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_3

Ahsanullah and Chan (1992), Balakrishnan and Ahsanullah (1993, 1995), Balakrishnan and Balasubramanian (1995), Nevzorov (1995), Raqab and Ahsanullah (2000), Raqab (2002) and Shakil and Ahsanullah (2011). The probability density function of upper record values is given in the following section.

3.2 Marginal and Joint Distribution of Upper Record Values

The distribution theory of upper record values ask for special attention. The upper records depends upon certain hazards which arise in their occurrence and hence are based upon the hazard rate function of the probability distribution followed by the sequence of observations. The probability distribution of upper record values is given in the following.

Suppose that we have a sequence of independent random variables $X_1, X_2, \ldots$ each having the same density function $f(x)$ and distribution function $F(x)$. Suppose further that the hazard rate function of nth member of the sequence is

$$r(x_n) = \frac{f(x)}{1 - F(x)};$$

as each member has same distribution. Further, the total hazard rate is

$$\begin{aligned} R(x) &= \int r(x_n)\, dx = \int \frac{f(x)}{1 - F(x)} dx \\ &= -\ln[1 - F(x)]. \end{aligned}$$

Based upon above notations Ahsanullah (2004) has given the joint density of n upper records as

$$f_{X_{U(1)},\ldots,X_{U(n)}}(x_1, \ldots, x_n) = \left[\prod_{i=1}^{n-1} r(x_i)\right] f(x_n). \tag{3.1}$$

The joint density function of n upper records given in (3.1) provide basis to derive the marginal density function of $X_{U(n)}$ and joint marginal density of two upper records $X_{U(m)}$ and $X_{U(n)}$ for $m < n$. Following Ahsanullah (2004), the marginal density function of $X_{U(n)}$ is given in the following.

Let $F_{U(n)}(x)$ be the distribution function of $X_{U(n)}$, then by definition we have

$$F_{X_{U(1)}}(x) = P\left(X_{U(1)} \le x\right) = F(x).$$

Again the distribution function of $X_{U(2)}$ is

$$\begin{aligned} F_{X_{U(2)}}(x) &= P\left(X_{U(2)} \le x\right) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\sum_{i=1}^{\infty}[F(u)]^{i-1}f(u)f(y)\,dudy \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{f(u)}{1-F(u)}f(y)\,dudy \\ &= \int_{-\infty}^{x}R(y)f(y)\,dy. \end{aligned}$$

The density function of $X_{U(2)}$ is readily written from above as

$$f_{X_{U(2)}}(x) = R(x)f(x)\,;\ -\infty < x < \infty.$$

Once again the distribution function of $X_{U(3)}$ is

$$\begin{aligned} F_{X_{U(3)}}(x) &= P\left(X_{U(3)} \le x\right) \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\sum_{i=1}^{\infty}[F(u)]^{i-1}R(u)f(u)f(y)\,dudy \\ &= \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{f(u)}{1-F(u)}R(u)f(y)\,dudy \\ &= \frac{1}{2!}\int_{-\infty}^{x}\left[R(y)\right]^{2}f(y)\,dy; \end{aligned}$$

and the density function of $X_{U(3)}$ is

$$f_{X_{U(3)}}(x) = \frac{1}{2!}[R(x)]^{2}f(x)\,;\ -\infty < x < \infty.$$

Proceeding in the same way, the distribution function of $X_{U(n)}$ is

$$\begin{aligned} F_{X_{U(n)}}(x) &= P\left(X_{U(n)} \le x\right) \\ &= \int_{-\infty}^{x}f(u_n)\,du_n\int_{-\infty}^{u_n}\frac{f(u_{n-1})}{1-F(u_{n-1})}du_{n-1} \\ &\quad \times\cdots\times\int_{-\infty}^{u_2}\frac{f(u_1)}{1-F(u_1)}du_1 \\ &= \frac{1}{(n-1)!}\int_{-\infty}^{x}\left[R(y)\right]^{n-1}f(y)\,dy. \end{aligned} \tag{3.2}$$

The density function of $X_{U(n)}$ is readily written from (3.2) as

$$f_{X_{U(n)}}(x) = \frac{1}{(n-1)!}[R(x)]^{n-1} f(x)$$
$$= \frac{1}{\Gamma(n)}[R(x)]^{n-1} f(x);\ -\infty < x < \infty. \tag{3.3}$$

The joint density function of two upper record values $X_{U(m)}$ and $X_{U(n)}$; $m < n$; has been given by Ahsanullah (2004) as

$$f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2)[R(x_1)]^{m-1}$$
$$[R(x_2) - R(x_1)]^{n-m-1}, \tag{3.4}$$

for $-\infty < x_1 < x_2 < \infty$.

If we make the transformation $v = R(x)$ in (3.3), then we can see that the density function of $R(x)$ is

$$f_V(v) = \frac{1}{\Gamma(n)} v^{n-1} e^{-v};\ v > 0, \tag{3.5}$$

that is $v = R(x)$ has Gamma distribution with shape parameter n. The joint density of $v_1 = R(x_1)$ and $v_2 = R(x_2)$ is obtained as below.

We have

$$f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2)[R(x_1)]^{m-1}$$
$$[R(x_2) - R(x_1)]^{n-m-1}.$$

Making the transformation $v_1 = R(x_1)$ and $v_2 = R(x_2)$ we have

$$r(x_1) = \frac{f(x_1)}{1 - F(x_1)} = \frac{f(x_1)}{e^{-v_1}} = f(x_1)\, e^{v_1}.$$

The Jacobbian of transformation from x_1 and x_2 to v_1 and v_2 is

$$|J| = \frac{1}{f(x_1) f(x_2)} e^{-v_1} e^{-v_2}.$$

The joint density of v_1 and v_2 is, therefore

$$f(v_1, v_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} f(x_1)\, e^{v_1} f(x_2)\, v_1^{m-1}$$
$$\times\ (v_2 - v_1)^{n-m-1} \frac{1}{f(x_1) f(x_2)} e^{-v_1} e^{-v_2}$$
$$= \frac{1}{\Gamma(m)\Gamma(n-m)} v_1^{m-1} (v_2 - v_1)^{n-m-1} e^{-v_2}$$

or

$$f(v_1, v_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} v_1^{m-1}(v_2 - v_1)^{n-m-1} e^{-v_2}, \tag{3.6}$$

for $0 < v_1 < v_2 < \infty$.

Again consider the joint density of $X_{U(m)}$ and $X_{U(n)}$ as

$$f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1}.$$

or

$$f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} [R(x_2)]^{n-m-1} \left[1 - \frac{R(x_1)}{R(x_2)}\right]^{n-m-1}.$$

Now making the transformation $w_1 = R(x_1)$ and $w_2 = R(x_1)/R(x_2)$, the joint density function of w_1 and w_2 is

$$f(w_1, w_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} f(x_1) e^{w_1} f(x_2) w_1^{m-1} \left(\frac{w_1}{w_2}\right)^{n-m-1} \times (1 - w_2)^{n-m-1} \frac{1}{f(x_1) f(x_2)} \frac{w_1}{w_2^2} e^{-w_1} e^{-w_1/w_2}$$

or

$$f(w_1, w_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} w_1^{n-1} \frac{(1 - w_2)^{n-m-1}}{w_2^{n-m+1}} e^{-w_1/w_2},$$

for $w_1 > 0$ and $0 < w_2 < 1$. The marginal density of w_2 is

$$\begin{aligned} f(w_2) &= \int_0^\infty f(w_1, w_2)\, dw_1 \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)} \frac{1}{w_2^{n-m+1}} (1 - w_2)^{n-m-1} \times \int_0^\infty w_1^{n-1} e^{-w_1/w_2} dw_1 \end{aligned}$$

or

$$f(w_2) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} w_2^{m-1} (1 - w_2)^{n-m-1}; \; 0 < w_2 < 1; \tag{3.7}$$

that is $w_2 = R(x_1)/R(x_2)$ has Beta distribution with parameters m and $n - m$.

Example 3.1 Find the distribution of upper record $X_{U(n)}$ and joint distribution of two upper records $X_{U(m)}$ and $X_{U(n)}$ for exponential distribution with density

$$f(x) = \alpha e^{-\alpha x};\ \alpha, x > 0.$$

Solution: The density function of $X_{U(n)}$ is given in (3.3) as

$$f_{X_{U(n)}}(x) = \frac{1}{\Gamma(n)}[R(x)]^{n-1} f(x);\ -\infty < x < \infty;$$

where $R(x) = -\ln[1 - F(x)]$. Now for given distribution we have

$$F(x) = \int_0^x f(u)\,du = \int_0^x \alpha e^{-\alpha u} du = 1 - e^{-\alpha x};\ x > 0.$$

So

$$R(x) = -\ln[1 - F(x)] = -\ln\left(e^{-\alpha x}\right) = \alpha x.$$

Hence the density function of $X_{U(n)}$ is

$$\begin{aligned} f_{X_{U(n)}}(x) &= \frac{1}{\Gamma(n)}(\alpha x)^{n-1}\alpha e^{-\alpha x} \\ &= \frac{\alpha^n}{\Gamma(n)} x^{n-1} e^{-\alpha x};\ x > 0, \end{aligned}$$

which is Gamma distribution with shape parameter n and scale parameter α.

Again the joint density function of $X_{U(m)}$ and $X_{U(n)}$ is given in (3.4) as

$$\begin{aligned} f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = {} & \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2)[R(x_1)]^{m-1} \\ & [R(x_2) - R(x_1)]^{n-m-1}. \end{aligned}$$

Now for given distribution we have

$$r(x_1) = \frac{f(x_1)}{1 - F(x_1)} = \frac{\alpha e^{-\alpha x_1}}{e^{-\alpha x_1}} = \alpha,$$

hence the joint density of $X_{U(m)}$ and $X_{U(n)}$ is

$$\begin{aligned} f_{X_{U(m)},X_{U(n)}}(x_1, x_2) &= \frac{\alpha^2 e^{-\alpha x_2}}{\Gamma(m)\Gamma(n-m)}(\alpha x_1)^{m-1}(\alpha x_2 - \alpha x_1)^{n-m-1} \\ &= \frac{\alpha^n}{\Gamma(m)\Gamma(n-m)} x_1^{m-1}(x_2 - x_1)^{n-m-1} e^{-\alpha x_2}, \end{aligned}$$

for $0 < x_1 < x_2 < \infty$.

Example 3.2 Obtain the marginal distribution of $X_{U(n)}$ and joint distribution of $X_{U(m)}$ and $X_{U(n)}$ for $m < n$ if a sequence of random variables have Weibull distribution with density

$$f(x) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

Solution: The density function of $X_{U(n)}$ is given in (3.3) as

$$f_{X_{U(n)}}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x);\ -\infty < x < \infty,$$

where $R(x) = -\ln\left[1 - F(x)\right]$. Now for given distribution we have

$$\begin{aligned} F(x) &= \int_0^x f(u)\,du = \int_0^x \alpha\beta x^{\beta-1}\exp\left(-\alpha u^{\beta}\right)du \\ &= 1 - \exp\left(-\alpha x^{\beta}\right);\ x > 0. \end{aligned}$$

So

$$\begin{aligned} R(x) &= -\ln\left[1 - F(x)\right] \\ &= -\ln\left[\exp\left(-\alpha x^{\beta}\right)\right] = \alpha x^{\beta}. \end{aligned}$$

The density function of $X_{U(n)}$ is therefore

$$\begin{aligned} f_{X_{U(n)}}(x) &= \frac{1}{\Gamma(n)}\left(\alpha x^{\beta}\right)^{n-1}\alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right) \\ &= \frac{\alpha^n\beta}{\Gamma(n)}x^{n\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta, n > 0. \end{aligned}$$

Again the joint density of $X_{U(m)}$ and $X_{U(n)}$ is given as

$$\begin{aligned} f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)}&r(x_1) f(x_2)\left[R(x_1)\right]^{m-1} \\ &\left[R(x_2) - R(x_1)\right]^{n-m-1}. \end{aligned}$$

Now for given distribution we have

$$\begin{aligned} r(x_1) &= \frac{f(x_1)}{1 - F(x_1)} = \\ &= \frac{\alpha\beta x_1^{\beta-1}\exp\left(-\alpha x_1^{\beta}\right)}{\exp\left(-\alpha x_1^{\beta}\right)} = \alpha\beta x^{\beta-1}, \end{aligned}$$

hence the joint density of $X_{U(m)}$ and $X_{U(n)}$ is

$$\begin{aligned}f_{X_{U(m)},X_{U(n)}}(x_1,x_2) &= \frac{1}{\Gamma(m)\,\Gamma(n-m)}\alpha\beta x_1^{\beta-1}\alpha\beta x_2^{\beta-1}\exp\left(-\alpha x_2^{\beta}\right)\\ &\quad\times\left(\alpha x_1^{\beta}\right)^{m-1}\left(\alpha x_2^{\beta}-\alpha x_1^{\beta}\right)^{n-m-1}\\ &= \frac{\alpha^2\beta^2}{\Gamma(m)\,\Gamma(n-m)}\exp\left(-\alpha x_2^{\beta}\right)\left(\alpha x_1^{\beta}\right)^{m-1}\\ &\quad\times\left(\alpha x_2^{\beta}-\alpha x_1^{\beta}\right)^{n-m-1}\end{aligned}$$

or

$$f_{X_{U(m)},X_{U(n)}}(x_1,x_2) = \frac{\alpha^n\beta^2 x_1^{\beta m-1}}{\Gamma(m)\,\Gamma(n-m)}\left(x_2^{\beta}-x_1^{\beta}\right)^{n-m-1}\exp\left(-\alpha x_2^{\beta}\right),$$

for $0 < x_1 < x_2 < \infty$. We can see that the distribution of $X_{U(n)}$ and joint distribution of $X_{U(m)}$ and $X_{U(n)}$ in case of Weibull distribution reduces to the same for exponential distribution when $\beta = 1$.

Example 3.3 Find the distribution of $X_{U(n)}$ and joint distribution of $X_{U(m)}$ and $X_{U(n)}$ for Lomax distribution with density

$$f(x) = \frac{\alpha}{(1+x)^{\alpha+1}};\ x > 0, \alpha > 1.$$

Solution: The distribution of $X_{U(n)}$ is given as

$$f_{X_{U(n)}}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x);\ -\infty < x < \infty;$$

where $R(x) = -\ln\left[1 - F(x)\right]$. Now for given distribution we have

$$\begin{aligned}F(x) &= \int_0^x f(u)\,du = \int_0^x \frac{\alpha}{(1+u)^{\alpha+1}}du\\ &= 1 - \frac{1}{(1+x)^{\alpha}};\ x > 0.\end{aligned}$$

So

$$\begin{aligned}R(x) &= -\ln\left[1 - F(x)\right]\\ &= -\ln\left[\frac{1}{(1+x)^{\alpha}}\right] = \alpha\ln\left[(1+x)\right].\end{aligned}$$

The density function of $X_{U(n)}$ is therefore

$$\begin{aligned} f_{X_{U(n)}}(x) &= \frac{1}{\Gamma(n)}\left[\alpha \ln(1+x)\right]^{n-1} \frac{\alpha}{(1+x)^{\alpha+1}} \\ &= \frac{\alpha^n}{\Gamma(n)} \frac{\left[\ln(1+x)\right]^{n-1}}{(1+x)^{\alpha+1}}. \end{aligned}$$

Again, the joint density of $X_{U(m)}$ and $X_{U(n)}$ is

$$\begin{aligned} f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = &\frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) \left[R(x_1)\right]^{m-1} \\ &\times \left[R(x_2) - R(x_1)\right]^{n-m-1}. \end{aligned}$$

Now for given distribution we have

$$\begin{aligned} r(x_1) &= \frac{f(x_1)}{1 - F(x_1)} = \\ &= \frac{\alpha (1+x_1)^{\alpha}}{(1+x_1)^{\alpha+1}} = \frac{\alpha}{(1+x_1)}, \end{aligned}$$

hence the joint density of $X_{U(m)}$ and $X_{U(n)}$ is

$$\begin{aligned} f_{X_{U(m)},X_{U(n)}}(x_1, x_2) = &\frac{\alpha^n}{\Gamma(m)\Gamma(n-m)} \frac{1}{(1+x_1)} \frac{1}{(1+x_2)^{\alpha+1}} \\ &\times \left[\ln(1+x_1)\right]^{m-1} \left[\ln\left(\frac{1+x_2}{1+x_1}\right)\right]^{n-m-1}, \end{aligned}$$

for $x_1 < x_2$.

3.3 Conditional Distributions of Record Values

In previous section we have discussed the marginal distribution of an upper record and joint distribution of two upper records. These distributions can be used to derive the conditional distributions of upper records. In the following we discuss the conditional distributions of upper records.

The marginal distribution of upper record $X_{U(n)}$ is given in (3.3) as

$$f_{X_{U(n)}}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x); \; -\infty < x < \infty,$$

and the joint distribution of $X_{U(m)}$ and $X_{U(n)}$ is given in (3.4) as

$$f_{X_{U(m)},X_{U(n)}}(x_1,x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} \times [R(x_2) - R(x_1)]^{n-m-1}.$$

Now the conditional distribution of $X_{U(n)}$ given $X_{U(m)} = x_1$ is

$$\begin{aligned} f_{X_{U(n)}|x_1}(x_2|x_1) &= \frac{f_{X_{U(m)},X_{U(n)}}(x_1,x_2)}{f_{X_{U(m)}}(x_1)} \\ &= \left[\frac{1}{\Gamma(m)\Gamma(n-m)} \frac{f(x_1)}{1-F(x_1)} f(x_2) \times [R(x_1)]^{m-1} [R(x_2)-R(x_1)]^{n-m-1}\right] \\ &\quad \Big/ \left[\frac{1}{\Gamma(m)} [R(x_1)]^{n-1} f(x_1)\right] \end{aligned}$$

or

$$f_{X_{U(n)}|x_1}(x_2|x_1) = \frac{1}{\Gamma(n-m)} \frac{f(x_2)}{1-F(x_1)} [R(x_2) - R(x_1)]^{n-m-1}. \tag{3.8}$$

Using $n = m + 1$; the conditional distribution of two contiguous records is

$$f_{X_{U(m+1)}|x_1}(x_{m+1}|x_m) = \frac{f(x_{m+1})}{1-F(x_m)}. \tag{3.9}$$

The conditional distributions are very useful in studying certain behaviors of records.

Example 3.4 Obtain the conditional distribution of $X_{U(n)}$ given $X_{U(m)} = x_1$ for Weibull distribution with density

$$f(x) = \alpha\beta x^{\beta-1} \exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

Solution: The conditional distribution of $X_{U(n)}$ given $X_{U(m)} = x_1$ is given as

$$f_{X_{U(n)}|x_1}(x_2|x_1) = \frac{1}{\Gamma(n-m)} \frac{f(x_2)}{1-F(x_1)} [R(x_2) - R(x_1)]^{n-m-1}.$$

For Weibull distribution we have

$$\begin{aligned} F(x) &= 1 - \exp\left(-\alpha x^{\beta}\right) \\ R(x) &= -\ln[1 - F(x)] = \alpha x^{\beta}. \end{aligned}$$

The conditional distribution is, therefore

$$f_{X_{U(n)}|x_1}(x_2|x_1) = \frac{1}{\Gamma(n-m)} \frac{\alpha\beta x_2^{\beta-1}\exp\left(-\alpha x_2^{\beta}\right)}{\exp\left(-\alpha x_1^{\beta}\right)} \times \left(\alpha x_2^{\beta} - \alpha x_1^{\beta}\right)^{n-m-1}$$

or

$$f_{X_{U(n)}|x_1}(x_2|x_1) = \frac{\alpha^{n-m}\beta x_2^{\beta-1}}{\Gamma(n-m)} \left(x_2^{\beta} - x_1^{\beta}\right)^{n-m-1} \times \exp\left[-\alpha\left(x_2^{\beta} - x_1^{\beta}\right)\right],$$

for $0 < x_1 < x_2 < \infty$. Using $n = m + 1$ the conditional distribution of two contiguous records is

$$f_{X_{U(m+1)}|x_m}(x_{m+1}|x_m) = \frac{\alpha\beta x_{m+1}^{\beta-1}\exp\left(-\alpha x_{m+1}^{\beta}\right)}{\exp\left(-\alpha x_m^{\beta}\right)},$$

3.4 Record Values as Markov Chain

In previous section we have derived the conditional distribution of $X_{U(n)}$ given $X_{U(m)}$. The conditional distribution can further be used to study another important property of record values that is the Markoven property. In the following we show that the record values follows the Markov chain just like the order statistics.

We know that a sequence of random variables $X_1, X_2, \ldots, X_m, X_n$ has Markov chain property if the conditional distribution of X_n given $X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m$ is same as the conditional distribution of X_n given $X_m = x_m$; that is if

$$f(x_s|X_1 = x_1, \ldots, X_r = x_r) = f(x_s|X_r = x_r).$$

The record values will follow the Markov chain if

$$f_{X_{U(n)}|x_1,\ldots,x_m}(x_n|x_1, \ldots, x_m) = f_{X_{U(n)}|x_m}(x_n|x_m).$$

Now from (3.8) we have

$$f_{X_{U(n)}|x_m}(x_n|x_m) = \frac{1}{\Gamma(n-m)} \frac{f(x_n)}{1-F(x_m)} \left[R(x_n) - R(x_m)\right]^{n-m-1}$$

Further, the joint distribution of m upper record values is given in (3.1) as

$$f_{X_{U(1)},\ldots,X_{U(m)}}(x_1,\ldots,x_m)=\left[\prod_{i=1}^{m-1} r(x_i)\right] f(x_m)$$

Also, the joint distribution of $X_{U(1)}, X_{U(2)},\ldots,X_{U(m)}$ and $X_{U(n)}$ is immediately written as

$$f_{X_{U(1)},\ldots,X_{U(m)},X_{U(n)}}(x_1,\ldots,x_m,x_n)=\frac{1}{\Gamma(n-m)}\left[\prod_{i=1}^{m} r(x_i)\right] f(x_n) \times [R(x_n)-R(x_m)]^{n-m-1}. \tag{3.10}$$

Now, the conditional distribution of $X_{U(n)}$ given $X_{U(1)}=x_1,\ldots,X_{U(m)}=x_m$ is obtained from (3.1) and (3.10) as

$$\begin{aligned} f_{X_{U(n)}|x_1,\ldots,x_m}(x_n|x_1,\ldots,x_m) &= \frac{f_{X_{U(1)},\ldots,X_{U(m)},X_{U(n)}}(x_1,\ldots,x_m,x_n)}{f_{X_{U(1)},\ldots,X_{U(m)}}(x_1,\ldots,x_m)} \\ &= \left[\frac{1}{\Gamma(n-m)}\left\{\prod_{i=1}^{m} r(x_i)\right\} f(x_n) \times [R(x_n)-R(x_m)]^{n-m-1}\right] \\ &\quad \Big/ \left[\left\{\prod_{i=1}^{m-1} r(x_i)\right\} f(x_m)\right] \end{aligned}$$

or

$$f_{X_{U(n)}|x_1,\ldots,x_m}(x_n|x_1,\ldots,x_m)=\frac{1}{\Gamma(n-m)}\frac{r(x_m)}{f(x_m)} f(x_n) \times [R(x_n)-R(x_m)]^{n-m-1}$$

or

$$f_{X_{U(n)}|x_1,\ldots,x_m}(x_n|x_1,\ldots,x_m)=\frac{1}{\Gamma(n-m)}\frac{f(x_n)}{1-F(x_m)} \times [R(x_n)-R(x_m)]^{n-m-1},$$

which is same as (3.8). Hence the upper record values follow the Markov chain. The transition probabilities of upper record values are computed below

$$\begin{aligned} P\left(X_{U(m+1)} \geq y|X_{U(m)} = x_m\right) &= \int_y^\infty f_{X_{U(m+1)}|x_1}\left(x_{m+1}|x_m\right) dx_{m+1} \\ &= \int_y^\infty \frac{f\left(x_{m+1}\right)}{1 - F\left(x_m\right)} dx_{m+1} \\ &= \frac{1 - F\left(y\right)}{1 - F\left(x_m\right)}. \end{aligned}$$

The transition probabilities can be computed for different parent distribution of the sequence. For example if the sequence has Weibull distribution with shape parameter β and scale parameter α then the transition probability of upper record is given as

$$P\left(X_{U(m+1)} \geq y|X_{U(m)} = x_m\right) = \frac{\exp\left(-\alpha y^\beta\right)}{\exp\left(-\alpha x_m^\beta\right)},$$

which can be computed for various values of α and β.

3.5 The K-Upper Record Values

We have seen that the upper record values of a sequence are based upon the maximum observation of that sequence. Often we are interested in records which are not based upon maximum of the sequence; say for example records which are based upon kth largest observation of that sequence. The record values which are based upon kth maximum of a sequence are called the *k-Upper Record Values*. The k-upper record values are based upon the *k-record time*. The k-record values are formally defined as below.

Let $\{X_n;\ n \geq 1\}$ be a sequence of independently and identically distributed random variables with an absolutely continuous distribution function $F(x)$ and density function $f(x)$. Let $X_{r:n}$ be the rth order statistics based upon a sample of size n. For a fixed $k \geq 1$ the kth upper record time $U_K(n)\,;\ n \geq 1$ is defined as $U_K(1) = 1$ and

$$U_K(n+1) = \min\left\{r > U_K(n) : X_{r:r+k-1} > X_{U_K(n),U_K(n)+k-1}\right\};\ n \in \mathbb{N}.$$

The kth upper record values are $X_{U_K(n):U_K(n)+k-1}$ and for the sake of simplicity will be denoted as $X_{U_K(n)}$. The joint density of n kth upper record values is given by Dziubdziela and Kopocinski (1976) as

$$\begin{aligned} f_{U_K(1),U_K(2),\ldots,U_K(n)}\left(x_1, x_2, \ldots, x_n\right) = k^n \left\{\prod_{i=1}^{n-1} \frac{f\left(x_i\right)}{1 - F\left(x_i\right)}\right\} \\ \times \left[1 - F\left(x_n\right)\right]^{k-1} f\left(x_n\right). \end{aligned} \tag{3.11}$$

The marginal density of $X_{U_K(n)}$ is readily written as

$$f_{U_K(n)}(x) = \frac{k^n}{\Gamma(n)} [R(x)]^{n-1} [1 - F(x)]^{k-1} f(x), \tag{3.12}$$

where $R(x) = -\ln[1 - F(x)]$. The joint density of two kth upper record is given by Dziubdziela and Kopocinski (1976) as

$$f_{U_K(m),U_K(m)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} \times [R(x_2) - R(x_1)]^{n-m-1} [1 - F(x_2)]^{k-1}, \tag{3.13}$$

where $-\infty < x_1 < x_2 < \infty$. Using (3.12) and (3.13), the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(m)} = x_1$ is readily written as

$$f_{U_K(n)|x_1}(x_2|x_1) = \left[\frac{k^n}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} \times [R(x_2) - R(x_1)]^{n-m-1} [1 - F(x_2)]^{k-1}\right] / \left[\frac{k^m}{\Gamma(m)} [R(x_1)]^{m-1} [1 - F(x_1)]^{k-1} f(x_1)\right]$$

or

$$f_{U_K(n)|x_1}(x_2|x_1) = \frac{k^{n-m}}{\Gamma(n-m)} \frac{f(x_2)}{[1 - F(x_1)]^k} [1 - F(x_2)]^{k-1} \times [R(x_2) - R(x_1)]^{n-m-1}. \tag{3.14}$$

Substituting $n = m + 1$ in (3.14), the conditional distribution of two contiguous kth upper records is

$$f_{U_K(n)|x_1}(x_2|x_1) = k \frac{f(x_2)}{[1 - F(x_1)]^k} [1 - F(x_2)]^{k-1}. \tag{3.15}$$

Further, the joint distribution of $X_{U_K(1)}, \ldots, X_{U_K(m)}$ and $X_{U_K(n)}$ is given as

$$f_{U_K(1),\ldots,U_K(m),U_K(n)}(x_1, \ldots, x_m, x_n) = \frac{k^n}{\Gamma(n-m)} \left[\prod_{i=1}^{m} r(x_i)\right] \times f(x_n) [1 - F(x_n)]^{k-1} \times [R(x_n) - R(x_m)]^{n-m-1}. \tag{3.16}$$

Now using (3.11) and (3.16), the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(1)} = x_1, \ldots, X_{U_K(m)} = x_m$ is

$$f_{U_K(n)|x_1,\ldots,x_m}(x_n|x_1,\ldots,x_m) = \frac{f_{U_K(1),\ldots,U_K(m),U_K(n)}(x_1,\ldots,x_m,x_n)}{f_{U_K(1),U_K(2),\ldots,U_K(n)}(x_1,x_2,\ldots,x_n)}$$
$$= \left[\frac{k^n}{\Gamma(n-m)}\left\{\prod_{i=1}^{m} r(x_i)\right\} f(x_n)\right.$$
$$\left.[1-F(x_n)]^{k-1}[R(x_n)-R(x_m)]^{n-m-1}\right] /$$
$$\left[k^m\left\{\prod_{i=1}^{m-1} r(x_i)\right\}[1-F(x_m)]^{k-1} f(x_m)\right]$$

or

$$f_{U_K(n)|x_1,\ldots,x_m}(x_n|x_1,\ldots,x_m) = \frac{k^{n-m}}{\Gamma(n-m)}\frac{f(x_n)}{[1-F(x_m)]^k}[1-F(x_n)]^{k-1} \times [R(x_n)-R(x_m)]^{n-m-1},$$

which is same as the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(m)} = x_1$. The kth upper record values therefore follow the Markov chain. The transition probability for kth upper record is computed below

$$P\left(X_{U_K(m+1)} \geq y|X_{U_K(m)} = x_m\right) = \int_y^\infty f_{U_K(m+1)|x_m}(x_{m+1}|x_m)\,dx_{m+1}$$
$$= \frac{k}{[1-F(x_m)]^k}\int_y^\infty f(x_{m+1}) \times \left[1-F(x_{m+1})\right]^{k-1} dx_{m+1}$$
$$= \left[\frac{1-F(y)}{1-F(x_m)}\right]^k.$$

We can see that the results of kth upper records reduces to simple upper records for $k = 1$.

Example 3.5 Find the distribution of $X_{U_K(n)}$, joint distribution of $X_{U_K(m)}$ and $X_{U_K(n)}$; $m < n$; and the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(m)} = x_1$ if the sequence of random variables follow the Weibull distribution with density

$$f(x) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^\beta\right);\ x, \alpha, \beta > 0.$$

Solution: The marginal distribution of $X_{U_K(n)}$ is given in (3.12) as

$$f_{U_K(n)}(x) = \frac{k^n}{\Gamma(n)}[R(x)]^{n-1}[1-F(x)]^{k-1} f(x).$$

Now for Weibull distribution we have

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \alpha\beta t^{\beta-1} \exp\left(-\alpha t^{\beta}\right) dt$$
$$= 1 - \exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

Hence

$$R(x) = -\ln\left[1 - F(x)\right] = \alpha x^{\beta}.$$

The density function of $X_{U_K(n)}$ is therefore given as

$$f_{U_K(n)}(x) = \frac{k^n}{\Gamma(n)} \left(\alpha x^{\beta}\right)^{n-1} \left[\exp\left(-\alpha x^{\beta}\right)\right]^{k-1}$$
$$\times\, \alpha\beta x^{\beta-1} \exp\left(-\alpha x^{\beta}\right)$$
$$= \frac{\alpha^n k^n \beta}{\Gamma(n)} x^{n\beta-1} \exp\left(-k\alpha x^{\beta}\right);\ x, \alpha, \beta, n, k > 0.$$

We can readily see that for $k = 1$ the density function of $X_{U_K(n)}$, given above, reduces to the density of $X_{U(n)}$ given in Example 3.2.

Again the joint density of $X_{U_K(m)}$ and $X_{U_K(n)}$ is given in (3.13) as

$$f_{U_K(m),U_K(m)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\,\Gamma(n-m)} r(x_1) f(x_2) \left[R(x_1)\right]^{m-1}$$
$$\times\ \left[R(x_2) - R(x_1)\right]^{n-m-1} \left[1 - F(x_2)\right]^{k-1}.$$

Now for Weibull distribution we have

$$r(x_1) = \frac{d}{dx_1} R(x_1) = \frac{d\left(\alpha x_1^{\beta}\right)}{dx_1} = \alpha\beta x_1^{\beta-1}.$$

Hence the joint distribution of $X_{U_K(m)}$ and $X_{U_K(n)}$ is

$$f_{U_K(m),U_K(m)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\,\Gamma(n-m)} \alpha\beta x_1^{\beta-1} \alpha\beta x_2^{\beta-1} \exp\left(-\alpha x_2^{\beta}\right)$$
$$\times \left(\alpha x_1^{\beta}\right)^{m-1} \left(\alpha x_2^{\beta} - \alpha x_1^{\beta}\right)^{n-m-1}$$
$$\times \left[\exp\left(-\alpha x_2^{\beta}\right)\right]^{k-1}$$

or

$$f_{U_K(m),U_K(m)}(x_1, x_2) = \frac{\alpha^n k^n \beta^2 x_1^{m\beta-1}}{\Gamma(m)\,\Gamma(n-m)} x_2^{\beta-1} \left(x_2^{\beta} - x_1^{\beta}\right)^{n-m-1}$$
$$\times\ \exp\left(-k\alpha x_2^{\beta}\right),$$

for $0 < x_1 < x_2 < \infty$. We again see that for $k = 1$ the above density reduces to the joint density of $X_{U(m)}$ and $X_{U(n)}$ given in Example 3.2. Finally the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(m)} = x_1$ is given in (3.14) as

$$f_{U_K(n)|x_1}(x_2|x_1) = \frac{k^{n-m}}{\Gamma(n-m)} \frac{f(x_2)}{[1-F(x_1)]^k} [1-F(x_2)]^{k-1} \times [R(x_2) - R(x_1)]^{n-m-1}.$$

For Weibull distribution the conditional distribution of $X_{U_K(n)}$ given $X_{U_K(m)} = x_1$ is

$$f_{U_K(n)|x_1}(x_2|x_1) = \frac{k^{n-m}}{\Gamma(n-m)} \frac{\alpha\beta x_2^{\beta-1} \exp\left(-\alpha x_2^{\beta}\right)}{\left[\exp\left(-\alpha x_1^{\beta}\right)\right]^k} \times \left[\exp\left(-\alpha x_2^{\beta}\right)\right]^{k-1} \left(\alpha x_2^{\beta} - \alpha x_1^{\beta}\right)^{n-m-1}$$

or

$$f_{U_K(n)|x_1}(x_2|x_1) = \frac{\alpha^{n-m} k^{n-m} \beta}{\Gamma(n-m)} x_2^{\beta-1} \left(x_2^{\beta} - x_1^{\beta}\right)^{n-m-1} \times \exp\left[-k\alpha\left(x_2^{\beta} - x_1^{\beta}\right)\right],$$

for $0 < x_1 < x_2 < \infty$.

3.6 Moments of Record Values

The marginal, joint and conditional distributions of upper record values are proper probability distributions and hence we can study some additional properties of these distributions. A relatively useful method to study the properties of any probability distribution is to compute its moments. In this section we present the simple, product and conditional moments of upper record values that can be used to study further properties of records based upon certain specific distributions. The basic definition of moments of upper records is given in the following.

The pth marginal moment of upper record value $X_{U(n)}$ is defined as

$$\mu_{(n)}^{p} = E\left(X_{U(n)}^{p}\right) = \int_{-\infty}^{\infty} x^p f_{X_{U(n)}}(x)\,dx = \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} x^p [R(x)]^{n-1} f(x)\,dx. \quad (3.17)$$

The quantity $\mu_{(n)}$ is called mean of $X_{U(n)}$. The variance of $X_{U(n)}$ is computed as

$$\sigma^2_{(n)} = \mu^2_{(n)} - \left[\mu^1_{(n)}\right]^2.$$

The (p, q)th product moment of $X_{U(m)}$ and $X_{U(n)}$ are defined as

$$\begin{aligned}\mu^{p,q}_{(m,n)} &= E\left(X^p_{U(m)} X^q_{U(n)}\right) \\ &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{X_{U(m)},X_{U(n)}}(x_1, x_2)\, dx_2 dx_1 \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} r(x_1) f(x_2)\left[R(x_1)\right]^{m-1} \\ &\quad\times \left[R(x_2) - R(x_1)\right]^{n-m-1} dx_2 dx_1. \end{aligned} \tag{3.18}$$

The covariance between two upper records is computed as

$$\sigma_{(m,n)} = \mu^{1,1}_{(m,n)} - \mu^{1,0}_{(m,n)}\mu^{0,1}_{(m,n)}. \tag{3.19}$$

The pth conditional moment of $X_{U(n)}$ given $X_{U(m)} = x_1$ is defined as

$$\begin{aligned}\mu^p_{(n|m)} &= E\left(X^p_{U(n)}|x_1\right) = \int_{x_1}^{\infty} x_2^p f_{X_{U(n)}|x_1}(x_2|x_1)\, dx_2 \\ &= \frac{1}{\Gamma(n-m)}\int_{x_1}^{\infty}\frac{f(x_2)}{1-F(x_1)}\left[R(x_2) - R(x_1)\right]^{n-m-1} dx_2. \end{aligned} \tag{3.20}$$

The conditional mean and conditional variance are easily computed from (3.20).

We can also define the moments of kth upper records on the parallel lines. Specifically, the pth moment of kth record based on a sequence $\{X_n;\ n \geq 1\}$ is defined as

$$\begin{aligned}\mu^p_{K(n)} &= E\left(X^p_{U_K(n)}\right) = \int_{-\infty}^{\infty} x^p f_{U_K(n)}(x)\, dx \\ &= \frac{k^n}{\Gamma(n)}\int_{-\infty}^{\infty} x^p f(x)\left[1 - F(x)\right]^{k-1}\left[R(x)\right]^{n-1} dx. \end{aligned} \tag{3.21}$$

The mean and variance of $X_{U_K(n)}$ is readily computed from (3.21). The (p, q)th product moments of $X_{U_K(m)}$ and $X_{U_K(n)}$ are defined as

$$\begin{aligned}\mu^{p,q}_{K(m,n)} &= E\left(X^p_{U_K(m)} X^q_{U_K(n)}\right) \\ &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{U_K(m),U_K(n)}(x_1, x_2)\, dx_2 dx_1 \\ &= \frac{k^n}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q r(x_1) f(x_2)\left[R(x_1)\right]^{m-1} \\ &\quad\times \left[R(x_2) - R(x_1)\right]^{n-m-1}\left[1 - F(x_2)\right]^{k-1} dx_2 dx_1. \end{aligned} \tag{3.22}$$

The covariance between two kth upper records is

$$\sigma_{K(m,n)} = \mu^{1,1}_{K(m,n)} - \mu^{1,0}_{K(m,n)}\mu^{0,1}_{K(m,n)}.$$

The conditional moments of kth record are computed as

$$\begin{aligned}\mu^{p}_{K(n|m)} &= E\left(X^{p}_{U_K(n)}|x_1\right) = \int_{x_1}^{\infty} x_2^p f_{U_K(n)|x_1}\left(x_2|x_1\right) dx_2 \\ &= \frac{k^{n-m}}{\Gamma\left(n-m\right)}\int_{x_1}^{\infty}\frac{f\left(x_2\right)}{\left[1-F\left(x_1\right)\right]^k}\left[R\left(x_2\right)-R\left(x_1\right)\right]^{n-m-1} \\ &\quad\times\left[1-F\left(x_2\right)\right]^{k-1} dx_2. \end{aligned} \tag{3.23}$$

The conditional mean and variance are easily computed from (3.23).

Example 3.6 Obtain the expression for pth marginal moments and (p, q)th product moments of upper record values if the sequence $\{X_n;\ n \geq 1\}$ has Weibull distribution with density

$$f\left(x\right) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

Solution: The pth moment of $X_{U(n)}$ is defined as

$$\mu^{p}_{(n)} = E\left(X^{p}_{U(n)}\right) = \int_{-\infty}^{\infty} x^p f_{X_{U(n)}}\left(x\right) dx,$$

where $f_{X_{U(n)}}\left(x\right)$ is density function of $X_{U(n)}$. Now for Weibull distribution, the density function of $X_{U(n)}$ is given in Example 3.2 as

$$f_{X_{U(n)}}\left(x\right) = \frac{\alpha^n\beta}{\Gamma\left(n\right)}x^{n\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta, n > 0.$$

Hence the pth moment of $X_{U(n)}$ is

$$\begin{aligned}\mu^{p}_{(n)} &= \frac{\alpha^n\beta}{\Gamma\left(n\right)}\int_0^{\infty} x^p x^{n\beta-1}\exp\left(-\alpha x^{\beta}\right) dx \\ &= \frac{\alpha^n\beta}{\Gamma\left(n\right)}\int_0^{\infty} x^{n\beta+p-1}\exp\left(-\alpha x^{\beta}\right) dx.\end{aligned}$$

Now making the transformation $w = \alpha x^{\beta}$ we have $x = \left(w/\alpha\right)^{1/\beta}$ and $dx = \frac{1}{\alpha\beta}\left(w/\alpha\right)^{1/\beta-1} dw$. Hence we have

$$\begin{aligned}\mu^{p}_{(n)} &= \frac{\alpha^n\beta}{\Gamma\left(n\right)}\int_0^{\infty}\left(\frac{w}{\alpha}\right)^{\frac{n\beta+p-1}{\beta}} e^{-w}\frac{1}{\alpha\beta}\left(\frac{w}{\alpha}\right)^{1/\beta-1} dw \\ &= \frac{1}{\alpha^{\frac{p}{\beta}}\Gamma\left(n\right)}\int_0^{\infty} w^{n+\frac{p}{\beta}-1}e^{-w} dw\end{aligned}$$

or

$$\mu_{(n)}^{p} = \frac{1}{\alpha^{\frac{p}{\beta}}\Gamma\left(n\right)}\Gamma\left(n+\frac{p}{\beta}\right).$$

The mean and variance are readily obtained from above by using $p = 1$ and $p = 2$ respectively. For $\beta = 1$ the expression for pth moment of upper record for exponential distribution is obtained as

$$\mu_{(n)}^{p} = \frac{1}{\alpha^{p}\Gamma\left(n\right)}\Gamma\left(n+p\right) = \frac{(n+p-1)!}{\alpha^{p}\left(n-1\right)!},$$

and the mean and variance of $X_{U(n)}$ for exponential distribution are

$$\mu_{(n)} = \frac{1}{\alpha} \text{ and } \sigma_{(n)} = \frac{n}{\alpha^2}$$

respectively. Again the product moments of $X_{U(m)}$ and $X_{U(n)}$ are computed as

$$\begin{aligned}\mu_{(m,n)}^{p,q} &= E\left(X_{U(m)}^{p}X_{U(n)}^{q}\right)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} x_1^p x_2^q f_{X_{U(m)},X_{U(n)}}\left(x_1,x_2\right)dx_1dx_2,\end{aligned}$$

where $f_{X_{U(m)},X_{U(n)}}\left(x_1,x_2\right)$ is joint density of $X_{U(m)}$ and $X_{U(n)}$. Now for Weibull distribution we have, from Example 3.2,

$$f_{X_{U(m)},X_{U(n)}}\left(x_1,x_2\right) = \frac{\alpha^n\beta^2x_1^{\beta m-1}}{\Gamma\left(m\right)\Gamma\left(n-m\right)}\left(x_2^{\beta}-x_1^{\beta}\right)^{n-m-1}\exp\left(-\alpha x_2^{\beta}\right),$$

for $0 < x_1 < x_2 < \infty$. The product moments are therefore

$$\begin{aligned}\mu_{(m,n)}^{p,q} &= \int_{0}^{\infty}\int_{0}^{x_2} x_1^p x_2^q \frac{\alpha^n\beta^2x_1^{\beta m-1}}{\Gamma\left(m\right)\Gamma\left(n-m\right)}\\ &\quad\times\left(x_2^{\beta}-x_1^{\beta}\right)^{n-m-1}\exp\left(-\alpha x_2^{\beta}\right)dx_1dx_2\\ &= \frac{\alpha^n\beta^2}{\Gamma\left(m\right)\Gamma\left(n-m\right)}\int_0^{\infty}x_2^q\exp\left(-\alpha x_2^{\beta}\right)I\left(x_1\right)dx_2, \qquad \text{(i)}\end{aligned}$$

where

$$I\left(x_1\right) = \int_0^{x_2} x_1^{\beta m+p-1}\left(x_2^{\beta}-x_1^{\beta}\right)^{n-m-1}dx_1$$

Now making the transformation $x_1 = ux_2$ we have $dx_1 = x_2 du$ and hence

$$\begin{aligned}
I(x_2) &= \int_0^1 (ux_2)^{\beta m+p-1}\left(x_2^\beta - u^\beta x_2^\beta\right)^{n-m-1} x_2 du \\
&= x_2^{\beta n+p-\beta}\int_0^1 u^{\beta m+p-1}\left(1-u^\beta\right)^{n-m-1} du \\
&= x_2^{\beta n+p-\beta}\frac{\Gamma(n-m)\Gamma\left(m+\frac{p}{\beta}\right)}{\beta\Gamma\left(n+\frac{p}{\beta}\right)}.
\end{aligned}$$

Using the value of $I(x_1)$ in (i) we have

$$\begin{aligned}
\mu_{(m,n)}^{p,q} &= \frac{\alpha^n\beta^2}{\Gamma(m)\Gamma(n-m)}\int_0^\infty x_2^q \exp\left(-\alpha x_2^\beta\right) x_2^{\beta n+p-\beta} \\
&\quad\times\frac{\Gamma(n-m)\Gamma\left(m+\frac{p}{\beta}\right)}{\beta\Gamma\left(n+\frac{p}{\beta}\right)} dx_2 \\
&= \frac{\alpha^n\beta\Gamma\left(m+\frac{p}{\beta}\right)}{\Gamma(m)\Gamma\left(n+\frac{p}{\beta}\right)}\int_0^\infty x_2^{\beta n+p+q-\beta}\exp\left(-\alpha x_2^\beta\right) dx_2.
\end{aligned}$$

Now making the transformation $\alpha x_2^\beta = w$ we have $x_2 = \left(\frac{w}{\alpha}\right)^{1/\beta}$ or $dx_2 = \frac{1}{\beta}\left(\frac{w}{\alpha}\right)^{1/\beta-1}\frac{1}{\alpha}dw$ and hence

$$\begin{aligned}
\mu_{(m,n)}^{p,q} &= \frac{\alpha^n\beta\Gamma\left(m+\frac{p}{\beta}\right)}{\Gamma(m)\Gamma\left(n+\frac{p}{\beta}\right)}\int_0^\infty\left(\frac{w}{\alpha}\right)^{n+\frac{p+q}{\beta}-1}\exp(-w)\frac{1}{\alpha^{1/\beta}\beta}w^{1/\beta-1}dw \\
&= \frac{\Gamma\left(m+\frac{p}{\beta}\right)}{\alpha^{(p+q+1/\beta)-1}\Gamma(m)\Gamma\left(n+\frac{p}{\beta}\right)}\int_0^\infty w^{n+\frac{p+q+1}{\beta}-2}\exp(-w)\,dw
\end{aligned}$$

or

$$\mu_{(m,n)}^{p,q} = \frac{\Gamma\left(m+\frac{p}{\beta}\right)\Gamma\left(n-1+\frac{p+q+1}{\beta}\right)}{\alpha^{(p+q+1/\beta)-1}\Gamma(m)\Gamma\left(n+\frac{p}{\beta}\right)}.$$

The expression for product moments for exponential distribution can be obtained by using $\beta = 1$ in above expression as

$$\mu_{(m,n)}^{p,q} = \frac{\Gamma(m+p)\Gamma(n+p+q)}{\alpha^{p+q}\Gamma(m)\Gamma(n+p)}.$$

Using $p = q = 1$ in above expansion, we have

$$\mu^{1,1}_{(m,n)} = \frac{m\,(n+1)}{\alpha^2}.$$

Also

$$\mu^{1,0}_{(m,n)} = \frac{m}{\alpha} \text{ and } \mu^{0,1}_{(m,n)} = \frac{n}{\alpha}.$$

The covariance between $X_{U(m)}$ and $X_{U(n)}$ for exponential distribution is therefore

$$\begin{aligned}\sigma_{(m,n)} &= \mu^{1,1}_{(m,n)} - \mu^{1,0}_{(m,n)}\mu^{0,1}_{(m,n)} \\ &= \frac{m\,(n+1)}{\alpha^2} - \left(\frac{m}{\alpha}\right)\left(\frac{n}{\alpha}\right) = \frac{m}{\alpha^2}.\end{aligned}$$

We see that the covariance does not depend upon n. Finally the Correlation Coefficient between two upper records for exponential distribution is

$$\rho_{(m,n)} = \frac{\sigma_{(m,n)}}{\sqrt{\sigma_{(m)}\sigma_{(n)}}} = \frac{m/\alpha^2}{\sqrt{\left(m/\alpha^2\right)\left(n/\alpha^2\right)}} = \sqrt{\frac{m}{n}}.$$

We can see that the correlation coefficient between upper records only depends upon m and n only.

Example 3.7 A sequence $\{X_n;\ n \geq 1\}$ has exponential distribution with density

$$f\,(x) = \alpha e^{-\alpha x};\ x, \alpha > 0.$$

Derive the expression for pth moment of $X_{U_K(n)}$ and (p, q)th joint moments of $X_{U_K(m)}$ and $X_{U_K(n)}$.

Solution: The pth moment of $X_{U_K(n)}$ is given as

$$\mu^p_{K(n)} = E\left(X^p_{U_K(n)}\right) = \int_{-\infty}^{\infty} x^p f_{U_K(n)}\,(x)\,dx,$$

where $f_{U_K(n)}\,(x)$ is density function of $X_{U_K(n)}$ and is given as

$$f_{U_K(n)}\,(x) = \frac{k^n}{\Gamma\,(n)}\,[R\,(x)]^{n-1}\,[1 - F\,(x)]^{k-1} f\,(x)\,.$$

For exponential distribution we have $R\,(x) = 1 - F\,(x) = \alpha x$ and hence the density function of $X_{U_K(n)}$ is

$$\begin{aligned}f_{U_K(n)}\,(x) &= \frac{k^n}{\Gamma\,(n)}\,(\alpha x)^{n-1}\left(e^{-\alpha x}\right)^{k-1}\alpha e^{-\alpha x} \\ &= \frac{\alpha^n k^n}{\Gamma\,(n)} x^{n-1} e^{-\alpha k x};\ x, k, \alpha > 0.\end{aligned}$$

The pth moment of $X_{U_K(n)}$ is therefore

$$\mu_{K(n)}^{p} = \frac{\alpha^n k^n}{\Gamma(n)} \int_0^\infty x^{n+p-1} e^{-\alpha k x} dx = \frac{1}{(\alpha k)^p \Gamma(n)} \Gamma(n+p).$$

The mean and variance of $X_{U_K(n)}$ are

$$\mu_{K(n)} = \frac{n}{\alpha k} \text{ and } \sigma_{K(n)}^2 = \frac{n}{\alpha^2 k^2}.$$

The (p, q)th product moments of $X_{U_K(m)}$ and $X_{U_K(n)}$ are computed as

$$\mu_{K(m,n)}^{p,q} = E\left(X_{U_K(m)}^p X_{U_K(n)}^q\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} x_1^p x_2^q f_{U_K(m),U_K(n)}(x_1, x_2)\, dx_1 dx_2 \qquad \text{(i)}$$

where $f_{U_K(m),U_K(n)}(x_1, x_2)$ is joint density of $X_{U_K(m)}$ and $X_{U_K(n)}$ and is given as

$$f_{U_K(m),U_K(n)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\Gamma(n-m)} r(x_1) f(x_2) [R(x_1)]^{m-1} \times [R(x_2) - R(x_1)]^{n-m-1} [1 - F(x_2)]^{k-1}.$$

Now for exponential distribution we have

$$f_{U_K(m),U_K(n)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \alpha\alpha e^{-\alpha x_2} (\alpha x_1)^{m-1} \times (\alpha x_2 - \alpha x_1)^{n-m-1} \left(e^{-\alpha x_2}\right)^{k-1} = \frac{\alpha^n k^n}{\Gamma(m)\Gamma(n-m)} x_1^{m-1} (x_2 - x_1)^{n-m-1} \times e^{-\alpha k x_2},$$

for $0 < x_1 < x_2 < \infty$. Using the joint density in (i) we have

$$\mu_{K(m,n)}^{p,q} = \int_0^\infty \int_0^{x_2} x_1^p x_2^q \frac{\alpha^n k^n}{\Gamma(m)\Gamma(n-m)} x_1^{m-1} \times (x_2 - x_1)^{n-m-1} e^{-\alpha k x_2} dx_1 dx_2$$

or

$$\mu_{K(m,n)}^{p,q} = \frac{\alpha^n k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} x_2^q e^{-\alpha k x_2} I(x_1)\, dx_1, \qquad \text{(ii)}$$

where

$$I(x_1) = \int_0^{x_2} x_1^{p+m-1} (x_2 - x_1)^{n-m-1} \, dx_1.$$

Now making the transformation $x_1 = ux_2$ we have

$$\begin{aligned} I(x_1) &= \int_0^1 (ux_2)^{p+m-1} (x_2 - ux_2)^{n-m-1} x_2 du. \\ &= x_2^{n+p-1} \int_0^1 u^{p+m-1} (1-u)^{n-m-1} \, du \\ &= x_2^{n+p-1} \frac{\Gamma(n-m)\Gamma(m+p)}{\Gamma(n+p)}. \end{aligned}$$

Using above result in (ii) we have

$$\begin{aligned} \mu_{K(m,n)}^{p,q} &= \frac{\alpha^n k^n}{\Gamma(m)\Gamma(n-m)} \frac{\Gamma(n-m)\Gamma(m+p)}{\Gamma(n+p)} \\ &\quad \times \int_0^\infty x_2^q e^{-\alpha k x_2} x_2^{n+p-1} dx_2 \\ &= \frac{\alpha^n k^n \Gamma(m+p)}{\Gamma(m)\Gamma(n+p)} \int_0^\infty x_2^q e^{-\alpha k x_2} x_2^{n+p-1} dx_2 \\ &= \frac{\alpha^n k^n \Gamma(m+p)}{\Gamma(m)\Gamma(n+p)} \int_0^\infty x_2^{n+p+q-1} e^{-\alpha k x_2} dx_2 \end{aligned}$$

or

$$\begin{aligned} \mu_{K(m,n)}^{p,q} &= \frac{\alpha^n k^n \Gamma(m+p)}{\Gamma(m)\Gamma(n+p)} \frac{1}{(\alpha k)^{n+p+q}} \Gamma(n+p+q) \\ &= \frac{\Gamma(m+p)\Gamma(n+p+q)}{(\alpha k)^{p+q}\Gamma(m)\Gamma(n+p)}. \end{aligned}$$

For $p = q = 1$ we have

$$\mu_{K(m,n)}^{1,1} = \frac{m(n+1)}{\alpha^2 k^2}.$$

The covariance between $X_{U_K(m)}$ and $X_{U_K(n)}$ is therefore

$$\begin{aligned} \sigma_{K(m,n)} &= \mu_{K(m,n)}^{1,1} - \mu_{K(m,n)}^{1,0} \mu_{K(m,n)}^{0,1} \\ &= \frac{m(n+1)}{\alpha^2 k^2} - \left(\frac{m}{\alpha k}\right)\left(\frac{n}{\alpha k}\right) = \frac{m}{\alpha^2 k^2}. \end{aligned}$$

We can see that $\sigma_{K(m,n)} = \sigma_{K(m)}^2$. Finally the correlation coefficient between $X_{U_K(m)}$ and $X_{U_K(n)}$ is

$$\rho_{(m,n)} = \frac{\sigma_{K(m,n)}}{\sqrt{\sigma_{K(m)}\sigma_{K(n)}}} = \frac{m/\alpha^2 k^2}{\sqrt{\left(m/\alpha^2 k^2\right)\left(n/\alpha^2 k^2\right)}} = \sqrt{\frac{m}{n}}.$$

We can see that the correlation coefficient between two kth upper record values for exponential distribution is same as the correlation coefficient between two simple upper record values.

Example 3.8 Obtain the expression for pth conditional moment of $X_{U(n)}$ given $X_{U(m)} = x_1$ if the sequence $\{X_n;\ n \geq 1\}$ has Weibull distribution with density

$$f(x) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

Hence or otherwise obtain the conditional mean and variance of $X_{U(n)}$ given $X_{U(m)} = x_1$ in case of exponential distribution.

Solution: The pth conditional moment of $X_{U(n)}$ given $X_{U(m)} = x_1$ is computed as

$$\mu^p_{(n|m)} = E\left(X^p_{U(n)}|x_1\right) = \int_{x_1}^{\infty} x_2^p f_{X_{U(n)}|x_1}(x_2|x_1)\,dx_2;$$

where $f_{X_{U(n)}|x_1}(x_2|x_1)$ is conditional distribution of $X_{U(n)}$ given $X_{U(m)} = x_1$ and for Weibull distribution is given in Example 3.4 as

$$f_{X_{U(n)}|x_1}(x_2|x_1) = \frac{\alpha^{n-m}\beta x_2^{\beta-1}}{\Gamma(n-m)}\left(x_2^{\beta} - x_1^{\beta}\right)^{n-m-1} \times \exp\left[-\alpha\left(x_2^{\beta} - x_1^{\beta}\right)\right].$$

The pth conditional moment is, therefore

$$\mu^p_{(n|m)} = \frac{\alpha^{n-m}\beta}{\Gamma(n-m)}\int_{x_1}^{\infty} x_2^{p+\beta-1}\left(x_2^{\beta} - x_1^{\beta}\right)^{n-m-1} \times \exp\left[-\alpha\left(x_2^{\beta} - x_1^{\beta}\right)\right]dx_2$$

Now making the transformation $\left(x_2^{\beta} - x_1^{\beta}\right) = w$ we have $x_2 = \left(w + x_1^{\beta}\right)^{1/\beta}$ and $dx_2 = \frac{1}{\beta}\left(w + x_1^{\beta}\right)^{1/\beta-1} du$ and hence the pth conditional moment is

$$\begin{aligned}\mu^p_{(n|m)} &= \frac{\alpha^{n-m}\beta}{\Gamma(n-m)}\int_0^{\infty}\left\{\left(w + x_1^{\beta}\right)^{1/\beta}\right\}^{p+\beta-1} w^{n-m-1}e^{-\alpha w} \\ &\quad \times \frac{1}{\beta}\left(w + x_1^{\beta}\right)^{1/\beta-1} dw \\ &= \frac{\alpha^{n-m}}{\Gamma(n-m)}\int_0^{\infty}\left(w + x_1^{\beta}\right)^{p/\beta} w^{n-m-1}e^{-\alpha w}dw.\end{aligned}$$

Now expanding $\left(w + x_1^{\beta}\right)^{p/\beta}$ we have

$$\mu^p_{(n|m)} = \frac{\alpha^{n-m}}{\Gamma(n-m)} \int_0^{\infty} \sum_{r=0}^{p/\beta} \frac{\Gamma(p/\beta+1)}{\Gamma(r+1)\,\Gamma(p/\beta-r+1)} \times w^{p/\beta-r} x_1^{\beta r} w^{n-m-1} e^{-\alpha w} dw$$

or

$$\mu^p_{(n|m)} = \frac{\alpha^{n-m}}{\Gamma(n-m)} \sum_{r=0}^{p/\beta} \frac{\Gamma(p/\beta+1)\, x_1^{\beta r}}{\Gamma(r+1)\,\Gamma(p/\beta-r+1)} \times \int_0^{\infty} w^{n+p/\beta-m-r-1} e^{-\alpha w} dw$$

or

$$\mu^p_{(n|m)} = \frac{1}{\Gamma(n-m)} \sum_{r=0}^{p/\beta} \frac{\alpha^{p/\beta+r}\Gamma(p/\beta+1)\, x_1^{\beta r}}{\Gamma(r+1)\,\Gamma(p/\beta-r+1)} \times \Gamma\left(n-m-r-\frac{p}{\beta}\right).$$

Using $\beta = 1$ the expression for conditional moments of upper records for exponential distribution is

$$\mu^p_{(n|m)} = \frac{1}{\Gamma(n-m)} \sum_{r=0}^{p} \frac{\alpha^{p+r} x_1^r \Gamma(p+1)}{\Gamma(r+1)\,\Gamma(p-r+1)} \times \Gamma(n-m-r-p).$$

The conditional mean is obtained by using $p = 1$ as

$$\mu_{(n|m)} = \frac{\alpha(\alpha x_1 + n - m - 2)}{(n-m-1)(n-m-2)}.$$

The conditional variance can be obtained easily by using $p = 2$ in expression of $\mu^p_{(n|m)}$.

3.7 Recurrence Relations for Moments of Record Values

In previous section we have discussed single, product and conditional moments of upper records and kth upper records. We have also given some examples to compute the moments of upper records. In several cases the explicit expression for moments of upper records can be derived but several cases may arise where explicit expression for moments of upper record is very complicated. Several researchers have established the recurrence relations to compute moments of specific order for upper records by using the information of lower order moments. In this section we will discuss a general method to derive the recurrence relations for single and product moments of upper records. We will also illustrate the use of that general expression to derive the recurrence relations for moments of upper record values for certain special distributions. We first present a general expression for recurrence relations of moments of upper records, due to Bieniek and Szynal (2002), in the following theorem.

Theorem 3.1 *Suppose a sequence of random variables* $\{X_n;\ n \geq 1\}$ *is available from an absolutely continuous distribution function* $F(x)$. *Suppose further that* $X_{U_K(n)}$ *be kth upper record of the sequence then following recurrence relation hold between moments of the records*

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx, \tag{3.24}$$

and

$$\begin{aligned} \mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} &\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \\ &\times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\times [1 - F(x_2)]^k dx_2 dx_1, \end{aligned} \tag{3.25}$$

where $R(x) = -\ln[1 - F(x)]$.

Proof The pth moment of kth upper record is

$$\begin{aligned} \mu^p_{K(n)} = E\left(X^p_{U_K(n)}\right) &= \int_{-\infty}^{\infty} x^p f_{U_K(n)}(x)\, dx \\ &= \frac{k^n}{\Gamma(n)} \int_{-\infty}^{\infty} x^p f(x) \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx. \end{aligned}$$

Integrating above equation by parts taking $f(x)\{1 - F(x)\}^{k-1}$ as function for integration we have

$$\begin{aligned} \mu^p_{K(n)} = \frac{k^n}{\Gamma(n)} \Bigg[& -x^p \{R(x)\}^{n-1} \frac{\{1 - F(x)\}^k}{k} \Bigg|_{-\infty}^{\infty} \\ & - \int_{-\infty}^{\infty} \Big\{ px^{p-1} \{R(x)\}^{n-1} + (n-1) x^p \{R(x)\}^{n-2} \\ & \times \frac{f(x)}{[1 - F(x)]} \Big\} \frac{-\{1 - F(x)\}^k}{k} dx \Bigg] \end{aligned}$$

$$= \frac{pk^n}{k\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx$$
$$+ \frac{(n-1)k^n}{k\Gamma(n)} \int_{-\infty}^{\infty} x^p f(x) [1 - F(x)]^{k-1} \{R(x)\}^{n-2} dx$$

or

$$\mu_{K(n)}^p = \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx$$
$$+ \frac{k^{n-1}}{\Gamma(n-1)} \int_{-\infty}^{\infty} x^p f(x) [1 - F(x)]^{k-1} \{R(x)\}^{n-2} dx.$$

Since

$$\mu_{K(n-1)}^p = \frac{k^{n-1}}{\Gamma(n-1)} \int_{-\infty}^{\infty} x^p f(x) [1 - F(x)]^{k-1} \{R(x)\}^{n-2} dx,$$

hence above equation can be written as

$$\mu_{K(n)}^p = \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx + \mu_{K(n-1)}^p$$

or

$$\mu_{K(n)}^p - \mu_{K(n-1)}^p = \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx;$$

which is (3.24).

Again consider the expression for product moments of kth upper records as

$$\mu_{K(m,n)}^{p,q} = E\left(X_{U_K(m)}^p X_{U_K(n)}^q\right)$$
$$= \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^q f_{U_K(m),U_K(n)}(x_1, x_2)\, dx_2 dx_1$$
$$= \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^q r(x_1) f(x_2) [R(x_1)]^{m-1}$$
$$\times [R(x_2) - R(x_1)]^{n-m-1} [1 - F(x_2)]^{k-1} dx_2 dx_1.$$

or

$$\mu_{K(m,n)}^{p,q} = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} x_1^p r(x_1) [R(x_1)]^{m-1} I(x_2)\, dx_1 \qquad \text{(i)}$$

where

$$I(x_2) = \int_{x_1}^{\infty} x_2^q f(x_2) [1 - F(x_2)]^{k-1} [R(x_2) - R(x_1)]^{n-m-1} dx_2.$$

Integrating above integral by parts using $f(x_2)\{1-F(x_2)\}^{k-1}$ for integration we have

$$
\begin{aligned}
I(x_2) = & -x_2^q[R(x_2)-R(x_1)]^{n-m-1}\frac{\{1-F(x_2)\}^k}{k}\bigg|_{x_1}^{\infty} \\
& + \frac{1}{k}\int_{x_1}^{\infty}\Big[qx_2^{q-1}[R(x_2)-R(x_1)]^{n-m-1} \\
& + (n-m-1)\,x_2^q[R(x_2)-R(x_1)]^{n-m-2} \\
& \times \frac{f(x_2)}{1-F(x_2)}\Big]\{1-F(x_2)\}^k\,dx_2
\end{aligned}
$$

or

$$
\begin{aligned}
I(x_2) = & \frac{q}{k}\int_{x_1}^{\infty} x_2^{q-1}[R(x_2)-R(x_1)]^{n-m-1}\,[1-F(x_2)]^k\,dx_2 \\
& + \frac{(n-m-1)}{k}\int_{x_1}^{\infty} x_2^q f(x_2)\,[R(x_2)-R(x_1)]^{n-m-2} \\
& \times [1-F(x_2)]^{k-1}dx_2.
\end{aligned}
$$

Now using the value of $I(x_2)$ in (i) we have

$$
\begin{aligned}
\mu_{K(m,n)}^{p,q} = & \frac{k^n}{\Gamma(m)\,\Gamma(n-m)}\int_{-\infty}^{\infty} x_1^p r(x_1)\,[R(x_1)]^{m-1} \\
& \times \left[\frac{q}{k}\int_{x_1}^{\infty} x_2^{q-1}[R(x_2)-R(x_1)]^{n-m-1}\,[1-F(x_2)]^k\,dx_2\right. \\
& + \frac{(n-m-1)}{k}\int_{x_1}^{\infty} x_2^q f(x_2)\,[R(x_2)-R(x_1)]^{n-m-2} \\
& \left.\times [1-F(x_2)]^{k-1}dx_2\right]dx_1
\end{aligned}
$$

or

$$
\begin{aligned}
\mu_{K(m,n)}^{p,q} = & \frac{qk^n}{k\Gamma(m)\,\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1)\,[R(x_1)]^{m-1} \\
& \times [R(x_2)-R(x_1)]^{n-m-1}\,[1-F(x_2)]^k\,dx_2dx_1 \\
& + \frac{(n-m-1)\,k^n}{k\Gamma(m)\,\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q r(x_1) f(x_2)\,[R(x_1)]^{m-1} \\
& \times [R(x_2)-R(x_1)]^{n-m-2}\,[1-F(x_2)]^{k-1}dx_2dx_1
\end{aligned}
$$

or

$$\mu^{p,q}_{K(m,n)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^k dx_2 dx_1 + \frac{k^{n-1}}{\Gamma(m)\Gamma(n-m-1)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q r(x_1) f(x_2)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-2}[1-F(x_2)]^{k-1} dx_2 dx_1.$$

Since

$$\mu^{p,q}_{K(m,n-1)} = \frac{k^{n-1}}{\Gamma(m)\Gamma(n-m-1)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q r(x_1) f(x_2)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-2}[1-F(x_2)]^{k-1} dx_2 dx_1.$$

Hence

$$\mu^{p,q}_{K(m,n)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^k dx_2 dx_1 + \mu^{p,q}_{K(m,n-1)}$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^k dx_2 dx_1;$$

which is (3.25) and hence the theorem.

Corollary *Suppose a sequence of random variables $\{X_n;\ n \geq 1\}$ is available from an absolutely continuous distribution function $F(x)$. Suppose further that $X_{U(n)}$ be upper record of the sequence then following recurrence relation hold between moments of the records*

$$\mu^p_{(n)} - \mu^p_{(n-1)} = \frac{p}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}\{R(x)\}^{n-1} dx; \tag{3.26}$$

and

$$\mu^{p,q}_{(m,n)} - \mu^{p,q}_{(m,n-1)} = \frac{q}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^k dx_2 dx_1. \tag{3.27}$$

where $R(x) = -\ln[1 - F(x)]$.

Proof The proof is straightforward by using $k = 1$ in (3.24) and (3.25).

The relations given in (3.24)–(3.27) are very useful in deriving recurrence relations for single and product moments of upper record values for special distributions. In the following we have given recurrence relations for single and product moments of kth upper records and upper records for some selected distributions.

3.7.1 The Uniform Distribution

The Uniform distribution is a simple yet powerful distribution. The density and distribution functions of Uniform distribution are respectively

$$f(x) = \frac{1}{b-a};\ a \le x \le b$$

and

$$F(x) = \frac{x-a}{b-a};\ a \le x < b.$$

The density and distribution function are related as

$$(b-x)f(x) = 1 - F(x)\,. \tag{3.28}$$

Bieniek and Szynal (2002) has used (3.28) to derive the recurrence relations for single and product moments of kth upper records of uniform distribution. We present these relations in the following.

We have from (3.24)

$$\begin{aligned}\mu^p_{K(n)} - \mu^p_{K(n-1)} &= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^k\{R(x)\}^{n-1}dx\\ &= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{k-1}\{R(x)\}^{n-1}\\ &\quad\times\{1-F(x)\}\,dx\end{aligned}$$

Now using (3.28) we have

$$\begin{aligned}\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pk^{n-1}}{\Gamma(n)}\int_a^b x^{p-1}\{1-F(x)\}^{k-1}\{R(x)\}^{n-1}\\ \times(b-x)f(x)\,dx\end{aligned}$$

or

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{bpk^{n-1}}{\Gamma(n)} \int_a^b x^{p-1} f(x) \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx$$
$$-\frac{pk^{n-1}}{\Gamma(n)} \int_a^b x^p f(x) \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx$$

or

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pb}{k} \mu^{p-1}_{K(n)} - \frac{p}{k} \mu^p_{K(n)}$$

or

$$\left(1 + \frac{p}{k}\right) \mu^p_{K(n)} = \frac{pb}{k} \mu^{p-1}_{K(n)} + \mu^p_{K(n-1)}$$

or

$$\mu^p_{K(n)} = \frac{pb}{k+p} \mu^{p-1}_{K(n)} + \frac{k}{k+p} \mu^p_{K(n-1)}. \tag{3.29}$$

Again from Theorem 3.1, we have

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1)$$
$$\times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1}$$
$$\times [1 - F(x_2)]^k dx_2 dx_1$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1)$$
$$\times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1}$$
$$\times [1 - F(x_2)]^{k-1} [1 - F(x)] dx_2 dx_1.$$

Now using (3.28) we have

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_a^b \int_{x_1}^b x_1^p x_2^{q-1} r(x_1)$$
$$\times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1}$$
$$\times [1 - F(x_2)]^{k-1} (b - x_2) f(x_2) dx_2 dx_1.$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qbk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_a^b \int_{x_1}^b x_1^p x_2^{q-1} r(x_1)$$

$$\times f(x_2)[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1}$$
$$\times\ [1-F(x_2)]^{k-1}\,dx_2dx_1-\frac{qk^{n-1}}{\Gamma(m)\,\Gamma(n-m)}$$
$$\times\int_a^b\int_{x_1}^b x_1^p x_2^q r(x_1) f(x_2)[R(x_1)]^{m-1}$$
$$\times[R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^{k-1}\,dx_2dx_1.$$

or

$$\mu^{p,q}_{K(m,n)}-\mu^{p,q}_{K(m,n-1)}=\frac{qb}{k}\mu^{p,q-1}_{K(m,n)}-\frac{q}{k}\mu^{p,q}_{K(m,n)}$$

or

$$\left(1+\frac{q}{k}\right)\mu^{p,q}_{K(m,n)}=\frac{qb}{k}\mu^{p,q-1}_{K(m,n)}+\mu^{p,q}_{K(m,n-1)}$$

or

$$\mu^{p,q}_{K(m,n)}=\frac{qb}{k+q}\mu^{p,q-1}_{K(m,n)}+\frac{k}{k+q}\mu^{p,q}_{K(m,n-1)}. \tag{3.30}$$

The recurrence relations for upper records can be obtained by substituting $k=1$ in (3.29) and (3.30) and are given by Ahsanullah (2004) as

$$\mu^{p}_{(n)}=\frac{pb}{p+1}\mu^{p-1}_{(n)}+\frac{1}{p+1}\mu^{p}_{(n-1)}. \tag{3.31}$$

and

$$\mu^{p,q}_{(m,n)}=\frac{qb}{q+1}\mu^{p,q-1}_{(m,n)}+\frac{1}{q+1}\mu^{p,q}_{(m,n-1)}. \tag{3.32}$$

3.7.2 Power Function Distribution

The Power function distribution is extended form of the Uniform distribution. The density and distribution function of a random variable X having Power function distribution are

$$f(x)=\frac{\theta(x-a)^{\theta-1}}{(b-a)^{\theta}};\ a\le x\le b,\theta\ge 1$$

and

$$F(x)=\left(\frac{x-a}{b-a}\right)^{\theta};\ a\le x<b,\theta\ge 1.$$

We can see that following relation holds between density and distribution function

$$(b-x)f(x)=\theta[1-F(x)]. \tag{3.33}$$

The recurrence relation for single moments of Power function distribution has been derived by Bieniek and Szynal (2002) by using (3.33). We present these relations in the following.

We have from (3.24)

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^k \{R(x)\}^{n-1} dx$$
$$= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} \times [1 - F(x)] dx.$$

Now using (3.33) we have

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pk^{n-1}}{\Gamma(n)} \int_a^b x^{p-1} \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} \times \frac{b-x}{\theta} f(x) dx$$

or

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pbk^{n-1}}{\theta\Gamma(n)} \int_a^b x^{p-1} \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx - \frac{pk^{n-1}}{\theta\Gamma(n)} \int_a^b x^p \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx$$

or

$$\mu^p_{K(n)} - \mu^p_{K(n-1)} = \frac{pb}{k\theta}\mu^{p-1}_{K(n)} - \frac{p}{k\theta}\mu^p_{K(n)}$$

or

$$\left(1 + \frac{p}{k\theta}\right)\mu^p_{K(n)} = \frac{pb}{k\theta}\mu^{p-1}_{K(n)} + \mu^p_{K(n-1)}$$

or

$$\mu^p_{K(n)} = \frac{pb}{k\theta + p}\mu^{p-1}_{K(n)} + \frac{k\theta}{k\theta + p}\mu^p_{K(n-1)}. \tag{3.34}$$

The recurrence relation for product moments of Power function distribution can be derived by using (3.25)

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^k dx_2 dx_1$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1}[1-F(x)]\, dx_2 dx_1.$$

Now using (3.33) we have

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_a^b \int_{x_1}^b x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1} \frac{(b-x_2)}{\theta} f(x_2)\, dx_2 dx_1,$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{qbk^{n-1}}{\theta\Gamma(m)\Gamma(n-m)} \int_a^b \int_{x_1}^b x_1^p x_2^{q-1} r(x_1) \times f(x_2)[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1} dx_2 dx_1 - \frac{qk^{n-1}}{\theta\Gamma(m)\Gamma(n-m)} \times \int_a^b \int_{x_1}^b x_1^p x_2^q r(x_1) f(x_2)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^{k-1} dx_2 dx_1,$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{qb}{k\theta}\mu_{K(m,n)}^{p,q-1} - \frac{q}{k\theta}\mu_{K(m,n)}^{p,q}$$

or

$$\left(1+\frac{q}{k\theta}\right)\mu_{K(m,n)}^{p,q} = \frac{qb}{k\theta}\mu_{K(m,n)}^{p,q-1} + \mu_{K(m,n-1)}^{p,q}$$

or

$$\mu_{K(m,n)}^{p,q} = \frac{qb}{k\theta+q}\mu_{K(m,n)}^{p,q-1} + \frac{k\theta}{k\theta+q}\mu_{K(m,n-1)}^{p,q}. \tag{3.35}$$

We can see that the recurrence relations for single and product moments of kth upper record values for Uniform distribution can be obtained from (3.34) and (3.35) by using $\theta = 1$ and are given in (3.29) and (3.30). Further, the recurrence relations for single and product moments of upper records for Power function distribution can be obtained from (3.34) and (3.35) by using $k = 1$ and are given as

$$\mu_{(n)}^{p} = \frac{pb}{p+\theta}\mu_{K(n)}^{p-1} + \frac{\theta}{p+\theta}\mu_{K(n-1)}^{p}, \tag{3.36}$$

and

$$\mu_{(m,n)}^{p,q} = \frac{qb}{q+\theta}\mu_{K(m,n)}^{p,q-1} + \frac{\theta}{q+\theta}\mu_{K(m,n-1)}^{p,q}. \tag{3.37}$$

3.7.3 The Burr Distribution

The density and distribution function of the Burr distribution are given as

$$f(x) = \frac{c\beta\lambda^{\beta}x^{c-1}}{(\lambda+x^{c})^{\beta+1}};\ x, c, \beta, \lambda > 0$$

and

$$F(x) = 1 - \left(\frac{\lambda}{\lambda+x^{c}}\right)^{\beta};\ x > 0.$$

The density and distribution function are related as

$$f(x) = \frac{c\beta x^{c-1}}{\lambda+x^{c}}\left[1-F(x)\right]. \tag{3.38}$$

The recurrence relations for single and product moments of kth upper records for Burr distribution have been derived by Pawlas and Szynal (1999) by using (3.38) in (3.24). We present these relations in the following.

Consider (3.24) as

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{k}\{R(x)\}^{n-1}\,dx \\ &= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{k-1}\left[1-F(x)\right] \\ &\quad \times \{R(x)\}^{n-1}\,dx.\end{aligned}$$

Now using (3.38) in above equation we have

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} = \frac{pk^{n-1}}{\Gamma(n)}\int_{0}^{\infty} x^{p-1}\{1-F(x)\}^{k-1}\left(\frac{\lambda+x^{c}}{c\beta x^{c-1}}\right) \\ \times f(x)\{R(x)\}^{n-1}\,dx\end{aligned}$$

or

$$\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} = \frac{pk^{n-1}}{c\beta\Gamma(n)} \int_0^\infty x^{p-1} \{1 - F(x)\}^{k-1} \left(\frac{\lambda}{x^{c-1}} + x \right) \times f(x) \{R(x)\}^{n-1} dx$$

or

$$\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} = \frac{\lambda pk^{n-1}}{c\beta\Gamma(n)} \int_0^\infty x^{p-c} f(x) \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx + \frac{pk^{n-1}}{c\beta\Gamma(n)} \int_0^\infty x^{p} f(x) \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx$$

or

$$\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} = \frac{\lambda p}{kc\beta} \mu_{K(n)}^{p-c} + \frac{p}{kc\beta} \mu_{K(n)}^{p}$$

or

$$\left(1 - \frac{p}{kc\beta}\right) \mu_{K(n)}^{p} = \mu_{K(n-1)}^{p} + \frac{\lambda p}{kc\beta} \mu_{K(n)}^{p-c}$$

or

$$\mu_{K(n)}^{p} = \frac{kc\beta}{kc\beta - p} \mu_{K(n-1)}^{p} + \frac{\lambda p}{kc\beta - p} \mu_{K(n)}^{p-c}. \tag{3.39}$$

The recurrence relation for single moments of upper records for Burr distribution can be obtained by using $k = 1$ in (3.39) and is given as

$$\mu_{(n)}^{p} = \frac{c\beta}{c\beta - p} \mu_{(n-1)}^{p} + \frac{\lambda p}{c\beta - p} \mu_{(n)}^{p-c}. \tag{3.40}$$

Further, the recurrence relation for single moments of kth upper records and simple upper records for Pareto distribution can be readily obtained from (3.39) and (3.40) by using $c = 1$ and are given as

$$\mu_{K(n)}^{p} = \frac{k\beta}{k\beta - p} \mu_{K(n-1)}^{p} + \frac{\lambda p}{k\beta - p} \mu_{K(n)}^{p-1} \tag{3.41}$$

and

$$\mu_{(n)}^{p} = \frac{\beta}{\beta - p} \mu_{(n-1)}^{p} + \frac{\lambda p}{\beta - p} \mu_{(n)}^{p-1}. \tag{3.42}$$

Again the recurrence relation for product moments is derived by using (3.25) as

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^k \, dx_2 dx_1,$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1}[1-F(x_2)] \, dx_2 dx_1.$$

Now using (3.38) in above equation we have

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{0}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1} \left(\frac{\lambda + x_2^c}{c\beta x_2^{c-1}}\right) f(x_2) \, dx_2 dx_1$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{c\beta\Gamma(m)\Gamma(n-m)} \int_{0}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \times \left(\frac{\lambda}{x_2^{c-1}} + x_2\right)[R(x_1)]^{m-1} r(x_1) f(x_2) \times [R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^{k-1} \, dx_2 dx_1$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{\lambda qk^{n-1}}{c\beta\Gamma(m)\Gamma(n-m)} \int_{0}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-c} r(x_1) f(x_2) \times [R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1} \times [1-F(x_2)]^{k-1} \, dx_2 dx_1 + \frac{qk^{n-1}}{c\beta\Gamma(m)\Gamma(n-m)} \times \int_{0}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q} r(x_1) f(x_2)[R(x_1)]^{m-1} \times [R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^{k-1} \, dx_2 dx_1$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{\lambda q}{kc\beta}\mu_{K(m,n)}^{p,q-c} + \frac{q}{kc\beta}\mu_{K(m,n)}^{p,q}$$

or

$$\mu_{K(m,n)}^{p,q} - \frac{q}{kc\beta}\mu_{K(m,n)}^{p,q} = \mu_{K(m,n-1)}^{p,q} + \frac{\lambda q}{kc\beta}\mu_{K(m,n)}^{p,q-c}$$

or

$$\left(1 - \frac{q}{kc\beta}\right)\mu_{K(m,n)}^{p,q} = \mu_{K(m,n-1)}^{p,q} + \frac{\lambda q}{kc\beta}\mu_{K(m,n)}^{p,q-c}$$

or

$$\mu_{K(m,n)}^{p,q} = \frac{kc\beta}{kc\beta - q}\mu_{K(m,n-1)}^{p,q} + \frac{\lambda q}{kc\beta - q}\mu_{K(m,n)}^{p,q-c}. \tag{3.43}$$

The recurrence relation for upper records of Burr distribution can be readily obtained from (3.43) by using $k = 1$ and is given as

$$\mu_{(m,n)}^{p,q} = \frac{c\beta}{c\beta - q}\mu_{(m,n-1)}^{p,q} + \frac{\lambda q}{c\beta - q}\mu_{(m,n)}^{p,q-c}. \tag{3.44}$$

Finally, the recurrence relation for product moments of kth upper records and upper records for Pareto distribution can be easily obtained from (3.43) and (3.44) by using $c = 1$.

3.7.4 The Exponential Distribution

The exponential distribution is very useful distribution in life testing. The density and distribution function of a random variable X having exponential distribution are respectively

$$f(x) = \alpha e^{-\alpha x};\ x, \alpha > 0$$

and

$$F(x) = 1 - e^{-\alpha x};\ x, \alpha > 0.$$

The density and distribution function are related as

$$f(x) = \alpha\left[1 - F(x)\right]. \tag{3.45}$$

The recurrence relation for single moments of kth upper records is derived by Pawlas and Szynal (1998) by using the relation (3.45) in (3.24) and is given below.

Consider (3.24) as

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k} \{R(x)\}^{n-1} dx \\ &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} \{1 - F(x)\} \\ &\quad \times \{R(x)\}^{n-1} dx.\end{aligned}$$

Now using (3.45) in above equation we have

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\alpha\Gamma(n)} \int_{0}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} f(x) \{R(x)\}^{n-1} dx \\ &= \frac{p}{k\alpha} \mu_{K(n)}^{p-1}\end{aligned}$$

or

$$\mu_{K(n)}^{p} = \mu_{K(n-1)}^{p} + \frac{p}{k\alpha} \mu_{K(n)}^{p-1}. \tag{3.46}$$

Again, the recurrence relation for product moments of kth upper records for Exponential distribution is derived below.

We have from (3.25)

$$\begin{aligned}\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} &= \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^{p} x_2^{q-1} r(x_1) \\ &\quad \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\quad \times [1 - F(x_2)]^{k} dx_2 dx_1\end{aligned}$$

or

$$\begin{aligned}\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} &= \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^{p} x_2^{q-1} r(x_1) \\ &\quad \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\quad \times [1 - F(x_2)]^{k-1} [1 - F(x_2)] dx_2 dx_1.\end{aligned}$$

Now using (3.45) in above equation we have

$$\begin{aligned}\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} &= \frac{qk^{n-1}}{\alpha\Gamma(m)\Gamma(n-m)} \int_{0}^{\infty} \int_{x_1}^{\infty} x_1^{p} x_2^{q-1} r(x_1) \\ &\quad \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\quad \times [1 - F(x_2)]^{k-1} f(x_2) dx_2 dx_1\end{aligned}$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{q}{k\alpha}\mu^{p,q-1}_{K(m,n)}$$

or

$$\mu^{p,q}_{K(m,n)} = \mu^{p,q}_{K(m,n-1)} + \frac{q}{k\alpha}\mu^{p,q-1}_{K(m,n)}. \tag{3.47}$$

The recurrence relations for single and product moments of upper records of exponential distribution can be obtained from (3.46) and (3.47) by using $k = 1$.

3.7.5 The Weibull Distribution

The density and distribution function of Weibull random variable are

$$f(x) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0$$

and

$$F(x) = 1 - \exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0.$$

The density and distribution function are related as

$$f(x) = \alpha\beta x^{\beta-1}\left[1 - F(x)\right]. \tag{3.48}$$

The relation (3.48) has been used by Pawlas and Szynal (2000) to derive the recurrence relation for moments of kth upper records and is given below.

Consider (3.24) as

$$\begin{aligned}
\mu^{p}_{K(n)} - \mu^{p}_{K(n-1)} &= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1 - F(x)\}^{k}\{R(x)\}^{n-1}dx \\
&= \frac{pk^{n-1}}{\Gamma(n)}\int_{-\infty}^{\infty} x^{p-1}\{1 - F(x)\}^{k-1}\{1 - F(x)\} \\
&\quad\times \{R(x)\}^{n-1}dx.
\end{aligned}$$

Now using (3.48) in above equation we have

$$\begin{aligned}
\mu^{p}_{K(n)} - \mu^{p}_{K(n-1)} &= \frac{pk^{n-1}}{\alpha\beta\Gamma(n)}\int_{0}^{\infty} x^{p-1}\{1 - F(x)\}^{k-1}\{R(x)\}^{n-1} \\
&\quad\times x^{1-\beta}f(x)\,dx \\
&= \frac{p}{k\alpha\beta}\mu^{p-\beta}_{K(n)}
\end{aligned}$$

or

$$\mu^{p}_{K(n)} = \mu^{p}_{K(n-1)} + \frac{p}{k\alpha\beta}\mu^{p-\beta}_{K(n)}. \tag{3.49}$$

We can see that $\beta = 1$, the recurrence relation (3.49) reduces to the recurrence relation for single moments of kth upper records for exponential distribution. The recurrence relation for single moments of upper records can be easily obtained from (3.49) by using $k = 1$.

The recurrence relation for product moments of kth upper records for Weibull distribution can be derived by using (3.25) as below.

We have

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^k\, dx_2 dx_1$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^{k-1}[1 - F(x_2)]\, dx_2 dx_1.$$

Now using (3.48) in above equation we have

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{qk^{n-1}}{\alpha\beta\Gamma(m)\Gamma(n-m)}\int_{0}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} r(x_1) \times [R(x_1)]^{m-1}[R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^{k-1} x_2^{1-\beta} f(x_2)\, dx_2 dx_1$$

or

$$\mu^{p,q}_{K(m,n)} - \mu^{p,q}_{K(m,n-1)} = \frac{q}{k\alpha\beta}\mu^{p,q-\beta}_{K(m,n)}$$

or

$$\mu^{p,q}_{K(m,n)} = \mu^{p,q}_{K(m,n-1)} + \frac{q}{k\alpha\beta}\mu^{p,q-\beta}_{K(m,n)}. \tag{3.50}$$

It can be seen that (3.50) reduces to (3.47) for $\beta = 1$ as the case should be. Finally, the recurrence relation for product moments of upper records for Weibull distribution can be obtained from (3.50) by using $k = 1$.

3.7.6 The Frechet Distribution

The density and distribution function of Frechet distribution are

$$f(x) = \frac{\alpha\beta}{x^{\beta+1}} \exp\left(-\frac{\alpha}{x^{\beta}}\right);\ x, \alpha, \beta > 0$$

and

$$F(x) = \exp\left(-\frac{\alpha}{x^{\beta}}\right).$$

The density and distribution function are related as

$$1 - F(x) = \frac{x^{\beta+1}}{\alpha\beta}\left\{\exp\left(\frac{\alpha}{x^{\beta}}\right) - 1\right\} f(x). \tag{3.51}$$

The recurrence relations for single and product moments of kth upper records for Frechet distribution are obtained by using (3.51) in (3.24) and (3.25). First consider (3.24) as

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k} \{R(x)\}^{n-1} dx \\ &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} \{1 - F(x)\} \\ &\quad \times \{R(x)\}^{n-1} dx.\end{aligned}$$

Now using (3.51) in above equation we have

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} \frac{x^{\beta+1}}{\alpha\beta} f(x) \\ &\quad \times \left\{\exp\left(\frac{\alpha}{x^{\beta}}\right) - 1\right\} \times \{R(x)\}^{n-1} dx\end{aligned}$$

or

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\alpha\beta\Gamma(n)} \int_{-\infty}^{\infty} x^{p+\beta} \exp\left(\frac{\alpha}{x^{\beta}}\right) f(x) \{1 - F(x)\}^{k-1} \\ &\quad \times \{R(x)\}^{n-1} dx - \frac{pk^{n-1}}{\alpha\beta\Gamma(n)} \int_{-\infty}^{\infty} x^{p+\beta} f(x) \\ &\quad \times \{1 - F(x)\}^{k-1} \{R(x)\}^{n-1} dx\end{aligned}$$

or

$$\begin{aligned}\mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\alpha\beta\Gamma(n)} \int_{-\infty}^{\infty} x^{p+\beta} \exp\left(\frac{\alpha}{x^{\beta}}\right) f(x) \{1 - F(x)\}^{k-1} \\ &\quad \times \{R(x)\}^{n-1} dx - \frac{p}{k\alpha\beta} \mu_{K(n)}^{p+\beta}\end{aligned}$$

Now expanding $\exp\left(\alpha/x^{\beta}\right)$ we have

$$\mu_{K(n)}^{p}-\mu_{K(n-1)}^{p}=\sum_{j=0}^{\infty}\frac{\alpha^{j-1}}{j!}\frac{pk^{n}}{k\beta\Gamma(n)}\int_{-\infty}^{\infty}x^{p+\beta-j\beta}f(x)\times\{1-F(x)\}^{k-1}\{R(x)\}^{n-1}dx-\frac{p}{k\alpha\beta}\mu_{K(n)}^{p+\beta}$$

or

$$\mu_{K(n)}^{p}-\mu_{K(n-1)}^{p}=\frac{p}{k\beta}\left\{\sum_{j=0}^{\infty}\frac{\alpha^{j-1}}{j!}\mu_{K(n)}^{p-\beta(j-1)}-\frac{1}{\alpha}\mu_{K(n)}^{p+\beta}\right\}. \tag{3.52}$$

The recurrence relations for single moments of upper records can be easily obtained from (3.52) by using $k=1$. Again consider (3.25) as

$$\mu_{K(m,n)}^{p,q}-\mu_{K(m,n-1)}^{p,q}=\frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^{p}x_2^{q-1}r(x_1)\times[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1}\times[1-F(x_2)]^{k}dx_2dx_1$$

or

$$\mu_{K(m,n)}^{p,q}-\mu_{K(m,n-1)}^{p,q}=\frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^{p}x_2^{q-1}r(x_1)\times[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1}\times[1-F(x_2)]^{k-1}[1-F(x_2)]dx_2dx_1.$$

Now using (3.51) in above equation we have

$$\mu_{K(m,n)}^{p,q}-\mu_{K(m,n-1)}^{p,q}=\frac{qk^{n-1}}{\alpha\beta\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^{p}x_2^{q+\beta}r(x_1)f(x_2)\times[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1}[1-F(x_2)]^{k-1}\times\left\{\exp\left(\frac{\alpha}{x_2^{\beta}}\right)-1\right\}dx_2dx_1$$

or

$$\mu_{K(m,n)}^{p,q}-\mu_{K(m,n-1)}^{p,q}=\frac{qk^{n-1}}{\alpha\beta\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^{p}x_2^{q+\beta}\exp\left(\frac{\alpha}{x_2^{\beta}}\right)\times r(x_1)f(x_2)[R(x_1)]^{m-1}[R(x_2)-R(x_1)]^{n-m-1}\times[1-F(x_2)]^{k-1}dx_2dx_1-\frac{qk^{n-1}}{\alpha\beta\Gamma(m)\Gamma(n-m)}$$

$$\times \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+\beta} r(x_1) f(x_2) [R(x_1)]^{m-1} \times [R(x_2) - R(x_1)]^{n-m-1} [1 - F(x_2)]^{k-1} dx_2 dx_1$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{qk^{n-1}}{\alpha\beta\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+\beta} \exp\left(\frac{\alpha}{x_2^{\beta}}\right) \times r(x_1) f(x_2) [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^{k-1} dx_2 dx_1 - \frac{q}{k\alpha\beta} \mu_{K(m,n)}^{p,q+\beta}.$$

Now expanding $\exp\left(\alpha/x^{\beta}\right)$ we have

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \sum_{j=0}^{\infty} \frac{\alpha^{j-1}}{j!} \frac{qk^{n-1}}{\beta\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+\beta-j\beta} \times r(x_1) f(x_2) [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^{k-1} dx_2 dx_1 - \frac{q}{k\alpha\beta} \mu_{K(m,n)}^{p,q+\beta}.$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \frac{q}{k\beta} \left\{ \sum_{j=0}^{\infty} \frac{\alpha^{j-1}}{j!} \mu_{K(m,n)}^{p,q-\beta(j-1)} - \frac{1}{\alpha} \mu_{K(m,n)}^{p,q+\beta} \right\}. \tag{3.53}$$

The recurrence relation for product moments of upper records can be readily obtained from (3.53) by using $k = 1$.

3.7.7 The Gumbel Distribution

The Gumbel distribution is a popular distribution in extreme value theory. The distribution is described in two situations, namely Gumbel maximum and Gumbel minimum distribution. The density and distribution function for Gumbel minimum distribution are

$$f(x) = \frac{1}{\sigma} \exp\left[\frac{x}{\sigma} - \exp\left(\frac{x}{\sigma}\right)\right]; \ -\infty < x < \infty, \ \sigma > 0$$

and

$$F(x) = 1 - \exp\left[-\exp\left(\frac{x}{\sigma}\right)\right]; \ -\infty < x < \infty, \ \sigma > 0.$$

The density and distribution function are related as

$$\sum_{j=0}^{\infty} (-1)^j \frac{x^j}{\sigma^{j-1} j!} f(x) = [1 - F(x)]. \tag{3.54}$$

The recurrence relation for single moments of kth upper records is derived by using the relation (3.54) in (3.24) and is given below.

Consider (3.24) as

$$\begin{aligned} \mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k} \{R(x)\}^{n-1} dx \\ &= \frac{pk^{n-1}}{\Gamma(n)} \int_{-\infty}^{\infty} x^{p-1} \{1 - F(x)\}^{k-1} \{1 - F(x)\} \\ &\quad \times \{R(x)\}^{n-1} dx. \end{aligned}$$

Now using (3.54) in above equation we have

$$\begin{aligned} \mu_{K(n)}^{p} - \mu_{K(n-1)}^{p} &= \sum_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{pk^{n-1}}{\Gamma(n)} \int_{0}^{\infty} x^{p+j-1} f(x) \{1 - F(x)\}^{k-1} \\ &\quad \times \{R(x)\}^{n-1} dx \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{p}{k} \mu_{K(n)}^{p+j-1}. \end{aligned}$$

or

$$\mu_{K(n)}^{p} = \mu_{K(n-1)}^{p} + \sum_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{p}{k} \mu_{K(n)}^{p+j-1}. \tag{3.55}$$

Again, the recurrence relation for product moments of kth upper records for Exponential distribution is derived below.

We have from (3.25)

$$\begin{aligned} \mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} &= \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^{p} x_2^{q-1} r(x_1) \\ &\quad \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\quad \times [1 - F(x_2)]^{k} dx_2 dx_1 \end{aligned}$$

or

$$\begin{aligned} \mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} &= \frac{qk^{n-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^{p} x_2^{q-1} r(x_1) \\ &\quad \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \\ &\quad \times [1 - F(x_2)]^{k-1} [1 - F(x_2)] dx_2 dx_1. \end{aligned}$$

Now using (3.54) in above equation we have

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \sum\nolimits_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{q k^{n-1}}{\Gamma(m)\,\Gamma(n-m)} \int_0^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q+j-1} r(x_1) \times [R(x_1)]^{m-1} [R(x_2) - R(x_1)]^{n-m-1} \times [1 - F(x_2)]^{k-1} f(x_2)\, dx_2 dx_1$$

or

$$\mu_{K(m,n)}^{p,q} - \mu_{K(m,n-1)}^{p,q} = \sum\nolimits_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{q}{k} \mu_{K(m,n)}^{p,q+j-1}$$

or

$$\mu_{K(m,n)}^{p,q} = \mu_{K(m,n-1)}^{p,q} + \sum\nolimits_{j=0}^{\infty} \frac{(-1)^j}{\sigma^{j-1} j!} \frac{q}{k} \mu_{K(m,n)}^{p,q+j-1}. \tag{3.56}$$

The recurrence relations for single and product moments of upper records of exponential distribution can be obtained from (3.55) and (3.56) by using $k = 1$.

Chapter 4
The Generalized Order Statistics

4.1 Introduction

In Chap. 1 we have given a brief overview of some of the possible models for ordered random variables. Further, in previous two chapters we have discussed, in detail, two popular models of ordered random variables, namely Order Statistics and Record Values. We have seen that the order statistics and record values have been studied by several authors in context of different underlying probability models.

The other models of ordered random variables given in Chap. 1 have not been studied in much details for specific probability distributions but they all have been combined in a more general model for ordered data known as *Generalized Order Statistics (GOS)*. Kamps (1995a) has proposed GOS as a unified models for ordered random variables which produce several models as a special case. Since its inception GOS has attracted number of statisticians as distribution specific results obtained for GOS can be used to obtain the results for other models of ordered random variables as special case. We formally define GOS and their joint distribution in the following.

4.2 Joint Distribution of GOS

Suppose a random sample of size $\{n; n \in \mathbb{N}\}$ is available from a distribution with cumulative distribution function $F(x)$ and let $k \geq 1$. Suppose further that the constants $m_1, m_2, \ldots, m_{n-1}$ are available such that $m_r \in \mathbb{R}$ and let

$$M_r = \sum_{j=r}^{n-1} m_j; \ 1 \leq r \leq n-1.$$

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_4

The numbers n, k and m_r are parameters of the model. Define γ_r as $\gamma_r = k + (n-r) + M_r$ such that $\gamma_r \geq 1$ for all $r \in \{1, 2, \ldots, n-1\}$. Finally let

$$\tilde{m} = (m_1, m_2, \ldots, m_{n-1}); \, if \; n \geq 2$$

then the random variables $X_{r:n,\tilde{m},k}$ are the Generalized Order Statistics (GOS) from the distribution $F(x)$, if their joint density function is of the form

$$f_{1,\ldots,n:n,\tilde{m},k}(x_1, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\{1 - F(x_n)\}^{k-1} f(x_n) \times \left[\prod_{i=1}^{n-1}\{1 - F(x_i)\}^{m_i} f(x_i)\right]; \tag{4.1}$$

and is defined on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.
If $m_1 = m_2 = \cdots = m_{n-1} = m$, then GOS are denoted as $X_{r:n,m,k}$.

Making the transformation $U_{r:n,\tilde{m},k} = F\left(X_{r:n,\tilde{m},k}\right)$, the random variables $U_{r:n,\tilde{m},k}$ are called the Uniform GOS with joint density function

$$f_{1,\ldots,n:n,\tilde{m},k}(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}(1 - u_i)^{m_i}\right] \times (1 - u_n)^{k-1}; \tag{4.2}$$

with $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n < 1$. The joint distribution of GOS given in (4.1) provides a comprehensive model for joint distribution of all models of ordered random variables for different values of the parameters involved. We have given the joint distribution of various models of ordered random variables as special cases of (4.1) in the following.

4.3 Special Cases of GOS

Kamps (1995b) have discussed GOS as a unified model for ordered random variables. This model contains all the models of GOS discussed in Chap. 1. We see in the following how GOS provide various models of ordered random variables as special cases.

1. Choosing $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$, such that $\gamma_r = n - r + 1$, density (4.1) reduces to

$$f_{1,\ldots,n:n,0,1}(x_1,\ldots,x_n) = n!\prod_{i=1}^{n} f(x_i),$$

which is joint density of Ordinary Order Statistics (OOS).

2. Choosing $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = \alpha - n + 1$, with $n - 1 < \alpha$ such that $\gamma_r = \alpha - r + 1$, the density (4.1) reduces to

$$f_{1,\ldots,n:n,0,\alpha-n+1}(x_1,\ldots,x_n) = \prod_{j=1}^{n}(\alpha - j + 1) \times [1 - F(x_n)]^{\alpha-n} \times \prod_{i=1}^{n} f(x_i),$$

which is joint density of OOS with non–integral sample size.

3. Choosing $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1; i = 1, 2, ..., n-1; k = \alpha_n$ for some real number $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\gamma_r = (n-r+1)\alpha_r$, the density (4.1) becomes

$$f_{1,\ldots,n:n,\tilde{m},\alpha_n}(x_1,...,x_n) = n!\left(\prod_{j=1}^{n-1}\alpha_j\right)\left[\prod_{i=1}^{n-1}\{1 - F(x_i)\}^{m_i} \times f(x_i)\right]\{1 - F(x_n)\}^{\alpha_n-1} \times f(x_n),$$

which is joint density of Sequential Order Statistics (SOS) based on the arbitrary distribution function

$$F_r(t) = 1 - [1 - F(t)]^{\alpha_r};\ 1 \le r \le n.$$

4. For $m_1 = m_2 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$, such that $\gamma_r = k$, the density (4.1) reduces to

$$f_{1,\ldots,n:n,-1,k}(x_1,...,x_n) = k^n\left[\prod_{i=1}^{n-1}\frac{f(x_i)}{1 - F(x_i)}\right] \times \{1 - F(x_n)\}^{k-1} f(x_n),$$

which is joint density of kth records. Choosing $k = 1$ we obtain joint density of records.

5. For positive real numbers $\beta_1, \beta_2, \ldots, \beta_n$, choosing $m_i = \beta_i - \beta_{i+1} - 1; i = 1, 2, \ldots, n-1$ and $k = \beta_n$; such that $\gamma_r = \beta_r$; the density (4.1) reduces to

$$f_{1,\ldots,n:n,\tilde{m},\beta_n}(x_1, \ldots, x_n) = \left(\prod_{j=1}^{n}\beta_j\right)\left[\prod_{i=1}^{n-1}\{1-F(x_i)\}^{m_i} f(x_i)\right]\{1-F(x_n)\}^{\beta_n-1} \times f(x_n),$$

which is joint density of Pfeifer's record values from non–identically distributed random variables based upon

$$F_r(t) = 1 - [1 - F(t)]^{\beta_r};\ 1 \le r \le n.$$

6. For positive real numbers $\beta_1, \beta_2, \ldots, \beta_n$, choosing

$$m_i = \beta_i k_i - \beta_{i+1} k_{i+1} - 1;\ i = 1, 2, \ldots, n-1$$

and $k = \beta_n k_n$; such that $\gamma_r = \beta_r k_r$; the density (4.1) reduces to

$$f_{1,\ldots,n:n,\tilde{m},\beta_n}(x_1, \ldots, x_n) = \left(\prod_{j=1}^{n} k_j\right) f(x_n) \times \left[\prod_{i=1}^{n-1}\{1-F(x_i)\}^{k_i-k_{i+1}-1} f(x_i)\right] \times \{1-F(x_n)\}^{k_n-1},$$

which is joint density of k_n–records from non–identically distributed random variables.

7. Choosing

$$m_1 = \cdots = m_{r_1-1} = m_{r_1+1} = \cdots = m_{n-1} = 0;$$

$m_{r_1} = n_1$ and $k = \nu - n_1 - n + 1$ such that

$$\gamma_r = \nu - n + 1;\ 1 \le r \le r_1$$

and

$$\gamma_r = \nu - n_1 - r + 1;\ r_1 < r \le n-1$$

the density (4.1) reduces to

$$f_{1,\ldots,n:n,\tilde{m},\beta_n}(x_1, \ldots, x_n) = \frac{\nu!(\nu - r_1 - n_1)!}{(\nu - r_1)!(\nu - n_1 - n)} \prod_{i=1}^{n} f(x_i) \times \left[1 - F\left(x_{r_1}\right)\right]^{n_1} [1 - F(x_n)]^{\nu - n_1 - n}$$

which is joint density of progressive type II censoring with two stages.

4.4 Some Notations

Since GOS provides a unified model for ordered random variables, it requires certain special notations. These notations are given below.

1. The constant C_{r-1} is defined as

$$C_{r-1} = \prod_{j=1}^{r} \gamma_j; r = 1, 2, \ldots, n$$

with $\gamma_n = k$. Hence we have

$$C_{n-1} = \prod_{j=1}^{n} \gamma_j = k \prod_{j=1}^{n-1} \gamma_j.$$

2. On the unit interval the functions $h_m(x)$ and $g_m(x)$, $m \in \mathbb{R}$, are defined as

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}; & m \neq -1 \\ -\ln(1-x) & ; m = -1 \end{cases}; \ x \in [0, 1)$$

$$g_m(x) = h_m(x) - h_m(0)$$

$$= \begin{cases} \frac{1}{m+1}\left[1-(1-x)^{m+1}\right]; & m \neq -1 \\ -\ln(1-x); & m = -1 \end{cases}; \ x \in [0, 1)$$

Using above representation, the joint density of Uniform GOS can be written as

$$f_{1,\ldots,n:n,\tilde{m},k}(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left[\prod_{i=1}^{n-1} \frac{d}{du_i} h_{m_i}(u_i)\right] \times (1-u_n)^{k-1}$$

Hence the functions $h_m(x)$ and $g_m(x)$ occur very frequently in context of GOS.

We now give the joint marginal distribution of first r GOS.

4.5 Joint Marginal Distribution of *r* GOS

The joint density function of n uniform GOS is given in (4.2) as

$$f_{1,\ldots,n:n,\tilde{m},k}(u_1, \ldots, u_n) = C_{n-1}\left[\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right] \times (1-u_n)^{k-1}.$$

The joint marginal density of r uniform GOS is readily written by integrating out variables $u_{r+1}, \ldots, u_n$ by induction as

$$f_{1,\ldots,r:n,\tilde{m},k}(u_1, \ldots, u_r) = C_{r-1}\left[\prod_{i=1}^{r-1}(1-u_i)^{m_i}\right](1-u_r)^{\gamma_r-1} \quad ; \; 0 \le u_1 \le \cdots \le u_r < 1, \tag{4.3}$$

which immediately yields following joint marginal distribution of GOS for any parent distribution $F(x)$

$$f_{1,\ldots,r:n,\tilde{m},k}(x_1, \ldots, x_r) = C_{r-1}\left[\prod_{i=1}^{r-1}\{1-F(x_i)\}^{m_i} f(x_i)\right] \times \{1-F(x_r)\}^{\gamma_r-1} f(x_r), \tag{4.4}$$

on the cone $F^{-1}(0) < x_1 \le \cdots \le x_r < F^{-1}(1)$.

The joint marginal distribution of r GOS given in (4.4) provide joint marginal distribution of other models of ordered random variables as special case. These special cases are given below.

1. *Ordinary Order Statistics*: using $m_1 = \cdots = m_{n-1} = 0$ and $k = 1$ in (4.4), we have the joint marginal distribution of r ordinary order statistics as

$$f_{1,\ldots,r:n,0,1}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r-1} f(x_i)\right] \times \{1-F(x_r)\}^{n-r} f(x_r).$$

2. *Fractional Order Statistics*: using $m_1 = \cdots = m_{n-1} = 0$ and $k = \alpha - n + 1$ in (4.4), we have the joint marginal distribution of r fractional order statistics as

$$f_{1,\ldots,r:n,0,\alpha-n+1}(x_1, \ldots, x_r) = \prod_{j=1}^{r}(\alpha - j + 1)\left[\prod_{i=1}^{r-1} f(x_i)\right] \times \{1-F(x_r)\}^{\alpha-r} f(x_r).$$

3. *Sequential Order Statistics*: The joint distribution of r sequential order statistics is obtained by using $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1}$ and $k = \alpha_n$ in (4.4) as

$$f_{1,\ldots,r:n,\tilde{m},\alpha_n}(x_1, \ldots, x_n) = \frac{n!}{(n-r)!}\prod_{j=1}^{r}\alpha_j\left[\prod_{i=1}^{r-1} f(x_i) \times \{1-F(x_i)\}^{m_i}\right] \times \{1-F(x_r)\}^{\alpha_r(n-r+1)-1} f(x_r).$$

4. *Record Values*: using $m_1 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$ in (4.4), the joint marginal distribution of r $k-$record values is obtained as

$$f_{1,\ldots,r:n,-1,k}(x_1, \ldots, x_r) = k^r \left[\prod_{i=1}^{r-1} \frac{f(x_i)}{1 - F(x_i)}\right] \times \{1 - F(x_r)\}^{k-1} f(x_r).$$

5. *Pfeifer Record Values*: using $m_i = \beta_i - \beta_{i+1} - 1$; $k \in \beta_n$; as

$$f_{1,\ldots,r:n,-1,k}(x_1, \ldots, x_r) = \prod_{j=1}^{r} \beta_j \left[\prod_{i=1}^{r-1} \{1 - F(x_i)\}^{m_i} f(x_i)\right] \times \{1 - F(x_r)\}^{\beta_r - 1} f(x_r).$$

Other special cases can also be obtained from (4.4). The joint marginal distribution of r uniform GOS given in (4.3) can be used to obtain the marginal distribution of rth GOS and joint marginal distribution of rth and sth GOS. We have given these distributions in the following but we first give a Lemma due to Kamps (1995b).

Lemma 4.1 *We define the quantity A_j as*

$$A_j = \int_{u_{r-j-1}}^{u_r} \cdots \int_{u_{r-2}}^{u_r} \prod_{i=1}^{r-1} h_m^{/}(u_i) du_{r-1} \cdots du_{r-j} = \frac{1}{j!} \prod_{i=1}^{r-j-1} h_m^{/}(u_i) \left\{h_m(u_r) - h_m\left(u_{r-j-1}\right)\right\}^j.$$

We now give the marginal distribution of rth GOS.

4.6 Marginal Distribution of a Single GOS

The joint marginal distribution of r uniform GOS is given in (4.3) as

$$f_{1,\ldots,r:n,\tilde{m},k}(u_1, \ldots, u_r) = C_{r-1} \left[\prod_{i=1}^{r-1} (1 - u_i)^{m_i}\right] (1 - u_r)^{\gamma_r - 1} \; ; \; 0 \le u_1 \le \cdots \le u_r < 1.$$

Assuming $m_1 = \cdots = m_{n-1} = m$, the joint distribution is

$$f_{1,\ldots,r:n,m,k}(u_1,\ldots,u_r) = C_{r-1}\left[\prod_{i=1}^{r-1}(1-u_i)^m\right](1-u_r)^{\gamma_r-1}$$
$$;\ 0 \le u_1 \le \cdots \le u_r < 1.$$

The marginal distribution of rth uniform GOS can be obtained from above by integrating out $u_1, u_2, \ldots, u_{r-1}$ as under

$$\begin{aligned}
f_{r:n,m,k}(u_r) &= \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r} f_{1,\ldots,r:n,m,k}(u_1,\ldots,u_r)du_{r-1}\cdots du_1 \\
&= \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r} C_{r-1}\left[\prod_{i=1}^{r-1}(1-u_i)^m\right](1-u_r)^{\gamma_r-1} \\
&\quad \times du_{r-1}\cdots du_1 \\
&= C_{r-1}(1-u_r)^{\gamma_r-1}\int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\left[\prod_{i=1}^{r-1}(1-u_i)^m\right] \\
&\quad \times du_{r-1}\cdots du_1 \\
&= C_{r-1}(1-u_r)^{\gamma_r-1}\int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\prod_{i=1}^{r-1}h_m^{/}(u_i)du_{r-1}\cdots du_1
\end{aligned}$$

Now using the Lemma 4.1 with $j = r-1$; and noting that $u_0 = 0$; we have:

$$\begin{aligned}
A_{r-1} &= \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\prod_{i=1}^{r-1}h_m^{/}(u_i)du_{r-1}\cdots du_1 \\
&= \frac{1}{(r-1)!}\{h_m(u_r)-h_m(0)\}^{r-1} = \frac{1}{(r-1)!}g_m^{r-1}(u_r).
\end{aligned}$$

Hence the marginal distribution of rth uniform GOS is

$$f_{r:n,m,k}(u_r) = \frac{C_{r-1}}{(r-1)!}(1-u_r)^{\gamma_r-1}g_m^{r-1}(u_r). \tag{4.5}$$

The marginal density of rth GOS for any parent distribution is readily written from (4.5); by noting that for any distribution the random variable $F(x)$ is always uniform; as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)]. \tag{4.6}$$

The special cases can be readily written from (4.6). Specifically the marginal distribution of rth $k-$record value is

$$f_{r:n,-1,k}(x) = \frac{k^r}{(r-1)!}f(x)\{1-F(x)\}^{k-1}[-\ln\{1-F(x)\}]^{r-1}; \tag{4.7}$$

which for $k = 1$ reduces to distribution of rth record value as

$$f_{r:n,-1,k}(x) = \frac{1}{(r-1)!} f(x)[-\ln\{1-F(x)\}]^{r-1}. \tag{4.8}$$

Further from (4.6) it can be readily seen that GOS $X_{r:n,m,k}$ and $X_{r:n',-m-2,k'}$ are identically distributed where

$$k - k' = -\big(n + n' - r - 1\big)(m+1) \,\&\, k', n' \in \mathbb{N}.$$

Also we can see from (4.5) that $U_{r:n,m,k+m+1}$ and $U_{r:n+1,m,k}$ are identically distributed.

We now give the joint distribution of rth and sth GOS in the following.

4.7 Joint Distribution of Two GOS

We derive the joint distribution of rth and sth GOS ; with $r < s$; as under. The joint marginal distribution of first s uniform GOS is given from (4.3) as

$$f_{1,\ldots,s:n,m,k}(u_1, \ldots, u_s) = C_{s-1}\left[\prod_{i=1}^{s-1}(1-u_i)^m\right](1-u_s)^{\gamma_s-1}$$
$$;\ 0 \le u_1 \le \cdots \le u_s < 1,$$

or

$$f_{1,\ldots,s:n,m,k}(u_1, \ldots, u_s) = C_{s-1}\left[\prod_{i=1}^{s-1} h_m'(u_i)\right](1-u_s)^{\gamma_s-1}$$
$$;\ 0 \le u_1 \le \cdots \le u_s < 1.$$

The joint distribution of rth and sth GOS is obtained by integrating out $u_r = u_1$ and $u_s = u_2$ as

$$\begin{aligned} f_{r,s:n,m,k}(u_1, u_2) &= \int_0^{u_r} \cdots \int_{u_{r-2}}^{u_r} \int_{u_r}^{u_s} \cdots \int_{u_{s-2}}^{u_s} f_{1,\ldots,s:n,m,k}(u_1, \ldots, u_s) \\ &\quad \times du_{s-1} \cdots du_{r+1} du_{r-1} \cdots du_1 \\ &= \int_0^{u_r} \cdots \int_{u_{r-2}}^{u_r} \int_{u_r}^{u_s} \cdots \int_{u_{s-2}}^{u_s} C_{s-1}\left[\prod_{i=1}^{s-1} h_m'(u_i)\right] \\ &\quad \times (1-u_s)^{\gamma_s-1} du_{s-1} \cdots du_{r+1} du_{r-1} \cdots du_1 \end{aligned}$$

or

$$
\begin{aligned}
f_{r,s:n,m,k}(u_1,u_2) = C_{s-1}(1-u_s)^{\gamma_s-1} \\
& \times \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\int_{u_r}^{u_s}\cdots\int_{u_{s-2}}^{u_s}\prod_{i=1}^{s-1}h_m^{/}(u_i) \\
& \times du_{s-1}\cdots du_{r+1}du_{r-1}\cdots du_1
\end{aligned}
$$

or

$$
\begin{aligned}
f_{r,s:n,m,k}(u_1,u_2) = C_{s-1}(1-u_s)^{\gamma_s-1}(1-u_r)^m \\
& \times \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\prod_{i=1}^{r-1}h_m^{/}(u_i) \\
& \times \left[\int_{u_r}^{u_s}\cdots\int_{u_{s-2}}^{u_s}\prod_{i=r+1}^{s-1}h_m^{/}(u_i)du_{s-1}\cdots du_{r+1}\right] \\
& \times du_{r-1}\cdots du_1
\end{aligned}
$$

or

$$
\begin{aligned}
f_{r,s:n,m,k}(u_1,u_2) &= C_{s-1}(1-u_s)^{\gamma_s-1}(1-u_r)^m \\
& \times \int_0^{u_r}\cdots\int_{u_{r-2}}^{u_r}\prod_{i=1}^{r-1}h_m^{/}(u_i) \\
& \times I(s)du_{r-1}\cdots du_1,
\end{aligned}
\tag{4.8}
$$

where

$$
I(s) = \int_{u_r}^{u_s}\cdots\int_{u_{s-2}}^{u_s}\prod_{i=r+1}^{s-1}h_m^{/}(u_i)du_{s-1}\cdots du_{r+1}.
$$

Now using Lemma 4.1 with $s=r$ and $j=s-r-1$ we have

$$
\begin{aligned}
A_{s-r-1} &= I(s) = \int_{u_r}^{u_s}\cdots\int_{u_{s-2}}^{u_s}\prod_{i=r+1}^{s-1}h_m^{/}(u_i)du_{s-1}\cdots du_{r+1} \\
&= \frac{1}{(s-r-1)!}\{h_m(u_s)-h_m(u_r)\}^{s-r-1}
\end{aligned}
$$

Using above result in (4.8) we have

$$\begin{aligned} f_{r,s:n,m,k}(u_1, u_2) &= C_{s-1}(1-u_s)^{\gamma_s-1}(1-u_r)^m \\ &\times \frac{1}{(s-r-1)!}\{h_m(u_s) - h_m(u_r)\}^{s-r-1} \\ &\times \int_0^{u_r} \cdots \int_{u_{r-2}}^{u_r} \prod_{i=1}^{r-1} h_m^{/}(u_i) du_{r-1} \cdots du_1 \\ &= C_{s-1}(1-u_s)^{\gamma_s-1}(1-u_r)^m \frac{1}{(s-r-1)!} \\ &\times \{h_m(u_s) - h_m(u_r)\}^{s-r-1} I(r) \end{aligned}$$

Again using Lemma 4.1 with $j = r-1$; and noting that $u_0 = 0$; we have

$$\begin{aligned} A_{r-1} &= I(r) = \int_0^{u_r} \cdots \int_{u_{r-2}}^{u_r} \prod_{i=1}^{r-1} h_m^{/}(u_i) \\ &\times du_{r-1} \cdots du_1 \\ &= \frac{1}{(r-1)!}\{h_m(u_r) - h_m(0)\}^{r-1} \end{aligned}$$

or

$$A_{r-1} = \frac{1}{(r-1)!} g_m^{r-1}(u_r).$$

Hence the joint density of rth and sth uniform GOS is

$$\begin{aligned} f_{r,s:n,m,k}(u_1, u_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}(1-u_1)^m g_m^{r-1}(u_1) \\ &\times (1-u_2)^{\gamma_s-1}\{h_m(u_2) - h_m(u_1)\}^{s-r-1}. \end{aligned} \tag{4.9}$$

The joint density of rth and sth GOS from any parent distribution is readily written as

$$\begin{aligned} f_{r,s:n,m,k}(x_1, x_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2) \\ &\times \{1 - F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ &\times \{1 - F(x_2)\}^{\gamma_s-1}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}. \end{aligned} \tag{4.10}$$

The joint density of two contiguous GOS is immediately written as

$$\begin{aligned} f_{r,r+1:n,m,k}(x_1, x_2) &= \frac{C_r}{(r-1)!} f(x_1) f(x_2)\{1 - F(x_1)\}^m \\ &\times g_m^{r-1}\{F(x_1)\}\{1 - F(x_2)\}^{\gamma_s-1}. \end{aligned} \tag{4.11}$$

Further, the joint density of smallest and largest GOS is

$$f_{1,n:n,m,k}(x_1, x_2) = \frac{C_{n-1}}{(n-2)!} f(x_1) f(x_2) \{1 - F(x_1)\}^m \times \{1 - F(x_2)\}^{k-1} [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{n-2}. \tag{4.12}$$

The expression for special cases can be immediately written from (4.10), (4.11) and (4.12). Specifically the joint density of rth and sth k record values is

$$f_{r,s:n,-1,k}(x_1, x_2) = \frac{k^s}{(r-1)!(s-r-1)!} \left\{ \frac{f(x_1)}{1 - F(x_1)} \right\} f(x_2) \times [-\ln\{1 - F(x_1)\}]^{r-1} \{1 - F(x_2)\}^{k-1} \times [\ln\{1 - F(x_1)\} - \ln\{1 - F(x_2)\}]^{s-r-1}. \tag{4.13}$$

Other special cases can also be obtained in similar way.

Example 4.1 A random sample of size n is drawn from standard exponential distribution with density function

$$f(x) = e^{-x}; x > 0.$$

Obtain the distribution of rth Generalized Order Statistics and joint distribution of rth and sth Generalized Order Statistics.

Solution: We have $f(x) = e^{-x}; x > 0$ and hence

$$F(x) = \int_0^x e^{-t} dt = 1 - e^{-x}; x > 0.$$

The density function of rth GOS is given in (4.6) as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x) \{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)],$$

where

$$g_m(u) = \frac{1}{m+1}\left[1 - (1-u)^{m+1}\right]$$

so

$$g_m[F(x)] = \frac{1}{m+1}\left[1 - \{1 - F(x)\}^{m+1}\right].$$

Now for given distribution we have

$$\begin{aligned} g_m[F(x)] &= \frac{1}{m+1}\left[1 - e^{-(m+1)x}\right] \\ \text{or } g_m^{r-1}[F(x)] &= \left[\frac{1}{m+1}\left\{1 - e^{-(m+1)x}\right\}\right]^{r-1} \\ &= \frac{1}{(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} e^{-(m+1)ix} \end{aligned}$$

Hence the density function of rth GOS for standard exponential distribution is

$$\begin{aligned} f_{r:n,m,k}(x) &= \frac{C_{r-1}}{(r-1)!} f(x)\{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)] \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} e^{-x}\left(e^{-x}\right)^{\gamma_r - 1} \\ &\quad \times \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} e^{-(m+1)ix} \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} e^{-\gamma_r x} \sum_{i=0}^{r-1} (-1)^i \\ &\quad \times \binom{r-1}{i} e^{-(m+1)ix} \end{aligned}$$

or

$$\begin{aligned} f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \\ \times \exp[-\{(m+1)i + \gamma_r\}x], \end{aligned}$$

for $x > 0$. Again the joint density of rth and sth GOS is

$$\begin{aligned} f_{r,s:n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2)[1 - F(x_1)]^m \\ \times g_m^{r-1}[F(x_1)][1 - F(x_2)]^{\gamma_s - 1} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}; \end{aligned}$$

where

$$h_m\{F(x)\} = -\frac{1}{m+1}\{1 - F(x)\}^{m+1}.$$

For standard exponential distribution we have

$$h_m[F(x_2)] = -\frac{1}{m+1}e^{-(m+1)x_2}$$
$$h_m[F(x_1)] = -\frac{1}{m+1}e^{-(m+1)x_1}$$
$$g_m^{r-1}[F(x_1)] = \frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}e^{-(m+1)ix_1}$$

Now using these values; the joint density of rth and sth GOS for standard exponential distribution is

$$\begin{aligned}
f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}e^{-x_1}e^{-x_2}e^{-mx_1}\frac{1}{(m+1)^{r-1}}\\
&\quad\times\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}e^{-(m+1)ix_1}\left(e^{-x_2}\right)^{\gamma_s-1}\\
&\quad\times\left[-\frac{1}{m+1}e^{-(m+1)x_2}+\frac{1}{m+1}e^{-(m+1)x_1}\right]^{s-r-1}\\
&= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\cdot e^{-\gamma_s x_2}\frac{1}{(m+1)^{r-1}}\\
&\quad\times\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}e^{-(m+1)(i+1)x_1}\frac{1}{(m+1)^{s-r-1}}\\
&\quad\times\left\{e^{-(m+1)x_1}+e^{-(m+1)x_2}\right\}^{s-r-1}
\end{aligned}$$

or

$$\begin{aligned}
f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}e^{-\gamma_s x_2}\\
&\quad\times\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}e^{-(m+1)(i+1)x_1}\sum_{j=0}^{s-r-1}(-1)^j\\
&\quad\times\binom{s-r-1}{j}e^{-(m+1)(s-r-j-1)x_1}e^{-(m+1)jx_2}
\end{aligned}$$

or

$$\begin{aligned}
f_{r,s:n,m,k}(x_1,x_2) = &\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\\
&\times\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\\
&\times e^{-(m+1)(s-r-j+i)x_1}e^{-[(m+1)j+\gamma_s]x_2},
\end{aligned}$$

$0 < x_1 < x_2 < \infty$; as required.

Example 4.2 A random sample of size n is drawn from standard Weibull distribution with density function

$$f(x) = \beta x^{\beta-1} \exp\left(-x^{\beta}\right);\ x, \beta > 0.$$

Obtain the distribution of rth $k-$Record value and joint distribution of rth and sth k record values.

Solution: The distribution of rth k–Record value is given as

$$f_{U_K(r)}(x) = \frac{k^r}{(r-1)!} f(x)\{1 - F(x)\}^{k-1}[-\ln\{1 - F(x)\}]^{r-1}.$$

For given distribution we have

$$\begin{aligned} f(x) &= \beta x^{\beta-1} \exp\left(-x^{\beta}\right);\ x, \beta > 0; \\ F(x) &= \int_0^x f(t)dt = \int_0^x \beta t^{\beta-1} \exp\left(-t^{\beta}\right)dt \\ &= 1 - \exp\left(-x^{\beta}\right);\ x, \beta > 0. \end{aligned}$$

So $-\ln[1 - F(x)] = x^{\beta}$.
Using above results the density of rth k–Record value is

$$\begin{aligned} f_{U_K(r)}(x) &= \frac{k^r}{(r-1)!}\beta x^{\beta-1} \exp\left(-x^{\beta}\right)\left\{\exp\left(-x^{\beta}\right)\right\}^{k-1}\left(x^{\beta}\right)^{r-1} \\ &= \frac{k^r}{(r-1)!}\beta x^{r\beta-1} \exp\left(-kx^{\beta}\right). \end{aligned}$$

For $k = 1$ we have density of rth Record value as

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!}\beta x^{r\beta-1} \exp\left(-x^{\beta}\right).$$

Again, the joint density of rth and sth k–Record value is

$$\begin{aligned} f_{U_K(r),U_K(s)}(x_1, x_2) = {} & \frac{k^s}{(r-1)!(s-r-1)!} r(x_1) \\ & \times f(x_2)[R(x_1)]^{r-1}\{1 - F(x_2)\}^{k-1} \\ & \times [R(x_2) - R(x_1)]^{s-r-1}, \end{aligned}$$

where $R(x) = -\ln[1 - F(x)]$; $r(x) = R^{/}(x) = \frac{f(x)}{1-F(x)}$.

Now for given distribution we have

$$r(x_1) = \frac{f(x_1)}{1 - F(x_1)} = \frac{\beta x_1^{\beta-1} \exp\left(-x_1^{\beta}\right)}{\exp\left(-x_1^{\beta}\right)} = \beta x_1^{\beta-1},$$
$$R(x_1) = -\ln[1 - F(x_1)] = x_1^{\beta}.$$

So

$$R(x_2) - R(x_1) = x_2^{\beta} - x_1^{\beta}.$$

Using these values; we have the joint density of rth and s th k–Record values as

$$\begin{aligned} f_{U_K(r),U_K(s)}(x_1, x_2) &= \frac{k^s}{(r-1)!(s-r-1)!} \beta x_1^{\beta-1} \beta x_2^{\beta-1} \exp\left(-x_2^{\beta}\right) \\ &\quad \times \left(x_1^{\beta}\right)^{r-1} \left\{\exp\left(-x_2^{\beta}\right)\right\}^{k-1} \left(x_2^{\beta} - x_1^{\beta}\right)^{s-r-1} \\ &= \frac{\beta^2 k^s}{(r-1)!(s-r-1)!} x_1^{r\beta-1} x_2^{\beta-1} \exp\left(-k x_2^{\beta}\right) \\ &\quad \times \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} \left(x_2^{\beta}\right)^{s-r-1-i} \left(x_1^{\beta}\right)^i \end{aligned}$$

or

$$\begin{aligned} f_{U_K(r),U_K(r)}(x_1, x_2) = \frac{\beta^2 k^s}{(r-1)!(s-r-1)!} \exp\left(-k x_2^{\beta}\right) \\ \times \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} x_2^{\beta(s-r-i)-1} x_1^{r\beta+i\beta-1}, \end{aligned}$$

for $x_1 < x_2$. Using $k = 1$ the joint density of rth and sth Record value is easily obtained as

$$\begin{aligned} f_{X_{U(r)},X_{U(s)}}(x_1, x_2) = \frac{\beta^2}{(r-1)!(s-r-1)!} \exp\left(-x_2^{\beta}\right) \\ \times \sum_{i=0}^{s-r-1} (-1)^i \binom{s-r-1}{i} x_2^{\beta(s-r-i)-1} x_1^{r\beta+i\beta-1}, \end{aligned}$$

for $x_1 < x_2$.

4.8 Distribution Function of GOS and Its Properties

The distribution function of GOS is very useful in exploring certain properties of GOS. The distribution function of GOS can be written in different forms and each of the form can be further used to study the properties of GOS as given by Kamps (1995b). We will discuss the distribution function of GOS in the following.

Consider the density function of uniform GOS and GOS from any parent distribution as

$$\varphi_{r,n}(u) = \frac{C_{r-1}}{(r-1)!}(1-u)^{\gamma_r - 1} g_m^{r-1}(u) \tag{4.14}$$

and

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)]. \tag{4.15}$$

Kamps (1995b) have shown that the distribution function of uniform GOS can be written as

$$\Phi_{r,n}(u) = 1 - C_{r-1}(1-u)^{\gamma_r} \sum_{j=0}^{r-1} \frac{1}{j! C_{r-j-1}} g_m^j(u).$$

Using the probability integral transform, the distribution function of GOS from any parent distribution $F(x)$ is readily written as

$$F_{X(r:n,m,k)}(x) = 1 - C_{r-1}[1-F(x)]^{\gamma_r} \sum_{j=0}^{r-1} \frac{1}{j! C_{r-j-1}} g_m^j[F(x)]. \tag{4.16}$$

Burkschat et al. (2003) have shown that the distribution function of GOS can be written in the following form

$$F_{X(r:n,m,k)}(x) = 1 - C_{r-1} \int_0^{1-F(x)} G_{r,r}^{r,0}\left[y \middle| \begin{matrix} \gamma_1, \dots \gamma_r \\ \gamma_1 - 1, \dots \gamma_r - 1 \end{matrix} \right] dy \tag{4.17}$$

where $G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_p \end{matrix} \right)$ is Meijer's G–function.

The distribution function of rth GOS can also be presented in the form of Incomplete Beta function ratio as under

$$\begin{aligned} F_{X(r:n,m,k)}(x) &= \int_{-\infty}^{x} f_{r:n,m,k}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{x} f(t)\{1-F(t)\}^{\gamma_r - 1} g_m^{r-1}[F(t)]dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{x} f(t)\{1-F(t)\}^{\gamma_r - 1} \\ &\quad \times \left[\frac{1}{m+1}\left(1 - \{1-F(t)\}^{m+1}\right) \right]^{r-1} dt \end{aligned}$$

Now making the transformation $w = \left(1 - \{1 - F(t)\}^{m+1}\right)$ we have

$$F_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)(m+1)^r} \int_{-\infty}^{\alpha[F(x)]} w^{r-1}(1-w)^{[\gamma_r/(m+1)]-1} dw,$$

where $\alpha[F(x)] = 1 - \{1 - F(x)\}^{m+1}$. So the distribution function of GOS is

$$\begin{aligned} F_{X(r:n,m,k)}(x) &= \frac{C_{r-1}}{\Gamma(r)(m+1)^r} B_{\alpha[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right) \\ &= \frac{C_{r-1}}{\Gamma(r)(m+1)^r} B\left(r, \frac{\gamma_r}{m+1}\right) I_{\alpha[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right), \end{aligned}$$

where $I_x(a,b)$ is incomplete Beta function ratio. The distribution function of GOS may further be simplified as

$$F_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)(m+1)^r} \frac{\Gamma(r)\Gamma\left(\frac{\gamma_r}{m+1}\right)}{\Gamma\left(r + \frac{\gamma_r}{m+1}\right)} I_{\alpha[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right).$$

Now using the relation

$$\begin{aligned} \Gamma\left(r + \frac{\gamma_r}{m+1}\right) &= \left(\frac{k}{m+1} + n - 1\right) \cdots \left(\frac{k}{m+1} + n - r\right) \Gamma\left(\frac{\gamma_r}{m+1}\right) \\ &= \frac{C_{r-1}}{(m+1)^r} \Gamma\left(\frac{\gamma_r}{m+1}\right), \end{aligned}$$

we have

$$F_{X(r:n,m,k)}(x) = I_{\alpha[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right), \tag{4.18}$$

The above relation can be used to obtain distribution function of special cases for example using $m = 0$ and $k = 1$ in (4.18) we have

$$F_{X(r:n,0,1)}(x) = I_{F(x)}(r, n - r + 1);$$

which is (2.6), the distribution function of ordinary order statistics.

The distribution function of GOS has certain recurrence relations which are really useful in computing probabilities for GOS from any parent distribution. We give a useful recurrence relation between distribution functions of uniform GOS in the following theorem.

Theorem 4.1 *The distribution functions of uniform GOS are related as*

$$\Phi_{r:n}(x) - \Phi_{r-1:n}(x) = -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x) \tag{4.19}$$

Proof We have

$$\Phi_{r:n}(x) = \int_0^x \varphi_{r:n}(t)dt = \int_0^x \frac{C_{r-1}}{(r-1)!}(1-t)^{\gamma_r-1} g_m^{r-1}(t)dt$$

Now integrating by parts treating $(1-t)^{\gamma_r-1}$ for integration we have

$$\begin{aligned}\Phi_{r:n}(x) &= -\frac{C_{r-1}}{(r-1)!}\cdot\frac{(1-t)^{\gamma_r}}{\gamma_r}\cdot g_m^{r-1}(t)\Big|_0^x \\ &\quad + \int_0^x \left\{\frac{C_{r-1}}{(r-2)!}\frac{(1-t)^{\gamma_r}}{\gamma_r} g_m^{r-2}(t) g_m^{/}(t)\right\}dt \\ &= -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x) \\ &\quad + \int_0^x \frac{C_{r-2}}{(r-2)!}(1-t)^{\gamma_r} g_m^{r-2}(t)(1-t)^m dt\end{aligned}$$

or

$$\begin{aligned}\Phi_{r,n}(x) &= -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x) \\ &\quad + \int_0^x \frac{C_{r-2}}{(r-2)!}(1-t)^{\gamma_r+m} g_m^{r-2}(t)dt \\ &= -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x) \\ &\quad + \int_0^x \frac{C_{r-2}}{(r-2)!}(1-t)^{\gamma_{r-1}-1} g_m^{r-2}(t)dt\end{aligned}$$

or

$$\Phi_{r,n}(x) = -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x) + \Phi_{r-1:n}(x),$$

or

$$\Phi_{r:n}(x) - \Phi_{r-1:n}(x) = -\frac{C_{r-2}}{(r-1)!}(1-x)^{\gamma_r} g_m^{r-1}(x),$$

as required.

The relationship between distribution function of GOS from any parent distribution is readily written as

$$F_{X(r:n,m,k)}(x) - F_{X(r-1:n,m,k)}(x) = -\frac{C_{r-2}}{(r-1)!}[1-F(x)]^{\gamma_r} g_m^{r-1}[F(x)]. \quad (4.20)$$

The corresponding relationships for special cases can be readily obtained, for example the recurrence relation for distribution functions of ordinary order statistics is obtained from (4.20) by using $m = 0$ and $k = 1$ as

$$F_{X(r:n,0,1)}(x) - F_{X(r-1:n,0,1)}(x) = -\frac{C_{r-2}}{(r-1)!}[1 - F(x)]^{n-r+1}[F(x)]^{r-1}.$$

Recurrence relations for other special cases can be obtained in similar way.

4.9 GOS as Markov Chain

We have seen that the ordinary order statistics and record values form the Markov chain with certain transition probabilities. In the following we will see that the GOS also form the Markov chain with certain transition probability. The Markovian property of GOS can be easily proved by looking at the conditional distributions derived below.

The marginal distribution of rth GOS and joint marginal distribution of rth and sth GOS are given in (4.6) and (4.10) as

$$f_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)] \tag{4.21}$$

and

$$\begin{aligned} f_{r,s:n,m,k}(x_1, x_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2)\{1 - F(x_1)\}^m \\ &\quad \times g_m^{r-1}\{F(x_1)\}\{1 - F(x_2)\}^{\gamma_s - 1} \\ &\quad \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}. \end{aligned}$$

Using above two equations, the conditional distribution of sth GOS given rth GOS is readily written as

$$f_{s|r:n,m,k}(x_2|x_1) = \frac{f_{r,s:n,m,k}(x_1, x_2)}{f_{r:n,m,k}(x_1)}$$

$$\begin{aligned} &= \frac{C_{s-1} f(x_2)\{1 - F(x_2)\}^{\gamma_s - 1}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}}{C_{r-1}(s-r-1)!\{1 - F(x)\}^{\gamma_r - 1 - m}} \\ &= \frac{C_{s-1} f(x_2)\{1 - F(x_2)\}^{\gamma_s - 1}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}}{C_{r-1}(s-r-1)!\{1 - F(x)\}^{\gamma_{r+1}}}. \end{aligned} \tag{4.22}$$

Again the joint density of first r GOS is given in (4.4) as

$$\begin{aligned} f_{1,2,\ldots,r:n,m,k}(x_1, x_2, \ldots, x_r) = C_{r-1}&\left[\prod_{i=1}^{r-1}\{1 - F(x_i)\}^m f(x_i)\right] \\ &\times \{1 - F(x_r)\}^{\gamma_r - 1} f(x_r). \end{aligned} \tag{4.23}$$

Also the joint density of first r and sth GOS is

$$f_{1,2,\ldots,r,s:n,m,k}(x_1, x_2, \ldots, x_r) = \frac{C_{s-1}}{(s-r-1)!}\left[\prod_{i=1}^{r}\{1-F(x_i)\}^m f(x_i)\right] \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \times \{1-F(x_s)\}^{\gamma_s - 1} f(x_s). \tag{4.24}$$

Using (4.4) and (4.24), the conditional distribution of sth GOS given information of first r GOS is

$$f_{s|1,2,\ldots,r:n,m,k}(x_s|x_1, \ldots, x_r) = \frac{f_{1,\ldots,r,s:n,m,k}(x_1, \ldots, x_r, x_s)}{f_{1,\ldots,r:n,m,k}(x_1)}$$

$$= \frac{\frac{C_{s-1}}{(s-r-1)!}\left[\prod_{i=1}^{r}\{1-F(x_i)\}^m f(x_i)\right]\{1-F(x_s)\}^{\gamma_s-1} f(x_s)}{C_{r-1}\left[\prod_{i=1}^{r-1}\{1-F(x_i)\}^m f(x_i)\right]\{1-F(x_r)\}^{\gamma_r-1} f(x_r)}$$
$$= \frac{C_{s-1} f(x_2)\{1-F(x_2)\}^{\gamma_s-1}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}}{C_{r-1}(s-r-1)!\,\{1-F(x)\}^{\gamma_r-1-m}}$$
$$= \frac{C_{s-1} f(x_2)\{1-F(x_2)\}^{\gamma_s-1}\,[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}}{C_{r-1}(s-r-1)!\,\{1-F(x)\}^{\gamma_{r+1}}}; \tag{4.25}$$

which is same as the conditional distribution of sth GOS given rth GOS and hence the GOS form the Markov Chains.

The transition probability of GOS are obtained by using the conditional distribution of two contiguous GOS. The conditional distribution of sth GOS given rth GOS is given in (4.22) as

$$f_{s|r:n,m,k}(x_2|x_1) = \frac{C_{s-1} f(x_2)\{1-F(x_2)\}^{\gamma_s-1}}{C_{r-1}(s-r-1)!\,\{1-F(x)\}^{\gamma_{r+1}}} \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}.$$

Using $s = r + 1$ in above equation the conditional distribution of $(r + 1)$th GOS given rth GOS is

$$f_{r+1|r:n,m,k}(x_{r+1}|x_r) = \frac{C_r f(x_{r+1})\{1-F(x_{r+1})\}^{\gamma_{r+1}-1}}{C_{r-1}!\,\{1-F(x_r)\}^{\gamma_{r+1}}}$$
$$= \gamma_{r+1}\left[\frac{1-F(x_{r+1})}{1-F(x_r)}\right]^{\gamma_{r+1}-1}\frac{f(x_{r+1})}{1-F(x_r)}. \tag{4.26}$$

The transition probability of GOS is therefore

$$\begin{aligned} P\left(X_{r+1:n,m,k} \geq y | X_{r:n,m,k} = x\right) &= \int_y^\infty f_{r+1|r:n,m,k}(x_{r+1}|x_r = x)dx_{r+1} \\ &= \int_y^\infty \gamma_{r+1}\left[\frac{1-F(x_{r+1})}{1-F(x)}\right]^{\gamma_{r+1}-1} \\ &\quad \times \frac{f(x_{r+1})}{1-F(x)}dx_{r+1} \end{aligned}$$

or

$$\begin{aligned} P\left(X_{r+1:n,m,k} \geq y | X_{r:n,m,k} = x\right) = \frac{\gamma_{r+1}}{\{1-F(x)\}^{\gamma_{r+1}}}\int_y^\infty f(x_{r+1}) \\ \times \{1-F(x_{r+1})\}^{\gamma_{r+1}-1}dx_{r+1} \end{aligned}$$

or

$$P\left(X_{r+1:n,m,k} \geq y | X_{r:n,m,k} = x\right) = \left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_{r+1}}. \tag{4.27}$$

The transition probability for special cases can be readily obtained from (4.27). For example using $m = 0$ and $k = 1$, the transition probability for ordinary order statistics is obtained as

$$P\left(X_{r+1:n,0,1} \geq y | X_{r:n,0,1} = x\right) = \left[\frac{1-F(y)}{1-F(x)}\right]^{n-r}.$$

Transition probability for other special cases can be obtained in similar way.

4.10 Moments of GOS

The distribution of rth GOS is as like any conventional distribution and hence properties of the rth GOS can be explored by computing its moments. The moments computed for rth GOS can be used to obtain the moments of special cases by using specific values of the parameters. In the following we will discuss the moments of GOS for any parent probability distribution. Recall that the density function of rth GOS for the distribution $F(x)$ is given as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)].$$

The expected value of rth GOS; $X_{r:n,m,k}$; is defined as

$$\mu_{r:n,m,k} = E\left(X_{r:n,m,k}\right) = \int_{-\infty}^{\infty} x f_{r:n,m,k}(x)dx$$
$$= \int_{-\infty}^{\infty} x \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx. \qquad (4.28)$$

The expected value of some function of rth GOS is given as

$$E\left[t\left(X_{r:n,m,k}\right)\right] = \int_{-\infty}^{\infty} t(x) f_{r:n,m,k}(x)dx$$
$$= \int_{-\infty}^{\infty} t(x) \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx.$$

Again the pth raw moment of rth GOS; $\mu^p_{r:n,m,k}$; is computed as

$$\mu^p_{r:n,m,k} = E\left(X^p_{r:n,m,k}\right) = \int_{-\infty}^{\infty} x^p f_{r:n,m,k}(x)dx$$
$$= \int_{-\infty}^{\infty} x^p \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx; \qquad (4.29)$$

and the pth raw moment of any function of rth GOS is given as

$$E\left[\left\{t\left(X_{r:n,m,k}\right)\right\}^p\right] = \int_{-\infty}^{\infty} \{t(x)\}^p f_{r:n,m,k}(x)dx$$
$$= \int_{-\infty}^{\infty} \{t(x)\}^p \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx.$$

Using the probability integral transformation, the pth moment of rth GOS can also be written as

$$\mu^p_{r:n,m,k} = E\left(X^p_{r:n,m,k}\right) = \frac{C_{r-1}}{(r-1)!} \int_0^1 \left\{F^{-1}(t)\right\}^p \varphi_{r:n}(t)dt$$
$$= \frac{C_{r-1}}{(r-1)!} \int_0^1 \left\{F^{-1}(t)\right\}^p (1-t)^{\gamma_r-1} g_m^{r-1}(t)dt, \qquad (4.30)$$

where $t = F(x)$ and $x = F^{-1}(t)$ is the inverse function.

The joint density of rth and sth GOS provide basis for computation of product moments of two GOS. The joint density of rth and sth GOS is given in (4.10) as

$$f_{r,s:n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2)\{1 - F(x_1)\}^m \\ \times g_m^{r-1}\{F(x_1)\}\{1 - F(x_2)\}^{\gamma_s - 1} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}.$$

Using the joint density, the product moment of rth and sth GOS is computed as

$$\begin{aligned} \mu_{r,s:n,m,k} &= E\left(X_{r:n,m,k} X_{s:n,m,k}\right) = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1 x_2 f_{r,s:n,m,k}(x_1, x_2) dx_2 dx_1 \\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1 x_2 f(x_1) f(x_2)\{1 - F(x_1)\}^m \\ &\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ &\times \{1 - F(x_2)\}^{\gamma_s - 1} dx_2 dx_1. \end{aligned} \tag{4.31}$$

The (p, q)th raw moment of rth and sth GOS is readily written as

$$\begin{aligned} \mu_{r,s:n,m,k}^{p,q} &= E\left(X_{r:n,m,k}^p X_{s:n,m,k}^q\right) = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^q f_{r,s:n,m,k}(x_1, x_2) dx_2 dx_1 \\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^q f(x_1) f(x_2)\{1 - F(x_1)\}^m \\ &\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ &\times \{1 - F(x_2)\}^{\gamma_s - 1} dx_2 dx_1. \end{aligned} \tag{4.32}$$

The (p, q)th central moment of rth and sth GOS is given as

$$\sigma_{r,s:n,m,k}^{p,q} = E\left[\left\{X_{r:n,m,k} - \mu_{r:n,m,k}\right\}^p \left\{X_{s:n,m,k} - \mu_{s:n,m,k}\right\}^q\right].$$

The covariance between rth and sth GOS is readily computed from above as

$$\sigma_{r,s:n,m,k} = E\left[\left\{X_{r:n,m,k} - \mu_{r:n,m,k}\right\}\left\{X_{s:n,m,k} - \mu_{s:n,m,k}\right\}\right].$$

The correlation coefficient can also be computed easily.

Example 4.3 A random sample is drawn from standard exponential distribution with density function

$$f(x) = e^{-x}; x > 0.$$

Obtain expression for single and product moments of GOS for this distribution.

Solution: The distribution of rth GOS is

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)].$$

For given distribution we have $f(x) = e^{-x}$ and $F(x) = 1 - e^{-x}$. Now we have

$$g_m^{r-1}[F(x)] = \left(\frac{1}{m+1}\left[1 - \{1 - F(x)\}^{m+1}\right]\right)^{r-1}$$
$$= \frac{1}{(m+1)^{r-1}}\left[1 - e^{-(m+1)x}\right]^{r-1}.$$

The distribution of $X_{r:n,m,k}$ is therefore

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} e^{-\gamma_r x}\left\{1 - e^{-(m+1)x}\right\}^{r-1}.$$

Now the pth moment of $X_{r:n,m,k}$ is

$$\mu_{r:n,m,k}^{p} = E\left(X_{r:n,m,k}^{p}\right) = \int_{-\infty}^{\infty} x^p f_{r:n,m,k}(x)dx$$
$$= \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^p e^{-\gamma_r x}\left\{1 - e^{-(m+1)x}\right\}^{r-1}dx$$
$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}$$
$$\times \int_0^{\infty} x^p e^{-[(m+1)i+\gamma_r]x}dx$$
$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}$$
$$\times \frac{1}{\{(m+1)i+\gamma_r\}^{p+1}}\Gamma(p+1).$$

The Mean of $X_{r:n,m,k}$ is

$$\mu_{r:n,m,k} = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{\{(m+1)i+\gamma_r\}^2}.$$

Again the joint density of $X_{r:n,m,k}$ and $X_{s:n,m,k}$ is

$$f_{r,s:n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1)f(x_2)\{1 - F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}\{1 - F(x_2)\}^{\gamma_s - 1}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}.$$

Now for given distribution we have

$$
\begin{aligned}
f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} e^{-x_1} e^{-x_2} e^{-mx_1} \\
&\quad \times \left[\frac{1}{m+1}\left\{1-e^{-(m+1)x}\right\}\right]^{r-1} e^{-(\gamma_s-1)x_2} \\
&\quad \times \left[\frac{1}{m+1}e^{-(m+1)x_1} - \frac{1}{m+1}e^{-(m+1)x_2}\right]^{s-r-1} \\
&= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!} \\
&\quad \times \sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\
&\quad \times e^{-(m+1)(s-r-j+i)x_1} e^{-[(m+1)j+\gamma_s]x_2}.
\end{aligned}
$$

The product moments of order (p,q) are

$$
\begin{aligned}
\mu^{p,q}_{r,s:n,m,k} &= E\left(X^p_{r:n,m,k}X^q_{s:n,m,k}\right) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{r,s:n,m,k}(x_1,x_2)dx_2dx_1 \\
&= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!} \\
&\quad \times \sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\
&\quad \times \int_0^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q e^{-(m+1)(s-r-j+i)x_1} e^{-[(m+1)j+\gamma_s]x_2} dx_2dx_1
\end{aligned}
$$

or

$$
\begin{aligned}
\mu^{p,q}_{r,s:n,m,k} &= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!} \\
&\quad \times \sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\
&\quad \times \frac{1}{(p+1)\{(m+1)j+\gamma_s\}^{p+q+2}}\Gamma(p+q+2) \\
&\quad \times {}_2F_1\left[p+1,\, p+q+2;\, p+2;\, \frac{(s-r-j+i)(m+1)}{(m+1)j+\gamma_s}\right].
\end{aligned}
$$

The Covariance can be obtained by using above results.

Example 4.4 A random sample is drawn from the Pareto distribution with density function

$$f(x) = \frac{ac}{x^{a+1}}; x > c^{1/a}, a, c > 0.$$

Show that the pth moment of rth GOS for this distribution is given as

$$\mu_{r:n,m,k}^{p} = E\left(X_{r:n,m,k}^{p}\right) = c^{p/a}\frac{C_{r-1}(k)}{C_{r-1}(k-p/a)},$$

where notations have their usual meanings.

Solution: The pth moment of rth GOS is given as

$$\begin{aligned}\mu_{r:n,m,k}^{p} = E\left(X_{r:n,m,k}^{p}\right) &= \int_{-\infty}^{\infty} x^p f_{r:n,m,k}(x)dx \\ &= \int_{-\infty}^{\infty} x^p \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx.\end{aligned}$$

Using the probability integral transform, the pth moment is given as

$$\begin{aligned}\mu_{r:n,m,k}^{p} = E\left(X_{r:n,m,k}^{p}\right) &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p \varphi_{r:n}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p (1-t)^{\gamma_r-1} g_m^{r-1}(t)dt.\end{aligned}$$

Now for given distribution we have

$$f(x) = \frac{ac}{x^{a+1}}; x > c^{1/a}$$

So

$$\begin{aligned}F(x) &= \int_{c^{1/a}}^{x} f(t)dt = \int_{c^{1/a}}^{x} \frac{ac}{t^{a+1}}dt \\ &= 1 - \frac{c}{x^a}; x > c^{1/a}.\end{aligned}$$

Also by using $t = F(x)$ we have

$$t = 1 - \frac{c}{x^a} \Longrightarrow x = F^{-1}(t) = \left\{\frac{1}{c}(1-t)\right\}^{-1/a}.$$

Hence the pth moment of rth GOS for Pareto distribution is:

$$\begin{aligned}\mu^p_{r:n,m,k} &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p (1-t)^{\gamma_r-1} g_m^{r-1}(t)dt\\ &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left[\left\{\frac{1}{c}(1-t)\right\}^{-1/a}\right]^p (1-t)^{\gamma_r-1} g_m^{r-1}(t)dt\\ &= \frac{C_{r-1}}{(r-1)!}c^{p/a}\int_0^1 (1-t)^{(k-p/a)+(n-r)(m+1)} g_m^{r-1}(t)dt\end{aligned}$$

or

$$\begin{aligned}\mu^p_{r:n,m,k} &= c^{p/a}\frac{C_{r-1}(k)}{C_{r-1}(k-p/a)}\frac{C_{r-1}(k-p/a)}{(r-1)!}\\ &\quad\times\int_0^1 (1-t)^{(k-p/a)+(n-r)(m+1)} g_m^{r-1}(t)dt\\ &= c^{p/a}\frac{C_{r-1}(k)}{C_{r-1}(k-p/a)};\end{aligned}$$

as

$$\frac{C_{r-1}(k-p/a)}{(r-1)!}\int_0^1 (1-t)^{(k-p/a)+(n-r)(m+1)} g_m^{r-1}(t)dt = 1,$$

as required.

4.11 Recurrence Relations for Moments of GOS

The moments for GOS from any parent distribution $F(x)$ can be used to obtain the expression for special cases by using certain values of the parameters involved. As we have seen in previous two chapters that the moments of ordinary order statistics and record values are connected via certain relations. We have also seen that those relations can be used to compute higher order single and product moments by using information of lower order moments. In this section we will discuss the recurrence relations between single and product moments of GOS and we will see that the recurrence relations for single and product moments discussed in previous two chapters turn out to be special cases for recurrence relations discussed below.

Various properties of GOS are based upon the probability integral transform of any probability distribution. Some of the properties are given in the following by first noting that the density of Uniform GOS is given as

$$\varphi_{r:n}(x) = f_{U(r:n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}(1-x)^{\gamma_r-1} g_m^{r-1}(x),$$

and the density function of rth GOS from any distribution is given as

$$f_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)].$$

The density function of rth GOS from any parent distribution; with distribution function $F(x)$; is readily derived from the density of Uniform GOS by using the relation

$$f_{X(r:n,m,k)}(x) = f_{U(r:n,m,k)}\{F(x)\} f(x) = \varphi\{F(x)\} f(x), \tag{4.33}$$

where $f(x)$ is the density function. The density function of uniform GOS are connected through various relationships. Some of these relationships are given below:

$$\gamma_{r+1}\varphi_{r:n}(x) + r(m+1)\varphi_{r+1:n}(x) = \gamma_1 \varphi_{r:n-1}(x) \tag{i}$$

$$\gamma_{r+1}\{\varphi_{r+1:n}(x) - \varphi_{r:n}(x)\} = \gamma_1\{\varphi_{r+1:n}(x) - \varphi_{r:n-1}(x)\} \tag{ii}$$

$$\gamma_1\{\varphi_{r:n}(x) - \varphi_{r:n-1}(x)\} = r(m+1)\{\varphi_{r:n}(x) - \varphi_{r+1:n}(x)\} \tag{iii}$$

The first relation is readily proved by noting that

$$\begin{aligned} C_{r(n)} &= \{k + (n-r-1)(m+1)\} C_{r-1(n)} \\ C_{r(n)} &= \{k + (n-1)(m+1)\} C_{r-1(n-1)}; \end{aligned}$$

$C_{r(n)}$ is C_r based upon the sample of size n etc. Now consider left hand side of (i) as

$$\begin{aligned} &\gamma_{r+1}\varphi_{r:n}(x) + r(m+1)\varphi_{r+1:n}(x) \\ &= \{k + (n-r-1)(m+1)\} \\ &\quad \times \left[\frac{C_{r-1(n)}}{(r-1)!}(1-x)^{k+(n-r)(m+1)-1} g_m^{r-1}(x)\right] \\ &\quad + r(m+1) \\ &\quad \times \left[\frac{C_{r(n)}}{r!}(1-x)^{k+(n-r-1)(m+1)-1} g_m^{r-1}(x)\right] \\ &= \frac{C_{r(n)}}{(r-1)!}(1-x)^{k+(n-r)(m+1)-1} g_m^{r-1}(x) + (m+1) \\ &\quad \times \left[\frac{C_{r(n)}}{(r-1)!}(1-x)^{k+(n-r-1)(m+1)-1} g_m^{r-1}(x)\right] \end{aligned}$$

or

$$\begin{aligned}
&\gamma_{r+1}\varphi_{r:n}(x) + r(m+1)\varphi_{r+1:n}(x) \\
&= \frac{C_{r(n)}}{(r-1)!}(1-x)^{k+(n-r-1)(m+1)-1} g_m^{r-1}(x) \\
&\times \left[(1-x)^{m+1} + (m+1)g_m(x)\right] \\
&= \frac{C_{r(n)}}{(r-1)!}(1-x)^{k+(n-r-1)(m+1)-1} g_m^{r-1}(x)
\end{aligned}$$

or

$$\begin{aligned}
&\gamma_{r+1}\varphi_{r:n}(x) + r(m+1)\varphi_{r+1:n}(x) \\
&= \frac{\{k+(n-1)(m+1)\}C_{r-1(n-1)}}{(r-1)!} \\
&\times (1-x)^{k+(n-r-1)(m+1)-1} g_m^{r-1}(x) \\
&= \gamma_1\varphi_{r:n-1}(x),
\end{aligned}$$

as required. Other two relationships can be proved in same way. The relationships between density function of uniform GOS provide relationship between density functions of GOS for any parent distribution. These relationships are given below:

$$\gamma_{r+1} f_{r:n,m,k}(x) + r(m+1) f_{r+1:n,m,k}(x) = \gamma_1 f_{r:n-1,m,k}(x) \tag{i}$$

$$\gamma_{r+1}\left\{f_{r+1:n,m,k}(x) - f_{r:n,m,k}(x)\right\} = \gamma_1\left\{f_{r+1:n,m,k}(x) - f_{r:n-1,m,k}(x)\right\} \tag{ii}$$

$$\gamma_1\left\{f_{r:n,m,k}(x) - f_{r:n-1,m,k}(x)\right\} = r(m+1)\left\{f_{r:n,m,k}(x) - f_{r+1:n,m,k}(x)\right\} \tag{iii}$$

Using above relations we can immediately write following relations between moments of GOS

$$\gamma_{r+1}\mu^p_{r:n,m,k} + r(m+1)\mu^p_{r+1:n,m,k} = \gamma_1\mu^p_{r:n-1,m,k} \tag{i}$$

$$\gamma_{r+1}\left\{\mu^p_{r+1:n,m,k} - \mu^p_{r:n,m,k}\right\} = \gamma_1\left\{\mu^p_{r+1:n,m,k} - \mu^p_{r:n-1,m,k}\right\} \tag{ii}$$

$$\gamma_1\left\{\mu^p_{r:n,m,k} - \mu^p_{r:n-1,m,k}\right\} = r(m+1)\left\{\mu^p_{r:n,m,k} - \mu^p_{r+1:n,m,k}\right\}. \tag{iii}$$

The second relation can be alternatively written as

$$\gamma_{r+1}\left\{\mu^p_{r+1:n,m,k} - \mu^p_{r:n,m,k}\right\} = \gamma_1\left\{\mu^p_{r+1:n,m,k} - \mu^p_{r:n-1,m,k}\right\}$$

or

$$\gamma_{r+1}\mu^p_{r+1:n,m,k} - \gamma_1\mu^p_{r+1:n,m,k} = \gamma_{r+1}\mu^p_{r:n,m,k} - \gamma_1\mu^p_{r:n-1,m,k}$$

or

$$(\gamma_{r+1} - \gamma_1)\mu^p_{r+1:n,m,k} = \gamma_{r+1}\mu^p_{r:n,m,k} - \gamma_1\mu^p_{r:n-1,m,k}$$

or

$$-r(m+1)\mu^p_{r+1:n,m,k} = \gamma_{r+1}\mu^p_{r:n,m,k} - \gamma_1\mu^p_{r:n-1,m,k}$$

or

$$\gamma_{r+1}\mu^p_{r:n,m,k} = \gamma_1\mu^p_{r:n-1,m,k} - r(m+1)\mu^p_{r+1:n,m,k}.$$

This relation reduces to corresponding relation for simple order statistics for $m = 0$ and $k = 1$. The relationship between moments of GOS given above hold for any distribution. Distribution specific relationships between single and product moments of GOS have been studied by various authors. The distribution specific relationships between single and product moments are readily derived by using a general result given by Athar and Islam (2004). We have given the result in a theorem below.

Theorem 4.2 *Suppose a sequence of random variables* $\{X_n; n \geq 1\}$ *is available from an absolutely continuous distribution function* $F(x)$. *Suppose further that* $X_{r:n,m,k}$ *be* rth *GOS of the sequence then following recurrence relation hold between moments of the GOS*

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx; \quad (4.34)$$

and

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = {} & \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ & \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ & \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ & \times \{1-F(x_2)\}^{\gamma_s} dx_2 dx_1. \end{aligned} \quad (4.35)$$

Proof We have

$$\begin{aligned}\mu^p_{r:m,n,k} &= E\left(X^p_{r:n,m,k}\right) = \int_{-\infty}^{\infty} x^p f_{r:n,m,k}(x)dx \\ &= \int_{-\infty}^{\infty} x^p \frac{C_{r-1}}{(r-1)!} f(x)[1-F(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)]dx \\ &= \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} x^p f(x)[1-F(x)]^{\gamma_r-1} g_m^{r-1}[F(x)]dx.\end{aligned}$$

Integrating above equation by parts taking $f(x)[1-F(x)]^{\gamma_r-1}$ as function for integration we have

$$\mu^p_{r:n,m,k} = \frac{C_{r-1}}{(r-1)!}\Bigg[-x^p g_m^{r-1}\{F(x)\}\frac{\{1-F(x)\}^{\gamma_r}}{\gamma_r}\Bigg|_{-\infty}^{\infty}$$
$$-\int_{-\infty}^{\infty}\{px^{p-1}g_m^{r-1}[F(x)] + (r-1)x^p g_m^{r-2}[F(x)]$$
$$\times[1-F(x)]^m f(x)\}\frac{-\{1-F(x)\}^{\gamma_r}}{\gamma_r}dx\Bigg]$$
$$= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]dx$$
$$+\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^p f(x)[1-F(x)]^{\gamma_r+m}g_m^{r-2}[F(x)]dx$$
$$= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]dx$$
$$+\frac{C_{r-2}}{(r-2)!}\int_{-\infty}^{\infty}x^p f(x)[1-F(x)]^{\gamma_{r-1}-1}g_m^{r-2}[F(x)]dx$$

Since

$$\mu^p_{r-1:n,m,k} = \frac{C_{r-2}}{(r-2)!}\int_{-\infty}^{\infty}x^p f(x)[1-F(x)]^{\gamma_{r-1}-1}g_m^{r-2}[F(x)]dx,$$

hence above equation can be written as

$$\mu^p_{r:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]dx + \mu^{/p}_{r-1:n,m,k},$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}x^{p-1}\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]dx,$$

as required. We can readily see; from (4.34); that for $p = 1$ following recurrence relationship exist between expectations of GOS

$$\mu_{r:n,m,k} - \mu_{r-1:n,m,k} = \frac{C_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty}\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]dx. \tag{4.36}$$

We also have an alternative representation for recurrence relation between single moments of GOS based upon probability integral transform of (4.34) as under

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^1\{F^{-1}(t)\}^{p-1}\{F^{-1}(t)\}^{/}$$
$$\times(1-t)^{\gamma_r}g_m^{r-1}(t)dt. \tag{4.37}$$

The representation (4.37) is very useful in deriving relations for specific distributions.
Again consider

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} &= E\left(X^p_{r:n,m,k}X^q_{s:n,m,k}\right) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f_{r,s:n,m,k}(x_1,x_2)dx_2dx_1\\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)\{1-F(x_1)\}^m\\ &\quad\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\ &\quad\times\{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty} x_1^p f(x_1)\{1-F(x_1)\}^m\\ &\times g_m^{r-1}\{F(x_1)\}I(x_2)dx_1;\end{aligned}\tag{4.38}$$

where

$$I(x_2) = \int_{x_1}^{\infty} x_2^q f(x_2)\{1-F(x_2)\}^{\gamma_s-1}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}dx_2.$$

Integrating above integral by parts using $f(x_2)\{1-F(x_2)\}^{\gamma_s-1}$ for integration we have

$$\begin{aligned}I(x_2) = &-x_2^q[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\frac{\{1-F(x_2)\}^{\gamma_s}}{\gamma_s}\Bigg|_{x_1}^{\infty}\\ &+\frac{1}{\gamma_s}\int_{x_1}^{\infty}\Big[qx_2^{q-1}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\ &+(s-r-1)x_2^q[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-2}\\ &\times\{1-F(x_2)\}^m f(x_2)\Big]\{1-F(x_2)\}^{\gamma_s}dx_2\end{aligned}$$

or

$$\begin{aligned}I(x_2) = \frac{q}{\gamma_s}&\int_{x_1}^{\infty} x_2^{q-1}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\ &\times\{1-F(x_2)\}^{\gamma_s}dx_2 + \frac{(s-r-1)}{\gamma_s}\int_{x_1}^{\infty} x_2^q f(x_2)\\ &\times[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-2}\{1-F(x_2)\}^{\gamma_s+m}dx_2.\end{aligned}\tag{4.39}$$

Now using the value of $I(x_2)$ from (4.39) in (4.38) we have

$$\mu_{r,s:n,m,k}^{p,q} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty} x_1^p f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times \left[\frac{q}{\gamma_s}\int_{x_1}^{\infty} x_2^{q-1}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\right.$$
$$\times \{1-F(x_2)\}^{\gamma_s}dx_2 + \frac{(s-r-1)}{\gamma_s}\int_{x_1}^{\infty} x_2^q f(x_2)$$
$$\left.\times [h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-2}\{1-F(x_2)\}^{\gamma_s+m}dx_2\right]dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)\{1-F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s}dx_2dx_1 + \frac{(s-r-1)C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\times \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-2}\{1-F(x_2)\}^{\gamma_s+m}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)\{1-F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s}dx_2dx_1 + \frac{C_{s-2}}{(r-1)!(s-r-2)!}$$
$$\times \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-2}\{1-F(x_2)\}^{\gamma_{s-1}-1}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} f(x_1)\{1-F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s}dx_2dx_1 + \mu_{r,s-1:n,m,k}^{p,q},$$

or

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s} dx_2 dx_1;$$

as required.

We now present recurrence relations for single and product moments of GOS for some special distributions.

4.11.1 Exponential Distribution

The density and distribution function of Exponential random variable are given as

$$f(x) = \alpha e^{-\alpha x};\ \alpha, x > 0$$

and

$$F(x) = 1 - e^{-\alpha x}.$$

We note that

$$f(x) = \alpha[1 - F(x)]. \tag{4.40}$$

Consider (4.34)

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!} \int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx \\ = \frac{pC_{r-1}}{\gamma_r(r-1)!} \int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r-1} \\ \times [1-F(x)] g_m^{r-1}[F(x)]dx.$$

Using (4.40) in (4.34) following recurrence relation between single moments of GOS has been obtained by Ahsanullah (2000)

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\alpha\gamma_r(r-1)!} \int_{-\infty}^{\infty} x^{p-1} f(x)\{1-F(x)\}^{\gamma_r-1} \\ \times g_m^{r-1}[F(x)]dx$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{p}{\alpha\gamma_r}\mu^{p-1}_{r:n,m,k},$$

or

$$\mu^{p}_{r:n,m,k} = \mu^{p}_{r-1:n,m,k} + \frac{p}{\alpha\gamma_r}\mu^{p-1}_{r:n,m,k} \tag{4.41}$$

The relationship (4.41) reduces to corresponding relationship for simple order statistics given in (2.58) for $m = 0$ and $k = 1$. Again consider (4.35) as

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s} dx_2 dx_1.\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s-1}[1-F(x_2)] dx_2 dx_1.\end{aligned}$$

Now using (4.40) in above equation we have

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\alpha\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ \times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s-1} dx_2 dx_1\end{aligned}$$

or

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r,s:n,m,k}$$

or

$$\mu^{p,q}_{r,s:n,m,k} = \mu^{p,q}_{r,s-1:n,m,k} + \frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r,s:n,m,k}. \tag{4.42}$$

Using $s = r + 1$ in (4.42) we have following relation between product moments of two contiguous GOS

$$\mu^{p,q}_{r,r+1:n,m,k} = \mu^{p+q}_{r:n,m,k} + \frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r,r+1:n,m,k}.$$

The recurrence relationship given in (4.42) reduces to relationship (2.59) for $m = 0$ and $k = 1$. Further, for $m = -1$, the relationship (4.42) reduces to recurrence relation for product moment of record values given in (3.47).

4.11.2 The Rayleigh Distribution

The density and distribution function of Rayleigh distribution are

$$f(x) = 2\alpha x \exp\left(-\alpha x^2\right);\ \alpha, x > 0;$$

and

$$F(x) = 1 - \exp\left(-\alpha x^2\right).$$

The density and distribution function are related as

$$f(x) = 2\alpha x[1 - F(x)]. \tag{4.43}$$

Using (4.43) in (4.34) Mohsin et al. (2010) has derived recurrence relation for single moments of GOS for Rayleigh distribution as under

$$\begin{aligned}\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1 - F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx \\ &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1 - F(x)\}^{\gamma_r - 1} \\ &\quad \times [1 - F(x)] g_m^{r-1}[F(x)]dx.\end{aligned}$$

Now using (4.43) in above equation we have

$$\begin{aligned}\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{2\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-2} f(x) \\ \times \{1 - F(x)\}^{\gamma_r - 1} g_m^{r-1}[F(x)]dx;\end{aligned}$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{p}{2\alpha\gamma_r}\mu^{p-2}_{r:n,m,k}$$

or

$$\mu^p_{r:n,m,k} = \mu^p_{r-1:n,m,k} + \frac{p}{2\alpha\gamma_r}\mu^{p-2}_{r:n,m,k}. \tag{4.44}$$

Again consider (4.35) as

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s-1}[1-F(x_2)]dx_2dx_1.$$

Now using (4.43) in above equation we have

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{qC_{s-1}}{2\alpha\gamma_s(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x_1}^{\infty} x_1^p x_2^{q-2} \\ \times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ \times \{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1.$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{q}{2\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q-2}$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \mu_{r,s-1:n,m,k}^{p,q} + \frac{q}{2\alpha\gamma_s}\mu_{r,s:n,m,k}^{p,q-2}. \tag{4.45}$$

The recurrence relation for single and product moments for special cases can be readily obtained from (4.43) and (4.44).

4.11.3 Weibull Distribution

The density and distribution function for Weibull random variable are

$$f(x) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right);\ x, \alpha, \beta > 0;$$

and

$$F(x) = 1 - \exp\left(-\alpha x^{\beta}\right).$$

We also have

$$f(x) = \alpha\beta x^{\beta-1}[1-F(x)]. \tag{4.46}$$

Now using (4.46) in (4.34) we have

$$\mu_{r:n,m,k}^{p} - \mu_{r-1:n,m,k}^{p} = \frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!} \int_0^{\infty} x^{p-\beta} f(x) \\ \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx.$$

or

$$\mu_{r:n,m,k}^{p} - \mu_{r-1:n,m,k}^{p} = \frac{p}{\alpha\beta\gamma_r}\mu_{r:n,m,k}^{(p-\beta)},$$

or

$$\mu_{r:n,m,k}^{p} = \mu_{r-1:n,m,k}^{p} + \frac{p}{\alpha\beta\gamma_r}\mu_{r:n,m,k}^{(p-\beta)}. \tag{4.47}$$

The relation (4.47) reduces to relation (4.41) for $\beta = 1$ and to (4.44) for $\beta = 2$ as expected. The recurrence relation between product moments can be derived by using (4.35) as

$$\begin{aligned}\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ &\times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ &\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ &\times \{1-F(x_2)\}^{\gamma_s-1}\{1-F(x_2)\}dx_2dx_1.\end{aligned}$$

Now using (4.46) in above equation we have

$$\begin{aligned}\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\beta} \\ &\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ &\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ &\times \{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1\end{aligned}$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{q}{\alpha\beta\gamma_s}\mu_{r,s:n,m,k}^{p,q-\beta},$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \mu_{r,s-1:n,m,k}^{p,q} + \frac{q}{\alpha\beta\gamma_s}\mu_{r,s:n,m,k}^{p,q-\beta}. \tag{4.48}$$

Further, by using $s = r + 1$, the recurrence relation between product moments of two contiguous GOS for Weibull distribution is obtained as

$$\mu_{r,r+1:n,m,k}^{p,q} = \mu_{r:n,m,k}^{p+q} + \frac{q}{\beta\gamma_s}\mu_{r,r+1:n,m,k}^{p,q-\beta}.$$

The corresponding relationships for Order Statistics and $k-$Upper Record Values can be readily obtained for $(m = 0; k = 1)$ and $m = -1$.

4.11.4 Power Function Distribution

The Power function distribution has density and distribution function as

$$f(x) = \frac{\theta(x-a)^{\theta-1}}{(b-a)^{\theta}}; a \le x \le b, \theta \ge 1$$

and

$$F(x) = \left(\frac{x-a}{b-a}\right)^{\theta}; a \le x < b, \theta \ge 1.$$

We can see that following relation holds between density and distribution function

$$(b-x)f(x) = \theta[1-F(x)]. \tag{4.49}$$

The recurrence relations for single and product moments of GOS for Power function distribution are derived by using (4.49) in (4.34) and (4.35). From (4.34) we have

$$\begin{aligned}\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx \\ &= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r-1} \\ &\quad \times [1-F(x)]g_m^{r-1}[F(x)]dx.\end{aligned}$$

Using (4.49) in above equation we have

$$\begin{aligned}\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}(b-x)f(x) \\ \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx\end{aligned}$$

or

$$\begin{aligned}\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{bpC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}(b-x)f(x) \\ \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx \\ -\frac{pC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p}f(x) \\ \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx\end{aligned}$$

or

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{bp}{\theta\gamma_r}\mu^{p-1}_{r:n,m,k} - \frac{p}{\theta\gamma_r}\mu^{p}_{r:n,m,k}$$

or

$$\left(1+\frac{p}{\theta\gamma_r}\right)\mu^{p}_{r:n,m,k}=\frac{bp}{\theta\gamma_r}\mu^{p-1}_{r:n,m,k}+\mu^{p}_{r-1:n,m,k}$$

or

$$\mu^{p}_{r:n,m,k}=\frac{1}{p+\theta\gamma_r}\left(pb\mu^{p-1}_{r:n,m,k}+\theta\gamma_r\mu^{p}_{r-1:n,m,k}\right). \tag{4.50}$$

We can see that for $m=-1$, the recurrence relation (4.50) reduces to (3.34) as it should be. Again consider (4.35) as

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k}-\mu^{p,q}_{r,s-1:n,m,k}=&\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}\\&\times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}\\&\times[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\&\times\{1-F(x_2)\}^{\gamma_s-1}\{1-F(x_2)\}dx_2dx_1.\end{aligned}$$

Now using (4.49) in above equation we have

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k}-\mu^{p,q}_{r,s-1:n,m,k}=&\frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}\\&\times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}\\&\times[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\&\times\{1-F(x_2)\}^{\gamma_s-1}(b-x_2)f(x_2)dx_2dx_1\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k}-\mu^{p,q}_{r,s-1:n,m,k}=&\frac{bqC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q-1}\\&\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}\\&\times[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\&\times\{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1\\&-\frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1^p x_2^{q}\\&\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}\\&\times[h_m\{F(x_2)\}-h_m\{F(x_1)\}]^{s-r-1}\\&\times\{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1\end{aligned}$$

or

$$\mu^{p,q}_{r,s:n,m,k}-\mu^{p,q}_{r,s-1:n,m,k}=\frac{bq}{\theta\gamma_s}\mu^{p,q-1}_{r,s:n,m,k}-\frac{q}{\theta\gamma_s}\mu^{p.q}_{r,s:n,m,k}$$

or

$$\left(1+\frac{q}{\theta\gamma_s}\right)\mu^{p,q}_{r,s:n,m,k}=\frac{bq}{\theta\gamma_s}\mu^{p,q-1}_{r,s:n,m,k}+\mu^{p,q}_{r,s-1:n,m,k}$$

or

$$\mu^{p,q}_{r,s:n,m,k}=\frac{bq}{q+\theta\gamma_s}\mu^{p,q-1}_{r,s:n,m,k}+\frac{\theta\gamma_s}{q+\theta\gamma_s}\mu^{p,q}_{r,s-1:n,m,k}. \tag{4.51}$$

Using $s=r+1$ in above equation we have following recurrence relation for product moments of two contiguous GOS for Power function distribution

$$\mu^{p,q}_{r,r+1:n,m,k}=\frac{1}{q+\theta\gamma_s}\left(bq\mu^{p,q-1}_{r,r+1:n,m,k}+\theta\gamma_s\mu^{p+q}_{r:n,m,k}\right). \tag{4.52}$$

The recurrence relations (4.51) and (4.52) reduces to recurrence relations for product moments of upper record values for $m=-1$. Further, using $\theta=1$ in (4.50), (4.51) and (4.52) we can readily derive the recurrence relations for single and product moments of GOS for Uniform distribution.

4.11.5 Marshall-Olkin-Weibull Distribution

The Marshall-Olkin-Weibull distribution is an extension of Weibull distribution. The density and distribution function of Marshall-Olkin-Weibull distribution are

$$f(x)=\frac{\lambda\theta x^{\theta-1}\exp\left(-x^{\theta}\right)}{\left[1-(1-\lambda)\exp\left(-x^{\theta}\right)\right]^2};\, x,\lambda,\theta>0$$

and

$$F(x)=1-\frac{\lambda\exp\left(-x^{\theta}\right)}{\left[1-(1-\lambda)\exp\left(-x^{\theta}\right)\right]^2}.$$

The density and distribution function of Marshall-Olkin-Weibull distribution are related as

$$1-F(x)=\frac{1}{\theta x^{\theta-1}}\left[1-(1-\lambda)\exp\left(-x^{\theta}\right)\right]f(x). \tag{4.53}$$

We can see that the Marshall-Olkin-Weibull distribution reduces to Weibull distribution for $\lambda=1$. Athar et al. (2012) have derived the recurrence relations for single and product moments of GOS for Marshall-Olkin-Weibull distribution by using (4.53). We present these relations in the following.

Consider (4.34) as

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx$$
$$= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r-1} \times [1-F(x)]g_m^{r-1}[F(x)]dx.$$

Using (4.53) in above equation we have

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-\theta} f(x)\{1-F(x)\}^{\gamma_r-1} \times \left[1-(1-\lambda)\exp\left(-x^{\theta}\right)\right]g_m^{r-1}[F(x)]dx$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-\theta} f(x)\{1-F(x)\}^{\gamma_r-1} \times g_m^{r-1}[F(x)]dx - \frac{p(1-\lambda)C_{r-1}}{\theta\gamma_r(r-1)!} \times \int_{-\infty}^{\infty} x^{p-\theta}\exp\left(-x^{\theta}\right) f(x) \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx$$

Now expanding $\exp\left(-x^{\theta}\right)$ in power series we have

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\theta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-\theta} f(x)\{1-F(x)\}^{\gamma_r-1} \times g_m^{r-1}[F(x)]dx - \frac{p(1-\lambda)}{\theta\gamma_r} \times \sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} x^{p-\theta(1-h)} f(x) \times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{p}{\theta\gamma_r}\mu^{p-\theta}_{r:n,m,k} - \frac{p(1-\lambda)}{\theta\gamma_r}\sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\mu^{p-\theta(1-h)}_{r:n,m,k}$$

or

$$\mu^{p}_{r:n,m,k} = \mu^{p}_{r-1:n,m,k} + \frac{p}{\theta\gamma_r}\mu^{p-\theta}_{r:n,m,k} - \frac{p(1-\lambda)}{\theta\gamma_r}\sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\mu^{p-\theta(1-h)}_{r:n,m,k}. \tag{4.54}$$

We can readily see that the relation (4.54) reduces to (4.47) for $\lambda = 1$. The recurrence relation for product moments is derived by using (4.53) in (4.35) as under

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = {} & \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1} \\ & \times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ & \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ & \times \{1-F(x_2)\}^{\gamma_s-1}\{1-F(x_2)\}dx_2dx_1.\end{aligned}$$

Now using (4.53) in above equation we have

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = {} & \frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\theta} \\ & \times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ & \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ & \times \{1-F(x_2)\}^{\gamma_s-1} \\ & \times \left[1-(1-\lambda)\exp\left(-x^{\theta}\right)\right]dx_2dx_1.\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = {} & \frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\theta} \\ & \times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\} \\ & \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1} \\ & \times \{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1 \\ & - \frac{q(1-\lambda)C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\theta} \\ & \times \exp\left(-x^{\theta}\right)f(x_1)f(x_2)\{1-F(x_1)\}^m \\ & \times g_m^{r-1}\{F(x_1)\}\{1-F(x_2)\}^{\gamma_s-1} \\ & \times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}dx_2dx_1\end{aligned}$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{q}{\theta\gamma_s}\mu_{r,s:n,m,k}^{p,q-\theta} - \frac{q(1-\lambda)C_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}$$
$$\times \sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\theta(1-h)}$$
$$\times f(x_1)f(x_2)\{1-F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}\{1-F(x_2)\}^{\gamma_s-1}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q} - \mu_{r,s-1:n,m,k}^{p,q} = \frac{q}{\theta\gamma_s}\mu_{r,s:n,m,k}^{p,q-\theta} - \frac{q(1-\lambda)}{\theta\gamma_s}$$
$$\times \sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\mu_{r,s:n,m,k}^{p,q-\theta(1-h)}$$

or

$$\mu_{r,s:n,m,k}^{p,q} = \mu_{r,s-1:n,m,k}^{p,q} + \frac{q}{\theta\gamma_s}\mu_{r,s:n,m,k}^{p,q-\theta}$$
$$-\frac{q(1-\lambda)}{\theta\gamma_s}\sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\mu_{r,s:n,m,k}^{p,q-\theta(1-h)}. \quad (4.55)$$

The recurrence relation (4.55) reduces to (4.48) for $\lambda = 1$ as it should be.

4.11.6 The Kumaraswamy Distribution

The Kumaraswamy distribution is a simple yet powerful distribution. The density and distribution function of Kumaraswamy distribution are

$$f(x) = \alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1};\ \alpha,\beta > 0,\ 0 \le x \le 1$$

and

$$F(x) = 1-(1-x^\alpha)^\beta.$$

The density and distribution function are related as

$$1-F(x) = \frac{1}{\alpha\beta}\left[x^{-(\alpha-1)} - x\right]f(x). \quad (4.56)$$

Kumar (2011) has derived the recurrence relations for single and product moments of GOS for Kumaraswamy distribution. We give these relations in the following.

Consider (4.34) as

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx$$
$$= \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r-1}$$
$$\times [1-F(x)]g_m^{r-1}[F(x)]dx.$$

Now using (4.56) in above equation we have

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r-1}$$
$$\times \left[x^{-(\alpha-1)} - x\right] f(x) g_m^{r-1}[F(x)]dx$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-\alpha} f(x)$$
$$\times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx$$
$$-\frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p} f(x)$$
$$\times \{1-F(x)\}^{\gamma_r-1} g_m^{r-1}[F(x)]dx$$

or

$$\mu^p_{r:n,m,k} - \mu^p_{r-1:n,m,k} = \frac{p}{\alpha\beta\gamma_r}\mu^{p-\alpha}_{r:n,m,k} - \frac{p}{\alpha\beta\gamma_r}\mu^{p}_{r:n,m,k}$$

or

$$\left(1+\frac{p}{\alpha\beta\gamma_r}\right)\mu^p_{r:n,m,k} = \mu^p_{r-1:n,m,k} + \frac{p}{\alpha\beta\gamma_r}\mu^{p-\alpha}_{r:n,m,k}$$

or

$$\mu^p_{r:n,m,k} = \frac{1}{p+\alpha\beta\gamma_r}\left(p\mu^{p-\alpha}_{r:n,m,k} + \alpha\beta\gamma_r\mu^p_{r-1:n,m,k}\right) \tag{4.57}$$

The recurrence relations for single moments of GOS for standard uniform distribution can be obtained from (4.57) by setting $\alpha = \beta = 1$. Again consider (4.35) as

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1}$$
$$\times f(x_1)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s-1}\{1-F(x_2)\}dx_2dx_1.$$

Using (4.56) in above equation we have

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-1}$$
$$\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s-1}\Big[x_2^{-(\alpha-1)} - x_2\Big]dx_2 dx_1$$

or

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q-\alpha}$$
$$\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s-1}dx_2 dx_1$$
$$-\frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^{q}$$
$$\times f(x_1)f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times [h_m\{F(x_2)\} - h_m\{F(x_1)\}]^{s-r-1}$$
$$\times \{1-F(x_2)\}^{\gamma_s-1}dx_2 dx_1$$

or

$$\mu^{p,q}_{r,s:n,m,k} - \mu^{p,q}_{r,s-1:n,m,k} = \frac{q}{\alpha\beta\gamma_s}\mu^{p,q-\alpha}_{r,s:n,m,k} - \frac{q}{\alpha\beta\gamma_s}\mu^{p,q}_{r,s:n,m,k}$$

or

$$\left(1+\frac{q}{\alpha\beta\gamma_s}\right)\mu^{p,q}_{r,s:n,m,k} = \frac{q}{\alpha\beta\gamma_s}\mu^{p,q-\alpha}_{r,s:n,m,k} + \mu^{p,q}_{r,s-1:n,m,k}$$

or

$$\mu^{p,q}_{r,s:n,m,k} = \frac{1}{q+\alpha\beta\gamma_s}\left(q\mu^{p,q-\alpha}_{r,s:n,m,k} + \alpha\beta\gamma_s\mu^{p,q}_{r,s-1:n,m,k}\right). \tag{4.58}$$

Using $\alpha = \beta = 1$ in (4.58) we can obtain the recurrence relations for product moments of GOS for Uniform distribution.

Example 4.5 A random sample is available from Uniform distribution over the interval [0, 1]. Show that

$$\mu_{r:n,m,k} - \mu_{r-1:n,m,k} = \frac{C_{r-2}(k)}{C_{r-1}(k+1)},$$

where notations have their usual meanings.

Solution: We have following recurrence relation for moments of GOS:

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1}\left\{F^{-1}(t)\right\}^{/} \times (1-t)^{\gamma_r} g_m^{r-1}(t)dt.$$

For $p = 1$ we have:

$$\mu_{r:n,m,k} - \mu_{r-1:n,m,k} = \frac{C_{r-1}}{\gamma_r(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{/}(1-t)^{\gamma_r} g_m^{r-1}(t)dt.$$

For standard uniform distribution we have

$$f(x) = 1; 0 < x < 1$$
$$\text{and } F(x) = \int_0^x f(t)dt = x$$

so $t = F(x)$ gives $t = x$ and hence

$$x = F^{-1}(t) = t, \left\{F^{-1}(t)\right\}^{/} = 1.$$

So we have

$$\begin{aligned}
\mu_{r:n,m,k} - \mu_{r-1:n,m,k} &= \frac{C_{r-2}}{(r-1)!}\int_0^1 (1-t)^{\gamma_r} g_m^{r-1}(t)dt \\
&= \frac{C_{r-2}}{(r-1)!}\int_0^1 (1-t)^{(k+1)+(n-r)(m+1)-1} g_m^{r-1}(t)dt \\
&= \frac{C_{r-2}(k)}{C_{r-1}(k+1)} \times \frac{C_{r-1}(k+1)}{(r-1)!} \\
&\quad \times \int_0^1 (1-t)^{(k+1)+(n-r)(m+1)-1} g_m^{r-1}(t)dt \\
&= \frac{C_{r-2}(k)}{C_{r-1}(k+1)},
\end{aligned}$$

as

$$\frac{C_{r-1}(k+1)}{(r-1)!}\int_0^1 (1-t)^{(k+1)+(n-r)(m+1)-1} g_m^{r-1}(t)dt = 1,$$

as required.

4.12 Relation for Moments of GOS for Special Class of Distributions

We have shown that following recurrence relation hold between single moments of GOS for any parent distribution

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{1-F(x)\}^{\gamma_r} g_m^{r-1}[F(x)]dx. \quad (4.32)$$

Using probability integral transform, above relation can be written as

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{0}^{1} \left\{F^{-1}(t)\right\}^{p-1}\left\{F^{-1}(t)\right\}^{/} \times (1-t)^{\gamma_r} g_m^{r-1}(t)dt. \quad (4.59)$$

The relation (4.59) is very useful in deriving recurrence relations between single moments of GOS for certain class of distributions depending upon choice of $\left\{F^{-1}(t)\right\}^{/}$. In following theorem; given by Kamps (1995b); we present a general recurrence relation between single moments of GOS for class of distributions defined as

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{(q-p_1)(m+1)+s_1-1} g_m^{p_1}(t) \text{ on } (0,1). \quad (4.60)$$

Theorem 4.3 *If the class of distributions is defined as*

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{(q-p_1)(m+1)+s_1-1} g_m^{p_1}(t) \textit{ on } (0,1);$$

with $d > 0$, $p_1, q \in \mathbb{Z}$ *and* $s_1 > 1-k$ *then following recurrence relation hold between moments of GOS*

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = pK\mu^{(p-1)}_{r+p_1:n+q,m,k+s}; \quad (4.61)$$

where

$$K = \frac{1}{d}\frac{C_{r-2}(n,k)}{C_{r+p_1-1}(n+q,k+s)}\frac{(r+p_1-1)!}{(r-1)!} = \mu_{r:n,m,k} - \mu_{r-1:n,m,k}.$$

Proof We have following recurrence relation for moments of GOS

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{0}^{1} \left\{F^{-1}(t)\right\}^{p-1}\left\{F^{-1}(t)\right\}^{/} \times (1-t)^{\gamma_r} g_m^{r-1}(t)dt.$$

Now using the representation of $\left\{F^{-1}(t)\right\}^{/}$ for given class we have

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{pC_{r-1}}{\gamma_r(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1} \times \frac{1}{d}(1-t)^{(q-p_1)(m+1)+s_1-1}(1-t)^{k+(n-r)(m+1)} \times g_m^{p_1}(t)g_m^{r-1}(t)dt.$$

or

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{1}{d}\frac{pC_{r-2}(n,k)}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1} g_m^{r+p_1-1}(t) \times (1-t)^{(k+s_1)+\{(n+q)-(r+p_1)\}(m+1)-1}dt$$

or

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = \frac{1}{d}\frac{pC_{r-2}(n,k)}{(r-1)!}\times\frac{C_{r+p_1-1}(n+q,k+s_1)}{C_{r+p_1-1}(n+q,k+s_1)} \times \frac{(r+p_1-1)!}{(r+p_1-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1} \times (1-t)^{(k+s_1)+\{(n+q)-(r+p_1)\}(m+1)-1} g_m^{r+p_1-1}(t)dt$$

or

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = p\frac{1}{d}\frac{C_{r-2}(n,k)}{C_{r+p_1-1}(n+q,k+s_1)}\frac{(r+p_1-1)!}{(r-1)!} \times \frac{C_{r+p-1}(n+q,k+s_1)}{(r+p_1-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1} \times (1-t)^{(k+s_1)+\{(n+q)-(r+p_1)\}(m+1)-1} g_m^{r+p_1-1}(t)dt$$

or

$$\mu^{p}_{r:n,m,k} - \mu^{p}_{r-1:n,m,k} = pK\mu^{(p-1)}_{r+p_1:n+q,m,k+s_1}.$$

Also from (4.35) we have for $p = 1$:

$$K = \mu_{r:n,m,k} - \mu_{r-1:n,m,k},$$

which completes the proof.

Above theorem readily produce the corresponding recurrence relation for Order Statistics for $m = 0$ and $k = 1$. In that case the class of distributions is

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}t^{p_1}(1-t)^{q-p_1-1},$$

and the recurrence relation is

$$\mu^{p}_{r:n} - \mu^{p}_{r-1:n} = pC(r, n, p_1, q)\mu^{(p-1)}_{r+p_1,n+q};$$

where

$$C(r, n, p, q) = \mu_{r:n} - \mu_{r-1:n} = \frac{1}{d}\frac{\binom{n}{r-1}}{(r+p)\binom{n+q}{r+p}}.$$

Above relation is same as we have derived in (2.33).

Further, if we have a real valued function defined as

$$\frac{d}{dt}h(t) = \frac{1}{d}(1-t)^{(q-p_1)(m+1)+s_1-1}g_m^{p_1}(t) \text{ on } (0, 1),$$

and distribution function of GOS is expressible as

$$F^{-1}(t) = \left[\begin{array}{l}\exp\{h(t)\}, \beta = 0\\ \{\beta h(t)\}^{1/\beta}, \beta > 0\end{array}\right.$$

then following recurrence relation holds between moments of GOS for this class of distributions

$$\mu^{(p+\beta)}_{r:n,m,k} - \mu^{(p+\beta)}_{r-1:n,m,k} = (p_1+\beta)K\mu^{p}_{r+p_1:n+q,m,k+s_1}, \tag{4.62}$$

where K is defined earlier.

The class of distributions given in (4.60) provide several distributions for various choices of the constants involved. For example choosing $p_1 = q = s_1 = 0$ in (4.60) we have

$$\left\{F^{-1}(t)\right\}^{/} = \frac{1}{d}(1-t)^{-1}$$

which gives

$$F^{-1}(t) = \frac{1}{d}\ln\left(\frac{1}{1-t}\right).$$

Solving $F^{-1}(t) = x$ for x we have

$$F(x) = 1 - e^{-dx},$$

which is Exponential distribution. Hence the recurrence relations for single moments of GOS can be directly obtained from (4.61) by setting $p_1 = q = s_1 = 0$.

Chapter 5
Dual Generalized Order Statistics

5.1 Introduction

In previous chapters we have discussed some popular models of ordered random variables when the sample is arranged in increasing order. The simplest of these models is Order Statistics and is discussed in detail in Chap. 2. The comprehensive model for ordered random variables arranged in ascending order is Generalized Order Statistics and is discussed in detail in Chap. 4. The generalized order statistics produce several models for ordered random variables arranged in ascending order as special case for various choices of the parameters involved. Often it happen that the sample is arranged in descending order for example the life length of an electric bulb arranged from highest to lowest. In such situations the distributional properties of variables can not be studied by using the models of ordered random variables discussed earlier. The study of distributional properties of such random variables is studied by using the inverse image of Generalized Order Statistics and is popularly known as *Dual Generalized Order Statistics* (*DGOS*). The dual generalized order statistics was introduced by Burkschat et al. (2003) as a unified model for descendingly ordered random variables. In this chapter we will discuss dual generalized order statistics in detail with its special cases.

5.2 Joint Distribution of Dual GOS

The Dual Generalized Order Statistics (DGOS) or sometime called Lower Generalized Order Statistics (LGOS) is a combined mechanism of studying random variables arranged in descending order. The technique was introduced by Burkschat et al. (2003) and is defined in the following.

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_5

Suppose a random sample of size n is available from a distribution with cumulative distribution function $F(x)$ and let k and m_j be real numbers such that

$$k \geq 1,\ m_1,\ m_2,\ \ldots,\ m_{n-1} \in \mathbb{R}.$$

Also let

$$M_r = \sum_{j=r}^{n-1} m_j;\ 1 \leq r \leq n-1$$

Define γ_r as $\gamma_r = k + (n-r) + M_r$ such that $\gamma_r \geq 1$ for all $r \in \{1, 2, \ldots, n-1\}$. Finally let

$$\tilde{m} = (m_1, m_2, \ldots, m_{n-1});\ \textit{if}\ n \geq 2$$

then the random variables $X_{r:n,\tilde{m},k}$ are the *Dual Generalized Order Statistics* (*DGOS*) from the distribution $F(x)$, if their joint density function is of the form

$$f_{1(d),2(d),\ldots,n(d):n,\tilde{m},k}(x_1, x_2, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\{F(x_n)\}^{k-1}f(x_n) \times \left[\prod_{i=1}^{n-1}\{F(x_i)\}^{m_i}f(x_i)\right] \tag{5.1}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_n > F^{-1}(0)$. If $m_1, m_2, \ldots, m_{n-1} = m$, then the joint density of DGOS is given as

$$f_{1(d),2(d),\ldots,n(d):n,m,k}(x_1, x_2, \ldots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\{F(x_n)\}^{k-1}f(x_n) \times \left[\prod_{i=1}^{n-1}\{F(x_i)\}^{m}f(x_i)\right]. \tag{5.2}$$

Making the transformation $U_{r:n,\tilde{m},k} = F\left(X_{r:n,\tilde{m},k}\right)$, the random variables $U_{r:n,\tilde{m},k}$ are called the Uniform DGOS with joint density function as

$$f_{1(d),2(d),\ldots,n(d):n,m,k}(u_1, u_2, \ldots, u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left[\prod_{i=1}^{n-1}u_i^m\right]u_n^{k-1}; \tag{5.3}$$

with $1 > u_1 \geq u_2 \geq \cdots \geq u_n \geq 0$.

The joint distribution of DGOS given in (5.1) provides a comprehensive model for joint distribution of all forms of descendingly ordered random variables using different values of the parameters involved. We have given the joint distribution of various models of ordered random variables as special case of (5.2) in the following.

5.3 Special Cases of Dual GOS

We see in the following how DGOS provide various models of ordered random variables as special case.

1. Choosing $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$, such that $\gamma_r = n - r + 1$, density (5.2) reduces to

$$f_{1(d),\ldots,n(d):n,0,1}(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i),$$

which is joint density of Ordinary Order Statistics (OOS).

2. Choosing $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = \alpha - n + 1$, with $n - 1 < \alpha$ such that $\gamma_r = \alpha - r + 1$, the density (5.2) reduces to

$$f_{1(d),\ldots,n(d):n,0,\alpha-n+1}(x_1, \ldots, x_n) = \prod_{j=1}^{n} (\alpha - j + 1) \times [F(x_n)]^{\alpha-n} \times \prod_{i=1}^{n} f(x_i),$$

which is joint density of OOS with non-integral sample size.

3. Choosing $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$; $i = 1, 2, \ldots, n - 1$; $k = \alpha_n$ for some real number $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\gamma_r = (n - r + 1)\alpha_r$, the density (5.2) becomes

$$f_{1(d),\ldots,n(d):n,\tilde{m},\alpha_n}(x_1, \ldots, x_n) = n! \left(\prod_{j=1}^{n-1} \alpha_j \right) \times \left[\prod_{i=1}^{n-1} \{F(x_i)\}^{m_i} f(x_i) \right] \times \{F(x_n)\}^{\alpha_n - 1} f(x_n),$$

which is joint density of Lower Sequential Order Statistics (SOS) based on the arbitrary distribution function

$$F_r(t) = 1 - [F(t)]^{\alpha_r};\ 1 \le r \le n.$$

4. For $m_1 = m_2 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$, such that $\gamma_r = k$, the density (5.2) reduces to

$$f_{1(d),\ldots,n(d):n,-1,k}(x_1,\ldots,x_n) = k^n \left[\prod_{i=1}^{n-1} \frac{f(x_i)}{F(x_i)}\right] \times \{F(x_n)\}^{k-1} f(x_n),$$

which is joint density of kth lower records. Choosing $k = 1$ we obtain joint density of lower records.

5. For positive real numbers $\beta_1, \beta_2, \ldots, \beta_n$, choosing $m_i = \beta_i - \beta_{i+1} - 1;\ i = 1, 2, \ldots, n-1$ and $k = \beta_n$; such that $\gamma_r = \beta_r$; the density (5.2) reduces to

$$f_{1(d),\ldots,n(d):n,\tilde{m},\beta_n}(x_1,\ldots,x_n) = \left(\prod_{j=1}^{n} \beta_j\right) \{F(x_n)\}^{\beta_n - 1} f(x_n) \times \left[\prod_{i=1}^{n-1} \{F(x_i)\}^{m_i} f(x_i)\right],$$

which is joint density of Pfeifer's lower record values from non-identically distributed random variables based upon

$$F_r(t) = 1 - [F(t)]^{\beta_r};\ 1 \le r \le n.$$

Other special cases can be obtained by using specific values of the parameters in density (5.2).

5.4 Some Notations for Dual GOS

We discussed some notations that repeatedly arises in study of GOS. The counterpart notations for dual GOS can also be described on the same way. The constants C_{r-1} which appear in GOS remains same for dual GOS. We give additional notations in the following.

On the unit interval the functions $h_{m(d)}(x)$ and $g_{m(d)}(x)$, $m \in \mathbb{R}$, are defined as

$$h_{m(d)}(x) = \begin{cases} \frac{x^{m+1}}{m+1}; & m \neq -1 \\ \ln x; & m = -1 \end{cases} \; ; \; x \in [0, 1)$$

$$\begin{aligned} g_{m(d)}(x) &= h_{m(d)}(1) - h_{m(d)}(x) \\ &= \begin{cases} \frac{1}{m+1}\left[1 - x^{m+1}\right]; & m \neq -1 \\ -\ln x; & m = -1 \end{cases} \; ; \; x \in [0, 1) \end{aligned}$$

Using above representation, the joint density of Uniform dual GOS can be written as

$$f_{1(d),\ldots,n(d):n,\tilde{m},k}(u_1, \ldots, u_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left[\prod_{i=1}^{n-1} \frac{d}{du_i} h_{m_i(d)}(u_i)\right] u_n^{k-1}.$$

Hence the functions $h_{m(d)}(x)$ and $g_{m(d)}(x)$ occur very frequently in context of dual GOS.

We now give the joint marginal distribution of first r dual GOS.

5.5 Joint Marginal Distribution of *r* Dual GOS

The joint density function of n uniform GOS is given in (5.3) as

$$f_{1(d),\ldots,n(d):n,\tilde{m},k}(u_1, \ldots, u_n) = C_{n-1}\left(\prod_{i=1}^{n-1} u_i^{m_i}\right) u_n^{k-1}.$$

The joint marginal density of r uniform dual GOS is readily written by integrating out variables $u_{r+1}, \ldots, u_n$ by induction as

$$f_{1(d),\ldots,r(d):n,\tilde{m},k}(u_1, \ldots, u_r) = C_{r-1}\left(\prod_{i=1}^{r-1} u_i^{m_i}\right) u_r^{\gamma_r - 1}$$
$$; \; 1 > u_1 \geq u_2 \geq \cdots \geq u_r \geq 0; \tag{5.4}$$

which immediately yields following joint marginal distribution of dual GOS for any parent distribution $F(x)$

$$f_{1(d),\ldots,r(d):n,\tilde{m},k}(x_1, \ldots, x_r) = C_{r-1}\left[\prod_{i=1}^{r-1} F(x_i)^{m_i} f(x_i)\right] F(x_r)^{\gamma_r - 1} f(x_r), \tag{5.5}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \cdots \geq x_r > F^{-1}(0)$.

The joint marginal distributions of r dual GOS given in (5.5) can be used to obtain the joint marginal distribution of other models of descendingly ordered random variables as special case. Some of these special cases are given below.

1. *Reversed Order Statistics*: using $m_1 = \cdots = m_{n-1} = 0$ and $k = 1$ in (5.5), we have the joint marginal distribution of r reversed ordinary order statistics as

$$f_{1(d),\ldots,r(d):n,0,1}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r-1} f(x_i)\right] \times \{F(x_r)\}^{n-r} f(x_r).$$

2. *Fractional Reversed Order Statistics*: using $m_1 = \cdots = m_{n-1} = 0$ and $k = \alpha - n + 1$ in (5.5), we have the joint marginal distribution of r fractional reversed order statistics as

$$f_{1(d),\ldots,r(d):n,0,\alpha-n+1}(x_1, \ldots, x_r) = \prod_{j=1}^{r}(\alpha - j + 1)\left[\prod_{i=1}^{r-1} f(x_i)\right] \times \{F(x_r)\}^{\alpha-r} f(x_r).$$

3. *Reversed Sequential Order Statistics*; The joint distribution of r reversed sequential order statistics is obtained by using $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1}$ and $k = \alpha_n$ in (5.5) as

$$f_{1(d),\ldots,r(d):n,\tilde{m},\alpha_n}(x_1, \ldots, x_n) = \frac{n!}{(n-r)!}\prod_{j=1}^{r}\alpha_j \times \left[\prod_{i=1}^{r-1}\{F(x_i)\}^{m_i} f(x_i)\right] \times \{F(x_r)\}^{\alpha_r(n-r+1)-1} f(x_r).$$

4. *Lower Record Values*: using $m_1 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$ in (5.5), the joint marginal distribution of r k- lower record values is obtained as

$$f_{1(d),\ldots,r(d):n,-1,k}(x_1, \ldots, x_r) = k^r\left[\prod_{i=1}^{r-1}\frac{f(x_i)}{F(x_i)}\right] \times \{F(x_r)\}^{k-1} f(x_r).$$

5. *Pfeifer Lower Record Values*: using $m_i = \beta_i - \beta_{i+1} - 1$; $k \in \beta_n$; as

$$f_{1(d),\ldots,r(d):n,-1,k}(x_1, \ldots, x_r) = \prod_{j=1}^{r}\beta_j\left[\prod_{i=1}^{r-1}\{F(x_i)\}^{m_i} f(x_i)\right] \times \{F(x_r)\}^{\beta_r-1} f(x_r).$$

Other special cases can also be obtained from (5.5). The joint marginal distribution of r uniform dual GOS given in (5.4) can be used to obtain the marginal distribution of rth dual GOS and joint marginal distribution of rth and sth dual GOS. We have given these distributions in the following.

Lemma 5.1 *We define the quantity $A_{j(d)}$ as*

$$A_{j(d)}=\int_{u_r}^{u_{r-j-1}}\cdots\int_{u_r}^{u_{r-2}}\prod_{i=1}^{r-1}h^{/}_{m(d)}(u_i)du_{r-1}\cdots du_{r-j}$$
$$=\frac{1}{j!}\prod_{i=1}^{r-j-1}h^{/}_{m(d)}(u_i)\left\{h_{m(d)}\left(u_{r-j-1}\right)-h_{m(d)}(u_r)\right\}^{j}.$$

We now give the marginal distribution of rth dual GOS.

5.6 Marginal Distribution of a Single Dual GOS

We have given the joint marginal distribution of r uniform dual GOS in (5.6) as

$$f_{1(d),\ldots,r(d):n,\tilde{m},k}(u_1,\ldots,u_r)=C_{r-1}\left(\prod_{i=1}^{r-1}u_i^{m_i}\right)u_r^{\gamma_r-1}$$
$$;\ 1>u_1\geq u_2\geq\cdots\geq u_r\geq 0; \tag{5.6}$$

Assuming $m_1=\cdots=m_{n-1}=m$; the joint distribution is

$$f_{1(d),\ldots,r(d):n,m,k}(u_1,\ldots,u_r)=C_{r-1}\left(\prod_{i=1}^{r-1}u_i^{m}\right)u_r^{\gamma_r-1}\ .$$

Now the marginal distribution of rth uniform dual GOS can be obtained from above by integrating out $u_1,u_2,\ldots,u_{r-1}$ as under

$$f_{r(d):n,m,k}(u_r)=\int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}f_{1(d),\ldots,r(d):n,m,k}(u_1,\ldots,u_r)du_{r-1}\cdots du_1$$
$$=\int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}C_{r-1}\left(\prod_{i=1}^{r-1}u_i^{m}\right)u_r^{\gamma_r-1}du_{r-1}\cdots du_1$$
$$=C_{r-1}u_r^{\gamma_r-1}\int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}\left(\prod_{i=1}^{r-1}u_i^{m}\right)du_{r-1}\cdots du_1$$
$$=C_{r-1}u_r^{\gamma_r-1}\int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}\prod_{i=1}^{r-1}h^{/}_{m(d)}(u_i)du_{r-1}\cdots du_1$$

Now using the Lemma 5.1 with $j = r - 1$; and noting that $u_0 = 1$; we have:

$$A_{r-1(d)} = \int_{u_r}^{1} \cdots \int_{u_r}^{u_{r-2}} \prod_{i=1}^{r-1} h^{/}_{m(d)}(u_i) du_{r-1} \cdots du_1$$
$$= \frac{1}{(r-1)!}\{h_m(1) - h_m(u_r)\}^{r-1} = \frac{1}{(r-1)!} g^{r-1}_{m(d)}(u_r).$$

Hence the marginal distribution of rth uniform dual GOS is

$$f_{r(d):n,m,k}(u_r) = \frac{C_{r-1}}{(r-1)!} u_r^{\gamma_r - 1} g^{r-1}_{m(d)}(u_r). \tag{5.7}$$

The marginal density of rth dual GOS for any parent distribution is easily written from (5.7) and is given as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g^{r-1}_{m(d)}[F(x)]. \tag{5.8}$$

The distribution of special cases are readily written from (5.8). The marginal distribution of rth reversed order statistics is obtained by using $m = 0$ and $k = 1$ and is given as

$$f_{r(d):n,m,k}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\{F(x)\}^{n-r}\{1 - F(x)\}^{r-1}, \tag{5.9}$$

$F^{-1}(0) < x < F^{-1}(1)$. We can see that the distribution of rth reversed order statistics is same as the distribution of $(n - r + 1)$th ordinary order statistics from the distribution $F(x)$. Again using $m = -1$ the marginal distribution of rth $k-$lower record value is

$$f_{r(d):n,-1,k}(x) = \frac{k^r}{(r-1)!} f(x)\{F(x)\}^{k-1}[-\ln\{F(x)\}]^{r-1}, \tag{5.10}$$

which for $k = 1$ reduces to distribution of rth lower record value as

$$f_{r(d):n,-1,k}(x) = \frac{1}{(r-1)!} f(x)[-\ln\{F(x)\}]^{r-1}. \tag{5.11}$$

We now give the joint distribution of rth and sth dual GOS for $r < s$ in the following.

5.7 Joint Distribution of Two Dual GOS

The joint marginal distribution of first s uniform GOS is given from (5.12) as

$$f_{1(d),\ldots,s(d):n,\tilde{m},k}(u_1,\ldots,u_s) = C_{s-1}\left(\prod_{i=1}^{s-1} u_i^{m_i}\right)u_s^{\gamma_s-1}$$
$$;\ 1 > u_1 \geq u_2 \geq \cdots \geq u_s \geq 0; \qquad (5.12)$$

or

$$f_{1(d),\ldots,s(d):n,m,k}(u_1,\ldots,u_s) = C_{s-1}\left[\prod_{i=1}^{s-1} h^{/}_{m(d)}(u_i)\right]u_s^{\gamma_s-1}$$
$$;\ 1 > u_1 \geq u_2 \geq \cdots \geq u_s \geq 0.$$

The joint distribution of rth and sth dual GOS is obtained by integrating out $u_r = u_1$ and $u_s = u_2$ as

$$\begin{aligned} f_{r(d),s(d):n,m,k}(u_1,u_2) &= \int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}\int_{u_s}^{u_r}\cdots\int_{u_s}^{u_{s-2}} f_{1(d),\ldots,s(d):n,m,k}(u_1,\ldots,u_s) \\ &\quad \times du_{s-1}\cdots du_{r+1}du_{r-1}\cdots du_1 \\ &= \int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}\int_{u_s}^{u_r}\cdots\int_{u_s}^{u_{s-2}} C_{s-1}\left[\prod_{i=1}^{s-1} h^{/}_{m(d)}(u_i)\right] \\ &\quad \times u_s^{\gamma_s-1}du_{s-1}\cdots du_{r+1}du_{r-1}\cdots du_1 \end{aligned}$$

or

$$\begin{aligned} f_{r(d),s(d):n,m,k}(u_1,u_2) &= C_{s-1}u_s^{\gamma_s-1} \\ &\quad \times \int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}}\int_{u_s}^{u_r}\cdots\int_{u_s}^{u_{s-2}} \prod_{i=1}^{s-1} h^{/}_{m(d)}(u_i) \\ &\quad \times du_{s-1}\cdots du_{r+1}du_{r-1}\cdots du_1 \end{aligned}$$

or

$$\begin{aligned} f_{r(d),s(d):n,m,k}(u_1,u_2) &= C_{s-1}u_s^{\gamma_s-1}u_r^{m} \\ &\quad \times \int_{u_r}^{1}\cdots\int_{u_r}^{u_{r-2}} \prod_{i=1}^{r-1} h^{/}_{m(d)}(u_i) \end{aligned}$$

$$\times \left[\int_{u_s}^{u_r} \cdots \int_{u_s}^{u_{s-2}} \prod_{i=r+1}^{s-1} h_{m(d)}^{/}(u_i) du_{s-1} \cdots du_{r+1} \right]$$
$$\times du_{r-1} \cdots du_1$$

or

$$f_{r(d),s(d):n,m,k}(u_1, u_2) = C_{s-1} u_s^{\gamma_s - 1} u_r^m \int_{u_r}^{1} \cdots \int_{u_r}^{u_{r-2}} \prod_{i=1}^{r-1} h_m^{/}(u_i)$$
$$\times I(s) du_{r-1} \cdots du_1; \tag{5.13}$$

where

$$I(s) = \int_{u_s}^{u_r} \cdots \int_{u_s}^{u_{s-2}} \prod_{i=r+1}^{s-1} h_{m(d)}^{/}(u_i) du_{s-1} \cdots du_{r+1}.$$

Now using Lemma 5.1 with $s = r$ and $j = s - r - 1$ we have

$$A_{s-r-1} = I(s) = \int_{u_s}^{u_r} \cdots \int_{u_s}^{u_{s-2}} \prod_{i=r+1}^{s-1} h_{m(d)}^{/}(u_i) du_{s-1} \cdots du_{r+1}$$
$$= \frac{1}{(s-r-1)!} \left\{ h_{m(d)}(u_r) - h_{m(d)}(u_s) \right\}^{s-r-1}$$

Using above result in (5.13) we have

$$f_{r(d),s(d):n,m,k}(u_1, u_2) = C_{s-1} u_s^{\gamma_s - 1} u_r^m \frac{1}{(s-r-1)!}$$
$$\times \left\{ h_{m(d)}(u_r) - h_{m(d)}(u_s) \right\}^{s-r-1}$$
$$\times \int_{u_r}^{1} \cdots \int_{u_r}^{u_{r-2}} \prod_{i=1}^{r-1} h_{m(d)}^{/}(u_i) du_{r-1} \cdots du_1$$
$$= C_{s-1} u_s^{\gamma_s - 1} u_r^m \frac{1}{(s-r-1)!}$$
$$\times \left\{ h_{m(d)}(u_r) - h_{m(d)}(u_s) \right\}^{s-r-1} I(r)$$

Again using Lemma 5.1 with $j = r - 1$; and noting that $u_0 = 1$; we have

$$A_{r-1} = I(r) = \int_{u_r}^{1} \cdots \int_{u_r}^{u_{r-2}} \prod_{i=1}^{r-1} h_{m(d)}^{/}(u_i) du_{r-1} \cdots du_1$$
$$= \frac{1}{(r-1)!} \left\{ h_{m(d)}(1) - h_{m(d)}(u_r) \right\}^{r-1}$$

or

$$A_{r-1} = \frac{1}{(r-1)!} g_{m(d)}^{r-1}(u_r).$$

Hence the joint density of rth and sth uniform dual GOS is

$$f_{r(d),s(d):n,m,k}(u_1, u_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} u_1^m g_{m(d)}^{r-1}(u_1) \times u_2^{\gamma_s - 1} \left\{h_{m(d)}(u_1) - h_{m(d)}(u_2)\right\}^{s-r-1}. \tag{5.14}$$

The joint density of rth and sth dual GOS from any parent distribution is readily written as

$$f_{r(d),s(d):n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2) \{F(x_1)\}^m \times g_m^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s - 1} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}, \tag{5.15}$$

$F^{-1}(0) < x_2 < x_1 < F^{-1}(1)$. The joint density of two contiguous dual GOS is immediately written as

$$f_{r(d),r+1(d):n,m,k}(x_1, x_2) = \frac{C_r}{(r-1)!} f(x_1) f(x_2) \{F(x_1)\}^m \times g_{m(d)}^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s - 1}. \tag{5.16}$$

Further, the joint density of smallest and largest dual GOS is

$$f_{1(d),n(d):n,m,k}(x_1, x_2) = \frac{C_{n-1}}{(n-2)!} f(x_1) f(x_2) \{F(x_1)\}^m \{F(x_2)\}^{k-1} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{n-2}. \tag{5.17}$$

The expression for special cases can be immediately written from (5.15), (5.16) and (5.17). Specifically the joint density of rth and sth k-lower record values is

$$f_{r(d),s(d):n,-1,k}(x_1, x_2) = \frac{k^s}{(r-1)!(s-r-1)!} \left\{\frac{f(x_1)}{F(x_1)}\right\} f(x_2) \times [-\ln\{F(x_1)\}]^{r-1} \{F(x_2)\}^{k-1} \times [\ln\{F(x_1)\} - \ln\{F(x_2)\}]^{s-r-1}. \tag{5.18}$$

Other special cases can also be obtained in similar way.

Example 5.1 A random sample is drawn from the Inverse Weibull distribution with density function

$$f(x) = \beta x^{-(\beta+1)} \exp\left(-x^{-\beta}\right); x > 0.$$

Obtain the distribution of rth Dual GOS and the joint distribution of rth and sth Dual GOS for this distribution.

Solution: The density function of rth Dual GOS is given in (5.19) as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)]. \tag{5.19}$$

where

$$g_{m(d)}(u) = \frac{1-u^{m+1}}{m+1}$$

Now for given distribution we have

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x \beta t^{-(\beta+1)} \exp\left(-t^{-\beta}\right) dt \\ &= \exp\left(-x^{-\beta}\right); \, x > 0. \end{aligned}$$

So

$$\begin{aligned} g_{m(d)}[F(x)] &= \frac{1}{m+1}\left[1 - \{F(x)\}^{m+1}\right] \\ &= \frac{1}{m+1}\left[1 - \left\{\exp\left(-x^{-\beta}\right)\right\}^{m+1}\right] \end{aligned}$$

or

$$g_{m(d)}[F(x)] = \frac{\left[1 - \exp\left\{\left(-(m+1)x^{-\beta}\right)\right\}\right]}{m+1}.$$

Hence

$$\begin{aligned} g_{m(d)}^{r-1}[F(x)] &= \left[\frac{1}{m+1}\left\{1 - \exp\left(-(m+1)x^{-\beta}\right)\right\}\right]^{r-1} \\ &= \frac{1}{(m+1)^{r-1}}\left\{1 - \exp\left(-(m+1)x^{-\beta}\right)\right\}^{r-1} \\ &= \frac{1}{(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \exp\left\{-(m+1)jx^{-\beta}\right\} \end{aligned}$$

Using above value in (5.19) we have the density of rth Dual GOS as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}\beta x^{-(\beta+1)} \exp\left(-x^{-\beta}\right)\left[\exp\left(-x^{-\beta}\right)\right]^{\gamma_r-1}$$
$$\times \frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\exp\left\{-(m+1)jx^{-\beta}\right\}$$

or

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}\beta x^{-(\beta+1)} \exp\left(-x^{-\beta}\right)\exp\left\{-(\gamma_r-1)x^{-\beta}\right\}$$
$$\times \frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\exp\left\{-(m+1)jx^{-\beta}\right\}$$
$$= \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}$$
$$\times \beta x^{-(\beta+1)}\exp\left[-\{(m+1)j+\gamma_r\}x^{-\beta}\right].$$

Again the joint density of rth and sth Dual GOS is

$$f_{r(d),s(d):n,m,k}(x_1,x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}f(x_1)f(x_2)[F(x_1)]^m g_{m(d)}^{r-1}\{F(x_1)\}$$
$$\times[F(x_2)]^{\gamma_s-1}\left[h_{m(d)}\{F(x_1)\}-h_{m(d)}\{F(x_2)\}\right]^{s-r-1}.$$

where $h_{m(d)}(u) = u^{m+1}/(m+1)$.

Now for given distribution we have

$$h_{m(d)}[F(x_2)] = \frac{1}{m+1}\left\{\exp\left(-x_2^{-\beta}\right)\right\}^{m+1} = \frac{1}{m+1}\exp\left(-(m+1)x_2^{-\beta}\right)$$
$$h_{m(d)}[F(x_1)] = \frac{1}{m+1}\left\{\exp\left(-x_1^{-\beta}\right)\right\}^{m+1} = \frac{1}{m+1}\exp\left(-(m+1)x_1^{-\beta}\right)$$

Hence the joint density of rth and sth Dual GOS is

$$f_{r(d),s(d):n,m,k}(x_1,x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\beta x_1^{-(\beta+1)}\beta x_1^{-(\beta+1)}\exp\left(-x_1^{-\beta}\right)$$
$$\times \exp\left(-x_2^{-\beta}\right)\left[\exp\left(-x_1^{-\beta}\right)\right]^m\left[\exp\left(-x_2^{-\beta}\right)\right]^{\gamma_s-1}$$
$$\times\left[\frac{1}{(m+1)^{r-1}}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\exp\left(-(m+1)jx^{-\beta}\right)\right]$$
$$\times\left[\frac{\exp\left\{-(m+1)x_1^{-\beta}\right\}}{m+1}-\frac{\exp\left\{-(m+1)x_2^{-\beta}\right\}}{m+1}\right]^{s-r-1}$$

or

$$f_{r(d),s(d):n,m,k}(x_1,x_2)=\frac{\beta^2 C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}x_1^{-(\beta+1)}x_2^{-(\beta+1)}$$
$$\times\exp\left(-x_1^{-\beta}\right)\exp\left(-x_2^{-\beta}\right)\exp\left(-x_1^{-m\beta}\right)\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}$$
$$\times\exp\left(-(m+1)jx^{-\beta}\right)\exp\left\{-(\gamma_s-1)x_2^{-\beta}\right\}\sum_{k=0}^{s-r-1}(-1)^k$$
$$\times\binom{s-r-1}{k}\exp\left(-(m+1)(s-r-k-1)x_1^{-\beta}\right)$$
$$\times\exp\left(-(m+1)kx_2^{-\beta}\right)$$

or

$$f_{r(d),s(d):n,m,k}(x_1,x_2)=\frac{\beta^2 C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}$$
$$\times\sum_{k=0}^{s-r-1}\sum_{k=0}^{r-1}(-1)^{j+k}\binom{r-1}{j}\binom{s-r-1}{j}x_1^{-(\beta+1)}x_2^{-(\beta+1)}$$
$$\times\exp\left[-\{(m+1)(s-r-k+j)\}x_1^{-\beta}\right]$$
$$\times\exp\left[-\{(m+1)k+\gamma_s\}x_2^{-\beta}\right].$$

5.8 Conditional Distributions for Dual GOS

The marginal distribution of rth dual GOS and joint distribution of rth and sth dual GOS are given in (5.20) and (5.21) respectively as

$$f_{r(d):n,m,k}(x_1)=\frac{C_{r-1}}{(r-1)!}f(x_1)\{F(x_1)\}^{\gamma_r-1}g_{m(d)}^{r-1}[F(x_1)]. \tag{5.20}$$

and

$$f_{r(d),s(d):n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}f(x_1)f(x_2)\{F(x_1)\}^m$$
$$\times g_m^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s-1}$$
$$\times\left[h_{m(d)}\{F(x_1)\}-h_{m(d)}\{F(x_2)\}\right]^{s-r-1}. \tag{5.21}$$

The conditional distribution of sth dual GOS given information of rth dual GOS is readily written as under

$$
\begin{aligned}
f_{s(d)|r(d):n,m,k}(x_2|x_1) &= \frac{f_{r(d),s(d):n,m,k}(x_1,x_2)}{f_{r(d):n,m,k}(x_1)} \\
&= \left[\frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2)\{F(x_1)\}^m \right.\\
&\quad \times g_m^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s-1} \\
&\quad \left. \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}\right] \\
&\quad \Bigg/ \left[\frac{C_{r-1}}{(r-1)!} f(x_1)\{F(x_1)\}^{\gamma_r-1} g_{m(d)}^{r-1}\{F(x_1)\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{C_{s-1} f(x_2)\{F(x_2)\}^{\gamma_s-1}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}}{C_{r-1}(s-r-1)!\{F(x_1)\}^{\gamma_r-1-m}} \\
&= \frac{C_{s-1} f(x_2)\{F(x_2)\}^{\gamma_s-1}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}}{C_{r-1}(s-r-1)!\{F(x_1)\}^{\gamma_{r+1}}}.
\end{aligned} \tag{5.22}
$$

Again the joint density function of r dual GOS is given in (5.23) as

$$
f_{1(d),\ldots,r(d):n,m,k}(x_1,\ldots,x_r) = C_{r-1}\left[\prod_{i=1}^{r-1} F(x_i)^{m_i} f(x_i)\right] F(x_r)^{\gamma_r-1} f(x_r). \tag{5.23}
$$

Further, the joint density of first r and sth dual GOS is given as

$$
\begin{aligned}
f_{1(d),\ldots,r(d),s(d):n,\tilde{m},k}(x_1,\ldots,x_r,x_s) &= \frac{C_{s-1}}{(s-r-1)!}\left[\prod_{i=1}^{r} F(x_i)^{m_i} f(x_i)\right] \\
&\quad \times \left[h_{m(d)}\{F(x_r)\} - h_{m(d)}\{F(x_s)\}\right]^{s-r-1} \\
&\quad \times F(x_s)^{\gamma_s-1} f(x_s).
\end{aligned}
$$

The conditional distribution of sth dual GOS given first r dual GOS is therefore

$$
f_{s(d)|1(d),\ldots,r(d):n,m,k}(x_s|x_1,\ldots,x_r) = \frac{f_{1(d),\ldots,r(d),s(d):n,\tilde{m},k}(x_1,\ldots,x_r)}{f_{1(d),\ldots,r(d):n,\tilde{m},k}(x_1,\ldots,x_r)}
$$

$$
= \frac{C_{s-1} f(x_s)\{F(x_s)\}^{\gamma_s-1}\left[h_{m(d)}\{F(x_r)\} - h_{m(d)}\{F(x_s)\}\right]^{s-r-1}}{C_{r-1}(s-r-1)!\{F(x_r)\}^{\gamma_{r+1}}}, \tag{5.24}
$$

which is (5.22). Hence the dual GOS form the Markov Chain.

The transition probability for dual GOS is obtained by first obtaining the conditional distribution of $(r+1)$th dual GOS given rth dual GOS. This conditional distribution is readily written from (5.22) by using $s = r + 1$ and is

$$f_{r+1(d)|r(d):n,m,k}(x_1, x_2) = \frac{C_r f(x_{r+1})\{F(x_{r+1})\}^{\gamma_{r+1}-1}}{C_{r-1}(s-r-1)!\{F(x_r)\}^{\gamma_{r+1}}}$$
$$= \gamma_{r+1}\left[\frac{F(x_{r+1})}{F(x_r)}\right]^{\gamma_{r+1}-1}\frac{f(x_{r+1})}{F(x_r)}. \quad (5.25)$$

The transition probability for dual GOS is therefore

$$P\big(X_{r+1(d):n,m,k} \geq y | X_{r(d):n,m,k} = x\big) = \int_y^\infty f_{r+1(d)|r(d):n,m,k}(x_1, x_2)\, dx_{r+1}$$
$$= \gamma_{r+1}\int_y^\infty \left[\frac{F(x_{r+1})}{F(x_r)}\right]^{\gamma_{r+1}-1}\frac{f(x_{r+1})}{F(x_r)} dx_{r+1}$$
$$= \frac{\gamma_{r+1}}{\{F(x_r)\}^{\gamma_{r+1}}}\int_y^\infty f(x_{r+1})$$
$$\times\ \{F(x_{r+1})\}^{\gamma_{r+1}-1} dx_{r+1}$$

or

$$P\big(X_{r+1(d):n,m,k} \geq y | X_{r(d):n,m,k} = x\big) = \left[\frac{F(y)}{F(x)}\right]^{\gamma_{r+1}}. \quad (5.26)$$

The transition probabilities for special cases are easily obtained from (5.26). For example the transition probability for reversed order statistics is obtained by using $m = 0$ and $k = 1$ and is given as

$$P\big(X_{r+1(d):n,0,1} \geq y | X_{r(d):n,0,1} = x\big) = \left[\frac{F(y)}{F(x)}\right]^{n-r}.$$

Again the transition probability for k-lower records is obtained by setting $m = -1$ in (5.26) and is given as

$$P\big(X_{r+1(d):n,-1,k} \geq y | X_{r(d):n,-1,k} = x\big) = \left[\frac{F(y)}{F(x)}\right]^{k}.$$

For example if the sequence follow the Weibull distribution with density

$$f(x) = \alpha\beta x^{\beta-1}\exp\big(-\alpha x^{\beta}\big);\ x, \alpha, \beta > 0$$

then the transition probability for reversed order statistics is

$$P\big(X_{r+1(d):n,0,1} \geq y | X_{r(d):n,0,1} = x\big) = \left[\frac{1-\exp\big(-\alpha y^{\beta}\big)}{1-\exp\big(-\alpha x^{\beta}\big)}\right]^{n-r},$$

and can be computed for various choices of the parameters involved.

5.9 Lower Record Values

In Sect. 5.2 we have seen that the dual GOS provides several models for descendingly ordered random variables as special case for various choices of the parameters. Among these special models two popular models for descendingly ordered random variables are of special interest. These models include *Reversed Order Statistics*; which has been discussed in Chap. 2; and *Lower Record Values*. We now discuss Lower Record Values in the following.

Record values are popular area of study within the context of ordered random variables. In Chap. 4 we have discussed in detail the concept of upper record values which appear as values that are greater than the earlier records. Often it happen that records are formed when values smaller than already available records form new records. Such records are called the *Lower Record Values*. The lower record values are defined below.

Let $\{X_n; n \geq 1\}$ be a sequence of independently and identically distributed random variables with an absolutely continuous distribution function $F(x)$ and density function $f(x)$. Let $X_{r(re):n}$ be the rth reversed order statistics based upon a sample of size n. For a fixed $k \geq 1$ the k-lower record time $L_K(n); n \geq 1$ is defined as $U_K(1) = 1$ and

$$L_K(n+1) = \min\big\{r > U_K(n) : X_{r(re):r+k-1} < X_{L_K(n),L_K(n)+k-1}\big\}; \, n \in \mathbb{N}.$$

The kth lower record values are $X_{L_K(n):L_K(n)+k-1}$ and for the sake of simplicity will be denoted as $X_{L_K(n)}$. The joint density of n k-lower record values is given by Ahsanullah (1995) as

$$f_{L_K(1),L_K(2),\ldots,L_K(n)}(x_1,\ldots,x_n) = k^n\left\{\prod_{i=1}^{n-1}\frac{f(x_i)}{F(x_i)}\right\} \times [F(x_n)]^{k-1} f(x_n). \tag{5.27}$$

When $k = 1$ then the k-lower record records reduces to lower records and are defined as under.

Suppose $X_1, X_2, \ldots$ be a sequence of independently and identically distributed random variables from the distribution function $F(x)$. Suppose $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We call X_j a Lower Record Value of the sequence $\{X_n, n \geq 1\}$ if $Y_j < Y_{j-1}$. From this definition it is clear that X_1 is a lower record value. We also associate the indices to each record value with which they occur. These indices are called the record time $\{L(n)\}, n > 0$ where

$$U(n) = \min\big\{j|j > L(n-1), X_j < X_{L(n-1)}, n > 1\big\}.$$

We can readily see that $L(1) = 1$. We will denote the lower record values by $X_{L(n)}$.

Using $k = 1$ in (5.27), the joint density of n lower records is readily written as

$$f_{X_{L(1)},\ldots,X_{L(n)}}(x_1, \ldots, x_n) = \left\{\prod_{i=1}^{n-1} \frac{f(x_i)}{F(x_i)}\right\} f(x_n). \tag{5.28}$$

The marginal density of nth k-lower record value is immediately obtained from (5.27) and is given as

$$f_{L_K(n)}(x) = \frac{k^n}{\Gamma(n)} f(x)\{F(x)\}^{k-1}[-\ln\{F(x)\}]^{n-1},$$

or

$$f_{L_K(n)}(x) = \frac{k^n}{\Gamma(n)} f(x)\{F(x)\}^{k-1}[H(x)]^{n-1}, -\infty < x < \infty, \tag{5.29}$$

where $H(x) = -\ln\{F(x)\}$. The marginal density of nth lower record value is readily obtained from (5.29) by using $k = 1$ and is given by Ahsanullah (1995) as

$$f_{X_L(n)}(x) = \frac{1}{\Gamma(n)} f(x)[H(x)]^{n-1}, -\infty < x < \infty.$$

Again the joint density of two k-lower record values is given by Ahsanullah (1995) as

$$f_{L_K(m),L_K(n)}(x_1, x_2) = \frac{k^n}{\Gamma(m)\Gamma(n-m)} \frac{f(x_1)}{F(x_1)} f(x_2)[H(x_1)]^{m-1} \\ \times [H(x_2) - H(x_1)]^{n-m-1}[F(x_2)]^{k-1}, \tag{5.30}$$

where $-\infty < x_n < x_m < \infty$. The joint density of mth and nth lower records is easily written from (5.30) by setting $k = 1$ and is given as

$$f_{X_L(m),X_L(n)}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \frac{f(x_1)}{F(x_1)} f(x_2)[H(x_1)]^{m-1} \\ \times [H(x_2) - H(x_1)]^{n-m-1}. \tag{5.31}$$

Using (5.29) and (5.30) we obtain following conditional distribution of n th lower record given mth lower record.

$$f_{L_K(n)|L_K(m)=x_1}(x_2|x_1) = \frac{k^{n-m}}{\Gamma(n-m)} \frac{f(x_2)}{F(x_1)} \left\{ \frac{F(x_2)}{F(x_1)} \right\}^{k-1} \times [H(x_2) - H(x_1)]^{n-m-1}. \tag{5.32}$$

The conditional distribution given in (5.32) can be directly obtained from (5.22) by using $m = -1$. The conditional distribution of $(m+1)$ th k-lower record given mth k-lower record is obtained from (5.32) by using $n = m + 1$ and is given as

$$f_{L_K(n)|L_K(m)=x_1}(x_2|x_1) = k\frac{f(x_2)}{F(x_1)} \left\{ \frac{F(x_2)}{F(x_1)} \right\}^{k-1},$$

and since lower records follow a Markov Chain, the transition probability is immediately obtained from (5.26) by using $m = -1$.

We have seen in Chap. 3 that the quantities $R(x) = -\ln[1 - F(x)]$ which appear in context of upper records have nice distributional properties. We now give some distributional properties for the quantities $H(x) = -\ln[F(x)]$ which appear in the context of lower records. The distributional properties of $H(x)$ are given below.

The distribution of nth lower record and joint distribution of mth and nth lower records are

$$f_{X_L(n)}(x) = \frac{1}{\Gamma(n)} f(x)[H(x)]^{n-1}, -\infty < x < \infty.$$

and

$$f_{X_L(m),X_L(n)}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \frac{f(x_1)}{F(x_1)} f(x_2)[H(x_1)]^{m-1} \times [H(x_2) - H(x_1)]^{n-m-1}.$$

Now making the transformation $v = H(x) = -\ln[F(x)]$ in the density function of nth lower record we have $F(x) = e^{-v}$ and hence $f(x)dx = e^{-v}dv$. The density function of v is therefore

$$f_V(v) = \frac{1}{\Gamma(n)} v^{n-1} e^{-v}, \ v > 0.$$

We see that the distribution of $H(x) = -\ln[F(x)]$ is Gamma with shape parameter n. In Chap. 3 we see that $R(x) = -\ln[1 - F(x)]$ also has the same distribution. We therefore conclude that the quantities $R(x)$ and $H(x)$ are same in distribution.

Again making the transformation $v_1 = H(x_1)$ and $v_2 = H(x_2)$ in joint density of mth and nth lower records, we have

$$F(x_1) = e^{-v_1} \text{ and } F(x_2) = e^{-v_2}$$

and the Jacobbian of transformation is

$$|J| = \frac{1}{f(x_1)f(x_2)} e^{-v_1} e^{-v_2}.$$

The joint density of v_1 and v_2 is therefore

$$f_{V_1,V_2}(v_1, v_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \frac{f(x_1)}{e^{-v_1}} f(x_2) v_1^{m-1} \\ \times \ (v_2 - v_1)^{n-m-1} \frac{1}{f(x_1)f(x_2)} e^{-v_1} e^{-v_2}$$

or

$$f_{V_1,V_2}(v_1, v_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} v_1^{m-1} (v_2 - v_1)^{n-m-1} e^{-v_2},$$

which is same as joint distribution of $R(x_1)$ and $R(x_2)$ given in (3.6). It can be easily shown that the distribution of the ratio $w = H(x_1)/H(x_2)$ is Beta distribution with parameters m and $n - m$.

Example 5.2 Obtain the distribution of nth lower record, joint distribution of mth and nth lower records and conditional distribution of nth lower record given mth lower record if sequence of random variables follow Inverse Weibull distribution with density

$$f(x) = \beta x^{-(\beta+1)} \exp\left(-x^{-\beta}\right); \ x, \ \beta > 0.$$

Solution: The density function of nth lower record is

$$f_{L_K(n)}(x) = \frac{1}{\Gamma(n)} f(x)[H(x)]^{n-1},$$

where $H(x) = -\ln[F(x)]$. Now for given distribution we have

$$F(x) = \int_0^x f(t)dt = \exp\left(-x^{-\beta}\right),$$

hence

$$H(x) = -\ln[F(x)] = x^{-\beta}.$$

The distribution of nth lower record is therefore

$$f_{L_K(n)}(x) = \frac{1}{\Gamma(n)} \beta x^{-(\beta+1)} \exp\left(-x^{-\beta}\right) \left(x^{-\beta}\right)^{n-1} \\ = \frac{1}{\Gamma(n)} \beta x^{-(\beta n+1)} \exp\left(-x^{-\beta}\right); \ x > 0.$$

Again the joint density of mth and nth lower record is given as

$$f_{X_{L(m)},X_{L(n)}}(x_1,x_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\frac{f(x_1)}{F(x_1)}f(x_2)[H(x_1)]^{m-1}\times[H(x_2)-H(x_1)]^{n-m-1}.$$

For given distribution we have

$$f_{X_{L(m)},X_{L(n)}}(x_1,x_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\frac{\beta x_1^{-(\beta+1)}\exp\left(-x_1^{-\beta}\right)}{\exp\left(-x_1^{-\beta}\right)}\beta x_2^{-(\beta+1)}\times\exp\left(-x_2^{-\beta}\right)\left(x_1^{-\beta}\right)^{m-1}\left(x_2^{-\beta}-x_1^{-\beta}\right)^{n-m-1}$$

or

$$f_{X_{L(m)},X_{L(n)}}(x_1,x_2)=\frac{\beta^2}{\Gamma(m)\Gamma(n-m)}x_1^{-(\beta m+1)}x_2^{-(\beta+1)}\times\left(x_2^{-\beta}-x_1^{-\beta}\right)^{n-m-1}\exp\left(-x_2^{-\beta}\right),$$

$-\infty<x_2<x_1<\infty$. Finally, the conditional distribution of n th lower record given mth lower record is

$$f_{X_{L(n)}|X_{L(m)}=x_1}(x_2|x_1)=\frac{1}{\Gamma(n-m)}\frac{f(x_2)}{F(x_1)}[H(x_2)-H(x_1)]^{n-m-1},$$

which for given distribution is

$$f_{X_{L(n)}|X_{L(m)}=x_1}(x_2|x_1)=\frac{1}{\Gamma(n-m)}\beta x_2^{-(\beta+1)}\left(x_2^{-\beta}-x_1^{-\beta}\right)^{n-m-1}\times\exp\left\{-\left(x_2^{-\beta}-x_1^{-\beta}\right)\right\},$$

$-\infty<x_2<x_1<\infty$.

5.10 Distribution Function of Dual GOS and Its Properties

The distribution function of dual GOS play important role in studying certain properties of dual GOS. Just like the distribution function of GOS the distribution function of dual GOS also has nice properties and are discussed in Burkschat et al. (2003). We give these properties of distribution function of dual GOS in the following.

The density function of rth dual GOS is given in (5.20) as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)].$$

The distribution function of rth dual GOS is

$$\begin{aligned} F_{r(d):n,m,k}(x) &= \int_{-\infty}^{x} f_{r(d):n,m,k}(t)dt \\ &= \int_{-\infty}^{x} \frac{C_{r-1}}{(r-1)!} f(t)\{F(t)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(t)]dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{x} f(t)\{F(t)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(t)]dt. \end{aligned}$$

The distribution functions of dual GOS are nicely related. One of important relation is obtained below. For this we integrate $F_{r(d):n,m,k}(x)$ by parts taking $\{F(t)\}^{\gamma_r - 1}$ for integration to get

$$\begin{aligned} F_{r(d):n,m,k}(x) &= \frac{C_{r-1}}{(r-1)!} g_{m(d)}^{r-1}[F(t)] \frac{\{F(t)\}^{\gamma_r}}{\gamma_r}\bigg|_{-\infty}^{x} + \frac{C_{r-1}}{(r-2)!} \\ &\quad \times \int_{-\infty}^{x} \left\{ \frac{\{F(t)\}^{\gamma_r}}{\gamma_r} g_{m(d)}^{r-2}[F(t)] f(t)\{F(t)\}^{m} \right\} dt \\ &= \frac{C_{r-2}}{(r-1)!} g_{m(d)}^{r-1}[F(x)]\{F(x)\}^{\gamma_r} + \frac{C_{r-2}}{(r-2)!} \\ &\quad \times \int_{-\infty}^{x} f(t)\{F(t)\}^{\gamma_r + m} g_{m(d)}^{r-2}[F(t)]dt \end{aligned}$$

or

$$\begin{aligned} F_{r(d):n,m,k}(x) &= \frac{C_{r-2}}{(r-1)!} g_{m(d)}^{r-1}[F(x)]\{F(x)\}^{\gamma_r - 1} \frac{F(x)}{f(x)} f(x) \\ &\quad + \frac{C_{r-2}}{(r-2)!} \int_{-\infty}^{x} f(t)\{F(t)\}^{\gamma_{r-1} - 1} g_{m(d)}^{r-2}[F(t)]dt \end{aligned}$$

or

$$F_{r(d):n,m,k}(x) = \frac{F(x)}{\gamma_r f(x)} f_{r(d):n,m,k}(x) + F_{r-1(d):n,m,k}(x)$$

or

$$\gamma_r \left\{ F_{r(d):n,m,k}(x) - F_{r-1(d):n,m,k}(x) \right\} = \frac{F(x)}{f(x)} f_{r(d):n,m,k}(x). \tag{5.33}$$

The relation (5.33) is a useful relation which shows that the difference between distribution functions of two contiguous dual GOS is proportional to the density function of a dual GOS.

Burkschat et al. (2003) have shown that the distribution function of dual GOS can be written in the following form

$$F_{r(d):n,m,k}(x) = C_{r-1} \int_0^{F(x)} G_{r,r}^{r,0}\left[y \middle| \begin{matrix} \gamma_1, \dots \gamma_r \\ \gamma_1 - 1, \dots \gamma_r - 1 \end{matrix} \right] dy \tag{5.34}$$

where $G_{p,q}^{m,n}\left(x \middle| \begin{smallmatrix} a_1,\dots,a_p \\ b_1,\dots,b_p \end{smallmatrix} \right)$ is Meijer's G-function.

The distribution function of rth dual GOS can also be presented in the form of Incomplete Beta function ratio as under

$$\begin{aligned} F_{r(d):n,m,k}(x) &= \int_{-\infty}^{x} f_{r(d):n,m,k}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{x} f(t)\{F(t)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(t)]dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{x} f(t)\{F(t)\}^{\gamma_r - 1} \\ &\times \left[\frac{1 - \{F(t)\}^{m+1}}{m+1} \right]^{r-1} dt \end{aligned}$$

Now making the transformation $w = \{F(t)\}^{m+1}$ we have

$$F_{r(d):n,m,k}(x) = \frac{C_{r-1}}{\Gamma(r)(m+1)^r} \int_{-\infty}^{\alpha[F(x)]} w^{\{\gamma_r/(m+1)\}-1}(1-w)^{r-1} dw;$$

where $\alpha^*[F(x)] = \{F(x)\}^{m+1}$. So the distribution function of dual GOS is

$$\begin{aligned} F_{r(d):n,m,k}(x) &= \frac{C_{r-1}}{\Gamma(r)(m+1)^r} B_{\alpha^*[F(x)]}\left(r, \frac{\gamma_r}{m+1} \right) \\ &= \frac{C_{r-1}}{\Gamma(r)(m+1)^r} B\left(r, \frac{\gamma_r}{m+1} \right) I_{\alpha^*[F(x)]}\left(r, \frac{\gamma_r}{m+1} \right); \end{aligned}$$

where $I_x(a,b)$ is incomplete Beta function ratio. The distribution function of dual GOS may further be simplified as

$$F_{X(r:n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)(m+1)^r} \frac{\Gamma(r)\Gamma\left(\frac{\gamma_r}{m+1}\right)}{\Gamma\left(r + \frac{\gamma_r}{m+1}\right)} I_{\alpha^*[F(x)]}\left(r, \frac{\gamma_r}{m+1} \right).$$

Now using the relation

$$\begin{aligned} \Gamma\left(r + \frac{\gamma_r}{m+1} \right) &= \left(\frac{k}{m+1} + n - 1 \right) \cdots \left(\frac{k}{m+1} + n - r \right) \Gamma\left(\frac{\gamma_r}{m+1} \right) \\ &= \frac{C_{r-1}}{(m+1)^r} \Gamma\left(\frac{\gamma_r}{m+1} \right); \end{aligned}$$

we have

$$F_{r(d):n,m,k}(x) = I_{\alpha^*[F(x)]}\left(r, \frac{\gamma_r}{m+1}\right). \tag{5.35}$$

The above relation can be used to obtain distribution function of special cases by using various values of the parameters involved.

5.11 Moments of Dual GOS

The moments of dual GOS are computed by using the density function of r th dual GOS. We know that the density function of rth dual GOS is given as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)].$$

The expected value of rth dual GOS; $X_{r(d):n,m,k}$; is defined as

$$\begin{aligned} \mu_{r:n,m,k} = E\left(X_{r(d):n,m,k}\right) &= \int_{-\infty}^{\infty} x f_{r(d):n,m,k}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} x f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)]dx. \end{aligned} \tag{5.36}$$

The expected value of some function of rth dual GOS is

$$\begin{aligned} E\left[t\left(X_{r(d):n,m,k}\right)\right] &= \int_{-\infty}^{\infty} t(x) f_{r:n,m,k}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} t(x) f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)]dx. \end{aligned}$$

Again the pth raw moment of rth dual GOS; $\mu^{p}_{r(d):n,m,k}$; is computed as

$$\begin{aligned} \mu^{p}_{r(d):n,m,k} = E\left(X^{p}_{r(d):n,m,k}\right) &= \int_{-\infty}^{\infty} x^{p} f_{r(d):n,m,k}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} x^{p} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)]dx; \end{aligned} \tag{5.37}$$

Using the probability integral transformation, the pth moment of rth dual GOS can also be written as

$$\begin{aligned} \mu^{p}_{r(d):n,m,k} = E\left(X^{p}_{r(d):n,m,k}\right) &= \frac{C_{r-1}}{(r-1)!} \int_{0}^{1} \left\{F^{-1}(t)\right\}^{p} \varphi_{r(d):n}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!} \int_{0}^{1} \left\{F^{-1}(t)\right\}^{p} t^{\gamma_r - 1} g_{m(d)}^{r-1}(t)dt; \end{aligned} \tag{5.38}$$

where $t = F(x), x = F^{-1}(t)$ is the inverse function and $\varphi_{r(d):n}(x)$ is density function of uniform Dual GOS.

The joint density of rth and sth dual GOS provide basis for computation of product moments of two GOS. The joint density of rth and sth dual GOS is given as

$$f_{r(d),s(d):n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2) \{F(x_1)\}^m \times g_{m(d)}^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s - 1} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}.$$

The (p, q)th raw moment of rth and sth dual GOS is readily written as

$$\begin{aligned} \mu_{r(d),s(d):n,m,k}^{p,q} &= E\left(X_{r(d):n,m,k}^{p} X_{s(d):n,m,k}^{q}\right) \\ &= \int_{-\infty}^{\infty} \int_{x_2}^{\infty} x_1^p x_2^q f_{r(d),s(d):n,m,k}(x_1, x_2) dx_1 dx_2 \\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} \int_{x_2}^{\infty} x_1^p x_2^q f(x_1) f(x_2) \{F(x_1)\}^m \\ &\quad \times g_{m(d)}^{r-1}\{F(x_1)\}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\quad \times \{F(x_2)\}^{\gamma_s - 1} dx_1 dx_2. \end{aligned} \tag{5.39}$$

The (p, q)th central moment of rth and sth dual GOS is given as

$$\sigma_{r(d),s(d):n,m,k}^{p,q} = E\left[\left\{X_{r(d):n,m,k} - \mu_{r(d):n,m,k}\right\}^p \left\{X_{s(d):n,m,k} - \mu_{s(d):n,m,k}\right\}^q\right].$$

The covariance between rth and sth dual GOS is readily computed from above as

$$\sigma_{r(d),s(d):n,m,k} = E\left[\left\{X_{r(d):n,m,k} - \mu_{r(d):n,m,k}\right\}\left\{X_{s(d):n,m,k} - \mu_{s(d):n,m,k}\right\}\right].$$

The correlation coefficient can also be computed easily.

Example 5.3 A random sample is drawn from standard Inverse Rayleigh distribution with density function

$$f(x) = \frac{2}{x^3} \exp\left(-x^{-2}\right); x > 0.$$

Obtain expression for single and product moments of dual GOS for this distribution.

Solution: The distribution of rth dual GOS is

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)].$$

For given distribution we have

$$F(x) = \int_0^x f(t)dt = \int_0^x \frac{2}{t^3} \exp\left(-t^{-2}\right)dt = \exp\left(-x^{-2}\right), x > 0.$$

Now we have

$$\begin{aligned} g_{m(d)}^{r-1}[F(x)] &= \left(\frac{1}{m+1}\left[1 - \{F(x)\}^{m+1}\right]\right)^{r-1} \\ &= \frac{\left[1 - \exp\left\{-(m+1)x^{-2}\right\}\right]^{r-1}}{(m+1)^{r-1}}. \end{aligned}$$

The distribution of $X_{r(d):n,m,k}$ is therefore

$$\begin{aligned} f_{r(d):n,m,k}(x) &= \frac{C_{r-1}}{(r-1)!} \frac{2}{x^3} \exp\left(-\gamma_r x^{-2}\right) \\ &\quad \times \frac{\left[1 - \exp\left\{-(m+1)x^{-2}\right\}\right]^{r-1}}{(m+1)^{r-1}} \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{2}{x^3} \exp\left(-\gamma_r x^{-2}\right) \\ &\quad \times \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \exp\left\{-(m+1)jx^{-2}\right\} \end{aligned}$$

or

$$\begin{aligned} f_{r(d):n,m,k}(x) &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \\ &\quad \times \frac{2}{x^3} \exp\left[-\{(m+1)j + \gamma_r\}x^{-2}\right]. \end{aligned}$$

The pth moment of $X_{r(d):n,m,k}$ is therefore

$$\mu^{p}_{r(d):n,m,k} = E\left(X^{p}_{r(d):n,m,k}\right) = \int_{-\infty}^{\infty} x^{p} f_{r(d):n,m,k}(x)dx$$
$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j}$$
$$\times \int_{0}^{\infty} 2x^{p-3} \exp\left[-\{(m+1)j+\gamma_r\}x^{-2}\right]$$
$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j}$$
$$\times \{(m+1)i+\gamma_r\}^{p/2-1} \Gamma\left(1-\frac{p}{2}\right).$$

The Mean of $X_{r(d):n,m,k}$ is

$$\mu_{r(d):n,m,k} = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{j}$$
$$\times \sqrt{\frac{\pi}{\{(m+1)i+\gamma_r\}}}.$$

Again the joint density of $X_{r(d):n,m,k}$ and $X_{s(d):n,m,k}$ is

$$f_{r(d),s(d):n,m,k}(x_1,x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2) \{F(x_1)\}^{m}$$
$$\times g^{r-1}_{m(d)}\{F(x_1)\}\{F(x_2)\}^{\gamma_s-1}$$
$$\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}.$$

Now for given distribution we have

$$f_{r(d),s(d):n,m,k}(x_1,x_2) = \frac{4C_{s-1}}{(r-1)!(s-r-1)!} \frac{1}{x_1^3 x_2^3} e^{-x_1^{-2}} e^{-x_2^{-2}} e^{-mx_1^{-2}}$$
$$\times \left[\frac{1}{m+1}\left\{1-e^{-(m+1)x_1^{-2}}\right\}\right]^{r-1}$$
$$\times e^{-(\gamma_s-1)x_2^{-2}} \left[\frac{e^{-(m+1)x_1^{-2}}}{m+1} - \frac{e^{-(m+1)x_2^{-2}}}{m+1}\right]^{s-r-1}$$

or

$$f_{r(d),s(d):n,m,k}(x_1,x_2)=\frac{4C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\\ \times\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\\ \times\frac{1}{x_1^3x_2^3}e^{-(m+1)(s-r-j+i)x_1^{-2}}e^{-\left[(m+1)j+\gamma_s\right]x_2^{-2}}.$$

The product moments of order (p,q) are therefore

$$\mu_{r(d),s(d):n,m,k}^{p,q}=E\left(X_{r(d):n,m,k}^{p}X_{s(d):n,m,k}^{q}\right)\\ =\int_{-\infty}^{\infty}\int_{x_2}^{\infty}x_1^px_2^qf_{r(d),s(d):n,m,k}(x_1,x_2)dx_1dx_2\\ =\frac{4C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\\ \times\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\int_{-\infty}^{\infty}\int_{x_2}^{\infty}x_1^px_2^q\\ \times\frac{1}{x_1^3x_2^3}e^{-(m+1)(s-r-j+i)x_1^{-2}}e^{-\left[(m+1)j+\gamma_s\right]x_2^{-2}}dx_2dx_1$$

or

$$\mu_{r,s:n,m,k}^{p,q}=\frac{4C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\\ \times\sum_{j=0}^{s-r-1}\sum_{i=0}^{r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\\ \times\frac{1}{(1-p/2)\{(m+1)j+\gamma_s\}^{2-(p+q)/2}}\Gamma\left(2-\frac{p+q}{2}\right)\\ \times{}_2F_1\left[1-\frac{q}{2},2-\frac{p+q}{2};2-\frac{q}{2};\frac{(m+1)j+\gamma_s}{(s-r-j+i)(m+1)}\right].$$

The Covariance can be obtained by using above results.

Example 5.4 A random sample is drawn from the distribution

$$f(x)=\frac{a}{c^a}x^{a-1};\ 0<x<c;\ a,c>0.$$

Show that the pth moment of rth dual GOS for this distribution is given as

$$\mu^p_{r(d):n,m,k} = E\left(X^p_{r(d):n,m,k}\right) = c^p \frac{C_{r-1}(k)}{C_{r-1}(k+p/a)};$$

where notations have their usual meanings.

Solution: The pth moment of rth dual GOS is given as

$$\begin{aligned}\mu^p_{r(d):n,m,k} &= E\left(X^p_{r(d):n,m,k}\right) = \int_{-\infty}^{\infty} x^p f_{r(d):n,m,k}(x)dx \\ &= \int_{-\infty}^{\infty} x^p \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r-1} g^{r-1}_{m(d)}[F(x)]dx.\end{aligned}$$

Using the probability integral transform, the pth moment is given as

$$\begin{aligned}\mu^p_{r(d):n,m,k} &= E\left(X^p_{r(d):n,m,k}\right) = \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p \varphi_{r(d):n,m,k}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p t^{\gamma_r-1} g^{r-1}_{m(d)}(t)dt.\end{aligned}$$

Now for given distribution we have

$$f(x) = \frac{a}{c^a}x^{a-1};\ 0 < x < c.$$

So

$$\begin{aligned}F(x) &= \int_0^x f(t)dt = \int_0^x \frac{a}{c^a}x^{a-1}dt \\ &= \left(\frac{x}{c}\right)^a;\ 0 < x < c.\end{aligned}$$

Also by using $t = F(x)$ we have

$$t = \frac{x^a}{c^a} \Longrightarrow x = F^{-1}(t) = ct^{1/a}.$$

Hence the pth moment of rth dual GOS is:

$$\begin{aligned}\mu^p_{r(d):n,m,k} &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^p t^{\gamma_r-1} g^{r-1}_{m(d)}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!}\int_0^1 \left(ct^{1/a}\right)^p t^{\gamma_r-1} g^{r-1}_{m(d)}(t)dt \\ &= \frac{C_{r-1}}{(r-1)!}c^p \int_0^1 t^{(k+p/a)+(n-r)(m+1)} g^{r-1}_{m(d)}(t)dt\end{aligned}$$

or

$$\mu^{p}_{r:n,m,k} = c^{p}\frac{C_{r-1}(k)}{C_{r-1}(k+p/a)}\frac{C_{r-1}(k+p/a)}{(r-1)!} \times \int_0^1 t^{(k+p/a)+(n-r)(m+1)} g^{r-1}_{m(d)}(t)dt = c^{p}\frac{C_{r-1}(k)}{C_{r-1}(k+p/a)};$$

as

$$\frac{C_{r-1}(k+p/a)}{(r-1)!}\int_0^1 t^{(k+p/a)+(n-r)(m+1)} g^{r-1}_{m(d)}(t)dt = 1,$$

as required.

5.12 Recurrence Relations for Moments of Dual GOS

The moments of Dual GOS are discussed in the previous section. For certain distributions the moments can be expressed in a simple and compact way but for others the expressions are sometime too much complex and hence computing the moments for real data is tedious. The problem can be overcomed by obtaining recurrence relations between moments of Dual GOS and hence these recurrence relations can be used to compute the higher order moments from the lower order moments. We discuss these recurrence relations between moments of Dual GOS in the following.

The density function of rth Dual GOS is given as

$$f_{r(d):n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g^{r-1}_{m(d)}[F(x)].$$

We can establish two types of recurrence relations between moments of Dual GOS, one which are distribution free and one which are distribution specific. The distribution free relationships are more general in their applications. The distribution free relationships between moments of Dual GOS can be readily established by noting following relationships between density functions of Dual GOS

$$\gamma_{r+1} f_{r(d):n,m,k}(x) + r(m+1) f_{r+1(d):n,m,k}(x) = \gamma_1 f_{r(d):n-1,m,k}(x) \quad \text{(i)}$$

$$\gamma_{r+1}\left\{f_{r+1(d):n,m,k}(x) - f_{r(d):n,m,k}(x)\right\} = \gamma_1\left\{f_{r+1(d):n,m,k}(x) - f_{r(d):n-1,m,k}(x)\right\} \quad \text{(ii)}$$

$$\gamma_1\left\{f_{r(d):n,m,k}(x) - f_{r(d):n-1,m,k}(x)\right\} = r(m+1)\left\{f_{r(d):n,m,k}(x) - f_{r+1(d):n,m,k}(x)\right\} \quad \text{(iii)}$$

Using above relations we can immediately write following relations between moments of dual GOS

$$\gamma_{r+1}\mu^{p}_{r(d):n,m,k} + r(m+1)\mu^{p}_{r+1(d):n,m,k} = \gamma_1\mu^{p}_{r(d):n-1,m,k} \tag{i}$$

$$\gamma_{r+1}\left\{\mu^{p}_{r+1(d):n,m,k} - \mu^{p}_{r(d):n,m,k}\right\} = \gamma_1\left\{\mu^{p}_{r+1(d):n,m,k} - \mu^{p}_{r(d):n-1,m,k}\right\} \tag{ii}$$

$$\gamma_1\left\{\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r(d):n-1,m,k}\right\} = r(m+1)\left\{\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r+1(d):n,m,k}\right\}. \tag{iii}$$

The second relation can be alternatively written as

$$r(m+1)\mu^{p}_{r+1(d):n,m,k} = \gamma_1\mu^{p}_{r(d):n-1,m,k} - \gamma_{r+1}\mu^{p}_{r(d):n,m,k}.$$

We can see that these relations are exactly same as the relations between moments of GOS and hence these two models of ordered random variables are same in expectations.

Distribution specific relationships between single and product moments of Dual GOS have been studied by various authors. The distribution specific recurrence relations between moments of Dual GOS are easily obtained by using a general result given by Khan et al. (2009). We have given the result in following theorem.

Theorem 5.1 *Suppose a sequence of random variables $\{X_n;\, n \geq 1\}$ is available from an absolutely continuous distribution function $F(x)$. Suppose further that $X_{r(d):n,m,k}$ be rth Dual GOS of the sequence then following recurrence relation hold between moments of the Dual GOS*

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx; \tag{5.40}$$

and

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -&\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1}\\ &\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}\\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}\\ &\times \{F(x_2)\}^{\gamma_s} dx_2 dx_1.\end{aligned} \tag{5.41}$$

Proof We have

$$\begin{aligned}\mu^{p}_{r(d):m,n,k} &= E\left(X^{p}_{r(d):n,m,k}\right) = \int_{-\infty}^{\infty} x^p f_{r(d):n,m,k}(x)dx\\ &= \int_{-\infty}^{\infty} x^p \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r-1} g^{r-1}_{m(d)}[F(x)]dx\\ &= \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} x^p f(x)\{F(x)\}^{\gamma_r-1} g^{r-1}_{m(d)}[F(x)]dx.\end{aligned}$$

Integrating above equation by parts taking $f(x)\{F(x)\}^{\gamma_r - 1}$ as function for integration we have

$$\mu^p_{r:n,m,k} = \frac{C_{r-1}}{(r-1)!}\Bigg[x^p g^{r-1}_{m(d)}\{F(x)\}\frac{\{F(x)\}^{\gamma_r}}{\gamma_r}\Bigg|^{\infty}_{-\infty}$$
$$- \int_{-\infty}^{\infty}\Big\{px^{p-1}g^{r-1}_{m(d)}[F(x)] - (r-1)x^p g^{r-2}_{m(d)}[F(x)]$$
$$\times [F(x)]^m f(x)\Big\}\frac{\{F(x)\}^{\gamma_r}}{\gamma_r}dx\Bigg]$$
$$= -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx$$
$$+ \frac{(r-1)C_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^p f(x)\{F(x)\}^{\gamma_r + m} g^{r-2}_{m(d)}[F(x)]dx$$
$$= -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx$$
$$+ \frac{C_{r-2}}{(r-2)!}\int_{-\infty}^{\infty} x^p f(x)\{F(x)\}^{\gamma_{r-1}-1} g^{r-2}_{m(d)}[F(x)]dx$$

Since

$$\mu^p_{r-1(d):n,m,k} = \frac{C_{r-2}}{(r-2)!}\int_{-\infty}^{\infty} x^p f(x)\{F(x)\}^{\gamma_{r-1}-1} g^{r-2}_{m(d)}[F(x)]dx,$$

hence above equation can be written as

$$\mu^p_{r(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx + \mu^p_{r-1(d):n,m,k},$$

or

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx,$$

as required. We can readily see; from (5.40); that for $p = 1$ following recurrence relationship exist between expectations of Dual GOS

$$\mu_{r(d):n,m,k} - \mu_{r-1(d):n,m,k} = -\frac{C_{r-1}}{\gamma_r (r-1)!}\int_{-\infty}^{\infty} \{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx. \qquad (5.42)$$

We also have an alternative representation for recurrence relation between single moments of GOS based upon probability integral transform of (5.40) as under

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r (r-1)!}\int_0^1 \left\{F^{-1}(t)\right\}^{p-1}\left\{F^{-1}(t)\right\}^{/}$$
$$\times\; t^{\gamma_r} g^{r-1}_{m(d)}(t)dt. \qquad (5.43)$$

The representation (5.43) is very useful in deriving relations for specific distributions.

Again consider

$$\mu^{p,q}_{r(d),s(d):n,m,k} = E\left(X^p_{r(d):n,m,k}X^q_{s(d):n,m,k}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f_{r(d),s(d):n,m,k}(x_1,x_2)dx_2dx_1$$
$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q f(x_1)f(x_2)\{F(x_1)\}^m$$
$$\times\ g^{r-1}_{m(d)}\{F(x_1)\}\left[h_{m(d)}\{F(x_1)\} - h_m\{F(x_2)\}\right]^{s-r-1}$$
$$\times\ \{1-F(x_2)\}^{\gamma_s-1}dx_2dx_1$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty} x_1^p f(x_1)\{F(x_1)\}^m$$
$$\times\ g^{r-1}_{m(d)}\{F(x_1)\}I(x_2)dx_1; \tag{5.44}$$

where

$$I(x_2) = \int_{-\infty}^{x_1} x_2^q f(x_2)\{F(x_2)\}^{\gamma_s-1}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}dx_2.$$

Integrating above integral by parts using $f(x_2)\{F(x_2)\}^{\gamma_s-1}$ for integration we have

$$I(x_2) = x_2^q\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}\frac{\{F(x_2)\}^{\gamma_s}}{\gamma_s}\Bigg|_{-\infty}^{x_1}$$
$$-\frac{1}{\gamma_s}\int_{-\infty}^{x_1}\Big[qx_2^{q-1}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$-(s-r-1)x_2^q\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-2}$$
$$\times\{1-F(x_2)\}^m f(x_2)\Big]\{F(x_2)\}^{\gamma_s}dx_2$$

or

$$I(x_2) = -\frac{q}{\gamma_s}\int_{x_1}^{\infty} x_2^{q-1}\left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$\times\ \{F(x_2)\}^{\gamma_s}dx_2 + \frac{(s-r-1)}{\gamma_s}\int_{-\infty}^{x_1} x_2^q f(x_2)$$
$$\times\ \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-2}\{F(x_2)\}^{\gamma_s+m}dx_2. \tag{5.45}$$

Now using the value of $I(x_2)$ from (5.45) in (5.44) we have

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty} x_1^p f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times \Bigg[-\frac{q}{\gamma_s}\int_{-\infty}^{x_1} x_2^{q-1}\big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-1}$$
$$\times \ \{F(x_2)\}^{\gamma_s} dx_2 + \frac{(s-r-1)}{\gamma_s}\int_{-\infty}^{x_1} x_2^q f(x_2)\{F(x_2)\}^{\gamma_s+m}$$
$$\times \big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-2} dx_2\Bigg] dx_1$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\{F(x_1)\}^m$$
$$\times \ g^{r-1}_{m(d)}\{F(x_1)\}\big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-1}$$
$$\times \ \{F(x_2)\}^{\gamma_s} dx_2 dx_1 + \frac{(s-r-1)C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^q f(x_1) f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times \ \big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-2}\{F(x_2)\}^{\gamma_s+m} dx_2 dx_1$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\{F(x_1)\}^m$$
$$\times \ g^{r-1}_{m(d)}\{F(x_1)\}\big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-1}$$
$$\times \ \{F(x_2)\}^{\gamma_s} dx_2 dx_1 + \frac{C_{s-2}}{(r-1)!(s-r-2)!}\int_{-\infty}^{\infty}\int_{x_1}^{\infty} x_1^p x_2^q$$
$$\times \ f(x_1) f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}\{F(x_2)\}^{\gamma_{s-1}-1}$$
$$\times \ \big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-2} dx_2 dx_1$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} f(x_1)\{F(x_1)\}^m$$
$$\times \ g^{r-1}_{m(d)}\{F(x_1)\}\big[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\big]^{s-r-1}$$
$$\times \ \{F(x_2)\}^{\gamma_s} dx_2 dx_1 + \mu^{p,q}_{r(d),s-1(d):n,m,k};$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1}$$
$$\times\ f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times\ \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$\times\ \{F(x_2)\}^{\gamma_s} dx_2 dx_1,$$

as required.

We now present recurrence relations for single and product moments of GOS for some special distributions.

5.12.1 Reflected Exponential Distribution

The density and distribution function of Reflected Exponential random variable are given as

$$f(x) = \alpha e^{\alpha x};\ x < 0, \alpha > 0$$

and

$$F(x) = e^{\alpha x}.$$

We note that

$$f(x) = \alpha F(x). \tag{5.46}$$

Consider (5.40)

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx$$
$$= -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r-1}$$
$$\times\ F(x) g^{r-1}_{m(d)}[F(x)]dx$$

Using (5.46) in (5.40) following recurrence relation between single moments of Dual GOS has been obtained by Ahsanullah (2000)

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1} f(x)\{F(x)\}^{\gamma_r-1}$$
$$\times\ g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{p}{\alpha\gamma_r}\mu^{p-1}_{r(d):n,m,k},$$

or

$$\mu^{p}_{r(d):n,m,k} = \mu^{p}_{r-1(d):n,m,k} - \frac{p}{\alpha\gamma_r}\mu^{p-1}_{r(d):n,m,k} \tag{5.47}$$

The relationship (5.47) reduces to corresponding relationship for reversed order statistics for $m = 0$ and $k = 1$ and for lower record values for $m = -1$. Again consider (5.41) as

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s} dx_2 dx_1\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s-1} F(x_2) dx_2 dx_1.\end{aligned}$$

Now using (5.46) in above equation we have

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\alpha\gamma_s(r-1)!(s-r-1)!}&\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times f(x_1)f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s-1} dx_2 dx_1.\end{aligned}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r(d),s(d):n,m,k}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \mu^{p,q}_{r(d),s-1(d):n,m,k} - \frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r(d),s(d):n,m,k}. \tag{5.48}$$

Using $s = r + 1$ in (5.48) we have following relation between product moments of two contiguous Dual GOS

$$\mu^{p,q}_{r(d),r+1(d):n,m,k} = \mu^{p+q}_{r(d):n,m,k} - \frac{q}{\alpha\gamma_s}\mu^{p,q-1}_{r(d),r+1(d):n,m,k}.$$

The recurrence relationship given in (5.48) reduces to relationship for reversed order statistics for $m = 0$ and $k = 1$. Further, for $m = -1$, the relationship (5.48) reduces to recurrence relation for product moment of lower record values.

5.12.2 *The Inverse Rayleigh Distribution*

The density and distribution function of Inverse Rayleigh distribution are

$$f(x) = \frac{2\alpha}{x^3}\exp\left(-\frac{\alpha}{x^2}\right);\ \alpha, x > 0;$$

and

$$F(x) = \exp\left(-\frac{\alpha}{x^2}\right).$$

The density and distribution function are related as

$$f(x) = \frac{2\alpha}{x^3}F(x). \tag{5.49}$$

Using (5.49) in (5.40) the recurrence relation for moments of single Dual GOS can be derived as under. Consider (5.40) as

$$\begin{aligned}\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} &= -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r} g^{r-1}_{m(d)}[F(x)]dx \\ &= -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r-1} \\ &\quad\times F(x) g^{r-1}_{m(d)}[F(x)]dx\end{aligned}$$

Using (5.49) in above equation we have

$$\begin{aligned}\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r-1} \\ \times \frac{x^3}{2\alpha} f(x) g^{r-1}_{m(d)}[F(x)]dx\end{aligned}$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{2\alpha\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p+2} f(x) \times \{F(x)\}^{\gamma_r-1} g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{p}{2\alpha\gamma_r}\mu^{p+2}_{r(d):n,m,k}$$

or

$$\mu^{p}_{r(d):n,m,k} = \mu^{p}_{r-1(d):n,m,k} - \frac{p}{2\alpha\gamma_r}\mu^{p+2}_{r(d):n,m,k}. \tag{5.50}$$

Again consider (5.41) as

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \times \{F(x_2)\}^{\gamma_s} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \times \{F(x_2)\}^{\gamma_s-1} F(x_2) dx_2 dx_1.$$

Now using (5.49) in above equation we have

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{2\alpha\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q+2} \times f(x_1)f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \times \{F(x_2)\}^{\gamma_s-1} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{q}{2\alpha\gamma_s}\mu^{p,q+2}_{r(d),s(d):n,m,k}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \mu^{p,q}_{r(d),s-1(d):n,m,k} - \frac{q}{2\alpha\gamma_s}\mu^{p,q+2}_{r(d),s(d):n,m,k}. \tag{5.51}$$

We can obtain the recurrence relation for moments for special cases by using specific values of parameters in (5.50) and (5.51). For example the recurrence relation for single moments of lower order statistics can be obtained by using $m = -1$ and are given as

$$\mu^{p}_{r(d):n,-1,k} = \mu^{p}_{r-1(d):n,-1,k} - \frac{p}{2\alpha k}\mu^{p+2}_{r(d):n,-1,k}$$

and

$$\mu^{p,q}_{r(d),s(d):n,-1,k} = \mu^{p,q}_{r(d),s-1(d):n,-1,k} - \frac{q}{2\alpha\gamma_s}\mu^{p,q+2}_{r(d),s(d):n,-1,k}.$$

Recurrence relations for other special cases can be readily written.

5.12.3 The Inverse Weibull Distribution

The density and distribution function for Inverse Weibull distribution

$$f(x) = \frac{\alpha\beta}{x^{\beta+1}}\exp\left(-\frac{\alpha}{x^{\beta}}\right); x, \alpha, \beta > 0;$$

and

$$F(x) = \exp\left(-\frac{\alpha}{x^{\beta}}\right).$$

We also have

$$f(x) = \frac{\alpha\beta}{x^{\beta+1}}F(x). \tag{5.52}$$

Pawlas and Szynal (2001) derived the recurrence relations for single and product moments of Dual GOS for Inverse Weibull distribution. These relations are given below.

Using (5.52) in (5.40) we have

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r-1} \times \frac{x^{\beta+1}}{\alpha\beta}f(x)g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_{-\infty}^{\infty} x^{p+\beta}f(x) \times \{F(x)\}^{\gamma_r-1}g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{p}{2\alpha\gamma_r}\mu^{p+\beta}_{r(d):n,m,k}$$

or

$$\mu^{p}_{r(d):n,m,k} = \mu^{p}_{r-1(d):n,m,k} - \frac{p}{2\alpha\gamma_r}\mu^{p+\beta}_{r(d):n,m,k}. \tag{5.53}$$

Again consider (5.41) as

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ \times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ \times \{F(x_2)\}^{\gamma_s} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ \times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ \times \{F(x_2)\}^{\gamma_s-1} F(x_2) dx_2 dx_1.$$

Now using (5.52) in above equation we have

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\alpha\beta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q+\beta} \\ \times f(x_1)f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ \times \{F(x_2)\}^{\gamma_s-1} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{q}{\alpha\beta\gamma_s}\mu^{p,q+\beta}_{r(d),s(d):n,m,k}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \mu^{p,q}_{r(d),s-1(d):n,m,k} - \frac{q}{\alpha\beta\gamma_s}\mu^{p,q+\beta}_{r(d),s(d):n,m,k}. \tag{5.54}$$

The recurrence relations (5.53) and (5.54) reduces to relations (5.50) and (5.51) for $\beta = 2$ as expected. Further, by using $s = r + 1$, the recurrence relation between product moments of two contiguous Dual GOS for Inverse Weibull distribution is obtained as

$$\mu^{p,q}_{r(d),r+1(d):n,m,k} = \mu^{p+q}_{r(d):n,m,k} - \frac{q}{\alpha\beta\gamma_s}\mu^{p,q+\beta}_{r(d),r+1(d):n,m,k}.$$

The recurrence relations for reversed Order Statistics and k-Lower Record Values can be readily obtained for ($m = 0$; $k = 1$) and $m = -1$.

5.12.4 The Power Function Distribution

The density and distribution function of Power Function distribution are

$$f(x) = \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha};\ 0 < x < \theta,\ \alpha > -1$$

and

$$F(x) = \left(\frac{x}{\theta}\right)^{\alpha+1};\ 0 < x < \theta.$$

The density and distribution function are related as

$$f(x) = \frac{\alpha+1}{x} F(x). \tag{5.55}$$

Athar and Faizan (2011) have derived the recurrence relations for single and product moments of Dual GOS by using (5.55) in (5.40) and (5.41). We have given these relations in the following.

Using (5.55) in (5.40) we have

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r (r-1)!} \int_{-\infty}^{\infty} x^{p-1} \{F(x)\}^{\gamma_r - 1} \times \frac{x}{\alpha+1} f(x) g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{(\alpha+1)\gamma_r (r-1)!} \int_{-\infty}^{\infty} x^{p} f(x) \times \{F(x)\}^{\gamma_r - 1} g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{p}{(\alpha+1)\gamma_r} \mu^{p}_{r(d):n,m,k}$$

or

$$\mu^{p}_{r(d):n,m,k} \left\{ \frac{(\alpha+1)\gamma_r + p}{(\alpha+1)\gamma_r} \right\} = \mu^{p}_{r-1(d):n,m,k}$$

or

$$\mu^{p}_{r(d):n,m,k} = \left\{ \frac{(\alpha+1)\gamma_r}{(\alpha+1)\gamma_r + p} \right\} \mu^{p}_{r-1(d):n,m,k}. \tag{5.56}$$

Again consider (5.41) as

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1}$$
$$\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$\times \{F(x_2)\}^{\gamma_s} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1}$$
$$\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$\times \{F(x_2)\}^{\gamma_s-1} F(x_2) dx_2 dx_1.$$

Using (5.55) in above equation we have

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{(\alpha+1)\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q}$$
$$\times f(x_1)f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\}$$
$$\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1}$$
$$\times \{F(x_2)\}^{\gamma_s-1} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{q}{(\alpha+1)\gamma_s}\mu^{p,q}_{r(d),s(d):n,m,k}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k}\left\{\frac{(\alpha+1)\gamma_s+q}{(\alpha+1)\gamma_s}\right\} = \mu^{p,q}_{r(d),s-1(d):n,m,k}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \left\{\frac{(\alpha+1)\gamma_s}{(\alpha+1)\gamma_s+q}\right\}\mu^{p,q}_{r(d),s-1(d):n,m,k}. \tag{5.57}$$

The recurrence relations for special cases can be readily obtained from (5.56) and (5.57).

5.12.5 *The General Class of Inverted Distributions*

The Inverted Rayleigh and Inverted Weibull distributions belong to a more general class of inverted distributions. The density function of this inverted class is

$$f(x) = \frac{\theta\lambda'(x)}{\lambda^2(x)} \exp\left\{-\frac{\theta}{\lambda(x)}\right\}; \ \alpha < x < \beta,$$

where α and β are suitable numbers and $\lambda(x)$ is a non-negative, strictly increasing and differentiable function of x such that: $\lambda(x) \to 0$ and $x \to \alpha$ and $\lambda(x) \to \infty$ and $x \to \beta$. The distribution function corresponding to above density is

$$F(x) = \exp\left\{-\frac{\theta}{\lambda(x)}\right\}; \ \alpha < x < \beta.$$

The Inverted Rayleigh and Inverted Weibull distributions appear as special case of the inverted class for $\lambda(x) = x^2$ and $\lambda(x) = x^\beta$ respectively. The density and distribution functions are related as

$$f(x) = \frac{\theta\lambda'(x)}{\lambda^2(x)} F(x). \tag{5.58}$$

Using the relation (5.58), Kotb et al. (2013) have derived the recurrence relations for single and product moments of Dual GOS for the general class of Inverted distributions . These recurrence relations are given in the following.

Using (5.58) in (5.40) we have

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\gamma_r(r-1)!} \int_{-\infty}^{\infty} x^{p-1}\{F(x)\}^{\gamma_r - 1} \times \frac{\lambda^2(x)}{\theta\lambda'(x)} f(x) g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\theta\gamma_r(r-1)!} \int_{-\infty}^{\infty} \left\{x^{p-1}\frac{\lambda^2(x)}{\lambda'(x)}\right\} f(x) \times \{F(x)\}^{\gamma_r - 1} g^{r-1}_{m(d)}[F(x)]dx$$

or

$$\mu^p_{r(d):n,m,k} - \mu^p_{r-1(d):n,m,k} = -\frac{pC_{r-1}}{\theta\gamma_r(r-1)!} \int_{-\infty}^{\infty} \phi(x) f(x) \times \{F(x)\}^{\gamma_r - 1} g^{r-1}_{m(d)}[F(x)]dx,$$

where $\phi(x) = \left\{x^{p-1}\frac{\lambda^2(x)}{\lambda'(x)}\right\}$. Above relation can be written as

$$\mu^{p}_{r(d):n,m,k} - \mu^{p}_{r-1(d):n,m,k} = -\frac{p}{\theta\gamma_r}\mu^{\phi(x)}_{r(d):n,m,k}$$

or

$$\mu^{p}_{r(d):n,m,k} = \mu^{p}_{r-1(d):n,m,k} - \frac{p}{\theta\gamma_r}\mu^{\phi(x)}_{r(d):n,m,k}. \tag{5.59}$$

The recurrence relation (5.59) reduces to (5.50) for $\lambda(x) = x^2$ and it reduces to (5.53) for $\lambda(x) = x^{\beta}$ as expected.

Again consider (5.41) as

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = &-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s} dx_2 dx_1.\end{aligned}$$

or

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = &-\frac{qC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times f(x_1)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s-1} F(x_2) dx_2 dx_1.\end{aligned}$$

Now using (5.58) in above equation we have

$$\begin{aligned}\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = &-\frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1} x_1^p x_2^{q-1} \\ &\times \frac{\lambda^2(x_2)}{\lambda'(x_2)} f(x_1) f(x_2)\{F(x_1)\}^m g^{r-1}_{m(d)}\{F(x_1)\} \\ &\times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} \\ &\times \{F(x_2)\}^{\gamma_s-1} dx_2 dx_1.\end{aligned}$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!} \times \int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\left\{x_1^p x_2^{q-1}\frac{\lambda^2(x_2)}{\lambda'(x_2)}\right\} f(x_1)f(x_2) \times \{F(x_1)\}^m g_{m(d)}^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s-1} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} dx_2 dx_1.$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{qC_{s-1}}{\theta\gamma_s(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}\phi(x_1,x_2)f(x_1) \times f(x_2)\{F(x_1)\}^m g_{m(d)}^{r-1}\{F(x_1)\}\{F(x_2)\}^{\gamma_s-1} \times \left[h_{m(d)}\{F(x_1)\} - h_{m(d)}\{F(x_2)\}\right]^{s-r-1} dx_2 dx_1.$$

where $\phi(x_1, x_2) = \left\{x_1^p x_2^{q-1}\frac{\lambda^2(x_2)}{\lambda'(x_2)}\right\}$. Above recurrence relation can be written as

$$\mu^{p,q}_{r(d),s(d):n,m,k} - \mu^{p,q}_{r(d),s-1(d):n,m,k} = -\frac{q}{\theta\gamma_s}\mu^{\phi(x_1,x_2)}_{r(d),s(d):n,m,k},$$

or

$$\mu^{p,q}_{r(d),s(d):n,m,k} = \mu^{p,q}_{r(d),s-1(d):n,m,k} - \frac{q}{\theta\gamma_s}\mu^{\phi(x_1,x_2)}_{r(d),s(d):n,m,k}. \tag{5.60}$$

Recurrence relation (5.60) reduces to (5.51) for $\lambda(x) = x^2$ and to (5.54) for $\lambda(x) = x^\beta$ as the case should be.

5.13 Relationship Between GOS and Dual GOS

The GOS and Dual GOS present two models of ordered random variables. Since the models represent random variables arranged in ascending and descending order respectively, there is essentially some relationships between these two models and hence between special cases. Arnold et al. (2008) has shown that ordinary order statistics from some distribution function $F(x)$ are related as

$$X_{r:n} \sim -X_{n-r+1:n}\ ;\ 1 \le r \le n.$$

Arnold et al. (1998) have also given following relation between upper and lower records of a sequence of random variables

$$X_{U(n)} \sim -X_{L(n)}\ ;\ n \in \mathbb{N}.$$

The extended results for relationship between GOS and Dual GOS have been discussed by Burkschat et al. (2003) which produce the above two relations as special case. These relationships are given below but we first give some common notations.

We will denote the GOS from any continuous distribution $F(x)$ by $X_{r:n,m,k}$ and the Dual GOS by $X_{r(d):n,m,k}$. The uniform GOS and Dual GOS will be denoted by $U_{r:n,m,k}$ and $U_{r(d):n,m,k}$ respectively. The first relation between GOS and Dual GOS is evident from their distribution functions given in (4.17) and (5.34). From these two equations it is obvious that distribution functions of both the models can be written in the form of Meijer G-function and this relation is given in Burkschat et al. (2003).

An important relationship which relates GOS and Dual GOS is based upon the probability integral transform of both the models. This relationship is based upon GOS and Dual GOS from two different distribution functions $F(x)$ and $G(x)$ and is given as

$$F\big(X_{r(d):n,m,k}\big) \sim \big\{1 - G\big(X_{r:n,m,k}\big)\big\}\ ;\ 1 \le r \le n. \tag{5.61}$$

The corresponding relationship for uniform GOS and Dual GOS is immediately written as

$$U_{r(d):n,m,k} \sim \big(1 - U_{r:n,m,k}\big)\ ; 1 \le r \le n. \tag{5.62}$$

The relationship (5.62) provide basis for the following relationship which enable us to give distribution of GOS from uniform Dual GOS and distribution of Dual GOS from uniform GOS

$$X_{r(d):n,m,k} \sim F^{-1}\big(1 - U_{r:n,m,k}\big), \tag{5.63}$$

and

$$X_{r:n,m,k} \sim F^{-1}\big(1 - U_{r(d):n,m,k}\big). \tag{5.64}$$

The corresponding results for special cases can be readily written from above relationships.

Chapter 6
Some Uses of Ordered Random Variables

6.1 Introduction

In previous chapters we have discussed popular models of ordered random variables alongside their common distributional properties. Ordered random variables are useful in several areas of statistics like reliability analysis and censoring. In this chapter we will discuss some common uses of ordered random variables.

6.2 Concomitants of Ordered Random Variables

Sofar in this book we have discussed the concepts of various models of ordered random variables when a sample is available from some distribution $F(x)$. The comprehensive models of Generalized Order Statistics and Dual Generalized Order Statistics have been discussed in detail in Chaps. 4 and 5 which contain several special models for various choices of the parameters involved. Often it happen that the sample of size n is available from some bivariate distribution $F(x, y)$ and the sample is ordered with respect to one of the variable by using any of the models of ordered random variables. In such situations the other variable is automatically shuffled and is called the *Concomitants of Ordered Variable*. The concomitants of ordered random variables are accompanying variables which occur naturally when sample is available from a bivariate distribution and is arranged with respect to one variable. The concept of concomitants of ordered random variables can be extended to situations when sample is available from some multivariate distribution and is arranged with respect to one of the variable. In such situations all other variables are called the multivariate concomitants. The concomitants of ordered random variables are random variables and have formal distributional properties. The distributional properties of concomitants of order statistics have been discussed in Arnold et al. (2008) and distributional properties of concomitants of generalized order statistics

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_6

have been discussed in Ahsanullah and Nevzorov (2001). In this chapter we will discuss distributional properties of concomitants for various models of ordered random variables.

6.2.1 Concomitants of Generalized Order Statistics

The Concomitants of Generalized Order Statistics arise when we have sample from some bivariate or multivariate distribution and the sample is arranged on the basis of GOS of one of the variable. We will discuss in detail the concomitants of GOS when sample is available from some bivariate distribution. The idea can be easily extended to multivariate case. The concomitants of GOS are formally defined below.

Suppose (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n) be a random sample of size n from some bivariate distribution $F(x, y)$. Suppose further that the sample is ordered with respect to variable X, that is $X_{r:n,m,k}$ is the rth GOS for marginal distribution of X. The automatically shuffled variable Y is called the Concomitant of Generalized Order Statistics and is denoted as $Y_{[r:n,m,k]}$. The distribution rth concomitant of GOS is given by Ahsanullah and Nevzorov (2001) as

$$f_{[r:n,m,k]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{r:n,m,k}(x)\, dx, \tag{6.1}$$

where $f(y|x)$ is conditional distribution of Y given x and $f_{r:n,m,k}(x)$ is the marginal distribution of rth GOS for random variable X. The joint distribution of rth and sth concomitants of GOS is

$$f_{[r,s:n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(y_1|x_1) f(y_2|x_2) f_{r,s:n,m,k}(x_1, x_2)\, dx_2 dx_1, \tag{6.2}$$

where $f_{r,s:n,m,k}(x_1, x_2)$ is joint distribution of r th and sth GOS for random variable X.

The marginal distribution of rth concomitant of GOS and joint distribution of rth and sth concomitant of GOS provide basis to study basic properties of the concomitants of GOS. Formally, the pth moment of rth concomitant of GOS is readily written as

$$E\left(Y^p_{[r:n,m,k]}\right) = \int_{-\infty}^{\infty} y^p f_{[r:n,m,k]}(y)\, dy, \tag{6.3}$$

and the joint (p, q)th moment of rth and sth concomitants is computed as

$$E\left(Y^p_{1[r:n,m,k]}, Y^q_{2[r:n,m,k]}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1^p y_2^q f_{[r,s:n,m,k]}(y_1, y_2)\, dy_1 dy_2. \tag{6.4}$$

The mean, variance and covariance are easily obtained from above expressions.

The idea of concomitants of GOS can be easily extended to the case when sample is available from some multivariate distribution, say $F\left(x, y_1, y_2, \ldots, y_p\right)$ and the sample is ordered with respect to variable X. The vector $\mathbf{y} = \left[Y_1, Y_2, \ldots, Y_p\right]^{/}$ in this case is the vector of concomitants of GOS. The distribution of rth multivariate concomitant in this case is given as

$$f_{[r:n,m,k]}\left(\mathbf{y}\right) = \int_{-\infty}^{\infty} f\left(\mathbf{y}|x\right) f_{r:n,m,k}\left(x\right) dx, \tag{6.5}$$

where $f\left(\mathbf{y}|x\right)$ is the conditional distribution of vector $\mathbf{y}$ given x.

Example 6.1 A random sample of size n is drawn from the density

$$f\left(x, y\right) = f\left(x\right) f\left(y\right)\left[1 + \alpha\left\{2F\left(x\right) - 1\right\}\left\{2F\left(y\right) - 1\right\}\right].$$

Obtain distribution of rth concomitant of GOS for this distribution alongside the expression for pth moment of rth concomitant (Beg and Ahsanullah 2008).

Solution: The distribution of rth concomitant of GOS is given as

$$f_{[r:n,m,k]}\left(y\right) = \int_{-\infty}^{\infty} f\left(y|x\right) f_{r:n,m,k}\left(x\right) dx,$$

where $f_{r:n,m,k}\left(x\right)$ is distribution of rth GOS for random variable X and is given as

$$f_{r:n,m,k}\left(x\right) = \frac{C_{r-1}}{(r-1)!} f\left(x\right)\left\{1 - F\left(x\right)\right\}^{\gamma_r - 1} g_m^{r-1}\left[F\left(x\right)\right].$$

Further, the conditional distribution of Y given x is

$$\begin{aligned} f\left(y|x\right) &= \frac{f\left(x, y\right)}{f\left(x\right)} = f\left(y\right)\left[1 + \alpha\left\{2F\left(x\right) - 1\right\}\left\{2F\left(y\right) - 1\right\}\right] \\ &= f\left(y\right) + \alpha f\left(y\right)\left\{2F\left(x\right) - 1\right\}\left\{2F\left(y\right) - 1\right\}. \end{aligned}$$

Now distribution of rth concomitant of GOS is

$$\begin{aligned} f_{[r:n,m,k]}\left(y\right) = &\int_{-\infty}^{\infty}\left[f\left(y\right) + \alpha f\left(y\right)\left\{2F\left(x\right) - 1\right\}\left\{2F\left(y\right) - 1\right\}\right] \\ &\times \frac{C_{r-1}}{(r-1)!} f\left(x\right)\left\{1 - F\left(x\right)\right\}^{\gamma_r - 1} g_m^{r-1}\left[F\left(x\right)\right] dx \end{aligned}$$

or

$$f_{[r:n,m,k]}(y) = f(y)\int_{-\infty}^{\infty}\frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)]\,dx$$
$$+\alpha f(y)\{2F(y)-1\}\int_{-\infty}^{\infty}\{2F(x)-1\}\frac{C_{r-1}}{(r-1)!}f(x)$$
$$\times\ \{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)]\,dx$$

or

$$f_{[r:n,m,k]}(y) = f(y)+\alpha f(y)\{2F(y)-1\}\int_{-\infty}^{\infty}[1-2\{1-F(x)\}]$$
$$\times\frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)]\,dx$$

or

$$f_{[r:n,m,k]}(y) = f(y)+\alpha f(y)\{2F(y)-1\}\left[\int_{-\infty}^{\infty}\frac{C_{r-1}}{(r-1)!}f(x)\right.$$
$$\times\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}[F(x)]\,dx$$
$$\left.-2\int_{-\infty}^{\infty}\frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r}g_m^{r-1}[F(x)]\,dx\right]$$

or

$$f_{[r:n,m,k]}(y) = f(y)+\alpha f(y)\{2F(y)-1\}\left[1-2\int_{-\infty}^{\infty}\frac{C_{r-1}}{(r-1)!}f(x)\right.$$
$$\left.\times\{1-F(x)\}^{\gamma_r+1-1}g_m^{r-1}[F(x)]\,dx\right]$$

or

$$f_{[r:n,m,k]}(y) = f(y)+\alpha f(y)\{2F(y)-1\}\left[1-2\frac{C_{r-1}}{C_{r-1}(1)}\int_{-\infty}^{\infty}\frac{C_{r-1}(1)}{(r-1)!}\right.$$
$$\left.\times\ f(x)\{1-F(x)\}^{\gamma_r+1-1}g_m^{r-1}[F(x)]\,dx\right]$$

or

$$f_{[r:n,m,k]}(y) = f(y)+\alpha f(y)\{2F(y)-1\}\left(1-2\frac{C_{r-1}}{C_{r-1}(1)}\right),$$

where $C_{r-1}(1)=\prod_{j=1}^{r}\left(\gamma_j+1\right)$. The density function or rth concomitant of GOS can further be written as

$$f_{[r:n,m,k]}(y) = f(y) + \alpha f(y)\{2F(y) - 1\}\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right)$$
$$= f(y)\left[1 + \alpha\{2F(y) - 1\}\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right)\right].$$

Now using the fact that $2F(y) - 1 = 1 - 2\{1 - F(y)\} = 1 - 2\bar{F}(y)$ the density function of rth concomitant can be written as

$$f_{[r:n,m,k]}(y) = f(y)\left[1 + \alpha\left\{1 - 2\bar{F}(y)\right\}\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right)\right]$$
$$= f(y)\left[1 + \alpha\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right) - 2\alpha\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right)\bar{F}(y)\right]$$
$$= f(y)\left[d_{1,r} + d_{2,r}\bar{F}(y)\right], \tag{6.6}$$

where

$$d_{1,r} = 1 + \alpha\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right) \text{ and } d_{2,r} = -2\alpha\left(1 - 2\frac{C_{r-1}}{C_{r-1}(1)}\right).$$

The density (6.6) provide general expression for distribution of rth concomitant of GOS for *Farlie-Gumbel-Morgenstern* family of distributions discussed by Morgenstern (1956), Farlie (1960) and Gumbel (1960).

The expression for pth moment of rth concomitant of GOS is

$$E\left(Y^p_{[r:n,m,k]}\right) = \int_{-\infty}^{\infty} y^p f_{[r:n,m,k]}(y)\,dy$$
$$= \int_{-\infty}^{\infty} y^p f(y)\left[d_{1,r} + d_{2,r}\bar{F}(y)\right]dy$$
$$= d_{1,r}\int_{-\infty}^{\infty} y^p f(y)\,dy + d_{2,r}\int_{-\infty}^{\infty} y^p f(y)\,\bar{F}(y)\,dy$$

or

$$E\left(Y^p_{[r:n,m,k]}\right) = d_{1,r}\mu^p_Y + d_{2,r}\int_{-\infty}^{\infty} y^p f(y)\{1 - F(y)\}\,dy$$
$$= d_{1,r}\mu^p_Y + d_{2,r}\mu^p_Y - d_{2,r}\int_{-\infty}^{\infty} y^p f(y)\,F(y)\,dy,$$

where $\mu^p_Y = E(Y^p)$. The expression for pth moment can further be simplified as

$$E\left(Y^p_{[r:n,m,k]}\right) = d_{1,r}\mu^p_Y + d_{2,r}\mu^p_Y - \frac{d_{2,r}}{2}\mu^p_{Y(2:2)}$$
$$= \mu^p_Y\left(d_{1,r} + d_{2,r}\right) - \frac{d_{2,r}}{2}\mu^p_{Y(2:2)}, \tag{6.7}$$

where $\mu^p_{Y(2:2)}$ is pth moment of maximum of Y in a sample of size 2. Expression (6.7) can be used to compute moments of concomitants of GOS for various members of the family. For example if we use exponential distribution with

$$f(y) = e^{-y} \text{ and } F(y) = 1 - e^{-y}$$

then the density function of rth concomitant of GOS is

$$\begin{aligned} f_{[r:n,m,k]}(y) &= e^{-y}\left[d_{1,r} + d_{2,r}e^{-y}\right] \\ &= d_{1,r}e^{-y} + d_{2,r}e^{-2y}. \end{aligned}$$

The expression for pth moment of rth concomitant of GOS is immediately written as

$$\begin{aligned} E\left(Y^p_{[r:n,m,k]}\right) &= d_{1,r}\int_0^\infty y^p e^{-y}dy + d_{2,r}\int_0^\infty y^p e^{-2y}dy \\ &= d_{1,r}\Gamma(p+1) + d_{2,r}\frac{1}{2^{p+1}}\Gamma(p+1) \\ &= \left(d_{1,r} + \frac{d_{2,r}}{2^{p+1}}\right)\Gamma(p+1), \end{aligned}$$

as given by BuHamra and Ahsanullah (2013). The mean and variance can be readily obtained from above.

6.2.2 *Concomitants of Order Statistics*

The distribution of concomitants of GOS discussed in the previous section provide basis for distribution of concomitants for sub models. Specifically, the concomitants of order statistics has attracted several statisticians. The concomitants of order statistics are formally defined below.

Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample from a bivariate distribution $F(x, y)$ and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics of variable X, the variable Y in this case is called the *Concomitants of Order Statistics*. The density function of rth concomitant of order statistics is given by Arnold et al. (2008) as

$$f_{[r:n]}(y) = \int_{-\infty}^\infty f(y|x) f_{r:n}(x)\, dx, \tag{6.8}$$

where $f_{r:n}(x)$ is the marginal distribution of rth order statistics for X. The joint distribution of rth and sth concomitants is

$$f_{[r,s:n]}(y_1, y_2) = \int_{-\infty}^\infty \int_{x_1}^\infty f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2)\, dx_2 dx_1, \tag{6.9}$$

where $f_{r,s:n}(x_1, x_2)$ is joint distribution of rth and sth order statistics for X.

The moments for concomitants of order statistics can be easily computed from (6.8) and (6.9). The distribution of concomitants of order statistics can be obtained from distribution of concomitants of GOS by using $m = 0$ and $k = 1$.

Example 6.2 A random sample of size n is drawn from the density

$$f(x, y) = f(x) f(y) [1 + \alpha \{2F(x) - 1\} \{2F(y) - 1\}] .$$

Obtain distribution of rth concomitant of order statistics for this distribution alongside the expression for pth moment of rth concomitant.

Solution: The distribution of rth concomitant of order statistics is

$$f_{[r:n,m,k]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{r:n}(x) dx,$$

where $f_{r:n}(x)$ is distribution of rth order statistics for X given as

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} \\ &= C_{r,n} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} , \end{aligned}$$

where $C_{r,n} = \frac{n!}{(r-1)!(n-r)!}$. Now the conditional distribution of Y given x is

$$\begin{aligned} f(y|x) &= \frac{f(x, y)}{f(x)} = f(y) [1 + \alpha \{2F(x) - 1\} \{2F(y) - 1\}] \\ &= f(y) + \alpha f(y) \{2F(x) - 1\} \{2F(y) - 1\} . \end{aligned}$$

The distribution of rth concomitant of order statistics is

$$\begin{aligned} f_{[r:n]}(y) = \int_{-\infty}^{\infty} & [f(y) + \alpha f(y) \{2F(x) - 1\} \{2F(y) - 1\}] \\ & \times C_{r,n} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} dx \end{aligned}$$

or

$$\begin{aligned} f_{[r:n]}(y) = f(y) \int_{-\infty}^{\infty} & C_{r,n} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} dxdx \\ & + \alpha f(y) \{2F(y) - 1\} \int_{-\infty}^{\infty} \{2F(x) - 1\} C_{r,n} f(x) \\ & \times [F(x)]^{r-1} [1 - F(x)]^{n-r} dx \end{aligned}$$

or

$$f_{[r:n]}(y) = f(y) + \alpha f(y)\{2F(y) - 1\} \int_{-\infty}^{\infty} [1 - 2\{1 - F(x)\}] \\ \times C_{r,n} f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} dx$$

or

$$f_{[r:n,m,k]}(y) = f(y) + \alpha f(y)\{2F(y) - 1\} \left[\int_{-\infty}^{\infty} C_{r,n} f(x) [F(x)]^{r-1} \right. \\ \times [1 - F(x)]^{n-r} dx - 2\int_{-\infty}^{\infty} C_{r,n} f(x) \\ \left. [F(x)]^{r-1} [1 - F(x)]^{n-r+1} dx\right]$$

or

$$f_{[r:n]}(y) = f(y) + \alpha f(y)\{2F(y) - 1\} \left[1 - 2\int_{-\infty}^{\infty} C_{r,n} f(x) \right. \\ \left. \times [F(x)]^{r-1} [1 - F(x)]^{(n+1)-r} dx\right]$$

or

$$f_{[r:n]}(y) = f(y) + \alpha f(y)\{2F(y) - 1\} \left[1 - 2\frac{C_{r,n}}{C_{r,n+1}} \int_{-\infty}^{\infty} C_{r,n+1} \right. \\ \left. \times f(x) [F(x)]^{r-1} [1 - F(x)]^{(n+1)-r} dx\right],$$

where $C_{r,n+1} = \frac{(n+1)!}{(r-1)!(n+1-r)!}$. The density function can further be simplified as under

$$\begin{aligned} f_{[r:n]}(y) &= f(y) + \alpha f(y)\{2F(y) - 1\}\left(1 - 2\frac{C_{r,n}}{C_{r,n+1}}\right) \\ &= f(y) + \alpha f(y)\{2F(y) - 1\}\left(1 - 2\frac{n - r + 1}{n + 1}\right) \\ &= f(y)\left[1 + \alpha\{2F(y) - 1\}\left(\frac{2r - n - 1}{n + 1}\right)\right]. \end{aligned}$$

The density function of rth concomitant of order statistics can be obtained from (6.6) by using $m = 0$ and $k = 1$. The expression for pth moment of rth concomitant of order statistics is

$$
\begin{aligned}
E\left(Y_{[r:n]}^{p}\right) &= \int_{-\infty}^{\infty} y^{p} f_{[r:n]}(y)\,dy \\
&= \int_{-\infty}^{\infty} y^{p} f(y)\left[1+\alpha\left\{2F(y)-1\right\}\left(\frac{2r-n-1}{n+1}\right)\right]dy \\
&= \int_{-\infty}^{\infty} y^{p} f(y)\,dy + \alpha\left(\frac{2r-n-1}{n+1}\right) \\
&\quad \times \int_{-\infty}^{\infty} y^{p} f(y)\left\{2F(y)-1\right\}dy
\end{aligned}
$$

or

$$
\begin{aligned}
E\left(Y_{[r:n,m,k]}^{p}\right) &= \mu_{Y}^{p} + \alpha\left(\frac{2r-n-1}{n+1}\right)\int_{-\infty}^{\infty} y^{p} 2f(y)\,F(y)\,dy \\
&\quad - \alpha\left(\frac{2r-n-1}{n+1}\right)\int_{-\infty}^{\infty} y^{p} f(y)\,dy \\
&= \mu_{Y}^{p} - \alpha\left(\frac{2r-n-1}{n+1}\right)\mu_{Y}^{p} + \alpha\left(\frac{2r-n-1}{n+1}\right)\mu_{Y(2:2)}^{p},
\end{aligned}
$$

where $\mu_{Y}^{p} = E\left(Y^{p}\right)$ and $\mu_{Y(2:2)}^{p}$ is pth moment of maximum of Y in a sample of size 2. Above expression can be used to compute moments of concomitants of order statistics for various members of the family.

6.2.3 Concomitants of Upper Record Values

In Chap. 3 we have discussed in detail the distributional properties of upper record values when the sequence of random variables is available from a continuous univariate distribution function $F(x)$. When sample is available from a bivariate distribution $F(x, y)$ and is arranged with respect to upper records of variable X then the variable Y is called the concomitant of upper record values. The marginal density function of nth concomitant of $k-$record values and the joint distribution of nth and mth concomitants of $k-$record values are given by Ahsanullah (2008) as

$$
f_{[U_K(n)]}(y) = \int_{-\infty}^{\infty} f(y|x)\, f_{U_K(n)}(x)\,dx, \tag{6.10}
$$

and

$$
f_{[U_K(n),U_K(m)]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(y_1|x_1)\, f(y_2|x_2)\, f_{U_K(n),U_K(m)}(x_1, x_2)\,dx_2 dx_1, \tag{6.11}
$$

where $f_{U_K(n)}(x)$ is the marginal distribution of nth k-record values and $f_{U_K(n),U_K(m)}(x_1, x_2)$ is joint distribution of nth and mth k-record values of X given in (3.12) and (3.13) respectively. The distribution of concomitants of record values can be obtained by using $k = 1$ in (6.10) and (6.11). The moments of concomitants can be obtained from (6.10) and (6.11).

Several authors have studied the concomitants of record values for certain class of distributions. We have given some examples below.

Example 6.3 The joint distribution of X and Y is

$$f(x, y) = \beta\theta_1\theta_2 x^{2\theta_1-1} y^{\theta_2-1} \exp\left\{-x^{\theta_1}\left(\beta + y^{\theta_2}\right)\right\}; x, y, \theta_1, \theta_2, \beta > 0.$$

Obtain the distribution of nth concomitant of upper record value for this distribution (Ahsanullah et al. 2010).

Solution: The distribution of nth concomitant of upper record values is given as

$$f_{\left[Y_{U(n)}\right]}(y) = \int_{-\infty}^{\infty} f(y|x)\, f_{X_{U(n)}}(x)\, dx,$$

where $f_{X_{U(n)}}$ is given in (3.3). Now for given distribution we have

$$\begin{aligned} f(x) &= \int_0^{\infty} f(x, y)\, dy \\ &= \int_0^{\infty} \beta\theta_1\theta_2 x^{2\theta_1-1} y^{\theta_2-1} \exp\left\{-x^{\theta_1}\left(\beta + y^{\theta_2}\right)\right\} dy \\ &= \beta\theta_1\theta_2 x^{2\theta_1-1} \exp\left(-\beta x^{\theta_1}\right) \int_0^{\infty} y^{\theta_2-1} \exp\left(-x^{\theta_1} y^{\theta_2}\right) dy \\ &= \beta\theta_1 x^{\theta_1-1} \exp\left(-\beta x^{\theta_1}\right),\ x, \theta_1, \beta > 0. \end{aligned}$$

Also

$$F(x) = \int_0^x f(t)\, dt = 1 - \exp\left(-\beta x^{\theta_1}\right)$$

and

$$f_{X_{U(n)}}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x),$$

where $R(x) = -\ln\left[1 - F(x)\right] = \beta x^{\theta_1}$. So we have

$$\begin{aligned} f_{X_{U(n)}}(x) &= \frac{1}{\Gamma(n)}\left(\beta x^{\theta_1}\right)^{n-1} \beta\theta_1 x^{\theta_1-1} \exp\left(-\beta x^{\theta_1}\right) \\ &= \frac{\theta_1}{\Gamma(n)}\beta^n x^{n\theta_1-1} \exp\left(-\beta x^{\theta_1}\right);\ x, n, \beta, \theta_1 > 0. \end{aligned}$$

Now the conditional distribution of Y given X is

$$f(y|x) = \theta_2 x^{\theta_1} y^{\theta_2 - 1} \exp\left(-x^{\theta_1} y^{\theta_2}\right);\ x, y, \theta_1, \theta_2 > 0,$$

so the distribution of nth concomitant of record value is

$$\begin{aligned} f_{[Y_{U(n)}]}(y) &= \int_0^\infty \theta_2 x^{\theta_1} y^{\theta_2 - 1} \exp\left(-x^{\theta_1} y^{\theta_2}\right) \\ &\quad \times \frac{\theta_1}{\Gamma(n)} \beta^n x^{n\theta_1 - 1} \exp\left(-\beta x^{\theta_1}\right) dx \\ &= \frac{\theta_1 \theta_2}{\Gamma(n)} \beta^n y^{\theta_2 - 1} \int_0^\infty x^{n\theta_1 + \theta_1 - 1} \exp\left\{-x^{\theta_1}\left(\beta + y^{\theta_2}\right)\right\} dx, \end{aligned}$$

now making the transformation $x^{\theta_1}\left(\beta + y^{\theta_2}\right) = w$ and integrating we have

$$f_{[Y_{U(n)}]}(y) = \frac{n\theta_2 \beta^n y^{\theta_2 - 1}}{\left(\beta + y^{\theta_2}\right)^{n+1}};\ y, n, \theta_2, \beta > 0.$$

Now pth moment of nth concomitant is

$$\begin{aligned} E\left(Y^p_{[U(n)]}\right) &= \int_0^\infty y^p f_{[Y_{U(n)}]}(y)\, dy = \int_0^\infty y^p \frac{n\theta_2 \beta^n y^{\theta_2 - 1}}{\left(\beta + y^{\theta_2}\right)^{n+1}} dy \\ &= n\theta_2 \beta^n \int_0^\infty \frac{y^{\theta_2 + p - 1}}{\left(\beta + y^{\theta_2}\right)^{n+1}} dy \end{aligned}$$

or

$$E\left(Y^p_{[U(n)]}\right) = \frac{\beta^{p/\theta_2} \Gamma(n - p/\theta_2)\, \Gamma(p/\theta_2 + 1)}{\Gamma(n)};\ p < n\theta_2.$$

The mean and variance can be obtained from above.

6.2.4 Concomitants of Dual GOS

The concomitants of ordered random variables can be extended to the case of Dual GOS and are defined as below.

Suppose a random sample of size n is available from $F(x, y)$ and distribution of rth dual GOS is obtained for X. The variable Y in this case is called the concomitant of dual GOS. The density function of rth concomitant of dual GOS is readily written on the lines of distribution of concomitants of GOS as

$$f_{[r(d):n,m,k]}(y) = \int_{-\infty}^\infty f(y|x)\, f_{r(d):n,m,k}(x)\, dx, \tag{6.12}$$

where $f_{r(d):n,m,k}(x)$ is the marginal distribution of rth dual GOS for X. The joint distribution of rth and sth concomitants of dual GOS is

$$f_{[r(d),s(d):n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(y_1|x_1)\, f(y_2|x_2) \times f_{r(d),s(d):n,m,k}(x_1, x_2)\, dx_2 dx_1, \tag{6.13}$$

where $f_{r,s:n,m,k}(x_1, x_2)$ is joint distribution of rth and sth GOS for random variable X. Expression for single and product moments can be easily obtained from (6.12) and (6.13).

The distribution of concomitants of dual GOS provides distribution of concomitants for special models by using specific values of the parameters involved. For example by using $m = -1$ in (6.12) and (6.13) we obtain the distribution of concomitants of lower record values and is given as

$$f_{[L_K(n)]}(y) = \int_{-\infty}^{\infty} f(y|x)\, f_{L_K(n)}(x)\, dx, \tag{6.14}$$

where $f_{L_K(n)}(x)$ is distribution of nth $k-$lower records.

Example 6.4 A random sample of size n is drawn from the density

$$f(x, y) = f(x)\, f(y)\, [1 + \alpha\{2F(x) - 1\}\{2F(y) - 1\}].$$

Obtain distribution of rth concomitant of dual GOS for this distribution alongside the expression for pth moment of rth concomitant.

Solution: The distribution of rth concomitant of GOS is given as

$$f_{[r(d):n,m,k]}(y) = \int_{-\infty}^{\infty} f(y|x)\, f_{r(d):n,m,k}(x)\, dx,$$

where $f_{r(d):n,m,k}(x)$ is distribution of rth dual GOS for random variable X and is given as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r - 1} g_{m(d)}^{r-1}[F(x)].$$

Further, the conditional distribution of Y given x is

$$\begin{aligned} f(y|x) = \frac{f(x, y)}{f(x)} &= f(y)[1 + \alpha\{2F(x) - 1\}\{2F(y) - 1\}] \\ &= f(y) + \alpha f(y)\{2F(x) - 1\}\{2F(y) - 1\} \\ &= f(y) + 2\alpha f(y)\{2F(y) - 1\} F(x) - \alpha f(y)\{2F(y) - 1\}. \end{aligned}$$

Now distribution of rth concomitant of GOS is

$$f_{[r(d):n,m,k]}(y) = \int_{-\infty}^{\infty} [f(y) + 2\alpha f(y)\{2F(y)-1\}F(x) - \alpha f(y)\{2F(y)-1\}] \times \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r-1} g_{m(d)}^{r-1}[F(x)]\,dx$$

or

$$\begin{aligned} f_{[r(d):n,m,k]}(y) = f(y)\int_{-\infty}^{\infty} \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r-1} g_{m(d)}^{r-1}[F(x)]\,dx \\ + 2\alpha f(y)\{2F(y)-1\}\int_{-\infty}^{\infty} F(x)\frac{C_{r-1}}{(r-1)!} f(x) \\ \times \{F(x)\}^{\gamma_r-1} g_{m(d)}^{r-1}[F(x)]\,dx - \alpha f(y)\{2F(y)-1\} \\ \times \int_{-\infty}^{\infty} \frac{C_{r-1}}{(r-1)!} f(x)\{F(x)\}^{\gamma_r-1} g_{m(d)}^{r-1}[F(x)]\,dx \end{aligned}$$

or

$$\begin{aligned} f_{[r(d):n,m,k]}(y) = f(y) \;+ 2\alpha f(y)\{2F(y)-1\}\int_{-\infty}^{\infty} \frac{C_{r-1}}{(r-1)!} f(x) \\ \times \{F(x)\}^{\gamma_r+1-1} g_{m(d)}^{r-1}[F(x)]\,dx - \alpha f(y)\{2F(y)-1\} \end{aligned}$$

or

$$\begin{aligned} f_{[r(d):n,m,k]}(y) = f(y) + 2\alpha f(y)\{2F(y)-1\}\left[\frac{C_{r-1}}{C_{r-1}(1)}\int_{-\infty}^{\infty}\frac{C_{r-1}(1)}{(r-1)!}\right. \\ \left. \times f(x)\{F(x)\}^{\gamma_r+1-1} g_{m(d)}^{r-1}[F(x)]\,dx\right] - \alpha f(y)\{2F(y)-1\} \end{aligned}$$

or

$$f_{[r(d):n,m,k]}(y) = f(y) + \alpha f(y)\{2F(y)-1\}\frac{C_{r-1}}{C_{r-1}(1)} - \alpha f(y)\{2F(y)-1\},$$

where $C_{r-1}(1) = \prod_{j=1}^{r}(\gamma_j + 1)$. The density function or rth concomitant of dual GOS can further be written as

$$\begin{aligned} f_{[r(d):n,m,k]}(y) &= f(y)\left[1 + \alpha\{2F(y)-1\}\left(\frac{2C_{r-1}}{C_{r-1}(1)} - 1\right)\right] \\ &= f(y)\left[1 + \alpha C^*\{2F(y)-1\}\right], \end{aligned}$$

where $C^* = \left(\frac{2C_{r-1}}{C_{r-1}(1)} - 1\right)$. The density function can be further simplified as

$$\begin{aligned} f_{[r(d):n,m,k]}(y) &= f(y)\left[1+\alpha C^*\{2F(y)-1\}\right] \\ &= f(y)+\alpha C^*\{2f(y)F(y)-f(y)\} \\ &= f_{1:1}(y)+\alpha C^*\{f_{2:2}(y)-f_{1:1}(y)\}, \end{aligned} \tag{6.15}$$

where $f_{2:2}(y)$ is distribution of maximum order statistics for a sample of size 2 and $f_{1:1}(y) = f(y)$. The density (6.15) provide general expression for distribution of rth concomitant of dual GOS for *Farlie-Gumbel-Morgenstern* family of distributions discussed by Morgenstern (1956), Farlie (1960) and Gumbel (1960). The expression for pth moment of rth concomitant of dual GOS is readily written as

$$\begin{aligned} E\left(Y^p_{[r:n,m,k]}\right) &= \int_{-\infty}^{\infty} y^p f_{[r:n,m,k]}(y)\,dy \\ &= \int_{-\infty}^{\infty} y^p\left[f_{1:1}(y)+\alpha C^*\{f_{2:2}(y)-f_{1:1}(y)\}\right]dy \\ &= \int_{-\infty}^{\infty} y^p f(y)\,dy+\alpha C^*\int_{-\infty}^{\infty} y^p f_{2:2}(y)\,dy \\ &\quad - \alpha C^*\int_{-\infty}^{\infty} y^p f(y)\,dy \end{aligned}$$

or

$$\begin{aligned} E\left(Y^p_{[r:n,m,k]}\right) &= \mu_Y^p+\alpha C^*\mu^p_{Y(2:2)}-\alpha C^*\mu_Y^p \\ &= \mu_Y^p+\alpha C^*\left\{\mu^p_{Y(2:2)}-\mu_Y^P\right\}. \end{aligned} \tag{6.16}$$

Expression (6.16) can be used to compute moments of concomitants of dual GOS for various members of the family.

Example 6.5 Obtain the distribution of nth concomitant of lower record value for the distribution

$$f(x,y) = \frac{4\theta}{x^5y^3}\exp\left\{-\frac{1}{x^2}\left(\theta+\frac{1}{y^2}\right)\right\};\ x,y,\theta>0.$$

Also obtain expression for pth moment of rth concomitant of lower record value (Mohsin et al. 2009).

Solution: The distribution of rth concomitant of lower record values is

$$f_{[Y_L(n)]}(y) = \int_{-\infty}^{\infty} f(y|x)\,f_{X_L(n)}(x)\,dx,$$

where $f_{X_{L(n)}}(x)$ is given as

$$f_{X_L(n)}(x) = \frac{1}{\Gamma(n)}f(x)\left[H(x)\right]^{n-1},\ -\infty<x<\infty$$

and $H(x) = -\ln[F(x)]$. Now for given distribution we have

$$\begin{aligned} f(x) &= \int_0^\infty f(x,y)\,dy = \int_0^\infty \frac{4\theta}{x^5 y^3} \exp\left\{-\frac{1}{x^2}\left(\theta + \frac{1}{y^2}\right)\right\} dy \\ &= \frac{4\theta}{x^5} \exp\left(-\frac{\theta}{x^2}\right) \int_0^\infty \frac{1}{y^3} \exp\left(-\frac{1}{x^2 y^2}\right) dy \\ &= \frac{2\theta}{x^3} \exp\left(-\frac{\theta}{x^2}\right), \quad x, \theta > 0. \end{aligned}$$

and

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{2\theta}{t^3} \exp\left(-\frac{\theta}{t^2}\right) dt = \exp\left(-\frac{\theta}{x^2}\right).$$

So $H(x) = -\ln[F(x)] = \theta/x^2$. We therefore have

$$\begin{aligned} f_{X_{L(n)}}(x) &= \frac{1}{\Gamma(n)} \frac{2\theta}{x^3} \exp\left(-\frac{\theta}{x^2}\right) \left(\frac{\theta}{x^2}\right)^{n-1} \\ &= \frac{1}{\Gamma(n)} \frac{2\theta^n}{x^{2n+1}} \exp\left(-\frac{\theta}{x^2}\right). \end{aligned}$$

Also we have

$$f(y|x) = \frac{2}{x^2 y^3} \exp\left(-\frac{1}{x^2 y^2}\right)$$

and hence the distribution of nth concomitant is

$$\begin{aligned} f_{[Y_{L(n)}]}(y) &= \frac{4\theta^n}{\Gamma(n)} \frac{1}{y^3} \int_0^\infty \frac{1}{x^{2n+3}} \exp\left\{-\frac{1}{x^2}\left(\theta + \frac{1}{y^2}\right)\right\} dx \\ &= \frac{2n\theta^n}{y^3} \left(\frac{y^2}{1+\theta y^2}\right)^{n+1}, \quad y, n, \theta > 0. \end{aligned}$$

The expression for pth moment is

$$\begin{aligned} E\left(Y_{L(n)}^p\right) &= \int_0^\infty y^p f_{[Y_{L(n)}]}(y)\,dy \\ &= \int_0^\infty y^p \frac{2n\theta^n}{y^3} \left(\frac{y^2}{1+\theta y^2}\right)^{n+1} dy \end{aligned}$$

or

$$E\left(Y_{L(n)}^p\right) = \frac{1}{\theta^{n/2}\Gamma(n)} \Gamma\left(n + \frac{p}{2}\right) \Gamma\left(1 - \frac{p}{2}\right); \; p < 2.$$

We can see that the variance of the distribution of concomitants does not exist.

6.3 Ordered Random Variables in Statistical Inference

Ordered random variables have been used by various authors in drawing inferences about population parameters. Ordered random variables have been used in estimation as well as in hypothesis testing about population parameters. Ordered random variables has attracted several statisticians in drawing inferences about location scale populations. In this section we will discuss some common uses of ordered random variables in statistical inference.

6.3.1 Maximum Likelihood Estimation

Maximum likelihood method has been a useful method of estimation of population parameters due to its nice properties. The maximum likelihood method is based upon maximizing the likelihood function of the data with respect to unknown parameters, that is if a random sample of size n is available from the distribution $F(x; \boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is a $(p \times 1)$ vector of parameters then the parameters can be estimated by solving the likelihood equations

$$\frac{\partial \ln [L(\boldsymbol{x}; \boldsymbol{\theta})]}{\partial \theta_i} = 0\ ;\ i = 1, 2, \ldots, p$$

where $[L(\boldsymbol{x}; \boldsymbol{\theta})]$ is likelihood function of the data.

Situations do arise where the domain of the distribution involve a parameter and in such situations the maximum likelihood estimation can not be done in conventional way but is based upon the order statistics of the sample. For example if we have a sample of size n from the distribution

$$f(x; \theta) = \frac{1}{\theta}\ ;\ 0 < x < \theta,$$

then the maximum likelihood estimator is $\hat{\theta} = X_{n:n}$, the largest value of the sample.

Another way of using ordered random variables in maximum likelihood estimation is to use the joint distribution of the ordered data instead of using the joint distribution of the sample to obtain the maximum likelihood estimates. The idea is illustrated in the following by using the joint distribution of Generalized Order Statistics.

Suppose $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $F(x; \theta)$ and $X_{1:n,m,k},\ X_{2:n,m,k},\ \ldots,\ X_{n:n,m,k}$ be the corresponding GOS of the sample. The joint distribution of GOS is given in (4.1) as

$$f_{1,\ldots,n:n,\tilde{m},k}(x_1, \ldots, x_n; \boldsymbol{\theta}) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\{1 - F(x_n; \boldsymbol{\theta})\}^{k-1} f(x_n; \boldsymbol{\theta}) \times \left[\prod_{i=1}^{n-1}\{1 - F(x_i; \boldsymbol{\theta})\}^m f(x_i; \boldsymbol{\theta})\right];$$

and can be used as the likelihood function to estimate the population parameters. The maximum likelihood estimation based upon ordered random variables is done in conventional way, that is by differentiating the joint distribution with respect to unknown parameters and equating the derivatives to zero. That is, the maximum likelihood estimate of parameters based upon GOS are given by the solution of likelihood equations

$$\frac{\partial \ln\left[L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)\right]}{\partial\theta_i}=0\ ;\ i=1,2,\ldots,n \tag{6.17}$$

where $L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)$ is given in (4.1) above.

The maximum likelihood estimates of population parameters obtained by using joint distribution of GOS can be used to obtain the maximum likelihood estimates based upon special cases by using specific values of the parameters. For example using $m=0$ and $k=1$ in (6.17) we obtain the maximum likelihood estimates based upon order statistics and for $m=-1$ we obtain the maximum likelihood estimates based upon upper record values.

Example 6.6 A random sample of size n is drawn from the distribution

$$f(x;\theta)=\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)\ ;\ x,\theta>0.$$

Obtain maximum likelihood estimate of θ on the basis of GOS.

Solution: The joint distribution of GOS is given as

$$L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\{1-F(x_n;\boldsymbol{\theta})\}^{k-1}f(x_n;\boldsymbol{\theta})$$
$$\times\left[\prod_{i=1}^{n-1}\{1-F(x_i;\boldsymbol{\theta})\}^m f(x_i;\boldsymbol{\theta})\right].$$

Now for given distribution we have

$$f(x;\theta)=\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)\text{ and }F(x)=1-\exp\left(-\frac{x}{\theta}\right).$$

The joint density of GOS is therefore

$$L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{\exp\left(-\frac{x_i}{\theta}\right)\right\}^{k-1}\frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right)$$
$$\times\left[\prod_{i=1}^{n-1}\left\{\exp\left(-\frac{x_i}{\theta}\right)\right\}^m\frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right)\right]$$

or

$$L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)=C_{n-1}\frac{1}{\theta^n}\exp\left(-\frac{kx_i}{\theta}\right)\prod_{i=1}^{n-1}\left\{\exp\left(-\frac{(m+1)\,x}{\theta}\right)\right\}$$
$$=C_{n-1}\frac{1}{\theta^n}\exp\left(-\frac{kx_i}{\theta}\right)\exp\left(-\frac{m+1}{\theta}\sum_{i=1}^{n-1}x_i\right)$$

where $C_{n-1}=k\left(\prod_{j=1}^{n-1}\gamma_j\right)$. The log of likelihood function is

$$\ln\left[L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)\right]=\ln\left(C_{n-1}\right)-n\ln\theta-\frac{kx_i}{\theta}-\frac{m+1}{\theta}\sum\nolimits_{i=1}^{n-1}x_i.$$

Now

$$\frac{\partial\ln\left[L\left(\mathbf{x}_{n,m,k};\boldsymbol{\theta}\right)\right]}{\partial\theta}=-\frac{n}{\theta}+\frac{1}{\theta^2}\left\{kx_i+(m+1)\sum\nolimits_{i=1}^{n-1}x_i\right\}.$$

The maximum likelihood estimator of θ is the solution of

$$-\frac{n}{\hat{\theta}}+\frac{1}{\hat{\theta}^2}\left\{kx_i+(m+1)\sum\nolimits_{i=1}^{n-1}x_i\right\}=0$$

or

$$\frac{1}{\hat{\theta}^2}\left\{kx_i+(m+1)\sum\nolimits_{i=1}^{n-1}x_i\right\}=\frac{n}{\hat{\theta}}$$

or

$$\hat{\theta}=\frac{1}{n}\left\{kx_i+(m+1)\sum\nolimits_{i=1}^{n-1}x_i\right\}.$$

We can see that the maximum likelihood estimator based upon GOS reduces to conventional maximum likelihood estimator for $m=0$ and $k=1$.

6.3.2 The L-Moment Estimation

The L-moment estimation was introduced by Hosking (1990) and is based upon equating few moments of order statistics with corresponding sample moments and obtaining estimates of unknown population parameters. The L-moment estimation is a useful technique for estimation of population parameters when the distribution under study is a location-scale distribution. The L- moment estimation is described in the following.

Suppose a random sample of size n is available from the density $f\,(x;\theta)$ and $x_{1:n}\le x_{2:n}\le\cdots\le x_{n:n}$ be the corresponding order statistics. Suppose further that $E\,(X_{r:n})$ exist for $r=1,2,\ldots,n$ where

$$
\begin{aligned}
E\left(X_{r:n}\right) &= \frac{n!}{(r-1)!\,(n-r)!}\int_{\Re} xf\left(x\right)\left[F\left(x\right)\right]^{r-1}\left[1-F\left(x\right)\right]^{n-r}dx \\
&= \frac{n!}{(r-1)!\,(n-r)!}\int_{0}^{1}\left\{F^{-1}\left(u\right)\right\}u^{r-1}\left(1-u\right)^{n-r}du;
\end{aligned}
$$

and $F^{-1}(u)$ is inverse function and U is $U(0, 1)$ random variable. The L-moments are weighted sum of expected values of these order statistics and are computed as

$$
\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{r-k:r}\right). \tag{6.18}
$$

The "L" in L-moments is emphasizes that λ_r is a linear function of "Expected Order Statistics". The first few L-moments are given as

$$
\begin{aligned}
\lambda_1 &= E\left(X_{1:1}\right) = \bar{X} = \int_0^1\left\{F^{-1}\left(u\right)\right\}du \\
\lambda_2 &= \frac{1}{2}E\left(X_{2:2}-X_{1:2}\right) = \int_0^1\left\{F^{-1}\left(u\right)\right\}\left(2u-1\right)du \\
\lambda_3 &= \frac{1}{3}E\left(X_{3:3}-2X_{2:3}+X_{1:3}\right) \\
&= \int_0^1\left\{F^{-1}\left(u\right)\right\}\left(6u^2-6u+1\right)du
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_4 &= \frac{1}{4}E\left(X_{4:4}-3X_{3:4}+2X_{2:4}-X_{1:4}\right) \\
&= \int_0^1\left\{F^{-1}\left(u\right)\right\}\left(20u^3-30u^2+12u-1\right)du.
\end{aligned}
$$

The L-moments can also be computed as special case of Probability Weighted Moments of Greenwood et al. (1979) which are defined as

$$
\begin{aligned}
M_{p,r,s} &= E\left[X^p\left\{F\left(X\right)\right\}^r\left\{1-F\left(X\right)\right\}^s\right] \\
&= \int_{\Re} x^p\left\{F\left(x\right)\right\}^r\left\{1-F\left(x\right)\right\}^s f\left(x\right)dx \\
&= \int_0^1\left\{F^{-1}\left(u\right)\right\}^p\left\{u\right\}^r\left\{1-u\right\}^s f\left(u\right)du.
\end{aligned} \tag{6.19}
$$

For this we first define β_r as

$$\beta_r = M_{1,r,0} = \int_{\Re} xf(x)\{F(x)\}^r\,dx$$
$$= \int_0^1 \left\{F^{-1}(u)\right\} u^r du. \tag{6.20}$$

The L-moments can then be computed from β_r as

$$\lambda_1 = \beta_0 \tag{6.21}$$
$$\lambda_2 = 2\beta_1 - \beta_0 \tag{6.22}$$
$$\lambda_3 = 6\beta_2 - 5\beta_1 + \beta_0 \tag{6.23}$$
$$\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0. \tag{6.24}$$

The L-moment estimates can be obtained by equating population L-moments with corresponding sample L-moments, that is, the L-moment estimates are obtained by solving

$$\lambda_r = \hat{\lambda}_r\ ;\ r = 1, 2, \ldots, p. \tag{6.25}$$

where the sample L-moments are computed as

$$\hat{\lambda}_r = \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r-1}{k}\binom{r+k-1}{k}\hat{\beta}_k \tag{6.26}$$

where

$$\hat{\beta}_r = \frac{1}{n}\sum\nolimits_{j=r+1}^{n} \frac{(j-1)(j-2)\ldots(j-r)}{(n-1)(n-2)\ldots(n-r)} x_{j:n}. \tag{6.27}$$

First few $\hat{\lambda}_r$ are

$$\hat{\lambda}_1 = \hat{\beta}_0$$
$$\hat{\lambda}_2 = 2\hat{\beta}_1 - \hat{\beta}_0$$
$$\hat{\lambda}_3 = 6\hat{\beta}_2 - 5\hat{\beta}_1 + \hat{\beta}_0$$
$$\hat{\lambda}_4 = 20\hat{\beta}_3 - 30\hat{\beta}_2 + 12\hat{\beta}_1 - \hat{\beta}_0.$$

and first few $\hat{\beta}_r$ are given as

$$\hat{\beta}_0 = \frac{1}{n}\sum\nolimits_{i=1}^{n} x_{1:n} = \bar{X}$$
$$\hat{\beta}_1 = \frac{1}{n}\sum\nolimits_{j=2}^{n} \frac{(j-1)}{(n-1)} x_{j:n}$$

$$\hat{\beta}_2 = \frac{1}{n}\sum\nolimits_{j=3}^{n} \frac{(j-1)(j-2)}{(n-1)(n-2)} x_{j:n}$$
$$\hat{\beta}_3 = \frac{1}{n}\sum\nolimits_{j=4}^{n} \frac{(j-1)(j-2)(j-3)}{(n-1)(n-2)(n-3)} x_{j:n}.$$

We see that the L-moment estimation is like conventional moment estimation with the difference that the simple moments are replaced by the L- moments.

Example 6.7 Suppose we have a random sample from the density

$$f(x;\alpha,\theta) = \frac{1}{\theta}\exp\left\{-\frac{1}{\theta}(x-\alpha)\right\}\ ;\ x \ge \alpha, \theta > 0.$$

Obtain L-moment estimate of the parameters.

Solution: We know that the L-moment estimates are solution of

$$\lambda_r = \hat{\lambda}_r\ ;\ r = 1, 2, \ldots, k.$$

Since given distribution has two parameters therefore L-moment estimates are solution of

$$\lambda_1 = \hat{\lambda}_1 \text{ and } \lambda_2 = \hat{\lambda}_2.$$

Now we find L-moments of the distribution. We have

$$F(x) = \int_{\alpha}^{x} f(t)\,dt = \int_{\alpha}^{x} \frac{1}{\theta}\exp\left\{-\frac{1}{\theta}(t-\alpha)\right\}dt$$
$$= 1 - \exp\left\{-\frac{1}{\theta}(x-\alpha)\right\}.$$

So $F(x) = u$ gives

$$1 - \exp\left\{-\frac{1}{\theta}(x-\alpha)\right\} = u$$

or

$$x = F^{-1}(u) = \alpha - \theta \log(1-u).$$

Now we have

$$\beta_0 = \int_0^1 \left\{F^{-1}(u)\right\}du = \int_0^1 \{\alpha - \theta\log(1-u)\}\,du$$
$$= \alpha\int_0^1 du - \theta\int_0^1 \log(1-u)\,du = \alpha + \theta.$$

Again

$$\beta_1 = \int_0^1 \left\{F^{-1}(u)\right\} udu = \int_0^1 \{\alpha - \theta \log(1-u)\} udu$$
$$= \alpha \int_0^1 udu - \theta \int_0^1 u \log(1-u)\, du = \frac{\alpha}{2} + \frac{3\theta}{4}.$$

So by using (6.22) and (6.23) we have

$$\lambda_1 = \beta_0 = \alpha + \theta$$
$$\lambda_2 = 2\beta_1 - \beta_0 = 2\left(\frac{\alpha}{2} + \frac{3\theta}{4}\right) - (\alpha + \theta) = \frac{\theta}{2}.$$

The L-moment estimates are therefore the solution of

$$\hat{\alpha} + \hat{\theta} = \hat{\lambda}_1 \text{ and } \hat{\theta}/2 = \hat{\lambda}_2;$$

which gives

$$\hat{\alpha} = \hat{\lambda}_1 - 2\hat{\lambda}_2 \text{ and } \hat{\theta} = 2\hat{\lambda}_2.$$

We see that the L-moment estimates are easy to compute.

6.3.3 Ordered Least Square Estimation

The Least Square Estimation is a popular method of estimation for parameter of statistical models. The method can also be used to compute parameters of probability distributions. Lloyed (1952) introduced the use of least square estimation to estimate location and scale parameters of probability distributions by using the order statistics of a sample of size n from distribution $F(x)$ with location and scale parameters μ and σ. The ordered least square method is illustrated in the following.

Let $X_1, X_2, \ldots, X_n$ be a sample from distribution $F(x; \mu, \sigma)$ where μ and σ are location and scale parameters respectively and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the corresponding order statistics of the sample and define

$$Z_{r:n} = \frac{X_{r:n} - \mu}{\sigma}$$

with $E(Z_{r:n}) = \alpha_r$, $Var(Z_{r:n}) = V_{r,s}$ and $Cov(Z_{r:n}, Z_{s:n}) = V_{r,s}$. Now using $X_{r:n} = \mu + \sigma Z_{r:n}$ we have

$$E(X_{r:n}) = \mu + \sigma\alpha_r$$
$$Var(X_{r:n}) = \sigma^2 V_{r,r} \text{ and } Cov(X_{r:n}, X_{s:n}) = \sigma^2 V_{r,s}.$$

Now writing $\mathbf{z} = \left[Z_{1:n} \; Z_{2:n} \; \ldots\ldots \; Z_{n:n} \right]^{/}$ we have $\mathbf{x} = \mu\mathbf{1} + \sigma\mathbf{z}$,

$$E(\mathbf{z}) = \alpha\ ,\ Cov(\mathbf{z}) = \mathbf{V},$$

$$E(\mathbf{x}) = \mu\mathbf{1} + \sigma\alpha \text{ and } Cov(\mathbf{x}) = \sigma^2\mathbf{V}.$$

where $\mathbf{1}$ is $(n \times 1)$ vector of $1's$, α is $(n \times 1)$ vector of $E(Z_{r:n})$ and $\mathbf{V}$ is $(n \times n)$ matrix of variances and covariances of $Z_{r:n}$. Now using the fact that least square estimate of parameters of the model $\mathbf{y} = \mathbf{X}\beta + \varepsilon$ with $Cov(\varepsilon) = \mathbf{V}$ are given as

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^{/}\mathbf{V}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{/}\mathbf{V}^{-1}\mathbf{y}$$

the estimates of μ and σ are obtained as under.

From $E(\mathbf{x}) = \mu\mathbf{1} + \sigma\alpha$ we have

$$E(\mathbf{x}) = \left[\mathbf{1}\ \alpha\right]\begin{bmatrix}\mu\\ \sigma\end{bmatrix} = \mathbf{X}\beta,$$

so

$$\begin{aligned}\begin{bmatrix}\hat{\mu}\\ \hat{\sigma}\end{bmatrix} &= \left(\left[\mathbf{1}\ \alpha\right]^{/}\mathbf{V}^{-1}\left[\mathbf{1}\ \alpha\right]\right)^{-1}\left[\mathbf{1}\ \alpha\right]^{/}\mathbf{V}^{-1}\mathbf{y}\\ &= \begin{bmatrix}\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1} & \mathbf{1}^{/}\mathbf{V}^{-1}\alpha\\ \mathbf{1}^{/}\mathbf{V}^{-1}\alpha & \alpha^{/}\mathbf{V}^{-1}\alpha\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y}\\ \alpha^{/}\mathbf{V}^{-1}\mathbf{y}\end{bmatrix}\end{aligned}$$

or

$$\begin{aligned}\begin{bmatrix}\hat{\mu}\\ \hat{\sigma}\end{bmatrix} &= \frac{1}{\Delta}\begin{bmatrix}\alpha^{/}\mathbf{V}^{-1}\alpha & -\mathbf{1}^{/}\mathbf{V}^{-1}\alpha\\ -\mathbf{1}^{/}\mathbf{V}^{-1}\alpha & \mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y}\\ \alpha^{/}\mathbf{V}^{-1}\mathbf{y}\end{bmatrix}\\ &= \frac{1}{\Delta}\begin{bmatrix}\left(\alpha^{/}\mathbf{V}^{-1}\alpha\right)\left(\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y}\right) - \left(\mathbf{1}^{/}\mathbf{V}^{-1}\alpha\right)\left(\alpha^{/}\mathbf{V}^{-1}\mathbf{y}\right)\\ \left(\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1}\right)\left(\alpha^{/}\mathbf{V}^{-1}\mathbf{y}\right) - \left(\mathbf{1}^{/}\mathbf{V}^{-1}\alpha\right)\left(\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y}\right)\end{bmatrix}\end{aligned}$$

or

$$\hat{\mu} = \frac{1}{\Delta}\left\{-\alpha^{/}\mathbf{V}^{-1}\left(\mathbf{1}\alpha^{/} - \alpha\mathbf{1}^{/}\right)\mathbf{V}^{-1}\mathbf{y}\right\} \tag{6.28}$$

and

$$\hat{\sigma} = \frac{1}{\Delta}\left\{\mathbf{1}^{/}\mathbf{V}^{-1}\left(\mathbf{1}\alpha^{/} - \alpha\mathbf{1}^{/}\right)\mathbf{V}^{-1}\mathbf{y}\right\}, \tag{6.29}$$

where $\Delta = \left(\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1}\right)\left(\alpha^{/}\mathbf{V}^{-1}\alpha\right) - \left(\mathbf{1}^{/}\mathbf{V}^{-1}\alpha\right)^2$. Also we have

$$\begin{aligned}Var\left(\hat{\mu}\right) &= \frac{1}{\Delta}\left(\alpha^{/}\mathbf{V}^{-1}\alpha\right),\ Var\left(\hat{\sigma}\right) = \frac{1}{\Delta}\left(\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1}\right)\\ Cov\left(\hat{\mu},\hat{\sigma}\right) &= -\frac{1}{\Delta}\left(\mathbf{1}^{/}\mathbf{V}^{-1}\alpha\right).\end{aligned} \tag{6.30}$$

If the parent distribution is symmetrical then we have $\alpha_r = -\alpha_{n+r-1}$ and hence we have $\mathbf{1}'\mathbf{V}^{-1}\alpha = \mathbf{0}$ and hence the ordered least square estimators reduces to

$$\hat{\mu} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \text{ and } \hat{\sigma} = \frac{\alpha'\mathbf{V}^{-1}\mathbf{y}}{\alpha'\mathbf{V}^{-1}\alpha}. \tag{6.31}$$

The variances in this case reduces to

$$Var\left(\hat{\mu}\right) = \frac{\sigma^2}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \text{ and } \hat{\sigma} = \frac{\sigma^2}{\alpha'\mathbf{V}^{-1}\alpha}. \tag{6.32}$$

The covariance in this case is zero.

The Lloyed's ordered least square estimation method can be extended by using mean vector and covariance matrix of any model of ordered random variables.

Example 6.8 A random sample of size n is drawn from the distribution

$$f\left(x\right) = \frac{1}{\sigma}\ ;\ \mu - \frac{1}{2}\sigma < x < \mu + \frac{1}{2}\sigma.$$

Obtain the ordered least square estimators of μ and σ by using order statistics.

Solution: We see that the distribution is symmetrical as $f\left(\mu - x\right) = f\left(\mu + x\right)$ therefore the ordered least square estimators of parameters μ and σ are given as

$$\hat{\mu} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \text{ and } \hat{\sigma} = \frac{\alpha'\mathbf{V}^{-1}\mathbf{y}}{\alpha'\mathbf{V}^{-1}\alpha},$$

where α is mean vector and $\mathbf{V}$ is covariance matrix of standardized order statistics

$$Z_{r:n} = \frac{X_{r:n} - \mu}{\sigma}.$$

Now we first obtain the distribution of standardized rth order statistics and joint distribution of standardized rth and sth order statistics. For this we first obtain the distribution of $Z = \left(X - \mu\right)/\sigma$ which is given as

$$f\left(x\right) = 1\ ;\ \ -\frac{1}{2} < z < \frac{1}{2}.$$

Also

$$F\left(z\right) = \int_{-1/2}^{z} f\left(t\right)dt = \int_{-1/2}^{z} dt = z + \frac{1}{2}.$$

Now distribution of rth order statistics for Z is

$$f_{r:n}(z) = \frac{n!}{(r-1)!\,(n-r)!} f(z)\,[F(z)]^{r-1}\,[1-F(z)]^{n-r}$$
$$= \frac{n!}{(r-1)!\,(n-r)!}\left(z+\frac{1}{2}\right)^{r-1}\left(1-z-\frac{1}{2}\right)^{n-r}.$$

Again the joint distribution of rth and sth order statistics for z is

$$f_{r,s:n}(z_1,z_2) = C_{r,s:n} f(z_1)\,f(z_2)\,[F(z_1)]^{r-1}\,[F(z_2)-F(z_1)]^{s-r-1} \times [1-F(z_2)]^{n-s}$$
$$= C_{r,s:n}\left(z_1+\frac{1}{2}\right)^{r-1}(z_2-z_1)^{s-r-1}\left(1-z_2-\frac{1}{2}\right)^{n-s},$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. We now find mean vector and covariance matrix of $\mathbf{z}$ as under

$$\alpha_r = E(Z_{r:n}) = \int_{-1/2}^{1/2} z f_{r:n}(z)\,dz$$
$$= \frac{n!}{(r-1)!\,(n-r)!}\int_{-1/2}^{1/2} z\left(z+\frac{1}{2}\right)^{r-1}\left(1-z-\frac{1}{2}\right)^{n-r} dz.$$

Now making the transformation $z+\frac{1}{2}=w$ we have

$$\alpha_r = \frac{n!}{(r-1)!\,(n-r)!}\int_0^1\left(w-\frac{1}{2}\right)w^{r-1}(1-w)^{n-r}\,dz$$
$$= \frac{n!}{(r-1)!\,(n-r)!}\int_0^1 w^r(1-w)^{n-r}\,dz$$
$$-\frac{1}{2}\frac{n!}{(r-1)!\,(n-r)!}\int_0^1 w^{r-1}(1-w)^{n-r}\,dz$$
$$= \frac{n!}{(r-1)!\,(n-r)!}B(r+1,n-r+1)$$
$$-\frac{1}{2}\frac{n!}{(r-1)!\,(n-r)!}B(r,n-r+1)$$

or

$$\alpha_r = \frac{n!}{(r-1)!\,(n-r)!}\times\frac{r!\,(n-r)!}{(n+1)!}$$
$$-\frac{1}{2}\frac{n!}{(r-1)!\,(n-r)!}\times\frac{(r-1)!\,(n-r)!}{n!}$$
$$= \frac{r}{n+1}-\frac{1}{2}.$$

Again

$$
\begin{aligned}
V_{r,s} &= E\left\{\left(Z_{r:n}-\alpha_r\right)\left(Z_{s:n}-\alpha_s\right)\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{z_2}\left(z_1-\alpha_r\right)\left(z_2-\alpha_s\right) f_{r,s:n}\left(z_1,z_2\right) dz_1 dz_2 \\
&= C_{r,s:n}\int_{-1/2}^{1/2}\int_{-1/2}^{z_2}\left(z_1-\alpha_r\right)\left(z_2-\alpha_s\right)\left(z_1+\frac{1}{2}\right)^{r-1} \\
&\quad \times\ \left(z_2-z_1\right)^{s-r-1}\left(1-z_2-\frac{1}{2}\right)^{n-s} dz_1 dz_2,
\end{aligned}
$$

which after simplification becomes

$$
\begin{aligned}
V_{r,s} &= \frac{r\left(n-s+1\right)}{\left(n+1\right)^2\left(n-s\right)}\ ;\ r \le s \\
&= 0 \textit{ Otherwise}
\end{aligned}
$$

Now we have

$$
\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = -\frac{1}{2\left(n+1\right)}\begin{bmatrix} n-1 \\ n-3 \\ \vdots \\ 3-n \\ 1-n \end{bmatrix}
$$

and

$$
\mathbf{V} = \left[V_{r,s}\right] = \frac{1}{\left(n+1\right)^2\left(n-2\right)}\begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ 0 & 2\left(n-1\right) & 2\left(n-2\right) & \cdots & 2 \\ 0 & 0 & 3\left(n-2\right) & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.
$$

Also

$$
\mathbf{V}^{-1} = \left(n+1\right)\left(n+2\right)\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}.
$$

Further

$$
\begin{aligned}
\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1} &= 2\left(n+1\right)\left(n+2\right) \\
\alpha^{/}\mathbf{V}^{-1}\alpha &= \frac{1}{2}\left(n+2\right)\left(n-1\right)
\end{aligned}
$$

$$\alpha^{/}\mathbf{V}^{-1}\mathbf{y} = \frac{(n+1)(n+2)}{2}(y_{n:n} - y_{1:n})$$
$$\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y} = (n+1)(n+2)(y_{n:n} + y_{1:n})$$

The ordered least square estimators are therefore

$$\hat{\mu} = \frac{\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{y}}{\mathbf{1}^{/}\mathbf{V}^{-1}\mathbf{1}} = \frac{1}{2}(y_{n:n} + y_{1:n})$$

and

$$\hat{\sigma} = \frac{\alpha^{/}\mathbf{V}^{-1}\mathbf{y}}{\alpha^{/}\mathbf{V}^{-1}\alpha} = \frac{n+1}{n-1}(y_{n:n} - y_{1:n}).$$

The ordered least square estimation is computer extensive and in certain cases we need to use numerical methods to obtain the ordered least square estimators as analytical solutions are very complicated.

6.3.4 Bayes Estimation Using Ordered Variables

Bayes estimation is a popular method of parameter estimation when parameters are random variables. The conventional Bayes method is based upon computing parameter estimates by using posterior distribution of the parameter given as

$$f(\theta|\mathbf{x}) = \frac{f(\mathbf{x}|\theta)\, g(\theta)}{\int_{\Re} f(\mathbf{x}|\theta)\, g(\theta)\, d\theta}, \tag{6.33}$$

where $f(\mathbf{x}|\theta)$ is joint distribution of data and $g(\theta)$ is prior distribution of the parameter. In computing Bayes estimates by using ordered random variables the joint distribution of data in (6.33) is replaced by the joint distribution of any model of ordered random variables. For example if the joint distribution of GOS given in (4.1) is replaced in (6.33) then Bayes estimation is based upon GOS.

Example 6.9 A random sample of size n is drawn from the distribution

$$f(x;\theta) = \theta e^{-\theta x}\ ;\ x, \theta > 0.$$

Compute Bayes estimator of θ by using GOS if prior distribution of θ is

$$f(\theta;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\ ;\ \theta, \alpha, \beta > 0.$$

Solution: The Bayes estimator of θ is the mean of posterior distribution of θ given as

$$f\left(\theta|\mathbf{x}_{n,m,k}\right)=\frac{f\left(\mathbf{x}_{n,m,k}|\theta\right)g\left(\theta\right)}{\int_{\Re}f\left(\mathbf{x}_{n,m,k}|\theta\right)g\left(\theta\right)d\theta},$$

where $f\left(\mathbf{x}_{n,m,k}|\theta\right)$ if the joint distribution of GOS given as

$$f\left(\mathbf{x}_{n,m,k};\theta\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\{1-F\left(x_n;\theta\right)\}^{k-1}f\left(x_n;\theta\right)\\ \times\left[\prod_{i=1}^{n-1}\{1-F\left(x_i;\theta\right)\}^{m}f\left(x_i;\theta\right)\right].$$

Now for given distribution we have

$$f\left(x;\theta\right)=\theta e^{-\theta x}\text{ and }F\left(x\right)=1-e^{-\theta x}.$$

The joint density of GOS is therefore

$$f\left(\mathbf{x}_{n,m,k};\theta\right)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{e^{-\theta x_i}\right\}^{k-1}\theta e^{-\theta x_i}\left[\prod_{i=1}^{n-1}\left\{e^{-\theta x_i}\right\}^{m}\theta e^{-\theta x_i}\right]$$

or

$$\begin{aligned}f\left(\mathbf{x}_{n,m,k};\theta\right)&=C_{n-1}\theta^{n}e^{-k\theta x_i}\prod_{i=1}^{n-1}e^{-(m+1)\theta x_i}\\&=C_{n-1}\theta^{n}\exp\left[-\theta\left\{kx_i+(m+1)\left(\sum_{i=1}^{n-1}x_i\right)\right\}\right]\\&=C_{n-1}\theta^{n}\exp\left(-\theta u\right)\end{aligned}$$

where $C_{n-1}=k\left(\prod_{j=1}^{n-1}\gamma_j\right)$ and $u=\left\{kx_i+(m+1)\left(\sum_{i=1}^{n-1}x_i\right)\right\}$.

Now

$$\begin{aligned}f\left(\mathbf{x}_{n,m,k};\theta\right)g\left(\theta\right)&=C_{n-1}\theta^{n}\exp\left(-\theta w\right)\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\theta^{\alpha-1}e^{-\beta\theta}\\&=C_{n-1}\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\theta^{n+\alpha-1}\exp\left\{-\theta\left(\beta+u\right)\right\},\end{aligned}$$

also

$$\begin{aligned}\int_0^{\infty}f\left(\mathbf{x}_{n,m,k};\theta\right)g\left(\theta\right)d\theta&=\int_0^{\infty}C_{n-1}\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)}\theta^{n+\alpha-1}\exp\left\{-\theta\left(\beta+u\right)\right\}d\theta\\&=\frac{C_{n-1}\beta^{\alpha}\Gamma\left(n+\alpha\right)}{\left(\beta+u\right)^{n+\alpha}\Gamma\left(\alpha\right)},\end{aligned}$$

hence

$$f\left(\theta|\mathbf{x}_{n,m,k}\right) = \frac{(\beta+u)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}\exp\{-\theta(\beta+u)\}.$$

The Bayes estimator of θ is therefore

$$\begin{aligned}\hat{\theta} &= E_\theta(\theta) = \int_{\Re}\theta f\left(\theta|\mathbf{x}_{n,m,k}\right)d\theta \\ &= \int_0^\infty \theta\frac{(\beta+u)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}\exp\{-\theta(\beta+u)\}\,d\theta \\ &= \frac{(\beta+u)^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^\infty \theta^{n+\alpha}\exp\{-\theta(\beta+u)\}\,d\theta.\end{aligned}$$

Integrating we have

$$\begin{aligned}\hat{\theta} &= \frac{\Gamma(n+\alpha+1)}{(\beta+u)\Gamma(n+\alpha)} = \frac{n+\alpha}{\beta+u} \\ &= \frac{n+\alpha}{\beta+\left\{kx_i+(m+1)\left(\sum_{i=1}^{n-1}x_i\right)\right\}}.\end{aligned}$$

We can see that the estimator reduces to conventional Bayes estimator for $m=0$ and $k=1$.

Chapter 7
Characterizations of Distribution

7.1 Introduction

Characterization of probability distributions play important role in probability and statistics. Before a probability distribution is applied to a real set of data it is necessary to know the distribution that fits the data by characterization. A probability distribution can be characterized by various methods, see for example Ahsanullah et al. (2014). In this chapter we will characterize probability distributions by various properties of ordered data. We will consider order statistics, record values and generalized order statistics to characterize the probability distributions.

7.2 Characterization of Distributions by Order Statistics

We assume that we have n (fixed) number of observations from an absolutely continuous distribution with cdf F(x) and pdf f(x). Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding order statistics. We will use order statistics for various characterization of probability distributions.

7.2.1 Characterization of Distributions by Conditional Expectations

We assume that E(X) exists. We consider that $E(X_{j,n}|X_{i,n} = x) = ax + b \;\; j > i \geq 1$. Fisz (1958) considered the characterization of exponential distribution by considering j = 2, i = 1 and $a = 1$. Roger (1963) characterized the exponential distribution by considering j = i + 1 and $a = 1$. Ferguson (1963) characterized following distributions with j = i + 1.

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_7

(i) Exponential distribution with $a = 1$
(ii) Pareto distribution with $a > 1$
(iii) Power function distribution with $a < 1$.

The following useful theorem is given by Gupta and Ahsanullah (2004).

Theorem 7.1 *Under some mild conditions on* $\Psi(x)$ *and* g(x) *the relation*

$$E(\psi(X_{i+s,n})|X_{i,n} = x) = g(x), s \geq 1 \tag{7.1}$$

Uniquely determines the distribution F(x).

The relation (7.1) for s = 1 will lead to the equation

$$r(x) = \frac{g\prime(x)}{(n-i)(g(x) - \psi(x))} \tag{7.2}$$

Here r(x) = f(x)/(1 − F(x)), the hazard rate of X. If $\Psi(x) = x$ *and g(x) = ax + b, then we obtain from (7.2)*

$$r(x) = \frac{a}{(n-i)((a-1)x + b)} \tag{7.3}$$

From (7.3) we have

(i) If $a = 1$*, then r(x) = constant and X has the exponential distribution with* $F(x) = 1 - e^{-\lambda(x-\mu)}, x \geq \mu$*, and* $\lambda = \frac{1}{b(n-i)}, x \geq \mu, b > 0$.
(ii) If $a > 1$*, then X will have the Pareto distribution with* $F(x) = 1 - (x - \frac{b}{a-1})^{-\frac{a}{(a-1)(n-i)}}, x \geq \frac{b}{a-1}$
(iii) If $a < 1$*, then X will have power function distribution with* $F(x) = 1 - (\frac{b}{1-a} - x)^{\frac{a}{(1-a)(n-i)}}, 0 \leq x \leq \frac{b}{1-a}$.

Wesolowski and Ahsanullah (2001) have extended the result of Ferguson's (1963), which we have given in the following theorem.

Theorem 7.2 *Suppose that X is an absolutely continuous random variable with cumulative distribution function* $F(x)$ *and probability distribution function f(x). If* $E(X_{k+2,n}) < \infty$*, 1 ≤ k ≤ n − 2, n > 2, then* $E(X_{k+2,n}|X_{k,n} = x) = ax + b$ *iff*

(i) $a > 1$, $F(x) = 1 - (\frac{\mu+\delta}{x+\delta})^{\theta}$, $x \geq \mu, \theta > 1$

where μ *is a real number,* $\delta = b/(a-1)$ *and*

$$\theta = \frac{a(2n-2k-1) + \sqrt{a^2 + 4a(n-k)(n-k-1)}}{2(a-1)(n-k)(n-k-1)}$$

(ii) $a = 1$, $F(x) = 1 - e^{-\lambda(x-\mu)}, x \geq \mu$,

$$b = \frac{2n-2k-1}{\lambda(n-k)(n-k-1)}, \lambda > 0$$

(iii) $a < 1$, $F(x) = 1 - (\frac{\nu - x}{\nu + \mu})^{\theta}, 0 \le x \le \nu, \nu = \frac{b}{1-a}$ *and*

$$\theta = \frac{a(2n-2k-1) + \sqrt{a^2 + 4a(n-k)(n-k-1)}}{2(1-a)(n-k)(n-k-1)}.$$

Deminska and Wesolowski (1998) gave another general result which we give in following theorem.

Theorem 7.3 *Suppose that X is an absolutely continuous random variables with cumulative distribution function F(x) and probability distribution function f(x). If* $E(X_{i+r,n}) < \infty, 1 \le i \le n-r, r \ge 1, n \ge 2$, *then* $E(X_{i+r}|X_{i,n} = x) = ax + b$ *iff*

(i) $a > 1, F(x) = 1 - (\frac{\mu+\delta}{x+\delta})^{\theta}, \ge \mu, \theta > \frac{1}{n-k-r+1}$

where μ *is a real number,*

$$a = \frac{n(n-k)!}{(n-k-r)!} \sum_{m=0}^{r-1} \frac{1}{m!(r-1-m)!} \frac{(-1)^m}{[\theta(n-k-r+1+m)-1]}$$

$$b = \delta \frac{\theta(n-k)}{(n-k-r)!} \sum_{m=0}^{r-1} \frac{1}{m!(r-1-m)!} \frac{(-1)^m}{n(n-r+1+m)!\theta(n-k-r+1+m)[\theta(n-k-r+1+m)-1]}$$

(ii) $a < 1, F(x) = 1 - (\frac{\upsilon - x}{\upsilon - \mu})^{\theta}, \mu \le x \le \nu.$

$$b = \nu \frac{\theta(n-k)}{(n-k-r)!} \sum_{m=0}^{r-1} \frac{1}{m!(r-1-m)!} \frac{(-1)^m}{\theta(n-k-r+1+m)[\theta(n-k-r+1+m)+1]}$$

$$a = \frac{\theta(n-k)}{(n-k-r)!} \sum_{m=0}^{r-1} \frac{1}{m!(r-1-m)!} \frac{(-1)^m}{[\theta(n-k-r+1+m)+1]}$$

(iii) $a = 1, F(x) = 1 - e^{-\lambda(x-\mu)}, x \ge \mu,$

$$b = \frac{(n-k)!}{\lambda(n-k-r)!} \sum_{m=0}^{r-1} \frac{1}{m!(r-1-m)!} \frac{(-1)^m}{(n-k-r+1+m)^2}$$

Consider the extended sample case. Suppose in addition to n sample observations, we take another m observations from the same distribution. We order the m+n observations. The combined order statistics is $X_{1:m+n} \le X_{2:m+n} < \cdots < X_{m+n:m+n}$. *We assume* $F(x)$ *is the cdf of the observations.*

Ahsanullah and Nevzorov (1997) proved the following theorem, Theorem 7.4 if $E(X_{1:n}|X_{1:m+n} = x) = ax + m(x)$, *then*

(i) For $a = 1$, *F(x) is exponential with* $F(x) = 1 - exp(-x), x > 0$ *and* $m(x) = \frac{m}{n(m+n)}$

(ii) *For* $a > 1$, *F(x) is Pareto with* $F(x) = 1 - (x - 1)^{-\partial}$, $x > 1$, $\partial > 0$ *and* $m(x) = \frac{m(x-1)}{(m+n)(m\partial+1)}$

(iii) *For* $a < 1$, *F(x) is Power function with* $F(x) = 1—(1 - x)^{\partial}, 0 < x < 1, \partial > 0$ *and* $m(x) = \frac{m(1-x)}{(m+n)(m\partial+1)}$.

7.2.2 Characterization by Identical Distribution

It is known that for exponential distribution $X_{1:n}$ and X are identically distributed. Desu (1971) proved that if $X_{1:n}$ and X are equally distributed for all *n*, then the distribution of X is exponential. Ahsanullah (1977) proved that with a mild condition on F(x) if for a fixed n, $X_{1:n}$ and X are identically distributed, then the distribution of X is exponential Ahsanullah (1978a, b, 1984) prove that the

identical distributions of $D_{i+1:n}$ and $X_{n-I:n}$ characterize the exponential distribution.

We will call a distribution function "new better than used, (NBU) if $1 - F(x + y) \leq (1 - F(x))(1 - F(y))$ for all $x, y \geq 0$" and "new worse than used (NWU) if $1 - F(x + y) \geq (1 - F(x))(1 - F(y))$".

For all $x, y \geq 0$. We say that F belongs to class c_0 if F is either NBU or NWU. We say that F belongs to the class c if the hazard rate $(f(x)/(1 - F(x))$ is either increasing or decreasing.

The following theorem by Ahsanullah (1976) gives a characterization of exponential distribution based on the equality of two spacing.

Theorem 7.4 *Let X be a non-negative random variable with an absolutely continuous cumulative distribution function F(x) that is strictly increasing in* $[0, \infty)$ *and having probability density function f(x), then the following two conditions are identical.*

(a) F(x) has an exponential distribution with $F(x) = \exp(-\lambda x), x \geq 0, \lambda > 0$ *for some i, j,* $\underline{i \leq} < j < n$ *the statistics* $D_{j:n}$ *and* $D_{i:n}$ *are identically distributed and F belongs to the class C.*

Proof We have already seen (a) ⇒ (*b*). We will give here the proof of (b) ⇒ (a).

The conditional pdf of $D_{j:n}$ given $X_{i:n} = x$ is given by

$$f_{D_{i,n}}(d|X_{i,n}) = k\int_d^{\infty} (\overline{F}(x) - \overline{F}(x+s))(\overline{F}(x))^{-(i-i-1)} \cdot (\overline{F}(x+s+\frac{d}{n-j})/(\overline{F}(x))^{-1})^{n-j-1}$$

$$\frac{f(x+s)}{\overline{F}(x)}\frac{f(x+s+\frac{d}{n-j})}{\overline{F}(x)}ds \tag{7.4}$$

where $k = \frac{(n-i)!}{(j-i-1)!(n-j)!}$.

Integrating the above equation with respect to d from d to ∞, we obtain

$$\overline{F}_{D_{j,n}}(d||X_{i,n}) = k\int_d^{\infty} (\overline{F}(x) - \overline{F}(x+s))(\overline{F}(x))^{-(i-i-1)} \cdot (\overline{F}(x+s+\frac{d}{n-j})/(\overline{F}(x))^{-1})^{n-j-1}$$

$$\frac{f(x+s)}{\hat{F}(x)}ds$$

The conditional probability density function f $_{i:n}$ of D$_{i:n}$ given X$_{i,n}-$ x is given by

$$f_{D_{i+1:n}}(d|X_{i,n}=x) = (n-i)\frac{(\overline{F}(d+\frac{x}{n-r}))^{n-i-1}}{(\overline{F}(x))^{n-i}}\frac{f(u+\frac{x}{n-i})}{\overline{F}(x)}$$

The corresponding cdf F$_{i:n}$(x) is giving by $1 - \text{F}_{D_{i+1:n}} = \frac{(\bar{F}(d+\frac{x}{n-r}))^{n-i}}{(\bar{F}(x))^{n-i}}$.

Using the relations

$$\frac{1}{k}\int_0^{\infty}(\frac{\overline{F}(x+s)}{\overline{F}(x)})^{n-j}(\frac{\overline{F}(x)-\overline{F}(x+s)}{\overline{F}(x)})^{j-i-1}\frac{f(x+s)}{\overline{F}(x)}ds$$

and the equality of the distribution of D$_{\text{j:n}}$ and D$_{\text{i:n}}$ given X$_{\text{i:n}}$, we obtain

$$\int_0^{\infty}(\frac{\overline{F}(x+s)}{\overline{F}(x)})^{n-j}(\frac{\overline{F}(x)-\overline{F}(x+s)}{\overline{F}(x)})^{j-i-1}G(x,d,s)\frac{f(x+s)}{\overline{F}(x)}ds = 0 \tag{7.5}$$

where

$$G(x,d,s) = (\frac{\overline{F}(x+\frac{d}{n-i})}{\overline{F}(x)})^{n-i} - (\frac{\overline{F}(x+s+\frac{d}{n-j})}{\overline{F}(x+s)})^{n-j}. \tag{7.6}$$

Differentiating (7.5) with respect to s, we obtain

$$\frac{\partial}{\partial s}G(x,s,d) = (\frac{\bar{F}(x+s+\frac{d}{n-j})}{\bar{F}(x+s)})^{n-j}(r(x+s+\frac{d}{n-i}) - r(x+s)) \tag{7.7}$$

(i) If F has IHR, then G(x, s, d) is increasing with s. Thus (7.5) to be true, we must have G(x, 0, d) $\le G(x,s,d) \le$ 0. If F has IFR, then $ln\bar{F}$ is concave and

$$\ln(\bar{F}(x+\frac{d}{nn-i}) \ge \frac{j-i}{n-i}\ln(\bar{F}(x)) + \frac{n-j}{n-i}\ln(\bar{F}(x+\frac{d}{n-j}))$$

i.e.

$$(\bar{F}(x+\frac{d}{nn-i}))^{n-i} \ge (\bar{F}(x))^{j-i}(\bar{F}(x+\frac{d}{n-j}))^{n-j}.$$

Thus G(x, 0, d) $\geq$ 0. Thus (7.5) to be true we must have G(x, 0, d) = 0 for all d and any given x.

(ii) If F has DHR, then similarly we get G(x, 0, d) = 0. Taking x = o, we obtain from G(x, 0, d) as

$$(\bar{\bar{F}}(\frac{d}{n-i})^{n-i} \quad = (\bar{F}(\frac{d}{n-j}))^{n-j} \tag{7.8}$$

for all d $\geq$ 0 and some i, j, n with $1 \leq i < j < n$.

Using $\varphi(d) = \ln(\bar{F}(d))$ we obtain (n − i) $\varphi(\frac{d}{n-i}) = (n - j\varphi)(\frac{d}{n-j})$

Putting $\frac{d}{n-i} = t$, we obtain

$$\varphi t(t) = \frac{jn-j}{n-i}\varphi t(\frac{n-i}{n-j}t) \tag{7.9}$$

The non zero solution of (7.9) is

$$\varphi(x) = x \tag{7.10}$$

for all x $\geq$ 0.

Using the boundary conditions F(x) = 0 and $_{F(\infty)1}$, we obtain

$$F(x) = 1 - e^{-\lambda x}, x \geq 0, \lambda > 0. \tag{7.11}$$

for all x $\geq$ 0 and $\lambda > 0$.

In the Theorem 7.5 under the assumption of finite expectation of X, the equality of the spacings can be replaced by the equality of the expectations.

7.2.3 *Characterization by Independence Property*

Fisz (1958) gave the following characterization theorem based upon independence property.

Theorem 7.5 *If X_1 and X_2 are independent and identically distributed with continuous cumulative distribution function F(x). Then $X_{2:2} - X_{1:2}$ is independent of $X_{1:2}$ if and only if F(x) = 1 − exp(−λx), x ≥ 0 and* $\lambda > 0$.

Proof The "if" condition is easy to show. We will show "the only if" condition.

We have

$$P(X_{2:2} - X_{1:2} > y|X_{1:2} = x) = P(X_{2:2} > x + y|X_{1:2} = x) = \frac{1 - F(x + y)}{1 - F(x)}.$$

Since $\frac{1-F(x+y)}{1-F(x)}$ is independent of X, we must have $\frac{1-F(x+y)}{1-F(x)}$ = g(y) where g(y) is a function of y only. Taking x $\Rightarrow$ 0, we obtain from the above equation

$$1 - F(x + y) = (1 - F(x))(1 - F(y)) \tag{7.12}$$

for all λ > 0 and almost all x≥ 0.

The non zero solution of the above equation with the boundary conditions F(0) = 0 and $F(\infty) = 1$ is F(x) = 1 − exp(−λx) for all λ > 00 and almost all x ≥0.

The following theorem is a generalization of Fisz (1958) result.

Theorem 7.6 *If X_1 $X_{2,}$ . . .X_n are independent and identically distributed random variables with continuous cumulative distribution function F(x). Then $X_{n:n}$ $-X_{1:n}$ and $X_{1:n}$ are independent of if and only if F(x) = 1 − exp(− λx), x ≥ 0 and* λ > 0.

The proof of the only if condition will lead to the following differential equation

$$\left(\frac{(1 - F(x) - (1 - F(y + x))}{1 - F(x)}\right)^{n-1} = g(y)$$

where g(y) is a function of y for all x ≥ 0 a and almost all y ≥ 0. The following theorem gave a more general result.

Theorem 7.7 *Suppose that X_1 $X_{2,}$ . . .X_n are independent and identically distributed random variables with continuous cumulative distribution function F(x). Then $X_{j:n} - X_{i:n}$ and $X_{1:n}$ are independent if and only if F(x) = 1 − exp(− λx), x ≥ 0 and* λ > 0.

The proof of the only if condition will be similar to the proof of the following theorem of Ahsanullah and Kabir (1973).

Theorem 7.8 *Suppose that X_1 $X_{2...,}X_n$ are independent and identically distributed with continuous cumulative distribution function F(x) and probability density function f(x). Then $\frac{X_{j:n}}{X_{i:n}}$ and $X_{i,n}$ are independent if and only if $F(x) = x^{-\beta}$ for all $x \geq 1$ and $\beta > 0$.*

Proof We will give here the proof of the only if condition.

Writing V = $X_{j:n}|X_{i:n}$ and U = $X_{i:n}$, the conditional pdf h(v) of V given U = u can be written as

$$H(v) = c\left(\frac{F(uv) - F(u)}{1 - F(u)}\right)^{j-i-1}\left(\frac{1 - F(uv)}{1 - F(u)}\right)^{s-r-1}\frac{uf(uv)}{1 - F(u)} \tag{7.13}$$

where $c = \frac{(n-i)!}{(j-i-1)!(n-j)!}$.

Since U and V are independent h(v) will be independent of u. Thus we will need for some j and i, 1≤ i < j≤ n$(1 - q(u,v))^{j-i-1}(q(u,v))^{n-j}\frac{\delta q(u,v)}{\delta v}$ to be independent of u. where $q(uv) = \frac{1-F(uv)}{1-F(u)}$.

The independence of (7.13) of u will lead to the functional equation

$$1 - F(uv) = (1 - F(u))(1 - F(v)) \tag{7.14}$$

For all $u \geq$ *and* $v \geq 1$.

The solution of the Eq. (7.14) with the boundary condition F(1) = 0 and $F(\infty) = 1$ is

$$F(x) = 1 - (\frac{1}{x})^{\beta} \tag{7.15}$$

where for all $x \geq 1$ and $\beta > 0$

The following theorem gives a characterization of the power function distribution.

Theorem 7.9 *Suppose that* $X_1, X_{2,...,}X_n$ *are independent and identically distributed with continuous cumulative distribution function F(x) and probability density function f(x), then* $\frac{X_{i:n}}{X_{j:n}}$ *and* $X_{i,n}$ *are independent if and only if* $F(x) = x^{\beta}$*, for all o*$\leq x \leq 1$ *and* $\beta > 0$.

The proof of only if condition is similar to Theorem 7.8.

7.3 Characterization of Distributions by Record Values

Suppose {X_i, i = 1,2,...} be a sequence of independent and identically distributed random variables with cdf F(x) and pdf f(x). We assume E(X_i) exists. Let X(n), n $\geq$ 1 be the corresponding upper records.

7.3.1 Characterization Using Conditional Expectations

We have the following theorem to determine F(x) based on the conditional expectation.

Theorem 7.10 *The condition* $E(\psi(X(k+s)|X(k) = z) = g(z)$ *where k,s ≥1 and* $\psi(x)$ *is a continuous function, determines the distribution F(x) uniquely*

Proof Consider

$$E(\psi(X(k+s)|X(k) = z) = \int_z^{\infty} \frac{\psi(x)(R(x) - R(z))^{s-1}}{\bar{F}(z)} f(x)dx \tag{7.16}$$

where R(x) = $-\ln\bar{F}(x)$.

Case s = 1

Using the Eq. (7.16), we obtain

$$\int_z^{\infty} \psi(x)f(x)dx = g(z)\bar{F}(z) \tag{7.17}$$

Differentiating both sides of (7.17) with respect to z and simplifying, we obtain

$$r(z) = \frac{f(z)}{\bar{F}(z)} = \frac{g\prime(z)}{g(z) - \psi(z)} \tag{7.18}$$

where r(z) is the failure rate of the distribution. Hence the result.

If $\psi(x) = x$ and g(x) $=$ ax + b, a, b $\geq$ 0, then

$$r(x) = \frac{a}{(a-1)x+b} \tag{7.19}$$

If $a \neq 1$, then $F(x) - 1 - ((a-1)x+b)^{-\frac{a}{a-1}}$, which is the power function distribution for a <1 and the Pareto distribution with a > 1. For $a = 1$, (7.19) will give exponential distribution. Nagaraja (1977) gave following characterization theorem.

Theorem 7.11 *Let F be a continuous cumulative distribution function. If, for some constants a and b, E(X(n)|X(n − 1)=x) = ax+b, then except for a change of location and scale,*

(i) $F(x) = 1-(-x)^{\theta}$, $x<0$, *if* $0< a <1$
(ii) $F(x)= 1-e^{-x}$, $x \geq 0$, *if* $a = 1$
(iii) $F(x)= 1-x^{\theta}$, $x >1$ *if* $a >1$,

where $\theta = a/(1-a)$. *Here* $a >0$.

Proof of Theorem 7.11
In this case , we obtain

$$\int_z^{\infty} \psi(x)(R(x) - R(z)f(x)dx = g(z)\bar{F}(z) \tag{7.20}$$

Differentiating both sides of the equation with respect to z, we obtain

$$-\int_z^{\infty} \psi(x)f(z)dx = g\prime(z)\frac{(\bar{F}(z))^2}{f(z)} - g(z)\bar{F}(z) \tag{7.21}$$

Differentiating both sides of (7.21) with respect to z and using the relation $\frac{f\prime(z)}{f(z)} = \frac{r\prime(z)}{r(z)} - r(z)$ we obtain on simplification

$$g'(z)\frac{r'(z)}{r(z)} + 2g'(z)r(z) = g''(z) + (r(z))^2(g(z) - \psi(z)) \tag{7.22}$$

Thus r'(z) is expressed in terms of r(z) and known functions. The solution of r(x) is unique (for details see Gupta and Ahsanullah (2004)).

Putting $\psi(x) = x$ and g(x) = ax + b, we obtain from (7.22)

$$a\frac{r'(z)}{r(z)} + 2ar(z) = (r(z))^2((a-1)a+b) \tag{7.23}$$

The solution of (7.23) is r(x) = $\frac{a+\sqrt{a}}{(a-1)\,x+b}$. Thus X will have (i) exponential distribution if a=1, (ii) power function distribution if $a<1$ and (iii) Pareto distribution if $a>1$.

Ahsanullah and Wesolowski (1998) extended the result Theorem 7.11 for non adjacent record values. Their result is given in the following theorem.

Theorem 7.12 *If E(X(n+2)|X(n)) = a X(n) + b, n≥1 where the constants a and b , then if:*

(a) a =1, then X_i has the exponential distribution,
(b) $a<1$, then X_i has the power function distribution
(c) $a>1$, X_i has the Pareto distribution

Proof of Theorem 7.12 for s > 2.
In this case, the problem becomes more complicated because of the nature of the resulting differential equation

Lopez-Blazquez and Moreno-Rebollo (1997) also gave characterizations of distributions by using the following linear property

$$E(X(k)|X(k+s)=z)=az+b, k\geq 1\,k, s\geq 1,$$

Raqab (2002) and Wu (2004) considered this problem for non-adjacent record values under some stringent smoothness assumptions on the distribution function $F(x)$. Dembinska and Wesolowski (2000) characterized the distribution by means of the relation $E(X(s+k)||X(k)=z)=az+b$, for $k\geq 1, s\geq 1$.

They used a result of Rao and Shanbhag (1994) which deals with the solution of extended version of integrated Cauchy functional equation. It can be pointed out earlier that Rao and Shanbhag's result is applicable only when the conditional expectation is a linear function.

Theorem 7.13 *Let $\{X_n, n\geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F which is absolutely continuous with pdf f. Assume that F(0) = 0 and F(x) > 0 for all x > 0. Then X_n to have the cdf, $F(x)=1-e^{-x/\sigma}, x\geq 0, \sigma>0$, it is necessary and sufficient that X(n) –X(n–1) and X(n–1), $n\geq 2$, are independent.*

Proof It is easy to establish that if X_n has the cdf, $F(x)=1-e^{-x/\sigma}, x\geq 0, \sigma>0$, then X(n) –X(n – 1) and X(n – 1) are independent. Suppose that X(n) – X(n-1) and X(n), n ≥ 1, are independent. Now the joint pdf f(z, u) of Z = X(n) – X(n-1) and U = X(n) can be written as

$$\begin{aligned} f(z,u) &= \frac{[R(u)]^{n-1}}{\Gamma(n)}\, r(u) f(u+z), 0<u, z<\infty. \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{7.24}$$

But the pdf f_n (u) of X(n) can be written as

$$f_{n-1}(u) = \frac{[R(u)]^{n-1}}{\Gamma(n)} f(u), 0 < u < \infty,$$
$$= 0, \quad \text{otherwise.} \tag{7.25}$$

Since Z and U are independent, we get from (7.24) and (7.25)

$$\frac{f(u+z)}{\bar{F}(u)} = g(z), \tag{7.26}$$

where g(z) is the pdf of u. Integrating (7.26) with respect z from 0 to z_1, we obtain on simplification

$$\bar{F}(u) - \bar{F}(u+z_1) = \bar{F}(u)G(z_1) \,. \tag{7.27}$$

Since $G(z_1) = \int_0^{z_1} g(z)dz$. Now $u \to 0^+$ and using the boundary condition $\bar{F}(0) = 1$, we see that $G(z_1) = F(z_1)$. Hence we get from (7.27)

$$\bar{F}(u+z_1) = \bar{F}(u)\,\bar{F}(z_1). \tag{7.28}$$

The only continuous solution of (7.28) with the boundary conditions $\bar{F}(0) = 0$ and $\bar{F}(\infty) = 1$,is

$$\bar{F}(x) = e^{-\sigma^{-1}x}, x \geq 0 \tag{7.29}$$

where σ is an arbitrary positive real number.

The following theorem is a generalization of the Theorem 7.13.

Theorem 7.14 *Let $\{X_n, n \geq 1\}$ be independent and identically distributed with common distribution function F which is absolutely continuous and F(0) = 0 and F(x) < 1 for all x > 0. Then X_n has the cdf, $F(x) = 1 - e^{-\sigma x}, x \geq 0, \sigma > 0$, it is necessary and sufficient that X(n) –X(m) and X(m), n > m ≥ 1, are independent.*

Proof The necessary condition is easy to establish. To prove the sufficient condition, we need the following lemma.

Lemma 7.1 *Let F(x) be an absolutely continuous function and $\bar{F}(x) > 0$, for all x > 0. Suppose that $\bar{F}(u+v)(\bar{F}(v))^{-1} = \exp\{-q(u,v)\}$ and $h(u,v) = \{q(u,v)\}^r \exp\{-q(u,v)\} \frac{\partial}{\partial u} q(u,v)$, for r ≥ 0. Further if h(u, v) ≠ 0, and $\frac{\partial}{\partial u} q(u,v) \neq 0$ for any positive u and v. If h(u, v) is independent of v, then q(u, v) is a function of u only.*

We refer to Ahsanullah and Kabir (1974) for the proof of the lemma.

Proof: of the sufficiency part of Theorem 7.14.

The conditional pdf of Z = X(n) – X(m) given V(m) = x is $f(z|X(m) = x) = \frac{1}{\Gamma(n-m)} [R(z+x) - R(x)]^{n-m-1} \frac{f(z+x)}{\bar{F}(x)}$, $0 < z < \infty, 0 < x < \infty$.

Since Z and X(m) are independent, we will have for all z > 0,

$$(R(z+x) - R(x))^{n-m-1} \frac{f(z+x)}{\bar{F}(x)} \tag{7.30}$$

as independent of x. Now let $R(z + x) - R(x) = -\ln \frac{\bar{F}(z+x)}{\bar{F}(x)} = q(z, x)$*, say.*

Writing (7.30) in terms of q(z, x), we get

$$[q(z, x)]^{n-m-1} \exp\{-q(z, x)\} \frac{\partial}{\partial z} q(z, x), \tag{7.31}$$

as independent of x. Hence by the Lemma 7.1 we have

$$-\ln\{\bar{F}(z + x)(\bar{F}(x))^{-1}\} = q(z + x) = c(z), \tag{7.32}$$

where c(z) is a function of z only. Thus

$$\bar{F}(z + x)(\bar{F}(x))^{-1} = c_1(z), \tag{7.33}$$

and $c_1(z)$ *is a function of z only.*

The relation (7.33) is true for all $z \geq 0$ *and any arbitrary fixed positive number x. The continuous solution of (7.33) with the boundary conditions,* $\bar{F}(0) = 1$ *and* $\bar{F}(\infty) = 0$ *is*

$$\bar{F}(x) = \exp(-x\sigma^{-1}), \tag{7.34}$$

for $x \geq 0$ *and any arbitrary positive real number* σ*. The assumption of absolute continuity of F(x) in the Theorem can be replaced by the continuity of F(x).*

Chang (2007) gave an interesting characterization of the Pareto distribution. Unfortunately the statement and the proof of the theorem were wrong. Here we will give a correct statement and proof of his theorem.

Theorem 7.15 *Let* $\{X_n, n \geq 1\}$ *be independent and identically distributed with common distribution function F which is absolutely continuous and F(1) = 0 and F(x) < 1 for all x > 1. Then* X_n *has the cdf,* $F(x) = 1 - x^{-\theta}, x \geq 1, \theta > 0$*, it is necessary and sufficient that* $\frac{X(n)}{X(n+1)-X(n)}$ *and X(m),* $n \geq 1$ *are independent.*

Proof If $F(x) = 1 - x^{-\theta}, x \geq 1, \theta > 0$, then the joint pdf $f_{n,n+1}(x,y)$ of X(n) and X(n + 1) is

$$f_{n,n+1}(x, y) = \frac{1}{\Gamma(n)} \frac{\theta^{n+1}(\ln x)^{n-1}}{xy^{\theta+1}}, 1 < x < y < \infty, \theta > 0.$$

Using the transformation, U = X(m) and V = $\frac{X(n)}{X(n+1)-X(n)}$. The joint pdf f_{UV}(u, v) can be written as

$$f_{W,V}(w, v) = \frac{1}{\Gamma(n)} \frac{\theta^{n+1}(\ln u)^{n-1}}{u^{\theta+3}} \left(\frac{v}{1+v}\right)^{\theta+1}, 1 < u, v < \infty, \theta > 0. \tag{7.35}$$

Thus U and V are independent.

The proof of sufficiency.

The joint pdf of U and V can be written as

$$f_{W,V}(u,v) = \frac{(R(u))^{n-1}}{\Gamma(n)} r(u) f(\frac{1+v}{v}u)\frac{u}{V^2}, 1 < u, v < \infty, \tag{7.36}$$

where R(x) = −ln(1-F(x)) and r(x) = $\frac{d}{dx}R(x)$.

We have the pdf f_U(u) of U as $f_U(u) = \frac{(R(u))^{n-1}}{\Gamma(n)} f(u)$. Since U and V are independent, we must have the pdf f_V(v) of V as

$$f_V(v) = f(\frac{1+v}{v}u)\frac{u}{v^2}\frac{1}{1-F(u)}, 0 < v < \infty. \tag{7.37}$$

Integrating above pdf from v_0 to ∞, we obtain

$$1 - F(v_0) = \frac{1 - F(\frac{1+v_0}{v_0}u)}{1 - F(u)} \tag{7.38}$$

Since F(v_0) is independent of U, we must have

$$\frac{1 - F(\frac{1+v_0}{v_0}u)}{1 - F(u)} = G(v_0) \tag{7.39}$$

where G(v_0) is independent of u.

Letting u → 1, we obtain G(v_0) = $1 - F(\frac{1+v_0}{v_0})$.

We can rewrite (7.39) as

$$1 - F(\frac{1+v_0}{v_0}u) = (1 - F(\frac{1+v_0}{v_0})(1 - F(u)) \tag{7.40}$$

Since the above equation is true all u $\geq$ 1 and almost all v_0 $\geq$1, we must have F(x) = $1 - x^{\beta}$. Since F(1) = 0 and $F(\infty) = 1$, we must have F(x) = $1 - x^{-\theta}$, x $\geq$ 1 and $\theta > 0$.

The following theorem is a generalization of Chang's (2007) result.

Theorem 7.16 *Let {X_n, n $\geq$ 1} be independent and identically distributed with common distribution function F which is absolutely continuous and F(1) = 0 and F(x) < 1 for all x > 0. Then X_n has the cdf,* $F(x) = 1 - x^{-\theta}, x \geq 1, \theta > 0$, *it is necessary and sufficient that* $\frac{X(m)}{X(n)-X(m)}$, *1 $\leq$ m < n and X(m) are independent.*

Proof The joint pdf $f_{m,n}$(x, y) of X(n) and X(m) is

$$f_{m,n}(x,y) = \frac{(R(x))^{m-1}}{\Gamma(m)}\frac{(R(y) - R(x))^{n-m-1}}{\Gamma(n-m)} r(x) f(y), \tag{7.41}$$

We have F(x) = $1 - x^{-\theta}$, R(x) = $\theta \ln x$, $r(x) = \frac{\theta}{x}$, thus we obtain

$$f_{m,n}(x,y) = \frac{(\theta \ln x)^{m-1}}{\Gamma(m)}\frac{(\ln y - \ln x)^{n-m-1}}{\Gamma(n-m)}\frac{1}{xy^{\theta+1}}. \tag{7.42}$$

where 1≤ x < y < ∞, $\theta > 0$.

Using the transformation U = X(m) and V = $\frac{X(m)}{X(n)-X(m)}$, we obtain the pdf $f_{U,V}$(u, v) of U and V as

$$f_{U,V}(u, v = \frac{\theta^n (\ln u)^{n-1}}{\Gamma(n)} \frac{(\ln(\frac{1+v}{v}))^{n-m-1}}{\Gamma(n-m)} \frac{v^{\theta-1}}{u^{\theta+1}(1+v)^{\theta+1}}$$

Thus X(m) and $\frac{X(m)}{X(n)-X(m)}$ are independent.

Proof of sufficiency.

Using U = X(m) and V = $\frac{X(m)}{X(n)-X(m)\_}$, we can obtain the pdf f_{UV}(u, v) of U and V from (7.42) as

$$f_{U,V}(u, v) = \frac{(R(u))^{m-1}}{\Gamma(m)} \frac{(R(\frac{u(1+v)}{v}) - R(u))^{n-m-1}}{\Gamma(n-m)} r(u) \frac{u}{v^2} f(\frac{u(1+v)}{v}), \tag{7.43}$$

We can write the conditional pdf $f_{V|U}$(v|u) of V|U as

$$f_{V|U}(v|u) = \frac{(R(\frac{u(1+v)}{v}) - R(u))^{n-m-1}}{\Gamma(n-m)} \frac{uf(\frac{u(1+v)}{v})}{v^2 \bar{F}(u)}, 1 < u < \infty, 0 \le v < \infty. \tag{7.44}$$

Using the relation R(x) = –ln $\bar{F}(x)$, we obtain from (7.44) that

$$f_{V|U}(v|u) = \frac{(-\ln(\frac{\bar{F}(\frac{u(1+v)}{v})}{\bar{F}(u)}))^{n-m-1}}{\Gamma(n-m)} \frac{d}{dv}(\frac{\bar{F}(\frac{u(1+v)}{v})}{\bar{F}(u)}), 1 < u < \infty, 0 < v < \infty. \tag{7.45}$$

Since V and U are independent, we must have $\frac{\bar{F}(\frac{u(1+v)}{v})}{\bar{F}(u)}$ independent of U.

Let

$$\frac{\bar{F}(\frac{u(1+v)}{v})}{\bar{F}(u)} = G(v), \tag{7.46}$$

Letting $u \to 1$, we obtain

$$\bar{F}(\frac{u(1+v)}{v}) = \bar{F}(u)\bar{F}(\frac{1+v}{v}), \tag{7.47}$$

For all u, 1< $u < \infty$ and all v, 0 $< v < \infty$.

The continuous solution of (7.47) with the boundary conditions F(0) = 0 and F(∞) = 1 is F(x) = $1 - x^{-\theta}$, $x \ge 1$ and $\theta > 0$.

Remark 7.1 If X_k, k ≥ 1 has an absolutely continuous distribution function F with pdf f and F(0) = 0. If $I_{n,n+1}$ and $I_{n-1,n}$, n ≥ 1, are identically distributed and F belongs to C, then X_k has the cdf F(x) = $1 - e^{-\varsigma x}$, $x \ge 0, \sigma > 0$, k ≥ 1.

7.3.2 *Characterization Based on Identical Distribution*

If F is the distribution function of a non-negative random variable, we will call F is "new better than used" (NBU) if for x, y $\geq$ 0, $\bar{F}(x + y) \leq \bar{F}(x)\,\bar{F}(y)$, and F is "new worse than used" (NWU) if for x, y $\geq$ 0, $\bar{F}(x + y) \geq \bar{F}(x)\,\bar{F}(y)$. We will say F belongs to the class C_1 if either F is NBU or NWU. We say F belongs to the class C_2 if the hazard rate r(x) = $\frac{f(x)}{1-F(x)}$ increases monotonically for all x.

Theorem 7.17 *Let X_n, $n \geq 1$ be a sequence of i.i.d. random variables which has absolutely continuous distribution function F with pdf f and F(0) = 0. Assume that F(x) < 1 for all x > 0. If X_n belongs to the class C_1 and $I_{n-1,n}$ = X(n)–X(n–1), n>1, has an identical distribution with X_k, $k \geq 1$, then X_k* has the cdf F(x) = 1− $e^{-x/\sigma}$, $x \geq 0, \sigma > 0,$

Proof The if condition is easy to establish. We will proof here the only if condition.
The pdf $f_{n-1,n}$ of $I_{n-1,n}$ can be written as

$$\mathbf{f}_{n-1,n(x,y)} = \int_0^\infty \frac{[R(u)]^{n-1}}{\Gamma(n)}\, r(u) f(u + z)\, du\ , \mathrm{z} \geq 0$$
$$= 0, \qquad \text{otherwise.} \tag{7.48}$$

By the assumption of the identical distribution of $I_{n-1,n}$ and X_k, we must have

$$\int_0^\infty [R(u)]^{n-1} \frac{r(u)}{\Gamma(n)} f(u+z)\, du = f(z), \text{ for all z > o.} \tag{7.49}$$

Substituting

$$\int_0^\infty [R(u)]^{n-1} f(u)\, du = \Gamma(n), \tag{7.50}$$

we have

$$\int_0^\infty [R(u)]^{n-1} r(u) f(u + z)\, du = f(z) \int_0^\infty [R(u)]^{n-1} f(u)\, du\ , z > 0. \tag{7.51}$$

Thus

$$\int_0^\infty [R(u)]^{n-1}\, f(u)\, [f(u + z)\, (\bar{F}(u))^{-1} - f(z)]du = 0, \mathrm{z} > 0. \tag{7.52}$$

Integrating the above expression with respect to z from z_1 to ∞, we get from (7.52)

$$\int_0^{\infty} [R(u)]^{n-1} f(u)[\bar{F}(u + z_1)(\bar{F}(u))^{-1} - \bar{F}(z_1)]du = 0, \mathrm{z}_1 > 0. \quad (7.53)$$

If F(x) is NBU, then (7.53) is true if

$$\bar{F}(u + z_1)(\bar{F}(u))^{-1} = \bar{F}(z_1) > 0, \text{for all u and almost all } z_1 > 0. \quad (7.54)$$

The only continuous solution of (7.54) with the boundary conditions $\bar{F}(0) = 1$ and $\bar{F}(\infty) = 0$ is $\bar{F}(x) = e^{-x/\sigma}, +$ where σ is an arbitrary real positive number. Similarly, if F is NWU then (7.44) is true if (7.54) is satisfied and X_k has the cdf $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0, \sigma > 0$, $k \geq 1$.

Theorem 7.18 *Let X_n, $n \geq 1$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function F(x) with f(x) as the corresponding density function. If $F \in C_2$ and for some fixed n, m, $1 \leq m < n < \infty$, $I_{m,n} \overset{d}{=} X(n-m-1)$, then X_k has the cdf $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0, \sigma > 0$, $k \geq 1$.*

Proof The pdfs $f_1(x)$ of R_{n-m} and $f_2(x)$ of $I_{m,n} = (R_n - R_m)$ can be written as

$$f_1(x) = \frac{1}{\Gamma(n-m)} [R(x)]^{n-m-1} f(x), \text{for } 0 < x < \infty, \quad (7.55)$$

and

$$f_2(x) = \int_0^{\infty} \frac{[R(u)]^{m-1}}{\Gamma(m)} \frac{[R(x+u) - R(x)]^{n-m-1}}{\Gamma(n-m)} r(u) f(u+x)\, du, 0 < x < \infty. \quad (7.56)$$

Integrating (7.55) and (7.56) with respect to x from o to x_o, we get

$$F_1(x_0) = 1 - g_1(x_0), \quad (7.57)$$

where

$$g_1(x_0) = \sum_{j=1}^{n-m} \frac{[R(x_0)]^{j-1}}{\Gamma(j)} e^{-R(x_o)},$$

and

$$F_2(x_o) = 1 - g_2(x_{o,u}), \quad (7.58)$$

where

$$g_2(x_0, u) = \sum_{j=1}^{n-m} \frac{[R(u + x_o) - R(u)]^{j-1}}{\Gamma(j)} \exp\{-(R(u + x_o) - R(u))\}.$$

Now equating (7.57) and (7.58), we get

$$\int_0^\infty \frac{[R(y)]^{m-1}}{\Gamma(m)} f(u)[g_2(u, x_0) - g_1(x_0)]du = 0, x_0 > 0. \quad (7.59)$$

Now $g_2(x_0) = 0 = g_1(0)$ and

$$0 = \frac{[R(u) - R(u)]^{n-m-1}}{\Gamma(n-m)} \exp\{-(R(u + x_o) - R(u)\} [r(x_o) - r(u + x_o)].$$

Thus if $F \in C_2$ then (7.59) is true if

$$r(u + x_o) = r(u) \quad (7.60)$$

for almost all u and any fixed $x_o \geq 0$. Hence X_k has the cdf $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0, \sigma > 0$, $k \geq 1$. Here σ is an arbitrary positive real number. Substituting m = n – 1, we get $I_{n-1,n} \overset{d}{=} X_1$ as a characteristic property of the exponential distribution.

Theorem 7.19 *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous distribution function F(x) and the corresponding density function f(x). If F belongs to C_2 and for some m, m > 1, X(m) and X(m – 1) + U are identically distributed, where U is independent of X(m) and X(m – 1) is distributed as X_n's, then X_k has the cdf $F(x) = 1 - e^{-x/\sigma}$, $x \geq 0, \sigma > 0$.*

Proof The pdf $f_m(x)$ of R_m, $m \geq 1$, can be written as

$$\begin{aligned} f_m(y) &= \tfrac{[R(y)]^m}{\Gamma(m+1)} f(y), 0 < y < \infty, \\ &= \tfrac{d}{dy}\left(-\bar{F}(y) \int_0^y \tfrac{[R(x)]^{m-1}}{\Gamma(m)} r(x)\, dx + \int_0^y \tfrac{[R(x)]^m}{\Gamma(m)} f(x)\, dx\right), \end{aligned} \quad (7.61)$$

The pdf $f_2(y)$ of X(n – 1) + U can be written as

$$\begin{aligned} f_2(y) &= \int_0^y \tfrac{[R(x)]^{m-1}}{\Gamma(m)} f(y-x) f(x) dy \\ &= \tfrac{d}{dy}\left(-\tfrac{[R(x)]^{m-1}}{\Gamma(m)} \bar{F}(y-x) f(x) dx + \int_0^y \tfrac{[R(x)]^{m-1}}{\Gamma(m)} f(x)\, dx\right). \end{aligned} \quad (7.62)$$

Equating (7.61) and (7.62), we get on simplification

$$\int_0^y \frac{[R(x)]^{m-1}}{\Gamma(m-1)} f(x) H_1(x, y)\, dx = 0, \quad (7.63)$$

where $H_1(x, y) = \bar{F}(y - x) - \bar{F}(y)\,(\bar{F}(x))^{-1}$, $0 < x < y < \infty$. Since $F \in C_1$, therefore for (7.63) to be true, we must have

$$H_1(x, y) = 0 \tag{7.64}$$

for almost all x, $0 < x < y < \infty$. This implies that

$$\bar{F}(y - x)\,\bar{F}(x) = \bar{F}(y), \tag{7.65}$$

for almost all x, $0 < x < y < \infty$. The only continuous solution of (7.65) with the boundary conditions $\overline{F}(0) = 1$, *and* $\overline{F}(\infty) = 0$, *is*

$$\overline{F}(x) = e^{-x\sigma^{-1}}, \tag{7.66}$$

where σ is an arbitrary positive number.

Remark 7.2 The Theorem 7.19 can be used to obtain the following known results of a two parameter exponential distribution $\bar{F}(x) = \exp\{-\sigma^{-1}(x - \mu)\}$).
$E(X(n)) = \mu + n\sigma$
$Var(X(n)) = n\,\sigma^2$
$Cov(X(m)X(n)) = m\,\sigma^2$, $m < n$.

7.4 Characterization of Distributions by Generalized Order Statistics

Kamp (1995a) introduced Generalized Order Statistics (gos) and many interesting properties of it. The characterizations given in order statistics and record values can be obtained as special cases of gos. For example consider the following conditional characterization theorem given by Beg et al. (2013).

Theorem 7.20 *If for given two consecutive values of r, r+1 and an integer s with $1 \le r < s < n$, $E(\psi(X(s, n, m, k)|X(r, n, m, k) - x) = g_r(x)$ where $\psi(x)$ is a continuous function and $g_r(x)$ is a differential function of x, then $1 - F(x) = c\exp(-\int M(x)dx)$. $M(x) = \frac{g'_{r(}(x)}{\gamma_{r+1}(g_{r+}(x) - g_r(x))}$ and c is determined by the condition $\frac{1}{c} = \int f(x)dx$. If $\psi(x) = x$ and $g_r(x) = ax + b$, then we obtain for*

(i) $a = 1$, exponential distribution
(ii) $a > 1$, Pareto distribution'
(iii) $a < 1$, Power function distribution.

This theorem for s = r + 1 was proved by Ahsanullah and Bairamov (2004)
If s = r + 1 and m = 0, then from this theorem we will get Theorem 7.1. If s = r + 1 and m = 0, then we will get the Theorem 7.2. If s > r and m = o, then we will have the Theorem 7.3.

If $s = n$ *and* $r = n - 1$*, then we will have Theorem 7.11. If* $s = n + 2$ *and* $r = n$*, we will get the Theorem 7.12. If s> n and r=n, r, then we get Theorem 7.13.*

Let D(1, n, m, k) = $\gamma_1(X(1, n, m, k)$ *and*

D(r, n, m, k) =$\gamma_r(X(r+1, n, m, k) - X(r, n, m, k)$

Kamp and Gather (1997) proved the following theorem.

Theorem 7.21 *Let F (x) be absolutely continuous and let corresponding pdf be f(x). If F(0) = 0 and F(x) is strictly increasing on (0,* ∞*) with E(X) exists and either F(X) is IFR or DFR. Then F(x) = 1 – exp(–λx) iff there exists integers r and n,* $1 \leq r < n$ *such that E(D(r, n, m, k))= E(D(r + 1, n, m, k).*

The theorem is also true if the expected values are replaced by the equality in distributions. Recently Rasouli et al. (2016) gave the following theorem.

Theorem 7.22 *Let X be an absolutely continuous with cdf F(x) and E(X) exists. Then E(X(s, n, m, k)| X(r, n, n, k) = x) = ax +b iff*

(1) $a = 1$*, F(x) is exponential*
(2) a > 1, F(x) is Pareto
(3) a < 1, F(x) is power function distribution.

The results given by Deminska and Wesolowski (1998), Wesolowski and Ahsanullah (1997) for order statistics and Ahsanullah and Wesolowski (1998) for record values are special cases of this theorem.

Chapter 8
Extreme Value Distributions

8.1 Introduction

Extreme value distributions arise in probability theory as limit distributions of maximum or minimum of n independent and identically distributed random variables with some normalizing constants. For example if $X_{1,}X_2, \ldots, X_n$ are n independent and identically distributed random variables, then the largest normalized order statistic $X_{n,n,}$ will converge to one of the following three distributions if it has a nondegenerate distribution as $n \to \infty$.

(1) Type 1: (Gumbel) $F(x) = \exp(-e^{-x})$ for all x, $-\infty < x < \infty$
(2) Type 2: (Frechet) $F(x) = \exp(-x^{-\delta})$, $x \geq 0$, $\delta > 0$
(3) Type 3: (Weibull) $F(x) = \exp(-(-x)^{-\delta})$, $x \leq 0$, $\delta > 0$

Since the smallest order statistic $X_{1:n} = Y_{n:n}$, where $Y = -X$, $X_{1:n}$ with some appropriate normalizing constants will also converge to one of the above three limiting distributions if we change x to −x in (1), (2) and (3). Gumbel (1958) has given various applications of these distributions.

Suppose $X_{1,}$ $X_2, \ldots X_n$ be i.i.d. random variable having the distribution function F(X) with $F(x) = 1 - e^{-x}$. Then with normalizing constant $a_n = \ln n$ and $b_n = 1$, $P(X_{n.n,} < a_n + b_n x) = P(X_{n,n} \leq \ln n + x) = (1 - e^{-(\ln n + x)})^n = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}$ as $n \to \infty$. Thus the limiting distribution of $X_{n:n}$ when X's are distributed as exponential with unit mean is Type 1 extreme value distribution as given above. It can be shown that Type 1 (Gumbel distribution) is the limiting distribution of $X_{n,n}$ when F(x) is normal, log normal, logistic, gamma etc. The type 2 and type 3 distributions can be transformed to Type 1 distribution by the transformations ln X and—ln X respectively. We will denote the Type 1 distribution as T_{10} and Type 2 and Type 3 distribution as $T_{2\delta}$ and $T_{3\delta}$ respectively. If the $X_{n,n}$ of n independent random variables from a distribution F when normalized has the limiting distribution T, then we will say that F belongs to the domain of attraction of T and write $F \in D(T)$.

M.Q. Shahbaz et al., *Ordered Random Variables: Theory and Applications*,
Atlantis Studies in Probability and Statistics 9, DOI 10.2991/978-94-6239-225-0_8

The extreme value distributions were originally introduced by Fisher and Tippet (1928). These distributions have been used in the analysis of data concerning floods, extreme sea levels and air pollution problems for details see Gumbel (1958), Horwitz (1980), Jenkinson (1955) and Roberts (1979).

8.2 The PDF's of the Extreme Values Distributions of $X_{n,n}$

8.2.1 Type 1 Extreme Value Distribution

The probability density function of type 1 extreme value distribution (T_{10}) is given in Fig. 8.1.

The type I extreme value distribution is unimodal with mode at 0 and the points of inflection are at $\pm \ln\left((3+\sqrt{5})/2\right)$. The pth percentile η_p, $(0 < p < 1)$ of the curve can be calculated by the relation $\eta_p = -\ln(-\ln p)$. The median of X is—lnln2. The moment generating function $M_{10}(t)$, of this distribution for some t, $0 < |t| < \delta$, is $M_{10}(t) = \int_{-\infty}^{\infty} e^{tx} e^{-x} e^{-e^{-x}} dx = e^t \Gamma(1-t)$. The mean $= \gamma$, the Euler's constant and the variance $= \pi^2/6$.

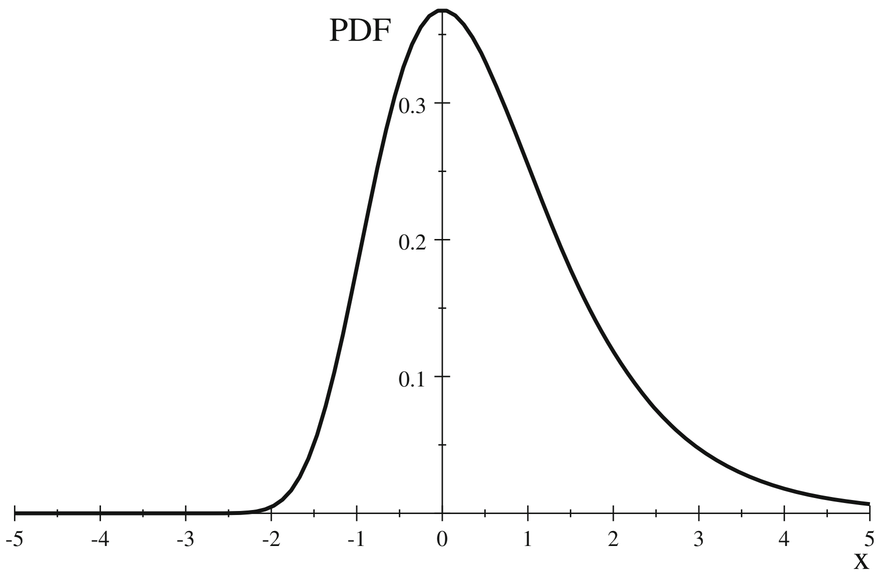

Fig. 8.1 PDF of T_{10}

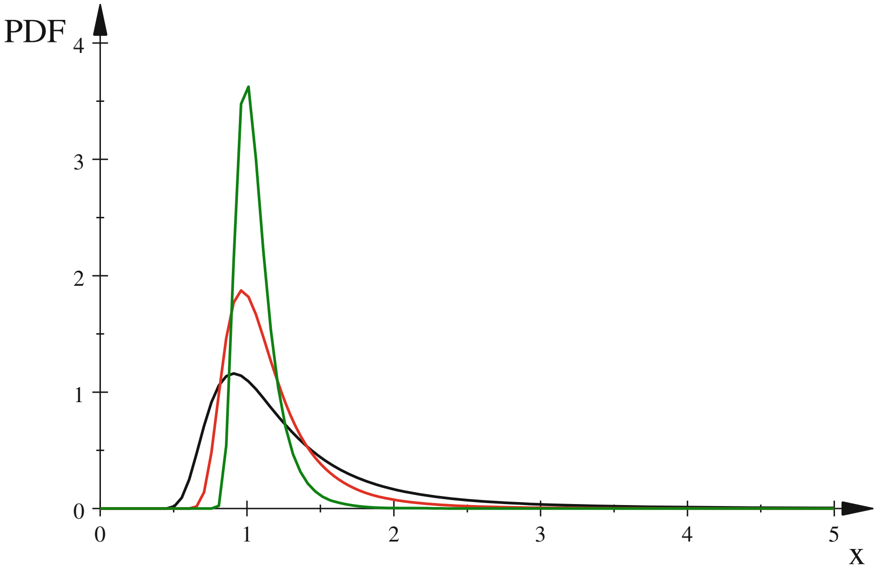

Fig. 8.2 PDFs. $T_{2,3}$-*Black*, $T_{2,5}$-*Red*, $T_{2,10}$-*Green*

8.2.2 Type 2 Extreme Value Distribution

The probability density functions of $T_{23,} T_{2,5}$ and $T_{2.10}$ are given in Fig. 8.2.

For $T_{2,\delta}$, mode $= (\frac{\delta}{1+\delta})^{1/\delta}$, median $= (\frac{1}{\ln 2})^{1/\delta}$, mean $= \Gamma(1 - \frac{1}{\delta})$, $\delta > 1$ and variance $= \Gamma(1 - \frac{2}{\delta}) - (\Gamma(1 - \frac{1}{\delta}))^2$, $\delta > 2$.

8.2.3 Type 3 Extreme Value Distribution

The probability density functions of type 3 extreme value distributions for $\delta = 3, 5$ and 10 are given in Fig. 8.3. Note for $\delta = 1$, T_{31} is the reverse exponential distribution.

The mode of the type 3 distribution is at $\left(\frac{\delta-1}{\delta}\right)^{\frac{1}{\delta}}$. For type 3 distribution, E(X) = $\Gamma\left(1 + \frac{1}{\delta}\right)$ and Var(X)= $\Gamma(1 + \frac{2}{\delta}) - \left(\Gamma\left(1 + \frac{1}{\delta}\right)\right)^2$.

Table 8.1 gives the percentile points of T_{10}, T_{21}, T_{31} and T_{32} for some selected values of p.

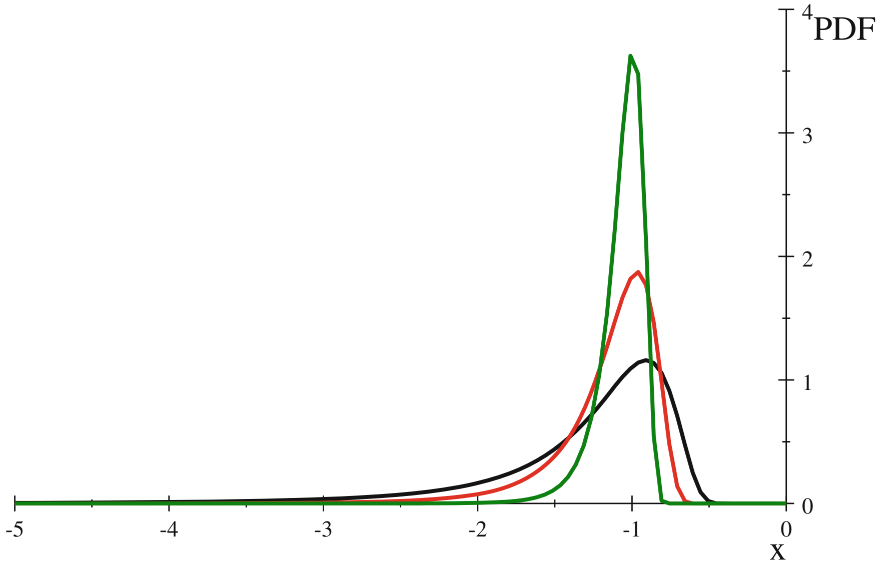

Fig. 8.3 / PDFs $T_{3,3}$-*Black*, $T_{3,5}$-*Red*, $T_{3,10}$-*Green*

Table 8.1 Percentile points of T_{10}, T_{21}, T_{31} and T_{32}

P	T_{10}	T_{21}	T_{31}	T_{32}
0.1	−0.83403	0.43429	−2.30259	−1.51743
0.2	−0.47589	0.62133	−1.60844	−1.26864
0.3	−0.18563	0.83058	−1.20397	−1.09726
0.4	0.08742	1.09136	−0.91629	−0.95723
0.5	0.36651	1.44270	−0.69315	−0.83255
0.6	0.67173	1.95762	−0.51083	−0.71472
0.7	1.03093	2.80367	−0.35667	−0.59722
0.8	1.49994	4.48142	−0.22314	−0.47239
0.9	2.2504	9.49122	−0.10536	−0.32459

8.3 Domain of Attraction of $X_{n,n}$

In this section we will study the domain of attraction of various distributions. The maximum order statistics $X_{n,n}$ of n independent and identically distributed random variables will considered. We will say that $X_{n,n}$ will belong to the domain of attraction of T(x) if the $\lim_{n\to\infty} P(X_{n:n} \le a_n + b_n x) = \mathrm{T(x)}$ for some sequence of normalizing constants a_n and b_n.
The following lemma will be helpful in proving the theorems of the domain of attractions.

Lemma 8.1 *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with distribution function F. Consider a sequence $(e_n, n \geq 1\}$ of real numbers. Then for any ξ, $0 \leq \xi < \infty$, the following two statements are equivalent*

(i) $\lim_{n\to\infty} n(\bar{F}(e_n)) = \xi$

(ii) $\lim_{n\to\infty} P\left(X_{n,n} \leq e_n\right) = e^{-\xi}$.

8.3.1 Domain of Attraction of Type I Extreme Value Distribution

The following theorem is due to Gnedenko (1943).

Theorem 8.1 *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d. random variables with distribution function F and* $e(F) = sup\{x : F(x) < 1\}$. *Then* $F \in T_{10}$ *iff there exists a positive function g(t) such that*

$$\lim_{t\to e(F)} \frac{\bar{\bar{F}}(t + xg(t))}{\bar{\bar{F}}(t)} = e^{-x},\ \bar{F} = 1 - F \textit{ for all real } x.$$

The following Lemma (see Von Mises (1936)) gives a sufficient condition for the domain of attraction of Type 1 extreme value distribution for $X_{n:n}$.

Lemma 8.2 *Suppose that the distribution function F has a derivative on* $(c_{0,}e(F))$ *for some* c_0, $0 < c_0 < e(F)$, *then* $\lim_{x\uparrow e(F)} \frac{f(x)}{\bar{F}(x)} = c, c > 0$, *then* $F \in D(T_{10})$.

Example 8.1 The exponential distribution $F(x) = 1 - e^{-x}$ satisfies the sufficient condition, since $\lim_{x\to\infty} \frac{f(x)}{\bar{F}(x)} = 1$. For the logistic distribution $F(x) = \frac{1}{1+e^{-x}}$, $\lim_{x\to\infty} \frac{f(x)}{\bar{F}(x)} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$. Thus the logistic distribution satisfies the sufficient condition.

Example 8.2 For the standard normal distribution with $x > 0$, (see Abramowitz and Stegun (1968 p. 932)

$\bar{F}(x) = \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} h(x)$, where $h(x) = 1 - \frac{1}{x^2} + \frac{1.3}{x^4} + \cdots + \frac{(-1)^n 1.3.....(2n-1)}{x^{2n}} + R_n$ and $R_n = (-1)^{n+1} 1.3....(2n+1) = \int_x^\infty \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}u^{2n+2}} du$ which is less in absolute value than the first neglected term.

It can be shown that $g(t) = 1/t + 0(t^3)$. Thus

$$\lim_{t\to\infty} \frac{\bar{F}(t + xg(t))}{\bar{F}(t)} = \lim_{t\to\infty} \frac{te^{\frac{t^2}{2}}}{(t + xg(t))\, e^{\frac{1}{2}(t+xg(t))^2}} \frac{h(t + xg(t))}{h(t)} = \lim_{t\to\infty} \frac{te^{-xm(x)}}{t + xg(t)},$$

where $m(t, x) = g(t)(1 + \frac{1}{2}g(t))$. Since as $t \to \infty$, $m(t, x) \to 1$, we $\lim_{t\to\infty} \frac{\bar{F}(t+xg(t))}{\bar{F}(t)} = e^{-x}$. Thus normal distribution belongs to the domain of attraction of Type I distribution.

Since $\lim_{x\to\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}x\bar{F}(x)} = \lim_{x\to\infty} h(x) = 1$, the standard normal distribution does not satisfy the von Mises sufficient condition for the domain of attraction of the type I distribution.

We can take $a_n = \frac{1}{b_n} - \frac{b_n}{2}(\ln\ln n + 4\pi)$ and $b_n = (2\ln\ln n)^{-1/2}$. However this choice of a_n and b_n is not unique. The rate of convergence of $P(X_{n,n} \le a_n + b_n x)$ to $T_{10}(x)$ depends on the choices of a_n and b_n.

8.3.2 Domain of Attraction of Type 2 Extreme Value Distribution

Theorem 8.2 *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d random variables with distribution function F and e(F) = sup {x: F(x) < 1}. If* $e(F) = \infty$, *then* $F \in T_{2\delta}$ *iff* $\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-\delta}$ *for x > 0 and some constant* $\delta > 0$.

Proof Let $a_n = \inf\{x : \bar{F}(x) \le \frac{1}{n}\}$, then $a_n \to \infty$ as $n \to \infty$. Thus $\lim_{n\to\infty} n(\bar{F}(a_n x)) = \lim_{n\to\infty} n(\bar{F}(a_n))\frac{\bar{F}(a_n x)}{\bar{F}(a_n)} = x^{-\delta} \lim_{n\to\infty} n\bar{F}(a_n)$.

It is easy to show that $\lim_{n\to\infty} n\bar{F}(a_n) = 1$. Thus $\lim_{n\to\infty} n(\bar{F}(a_n x)) = x^{-\delta}$ and the proof the only if condition of the Theorem follows from Lemma 8.1.

Example 8.3 For the Pareto distribution with $\bar{F}(x) = \frac{1}{x^\delta}$, $\delta > 0, 0 \le x < \infty$ $\lim_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = \frac{1}{x^\delta}$. Thus the Pareto distribution belongs to $T_{2\delta}$.

The following theorem gives a necessary and sufficient condition for the domain of attraction Type 3 distribution for $X_{n:n}$ when $e(F) < \infty$.

Theorem 8.3 *Let* $X_1, X_2, ..$ *be a sequence of i.i.d random variables with distribution function F and e(F) = sup {x: F(x) < 1}. If* $e(F) < \infty$, *then* $F \in T_{2\delta}$ *iff*

$\lim_{t\to\infty} \frac{\bar{F}(e(F)-\frac{1}{tx})}{\bar{F}(e(F)-\frac{1}{t})} = x^{-\delta}$ *for x > 0 and some constant* $\delta > 0$.

Proof Similar to Theorem 8.2

Example 8.4 The truncated Pareto distribution $f(x) = \frac{\delta}{x^{\delta+1}} \cdot \frac{1}{1=b^{-\delta}}$, $1 \le x < b, b > 1$, $\lim_{t\to\infty} \frac{\bar{F}(e(F)-\frac{1}{tx})}{\bar{F}(e(F)-\frac{1}{t})} = \lim_{t\to\infty} \frac{\bar{F}(b-\frac{1}{tx})}{\bar{F}(b-\frac{1}{t})} = \lim_{t\to\infty} \frac{(b-\frac{1}{tx})^{-\delta} - b^{-\delta}}{(b-\frac{1}{t})^{-\delta} - b^{-\delta}} = x^{-1}$. Thus the truncated Pareto distribution belongs to the domain of attraction of Type T_{21} distribution.

The following Lemma (see von Mises (1936)) gives a sufficient condition for the domain of attraction of Type 2 extreme value distribution for $X_{n,n}$.

Lemma 8.3 *Suppose the distribution function F is absolutely continuous in* c_0, e(F) *for some* c_0, $0 < c_0 < e(F)$, *then if* $\lim_{x\uparrow e(F)} \frac{xf(x)}{\bar{F}(x)} = \delta, \delta > 0$, *then* $F \in D(T_{2\delta})$.

Example 8.5 The truncated Pareto distribution $f(x) = \frac{\delta}{x^{\delta+1}} \cdot \frac{1}{1-b^{-\delta}}$, $1 \le x < b$, $b > 1$, $\lim_{x\to\infty} \frac{xf(x)}{\bar{F}(x)} = \lim_{x\to b} \frac{\delta x^{-\delta}}{x^{-\delta}-b^{-\delta}} = \infty$. Thus the truncated Pareto distribution does not satisfy the von Mises sufficient condition. However it belongs to the domain of attraction of the type 2 extreme value distribution, because $\lim_{t\to\infty} \frac{\bar{F}(e(F)-\frac{1}{tx})}{\bar{F}(e(F)-\frac{1}{t})} = x^{-\delta}$ for $x > 0$ and some constant $\delta > 0$.

8.3.3 *Domain of Attraction of Type 3 Extreme Value Distribution*

The Following Theorem Gives a Necessary and Sufficient Condition For the Domain of Attraction of Type 3 Distribution For $X_{n,n}$

Theorem 8.4 *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d. random variables with distribution function F and* $e(F) = \sup\{x : F(x) < 1\}$. *If* $e(F) < \infty$, *then* $F \in T_{3\delta}$ *iff* $\lim_{t\to 0^+} \frac{\bar{F}(e(F)+tx)}{\bar{F}(e(F)-t)} = (-x)^{\delta}$ *for* $x < 0$ *and some constant* $\delta > 0$.

Proof Similar to Theorem 8.3.

Example 8.6 Suppose X is an exponential distribution truncated at $x = b > 0$. The pdf of X is $f(x) = \frac{e^{-x}}{F(b)}$, then for $x < 0$, $P(X_{n:n} \le b + \frac{x(e^b-1)}{n}) = \left(\frac{1-e^{-(b+\frac{x(e^b-1)}{n})}}{1-e^{-b}}\right)^n$ $\to e^x$ as $n \to \infty$.
Thus the truncated exponential distribution belongs to T_{31}.

Example 8.7 Since $\lim_{t\to 0^+} \frac{\bar{F}(e(F)+tx)}{\bar{F}(e(F)-t)} = \lim_{t\to 0^+} \frac{e^{-(b+tx)}-e^{-b}}{e^{-(b-t)}-e^{-b}} = -x$, the truncated exponential distribution satisfies the necessary and sufficient condition for the domain of attraction of type 3 distribution for maximum.

The following Lemma due to von Mises gives a sufficient condition for the domain of attraction of type 3 distribution for $X_{n,n}$.

Lemma 8.4 *Suppose the distribution function F is absolutely continuous in* $[c_0, e(F)]$ *for some* c_0, $0 < c_0 < e(F) < \infty$, *then if* $\lim_{x\uparrow e(F)} \frac{(e(F)-x)f(x)}{\bar{F}(x)} = \delta$, $\delta > 0$., *then* $F \in D(T_{3\delta})$.

Proof Similar to Lemma 8.3.

Example 8.8 Suppose X is an exponential distribution truncated at $x = b > 0$, then the pdf of X is $f(x) = \frac{e^{-x}}{F(b)}$. Now $\lim_{x\uparrow e(F)} \frac{(e(F)-x)f(x)}{\bar{F}(x)} = \lim_{x\uparrow b} \frac{(b-x)e^{-x}}{e^{-x}-e^{-b}} = 1$, thus the truncated exponential distribution satisfies the von Mises sufficient condition for the domain of attraction to type 3 distribution.

The normalizing constants of $X_{n:n}$ are not unique for any distribution. From the table it is evident that two different distributions (exponential and logistic) belong to the domain of attraction of the same distribution and have the same normalizing constants. The normalizing constants depend on F and the limiting distribution. It may happen that $X_{n:n}$ with any normalizing constants may not converge in distribution to a non degenerate limiting distribution but $W_{n:n}$ where $W = u(X)$, a function of X, with some normalizing constants, may converge in distribution to one of the three limiting distributions. We can easily verify that the rv X whose pdf, $f(x) = \frac{1}{x(\ln x)^2}, x \geq e$ does not satisfy the necessary and sufficient conditions for the convergence in distribution of $X_{n:n}$ to any of the extreme value distributions. Suppose $W = \ln X$, then $F_W(x) = 1 - 1/x$ for $x > 1$. Thus with $a_n = 0$ and $b_n = 1/n$, $P(W_{n:n}. \leq x) \to T_{31}$ as $n \to \infty$.

Following Pickands (1975), the following theorem gives a necessary and sufficient condition for the domain of attraction of $X_{n:n}$ from a continuous distribution.

Theorem 8.5 *For a continuous random variable the necessary and sufficient condition for* $X_{n:n}$ *to belong to the domain of attraction of the extreme value distribution of the maximum is*

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1 \text{ if } F \in T_{10},$$

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 2^{1/\delta} \text{if } F \in T_{2\delta}$$

and

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 2^{-1/\delta} \text{if } F \in T_{3\delta}$$

Example 8.9 For the exponential distribution, $E(0, \sigma)$, with pdf $f(x) = \sigma^{-1} e^{-\sigma^{-1}x}$, $x \geq 0$, $F^{-1}(x) = -\sigma^{-1}\ln(1-x)$ and $\lim_{c\to 0} \frac{F^{-1}(1-c)-F^{-1}(1-2c)}{F^{-1}(1-2c)-F^{-1}(1-4c)} = \lim_{c\to 0} \frac{-\ln\{1-(1-c)\}+\ln\{1-(1-2c)\}}{-\ln\{1-(1-2c)\}+\ln\{1-(1-4c)\}} = 1$. Thus the domain of attraction of $X_{n:n}$ from the exponential distribution, $E(0, \sigma)$, is T_{10}.

For the Pareto distribution, $P(0, 0, \alpha)$ with pdf $f(x) = \alpha x^{-(\alpha+1)}, x > 1, \alpha > 0$, $F^{-1}(x) = (1-x)^{-1/\alpha}$ and

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = \lim_{c\to 0} \frac{c^{-1/\alpha} - (2c)^{-1/\alpha}}{(2c)^{-1/\alpha} - (4c)^{=1/\alpha}} = 2^{1/\alpha}.$$

Hence the domain of attraction of $X_{n:n}$ from the Pareto distribution, $P(0, 0, \alpha)$ is $T_{2\alpha}$.

For the uniform distribution, $U(-1/2, 1/2)$, with pdf $f(x) = 1, -\frac{1}{2} < x < \frac{1}{2}$, $F^{-1}(x) = x - 1/2$. We have

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} == \lim_{c\to 0} \frac{1-c-(1-2c)}{(1-2c)-(1-4c)} = 2^{-1}.$$

Consequently the domain of attraction of of $X_{n:n}$ from the uniform distribution, $U(-1/2, 1/2)$ is T_{31}.

It may happen that $X_{n:n}$ from a continuous distribution does not belong to the domain of attraction of any one of three distributions. In that case $X_{n:n}$ has a degenerate limiting distribution. Suppose the rv X has the pdf f(x), where $f(x) = \frac{1}{x(\ln x)^2}, x \geq e$. $F^{-1}(x) = e^{\frac{1}{1-x}}, 0 < x < 1$.

Then $\lim\limits_{c\to 0} \frac{F^{-1}(1-c)-F^{-1}(1-2c)}{F^{-1}(1-2c)-F^{-1}(1-4c)} = \lim\limits_{c\to 0} \frac{e^{\frac{1}{c}}-e^{\frac{1}{2c}}}{e^{\frac{1}{2c}}-e^{\frac{1}{4c}}} = \lim\limits_{c\to 0} \frac{e^{\frac{1}{c}}-1}{1-e^{\frac{1}{2c}}} = \lim\limits_{c\to 0} \frac{2e^{\frac{1}{c}}}{e^{\frac{1}{2c}}} = \lim\limits_{c\to 0} 2e^{\frac{1}{2c}} = \infty$

Thus the limit does not exit. Hence the rv X does not satisfy the necessary and sufficient condition given in Theorem 8.4.

Theorems 8.1, 8.2, 8.3, 8.4 and 8.5 are also true for discrete distributions. If $X_{n:n}$ is from discrete random variable with finite number of points of support, then $X_{n:n}$ can not converge to one of the extreme value distributions. Thus $X_{n:n}$ from binomial and discrete uniform distribution will converge to degenerate distributions. The following Lemma (Galambos (1987), p. 85) is useful to determine whether $X_{n:n}$ from a discrete distribution will have a degenerate distribution.

Lemma 8.5 *Suppose X is a discrete random variable with infinite number of points in its support and taking values on non negative integers with* $P(X = k) = p_k$. *Then a necessary condition for the convergence of* $P(X_{nn} \leq a_n + b_n x)$, *for a suitable sequence of* a_n and b_n, *to one of the three extreme value distributions is* $\lim\limits_{k\to\infty} \frac{p_k}{P(X\geq k)} = 0$.

For the geometric distribution, $P(X = k) = p(1-p)^{k-1}, k \geq 1, 0 < p < 1$, $\frac{p_k}{P(X\geq k)} = p$. Thus X_{nn} from the geometric distribution will have degenerate distribution as limiting distribution of X_{nn}.

Consider the distribution: $P(X = k) = \frac{1}{k(k+1)}, k = 1, 2, \ldots$, then $P(X \geq k) = \frac{1}{k}$ and $\lim\limits_{k\to\infty} \frac{p_k}{P(X\geq k)} = \lim\limits_{k\to\infty} \frac{1}{k+1} = 0$. But $\lim\limits_{t\to\infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-1}$. Thus X belongs to the domain of attraction of T_{21}. The normalizing constants are $a_n = 0$ and $b_n = n$.
However the condition $\lim\limits_{k\to\infty} \frac{p_k}{P(X\geq k)} = 0$ is necessary but not sufficient.
Consider the discrete probability distribution whose $P(X = k) = \frac{c}{k(\ln(k+1))^6}, k = 1, 2, \ldots$ where $1/c = \sum_{k=1}^{\infty} \frac{1}{k(\ln(k+1))^6} \cong 9.3781$.

Since $1 - \sum_{k=1}^{n} \frac{1}{k(\ln(k+1))^6} \propto \frac{1}{(\ln n)^5}, \frac{P(X=n)}{1-\sum_{k=1}^{n=1} P(X=k)} \to 0$ an $n\to \infty$.

But this probability distribution does not satisfy the necessary and sufficient conditions for the convergence of $X_{n,n}$ to the extreme value distributions.

We can use the following lemma to calculate the normalizing constants for various distributions belonging to the three domain of attractions of T(x).

Lemma 8.6 *Suppose* $P(X_{n:n} \leq a_n + b_n x) \to T(x)$ *as* $n \to \infty$, *then*

(i) $a_n = F^{-1}(1-\frac{1}{n}), b_n = F^{-1}(1-\frac{1}{ne}) - F^{-1}(1-\frac{1}{n})$ *if* $T(x) = T_{10}(x)$,
(ii) $a_n = 0, b_n = F^{-1}(1-\frac{1}{n})$ *if* $T(x) = T_{2\delta}(x)$,
(iii) $a_n = F^{-1}(1), b_n = F^{-1}(1) - F^{-1}(1-\frac{1}{n})$ *if* $T(x) = T_{3\delta}(x)$

We have seen that the normalizing constants are not unique. However we can use the following Lemma to select simpler normalizing constants.

Lemma 8.7 *Suppose* a_n and b_n *is a sequence of normalizing constants for* $X_{n:n}$ *for the convergence to the domain of attraction of any one of the extreme value distributions. If* a_n^* and b_n^* *is another sequence such that* $\lim_{n\to\infty} \frac{a_n - b_n^*}{b_n} = 0$ *and* $\lim_{n\to\infty} \frac{b_n^*}{b_n} = 0$*, then* a_n^* andb_n^* *can be substituted for as the normalizing constants* a_n and b_n *for* $X_{n,n}$.

Example 8.10 We have seen (see Table 8.2) that for the Cauchy distribution with pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$ the normalizing constants as $a_n = 0$ and $b_n = \cot(\pi/n)$. However we can take $a_n^* = 0$ *and* $b_n^* = \frac{n}{\pi}$.

The following tables gives the normalizing constants for some well known distributions belonging the domain of attraction of the extreme value distributions.

8.4 The PDF's of Extreme Value Distributions for $X_{1:n}$

Let us consider $X_{1:n}$ of n i.i.d. random variables. Suppose $P(X_{1:n} \le \underline{c}_n + d_n x) \to H(x)$ as $n \to \infty$, then the following three types of distributions are possible for H(x).

Type 1 distribution $H_{10}(x) = 1 - e^{-e^x}$, $-\infty < x < \infty$:;

Type 2 distribution $H_{2\delta}(x) = 1 - e^{-(-x)^{-\delta}}$, $x < 0, \delta > 0$.

Type 3 distribution $H_{3\delta}(x) = 1 - e^{-x^\delta}$, $x > 0, \delta > 0$.

It may happen that $X_{n:n}$ and $X_{1:n}$ may belong to different types of extreme value distributions. For example consider the exponential distribution, $f(x) = e^{-x}$, $x > 0$. The $X_{n:n}$ belongs to the domain of attraction of the type 1 distribution of the maximum, T_{10}. Since $P(X_{1:n} > n^{-1}x) = e^{-x}$, $X_{1:n}$ belongs to the domain of attraction of Type 2 distribution of the minimum, H_{21}.It may happen that $X_{n:n}$ does not belong to any one of the three limiting distributions of the maximum but $X_{1:n}$ belong to the domain of attraction of one of the limiting distribution of the minimum. Consider the rv X whose pdf, $f(x) = \frac{1}{x(\ln x)^2}$, $x \ge e$. We have seen that F does not satisfy the necessary and sufficient conditions for the convergence in distribution of $X_{n:n}$ to any of the extreme value distributions. However it can be shown that $P(X_{1:n} > \alpha_n + \beta_n x) \to e^{-x}$ as $n \to \infty$ for $\alpha_n = e$ and $\beta_n = e^{-\frac{n-1}{n}} - e$. Thus the $X_{1:n}$ belongs to the domain of attraction of $H_{21.}$

If X is a symmetric random variable and $X_{n:n}$ belongs to the domain of attraction of $T_i(x)$, then $X_{1:n}$ will belong to the domain of attraction of the corresponding $H_i(x)$, i =1, 2, 3.

Table 8.2 Normalizing Constants and the domain of attraction ofr $X_{n:n}$

Distribution	f(x)	a_n	b_n	Domain
Beta	$cx^{\alpha-1}(1-x)^{\beta-1}$ $c=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ $\alpha>0,\beta>0$ $0<x<1$	1	$\left(\frac{\beta}{nc}\right)^{1/\beta}$	$T_{3\beta}$
Cauchy	$\frac{1}{\pi(1+x^2)}$, $-\infty<x<\infty$	0	$\cot(\frac{\pi}{n})$	T_{21}
Discrete Pareto	$P(X=k)=[k]^{\theta}-[k+1]^{\theta},\theta>0$, $k\geq 1$, [] represents the greatest integer contained in	0	$n^{1/\theta}$	$T_{2\theta}$
Exponential	$\sigma e^{-\sigma x}, 0<x<\infty,\sigma>0$	$\frac{1}{\sigma}\ln n$	$\frac{1}{\sigma}$	T_{10}
Gamma	$\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$, $0<x<\infty$	$\ln n+\ln\Gamma(\alpha)-(\alpha-1)\ln\ln n$	1	T_{10}
Laplace	$\frac{1}{2}e^{-\lvert x\rvert}$, $-\infty<x<\infty$	$\ln\left(\frac{n}{2}\right)$	1	T_{10}
Logistic	$\frac{e^{-x}}{(1+e^{-x})^2}$	ln n	1	T_{10}
Lognorma 1	$\frac{1}{x\sqrt{2\pi}}e^{-\frac{1}{2}(\ln x)^2} 0<x<\infty$	e^{α_n}, $\alpha_n=\frac{1}{\beta_n}-\frac{\beta_n D_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $\beta_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}e^{\alpha_n}$	T_{10}
Normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, -\infty<x<\infty$	$\frac{1}{\beta_n}-\frac{\beta_n D_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $\beta_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}$	T_{10}
Pareto	$\alpha x^{-(\alpha+1)}$ $x>1,\alpha>0$	0	$n^{1/\alpha}$	$T_{2\alpha}$
Power Function	$\alpha x^{\alpha-1}$, $0<x<1,\alpha>0$	1	$\frac{1}{n\alpha}$	T_{31}
Rayleigh	$\frac{2x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}, x\geq 0$	$\sigma(\ln n)^{\frac{1}{2}}$	$\frac{\sigma}{2}(\ln n)^{-\frac{1}{2}}$	T_{10}
t distribution	$\frac{k}{\left(1+\frac{x^2}{\upsilon}\right)^{(\upsilon+1)/2}}$ $k=\frac{\Gamma((\upsilon+1)/2)}{(\pi\upsilon)^{1/2}\Gamma(\upsilon/2)}$	0	$\left(\frac{kn}{\upsilon}\right)^{1/\upsilon}$	$T_{2\upsilon}$
Truncated Exponential	Ce^{-x}, $C=1/(1-e^{-e(F)})$, $0<x<e(F)<\infty$	E(F)	$\frac{e^{e(F)}-1}{n}$	T_{31}
Type 1	$e^{-x}e^{-e^{-x}}$	ln n	1	T_{10}
Type 2	$\alpha x^{-(\alpha+1)}e^{-x^{-\alpha}}$ $x>0,\alpha>0$	0	$n^{1/\alpha}$	$T_{2\alpha}$
Type 3	$\alpha(-x)^{\alpha-1}$ $.e^{-(-x)^{\alpha}}, x<0$, $\alpha>0$	0	$n^{-1/\alpha}$	$T_{3\alpha}$
Uniform	$1/\theta, 0<x<\theta$	θ	θ/n	T_{31}
Weibull	$\alpha x^{\alpha-1}e^{-x^{\alpha}}$, $x>0,\alpha>0$	$(\ln n)^{1/\alpha}$	$\frac{(\ln n)^{\frac{1-\alpha}{\alpha}}}{\alpha}$	$T_{1\alpha}$

8.5 Domain of Attractions for $X_{1:n}$

The following Lemma is needed to prove the necessary and sufficient conditions for the convergence of X_{1n} to one of the three limiting distributions.

Lemma 8.8 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables with distribution function F. Consider a sequence* $(e_n, n \geq 1\}$ *of real numbers. Then for any* $\xi, 0 \leq \xi < \infty$, *the following two statements are equivalent*

(iii) $\lim_{n\to\infty} n(F(e_n)) = \xi$

(iv) $\lim_{n\to\infty} P\left(X_{n,n} > e_n\right) = e^{-\xi}$.

Proof: The proof of the Lemma follows from Lemma 2.1.1 by considering the fact $P(X_{1n} > e_n) = (1 - F(e_n))^n$

8.5.1 *Domain of Attraction for Type 1 Extreme Value Distribution for* $X_{1:n}$

The following theorem gives a necessary and sufficient condition for the convergence of X_{1n} to $H_{10}(x)$.

Theorem 8.6 *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d. random variables with distribution function F. Assume further that* $E(X|X \leq t)$ *is finite for some* $t > \alpha(F)$ *and* $h(t) = E(t - X|X \leq t)$, *then* $F \in H_{10}$ *iff* $\lim_{t\to\alpha(F)} \frac{F(t+xh(t))}{F(t)} = e^x$ *for all real x.*

Proof Similar to Theorem 8.1.

Example 8.10 Consider the logistic distribution with $F(x) = \frac{1}{1+e^{-x}}, -\infty < x < \infty$. Now

$$h(t) = E(t - x|X \leq t) = t - (1 + e^{-t}) \int_{-\aleph}^{t} \frac{xe^{-x}}{(1 + e^{-x})^2} dx = (1 + e^{-t}) \ln(1 + e^t).$$

It can easily be shown that $h(t) \to 1$ as $t \to -\infty$. We have $\lim_{t\to\alpha(F)} \frac{F(t+xh(t))}{F(t)} = \lim_{t\to-\infty} \frac{1+e^{-t}}{1+e^{-(t+xh(t))}} = \lim_{t\to-\infty} \frac{e^{t+xh(t)}+e^{xh(t)}}{1+e^{t+xh(t)}} = e^x$. Thus $X_{1:n}$ from logistic distribution belongs to the domain of $H_{10.}$

8.5.2 *Domain of Attraction of Type 2 Distribution for* $X_{1:n}$

Theorem 8.7 *Let* X_1, *is from a distribution function F then* $F \in H_{2\delta}$ *iff* $\alpha(F) = -\infty$ and $\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = x^\delta$ *for all* $x > 0$.

Proof Suppose $H_{2\delta}(x) = 1 - e^{-(-x)^{-\delta}}, x < 0, \delta > 0$, then we have

$$\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = \lim_{t\to-\infty} \frac{1 - e^{-(-tx)^{-\delta}}}{1 - e^{-(-t)^{=\delta}}} = \lim_{t\to-\infty} \frac{\delta x(-tx)^{-(\delta+1)} e^{-(-tx)^{-\delta}}}{\delta(-t)^{-(\delta+1)} e^{-(-t)^{=\delta}}} = x^{-\delta}, \delta > 0.$$

Let $\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = x^{-\delta}, \delta > 0$. We can write $a_n = inf\{x : \bar{F}(x) \le \frac{1}{n}\}$, then $a_n \to -\infty$ as $n \to \infty$. Thus $\lim_{n\to-\infty} n(F(a_n x)) = \lim_{n\to-\infty} n(F(a_n)) \frac{F(a_n x)}{F(a_n)} = x^{-\delta} \lim_{n\to-\infty} nF(a_n)$.

It is easy to show that $\lim_{n\to-\infty} nF(a_n) = 1$. Thus $\lim_{n\to-\infty} n(F(a_n x)) = x^{-\delta}$ and the proof of the follows.

Example 8.11 For the Cauchy distribution F(x) $\lim_{t\to\alpha(F)} \frac{F(tx)}{F(t)} = \lim_{t\to-\infty} \frac{\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(tx)}{\frac{1}{2}+-\frac{1}{\pi}\tan^{-1}(i)} =$
$\lim_{t\to-\infty} \frac{x(1+t^2)}{1+(tx)^2} = x^{-1}. = \frac{1}{2} + \tan^{-1}(x)$.
Thus Cauchy distribution belongs to the domain of attraction of H_{21}.

8.5.3 Domain of Attraction of Type 3 Extreme Value Distribution

Theorem 8.8 *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d random variables with distribution function F then* $F \in H_{3\delta}$ *iff* $\alpha(F)$ *is finite and* $\lim_{t\to0} \frac{F(\alpha(F)+tx)}{F(\alpha(F)+t)} = x^{\delta}$, $\delta >$ *0 and for all x > 0.*

Proof The proof is similar to Theorem 8.3.

Example 8.12 Suppose X has the uniform distribution with $F(x) = x, 0 < x < 1$.Then
$\lim_{t\Rightarrow0} \frac{F(tx)}{F(t)} = x$. Thus then $F \in H_{31}$.

Following Pickands (1975), the following theorem gives a necessary and sufficient condition for the domain of attraction of $X_{1:n}$ from a continuous distribution.

Theorem 8.9 *For a continuous random variable the necessary and sufficient condition for* $X_{1:n}$ *to belong to the domain of attraction of the extreme value distribution of the minimums*

$$\lim_{c\to0} \frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = 1 \text{ if } F \in H_{10},$$

$$\lim_{c\to0} \frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = 2^{1/\delta} \text{ if } F \in H_{2\delta} \text{ and}$$

$$\lim_{c\to0} \frac{F^{-1}(c) - F^{-1}(2c)}{F^{-1}(2c) - F^{-1}(4c)} = 2^{-1/\delta} \text{ if } F \in H_{3\delta}$$

Example 8.13 For the logistic distribution with $F(x) = \frac{1}{1+e^{-x}}$, $F^{-1}(x) = \ln x - \ln(1-x)$

$\lim_{c\to 0} \frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \lim_{c\to 0} \frac{\ln c-\ln(1-c)-\ln 2c+\ln(1-2c)}{\ln 2c-\ln(1-2c)-\ln 4c+\ln(1-4c)} = 1$. Thus the domain of attraction of $X_{1:n}$ from the logistic distribution is T_{10}.

For the Cauchy distribution with $F(x) = \frac{1}{2} + \tan^{-1}(x)$.

We have $F^{-1}(x) = \tan\pi(x - \frac{1}{2}) = -\frac{1}{\pi x}$ for small x.

Thus $\lim_{c\to 0} \frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \frac{\frac{1}{2\pi c}-\frac{1}{\pi c}}{\frac{1}{4\pi c}-\frac{1}{2\pi c}} = 2.$

Thus the domain of attraction of $X_{1:n}$ from the Cauchy distribution is T_{21}.

For the exponential distribution, $E(0, \sigma)$, with pdf $f(x) = \sigma^{-1}e^{-\sigma^{-1}x}$, $x \geq 0$, $F^{-1}(x) = -\sigma^{-1}\ln(1-x)$ and $\lim_{c\to 0} \frac{F^{-1}(c)-F^{-1}(2c)}{F^{-1}(2c)-F^{-1}(4c)} = \lim_{c\to 0} \frac{-\ln\{1-c)+\ln\{1-2c)}{-\ln\{1-2c)+\ln\{1-4c)} = 2^{-1}$. Thus the domain of attraction of $X_{1:n}$ from the exponential distribution, $E(0, \sigma)$, is T_{31}.

We can use the following lemma to calculate the normalizing constants for various distributions belonging to the domain of attractions of H(x).

Lemma 8.9 *Suppose* $P(X_{1:n} \leq c_n + d_n x) \to H(x)$ *as* $n \to \infty$, *then*

(i) $c_n = F^{-1}(\frac{1}{n}), d_n = F^{-1}(\frac{1}{n}) - F^{-1}(\frac{1}{ne})$ *if* $H(x) = H_{10}(x)$,
(ii) $c_n = 0, b_n = |F^{-1}(\frac{1}{n})|$ *if* $H(x) = H_{2\delta}(x)$,
(iii) $c_n = \alpha(F), b_n = F^{-1}(\frac{1}{n}) - \alpha(F)$ *if* $H(x) = H_{3\delta}(x)$

We have seen that the normalizing constants are not unique for $X_{n:n}$. *The same is also true for the* $X_{1:n}$.

Example 8.14 For the logistic distribution with $F(x) = \frac{1}{1+e^{-x}}$, $X_{1:n}$ when normalized converge in distribution to Type $1(H_{10})$ distribution. The normalizing constants are $c_n = F^{-1}(\frac{1}{n}) = \ln\left(\frac{1/n}{1-(1/n)}\right) \cong -\ln n$ *and* $d_n = F^{-1}(\frac{1}{n}) - F^{-1}(\frac{1}{ne}) = 1$.

For Cauchy distribution with $F(x)= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$, X_{1n} when normalized converges in distribution to Type 2 (H_{21}) distribution. The normalizing constants are $c_n = 0$ and $d_n = |F^{-1}(\frac{1}{n})| = \tan\pi(\frac{1}{2} - \frac{1}{n}), n > 2$

For the uniform distribution with F(x) = x, X_{1n} when normalized converge in distribution to Type 3 (H_{31}) distribution. The normalizing constants are $c_n = 0, b_n = F^{-1}(\frac{1}{n}) = \frac{1}{n}$ (see Table 8.3).

Table 8.3 Normalizing constants and domain of attraction of $X_{1:n}$

Distribution	f(x)	C_n	D_n	Domain
Beta	$cx^{\alpha-1}(1-x)^{\beta-1}$ $\alpha>0, \beta>0$ $c=\frac{1}{B(\alpha,\beta)}$ $0<x<1$	0	$\left(\frac{c\alpha}{n}\right)^{1/\alpha}$ $c=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$H_{3\alpha}$
Cauchy	$\frac{1}{\pi(1+x^2)}$, $-\infty<x<\infty$	0	$\cot(\frac{\pi}{n})$	H_{21}
Exponential	$\sigma e^{-\sigma x}$, $0<x<\infty$, $\sigma>0$	0	$\frac{1}{n\sigma}$	H_{31}
Gamma	$\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$, $0<x<\infty$	0	$\frac{\Gamma(\alpha)}{n}$	H_{31}
Laplace	$\frac{1}{2}e^{-\lvert x\rvert}$, $-\infty<x<\infty$	$\ln\left(\frac{n}{2}\right)$	1	H_{10}
Logistic	$\frac{e^{-x}}{(1+e^{-x})^2}$	-ln n	1	H_{10}
Lognorma 1	$\frac{1}{x\sqrt{2\pi}}e^{-\frac{1}{2}(\ln x)^2}$ $0<x<\infty$	$e^{n\alpha_n}$, $\alpha_n=\frac{1}{b_n}-\frac{b_nD_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $b_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}e^{\alpha_n}$	H_{10}
Normal	$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $-\infty<x<\infty$	$\frac{1}{c_n}-\frac{c_nD_n}{2}$, $D_n=\ln\ln n+\ln 4\pi$ $b_n=(2\ln n)^{-1/2}$	$(2\ln n)^{-1/2}$	H_{10}
Pareto	$\alpha x^{-(\alpha+1)}$ $x>1, \alpha>0$	0	$\left(\frac{n}{n-1}\right)^{1/\alpha}$	H_{21}
Power Function	$\alpha x^{\alpha-1}$, $0<x<1, \alpha>0$	0	$\frac{1}{n^{1/\alpha}}$	H_{31}
Rayleigh	$\frac{2x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}$, $x\geq 0$	0	$\sigma\sqrt{\frac{2}{n}}$	H_{32}
T distribution	$\frac{k}{\left(1+\frac{x^2}{\upsilon}\right)^{(\upsilon+1)/2}}$ $k=\frac{\Gamma((\upsilon+1)/2)}{(\pi\upsilon)^{1/2}\Gamma(\upsilon/2)}$	0	$\left(\frac{kn}{\upsilon}\right)^{1/\upsilon}$	$H_{2\upsilon}$
Type 1 (for minimum)	$e^xe^{-e^x}$	-ln n	1	H_{10}
Type 2 (for minimum)	$\alpha(-x)^{-(\alpha+1)}e^{-x^{-\alpha}}$ $x<0, \alpha>0$	0	$n^{1/\alpha}$	$H_{2\alpha}$
Type 3 (for minimum)	$\alpha x^{\alpha-1}.e^{x^\alpha}$ $x>0, \alpha>0$	0	$n^{-1/\alpha}$	$H_{3\alpha}$
Uniform	$1/\theta$, $0<x<\theta$	0	θ/n	H_{31}
Weibull	$\alpha x^{\alpha-1}e^{-x^\alpha}$, $x>0, \alpha>0$	0	$\frac{1}{n^{1/\alpha}}$	$H_{3\alpha}$

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