SOLID MECHANICS AND ITS APPLICATIONS

Uri Kirsch

Design-Oriented Analysis of Structures

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Design-Oriented Analysis of Structures

A Unified Approach

by

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To my wife, Aira

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Preface

This book was developed while I was teaching graduate courses on analysis, design and optimization of structures, in the United States, Europe and Israel. Structural analysis is a main part of any design problem, and the analysis often must be repeated many times during the design process. Much work has been done on design-oriented analysis of structures recently and many studies have been published. The purpose of the book is to collect together selected topics of this literature and to present them in a unified approach. It meets the need for a general text covering the basic concepts and methods as well as recent developments in this area. This should prove useful to students, researchers, consultants and practicing engineers involved in analysis and design of structures. Previous books on structural analysis do not cover most of the material presented in the book.

The book deals with the problem of multiple repeated analyses (reanalysis) of structures that is common to numerous analysis and design tasks. Reanalysis is needed in many areas such as structural optimization, analysis of damaged structures, nonlinear analysis, probabilistic analysis, controlled structures, smart structures and adaptive structures. It is related to a wide range of applications in such fields as Aerospace Engineering, Civil Engineering, Mechanical Engineering and Naval Architecture.

In a typical structural design process, the analysis must be repeated numerous times due to changes in the size of elements, the material properties, the geometry of the structure (coordinates of joints), the topology (number and orientation of elements and joints) and support conditions. The high computational cost involved in repeated analyses is one of the main obstacles in the solution of structural optimization problems, and only methods that do not involve many time consuming analyses are useful. In structural damage analysis, it is necessary to analyze the structure for various changes. It is difficult to determine *a priori* what damage scenarios should be checked, and numerous analyses are required to evaluate various hypothetical scenarios. In nonlinear analysis the set of updated linear equations must be solved repeatedly many times during the solution process.

Design-oriented analysis is intended for efficient and accurate repeated analyses of structures. The book introduces effective computational procedures for reanalysis. The necessary background material on structural analysis needed in the rest of the book is summarized in the first two chapters. However, the reader is expected to be familiar

with the basic concepts of matrix analysis of structures. Various analysis models are considered in the book, including linear and nonlinear analysis, eigenproblems and design sensitivity analysis. The text does not present a survey on reanalysis methods. Rather, part 1 concentrates on various concepts and methods that form the basis of the unified approach presented in part 2. To clarify the presentation, many illustrative examples and numerical results are demonstrated. No specific system of units is used in the examples, However, in some examples actual dimensions of the structure and specified magnitude of forces have been used.

In part 1 (Chapters 1–6) the basic concepts of design-oriented analysis are introduced and various reanalysis methods are developed. In part 2 (Chapters 7–12) the concepts and the methods presented in part 1 are integrated into a unified approach for effective reanalysis of structures. Recent developments and applications in this area are discussed in this part of the book. Some sections of the book are necessary for continuity, while others are needed only for those interested in greater depth in a particular topic. Many sections are independent and can be omitted, or their order can be changed.

The approach presented in the book is suitable for a wide range of applications. It combines several advantages in terms of generality, ease-of implementation, flexibility, efficiency and accuracy. The approach is suitable for various types of changes in the structure and different types of structures. The solution procedure uses the stiffness analysis formulation and it can be integrated into available finite element programs. Calculation of derivatives is not required, and the approach is most attractive in cases where derivatives are not readily available or not easy to calculate. The accuracy of the results, and the efficiency of the calculations can be controlled by the level of simplification and the amount of information considered. Highly accurate results can be achieved at the expense of more computational effort by considering high-order approximations. On the other hand, very efficient solutions can be obtained by simplified low-order approximations. In certain cases exact solutions can be achieved with a small computational effort.

Chapter 1 presents introductory material on analysis and reanalysis of structures. Various types of changes in the structure are discussed, including changes in the structural model itself, and the scope of the text is described.

In Chapter 2 some background material on analysis of structures is introduced. Linear elastic analysis, analysis of continuum structures, nonlinear analysis and dynamic analysis are briefly described.

Chapter 3 deals with the statement of reanalysis problems. Formulations of linear, nonlinear and eigenproblem reanalysis are presented and various direct as well as approximate reanalysis methods are reviewed.

Direct methods, giving exact closed-form solutions, are presented in Chapter 4. These methods are efficient in situations where a relatively small proportion of the structure is changed (e.g., changes in cross sections of only a small number elements).

Chapter 5 presents the most simple and most efficient local approximations. These include the common Taylor series, the binomial series, simplified first-order approximations and improved series approximations.

In Chapter 6 we describe global approximations such as polynomial-fitting techniques, the response surface approach, reduced basis methods and the conjugate gradient method. These approximations are usually obtained by analyzing the structure at a number of design points, and they are valid for large changes in the structure.

Chapter 7 presents the combined approximations approach. The basic concepts of combining various methods into a unified solution approach are introduced. The advantage is that the efficiency of local approximations and the improved accuracy of global approximations are combined to obtain effective solution procedures. Some typical cases, where exact or accurate solutions can be achieved, are developed.

Chapter 8 describes simplified solution procedures that can be derived from the general approach. The various procedures include approximate and most efficient techniques as well as direct methods that provide exact solutions. We can view some conventional approximations and direct methods as particular cases of the general approach presented.

In Chapter 9 we discuss reanalysis for topological and geometrical changes. Developing reanalysis methods for such changes is most challenging, since the structural model itself is changed. Both approximate and exact solutions are demonstrated for various cases of deletion and addition of elements and joints, as well as changes in the joint coordinates. Solution procedures are developed for the most challenging problem where the number of degrees of freedom is changed.

In Chapter 10 we develop procedures for calculating the response derivatives with respect to design variables for designs where results of exact analysis are not available. Accurate derivatives can be obtained for such designs with a reduced computational effort.

Nonlinear reanalysis is discussed in Chapter 11. The unified approach presented is most suitable for solving efficiently the updated linear equations in such problems.

Vibration reanalysis by the unified approach is developed in Chapter 12. Solutions are demonstrated for problems of eigenvector reanalysis, where we evaluate the mode shapes, and eigenvalue reanalysis, where we calculate the eigenvalues.

Uri Kirsch

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PART ONE

CONCEPTS AND METHODS

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1 Introduction

1.1 ANALYSIS AND REANALYSIS

1.1.1 Structural Analysis

Structural analysis is a most exciting field of activity, but it is clearly only a support activity in the larger field of structural design. Analysis is a main part of the formulation and the solution of any design problem, and it often must be repeated many times during the design process. The analysis process helps to identify improved designs with respect to performance and cost.

Referring to behavior under working loads, the objective of the analysis of a given structure is to determine the internal forces, stresses and displacements under application of the given loading conditions. In order to evaluate the response of the structure it is necessary to establish an analytical model, which represents the structural behavior under application of the loadings. An acceptable model must describe the physical behavior of the structure adequately, and yet be simple to analyze. That is, the basic assumptions of the analysis will ensure that the model represents the problem under consideration and that the idealizations and approximations used result in a simplified solution. This latter property is essential particularly in the design of complex or large systems.

Two categories of mathematical models are often considered:

- Lumped-parameter (discrete-system) models.
- Continuum-mechanics-based (continuous-system) models

The solution of discrete analysis models usually involves the following steps:

- *Idealization of the system into a form that can be solved.* The actual structure is idealized as an assemblage of elements that are interconnected at the joints.
- *Formulation of the mathematical model.* The equilibrium requirements of each element are first established in terms of unknown displacements, and the element interconnection requirements are then used to establish the set of simultaneous analysis equations for the unknown displacements.
- *Solution of the model.* The response is calculated by solving the simultaneous equations for the unknown displacements; the internal forces and stresses of each element are calculated by using the element equilibrium requirements.

The overall effectiveness of an analysis depends to a large degree on the numerical procedures used for the solution of the equilibrium equations [1]. The accuracy of the analysis can, in general, be improved if a more refined model is used. In practice, there is a tendency to employ more and more refined models to approximate the actual structure. This means that the cost of an analysis and its practical feasibility depend to a considerable degree on the algorithms available for the solution of the resulting equations. Because of the requirement that large systems be solved, much research effort has been invested in equation solution algorithms.

The time required for solving the equilibrium equations can be a high percentage of the total solution time, particularly in nonlinear analysis or in dynamic analysis, when the solution must be repeated many times. An analysis may not be possible if the solution procedures are too costly or unstable.

Linear Elastic Analysis. In elastic analysis the forces must satisfy the conditions of equilibrium, and produce deformations compatible with the continuity of the structure and the support conditions. That is, any method must ensure that both conditions of equilibrium and compatibility are satisfied. In linear analysis we assume that displacements (translations or rotations) vary linearly with the applied forces. That is, any increment in a displacement is proportional to the force causing it. This assumption is based on the following two conditions:

a. The material of the structure is elastic and obeys Hooke's law.
b. All deformations are assumed to be small, so that the displacements do not significantly affect the geometry of the structure and hence do not alter the forces in the members. Thus, the changes in the geometry are small and can be neglected.

The majority of actual structures are designed to undergo only small and linear deformations. In such cases the principle of superposition can be applied.

Linear elastic analysis involves the solution of a set of simultaneous linear equations. The displacement (stiffness) method [2, 3], which is the most commonly used analysis method, is considered throughout this text. The method is described in Section 2.1.

Continuum Structures. Elastic analysis of continuum structures, such as plates and shells, is usually performed by the numerical finite elements method [1, 4]. This method can be regarded as an extension of the displacement method to two- and three-dimensional continuum structures. The actual continuum is replaced by an equivalent idealized structure composed of discrete elements, referred to as finite elements, connected together at a number of nodes. By assuming displacement fields or stress patterns within an element, we can derive a stiffness matrix. A set of simultaneous algebraic equations is formed by applying conditions of equilibrium at every node of the idealized structure. The solution gives the nodal displacements, which in turn are used to determine the stresses. The finite elements method is described in Section 2.2.

Nonlinear Analysis. Considering separately material nonlinear effects and geometric nonlinear effects, we can categorize different nonlinear analysis problems. A nonlinear relationship between the applied forces and the resulting displacements exists under either of two conditions:

a. The stress-strain relation of the material of the structure is nonlinear (material non-linearity). Material non-linearity is considered in plastic analysis.
b. The stress-strain relation is within the linear-elastic range, but the geometry of the structure changes significantly during an application of the loads (geometrical non-linearity).

Nonlinear analysis of structures is usually carried out in an iterative process. The study of nonlinear behavior of structures includes plastic analysis and buckling of structures. The basic problem in nonlinear analysis is the solution of a set of nonlinear equations. Depending on the history of the loading, the stiffness of the structure may be softening or stiffening, the equilibrium path may be stable or unstable, and the structure itself may be at a stage of loading or unloading. All such phenomena are typified by the occurrence of critical points such as the limit points and snap-back points in the load-deflection curves.

The solution process of nonlinear analysis problems can be carried out by different methods [5, 6], including an incremental (Euler) solution scheme, the iterative Newton-Raphson method and combined incremental/iterative solutions (full or modified Newton-Raphson or the initial stress method). A requirement for the solution method is its ability to overcome the numerical problems associated with various types of behavior. Nonlinear analysis is briefly described in Section 2.3.

Dynamic Analysis. The purpose of dynamic analysis is to determine internal forces, stresses and displacements under application of dynamic (time varying) loads [7]. In general, the structural response to any dynamic loading is expressed in terms of the displacements of the structure. The eigenproblem is to find the free-vibration frequencies and the mode shapes of the vibrating system. The solution of the eigenproblem for large structures is often the most costly phase of a dynamic response analysis, and calculation of the eigenvalues and eigenvectors requires much computational effort. The discretized model of a complicated system may have numerous degrees of freedom. Therefore, it is customary in dynamic analysis to reduce the equations of motion to a much smaller number before the dynamic response is calculated [8]. The reduction of degrees of freedom is most important because the solution must be performed successively at many different times to generate the time history of the response. Dynamic analysis is briefly described in Section 2.4.

1.1.2 Design Variables

A structural system can be described by a set of quantities, some of which are viewed as variables during the design process. Those quantities defining a structural system that are fixed during the design are called pre-assigned parameters. Those quantities that are

not pre-assigned are called design variables. The pre-assigned parameters together with the design variables completely describe a design. Quantities are designated as pre-assigned parameters for a variety of reasons. It may be that the designer is not free to choose certain parameters, or it may be known from experience that a particular value of the parameter produces good results. From a physical point of view, the design variables that are varied during the design process may represent [9]:

a. the mechanical or physical properties of the material;
b. the topology of the structure, i.e., the pattern of connection of members (number and orientation of elements and joints);
c. the geometry of the structure (coordinates of joints);
d. the cross-sectional dimensions or the sizes of elements.

From a mathematical point of view, it is important to distinguish between continuous and discrete design variables. In cases of discrete variables with a large number of values uniformly distributed over a given interval, use of a continuous variable representation is often satisfactory, followed by selection of the nearest available discrete value. When a strictly discrete design variable is handled in this way, it is categorized as pseudo-discrete. However, it should be recognized that situations arise when it will be essential to employ discrete or integer variables; for example, the number of elements in the structure is an integer.

Material selection presents a special problem with conventional materials, as they have discrete properties, i.e., a choice is to be made from a discrete set of variables. Application of high-performance composite materials in structural components has encouraged further consideration of material properties as design variables. For example, in fiber composites the volume fraction of fibers or the modulus of elasticity in the longitudinal direction of carbon fibers could be considered as design variables.

Topological optimization is perhaps the most challenging class of problems in structural optimization because there exists an infinite number of possible topologies, which are difficult to classify and quantify. At the same time, topological optimization is of considerable importance because it leads to significant material savings. The topology of the structure can be optimized automatically in many cases when elements are allowed to reach zero size. This permits elimination of some uneconomical elements during the optimization process. In other cases, however, it may be necessary to represent some design variables as integer variables and to declare the existence or absence of a structural element. An example of an integer topological variable is a truss member joining two nodes, which is limited to the values 1 (the member exists), or 0 (the member is absent). Other examples of integer topological variables include the number of spans in a bridge, the number of columns supporting a roof system, or the number of elements in a grillage system.

Geometrical variables may represent, for example, the coordinates of joints in a truss or in a frame. Other examples for this class of variable include the location of supports in a bridge, the length of spans in a continuous beam, and the height of a shell structure. In general, the geometry of the structure is represented by continuous variables.

Cross-sectional dimensions are the simplest design variables. The cross-sectional area of a truss member, the moment of inertia of a flexural member, or the thickness of a plate are some examples of this class of design variable. In certain cases a single design variable is adequate to describe the cross section, but a more detailed design with several design variables for each cross section may be necessary. For example, if the axial buckling of members is considered, the cross-sectional dimensions which define the area and the moment of inertia can be taken as design variables. In practical design, cross-sectional variables may be restricted to some discrete values, e.g. the areas of commercial steel section shapes.

It should be noted that a change in the cross sections or in the geometry might affect the topology. For example, the topology will be changed due to zero areas during sizing modifications or the coalescence of joints during geometrical modifications. In addition, the geometry might be affected by topological changes due to elimination of members and joints.

1.1.3 Changes in the Structural Model

Changes in the design often affect only the numerical values of the coefficients of the analysis equations. However, in some cases of topological changes, members and joints are deleted or added and the structural model is allowed to vary during the design process. The following cases of topological changes are considered in this text [10]:

a. Deletion and addition of members, where the number of Degrees of Freedom (DOF) is unchanged (Figure 1.1*a*). In this case the number of analysis equations is also unchanged and only the numerical values of the coefficients of the equations are modified.
b. Deletion and addition of members, and deletion of some joints, where the number of DOF is decreased (Figure 1.1*b*). In this case it is necessary to change the analysis model such that the deleted DOF are not included in the modified analysis equations.
c. Deletion and addition of members, and addition of some joints, where the number of DOF is increased (Figure 1.1*c*). In this case it is necessary to augment the analysis model such that the new degrees of freedom are included in the modified analysis equations.

The resulting structures may be classified as follows:

a. Stable (S) structures. This is the typical case where the modified equilibrium equations can be solved.
b. Conditionally Unstable (CU) structures. In this case the forces in the structure satisfy equilibrium conditions for a specific loading. That is, the structure can carry only specific loading conditions, and it is unstable for another loading condition.
c. Unstable (U) structures, where the structure or part of it is unstable for a general loading condition. In this case the modified equilibrium equations cannot be solved and a collapse of the structure will occur.

To illustrate the various cases, consider the initial ten-bar truss shown in Figure 1.2 subjected to two concentrated loads. The number of unknown forces is fourteen (ten member forces and four support reactions) and the number of independent equilibrium conditions is twelve (twice the number of joints). The truss is statically indeterminate having two redundant members, thus various cases of deletion of two members could be considered. For all forty-five possible cases of deletion of two members, the resulting structure might be stable, conditionally unstable or unstable. Only twenty-nine of the resulting structures are stable (Figure 1.3), whereas four structures are conditionally unstable (Figure 1.4) and twelve structures are unstable (Figure 1.5). Some of the conditionally unstable structures could be transformed into stable structures by deleting or adding zero force members. It is instructive to note that conditionally unstable structures could be obtained also by addition of some members and joints. In addition, various stable and conditionally unstable structures could be obtained by deletion of three or four members, as can be observed in Figures. 1.6 and 1.7.

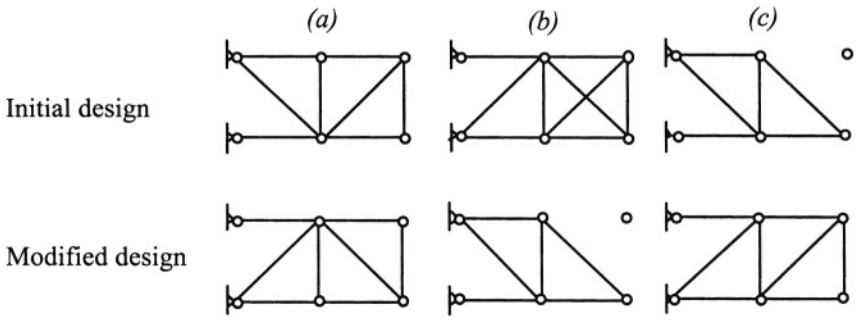

Fig. 1.1. Types of topological changes.

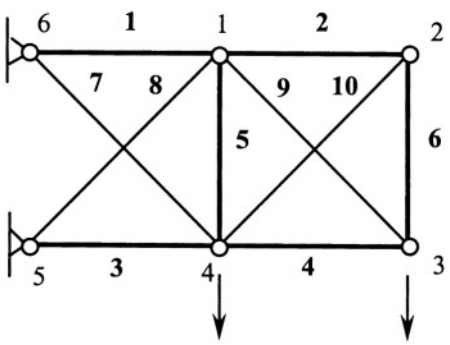

Fig. 1.2. Ten-bar truss.

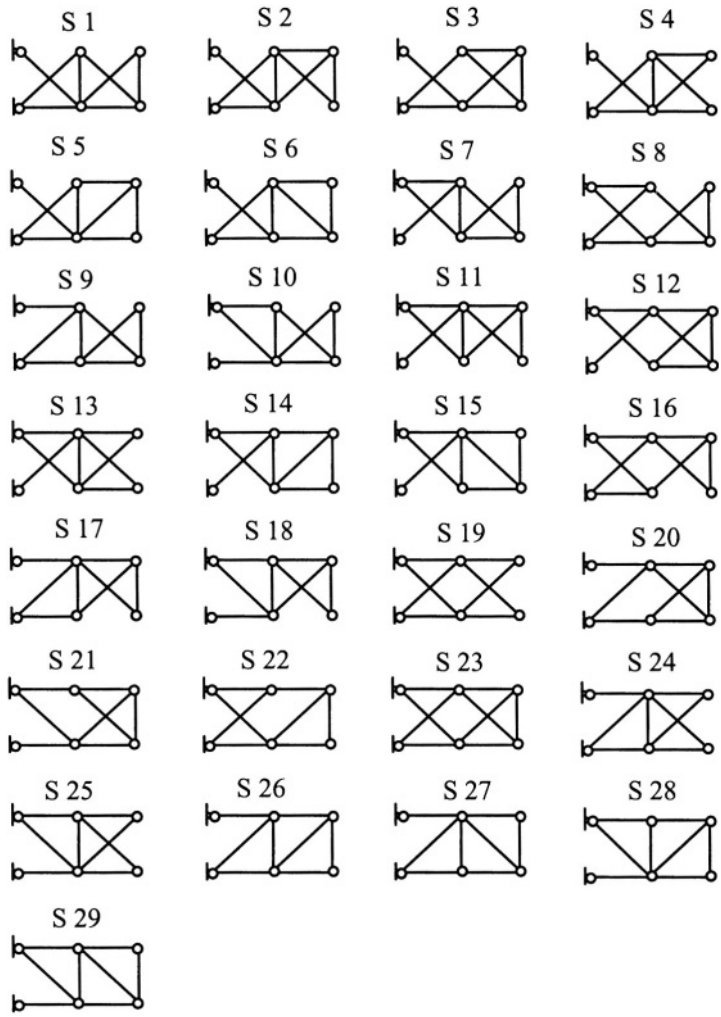

Fig. 1.3. Stable structures obtained by deletion of two members.

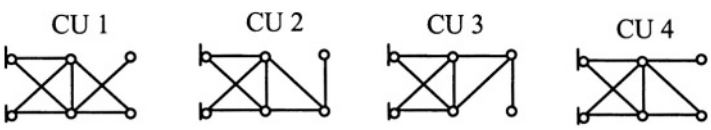

Fig. 1.4. Conditionally unstable structures obtained by deletion of two members.

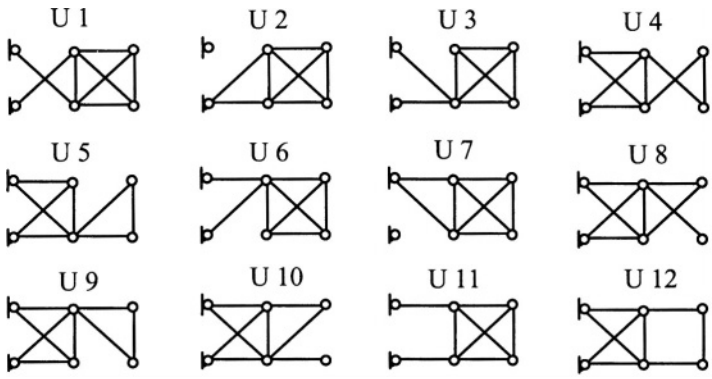

Fig. 1.5. Unstable structures obtained by deletion of two members.

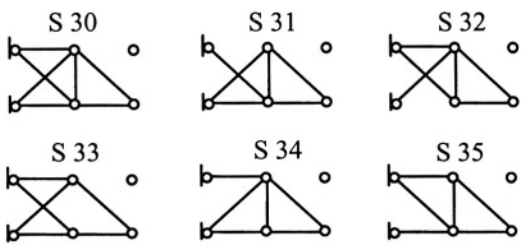

Fig. 1.6. Stable structures obtained by deletion of three or four members.

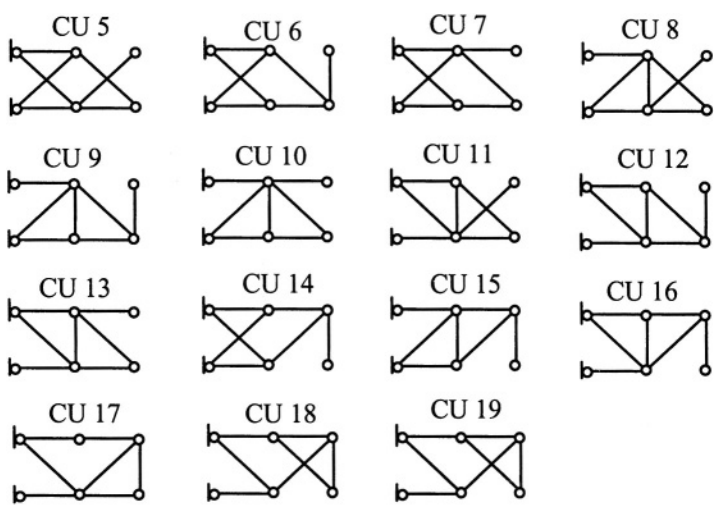

Fig. 1.7. Conditionally unstable structures obtained by deletion of three/four members.

1.1.4 Reanalysis of Structures

Repeated analysis or reanalysis is needed in various problems of structural analysis, design and optimization. In general, the structural response cannot be expressed explicitly in terms of the structure properties, and structural analysis involves solution of a set of simultaneous equations. Reanalysis methods are intended to analyze efficiently structures that are modified due to changes in the design. The object is to evaluate the structural response (e.g. displacements, forces and stresses) for such changes without solving the complete set of modified simultaneous equations. The solution procedures usually use the original response of the structure. Some common problems, where multiple repeated analyses are needed, are described in the following.

- In structural optimization the solution is iterative and consists of repeated analyses followed by redesign steps. The high computational cost involved in repeated analyses of large-scale problems is one of the main obstacles in the solution process. In many problems the analysis part will require most of the computational effort, therefore only methods that do not involve numerous time-consuming implicit analyses might prove useful. Reanalysis methods, intended to reduce the computational cost, have been motivated by some typical difficulties involved in the solution process. The number of design variables is usually large, and various failure modes under each of several load conditions are often considered. The constraints are implicit functions of the design variables, and evaluation of the constraint values for any assumed design requires the solution of a set of simultaneous analysis equations.
- In structural damage analysis, it is necessary to analyze the structure for various changes due to deterioration, poor maintenance, damage, or accidents. In general many hypothetical damage scenarios, describing various types of damage, should be considered. These include partial or complete damage in various elements of the structure and changes in the support conditions. Numerous analyses are required to assess the adequacy of redundancy and to evaluate various hypothetical damage scenarios for various types of damage.
- In the design of the construction stages of complex structures, it might be necessary to analyze repeatedly structures that are modified during the construction. The modified structures are subjected to different loading conditions. The changes in the structure may include additional elements and different support conditions.
- Nonlinear analysis of structures is usually carried out in an iterative process. The solution process can be performed by different methods but, in general, a set of updated linear equations must be solved repeatedly. Similarly, many of the vibration (or eigenproblem) solution techniques are based on matrix iteration methods. To calculate the mode shapes it is necessary to solve repeatedly a set of updated analysis equations.
- Reanalysis methods might prove useful in other applications such as probabilistic analysis, controlled structures, smart structures, adaptive structures, and for conceptual design problems.

1.2 SCOPE OF TEXT

This text consists of two parts. In part 1 (Chapters 1 – 6), the basic concepts of design-oriented analysis are introduced and various reanalysis methods are developed. In part 2 (Chapters 7 – 12) the concepts and the methods presented in part 1 are integrated into a unified approach for effective analysis and reanalysis of structures. The most recent developments and applications in this area are discussed in this part of the book.

In Chapter 2 some background material on analysis of structures is introduced and the following analysis problems are briefly described:

a. Section 2.1 describes linear elastic analysis, where displacements are assumed to vary linearly with the applied forces. This assumption is based on the two conditions that the material is elastic and obeys Hooke's law, and the changes in the geometry are so small that they can be neglected. Linear analysis involves the solution of a set of simultaneous linear equations. The displacement (stiffness) method, which is the most commonly used analysis method, is considered throughout the book.

b. Elastic analysis of continuum structures such as plates and shells is presented in Section 2.2, and the numerical finite element method is briefly described. To apply the method, it is necessary first to convert the continuum into a system with a finite number of unknowns so that the problem can be solved numerically. The structure is divided into finite elements and the elements are assumed to be interconnected at a discrete number of nodal points situated on the element boundaries. A displacement function, in terms of nodal-displacement parameters, is then chosen to represent the displacement field within each element. The stiffness matrix and the load matrix are derived and the unknown nodal displacements are computed by solving the set of equilibrium equations

c. Section 2.3 presents nonlinear analysis, where the displacements are nonlinear functions of the applied forces. The nonlinear relationship is a result of inelastic material (material non-linearity) or significant changes in the geometry (geometrical non-linearity). The basic problem in nonlinear analysis is the solution of a set of nonlinear equations. The solution is usually carried out by a combination of incremental and iterative techniques, and involves repeated solution of linearized equations.

d. Section 2.4 describes dynamic analysis, where displacements and stresses under application of dynamic loads are determined. The discretized model of a complicated structural system may have numerous degrees of freedom. Therefore, it is customary to reduce the number of equations of motion to a much smaller number before dynamic response is calculated. The solution of the eigenproblem for large structures is one of the costly phases of a dynamic response analysis.

Chapter 3 deals with the statement of reanalysis problems. Given an initial structure and its response, the reanalysis problem considered in the text is to evaluate efficiently and accurately the modified response for successive changes in the structure properties

without solving the complete set of modified analysis equations. Formulations of linear, nonlinear and vibration reanalysis problems are presented in Section 3.1, and various reanalysis methods are reviewed in Section 3.2.

Direct methods, giving exact closed-form solutions, are presented in Chapter 4. These methods are efficient in cases of low-rank modifications in the stiffness matrix. In particular, such methods are applicable to situations where a relatively small proportion of the structure is changed and the number of modified elements in the stiffness matrix is limited (e.g., changes in cross section areas of only a small number elements). Direct methods are usually based on the Sherman-Morrison and Woodbury formulae for the update of the inverse of a matrix. Solutions obtained by the formulae are discussed in Section 4.1, and methods for direct calculation of the response that are based on the formulae are presented in Section 4.2.

Chapter 5 presents the most simple and most efficient local approximations such as the Taylor series, the binomial series, simplified first-order approximations and improved series approximations. Local approximations are based on information calculated at a single point. Accurate results can often be achieved by these methods with a small computational effort for small changes in the structure. In Section 5.1 the common Taylor series and the binomial series approximations are presented and the relationship between the two basic approximations is discussed. The two approximations differ in the efficiency of the calculations and in the ease of implementation, but they become equivalent under certain conditions. In Section 5.2 various simplified first-order approximations, using intermediate variables, are presented. The common direct and reciprocal approximations, as well as conservative and convex approximations, are discussed and compared. Intermediate variables can improve the accuracy of conventional series approximations with little computational effort. Some improved series approximations are discussed in Section 5.3. Conventional series approximations can be improved significantly by scaling procedures. The concept of scaling is introduced and methods for improved-scaled approximations, using simple procedures for selecting the scaling multiplier, are developed.

Chapter 6 describes global approximations such as polynomial-fitting techniques, the response surface approach, reduced-basis methods and the conjugate gradient method. These approximations are obtained by analyzing the structure at a number of design points, and they are valid for large changes in the structure. In general, global approximations provide accurate results but they may require much computational effort, particularly in cases of large numbers of design variables. Polynomial-fitting techniques tend to give accurate solutions in cases where the response functions can be well approximated by low-order polynomials. The quadratic fitting, the cubic fitting, least-square solutions and the response surface approach are discussed in Section 6.1. Reduced-basis methods, introduced in Section 6.2, demonstrate how analysis problems with large numbers of degrees of freedom may be transformed into much smaller problems with corresponding small systems of equations. Reanalysis by the conjugate gradient method is discussed in Section 6.3.

Chapter 7 presents a combined approximations approach. The basic concepts of combining various methods into a unified solution approach are introduced. The

advantage is that the efficiency of local approximations (series expansion) and the improved quality of global approximations (the reduced basis method) are combined to obtain effective solution procedures. The effectiveness of the approach depends, to a great extent, on the appropriate choice of the basis vectors. The problem of determining these vectors is discussed, and generation of vectors for which the reduced set of analysis equations becomes uncoupled is demonstrated. Some typical cases where accurate and exact solutions can be achieved by the combined approximations approach are developed, equivalence of the approach presented with a preconditioned conjugate gradient method is demonstrated, and various methods of error evaluation are presented.

Chapter 8 describes simplified solution procedures that can be derived from a general solution approach. We can view some conventional approximations and direct methods as particular cases of the general approach presented. The preferred procedure for a specific application depends on efficiency and accuracy considerations. In general, better accuracy can be achieved at the expense of more computational effort. Approximate solution procedures are presented in Section 8.1. Application of low-order approximations in structural optimization and analysis of damaged structures is illustrated, the efficiency of the various procedures is demonstrated, and limitations on the design changes are discussed. We demonstrate in Section 8.2 a procedure to obtain exact solutions by the unified approach for simultaneous rank-one changes in the stiffness matrix. This procedure is efficient when the number of changed members in the structure is much smaller than the number of degrees of freedom. It is shown that solutions obtained by the approach presented are equivalent to those given by the Sherman-Morrison-Woodbury formulae.

Reanalysis for topological and geometrical changes is discussed in Chapter 9. Developing methods for such changes is most challenging, since the structural model itself is changed and the resulting response might be significantly different. Most approximate reanalysis methods are not suitable for such changes and provide inadequate or meaningless results. In this chapter reanalysis procedures for various topological and geometrical changes are introduced. Approximate and exact solutions are demonstrated for various cases of deletion and addition of elements and joints, as well as changes in the joint coordinates. In particular, we develop solution procedures for the most challenging problem where the number of degrees of freedom is changed.

Design sensitivity analysis deals with the calculation of changes in the response of the structure resulting from changes in the design variables describing the structure. The derivatives of the response vector with respect to the system parameters are referred to as the sensitivity coefficients. The latter are used in various design and optimization problems. Most approximations that are adequate for structural reanalysis are not sufficiently accurate for sensitivity analysis. In Chapter 10 we develop procedures for calculating the response derivatives with respect to design variables, for designs where results of exact analysis are not available. Exact analytical derivatives, obtained by the direct method and the adjoint-variable method, are presented in Section 10.1. Some procedures for calculation of approximate derivatives are developed in Section 10.2 and results obtained by various methods are compared in Section 10.3. Calculation of second-order derivatives is demonstrated in Section 10.4 and computational procedures

are presented in Section 10.5. Accurate derivatives are obtained for significant changes in the design with a reduced computational effort.

Chapter 11 demonstrates solution of nonlinear reanalysis problems. These problems can be solved by different methods but no matter what method is used a set of updated linear equations must be solved repeatedly during the solution process. It is shown that the combined approximations approach is most suitable for calculating efficiently the updated linear equations in nonlinear analysis problems.

Vibration reanalysis by the combined approximations approach is developed in Chapter 12. Solutions of eigenvector reanalysis, where we evaluate the mode shapes, and eigenvalue reanalysis, where we calculate the eigenvalues, are demonstrated.

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2 Structural Analysis

Some background material on analysis of structures is given in this chapter. The objective of structural analysis is usually to determine the internal forces, stresses and displacements of a structure under application of the working loads. The forces must satisfy the conditions of equilibrium and produce deformations compatible with the continuity of the structure and the support conditions.

Any method of elastic analysis must ensure that both conditions of equilibrium and compatibility are satisfied. In linear elastic analysis, presented in Section 2.1, it is assumed that displacements vary linearly with the applied forces. That is, any increment in a displacement is proportional to the force causing it. Linear elastic analysis involves the solution of a set of simultaneous linear equations. The displacement (stiffness) method, which is the most commonly used analysis method, is considered throughout this text. Various topics associated with the method are discussed in the comprehensive literature related to structural analysis (e.g. [1, 2]).

Elastic analysis of continuum structures, such as plates and shells, using the numerical finite element method, is given in Section 2.2. The method can be regarded as an extension of the displacement method to two- and three-dimensional continuum structures. The actual continuum is replaced by an equivalent idealized structure composed of discrete elements, referred to as finite elements, connected together at a number of nodes. By assuming displacement fields or stress patterns within an element, we can derive a stiffness matrix. A set of simultaneous algebraic equations is formed by applying conditions of equilibrium at every node of the idealized structure. The solution gives the nodal displacements, which in turn are used to determine the stresses. The finite elements method is discussed in various texts related to this topic (e.g. [3, 4]).

Nonlinear analysis, where the displacements are nonlinear functions of the applied forces, is described in Section 2.3. The nonlinear relationship is a result of inelastic material (material non-linearity) or significant changes in the geometry (geometrical non-linearity). A basic problem in nonlinear analysis is the solution of the nonlinear set of equations. The solution is usually carried out by a combination of incremental and iterative techniques, and involves repeated solution of linearized equations. A requirement for the solution method is its ability to overcome the numerical problems associated with various types of behavior. Nonlinear analysis is discussed in various texts related to this topic (e.g. [5, 6]).

Dynamic analysis, where displacements, stresses, and forces under application of dynamic (time varying) loads are determined, is discussed in Section 2.4. In general, the structural response to any dynamic loading is expressed in terms of the displacements of the structure. The solution of the eigenproblem for large structures is often the most costly phase of a dynamic response analysis, and calculation of the eigenvalues and eigenvectors requires much computational effort. It is customary in dynamic analysis to reduce the number of equations of motion to a much smaller number before dynamic response is calculated. Reduced-basis methods are presented later in Section 6.2. Detailed discussion on dynamic analysis is given elsewhere (e.g. [3, 7]).

2.1 LINEAR ANALYSIS OF FRAMED STRUCTURES

Framed structures are systems consisting of members that are long in comparison to the dimensions of their cross section. Typical framed structures are beams, grids, and plane and space trusses and frames. In linear elastic analysis we assume that displacements (translations or rotations) vary linearly with the applied forces, that is, any increment in a displacement is proportional to the force causing it. This assumption is based on the following two conditions:

a. The material of the structure is elastic and obeys Hooke's law.
b. All deformations are assumed to be small, so that the displacements do not significantly affect the geometry of the structure and hence do not alter the forces in the members. Thus, the changes in the geometry are small and can be neglected.

The majority of actual structures are designed to undergo only small and linear deformations. In such cases the principle of superposition can be applied. In elastic analysis we refer to behavior under working loads. The objective of the analysis of a given structure is to determine the internal forces, stresses and displacements under the given loads. The forces must satisfy the conditions of equilibrium and produce deformations compatible with the continuity of the structure and the support conditions. That is, any method of elastic analysis must ensure that both conditions of equilibrium and compatibility are satisfied.

In this section the displacement (stiffness) method is described. In this method joint displacements, chosen as the analysis unknowns, are determined from the conditions of equilibrium. The internal forces and the stresses are then determined by superposition of the effects of the external loads and the separate joint displacements.

2.1.1 Basic Relations

The relations presented here form the basis for elastic analysis by various methods. The equilibrium equations are*

* The following symbols have been used throughout this text: bold letters represent matrices or column vectors; superscripts T represent transposed matrices or vectors; subscripts d denote diagonal matrices.

$$\mathbf{C}\,\mathbf{N} = \mathbf{R} \tag{2.1}$$

where the elements of matrix **C** depend on the undeformed geometry of the structure; **N** is the vector of members' forces; and the vector **R** represents the external loads. The constitutive law is

$$\mathbf{K}_d\,\mathbf{e} = \mathbf{N} \tag{2.2}$$

where $\mathbf{K}_d$ is a diagonal matrix of member stiffnesses and **e** is the vector of member displacements. The compatibility equations relate the member displacements **e** to the nodal displacements **r** by

$$\mathbf{Q}\,\mathbf{r} = \mathbf{e} \tag{2.3}$$

where

$$\mathbf{Q} = \mathbf{C}^T \tag{2.4}$$

Various analysis methods can be derived from these basic relations. The displacement (stiffness) method is considered in this text.

Substituting Eqs. (2.2), (2.3) and (2.4) into Eq. (2.1) gives the displacement method equilibrium equations

$$\mathbf{Q}^T\mathbf{K}_d\,\mathbf{Q}\,\mathbf{r} = \mathbf{R} \tag{2.5}$$

Denoting the system stiffness matrix by **K**, where

$$\mathbf{K} = \mathbf{Q}^T\mathbf{K}_d\,\mathbf{Q} \tag{2.6}$$

and substituting into Eq. (2.5) yields

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \tag{2.7}$$

Example 2.1. The purpose of this example is to demonstrate the basic relations of elastic analysis. Consider the four-bar truss shown in Figure 2.1 [8]; the equilibrium equations (2.1) are

$$\begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & -1 \\ \sqrt{2}/2 & 1 & \sqrt{2}/2 & 0 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \sqrt{2} \begin{Bmatrix} 10 \\ 10 \end{Bmatrix}$$

and the constitutive law of Eq. (2.2) is

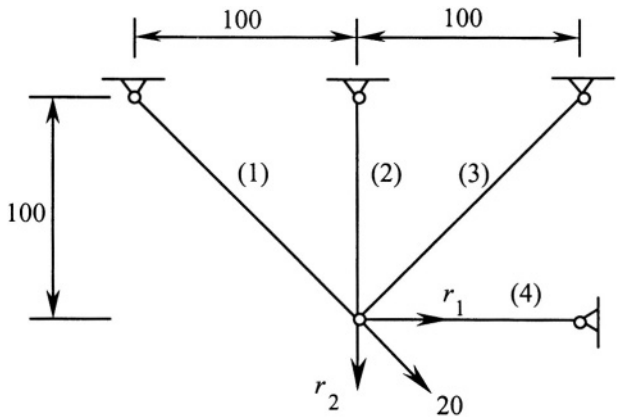

Fig. 2.1. Four-bar truss.

$$\frac{E}{100}\begin{bmatrix} A_1/\sqrt{2} & & & \\ & A_2 & & \\ & & A_3/\sqrt{2} & \\ & & & A_4 \end{bmatrix}\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}$$

where E is the modulus of elasticity and A_i is the cross-sectional area of the ith member. The compatibility equations (2.3) are

$$\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix}$$

and the displacement method equilibrium equations (2.7) are

$$\frac{E}{100}\begin{bmatrix} (\sqrt{2}/4)(A_1 + A_3) + A_4 & (\sqrt{2}/4)(A_1 - A_3) \\ (\sqrt{2}/4)(A_1 - A_3) & (\sqrt{2}/4)(A_1 + A_3) + A_2 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \sqrt{2}\begin{Bmatrix} 10 \\ 10 \end{Bmatrix}$$

2.1.2 Solution by the Displacement Method

In the displacement method, joint displacements, chosen as the analysis unknowns, are determined from the conditions of equilibrium. The internal forces and stresses are then determined by superposition of the effects of the external loads and the separate joint displacements.

The equilibrium equations to be solved by the displacement method are

$$\mathbf{K}\,\mathbf{r} + \mathbf{R}_L = \mathbf{R}_E \tag{2.8}$$

where $\mathbf{K}$ = stiffness matrix and the elements K_{ij} represent the force in the ith coordinate due to unit displacement in the jth coordinate (K_{ij} are computed in the restrained structure and i and j are coordinates corresponding to displacement degrees of freedom); $\mathbf{r}$ = the vector of unknown displacements; $\mathbf{R}_L$ = the vector of forces corresponding to the unknown displacements in the restrained structure; $\mathbf{R}_E$ = the vector of external loads corresponding to the unknown displacements ($\mathbf{R}_E = \mathbf{0}$ if there are no loads acting in the direction of degrees of freedom). The load vector is defined as $\mathbf{R} = \mathbf{R}_E - \mathbf{R}_L$, and Eq. (2.8) becomes [see Eq. (2.7)]

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \tag{2.9}$$

The vector of unknown displacements $\mathbf{r}$ is computed by solving the set of simultaneous equations (2.9). Final member forces $\mathbf{N}$ at any desired points in the structure are given by the following superposition equations

$$\mathbf{N} = \mathbf{N}_L + \mathbf{N}_r\,\mathbf{r} \tag{2.10}$$

where $\mathbf{N}_L$ = vector of forces due to loads in the restrained structure and $\mathbf{N}_r$ = matrix of forces due to unit value of the components of $\mathbf{r}$ in the restrained structure.

In many cases, where the loads act only in the directions of the nodal displacements, $\mathbf{N}_L = \mathbf{0}$ and the stresses $\boldsymbol{\sigma}$ can be determined from Eq. (2.10) by

$$\boldsymbol{\sigma} = \mathbf{S}\,\mathbf{r} \tag{2.11}$$

where $\mathbf{S}$ is the stress transformation matrix.

All equations are related to the action of a single loading. In the case of several loading conditions, all vectors will be transformed into matrices so that each of their columns will correspond to a certain loading condition. The elements of $\mathbf{K}$, $\mathbf{N}_r$ and $\mathbf{S}$ are functions of the material properties, the geometry of the structure and members' cross-sections. If the loads on the structure are predetermined, the elements of $\mathbf{R}$ and $\mathbf{N}_L$ depend only on the geometry of the structure.

Example 2.2. To illustrate solution by the displacement method, consider the continuous beam shown in Figure 2.2*a* [8]. The beam has a constant flexural rigidity *EI, E* is the modulus of elasticity, *I* is the moment of inertia, *L* represents the spans, *P* represents the loads and the object is to find the forces at the left-end support, N_1 and N_2. The structure has two degrees of freedom, the two support rotations r_1 and r_2, which are the unknown displacements. The coefficients computed in the restrained structure are (Figures 2.2*b* and 2.2*c*)

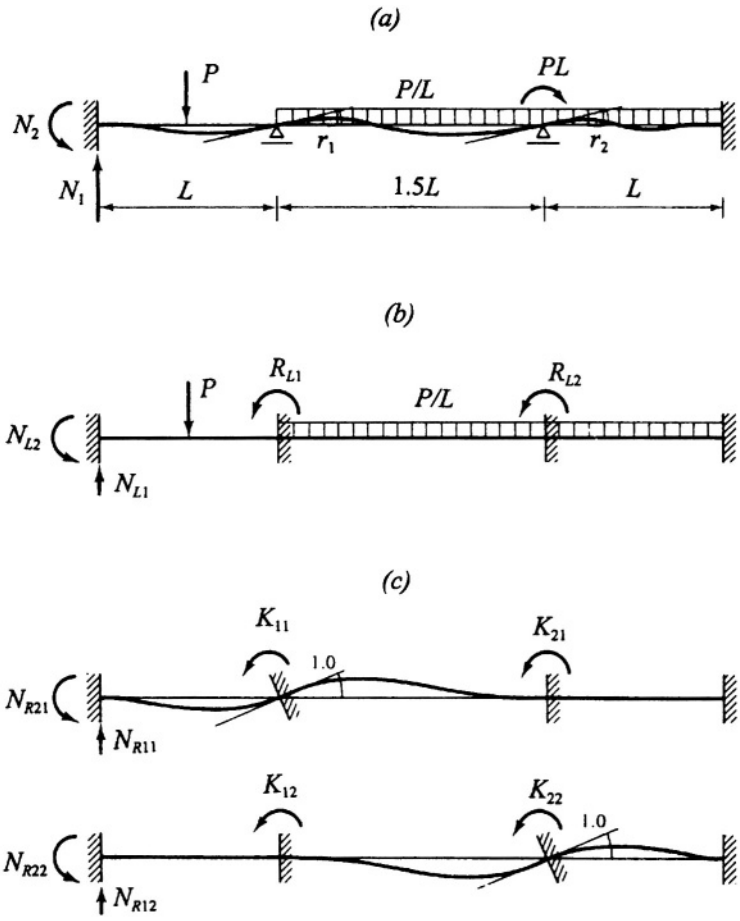

Fig. 2.2. *a.* Continuous beam example. *b.* Loads on the restrained structure. *c.* Unit displacements on the restrained structure.

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \frac{4}{3}\frac{EI}{L}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{R}_L = \begin{Bmatrix} R_{L1} \\ R_{L2} \end{Bmatrix} = \frac{PL}{48}\begin{Bmatrix} 3 \\ -5 \end{Bmatrix} \qquad \mathbf{R}_E = \begin{Bmatrix} 0 \\ -PL \end{Bmatrix}$$

$$\mathbf{R} = \mathbf{R}_E - \mathbf{R}_L = -\frac{PL}{48}\begin{Bmatrix} 3 \\ 43 \end{Bmatrix}$$

$$\mathbf{N}_L = \begin{Bmatrix} N_{L1} \\ N_{L2} \end{Bmatrix} = \begin{Bmatrix} P/2 \\ PL/8 \end{Bmatrix} \qquad \mathbf{N}_r = \begin{bmatrix} N_{r11} & N_{r12} \\ N_{r21} & N_{r22} \end{bmatrix} = \frac{2EI}{L^2}\begin{bmatrix} 3 & 0 \\ L & 0 \end{bmatrix}$$

Substituting into Eq. (2.9) we find the unknown displacements

$$\frac{4}{3}\frac{EI}{L}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = -\frac{PL}{48}\begin{Bmatrix} 3 \\ 43 \end{Bmatrix} \qquad \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{PL^2}{384EI}\begin{Bmatrix} 7 \\ -53 \end{Bmatrix}$$

The resulting forces **N** are computed by Eq. (2.10) as

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \frac{P}{192}\begin{Bmatrix} 117 \\ 31L \end{Bmatrix}$$

2.2 CONTINUUM STRUCTURES

Analysis of continuum structures is usually carried out by the finite element method. To apply the method, it is necessary first to convert the continuum into a system with a finite number of unknowns so that the problem can be solved numerically. This procedure involves the following steps:

a. The structure is divided into finite elements defined by fictitious lines or surfaces.
b. The elements are assumed to be interconnected at discrete nodal points situated on the element boundaries. The degrees of freedom at the nodes, called nodal-displacement parameters, normally refer to the displacements at the nodes.
c. A displacement function, in terms of nodal-displacement parameters, is chosen to represent the displacement field within each element. Based on the displacement function, a stiffness matrix is written to relate the nodal forces to nodal-displacement parameters. The principle of virtual work or the principle of minimum total potential energy can be used to do this.

The choice of a good displacement function is most important, as badly chosen functions will lead to inaccurate analysis results. The displacement function must have the same number of unknown constants as the total number of degrees of freedom of the element. It must be balanced with respect to the coordinate axes and it must allow the element to undergo rigid-body movement without any internal strain. The displacement function also must be able to represent states of constant stress or strain. Otherwise, the stresses will not converge to a continuous function as progressively smaller elements are used in the idealization of the structure. It also must satisfy the

compatibility of displacements along the boundaries with adjacent elements. That is, the function values, and sometimes their first derivatives, are required to be continuous along the boundaries. If all of the above conditions are satisfied, the idealization of the whole system will generally provide a lower bound to the strain energy, and convergence can be guaranteed as the mesh size is successively reduced. However, a number of elements that do not completely satisfy the conditions of compatibility along the boundaries have been successfully used, even though convergence is not assured.

Once the displacement function has been determined, it is possible to obtain all the strains and stresses within the element and to formulate the stiffness matrix and a consistent load matrix. The load matrix represents the equivalent nodal forces, which replace the action of external distributed loads.

Consider a linear elastic two-dimensional element (see Figure 2.3), for which the displacement function **f** can be written in the form

$$\mathbf{f} = \mathbf{P}\,\mathbf{H} \tag{2.12}$$

where **f** may have three components (for a three-dimensional body), two translation components u, v (for plane stress, Figure 2.3*a*), or simply be equal to the transverse deflection w (for a plate in bending, Figure 2.3*b*); **P** is a function of the coordinates x and y only; and **H** is a vector of undetermined constants. Note that the element of Figure 2.3*a* has six degrees of freedom representing the translations u and v for each node, while the element of Figure 2.3*b* has nine degrees of freedom (vertical deflection w, and two rotations at each of the three nodes).

Applying Eq. (2.12) repeatedly to the nodes of the element one after the other, we obtain the following set of equations relating the nodal parameters $\mathbf{r}_e$ to the constants **H**

$$\mathbf{r}_e = \mathbf{C}\,\mathbf{H} \tag{2.13}$$

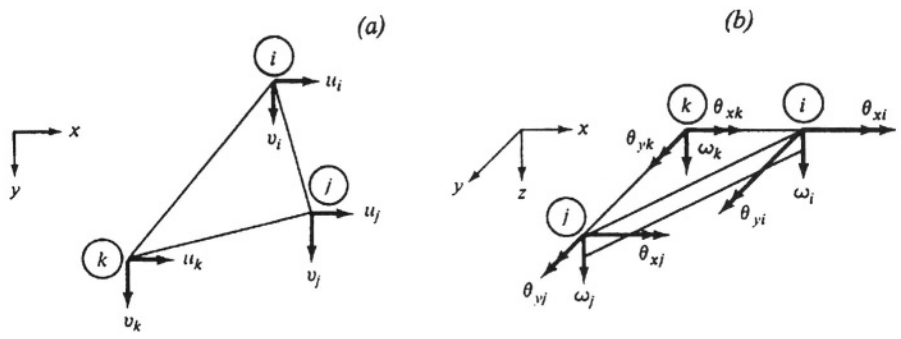

Fig. 2.3. Displacement parameters of a simple two-dimensional triangular element: *a*. Plane stress. *b*. Bending element.

The elements of matrix **C** are functions of the relevant nodal coordinates. From Eq. (2.13), the undetermined constants **H** can be expressed as

$$\mathbf{H} = \mathbf{C}^{-1}\,\mathbf{r}_e \tag{2.14}$$

Substituting Eq (2.14) into Eq. (2.12), we have

$$\mathbf{f} = \mathbf{P}\,\mathbf{C}^{-1}\,\mathbf{r}_e \tag{2.15}$$

In many cases, the displacement function is constructed directly in terms of the nodal parameters

$$\mathbf{f} = \mathbf{L}\,\mathbf{r}_e \tag{2.16}$$

where **L** is a function of *x, y* and the coordinates of the nodes. Comparing Eq. (2.15) with Eq. (2.16), it is clear that

$$\mathbf{L} = \mathbf{P}\,\mathbf{C}^{-1} \tag{2.17}$$

The vector $\boldsymbol{\varepsilon}$ of generalized strain (normal strain, shear strain, bending or twisting curvature) can be expressed in the form

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{r}_e \tag{2.18}$$

where the elements of **B** are derived by appropriate differentiation of **L** [Eq. (2.16)] with respect to *x* and *y*. The vector $\boldsymbol{\sigma}$ of generalized stresses is given by

$$\boldsymbol{\sigma} = \mathbf{d}\,\boldsymbol{\varepsilon} \tag{2.19}$$

where **d** is a symmetric elasticity matrix, representing the material properties of the element. Substituting Eq. (2.18) into Eq. (2.19) yields

$$\boldsymbol{\sigma} = \mathbf{d}\,\mathbf{B}\,\mathbf{r}_e \tag{2.20}$$

The product **d B** is called the stress matrix **S**, and Eq. (2.20) can be written as

$$\boldsymbol{\sigma} = \mathbf{S}\,\mathbf{r}_e \tag{2.21}$$

To formulate the stiffness and the consistent load matrices, we consider an element subjected to concentrated forces $\mathbf{Q}_e$ at the nodes together with the uniformly distributed loads **q** per unit area. Writing the expression for the total potential energy of the element, differentiating with respect to the nodal parameters one after another and then (using the principle of minimum total potential energy) setting them equal to zero, we obtain the set of simultaneous equations

$$\left(\int_v \mathbf{B}^T \mathbf{d}\,\mathbf{B}\,dV\right)\mathbf{r}_e = \mathbf{Q}_e + \int_\Delta \mathbf{L}^T \mathbf{q}\,d\Delta \tag{2.22}$$

where the two integrals are over the volume V and the area of the element Δ. Equation (2.22) can be written as

$$\mathbf{K}_e\,\mathbf{r}_e = \mathbf{R}_e \tag{2.23}$$

where the element stiffness matrix $\mathbf{K}_e$ is defined by

$$\mathbf{K}_e = \int_v \mathbf{B}^T \mathbf{d}\,\mathbf{B}\,dV \tag{2.24}$$

and the consistent load vector $\mathbf{R}_e$ is given by

$$\mathbf{R}_e = \mathbf{Q}_e + \mathbf{Q}_{qe} \tag{2.25}$$

$\mathbf{Q}_{qe}$ being the consistent load vector for distributed loads $\mathbf{q}$

$$\mathbf{Q}_{qe} = \int_\Delta \mathbf{L}^T \mathbf{q}\,d\Delta \tag{2.26}$$

The stiffness relationship can also be derived by the principle of virtual work, and Eq. (2.26) can be obtained by equating the virtual work done by the equivalent nodal forces and virtual work done by the distributed loads for the same set of permissible virtual displacements.

Consider the simple element of Figure 2.3*a* with the following displacement function [see. Eq. (2.12)]

$$\mathbf{f} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \\ H_6 \end{Bmatrix} \tag{2.27}$$

Writing Eq. (2.27) for each of the nodes, we have

$$\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} \qquad \begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} H_4 \\ H_5 \\ H_6 \end{Bmatrix} \tag{2.28}$$

The constants **H** can be expressed in terms of nodal displacements [Eq. (2.14)] by inversion of Eq. (2.28)

$$\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix} \qquad \begin{Bmatrix} H_4 \\ H_5 \\ H_6 \end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix} \tag{2.29}$$

where *a, b, c,* Δ are constants expressed in terms of the nodal coordinates $x_i, y_i, x_j, y_j, x_k, y_k$ (Δ is the area of the element). From Eq. (2.29) we may find the matrix $\mathbf{C}^{-1}$ corresponding to the vector of nodal-displacement parameters, $\mathbf{r}_e$. Thus $\mathbf{C}^{-1}$ and $\mathbf{r}_e$ are given by

$$\mathbf{C}^{-1} = \frac{1}{2\Delta}\begin{bmatrix} a_i & 0 & a_j & 0 & a_k & 0 \\ b_i & 0 & b_j & 0 & b_k & 0 \\ c_i & 0 & c_j & 0 & c_k & 0 \\ 0 & a_i & 0 & a_j & 0 & a_k \\ 0 & b_i & 0 & b_j & 0 & b_k \\ 0 & c_i & 0 & c_j & 0 & c_k \end{bmatrix} \qquad \mathbf{r}_e = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} \tag{2.30}$$

From Eqs (2.27) and (2.30) we may find the matrix **L** [see Eq. (2.17)]

$$\mathbf{L} = \frac{1}{2\Delta}\begin{bmatrix} (a_i + b_i x + c_i y) & 0 & (a_j + b_j x + c_j y) & 0 & (a_k + b_k x + c_k y) & 0 \\ 0 & (a_i + b_i x + c_i y) & 0 & (a_j + b_j x + c_j y) & 0 & (a_k + b_k x + c_k y) \end{bmatrix} \tag{2.31}$$

The strains for a plane problem are given by

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.32}$$

Differentiation of the displacement function

$$\mathbf{f} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \mathbf{L}\,\mathbf{r}_e \tag{2.33}$$

gives the following expressions for $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{2\Delta}(b_i u_i + b_j u_j + b_k u_k) \\ \dfrac{1}{2\Delta}(c_i v_i + c_j v_j + c_k v_k) \\ \dfrac{1}{2\Delta}(c_i u_i + b_i v_i + c_j u_j + b_j v_j + c_k u_k + b_k v_k) \end{Bmatrix} \tag{2.34}$$

The matrix $\mathbf{B}$ relating the strains $\boldsymbol{\varepsilon}$ to the nodal displacements $\mathbf{r}_e$ is thus given by [see Eq.(2.18)]

$$\mathbf{B} = \frac{1}{2\Delta}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \tag{2.35}$$

Since the elements of $\mathbf{B}$ are constants, the strains inside the element must all be constant. This type of element is often called constant strain triangular element.

The elasticity matrix $\mathbf{d}$ [Eq. (2.19)] can be shown to be

$$\mathbf{d} = \begin{bmatrix} C_1 & C_1 C_2 & 0 \\ C_1 C_2 & C_1 & 0 \\ 0 & 0 & C_{12} \end{bmatrix} \tag{2.36}$$

where

$C_1 = E/(1 - \upsilon^2)$ $\quad C_2 = \upsilon$ for plane stress,

$C_1 = \dfrac{E(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}$ $\quad C_2 = \dfrac{\upsilon}{1-\upsilon}$ for plane strain,

$C_{12} = C_1 (1\text{-}C_2)/2$ for both cases,

υ = Poisson's ratio.

Since the elements of the stress matrix $\mathbf{S} = \mathbf{d}\,\mathbf{B}$ [see Eq. (2.20)] are constants, stresses within an element will also be constant and the result would be stress discontinuities from one element to the next. In practice, this can be overcome either by considering the

stress values at the centroid of each element, or by using a method of averaging, in which the stresses of all the elements surrounding a node are summed and then averaged. These average stresses are then assigned to the node concerned.

The stiffness matrix, $\mathbf{K}_e$, can now be computed by Eq. (2.24). Since the elements of **B** and **d** are constants, the integration is equivalent to multiplying the integrand by the volume of the element, *V*,

$$\mathbf{K}_e = V\,\mathbf{B}^T\mathbf{d}\,\mathbf{B} \tag{2.37}$$

For uniformly distributed body forces in the *x* and *y* directions of magnitude q_x and q_y per unit area, the consistent load vector, $\mathbf{Q}_{qe}$, is given by [see Eqs. (2.26) and (2.31)]

$$\mathbf{Q}_{qe} = \int_{\Delta} \mathbf{L}^T \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} dx\,dy = \frac{\Delta}{3} \begin{Bmatrix} q_x \\ q_y \\ q_x \\ q_y \\ q_x \\ q_y \end{Bmatrix} \tag{2.38}$$

namely, 1/3 of the total load acting on the element is assigned to each node.

Consider the square isotropic plate of constant thickness $h = 1.0$ shown in Figure 2.4*a* [1], which is free along three edges and fixed at the other edge. The displacement conditions are $u = v = 0$ at the fixed edge, and $u = 0$ along the line of symmetry. Poisson's ratio is $\upsilon = 0.25$, the modulus of elasticity is $E = 1.0$, and the object is to find the stress distribution in the plate. Using the finite-element idealization shown in Figure 2.4*b*, the unknown displacements for the right-hand half of the plate are

$$\mathbf{r}^T = \{v_1, v_2, u_4, v_4, u_5, v_5, u_7, v_7, u_8, v_8\} \tag{2.39}$$

and the corresponding external applied force vector is

$$\mathbf{R}^T = \{0.5, 0, 0, 1.0, 0, 0, 0, 0.5, 0, 0\}$$

The stiffness matrices of the elements $\mathbf{K}_e$ are computed by Eq. (2.37), and the assembled matrix **K** is then computed by assuming the contribution of the elements. Solving the equations

$$\mathbf{K}\mathbf{r} = \mathbf{R} \tag{2.40}$$

we find the nodal displacements **r** shown in Figure 2.4*c*. The stress in each element is then calculated by Eq. (2.20). The element stresses are shown in Figure 2.4*d*.

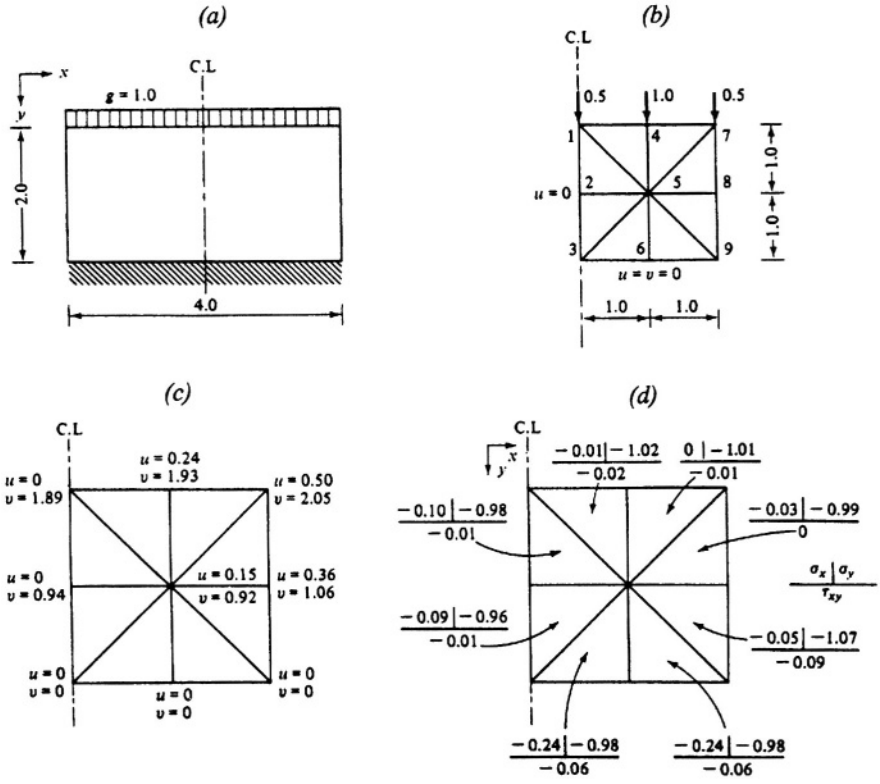

Fig. 2.4. *a*. Plate dimensions and loading. *b*. Finite-element idealization. *c*. Nodal displacements. *d*. Element stresses.

2.3 NONLINEAR ANALYSIS

In practical nonlinear analysis, the external forces are introduced in stages. Solution of the nonlinear set of equations is usually carried out by an incremental/iterative technique, such as a predictor-corrector method. This is accomplished by solving the equations for successive values of a load or displacement parameter, such that the solution corresponding to a particular value of the parameter is used to calculate a suitable approximation (predictor) for the displacements **r** at a different value of the parameter. This approximation is then chosen as an initial estimate of **r** in a corrective-iterative procedure such as the Newton-Raphson technique. The solution process can be carried out by different methods (see, e.g. [5]), including an incremental (Euler) scheme, the iterative Newton-Raphson method, and combined incremental/iterative solutions. Full or modified Newton-Raphson methods are often considered. In the latter

method, the same tangent stiffness matrix is used in the iteration cycles in order to save computing a new matrix in each cycle.

Geometrically nonlinear analysis is presented in Section 2.3.1 and material non-linearity is briefly described in Section 2.3.2.

2.3.1 Geometrical Non-linearity

For illustrative purposes assume an iterative solution process, using the Newton-Raphson method. Starting with linear analysis, we first calculate the initial displacements $\mathbf{r}_0$ by the linear analysis equations

$$\mathbf{K}_0 \mathbf{r}_0 = \mathbf{R}_0 \tag{2.41}$$

where $\mathbf{K}_0$ is the given elastic stiffness matrix and $\mathbf{R}_0$ is the given external force vector. The matrix $\mathbf{K}_0$ is often given in the decomposed form

$$\mathbf{K}_0 = \mathbf{U}_0^T \mathbf{U}_0 \tag{2.42}$$

where $\mathbf{U}_0$ is an upper triangular matrix. The member forces $\mathbf{N}$ are calculated for the deformed geometry. Considering both the compatibility equations and the constitutive law, the member forces are some nonlinear functions $\mathbf{N}(\mathbf{r})$ of the displacements $\mathbf{r}$

$$\mathbf{N} = \mathbf{N}(\mathbf{r}) \tag{2.43}$$

The forces computed by Eq. (2.43) are now not in equilibrium with the external forces $\mathbf{R}_0$. The internal forces $\mathbf{R}_I$, corresponding to the member forces $\mathbf{N}$, are determined by the equilibrium equations of the deformed geometry

$$\mathbf{R}_I = \mathbf{C}(\mathbf{r})\, \mathbf{N} \tag{2.44}$$

where the elements of matrix $\mathbf{C}(\mathbf{r})$ depend on the deformed geometry.

The out-of-balance (residual) force vector $\delta\mathbf{R}$ and the tangent stiffness matrix $\mathbf{K}_T$ are defined as

$$\delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I \tag{2.45}$$

$$\mathbf{K}_T = \mathbf{K}_0 + \mathbf{K}_G \tag{2.46}$$

where $\mathbf{K}_G = \mathbf{K}_G(\mathbf{N}, \mathbf{r})$ is the geometric stiffness matrix, whose elements are functions of the deformed geometry and the member forces. The out-of-balance forces correspond to a load vector that is not yet balanced by element stresses, and hence an increment in the nodal-point displacements is required. The elements of the tangent stiffness matrix can be calculated by the expression

$$\mathbf{K}_T = \partial \mathbf{R}_I / \partial \mathbf{r} \tag{2.47}$$

It can be observed from Eqs. (2.44) and (2.47) that the elements of $\mathbf{K}_T$ are some functions of $\mathbf{N}$ and $\mathbf{r}$. The vector of displacements $\delta\mathbf{r}$ due to the out-of-balance forces $\delta\mathbf{R}$ is calculated by the modified equilibrium equations, written for the deformed geometry,

$$\mathbf{K}_T \, \delta\mathbf{r} = \delta\mathbf{R} \tag{2.48}$$

Starting with the initial displacements $\mathbf{r}_0$, we update the displacements iteratively by

$$\mathbf{r} = \mathbf{r}_0 + \delta\mathbf{r} \tag{2.49}$$

redefining $\mathbf{r}_0$ as the updated displacement at the previous cycle.

The most frequently used iteration scheme for the solution of the nonlinear analysis equations is the Newton Raphson method. In the solution process the following quantities are calculated repeatedly:

- the member forces $\mathbf{N}$ [Eq. (2.43)];
- the corresponding internal force vector $\mathbf{R}_I$ [Eq. (2.44)];
- the out-of-balance force vector $\delta\mathbf{R}$ [Eq. (2.45)];
- the tangent stiffness matrix $\mathbf{K}_T$ [Eq. (2.47)];
- the displacements due to the out-of-balance forces $\delta\mathbf{r}$ [Eq. (2.48)]; and
- the updated displacements $\mathbf{r}$ [Eq. (2.49)]

These calculations are repeated until convergence occurs. Typical convergence conditions, related to the norms of $\delta\mathbf{r}$ and $\delta\mathbf{R}$, are

$$\begin{aligned} \|\delta\mathbf{R}\| &\le \varepsilon_R \\ \|\delta\mathbf{r}\| &\le \varepsilon_r \end{aligned} \tag{2.50}$$

where ε_R and ε_r are some small predetermined parameters.

This procedure is a simplified version of the Newton-Raphson method. Different variations and improvements might be considered [5]. These include an incremental (Euler) solution, retaining the original decomposed tangential matrix during several iterations, or combined incremental-iterative methods. The correct evaluation of the tangent stiffness matrix is important. However, because of the expense involved in evaluating and factoring a new tangent stiffness matrix, it can be more efficient to evaluate the matrix only at certain times. In the modified Newton-Raphson method a new tangent stiffness matrix is established only at the beginning of each load step.

2.3.2 Material Non-linearity

Material non-linearity can arise when the stress-strain relationship of the material is nonlinear in the elastic and/or in the plastic range. For a bilinear moment-curvature relationship in plastic analysis of frames, for example, the structure behaves linearly until the first plastic hinge has developed. Under increasing load, the structure continues to behave linearly, generally with a reduced stiffness, until a second hinge is formed. The same behavior continues under increasing load until sufficient hinges have developed to form a failure mechanism.

In the simplest incremental method the applied forces $\mathbf{R}$ are divided into increments. The load increments are applied one at a time and an elastic analysis is carried out. For the ith load increment $\Delta\mathbf{R}_i$, the equilibrium equations $\mathbf{K}_i\ \Delta\mathbf{r}_i = \Delta\mathbf{R}_i$ are solved. The stiffness matrix $\mathbf{K}_i$ depends upon the stress level reached in the preceding increment. Thus, for the ith increment, the modulus of elasticity is the slope of the stress-strain diagram at the stress level reached in the increment i-1. The displacements obtained by the solution of the equilibrium equations for each load increment are summed to give the final displacements. The advantage of the incremental method is its simplicity. It can also be used for geometrically nonlinear analysis. For this purpose, the stiffness matrix $\mathbf{K}_i$ for the increment i is based on the geometry of the structure and the internal forces determined in the preceding increment, i-1.

The Newton-Raphson or the modified Newton-Raphson methods can also be used to analyze structures with material non-linearity. In the Newton-Raphson method the full load is introduced, and an approximate solution is obtained and corrected by a series of iterations. A new tangent stiffness matrix is used in the solution of the linear equations in each iteration. In the modified Newton-Raphson method the load is introduced in stages. To avoid generating a new stiffness matrix in each iteration cycle, the tangent stiffness matrix determined in the first cycle for each load stage is employed in all subsequent cycles, before proceeding to the next load stage. That is, a new tangent stiffness matrix is introduced only in the first cycle of each new load increment.

2.4 DYNAMIC ANALYSIS

An elastic structure disturbed from its equilibrium condition by the application and removal of forces will oscillate about its position of static equilibrium. Thus, the displacements will vary periodically between specific limits in either direction. The distance of either of these limits from the position of equilibrium is the amplitude of the vibration. We may distinguish between the following two types of motion:

- Free-vibration motion, where no external forces act on the structure, and the motion may continue with the same amplitude for an indefinitely long time.
- Damped free-vibration, where forces tending to oppose the motion act on the structure. In practice there are always such forces, which cause the amplitude to diminish gradually until the motion ceases.

For ease of exposition, consider a structure with no damping. The equations of motion are

$$\mathbf{M}\ddot{\mathbf{r}}(t) + \mathbf{K}\mathbf{r}(t) = \mathbf{R}(t) \tag{2.51}$$

where **M** is the mass matrix and **K** is the stiffness matrix, the elements of both matrices being functions of the structure properties. The unknown displacement vector $\mathbf{r}(t)$, the acceleration vector $\ddot{\mathbf{r}}(t)$ and the load vector $\mathbf{R}(t)$ are functions of the time variable t.

Mathematically, Eq. (2.51) represents a system of linear differential equations of second order and, in principle, the solution can be obtained by standard procedures for the solution of differential equations with constant coefficients. In practical analysis, the common procedures can be divided into two methods of solution (the choice of one method or the other is determined by their relative numerical effectiveness):

- *Direct integration,* where Eqs. (2.51) are integrated using a numerical step-by-step procedure. The term direct means that prior to the numerical integration, no transformation of the equations into different form is carried out. Direct numerical integration is based on the idea that the equations are satisfied only at discrete time intervals. In addition, the method is based on the assumption that a variation of displacements, velocities and accelerations within each time interval has a certain form. The form of this assumption determines the accuracy, stability, and cost of the solution procedure.
- *Mode superposition,* where the equilibrium equations are transformed into a form in which the step-by-step solution is less costly. This method may be more effective if the integration must be carried out for many time steps.

The discretized model of a complicated structural system may have many degrees of freedom. Therefore, it is customary in dynamic analysis to reduce the equations of motion to a much smaller number before dynamic response is calculated. In dynamic problems, the reduction of degrees of freedom is more important than in static problems, because the solution must be performed successively at many different times, to generate the time history of the response. In this section, a general formulation of dynamic analysis is considered. Reduction methods are discussed in Section 6.2.

Some of the methods used in dynamic analysis are particularly useful when the applied load causing the dynamic response is of the form

$$\mathbf{R}(t) = \mathbf{R}\, f(t) \tag{2.52}$$

where the external load distribution, **R**, may have any form but remains constant, and only its amplitude $f(t)$ varies with time.

2.4.1 The Eigenproblem

Ignoring the notation of the time variable (t), we obtain the equations of motion for a freely vibrating undamped system by omitting the load vector from Eq. (2.51)

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{0} \tag{2.53}$$

The problem of vibration analysis consists of determining the conditions under which the equilibrium conditions of Eq.(2.53) are satisfied.

Assuming that the free-vibration motion is simple harmonic, we find

$$\ddot{\mathbf{r}} = -\omega^2 \mathbf{r} \tag{2.54}$$

where ω is the circular frequency. Substituting Eq. (2.54) into Eq. (2.53) and rearranging gives

$$\mathbf{K}\mathbf{r} = \omega^2 \mathbf{M}\,\mathbf{r} = \lambda \mathbf{M}\,\mathbf{r} \tag{2.55}$$

where the quantities $\omega^2 = \lambda$ are the eigenvalues indicating the square of the free-vibration frequencies, while the corresponding displacement vectors $\mathbf{r}$ express the eigenvectors, or mode shapes of the vibrating system. For a system having *n* degrees of freedom, the frequency vector $\boldsymbol{\omega}^T = \{\omega_1, \omega_2, \omega_3, ..., \omega_n\}$ represents the frequencies of the *n* modes of vibration possible in the system. The mode having the lowest frequency is called the first mode; the next higher frequency is the second mode, etc.

Equation (2.55) represents the generalized eigenproblem (e.g., see [3, 7]). If the solution is considered in order to obtain eigenvalues and eigenvectors the problem is referred to as an eigenproblem, whereas if only eigenvalues are to be calculated, the problem is called an eigenvalue problem.

A dynamic response calculation is substantially more costly than a static analysis. Whereas in a static analysis the solution is obtained in one step, in dynamic analysis the solution is required at a number of discrete points over the time interval considered. Considering a mode-superposition analysis, the main computational effort is spent in the solution of the eigenprolem, which also requires considerably more effort than a static analysis. Since exact solution of the eigenproblem can be prohibitively expensive, approximate solution techniques have been developed, primarily to calculate the lowest eigenvalues and corresponding eigenvectors, when the order of the system is large.

The shape of the vibrating system can be determined by solving for all the displacements in terms of any one coordinate. For convenience the displacement vector associated with the *m*th mode of vibration is usually expressed in dimensionless form by dividing all the components by one reference component (usually the largest). The resulting vector, called the *m*th mode shape $\boldsymbol{\Phi}_m$, is given by

$$\boldsymbol{\Phi}_m = \begin{Bmatrix} \Phi_{1m} \\ \Phi_{2m} \\ \vdots \\ \Phi_{nm} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \Phi_{2m} \\ \vdots \\ \Phi_{nm} \end{Bmatrix} \tag{2.56}$$

where Φ_{tm} is taken as the reference component. The square matrix made up of the n mode shapes is represented by $\mathbf{\Phi}$

$$\mathbf{\Phi} = \mathbf{\Phi}_1, \mathbf{\Phi}_2, ..., \mathbf{\Phi}_n \tag{2.57}$$

A common eigenproblem is the one to be solved in vibration mode superposition analysis. The generalized eigenproblem considered in this case is

$$\mathbf{K}\,\mathbf{\Phi} = \lambda\,\mathbf{M}\,\mathbf{\Phi} \tag{2.58}$$

where the eigenvalues are the free vibration frequencies squared, and the eigenvectors are the corresponding mode shape vectors. Some properties of the eigenproblem are discussed in the following. An important property of the eigenvalues of the problem is that they are the roots of the characteristic polynomial $\det(\mathbf{K} - \lambda\mathbf{M})$. Consider the eigenpair λ_i, $\mathbf{\Phi}_i$ satisfying

$$\mathbf{K}\mathbf{\Phi}_i = \lambda_i \mathbf{M}\mathbf{\Phi}_i \tag{2.59}$$

and also the relations

$$\mathbf{\Phi}_i^T \mathbf{M}\mathbf{\Phi}_i = 1 \tag{2.60}$$

which fixes the lengths of the eigenvectors. Then the eigenvectors satisfy the orthogonality conditions

$$\begin{aligned} \mathbf{\Phi}_i^T \mathbf{M}\mathbf{\Phi}_j &= \delta_{ij} \\ \mathbf{\Phi}_i^T \mathbf{K}\mathbf{\Phi}_j &= \lambda_i \delta_{ij} \end{aligned} \tag{2.61}$$

where δ_{ij} is the Kronecker delta, for which $\delta_{ij} = 0$ $(i \neq j)$ and $\delta_{ii} = 1$. Using these relations, we may write the following conditions that the eigenvctors must satisfy

$$\begin{aligned} \mathbf{\Phi}^T \mathbf{K}\mathbf{\Phi} &= \mathbf{\Lambda} \\ \mathbf{\Phi}^T \mathbf{M}\mathbf{\Phi} &= \mathbf{I} \end{aligned} \tag{2.62}$$

where $\mathbf{\Phi}$ is a matrix of the eigenvectors and $\mathbf{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues. The orthogonality relations discussed here hold also for the eigenvectors of the problems encountered in buckling analysis presented in Section 2.5.

For the eigenproblem of Eq. (2.58), the object is often to calculate the smallest eigenvalues and the corresponding eigenvectors. All solution methods are iterative.

Once an eigenvalue is known, the corresponding eigenvector can be obtained without iteration from $(\mathbf{K} - \lambda_i \mathbf{M})\ \boldsymbol{\Phi}_i = 0$. In addition, if we have evaluated $\boldsymbol{\Phi}_i$ by iteration, we can obtain the required eigenvalue by the Rayleigh quotient

$$\lambda_i = \boldsymbol{\Phi}_i^T \mathbf{K} \boldsymbol{\Phi}_i$$
$$\boldsymbol{\Phi}_i^T \mathbf{M} \boldsymbol{\Phi}_i = 1 \tag{2.63}$$

A basic question in considering an effective solution method is whether we should first solve for the eigenvalue and then calculate the eigenvector, or vice versa, or whether it is more economical to solve for both simultaneously. The answer to this question depends on various properties of the problem under consideration. The effectiveness of a solution method depends on the possibility of a reliable use of the procedure and the cost of solution, determined essentially by the number of high-speed storage operations and an efficient use of backup storage devices.

The common solution methods can be subdivided into the following groups, corresponding to the basic properties used in the solution process [3.]:

- Vector iteration methods (e.g. inverse iteration).
- Transformation methods(e.g. Jacobi iteration).
- Polynomial iteration methods.
- Methods that employ the sequence property of the characteristic polynomials.

There are many variants of these procedures. The Lanczos method and the subspace iteration method use a combination of the properties used in the above methods.

Many of the vibration (or eigenproblem) solution techniques are based on the matrix iteration approach. The use of iteration to evaluate the vibration mode of a structure is a very old concept that originally was called the Stodola method. To calculate the first-mode shape, Eq. (2.55) can be rewritten in an iterative form. Since only the shape is needed, the frequency is dropped from this equation to obtain

$$\mathbf{K}\bar{\mathbf{r}}^{(k)} = \mathbf{M}\mathbf{r}^{(k-1)} \tag{2.64}$$

where k denotes the iteration number, $\mathbf{r}^{(k-1)}$ is the displacement vector in the previous iteration and $\bar{\mathbf{r}}^{(k)}$ is the resulting improved shape. Defining the dynamic matrix $\mathbf{D}$ as

$$\mathbf{D} = \mathbf{K}^{-1}\,\mathbf{M} \tag{2.65}$$

and substituting into Eq. (2.64) yields

$$\bar{\mathbf{r}}^{(k)} = \mathbf{D}\,\mathbf{r}^{(k-1)} \tag{2.66}$$

To initiate the iteration procedure for evaluating the first mode shape, a trial displacement vector $\mathbf{r}^{(0)}$ is assumed that is a reasonable estimate of this shape. The improved iteration vector is then obtained by normalizing the shape $\bar{\mathbf{r}}^{(k)}$. There are various ways to obtain convenient normalized vectors. Normalizing the shape $\bar{\mathbf{r}}^{(k)}$ by dividing it by an arbitrary reference element of the vector $ref\left(\bar{\mathbf{r}}^{(k)}\right)$ gives

$$\mathbf{r}^{(k)} = \frac{\bar{\mathbf{r}}^{(k)}}{ref\left(\bar{\mathbf{r}}^{(k)}\right)} \tag{2.67}$$

This operation has the effect of scaling the reference element of the vector to unity. In general, the vector is normalized with respect to its largest element. It has been proven that the iteration process converges to the first-mode shape [3, 7]. Other mode shapes can be calculated by similar procedures. By repeating the process sufficiently, we can improve the mode-shape approximation to any desired level of accuracy. That is, after s cycles

$$\bar{\mathbf{r}}^{(s)} = \frac{1}{\omega_1^2}\mathbf{r}^{(s-1)} \cong \frac{1}{\omega_1^2}\boldsymbol{\Phi}_1 \tag{2.68}$$

where the proportionality between $\bar{\mathbf{r}}^{(s)}$ and $\mathbf{r}^{(s-1)}$ can be achieved to any specified accuracy and the resulting shape is accepted as the first mode shape. The frequency can be obtained from Eq. (2.68) by selecting the degree of freedom having the maximum displacement

$$\omega_1^2 = \frac{\max\left(\mathbf{r}^{(s-1)}\right)}{\max\left(\bar{\mathbf{r}}^{(s)}\right)} = \frac{1}{\max\left(\bar{\mathbf{r}}^{(s)}\right)} \tag{2.69}$$

Matrix iteration can be used to evaluate higher order modes as well, by assuming shapes that contain no lower-mode components. In the iteration procedure of Eqs. (2.66), (2.67) the stiffness matrix $\mathbf{K}$ is often given from the initial analysis in the decomposed form of Eq. (2.42), taking advantage of its narrow banded character.

2.5 COLLAPSE AND BUCKLING ANALYSIS

The objective of a nonlinear analysis is in many cases to estimate the maximum load that a structure can support prior to structural instability or collapse. In the analysis the load distribution on the structure is known, but the load magnitude that the structure can sustain is unknown.

The collapse analysis of a structure requires, in general, an incremental load analysis, which should include the geometrical and material non-linearities. Structural imperfections can also have a major effect on the predicted load-carrying capacity of the structure.

For the calculation of the linearized buckling load, it is assumed that the elements in the stiffness matrix vary linearly. The linearized buckling analysis gives a reasonable estimate of the collapse load only if the collapse displacements are relatively small, and any changes in the material properties do not significantly violate the assumption of linearity. The stiffness matrix and the externally applied loads are expressed as

$$\mathbf{K} = \mathbf{K}_{t-dt} + \lambda(\mathbf{K}_t - \mathbf{K}_{t-dt}) \tag{2.70}$$

$$\mathbf{R} = \mathbf{R}_{t-dt} + \lambda(\mathbf{R}_t - \mathbf{R}_{t-dt}) \tag{2.71}$$

where subscripts *t-dt* and *t* denote times (i.e. load levels), and $\lambda > 1$ is a scaling factor. At collapse or buckling the tangent stiffness matrix is singular, and hence the condition for calculating λ is

$$Det\ \mathbf{K} = 0 \tag{2.72}$$

or, equivalently

$$\mathbf{K}\ \mathbf{\Phi} = 0 \tag{2.73}$$

where $\mathbf{\Phi}$ is a nonzero vector. Substituting Eq. (2.70) into Eq. (2.73), we obtain the eigenproblem

$$[\mathbf{K}_{t-dt} + \lambda(\mathbf{K}_t - \mathbf{K}_{t-dt})]\ \mathbf{\Phi} = 0 \tag{2.74}$$

or, after rearranging

$$\mathbf{K}_{t-dt}\ \mathbf{\Phi} = \lambda(\mathbf{K}_{t-dt} - \mathbf{K}_t)\ \mathbf{\Phi} \tag{2.75}$$

The eigenvalues λ_i give the buckling loads, and the eigenvectors $\mathbf{\Phi}_i$ represent the corresponding buckling modes. Since only the smallest positive eigenvalues are of interest, rewrite Eq. (2.75) as

$$\mathbf{K}_t\ \mathbf{\Phi} = \gamma\ \mathbf{K}_{t-dt}\mathbf{\Phi} \tag{2.76}$$

where

$$\gamma = (\lambda - 1)/\lambda \tag{2.77}$$

The eigenvalues γ are all positive, and usually only the smallest values γ_1 are of interest. Having evaluated γ_1, the value of λ_1 is obtained from Eq. (2.77), and then the buckling (or collapse) load is given by Eq. (2.71)

$$\mathbf{R}_{buckling} = \mathbf{R}_{t-dt} + \lambda_1(\mathbf{R}_t - \mathbf{R}_{t-dt}) \tag{2.78}$$

This analysis can be performed equally well when geometrical or material non-linearities are considered.

2.6 REFERENCES

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3 Reanalsis of Structures

3.1 FORMULATION OF REANALYSIS PROBLEMS

3.1.1 Linear Reanalysis

Linear reanalysis is encountered in numerous design and optimization problems. The formulation presented in this section is general, and covers a wide range of problems [1]. Assuming the displacement method of analysis (as given in summary form in Section 2.1.2), we can state a typical reanalysis problem as follows:

a. Given an initial design, the corresponding stiffness matrix $\mathbf{K}^{\bullet}$, and the load vector $\mathbf{R}^{\bullet}$, the displacements $\mathbf{r}^{\bullet}$ are computed by the equilibrium equations [Eq. (2.9)]

$$\mathbf{K}^{\bullet}\,\mathbf{r}^{\bullet} = \mathbf{R}^{\bullet} \tag{3.1}$$

where the symmetric positive-definite stiffness matrix $\mathbf{K}^{\bullet}$ is usually given from the initial analysis in the decomposed form

$$\mathbf{K}^{\bullet} = \mathbf{U}^{\bullet T}\mathbf{U}^{\bullet} \tag{3.2}$$

and $\mathbf{U}^{\bullet}$ is an upper triangular matrix.

b. Assume a change in the structure and corresponding changes $\Delta\mathbf{K}$ in the stiffness matrix and $\Delta\mathbf{R}$ in the load vector. The modified stiffness matrix $\mathbf{K}$ and the modified load vector $\mathbf{R}$ are given by

$$\mathbf{K} = \mathbf{K}^{\bullet} + \Delta\mathbf{K} \tag{3.3}$$

$$\mathbf{R} = \mathbf{R}^{\bullet} + \Delta\mathbf{R} \tag{3.4}$$

In general, the elements of the stiffness matrix $\mathbf{K}$ are some explicit functions of the design variables. The changes $\Delta\mathbf{K}$ are functions of the members' cross-sections, the material properties, the geometry and the topology of the structure. The elements of the load vector $\mathbf{R}$ are usually assumed to be independent of the design variables,

that is, $\Delta\mathbf{R} = \mathbf{0}$. In certain cases where changes in the load vector are considered, the elements of the vector of changes $\Delta\mathbf{R}$ depend on the geometry and the topology of the structure.

c. The object is to estimate the modified displacements $\mathbf{r}$ due to the changes in the structure, without solving the complete set of modified analysis equations

$$(\mathbf{K}^* + \Delta\mathbf{K})\,\mathbf{r} = (\mathbf{R}^* + \Delta\mathbf{R}) \tag{3.5}$$

Once the displacements have been evaluated, the stresses σ are readily calculated using the explicit stress-displacement relations of Eq. (2.11). Thus, reanalysis methods essentially replace the formal solution of the analysis equations (3.5).

In this formulation the initial stiffness matrix $\mathbf{K}^*$ the load-vector $\mathbf{R}^*$, and the corresponding displacements $\mathbf{r}^*$ are given from initial analysis of the structure. It is shown in the following that the reanalysis formulation might prove useful also in cases where results of exact analysis are not available, that is, the initial displacements are not known. The object in this case is to evaluate $\mathbf{r}$ expressed in the form of Eq. (3.5) for the stiffness matrix $\mathbf{K}$, where matrices $\mathbf{K}^*$ and $\Delta\mathbf{K}$ can be chosen such that the solution is simple and effective. The simplest approach is to choose $\mathbf{K}^*$ as a diagonal matrix $\mathbf{K}_d$ consisting of the diagonal elements of $\mathbf{K}$. That is, matrices $\mathbf{K}^*$, and $\Delta\mathbf{K}$ are defined as

$$\begin{aligned} \mathbf{K}^* &= \mathbf{K}_d = \text{diagonal}\,(\mathbf{K}) \\ \Delta\mathbf{K} &= \mathbf{K} - \mathbf{K}^* = \mathbf{K} - \mathbf{K}_d \end{aligned} \tag{3.6}$$

and the initial displacements are given directly by

$$\mathbf{r}^* = (\mathbf{K}_d)^{-1}\,\mathbf{R} \tag{3.7}$$

It will be shown later (example 7.4) that despite this poor selection of the initial displacements $\mathbf{r}^*$ accurate results can be achieved by the reanalysis approach presented in Chapter 7.

3.1.2 Nonlinear Reanalysis

The basic equations of nonlinear analysis are summarized in Section 2.3. In a typical geometrically nonlinear analysis the following set of equations is repeatedly solved [Eqs. (2.46), (2.48)]

$$(\mathbf{K}_0 + \mathbf{K}_G)\,\delta\mathbf{r} = \mathbf{K}_T\,\delta\mathbf{r} = \delta\mathbf{R} \tag{3.8}$$

where $\mathbf{K}_0$ is the given elastic stiffness matrix; $\mathbf{K}_G$ is the geometric stiffness matrix whose elements are changed from one iteration cycle to another; $\mathbf{K}_T$ is the tangent

stiffness matrix; and $\delta\mathbf{r}$ is the vector of displacements due to the out-of-balance (residual) forces $\delta\mathbf{R}$. The latter forces are given by Eq. (2.45) as

$$\delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I \tag{3.9}$$

where $\mathbf{R}_0$ is the given external force vector and $\mathbf{R}_I$ is the vector of internal forces, whose elements are functions of the deformed geometry and are changed from one iteration cycle to another.

Starting with the initial displacements $\mathbf{r}_0$, computed by [Eq. (2.41)]

$$\mathbf{K}_0 \mathbf{r}_0 = \mathbf{R}_0 \tag{3.10}$$

we update the displacements $\mathbf{r}$ iteratively by the recurrence relation [Eq. (2.49)]

$$\mathbf{r} = \mathbf{r}_0 + \delta\mathbf{r} \tag{3.11}$$

In a typical iterative solution process using the Newton-Raphson method, the out-of-balance force vector $\delta\mathbf{R}$, the geometric stiffness matrix $\mathbf{K}_G$, the displacements due to the out-of-balance forces $\delta\mathbf{r}$ and the updated displacements $\mathbf{r}$ are calculated repeatedly for a number of iteration cycles until convergence occurs.

The reanalysis approach described in this text can be used for efficient calculation of the displacements $\delta\mathbf{r}$ at each iteration cycle, where both the geometric stiffness matrix $\mathbf{K}_G$ and the out-of-balance forces $\delta\mathbf{R}$ are modified. Defining

$$\begin{aligned} \Delta\mathbf{K}_0 &= \mathbf{K}_G \\ \Delta\mathbf{R}_0 &= -\mathbf{R}_I \end{aligned} \tag{3.12}$$

then from Eqs. (3.8) and (3.9)

$$(\mathbf{K}_0 + \Delta\mathbf{K}_0)\,\delta\mathbf{r} = \mathbf{R}_0 + \Delta\mathbf{R}_0 \tag{3.13}$$

It can be observed that the modified equations (3.5) and (3.13) are of similar form. That is, the problem of nonlinear analysis [Eq. (3.13)] can be stated in the form of linear reanalysis [Eq. (3.5)].

In general nonlinear reanalysis, the matrix of stiffness changes $\Delta\mathbf{K}$ and the vector of force changes $\Delta\mathbf{R}$ consist of the following two parts:

a. The changes $\Delta\mathbf{K}_0$ and $\Delta\mathbf{R}_0$ due to the nonlinear behavior, as given by Eqs. (3.12).

b. The changes $\Delta\mathbf{K}_D$ and $\Delta\mathbf{R}_D$ in the elastic stiffness matrix and in the external force vector, respectively, due to changes in the design variables.

Thus, the changes $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$ are expressed as

$$\Delta\mathbf{K} = \Delta\mathbf{K}_0 + \Delta\mathbf{K}_D$$
$$\Delta\mathbf{R} = \Delta\mathbf{R}_0 + \Delta\mathbf{R}_D \tag{3.14}$$

The modified equations to be solved at each iteration cycle are

$$(\mathbf{K}_0 + \Delta\mathbf{K})\,\delta\mathbf{r} = (\mathbf{R}_0 + \Delta\mathbf{R}) \tag{3.15}$$

Starting with linear analysis, solution of the nonlinear reanalysis problem involves the following two parts:

a. We first calculate the initial displacements by the linear equilibrium equations (3.10). Nonlinear analysis is then carried out by the iterative procedure described in Section 2.3.1. At each iteration cycle it is necessary to solve repeatedly the linearized equations (3.13). The reanalysis approach presented in Chapter 7 can be used for this purpose.

b. For the changes $\Delta\mathbf{K}_D$ and $\Delta\mathbf{R}_D$ due to a change in the design, the nonlinear reanalysis process is similar to that of nonlinear analysis, with $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$ [Eq. (3.14)] replacing $\Delta\mathbf{K}_0$ and $\Delta\mathbf{R}_0$, respectively [Eq. (3.12)]. The elements of the matrix $\Delta\mathbf{K}_D$ and the vector $\Delta\mathbf{R}_D$ are constant during the solution process, whereas the elements of the geometric stiffness matrix $\Delta\mathbf{K}_0 = \mathbf{K}_G$ and the vector of internal forces $\Delta\mathbf{R}_0 = -\mathbf{R}_I$ are updated after each iteration cycle.

The solution process is carried out for various changes $\Delta\mathbf{K}_D$ and $\Delta\mathbf{R}_D$ in the design as necessary, with the initial decomposed $\mathbf{K}_0$ being unchanged. Solution of nonlinear reanalysis problems is discussed in Chapter 11.

3.1.3 Vibration Reanalysis

In a typical eigenproblem, the following set of initial analysis equations is solved for an initial structure [see Eq. (2.55)]

$$\mathbf{K}_0\mathbf{r}_0 = \lambda_0\mathbf{M}_0\mathbf{r}_0 \tag{3.16}$$

where $\mathbf{K}_0$ is the stiffness matrix, $\mathbf{M}_0$ is the mass matrix, $\lambda_0 = \omega_0^2$ are the eigenvalues, ω_0 is the circular free-vibration frequency, and $\mathbf{r}_0$ are the corresponding displacement vectors (the eigenvectors) or mode shapes of the vibrating system. Assume a change in the design and a corresponding change in the stiffness and mass matrices such that the modified matrices are expressed as

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}$$
$$\mathbf{M} = \mathbf{M}_0 + \Delta\mathbf{M} \tag{3.17}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ are the changes in the stiffness and mass matrices, respectively, due to the change in the design.

The modified analysis equations are given by

$$\mathbf{K}\,\mathbf{r} = \lambda\,\mathbf{M}\,\mathbf{r} \tag{3.18}$$

where $\lambda = \omega^2$ are the modified eigenvalues, ω is the circular frequency of the modified structure, and $\mathbf{r}$ are the corresponding displacement vectors (the mode shapes) or modified eigenvectors of the vibrating system.

Denoting the right hand side vectors of Eqs. (3.16) and (3.18) as

$$\mathbf{R}_0 = \lambda_0\mathbf{M}_0\mathbf{r}_0$$
$$\mathbf{R} = \lambda\,\mathbf{M}\,\mathbf{r} \tag{3.19}$$

and substituting Eqs. (3.19) into Eqs. (3.16) and (3.18) we find

$$\mathbf{K}_0\,\mathbf{r}_0 = \mathbf{R}_0 \tag{3.20}$$

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \tag{3.21}$$

The vibration reanalysis problem under consideration can be stated as follows. Suppose the initial values $\mathbf{K}_0$ and $\mathbf{M}_0$, the initial eigenvectors $\mathbf{r}_0$ and eigenvalues λ_0 are first calculated. The object is to evaluate efficiently and accurately the modified eigenvectors $\mathbf{r}$ and eigenvalues λ due to various changes in the design, such that the modified analysis equations (3.21) are satisfied. In this formulation, the elements of $\mathbf{K}$ and $\mathbf{M}$ are explicit functions of the design variables whereas the elements of the right hand side vector $\mathbf{R}$ are functions of $\mathbf{M}$, $\mathbf{r}$ and λ.

The modified analysis equations (3.21) can be expressed in a form similar to that of Eq. (3.5)

$$(\mathbf{K}_0 + \Delta\mathbf{K})\,\mathbf{r} = (\mathbf{R}_0 + \Delta\mathbf{R}) \tag{3.22}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$ are defined as [see Eqs. (3.17) and (3.19)]

$$\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0$$
$$\Delta\mathbf{R} = \mathbf{R} - \mathbf{R}_0 = \lambda\,\mathbf{M}\,\mathbf{r} - \lambda_0\mathbf{M}_0\mathbf{r}_0 \tag{3.23}$$

It can be observed that the modified equations (3.5) and (3.22) are of similar form and, therefore, the vibration reanalysis problem can be stated in the form of linear reanalysis.

The reanalysis approach presented in this text can be used to solve eigenvalue problems when results of initial analysis are not available. The object is to solve the modified equations (3.22), where results of previous analysis are not known. Assuming the initial diagonal stiffness matrix $\mathbf{K}_0 = \mathbf{M}$ then from Eq. (3.17) we have

$$\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0 = \mathbf{K} - \mathbf{M} \tag{3.24}$$

Assuming some reasonable initial displacements $\mathbf{r}_0$, we can determine the modified displacement $\mathbf{r}$ by the reanalysis approach presented in Chapter 7. Solution of vibration reanalysis problems is discussed in Chapter 12.

Collapse and buckling reanalysis (Section 2.5) can be formulated in a similar way. The different mathematical expressions for the various reanalysis problems presented in this section are summarized in Table 3.1.

Table 3.1 Mathematical expressions for various reanalysis problems

Problem	Linear	Nonlinear	Eigenproblem
Modified equations	(3.5) $(\mathbf{K}^*+\Delta\mathbf{K})\mathbf{r}=(\mathbf{R}^*+\Delta\mathbf{R})$	(3.15) $(\mathbf{K}_0+\Delta\mathbf{K})\delta\mathbf{r}=(\mathbf{R}_0+\Delta\mathbf{R})$	(3.22) $(\mathbf{K}_0+\Delta\mathbf{K})\mathbf{r}=(\mathbf{R}_0+\Delta\mathbf{R})$
$\Delta\mathbf{K}$	$\mathbf{K} - \mathbf{K}^*$	$\mathbf{K}_G + \Delta\mathbf{K}_D$	$\mathbf{K} - \mathbf{K}_0$
$\Delta\mathbf{R}$	$\mathbf{R} - \mathbf{R}^*$	$-\mathbf{R}_I + \Delta\mathbf{R}_D$	$\lambda\mathbf{M}\,\mathbf{r} - \lambda_0\mathbf{M}_0\mathbf{r}_0$

3.2 REANALYSIS METHODS

Several comprehensive reviews on reanalysis methods have been published [e.g. 2-4]. The various methods may be divided into the following two general categories:

a. Direct methods, giving exact closed-form solutions and applicable to situations where a relatively small part of the structure is changed.
b. Approximate methods, giving approximate solutions, with the accuracy being dependent on the type of changes. Approximate methods are usually suitable for situations where changes occur for large parts of the structure.

The two classes of reanalysis methods are reviewed in this section.

3.2.1 Direct Methods

Direct reanalysis methods are efficient for low-rank changes in the stiffness matrix. In particular, these methods are applicable to situations where a relatively small proportion of the structure is changed and the changes in the stiffness matrix can be represented by a small sub-matrix (for example, in cases where the cross sections of only a few members are changed). Direct methods are inefficient when the sub-matrix of changes in the system stiffness matrix is large.

Direct methods are usually based on the Sherman-Morrison [5] and Woodbury [6] formulae for the update of the inverse of a matrix. Surveys on these methods are given elsewhere [7 - 9]. A comprehensive historical survey of the origin of these formulae is presented in [8]. It has been shown [9] that various reanalysis methods may be viewed as variants of these formulae. Several methods for calculating the modified response due to changes in the structure were proposed in the late sixties and the early seventies. Most of these improved methods are based on the Sherman-Morrison identity [e.g. 10 - 12]. Direct methods are described in Chapter 4.

The *Combined Approximations* (CA) method presented in Chapter 7 provides exact solutions under certain conditions. It is shown in Sections 8.2.2 and 8.2.3 that in such cases exact solutions achieved by the CA method and Sherman-Morrison-Woodbury formulae are equivalent.

Other direct methods include the *Virtual Distortion Method* (VDM, [13 - 14]) and the *Theorems of Structural Variation* (TSV, [15 - 20]). These two methods, also called load-based methods [9], require collinear loads to be applied to the modified members in order to compute an influence matrix. In the VDM, a reduced set of equations is then solved for a set of scalar multipliers of the influence vectors. In the TSV approach the modified member forces are given in terms of the original forces and forces due to unit loadings, while the nodal displacements are given in terms of the original displacements and displacements due to unit loadings.

3.2.2 Approximate Methods

Approximate reanalysis methods are suitable for cases of changes in large parts or all of the structure. These methods have been used extensively in structural optimization to reduce the number of exact analyses and the overall computational cost during the solution process. Reduction of the computational cost, in turn, allows the solution of practical design problems.

In general, the following factors are considered in choosing an approximate reanalysis method for a specific application:

a. The accuracy of the calculations (the quality of the approximations).
b. The computational effort involved (the efficiency of the method).
c. The ease-of-implementation.

The implementation effort is weighted against the performance of the algorithms as reflected in their computational efficiency and accuracy. The quality of the results and the efficiency of the calculations are usually two conflicting factors. That is, better

approximations are often achieved at the expense of more computational effort. In various applications the different levels of analysis range from inexpensive and inaccurate to costly and accurate.

The common approximations can be divided into the following classes [4, 21]:

a. *Local approximations* (called also *single-point approximations*), such as the first-order Taylor series expansion or the binomial series expansion about a given design point. Local approximations are based on information calculated at a single point. These methods are very efficient but they are effective only for small changes in the design variables. For large changes in the design the accuracy of the approximations often deteriorates and the results may become meaningless. That is, the approximations are valid only in the vicinity of a design point. To improve the quality of the results, reciprocal cross-sectional areas have been assumed as design variables [22, 23]. A hybrid form of the direct and reciprocal approximations, which is more conservative than either, can also be introduced [24]. This approximation has the advantage of being convex [25], but it has been found that the hybrid approximation tends to be less accurate than either the direct or the reciprocal approximation. More accurate convex approximations can be introduced by the method of moving asymptotes [26], but the quality of the results might be dependent on the selection of these asymptotes. Another possibility to improve the quality of the results is to consider second-order approximations [27, 28] but this considerably increases the computational effort. Local approximations are discussed in Chapter 5.

b. *Global approximations* (called also *multipoint approximations*), such as polynomial fitting, response surface or reduced basis methods [29 - 33]. These approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). In response surface methods [e.g. 30, 31], the response functions are replaced by simple functions (polynomials), which are fitted to data computed at a set of selected design points. So far, the use of response surface methods has been limited to problems with a relatively small number of design variables. In reduced basis methods [32, 33] the response of a large system, which was originally described by a large number of degrees of freedom, is approximated by a linear combination of a few pre-selected basis vectors. The problem is then stated in terms of a small number of unknown coefficients of the basis vectors. The approach is most effective when highly accurate approximations can be introduced by the reduced and much smaller system of equations. A basic question in using reduced basis methods is the choice of an appropriate set of the basis vectors. Response vectors of previously analyzed designs could be used, but an *ad hoc* or intuitive choice may not lead to satisfactory approximations. In addition, calculation of the basis vectors requires several exact analyses of the structure for the basis designs, which involves extensive computational effort. In summary, global approximations may require much computational effort, particularly in problems with large numbers of design variables. Global approximations are introduced in Chapter 6.

c. *Combined approximations.* In the second part of this text, we develop a third class of approximations, called Combined Approximations (CA). In this approach we attempt to give global qualities to local approximations. This can be achieved by considering the terms of local (series) approximations as basis vectors in a global (reduced basis) expression. It has been shown that this choice of basis vectors provides accurate results. The advantage is that the efficiency of local approximations and the improved quality of global approximations are combined to obtain an effective solution procedure. The CA is a general approach, but it also can be viewed as a specific method, where the binomial series terms are used as basis vectors in specific reduced basis approximations. The approach is versatile, providing different options and possibilities in applications. The main developments of the approach are briefly reviewed in this section. The basic concepts of the approach are discussed in Chapter 7 and some simplified solution procedures derived from the approach are described in Chapter 8.

Initially, the main objective in developing the CA method was to simplify design optimization procedures for practical structures. Later, it was found that the method might prove useful not only in structural optimization but also in various analysis and design tasks. In particular, solutions for the following classes of problems have been developed:

a. *Linear analysis.* Solutions of complete analysis, various reanalysis problems (including topological and geometrical changes, accurate and exact solutions) and design sensitivity analysis problems.
b. *Nonlinear analysis.* Solutions of both complete analysis and reanalysis, for geometrically nonlinear problems.
c. *Vibration analysis.* Solutions of both complete analysis and reanalysis of typical eigenproblems.

Several studies in the early eighties [34, 35] showed that improved local approximations could be achieved by scaling the initial design such that the changes in the design variables are reduced. The advantage is that the solution is still based on results of a single exact analysis. Scaling procedures significantly improve the accuracy of the results with little computational effort. It has been demonstrated that scaling is useful for various types of design variables and response functions. Several criteria for selecting the scaling multiplier have been proposed, based on geometrical [34] and mathematical [35] considerations. In the early nineties it was found [36 - 38] that scaling of both the initial design and the modified approximate displacements can be expressed in a reduced basis form, using transformations of variables. Scaling concepts are described in Section 5.3.

Extending the concept of scaling to include also the approximate displacements, in addition to the initial design, significantly improved the results. The effectiveness of the CA method in approximations of displacements in problems of cross-section optimization as well as geometrical and topological optimization has been demonstrated

[36 - 44]. It has been shown [40] that accurate approximations can be achieved by the method for very large changes in the design variables by considering only first-order approximations (only two basis vectors). Good approximate results have been demonstrated for displacements, stresses and forces.

In the mid nineties, it was shown that the CA method provides the exact solution in certain cases [41]. Specifically, exact solutions are achieved for some cross-sectional variables if for any change in a member only one basis vector (the first-order term) is considered. For simultaneous changes in several members, exact solutions are achieved if for each changed member a corresponding basis vector is considered. This result is valid also for all types of topological changes in the structure, namely, elimination and addition of members and joints [42]. Exact and accurate solutions for all types of topological changes [44] and geometrical changes [45] were demonstrated.

It has been shown [9] that exact solutions achieved by the CA method and the Sherman-Morrison-Woodbury formulae are equivalent. The CA method and the Sherman-Morrison formula both find that the change in nodal displacements due to a change in the cross section area of a truss member is a multiple of the response to a pair of collinear forces acting at the ends of the member. It has been shown also that this result can be generalized to any structural member such as a frame element or a plate element. Exact reanalysis by the CA method and its relationship with the Sherman-Morrison-Wooodbury formulae are discussed in Section 8.2. The performance of the CA method in reanalysis of damaged structures, including topological changes in dynamic reanalysis of grillage and frame structures, was studied in the late nineties [46 - 48]. Nearly exact solutions were achieved for significant changes in the structural response as a result of elimination of primary elements in bridge structures. Applications of the CA method in structural optimization and reanalysis of damaged structures are demonstrated in Chapter 8. Solution of reanalysis problems for geometrical and topological changes is discussed in Chapter 9.

Most approximations that might be adequate for structural reanalysis are not sufficiently accurate for *design sensitivity analysis.* It has been shown [49, 50] that the CA method can be used also for effective approximations of the response derivatives for designs where results of exact analysis are not available. Accurate results have been achieved by either the direct method or the adjoint-variable method for calculating derivatives. Design sensitivity analysis by the CA method is discussed in Chapter 10.

It has been shown in the late nineties that by using a Gram-Schmidt orthonormalization procedure a new set of basis vectors can be generated such that the reduced set of analysis equations becomes uncoupled [51, 52]. For any assumed number of basis vectors, the results obtained by considering either the original set of basis vectors or the new set of uncoupled basis vectors, are identical. The advantage in using the latter vectors is that all expressions for evaluating the displacements become explicit functions of the parameters of the structure. As a result, additional vectors can be considered without modifying the calculations that have been already carried out. In addition, the uncoupled system is more well-conditioned.

The CA method has been used successfully in the solution of nonlinear analysis and dynamic reanalysis problems. It has been shown [51] that the method is good for

calculating the modified displacements at each iteration cycle of nonlinear analysis. Accurate results and significant savings in computational effort have been reported in reanalysis of eigenproblems [47, 53, 54], Application of the method in nonlinear reanalysis is demonstrated in Chapter 11 and solution of vibration problems is discussed in Chapter 12.

Some considerations related to the accuracy of the results and convergence of the solution process have been studied [55, 56]. It has been shown that a preconditioned conjugate gradient method and the CA method provide theoretically identical results. As a result, some convergence criteria and error expressions developed for conjugate gradient methods can be used for the CA method. The method has been successfully applied to both low-rank and moderately high-rank modifications to structures [57]. Accurate results were reported for large scale systems [58] and reliability analysis problems [59].

In summary, the CA method is based on a general approach, which is suitable for various structural analysis and reanalysis problems, including sensitivity analysis, nonlinear analysis and dynamic analysis. The method has been used for various types of changes in the structure, including cross-sectional, topological and geometrical changes. Accurate results can be achieved efficiently for significant changes in the structure, and exact solutions are obtained in certain cases. Finally, the efficiency of the calculations and the accuracy of the results can be controlled by the level of simplification and the amount of information considered. That is, better accuracy can be achieved at the expense of more computational effort.

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4 Direct Methods

Direct (closed form) reanalysis methods are efficient for low-rank changes in the stiffness matrix. In particular, these methods are applicable to situations where a relatively small proportion of the structure is changed and the changes in the stiffness matrix can be represented by a small sub-matrix. Direct methods are inefficient when the sub-matrix of changes in the system stiffness matrix is of high-rank or large.

The rank of a matrix **A** (rank **A**) is the dimension of the linear space spanned by its columns. Rank **A** is equal to the maximum number of linearly independent columns (or rows) of **A**. Rank **A** is also equal to the order of the square sub-matrix of **A** of greatest order whose determinant does not vanish.

Direct methods are usually based on the Sherman-Morrison [1] and Woodbury [2] formulae for the update of the inverse of a matrix. Surveys on these methods are given elsewhere [3 - 5]. A comprehensive historical survey of the origin of these formulae is presented in [4]. It has been shown [5] that various reanalysis methods may be viewed as variants of these formulae. When the stiffness matrix is modified by a rank-one increment, the solution can be updated inexpensively with the Sherman-Morrison formula by solving the initial analysis equations with a different right-hand side vector, which is a factor of the matrix increment. Similarly, solution for a higher-rank change in the stiffness matrix can be carried out by superposition of rank-one changes. This is reflected in the Woodbury formula.

The Sherman-Morrison formula for exact solutions for a single rank-one change is introduced in Section 4.1. The Woodbury formula for exact solutions in cases of multiple rank-one changes is presented in Section 4.2. Exact solutions for general changes in the design are developed in Section 4.3.

4.1 A SINGLE RANK-ONE CHANGE

For a change of rank-one in the $n \times n$ stiffness matrix $\mathbf{K}^*$, the matrix of changes $\Delta\mathbf{K}$ can be expressed in terms of the vectors $\mathbf{v}$, $\mathbf{w}$ and a scalar η as

$$\Delta\mathbf{K} = \mathbf{v}\,\mathbf{w}^T = \eta\,\mathbf{v}\,\mathbf{v}^T \tag{4.1}$$

where the scalar η is positive or negative, depending on the sign of the stiffness change, and

$$\mathbf{w} = \eta\, \mathbf{v} \tag{4.2}$$

The Sherman-Morrison (S-M) formula, giving the change in the inverse of a matrix due to a rank-one change $\Delta\mathbf{K}$, can be expressed in terms of $\mathbf{v}$ and $\mathbf{w}$ as

$$\left(\mathbf{K}^* + \Delta\mathbf{K}\right)^{-1} = \left(\mathbf{K}^* + \mathbf{v}\mathbf{w}^T\right)^{-1} = \mathbf{K}^{*-1} - \mathbf{K}^{*-1}\mathbf{v}\left(1 + \mathbf{w}^T\mathbf{K}^{*-1}\mathbf{v}\right)^{-1}\mathbf{w}^T\mathbf{K}^{*-1} \tag{4.3}$$

Using the relation of Eq. (4.2), we can express the S-M formula in terms of $\mathbf{v}$ and η as

$$\left(\mathbf{K}^* + \eta\,\mathbf{v}\,\mathbf{v}^T\right)^{-1} = \mathbf{K}^{*-1} - \mathbf{K}^{*-1}\mathbf{v}\left(1 + \eta\,\mathbf{v}^T\mathbf{K}^{*-1}\mathbf{v}\right)^{-1}\eta\,\mathbf{v}^T\mathbf{K}^{*-1} \tag{4.4}$$

Define the vector $\mathbf{t}$ by

$$\mathbf{t} = \mathbf{K}^{*-1}\mathbf{v} \tag{4.5}$$

Post-multiplying Eq. (4.4) by the load vector $\mathbf{R}^*$ and substituting Eqs. (3.1) and (4.5), we obtain the S-M formula

$$\mathbf{r} = \left(\mathbf{K}^* + \eta\,\mathbf{v}\,\mathbf{v}^T\right)^{-1}\mathbf{R}^* = \mathbf{r}^* - \eta\,\mathbf{t}\left(1 + \eta\,\mathbf{v}^T\mathbf{t}\right)^{-1}\mathbf{v}^T\mathbf{r}^* \tag{4.6}$$

If

$$a = \eta\left(1 + \eta\,\mathbf{v}^T\mathbf{t}\right)^{-1}\mathbf{v}^T\mathbf{r}^* \tag{4.7}$$

Eq. (4.6) becomes

$$\mathbf{r} = \mathbf{r}^* - a\,\mathbf{t} \tag{4.8}$$

In summary, solution by the S-M formula for evaluating the modified displacements, due to a rank-one change $\Delta\mathbf{K}$ in the stiffness matrix $\mathbf{K}^*$, involves the following steps:

a. The vector $\mathbf{t}$ is calculated by Eq. (4.5). It can be observed that this step is equivalent to the solution of the initial analysis equations with a different right-hand side vector.
b. The scalar a is determined by Eq. (4.7).
c. The modified displacements are calculated by Eq. (4.8).

4.2 MULTIPLE RANK-ONE CHANGES

For m rank-one changes $\Delta\mathbf{K}_i$ (i = 1, 2, ..., m) in the $n \times n$ stiffness matrix, the total change in stiffness $\Delta\mathbf{K}$ can be expressed in terms of the matrices **V, W** and a diagonal matrix **H** as

$$\Delta\mathbf{K} = \Delta\mathbf{K}_1 + \Delta\mathbf{K}_2 + ... + \Delta\mathbf{K}_m = \mathbf{V}\,\mathbf{W}^T = \mathbf{V}\,\mathbf{H}\,\mathbf{V}^T \tag{4.9}$$

where matrices **V** and **W** are of order $n \times m$, matrix **H** is of order $m \times m$ and

$$\mathbf{W} = \mathbf{V}\,\mathbf{H} \tag{4.10}$$

The Woodbury formula, giving the change in the inverse of a matrix due to a rank-m change $\Delta\mathbf{K}$, is given in terms of **V** and **W** as

$$\left(\mathbf{K}^* + \Delta\mathbf{K}\right)^{-1} = \left(\mathbf{K}^* + \mathbf{V}\mathbf{W}^T\right)^{-1} = \mathbf{K}^{*-1} - \mathbf{K}^{*-1}\mathbf{V}\left(\mathbf{I} + \mathbf{W}^T\mathbf{K}^{*-1}\mathbf{V}\right)^{-1}\mathbf{W}^T\mathbf{K}^{*-1} \tag{4.11}$$

Using Eq. (4.10), we can express the Woodbury formula in terms of **V** and **H** as

$$\left(\mathbf{K}^* + \mathbf{V}\,\mathbf{H}\,\mathbf{V}^T\right)^{-1} = \mathbf{K}^{*-1} - \mathbf{K}^{*-1}\mathbf{V}\left(\mathbf{I} + \mathbf{H}\,\mathbf{V}^T\mathbf{K}^{*-1}\mathbf{V}\right)^{-1}\mathbf{H}\,\mathbf{V}^T\mathbf{K}^{*-1} \tag{4.12}$$

Define the matrix **T** by

$$\mathbf{T} = \mathbf{K}^{*-1}\mathbf{V} \tag{4.13}$$

Post-multiplying Eq. (4.12) by $\mathbf{R}^*$ and substituting Eqs. (3.1) and (4.13), we obtain the Woodbury formula

$$\mathbf{r} = \left(\mathbf{K}^* + \mathbf{V}\,\mathbf{H}\,\mathbf{V}^T\right)^{-1}\mathbf{R}^* = \mathbf{r}^* - \mathbf{T}\left(\mathbf{I} + \mathbf{H}\,\mathbf{V}^T\mathbf{T}\right)^{-1}\mathbf{H}\mathbf{V}^T\mathbf{r}^* \tag{4.14}$$

Denoting the right-hand term in parentheses, called the capacitance matrix, by **C**

$$\mathbf{C} = \mathbf{I} + \mathbf{H}\mathbf{V}^T\mathbf{T} \tag{4.15}$$

defining the vector **A** as

$$\mathbf{A} = (\mathbf{C})^{-1}\mathbf{H}\mathbf{V}^T\mathbf{r}^* \tag{4.16}$$

and substituting into Eq. (4.14), we find the modified displacements

$$\mathbf{r} = \mathbf{r}^* - \mathbf{TA} \tag{4.17}$$

In summary, solution by the Woodbury formula for evaluating the modified displacements, due to m rank-one changes in the stiffness matrix, involves the following steps:

a. The matrix **T** is calculated by Eq. (4.13).
b. The vector **A** is determined by Eq. (4.16).
c. The modified displacements are calculated by Eq. (4.17).

It can be observed that calculation of matrix **T** by Eq. (4.13) is equivalent to the solution of the initial analysis equations with m different right-hand side vectors. In addition, calculation of the vector **A** involves solution of the $m \times m$ set of Eq. (4.16). When the rank of m is small compared to the order of **K**, the main cost is the solution of Eq. (4.13). For a banded matrix **K** of order n and bandwidth b, this will require about mnb multiplications. The factorization of the matrix will require about b^2n multiplications. Therefore, the above solution process is effective only when the ratio m/b is a relatively small fraction.

Example 4.1 - Exact Solutions by the S-M Formula. To illustrate the calculation of the exact displacements by the S-M Formula, consider the ten-bar truss shown in Figure 4.1. The truss is subjected to a single loading condition of two concentrated loads, the modulus of elasticity is E =30 000 and the eight analysis unknowns are the horizontal (to the right) and the vertical (downward) displacements at joints 1, 2, 3 and 4, respectively. Assuming the initial cross-sectional areas $\mathbf{X} = \mathbf{1.0}$, we will present the solution for two separate changes.

Case 1. For a change $\Delta X_1 = 1.0$ in member 1.
Case 2. For a change $\Delta X_2 = 1.0$ in member 2.

The corresponding matrices of change, $\Delta\mathbf{K}_1$ and $\Delta\mathbf{K}_2$ are given by

$$\Delta\mathbf{K}_1 = \frac{E\Delta X_1}{L_1}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \eta_1 \mathbf{v}_1 \mathbf{v}_1^T$$

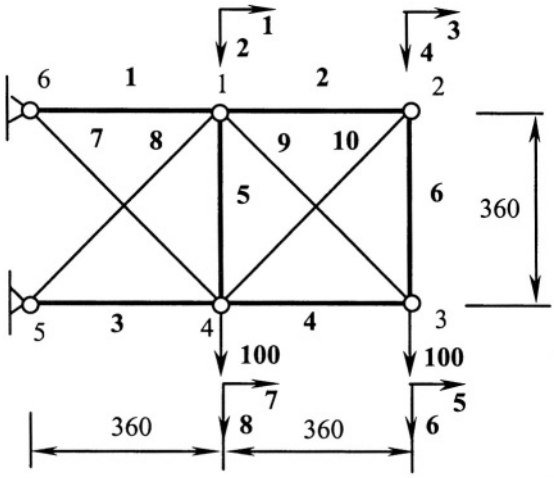

Fig. 4.1. Ten-bar truss.

$$\Delta \mathbf{K}_2 = \frac{E\Delta X_2}{L_2}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \eta_2 \mathbf{v}_2 \mathbf{v}_2^T$$

where L_1, L_2 are the members lengths and

$$\eta_1 = 83.333 \qquad \mathbf{v}_1^T = \{1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

$$\eta_2 = 83.333 \qquad \mathbf{v}_2^T = \{1,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

The vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are computed by Eq. (4.5)

$$\mathbf{t}_1 = \mathbf{K}^{*-1}\mathbf{v}_1 = \begin{Bmatrix} 0.0106 \\ 0.0066 \\ 0.0107 \\ 0.0179 \\ -0.0013 \\ 0.0181 \\ -0.0014 \\ 0.0054 \end{Bmatrix} \qquad \mathbf{t}_2 = \mathbf{K}^{*-1}\mathbf{v}_2 = \begin{Bmatrix} -0.0001 \\ -0.0006 \\ -0.0109 \\ -0.0066 \\ 0.0011 \\ -0.0054 \\ -0.0001 \\ 0.0006 \end{Bmatrix}$$

and the scalars a_1 and a_2 are computed by Eq. (4.7)

$$a_1 = \eta_1\left(1 + \eta_1 \mathbf{v}_1^T \mathbf{t}\right)^{-1} \mathbf{v}_1^T \mathbf{r}^* = 103.7468$$

$$a_2 = \eta_2\left(1 + \eta_2 \mathbf{v}_2^T \mathbf{t}\right)^{-1} \mathbf{v}_2^T \mathbf{r}^* = -21.1721$$

The final modified displacements obtained by the S-M formula for both cases are computed by Eq. (4.8)

Case 1

$$\mathbf{r}^T = (\mathbf{r}^* - a_1\mathbf{t}_1)^T = \{1.24, 4.89, 1.71, 10.79, -3.04, 11.26, -2.31, 5.45\}$$

Case 2

$$\mathbf{r}^T = (\mathbf{r}^* - a_2\mathbf{t}_2)^T = \{2.34, 5.57, 2.60, 12.51, -3.15, 13.02, -2.46, 6.02\}$$

4.3 GENERAL PROCEDURE

Assume the general case where the incremental stiffness matrix $\Delta\mathbf{K}$ can be compressed, by eliminating zero columns and rows, to form a reduced incremental matrix $\Delta\mathbf{K}_R$ of size equal to the number of changed columns (or rows) in matrix **K.** The relation between $\Delta\mathbf{K}$ and $\Delta\mathbf{K}_R$ is given by

$$\Delta\mathbf{K} = \mathbf{b}^T \Delta\mathbf{K}_R \mathbf{b} \tag{4.18}$$

where **b** is a Boolean matrix with linearly independent rows, each of which contains all zeroes except for one unit value, located at the column number where a change in **K** occurs.

It has been shown [6] that the following formula for computing $\mathbf{K}^{-1}$ can be derived from the Sherman-Morrison identity

$$\mathbf{K}^{-1} = (\mathbf{K}^{*} + \mathbf{b}^{T}\,\Delta\mathbf{K}_{R}\,\mathbf{b})^{-1} = \mathbf{K}^{*-1} - \mathbf{K}^{*-1}\mathbf{b}^{T}(\mathbf{I} + \Delta\mathbf{K}_{R}\mathbf{b}\mathbf{K}^{*-1}\mathbf{b}^{T})^{-1}\Delta\mathbf{K}_{R}\mathbf{b}\mathbf{K}^{*-1} \qquad (4.19)$$

where I is the identity matrix. Note that both matrices $\mathbf{I}$ and $\Delta\mathbf{K}_{R}\mathbf{b}\mathbf{K}^{*-1}\mathbf{b}^{T}$ are of size equal to that of $\Delta\mathbf{K}_{R}$. In addition, for $\mathbf{V}=\mathbf{b}^{T}$ and $\mathbf{W}^{T}=\Delta\mathbf{K}_{R}\,\mathbf{b}$, Eqs. (4.11) and (4.19) are equivalent.

The following procedure [6] utilizes symmetry and positive definiteness properties to compute the modified displacements directly. Assume that the initial stiffness matrix $\mathbf{K}^{*}$ is available from the initial analysis in the decomposed form

$$\mathbf{K}^{*} = \mathbf{U}^{*T}\mathbf{U}^{*} \qquad (4.20)$$

where $\mathbf{U}^{*}$ is an upper triangular matrix. Denote

$$\mathbf{r} = \mathbf{r}^{*} + \Delta\mathbf{r} \qquad (4.21)$$

where $\Delta\mathbf{r}$ is the change in the displacements. Post multiplying Eq. (4.19) by $\mathbf{R}^{*}$ and substituting Eqs. (3.1) and (4.21) yields

$$\Delta\mathbf{r} = -\mathbf{K}^{*-1}\mathbf{b}^{T}(\mathbf{I} + \Delta\mathbf{K}_{R}\mathbf{b}\,\mathbf{K}^{*-1}\mathbf{b}^{T})^{-1}\Delta\mathbf{K}_{R}\mathbf{b}\,\mathbf{r}^{*} \qquad (4.22)$$

The reduced unsymmetrical matrix $(\mathbf{I} + \Delta\mathbf{K}_{R}\mathbf{b}\,\mathbf{K}^{*-1}\mathbf{b}^{T})$ can readily be shown to be nonsingular, even when $\Delta\mathbf{K}_{R}$ is singular. Using $\mathbf{U}^{*}$ from Eq. (4.20) we define the symmetric influence matrix of unit changes, $\mathbf{Q}$, by

$$\mathbf{Q} = \mathbf{b}\,\mathbf{K}^{*-1}\mathbf{b}^{T} = \mathbf{b}\,\mathbf{U}^{*-1}(\mathbf{U}^{*T})^{-1}\mathbf{b}^{T} = \mathbf{Z}^{T}\mathbf{Z} \qquad (4.23)$$

where the rectangular matrix $\mathbf{Z}$ is defined as

$$\mathbf{Z} = (\mathbf{U}^{*T})^{-1}\mathbf{b}^{T} \qquad (4.24)$$

The matrix $\mathbf{Z}$ can be produced by a forward-substitution process performed on $\mathbf{b}^{T}$.

Using Eqs. (4.23) and (4.24), we can write Eq. (4.22) as

$$\Delta\mathbf{r} = -\mathbf{U}^{*-1}\mathbf{Z}(\mathbf{I} + \Delta\mathbf{K}_{R}\mathbf{Q})^{-1}\Delta\mathbf{K}_{R}\mathbf{b}\,\mathbf{r}^{*} \qquad (4.25)$$

This equation can be written in convenient positive-definite form by extracting the matrix $\mathbf{Q}$ as a common factor from the matrix to be inverted, as follows

$$\Delta \mathbf{r} = -\mathbf{U}^{*-1}\mathbf{Z}\mathbf{Q}^{-1}(\mathbf{Q}^{-1} + \Delta\mathbf{K}_R)^{-1}\Delta\mathbf{K}_R\mathbf{b}\,\mathbf{r}^* \tag{4.26}$$

Matrices $\mathbf{Q}$ and $(\mathbf{Q}^{-1} + \Delta\mathbf{K}_R)$ can be shown to be positive definite, thus pivoting is unnecessary in the triangularization. An optimal order of calculation is proposed elsewhere [6, 7] and illustrated in the following numerical example.

Example 4.2. To illustrate calculation of exact displacements by the procedure of Eq. (4.26), consider a structure with the following initial values

$$\mathbf{K}^* = \begin{bmatrix} 9 & -6 & 0 \\ -6 & 8 & -2 \\ 0 & -2 & 2 \end{bmatrix} \qquad \mathbf{U}^* = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}^* = \begin{Bmatrix} 0 \\ 24 \\ 84 \end{Bmatrix} \qquad \mathbf{r}^* = \begin{Bmatrix} 36 \\ 54 \\ 96 \end{Bmatrix}$$

Changes are made to the structure as follows

$$\Delta\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix} \qquad \Delta\mathbf{K}_R = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To determine $\Delta\mathbf{r}$ by Eq. (4.26), we calculate the following intermediate values

$$\mathbf{r}_R = \mathbf{b}\,\mathbf{r}^* = \begin{Bmatrix} 54 \\ 96 \end{Bmatrix} \qquad \mathbf{r}_1 = \Delta\mathbf{K}_R\mathbf{r}_R = \begin{Bmatrix} -210 \\ 210 \end{Bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0.5 & 1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad \mathbf{Q}^{-1} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

The matrix $\mathbf{P} = (\mathbf{Q}^{-1} + \Delta\mathbf{K}_R)$ is factorized into $\mathbf{P} = \mathbf{U}_P^T\mathbf{U}_P$

$$\mathbf{U}_P = \begin{bmatrix} 3 & -2.33 \\ 0 & 0.33(14)^{1/2} \end{bmatrix}$$

and the vectors, $\mathbf{r}_{11}$, $\mathbf{r}_{111}$ are calculated by

$$\mathbf{r}_{11} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\mathbf{r}_1 = \begin{Bmatrix} -60 \\ 60 \end{Bmatrix}$$

$$\mathbf{r}_{111} = \mathbf{Z}\,\mathbf{r}_{11} = \begin{Bmatrix} 0 \\ -30 \\ 30 \end{Bmatrix}$$

The final results for Eqs. (4.26) and (4.21) are

$$\Delta\mathbf{r}^T = -\{0,\ 0,\ 30\} \qquad \mathbf{r}^T = (\mathbf{r}^* + \Delta\mathbf{r})^T = \{36,\ 54,\ 66\}$$

4.4 REFERENCES

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5 Local Approximations

Local (single-point) approximations of displacements and stresses are often used in various design and optimization problems. Some of these approximations are presented in this chapter. In Section 5.1, the common Taylor series and the binomial series approximations are described. The relationship between the two series approximations is discussed and it is shown that for the common case of homogeneous displacement functions they are equivalent.

Various first-order approximations, using intermediate variables, are presented in Section 5.2. These approximations may significantly improve the accuracy of the results with almost no additional computational effort. The approximations presented include the common direct and reciprocal approximations, as well as conservative and convex approximations.

The basic concepts of scaling of a design are introduced in Section 5.3. A method for improving the accuracy of the approximations by scaling of the initial design is developed, using either geometrical or mathematical considerations for selecting the scaling multiplier. It is shown that the accuracy of conventional series approximations can be improved significantly with little computational effort. Scaling of both the initial design and some terms of the resulting approximations is then presented and it is shown that these scaled approximations can be expressed in a reduced basis form.

5.1 SERIES EXPANSION

A typical series expansion of the displacements $\mathbf{r}$ can be expressed as

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \ldots \mathbf{r}_s \tag{5.1}$$

where $\mathbf{r}_i$ ($i = 1, \ldots, s$) are the series terms and s is the number of terms considered. The common Taylor series and binomial series approximations are described in this section.

5.1.1 The Taylor Series

Taylor series is one of the most commonly used approximations in optimization of structures. For a vector $\mathbf{X}$ of design variables the first three terms of the series, obtained by expanding $\mathbf{r}$ about an initial design point $\mathbf{X}^*$, are given by

$$\mathbf{r}_1 = \mathbf{r}^*$$

$$\mathbf{r}_2 = \nabla \mathbf{r}^* (\mathbf{X} - \mathbf{X}^*) = \nabla \mathbf{r}^* \Delta \mathbf{X} \tag{5.2}$$

$$r_{3j} = 1/2 \, \Delta \mathbf{X}^T \mathbf{H}_j^* \Delta \mathbf{X}$$

where $\mathbf{r}^*$ = the vector of initial displacements, $\nabla \mathbf{r}^*$ = the matrix of first derivatives of $\mathbf{r}$ with respect to $\mathbf{X}$ at $\mathbf{X}^*$; $\mathbf{H}_j^*$ = the matrix of second derivatives of the jth component of $\mathbf{r}$ with respect to $\mathbf{X}$ at $\mathbf{X}^*$; $\Delta \mathbf{X} = \mathbf{X} - \mathbf{X}^*$ = the vector of changes in the design variables; and r_{3j} = the jth component of vector $\mathbf{r}_3$.

The main difficulty in using Taylor series is the calculation of high-order derivatives. The first-order (two term) approximations require evaluation of the first derivatives of displacements $\nabla \mathbf{r}^*$, which can readily be calculated [see Chapter 10]. Once the matrix of first derivatives is available, each reanalysis involves only calculation of the product $\nabla \mathbf{r}^* \Delta \mathbf{X}$. This is probably the most efficient reanalysis model.

If the accuracy of the first-order approximations is insufficient, second-order methods might be needed. These methods can be divided into two groups:

a. Methods based on calculation of the complete $\mathbf{H}_j^*$ matrices. An advantage of this approach is that all available second-order information is used. However, the computational effort involved in this calculation might be prohibitive.

b. Methods based on consideration of only the diagonal elements of matrices $\mathbf{H}_j^*$ [e.g. 1, 2]. Neglecting the off-diagonal elements of matrices $\mathbf{H}_j^*$ will considerably reduce the computational effort for the second order approximations. In addition, the use of diagonal second order derivatives will provide separable approximations which are desirable in some applications.

5.1.2 The Binomial Series

Consider the modified analysis equations (3.5), which can be rearranged to read

$$\mathbf{K}^* \mathbf{r} = \mathbf{R} - \Delta \mathbf{K} \, \mathbf{r} \tag{5.3}$$

Writing this equation as the recurrence relation

$$\mathbf{K}^* \mathbf{r}^{(k+1)} = \mathbf{R} - \Delta \mathbf{K} \, \mathbf{r}^{(k)} \tag{5.4}$$

where $\mathbf{r}^{(k+1)}$ is the value of $\mathbf{r}$ after the kth cycle, and assuming the initial value $\mathbf{r}^{(1)} = \mathbf{r}^*$, we obtain the following series

$$\mathbf{r} = (\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \ldots)\,\mathbf{r}_1 \tag{5.5}$$

where $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{B}$ and the vector $\mathbf{r}_1$ are defined by

$$\begin{aligned} \mathbf{B} &= \mathbf{K}^{*-1}\,\Delta\mathbf{K} \\ \mathbf{r}_1 &= \mathbf{K}^{*-1}\,\mathbf{R} \end{aligned} \tag{5.6}$$

That is, the first three terms of the series are given by

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{K}^{*-1}\,\mathbf{R} \\ \mathbf{r}_2 &= -\mathbf{B}\,\mathbf{r}_1 \\ \mathbf{r}_3 &= \mathbf{B}^2\,\mathbf{r}_1 = -\mathbf{B}\,\mathbf{r}_2 \end{aligned} \tag{5.7}$$

We see that the series terms can be calculated by the recurrence relation

$$\mathbf{r}_i = -\mathbf{B}\,\mathbf{r}_{i-1} \qquad (i = 2, \ldots, s) \tag{5.8}$$

In many applications the load vector is unchanged, that is $\Delta\mathbf{R} = \mathbf{0}$ and $\mathbf{R} = \mathbf{R}^*$. In such cases the first term $\mathbf{r}_1 = \mathbf{r}^*$ is already given from initial analysis of the structure. It will be shown in Section 7.1.1 that calculation of the series terms $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_s$ by Eq. (5.8) involves only forward and backward substitutions in cases where $\mathbf{K}^*$ is given from the initial analysis in the decomposed form of Eq. (3.2).

The series of Eq. (5.5) can be obtained in an alternative way as follows. Premultyplying Eq. (3.5) by $\mathbf{K}^{*-1}$ and substituting Eqs. (5.6) yields

$$(\mathbf{I} + \mathbf{B})\,\mathbf{r} = \mathbf{r}_1 \tag{5.9}$$

Premultiplying Eq. (5.9) by $(\mathbf{I} + \mathbf{B})^{-1}$ and expanding

$$(\mathbf{I} + \mathbf{B})^{-1} \cong \mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \ldots \tag{5.10}$$

gives the binomial series of Eq. (5.5). It will be shown in Section 5.1.3 that the terms of the binomial series [Eq. (5.7)] are equivalent to those of the Taylor series [Eq. (5.2)] for homogeneous displacement functions.

Both the Taylor series and the binomial series are based on information on a single design. As a result, the accuracy of the results might be insufficient for large changes in the design, where problems of slow convergence or divergence of the series may be encountered. Several methods have been proposed to improve the series convergence. These include the Jacobi iteration, block Gauss-Seidel iteration, dynamic acceleration and scaling of the initial design [3 - 5]. It will be shown in Section 5.3 that some of

these methods significantly improve the accuracy of the results with little computational effort. The accuracy can be improved further by considering the binomial series terms as basis vectors in the reduced basis method, as will be shown in Chapter 7.

The advantage of using the binomial series is that, unlike the Taylor series, calculation of derivatives is not required and high-order terms can readily be calculated. This makes the method more attractive in various applications where derivatives are not available or difficult to calculate. Assuming for example first-order (two-term) approximations, we must repeat only the calculation of $\mathbf{B}\,\mathbf{r}^*$ for each trial design. This requires calculation of a single vector by forward and backward substitutions.

In some particular cases, repeated calculation of the series terms involves almost no computational effort. Consider for example the common case of approximations along the line defined by

$$\mathbf{K} = \mathbf{K}^* + \Delta\mathbf{K} = \mathbf{K}^* + \alpha\,\Delta\mathbf{K}^* \tag{5.11}$$

where

$$\Delta\mathbf{K} = \alpha\,\Delta\mathbf{K}^* \tag{5.12}$$

The elements of matrix $\Delta\mathbf{K}^*$ are constant, representing the given direction of movement, and the scalar a is the step size variable. If $\mathbf{B}^* = \mathbf{K}^{*-1}\Delta\mathbf{K}^*$ [see Eq. (5.6)] and $\mathbf{R} = \mathbf{R}^*$, the terms of the binomial series [Eqs. (5.7)] become

$$\mathbf{r}_1 = \mathbf{r}^* \qquad \mathbf{r}_2 = \alpha\,\mathbf{r}_2^* \qquad \mathbf{r}_3 = \alpha^2\,\mathbf{r}_3^* \tag{5.13}$$

where the elements of vectors $\mathbf{r}_2^*$ and $\mathbf{r}_3^*$ are constant, defined by

$$\begin{aligned} \mathbf{r}_2^* &= -\,\mathbf{B}^*\,\mathbf{r}_1^* \\ \mathbf{r}_3^* &= -\,\mathbf{B}^*\,\mathbf{r}_2^* \end{aligned} \tag{5.14}$$

That is, once $\mathbf{r}_2^*$ and $\mathbf{r}_3^*$ are calculated, then for any assumed α it is necessary to calculate only the products $\alpha\,\mathbf{r}_2^*$ and $\alpha^2\,\mathbf{r}_3^*$.

For a change of rank-one in the stiffness matrix, the matrix of changes $\Delta\mathbf{K}$ can be expressed in the form of Eq. (5.12) where the elements of matrix $\Delta\mathbf{K}^*$ are again constant, and the scalar α is the step size variable.

5.1.3 Homogeneous Functions

In many structural design problems the displacements, the stresses and the forces are homogeneous functions of the design variables [6]. It will be shown in this section that in such cases the Taylor series and the binomial series become equivalent. Assume that

the displacements $\mathbf{r}$ are homogeneous functions of degree m in the vector of design variables $\mathbf{X}$, for which we have by definition

$$\mathbf{r}(\mu \mathbf{X}^*) = \mu^m \mathbf{r}(\mathbf{X}^*) \tag{5.15}$$

where μ is a scalar multiplier. Euler theorem on homogeneous functions states that

$$\nabla \mathbf{r}^* \mathbf{X}^* = m \mathbf{r}^* \tag{5.16}$$

It is instructive to note that the derivatives of homogeneous functions of degree m are homogeneous functions of degree m-1, that is

$$\nabla \mathbf{r}(\mu \mathbf{X}^*) = \mu^{m-1} \nabla \mathbf{r}(\mathbf{X}^*) \tag{5.17}$$

These properties can be used to obtain simplified local approximations of displacements [6, 7].

Assuming the first-order Taylor series expansion of the displacements $\mathbf{r}$

$$\mathbf{r} = \mathbf{r}^* + \nabla \mathbf{r}^* (\mathbf{X} - \mathbf{X}^*) \tag{5.18}$$

and substituting Eq. (5.16) into Eq. (5.18), we find

$$\mathbf{r} = (1 - m)\mathbf{r}^* + \nabla \mathbf{r}^* \mathbf{X} \tag{5.19}$$

Consider, for example, the common case of cross-sectional design variables. If the elements of the stiffness matrix are linear functions of $\mathbf{X}$ and the elements of the load vector are constant ($\mathbf{R} = \mathbf{R}^*$,) then the displacements are homogeneous functions of degree $m = -1$ in $\mathbf{X}$, and Eq. (5.19) becomes

$$\mathbf{r} = 2\mathbf{r}^* + \nabla \mathbf{r}^* \mathbf{X} \tag{5.20}$$

Differentiating Eq. (3.1) with respect to a design variable X and rearranging gives

$$\nabla \mathbf{r}^* = -\mathbf{K}^{*-1} \nabla \mathbf{K}^* \mathbf{r}^* \tag{5.21}$$

For a change $\Delta X = X - X^*$ of a single variable the stiffness matrix $\mathbf{K}$ can be expressed as

$$\mathbf{K} = \mathbf{K}^* + \mathbf{K}^0 \Delta X \tag{5.22}$$

where the elements of matrix $\mathbf{K}^0$ are constant. From Eq. (5.22), the elements of matrix $\nabla \mathbf{K}$ are constants given by

$$\nabla \mathbf{K} = \mathbf{K}^0 = \nabla \mathbf{K}^* \tag{5.23}$$

Substituting Eqs. (5.21) and (5.23) into Eq. (5.18) yields

$$\mathbf{r} = \mathbf{r}^* - \mathbf{K}^{*-1}\mathbf{K}^0\mathbf{r}^*\Delta X \tag{5.24}$$

From Eq. (5.22) matrix $\Delta\mathbf{K}$ can be expressed as

$$\Delta \mathbf{K} = \mathbf{K}^0 \Delta X \tag{5.25}$$

Substituting Eqs. (5.6) and (5.25) into Eq. (5.24) yields

$$\mathbf{r} = (\mathbf{I} - \mathbf{B})\,\mathbf{r}^* \tag{5.26}$$

This expression is identical to the first two terms of Eq. (5.5). That is, for the common case of homogeneous displacement functions of degree m = -1, the first-order Taylor series and the first-order binomial series are equivalent.

5.2 INTERMEDIATE VARIABLES

Various approximations can be improved by using intermediate variables, Y_i, defined by

$$Y_i = Y_i(X_i) \tag{5.27}$$

A typical example is the general form

$$Y_i = X_i^p \tag{5.28}$$

where p is a predetermined parameter. One of the more popular intermediate variables is the reciprocal of X_i [8]

$$Y_i = \frac{1}{X_i} \tag{5.29}$$

The reason for this is that displacements and stresses in determinate structures are often linear functions of the reciprocal variables. For statically indeterminate structures, the use of these variables also proves to be a useful device to obtain better approximations, because the response functions are nearly linear functions of the variables. Intermediate variables are often most effective for homogeneous functions [4, 6].

One disadvantage of the reciprocal approximation is that it becomes infinite if any X_i is zero. This difficulty can be overcome by the simple transformation [9]

$$Y_i = \frac{1}{X_i + \delta X_i} \tag{5.30}$$

where the values of δX_i are typically small compared to representative values of the corresponding X_i's.

Consider the first order Taylor series expansion of a constraint function $g(\mathbf{X})$ in terms of the design variables X_i, denoted as a direct approximation g_D

$$g_D \cong g^* + \sum_{i=1}^{n} \frac{\partial g^*}{\partial X_i}(X_i - X_i^*) \tag{5.31}$$

where n is the number of design variables. To improve the quality of the results, g can be expressed in terms of the reciprocal variables Y_i [Eq. (5.29)]. The resulting expression, denoted as a reciprocal approximation g_R, is given by

$$g_R \cong g^* + \sum_{i=1}^{n} \left(\frac{\left(\frac{\partial g^*}{\partial X_i}\right)}{\left(\frac{\partial Y_i}{\partial X_i}\right)} \right)(Y_i - Y_i^*) = g^* + \sum_{i=1}^{n} \frac{X_i^*}{X_i}\frac{\partial g^*}{\partial X_i}(X_i - X_i^*) \tag{5.32}$$

5.2.1 Conservative and Convex Approximations

In some applications it is desirable to introduce conservative approximations $g_C(\mathbf{X}) \leq 0$ for the constraint $g(\mathbf{X}) \leq 0$. Since the constraint is expressed as $g \leq 0$, g_C is a conservative approximation if it is more positive than g. Conservative approximations are useful, for example, in optimal design procedures where it is desired that all intermediate solutions lie in the feasible region. Such procedures have an advantage from a practical point of view in that the solution process can be stopped at any time with a resulting feasible solution. A hybrid form of the direct and reciprocal approximations, which is more conservative than either, can be introduced [10]. Subtracting the reciprocal from the direct approximation, we derive the form of this conservative approximation:

$$g_D - g_R = \sum_{i=1}^{n} \frac{\partial g^*}{\partial X_i}\frac{(X_i - X_i^*)^2}{X_i} \tag{5.33}$$

The sign of each term in the sum is determined by the sign of the ratio $(\partial g^*/\partial X_i)/X_i$ which is also the sign of the product $X_i(\partial g^*/\partial X_i)$. It is possible to create a

conservative approximation g_C, which includes the more positive term for each design variable

$$g_C \cong g^* + \sum_{i=1}^{n} \alpha_i \frac{\partial g^*}{\partial X_i}(X_i - X_i^*) \tag{5.34}$$

where

$$\alpha_i = \begin{cases} 1 & if \quad X_i \dfrac{\partial g^*}{\partial X_i} \geq 0 \\ \dfrac{X_i^*}{X_i} & if \quad X_i \dfrac{\partial g^*}{\partial X_i} < 0 \end{cases} \tag{5.35}$$

It should be noted that this approach does not ensure that the approximate constraint is conservative with respect to the true constraint. The conservative approximation is only more conservative than both the linear and the reciprocal approximations.

A function $f(\mathbf{X})$ is said to be convex if, on the line connecting every pair of points $\mathbf{X}_1$ and $\mathbf{X}_2$ in its domain of definition, the value of the function is less than or equal to a linear interpolation $f(\mathbf{X}_1)$ and $f(\mathbf{X}_2)$. The conservative approximation has the advantage of being a convex approximation [11], However, it has been found that this approximation tends to be less accurate than either the direct or the reciprocal approximation.

The Method of Moving Asymptotes (MMA) is intended to introduce more accurate approximations [12]. Using this method, we define the intermediate variables such that the degree of convexity, and hence conservativeness, of the approximation can be adjusted. Instead of using direct and reciprocal variables, the method employs one of the intermediate variables

$$Y_i = \frac{1}{X_i - L_i}$$
$$Y_i = \frac{1}{U_i - X_i} \tag{5.36}$$

where L_i and U_i are specified parameters that may be changed. Based on this transformation, the moving asymptotes approximation g_M is formulated as

$$g_M \cong \overline{g}^* + \sum_{i=1}^{n} \frac{\partial g^*}{\partial X_i}\left(\frac{\alpha_i}{U_i - X_i} + \frac{\beta_i}{X_i - L_i}\right) \tag{5.37}$$

where

$$\alpha_i = \begin{cases} (U_i - X_i^*)^2 & \text{if} \quad \partial g^* / \partial X_i > 0 \\ 0 & \text{if} \quad \partial g^* / \partial X_i \le 0 \end{cases}$$

$$\beta_i = \begin{cases} 0 & \text{if} \quad \partial g^* / \partial X_i \ge 0 \\ -(X_i^* - L_i)^2 & \text{if} \quad \partial g^* / \partial X_i < 0 \end{cases} \tag{5.38}$$

$$\overline{g}^* = g^* - \sum_{i=1}^{n} \frac{\partial g^*}{\partial X_i} \left(\frac{\alpha_i}{U_i - X_i^*} + \frac{\beta_i}{X_i^* - L_i} \right) \tag{5.39}$$

Since all the coefficients in Eq. (5.37) are non-negative, the approximations for g are convex functions. The approximated functions are driven by the selected values for the parameters L_i and U_i which act as asymptotes.

The moving asymptotes approximation is general; the direct approximation and the conservative approximation can be viewed as the following special cases:

- For $L_i \to -\infty$, $U_i \to \infty$, no intermediate variables are considered, and the direct formulation of Eq. (5.31) is obtained.
- For $L_i = 0$, $U_i \to \infty$, the conservative formulation of Eq. (5.34) is obtained.

Other values of L_i and U_i are possible, and these values may even be modified during the solution process. However, it is not at all straightforward to find suitable values for the asymptotes. To avoid the possibility of any unexpected division by zero, move limits X_i^L and X_i^U can be chosen such that $L_i < X_i^L$ and $X_i^U < U_i$. The closer L_i and U_i are chosen to X_i^*, the more curvature is given to the approximate function, and the more conservative becomes the approximation of the original function. By choosing L_i and U_i far away from X_i^*, the approximate function becomes close to linear.

Both formulations of Eqs. (5.34) and (5.37) are based on first-order convex approximations that attempt to simulate curvature of the functions. The MMA method offers more flexibility through the moving asymptotes L_i and U_i.

5.2.2 Intermediate Response Functions

In statically determinate structures the element forces are independent of the cross-sectional variables while the stresses are not. In statically indeterminate structures the element forces might be nearly linear functions of the design variables even when the stresses are highly nonlinear functions of the variables. As a result, improved approximations of stresses can be developed by using element forces as intermediate response quantities. That is, instead of using first-order approximations of the stresses σ_j

$$\sigma_j = \sigma_j^* + \sum_{i=1}^{n} \frac{\partial \sigma_j^*}{\partial X_i}(X_i - X_i^*) \tag{5.40}$$

it is possible to use the forces as intermediate response quantities to obtain [13]

$$\sigma_j = \frac{N_j}{X_j} = \frac{N_j^* + \sum_{i=1}^{n} \frac{\partial N_j^*}{\partial X_i}(X_i - X_i^*)}{X_j} \tag{5.41}$$

Note that the approximation of Eq. (5.41) is exact for statically determinate structures. In general, Eq. (5.41) produces a more accurate approximation than Eq. (5.40) because it captures the cross coupling between the effect of X_i and X_j on the stress. This cross coupling is present because the stress is a nonlinear function of the element force N_j, which is a function of the X_i's, and the element variable X_j. This can be seen by examining the second partial derivatives of Eqs. (5.40) and (5.41). The second partial derivative of Eq. (5.40) is zero while the second derivative of Eq. (5.41) is

$$\frac{\partial^2 \sigma_j}{\partial X_i \partial X_j} = -\frac{1}{X_j^2}\frac{\partial N_j^*}{\partial X_i} \tag{5.42}$$

Example 5.1 - First-Order Approximations Using Intermediate Variables. To illustrate various first-order approximations using intermediate variables, consider the constraints [4]

$$g_1 \equiv X_1^2/20 - X_2 + 1 \le 0$$
$$g_2 \equiv X_2^2/20 - X_1 + 1 \le 0$$

Assuming the initial point $\mathbf{X}^{*T} = \{6, 3\}$, we obtain the following expressions.

a. Direct-linear approximations [Eq. (5.31)]

$$g_1 \cong -0.8 + 0.6X_1 - X_2 \le 0$$
$$g_2 \cong 0.55 - X_1 + 0.3X_2 \le 0 \qquad (a)$$

b. Conservative-convex approximations [Eq. (5.34)]

$$g_1 \cong 0.6X_1 + 9/X_2 - 6.8 \le 0$$
$$g_2 \cong 36/X_1 + 0.3X_2 - 11.45 \le 0 \qquad (b)$$

c. Moving asymptotes approximations [(5.37)]

$$g_1 \cong (0.4 - 0.6U_1 + L_2) + 0.6\frac{(U_1 - 6)^2}{U_1 - X_1} + \frac{(3 - L_2)^2}{X_2 - L_2} \le 0$$

$$g_2 \cong (9.65 + L_1 - 0.3U_2) + \frac{(6 - L_1)^2}{X_1 - L_1} + 0.3\frac{(U_2 - 3)^2}{U_2 - X_2} \le 0 \qquad (c)$$

The various approximations are demonstrated in Figure 5.1. For the MMA, the closer L_i and U_i are chosen to X_i^*, the more curvature is given to the approximate functions, and the more conservative become the approximations of the original constraints.

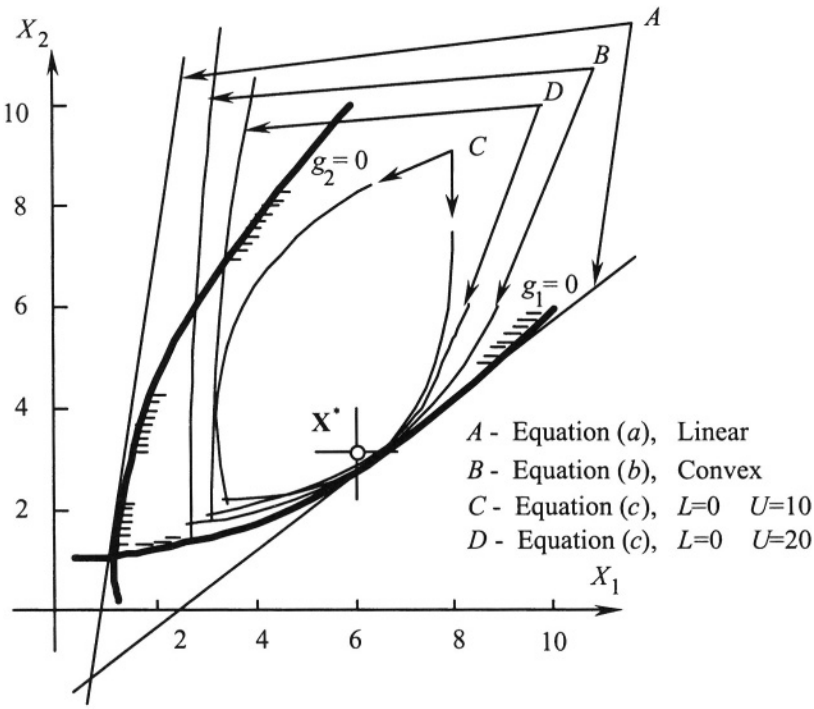

Fig. 5.1. First-order approximations of constraints.

5.3 IMPROVED SERIES APPROXIMATIONS

Concepts of scaling presented in this section can be used to significantly improve the accuracy of series approximations. In addition, these concepts help to understand the physical meaning of the results obtained by the CA method presented in Chapter 7, and

explain the good accuracy of the approximations. The following two types of scaling will be considered in this section:

a. *The uniform scaling of the initial design,* where the initial stiffness matrix is multiplied by a scalar multiplier, and the exact displacements after scaling can readily be determined. This type of scaling is most effective in cases where the scaled design represents a better initial point then the original design.
b. *Scaling of displacements,* where some of the terms of the approximate displacements are multiplied by scalar multipliers to improve the accuracy of the results.

5.3.1 Scaling of the Initial Design

Scaling of the initial stiffness matrix is carried out by multiplying the initial stiffness matrix $\mathbf{K}^*$ by a positive scaling multiplier μ to obtain the modified matrix

$$\mathbf{K} = \mu \mathbf{K}^* \tag{5.43}$$

From Eqs. (3.1), (3.5) and (5.43), it is clear that for $\mathbf{R} = \mathbf{R}^*$ the exact displacements after scaling can be calculated directly by

$$\mathbf{r} = \mu^{-1} \mathbf{r}^* \tag{5.44}$$

where $\mathbf{r}^*$ is the vector of initial displacements. It should be noted that Eq. (5.43) represents only an algebraic operation and it does not require linear dependence of the stiffness matrix on the design variables. Only when all elements of $\mathbf{K}$ are linear functions of the design variables $\mathbf{X}$, will the matrix $\mu\mathbf{K}^*$ correspond to the design $\mathbf{X}=\mu\mathbf{X}^*$. In the general case where the elements of $\mathbf{K}$ are some nonlinear functions of $\mathbf{X}$, the scaled stiffness matrix might not have the usual physical meaning. The exact stresses after scaling are calculated by substituting the scaled displacements (5.44) into the stress-displacement relations.

For the reanalysis problem under consideration the object is to evaluate the approximate displacements for various changes $\Delta\mathbf{K}$ in the initial design. To improve the accuracy of the binomial approximations [Eq. (5.5)] it is possible to assume a scaled stiffness matrix $\mu\mathbf{K}^*$ as an initial point instead of $\mathbf{K}^*$. Then, the modified displacements become some functions of μ, as will be shown in the following.

Consider first the common case where the modified design is a scaled design $\mu\mathbf{K}^*$, as given by Eq. (5.43). Then $\Delta\mathbf{K} = (\mu - 1)\mathbf{K}^*$ and matrix $\mathbf{B}$ [Eq. (5.6)] becomes

$$\mathbf{B} = \mathbf{K}^{*-1} \Delta\mathbf{K} = (\mu - 1)\,\mathbf{I} \tag{5.45}$$

where $\mathbf{I}$ is the identity matrix. For $\mathbf{R} = \mathbf{R}^*$ we have $\mathbf{r}_1 = \mathbf{r}^*$ and the remaining terms of the binomial series become linearly dependent [Eqs. (5.7)]

$$\mathbf{r}_2 = -(\mu - 1)\ \mathbf{r}^*$$
$$\mathbf{r}_3 = (\mu - 1)^2\ \mathbf{r}^* \tag{5.46}$$

Thus, the exact modified displacements are determined directly by Eq. (5.44).

Consider the general case where the modified design is not a scaled design. Rewrite the expression of $\mathbf{K}$ [Eq. (3.3)] in terms of μ as [see Figure 5.2]

$$\mathbf{K} = \mathbf{K}^* + \Delta\mathbf{K} = \mu\mathbf{K}^* + \Delta\mathbf{K}_\mu \tag{5.47}$$

If the scaled design $\mu\mathbf{K}^*$ is assumed as an initial point instead of $\mathbf{K}^*$, then the modified stiffness matrix $\mathbf{K}$ can be expressed in terms of the corresponding changes $\Delta\mathbf{K}_\mu$ instead of the original changes $\Delta\mathbf{K}$. From Eq. (5.47), $\Delta\mathbf{K}_\mu$ is given by

$$\Delta\mathbf{K}_\mu = (1 - \mu)\,\mathbf{K}^* + \Delta\mathbf{K} \tag{5.48}$$

Assume $\mu\mathbf{K}^*$, $\Delta\mathbf{K}_\mu$ and $\mu^{-1}\,\mathbf{r}^*$ as initial values instead of $\mathbf{K}^*$ $\Delta\mathbf{K}$ and $\mathbf{r}^*$, respectively. Then for $\mathbf{R} = \mathbf{R}^*$, the binomial series [Eq. (5.5)] becomes

$$\mathbf{r} = \frac{1}{\mu}(\mathbf{I} - \mathbf{B}_\mu + \mathbf{B}_\mu^2 - \ldots)\mathbf{r}^* \tag{5.49}$$

where $\mathbf{B}_\mu$ is defined as

$$\mathbf{B}_\mu = \frac{1-\mu}{\mu}\mathbf{I} + \frac{1}{\mu}\mathbf{B} \tag{5.50}$$

and $\mathbf{B}$ is given by Eq. (5.6). It can be observed that for $\mu = 1$ matrix $\mathbf{B}_\mu$ becomes $\mathbf{B}$ and Eqs. (5.47) and (5.49) are reduced to Eqs. (3.3) and (5.5), respectively. Substituting Eq. (5.50) into Eq. (5.49) and rearranging yields

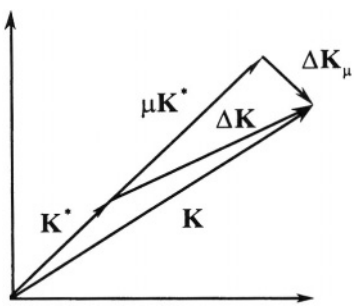

Fig. 5.2. Scaling of the initial stiffness matrix.

$$\mathbf{r} = y_1(\mu)\,\mathbf{r}_1 + y_2(\mu)\,\mathbf{r}_2 + , ..., y_s\,(\mu)\,\mathbf{r}_s \tag{5.51}$$

where $\mathbf{r}_1$ is given by Eq. (5.6), $\mathbf{r}_2$, .. , $\mathbf{r}_s$, are defined by Eqs. (5.8), and the coefficients $y_1(\mu), y_2(\mu), ..., y_s\,(\mu)$, are given by (for $s = 5$)

$$\begin{aligned}
y_1(\mu) &= 1/\mu + (\mu-1)/\mu^2 + (\mu-1)^2/\mu^3 + (\mu-1)^3/\mu^4 + (\mu-1)^4/\mu^5 \\
y_2(\mu) &= 1/\mu^2 + 2(\mu-1)/\mu^3 + 3(\mu-1)^2/\mu^4 + 4(\mu-1)^3/\mu^5 \\
y_3(\mu) &= 1/\mu^3 + 3(\mu-1)/\mu^4 + 6(\mu-1)^2/\mu^5 \\
y_4(\mu) &= 1/\mu^4 + 4(\mu-1)/\mu^5 \\
y_5(\mu) &= 1/\mu^5
\end{aligned} \tag{5.52}$$

The approximations of Eq. (5.51) are termed Scaled Approximations (SA). The modified displacements are now some functions of the multiplier μ, and it is possible to choose μ such that the accuracy of the approximations is improved.

Selecting the Scaling Multiplier. Procedures for selecting the scaling multiplier have been proposed in several studies [3-5]. Criteria for selecting μ based on either geometrical or mathematical considerations are demonstrated in the following.

To establish a geometrical criterion for selecting the scaling multiplier μ, express the angle θ between the vector of the modified design variables $\mathbf{X}$ and the vector of the initial design variables $\mathbf{X}^*$ by [see Figure (5.3)]

$$\cos\theta = (\mathbf{X}^T\,\mathbf{X}^*) / (|\mathbf{X}|\,|\mathbf{X}^*|) \tag{5.53}$$

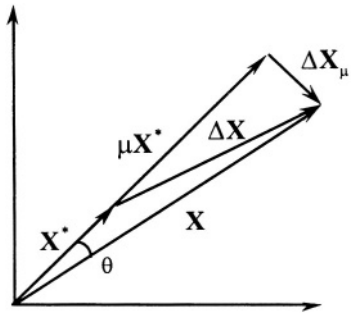

Fig. 5.3. Geometrical basis for selecting the μ multiplier.

where $|\mathbf{X}|$ denotes absolute value of $\mathbf{X}$. The design variables $\mathbf{X}$ can be expressed as

$$\mathbf{X} = \mathbf{X}^* + \Delta\mathbf{X} = \mu\mathbf{X}^* + \Delta\mathbf{X}_\mu \tag{5.54}$$

If μ is selected such that the change in the design variables $\Delta\mathbf{X}_\mu$ is reduced, then the accuracy of the approximate modified displacements are generally improved. Choosing for example $\Delta\mathbf{X}_\mu$ perpendicular to the bisector of θ, we find from the condition $|\mu\mathbf{X}^*|=|\mathbf{X}|$ that

$$\mu = \left(\frac{\mathbf{X}^T\mathbf{X}}{\mathbf{X}^{*T}\mathbf{X}^*} \right)^{1/2} \tag{5.55}$$

It will be shown later in this chapter by numerical examples that by using this geometrical criterion, the approximations of Eq. (5.51) provide much better results than the binomial series without scaling ($\mu = 1$). Other possibilities for selecting μ on the basis of geometrical considerations (such as $\Delta\mathbf{X}_\mu$ perpendicular to $\mathbf{X}^*$ or to $\mathbf{X}$) are demonstrated elsewhere [3].

An alternative mathematical approach for selecting μ is to minimize the Euclidean norm of $\mathbf{B}_\mu$, that is

$$\| \mathbf{B}_\mu \| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} B_{\mu ij}^2 \right)^{1/2} \rightarrow \text{minimum} \tag{5.56}$$

where n denotes the dimension of matrix $\mathbf{B}$. One drawback of using the criterion of Eq. (5.56) is that the elements of matrix $\mathbf{B}$ must be calculated. Since this operation involves much computational effort, we may use an alternative criterion which minimizes the Euclidean norm of the second term in the series of Eq. (5.49), that is [5]

$$\|\mathbf{B}_\mu \mathbf{r}^*\| \rightarrow \text{minimum} \tag{5.57}$$

Substituting $\mathbf{B}_\mu$ from Eq. (5.50) into Eq. (5.57), differentiating and setting the result equal to zero, yields

$$\mu = \frac{a}{b} \tag{5.58}$$

where

$$a = \sum_{i=1}^{n} (r_{1i} - r_{2i})^2$$

$$b = \sum_{i=1}^{n} (r_{1i}^2 - r_{1i} r_{2i}) \tag{5.59}$$

The terms r_{1i} are the elements of $\mathbf{r}_1 = \mathbf{r}^*$ and the terms r_{2i} are the elements of $\mathbf{r}_2$ that are already given by Eq. (5.7). The effect of μ on the quality of the approximations is demonstrated by the following numerical examples.

Example 5.2 - Scaled Approximations, Cross Section Changes. The object of this example is to demonstrate the accuracy of the results achieved by scaled approximations for cross section changes. Consider the ten-bar truss shown in Figure 5.4 subjected to a single loading condition of two concentrated loads. The design variables are the members' cross-sectional areas, the initial cross section areas $\mathbf{X}^*$ are equal to unity, the modulus of elasticity is 30000, and the eight analysis unknowns are the horizontal and vertical displacements at joints 1, 2, 3 and 4, respectively. The stress constraints are $-25.0 \le \sigma \le 25.0$, and the minimum size constraints are $0.001 \le \mathbf{X}$. Assuming the weight as an objective function, we find the optimal (minimum weight) design [4]

$$\mathbf{X}_{opt}^T = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

The line from the initial design to the optimal design is given by

$$\mathbf{X} = \mathbf{X}^* + \alpha \, \Delta\mathbf{X}^*$$

where $\Delta\mathbf{X}^*$ is defined as

$$\Delta\mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

In this example, results will be demonstrated for various α values. Summarized in Tables 5.1 and 5.2 are results calculated by the following two methods, assuming first-order (two-term) and second-order (three-term) approximations:

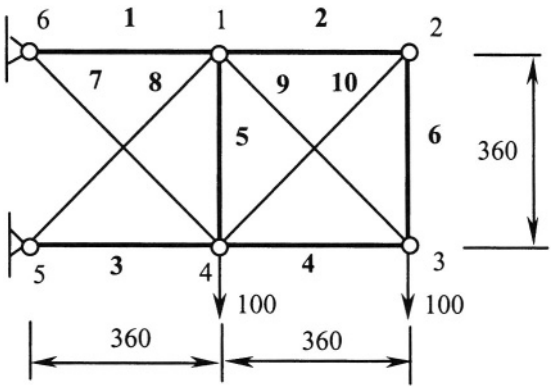

Fig. 5.4. Ten-bar truss.

- SA_G, Scaled Approximations using a Geometrical criterion [Eqs. (5.49), (5.55)].
- SA_M, Scaled Approximations using a Mathematical criterion [Eqs. (5.49), (5.58)].

As expected, the errors involved in the approximations increase with α and smaller errors are obtained for second-order approximations (Table 5.2). Results obtained by the binomial approximations [Eq. (5.5) which are equivalent to the Taylor series in the present case of homogeneous displacement functions] are not shown since very large errors were obtained even for the smallest change $\alpha = 0.2$.

Table 5.1. First-order scaled approximations, cross-section changes.

		Displacements							
α	Method	1	2	3	4	5	6	7	8
0.2	SA_G	0.78	2.39	1.12	5.55	-1.39	5.88	-0.84	2.66
	SA_M	0.95	2.60	1.26	5.98	-1.50	6.28	-1.02	2.86
	Exact	0.97	2.62	1.32	6.21	-1.61	6.55	-1.03	2.88
0.4	SA_G	0.42	1.49	0.67	3.51	-0.88	3.75	-0.46	1.69
	SA_M	0.59	1.69	0.81	3.91	-0.98	4.13	-0.63	1.87
	Exact	0.61	1.73	0.89	4.21	-1.12	4.49	-0.65	1.90
0.8	SA_G	0.21	0.86	0.37	2.05	-0.51	2.20	-0.24	0.99
	SA_M	0.33	1.00	0.47	2.31	-0.58	2.45	-0.36	1.11
	Exact	0.36	1.06	0.56	2.61	-0.71	2.82	-0.37	1.10

Table 5.2. Second-order scaled approximations, cross-section changes.

		Displacements							
α	Method	1	2	3	4	5	6	7	8
0.2	SA_G	1.05	2.71	1.40	6.46	-1.68	6.80	-1.12	2.98
	SA_M	0.96	2.62	1.31	6.16	-1.57	6.49	-1.04	2.89
	Exact	0.97	2.62	1.32	6.21	-1.61	6.55	-1.03	2.88
0.4	SA_G	0.71	1.83	0.98	4.51	-1.21	4.77	-0.76	2.03
	SA_M	0.61	1.72	0.86	4.11	-1.07	4.36	-0.66	1.92
	Exact	0.61	1.73	0.89	4.21	-1.12	4.49	-0.65	1.90
0.8	SA_G	0.43	1.11	0.62	2.81	-0.78	2.99	-0.47	1.23
	SA_M	0.35	1.02	0.52	2.49	-0.65	2.65	-0.38	1.15
	Exact	0.36	1.06	0.56	2.61	-0.71	2.82	-0.37	1.10

Example 5.3. Scaled Approximations, Geometrical Changes. To demonstrate the accuracy of the results achieved by scaled approximations for geometrical changes, consider the initial geometry of the ten-bar truss shown in Figure 5.4 with a fixed uniform cross-sectional area of unity for all members. Assuming the depth of the truss

as a single geometrical variable, Y, approximate displacements evaluated by the SA_G and the SA_M for $Y = 540$ and $Y = 720$ (increase in the depth of the truss by 50% and 100%, respectively) are summarized in Tables 5.3 and 5.4. Again, higher accuracy is achieved for smaller α values and for second-order approximations (Table 5.4). In addition, better results were obtained by the SA_M method.

Table 5.3. First-order scaled approximations, geometrical changes.

		Displacements							
Y	Method	1	2	3	4	5	6	7	8
540	SA_G	1.44	3.79	1.68	7.66	-1.99	8.14	-1.55	4.20
	SA_M	1.49	3.89	1.74	7.81	-2.07	8.35	-1.60	4.35
	Exact	1.55	3.94	1.82	7.84	-2.18	8.47	-1.66	4.44
720	SA_G	1.16	3.26	1.33	6.47	-1.58	6.87	-1.25	3.61
	SA_M	1.15	3.52	1.31	6.59	-1.61	7.12	-1.26	3.97
	Exact	1.15	3.67	1.34	6.60	-1.66	7.36	-1.25	4.24

Table 5.4. Second-order scaled approximations, geometrical changes.

		Displacements							
Y	Method	1	2	3	4	5	6	7	8
540	SA_G	1.56	3.72	1.89	8.43	-2.11	8.75	-1.64	4.01
	SA_M	1.45	3.77	1.69	7.72	-1.99	8.18	-1.55	4.17
	Exact	1.55	3.94	1.82	7.84	-2.18	8.47	-1.66	4.44
720	SA_G	1.17	2.79	1.41	6.33	-1.59	6.57	-1.23	3.01
	SA_M	1.15	3.30	1.31	6.47	-1.58	6.89	-1.24	3.67
	Exact	1.15	3.67	1.34	6.60	-1.66	7.36	-1.25	4.24

5.3.2 Scaling of Displacements

It has been noted that scaling of the initial design may improve the accuracy of the results. It is shown in this section how the approximate displacements can be expressed in a reduced basis form, by extending the concept of scaling to include additional multipliers.

Assuming, for example, three terms of the scaled approximations [Eq. (5.51)], the resulting displacements are given by

$$\mathbf{r} = y_1(\mu)\,\mathbf{r}_1 + y_2(\mu)\,\mathbf{r}_2 + y_3(\mu)\,\mathbf{r}_3 \tag{5.60}$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, are defined by Eqs. (5.6), (5.8) and the coefficients $y_1(\mu)$, $y_2(\mu)$, $y_3(\mu)$ are given by [see Eq. (5.52)]

$$y_1(\mu) = 1/\mu + (\mu - 1)/\mu^2 + (\mu - 1)^2/\mu^3$$

$$y_2(\mu) = 1/\mu^2 + 2(\mu - 1)/\mu^3 \tag{5.61}$$

$$y_3(\mu) = 1/\mu^3$$

Multiplying the first term and the second term of Eq. (5.60) by scaling multipliers Λ_1 and Λ_2, respectively, we find

$$\mathbf{r} = \Lambda_1 y_1(\mu)\, \mathbf{r}_1 + \Lambda_2 y_2(\mu)\, \mathbf{r}_2 + y_3(\mu)\, \mathbf{r}_3 \tag{5.62}$$

Substituting Eq. (5.61) into Eq. (5.62) gives

$$\mathbf{r} = \Lambda_1[\mu^{-3}(3\mu^2 - 3\mu + 1)]\ \mathbf{r}_1 + \Lambda_2[\mu^{-3}(3\mu - 2)]\ \mathbf{r}_2 + \mu^{-3}\ \mathbf{r}_3 \tag{5.63}$$

Denoting

$$y_1 = \Lambda_1[\mu^{-3}(3\mu^2 - 3\mu + 1)]$$

$$y_2 = \Lambda_2[\mu^{-3}(3\mu - 2)] \tag{5.64}$$

$$y_3 = \mu^{-3}$$

and substituting Eq. (5.64) into Eq. (5.63) gives

$$\mathbf{r} = y_1\, \mathbf{r}_1 + y_2\, \mathbf{r}_2 + y_3\, \mathbf{r}_3 = \mathbf{r}_B\, \mathbf{y} \tag{5.65}$$

where the matrix $\mathbf{r}_B$ and the vector $\mathbf{y}$ are defined as

$$\mathbf{r}_B = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$$

$$\mathbf{y}^T = \{y_1, y_2, y_3\} \tag{5.66}$$

That is, scaling of the initial design and the first two terms of the scaled approximations of Eq. (5.60) gives the expression of Eq. (5.65). This equation can be viewed as a reduced basis expression, as will be shown in Section 6.2. Once the coefficients $\mathbf{y}$ have been calculated, the scaling parameters μ, Λ_1, Λ_2, can be uniquely determined by Eqs. (5.64).

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6 Global Approximations

Global (multipoint) approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). This type of approximation may require much computational effort, particularly in problems with large numbers of design variables. This difficulty can be alleviated by the approach presented in Chapter 7.

Polynomial fitting and *response surface* methods are introduced in Section 6.1. In response surface methods, the response functions are replaced by simple functions (polynomials), which are fitted to data computed at a set of selected design points. So far in practice, the use of these methods has been limited to problems with a few design variables.

Reduced-basis methods are presented in Section 6.2. Using this approach, we approximate the response of a large system, which is originally described by many degrees of freedom, by a linear combination of a few pre-selected basis vectors. The problem is then stated in terms of a small number of unknown coefficients of the basis vectors. This approach is most effective in cases where highly accurate approximations can be achieved by the reduced system of equations. A basic question in using reduced basis methods relates to the choice of an appropriate set of the basis vectors. Response vectors of previously analyzed designs could be used, but an *ad hoc* or intuitive choice of these vectors may not lead to satisfactory approximations. In addition, calculation of the basis vectors requires several exact analyses of the structure for the basis design points, which might involve extensive computational effort. A method for selecting the basis vectors that provides efficient and accurate results is presented in Section 7.1.1.

The conjugate gradient method described in Section 6.3 is an iterative method for solving a set of linear equations. The problem can be stated equivalently as the minimization of a quadratic function. The method generates a set of conjugate vectors such that the solution requires little storage and computation. If the quadratic function is minimized sequentially in an n dimensional space, once along each of a set of n conjugate directions, the minimum will be found at or before the nth step, regardless of the starting point. *A preconditioned conjugate gradient method,* intended to accelerate convergence of ill-conditioned problems by transformation of the set of linear equations, is then developed.

6.1 POLYNOMIAL FITTING AND RESPONSE SURFACE

When the number of design variables is small it might be practical to analyze the structure at a number of design points, and use the response at those points to construct a polynomial approximation to the response at other points. Polynomial approximations obtained by analyzing the structure at a number of design points are global approximations. Obtaining such approximations can be quite expensive for problems with large numbers of design variables. For example, if the object is to fit the structural response by a quadratic polynomial, it is necessary to analyze the structure for at least $n(n+1)/2$ design points (typically, many more are required to ensure a robust approximation), where n is the number of design variables [1].

The most common global approximation is the response surface approach. Using this approach, we compute the response functions at a number of points, and then fit an analytical response surface, such as a polynomial, to the data. Construction of a response surface often relies on the theory of experiments [2] and is an iterative process that begins with the assumption of the analytical form of the response surface, for example, a quadratic polynomial. The approximation contains a number of unknown parameters, such as polynomial coefficients, that must be adjusted to match the function to be approximated. Analyses are performed at a number of selected design points, and a least square solution is typically used to extract the parameter values from the analysis results. Then the response surface is used to predict the response at a number of selected test points, and statistical measures are used to assess the accuracy of the response surface. If the fit is not satisfactory, the process is repeated, and further experiments are made, or the model is improved by removing and/or adding terms.

Response surface techniques have not been used extensively in structural analysis (see [3] for applications in structural optimization). This may be due to the fact that the number of analyses required in constructing the response surface increases dramatically with the number of design variables. The method is practical only for problems with a small number of design variables.

6.1.1 Polynomial Fitting

Polynomial-fitting techniques are most efficient and tend to give solutions with sufficient accuracy when the response functions can be well approximated by low-order polynomials. Polynomial fitting is often used to obtain explicit approximations of the displacements along a line. While series expansions are based on a single exact analysis, polynomial fitting techniques usually require analyses or calculation of the displacement derivatives for several designs. Since more information is used in this class of approximate methods, compared with series expansion, the quality of the approximations is higher at the expense of more computational effort.

Quadratic Fitting. For quadratic fitting we assume that the function $f(\alpha)$ of the single variable α can be approximated by the quadratic function

$$Q(\alpha) = a + b\alpha + c\alpha^2 \tag{6.1}$$

The constant coefficients *a, b* and *c* can be determined by computing f_1, f_2, f_3, the value of $f(\alpha)$ at three different α values, α_1, α_2, α_3, and solving the equations

$$\begin{aligned} f_1 &= a + b\alpha_1 + c\alpha_1^2 \\ f_2 &= a + b\alpha_2 + c\alpha_2^2 \\ f_3 &= a + b\alpha_3 + c\alpha_3^2 \end{aligned} \tag{6.2}$$

If we use $\alpha_1 = 0$, $\alpha_2 = \Delta\alpha$, and $\alpha_3 = 2\Delta\alpha$, where $\Delta\alpha$ is a pre-selected step size, Eqs. (6.2) become

$$\begin{aligned} f_1 &= a \\ f_2 &= a + b\Delta\alpha + c(\Delta\alpha)^2 \\ f_3 &= a + 2b\Delta\alpha + 4c(\Delta\alpha)^2 \end{aligned} \tag{6.3}$$

Solving for *a, b,* and *c,* we find

$$a = f_1 \qquad b = \frac{4f_2 - 3f_1 - f_3}{2\Delta\alpha} \qquad c = \frac{f_3 + f_1 - 2f_2}{2(\Delta\alpha)^2} \tag{6.4}$$

Similarly, quadratic fitting of the displacement vector **r** can be approximated by

$$\mathbf{r}(\alpha) = \mathbf{a} + \mathbf{b}\alpha + \mathbf{c}\alpha^2 \tag{6.5}$$

where the constants **a, b, c** can be determined from results of analyses of three designs. Using for example results of exact analyses at $\alpha^* = 0$, $\alpha^{**} = 0.5$, $\alpha^{***} = 1$, substituting the corresponding values $\mathbf{r}^*, \mathbf{r}^{**}, \mathbf{r}^{***}$ into Eq. (6.5) and solving for **a, b, c** gives

$$\mathbf{r} = \mathbf{r}^* + (-3\mathbf{r}^* + 4\mathbf{r}^{**} - \mathbf{r}^{***})\,\alpha + (2\mathbf{r}^* - 4\mathbf{r}^{**} + 2\mathbf{r}^{***})\alpha^2 \tag{6.6}$$

Assuming conditions that are based on calculation of displacements and displacement derivatives at $\alpha^* = 0$, and displacements at $\alpha^{**} = 1$, i.e.,

$$\left.\begin{aligned} \mathbf{r}^* &= \mathbf{a} \\ \frac{\partial \mathbf{r}^*}{\partial \alpha} &= \mathbf{b} \end{aligned}\right\} \qquad \text{for } \alpha = \alpha^* = 0 \tag{6.7}$$

$$\mathbf{r}^{**} = \mathbf{a} + \mathbf{b} + \mathbf{c} \qquad \text{for } \alpha = \alpha^{**} = 1 \tag{6.8}$$

and substituting into Eq. (6.5), we find

$$\mathbf{r}(\alpha) = \mathbf{r}^{*} + \frac{\partial \mathbf{r}^{*}}{\partial \alpha}\alpha + \left(\mathbf{r}^{**} - \mathbf{r}^{*} - \frac{\partial \mathbf{r}^{*}}{\partial \alpha}\right)\alpha^{2} \tag{6.9}$$

Cubic Fitting. If the derivatives of f with respect to α, $df/d\alpha$, are readily computed, a two-point cubic interpolation can be used. The function $f(\alpha)$ is approximated by

$$C(\alpha) = a + b\alpha + c\alpha^{2} + d\alpha^{3} \tag{6.10}$$

The parameters a, b, c, and d can be determined by solving the following equations for points $\alpha = A$ and $\alpha = B$

$$\begin{aligned} f(A) &= f_A = a + bA + cA^2 + dA^3 \\ f(B) &= f_B = a + bB + cB^2 + dB^3 \\ (df/d\alpha)_A &= f'_A = b + 2cA + 3dA^2 \\ (df/d\alpha)_B &= f'_B = b + 2cB + 3dB^2 \end{aligned} \tag{6.11}$$

Assuming $A = 0$ and $B = \Delta\alpha$, and solving Eqs. (6.11), we find

$$\begin{aligned} a &= f_A \\ b &= f'_A \\ c &= -\frac{1}{\Delta\alpha}\left(\frac{3f_A - 3f_B}{\Delta\alpha} + 2f'_A + f'_B\right) \\ d &= \frac{1}{(\Delta\alpha)^2}\left(\frac{2f_A - 2f_B}{\Delta\alpha} + f'_A + f'_B\right) \end{aligned} \tag{6.12}$$

The constants for cubic fitting of the displacement vector $\mathbf{r}$

$$\mathbf{r}(\alpha) = \mathbf{a} + \mathbf{b}\alpha + \mathbf{c}\alpha^{2} + \mathbf{d}\alpha^{3} \tag{6.13}$$

can be determined in a similar way from the values of displacements and displacement derivatives at two points, $\mathbf{r}^{*}$, $\partial \mathbf{r}^{*}/\partial\alpha$ (for $\alpha^{*} = 0$) and $\mathbf{r}^{**}$, $\partial \mathbf{r}^{**}/\partial\alpha$ (for $\alpha^{**} = 1$).

Example 6.1 – Comparison of Various Approximations. Consider the three-bar truss shown in Figure 6.1. The modulus of elasticity is 30 000 and the cross-sectional areas are equal to unity. The geometric variables Y_1, Y_2 represent the location of the free node. Two cases of changes along a line are considered [4]:

Case *a*

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} + \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} \alpha \qquad 0 \le \alpha \le 1$$

Case *b*

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 100 \end{Bmatrix} \alpha \qquad 0 \le \alpha \le 1$$

Figure 6.2 shows the results obtained for the horizontal displacement for the two cases by the following methods:

- *A* = exact solution.
- *B* = first-order Taylor series [Eq. (5.2)].
- *C* = second-order Taylor series [Eq. (5.2)].
- *D* = quadratic fitting [Eq. (6.9)].
- *E* = quadratic fitting [Eq. (6.6)].
- *F* = cubic fitting [Eq. (6.13)].
- *G* = first-order binomial series [Eq. (5.5)].

It can be seen that the first-order approximations (methods *B* and *G*) provide poor results. Better accuracy is obtained by second-order approximations (method *C*), and the best results are achieved by the polynomial fitting techniques (methods *D, E, F*).

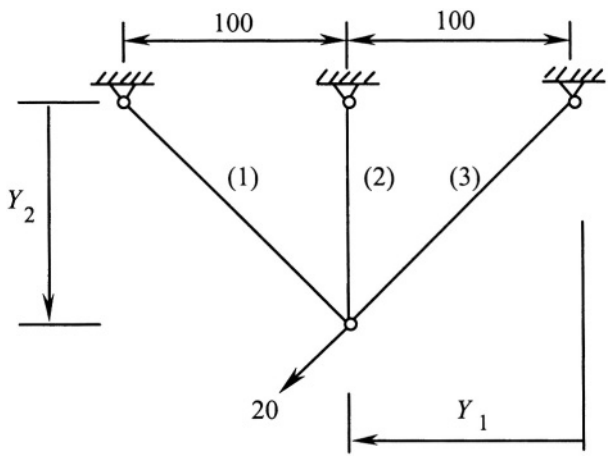

Fig. 6.1. Three-bar truss.

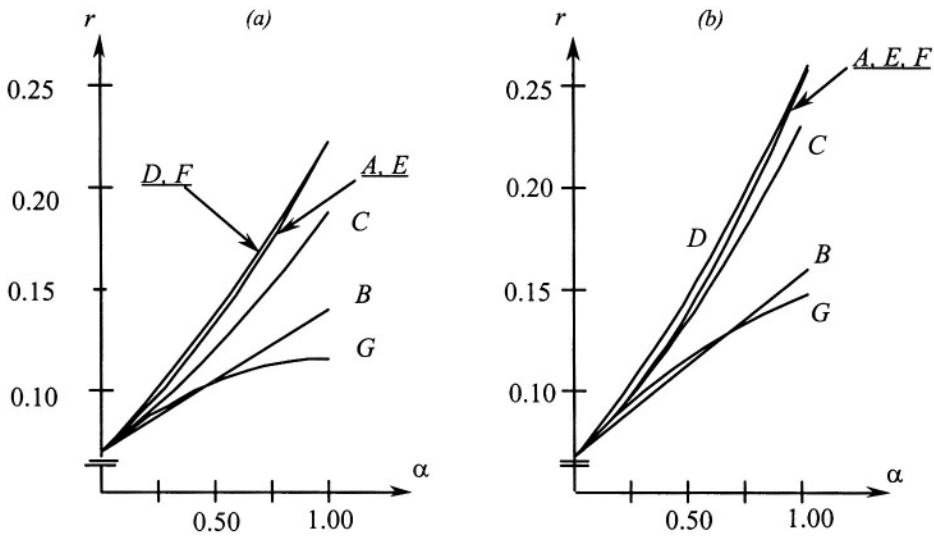

Fig. 6.2. Results, various approximations.

6.1.2 Least-Square Solutions

It is possible to evaluate the displacements by solving a reduced set of equations obtained from a least-square solution. Assume, for example, that the displacement vector $\mathbf{r}$ of a new design can be approximated by a linear combination of s displacement vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$ of previously analyzed designs (s is assumed to be much smaller than the number of degrees of freedom n)

$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \ldots + y_s\mathbf{r}_s = \mathbf{r}_B\,\mathbf{y} \tag{6.14}$$

where $\mathbf{r}_B$ is the $n \times s$ matrix of the displacement vectors and $\mathbf{y}$ is a vector of coefficients to be determined, defined as

$$\mathbf{r}_B = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s]$$

$$\mathbf{y}^T = \{y_1, y_2, \ldots, y_s\} \tag{6.15}$$

The errors in the equilibrium equations for the approximation of Eq. (6.14) are given by the residual load vector

$$\Delta\mathbf{R} = \mathbf{K}\,\mathbf{r} - \mathbf{R} = \mathbf{K}\,\mathbf{r}_B\,\mathbf{y} - \mathbf{R} \tag{6.16}$$

Following the least squared concept to minimize the square of the errors $\Delta\mathbf{R}$, we may write the quadratic form $q(\mathbf{y})$

$$q(\mathbf{y}) = \Delta\mathbf{R}^T\Delta\mathbf{R} \tag{6.17}$$

Substituting Eq. (6.16) into Eq. (6.17), differentiating with respect to $\mathbf{y}$ and setting the result equal to zero, we obtain the set of $s \times s$ linear equations

$$\mathbf{A}\,\mathbf{y} = \mathbf{b} \tag{6.18}$$

where the $s \times s$ matrix $\mathbf{A}$ and the $s \times 1$ vector $\mathbf{b}$ are given by

$$\mathbf{A} = (\mathbf{K}\,\mathbf{r}_B)^T(\mathbf{K}\,\mathbf{r}_B) \qquad \mathbf{b} = (\mathbf{K}\,\mathbf{r}_B)^T\mathbf{R} \tag{6.19}$$

The solution process to find $\mathbf{r}$ involves the following steps [5]:

a. The modified stiffness matrix $\mathbf{K}$ is introduced.
b. The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are determined by Eqs. (6.19).
c. The coefficients $\mathbf{y}$ are calculated by solving the set of Eq. (6.18).
d. The displacements $\mathbf{r}$ are evaluated by Eq. (6.14).

It should be noted that although $\Delta\mathbf{R}$ may be very small, the error in the solution might still be large. On the other hand, for an accurate solution $\Delta\mathbf{R}$ must be small. Therefore, a small residual $\Delta\mathbf{R}$ is a necessary but not a sufficient condition for an accurate solution. To obtain more information on the solution errors the corresponding residual displacements vector $\Delta\mathbf{r}$ is expressed as

$$\Delta\mathbf{r} = \mathbf{K}^{-1}\,\Delta\mathbf{R} \tag{6.20}$$

An analysis can be performed that uses the condition number of $\mathbf{K}$, defined by the ratio of the maximum and minimum eigenvalues, to evaluate the solution errors. It has been shown [6] that a large condition number means that solution errors are more likely.

Example 6.2 – Least-Square Solutions. Consider the ten-bar truss shown in Figure 6.3 subjected to two loads. The modulus of elasticity and the initial cross-section areas $\mathbf{X}^*$ are equal to unity. The eight unknowns are the horizontal and vertical displacements at joints 1, 2, 3 and 4, respectively. The stress constraints are $\mathbf{-25.0 \le \sigma \le 25.0}$, and the minimum size constraints are $\mathbf{0.1 \le X}$. Assuming the weight as an objective function, we find that the cross-section areas for the optimal (minimum weight) design are [7]

$$\mathbf{X}_{opt}^T = \{7.94, 0.1, 8.06, 3.94, 0.1, 0.1, 5.75, 5.57, 5.57, 0.1\}$$

The following two cases of modified designs relative to $\mathbf{X^*=1.0}$ are considered:

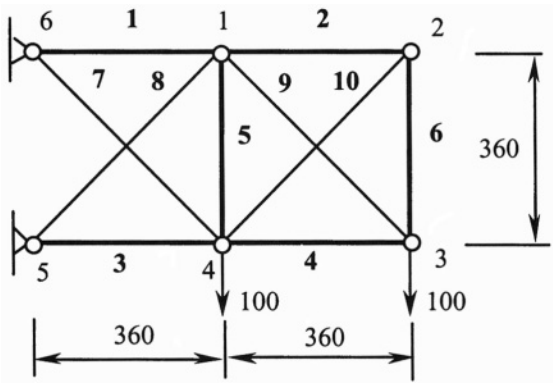

Fig. 6.3. Ten-bar truss.

a. The nearly scaled design of $\mathbf{X}^*$

$$\mathbf{X}^T = \{10, 10, 10, 10, 8, 8, 8, 8, 8, 8\}$$

b. The far-from-scaled optimal design $\mathbf{X}_{opt}$.

For second-order approximations [two terms of Eq. (6.14)] the displacements obtained by the Least-Square (LS) solutions are summarized in Table 6.1. Accurate results are obtained in case *a* for large changes (up to 900%) in the design variables. The high quality of the results can be explained by the fact that the modified design is relatively close to a scaled design. The results obtained in case *b* are less accurate, due to the nature of change in cross sections (simultaneous changes of up to +905% and -90%). High-order terms could be used to improve the accuracy.

Table 6.1. Second-order Least-Square (LS) approximations, ten-bar truss.

Case	Method	Displacements							
a	LS	19.5	53.2	23.5	115.5	-26.5	120.5	-20.5	57.5
	Exact	19.5	53.2	23.5	115.5	-26.5	120.5	-20.5	57.5
b	LS	22.8	67.4	35.6	172.5	-47.6	185.4	-24.6	75.9
	Exact	25.0	75.0	40.5	184.4	-50.0	200.0	-25.0	75.0

6.2 REDUCED BASIS

It has been noted that in many structural analysis and design problems, a large system of simultaneous equations must be solved repeatedly in order to evaluate the response of the structure. This process will involve much computational effort, particularly for large-scale, nonlinear and time-dependent (dynamic) problems.

The basic idea of the reduced-basis approach is that of transforming a problem with a large number of Degrees of Freedom (DOF) into one with a much smaller number of DOF. The response of the system, which was originally described by a large number of DOF, is approximated by a linear combination of a few pre-selected basis vectors. The problem is then stated in terms of a small number of unknown coefficients of the basis vectors. The resulting analysis model is more efficient, since only the corresponding small system of equations must be solved during reanalysis. This approach is most effective in cases where highly accurate approximations can be achieved by the reduced system of equations.

The solution process involves the following two main stages:

a. Generation of a number of basis vectors for approximating the response of the system.
b. Determination of the unknown coefficients of the basis vectors, using a variational technique.

The reduced-basis approach has been used successfully for various applications. The status and some developments of reduced-basis methods and their application to various mechanics problems are reviewed and summarized elsewhere [8]. Applications in several areas such as eigenvalue problems, nonlinear vibrations, and dynamic analysis are discussed in that review.

The reduced-basis approach for static analysis is presented in Section 6.2.1. We derive the reduced set of equations and discuss considerations for selecting the basis vectors. The more complicated problem of dynamic analysis is considered in Section 6.2.2. We review some reduced basis models and describe a method for generating the basis vectors.

6.2.1 Static Analysis

The Reduced Set of Equations. In the approach presented in this section [9], we assume that the displacement vector $\mathbf{r}$ of a new design can be approximated by a linear combination of pre-selected s linearly independent basis vectors, also called global approximation vectors, $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$ [see Eq. (6.14)]

$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \ldots + y_s\mathbf{r}_s = \mathbf{r}_B\,\mathbf{y} \tag{6.21}$$

where we assume that s is much smaller than the number of degrees of freedom n, $\mathbf{r}_B$ is the $n \times s$ matrix of the basis vectors and $\mathbf{y}$ is a vector of coefficients to be determined [Eqs.(6.15)]

$$\mathbf{r}_B = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s]$$

$$\mathbf{y}^T = \{y_1, y_2, \ldots, y_s\} \tag{6.22}$$

The space spanned by the global approximation vectors (matrix $\mathbf{r}_B$), is usually referred to as the reduced basis subspace.

The justification for using this approach is that the large number of degrees of freedom describing the response of the system is often dictated by such considerations as complex topology or numerous changes in the system properties, rather than by the complexity of the response. The power of the reduced basis method derives from the fact that for many systems of practical interest, the transformation of Eq. (6.21) can provide highly accurate approximations of $\mathbf{r}$, even when s is very much smaller than n.

The modified analysis equations (3.5) are now approximated by a smaller system of equations in the new unknowns $\mathbf{y}$. Substituting Eq. (6.21) into Eq. (3.5) and premultiplying the resultant equation by $\mathbf{r}_B^T$ gives the $s \times s$ system

$$\mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \mathbf{y} = \mathbf{r}_B^T \mathbf{R} \tag{6.23}$$

Introducing the notation

$$\begin{aligned} \mathbf{K}_R &= \mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \\ \mathbf{R}_R &= \mathbf{r}_B^T \mathbf{R} \end{aligned} \tag{6.24}$$

and substituting Eq. (6.24) into Eq. (6.23), we obtain

$$\mathbf{K}_R \mathbf{y} = \mathbf{R}_R \tag{6.25}$$

The $s \times s$ matrix $\mathbf{K}_R$ is full but is symmetric and much smaller in size than the $n \times n$ matrix $\mathbf{K}$ of the original system. That is, rather than computing the exact solution by solving the large $n \times n$ system in Eq. (3.5), we first solve the smaller $s \times s$ system in Eq. (6.25) for $\mathbf{y}$, and then evaluate the approximate displacements $\mathbf{r}$ for the computed $\mathbf{y}$ by Eq. (6.21).

The reduced set of Eqs. (6.25) can be obtained from the total potential energy expression

$$U = 1/2\, \mathbf{r}^T \mathbf{K} \mathbf{r} - \mathbf{r}^T \mathbf{R} \tag{6.26}$$

Substituting Eq. (6.21) into Eq. (6.26) gives

$$U = \mathbf{y}^T \mathbf{r}_B^T (1/2\, \mathbf{K} \mathbf{r}_B \mathbf{y} - \mathbf{R}) \tag{6.27}$$

Differentiating Eq. (6.27) with respect to $\mathbf{y}$, setting the result equal to zero and using the symmetry of $\mathbf{K}$, we obtain the following conditions for minimum potential energy

$$\mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \mathbf{y} - \mathbf{r}_B^T \mathbf{R} = \mathbf{0} \tag{6.28}$$

From Eq. (6.24), these conditions are equivalent to the reduced set of Eqs.(6.25).

Selecting the Basis Vectors. The reduced-basis method is often based on analyzing the structure at a number of points, and it may thus be classified as a global approximation method. The effectiveness of the method depends, to a great extent, on the appropriate choice of the basis vectors (the global approximation vectors) $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$, which span the reduced basis subspace. Displacement vectors of previously analyzed designs could be used, but it should be emphasized that an *ad hoc* or intuitive choice of such vectors may not lead to satisfactory approximations. In addition, calculation of the basis vectors requires several exact analyses of the structure for the basis design points, which might involve extensive computational effort.

Proper selection of the basis vectors is perhaps the most important factor affecting the successful application of the reduced-basis method. Of course, an ideal set of basis vectors will provide accurate results with a small computational effort. Specifically, the following criteria must be satisfied in arriving at the basis vectors to be used [10]:

- Linear independence.
- Low computational expense in their generation, and simplicity of automatic selection of their number.
- Good approximation properties, in the sense of high accuracy of the solution obtained by using these vectors.
- Simplicity of obtaining the system response characteristics using these vectors.

The first criterion is necessary for convergence of the approximation process and the latter three criteria govern the computational efficiency of the method and its effectiveness. The proper selection of the basis vectors depends on the system response characteristics being approximated, as well as the particular application.

6.2.2 Dynamic Analysis

From the viewpoint of computational effort, the reduction of degrees of freedom is more important in dynamic problems than in static problems, because the solution must be performed successively at many different times to generate the time history of the response. Since the discretized model of a complicated structural system may have many degrees of freedom, it is customary to reduce the equations of motion before the dynamic response is calculated.

In a mode-superposition analysis, the main computational effort is spent in solving the eigenproblem. Since exact solution of the eigenproblem can be prohibitively expensive, approximate solution techniques have been developed, primarily to calculate the lowest eigenvalues and corresponding eigenvectors, when the order of the system is large. Those degrees of freedom that are not required to appear in the global finite element model are eliminated by static condensation. In the calculation of frequencies and mode shapes, the basic assumption of static condensation is that the mass of the structure can be lumped at only some specific degrees of freedom without much effect on the accuracy of the results. The Rayleigh-Ritz analysis is a general technique for finding approximations to the lowest eigenvalues and corresponding eigenvectors of the eigenproblem.

A comprehensive review with numerous references on reduced-basis methods is given elsewhere [8]. The earliest applications of these methods have been to eigenvalue problems, and date back to the early 1960's. At that time, the calculation of the eigenvalues and eigenvectors of large systems by the available algorithms required much computational effort. The earliest reduction method applied to linear dynamic problems is the classical modal superposition technique, in which the global approximation vectors are selected to be linear vibration modes. In the analysis of linear structures the response is often expressed in terms of the undamped free vibration mode shapes, using only the lower modes. The main analytical problem then becomes the evaluation of the mode shapes, and the problem of reducing the number of degrees of freedom is transferred to this phase of the analysis.

The different reduced-basis methods for eigenproblems can be classified according to the selection of the basis vectors into two general categories:

a. Single step methods, where both the basis vectors and the reduced equations are generated at a single step and then used to evaluate the approximate eigenvectors and eigenvalues. Examples for this class of methods include Rayleigh-Ritz reduction (projection method), static condensation, and dynamic substructuring or component mode synthesis techniques.

b. Multi-step methods, or generalized reduction methods, in which the basis vectors and the reduced equations are modified in successive iterations. In some of these methods the initial matrix consists of very simple vectors. Examples of these methods include the Lanczos method and subspace iteration.

Two of the most widely used modal methods for transient structural analysis are the mode-displacement method and the mode-acceleration method. In these two methods the dynamic response is approximated by a linear combination of modal displacements and modal accelerations, respectively. To improve the convergence rate of modal methods, a higher-order modal method can be used. In an effort to avoid the cost associated with calculating the eigenmodes, other choices have been proposed for the reduced-basis subspace. These include Taylor subspace, in which the global approximation vectors are the various-order time derivatives of the response, Lagrange subspace, in which the response vectors at various times are selected as the approximation vectors, Runge-Kutta subspace and Krylov subspace. A comprehensive discussion and references on these and other methods are given elsewhere [8].

Using transformations similar to those of Eq. (6.21), we can express the total displacement vector $\mathbf{r}$ in terms of the free-vibration mode shapes $\mathbf{\Phi}_m$ as

$$\mathbf{r} = \sum_{m=1}^{n} \mathbf{\Phi}_m y_m = \mathbf{\Phi}\mathbf{y} \tag{6.29}$$

where n is the number of degrees of freedom, y_m is the modal amplitude, and the components of $\mathbf{y}$ are the normal coordinates of the structure.

Some problems may occur if exact eigenvectors are used in the mode superposition analysis. First, the solution of the eigenproblem for large structures is often the most costly phase of a dynamic response analysis. In addition, the number of eigenvectors required to obtain an accurate dynamic solution is not known until after the eigenproblem is solved.

The Reduced Problem. Considering again the equations of motion (2.51) and ignoring the notation of the time variable *(t)* we find

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R} \tag{6.30}$$

A standard model reduction process for the system of Eqs. (6.30) can be described as the simple coordinate transformation of Eq. (6.21). Substituting the latter equation into Eq. (6.30) and pre-multiplying the resultant equation by $\mathbf{r}_B^T$ we obtain

$$\mathbf{M}_R\ddot{\mathbf{y}} + \mathbf{K}_R\mathbf{y} = \mathbf{R}_R \tag{6.31}$$

where the reduced quantities are defined as

$$\begin{aligned} \mathbf{K}_R &= \mathbf{r}_B^T\mathbf{K}\mathbf{r}_B \\ \mathbf{M}_R &= \mathbf{r}_B^T\mathbf{M}\mathbf{r}_B \\ \mathbf{R}_R &= \mathbf{r}_B^T\mathbf{R} \end{aligned} \tag{6.32}$$

The transformation matrix $\mathbf{r}_B$ could be a modal matrix containing normal modes of the system. The model reduction process is then just the standard modal superposition technique (the mode-displacement method), and Eq. (6.31) is a set of uncoupled linear ordinary differential equations. More generally, $\mathbf{r}_B$ could contain general Ritz vectors or so-called "assumed modes" [11].

The Rayligh-Ritz approximation method (see e.g. [12]) has been used widely to reduce the dimension of the equations of motion. The method is based on the assumption that the displacement vector can be expressed in terms of a set of assumed shapes. The latter are taken as approximations of corresponding true vibration mode shapes $\mathbf{\Phi}_m$. *Orthogonal Ritz vectors,* which are not the eigenvectors of the system, have been used successfully in the reduction process [13]. The crucial step in the analysis is the choice of the Ritz coordinates, which must be efficient in the sense that a relatively small number of assumed shapes will be sufficient for reliable analysis of the dynamic response.

One approach to dealing with the problem of the adequacy of the assumed set of displacement vectors, known as subspace or simultaneous iteration, is to apply iterative improvement of the vectors. In this method, the trial vectors are all subjected to inverse iteration combined with some technique (such as Gram-Schmidt orthogonalization) that

forces convergence to independent shapes. The convergence is to the lowest undamped vibration mode shapes, and then coordinate transformation leads to an uncoupled set of modal equations. It frequently is assumed that this modal coordinate transformation is the most efficient method of dynamic response analysis because the independent modal equations can be solved separately and the total response is obtained by superposition (due to orthogonalization, all matrices are diagonal).

The *Krylov Sequence and Lanczos Method.* The basis vectors of the Krylov subspace consist of the following vectors

$$\mathbf{r}_1, \mathbf{D}\,\mathbf{r}_1, \mathbf{D}^2\mathbf{r}_1, \ldots, \mathbf{D}^{s-1}\mathbf{r}_1 \tag{6.33}$$

where s is the number of vectors considered, $\mathbf{r}_1$ is an arbitrary starting vector and matrix $\mathbf{D}$ is defined as $\mathbf{D} = \mathbf{K}^{-1}\mathbf{M}$ [see Eq. (2.65)]. When the starting vector in the Krylov subspace is selected to be the static response due to the given loads, the global approximation vectors are referred to as *Ritz vectors.* Various studies deal with the application of Ritz vectors to large eigenproblems and the assessment of the computational effort required for predicting the dynamic response of large systems by using different modal methods and the Lanczos method. Lanczos method dates back to 1950 [13]. It is equivalent to the discrete Rayleigh-Ritz reduction, with vectors in the Krylov sequence selected as the global approximation vectors. The method generates the latter vectors [Eq. (6.33)], and the sequence converges to the eigenvector corresponding to the smallest eigenvalue of the eigenproblem of Eq. (2.55).

The Lanczos coordinates are a very effective set of Ritz vectors, where the vectors are derived by a procedure that is similar in many respects to the matrix iteration analysis of the fundamental vibration modes [Eq. (2.66)]. In the derivation of the Lanczos coordinates, each step of the iteration sequence yields one Lanczos shape, whereas the standard matrix iteration procedure gives only the fundamental mode shape and the iteration serves only to improve the approximation to the true vibration shape.

The Lanczos algorithm involves supplementing the Krylov sequence with Gram-Schmidt orthogonalization process at each step. The result is a set of **M**-orthonormal vectors that is used to reduce the dimension of the dynamic equation set. These vectors do not have the full uncoupling property of the mode shapes, but they are much less expensive to generate.

Load Dependent Ritz Vectors. Ritz methods have long been used to approximate the dynamic response of structures and as model reduction techniques to reduce the size of large-scale models. The Ritz vectors are an attractive alternative to standard eigenvectors (normal modes), since Ritz vectors can be computed with significantly less computational effort. A proper choice of Ritz vectors, employed as starting vectors within some eigenvalue solvers, can significantly accelerate the iteration process of an eigenvalue solver.

Load dependent Ritz vectors are a particular class of Ritz vectors where information about the loading on the structure is used to generate the vectors. A procedure for

generating special Lanczos coordinates, called derived Ritz vectors [12, 14] is described in the following. The initial vector of the coordinate sequence is the deflected shape resulting from static application of the dynamic load distribution, and the subsequent vectors account for inertial effects on the dynamic response. The following notation is used in the derivation of the vectors:

- The preliminary deflected shape calculated as the first step in the derivation of each vector is denoted by $\mathbf{q}_i$, where the subscript is the number of the derived vector.
- After orthogonalization with the preceding vectors the vector is distinguished by a bar over the symbol.
- After normalization the final form of the derived vector is designated by the Ritz vector symbol $\boldsymbol{\psi}_i$.

The derivation of the vectors involves the following steps [12]. To derive the first vector, the static equilibrium equations

$$\mathbf{K}\,\mathbf{q}_1 = \mathbf{R} \tag{6.34}$$

are first solved for the preliminary deflected shape $\mathbf{q}_1$. In this solution, we use the banded form of **K.** The vector $\mathbf{q}_1$ is then normalized to obtain the first derived Ritz vector $\boldsymbol{\psi}_1$

$$\boldsymbol{\psi}_1 = \mathbf{q}_1 / \left(\mathbf{q}_1^T \mathbf{M} \mathbf{q}_1\right)^{1/2} \tag{6.35}$$

so that it provides a unit generalized mass, that is

$$\boldsymbol{\psi}_1^T \mathbf{M} \boldsymbol{\psi}_1 = 1 \tag{6.36}$$

Calculation of the second vector starts with solution of the equilibrium equations

$$\mathbf{K}\,\mathbf{q}_2 = \mathbf{M}\boldsymbol{\psi}_1 \tag{6.37}$$

to obtain the deflected shape $\mathbf{q}_2$ resulting from the inertial load $\mathbf{M}\boldsymbol{\psi}_1$ that is induced when the system is vibrating in the first vector shape $\boldsymbol{\psi}_1$. Then, the shape $\mathbf{q}_2$ is made mass orthogonal to the first vector $\boldsymbol{\psi}_1$ by the Gram-Schmidt procedure to obtain

$$\bar{\mathbf{q}}_2 = \mathbf{q}_2 - \left(\boldsymbol{\psi}_1^T \mathbf{M} \mathbf{q}_2\right) \boldsymbol{\psi}_1 \tag{6.38}$$

Finally, this shape is normalized to obtain the second derived Ritz vector

$$\boldsymbol{\psi}_2 = \bar{\mathbf{q}}_2 / \left(\bar{\mathbf{q}}_2^T \mathbf{M} \bar{\mathbf{q}}_2\right)^{1/2} \tag{6.39}$$

so that

$$\psi_2^T \mathbf{M} \psi_2 = 1 \tag{6.40}$$

Derivation of further vectors proceeds in essentially the same way. It has been shown [12] that when a vector is made orthogonal to the two preceding shapes, it automatically is orthogonal to all preceding shapes.

In summary, derivation of the $i+1$ vector ψ_{i+1} involves the following steps:

a. The deflected shape $\mathbf{q}_{i+1}$ is obtained by solving

$$\mathbf{K}\,\mathbf{q}_{i+1} = \mathbf{M}\,\psi_i \tag{6.41}$$

b. The shape $\mathbf{q}_{i+1}$ is made mass orthogonal with respect to the two preceding vectors by calculating

$$\bar{\mathbf{q}}_{i+1} = \mathbf{q}_{i+1} - \alpha_i \psi_i - \beta_i \psi_{i-1} \tag{6.42}$$

where

$$\alpha_i = \psi_i^T \mathbf{M} \mathbf{q}_{i+1} \tag{6.43}$$

$$\beta_i = \psi_{i-1}^T \mathbf{M} \mathbf{q}_{i+1} = \left(\bar{\mathbf{q}}_i^T \mathbf{M} \bar{\mathbf{q}}_i\right)^{1/2} \tag{6.44}$$

c. The shape is normalized to obtain ψ_{i+1} by calculting

$$\psi_{i+1} = \bar{\mathbf{q}}_{i+1} / \left(\bar{\mathbf{q}}_{i+1}^T \mathbf{M} \bar{\mathbf{q}}_{i+1}\right)^{1/2} \tag{6.45}$$

where the desired unit generalized mass is given by

$$\psi_{i+1}^T \mathbf{M} \psi_{i+1} = 1 \tag{6.46}$$

The derived Ritz vectors are used to perform a dynamic analysis in the same way as any other set of Ritz vectors. It has been shown that the unique orthogonality property of these Lanczos vectors makes it possible to organize the equations of motion in a special tri-diagonal form [12].

6.3 THE CONJUGATE GRADIENT METHOD

6.3.1 Solution Procedure

The Conjugate Gradient (CG) method can be used as an iterative method for solving the linear set of Eq. (3.5)

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \tag{6.47}$$

where $\mathbf{K}$ is an $n \times n$ symmetric and positive-definite matrix. This problem can be stated equivalently as minimization of the quadratic function

$$Q = 1/2\,\mathbf{r}^T\mathbf{K}\,\mathbf{r} - \mathbf{r}^T\mathbf{R} \tag{6.48}$$

A set of n nonzero vectors $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n$ is said to be conjugate with respect to a given $n \times n$ symmetric positive-definite matrix $\mathbf{K}$, if

$$\mathbf{S}_i^T\mathbf{K}\,\mathbf{S}_j = 0 \qquad \text{for all } i \neq j \tag{6.49}$$

It can be shown that any set of vectors satisfying this property is also linearly independent. A set of conjugate vectors possesses a powerful property [15]; namely, if the quadratic function Q is minimized sequentially, once along each of a set of n conjugate directions, the minimum of Q will be located at or before the nth step, regardless of the starting point. The order in which the directions are used is immaterial to this property.

There are many ways to choose the conjugate directions. The CG method is a conjugate directions method with a special property. In generating its set of conjugate vectors, it can compute a new vector by using only the previous vector and the current gradient. This property implies that the method requires little storage and computation. The method was developed in the 1950s as a method for finding exact solutions of symmetric positive definite systems [16]. Some years later the method came to be viewed as an iterative method that could give good approximate solutions to systems in much fewer than n steps [17, 18].

To solve the problem of Eq. (6.47) by the CG method, we start with the initial estimate $\mathbf{r}_0$ at the initial point $\mathbf{K} = \mathbf{K}_0$

$$\mathbf{r}_0 = \mathbf{K}_0^{-1}\,\mathbf{R} \tag{6.50}$$

The residual $\boldsymbol{\delta}_0$ of the linear system (6.47) is then calculated by

$$\boldsymbol{\delta}_0 = \mathbf{K}\,\mathbf{r}_0 - \mathbf{R} \tag{6.51}$$

The first direction of minimization, $\mathbf{S}_0$, is the *steepest descent direction* of the quadratic function Q [Eq. (6.48)], at the initial point $\mathbf{r}_0$, given by

$$\mathbf{S}_0 = -\delta_0 \tag{6.52}$$

A sequence of vectors $\mathbf{r}_k$ is then generated by the method. The vectors generated are given by the expression

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \alpha_k \mathbf{S}_k \tag{6.53}$$

where α_k is the one-dimensional minimizer of the quadratic function Q along $\mathbf{r}_k + \alpha_k \mathbf{S}_k$, given explicitly by

$$\alpha_k = \frac{\delta_k^T \delta_k}{\mathbf{S}_k^T \mathbf{K} \mathbf{S}_k} \tag{6.54}$$

The residual δ_{k+1} of the linear system of Eq. (6.47) is calculated by

$$\delta_{k+1} = \mathbf{K}\,\mathbf{r}_{k+1} - \mathbf{R} = \delta_k + \alpha_k\,\mathbf{K}\,\mathbf{S}_k \tag{6.55}$$

The $(k+1)$th direction, $\mathbf{S}_{k+1}$, is selected by

$$\mathbf{S}_{k+1} = -\delta_{k+1} + \beta_{k+1}\mathbf{S}_k \tag{6.56}$$

The scalar β_{k+1} is determined by the requirement that $\mathbf{S}_k$ and $\mathbf{S}_{k+1}$ must be conjugate with respect to $\mathbf{K}$ [see Eq. (6.49)]. By pre-multiplying Eq. (6.56) by $\mathbf{S}_k^T\mathbf{K}$ and imposing the condition $\mathbf{S}_k^T\mathbf{K}\mathbf{S}_{k+1} = 0$, we find the following expression for β_{k+1}

$$\beta_{k+1} = \frac{\delta_{k+1}^T \mathbf{K}\,\mathbf{S}_k}{\mathbf{S}_k^T \mathbf{K}\,\mathbf{S}_k} = \frac{\delta_{k+1}^T \delta_{k+1}}{\delta_k^T \delta_k} \tag{6.57}$$

In summary, for the calculated $\mathbf{r}_0$ [Eq. (6.50)], δ_0 [Eq. (6.51)] and $\mathbf{S}_0$ [Eq. (6.52)], the following calculations are carried out at each iteration:

- The scalar α_k is calculated by Eq. (6.54).
- The vector $\mathbf{r}_{k+1}$ is calculated by Eq. (6.53).
- The residual δ_{k+1} of the linear system of Eq. (6.47) is calculated by Eq. (6.55).
- The scalar β_{k+1} is calculated by Eq. (6.57).
- The direction $\mathbf{S}_{k+1}$ is calculated by Eq. (6.56).

Each search direction $\mathbf{S}_k$ and residual δ_k are constrained by the Krylov subspace of degree k for δ_0, defined as

$$\mathcal{D}(\delta_0, k) = \text{span}\,\{\delta_0, \mathbf{K}\delta_0, \ldots, \mathbf{K}^k \delta_0\} \tag{6.58}$$

The rate of convergence of the above procedure depends on the eigenvalue distribution of **K** and the initial approximation $\mathbf{r}_0$. The convergence is faster when the condition number of **K**, defined by the ratio of the maximum and minimum eigenvalues, is smaller and/or when **K** has clustered eigenvalues.

6.3.2 Preconditioned Conjugate Gradient

The CG properties are valid only in exact arithmetic. For ill-conditioned problems the convergence of the method might be slow, mainly due to round-off errors, when working with inexact arithmetic. For such problems, the conjugate directions are no longer exactly conjugate after some iteration cycles. It is possible to accelerate the rate of convergence by transformation of the linear system of Eqs. (6.47) such that the eigenvalue distribution of **K** is improved [19, 20]. The key to this process, which is known as preconditioning, is a change of variables from **r** to $\tilde{\mathbf{r}}$ via a nonsingular matrix **C** called the pre-conditioner

$$\mathbf{r} = \mathbf{C}\,\tilde{\mathbf{r}} \tag{6.59}$$

Substituting Eq. (6.59) into Eq. (6.47) and pre-multiplying the resulting equation by $\mathbf{C}^T$, we obtain the following new system of equations

$$(\mathbf{C}^T\mathbf{K}\,\mathbf{C})\,\tilde{\mathbf{r}} = \mathbf{C}^T\mathbf{R} \tag{6.60}$$

Define

$$\begin{aligned} \tilde{\mathbf{K}} &= \mathbf{C}^T\mathbf{K}\,\mathbf{C} \\ \tilde{\mathbf{R}} &= \mathbf{C}^T\,\mathbf{R} \end{aligned} \tag{6.61}$$

Substitution of Eqs. (6.61) into Eq. (6.60) gives the new system

$$\tilde{\mathbf{K}}\,\tilde{\mathbf{r}} = \tilde{\mathbf{R}} \tag{6.62}$$

Note that when **K** is symmetric and positive-definite and **C** has full rank, $\tilde{\mathbf{K}}$ is also symmetric and positive-definite. The convergence rate of the CG method applied to Eq. (6.62) will depend on the eigenvalues of the preconditioned matrix $\tilde{\mathbf{K}}$ rather than those of **K**. The aim is to choose the pre-conditioner **C** such that the condition number of $\tilde{\mathbf{K}}$ is much smaller than the condition number of the original matrix **K.** Alternatively, **C** could be chosen such that the eigenvalues of $\tilde{\mathbf{K}}$ are clustered.

It is possible to apply the CG method directly on the new system of Eq. (6.62). However, in this case it would be necessary to calculate and store the new (typically large) matrix $\tilde{\mathbf{K}}$ and to perform extensive computational effort at each iteration cycle.

Alternatively, the tilde notation can be used during the solution procedure applied to Eq. (6.62), also for the auxiliary vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{S}}_k$, where

$$\begin{aligned} \mathbf{r}_k &= \mathbf{C}\,\tilde{\mathbf{r}}_k \\ \mathbf{S}_k &= \mathbf{C}\,\tilde{\mathbf{S}}_k \end{aligned} \tag{6.63}$$

The relation between the residuals $\tilde{\boldsymbol{\delta}}_k$ and $\boldsymbol{\delta}_k$ is given by

$$\begin{aligned} \tilde{\boldsymbol{\delta}}_k &= \tilde{\mathbf{K}}\,\tilde{\mathbf{r}} - \tilde{\mathbf{R}} = \mathbf{C}^T\left(\mathbf{K}\mathbf{C}\mathbf{C}^{-1}\mathbf{r} - \mathbf{R}\right) = \mathbf{C}^T\left(\mathbf{K}\mathbf{r} - \mathbf{R}\right) = \mathbf{C}^T\boldsymbol{\delta}_k \\ \boldsymbol{\delta}_k &= (\mathbf{C}^T)^{-1}\,\tilde{\boldsymbol{\delta}}_k \end{aligned} \tag{6.64}$$

Define an auxiliary vector

$$\mathbf{z}_k = \mathbf{C}\,\mathbf{C}^T\,\delta_k \tag{6.65}$$

If no preconditioning is used, then $\mathbf{C}\mathbf{C}^T = \mathbf{I}$ and $\mathbf{z}_k = \boldsymbol{\delta}_k$. We can write the resulting Preconditioned Conjugate Gradient (PCG) method with the original variables $\mathbf{r}$, using modified formulas for α_k, β_{k+1} and $\mathbf{S}_{k+1}$

$$\alpha_k = \frac{\boldsymbol{\delta}_k^T\mathbf{z}_k}{\mathbf{S}_k^T\mathbf{K}\mathbf{S}_k} \tag{6.66}$$

$$\beta_{k+1} = \frac{\boldsymbol{\delta}_{k+1}^T\mathbf{z}_{k+1}}{\boldsymbol{\delta}_k^T\mathbf{z}_k} \tag{6.67}$$

$$\mathbf{S}_{k+1} = -\,\mathbf{z}_{k+1} + \beta_{k+1}\mathbf{S}_k \tag{6.68}$$

For the calculated $\mathbf{r}_0$ [Eq. (6.50)], $\boldsymbol{\delta}_0$ [Eq. (6.51)], $\mathbf{S}_0$ [Eq. (6.52)] and $\mathbf{z}_0$ [see Eq. (6.65)], each iteration cycle of the PCG method involves the following steps:

- The scalar α_k is calculated by Eq. (6.66).
- The vector $\mathbf{r}_{k+1}$ is calculated by Eq. (6.53).
- The residual $\boldsymbol{\delta}_{k+1}$ of the linear system (6.47) is calculated by Eq. (6.55).
- The vector $\mathbf{z}_{k+1}$ is calculated by $\mathbf{z}_{k+1} = \mathbf{C}\,\mathbf{C}^T\,\boldsymbol{\delta}_{k+1}$ [see Eq. (6.65)].
- The scalar β_{k+1} is calculated by Eq. (6.67).
- The direction $\mathbf{S}_{k+1}$ is calculated by Eq. (6.68).

Various pre-conditioners $\mathbf{C}$ have been proposed (e.g. [19]). The best choice of $\mathbf{C}$ could be the inverse factor $\mathbf{C} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is an upper triangular matrix given by the Cholesky factorization $\mathbf{K} = \mathbf{U}^T\mathbf{U}$. With this choice, the preconditioned matrix becomes equal to the identity matrix $\tilde{\mathbf{K}} = (\mathbf{U}^{-1})^T\mathbf{K}\,\mathbf{U}^{-1} = (\mathbf{U}^T)^{-1}(\mathbf{U}^T\mathbf{U})\,\mathbf{U}^{-1} = \mathbf{I}$. However, in this case much computational effort is needed to calculate matrix $\mathbf{C} = \mathbf{U}^{-1}$. Alternatively, matrix $\mathbf{C}$ can be chosen as [21]

$$\mathbf{C} = \mathbf{U}_0^{-1} \tag{6.69}$$

where $\mathbf{U}_0$ is an upper triangular matrix, given by the Cholesky factorization $\mathbf{K}_0 = \mathbf{U}_0^T\mathbf{U}_0$. In this case the only additional work, compared with the CG algorithm, is the computation of the auxiliary vector $\mathbf{z}_k$, i.e., the solution of the system of equations [see Eq. (6.65)]

$$\mathbf{K}_0\,\mathbf{z}_k = \boldsymbol{\delta}_k \tag{6.70}$$

for $\mathbf{z}_k$, which guarantees the preconditioning effect. Solution of Eq. (6.70) is easy, as we already dispose the Cholesky factor $\mathbf{U}_0$ of $\mathbf{K}_0$. The preconditioned matrix $\tilde{\mathbf{K}}$ for the chosen $\mathbf{C}$ is

$$\tilde{\mathbf{K}} = \mathbf{C}^T\mathbf{K}\,\mathbf{C} = (\mathbf{U}_0^{-1})^T\mathbf{K}\,\mathbf{U}_0^{-1} \tag{6.71}$$

It can be noted that for $\Delta\mathbf{K} = \mathbf{0}$ we obtain $\mathbf{K} = \mathbf{K}_0$ and the preconditioned matrix is equal to the identity matrix $\tilde{\mathbf{K}} = (\mathbf{U}_0^{-1})^T\,\mathbf{K}_0\,\mathbf{U}_0^{-1} = (\mathbf{U}_0^T)^{-1}(\mathbf{U}_0^T\,\mathbf{U}_0)\mathbf{U}_0^{-1} = \mathbf{I}$. If the eigenvalues of $\mathbf{K}$ are close to the eigenvalues $\mathbf{K}_0$, the preconditioned matrix is close to the identity matrix, and the convergence of the PCG is extremely fast. Obviously, the smaller or simpler (e.g., a rank-one change) is the change $\Delta\mathbf{K}$, the closer is $\mathbf{K}$ to $\mathbf{K}_0$.

It will be shown in Section 7.3.2 that the PCG method presented in this section and the CA method presented in Chapter 7 are equivalent [21].

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PART TWO

A UNIFIED APPROACH

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7 Combined Approximations (CA)

Given the initial stiffness matrix $\mathbf{K}^{\bullet}$ in the decomposed form of Eq. (3.2) and the initial displacement vector $\mathbf{r}^{\bullet}$, the problem considered in this chapter (see Section 3.1) is to estimate the modified displacements $\mathbf{r}$ due to changes in the design, without solving the complete set of modified analysis equations.

The Combined Approximations (CA) approach developed in this chapter is based on the integration of several concepts and methods, most of them presented in previous chapters. These include series expansion, reduced basis, matrix factorization and Gram-Schmidt orthonormalization. In the approach presented, the terms of the binomial series are used as high quality basis vectors in a reduced basis expression. The advantage is that efficient local approximations (series expansion) and accurate global approximations (the reduced basis method) are combined to achieve an effective solution procedure. Due to the nature of the selected basis vectors, high accuracy is often achieved by considering only a few vectors. Yet, the accuracy of the results can always be improved at the expense of more computational effort, by considering higher-order terms.

The main advantages of the CA approach, which can be studied in terms of several criteria, are as follows:

a. *Generality.* Various analysis models (linear, nonlinear, static, dynamic), different types of structures (trusses, frames, grillages, continuum structures) and all types of changes in the design (cross-sectional, geometrical, topological and material) may be considered. The changes can be of different extent, varying from changes in only a few variables to changes in all elements of the structure. As well, changes in the structural model itself may be considered,

b. *Accuracy of the results.* Accurate approximations are achieved for significant changes in the design of large structures. It is possible to improve the accuracy of the results by considering higher-order information. In certain cases exact solutions can be achieved.

c. *Efficiency.* Similar to local approximations, the calculations are based on results of a single exact analysis. The number of algebraic operations and the total CPU effort are usually much smaller than those needed to carry out complete analysis of the modified design.

d. *Flexibility.* The efficiency of the calculations and the accuracy of the results can be controlled by the level of simplification and the amount of information considered. Depending on the problem to be solved, various simplified versions of the approach may be considered. While simple approximations are often sufficient, better accuracy can always be achieved at the expense of more computational effort.

e. *Ease of implementation.* The method can readily be used with a general finite element system. Calculation of derivatives is not required and the solution steps are straightforward.

The CA approach is introduced in Section 7.1. Using the reduced-basis method described in Section 6.2, we transform a problem with a large number of degrees of freedom (DOF) into a problem with a much smaller number of DOF. Thereafter, only the corresponding small system of equations must be solved during reanalysis. The effectiveness of the method depends, to a great extent, on the appropriate choice of the basis vectors. In the approach presented, the first few terms of the binomial series expansion are considered as basis vectors. It is shown that this selection has several advantages in terms of generality, accuracy of the results, computational efficiency and ease of implementation.

It is shown in Section 7.2 that Gram-Schmidt orthogonalization with respect to **K**, generates a set of vectors for which the reduced set of analysis equations becomes uncoupled. The advantage then is that all expressions for evaluating the modified displacements become explicit functions of the design variables. As a result, additional terms can be considered without modifying the calculations that have been already carried out for previous terms. In addition, the uncoupled system is more well-conditioned.

Accurate solutions achieved by the CA approach are discussed in Section 7.3. The high accuracy achieved by the method with a small number of basis vectors for very large changes in the design is demonstrated and explained. It is shown that a Preconditioned Conjugate Gradient (PCG) method and the CA method provide identical results. Thus, various convergence criteria and error evaluation rules developed for PCG methods can be applied also for the CA method. Nearly exact solutions are obtained with a small number of basis vectors when the basis vectors come close to being linearly dependent. Such solutions are also achieved when the angle between the two vectors representing the initial design and the modified design is small, or when there is a low-rank change in the stiffness matrix.

7.1 COUPLED BASIS VECTORS

7.1.1 Determining the Basis Vectors

In the approach presented in this chapter the first few terms of the binomial series expansion are used as basis vectors. A typical series expansion of the displacements **r** can be expressed as

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \ldots \mathbf{r}_s \tag{7.1}$$

where $\mathbf{r}_i$ ($i = 1, ..., s$) are the series terms and s is the number of terms considered. Taylor series expansion is one of the most commonly used approximations in structural design and the terms of the series can be used as basis vectors. The major difficulty with this selection is that calculation of high-order derivatives is usually not practicable; therefore only linear (first-order) terms are considered, which limits the quality of the approximations. The first-order Taylor series terms have been used as basis vectors in some studies on the reduced-basis method (e.g. [1]).

It has been shown in Section 5.1.3 that the binomial series and the Taylor series are equivalent under certain circumstances. The advantage of using the binomial series is that, unlike the Taylor series, calculation of derivatives is not required. This makes the method more attractive in various applications where derivatives are not available. In addition, higher-order terms can readily be calculated.

The binomial series approximation is given by [Eq. (5.5)]

$$\mathbf{r} = (\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \mathbf{B}^3 + ...)\, \mathbf{r}_1 \tag{7.2}$$

where $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{B}$ and the vector $\mathbf{r}_1$ are defined by Eq. (5.6), and the basis vectors are calculated by the recurrence relation of Eq. (5.8)

$$\mathbf{B} = \mathbf{K}^{*-1}\, \Delta\mathbf{K}$$

$$\mathbf{r}_1 = \mathbf{K}^{*-1}\, \mathbf{R} \tag{7.3}$$

$$\mathbf{r}_i = -\mathbf{B}\, \mathbf{r}_{i-1} \qquad (i = 2, ..., s)$$

In many applications, the load vector is unchanged, that is $\Delta\mathbf{R} = \mathbf{0}$ and $\mathbf{R} = \mathbf{R}^*$. In such cases the first term $\mathbf{r}_1 = \mathbf{K}^{*-1}\mathbf{R}^* = \mathbf{r}^*$ is already given from initial analysis of the structure.

Matrix Factorization. In general, a square matrix $\mathbf{A}$ can be factorized into a product of two matrices

$$\mathbf{A} = \mathbf{L}\, \mathbf{U} \tag{7.4}$$

where the factors are triangular matrices. Specifically, $\mathbf{U}$ is an upper triangular matrix having pivots on its diagonal and $\mathbf{L}$ is a lower triangular matrix having 1's on its diagonal. Thus, the $\mathbf{L}\, \mathbf{U}$ factorization of Eq. (7.4) is unsymmetrical.

Divide $\mathbf{U}$ by a diagonal matrix $\mathbf{D}$ that contains the pivots (it is convenient to keep the same letter $\mathbf{U}$ for this new upper triangular matrix that has 1's on its diagonal). Then, the triangular factorization can be written as

$$\mathbf{A} = \mathbf{L}\, \mathbf{D}\, \mathbf{U} \tag{7.5}$$

where both $\mathbf{L}$ and $\mathbf{U}$ have 1's on their diagonal.

If $\mathbf{A}$ is a symmetric matrix, then $\mathbf{U} = \mathbf{L}^T$ and the factorization of Eq. (7.5) is also symmetric, that is

$$\mathbf{A} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T \tag{7.6}$$

Finally, in cases where $\mathbf{A}$ is a symmetric positive-definite matrix (that is, $\mathbf{X}^T\mathbf{A}\,\mathbf{X} > 0$ for every nonzero vector), then the factorization Eq. (7.6) can be written as the Choleski factorization $\mathbf{A} = \mathbf{L}\,\mathbf{L}^T$, or

$$\mathbf{A} = \mathbf{U}^T\mathbf{U} \tag{7.7}$$

where $\mathbf{U}$ is an upper triangular matrix.

Calculation of the Series Terms. It is shown in the following that calculation of the basis vectors $\mathbf{r}_i$ by Eq. (7.3) involves only forward and backward substitutions if $\mathbf{K}^*$ is given from the initial analysis in the decomposed form of Eq. (3.2) $\mathbf{K}^* = \mathbf{U}^{*T}\mathbf{U}^*$. The vector $\mathbf{r}_2$, for example, is calculated from

$$\mathbf{K}^*\mathbf{r}_2 = \mathbf{U}^{*T}\mathbf{U}^*\mathbf{r}_2 = -\Delta\mathbf{K}\,\mathbf{r}_1 \tag{7.8}$$

We first solve for the vector of unknowns $\mathbf{t}$ by the forward substitution

$$\mathbf{U}^{*T}\mathbf{t} = -\Delta\mathbf{K}\,\mathbf{r}_1 \tag{7.9}$$

$\mathbf{r}_2$ is then calculated by the backward substitution

$$\mathbf{U}^*\mathbf{r}_2 = \mathbf{t} \tag{7.10}$$

Similarly, the kth term $\mathbf{r}_k$ is calculated from

$$\mathbf{K}^*\mathbf{r}_k = -\Delta\mathbf{K}\,\mathbf{r}_{k-1} \tag{7.11}$$

7.1.2 Solution Procedure

Given the initial stiffness matrix $\mathbf{K}^*$ in the decomposed form of Eq. (3.2), the initial load vector $\mathbf{R}^*$ and the initial displacement vector $\mathbf{r}^*$, calculation of the modified displacements $\mathbf{r}$ by the CA method for any assumed changes $\Delta\mathbf{K}$ in the stiffness matrix and changes $\Delta\mathbf{R}$ in the load vector involves the following steps:

a. Calculate the modified stiffness matrix $\mathbf{K}$ and load vector $\mathbf{R}$ by Eqs. (3.3) and (3.4). Since the initial values $\mathbf{K}^*$ and $\mathbf{R}^*$ are already given, this step involves only calculation of $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$.

b. Calculate the basis vectors $\mathbf{r}_i$ $(i = 1, ..., s)$ by [Eqs. (7.3)]

$$\mathbf{r}_1 = \mathbf{K}^{*-1}\,\mathbf{R}$$
$$\mathbf{r}_i = -\mathbf{B}\,\mathbf{r}_{i-1} \qquad (i = 2, ..., s) \tag{7.12}$$

It has been shown in Section 7.1.1 that calculation of the basis vectors involves only forward and backward substitutions.

For large changes in the design, the elements of the basis vectors become very large due to large $\Delta\mathbf{K}$ values. To overcome numerical round off errors, it is possible to normalize a basis vector $\mathbf{r}_i$ by dividing it by an arbitrary reference element of the vector (say, the first element r_{1i}) to obtain the normalized vector $\mathbf{r}_{Ni}$

$$\mathbf{r}_{Ni} = \mathbf{r}_i / r_{1i} \tag{7.13}$$

This operation scales the first element of the vector to unity. It does not change the final solution, as shown in step *e* below.

c. Calculate the reduced stiffness matrix $\mathbf{K}_R$ and load vector $\mathbf{R}_R$ by Eqs. (6.24)

$$\mathbf{K}_R = \mathbf{r}_B^T \mathbf{K} \mathbf{r}_B$$
$$\mathbf{R}_R = \mathbf{r}_B^T\,\mathbf{R} \tag{7.14}$$

This calculation is straightforward and does not involve much computational effort.

d. Calculate the vector of coefficients $\mathbf{y}$ by solving the $s \times s$ system in Eqs. (6.25)

$$\mathbf{K}_R\,\mathbf{y} = \mathbf{R}_R \tag{7.15}$$

In cases where the number of basis vectors s is much smaller than the number of degrees of freedom n, it is necessary to solve only the smaller $s \times s$ system in Eq. (7.15) for $\mathbf{y}$ instead of computing the exact solution by solving the large $n \times n$ system in Eq. (3.5).

e. Evaluate the final displacements by [Eq.(6.21)]

$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + ... + y_s\mathbf{r}_s = \mathbf{r}_B\,\mathbf{y} \tag{7.16}$$

Equation (7.16) shows that the transformation of Eq. (7.13) does not change the final solution (but only the corresponding scalars y_i).

The above solution procedure is most effective in cases of low-order approximations (e.g. two or three basis vectors). Good accuracy is often achieved with only a small number of basis vectors, as is shown in this chapter by numerical examples.

Example 7.1 – Evaluation of Displacements, Forces and Stresses. In this example low-order approximations of displacements, stresses and forces for structural optimization are demonstrated. Consider the ten-bar truss shown in Figure 7.1. The design variables $\mathbf{X}$ are the member cross-sectional areas, the initial cross section areas $\mathbf{X}^*$ are all unity, the modulus of elasticity is 30000, and the eight analysis unknowns are the horizontal and vertical displacements at joints 1, 2, 3 and 4, respectively. The stress constraints are $-25.0 \le \sigma \le 25.0$, the minimum size constraints are $0.001 \le \mathbf{X}$ and the objective function represents the weight. The optimal (minimum weight) design is [2]

$$\mathbf{X}_{opt}^T = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

The line from the initial design to the optimal design is given by

$$\mathbf{X} = \mathbf{X}^* + \alpha \, \Delta\mathbf{X}^*$$

where α is the step size variable, and $\Delta\mathbf{X}^*$ is the direction vector defined as

$$\Delta\mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

Consider the modified design corresponding to $\alpha = 0.4$ with cross section areas

$$\mathbf{X}^T = \{3.8, 0.6, 3.8, 2.2, 0.6, 0.6, 2.867, 2.867, 2.867, 0.6\}$$

Results obtained with 2, 3, and 4 basis vectors are summarized in Tables 7.1, 7.2 and 7.3. The forces and the stresses are computed by the force-displacement and stress-displacement relations [Eqs. (2.10) and (2.11)]. It was found that acurate results are obtained with 4 basis vectors. Only two exact analyses are needed to reach the optimum when 3 basis vectors are used for reanalysis.

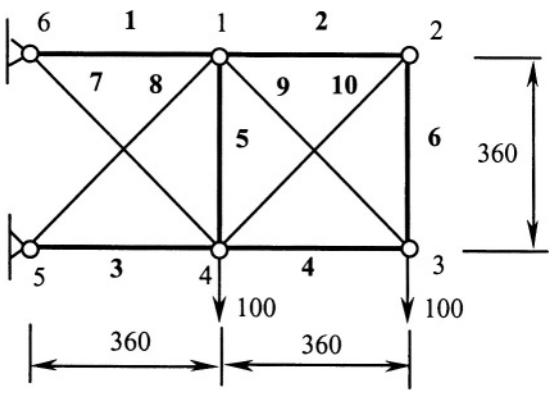

Fig. 7.1. Ten-bar truss.

Table 7.1. Displacements found by CA and exact methods for $\alpha = 0.4$.

Basis Vectors	Displacements							
2	0.59	1.78	0.84	4.14	-1.04	4.38	-0.64	1.98
3	0.61	1.71	0.88	4.20	-1.12	4.47	-0.65	1.91
4	0.61	1.73	0.89	4.21	-1.12	4.49	-0.65	1.90
Exact Solution	0.61	1.73	0.89	4.21	-1.12	4.49	-0.65	1.90

Table 7.2. Forces found by CA and exact methods for $\alpha = 0.4$.

Basis Vectors	Forces									
2	188.0	12.1	-201.3	-73.7	9.9	12.1	160.5	-141.8	115.5	-17.1
3	193.5	13.4	-206.6	-85.8	10.0	13.4	150.6	-131.6	122.6	-19.0
4	194.5	13.7	-205.5	-86.3	8.2	13.7	149.2	-133.6	122.0	-19.4
Exact Solution	194.5	13.7	-205.5	-86.3	8.2	13.7	149.2	-133.6	122.0	-19.4

Table 7.3. Stresses found by CA and exact methods for $\alpha = 0.4$.

Basis Vectors	Stresses									
2	49.4	20.2	-53.0	-33.5	16.6	20.2	56.0	-49.5	40.3	-28.6
3	50.9	22.3	-54.4	-39.0	16.7	22.3	52.5	-45.9	42.8	-31.6
4	51.2	23.0	-54.1	-39.2	13.7	23.0	52.0	-46.6	42.5	-32.4
Exact Solution	51.2	23.0	-54.1	-39.2	13.7	23.0	52.0	-46.6	42.5	-32.4

7.2 UNCOUPLED BASIS VECTORS

7.2.1 Determining the Basis Vectors

The Gram-Schmidt orthogonalization procedure can be used to generate a new set of basis vectors such that the reduced set of analysis equations [Eq, (7.15)] becomes uncoupled with respect to **K**. This procedure has been used for nonlinear analysis [3] and linear reanalysis of structures [4]. For any assumed number of basis vectors, the results obtained by considering the reduced set of equations, for either the original set of basis vectors or the new set of uncoupled vectors, are identical. The advantage in using the latter vectors is that all expressions for evaluating the displacements become

explicit functions of the design variables. As a result, additional vectors can be considered without modifying the calculations that already were carried out for previous terms. It is shown in this section how the new set of uncoupled basis vectors is generated.

It can be observed from Eq. (7.14) that the elements K_{Rij} of the reduced stiffness matrix $\mathbf{K}_R$ are given by

$$K_{R_{ij}} = \mathbf{r}_i^T \mathbf{K} \mathbf{r}_j \tag{7.17}$$

The object is to transform the reduced system of Eqs. (7.15) into an uncoupled set of equations. This can be done by generating a set of new vectors $\mathbf{V}_i$ $(i = 1, 2, \ldots, s)$, from the original vectors $\mathbf{r}_i$, such that for any two vectors $\mathbf{V}_i$ and $\mathbf{V}_j$

$$\mathbf{V}_i^T \mathbf{K} \mathbf{V}_j = \delta_{ij} \tag{7.18}$$

where δ_{ij} is the kronecker delta, for which

$$\begin{aligned} \delta_{ij} &= 0 \ (i \neq j) \\ \delta_{ii} &= 1 \end{aligned} \tag{7.19}$$

We say that the vectors $\mathbf{V}_i$ and $\mathbf{V}_j$ are orthogonal with respect to $\mathbf{K}$ if the condition of Eq. (7.18) is satisfied. The new basis vectors, which are linear combinations of the original vectors, are generated as follows.

The first normalized vector $\mathbf{V}_1$ is chosen as

$$\mathbf{V}_1 = | \mathbf{r}_1^T \mathbf{K} \mathbf{r}_1 |^{-1/2} \mathbf{r}_1 \tag{7.20}$$

To generate the second normalized vector $\mathbf{V}_2$, we first define the non-normalized vector $\overline{\mathbf{V}}_2$, which is a linear combination of $\mathbf{V}_1$ and $\mathbf{r}_2$, by

$$\overline{\mathbf{V}}_2 = \mathbf{r}_2 - \alpha \mathbf{V}_1 \tag{7.21}$$

where α is chosen such that the orthogonality condition [see Eq.(7.18)]

$$\overline{\mathbf{V}}_2^T \mathbf{K} \mathbf{V}_1 = 0 \tag{7.22}$$

is satisfied. Substituting Eq. (7.21) into the condition of Eq. (7.22) yields

$$\overline{\mathbf{V}}_2^T \mathbf{K} \mathbf{V}_1 = \mathbf{r}_2^T \mathbf{K} \mathbf{V}_1 - \alpha \mathbf{V}_1^T \mathbf{K} \mathbf{V}_1 = 0 \tag{7.23}$$

Since $\mathbf{V}_1^T \mathbf{K} \mathbf{V}_1 = 1$ [Eq. (7.20)], then Eq. (7.23) becomes

$$\alpha = \mathbf{r}_2^T \mathbf{K} \mathbf{V}_1 \tag{7.24}$$

Substituting Eq. (7.24) into Eq. (7.21) gives

$$\overline{\mathbf{V}}_2 = \mathbf{r}_2 - \left(\mathbf{r}_2^T \mathbf{K} \mathbf{V}_1\right) \mathbf{V}_1 \tag{7.25}$$

Finally, normalizing $\overline{\mathbf{V}}_2$ we obtain the second normalized vector

$$\mathbf{V}_2 = | \overline{\mathbf{V}}_2^T \mathbf{K} \overline{\mathbf{V}}_2 |^{-1/2} \overline{\mathbf{V}}_2 \tag{7.26}$$

Additional basis vectors are generated in a similar way. The resulting general expressions for all $i = 2,..., s$ vectors are

$$\begin{aligned} \overline{\mathbf{V}}_i &= \mathbf{r}_i - \sum_{j=1}^{i-1} \left(\mathbf{r}_i^T \mathbf{K} \mathbf{V}_j\right) \mathbf{V}_j \\ \mathbf{V}_i &= | \overline{\mathbf{V}}_i^T \mathbf{K} \overline{\mathbf{V}}_i |^{-1/2} \overline{\mathbf{V}}_i \end{aligned} \tag{7.27}$$

where $\overline{\mathbf{V}}_i$ and $\mathbf{V}_i$ are the ith non-normalized and normalized vectors, respectively.

Now, we will show how the new basis vectors $\mathbf{V}_i$ are used to evaluate the displacements. It is observed [Eq. (7.18)] that for the basis vectors $\mathbf{V}_i$ $(i = 1, 2, ..., s)$, the diagonal elements of the new reduced stiffness matrix equal unity and all other elements equal zero. That is, the new reduced stiffness matrix is the identity matrix **I.** Define the matrix of new basis vectors $\mathbf{V}_B$ and the vector of new coefficients $\mathbf{z}$ by

$$\begin{aligned} \mathbf{V}_B &= [\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_s] \\ \mathbf{z}^T &= \{z_1, z_2, \ldots, z_s\} \end{aligned} \tag{7.28}$$

Rather then the reduced system of Eq. (7.15) we have [see Eq. (7.14)]

$$\mathbf{I}\, \mathbf{z} = \mathbf{z} = \mathbf{V}_B^T \mathbf{R} \tag{7.29}$$

As expected, this system is uncoupled, and the coefficients $\mathbf{z}$ can be determined directly by Eq. (7.29). The final displacements are given by the explicit expression [see Eq.(7.16)]

$$\mathbf{r} = \mathbf{V}_B \mathbf{z} = \mathbf{V}_B \left(\mathbf{V}_B^T \mathbf{R}\right) \tag{7.30}$$

The displacements calculated by Eq. (7.30) can be expressed as an additively separable quadratic function of the basis vectors $\mathbf{V}_i$ by

$$\mathbf{r} = \sum_{i=1}^{s} \mathbf{V}_i \left(\mathbf{V}_i^T \mathbf{R} \right) \tag{7.31}$$

One advantage in using the new vectors is that all expressions for evaluating the displacements are explicit functions of the original basis vectors, and therefore, explicit functions of the design variables. This can be seen from the following expressions:

- The stiffness matrices $\Delta\mathbf{K}$, $\mathbf{K}$ and load vectors $\Delta\mathbf{R}$, $\mathbf{R}$ are explicit functions of the design variables.
- The basis vectors $\mathbf{r}_i$ are explicit functions of $\Delta\mathbf{K}$ and $\mathbf{R}$ [Eqs. (7.3)].
- The new basis vectors $\mathbf{V}_i$ are explicit functions of $\mathbf{r}_i$ and $\mathbf{K}$ [Eqs. (7.20) and (7.27)].
- The final displacements $\mathbf{r}$ are explicit functions of $\mathbf{V}_i$ and $\mathbf{R}$ [Eq. (7.31)].

In summary, calculation of any new basis vector $\mathbf{V}_i$ leads to an additional term in the displacements expression [Eq. (7.31)]. As a result, additional vectors can be considered without modifying the calculations that already were carried out.

It will be shown later by example 7.2 that, while the normalized vectors $\mathbf{V}_i$ are of similar magnitude, the values of the z_i coefficients and, therefore, the corresponding terms of the series of Eq. (7.31) are gradually decreased. Transformation of the binomial series terms [Eq. (7.3)] into the terms of the CA series [Eq. (7.31)] provides accurate solutions even in cases where the binomial series diverges.

7.2.2 *Solution Procedure*

Calculation of the modified displacements $\mathbf{r}$ for any assumed changes $\Delta\mathbf{K}$ in the stiffness matrix and changes $\Delta\mathbf{R}$ in the load vector involves the following steps:

a, b. These steps are identical to those described in Section 7.1.2 for the original basis vectors.

c. Generate the normalized basis vectors. The first vector is given by Eq. (7.20)

$$\mathbf{V}_1 = | \mathbf{r}_1^T \mathbf{K} \mathbf{r}_1 |^{-1/2} \mathbf{r}_1 \tag{7.32}$$

The additional normalized vectors $\mathbf{V}_i$ $(i = 2, ..., s)$ are generated by Eqs. (7.27)

$$\begin{aligned} \overline{\mathbf{V}}_i &= \mathbf{r}_i - \sum_{j=1}^{i-1} \left(\mathbf{r}_i^T \mathbf{K} \mathbf{V}_j \right) \mathbf{V}_j \\ \mathbf{V}_i &= | \overline{\mathbf{V}}_i^T \mathbf{K} \overline{\mathbf{V}}_i |^{-1/2} \overline{\mathbf{V}}_i \end{aligned} \tag{7.33}$$

where $\overline{\mathbf{V}}_i$ and $\mathbf{V}_i$ are the ith non-normalized and normalized vectors, respectively.

d. Evaluate the displacements by Eq. (7.31)

$$\mathbf{r} = \sum_{i=1}^{s} \mathbf{V}_i \left(\mathbf{V}_i^T \mathbf{R} \right) \tag{7.34}$$

The accuracy of the results for a specific number s of basis vectors can be evaluated by the methods described in Section 7.3.3. If the accuracy is insufficient, additional basis vectors are introduced by steps *b* and *c* and the updated displacements are evaluated by step *d*. Equations (7.33) and (7.34) show that additional vectors can be considered without modifying the calculations that were carried out already.

7.3 ACCURATE SOLUTIONS

7.3.1 Linearly Dependent Basis Vectors

As noted in Section 6.2.1, linear independence of the basis vectors is necessary for convergence of the reduced-basis approximations. The vector of approximate displacements [Eq. (7.16)] is a linear combination of the basis vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$. The latter vectors are said to be linearly independent if the relation

$$y_1 \mathbf{r}_1 + y_2 \mathbf{r}_2 + \ldots + y_s \mathbf{r}_s = \mathbf{0} \tag{7.35}$$

can be satisfied only for the trivial case, that is, only for the case where all the coefficients $y_1, y_2, \ldots, y_s$ are identically zero. If the relation is satisfied and at least one of the coefficients is different from zero, then the basis vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s$ are said to be linearly dependent, with the implication that one vector is a linear combination of the remaining vectors.

It is shown in the following that an exact solution is obtained by the CA method in cases where a newly created vector becomes a linear combination of the previous vectors [5]. For simplicity of presentation assume that $\mathbf{R} = \mathbf{R}^*$, and therefore, $\mathbf{r}_1 = \mathbf{r}^*$. To achieve a convenient expression for the exact solution of the modified design, premultiply Eq. (3.5) by $\mathbf{K}^{*-1}$ and substitute Eqs. (3.1) into the resulting equation to obtain

$$(\mathbf{I} + \mathbf{B})\, \mathbf{r} = \mathbf{r}^* \tag{7.36}$$

where matrix $\mathbf{B}$ is defined by Eq. (7.3). Premultipling Eq. (7.36) by $(\mathbf{I} + \mathbf{B})^{-1}$ gives the exact modified displacements

$$\mathbf{r} = (\mathbf{I} + \mathbf{B})^{-1} \mathbf{r}^* \tag{7.37}$$

To obtain a convenient expression for the approximate displacements in terms of the assumed s basis vectors, substitute the expressions of the basis vectors [Eq. (7.3)] into Eq. (7.16). We obtain

$$\mathbf{r} = y_1 \mathbf{r}^* - y_2 \mathbf{B} \mathbf{r}^* + y_3 \mathbf{B}^2\mathbf{r}^* - \ldots + y_s \mathbf{B}^{s-1}\mathbf{r}^* \tag{7.38}$$

Assuming that the approximate solution of Eq. (7.38) involving s terms is equal to the exact solution of Eq. (7.37), premultiplying both equations by $(\mathbf{I} + \mathbf{B})$ and rearranging we obtain the linear expression for an additional term $\mathbf{r}_{s+1}$

$$\mathbf{r}_{s+1} = \sum_{i=1}^{s} a_i \mathbf{r}_i \tag{7.39}$$

where a_i are scalar multipliers given by

$$\begin{aligned} a_1 &= (y_1 - 1)/y_s \\ a_i &= (y_i - y_{i-1})/y_s \qquad i = 2,\ 3,\ \ldots,\ s \end{aligned} \tag{7.40}$$

Equation (7.39) shows that when the reduced basis expression with s terms [Eq. (7.38)] is equal to the exact solution, then the $(s+1)$th basis vector is a linear combination of the previous s vectors. That is, the $s+1$ basis vectors are linearly dependent.

In general the CA method provides approximate solutions, but accurate solutions are often achieved with only a small number of basis vectors. It is expected that accurate (nearly exact) solutions will be achieved when the high-order basis vectors come close to being linearly dependent on all previous vectors. Two basis vectors $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ are close to being linearly dependent if

$$\cos \beta_{i,\,i+1} = (\mathbf{r}_i^T \mathbf{B}^* \mathbf{r}_i) / (|\mathbf{r}_i|\,|\mathbf{B}^*\mathbf{r}_i|) \approx 1 \tag{7.41}$$

where $\beta_{i,\,i+1}$ is the angle between the two vectors. Various numerical examples show that the basis vectors determined by the CA method satisfy the condition of Eq. (7.41), as the basis vectors index i is increased, even for very large changes in the design.

7.3.2 Equivalence of the CA Method and the PCG Method

The Conjugate Gradient (CG) method and the Preconditioned Conjugate Gradient (PCG) method were presented in Section 6.3. It is shown in this section that the PCG method and the CA method provide identical results [6] if matrix $\mathbf{C}$ is chosen as [see Eq. (6.69)]

$$\mathbf{C} = \mathbf{U}_0^{-1} \tag{7.42}$$

where $\mathbf{U}_0$ is an upper triangular matrix, given by the Cholesky factorization of the initial stiffness matrix $\mathbf{K}_0 = \mathbf{U}_0^T\mathbf{U}_0$. Applying k iterations of the CG method to Eq. (6.47) is equivalent to the minimization of the quadratic function Q [Eq. (6.48)] on the Krylov subspace of degree k, defined as

$$\mathcal{D}_k = \mathbf{r}_0 + \text{span}\ \{\delta_0, \mathbf{K}\,\delta_0, \mathbf{K}^2\delta_0, \ldots, \mathbf{K}^{k-1}\,\delta_0\} \tag{7.43}$$

where the residual vector δ_0 is given by Eq. (6.51). In the particular case where the CG method is applied to the preconditioned system [Eq. (6.62)] with $\mathbf{C} = \mathbf{U}_0^{-1}$ [Eq. (7.42)], the quadratic function

$$\tilde{Q} = 1/2\ \tilde{\mathbf{r}}^T\,\tilde{\mathbf{K}}\,\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^T\,\tilde{\mathbf{R}} \tag{7.44}$$

is minimized in the kth iteration on the Krylov subspace

$$\tilde{\mathcal{D}}_k = \tilde{\mathbf{r}}_0 + \text{span}\left\{\tilde{\delta}_0,\ \tilde{\mathbf{K}}\,\tilde{\delta}_0,\ \tilde{\mathbf{K}}^2\tilde{\delta}_0,\ \ldots,\ \tilde{\mathbf{K}}^{k-1}\,\tilde{\delta}_0\right\} \tag{7.45}$$

Denote $\mathbf{P} = \Delta\mathbf{K}\,\mathbf{K}_0^{-1}$. Assuming the initial point $\tilde{\mathbf{r}}_0 = 0$, then [see Eq. (6.61)] $\tilde{\delta}_0 = \tilde{\mathbf{R}} - \tilde{\mathbf{K}}\,\tilde{\mathbf{r}}_0 = \mathbf{C}^T\mathbf{R}$. Since $\mathbf{C}\mathbf{C}^T = \mathbf{K}_0^{-1}$, it is observed that [see Eqs. (3.3), (6.61)]

$$\begin{aligned}\tilde{\mathbf{K}}\,\tilde{\delta}_0 &= \mathbf{C}^T(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{C}\mathbf{C}^T\mathbf{R} = \mathbf{C}^T\mathbf{R} + \mathbf{C}^T\mathbf{P}\,\mathbf{R}\\ \tilde{\mathbf{K}}^2\,\tilde{\delta}_0 &= \mathbf{C}^T\mathbf{R} + 2\mathbf{C}^T\mathbf{P}\,\mathbf{R} + \mathbf{C}^T\mathbf{P}^2\mathbf{R}\end{aligned} \tag{7.46}$$

and so on, so that

$$\tilde{\mathcal{D}}_k = \mathbf{C}^{\mathrm{T}}\text{span}\left\{\mathbf{R},\ \mathbf{P}\,\mathbf{R},\ \mathbf{P}^2\mathbf{R},\ \ldots,\ \mathbf{P}^{k-1}\mathbf{R}\right\} \tag{7.47}$$

Hence the minimizer of $\tilde{Q}$ [Eq. (7.44)] on $\tilde{\mathcal{D}}_k$ [Eq. (7.47)] is of the form

$$\tilde{\mathbf{r}}_k = \mathbf{C}^{\mathrm{T}}\left(\alpha_0\ \mathbf{R} + \alpha_1\ \mathbf{P}\,\mathbf{R} + \alpha_2\mathbf{P}^2\,\mathbf{R} + \ldots + \alpha_{k-1}\mathbf{P}^{k-1}\,\mathbf{R}\right) \tag{7.48}$$

To return to the original variables, pre-multiply Eq. (7.48) [and thus Eq. (7.47)] by $\mathbf{C}$ to obtain [see Eq. (6.63)]

$$\mathcal{D}_k = \text{span}\left(\mathbf{K}_0^{-1}\,\mathbf{R},\ \mathbf{K}_0^{-1}\,\mathbf{P}\,\mathbf{R},\ \mathbf{K}_0^{-1}\mathbf{P}^2\mathbf{R},\ \ldots,\ \mathbf{K}_0^{-1}\mathbf{P}^{k-1}\mathbf{R}\right) \tag{7.49}$$

$$\mathbf{r}_k = \mathbf{K}_0^{-1}\left(\alpha_0\,\mathbf{R} + \alpha_1\ \mathbf{P}\,\mathbf{R} + \alpha_2\mathbf{P}^2\mathbf{R} + \ldots + \alpha_{k-1}\mathbf{P}^{k-1}\mathbf{R}\right) \tag{7.50}$$

In summary, in the *k*th iteration of the PCG method applied to Eq. (6.47), with the preconditioned matrix of Eq. (7.42) and the initial approximation $\tilde{\mathbf{r}}_0 = \mathbf{0}$, we minimize the quadratic function Q [Eq. (6.48)] on the Krylov subspace of Eq. (7.49). In the CA method, we introduce an $n \times k$ matrix $\mathbf{r}_B$ by means of vectors $\mathbf{r}_i$ defined recurrently by Eq. (7.3). The approximate solution of Eq. (6.47) is then obtained by solving the reduced system of Eq. (7.15), and interpolating by Eq. (7.16). But this is just the same as minimizing the quadratic functional Q from Eq. (6.48) on a subspace defined by Eq. (7.49). The conclusion is that the solution found by the CA method with an $n \times k$ matrix $\mathbf{r}_B$ is fully equivalent to k iterations of the PCG method with the preconditioned matrix of Eq. (7.42) and the initial value $\tilde{\mathbf{r}}_0 = \mathbf{0}$.

7.3.3 Error Evaluation

In this section convergence properties of the CG method and the CA (PCG) method are discussed. We introduce upper bound on errors for the CA method based on previously developed criteria for evaluating the errors in the CG method.

The CG Method. In exact arithmetic the CG method will terminate at the solution in at most n iterations. What is more remarkable is that when the distribution of the eigenvalues of $\mathbf{K}$ has certain favorable features, the method will identify the solution in much less than n iterations. In particular, it has been shown [e.g. 7, 8] that if $\mathbf{K}$ has only m distinct eigenvalues, then the CG method will terminate at the solution in at most m iterations. In addition, if the eigenvalues of $\mathbf{K}$ occur in m distinct clusters, the CG iterates will approximately solve the problem after m steps.

Define the usual energy norm

$$\| \mathbf{r} \|_{\mathbf{K}} = \left(\mathbf{r}^T \mathbf{K}\, \mathbf{r} \right)^{1/2} \tag{7.51}$$

If $\mathbf{K}$ has eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, the following estimate gives a useful characterization of the convergence behavior of the CG method [9]

$$\| \mathbf{r}_{k+1} - \mathbf{r}_{ex} \|_{\mathbf{K}}^2 \le \left(\frac{\lambda_{n-k} - \lambda_1}{\lambda_{n-k} + \lambda_1} \right)^2 \| \mathbf{r}_0 - \mathbf{r}_{ex} \|_{\mathbf{K}}^2 \tag{7.52}$$

Another, more approximate, convergence expression for the CG method is

$$\| \mathbf{r}_k - \mathbf{r}_{ex} \|_{\mathbf{K}} \le 2 \left(\frac{\sqrt{\psi(\mathbf{K})} - 1}{\sqrt{\psi(\mathbf{K})} + 1} \right)^k \| \mathbf{r}_0 - \mathbf{r}_{ex} \|_{\mathbf{K}} \tag{7.53}$$

where $\psi(\mathbf{K})$ is the Euclidean condition number of $\mathbf{K}$, defined by the ratio of the maximum and minimum eigenvalues

$$\psi(\mathbf{K}) = \lambda_{max} / \lambda_{min} \tag{7.54}$$

This bound often gives a large overestimate of the error, but it can be useful when the only information we have about **K** is estimates of the extreme eigenvalues.

The CA (PCG) Method. The series of basis vectors [Eq. (7.2)] converges if and only if [10]

$$\lim_{k \to \infty} \mathbf{B}^k = \mathbf{0} \tag{7.55}$$

which in turn holds if and only if $\rho(\mathbf{B}) < 1$, where $\rho(\mathbf{B})$ is the spectral radius (the largest eigenvalue) of matrix **B**.

To evaluate the errors involved in the binomial series approximations, we see that the sum of the additional terms in the series of Eq. (7.2), beyond the first *s* terms,

$$\Delta \mathbf{r} = \mathbf{B}^s \mathbf{r}_0 - \mathbf{B}^{s+1} \mathbf{r}_0 + \mathbf{B}^{s+2} \mathbf{r}_0 - \ldots \tag{7.56}$$

can be expressed as

$$\Delta \mathbf{r} = \mathbf{B}^s (\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \ldots) \mathbf{r}_0 \tag{7.57}$$

This expression is bounded from above by [11]

$$\Delta \mathbf{r} \le \| \mathbf{B} \|^s \frac{1}{1 - \| \mathbf{B} \|} \| \mathbf{r}_0 \| \tag{7.58}$$

Evidently, for large changes $\Delta \mathbf{K}$ (and corresponding large elements of **B**) this bound may become very large.

In the CA method, we solve the problem [Eqs. (7.14), (7.15)]

$$\mathbf{r}_B{}^T \mathbf{K} \mathbf{r}_B \mathbf{y} = \mathbf{r}_B{}^T \mathbf{R} \tag{7.59}$$

It was previously noted that the CA solution with *k* basis vectors is equivalent to *k* iterations of the PCG method. Applying results from the CG method, the error bound for *k* basis vectors in the CA method (equivalent to *k* steps in the CG method) is given by an expression similar to Eq. (7.53), with the preconditioned matrix $\tilde{\mathbf{K}} = (\mathbf{C}^T \mathbf{K} \mathbf{C})$ replacing **K**,

$$\| \mathbf{r}_k - \mathbf{r}_{ex} \|_{\tilde{K}} \le 2 \left(\frac{\sqrt{\psi(\tilde{\mathbf{K}})} - 1}{\sqrt{\psi(\tilde{\mathbf{K}})} + 1} \right)^k \| \mathbf{r}_0 - \mathbf{r}_{ex} \|_{\tilde{K}} \tag{7.60}$$

where

$$\| \mathbf{r} \|_{\tilde{\mathbf{K}}} = \left(\mathbf{r}^T \tilde{\mathbf{K}} \mathbf{r} \right)^{1/2} \tag{7.61}$$

Consider the approximate displacements expressed in the uncoupled form of Eq. (7.31). The errors in the results for a specific number of basis vectors s can be evaluated by assessing the size of the elements of the sth term

$$\mathbf{r}^{(s)} = \mathbf{V}_s \left(\mathbf{V}_s^T \mathbf{R} \right) \tag{7.62}$$

If the solution process converges, the size of the elements of the vector $\mathbf{r}^{(s)}$ in Eq. (7.62) can be used as a convergence criterion, namely

$$E_r \equiv \frac{\| \mathbf{r}^{(s)} \|}{\| \sum_{i=1}^{s} \mathbf{r}^{(i)} \|} = \frac{\| \mathbf{V}_s (\mathbf{V}_s^T \mathbf{R}) \|}{\| \sum_{i=1}^{s} \mathbf{V}_i (\mathbf{V}_i^T \mathbf{R}) \|} \leq E_r^U \tag{7.63}$$

where E_r^U is a small number and $\| \ \|$ is the Euclidean norm. Since the normalized vectors $\mathbf{V}_i$ are of similar magnitude whereas the values of the z_i coefficients are gradually decreased, an alternative convergence criterion is

$$E_z \equiv \frac{|z_s|}{\left| \sum_{i=1}^{s} z_i \right|} = \frac{\left| \mathbf{V}_s^T \mathbf{R} \right|}{\left| \sum_{i=1}^{s} \mathbf{V}_i^T \mathbf{R} \right|} \leq E_z^U \tag{7.64}$$

where E_z^U is again a small number.

7.3.4 Scaled and Nearly Scaled Designs

Scaled Designs. It is shown in this section that the basis vectors are linearly dependent in cases where the modified design is a scaled initial design. It was noted in Section 5.3 that scaling of the initial design is carried out by multiplying the initial stiffness matrix $\mathbf{K}^*$ by a positive scaling multiplier μ to obtain the modified matrix [Eq. (5.43)]

$$\mathbf{K} = \mu \mathbf{K}^* \tag{7.65}$$

For $\mathbf{R} = \mathbf{R}^*$ the exact displacements after scaling are given by [Eq. (5.44)]

$$\mathbf{r} = \mu^{-1} \mathbf{r}^* \tag{7.66}$$

In general, the elements of $\mathbf{K}$ are some nonlinear functions of the design variables. The condition of Eq. (7.65) represents only an algebraic operation and it does not require

linear dependence of **K** on the design variables. For the case where the modified design is given by the scaled stiffness matrix $\mu\mathbf{K}^*$ [Eq. (7.65)], the binomial series matrix **B** becomes

$$\mathbf{B} = \mathbf{K}^{*-1}\,\Delta\mathbf{K} = (\mu - 1)\,\mathbf{I} \tag{7.67}$$

where **I** is the identity matrix. It is observed that the resulting basis vectors [Eqs. (7.3)] become linearly dependent, that is

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}^* \\ \mathbf{r}_2 &= -(\mu - 1)\,\mathbf{r}^* \\ \mathbf{r}_3 &= (\mu - 1)^2\,\mathbf{r}^* \ldots \text{ etc.} \end{aligned} \tag{7.68}$$

Thus, consideration of the single basis vector $\mathbf{r}^*$ with a coefficient $y_1 = \mu^{-1}$ will provide the exact solution as given by Eq. (7.66).

Nearly Scaled Designs. Assume a change in the design so that the modified design can be expressed as

$$\mathbf{X} = \mathbf{X}^* + \Delta\mathbf{X} = \mathbf{X}^* + \alpha\,\Delta\mathbf{X}^* \tag{7.69}$$

where $\mathbf{X}^*$ is the vector of initial design variables, $\Delta\mathbf{X}$ is the vector of design changes, $\Delta\mathbf{X}^*$ is the direction of change and α is a scalar describing the magnitude of change. Both the direction of change and the magnitude of change may significantly affect the accuracy of the approximations. This effect can be quantified by the cosine of the angle θ between the vector of the modified design and the vector of initial design

$$\cos\theta = (\mathbf{X}^T\mathbf{X}^*) / (|\mathbf{X}|\,|\mathbf{X}^*|) \tag{7.70}$$

where $|\mathbf{X}|$ denotes the absolute value of **X**. Figure 7.2 shows that various designs, obtained by scaling a certain modified design, provide identical θ angles. For example, the two modified designs *A* at $\mathbf{X} = \mathbf{X}^* + 0.1\Delta\mathbf{X}^*$ and *D* at $\mathbf{X} = 10\,(\mathbf{X}^* + 0.1\Delta\mathbf{X}^*)$ correspond to an identical θ. It has been found [5] that high accuracy is achieved with a small number of basis vectors for designs *A* (representing a small change in the design) and *D* (representing a very large change in the design), as both correspond to a small θ value. More basis vectors are needed for designs *B* and *C* that correspond to larger θ. These observations are limited to the space formed by the vectors $\mathbf{X}^*$ and **X**. For the complete design space, smaller θ values do not always guarantee better approximations.

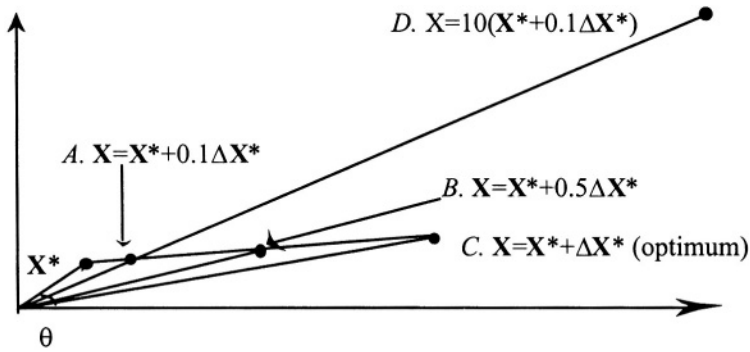

Fig. 7.2. Various modified designs shown in the space of $\mathbf{X}^*$ and $\Delta\mathbf{X}^*$.

Example 7.2 - Large Changes in the Design. The object of this example is to demonstrate the accuracy of the results achieved by the CA method for very large changes in the design. Consider again the classic ten-bar truss shown in Figure 7.1, subjected to a single loading condition of two concentrated loads. The design variables $\mathbf{X}$ are the member cross-sectional areas, the initial cross section areas $\mathbf{X}^*$ are all unity, the modulus of elasticity is 30000, and the eight analysis unknowns are the horizontal and vertical displacements at joints 1, 2, 3 and 4, respectively. The stress constraints for all ten members are $\mathbf{-25.0 \le \sigma \le 25.0}$, and the minimum size constraints are $\mathbf{0.001 \le X}$. Assuming the weight as an objective function, we find the optimal (minimum weight) design [2]

$$\mathbf{X}_{opt}^{T} = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

The line from the initial design to the optimal design is given by

$$\mathbf{X} = \mathbf{X}^* + \alpha\, \Delta\mathbf{X}^*$$

where $\Delta\mathbf{X}^*$ is defined as

$$\Delta\mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

For $\alpha = 1.0$ (the optimum) the change in the design is very large: members 1 and 3 are increased by 700%, member 4 is increased by 300%, members 7, 8, 9 are increased by 467%, and the topology is changed by effectively eliminating members 2, 5, 6, 10 and, therefore, joint 2 (displacements 3 and 4). Only two exact analyses are needed to reach the optimum when 3 basis vectors are used for reanalysis.

To illustrate the effect of various design changes on the accuracy of the results, three typical cases are considered (see designs *A, B* and *C* in Figure 7.2 and Table 7.4):

a. Small change in the design (up to -10% and +70%), $\alpha = 0.1$, $\theta = 14^0$. The modified design is given by $\mathbf{X} = \mathbf{X}^* + 0.1\Delta\mathbf{X}^*$ (Design *A*).

b. Large change in the design (up to -50% and +350%), $\alpha = 0.5$, $\theta = 34^0$. The modified design is given by $\mathbf{X} = \mathbf{X}^* + 0.5\Delta\mathbf{X}^*$ (Design *B*).

c. Very large change in the design (up to -99% and +700%), $\alpha = 1.0$, $\theta = 41^0$ (the optimum). The modified design is given by $\mathbf{X} = \mathbf{X}^* + \Delta\mathbf{X}^*$ (Design *C*).

Results obtained by the Conjugate Gradient (CG) and the CA methods are given in Tables 7.5, 7.6. For the given direction $\Delta\mathbf{X}^*$, the number of iterations needed to achieve a certain accuracy increases with α. For two-digit accuracy, 8 iterations are needed for convergence by the CG method. For the CA (PCG) method, only 3 basis vectors (iterations) are needed for $\alpha = 0.1$; 4 vectors are needed for $\alpha = 0.5$; and 5 vectors are needed for $\alpha = 1$. Define the error at every iteration step as

$$E_k = \frac{\| \mathbf{r}_k - \mathbf{r}_{ex} \|}{\| \mathbf{r}_{ex} \|}$$

where

$$\| \mathbf{r} \| = \sqrt{\Sigma (r_i)^2} \qquad \mathbf{r}_{ex} = \mathbf{K}^{-1}\mathbf{R}$$

The errors in the displacements for the various cases are shown in Table 7.7.

Table 7.8 shows that the condition numbers of $\tilde{\mathbf{K}}$ are much smaller than those of the corresponding $\mathbf{K}$. The larger the changes in the design are (larger α values), the larger are the condition numbers of both $\mathbf{K}$ and $\tilde{\mathbf{K}}$. In addition, the matrices $\tilde{\mathbf{K}}$ for various α values have clustered eigenvalues.

Table 7.4. Modified cross sections for various modified designs of the ten-bar truss.

Design	*A*	*B*	*C*
α	0.1	0.5	1.0
1	1.700	4.500	8.000
2	0.900	0.500	0.001
3	1.700	4.500	8.000
4	1.300	2.500	4.000
5	0.900	0.500	0.001
6	0.900	0.500	0.001
7	1.467	3.333	5.667
8	1.467	3.333	5.667
9	1.467	3.333	5.667
10	0.900	0.500	0.001

Consider again the results For design C ($\alpha = 1$). Figure 7.3 shows how the norm of the terms $\mathbf{r}_i$ of the series of basis vectors (the binomial series) is increased and the series diverges. Figure 7.4 shows the norms of the uncoupled basis vectors $\mathbf{V}_i$, and Figure 7.5 shows the coefficients z_i. While the norms of $\mathbf{V}_i$ are of similar order of magnitude, the coefficients z_i are gradually decreased. Thus, the CA terms $\mathbf{V}_i z_i$ are also gradually decreased. Figure 7.6 shows how the norm of the terms $\mathbf{V}_i z_i$ decreases and the series converges as the number of basis vectors is increased.

The values of $\cos \beta_{i,i+1}$ obtained for design C [see Eq. (7.41)] are $\cos \beta_{1,2} = 0.9989$, $\cos \beta_{2,3} = 0.9992$, $\cos \beta_{3,4} = 0.9994$, $\cos \beta_{4,5} = 0.9997$. Since the basis vectors determined by the CA method are close to being linearly dependent, accurate results are obtained even for very large changes in the design.

Table 7.5. Displacements found by the CG method.

Number of Iterations	2	3	4	5	6	7	8	Exact Solution
$\alpha = 0.1$	0.59	0.26	0.46	1.18	1.32	1.35	1.37	1.37
	1.62	2.87	3.15	3.22	3.50	3.54	3.56	3.56
	-0.36	-0.30	0.12	0.84	1.65	1.75	1.77	1.77
	1.39	3.24	6.03	7.59	8.22	8.25	8.25	8.25
	-0.59	-0.74	-1.26	-1.64	-2.19	-2.15	-2.10	-2.10
	2.48	4.17	6.08	7.64	8.67	8.63	8.65	8.65
	-0.23	-0.92	-0.84	-1.47	-1.49	-1.48	-1.45	-1.45
	2.11	3.05	3.80	3.95	3.85	3.90	3.89	3.89
$\alpha = 0.5$	0.30	0.10	0.38	0.41	0.44	0.52	0.52	0.52
	0.42	0.90	1.31	1.35	1.37	1.49	1.49	1.49
	-0.04	-0.09	-0.28	-0.25	0.28	0.77	0.77	0.77
	0.17	0.57	2.09	2.49	3.53	3.65	3.64	3.64
	-0.30	-0.56	-0.84	-0.81	-1.01	-0.97	-0.98	-0.98
	1.20	1.77	3.16	3.46	3.67	3.89	3.89	3.89
	-0.25	-0.38	-0.31	-0.49	-0.58	-0.55	-0.55	-0.55
	1.16	1.35	1.35	1.36	1.70	1.62	1.62	1.62
$\alpha = 1.0$	0.24	0.13	0.19	0.26	0.30	0.30	0.30	0.30
	0.24	0.50	0.64	0.71	0.90	0.90	0.90	0.90
	0.00	0.00	0.00	0.00	0.00	-0.09	0.49	0.49
	0.00	0.00	0.00	0.00	0.00	2.04	2.21	2.21
	-0.24	-0.50	-0.64	-0.71	-0.60	-0.60	-0.60	-0.60
	1.08	1.41	2.06	2.28	2.39	2.40	2.40	2.40
	-0.24	-0.31	-0.41	-0.34	-0.30	-0.30	-0.30	-0.30
	-1.08	1.23	1.01	0.94	0.90	0.90	0.90	0.90

The main conclusions drawn from this example are as follows:

a. The reduced basis coefficients can significantly change the convergence properties of the series of basis vectors (the binomial series). Accurate results are obtained by the CA method even in cases where the series of basis vectors diverges.

b. For any given direction vector $\Delta \mathbf{X}^*$, the magnitude of α determines the value of θ. For $\theta = 0$, an exact solution is achieved by scaling the initial displacements. For a given number of basis vectors and direction of change $\Delta \mathbf{X}^*$, the accuracy of the results depends on α. The larger α is, the larger is the angle θ and the more basis vectors will be needed to achieve adequate accuracy.

Table 7.6. Displacements found by the CA method.

Number of Basis Vectors	2	3	4	5	Exact Solution
$\alpha = 0.1$	1.36	1.37			1.37
	3.59	3.56			3.56
	1.76	1.77			1.77
	8.23	8.25			8.25
	-2.06	-2.10			-2.10
	8.62	8.65			8.65
	-1.44	-1.45			-1.45
	3.92	3.89			3.89
$\alpha = 0.5$	0.50	0.52	0.52		0.52
	1.53	1.46	1.49		1.49
	0.71	0.76	0.77		0.77
	3.56	3.63	3.64		3.64
	-0.89	-0.98	-0.98		-0.98
	3.77	3.87	3.89		3.89
	-0.54	-0.55	-0.55		-0.55
	1.70	1.64	1.62		1.62
$\alpha = 1.0$	0.28	0.29	0.29	0.30	0.30
	0.90	0.84	0.88	0.90	0.90
	0.41	0.45	0.47	0.49	0.49
	2.10	2.17	2.19	2.21	2.21
	-0.53	-0.61	-0.62	-0.60	-0.60
	2.24	2.34	2.37	2.40	2.40
	-0.30	-0.31	-0.31	-0.30	-0.30
	1.01	0.95	0.93	0.90	0.90

Table 7.7. Errors for CA and CG methods.

	$\mathbf{K}_R$ size	2	3	4	5	6	7	8
α=0.1	CA	0.01	0					
	CG	0.74	0.54	0.30	0.12	0.01	0.01	0
α=0.5	CA	0.03	0.01	0				
	CG	0.79	0.66	0.35	0.27	0.10	0	0
α=1.0	CA	0.07	0.03	0.02	0			
	CG	0.76	0.70	0.64	0.63	0.63	0.17	0

Table 7.8. Eigenvalues (λ) and Condition Number (ψ) for $\mathbf{K}$ and $\widetilde{\mathbf{K}}$.

	α=0.1		α=0.5		α=1.0	
Matrix	$\mathbf{K}$	$\widetilde{\mathbf{K}}$	$\mathbf{K}$	$\widetilde{\mathbf{K}}$	$\mathbf{K}$	$\widetilde{\mathbf{K}}$
λ	6.4	0.90	12.6	0.5	0.083	0.001
	52.4	0.90	48.7	0.5	0.142	0.001
	60.6	0.96	96.4	0.8	36	0.6
	147.2	1.20	126.7	2	117	3
	158.8	1.36	144.8	2.8	206	4.7
	249.8	1.47	366.2	3.3	558	5.7
	328.2	1.64	650.6	4.2	1066	7.4
	398.4	1.70	815.3	4.5	1354	8
ψ	62	1.89	64.8	9	16270	8000

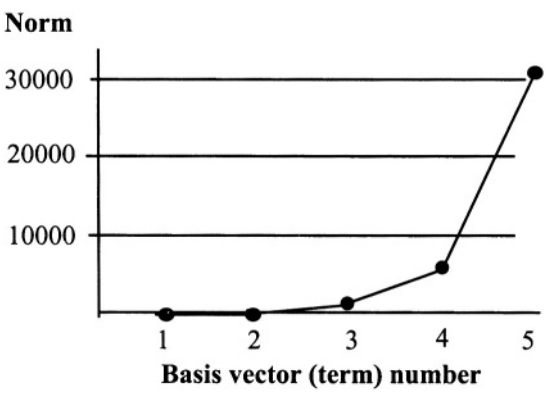

Fig. 7.3. Norm of the original basis vectors $\mathbf{r}_i$ (binomial series terms) for α = 1.0.

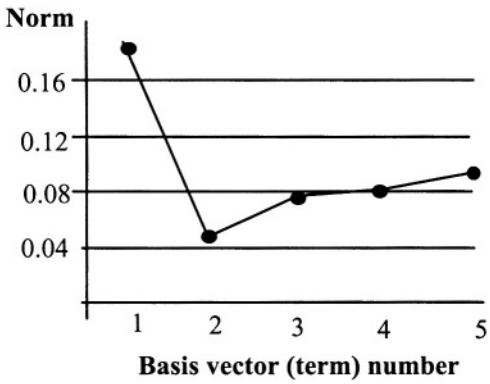

Fig. 7.4. Norm of the uncoupled normalized basis vectors $\mathbf{V}_i$ for $\alpha = 1.0$.

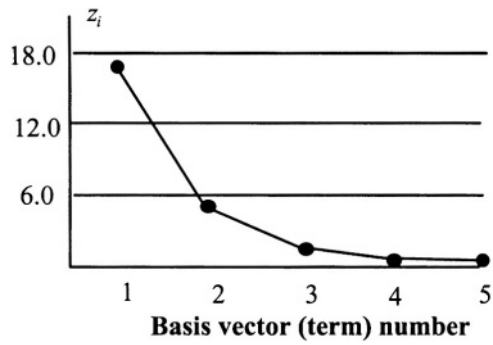

Fig. 7.5. Values of z_i coefficients for $\alpha = 1.0$.

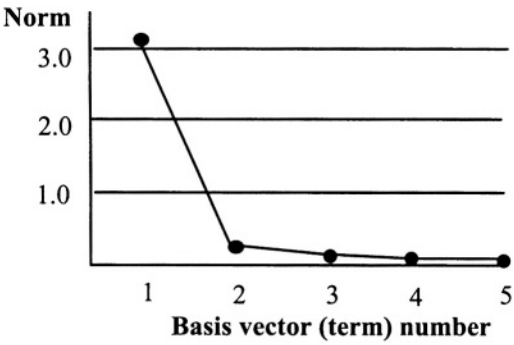

Fig. 7.6. Norm of the CA terms $\mathbf{V}_i z_i$ for $\alpha = 1.0$.

7.3.5 High-Order Approximations

The accuracy of low-order approximations might be insufficient for large changes in large-scale problems. In such cases a larger number of basis vectors (high-order terms) might be needed to improve the accuracy of the approximations. Results obtained by high-order approximations are presented in this section. The object of the examples presented is to illustrate the accuracy achieved by the CA method for some structures having a larger number of degrees of freedom. In all examples the design variables are the member cross-sectional areas and the initial cross section areas equal unity. The results indicate that the number of vectors needed to achieve a certain accuracy is not significantly increased with the size of the structure. Then, it is shown by a numerical example how the CA method can be used for complete analysis in cases where the results of an initial analysis are not available.

Example 7.3 - Various Structures. Results obtained with different numbers of basis vectors for several examples are demonstrated in the following. In all examples hundred cases of random changes in the cross-sections were assumed. The structures considered are as follows [4, 12]:

a. The 50-bar cantilever truss shown in Figure 7.7, subjected to a single tip load.
b. The 204-bar bridge truss shown in Figure 7.8, subjected to 3 concentrated loads.
c. Various large-scale rectangular space trusses made up of the double-layer segments consisting of two horizontal layers connected by diagonals shown in Fig. 7.9. The resulting structures are supported along the four edges and subjected to uniformly distributed loads. Results are demonstrated for 356-bar and 968-bar space trusses.

The numbers of basis vectors needed for the various structures to limit maximum errors to 1% and 0.1% are shown in Table 7.9. It is observed that the difference in the numbers of basis vectors required for the two cases of errors is small. In addition, the number of vectors is not significantly increased with the size of the structure.

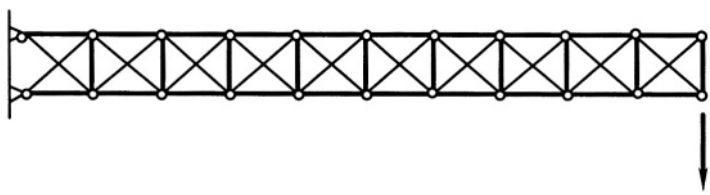

Fig.7.7. 50-bar truss.

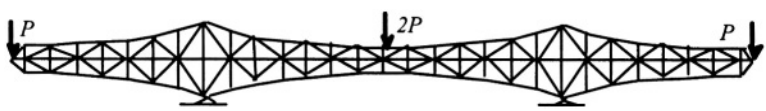

Fig. 7.8. 204-bar truss.

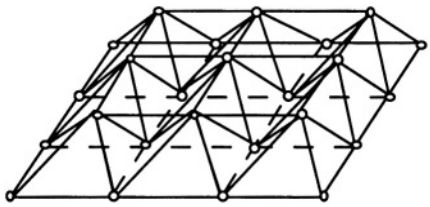

Fig. 7.9. Segments of double-layer space truss.

Table 7.9. Numbers of basis vectors needed to limit maximum errors to 1% and 0.1%

Structure	Maximum error 1%	Maximum error 0.1%
50-bar truss	5-6	6-7
204-bar truss	6-7	8-9
356-bar truss	8-9	9-10
968-bar truss	10-11	11-12

Example 7.4 - Complete Analysis. Initially, the CA method was intended for reanalysis of structures where results from initial analysis are known. It is shown in this example how the method can also be used in cases where the results of an initial analysis are not available. The object is to solve the analysis equations, expressed in the form of Eq. (3.5), where $\mathbf{K}$ (the stiffness matrix of the design to be analyzed) is known, matrices $\mathbf{K}^*$ and $\Delta\mathbf{K}$ are to be defined, and $\mathbf{r}^*$ is unknown. Since results of previous analysis are not available, the simplest approach is to choose $\mathbf{K}^*$ as a diagonal matrix $\mathbf{K}_d$ consisting of the diagonal elements of $\mathbf{K}$

$$\mathbf{K}^* = \mathbf{K}_d = \text{diagonal } (\mathbf{K})$$
$$\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}^* \qquad (a)$$

As a result the initial displacements corresponding to the initial stiffness matrix $\mathbf{K}^*$ are uncoupled and given directly by

$$\mathbf{r}^* = (\mathbf{K}_d)^{-1}\,\mathbf{R} \qquad (b)$$

It should be noted that $\mathbf{K}^*$ might represent several substructures, which are completely different from the actual structure represented by $\mathbf{K}$, as is shown in the following. Since the initial and the modified displacements are of different nature, the CA procedure may require high-order terms to achieve adequate accuracy.

Consider again the ten-bar truss shown in Figure 7.1, subjected to a single loading condition of two loads. The modulus of elasticity is 30 000 and the 8 analysis unknowns

are the horizontal and the vertical displacements at joints 1, 2, 3 and 4, respectively. The object is to analyze the structure for cross-sectional areas equal to unity.

With the initial structure as given by Eq. *(a),* the initial design consists of four different substructures with cross sections corresponding to the stiffness coefficients $\mathbf{K}^*$, as shown in Fig. 7.10. The initial displacements calculated by Eq. *(b)* are given by

$$\mathbf{r}^{*T} = \{0, 0, 0, 0, 0, 0.89, 0, 0.70\} \qquad (c)$$

The requested displacements obtained by considering various numbers of basis vectors are shown in Table 7.10. The results indicate that, despite the very poor initial displacements, the CA procedure converges to the exact solution.

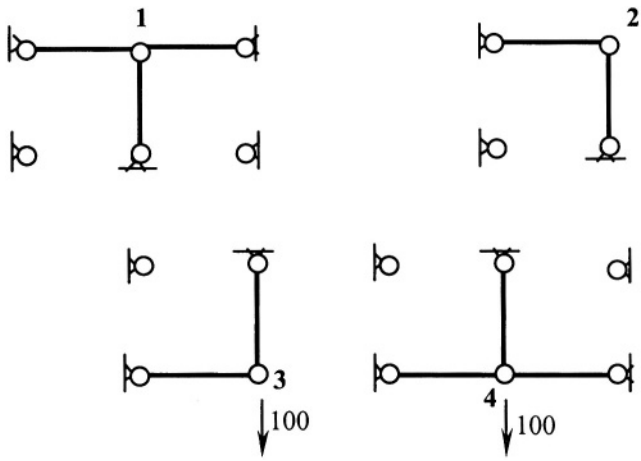

Fig. 7.10. Four different structures, representing the initial solution.

Table 7.10. Analysis by the CA method for cross-sectional areas of unity.

Number of Basis Vectors	2	3	4	5	6	7	Exact Solution
Displacements - **r**	0.45	0.45	2.25	2.32	2.30	2.34	2.34
	2.30	4.84	5.50	5.64	5.56	5.58	5.58
	-0.71	0.11	2.45	2.75	2.77	2.83	2.83
	3.24	6.92	11.72	12.50	12.65	12.65	12.65
	-0.89	-0.96	-3.06	-3.31	-3.24	-3.17	-3.17
	3.96	7.41	11.88	13.10	13.13	13.13	13.13
	0.00	-1.21	-2.58	-2.54	-2.50	-2.46	-2.46
	3.14	5.17	6.33	5.99	6.00	6.01	6.01

7.4 REFERENCES

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8 Simplified Solution Procedures

Various solution procedures that can be viewed as particular cases of the CA method are summarized in this chapter. As noted earlier, the accuracy of the results and the efficiency of the calculations are usually two conflicting factors. That is, better accuracy can be achieved at the expense of more computational effort by considering additional information. The CA method is most effective in cases where highly accurate approximations can be achieved by considering only a small number of basis vectors. Moreover, simplified procedures, such as the Binomial Approximations (BA) and the Scaled Approximations (SA), are often sufficiently accurate.

Some low-order approximations used in structural optimization and in the analysis of damaged structures are demonstrated in Sections 8.1.1 and 8.1.2. These include the first- and second-order approximations of the BA, SA and CA methods. Efficiency considerations are discussed in Section 8.1.3. It is shown that the number of algebraic operations and the computational cost involved in solution by the CA method are significantly smaller than those required for complete analysis. Limitations on design changes, intended to limit the errors occurring in low-order approximations, are presented in Section 8.1.4. The common design variable limits, often used for local approximations (such as the Taylor series and the BA), are not suitable for the SA and the CA methods. More rational limitations on the design changes are demonstrated for these methods.

A procedure to obtain exact solutions by the CA method for simultaneous rank-one changes in the stiffness matrix is demonstrated in Section 8.2. Solutions obtained by the CA method and the Sherman-Morrison-Woodbury formulae in such cases are equivalent. The procedure presented is efficient when the number of changed members in the structure is much smaller than the number of degrees of freedom.

8.1 LOW-ORDER APPROXIMATIONS

The low-order approximations considered in this section are summarized in Table 8.1. The BA method can be used in itself as an approximation of the displacements. This could be more efficient than using the series terms as basis vectors in the CA method since it is not necessary to calculate the multipliers **y.** Considering only the first three terms, the displacements computed by the BA method are given by [see Eq. (5.1)]

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \tag{8.1}$$

where the series terms are determined by Eq. (5.7). Calculation of the series terms involves only forward and backward substitutions and the method requires a small computational effort. As with other local approximations, this simple solution procedure is suitable only for small changes in the design. The accuracy of the approximations is insufficient or unacceptable for large changes in the structure, where problems of slow convergence or divergence of the series may be encountered.

The SA method involves slightly more calculations then the BA method, but the quality of the results is improved. Considering only three terms of the series, we evaluate the displacements by [see Eq. (5.51)]

$$\mathbf{r} = y_1(\mu)\mathbf{r}_1 + y_2(\mu)\mathbf{r}_2 + y_3(\mu)\mathbf{r}_3 \tag{8.2}$$

The series coefficients $y_i(\mu)$ are functions of a single scaling multiplier μ. We calculate the coefficients $y_i(\mu)$ by [see Eq. (5.52)]

$$\begin{aligned} y_1(\mu) &= \mu^{-3}(3\mu^2 - 3\mu + 1) \\ y_2(\mu) &= \mu^{-3}(3\mu - 2) \\ y_3(\mu) &= \mu^{-3} \end{aligned} \tag{8.3}$$

The scaling multiplier μ is calculated by one of the criteria described in Section 5.3.1. We calculate μ by the geometrical criterion of Eq. (5.55)

Table 8.1. Low-order approximations of displacements.

Procedure	Equation	Approximate displacements
BA	(8.1)	$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$
SA	(8.2)	$\mathbf{r} = y_1(\mu)\mathbf{r}_1 + y_2(\mu)\mathbf{r}_2 + y_3(\mu)\mathbf{r}_3$
CA	(8.7)	$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + y_3\mathbf{r}_3$

$$\mu = \left(\frac{\mathbf{X}^T \mathbf{X}}{\mathbf{X}^{*T} \mathbf{X}^*} \right)^{1/2} \tag{8.4}$$

where $\mathbf{X}^*$ and $\mathbf{X}$ are the vectors of the initial design variables and the modified design variables, respectively. Alternatively, using the mathematical criterion of Eq. (5.57), we calculate μ by Eq. (5.58)

$$\mu = \frac{a}{b} \tag{8.5}$$

where

$$a = \sum_{i=1}^{m} (r_{1i} - r_{2i})^2 \qquad b = \sum_{i=1}^{m} (r_{1i}^2 - r_{1i} r_{2i}) \tag{8.6}$$

The terms r_{1i} are the elements of $\mathbf{r}_1 = \mathbf{r}^*$ and the terms r_{2i} are the elements of $\mathbf{r}_2$ that are already given by Eq. (5.7).

The CA method described in Chapter 7 involves more calculations then the SA method, but the quality of the results is significantly improved. Considering only three basis vectors, we evaluate the displacements by [see Eq. (7.16)]

$$\mathbf{r} = y_1 \mathbf{r}_1 + y_2 \mathbf{r}_2 + y_3 \mathbf{r}_3 \tag{8.7}$$

Results achieved by the Combined Approximations of order 1 (CA1, considering two basis vectors) and the Combined Approximations of order 2 (CA2, considering three basis vectors) are demonstrated in this section.

8.1.1 Structural Optimization

In many problems low-order approximations of the displacements provide sufficiently accurate results. In most structural optimization problems it is necessary to analyze structures that are modified repeatedly due to successive changes in the design. The number of repeated analyses needed during the solution of structures with large numbers of variables and constraints is often a function of the problem size and might be very large. In addition, each repeated analysis involves solution of the implicit analysis equations and extensive computational effort. The high computational cost involved in repeated analyses of large-scale structures is one of the main obstacles in the solution process. Since the analysis part of design requires most of the computational effort, only design methods that do not involve many time-consuming analyses during optimization will prove useful. Application of low-order approximations in structural optimization is demonstrated in the following example.

Example 8.1 – Structural Optimization. To compare results achieved by various simplified procedures, consider again the classic ten-bar truss of example 7.2 shown in Figure 8.1. The truss is subjected to a single loading condition of two concentrated loads, the design variables are the member cross-sectional areas, the initial cross section areas equal unity, the modulus of elasticity is 30000, and the eight analysis unknowns are the horizontal and vertical displacements at joints 1, 2, 3 and 4, respectively. The stress constraints are $\mathbf{-25.0 \le \sigma \le 25.0}$, the minimum size constraints are $\mathbf{0.001 \le X}$, and the optimal (minimum weight) design is [1]

$$\mathbf{X}_{opt}^{T} = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

To illustrate results for various directions of change in the design space, the following three cases of changes $\Delta\mathbf{X}$ in the design variables are considered (see Figure 8.2, and Table 8.2):

a. large changes in all the design variables and $\theta = 0$;
b. large changes in the design variables for members 1, 4, 5 and 10 and a small θ;
c. small changes in the design variables for members 1, 4, 5 and 10 and a large θ;

where θ is the angle between the vectors of the modified design $\mathbf{X}$ and the initial design $\mathbf{X}^{*}$ [see Eq. (5.53) and Figure 5.3]. Results obtained by the first-order (two-terms) BA, Scaled Approximations using the geometrical criterion (SA_G) and CA methods are summarized in Table 8.3. As expected, for large changes in the design (cases *a* and *b*) the BA approximations diverge. In case *a* where the modified design is a scaled design, the exact solution is achieved by both the SA_G and the CA methods. In case *b,* good results are achieved by both methods for very large changes in the design. In case *c,* for a small change in the design variables and a large angle θ, good accuracy is achieved only by the CA method.

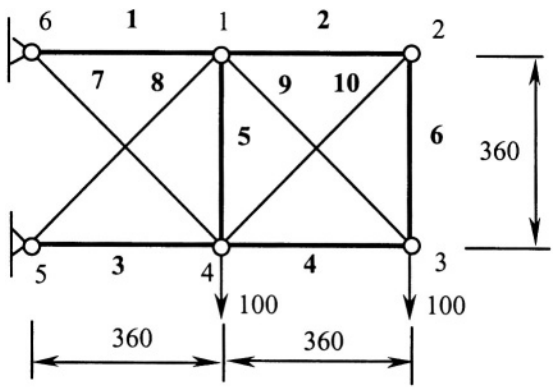

Fig. 8.1. Ten-bar truss.

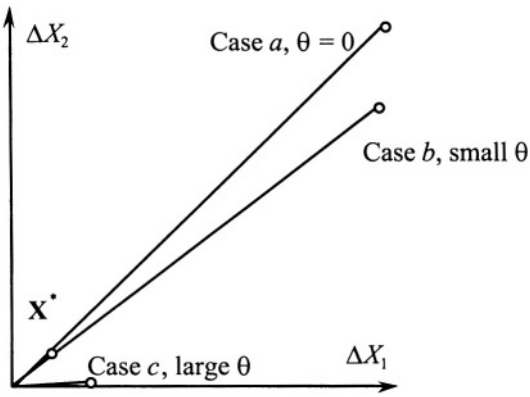

Fig. 8.2. Design lines for the various cases of Table 8.2.

Table 8.2. Data, various direction vectors: *a*. Large $\Delta\mathbf{X}$, $\theta = 0$. *b*. Large $\Delta\mathbf{X}$, small θ. *c*. Small $\Delta\mathbf{X}$, large θ.

Case	θ	ΔX_1 (members 1,4)	ΔX_2 (members 5,10)	ΔX_3 (members 2,3,6,7,8,9)
a	0	9.0	9.0	9.0
b	6.3	9.0	7.0	0
c	47.0	0.9	-0.9	0

Table 8.3. Displacements, first-order approximations, various direction vectors.

Case	Method	1	2	3	4	5	6	7	8
a	BA				Divergence				
	SA_G	0.23	0.56	0.28	1.26	-0.31	1.31	-0.25	0.60
	CA	0.23	0.56	0.28	1.26	-0.31	1.31	-0.25	0.60
	Exact	0.23	0.56	0.28	1.26	-0.31	1.31	-0.25	0.60
b	BA				Divergence				
	SA_G	0.22	0.62	0.27	1.34	-0.3	1.4	-0.24	0.67
	CA	0.23	0.64	0.28	1.39	-0.32	1.45	-0.25	0.69
	Exact	0.23	0.64	0.28	1.39	-0.32	1.45	-0.25	0.69
c	BA	0.21	6.31	0.22	12.02	-0.37	10.89	-0.27	4.95
	SA_G	0.80	5.93	0.94	10.47	-1.14	11.21	-0.87	6.58
	CA	1.13	33.16	1.16	52.59	-1.98	57.17	-1.38	37.09
	Exact	1.23	33.27	1.47	52.75	-1.69	57.29	-1.30	37.14

The line from the initial design to the optimal design is given by

$$\mathbf{X} = \mathbf{X}^* + \Delta\mathbf{X} = \mathbf{X}^* + \alpha\,\Delta\mathbf{X}^*$$

where $\Delta\mathbf{X}^*$ is the vector of design changes, defined as

$$\Delta\mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

Table 8.4. Approximations of displacements by the BA and the CA methods.

Method	BA		CA				Exact
Number of vectors	2	3	2	3	4	5	Solution
$\alpha = 0.01$ (small)	2.18	2.19	2.19				2.19
	5.26	5.28	5.28				5.28
	2.65	2.66	2.66				2.66
	11.95	11.99	11.99				11.99
	-3.00	-3.01	-3.01				-3.01
	12.42	12.46	12.46				12.46
	-2.29	-2.30	-2.30				-2.30
	5.68	5.69	5.69				5.69
$\alpha = 0.1$ (medium)	0.69	1.85	1.36	1.37			1.37
	2.38	4.28	3.59	3.56			3.56
	1.07	2.26	1.76	1.77			1.77
	5.59	9.96	8.23	8.25			8.25
	-1.40	-2.58	-2.06	-2.10			-2.10
	5.97	10.36	8.62	8.65			8.65
	-0.75	-1.94	-1.44	-1.45			-1.45
	2.69	4.61	3.92	3.89			3.89
$\alpha = 1.0$ (Large)	Divergence		0.28	0.29	0.29	0.30	0.30
			0.90	0.84	0.88	0.90	0.90
			0.41	0.45	0.47	0.49	0.49
			2.10	2.17	2.19	2.21	2.21
			-0.53	-0.61	-0.62	-0.60	-0.60
			2.24	2.34	2.37	2.40	2.40
			-0.30	-0.31	-0.31	-0.30	-0.30
			1.01	0.95	0.93	0.90	0.90

and α is the step-size variable. For $\alpha = 1.0$ (the optimum) the change in the design is very large: members 1 and 3 are increased by 700%, member 4 is increased by 300%, members 7, 8, 9 are increased by 467%, and the topology is changed by effectively eliminating members 2, 5, 6 and 10, and hence joint 2 (displacements 3 and 4).

To illustrate results for various magnitudes of change in the design variables, the following three cases of change are considered:

a. A small step size, $\alpha = 0.01$.
b. A medium step size, $\alpha = 0.1$.
c. A large step size, $\alpha = 1.0$ (the optimum).

Results obtained by the Binomial Approximations (BA) and the Combined Approximations (CA) are summarized in Table 8.4.

Considering the CA method, it is observed that the larger the step size is, more terms (basis vectors) are required to achieve a certain level of accuracy. For two-digit accuracy, only two terms are needed for $\alpha = 0.01$, 3 terms are needed for $\alpha = 0.1$ and 5 terms are needed for $\alpha = 1.0$.

Considering the BA method, 3 terms are needed for $\alpha = 0.01$, very slow convergence is obtained for $\alpha = 0.1$, and meaningless results due to divergence are obtained for $\alpha = 1.0$. Figures 8.3 and 8.4 show typical results for $\alpha = 1.0$ (the optimum) and different numbers of terms.

In summary, for the CA method good accuracy is achieved by low-order approximations (CA1 and CA2) even in cases where the BA method diverges.

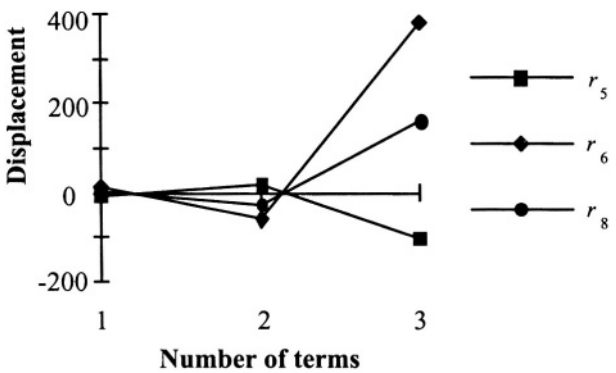

Fig. 8.3. Typical results for $\alpha = 1.0$, the BA method.

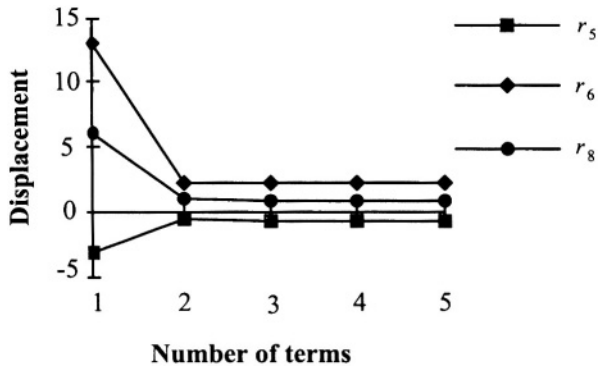

Fig. 8.4. Typical results for $\alpha = 1.0$, the CA method.

8.1.2 Reanalysis of Damaged Structures

In structural damage analysis, it is necessary to analyze the structure for various changes due to deterioration, poor maintenance, damage, or accidents. In general, it is difficult to determine *a priori* what damage scenarios should be checked and the computational effort required for a systematic comprehensive damage analysis might become prohibitive. Numerous analyses are required to assess the adequacy of redundancy and to evaluate various hypothetical damage scenarios for various types of damage. Because of all the possible combinations of damage in m elements for a structure consisting of n elements, the resulting number of analyses might be very large even for relatively small m and moderate scale structures. Table 8.5 shows the number of combinations of damage (approximate expressions) and typical numbers for small m ($m \leq 4$) and medium scale structures (n = 100 - 1000). Evidently, it is not practical to analyze all possible combinations. Even for significantly reduced numbers of analyses it is still necessary to use an efficient reanalysis procedure for analyzing numerous possible damage conditions.

The CA method can be used to analyze efficiently and accurately various possible situations of damage. Since the assumed damage scenarios usually involve changes in a small number of elements, results achieved by low-order CA are sufficiently accurate. Given the basis vectors for changes in various single members, it is then easy to evaluate the response of the structure for a change in any combination of several members. The method can be used for various cases of damage analysis, including:

Table 8.5. Combinations of damage in m elements of an n-element structure.

m	1	2	3	4
No. of combinations	n	$n^2/2$	$n^3/4$	$n^4/8$
$n = 100$	100	5×10^3	2.5×10^5	1×10^7
$n = 500$	500	125×10^3	312.5×10^5	781×10^7
$n = 1000$	1000	500×10^3	2500×10^5	12500×10^7

a. *Partial damage,* where the element is still capable of carrying some loading after damage has occurred. Deteriorated section properties and changed member forces can represent the effect of such damage of a member on the structural response. This approach can be used when analysis shows that a limit state analysis is needed. A reanalysis can be done for further incremental loadings. A member which reaches its ultimate force can be represented, for subsequent analysis, by zero cross section (or zero stiffness) and corresponding equivalent ultimate forces acting on its ends.

b. *Total damage* of an element, where the element is incapable of carrying any loading after damage, and it is therefore eliminated from the structure for subsequent analyses.

Example 8.2 - Analysis of a Damaged Grillage. A grillage model is often used to analyze load effects in common types of highway bridge structures [2]. This example shows damage in members and external supports of a grillage. Member damage could be due to corrosion, fracture or collision consequences. The support damage could be due to foundation scour, collision effects or seismic actions.

In grillage structures, each member is represented by at least two variables, i.e. the moment of inertia for bending and the moment of inertia for torsion. These variables might be dependent, but the CA method does not require any predetermined relationship. The problem considered in this example is as follows: Given the initial moments of inertia for bending $\mathbf{I}^{\bullet}$, and for torsion $\mathbf{J}^{\bullet}$, the corresponding stiffness matrix $\mathbf{K}^{\bullet}$, and the displacements $\mathbf{r}^{\bullet}$, evaluate the modified displacements $\mathbf{r}$ due to various changes in the cross sections $\Delta\mathbf{I}$, $\Delta\mathbf{J}$, corresponding to damage in some members and external supports.

Consider the grillage shown in Fig. 8.5, consisting of three longitudinal beams and supported by six clamped supports at joints D, E, F, G, H and I. Assume arbitrary units with material constants $E = G = 1.0$ and initial moments of inertia $\mathbf{I}^{\bullet} = \mathbf{J}^{\bullet} = \mathbf{1.0}$. The grillage is subjected to three vertical loads, $P_A = 5.0$, $P_B = P_C = 2.5$, acting upward at the three free joints A, B and C. The elements of the displacement vector are vertical displacement, rotation about the Y axis and rotation about the Z axis, at joints A, B, C, respectively. The total number of degrees of freedom is nine.

To illustrate results for changes in member sections, consider the following cases:

a. Reducing member 3 by 50%, $\Delta I_3 = \Delta J_3 = -0.5$.

b. Reducing member 5 by 50%, $\Delta I_5 = \Delta J_5 = -0.5$.

In both cases exact solutions are achieved by the CA2 procedure (only three basis vectors), as shown in Table 8.6 [3,4].

Results obtained by the CA2 method and a complete exact analysis for three cases of elimination of a single member are summarized in Table 8.7:

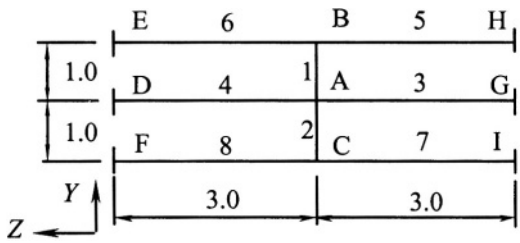

Fig. 8.5. Grillage of three longitudinal beams.

a. Elimination of member 1 (Figure 8.6*a*).
b. Elimination of member 3 (Figure 8.6*b*).
c. Elimination of member 5 (Figure 8.6*c*).

Elimination of member 3 or member 5 could be viewed as complete damage either of the element or the end supports of the member. It can be seen that exact solutions are achieved by the CA2 procedure for the cases of eliminating either member 1 or member 3, and that good accuracy is achieved for the case of eliminating member 5. Table 8.8 shows the approximate displacements obtained by the CA2 procedure when both members 3 and 5 are eliminated simultaneously (Figure 8.6*d*). Even for drastic changes in the displacements (see the initial displacements in Tables 8.6, 8.7 and 8.8), good accuracy has been achieved. Application of this approach to damage analysis scenarios with highway bridge foundations are demonstrated elsewhere [5].

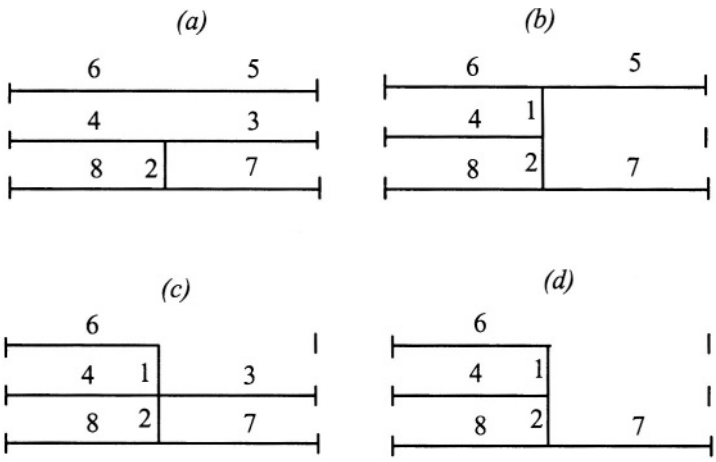

Fig. 8.6. Modified topologies, grillage of three longitudinal beams:
a. Elimination of member 1. *b.* Elimination of member 3.
c. Elimination of member 5. *d.* Elimination of members 3 and 5.

Table 8.6. Displacements, change in section properties of grillage.

			$\Delta I_3 = \Delta J_3 = -0.5$	$\Delta I_5 = \Delta J_5 = -0.5$
Joint	Direction	r*	Exact / CA2	Exact / CA2
A	X	3.875	4.343	4.285
	Y	0	-0.419	-0.122
	Z	0	0	-0.326
B	X	3.693	4.081	4.481
	Y	0	-0.114	-0.538
	Z	0.234	0.337	-0.116
C	X	3.693	4.081	3.817
	Y	0	-0.114	-0.033
	Z	-0.234	-0.337	-0.463

Table 8.7. Displacements, elimination of a single member.

Joint	Direction	Intact structure	Elimination Member 1	Elimination Member 3	Elimination Member 5	
		r*	Exact / CA2	Exact / CA2	CA2	Exact
A	X	3.875	4.603	5.207	5.395	5.394
	Y	0	0	-1.246	-0.491	-0.489
	Z	0	-0.684	0.000	-1.285	-1.300
B	X	3.693	2.815	4.797	6.765	6.782
	Y	0	0.000	-0.340	-2.158	-2.148
	Z	0.234	0.813	0.528	-1.306	-1.321
C	X	3.693	3.609	4.797	4.104	4.090
	Y	0	0	-0.340	-0.134	-0.133
	Z	-0.234	-0.491	-0.528	-1.110	-1.119

Table 8.8. Displacements, elimination of members 3 and 5.

Joint	Direction	r*	CA2	Exact
A	X	3.875	10.102	10.115
	Y	0	-4.002	-3.974
	Z	0	-3.138	-3.165
B	X	3.693	12.926	12.979
	Y	0	-5.417	-5.414
	Z	0.234	-2.495	-2.505
C	X	3.693	6.834	6.765
	Y	0	-1.091	-1.084
	Z	-0.234	-2.857	-2.951

Example 8.3 - Renalysis of a Damaged Frame [4, 6]. To evaluate the quality of the results by the CA1 procedure (only two basis vectors) for frames, the five-story two-bay steel frame subjected to three horizontal loads shown in Figure 8.7 is analyzed for three cases of damage at the connections A, B and C. These connection damages are assumed to be such that the connection may continue to transmit shear forces but not bending moments from beam to column.

This damage scenario is motivated by recent seismic events in California and Japan in which beam-to-column connections were damaged in steel buildings. The aim of this example is to investigate the consequences on structural behavior of the damage done to the connections. These analyses, which could also be done in advance of expensive field inspections to physically check the damage, are carried out using the efficient CA1 procedure for predicting responses and selecting connections whose failure would have the greatest effect on the frame behavior in the event of seismic occurrences.

It is assumed that in each case a hinge is formed at the location of connection damage (loss of flexural rigidity). In all three cases the exact solution is achieved by the CA1 procedure. That is, the complete analysis problem with (15 joints and 3 degrees of freedom per joint) 45 degrees of freedom is reduced to a problem with only two unknowns. Results obtained by the CA1 procedure for vibration analysis of a similar frame are demonstrated in example 12.3.

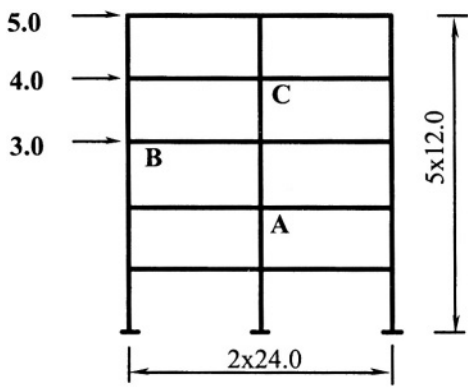

Fig.8.7. Five-story two-bay example.

8.1.3 Efficiency of the Calculations

The computational effort involved in the CA method is somewhat larger than that for the simple series approximations, such as the BA and the SA methods. The BA approximations involve only calculation of the series terms. It has been noted that calculation of each term involves only forward and backward substitutions, if $\mathbf{K}^*$ is given in the decomposed form of Eq. (3.2). The CA method requires in addition, calculation of the modified stiffness matrix $\mathbf{K}$ and some extra algebraic operations. These operations slightly increase the computational cost but accurate approximations can be achieved in cases where the BA and the SA methods provide poor or

meaningless results. If the uncoupled basis vectors $\mathbf{V}_i$ are used, it is also necessary to generate these additional vectors. As noted earlier, the uncoupled vectors are explicit functions of the original vectors. Therefore, once the original vectors are determined, calculation of the uncoupled vectors and the final displacements is straightforward.

The efficiency of reanalysis by the CA method, compared with complete analysis of the modified design, can be measured by various criteria, e.g. the CPU effort or the number of algebraic operations. It is then possible to relate the computational effort to various parameters such as the number of degrees of freedom, the number of basis vectors considered and the accuracy of the results. A small number of basis vectors is often sufficient to achieve adequate accuracy. It has been found that for small problems with moderate changes in the design two or three vectors are often sufficient. For medium problems with large changes, five or six vectors might be needed, and for large problems, nine or ten vectors are often sufficient.

Solution of various problems indicates that calculation of each basis vector involves about 2% of the CPU time needed for complete analysis. It is found that for five to six basis vectors the total CPU effort is reduced by more than 75%, compared with complete analysis of the modified design. Using the CA method for an accurate nonlinear analysis of a space frame with about 300 degrees of freedom, the resulting CPU effort was reduced by more than 60% compared to a complete analysis [7].

The number of algebraic operations needed to solve an $n \times n$ set of equations by complete analysis is $n^3/3$, where n is the number of displacement degrees of freedom. The number of operations needed by the CA method is $3n^2s + ns^2 + s^3/3$, where s is the number of basis vectors considered. The ratio P between the former and the latter numbers for various n and s values is shown in Figure 8.8. For example, for $n = 500$ and $s = 10$ the value of P is approximately 5. That is, the number of algebraic operations needed by the CA method is about 20% of the number needed for complete analysis.

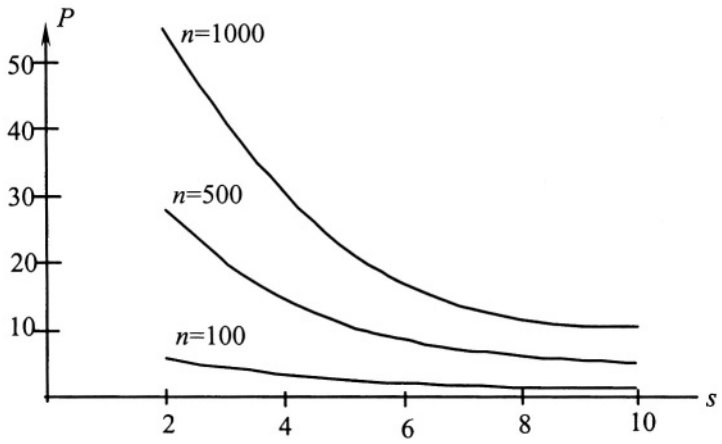

Fig. 8.8. The ratio P between the numbers of algebraic operations for complete analysis and the CA method

8.1.4 Limitations on Design Changes

To prevent significant errors occurring for low-order approximations due to large changes in the design variables, limitations on the design changes are often imposed. These limitations, called move limits, are commonly expressed as upper and lower limit constraints on the design variables. Several strategies for adjusting the common move limits in structural optimization have been proposed. One approach is to use an automated move limit strategy, based on effectiveness coefficients, which quantify a design variable's impact [8, 9]. Other approaches include sensitivity-based exponential methods [10, 11] and move limits based on global and local considerations [12].

The common move limit constraints are effective particularly for local series approximations. On the other hand, these constraints are not suitable for the combined approximations presented in Chapter 7. Alternative constraints, which are more rational and more effective, have been proposed [13] and are described in this section. Two types of move limit constraints for limiting the errors involved in various low-order approximations are presented and discussed in the following:

a. The common design variable move limits, which are explicit side constraints on the design variables. These constraints are the simplest, but they are suitable only for the conventional series approximations (e.g., the BA method) where accurate results can be achieved only for small changes in the design variables.
b. Angle move limits, which are nonlinear constraints on the angle between the vectors of the initial design and the modified design. These explicit and simple constraints are more suitable for the improved SA and CA methods.

Design Variable Move Limits. Considering conventional series approximations such as the Taylor series or the binomial series, the following move limit constraints are often assumed

$$\Delta \mathbf{X}^L \le \Delta \mathbf{X} \le \Delta \mathbf{X}^U \tag{8.8}$$

where $\Delta \mathbf{X}^L$ and $\Delta \mathbf{X}^U$ are predetermined lower and upper limits, respectively, on the design variable changes $\Delta \mathbf{X}$, given by

$$\Delta \mathbf{X} = \mathbf{X} - \mathbf{X}^* \tag{8.9}$$

The main advantage of using the move limits constraints is the simple and explicit nature of the side constraints. An alternative approach, used in trust region algorithms [14], is to restrict the solutions to some region around $\mathbf{X}^*$ by constraints of the form

$$\|\Delta \mathbf{X}\| \le \Delta \tag{8.10}$$

where Δ is the radius of the region to be restricted. Constraints of the type of Eqs. (8.8) or (8.10) are generally effective only for conventional local approximations, where small changes in the design variables are assumed. They are not effective for the SA and the CA methods applied to find accurate solutions for large changes.

Since the errors involved in the approximations are not known in advance, one problem in applying these constraints is that it is difficult to define appropriate move limits *a priori.* For computational efficiency in structural optimization problems it is desirable to initially choose large values for the move limits, so that the imposed limits will not slow convergence. As the design process progresses, the move limits can be gradually reduced. One reason for reducing the move limits is that the accuracy of the approximations is required to be higher as we get closer to the optimum. The move limits are reduced also in cases where successive solutions start to oscillate between vertices of the feasible region.

Angle Constraints. It was previously shown that accurate results are achieved for very large changes in the variables by the SA and CA methods in cases where the modified design is relatively close to a scaled design. This is the case if the angle θ between the vectors $\mathbf{X}$ and $\mathbf{X}^*$ (Figure 5.3) is small. This angle can be determined by Eq. (5.53). Since accurate results are expected for small angles θ, it might prove useful to apply the angle constraints

$$\theta^L \le \theta \le \theta^U \tag{8.11}$$

where θ^L and θ^U are predetermined limits. Figure 8.9 shows that the explicit constraints of Eq. (8.11) define a large conical region, rather than the relatively small region defined by the local approximation constraints [Eq. (8.8) box or Eq. (8.10) circle] in the neighborhood of $\mathbf{X}^*$.

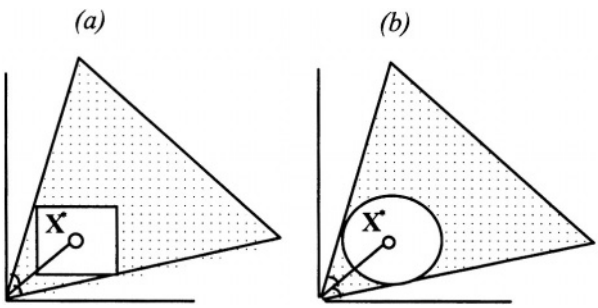

Fig.8.9. Various move-limit constraints.

8.2 EXACT SOLUTIONS

In general, the CA method provides approximate solutions for high-rank changes in the stiffness matrix. Two cases where exact solutions are obtained were presented in Chapter 7. Exact solutions are obtained if a basis vector becomes linearly dependent on all of the previous vectors (Section 7.3.1). Such solutions are obtained also if the modified design is a scaled design (Section 7.3.4). It is shown in this section that exact

solutions are obtained efficiently for a small number of simultaneous rank-one changes in the stiffness matrix. Solutions obtained by the CA method and the Sherman-Morrison-Woodbury formulae in such cases are equivalent.

8.2.1 Multiple Rank-One Changes

In this section we develop exact solutions for changes in a small number of elements. For $\mathbf{r}_1 = \mathbf{r}^*$ and $\mathbf{r}_2 = -\mathbf{B}\,\mathbf{r}^*$, the first and the second basis vectors are linearly dependent if

$$\mathbf{B}\,\mathbf{r}^* = y\,\mathbf{r}^* \tag{8.12)}$$

where y is a scalar different from zero and $\mathbf{r}^*$ is an arbitrary displacement vector. Substituting $\mathbf{B} = \mathbf{K}^{*-1}\,\Delta\mathbf{K}$ [Eq. 7.3)] into Eq. (8.12) and pre-multiplying by $\mathbf{K}^*$ yields

$$\Delta\mathbf{K}\,\mathbf{r}^* = y\,\mathbf{K}^*\mathbf{r}^* = y\,\mathbf{R}^* \tag{8.13}$$

It is observed that the condition of Eq. (8.13) is equivalent to the case of uniform scaling where $\Delta\mathbf{K} = y\,\mathbf{K}^*$.

Two successive basis vectors $\mathbf{B}^{k-1}\mathbf{r}^*$ and $\mathbf{B}^k\,\mathbf{r}^*$ are linearly dependent if

$$\mathbf{B}^k\,\mathbf{r}^* = y\,\mathbf{B}^{k-1}\,\mathbf{r}^* \tag{8.14}$$

for some scalar y different from zero. The condition of Eq. (8.14) is satisfied for arbitrary displacements $\mathbf{r}^*$ if

$$\mathbf{B}^k = y\,\mathbf{B}^{k-1} \tag{8.15}$$

For the second and the third basis vectors $\mathbf{B}\mathbf{r}^*$ and $\mathbf{B}^2\mathbf{r}^*$, respectively, this condition reduces to

$$\mathbf{B}^2 = y\,\mathbf{B} \tag{8.16}$$

The condition of Eq. (8.16) is satisfied in cases of rank-one changes in the stiffness matrix. Consider for example the typical case of a change in the cross-sectional area of a single truss member. An exact solution is obtained by the CA method with only two basis vectors as [15]

$$\mathbf{r} = \mathbf{r}^* - y_i\,\mathbf{B}_i\,\mathbf{r}^* \tag{8.17}$$

where the subscript i denotes changes in member i. That is, $\mathbf{B}_i$ corresponds to the change $\Delta\mathbf{K}_i$ in the stiffness matrix of member i. The expression of Eq. (8.17) can readily be extended to the general case of changes in m members, where the exact solution is given by [15]

$$\mathbf{r} = \mathbf{r}^* - \sum_{i=1}^{m} y_i \mathbf{B}_i \mathbf{r}^* \tag{8.18}$$

Exact solutions by Eq. (8.18) are efficient in cases where $m << n$, that is, the number of changes is much smaller than the number of degrees of freedom for the structure.

8.2.2 Equivalence of the CA Method and the S-M Formula

It is shown in this section that for a change in the cross-sectional area of a single truss member, the exact solutions obtained by the CA method [Eq. (8.17)] and the S-M formula [Eq. (4.8)] are equivalent [16].

For a change in a single truss member i, the expression for an exact solution by the CA method is [Eq. (8.17)]

$$\mathbf{r} = \mathbf{r}^* + \Delta\mathbf{r} = \mathbf{r}^* + y_i \mathbf{r}_i \tag{8.19}$$

where $\mathbf{r}_i$ is defined as

$$\mathbf{r}_i = - \mathbf{K}^{*-1} \Delta\mathbf{K}_i \mathbf{r}^* \tag{8.20}$$

and $\Delta\mathbf{K}_i$ is the matrix of changes in the stiffness due to a change in member i. The term $\Delta\mathbf{K}_i \mathbf{r}^*$ in Eq. (8.20) represents a pair of collinear forces on the ends of a modified truss member. Thus, the basis vector $\mathbf{r}_i$ may be viewed as an influence coefficient vector measuring the effect of the change $\Delta\mathbf{K}_i$ in element i on the displacement vector [Eq. (8.19)]. To determine the change in the displacement vector $\Delta\mathbf{r}$, consider the modified analysis equations (3.5). For $\mathbf{R} = \mathbf{R}^*$ the latter equations can be expressed as

$$(\mathbf{K}^* + \Delta\mathbf{K}_i)\, \Delta\mathbf{r} = - \Delta\mathbf{K}_i \mathbf{r}^* \tag{8.21}$$

Since the change $\Delta\mathbf{K}_i$ is of rank-one, the expression of Eq. (4.1) can be considered. Substituting Eqs. (4.1) and (4.5) into Eq. (8.20) gives

$$\mathbf{r}_i = - \eta\, \mathbf{t}\, \mathbf{v}^T \mathbf{r}^* \tag{8.22}$$

Substituting Eqs. (4.1), (8.19), (8.20) and (8.22) into Eq. (8.21) and pre-multiplying by $\mathbf{r}^{*T}$ yields

$$\mathbf{r}^{*T} (- \eta\, \mathbf{v}\, \mathbf{v}^T \mathbf{r}^* - \eta\, \mathbf{v}\, \mathbf{v}^T \eta\, \mathbf{t}\, \mathbf{v}^T \mathbf{r}^*) y_i = - \mathbf{r}^{*T} \eta\, \mathbf{v}\, \mathbf{v}^T \mathbf{r}^* \tag{8.23}$$

Solving Eq. (8.23) for y_i gives

$$y_i = 1/ (1 + \eta\, \mathbf{v}^T \mathbf{t}) \tag{8.24}$$

Substituting Eqs. (8.22) and (8.24) into Eq. (8.19) yields

$$\mathbf{r} = \mathbf{r}^* - \eta\, \mathbf{t}\, \mathbf{v}^T \mathbf{r}^* / (1 + \eta\, \mathbf{v}^T \mathbf{t}) \tag{8.25}$$

Substituting Eq. (4.7) into Eq. (8.25) gives Eq. (4.8). That is, solutions by the S-M formula [Eq. (4.8)] and the CA method [Eq. (8.17)] are equivalent.

8.2.3 Equivalence of the CA Method and the Woodbury Formula

For simultaneous changes in m truss members, it was shown in Section 8.2.1 that the exact solution by the CA method can be expressed as [Eq. (8.18)]

$$\mathbf{r} = \mathbf{r}^* + \Delta\mathbf{r} = \mathbf{r}^* + y_1 \mathbf{r}_1 + y_2 \mathbf{r}_2 + \ldots + y_m \mathbf{r}_m \tag{8.26}$$

where the basis vectors $\mathbf{r}_i$ $(i = 1,..., m)$ are given by [see Eq. (8.20)]

$$\mathbf{r}_i = - \mathbf{K}^{*-1} \Delta\mathbf{K}_i \mathbf{r}^* \tag{8.27}$$

Thus, Eq. (8.18) for a rank-m change in $\mathbf{K}$ represents a linear combination of rank-one changes in the displacements. From the Woodbury formula, as given by Eqs. (4.13) through (4.17), the change in the displacements $\Delta\mathbf{r}$ can be expressed as

$$\Delta\mathbf{r} = - \mathbf{T}\, \mathbf{A} = -a_1 \mathbf{r}_1 - a_2 \mathbf{r}_2 - \ldots - a_m \mathbf{r}_m \tag{8.28}$$

The solution steps using the Woodbury formula are summarized below.

- Each column of the matrix $\mathbf{T}$ is calculated by applying a pair of collinear forces representing one column of the matrix $\mathbf{V}$ and solving Eq. (4.13).
- The coefficients a_i are calculated by Eq. (4.16), which is an $m \times m$ linear system.
- The change in the displacements is calculated by Eq. (8.28).

It is seen that the CA method and the Woodbury formula involve exactly the same computations.

8.2.4 Solution Procedure

Consider the case of m rank-one simultaneous changes in the stiffness matrix (for example, simultaneous changes in m truss members). It was noted in Section 8.2.1 that an exact solution is obtained if one basis vector is selected for each changed member. This procedure is efficient when the number of changed members is much smaller than the number of degrees of freedom. If some of the basis vectors are linearly dependent, the exact solution is obtained with a smaller number of basis vectors.

Based on the expressions presented in Section 8.2.1, an exact reanalysis procedure is now developed. For the given initial values $\mathbf{K}^*$ and $\mathbf{r}^*$, the solution process involves the following steps [15]:

a. Calculation of the constant vectors $\mathbf{r}_i^*$. The basis vectors are calculated by Eq. (8.20)

$$\mathbf{r}_i = -\mathbf{K}^{*-1}\Delta\mathbf{K}_i\mathbf{r}^* \qquad i = 1, ..., m \tag{8.29}$$

where $\Delta\mathbf{K}_i$ is the contribution of the ith member to $\Delta\mathbf{K}$. It is observed that when the basis vectors are linear functions of the changes in cross sections ΔX_i, they can be expressed as

$$\mathbf{r}_i = \mathbf{r}_i^*\Delta X_i \qquad i = 1, ..., m \tag{8.30}$$

where $\mathbf{r}_i^*$ are constant vectors defined as

$$\mathbf{r}_i^* = \frac{\partial \mathbf{r}_i^*}{\partial X_i} \qquad i = 1, ..., m \tag{8.31}$$

That is, the vectors $\mathbf{r}_i^*$ are the constant derivative vectors $\partial \mathbf{r}_i^* / \partial X_i$. It was noted in Section 7.1.2 that multiplying a basis vector by any scalar does not change the approximate solution [but only the corresponding scalar y_i, see Eq. (7.13)]. Therefore, the constant vectors $\mathbf{r}_i^*$ can be considered as basis vectors instead of $\mathbf{r}_i$ for the whole solution process. In addition, calculation of the basis vectors involves only forward and backward substitutions, if $\mathbf{K}^*$ is given in a decomposed form of Eq. (3.2) from the initial analysis.

b. Calculation of the coefficients y_i. For any assumed set of changes $\Delta\mathbf{X}_i$ in m members, the matrix $\mathbf{K}$ is found by

$$\mathbf{K} = \mathbf{K}^* + \sum_{i=1}^{m} \Delta\mathbf{K}_i \tag{8.32}$$

where the matrices $\Delta\mathbf{K}_i$ are linear functions of $\Delta\mathbf{X}_i$. For the given basis vectors, the matrix $\mathbf{K}_R$ and the vector $\mathbf{R}_R$ are then found by Eq. (7.14), and the coefficients y_i are calculated by solving the set of $m \times m$ equations (7.15).

c. The final exact displacements are calculated by [see Eq. (8.18)]

$$\mathbf{r} = \mathbf{r}^* + \sum_{i=1}^{m} y_i\,\mathbf{r}_i = \mathbf{r}^* + \sum_{i=1}^{m} y_i\,\mathbf{r}_i^*\Delta X_i \tag{8.33}$$

Example 8.4 - Exact Solutions by the CA Method. To illustrate calculation of the exact displacements by the CA method, consider again the ten-bar truss shown in

Figure 8.10. Solution of this example by the S-M Formula was demonstrated in example 4.1. The truss is subjected to a single loading condition of two concentrated loads, the initial cross-sectional areas are **X= 1.0,** the modulus of elasticity is 30 000 and the eight analysis unknowns are the horizontal and the vertical displacements at joints 1, 2, 3 and 4, respectively.

The vectors $\mathbf{r}^*$ and $\mathbf{r}_i^*$ [Eq. (8.31)], calculated for changes in each of the 10 cross sections, are shown in Table 8.9. The final exact displacements can readily be determined for any assumed set of changes in the cross sections by multiplying the given vectors $\mathbf{r}_i^*$ by the corresponding ΔX_i and y_i.

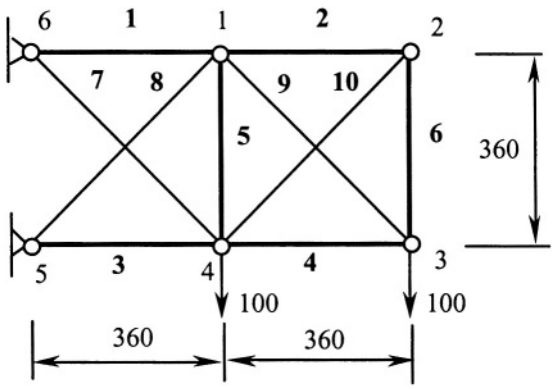

Fig. 8.10. Ten-bar truss.

Table 8.9. Basis vectors $\mathbf{r}^*$ and $\mathbf{r}_i^*$.

$\mathbf{r}^*$	$\mathbf{r}_1^*$	$\mathbf{r}_2^*$	$\mathbf{r}_3^*$	$\mathbf{r}_4^*$	$\mathbf{r}_5^*$	$\mathbf{r}_6^*$	$\mathbf{r}_7^*$	$\mathbf{r}_8^*$	$\mathbf{r}_9^*$	$\mathbf{r}_{10}^*$
2.34	-2.07	-0.01	-0.29	0.01	0.04	-0.01	-0.41	0.38	0.02	-0.02
5.58	-1.30	-0.02	-1.10	0.03	0.17	-0.02	-1.59	-1.79	0.09	-0.06
2.83	-2.10	-0.44	-0.26	-0.07	0.08	0.04	-0.37	0.34	-0.19	0.13
12.65	-3.50	-0.27	-3.70	-0.32	-0.02	0.21	-1.80	-1.60	-0.91	-0.75
-3.17	0.25	0.04	2.20	0.65	0.08	0.04	-0.37	0.34	-0.19	0.13
13.13	-3.53	-0.21	-3.67	-0.40	0.02	-0.22	-1.75	-1.64	-1.12	-0.61
-2.46	0.27	-0.01	2.17	0.01	0.04	-0.01	-0.41	0.38	0.02	-0.02
6.01	-1.05	0.02	-1.36	-0.03	-0.17	0.02	-1.96	-1.45	-0.09	0.06

Assume for example the following two separate changes:

Case 1. A change $\Delta X_1 = 1.0$ in member 1.
Case 2. A change $\Delta X_2 = 1.0$ in member 2.

The basis vectors for the two cases are given in Table 8.9, the scalars calculated by Eq. (8.24) are $y_1 = 0.531$ and $y_2 = 0.526$ (see calculation of $\boldsymbol{\eta}, \mathbf{v}, \mathbf{t}$ in example 4.1) and the final exact displacements, determined by Eq. (8.19), are:

Case 1

$$\mathbf{r} = \mathbf{r}^* + y_1\mathbf{r}_1 = \begin{Bmatrix} 2.34 \\ 5.58 \\ 2.83 \\ 12.65 \\ -3.17 \\ 13.13 \\ -2.46 \\ 6.01 \end{Bmatrix} + 0.531 \begin{Bmatrix} -2.07 \\ -1.30 \\ -2.10 \\ -3.50 \\ 0.25 \\ -3.53 \\ 0.27 \\ -1.05 \end{Bmatrix} = \begin{Bmatrix} 1.24 \\ 4.89 \\ 1.71 \\ 10.79 \\ -3.04 \\ 11.26 \\ -2.31 \\ 5.45 \end{Bmatrix}$$

Case 2

$$\mathbf{r} = \mathbf{r}^* + y_2\mathbf{r}_2 = \begin{Bmatrix} 2.34 \\ 5.58 \\ 2.83 \\ 12.65 \\ -3.17 \\ 13.13 \\ -2.46 \\ 6.01 \end{Bmatrix} + 0.526 \begin{Bmatrix} -0.01 \\ -0.02 \\ -0.44 \\ -0.27 \\ 0.04 \\ -0.21 \\ -0.01 \\ 0.02 \end{Bmatrix} = \begin{Bmatrix} 2.34 \\ 5.57 \\ 2.60 \\ 12.51 \\ -3.15 \\ 13.02 \\ -2.46 \\ 6.02 \end{Bmatrix}$$

Assuming simultaneous changes ΔX_1, ΔX_2 and ΔX_3 in members 1, 2 and 3, respectively, the following exact expression for the displacements is obtained [see Eq. (8.33) and Table 8.9]

$$\mathbf{r} = \begin{Bmatrix} 2.34 \\ 5.58 \\ 2.83 \\ 12.65 \\ -3.17 \\ 13.13 \\ -2.46 \\ 6.01 \end{Bmatrix} + y_1 \begin{Bmatrix} -2.07 \\ -1.30 \\ -2.10 \\ -3.50 \\ 0.25 \\ -3.53 \\ 0.27 \\ -1.05 \end{Bmatrix} \Delta X_1 + y_2 \begin{Bmatrix} -0.01 \\ -0.02 \\ -0.44 \\ -0.27 \\ 0.04 \\ -0.21 \\ -0.01 \\ 0.02 \end{Bmatrix} \Delta X_2 + y_3 \begin{Bmatrix} -0.29 \\ -1.10 \\ -0.26 \\ -3.70 \\ 2.20 \\ -3.67 \\ 2.17 \\ -1.36 \end{Bmatrix} \Delta X_3$$

Assuming the optimal changes in the three cross sections, $\Delta X_1 = 6.94$, $\Delta X_2 = -0.90$, and $\Delta X_3 = 7.06$, solution of the reduced CA problem gives $y_1 = 0.127$, $y_2 = 5.155$, $y_3 = 0.123$.

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9 Topological and Geometrical Changes

Changes in the topology and the geometry of the structure are encountered particularly in layout optimization problems [e.g. 1]. Layout optimization means the simultaneous selection of the optimal variables describing

- the topology (i.e. spatial sequence of members and joints);
- the geometry (i.e. the location of joints); and
- the cross-sectional dimensions (sizing).

Layout optimization is perhaps the most challenging class of problems in structural optimization because there exists a large number of possible topologies and they are difficult to classify and quantify. Moreover, at each point of the available space, potential members may run in a large number of directions. At the same time, layout optimization is of considerable importance because it results in much greater material savings than pure cross-section optimization.

Layout optimization is usually based on a ground structure, which is the union of all potential members. During the optimization procedure, non-optimal members are eliminated and the optimal size of the remaining members is determined. From the viewpoint of the ground structure approach, layout optimization can also be defined as a special case of cross-section sizing optimization in which cross-sectional areas may take on a zero value. An alternative to this approach is an incremental synthesis approach in which, starting with a few members, new members and joints are progressively added.

Because of the complexity in simultaneous optimization of the geometry, the topology and the cross sections, two classes of problems are often considered in optimization of the structural layout:

a. *Topological optimization,* where members are usually removed from the highly connected ground structure, to derive an optimum topology with the corresponding cross sections; that is, both the topological and the sizing variables are optimized simultaneously.

b. *Geometrical optimization,* where the coordinates of joints and the cross-sectional sizes are treated as design variables and optimized simultaneously.

In this chapter topological and geometrical changes of the structure are considered. Various changes in the topology are considered in Section 9.1. Developing a reanalysis method for topological changes is most challenging, since the structural model is changed and the resulting response might be significantly different from the initial response. Most approximate reanalysis methods are not suitable for such changes and provide inadequate or meaningless results. It is shown that accurate approximations can be achieved efficiently by the CA method for significant changes in the topology. Moreover, for some structures, such as trusses, exact solutions can be achieved by the solution procedure described in Section 8.2.4.

Geometrical changes are considered in Section 9.2. Accurate approximations for changes affecting a large number of members are discussed in Section 9.2.1. Exact solutions for changes affecting only a small number of members are demonstrated in Section 9.2.2.

The approach presented in this chapter is suitable for all types of structures. For simplicity of presentation small-scale truss structures are used. In all examples the cross-sectional areas are equal to unity.

9.1 TOPOLOGICAL CHANGES

Topological optimization has some basic difficulties involved in the solution process compared to fixed-layout optimization. One basic problem is that the structural model is itself allowed to vary during the design process. Discrete structures are generally characterized by the fact that the finite element model of the structure is not modified during the optimization process. In topological design, however, since members are added or deleted during the design process, both the finite element model and the set of design variables change. These phenomena greatly complicate the design and analysis interactions. Another difficulty is that the number of possible element-joint connectivities grows dramatically as the number of possible joint locations is increased. This number might be very large particularly in structures of practical size. These difficulties have motivated the use of approximate analysis in topological optimization.

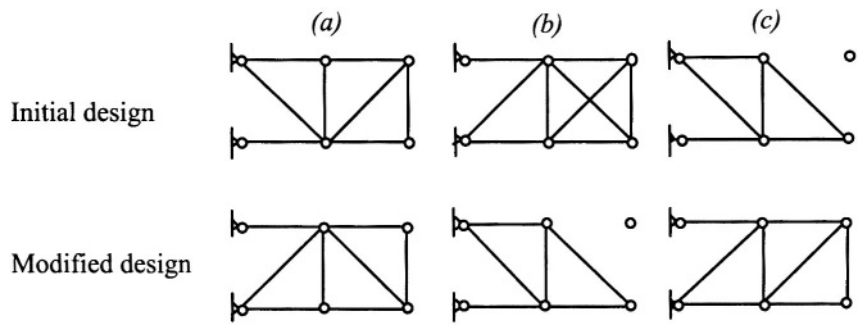

Fig. 9.1. Types of topological changes.

Most reanalysis methods are suitable for the simple cases where the number of Degrees of Freedom (DOF), or the number of analysis equations, is unchanged. The reanalysis approach considered in this section is suitable for problems where the number of DOF and the sizes of **K**, **r** and **R** are changed. The following typical cases of changes in the topology are considered (see Figure 9.1):

a. Deletion and addition of members, where the number of DOF is unchanged (Figure 9.1*a*). In this case, the number of analysis equations is also unchanged and only numerical values of the coefficients of the equations are modified. It is shown that exact solutions can be obtained efficiently by the CA method in cases where the number of changed members is relatively small.

b. Deletion and addition of members, and deletion of some joints, where the number of DOF is decreased (Figure 9.1*b*). It is shown that in this case approximate reanalysis by the CA method with a reduced number of unknowns provides accurate results even when the resulting structure is conditionally unstable.

c. Deletion and addition of members, and addition of some joints, where the number of DOF is increased (Figure 9.1*c*). In this case it is necessary to augment the analysis model such that the new degrees of freedom are included in the modified analysis equations. A general solution procedure is developed, where an exact modified initial analysis is first efficiently carried out. The modified initial analysis is then used for reanalysis of further topological modifications.

9.1.1 Number of DOF is Unchanged

In cases where the number of DOF is unchanged, the general formulation presented in Section 3.1 can be considered for various topological changes. Evaluation of modified displacements by the CA method for such cases is described in Chapter 7.

It was noted in Section 8.2 that exact solutions could be obtained efficiently by the CA method for low-rank changes in the stiffness matrix [2, 3]. Exact solutions obtained by the CA method for topological changes are demonstrated in this section. The solution procedure described in Section 8.2.4 can be used also in cases where members are deleted or added [3, 4]. The procedure is efficient in cases where the number of changed members is much smaller than the number of degrees of freedom.

As a typical example consider the case of deletion or addition of m truss members. The exact solution is achieved by the CA method if one basis vector is introduced for each changed member by Eq. (8.29)

$$\mathbf{r}_i = - \mathbf{K}^{*-1} \Delta\mathbf{K}_i \mathbf{r}^* \qquad i = 1, ..., m \tag{9.1}$$

where $\Delta\mathbf{K}_i$ is the contribution of the ith member to $\Delta\mathbf{K}$. The solution is given by Eq. (8.33)

$$\mathbf{r} = \mathbf{r}^* + \sum_{i=1}^{m} y_i \mathbf{r}_i \tag{9.2}$$

where $\mathbf{r}^*$ is the vector of initial displacements. When some of the basis vectors are linearly dependent, the exact solution is achieved for a smaller number of vectors.

Example 9.1 – Conditionally Unstable Structures. To illustrate exact solutions for modified designs representing conditionally unstable structures, consider again the initial ten-bar truss shown in Fig. 9.2 subjected to a single loading condition of two concentrated loads. The modulus of elasticity is 30000 and the eight analysis unknowns are the horizontal and the vertical displacements at joints 1, 2, 3 and 4, respectively. The following conditionally unstable modified structures are solved [5]:

a. Elimination of members 2+6 (Figure 9.3*a*).
b. Elimination of members 4+9 (Figure 9.3*b*).
c. Elimination of members 5+8+9 (Figure 9.3*c*).
d. Elimination of members 4+5+8+9 (Figure *93d*).

The exact solutions are found for all the above cases by second-order approximations (CA2, only 3 basis vectors), as summarized in Table 9.1.

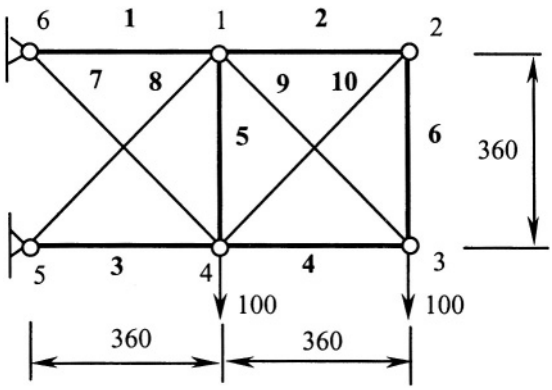

Fig. 9.2. Ten-Bar Truss.

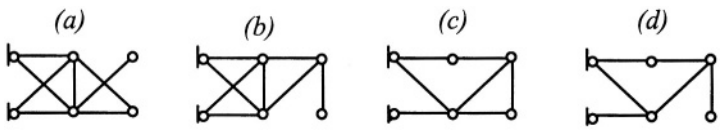

Fig. 9.3. Conditionally unstable structures obtained by elimination of members.

Table 9.1. Exact solutions obtained by second-order approximations (CA2).

Deleted members	Displacements							
	1	2	3	4	5	6	7	8
2+6	2.40	5.80	•	•	-3.60	15.18	-2.40	5.80
4+9	2.11	4.67	3.30	13.62	•	14.81	-1.35	5.57
5+8+9	•	•	2.40	19.76	-3.60	20.96	-3.60	10.38
4+5+8+9	•	•	2.40	19.76	•	20.96	-3.60	10.38

• Irrelevant results because of conditional instability

Example 9.2 – 50-Bar Truss. Consider the cantilever truss shown in Figure 9.4*a*. The truss is subjected to a single load at the tip, the member cross section areas are equal to unity, the modulus of elasticity is 10 000 and the 40 unknowns are the *X* direction (horizontal) and the *Y* direction (vertical) displacements in joints 2 through 21, respectively. Members are designated as follows: 1-10 = top chord; 11-20 = bottom chord; 21-40 = diagonals; 41-50 = verticals. Eliminating 10 diagonal members (members 22, 24, .., 40) to obtain the topology shown in Figure 9.4*b*, the resulting displacements and member forces found by the CA method and exact analysis are given in Tables 9.2 and 9.3 [3]. It is observed that results achieved by second-order approximations (CA2, only 3 basis vectors) are very close to the exact solution.

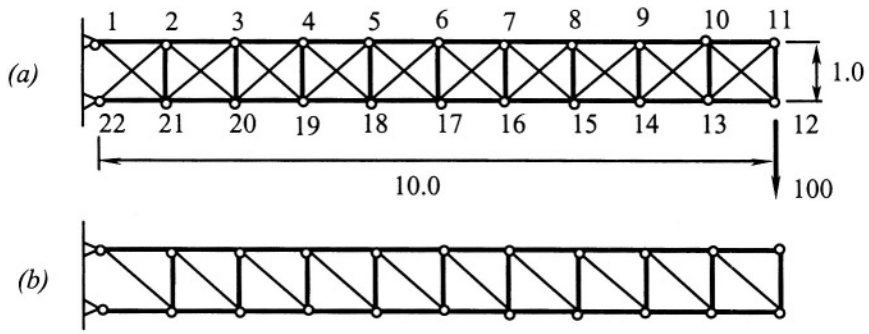

Fig.9.4. 50-bar truss: *a*. Initial topology *b*. Final topology.

Example 9.3 – 19-Bar Truss [4]. Consider the tower truss shown in Figure 9.5 with nineteen cross-sectional area variables, subjected to a single loading condition of two concentrated loads. The modulus of elasticity is 10000 and the twelve analysis unknowns are the horizontal and the vertical displacements at joints 2, 3, 4, 6, 7, and 8, respectively. With initial cross-sectional areas of unity and an initial nineteen-bar topology, the following four cases are solved by considering 0.001 (practically zero) cross-section areas for the eliminated members (see Figure 9.6):

Table 9.2 50-bar truss, displacements after elimination of members.

Joint	Direction	CA1	CA2	Exact
2	X	0.089	0.090	0.090
	Y	-0.143	-0.138	-0.138
3	X	0.169	0.170	0.170
	Y	-0.460	-0.457	-0.457
4	X	0.239	0.240	0.240
	Y	-0.938	-0.935	-0.935
5	X	0.300	0.300	0.300
	Y	-1.555	-1.554	-1.554
6	X	0.350	0.350	0.350
	Y	-2.293	-2.292	-2.292
7	X	0.390	0.390	0.390
	Y	-3.131	-3.130	-3.131
8	X	0.420	0.420	0.420
	Y	-4.048	-4.049	-4.049
9	X	0.440	0.440	0.440
	Y	-5.026	-5.027	-5.028
10	X	0.450	0.450	0.450
	Y	-6.044	-6.046	-6.046
11	X	0.450	0.450	0.450
	Y	-7.074	-7.074	-7.075
12	X	-0.550	-0.550	-0.550
	Y	-7.074	-7.074	-7.075
13	X	-0.540	-0.540	-0.540
	Y	-6.034	-6.036	-6.036
14	X	-0.520	-0.520	-0.520
	Y	-5.016	-5.017	-5.018
15	X	-0.490	-0.490	-0.490
	Y	-4.039	-4.039	-4.039
16	X	-0.450	-0.450	-0.450
	Y	-3.121	-3.120	-3.121
17	X	-0.400	-0.400	-0.400
	Y	-2.283	-2.282	-2.282
18	X	-0.341	-0.340	-0.340
	Y	-1.546	-1.544	-1.544
19	X	-0.271	-0.270	-0.270
	Y	-0.928	-0.925	-0.925
20	X	-0.191	-0.190	-0.190
	Y	-0.450	-0.447	-0.447
21	X	-0.101	-0.100	-0.100
	Y	-0.132	-0.128	-0.128

Table 9.3. 50-bar truss, member forces after elimination of members.

Member	CA1	CA2	Exact
1	890.2	900.0	900.2
2	802.7	799.2	800.2
3	701.4	700.5	700.2
4	601.5	600.3	600.1
5	501.4	500.3	500.1
6	401.3	400.3	400.1
7	301.3	300.4	300.1
8	201.4	200.3	200.0
9	101.1	99.9	100.0
10	-2.7	-0.4	0.0
11	-1010.3	-1000.2	-1000.3
12	-897.7	-901.0	-900.2
13	-798.9	-799.8	-800.2
14	-698.8	-700.0	-700.2
15	-598.7	-600.0	-600.1
16	-498.7	-500.0	-500.1
17	-398.6	-400.0	-400.1
18	-298.5	-300.0	-300.1
19	-198.9	-200.3	-200.0
20	-102.7	-100.4	-100.0
21	155.7	141.5	141.5
22	0.0	0.0	0.0
23	137.9	142.6	141.5
24	0.0	0.0	0.0
25	139.8	140.8	141.5
26	0.0	0.0	0.0
27	139.6	141.2	141.5
28	0.0	0.0	0.0
29	139.6	141.2	141.5
30	0.0	0.0	0.0
31	139.6	141.2	141.5
32	0.0	0.0	0.0
33	139.6	141.1	141.5
34	0.0	0.0	0.0
35	139.4	141.1	141.5
36	0.0	0.0	0.0
37	139.9	141.7	141.4
38	0.0	0.0	0.0
39	145.2	142.0	141.4
40	0.0	0.0	0.0
41	-107.5	-100.9	-100.0
42	-96.3	-100.5	-100.0
43	-97.5	-99.5	-100.0
44	-97.3	-99.7	-100.0
45	-97.4	-99.6	-100.0
46	-97.4	-99.6	-100.0
47	-97.3	-99.6	-100.0
48	-97.5	-100.0	-100.0
49	-101.6	-100.5	-100.0
50	-2.8	-0.3	0.0

a. Elimination of six members (Figure 9.6*a*).
b. Elimination of seven members (Figure 9.6*b*).
c. Elimination of nine members (Figure 9.6*c*).
d. Elimination of members as in case *c* and optimization of the remaining members to find the optimal cross section areas

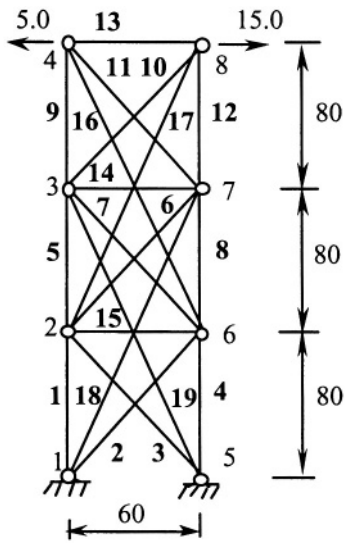

Fig. 9.5. 19-bar tower truss.

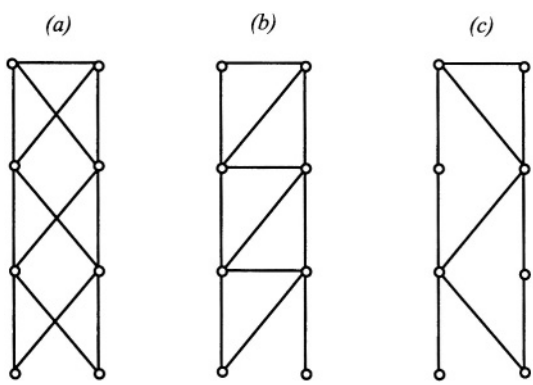

Fig. 9.6. Modified truss topologies.

Table 9.4. Displacements, 19-bar truss.

	Case *a*		Case *b*		Case *c*		Case *d*	
No	CA2	Exact	CA2	Exact	CA2	Exact	CA2	Exact
1	0.49	0.48	0.78	0.76	0.71	0.70	0.49	0.55
2	0.27	0.26	0.21	0.21	0.32	0.32	0.16	0.16
3	1.57	1.58	2.07	2.09	-	-	-	-
4	0.43	0.43	0.33	0.32	-	-	-	-
5	2.88	2.88	3.61	3.62	3.53	3.53	2.82	2.92
6	0.48	0.48	0.35	0.32	0.54	0.53	0.52	0.48
7	0.50	0.50	0.72	0.70	-	-	-	-
8	-0.27	-0.27	-0.32	-0.32	-	-	-	-
9	1.54	1.54	2.00	2.03	1.97	1.98	1.40	1.52
10	-0.42	-0.43	-0.51	-0.53	-0.42	-0.43	-0.30	-0.32
11	2.93	2.93	3.64	3.65	3.62	3.62	2.95	3.04
12	-0.48	-0.48	-0.63	-0.64	-0.42	-0.43	-0.50	-0.32

$$\mathbf{X}_{opt}^{T} = \{2.0, 0, 0.83, 1.33, 0.67, 0.83, 0, 1.33, 0.67, 0, 0.83, 0.1, 0.75, 0, 0, 0, 0, 0, 0\}$$

The results shown in Table 9.4 indicate that high accuracy is achieved by the CA2 procedure for most cases of these large topological changes.

9.1.2 Number of DOF is Decreased

This type of reanalysis problem is encountered in many topological optimization problems where some members and joints are deleted from an initial ground structure, which consists of numerous members and joints. As a result, the number of DOF is decreased and the number of analysis equations is changed.

The sizes of the stiffness matrix and the load vector are decreased according to the number of joints deleted from the structure. The modified stiffness matrix and the modified load vector can be expressed as

$$\mathbf{K} = \mathbf{K}^{*} + \Delta\mathbf{K} = \begin{bmatrix} \mathbf{K}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{9.3}$$

$$\mathbf{R} = \mathbf{R}^{*} + \Delta\mathbf{R} = \begin{Bmatrix} \mathbf{R}_r \\ \mathbf{0} \end{Bmatrix} \tag{9.4}$$

where $\mathbf{K}$ and $\mathbf{R}$ are the modified stiffness matrix and the modified load vector, respectively, of the complete set of equations, including the original degrees of freedom; and $\mathbf{K}_r$ and $\mathbf{R}_r$ are the stiffness sub-matrix and the load sub-vector, respectively, of the modified structure with the reduced number of degrees of freedom. The set of modified equations (3.5) to be solved is reduced to

$$\mathbf{K}_r \mathbf{r} = \mathbf{R}_r \tag{9.5}$$

where $\mathbf{r}_r$ is now a reduced vector of modified displacements. Despite the reduction in the size of the stiffness matrix, the number of modified analysis equations is often large and efficient reanalysis might prove useful.

Since some analysis equations become zero identities, stiffness analysis of the complete set of modified equations cannot be carried out. It is also noted that the resulting structure represented by Eq, (9.5) might be conditionally unstable. In which case, the modified stiffness matrix is singular and, again, exact analysis cannot be carried out. The reduced stiffness matrix $\mathbf{K}_R$ used by the CA method is usually not singular even in cases when the modified stiffness matrix $\mathbf{K}_r$ is singular. Therefore, approximate reanalysis by the CA method with a reduced number of unknowns may provide accurate results even when the modified matrix is singular. If the number of basis vectors considered is smaller than the number of degrees of freedom of the modified structure, the usual CA procedure will provide accurate results efficiently.

Example 9.4 - 2050-Bar Truss. Consider a minimum compliance (external work) truss topology design problem for a truss with a ground structure of 51×11 nodes. The potential bars connect all neighboring nodes as shown in Figure 9.7. All the nodes on the left-hand-side are fixed and the bottom node at the right-hand-side is subject to a vertical load. Altogether there are 2050 bars and 1100 degrees of freedom (nodal displacements). The optimal topology is shown in Figure 9.8 and the compliance of the optimal design is $6.2165 \cdot 10^{-6}$ [6].

Results achieved by the CA method are demonstrated for different systems defined by various values of the initial and the modified stiffness matrices. These matrices are given (through the design variables $\mathbf{X}$) by a parameter α according to $\mathbf{X} = \mathbf{X}^* + \alpha(\mathbf{X}_{opt} - \mathbf{X}^*)$ where $\mathbf{X}^*$ is a vector of initial constant numbers 1/2050 and $\mathbf{X}_{opt}$ is the optimal design vector from Figure 9.8. All design vectors satisfy the feasibility constraint $\sum X_i = 1$. Table 9.5 shows minimal and maximal values of the design variables for various α values, and indicates the significant changes in $\mathbf{X}$.

The condition numbers of the original matrix $\mathbf{K}$ and the preconditioned matrix $\tilde{\mathbf{K}}$ [see Eq. (7.54)] for various modified designs are shown in Table 9.6. It can be seen that the condition numbers of matrices $\tilde{\mathbf{K}}$ are much smaller than those of $\mathbf{K}$. The compliance and the errors for various initial designs, modified designs and numbers of basis vectors are shown in Tables 9.7 and 9.8. For the initial design $\alpha = 0$ and modified designs $\alpha = 0.1 - 0.5$, 10 - 15 iterations are needed to achieve accurate solutions. For the initial designs $\alpha = 0.1 - 0.5$ and modified designs $\alpha = 0.5 - 0.99$, only 3 iterations are needed to achieve accurate results.

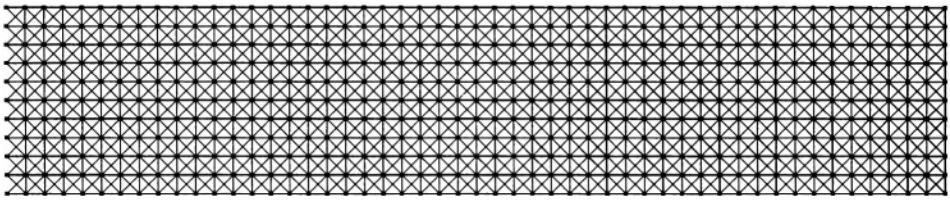

Fig. 9.7. Initial ground structure, 2050-bar truss.

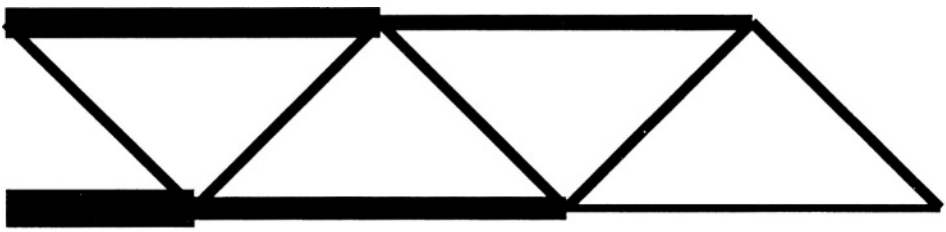

Fig. 9.8. Optimal topology, 2050-bar truss.

Table 9.5. Minimal and maximal **X** for various α values.

	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$	$\alpha = 0.99$	$\alpha = 1.0$
$\mathbf{X}_{min}$	0.0004878	0.000439	0.000243	4.9e-05	4.9e-06	2.7e-12
$\mathbf{X}_{max}$	0.0004878	0.014725	0.071672	0.12862	0.14143	0.14286

Table 9.6. Condition numbers of **K** and $\tilde{\mathbf{K}}$ for various modified designs.

Initial α	Modified α	$\psi(\mathbf{K})$	$\psi(\tilde{\mathbf{K}})$
0	0.1	1.05e+06	32.2
0	0.5	1.67e+06	281.8
0.1	0.5	1.67e+06	8.8
0.1	0.9	2.29e+06	78.5
0.5	0.9	2.29e+06	9.0
0.5	0.99	1.08e+07	98.7

Table 9.7. Compliance (times 10^6) for various numbers of basis vectors.

Initial α	Modified α	Basis vectors				Exact
		3	5	10	15	solution
0	0.1	36.48	39.15	39.55	39.55	39.55
0	0.5	8.88	10.76	11.41	11.44	11.44
0.1	0.5	11.44				11.44
0.1	0.9	6.84				6.84
0.5	0.9	6.84				6.84
0.5	0.99	6.27				6.27

Table 9.8. Errors for various numbers of basis vectors.

Initial α	Modified α	Basis vectors			
		3	5	10	15
0	0.1	2.79e-01	1.00e-01	9.02e-03	8.10e-04
0	0.5	4.73e-01	2.43e-01	4.62e-02	8.31e-03
0.1	0.5	7.47e-03			
0.1	0.9	1.24e-02			
0.5	0.9	3.88e-05			
0.5	0.99	5.50e-05			

9.1.3 Number of DOF is Increased

In cases where some members and joints are added to the initial structure, the number of DOF is increased, the number of analysis equations is changed, and the sizes of the stiffness matrix and the load vector are increased accordingly.

Let us define the augmented stiffness matrix $\mathbf{K}_A$ and the augmented load vector $\mathbf{R}_A$, with the increased number of DOF, by

$$\mathbf{K}_A = \begin{bmatrix} \mathbf{K}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \mathbf{R}_A = \begin{Bmatrix} \mathbf{R}^* \\ \mathbf{0} \end{Bmatrix} \tag{9.6}$$

Upon increasing the number of DOF, the matrix of changes $\Delta\mathbf{K}$ and the vector of changes $\Delta\mathbf{R}$ can be expressed in terms of corresponding sub-matrices and sub-vectors as

$$\Delta\mathbf{K} = \begin{bmatrix} \Delta\mathbf{K}_{00} & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix} \qquad \Delta\mathbf{R} = \begin{Bmatrix} \Delta\mathbf{R}_0 \\ \Delta\mathbf{R}_N \end{Bmatrix} \tag{9.7}$$

where $\Delta\mathbf{K}_{00}$ and $\Delta\mathbf{R}_0$ are the changes in stiffness coefficients and in the loads, respectively, corresponding to the original DOF; $\Delta\mathbf{K}_{NN}$ and $\Delta\mathbf{R}_N$ are the changes corresponding to the new DOF; and $\Delta\mathbf{K}_{0N}$ and $\Delta\mathbf{K}_{N0}$ are the changes in the coefficients corresponding to both the original and the new DOF.

The modified stiffness matrix and the modified load vector are given by

$$\begin{aligned}\mathbf{K} &= \mathbf{K}_A + \Delta\mathbf{K} \\ \mathbf{R} &= \mathbf{R}_A + \Delta\mathbf{R}\end{aligned} \tag{9.8}$$

where the new degrees of freedom are included in the set of modified analysis equations. The number of added DOF is usually small, compared with the original number of DOF. It is shown in the following that in such cases it is possible to calculate the modified displacements efficiently using the CA method.

Some procedures for using the CA method in cases where the number of DOF is increased have been previously proposed [3, 7]. The procedure presented in this section is more general and provides more accurate results [5], Upon increasing the number of DOF, it is necessary first to establish a Modified Initial Analysis (MIA), such that the new degrees of freedom are included in the analysis model. For the augmented stiffness matrix and the augmented load vector [Eq. (9.6)], the MIA model can be selected such that reanalysis will be convenient. Once the MIA is established, it is then possible to analyze structures modified due to addition or deletion of members, keeping the number of degrees of freedom unchanged.

The MIA is established as follows. The matrix of changes in the stiffness $\Delta\mathbf{K}$ is expressed first as a sum of the two matrices $\Delta\mathbf{K}_0$ and $\Delta\mathbf{K}_N$ by

$$\Delta\mathbf{K} = \Delta\mathbf{K}_0 + \Delta\mathbf{K}_N \tag{9.9}$$

These two matrices are defined in such a way that the modified initial analysis is easy to carry out. The modified initial stiffness matrix $\mathbf{K}_M$ is expressed as

$$\mathbf{K}_M = \mathbf{K}_A + \Delta\mathbf{K}_0 \tag{9.10}$$

Matrices $\Delta\mathbf{K}_0$ and $\Delta\mathbf{K}_N$ are defined as

$$\begin{aligned}\Delta\mathbf{K}_0 &= \alpha\begin{bmatrix} 0 & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix} \\ \Delta\mathbf{K}_N &= \begin{bmatrix} \Delta\mathbf{K}_{00} & (1-\alpha)\Delta\mathbf{K}_{0N} \\ (1-\alpha)\Delta\mathbf{K}_{N0} & (1-\alpha)\Delta\mathbf{K}_{NN} \end{bmatrix}\end{aligned} \tag{9.11}$$

where α is a scalar multiplier to be selected ($0 < \alpha \leq 1$). Substituting the expressions of $\mathbf{K}_A$ and $\Delta\mathbf{K}_0$ [Eqs. (9.6) and (9.11), respectively] into Eq. (9.10) yields

$$\mathbf{K}_M = \begin{bmatrix} \mathbf{K}^* & \alpha\Delta\mathbf{K}_{0N} \\ \alpha\Delta\mathbf{K}_{N0} & \alpha\Delta\mathbf{K}_{NN} \end{bmatrix} \tag{9.12}$$

The rationale of this selection is that, once the decomposed form of Eq. (3.2) is available, factorization of the modified initial stiffness matrix $\mathbf{K}_M$

$$\mathbf{K}_M = \mathbf{U}_M{}^T \mathbf{U}_M \tag{9.13}$$

is straightforward. Specifically, matrix $\mathbf{U}_M$ can be expressed as

$$\mathbf{U}_M = \begin{bmatrix} \mathbf{U}^* & \mathbf{U}_{0N} \\ \mathbf{0} & \mathbf{U}_{NN} \end{bmatrix} \tag{9.14}$$

where the elements of matrix $\mathbf{U}^*$ are already given. That is, the rows and columns corresponding to the original degrees of freedom are unchanged and only rows and columns corresponding to the new degrees of freedom are calculated. In general the number of added joints is small, and the factorization of Eq. (9.13) involves a small computational effort.

Concerning the selected value of α, it is observed that $\alpha = 1$ yields

$$\mathbf{K}_M = \begin{bmatrix} \mathbf{K}^* & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix} \tag{9.15}$$

One drawback of this selection is that matrix $\mathbf{K}_M$ is not necessarily positive definite and the factorization of Eq. (9.13) might not be possible. In such cases we can use the symmetric factorization

$$\mathbf{K}_M = \mathbf{L}_M \mathbf{D}_M \mathbf{L}_M{}^T \tag{9.16}$$

where $\mathbf{L}_M$ is a lower triangular matrix and $\mathbf{D}_M$ is a diagonal matrix. It should be noted that matrix $\mathbf{K}_M$ does not represent a real structure and experience has shown that the accuracy of the approximations deteriorates in this case.

In the following procedure, this difficulty is overcome by selecting a small α value such that matrix $\mathbf{K}_M$ [Eq. (9.12)] is a good approximation of the matrix $\mathbf{K}_A+\alpha\Delta\mathbf{K}$ [since $\alpha\Delta\mathbf{K}_{00} \ll \mathbf{K}^*$, see Eqs. (9.6), (9.7)]. With these definitions, the solution procedure involves the following two stages.

a. The modified initial analysis (MIA) is established. Assuming a small α value, we calculate and factorize the matrix $\mathbf{K}_M$ [Eqs. (9.12), (9.13)]. Since the decomposed form of Eq. (3.2) is available, this operation involves a small computational effort. The modified initial displacements $\mathbf{r}_M$ are then calculated by solving

$$\mathbf{K}_M \mathbf{r}_M = \mathbf{R} \tag{9.17}$$

This calculation involves only forward and backward substitutions.

b. Once the vector $\mathbf{r}_M$ has been determined, the displacements due to the remaining change in the stiffness matrix $\Delta\mathbf{K}_N$ [Eq. (9.11)] are calculated. The modified equations to be solved are

$$\mathbf{K}\,\mathbf{r} = (\mathbf{K}_M + \Delta\mathbf{K}_N)\,\mathbf{r} = \mathbf{R} \tag{9.18}$$

The solution process involves using the CA procedure described in Section 7.1 for the simple case where the number of DOF is unchanged, with $\mathbf{r}_M$, $\mathbf{K}_M$, $\mathbf{U}_M$ and $\mathbf{R}$ replacing $\mathbf{r}^*$, $\mathbf{K}^*$, $\mathbf{U}^*$ and $\mathbf{R}^*$, respectively, as initial values.

Example 9.5 - Number of DOF is Increased [5]. To illustrate reanalysis for the case of addition of members and joints, consider the initial six-bar truss shown in Figure 9.9*a.* The six unknowns are the horizontal and the vertical displacements at joints 1, 2 and 3, respectively. The initial displacement vector and the decomposed stiffness matrix are given by

$$\mathbf{r}^{*T} = \{1.20,\ 11.59,\ -4.80,\ 20.98,\ -3.60,\ 10.39\}$$

$$\mathbf{U}^* = \begin{bmatrix} 10.62 & 2.77 & -2.77 & -2.77 & 0 & 0 \\ & 10.25 & -2.12 & -2.12 & 0 & -8.13 \\ & & 10.03 & 1.72 & -8.31 & -1.72 \\ & & & 3.78 & 3.78 & -3.78 \\ & & & & 10.62 & 2.77 \\ & & & & & 4.67 \end{bmatrix}$$

Assume addition of one joint and four members to obtain the ten-bar truss shown in Figure 9.9*b.* Reanalysis is carried out in the following two stages:

a. The Modified Initial Analysis (MIA) is carried out. Selecting $\alpha = 0.001$, the initial decomposed stiffness matrix is given by $\mathbf{U}_M$ [Eq. (9.14)], where $\mathbf{U}^*$ is already given as above. Thus, it is necessary to calculate only the sub-matrices $\mathbf{U}_{0N}$, $\mathbf{U}_{NN}$ as

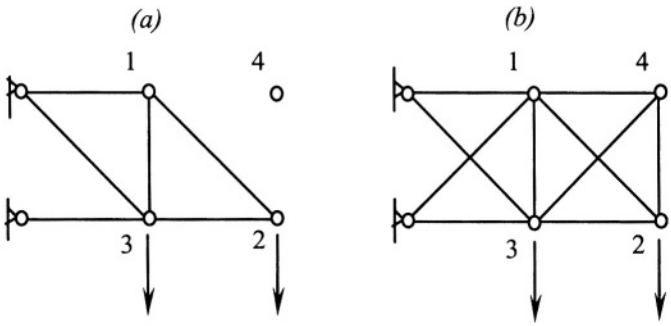

Fig. 9.9. *a*. Six-bar truss, initial topology. *b*. Modified topology.

$$\mathbf{U}_{0N} = \begin{bmatrix} -0.0078 & 0 \\ 0.0021 & 0 \\ -0.0017 & 0 \\ -0.0038 & -0.0220 \\ -0.0028 & 0.0106 \\ 0.0080 & -0.0305 \end{bmatrix} \qquad \mathbf{U}_{NN} = \begin{bmatrix} 0.3356 & -0.0872 \\ & 0.3220 \end{bmatrix}$$

For the $\mathbf{U}_M$ found from Eq. (9.14), calculation of the modified initial displacement vector $\mathbf{r}_M$ through Eq. (9.17) involves only forward and backward substitutions. The result is

$$\mathbf{r}_M^T = \{1.22, 11.74, -4.87, 21.28, -3.65, 10.52, 2.44, 20.06\}$$

It is observed that, due to the small change in stiffness, the displacements of the original degrees of freedom have changed only slightly.

b. The displacements due to the remaining change in the stiffness matrix $\Delta\mathbf{K}_N$ are calculated. Employing the CA procedure described in Section 7.1.2, with $\mathbf{r}_M$, $\mathbf{K}_M$, $\mathbf{U}_M$ and $\mathbf{R}$ replacing $\mathbf{r}^*$, $\mathbf{K}^*$, $\mathbf{U}^*$ and $\mathbf{R}^*$, respectively, as initial values, we achieve the exact solution with only three basis vectors (CA2, second-order approximations):

$$\mathbf{r}^T = \{2.34, 5.58, -3.17, 13.13, -2.46, 6.01, 2.82, 12.65\}$$

9.2 GEOMETRICAL CHANGES

In geometrical optimization the coordinates of joints and the cross-sectional sizes are treated as design variables and optimized simultaneously. In general, the design variables are assumed to be continuous, and numerical search algorithms are used to find the optimum. It is usually assumed that the topology is fixed, unless some of the joints coalesce during the solution process. The difficulties associated with the common problem of cross-section optimization (i.e., the implicit nature of the

constraints and the large numbers of variables and constraints) are magnified in problems of geometrical optimization. That is, the problem size is increased and the need for multiple repeated analyses is a major obstacle in applying structural optimization methods to large-scale structures.

Geometrical changes are conceptually similar to cross-section changes in the sense that the number of degrees of freedom is usually unchanged. However, since the displacements are highly nonlinear functions of the design variables, it is difficult to achieve accurate approximations. In addition, changes in the geometry significantly affect the response of the structure. Accurate and exact solutions for geometrical changes are demonstrated in this section.

It was noted in Section 7.3 that accurate approximations are obtained by the CA method in various cases. It is shown in Section 9.2.1 that accurate solutions can be achieved for geometrical changes if the basis vectors are close to being linearly dependent or, for nearly-scaled geometries, when the angle between the two vectors representing the initial design and modified design is small.

It is shown in Section 9.2.2 that exact solutions can be achieved efficiently for geometrical changes by viewing these changes as corresponding topological changes. Modifying the coordinates of a single joint, it is possible to obtain the exact solution for the new design by viewing the change in the geometry as two simultaneous changes in the topology: all members connected to the joint are deleted, and new members are added at the modified location [3, 8].

The approach presented is suitable for different types of structures and design variables. For illustrative purposes truss structures are considered in this section. Arbitrary units and cross sectional areas equal to unity are assumed in all examples.

9.2.1 Accurate Solutions

Example 9.6 - Nearly Dependent Basis Vectors (Ten-Bar Truss). Consider the ten-bar truss shown in Figure 9.10 with two geometrical variables, the depth D and the panel width W. With the initial values $W = D = 360$, then for $W = D = 720$ (increase of 100% in the length of all members), the exact solution is simply

$$\mathbf{r}^T = 2\,\mathbf{r}_0^T = \{4.68, 11.16, 5.64, 25.30, -6.34, 26.26, -4.92, 12.02\}$$

To demonstrate the accuracy of low-order approximations achieved by the CA method for various geometrical changes, assume the first-order (CA1, only 2 basis vectors) and the second-order (CA2, only 3 basis vectors) procedures for the following three cases of modified geometries [8]:

a, $W = 360$, $D = 540$ (increase of 50% in the depth).
b. $W = 360$, $D = 720$ (increase of 100% in the depth).
c. $W = 180$, $D = 720$ (decrease of 50% in the width and increase of 100% in the depth).

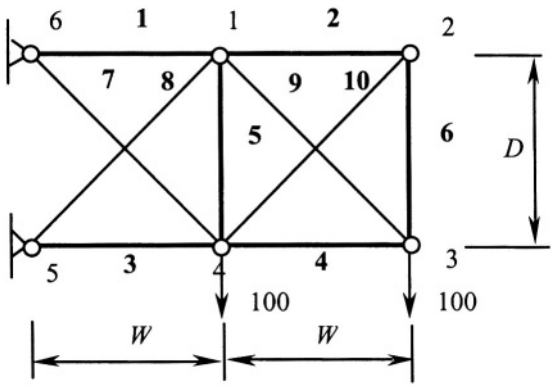

Fig. 9.10. Ten-bar truss.

From the results summarized in Table 9.9 it is observed that the changes in the geometry lead to significant changes in the displacements. The high accuracy of the results is explained by the fact that the basis vectors determined by the CA method are close to being linearly dependent [two basis vectors $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ are close to being linearly dependent if the condition of Eq. (7.41) is satisfied, where $\beta_{i,\,i+1}$ is the angle between the two vectors]. In Case *b,* for example, $\cos \beta_{1,2} = 0.9912$ and $\cos \beta_{2,3} = 0.9999$ for the three basis vectors of the CA2 procedure.

Table 9.9. First- and second-order approximations, ten-bar truss.

Case	W	D	Method	1	2	3	4	5	6	7	8
Initial	360	360	Exact	2.34	5.58	2.82	12.65	-3.17	13.13	-2.46	6.01
a	360	540	CA1	1.49	4.02	1.71	7.86	-2.06	8.40	-1.60	4.48
			CA2	1.53	3.93	1.81	7.83	-2.17	8.46	-1.64	4.45
			Exact	1.55	3.94	1.82	7.84	-2.18	8.47	-1.66	4.44
b	360	720	CA1	1.17	3.78	1.26	6.72	-1.61	7.29	-1.28	4.27
			CA2	1.14	3.67	1.34	6.62	-1.68	7.35	-1.24	4.25
			Exact	1.15	3.67	1.34	6.60	-1.66	7.36	-1.25	4.24
c	180	720	CA1	0.43	2.59	0.36	3.83	-0.59	4.27	-0.50	2.97
			CA2	0.31	2.52	0.43	3.86	-0.55	4.44	-0.35	2.98
			Exact	0.29	2.47	0.33	3.85	-0.42	4.53	-0.31	2.94

Example 9.7 - Nearly Scaled Designs (50-Bar Truss). Consider the initial geometry of the cantilever truss shown in Figure 9.11. The truss is subjected to a single load at the tip, the member cross section areas are equal to unity, the modulus of elasticity is 10 000 and the 40 unknowns are the X direction (horizontal) and the Y direction (vertical) displacements in joints 2 through 21, respectively. Two geometric variables are considered, the depth D and the panel width W. Exact solution is achieved with a single basis vector for all designs where the ratio between the depth and the width of the truss is unchanged. The reason is that the vertical and the horizontal joint coordinates are changed such that the geometry is scaled (the lengths of all members are changed by the same percentage and the direction cosines are unchanged).

With the initial design $D = W = 1.0$, the following two cases of changes in the geometry are solved [8]:

a. A change of 20% in the depth $D = 1.2$ (Figure 9.11 *a*).
b. A change of 100% in the depth $D = 2.0$ and 90% in the width $W = 1.9$ (Figure 9.11*b*).

The results are given in Table 9.10 for the CA1 procedure (first-order approximations, only two basis vectors). Comparing the results obtained for the two cases of geometrical changes, we see that better approximations are achieved in case *b,* involving larger changes in the geometry. This is attributed to the fact that the modified geometry is relatively close to a scaled geometry *(D = W)* for case *b,* for which the exact solution is achieved with a single basis vector.

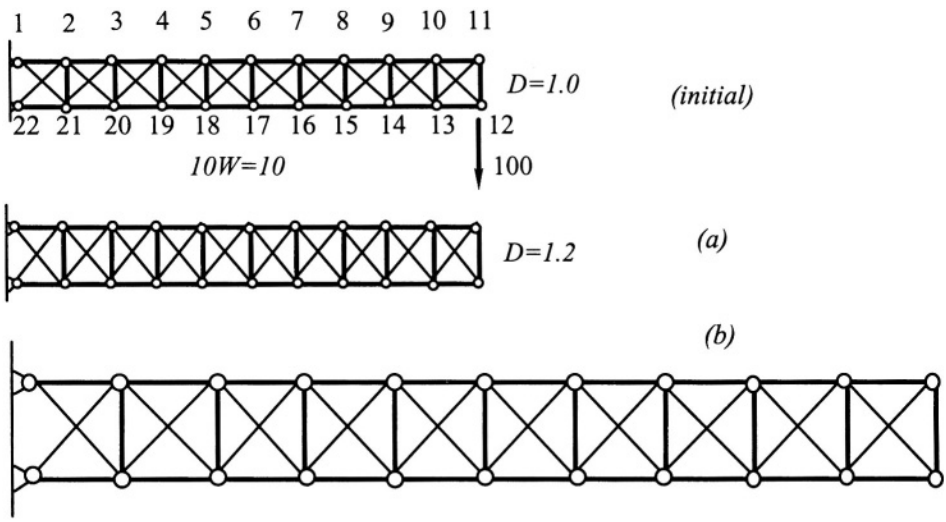

Fig. 9.11. Initial and modified geometries, 50-bar truss.

Table 9.10. First-order approximations of displacements, 50-bar truss.

		Case *a*		Case *b*	
Joint	Direction	CA1	Exact	CA1	Exact
2	X	0.09	0.08	0.20	0.20
	Y	0.11	0.08	0.25	0.24
3	X	0.16	0.15	0.38	0.38
	Y	0.35	0.28	0.90	0.88
4	X	0.22	0.21	0.54	0.54
	Y	0.69	0.60	1.90	1.87
5	X	0.26	0.27	0.68	0.67
	Y	1.12	1.01	3.21	3.18
6	X	0.29	0.31	0.79	0.79
	Y	1.60	1.51	4.79	4.74
7	X	0.32	0.35	0.88	0.88
	Y	2.13	2.07	6.58	6.54
8	X	0.34	0.38	0.95	0.96
	Y	2.69	2.69	8.54	8.50
9	X	0.35	0.40	1.01	1.01
	Y	3.27	3.36	10.63	10.60
10	X	0.35	0.41	1.04	1.04
	Y	3.86	4.05	12.81	12.79
11	X	0.35	0.42	1.05	1.05
	Y	4.45	4.75	15.02	15.02

Example 9.8 - 130-Bar Truss. To illustrate changes in the geometry of larger structures, consider the 130-bar truss shown in Figure 9.12. The ten-story three-bay truss is subjected to ten equal horizontal loads of magnitude 10.0. The following two types of change in the geometry are solved [8]:

Type a

Changes in the distance X defining the horizontal location of the 3rd column from the left (Figure 9.13*a*). For the initial design X=200, the following changes are considered:

Case *a*1: X= 250 (increase of 25%).
Case *a*2: X= 300 (increase of 50%).
Case *a*3: X= 350 (increase of 75%).

Type b

Symmetrical changes in the distance X defining the location of the outer joints (Figure 9.13*b*). For the initial design X=200, the following changes are considered:

Case *b*1: X= 100 (decrease of 50%).
Case *b*2: X= 50 (decrease of 75%).

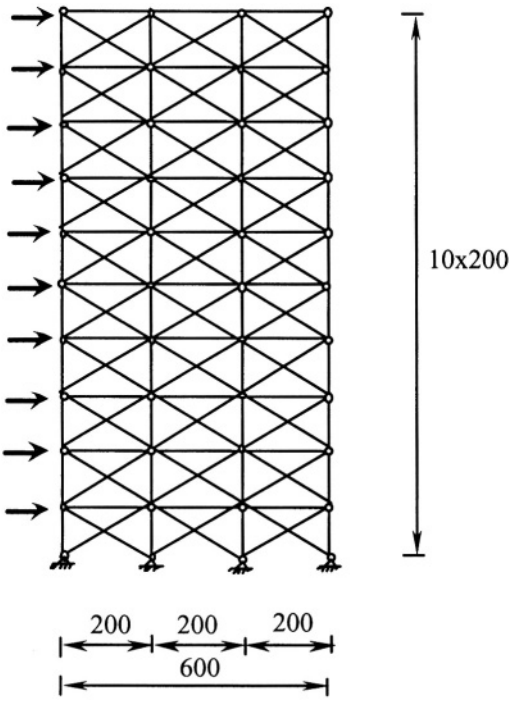

Fig. 9.12. Initial geometry, 130-bar truss.

The maximum horizontal displacements at the top left joint obtained by the CA and exact methods for various numbers of basis vectors are summarized in Table 9.11 and the corresponding errors in the approximations are shown in Table 9.12. It is observed that the larger is the change in the geometry the more the number of basis vectors needed to achieve accurate results. Similar errors are obtained for other displacements for the truss.

Table 9.11. Maximum horizontal displacements, 130-bar truss.

Type of change		*a*			*b*	
Case		*a*1	*a*2	*a*3	*b*1	*b*2
Number of basis vectors	2	14.06	13.12	11.42	13.37	10.67
	3	14.26	13.61	12.68	15.28	13.22
	4		13.82	13.33	15.61	15.31
	5			13.52		15.99
	6					16.18
Exact		14.26	13.83	13.56	15.63	16.28

Table 9.12. Errors [%] in maximum horizontal displacements, 130-bar truss.

Type of change		*a*			*b*	
Case		*a*1	*a*2	*a*3	*b*1	*b*2
	2	1.4	5.1	15.8	14.4	34.5
Number of	3	0	1.6	6.5	2.2	18.8
Basis vectors	4		0.1	1.7	0.1	16.0
	5			0.3		1.8
	6					0.7

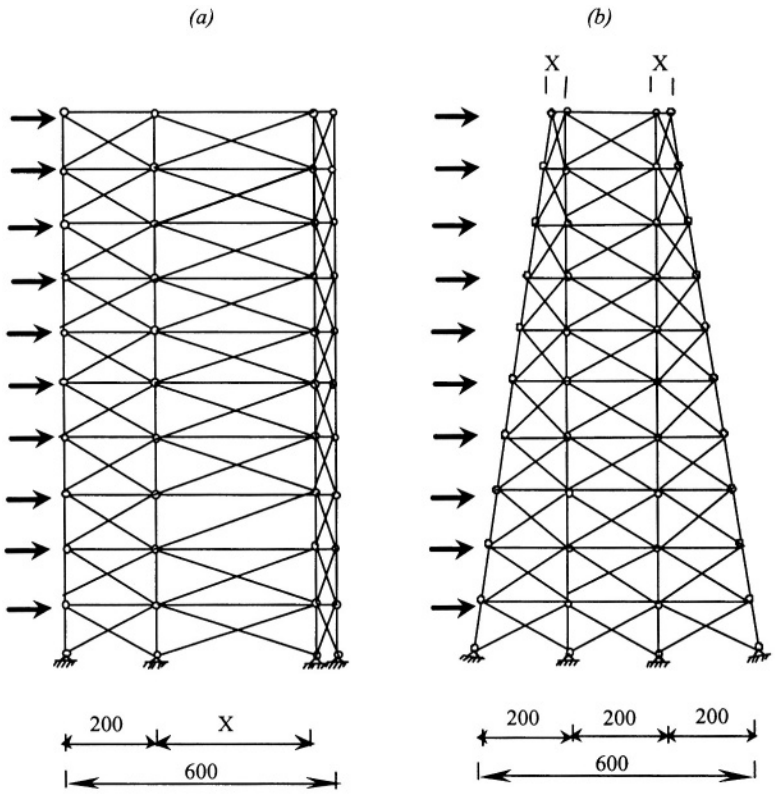

Fig. 9.13. Modified geometries, 130-bar truss.

Example* 9.9 - *Various Changes in the Geometry and the Topology. To illustrate the quality of the results for various changes in the geometry and the topology of the structure, consider the initial ten-bar truss design shown in Figure 9.14*a*. Three different cases of changes in the structural layout are considered [3]:

- Changing the topology by eliminating members 8, 9, 10 (Figure 9.14*b*).
- Changing the geometry by increasing the depth by 100% (Figure 9.14*c*).
- Changing the topology and the geometry by simultaneous elimination of members 8, 9, 10 and increasing the depth by 100% (Figure 9.13*d*).

Approximate results achieved by the CA1 and CA2 procedures are summarized in Tables 9.13, 9.14, 9.15. It is observed that good accuracy is achieved by the CA1 procedure. The results achieved by the CA2 procedure are very close to the exact solutions.

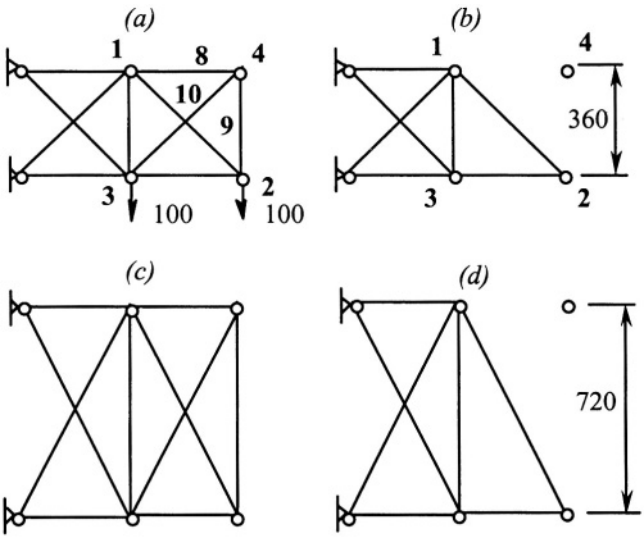

Fig. 9.14. *a.* Initial design, ten-bar truss. *b.* Elimination of members 8, 9, 10. *c.* Increasing the depth by 100%. *d.* Elimination of members 8, 9, 10 and increasing the depth by 100%.

Table 9.13. Results, eliminating members 8, 9, 10 (Figure 9.14*b*).

Method	Displacements							
CA1	2.32	5.47	-3.67	14.83	-2.48	6.11	•	•
CA2	2.40	5.79	-3.60	15.18	-2.40	5.79	•	•
Exact	2.40	5.79	-3.60	15.18	-2.40	5.79	•	•

• Joint 4 is not included.

Table 9.14. Results, increasing the depth (Figure 9.14*c*).

Method	Displacements							
CA1	1.17	3.78	-1.61	7.29	-1.28	4.27	1.26	6.72
CA2	1.14	3.67	-1.68	7.35	-1.24	4.25	1.34	6.62
Exact	1.15	3.67	-1.66	7.36	-1.25	4.24	1.34	6.60

Table 9.15. Results, eliminating members 8, 9, 10 and increasing the depth (Figure 9.14*d*).

Method	Displacements							
CA1	1.25	3.99	-1.89	8.59	-1.25	4.12	•	•
CA2	1.19	3.93	-1.81	8.79	-1.20	3.97	•	•
Exact	1.20	3.95	-1.80	8.80	-1.20	3.96	•	•

• Joint 4 is not included.

9.2.2 Exact solutions

Exact solutions can be achieved efficiently by the CA method for low-rank modifications in the structure, by viewing these changes as corresponding topological changes. For example, changing the coordinates of a single joint, we can obtain the exact solution for the new design by viewing the change in the geometry as the following two successive changes in the topology:

a. All members connected to that joint are deleted.
b. New members are added at the modified location of the joint.

It is noted that the number of basis vectors needed to achieve the exact solution is equal to the number of changed members. However, it is found that a smaller number of vectors are often sufficient.

Example 9.10 Change in a Small Number of Members. Consider the nine-bar truss with the initial geometry shown in Figure 9.15*a* [8]. To calculate displacements for the modified geometry shown in Figure 9.15*b*, members 8 and 9 connected to joint 4 are deleted from the structure and new members 10 and 11 are added at the new location of joint 4. The matrix of changes in the stiffness matrix is

$$\Delta\mathbf{K} = \Delta\mathbf{K}(8, 9) + \Delta\mathbf{K}(10, 11) = \begin{bmatrix} \Delta\mathbf{K}_{11} & \Delta\mathbf{K}_{12} \\ \Delta\mathbf{K}_{21} & \Delta\mathbf{K}_{22} \end{bmatrix}$$

where

$$\Delta \mathbf{K}_{11} = 83.333 \begin{bmatrix} -0.2844 & 0.3578 & 0 & 0 \\ 0.3578 & 0.1789 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta \mathbf{K}_{22} = 83.333 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2844 & 0.3578 \\ 0 & 0 & 0.3578 & 1.1789 \end{bmatrix}$$

$$\Delta \mathbf{K}_{21} = \Delta \mathbf{K}_{12}^{T} = 83.333 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2844 & -0.3578 & 0 & 0 \\ -0.3578 & -0.1789 & 0 & -1 \end{bmatrix}$$

Since four members have been changed, four basis vectors are needed to achieve the exact solution. In the present case, the matrices corresponding to the two vertical members 9 and 11 are linearly dependent and, therefore, only three basis vectors are required to obtain the exact displacements

$$\mathbf{r}^{T} = \{2.40,\ 5.80,\ -3.60,\ 15.19,\ -2.40,\ 5.80,\ -2.30,\ 15.19\}$$

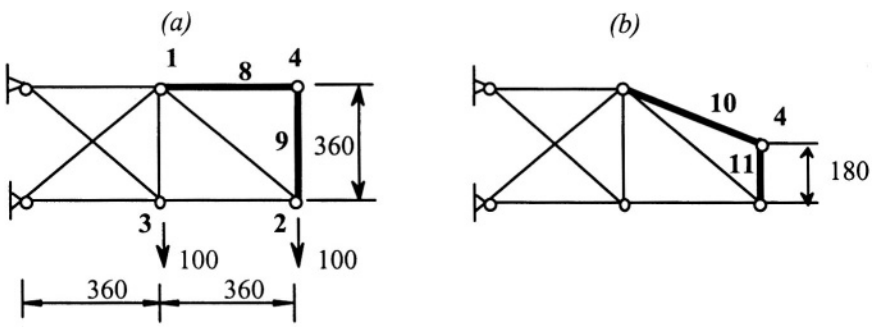

Fig. 9.15. *a*. Initial geometry, nine-bar truss. *b*. Change in geometry, viewed as elimination of members 8, 9 and addition of members 10, 11.

9.3 REFERENCES

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10 Design Sensitivity Analysis

Design sensitivity analysis deals with the calculation of changes in the response of the structure resulting from changes in the parameters describing the structure. The derivatives of the response vector with respect to the system parameters are usually referred to as the sensitivity coefficients. The latter are used to:

- predict the changes in the system response due to changes in the parameters;
- select a search direction in design optimization problems;
- construct explicit approximations of the constraint functions in terms of the structural parameters (e.g. first-order Taylor series approximations);
- generate approximations for the response of a modified system, including approximate reanalysis models;
- assess the effects of uncertainties in the structural properties (material properties, geometric parameters and other parameters) on the system response.

Methods of design sensitivity analysis can be divided into the following classes:

a. Analytical methods, which are widely used and often demonstrate good performance. However, implementation of these methods is difficult in some problems, such as shape optimization, where analytical derivatives of the stiffness matrix are not easy to obtain.

b. "Semi-analytical", or "quasi-analytical" methods, based on finite-difference evaluation of the right-hand-side vector, the so-called "pseudo load". These methods combine ease-of-implementation and computational efficiency, and they have been implemented in several finite element programs. However, the errors associated with the finite difference approximation of the right-hand-side vector can be substantial [1,2].

c. Finite-difference methods, which are the easiest to implement and therefore they are quite popular. For a problem with n design variables, finite difference derivative calculations require repetition of the analysis for $n+1$ different stiffness matrices. This procedure is usually not efficient compared to analytical and semi-analytical methods. Employing approximate reanalysis methods can reduce the large computational effort involved in finite difference calculations.

The two general methods used for calculating the sensitivity coefficients of the system response are:

- The direct method, which is based on the implicit differentiation of the analysis equations that describe the system response with respect to the desired parameters, and the solution of the resulting sensitivity equations.
- The adjoint-variable method, where an adjoint physical system is introduced whose solution permits the rapid evaluation of the desired sensitivity coefficients.

It has been noted [3] that a predominant contributor to the cost and time of the optimization of large structural systems is the calculation of derivatives. Moreover, the calculation of derivatives for a given design involves structural analysis of the design. There has been much interest in efficient procedures for calculating the sensitivity coefficients. However, most approximations that are adequate for structural reanalysis are not sufficiently accurate for sensitivity analysis.

The problem considered in this chapter is to evaluate the displacement derivatives for modified designs where the displacement response of the design is not available by exact analysis. Once the displacement derivatives are available, stress derivatives can be evaluated by direct differentiation of the stress-displacement relations. When results of exact analysis for the modified design are available, calculation of exact displacement derivatives is straightforward and approximations are usually not needed. It has been shown [4, 5] that the CA method presented in Chapter 7 can be used also for effective approximation of derivatives. Accurate results have been achieved efficiently by either the direct method or the adjoint-variable method.

Exact analytical derivatives, obtained by the direct method and the adjoint-variable method, are presented in Section 10.1. Three alternative sensitivity analysis methods for calculation of approximate derivatives by the CA method are developed in Section 10.2:

a. The Direct Approximation (DA) method.
b. The Adjoint-Variable Approximation (AVA) method.
c. The Finite-Difference Approximation (FDA) method.

Results obtained by the various methods are presented and compared in Section 10.3. Calculation of second-order derivatives is demonstrated in Section 10.4, and computational procedures for evaluating the approximate displacements, the first-order derivatives and the second-order derivatives for a modified design by the CA method are developed in Section 10.5.

10.1 EXACT ANALYTICAL DERIVATIVES

Exact analytical derivatives by the direct method and the adjoint-variable method are developed in this section. For simplicity of presentation a single design variable, denoted by α, is considered.

10.1.1 Direct Method

Differentiating Eq. (3.1) with respect to a design variable α and rearranging yields

$$\mathbf{K}^* \frac{\partial \mathbf{r}^*}{\partial \alpha} = \frac{\partial \mathbf{R}^*}{\partial \alpha} - \frac{\partial \mathbf{K}^*}{\partial \alpha} \mathbf{r}^* \tag{10.1}$$

The direct approach involves solution of Eq. (10.1) for $\partial \mathbf{r}^*/\partial \alpha$ and then taking the desired component. For multiple design variables, Eq. (10.1) must be solved repeatedly for each design variable. To obtain derivatives of a single displacement r_j we compute

$$\frac{\partial r_j^*}{\partial \alpha} = \mathbf{I}_j^T \frac{\partial \mathbf{r}^*}{\partial \alpha} \tag{10.2}$$

where $\mathbf{I}_j$ is a vector having unit value at the jth location and zeros elsewhere.

Equations (10.1) and (3.1) have the same coefficient matrix $\mathbf{K}^*$. If the decomposed form of Eq. (3.2) is available from initial analysis then only forward and backward substitutions are needed to solve for $\partial \mathbf{r}^*/\partial \alpha$.

In many problems the load vector $\mathbf{R}$ is assumed to be independent of the design variables. In such cases $\mathbf{R} = \mathbf{R}^*$ and Eq. (10.1) is reduced to the form

$$\mathbf{K}^* \frac{\partial \mathbf{r}^*}{\partial \alpha} = -\frac{\partial \mathbf{K}^*}{\partial \alpha} \mathbf{r}^* \tag{10.3}$$

10.1.2 Adjoint-Variable Method

Premultiplying Eq. (10.1) by $\mathbf{I}_j^T \mathbf{K}^{*-1}$ and substituting Eq. (10.2) yields

$$\frac{\partial r_j^*}{\partial \alpha} = \mathbf{I}_j^T \mathbf{K}^{*-1} \left(\frac{\partial \mathbf{R}^*}{\partial \alpha} - \frac{\partial \mathbf{K}^*}{\partial \alpha} \mathbf{r}^* \right) \tag{10.4}$$

The adjoint-variable vector $\boldsymbol{\xi}_j^*$ is defined as the solution of the set of equations

$$\mathbf{K}^* \boldsymbol{\xi}_j^* = \mathbf{I}_j \tag{10.5}$$

Substituting Eq. (10.5) into Eq. (10.4) gives

$$\frac{\partial r_j^*}{\partial \alpha} = \boldsymbol{\xi}_j^{*T} \left(\frac{\partial \mathbf{R}^*}{\partial \alpha} - \frac{\partial \mathbf{K}^*}{\partial \alpha} \mathbf{r}^* \right) \tag{10.6}$$

where use is made of the symmetry of $\mathbf{K}^*$. The adjoint-variable method involves solution of Eq. (10.5) for $\boldsymbol{\xi}_j^*$ and then calculation of $\partial r_j^* / \partial \alpha$ by Eq. (10.6). Assuming that the load vector is independent of the design variables then Eq. (10.6) is reduced to the form

$$\frac{\partial r_j^*}{\partial \alpha} = -\boldsymbol{\xi}_j^{*T} \frac{\partial \mathbf{K}^*}{\partial \alpha} \mathbf{r}^* \tag{10.7}$$

10.2 APPROXIMATE FIRST-ORDER DERIVATIVES

In the presentation that follows, the CA method presented in Section 7.1 is considered. Assume, for illustrative purposes, approximations along the line

$$\mathbf{K} = \mathbf{K}^* + \alpha \Delta \mathbf{K}^* \tag{10.8}$$

where α is a variable representing the step size and $\Delta \mathbf{K}^*$ is a given matrix of changes in the stiffness. Assume also that the load vector is independent of the design variables so that the basis vectors are given by [see Eq. (7.3)]

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1^* \\ \mathbf{r}_2 &= \alpha\ \mathbf{r}_2^* \\ &\cdot \\ &\cdot \\ \mathbf{r}_s &= \alpha^{s-1}\ \mathbf{r}_s^* \end{aligned} \tag{10.9}$$

where s is the number of terms considered and $\mathbf{r}_1^*, \mathbf{r}_2^*, \ldots, \mathbf{r}_s^*$ are vectors of constant elements, given by

$$\begin{aligned} \mathbf{r}_1^* &= \mathbf{r}^* \\ \mathbf{r}_2^* &= -\mathbf{K}^{*-1} \Delta \mathbf{K}^* \mathbf{r}_1^* = -\mathbf{B}^* \mathbf{r}_1^* \\ &\cdot \\ &\cdot \\ \mathbf{r}_s^* &= -\mathbf{K}^{*-1} \Delta \mathbf{K}^* \mathbf{r}_{s-1}^* = -\mathbf{B}^* \mathbf{r}_{s-1}^* \end{aligned} \tag{10.10}$$

and $\mathbf{B}^*$ is a matrix of constant elements, defined as [see Eq. (7.3)]

$$\mathbf{B}^* = \mathbf{K}^{*-1} \Delta \mathbf{K}^* \tag{10.11}$$

10.2.1 Direct Approximations (DA)

In the direct method, differentiation of Eq. (7.16) with respect to α yields

$$\frac{\partial \mathbf{r}}{\partial \alpha} = \frac{\partial \mathbf{r}_B}{\partial \alpha}\mathbf{y} + \mathbf{r}_B \frac{\partial \mathbf{y}}{\partial \alpha} \tag{10.12}$$

where $\mathbf{r}_B$ is the matrix of basis vectors and $\mathbf{y}$ is the vector of unknown coefficients. The derivatives $\partial \mathbf{y}/\partial \alpha$ are calculated by differentiating Eq. (7.15) and rearranging

$$\mathbf{K}_R \frac{\partial \mathbf{y}}{\partial \alpha} = \frac{\partial \mathbf{R}_R}{\partial \alpha} - \frac{\partial \mathbf{K}_R}{\partial \alpha}\mathbf{y} \tag{10.13}$$

where, from Eqs. (7.14) we have (for $\mathbf{R} = \mathbf{R}^*$)

$$\begin{aligned} \frac{\partial \mathbf{K}_R}{\partial \alpha} &= \frac{\partial \mathbf{r}_B}{\partial \alpha}^T \mathbf{K}\mathbf{r}_B + \mathbf{r}_B^T \frac{\partial \mathbf{K}}{\partial \alpha}\mathbf{r}_B + \mathbf{r}_B^T \mathbf{K}\frac{\partial \mathbf{r}_B}{\partial \alpha} \\ \frac{\partial \mathbf{R}_R}{\partial \alpha} &= \frac{\partial \mathbf{r}_B}{\partial \alpha}^T \mathbf{R} \end{aligned} \tag{10.14}$$

For the line of Eq. (10.8), the matrix $\partial \mathbf{r}_B/\partial \alpha$ becomes [see Eq. (10.9)]

$$\frac{\partial \mathbf{r}_B}{\partial \alpha} = [\mathbf{0}, \quad \mathbf{r}_2^*, \quad 2\alpha \mathbf{r}_3^*, \ldots, (s-1)\alpha^{s-2}\mathbf{r}_s^*] \tag{10.15}$$

Introduction of the modified stiffness matrix $\mathbf{K}$ by Eq. (10.8) is straightforward since the elements of matrix $\Delta \mathbf{K}^*$ are constant. In addition, the derivatives

$$\partial \mathbf{K}/\partial \alpha = \Delta \mathbf{K}^* \tag{10.16}$$

obtained by differentiation of Eq. (10.8) are also constant. In summary, given the initial stiffness matrix in the decomposed form of Eq. (3.2) and the initial displacements $\mathbf{r}^*$, evaluation of the approximate first-order derivatives by the DA method for any assumed α involves the following steps:

a. Determine the matrices $\partial \mathbf{r}_B/\partial \alpha$ and $\partial \mathbf{K}/\partial \alpha$ [Eqs. (10.15), (10.16)]. For the line of Eq. (10.8) this step involves only calculation of $2\alpha \mathbf{r}_3^*, \ldots, (s-1)\alpha^{s-2}\mathbf{r}_s^*$.

b. Calculate the matrix $\partial \mathbf{K}_R/\partial \alpha$ and the vector $\partial \mathbf{R}_R/\partial \alpha$ [Eq. (10.14)].

c. Calculate the vector $\partial \mathbf{y}/\partial \alpha$ by solving the set of $s \times s$ equations (10.13).

d. Evaluate the derivatives $\partial \mathbf{r}/\partial \alpha$ by Eq. (10.12).

10.2.2 Adjoint-Variable Approximations (AVA)

The approximate displacement derivatives can be evaluated by first solving the set of modified equations

$$\mathbf{K}\xi_j = \left(\mathbf{K}^* + \Delta\mathbf{K}\right)\xi_j = \mathbf{I} \tag{10.17}$$

for the adjoint-variable vector ξ_j. Since Eqs. (3.5) and (10.17) have the same coefficient matrix, we can apply a computational procedure similar to that used for evaluation of the approximate displacements, with $\mathbf{r}$ and $\mathbf{R}$ replaced by ξ_j and $\mathbf{I}_j$, respectively. Assuming that ξ_j^* is known from analysis of the initial design [Eq. (10.5)], and the approximate displacements are known from prior approximate analysis, evaluation of analytical derivatives by the AVA method for any modified design involves the following steps:

a. Calculate the basis vectors $\xi_2, \xi_3, \ldots, \xi_s$ and introduce matrix ξ_B of the basis vectors, where [see similar calculation of $\mathbf{r}$ by Eqs. (10.9) and (10.10)]

$$\xi_1 = \xi_j^* \qquad \xi_2 = -\mathbf{B}\,\xi_j^* \qquad \xi_s = \mathbf{B}^{s-1}\,\xi_j^*$$
$$\xi_B = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_s\} \tag{10.18}$$

Calculation the vectors $\xi_2, \xi_3, \ldots, \xi_s$ involves only forward and backward substitutions, if $\mathbf{K}^*$ is given in the decomposed form of Eq. (3.2).

b. Calculate the elements of $\mathbf{K}_R$ and $\mathbf{R}_R$ by [see Eqs. (7.14)]

$$\mathbf{K}_R = \xi_B^T \mathbf{K} \xi_B \qquad \mathbf{R}_R = \xi_B^T \mathbf{I}_j \tag{10.19}$$

c. Calculate the coefficients $\mathbf{y}$ by solving the set of $s \times s$ equations [see Eq. (7.15)]

$$\mathbf{K}_R\,\mathbf{y} = \mathbf{R}_R \tag{10.20}$$

d. Evaluate the vector ξ_j by [see Eq. (7.16)]

$$\xi_j = \xi_B\,\mathbf{y} \tag{10.21}$$

e. Evaluate the approximate derivatives $\partial r_j/\partial\alpha$ by [Eq. (10.7)]

$$\frac{\partial r_j}{\partial \alpha} = -\xi_j^T \frac{\partial \mathbf{K}}{\partial \alpha}\mathbf{r} \tag{10.22}$$

10.2.3 Finite Difference Approximations (FDA)

In cases where analytical derivatives of the displacements are not easy to implement, the finite difference (FD) method can be used. In this section, the FD method is considered only for purposes of comparison between methods.

In the forward-difference method, the displacement derivatives are computed from the exact displacements at the two points $\alpha = \alpha^*$ and $\alpha = \alpha^* + \Delta\alpha$ by

$$\frac{\partial \mathbf{r}^*}{\partial \alpha} = \frac{\mathbf{r}(\alpha^* + \Delta\alpha) - \mathbf{r}(\alpha^*)}{\Delta\alpha} \tag{10.23}$$

where $\Delta\alpha$ is a predetermined step-size. The accuracy can be improved by adopting the central-difference method

$$\frac{\partial \mathbf{r}^*}{\partial \alpha} = \frac{\mathbf{r}(\alpha^* + \Delta\alpha) - \mathbf{r}(\alpha^* - \Delta\alpha)}{2\Delta\alpha} \tag{10.24}$$

where the derivatives are computed from the exact displacements at the two points $\alpha = \alpha^* - \Delta\alpha$ and $\alpha = \alpha^* + \Delta\alpha$. When $\mathbf{r}(\alpha^*)$ is known, application of Eq. (10.23) involves only one additional calculation of the displacements at $\alpha^* + \Delta\alpha$ while Eq. (10.24) requires calculation at two points ($\alpha^* - \Delta\alpha$ and $\alpha^* + \Delta\alpha$).

The two FD methods of Eqs. (10.23) and (10.24) are compared in the following with the corresponding Finite Difference Approximations (FDA) methods. These methods use the approximate displacements $\mathbf{r}_a$ (computed by the CA method). The corresponding expressions are given by

$$\frac{\partial \mathbf{r}^*}{\partial \alpha} \cong \frac{\mathbf{r}_a(\alpha^* + \Delta\alpha) - \mathbf{r}_a(\alpha^*)}{\Delta\alpha} \tag{10.25}$$

$$\frac{\partial \mathbf{r}^*}{\partial \alpha} \cong \frac{\mathbf{r}_a(\alpha^* + \Delta\alpha) - \mathbf{r}_a(\alpha^* - \Delta\alpha)}{2\Delta\alpha} \tag{10.26}$$

Calculation of approximate derivatives by the FDA method [Eq. (10.25) or Eq. (10.26)] involves evaluation of the approximate displacements at two points.

Example 10.1 - Low-Order Approximations of Derivatives. To illustrate the accuracy of the results achieved by second-order approximations (CA2 – only three basis vectors), consider again the ten-bar truss shown in Fig. 10.1. The modulus of elasticity is 30000 and the eight analysis unknowns are the horizontal and the vertical displacements at joints 1, 2, 3 and 4, respectively. The initial cross-sectional areas are $\mathbf{X}^* = \mathbf{1.0}$, the stress constraints are $\mathbf{-25.0 \le \sigma \le 25.0}$, and the minimum size constraints are $\mathbf{0.001 \le X}$. Assuming the minimum weight as the design objective, the optimal design is [6]

$$\Delta \mathbf{X}^{T}_{opt} = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.667, 5.667, 5.667, 0.001\}$$

Assume the line from the initial design to the optimal design

$$\mathbf{X} = \mathbf{X}^{*} + \alpha \Delta \mathbf{X}^{*}$$

where α is a variable representing the step size and $\Delta \mathbf{X}^{*}$ is defined as

$$\Delta \mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

Exact and approximate displacement derivatives achieved by various methods are shown in Tables 10.1 through 10.4 for $\alpha = 0.5$ and $\alpha = 1.0$ (the optimum).

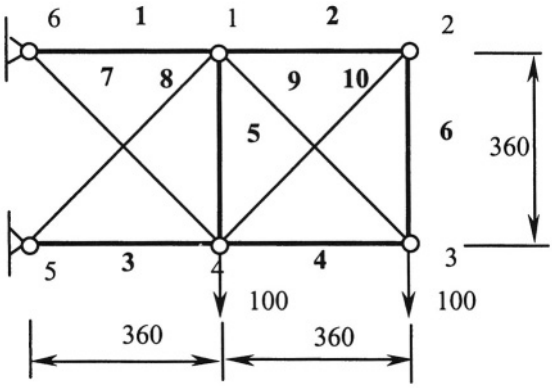

Fig. 10.1. Ten-bar truss.

Table 10.1. Displacement derivatives $\partial r_j^{*} / \partial \alpha$ by the DA method.

Displacement	$\alpha = 0.5$		$\alpha = 1.0$	
Number	DA	Exact	DA	Exact
1	0.81	0.79	0.25	0.24
2	2.16	2.09	0.72	0.64
3	1.04	1.01	•	•
4	4.93	4.89	•	•
5	1.23	1.24	0.46	0.47
6	5.17	5.10	1.84	1.76
7	0.85	0.87	0.27	0.29
8	2.36	2.42	0.80	0.88

• Joint 2 is effectively eliminated.

Table 10.2. Displacement derivatives $\partial r_j^* / \partial \alpha$ by the AVA method, $\alpha = 0.5$.

Displacement Number	AVA	Exact
1	0.81	0.79
2	2.06	2.09
3	1.18	1.01
4	4.88	4.89
5	1.25	1.24
6	5.11	5.10
7	0.87	0.87
8	2.44	2.42

Table 10.3. Displacement derivatives $\partial r_j^* / \partial \alpha$ by the FD method.

Displacement Number	$\alpha = 0.5$		$\alpha = 1.0$	
	FD Eq. (10.24)	Exact	FD Eq. (10.24)	Exact
1	0.81	0.79	0.26	0.24
2	2.14	2.09	0.69	0.64
3	1.03	1.01	•	•
4	4.98	4.89	•	•
5	1.26	1.24	0.50	0.47
6	5.2	5.10	1.89	1.76
7	0.88	0.87	0.31	0.29
8	2.46	2.42	0.93	0.88

Table 10.4. Displacement derivatives $\partial r_j^* / \partial \alpha$ by the FDA method.

Displacement Number	$\alpha = 0.5$		$\alpha = 1.0$	
	FDA Eq. (10.26)	Exact	FDA Eq. (10.26)	Exact
1	0.82	0.79	0.28	0.24
2	2.20	2.09	0.77	0.64
3	1.06	1.01	•	•
4	5.02	4.89	•	•
5	1.25	1.24	0.49	0.47
6	5.27	5.10	1.98	1.76
7	0.87	0.87	0.30	0.29
8	2.41	2.42	0.85	0.88

• Joint 2 is effectively eliminated.

10.3 COMPARISON OF RESULTS

10.3.1 Accuracy of the Calculations

The accuracy of the calculations by all methods depends on the accuracy of the approximate displacements. In the examples presented, good displacement approximations have been achieved for very large changes in the design variables. In general the accuracy of the DA and AVA methods is similar. However, the AVA method involves approximations of both $\mathbf{r}$ and ξ_j; therefore, poor results might be obtained by this method for some ξ_j values.

For the FDA method, the central-difference procedure provides better results than the forward-difference procedure. The truncation error is minimized by reducing the step-size $\Delta\alpha$, but a small step-size amplifies the round-off errors in the displacements. This is the step-size dilemma where a large step-size generates large truncation errors while a small step-size generates large round-off errors. Some considerations for choosing the forward-difference step-size are discussed elsewhere [7]. In structural optimization, truncation errors are not of major importance since it is often sufficient to find the average rate of change in the structural behavior and not necessarily the accurate local rate of change at a given point. Therefore, to eliminate round-off errors due to approximations it is recommended to increase the step-size.

10.3.2 Computational Efficiency

Evaluation of the approximate displacements requires calculation of the elements of matrix $\mathbf{r}_B$. As noted in Section 7.1.1, calculation of the basis vectors involves only forward and backward substitutions, if $\mathbf{K}^*$ is given in the decomposed form of Eq. (3.2). The additional calculations [the terms of Eq. (7.14), determination of $\mathbf{y}$ by solving the set of $s \times s$ equations (7.15), and multiplication of the basis vectors by $\mathbf{y}$] are straightforward. The FDA method requires additional evaluation of the approximate displacements at one or two points.

To compare the DA method and the AVA method, assume that the two methods involve a similar computational effort for calculating a single vector of derivatives. The adjoint-variable method is superior to the direct method when derivatives of a limited number of displacements must be calculated [8, 9]. Let m be the number of displacements considered, the adjoint-variable method requires calculation of m adjoint-vectors ξ_j. In the direct method, the number of vectors that must be determined is $n\ n_L$, where n is the number of variables and n_L is the number of loading conditions. Depending on the values of m and $n\ n_L$, one method is to be preferred over the other. In cases where $m < n\ n_L$, the direct method is less efficient than the adjoint-variable method. In most design optimization problems the number of displacements to be calculated may become large because stress constraints are also considered. Often, however, only derivatives of critical constraints are calculated. The number of critical constraints does not change significantly with the number of loading conditions, and is usually of the same order as the number of design variables. Therefore, the adjoint-variable method is often preferable for multiple loading conditions.

10.3.3 Ease-of-Implementation

All approximate methods require evaluation of the approximate displacements. That is, matrices $\Delta\mathbf{K}$ and $\mathbf{K}=\mathbf{K}^*+\Delta\mathbf{K}$ are first determined and then some quantities are calculated by simple algebraic operations. The FDA is the easiest method to implement since it requires only evaluation of the approximate displacements at one (forward-differences) or two (central-differences) additional points.

The AVA method requires calculation of ξ_j in a procedure similar to that used for evaluating the approximate displacements. In addition, it is necessary to calculate derivatives of the stiffness matrix with respect to design variables. When analytical derivatives $\partial\mathbf{K}/\partial\alpha$ are not easy to obtain, a semi-analytical method (finite-difference evaluation), which is easy-to-implement, can be used.

The DA method involves calculation of derivatives of both $\mathbf{K}$ and $\mathbf{r}_B$. Once again, in cases of cumbersome calculation of analytical derivatives, a semi-analytical method can be employed. However, as noted earlier, the errors associated with finite-difference approximations of these derivatives can be substantial.

10.4 SECOND-ORDER DERIVATIVES

Higher-order derivatives can be evaluated in a similar way to that for first-order derivatives. Calculation of approximate second-order derivatives by the direct method is demonstrated in this section. Differentiating Eq. (10.1) with respect to α and rearranging gives the following expression for the exact second-order derivatives

$$\mathbf{K}^* \frac{\partial^2 \mathbf{r}^*}{\partial \alpha^2} = \frac{\partial^2 \mathbf{R}^*}{\partial \alpha^2} - \frac{\partial^2 \mathbf{K}^*}{\partial \alpha^2}\mathbf{r}^* - 2\frac{\partial \mathbf{K}^*}{\partial \alpha}\frac{\partial \mathbf{r}^*}{\partial \alpha} \tag{10.27}$$

Equations (10.27), (10.1) and (3.1) have the same coefficient matrix $\mathbf{K}^*$. If the decomposed form of Eq. (3.2) is available then only forward and backward substitutions are needed to solve for $\partial^2\mathbf{r}^*/\partial\alpha^2$.

To obtain approximate second-order derivatives, Eq. (10.12) is differentiated with respect to α, giving

$$\frac{\partial^2 \mathbf{r}}{\partial \alpha^2} = \frac{\partial^2 \mathbf{r}_B}{\partial \alpha^2}\mathbf{y} + 2\frac{\partial \mathbf{r}_B}{\partial \alpha}\frac{\partial \mathbf{y}}{\partial \alpha} + \mathbf{r}_B \frac{\partial^2 \mathbf{y}}{\partial \alpha^2} \tag{10.28}$$

The second-order derivatives $\partial^2\mathbf{y}/\partial\alpha^2$ in Eq. (10.28) are obtained by differentiation of Eqs. (10.13) with respect to α. The result, after rearranging, is

$$\mathbf{K}_R \frac{\partial^2 \mathbf{y}}{\partial \alpha^2} = \frac{\partial^2 \mathbf{R}_R}{\partial \alpha^2} - \frac{\partial^2 \mathbf{K}_R}{\partial \alpha^2}\mathbf{y} - 2\frac{\partial \mathbf{K}_R}{\partial \alpha}\frac{\partial \mathbf{y}}{\partial \alpha} \tag{10.29}$$

The second-order derivatives $\partial^2\mathbf{K}_R/\partial\alpha^2$ and $\partial^2\mathbf{R}_R/\partial\alpha^2$ in Eq. (10.29) are obtained by differentiation of Eqs. (10.14) with respect to α, giving

$$\frac{\partial^2\mathbf{K}_R}{\partial\alpha^2} = \frac{\partial^2\mathbf{r}_B^T}{\partial\alpha^2}\mathbf{K}\mathbf{r}_B + \mathbf{r}_B^T\frac{\partial^2\mathbf{K}}{\partial\alpha^2}\mathbf{r}_B + \mathbf{r}_B^T\mathbf{K}\frac{\partial^2\mathbf{r}_B}{\partial\alpha^2} + 2\left(\frac{\partial\mathbf{r}_B^T}{\partial\alpha}\frac{\partial\mathbf{K}}{\partial\alpha}\mathbf{r}_B + \frac{\partial\mathbf{r}_B^T}{\partial\alpha}\mathbf{K}\frac{\partial\mathbf{r}_B}{\partial\alpha} + \mathbf{r}_B^T\frac{\partial\mathbf{K}}{\partial\alpha}\frac{\partial\mathbf{r}_B}{\partial\alpha}\right)$$

$$\frac{\partial^2\mathbf{R}_R}{\partial\alpha^2} = \frac{\partial^2\mathbf{r}_B^T}{\partial\alpha^2}\mathbf{R} \tag{10.30}$$

For the line defined by Eq. (10.8), the second-order derivatives $\partial^2\mathbf{r}_B/\partial\alpha^2$ and $\partial^2\mathbf{K}/\partial\alpha^2$ in Eq. (10.30) are obtained by differentiation of Eqs. (10.15) and (10.16) with respect to α, giving

$$\frac{\partial^2\mathbf{r}_B}{\partial\alpha^2} = [\mathbf{0}, \quad \mathbf{0}, \quad 2\mathbf{r}_3^*, \ldots, (s-1)(s-2)\alpha^{s-3}\mathbf{r}_s^*] \tag{10.31}$$

$$\partial^2\mathbf{K}/\partial\alpha^2 = \mathbf{0} \tag{10.32}$$

10.5 COMPUTATIONAL PROCEDURE

In a typical optimization process it is necessary to evaluate the response for numerous modified designs. Only for some of the designs it is necessary also to evaluate the response derivatives. When results of exact analysis for the modified design are available, calculation of exact response derivatives is straightforward and approximate procedures are usually not needed. The approach presented in this section is intended for problems where the displacement response of the modified design is not available by exact analysis. A computational procedure for evaluating the approximate displacements, the first-order derivatives, and the second-order derivatives for a modified design by the CA method is presented. It can be observed that calculations of various quantities involve similar algebraic operations [see Table 10.5].

Given the initial stiffness matrix in the decomposed form of Eq. (3.2) and the initial displacement vector $\mathbf{r}_1 = \mathbf{r}^*$ from exact analysis of the initial design, the solution process involves the following steps.

a. ***Introduction of the coefficient matrices.*** Calculate the modified matrix $\mathbf{K}$ by Eq. (3.3) and determine the matrices $\partial\mathbf{K}/\partial\alpha$ and $\partial^2\mathbf{K}/\partial\alpha^2$. For the line defined by Eq.

(10.8), the elements of matrix $\partial\mathbf{K}/\partial\alpha$ are constant [Eq. (10.16)] and the elements of matrix $\partial^2\mathbf{K}/\partial\alpha^2$ equal zero [Eq. (10.32)].

b. ***Introduction of the basis vectors.*** Calculate the basis vectors $\mathbf{r}_i$ $(i=2, 3, . , s)$ by Eq. (7.12). As noted earlier, this calculation involves only forward and backward substitutions. For the line of Eq. (10.8), the elements of the vectors $\mathbf{r}_i^*$ are constant [see Eqs. (10.9), (10.10)]. The elements of matrices $\partial\mathbf{r}_B/\partial\alpha$ and $\partial^2\mathbf{r}_B/\partial\alpha^2$ are also constant. That is, once the constant basis vectors have been determined, they apply for any α value.

c. ***Introduction of the reduced quantities.*** First, calculate the reduced matrix $\mathbf{K}_R$ and the reduced vector $\mathbf{R}_R$ by Eqs. (7.14). Then, calculate the matrix $\partial\mathbf{K}_R/\partial\alpha$ and the vector $\partial\mathbf{R}_R/\partial\alpha$ by Eq. (10.14). Finally, calculate the matrix $\partial^2\mathbf{K}_R/\partial\alpha^2$ and the vector $\partial^2\mathbf{R}_R/\partial\alpha^2$ by Eq. (10.30).

d. ***Calculation of the reduced unknowns.*** First, calculate the unknown coefficients $\mathbf{y}$ by solving the set of $s\times s$ equations (7.15). Then, calculate the vector $\partial\mathbf{y}/\partial\alpha$ by solving the set of $s\times s$ equations (10.13). Finally, calculate the vector $\partial^2\mathbf{y}/\partial\alpha^2$ by solving the set of $s\times s$ equations (10.29).

e. ***Calculation of the final unknowns.*** First, evaluate the final displacements by Eq. (7.16). Then, evaluate the derivatives $\partial\mathbf{r}/\partial\alpha$ by Eq. (10.12). Finally, evaluate the second-order derivatives $\partial^2\mathbf{r}/\partial\alpha^2$ by Eq. (10.28).

Table 10.5. Summary of algebraic operations.

Operation	Response	1st derivatives	2nd derivatives
Introducte coefficient matrices	$\mathbf{K}$	$\partial\mathbf{K}/\partial\alpha$	$\partial^2\mathbf{K}/\partial\alpha^2$
Introduce basis vectors	$\mathbf{r}_B$	$\partial\mathbf{r}_B/\partial\alpha$	$\partial^2\mathbf{r}_B/\partial\alpha^2$
Introduce reduced quantities	$\mathbf{K}_R$	$\partial\mathbf{K}_R/\partial\alpha$	$\partial^2\mathbf{K}_R/\partial\alpha^2$
	$\mathbf{R}_R$	$\partial\mathbf{R}_R/\partial\alpha$	$\partial^2\mathbf{R}_R/\partial\alpha^2$
Calculate reduced unknowns	$\mathbf{y}$	$\partial\mathbf{y}/\partial\alpha$	$\partial^2\mathbf{y}/\partial\alpha^2$
Calculate final unknowns	$\mathbf{r}$	$\partial\mathbf{r}/\partial\alpha$	$\partial^2\mathbf{r}/\partial\alpha^2$

Example 10.2 - Accurate Derivatives by the CA Method [5]. To illustrate the accuracy of the results achieved by the CA method, consider again the ten-bar truss of example 10.1 shown in Figure 10.1. Results of exact analysis are given for the initial design $\mathbf{X}^*=\mathbf{1.0}$. Assume the line from the initial design to the optimal design

$$\mathbf{X} = \mathbf{X}^* + \alpha \Delta \mathbf{X}^*$$

where α is a variable representing the step size and $\Delta \mathbf{X}^*$ is defined as

$$\Delta \mathbf{X}^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.667, 4.667, 4.667, -0.999\}$$

Evaluation of the displacements, the displacement first-derivatives and the displacement second-derivatives with respect to α are illustrated for $\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1.0$ (the optimum). The corresponding cross-section areas are summarized in Table 10.6. The results given in Tables 10.7 - 10.9 for various numbers of basis vectors show that 2-3 vectors provide good accuracy, and that the accuracy is improved even more by considering additional vectors. The percentage errors obtained for the displacements and displacement derivatives are summarized in Table 10.10. It is observed that the order of magnitude of the percentage of errors is the same for $\alpha = 0.5$ and $\alpha = 0.75$. The accuracy of the derivatives depends on the accuracy of the approximate displacements. Specifically, the errors in the displacements and in the displacement derivatives are of the same order of magnitude. In conclusion, for only 2-3 basis vectors, good accuracy is achieved for very large changes in the design, for both displacements and displacement derivatives. The second-order derivatives have not been calculated for $\alpha = 1.0$, since at this point joint 2 is effectively eliminated.

Table 10.6. Cross section areas of elements for $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1.0$.

Element	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$
1	4.500	6.250	8.000
2	0.501	0.251	0.001
3	4.500	6.250	8.000
4	2.500	3.250	4.000
5	0.501	0.251	0.001
6	0.501	0.251	0.001
7	3.334	4.500	5.667
8	3.334	4.500	5.667
9	3.334	4.500	5.667
10	0.501	0.251	0.001

Table 10.7. Approximate displacements and displacement derivatives for $\alpha = 0.5$.

Number of basis vectors	2	3	4	5	Exact
Displacements	0.50	0.52	0.52	0.52	0.52
	1.53	1.46	1.48	1.49	1.49
	0.71	0.76	0.77	0.77	0.77
	3.56	3.63	3.64	3.65	3.65
	-0.89	-0.98	-0.99	-0.98	-0.98
	3.77	3.87	3.89	3.90	3.90
	-0.54	-0.55	-0.55	-0.55	-0.55
	1.71	1.64	1.62	1.62	1.62
1st Derivatives	-0.80	-0.81	-0.80	-0.79	-0.79
	-2.16	-2.16	-2.11	-2.09	-2.09
	-1.05	-1.04	-1.02	-1.01	-1.01
	-4.96	-4.93	-4.91	-4.89	-4.89
	1.24	1.23	1.21	1.24	1.24
	-5.21	-5.17	-5.13	-5.10	-5.10
	0.85	0.85	0.86	0.87	0.86
	-2.37	-2.36	-2.39	-2.42	-2.42
2nd Derivatives	2.55	2.52	2.51	2.52	2.52
	6.17	6.35	6.29	6.34	6.34
	3.11	2.99	2.97	3.00	3.00
	14.02	13.79	13.78	13.81	13.81
	-3.52	-3.28	-3.27	-3.27	-3.27
	14.58	14.26	14.23	14.29	14.30
	-2.67	-2.64	-2.65	-2.64	-2.64
	6.66	6.82	6.86	6.81	6.81

Table 10.8. Approximate displacements and displacement derivatives for $\alpha = 0.75$.

Number of basis vectors	2	3	4	5	Exact
Displacements	0.36	0.37	0.37	0.38	0.38
	1.13	1.07	1.10	1.11	1.11
	0.52	0.56	0.58	0.59	0.59
	2.64	2.72	2.73	2.74	2.74
	-0.66	-0.75	-0.76	-0.74	-0.74
	2.81	2.91	2.94	2.95	2.95
	-0.38	-0.40	-0.39	-0.39	-0.39
	1.27	1.21	1.18	1.17	1.17
1st Derivatives	-0.41	-0.42	-0.41	-0.40	-0.40
	-1.17	-1.15	-1.12	-1.09	-1.09
	-0.56	-0.57	-0.55	-0.53	-0.53
	-2.71	-2.72	-2.70	-2.67	-2.67
	0.68	0.70	0.68	0.71	0.71
	-2.86	-2.87	-2.84	-2.80	-2.80
	0.44	0.44	0.45	0.46	0.46
	-1.30	-1.27	-1.30	-1.34	-1.34
2nd Derivatives	0.94	0.94	0.93	0.94	0.94
	2.46	2.48	2.44	2.50	2.50
	1.20	1.18	1.16	1.20	1.21
	5.62	5.55	5.54	5.58	5.58
	-1.41	-1.36	-1.34	-1.35	-1.35
	5.89	5.80	5.77	5.84	5.84
	-0.99	-0.99	-1.00	-0.98	-0.98
	2.68	2.70	2.73	2.67	2.67

Table 10.9. Approximate displacements and displacement derivatives for $\alpha = 1.0$.

Number of basis vectors	2	3	4	5	Exact
Displacements	0.28	0.29	0.29	0.30	0.30
	0.90	0.84	0.88	0.90	0.90
	0.41	0.45	0.47	0.49	•0.49
	2.10	2.17	2.19	2.21	•2.21
	-0.53	-0.61	-0.62	-0.60	-0.60
	2.24	2.34	2.37	2.40	2.40
	-0.30	-0.31	-0.31	-0.30	-0.30
	1.01	0.95	0.93	0.90	0.90
1st Derivatives	0.25	0.25	0.25	0.24	0.24
	0.74	0.72	0.69	0.64	0.64
	•	•	•	•	•
	•	•	•	•	•
	0.43	0.46	0.44	0.47	0.47
	1.81	1.84	1.81	1.74	1.74
	0.26	0.27	0.27	0.29	0.29
	0.82	0.80	0.81	0.87	0.88

• Joint 2 is effectively eliminated.

Table 10.10. Percentage errors in displacements and displacement derivatives.

Modified design	$\alpha = 0.5$		$\alpha = 0.75$	
Number of basis vectors	2	3	2	3
Displacements	4.15	0.81	6.06	1.90
	-2.86	1.80	-2.09	3.79
	7.54	1.78	11.34	3.99
	2.34	0.38	3.58	0.91
	8.60	-0.17	10.82	-0.48
	3.12	0.60	4.81	1.40
	2.16	-0.85	1.85	-2.00
	-5.43	-1.29	-8.43	-3.12
1st Derivatives	-0.84	-1.43	-1.21	-3.09
	-3.07	-2.89	-7.92	-5.89
	-4.13	-3.65	-5.86	-8.04
	-1.44	-0.85	-1.55	-1.77
	-0.46	0.56	3.79	1.10
	-2.06	-1.32	-2.41	-2.76
	1.88	1.40	4.28	2.74
	2.17	2.49	3.00	4.78
2nd Derivatives	-1.10	0.11	0.60	0.60
	2.63	-0.10	1.95	0.90
	-3.47	0.43	0.27	2.20
	-1.53	0.16	-0.86	0.41
	-7.52	-0.15	-4.27	-0.28
	-2.01	0.23	-0.86	0.75
	-1.12	-0.01	-0.55	-0.51
	2.09	-0.23	-0.43	-1.12

10.6 REFERENCES

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11 Nonlinear Reanalysis

It was noted in Section 2.3 that solution of a set of nonlinear equations for structural analysis can be carried out by different methods (e.g. [1]). In general, no matter what method is used, a set of updated linear equations must be repeatedly solved during the solution process. In addition, the structure stiffness matrix is often decomposed into upper and lower triangular matrices. As a result, the CA method is also most suitable for nonlinear analysis, as is shown in this chapter. The solution steps of geometric nonlinear analysis problems are summarized in Section 11.1. Application of the CA method in geometric nonlinear analysis is demonstrated in Section 11.2, and nonlinear reanalysis by the method is presented in Section 11.3.

11.1 GEOMETRIC NONLINEAR ANALYSIS

Geometric nonlinear analysis described in Section 2.3.1 involves the following steps. Starting with linear analysis, we first calculate the initial displacements $\mathbf{r}_0$ by the linear analysis equations [Eq. (2.41)]

$$\mathbf{K}_0 \mathbf{r}_0 = \mathbf{R}_0 \tag{11.1}$$

where $\mathbf{K}_0$ is the elastic stiffness matrix and $\mathbf{R}_0$ is the external force vector. In the solution process the following calculations are repeated until convergence occurs:

a. Calculate the member forces $\mathbf{N}$, which are functions of the displacements $\mathbf{r}$ [see Eq. (2.43)]

$$\mathbf{N} = \mathbf{N}(\mathbf{r}) \tag{11.2}$$

b. Calculate the corresponding internal force vector $\mathbf{R}_I$ by Eq. (2.44)

$$\mathbf{R}_I = \mathbf{C}(\mathbf{r})\,\mathbf{N} \tag{11.3}$$

where the elements of matrix $\mathbf{C}(\mathbf{r})$ depend on the deformed geometry.

c. Calculate the out-of-balance (residual) force vector $\delta\mathbf{R}$ by Eq. (2.45)

$$\delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I \tag{11.4}$$

d. Calculate the tangent stiffness matrix $\mathbf{K}_T$ by Eq. (2.47)

$$\mathbf{K}_T = \partial \mathbf{R}_I / \partial \mathbf{r} \tag{11.5}$$

e. Calculate the displacements due to the out-of-balance forces $\delta\mathbf{r}$ by Eq. (2.48)

$$\mathbf{K}_T \delta\mathbf{r} = \delta\mathbf{R} \tag{11.6}$$

f. Evaluate the updated displacements $\mathbf{r}$ by Eq. (2.49)

$$\mathbf{r} = \mathbf{r}_0 + \delta\mathbf{r} \tag{11.7}$$

redefining $\mathbf{r}_0$ as the updated displacement at the previous cycle.

11.2 NONLINEAR ANALYSIS BY THE CA METHOD

It was shown in previous chapters that the CA method is most effective in reanalysis of linear structures, where the changes in the stiffness matrix $\mathbf{K}$ depend only on the design variables. It is shown in this section that the method is suitable also for nonlinear analysis of structures, as demonstrated elsewhere [2]. In particular, the method can be used to calculate efficiently the displacements $\delta\mathbf{r}$ due to the out-of-balance forces at each iteration cycle. As a result, significant reductions in the computational effort involved in the repeated solution of the modified equations can be achieved.

Given the elastic stiffness matrix $\mathbf{K}_0$ and the corresponding initial displacements $\mathbf{r}_0$ calculated by the linear equilibrium equations (11.1), the modified equilibrium equations to be solved at each iteration cycle are given by [see Eqs. (2.46), (11.4) and (11.6)]

$$(\mathbf{K}_0 + \mathbf{K}_G)\,\delta\mathbf{r} = \mathbf{K}_T\,\delta\mathbf{r} = \delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I \tag{11.8}$$

where $\mathbf{K}_G$ is the geometric stiffness matrix whose elements change from one iteration cycle to another; $\mathbf{K}_T$ is the tangent stiffness matrix, accounting for elastic and geometric stiffnesses; $\mathbf{R}_0$ is the given external force vector; $\mathbf{R}_I$ is the vector of internal forces, whose elements are functions of the deformed geometry and change from one iteration cycle to another; and $\delta\mathbf{r}$ is the vector of displacements due to the out-of-balance forces $\delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I$.

Defining [see Eq. (3.12)]

$$\Delta \mathbf{K}_0 = \mathbf{K}_G$$
$$\Delta \mathbf{R}_0 = - \mathbf{R}_I \tag{11.9}$$

then Eq. (11.8) becomes [Eq. (3.13)]

$$(\mathbf{K}_0 + \Delta \mathbf{K}_0)\, \delta \mathbf{r} = \mathbf{R}_0 + \Delta \mathbf{R}_0 \tag{11.10}$$

Note that Eq. (3.5) for linear reanalysis and Eq. (11.10) are of similar form. That is, the problem of nonlinear analysis can be stated in the form of linear reanalysis where both the stiffness matrix and the load vector are changed.

To solve the problem of Eq. (11.10) by the CA method, a procedure similar to that described in Section 7.1.2 for linear reanalysis can be employed. Evaluation of the displacements $\delta \mathbf{r}$ involves the following steps:

a. Calculate the tangent stiffness matrix $\mathbf{K}_T = \mathbf{K}_0 + \Delta \mathbf{K}_0$ and the modified load vector $\delta \mathbf{R} = \mathbf{R}_0 + \Delta \mathbf{R}_0$. Since the initial values $\mathbf{K}_0$ and $\mathbf{R}_0$ are already given, this step involves only calculation of $\Delta \mathbf{K}_0$ and $\Delta \mathbf{R}_0$ (i.e., $\mathbf{K}_G$ and $-\mathbf{R}_I$, respectively).

b. Calculate the basis vectors $\delta \mathbf{r}_i$ ($i = 1,..., s$) by [see Eq. (7.12)]

$$\delta \mathbf{r}_1 = \mathbf{K}_0^{-1}(\mathbf{R}_0 + \Delta \mathbf{R}_0)$$
$$\delta \mathbf{r}_i = - \mathbf{B}\, \delta \mathbf{r}_{i-1} \qquad (i = 2, ..., s) \tag{11.11}$$

where s is the number of basis vectors considered and matrix $\mathbf{B}$ is defined as

$$\mathbf{B} = \mathbf{K}_0^{-1}\, \Delta \mathbf{K}_0 \tag{11.12}$$

As previously noted, the calculation of the basis vectors involves only forward and backward substitutions.

c. Calculate the reduced stiffness matrix $\mathbf{K}_R$ and the reduced load vector $\mathbf{R}_R$, by Eqs. (7.14) with $\mathbf{K}_T$ and $\delta \mathbf{R}$ replacing $\mathbf{K}$ and $\mathbf{R}$, respectively

$$\mathbf{K}_R = \delta \mathbf{r}_B^T \mathbf{K}_T\, \delta \mathbf{r}_B$$
$$\mathbf{R}_R = \delta \mathbf{r}_B^T\, \delta \mathbf{R} \tag{11.13}$$

where $\delta \mathbf{r}_B$ is the matrix of basis vectors [Eqs. (11.11)], defined as

$$\delta \mathbf{r}_B = [\delta \mathbf{r}_1, \delta \mathbf{r}_2, ..., \delta \mathbf{r}_s] \tag{11.14}$$

d. Evaluate the displacements $\delta\mathbf{r}$ in Eq. (11.10) by [see Eq. (7.16)]

$$\delta\mathbf{r} = y_1\delta\mathbf{r}_1 + y_2\delta\mathbf{r}_2 + \ldots + y_s\delta\mathbf{r}_s = \delta\mathbf{r}_B\,\mathbf{y} \tag{11.15}$$

where the vector of coefficients $\mathbf{y}^T = \{y_1, y_2, \ldots, y_s\}$ in Eq. (11.15) is calculated by solving the set of $s \times s$ equations [Eq. (7.15)]

$$\mathbf{K}_R\,\mathbf{y} = \mathbf{R}_R \tag{11.16}$$

It might prove useful to use the procedure described in Section 7.2.2 for uncoupled basis vectors. In such cases, the following procedure is considered instead of steps *c* and *d* above. Generate new set of basis vectors $\delta\mathbf{V}_i$ (i = 1, ..., *s*) by [see Eqs. (7.32), (7.33)]

$$\delta\mathbf{V}_1 = \left|\,\delta\mathbf{r}_1^T\mathbf{K}_T\delta\mathbf{r}_1\,\right|^{-1/2}\,\delta\mathbf{r}_1 \tag{11.17}$$

$$\begin{aligned}
\delta\overline{\mathbf{V}}_i &= \delta\mathbf{r}_i - \sum_{j=1}^{i-1}\left(\delta\mathbf{r}_i^T\mathbf{K}_T\delta\mathbf{V}_j\right)\delta\mathbf{V}_j \qquad i = 2, \ldots, s \\
\delta\mathbf{V}_i &= \left|\,\delta\overline{\mathbf{V}}_i^T\mathbf{K}_T\delta\overline{\mathbf{V}}_i\,\right|^{-1/2}\,\delta\overline{\mathbf{V}}_i
\end{aligned} \tag{11.18}$$

Then evaluate the displacements $\delta\mathbf{r}$ by [see Eq. (7.34)]

$$\delta\mathbf{r} = \delta\mathbf{V}_B\left(\delta\mathbf{V}_B^T\delta\mathbf{R}\right) = \sum_{i=1}^{s}\delta\mathbf{V}_i\left(\delta\mathbf{V}_i^T\delta\mathbf{R}\right) \tag{11.19}$$

where $\delta\mathbf{V}_B$ is the matrix of the new basis vectors

$$\delta\mathbf{V}_B = [\delta\mathbf{V}_1, \ldots, \delta\mathbf{V}_s] \tag{11.20}$$

If the accuracy is insufficient, additional basis vectors are introduced and the updated displacements $\delta\mathbf{r}$ are evaluated by Eq. (11.19). Consideration of an additional basis vector $\delta\mathbf{V}_{s+1}$ involves only calculation of the additional term $\delta\mathbf{V}_{s+1}\left(\delta\mathbf{V}_{s+1}^T\delta\mathbf{R}\right)$, added to the approximate displacements expression of Eq. (11.19).

11.3 NONLINEAR REANALYSIS BY THE CA METHOD

In this section, the CA method is used for nonlinear reanalysis, considering both changes due to the effect of geometrical non-linearity and changes in the design [3]. In this general case of nonlinear reanalysis, the matrix of changes $\Delta\mathbf{K}$ and the vector of changes $\Delta\mathbf{R}$ are expressed as [see Eq. (3.14)]

$$\Delta \mathbf{K} = \Delta \mathbf{K}_0 + \Delta \mathbf{K}_D \tag{11.21}$$

$$\Delta \mathbf{R} = \Delta \mathbf{R}_0 + \Delta \mathbf{R}_D \tag{11.22}$$

where the two parts of $\Delta \mathbf{K}$ and $\Delta \mathbf{R}$ are as follows:

a. the change $\Delta \mathbf{K}_0 = \mathbf{K}_G$ is the geometric stiffness matrix and the change $\Delta \mathbf{R}_0 = -\mathbf{R}$ is the internal force vector [see Eq. (11.9)]; and

b. the change $\Delta \mathbf{K}_D$ in the elastic stiffness matrix and the change $\Delta \mathbf{R}_D$ in the load vector due to changes in the design.

If the external forces are unchanged, then the change in the load vector becomes $\Delta \mathbf{R} = \Delta \mathbf{R}_0 = -\mathbf{R}_I$; i.e., the change $\mathbf{R}_I = \mathbf{C}(\mathbf{r})\,\mathbf{N} = \mathbf{R}_I(\mathbf{r})$ [Eqs. (11.2) and (11.3)] depends only on $\mathbf{r}$. Thus, the nonlinear reanalysis process consists of the following parts:

a. Start with linear analysis and first calculate the initial displacements by the linear equilibrium equations (11.1). Then, carry out nonlinear analysis by the iterative procedure described in Section 11.1. Use the CA method to solve the modified equilibrium equations (11.10) at each iteration cycle.

b. For a change in the design $\Delta \mathbf{K}_D$, the nonlinear reanalysis process is similar to that of nonlinear analysis, with the following different definition of the tangent stiffness matrix

$$\mathbf{K}_T = \mathbf{K}_0 + (\Delta \mathbf{K}_D + \Delta \mathbf{K}_0) \tag{11.23}$$

For any given change in the design, the elements of the matrix $\Delta \mathbf{K}_D$ are constant whereas the elements of the geometric stiffness matrix $\Delta \mathbf{K}_0 = \mathbf{K}_G$ are updated after each iteration. The CA method is used to solve the modified equations at each iteration cycle, as described in Section 11.1 for nonlinear analysis.

The solution process can be performed for various successive changes in the design $\Delta \mathbf{K}_D$ as necessary, with the initial decomposed $\mathbf{K}_0$ being unchanged.

Example 11.1 – Nonlinear Analysis. To illustrate the solution steps of nonlinear analysis, consider the simple two-bar truss shown in Figure 11.1, subjected to two external forces. The modulus of elasticity is 10 000 and the cross sectional area of both members is $A_1 = A_2 = 0.01$. The initial displacements $\mathbf{r}_0$ are first calculated by Eq. (11.1), where

$$\mathbf{K}_0 = \begin{bmatrix} EA_1/L_1 & \\ & EA_2/L_2 \end{bmatrix} = \begin{bmatrix} 5.0 & \\ & 10.0 \end{bmatrix} \qquad \mathbf{R}_0 = \begin{Bmatrix} R_{01} \\ R_{02} \end{Bmatrix} = \begin{Bmatrix} 10.0 \\ 5.0 \end{Bmatrix} \tag{a}$$

Substituting the given numerical data into Eq. (11.11) yields

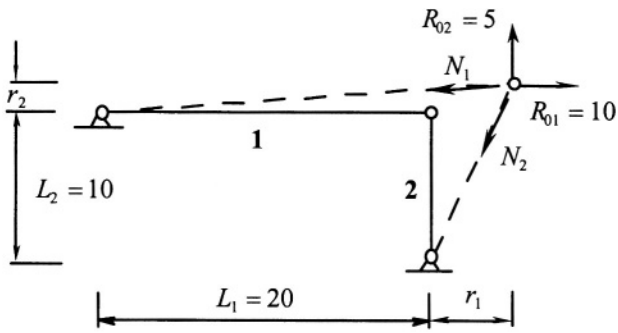

Fig. 11.1 Two-bar truss.

$$\mathbf{r}_0 = \begin{Bmatrix} r_{01} \\ r_{02} \end{Bmatrix} = \begin{Bmatrix} 2.0 \\ 0.5 \end{Bmatrix} \tag{b}$$

Assume small strains such that $r_1^2 / L_1^2 << r_1 / L_1$, where L_1, L_2 are the members lengths and r_1, r_2 are the nodal displacements shown in Figure 11.1. The force N_1 in member 1 can be expressed in terms of the displacements as

$$N_1 = EA_1\left(\left((L_1 + r_1)^2 + r_2^2\right)^{1/2} - L_1\right)/ L_1 \approx EA_1\left(r_1 / L_1 + 0.5r_2^2 / L_1^2\right) \tag{c}$$

Using a similar expression for the force N_2 in member 2 gives the following vector of member forces **N** [Eq. (11.2)]

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} EA_1\left(r_1 / L_1 + 0.5r_2^2 / L_1^2\right) \\ EA_2\left(r_2 / L_2 + 0.5r_1^2 / L_2^2\right) \end{Bmatrix} \tag{d}$$

The internal force vector $\mathbf{R}_I$ [Eq. (11.3)] consists of the horizontal and vertical components of the member forces, i.e.,

$$\mathbf{R}_I = \begin{Bmatrix} R_{I1} \\ R_{I2} \end{Bmatrix} = \begin{bmatrix} 1 & r_1 / L_2 \\ r_2 / L_1 & 1 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} N_1 + N_2 r_1 / L_2 \\ N_2 + N_1 r_2 / L_1 \end{Bmatrix} \tag{e}$$

and the out-of-balance forces $\delta\mathbf{R}$ [Eq. (11.4)] are given by

$$\delta\mathbf{R} = \begin{Bmatrix} \delta R_1 \\ \delta R_2 \end{Bmatrix} = \begin{Bmatrix} R_{01} - R_{I1} \\ R_{02} - R_{I2} \end{Bmatrix} = \begin{Bmatrix} R_{01} \\ R_{02} \end{Bmatrix} - \begin{Bmatrix} N_1 + N_2 r_1 / L_2 \\ N_2 + N_1 r_2 / L_1 \end{Bmatrix} \tag{f}$$

For $A_1=A_2=A$, the elements of the tangent stiffness matrix $\mathbf{K}_T$ [Eq. (11.5)] are

$$\mathbf{K}_T = \begin{bmatrix} \partial R_1/\partial r_1 & \partial R_1/\partial r_2 \\ \partial R_2/\partial r_1 & \partial R_2/\partial r_2 \end{bmatrix} =$$

$$EA\begin{bmatrix} 1/L_1 + r_2/L_2^2 + 1.5r_1^2/L_2^3 & r_1/L_2^2 + r_2/L_1^2 \\ r_1/L_2^2 + r_2/L_1^2 & 1/L_2 + r_1/L_1^2 + 1.5r_2^2/L_1^3 \end{bmatrix} \qquad (g)$$

The displacements $\delta\mathbf{r}$ due to $\delta\mathbf{R}$ [Eq. (11.6)] and, hence, the updated displacements $\mathbf{r}$ [Eq. (11.7)] are then calculated by [see Eqs. *(f),(g)*]

$$EA\begin{bmatrix} 1/L_1 + r_2/L_2^2 + 1.5r_1^2/L_2^3 & r_1/L_2^2 + r_2/L_1^2 \\ r_1/L_2^2 + r_2/L_1^2 & 1/L_2 + r_1/L_1^2 + 1.5r_2^2/L_1^3 \end{bmatrix} \begin{Bmatrix} \delta r_1 \\ \delta r_2 \end{Bmatrix} = \begin{Bmatrix} R_{01} \\ R_{02} \end{Bmatrix} - \begin{Bmatrix} N_1 + N_2 r_1/L_2 \\ N_2 + N_1 r_2/L_1 \end{Bmatrix} \qquad (h)$$

$$\mathbf{r} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} r_{01} \\ r_{02} \end{Bmatrix} + \begin{Bmatrix} \delta r_1 \\ \delta r_2 \end{Bmatrix} \qquad (i)$$

Starting with $\mathbf{r}_0^T = \{r_{01}, r_{01}\}$ [Eq. *(b)*], we obtain the results for three iteration cycles shown in Table 11.1. The iterative process converges very fast for this simple example.

Table 11.1. Results, Newton-Raphson iteration.

Iteration	**N** [Eq. (*d*)]	$\mathbf{R}_I$ [Eq. (*e*)]	$\delta\mathbf{R}$ [Eq. (*f*)]	$\delta\mathbf{r}$ [Eq. (*h*)]	**r** [Eq. (*i*)]
1	$\begin{Bmatrix} 10.0313 \\ 7.0 \end{Bmatrix}$	$\begin{Bmatrix} 11.4312 \\ 7.2508 \end{Bmatrix}$	$\begin{Bmatrix} -1.4312 \\ -2.2508 \end{Bmatrix}$	$\begin{Bmatrix} -0.1721 \\ -0.1794 \end{Bmatrix}$	$\begin{Bmatrix} 1.8279 \\ 0.3206 \end{Bmatrix}$
2	$\begin{Bmatrix} 9.1523 \\ 4.8766 \end{Bmatrix}$	$\begin{Bmatrix} 10.0437 \\ 5.0233 \end{Bmatrix}$	$\begin{Bmatrix} -0.0437 \\ -0.0233 \end{Bmatrix}$	$\begin{Bmatrix} -0.0072 \\ -0.0009 \end{Bmatrix}$	$\begin{Bmatrix} 1.8207 \\ 0.3197 \end{Bmatrix}$
3	$\begin{Bmatrix} 9.1163 \\ 4.8545 \end{Bmatrix}$	$\begin{Bmatrix} 10.0001 \\ 5.0002 \end{Bmatrix}$	$\begin{Bmatrix} -0.0001 \\ -0.0002 \end{Bmatrix}$	$\begin{Bmatrix} -0.0000 \\ -0.0000 \end{Bmatrix}$	$\begin{Bmatrix} 1.8207 \\ 0.3197 \end{Bmatrix}$

Table 11.2. Results, initial predictor and Newton-Raphson iteration.

Iteration	**N** [Eq. (*d*)]	$\mathbf{R}_I$ [Eq. (*e*)]	$\delta\mathbf{R}$ [Eq. (*f*)]	$\delta\mathbf{r}$ [Eq. (*h*)]	**r** [Eq. (*i*)]
1	$\begin{Bmatrix}7.9290\\2.8094\end{Bmatrix}$	$\begin{Bmatrix}8.3744\\2.8710\end{Bmatrix}$	$\begin{Bmatrix}1.6256\\2.1290\end{Bmatrix}$	$\begin{Bmatrix}0.2450\\0.1665\end{Bmatrix}$	$\begin{Bmatrix}1.8302\\0.3218\end{Bmatrix}$
2	$\begin{Bmatrix}9.1639\\4.8928\end{Bmatrix}$	$\begin{Bmatrix}10.0594\\5.0403\end{Bmatrix}$	$\begin{Bmatrix}-0.0594\\-0.0403\end{Bmatrix}$	$\begin{Bmatrix}-0.0095\\-0.0021\end{Bmatrix}$	$\begin{Bmatrix}1.8207\\0.3197\end{Bmatrix}$
3	$\begin{Bmatrix}9.1163\\4.8545\end{Bmatrix}$	$\begin{Bmatrix}10.0001\\5.0002\end{Bmatrix}$	$\begin{Bmatrix}-0.0001\\-0.0002\end{Bmatrix}$	$\begin{Bmatrix}-0.0000\\-0.0000\end{Bmatrix}$	$\begin{Bmatrix}1.8207\\0.3197\end{Bmatrix}$

An alternative procedure is to improve the initial displacements by calculating first the, so-called tangential predictor [1], Calculating the tangent stiffness matrix $\mathbf{K}_T$ for the given $\mathbf{r}_0$, we first calculate the displacements **r** by solving the set of equations

$$\mathbf{K}_T\,\mathbf{r} = \mathbf{R}_0 \qquad (j)$$

The result is $\mathbf{r}^T = \{1.5852, 0.1553\}$. The Newton-Raphson iteration then proceeds as shown in Table 11.2. Again, fast convergence is achieved for this simple example.

Example 11.2 - Nonlinear Reanalysis by the CA Method. The object of this example is to illustrate the solution steps by the CA method for nonlinear reanalysis using low-order approximations with the original set of basis vectors [Eqs. (11.11)].

Considering again the two-bar truss presented in example 11.1 and shown in Figure 11.1, we first calculate the initial displacements $\mathbf{r}_0$ by Eq. (11.1), where

$$\mathbf{K}_0 = \begin{bmatrix}5.0 & \\ & 10.0\end{bmatrix} \qquad \mathbf{R}_0 = \begin{Bmatrix}10.0\\5.0\end{Bmatrix} \qquad \mathbf{r}_0 = \begin{Bmatrix}2.0\\0.5\end{Bmatrix}$$

For the first iteration cycle, the modified equations to be solved are [Eq. (11.8)]

$$\mathbf{K}_T\,\delta\mathbf{r} = \delta\mathbf{R} = \mathbf{R}_0 - \mathbf{R}_I$$

The vector $\delta\mathbf{R}$ and the matrix $\mathbf{K}_T$ are calculated by [see Eqs. (11.9)]

$$\Delta \mathbf{R}_0 = -\mathbf{R}_I = -\begin{Bmatrix} 11.4312 \\ 7.2508 \end{Bmatrix}$$

$$\delta \mathbf{R} = \mathbf{R}_0 + \Delta \mathbf{R}_0 = -\begin{Bmatrix} 1.4312 \\ 2.2508 \end{Bmatrix}$$

$$\Delta \mathbf{K}_0 = \mathbf{K}_G = \begin{bmatrix} 1.1 & 2.125 \\ 2.125 & 0.5047 \end{bmatrix}$$

$$\mathbf{K}_T = \mathbf{K}_0 + \Delta \mathbf{K}_0 = \begin{bmatrix} 6.1 & 2.125 \\ 2.125 & 10.5047 \end{bmatrix}$$

The basis vectors are calculated by [see Eqs. (11.11), (11.12)]

$$\delta \mathbf{r}_1 = \mathbf{K}_0^{-1}(\mathbf{R}_0 + \Delta \mathbf{R}_0) = \begin{Bmatrix} -0.2862 \\ -0.2251 \end{Bmatrix}$$

$$\mathbf{B} = \mathbf{K}_0^{-1} \Delta \mathbf{K}_0 = \begin{bmatrix} 0.2200 & 0.4250 \\ 0.2125 & 0.0505 \end{bmatrix}$$

$$\delta \mathbf{r}_2 = -\mathbf{B} \delta \mathbf{r}_1 = \begin{Bmatrix} 0.1586 \\ 0.0722 \end{Bmatrix}$$

$$\delta \mathbf{r}_B = [\delta \mathbf{r}_1, \delta \mathbf{r}_2] = \begin{bmatrix} -0.2862 & 0.1586 \\ -0.2251 & 0.0722 \end{bmatrix}$$

The reduced stiffness matrix $\mathbf{K}_R$ and the reduced load vector $\mathbf{R}_R$, are calculated by [see Eqs. (11.13)]

$$\mathbf{K}_R = \delta \mathbf{r}_B^T \mathbf{K}_T \delta \mathbf{r}_B = \begin{bmatrix} 1.3058 & -0.5674 \\ -0.5674 & 0.2569 \end{bmatrix}$$

$$\mathbf{R}_R = \delta \mathbf{r}_B^T \delta \mathbf{R} = \begin{Bmatrix} 0.9163 \\ -0.3895 \end{Bmatrix}$$

The multipliers $\mathbf{y}$ and the displacements $\delta \mathbf{r}$ are calculated by [see Eqs. (11.15), (11.16)]

$$\mathbf{y} = \mathbf{K}_R^{-1}\mathbf{R}_R = \begin{Bmatrix} 1.0665 \\ 0.8394 \end{Bmatrix} \qquad \delta\mathbf{r} = \delta\mathbf{r}_B\mathbf{y} = \begin{Bmatrix} -0.1721 \\ -0.1794 \end{Bmatrix}$$

Since the number of basis vectors is equal to the number of degrees of freedom, the CA method provides the exact solution (see the first iteration in Table 11.1).

Assume a reduction of 50% in the cross section areas, the tangent stiffness matrix for the first iteration cycle is calculated by [see Eqs. (11.21), (11.23)]

$$\Delta\mathbf{K} = \Delta\mathbf{K}_0 + \Delta\mathbf{K}_D = \begin{bmatrix} 1.1 & 2.125 \\ 2.125 & 0.5047 \end{bmatrix} + \begin{bmatrix} -2.5 & \\ & -5.0 \end{bmatrix} = \begin{bmatrix} -1.4 & 2.125 \\ 2.125 & -4.4953 \end{bmatrix}$$

$$\mathbf{K}_T = \mathbf{K}_0 + \Delta\mathbf{K} = \begin{bmatrix} 3.6 & 2.125 \\ 2.125 & 5.5047 \end{bmatrix}$$

The out-of-balance forces $\delta\mathbf{R}$ are calculated by

$$\delta\mathbf{R} = \mathbf{R}_0 + \Delta\mathbf{R}_0 = \begin{Bmatrix} -1.4312 \\ -2.2508 \end{Bmatrix}$$

and the exact solution of $\delta\mathbf{r}$ is

$$\delta\mathbf{r} = \mathbf{K}_T^{-1}\delta\mathbf{R} = \begin{bmatrix} 3.6 & 2.125 \\ 2.125 & 5.5047 \end{bmatrix}^{-1} \begin{Bmatrix} -1.4312 \\ -2.2508 \end{Bmatrix} = \begin{Bmatrix} -0.2023 \\ -0.3308 \end{Bmatrix}$$

Assuming only two basis vectors, then from Eqs. (11.11), (11.12)

$$\delta\mathbf{r}_1 = \mathbf{K}_0^{-1}(\mathbf{R}_0 + \Delta\mathbf{R}_0) = \begin{Bmatrix} -0.2862 \\ -0.2251 \end{Bmatrix}$$

$$\mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K} = \begin{bmatrix} -0.2800 & 0.4250 \\ 0.2125 & -0.4495 \end{bmatrix}$$

$$\delta\mathbf{r}_2 = -\mathbf{B}\,\delta\mathbf{r}_1 = \begin{Bmatrix} 0.0155 \\ -0.0404 \end{Bmatrix}$$

The multipliers $\mathbf{y}$ and the displacements $\delta\mathbf{r}$ are calculated by [Eqs. (11.15), (11.16)]

$$\mathbf{y} = \mathbf{K}_R^{-1}\mathbf{R}_R = \begin{bmatrix} 0.8476 & 0.0511 \\ 0.0511 & 0.0072 \end{bmatrix}^{-1} \begin{Bmatrix} 0.9163 \\ 0.0686 \end{Bmatrix} = \begin{Bmatrix} 0.8840 \\ 3.2648 \end{Bmatrix}$$

$$\delta\mathbf{r} = \delta\mathbf{r}_B\mathbf{y} = \begin{Bmatrix} -0.2023 \\ -0.3308 \end{Bmatrix}$$

Again, since the number of basis vectors is equal to the number of degrees of freedom, the CA method provides the exact solution.

Example 11.3 – Nonlinear Analysis by the CA Method. To illustrate the solution steps of nonlinear analysis by the CA method, consider the four-bar truss shown in Figure 11.2, subjected to a single horizontal external force of 100 (all units are in Kilo-Newton and meters). The modulus of elasticity is $E = 20601\text{x}10^7$, the angle of member 3 is $\theta = 60^0$ and the cross sectional areas of the members are $A1 = A2 = A3 = 0.00128$ and $A4 = 0.000128$. The members lengths are denoted Li (i = 1, 2, 3, 4) and the four analysis unknowns are the horizontal and vertical displacements *(u2, w2, u*3, w3) at joints 2 and 3, respectively. The member forces Ni (i = 1, 2, 3, 4) are given in terms of the displacements by [Eq. (11.2)]

$$N1 = EA1\left(-w2/L1 + 0.5u2^2/L1^2\right)$$

$$N2 = EA2\left((u3-u2)/L2 + + 0.5(w3-w2)^2/L2^2\right)$$

$$N3 = EA3\left((u3\cos\theta - w3\sin\theta)/L3 + 0.5(u3\sin\theta + w3\cos\theta)^2/L3^2\right)$$

$$N4 = EA4\left(-w3/L4 + 0.5u3^2/L4^2\right)$$

The out-of-balance forces $\delta\mathbf{R}$ [Eq. (11.4)] are given by

$$\delta R_1 = R_{01} - \left(N1u2/L1^2 - N2/L2\right)$$

$$\delta R_2 = R_{02} - \left(-N1/L1 - N2w3/L2^2 + N2w2/L2^2\right)$$

$$\delta R_3 = R_{03} - (N2/L2 + N3\cos\theta/L3 + N3u3\sin^2\theta/L3^2 + N3\cos\theta w3\sin\theta/L3^2 + N4u3/L4^2)$$

$$\delta R_4 = R_{04} - (N2w3/L2^2 - N2w2/L2^2 - N3\sin\theta/L3 + N3u3\sin\theta\cos\theta/L3^2 + N3w3\cos^2\theta/L3^2 - N4/L4)$$

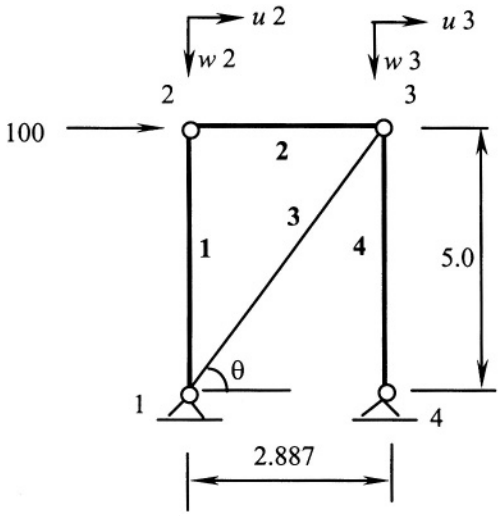

Fig. 11.2. Four-bar truss.

The initial displacements $\mathbf{r}_0$ are first calculated by Eq. (11.1)

$$\mathbf{r}_0^T = \{0.3381, 0, 0.3350, 0.1642\}$$

Starting with $\mathbf{r}_0^T$, we obtain the results by the Newton-Raphson iteration shown in Table 11.3. The iterative process converges very fast for this simple example.

To illustrate solution of the first iteration cycle by the CA method, we first calculate the basis vectors by [Eqs. (11.11), (11.12)]

$$\delta\mathbf{r}_1 = \mathbf{K}_0^{-1}(\mathbf{R}_0 + \Delta\mathbf{R}_0) \qquad \mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K}_0$$

$$\delta\mathbf{r}_2 = -\mathbf{B}\delta\mathbf{r}_1 \qquad \delta\mathbf{r}_3 = -\mathbf{B}\delta\mathbf{r}_2 \qquad \delta\mathbf{r}_4 = -\mathbf{B}\delta\mathbf{r}_3$$

The resulting matrix of basis vectors is

$$\delta\mathbf{r}_B = [\delta\mathbf{r}_1, \delta\mathbf{r}_2, \delta\mathbf{r}_3, \delta\mathbf{r}_4] = \begin{bmatrix} -.0665 & 0.1037 & -.0880 & 0.0931 \\ 0.0117 & -.0056 & 0.0079 & -.0069 \\ -.0709 & 0.1053 & -.0908 & 0.0954 \\ -.0230 & 0.0490 & -.0385 & 0.422 \end{bmatrix}$$

The multipliers $\mathbf{y}$ and the displacements $\delta\mathbf{r}$ are calculated by Eqs. (11.15), (11.16). The results obtained for 2, 3, and 4 basis vectors, and the exact solution, are summarized in Table 11.4. It is observed that accurate solution is achieved with only 2 basis vectors.

Table 11.3. Results, Newton-Raphson iteration.

	*u*2	*w*2	*u*3	*w*3	*N*1	*N*2	*N*3	*N*4
$\mathbf{r}=\mathbf{r}_0$	0.3381	0	0.3350	0.1642	603027	137958	1703000	-806844
$\delta\mathbf{r}$	-0.0022	0.0107	-0.0068	0.0099				
$\mathbf{r}+\delta\mathbf{r}$	0.3359	0.0107	0.3282	0.1741	28446	-287708	1153322	-861313
$\delta\mathbf{r}$	-0.0003	0	-0.0004	-0.0002				
$\mathbf{r}+\delta\mathbf{r}$	0.3356	0.0107	0.3278	0.1739	28155	-287584	1152406	-860404
$\delta\mathbf{r}$	0	0	0	0				
$\mathbf{r}+\delta\mathbf{r}$	0.3356	0.0107	0.3278	0.1739	28154	-287584	1152405	-860404

Table 11.4. Results, first iteration by the CA method.

Basis vectors	$\mathbf{y}$				$\delta\mathbf{r}$			
2	1.318	0.082			-.0023	0.0108	-.0069	0.0099
3	1.023	1.283	0.764		-.0022	0.0108	-.0068	0.0099
4	0.997	1.007	1.325	0.819	-.0022	0.0107	-.0068	0.0099
Exact					-.0022	0.0107	-.0068	0.0099

Example 11.4 -Nonlinear Analysis of A Space Frame [2]. To illustrate the results of nonlinear analysis of a space truss, consider the shallow geodesic dome shown in Figure 11.3. The dome consists of 156 space frame members, has a total of 61 joints with 24 boundary joints that are simply supported, and the total number of degrees of freedom is 24×3 + (61–24) ×6=294. The nonlinear analysis problem is solved by the following two procedures:

a. New basis vectors are generated at each solution of the modified equations.
b. The original basis vectors $\delta\mathbf{r}_i$ [Eqs. (11.11)] are assumed to be constant for each incremental load step, however the new basis vectors $\delta\mathbf{V}_i$ [Eqs. (11.17), (11.18)] and the tangent stiffness matrix $\mathbf{K}_T$ are modified for each iterative step.

In both cases, exact linear analysis is carried out only for the first iterative step of the first load increment. Assuming that the dome is subjected to a concentrated load at the apex, it is found that the load-displacement curves for the apex obtained by conventional nonlinear analysis and by the CA method are practically the same. The average number of basis vectors needed to achieve high accuracy is 5 - 6.

As to the efficiency of the CA method, it was found that solution by procedure *a* required about 40% of the CPU time needed by conventional nonlinear analysis while solution by procedure *b* required about 33% of the time [2]. The total number of iterations for both solutions by the two procedures is 538.

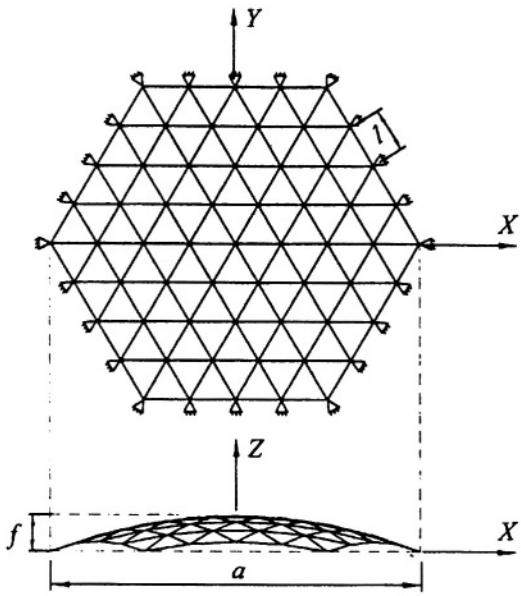

Fig. 11.3. Geodesic dome [2].

11.4 REFERENCES

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12 Vibration Reanalysis

Some topics related to vibration reanalysis are introduced in this chapter. Calculation of the first-mode shape by the matrix iteration approach is demonstrated in Section 12.1, and formulation of eigenproblem reanalysis is presented in Section 12.2. A distinction is made between the following two types of reanalysis problems:

a. Eigenvector reanalysis, where the object is to calculate the modified eigenvectors, or mode shapes, due to changes in the stiffness matrix and in the mass matrix. This type of problem, which is more difficult to solve than the eigenvalue reanalysis problem, can be solved by the CA method [1,2] as is shown in Section 12.3.

b. Eigenvalue reanalysis, where the object is to calculate the modified eigenvalues due to changes in the stiffness and mass matrices. In many optimal design problems we are interested only in the eigenvalues rather than the eigenvectors. It is often assumed that the mode shapes change only insignificantly due to changes in the design. The eigenvalue reanalysis problem, which can be solved by several methods, is discussed in Section 12.4.

12.1 VIBRATION ANALYSIS

Consider the generalized eigenproblem presented in Section 2.4.1 [Eq. (2.55)]

$$\mathbf{K}\,\mathbf{r} = \omega^2 \mathbf{M}\,\mathbf{r} = \lambda \mathbf{M}\,\mathbf{r} \tag{12.1}$$

where ω is the circular frequency, $\omega^2 = \lambda$ are the eigenvalues indicating the square of the free-vibration frequencies, while the corresponding displacement vectors $\mathbf{r}$ express the eigenvectors, or mode shapes of the vibrating system. For a system having n degrees of freedom, the frequency vector $\boldsymbol{\omega}^T = \{\omega_1, \omega_2, \omega_3, ..., \omega_n\}$ represents the frequencies of the n modes of vibration possible in the system.

To calculate the first-mode shape by the matrix iteration approach, Eq. (12.1) can be rewritten in an iterative form. Defining the dynamic matrix $\mathbf{D}$ as [Eq. (2.65)]

$$\mathbf{D} = \mathbf{K}^{-1}\,\mathbf{M} \tag{12.2}$$

we obtain the iterative expression [Eq. (2.66)]

$$\bar{\mathbf{r}}^{(k)} = \mathbf{D}\,\mathbf{r}^{(k-1)} \tag{12.3}$$

where k denotes the iteration number, $\mathbf{r}^{(k-1)}$ is the displacement vector in the previous iteration and $\bar{\mathbf{r}}^{(k)}$ is the resulting improved shape. To initiate the iteration procedure for evaluating the first mode shape, a trial displacement vector $\mathbf{r}^{(0)}$ is assumed that is a reasonable estimate of this shape. The improved iteration vector is then obtained by normalizing the shape $\bar{\mathbf{r}}^{(k)}$ by dividing it by a reference $ref\left(\bar{\mathbf{r}}^{(k)}\right)$ to obtain [Eq. 2.67)]

$$\mathbf{r}^{(k)} = \frac{\bar{\mathbf{r}}^{(k)}}{ref\left(\bar{\mathbf{r}}^{(k)}\right)} \tag{12.4}$$

In general, the vector is normalized with respect to its largest element. By repeating the process sufficiently, we can improve the mode-shape approximation to any desired level of accuracy. After s iteration cycles, the frequency can be obtained by [Eq. (2,69)]

$$\omega_1^2 = \lambda_1 = \frac{\max\left(\mathbf{r}^{(s-1)}\right)}{\max\left(\bar{\mathbf{r}}^{(s)}\right)} = \frac{1}{\max\left(\bar{\mathbf{r}}^{(s)}\right)} \tag{12.5}$$

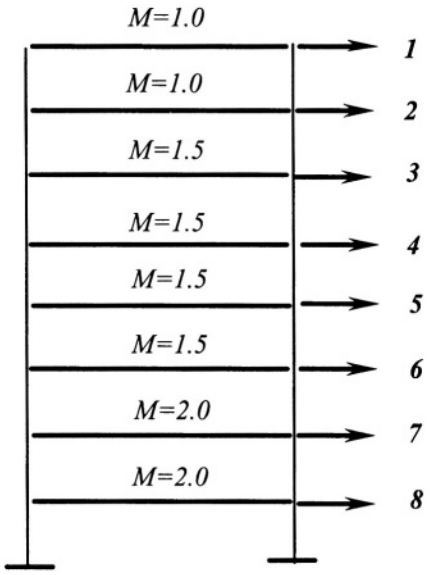

Fig. 12.1. Eight-story frame.

Example 12.1 - Calculating Mode Shapes by Matrix Iteration. Consider the eight-story frame shown in Figure 12.1 (all units are in kips, inches and seconds). The mass of the frame is lumped in the girders, with M values as shown. The girders are assumed to be non-deformable and the initial lateral stiffness of each of the eight stories is given by $EI/L^3 = 5.0$. Using the iterative procedure of Eqs. (12.3), (12.4) and assuming the initial displacement vector

$$\mathbf{r}^{(0)T} = \{1, 1, 1, 1, 1, 1, 1, 1\}$$

we found that the solution process converges in four iteration cycles, as shown in Table 12.1. The resulting frequency, calculated by Eq. (12.5), is $\omega = 1.253$.

Table 12.1. Results, calculation of the first-mode shape.

$\bar{\mathbf{r}}^{(1)}$	$\mathbf{r}^{(1)}$	$\bar{\mathbf{r}}^{(2)}$	$\mathbf{r}^{(2)}$	$\bar{\mathbf{r}}^{(3)}$	$\mathbf{r}^{(3)}$	$\bar{\mathbf{r}}^{(4)}$	$\mathbf{r}^{(4)}$
0.8000	1	0.6581	1	0.6392	1	0.6366	1
0.7833	0.9792	0.6414	0.9747	0.6226	0.9739	0.6199	0.9738
0.7500	0.9375	0.6084	0.9245	0.5897	0.9224	0.5870	0.9221
0.6917	0.8646	0.5520	0.8388	0.5336	0.8348	0.5311	0.8342
0.6083	0.7604	0.4740	0.7202	0.4566	0.7143	0.4543	0.7135
0.5000	0.6250	0.3769	0.5727	0.3616	0.5657	0.3596	0.5647
0.3667	0.4583	0.2642	0.4015	0.2523	0.3947	0.2507	0.3938
0.2000	0.2500	0.1363	0.2071	0.1296	0.2028	0.1287	0.2022

12.2 FORMULATION OF EIGENPROBLEM REANALYSIS

Consider an initial design with stiffness matrix $\mathbf{K}_0$ and mass matrix $\mathbf{M}_0$. The corresponding displacement vector (the eigenvector) $\mathbf{r}_0$ is calculated by solving the set of initial analysis equations (3.16)

$$\mathbf{K}_0\mathbf{r}_0 = \lambda_0\mathbf{M}_0\mathbf{r}_0 \tag{12.6}$$

where $\lambda_0 = \omega_0^2$ is the initial eigenvalue and ω_0 is the initial circular free-vibration frequency. Assume a change in the design and a corresponding change in the stiffness and mass matrices such that the modified matrices are expressed as [see Eq. (3.17)]

$$\begin{aligned} \mathbf{K} &= \mathbf{K}_0 + \Delta\mathbf{K} \\ \mathbf{M} &= \mathbf{M}_0 + \Delta\mathbf{M} \end{aligned} \tag{12.7}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$ are the changes in the stiffness and mass matrices, respectively.

The reanalysis problem can be stated as follows. Given the initial values $\mathbf{K}_0$ and $\mathbf{M}_0$, the initial eigenvector $\mathbf{r}_0$ and eigenvalue λ_0 are first calculated [Eq. (12.6)]. The object is to evaluate the modified eigenvector $\mathbf{r}$ and eigenvalue λ due to various changes in the design, such that the modified analysis equations are satisfied [see Eq. (3.18)]

$$\mathbf{K}\,\mathbf{r} = \lambda\,\mathbf{M}\,\mathbf{r} \tag{12.8}$$

Denoting the right-hand-side vectors of Eqs. (12.6) and (12.8) as [see Eq. (3.19)]

$$\begin{aligned} \mathbf{R}_0 &= \lambda_0 \mathbf{M}_0 \mathbf{r}_0 \\ \mathbf{R} &= \lambda\,\mathbf{M}\,\mathbf{r} \end{aligned} \tag{12.9}$$

we find [see Eqs. (3.20), (3.21)]

$$\mathbf{K}_0 \mathbf{r}_0 = \mathbf{R}_0 \tag{12.10}$$

$$\mathbf{K}\,\mathbf{r} = \mathbf{R} \tag{12.11}$$

In this formulation, the elements of $\mathbf{K}$ and $\mathbf{M}$ are explicit functions of the design variables, and the elements of the right-hand-side vector $\mathbf{R}$ are functions of $\mathbf{M}$, $\mathbf{r}$ and λ. It is noted that the modified analysis equations (12.11) can be expressed as in Eq. (3.5)

$$(\mathbf{K}_0 + \Delta\mathbf{K})\,\mathbf{r} = \mathbf{R}_0 + \Delta\mathbf{R} \tag{12.12}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$ are defined as [see Eqs. (12.7) and (12.9)]

$$\begin{aligned} \Delta\mathbf{K} &= \mathbf{K} - \mathbf{K}_0 \\ \Delta\mathbf{R} &= \mathbf{R} - \mathbf{R}_0 = \lambda\,\mathbf{M}\,\mathbf{r} - \lambda_0 \mathbf{M}_0 \mathbf{r}_0 \end{aligned} \tag{12.13}$$

Since the modified analysis equations (3.5) and (12.12) are of similar form, the CA method can readily be used to solve the latter equations efficiently.

12.3 REANALYSIS BY THE CA METHOD

In the CA method presented in Section 7.1.2, the displacements are approximated by [see Eq. (7.16)]

$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \ldots + y_s\mathbf{r}_s = \mathbf{r}_B\,\mathbf{y} \tag{12.14}$$

where $\mathbf{r}_B$ is the matrix of basis vectors

$$\mathbf{r}_B = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_s] \tag{12.15}$$

and $\mathbf{y}$ is a vector of coefficients to be determined

$$\mathbf{y}^T = \{y_1, y_2, \ldots, y_s\} \tag{12.16}$$

Substituting Eq. (12.14) into the modified analysis equations (12.8) and premultiplying by $\mathbf{r}_B^T$ yields

$$\mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \mathbf{y} = \lambda \mathbf{r}_B^T \mathbf{M} \mathbf{r}_B \mathbf{y} \tag{12.17}$$

Introducing the notation

$$\mathbf{K}_R = \mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \qquad \mathbf{M}_R = \mathbf{r}_B^T \mathbf{M} \mathbf{r}_B \tag{12.18}$$

and substituting into Eq. (12.17) gives

$$\mathbf{K}_R \mathbf{y} = \lambda \mathbf{M}_R \mathbf{y} \tag{12.19}$$

To evaluate the approximate displacements, the reduced eigenvalue problem of Eq. (12.19) is first solved for $\mathbf{y}$. The displacement elements of the eigenvector $\mathbf{r}$ are then evaluated for the calculated $\mathbf{y}$ by Eq. (12.14).

Given the initial values $\mathbf{K}_0$, $\mathbf{M}_0$, $\mathbf{r}_0$ and λ_0, from eigenvalue analysis of the initial structure, the reanalysis procedure to evaluate the modified displacements $\mathbf{r}$ and eigenvalue λ due to various changes in the design involves the following steps.

a. Select the basis vectors. This could be done as in linear reanalysis [Eq. (7.12)]

$$\mathbf{r}_1 = \mathbf{K}_0^{-1} \mathbf{R} \tag{12.20}$$

$$\mathbf{r}_i = -\mathbf{B} \mathbf{r}_{i-1} \qquad (i = 2, \ldots, s) \tag{12.21}$$

where

$$\mathbf{B} \equiv \mathbf{K}_0^{-1} \Delta \mathbf{K} \tag{12.22}$$

Since $\mathbf{R}$ is not known [see Eq. (12.9)], it is convenient to consider $\mathbf{R}_0$ instead of $\mathbf{R}$ in the calculation of $\mathbf{r}_1$. This selection greatly simplifies the calculation of the basis vectors, since the elements of $\mathbf{R}_0$ are constant. Thus, the first basis vector is already known from the initial analysis

$$\mathbf{r}_1 = \mathbf{K}_0^{-1}\mathbf{R}_0 = \mathbf{r}_0 \tag{12.23}$$

The additional vectors are calculated by Eq. (12.21). It is found that for this selection of $\mathbf{r}_1$, accurate results are achieved with a small number of basis vectors.

b. Calculate the reduced matrices $\mathbf{K}_R$ and $\mathbf{M}_R$ by Eqs. (12.18)

$$\mathbf{K}_R = \mathbf{r}_B^T\mathbf{K}\mathbf{r}_B \qquad \mathbf{M}_R = \mathbf{r}_B^T\mathbf{M}\mathbf{r}_B \tag{12.24}$$

c. Solve the reduced eigenproblem of Eq. (12.19)

$$\mathbf{K}_R\mathbf{y} = \lambda\mathbf{M}_R\mathbf{y} \tag{12.25}$$

where λ is the eigenvalue, and the coefficient vector $\mathbf{y}$ is the eigenvector. The iterative solution process at this step is similar to that of determining $\mathbf{r}$, except that a reduced system is considered.

d. Evaluate the final displacement elements of the eigenvector by Eq. (12.14)

$$\mathbf{r} = \mathbf{r}_B\mathbf{y} \tag{12.26}$$

This solution process involves less computational effort than that of the initial eigenvalue problem, since the matrix $\mathbf{K}_0$ is already given in a decomposed form and a reduced eigenvalue problem is solved instead of the larger original problem.

Example 12.2 - Reanalysis by the CA Method. To illustrate eigenvector reanalysis by the CA method, consider again the eight-story frame of example 12.1 shown in Figure 12.1 (all units are in kips, inches and seconds). The mass of the frame is lumped in the girders, with *M* values as shown. The girders are assumed to be nondeformable and the initial lateral stiffness of each of the eight stories is $EI/L^3 = 5.0$. Results are demonstrated for the first mode shape and the following modified design [1]

Stories 1 - 2	$EI/L^3 = 60.0$
Stories 3 - 6	$EI/L^3 = 55.0$
Stories 7 - 8	$EI/L^3 = 50.0$

That is, the stiffness matrix is increased non-uniformly by an order of magnitude. Specifically, the elements of the modified stiffness matrix are about 10-12 times larger then the elements of the initial matrix. The initial mode shape $\mathbf{r}_0$ is given in the last column of Table 12.1 and the resulting frequency, calculated by Eq. (12.5), is $\omega = 1.253$. To carry out reanalysis for the modified design by the CA method, the basis

vectors are first determined. The first basis vector $\mathbf{r}_1 = \mathbf{r}_0$ is already known from the initial analysis

$$\mathbf{r}_1^T = \mathbf{r}_0^T = \{1, 0.9738, 0.9221, 0.8342, 0.7135, 0.5647, 0.3938, 0.2022\}$$

The second basis vector, calculated by Eq. (12.21), is

$$\mathbf{r}_2^T = -\{10.3159, 10.0801, 9.6148, 8.7354, 7.5283, 6.0407, 4.3313, 2.2239\}$$

With only two basis vectors, the assumed initial coefficients vector $\mathbf{y}^T = \{1, 1\}$, and using the iterative procedure of Eqs. (12.3) and (12.4) to solve the reduced system of Eq. (12.25), the solution process converges in three iteration cycles to $\mathbf{y}^T = \{1, 0.0486\}$. The resulting first mode-shape displacements calculated by Eq. (12.26) are, in normalized form,

$$\mathbf{r}^T = \{1, 0.9705, 0.9121, 0.8215, 0.6971, 0.5437, 0.3675, 0.1887\}$$

An accurate solution is achieved with only three basis vectors

$$\mathbf{r}^T = \{1, 0.9700, 0.9109, 0.8205, 0.6963, 0.5433, 0.3675, 0.1887\}$$

If higher accuracy is needed, we can take more basis vectors.

Example 12.3 - Reanalysis of A Damaged Frame [2]. To evaluate the quality of the results by the CA1 procedure (only two basis vectors) for frames, the four-story two-bay steel frame shown in Figure 12.2 is solved. The 36 degrees of freedom (DOF) are the vertical displacement, the horizontal displacement, and the rotation at the 12 joints. The modulus of elasticity is 29 000 ksi, the columns are W14 × 109, and the beams are W24 × 84.

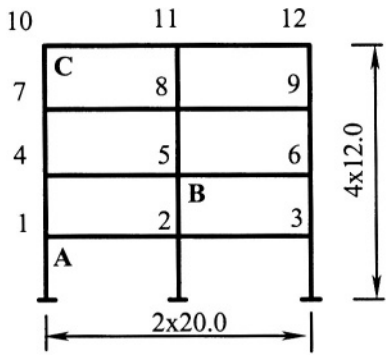

Fig.12.2. Four-story two-bay frame.

Table 12.2. Results for vibration mode 1.

DOF	Case A		Case B		Case C	
	CA1	Exact	CA1	Exact	CA1	Exact
1	0.0033	0.0033	0.0029	0.0029	0.0030	0.0030
2	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001
3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0033	0.0033	0.0029	0.0029	0.0031	0.0031
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0032	0.0033	0.0029	0.0029	0.0031	0.0030
8	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0073	0.0073	0.0069	0.0069	0.0070	0.0069
11	0.0001	0.0001	0.0002	0.0002	0.0001	0.0001
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
13	0.0073	0.0073	0.0069	0.0069	0.0069	0.0069
14	0.0000	0.0000	-0.0001	-0.0001	0.0000	0.0000
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
16	0.0073	0.0073	0.0069	0.0069	0.0069	0.0069
17	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001	-0.0001
18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
19	0.0100	0.0100	0.0101	0.0101	0.0099	0.0099
20	0.0001	0.0001	0.0002	0.0002	0.0001	0.0001
21	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
22	0.0100	0.0100	0.0101	0.0101	0.0099	0.0099
23	0.0000	0.0000	-0.0001	-0.0001	0.0000	0.0000
24	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
25	0.0100	0.0100	0.0101	0.0101	0.0099	0.0099
26	-0.0002	-0.0002	-0.0001	-0.0001	-0.0002	-0.0002
27	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
28	0.0113	0.0113	0.0116	0.0116	0.0117	0.0117
29	0.0001	0.0001	0.0002	0.0002	0.0001	0.0001
30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
31	0.0113	0.0113	0.0116	0.0116	0.0117	0.0117
32	0.0000	0.0000	-0.0001	-0.0001	0.0001	0.0001
33	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
34	0.0113	0.0113	0.0116	0.0116	0.0117	0.0117
35	-0.0002	-0.0002	-0.0001	-0.0001	-0.0002	-0.0002
36	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 12.3. Results for vibration mode 2.

DOF	Case A		Case B		Case C	
	CA1	Exact	CA1	Exact	CA1	Exact
1	0.0079	0.0082	0.0085	0.0085	0.0077	0.0078
2	-0.0001	-0.0002	-0.0001	0.0001	0.0000	0.0000
3	0.0000	-0.0001	0.0000	0.0000	0.0000	0.0000
4	0.0079	0.0082	0.0085	0.0086	0.0077	0.0079
5	-0.0001	0.0001	-0.0001	0.0000	-0.0001	-0.0001
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0079	0.0082	0.0085	0.0085	0.0077	0.0078
8	0.0002	0.0001	0.0002	0.0001	0.0002	0.0001
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0089	0.0100	0.0109	0.0101	-0.0093	-0.0099
11	-0.0003	-0.0004	-0.0003	-0.0004	0.0001	0.0001
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
13	0.0089	0.0099	0.0109	0.0101	0.0093	0.0099
14	-0.0001	0.0001	-0.0001	0.0001	-0.0002	-0.0002
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
16	0.0090	0.0099	0.0109	0.0101	0.0093	0.0099
17	0.0004	0.0004	0.0005	0.0003	0.0004	0.0004
18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
19	-0.0013	-0.0003	0.0014	0.0000	0.0000	0.0010
20	-0.0005	-0.0006	-0.0005	-0.0005	-0.0002	-0.0002
21	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000
22	-0.0013	-0.0004	0.0014	-0.0001	-0.0001	0.0009
23	-0.0001	0.0001	-0.0001	0.0001	-0.0004	-0.0004
24	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
25	-0.0013	-0.0003	0.0014	-0.0001	0.0002	0.0009
26	0.0005	0.0005	0.0006	0.0004	0.0005	0.0005
27	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
28	-0.0124	-0.0111	-0.0095	-0.0106	-0.0123	-0.0115
29	-0.0005	-0.0007	-0.0005	-0.0006	-0.0001	-0.0001
30	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001
31	-0.0124	-0.0111	-0.0095	-0.0106	-0.0122	-0.0114
32	0.0000	0.0001	-0.0001	0.0001	-0.0005	-0.0005
33	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
34	-0.0124	-0.0111	-0.0095	-0.0106	-0.0122	-0.0113
35	0.0006	0.0006	0.0007	0.0005	0.0006	0.0006
36	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Three cases of damage at the connections are considered, as represented by the possible beam-to-column hinge formation at locations A, B, and C shown in Figure 12.2; i.e., it is assumed that a hinge is formed at the location of damage (loss of rigidity). The connection damages are assumed to be such that the connection may continue to transmit shear forces but not bending moments from beam to column.

The results achieved by the CA1 procedure and exact reanalysis for the first two vibration modes for damage cases A, B, and C described above are summarized in Tables 12.2 and 12.3. It is observed that good accuracy is achieved for mode 1 and reasonable accuracy is obtained for mode 2.

12.4 EVALUATION OF MODIFIED EIGENVALUES

In many structural optimization problems it is necessary to consider dynamic frequency constraints. Optimal design of structures with frequency constraints is most useful in evaluating the dynamic characteristics [3], In most low-frequency vibration problems, the response of the structure to dynamic excitation is primarily a function of its fundamental frequency and mode shape. In such cases, the ability to manipulate the selected frequency can significantly improve the performance of the structure.

One of the most common problems in frequency optimization is the switching of vibration modes due modifications in the design of the structure. Mode switching causes convergence difficulties in structural optimization. In addition, structural optimization often results in very closely spaced natural frequencies. At the optimum, some structures exhibit repeated eigenvalues even though the initial design did not have any. Many of the generally available approximation methods are applicable to frequency functions, but several studies are devoted to approximations just for frequency problems [3]. It is found that the eigenvalues are highly nonlinear functions of the design variables.

Consider again the initial eigenproblem of Eq. (12.6)]

$$\mathbf{K}_0\mathbf{r}_0 = \lambda_0\mathbf{M}_0\mathbf{r}_0 \tag{12.27}$$

where $\lambda_0 = \omega_0^2$ is the eigenvalue of the initial structure. Assume a change in the design and a corresponding change in the stiffness and mass matrices such that the modified matrices are [Eq (12.7)]

$$\begin{aligned} \mathbf{K} &= \mathbf{K}_0 + \Delta\mathbf{K} \\ \mathbf{M} &= \mathbf{M}_0 + \Delta\mathbf{M} \end{aligned} \tag{12.28}$$

The modified analysis equations are given by

$$(\mathbf{K}_0 + \Delta\mathbf{K})(\mathbf{r}_0 + \Delta\mathbf{r}) = (\lambda_0 + \Delta\lambda)(\mathbf{M}_0 + \Delta\mathbf{M})(\mathbf{r}_0 + \Delta\mathbf{r}) \tag{12.29}$$

where $\Delta\mathbf{r}$ and $\Delta\lambda$ are the resulting changes in $\mathbf{r}$ and λ, respectively.

Several methods have been proposed to evaluate the modified eigenvalue $\lambda_0 + \Delta\lambda$ [3, 4]. Neglecting the quadratic and the cubic terms in Eq. (12.29) to arrive at a first-order approximation gives

$$(\mathbf{K}_0 - \lambda_0\mathbf{M}_0)\Delta\mathbf{r} + (\Delta\mathbf{K} - \lambda_0\Delta\mathbf{M})\mathbf{r}_0 = \Delta\lambda\mathbf{M}_0\mathbf{r}_0 \tag{12.30}$$

Premultiplying Eq. (12.30) by $\mathbf{r}_0^T$ and using Eq. (12.27) and the symmetry of matrices $\mathbf{K}_0$ and $\mathbf{M}_0$ the following first-order approximation is obtained

$$\Delta\lambda = \frac{\mathbf{r}_0^T(\Delta\mathbf{K} - \lambda_0\Delta\mathbf{M})\mathbf{r}_0}{\mathbf{r}_0^T\mathbf{M}_0\mathbf{r}_0} \tag{12.31}$$

Alternatively, premultiplying Eq. (12.29) by $(\mathbf{r}_0 + \Delta\mathbf{r})^T$ and neglecting high-order terms gives the following first-order approximation

$$\lambda_0 + \Delta\lambda = \frac{\mathbf{r}_0^T(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{r}_0}{\mathbf{r}_0^T(\mathbf{M}_0 + \Delta\mathbf{M})\mathbf{r}_0} \tag{12.32}$$

Equation (12.32) is the Rayleigh-quotient approximation to the perturbed eigenvalue, based on the original displacements $\mathbf{r}_0$. A better approximation can be obtained if it were possible to replace $\mathbf{r}_0$ by some approximate $\mathbf{r}$. In particular, the CA procedure described in Section 12.3 is most suitable for this purpose. With the approximate displacements $\mathbf{r}$ of Eq. (12.26), the modified eigenvalue is calculated by the Rayleigh-quotient

$$\lambda_0 + \Delta\lambda = \frac{\mathbf{r}^T(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{r}}{\mathbf{r}^T(\mathbf{M}_0 + \Delta\mathbf{M})\mathbf{r}} \tag{12.33}$$

Several eigenvalue reanalysis methods for modified structures are compared elsewhere [5], in terms of their computational accuracy and efficiency. It is shown that the CA method is the most suitable for large changes in the structure.

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