Mathematics Education in the Digital Era

Dragana Martinovic
Viktor Freiman
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Visual Mathematics and Cyberlearning

Springer

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MATHEMATICS EDUCATION IN THE DIGITAL ERA

Volume 1

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Introduction

Mathematics Education in the Digital Era (MEDEra) Series

The *Mathematics Education in the Digital Era* (MEDEra) is a new Springer book series co-edited by Dragana Martinovic, University of Windsor, Canada, and Viktor Freiman, Université de Moncton, Canada. With two annual volumes, it attempts to explore ways in which digital technologies change conditions for teaching and learning of mathematics. By paying attention also to educational debates, each volume will address one specific issue in mathematics education (e.g., visual mathematics and cyber-learning; inclusive and community based e-learning; teaching in the digital era) in an attempt to explore fundamental assumptions about teaching and learning mathematics in the presence of digital technologies.

This series aims to attract diverse readers including: researchers in mathematics education, mathematicians, cognitive scientists and computer scientists, graduate students in education, policy-makers, educational software developers, administrators and teachers-practitioners.

Among other things, the high quality scientific work published in this series will address questions related to the suitability of pedagogies and digital technologies for new generations of mathematics students who grew up with digital technologies and social networks. The series will also provide readers with deeper insight into how innovative teaching and assessment practices emerge, make their way into the classroom, and shape the learning and attitude towards mathematics of young students accustomed to various technologies.

The series will also look at how to bridge theory and practice to enhance the different learning styles of today's students, and turn their motivation and natural interest in technology into an additional support for meaningful mathematics learning. The series provides the opportunity for the dissemination of findings that address the effects of digital technologies on learning outcomes and their integration into effective teaching practices; the potential of mathematics educational software for the transformation of instruction and curricula; and the power of the e-learning of mathematics, as inclusive and community-based, yet personalized and hands-on.

Visual Mathematics and Cyberlearning – The First Book in the MEDEra Series

The first book in the MEDEra series, entitled *Visual Mathematics and Cyberlearning*, is co-edited by Dragana Martinovic, University of Windsor, Canada, Viktor Freiman, Université de Moncton, Canada, and Zekeriya Karadag, Bayburt University, Turkey. It offers a platform for dissemination of new ideas in visual mathematics and cyberlearning, addresses new developments in the field, and evokes new theoretical perspectives in mathematics education.

Recent studies describe the Net Generation as visual learners who thrive when surrounded with new technologies and whose needs can be met with the technological innovations. These new learners seek novel ways of studying, such as collaborating with peers, multitasking, as well as use of multimedia, the Internet, and other Information and Communication Technologies. How this can be used to present mathematics in new ways, as a contemporary subject that is engaging, exciting and enlightening?

For example, in the distributed environment of cyber space, mathematics learners play games, watch presentations on YouTube, create Java applets of mathematics simulations and exchange thoughts over the Instant Messaging tool. How should mathematics education resonate with these learners and technological novelties that excite them? How can educators make a meaningful use of dynamic, interactive, collaborative, and visual nature of new learning environments while having a deeper understanding of their potential advantages and limitations? Authors of nine chapters share their conceptual frameworks and research data that shed a light on innovative theories and practices in the field of visual mathematics and cyberlearning.

Jones, *Geraniou*, *and Tiropanis* study potential of Web 3.0 semantic tools that enhance mathematics discussion within collaborative, shared workspace by means of graphical argumentation and chat tools. Elementary students were given an opportunity to explore different patterns and combination of patterns by finding and augmenting an algebraic rule while working collaboratively in the *eXpresser* environment accompanied by visual support provided by *LASAD*. The authors reflect on innovative potential of cyberlearning to foster knowledge development and mathematical thinking.

Alagic and Alagic, on their turn, provide in-depth analysis of research mathematicians working together by means of large-scale computer supported collaborative learning tools that enrich networking opportunities and enhance self-regulated learning.

Çakır and Stahl describe socially situated interactional processes involved in collaborative online learning of mathematics. In the common online environment their Virtual Math Teams problem solve using chat, shared drawings and mathematics symbols and thus co-construct a deep mathematical understanding at the group level.

Güçler, *Hegedus*, *Robidoux*, *and Jackiw* investigate mathematical discourse of young learners involved in multi-modal mathematical inquiries. Using a context of dynamic geometry with haptic devices, authors claim that such integration fosters new learning experiences that lead to evolution of young learners' expression from informal to formal mathematical discourse.

Trninic and Abrahamson are interested in the role of embodied artefacts in the emergence of mathematical competence, viewed as independent from the physical world. By performing physically in the service of doing mathematics, students make observable what is otherwise hidden away 'in their heads'. In their chapter, the authors enrich investigations of embodied artefacts in light of increasingly ubiquitous monitor-sensor technologies, namely the Mathematical Imagery Trainer. Students work with proportions by moving their hands in an environment that changes its state in accord with the ratio of the hands' respective heights, then reflect on what they see on the computer screen, and analyze mathematically as particular case of proportionality.

Radford uses an approach where human cognition is conceptualized in non-dualistic, non-representational, and non-computational terms. The basic idea is that cognition is a feature of living material bodies characterized by a capacity for *responsive sensation*. As a result, human cognition can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, transforming, and making sense of the world. In his chapter, the author presents classroom experimental data involving 7–8-year-old students dealing with pattern recognition that lead to suggesting that a sensuous-based materialistic monistic view of cognition needs to attend not only to the plethora of sensorial modalities that teachers and students display while engaging in mathematical activities, but also to the manner in which sensorial modalities come to constitute more and more complex psychic *wholes* of sensorial and artefactual units.

Gadanidis and Namukasa discuss a case study of online mathematics learning for teachers through the lens of four affordances of new media: democratization, multimodality, collaboration and performance, which help to rethink and disrupt existing views of mathematics for teachers and for students.

LeSage used web-based video clips on rational numbers to provide pre-service teachers with accessible and flexible learning opportunities to support their individual learning needs. According to research findings from participants' narratives, careful consideration must be paid not only to the instructional design of video clips but also to support the development of pre-service teachers' pedagogical content knowledge and content knowledge of mathematics.

In the final chapter, *Martinovic*, *Freiman*, *and Karadag* show that diverse examples and deep insights given by the authors of the book chapters extend our understanding of the features and complexity of virtual mathematics tools suitable for visualization and exploration in the light of Activity and Affordance Theories, thus opening new perspectives in researching mathematics education in the digital era that can be investigated further in next volumes of the series.

Viktor Freiman and Dragana Martinovic, MEDEra Editors

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Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web

Keith Jones, Eirini Geraniou, and Thanassis Tiropanis

Abstract With current digital technologies there are a number of networked computer-based tools that provide ways for users, be they learners or teachers, to collaborate in tackling visual representations of mathematics, both algebraic and geometric. For learners, there are various ways of collaborating that can occur while the learners are tackling mathematical problems. In this chapter we use selected outcomes from recent innovative research on this aspect of learning and teaching mathematics with digital technologies to review the patterns of collaboration that can occur in terms of teacher and learner experience. Given that such patterns of collaboration are via current digital technologies, this chapter goes on to offer a view on the likely impact on the cyberlearning of mathematics of progress towards the next generation of Web technologies that seeks to make use of ideas related to the web of data and the semantic web. Such impact is likely to be in terms of enhancing the learning applications of digital technologies, improving ways of administrating the educational programmes that they support, and potentially enabling teachers to maintain involvement in technological development and use over the longer-term.

Keywords Argumentation • Algebra • Collaborative learning • Semantic web • Web 3.0

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Introduction

During the 30 years since the launch in 1981 of the IBM model 5150 personal computer (the PC that became the worldwide standard) there has been the implementation and enormous growth of communication between networked computers via the World Wide Web. The term Web 2.0 was coined by DiNucci (1999) to capture how Web technologies have developed since the beginning of the Web in 1991 such that users are able to interact and collaborate with each other in increasingly diverse ways. Since 2001, ideas about the nature of Web 3.0 (see, Berners-Lee, Hendler, & Lassila, 2001) have centred on features such as increasing personalisation and on the possible advent of what Berners-Lee calls the Semantic Web, a new vision of the Web where computers 'understand' the semantics (or meaning) of data and information on the World Wide Web.

Over the same time period governments across the world have been promoting digital technologies as powerful tools for education (for current information, see, for instance: Law, Pelgrum, & Plomp, 2008; EU Education, Audiovisual and Culture Executive Agency, 2011). As a result of commercial and Governmental initiatives, there are a range of computer-based networked tools that provide ways for users to interact and collaborate. In this chapter we use carefully-selected outcomes from recent innovative research involving digital technologies to review the various patterns of collaboration that can take place. The aim is to review the inter-person interaction and collaboration via digital technologies in terms of the experience of those involved. Given that such collaborations are based around current digital technologies and the current enactment of the World Wide Web, we use our research experience to offer a view on the likely impact on the 'cyberlearning' of mathematics of developments towards Web 3.0, especially developments relating to the notion of the Semantic Web.

Cyberlearning: From Web 1.0 to Web 2.0

The Web, from its beginnings in the early 1990s, has provided new and powerful ways for finding and accessing resources that could be used for learning. Regardless of the context (whether formal or informal), individuals have been able to use the Web to find content and software to support learning. The volume and types of resources that have become available on the Web have made it possible for people not only to find more content than ever before (and usually more efficiently too), but also collaboratively to publish additional content – and even to categorise it using taxonomies and tags. For many, this transition from a 'read-only' Web to a 'read-write' Web signifies the transition from Web 1.0 to Web 2.0 (although Berners-Lee, the inventor of the Web, always envisaged the Web as a means of connecting people; see, Laningham, 2006).

Web 2.0 technologies have enabled user-generated content, and powerful paradigms such as crowdsourcing (the outsourcing of tasks that might traditionally have been performed by employees, or a contractor, to an undefined group of people – a “crowd” – often via an open call). Such possibilities have been transformative by making the Web into a host for a large number of knowledge repositories (an example being Wikipedia) and by paving the way for a transition from a ‘Web of documents’ to a ‘Web of data’. In this way the Web is becoming a repository of data (in addition to documents) and people are able efficiently to aggregate the data that becomes available to create and provide new applications. Another key characteristic of the Web is that it has leveraged network effects, with resultant rapid growth. Such network effects have occurred because the more that content and data becomes available on the Web, the higher the value of the Web to the users and, in turn, the higher the volume of content and data that users are willing to contribute. At the same time, the Web has provided an environment for network effects to take place, examples being the growth of services like Wikipedia and of online social networks such as Facebook.

Key applications that enabled the transition to Web 2.0, according to Anderson (2007), include blogs, wikis, multimedia sharing, tagging and social bookmarking, audio blogging and podcasting, and RSS and syndication. These have been complemented by newer phenomena such as social networking, aggregation, data mashups, and collaboration services (Anderson). All these developments have meant that the use of Web 2.0 services for learning is becoming increasingly widespread. For example, the UK Higher Education sector has adopted technologies for publication repositories and wikis on an increasing scale; a survey in 2009 reported that 40 universities (out of 165) had adopted publication repository software systems and 14 had adopted wikis (see, Tiropanis, Davis, Millard, & Weal, 2009a).

The key value of Web 2.0 for education is in enhancing learning experiences, given its potential for personalisation, customisation and collaboration for knowledge creation (McLoughlin & Lee, 2007). Social software is increasingly an enabler for pedagogical innovation in terms of peer-to-peer learning, extended learning, cross-cultural collaborative work using student-generated content, and learner-centred instruction. The benefits of Web 2.0 technologies have been identified for a number of educational scenarios such as teacher-class communication and students’ participation in the collection and integration of learning material (Rollett et al., 2007). Most such benefits centre on the ease of reporting progress (e.g. through using blogs) and the efficiency of collaborative construction of complex reports (e.g. through using wikis for assignments). Criticism of Web 2.0 use in education often centres on the sometimes low quality of generated content and the way amateurishness can flourish; in addition, critics bemoan the time and knowledge investment that Web 2.0 technologies can require (see, Grosseck, 2009).

Overall, the unique value of Web 2.0 technologies seems to be in:

- Enabling information finding on a large scale, with the number of resources that learners can find on the Web growing every day.
- Supporting collaborative knowledge construction amongst a large number of people; using Web 2.0 services such as wikis and social software it is possible to mobilise communities across the world as Web 2.0 technologies can cope with the size and geographical distribution of these communities in efficient ways.

- Enabling collaboration among individuals for learning purposes; this is more easily and more efficiently accomplished using Web 2.0 software and, furthermore, when it comes to online collaboration, it is possible to achieve better matching among learners when geographical constraints are not a barrier.

These affordances that Web 2.0 technologies offer have also been incorporated into a number of areas of education including the teaching and learning of mathematics. It is to the findings of selected aspects of two pertinent current research projects that we turn next. In the next section we report on groups of lower secondary school pupils (aged 11–14) interacting and collaborating whilst tackling mathematical problems involving the visual and geometric representation of algebraic ideas. Our aim in doing this is to review the patterns of such collaboration in terms of learner and teacher experience.

Patterns of Collaboration: The Case of the MiGen and Metafora Projects

In this section, we illustrate how advances in technological tools can aid student collaboration by showing the patterns of collaboration in the cases of two such tools that were developed as part of the MiGen[1] and the Metafora[2] projects. Further below we elaborate on how these two tools, namely eXpresser and LASAD, impact on student collaboration. We begin by summarising the general possibilities for collaboration in exploratory learning environments.

Collaboration in Exploratory Learning Environments

Research (e.g. Cobb, Boufi, McClain, & Whitenack, 1997; Leonard, 2001; Linchevski & Kutscher, 1998) has revealed the considerable value of collaboration and classroom discourse towards students' cognitive development. When working in small groups, more students are likely to ask questions compared with whole class situations. In addition, students are more likely to reflect on their own work and attempt to make sense of the work of other students. Students who explain their ideas and solutions to their peers have greater success in their learning than those who do not (e.g. Cohen & Lotan, 1995; Lou et al., 1996). Through such

[1]The MiGen project is funded by the ESRC/EPSRC Teaching and Learning Research Programme (Technology Enhanced Learning; Award no: RES-139-25-0381). For more details about the project, see http://www.migen.org

[2]The Metafora project is co-funded by the European Union under the Information and Communication Technologies (ICT) theme of the 7th Framework Programme for R&D (FP7). For more information, visit http://www.metafora-project.org

interactions, students participate actively in learning with their peers and tend to adopt metacognitive skills, all of which is beneficial for learning (Biggs, 1985; Maudsley, 1979; Schoenfeld, 1992).

Recent research (e.g. Geraniou, Mavrikis, Hoyles, & Noss, 2011; Healy & Kynigos, 2010) is showing how the use of exploratory learning environments can support, and moreover enhance, students' knowledge development and interactions through individual as well as collaborative activities. A pattern of collaboration is therefore emerging where exploratory learning environments such as microworlds (a "subset of reality or a constructed reality whose structure matches that of a given cognitive mechanism so as to provide an environment where the latter can operate effectively", Papert, 1980, p. 204) are increasingly being used in the classroom. Microworlds aim to embed "important ideas in a form that students can readily explore", with the best having "an easy-to-understand set of operations that students can use to engage tasks of value to them, and in doing so, they come to understanding powerful underlying principles" (diSessa, 2000, p. 47). As such, microworlds can empower learners to engage with abstract ideas and explore not only the structure of objects, but also the relationships through investigating the underlying representations that enforce these relationships (Hoyles, 1993; Thompson, 1987). This can happen through individual student interactions as well as through discussions with their peers. In the particular case of the MiGen system, we show in the next sub-section some of the forms of discussion between students that can be supported by visual artefacts and dynamic objects that students can interact with and explore.

The MiGen System and the Metafora Platform

The MiGen system provides digital tools that support students' collaboration by allowing them to interact with each other as they tackle algebraic generalisation problems. The Metafora platform, currently under development, is being designed to offer visual means (including pictorial symbols) for students to use to plan their learning together and visualise their sub-tasks, stages of work, and required roles. The Metafora platform is also being designed to provide an argumentation space where students can discuss their findings and emerge with an agreed solution. In what follows we illustrate the different patterns of collaboration of students while they interact in the mathematical microworld of the MiGen system and when their discussions and structured arguments are further supported by the Metafora platform. We start by giving some information regarding the two projects.

The MiGen project aimed to tackle a well-known issue in mathematics education – the difficulty that students in lower secondary school (when aged 11–14) can have in coming to terms with algebraic generalisation. Such students are generally able to verbalise algebraic rules in natural language but can struggle to use the appropriate mathematical language (Warren & Cooper, 2008). In addition, students can often fail to see the rationale, let alone the power, of algebraic generalisa-

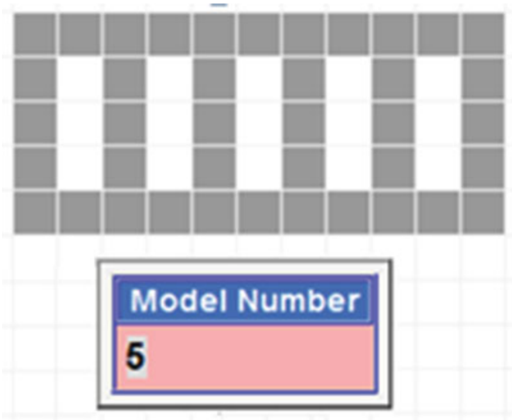

Fig. 1 An example of a TrainTrack in eXpresser

tion. In an effort to support such students in learning algebraic generalisation, a computational environment comprising a number of tools was developed. The core of the MiGen system is a microworld, named *eXpresser*, in which students build figural patterns of square tiles (as in Figs. 1, 2 and 3) and express the rules underlying the chosen patterns. The eXpresser is designed to provide students with a model for generalisation that could be used as a precursor to introducing algebra, one that helps them develop an algebraic 'habit of mind' (Cuoco, Goldenberg, & Mark, 1996). The sequence of student activity in eXpresser involves some free-play to explore the system, some introductory tasks to become familiar with its features, a generalisation task and a collaborative activity. The MiGen system also has an 'intelligent' component, namely *eGeneraliser*, which provides feedback to students throughout their interactions with the system (see, Gutierrez-Santos, Mavrikis, & Magoulas, 2010; Noss et al., 2012). A suite of tools, named the *Teacher Assistance Tools,* aim to help the teacher in monitoring students' progress, assisting with possible interventions and reviewing students' achievements to aid future lesson planning. One of these tools is the *Grouping Tool* which puts forward possible pairings of students for collaboration based on the similarities between students' constructions in the MiGen system (for details of the other tools, see, Gutierrez-Santos, Geraniou, Pearce-Lazard, & Poulovassilis, 2012; Pearce-Lazard, Poulovassilis, & Geraniou, 2010).

The Metafora platform, in comparison, includes a web-based argumentation tool called LASAD[3] that enables discussions to take place within groups of learners in a structured manner (Loll, Pinkwart, Scheuer, & McLaren, 2009; Scheuer, McLaren, Loll, & Pinkwart, 2009). This collaborative, shared workspace, together with graphical argumentation and chat tools, is used by students to share ideas, organise their thoughts, discuss and argue as they learn new concepts (Dragon, McLaren, Mavrikis, & Geraniou, 2011). In addition, other components of the Metafora platform analyse the students' work and provide feedback that supports collaboration

[3] http://cscwlab.in.tu-clausthal.de/lasad/

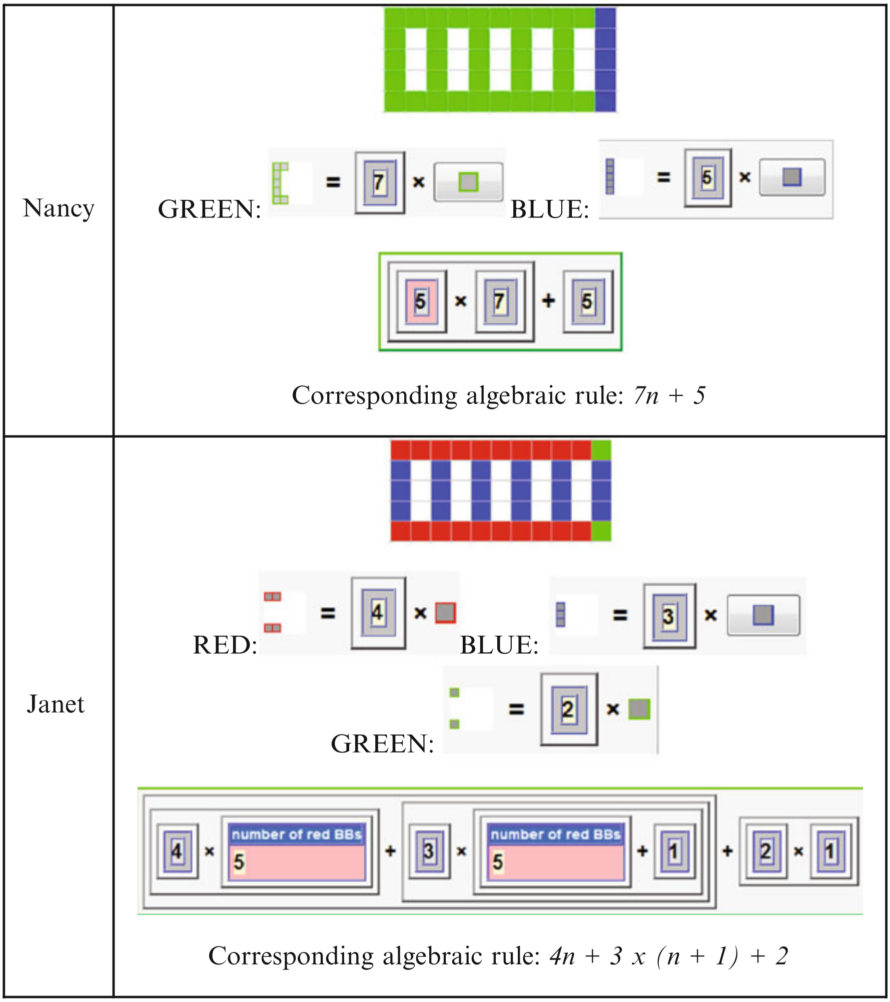

Fig. 2 Nancy and Janet's TrainTrack models

and helps students make progress while they grapple with the challenge. The system also identifies situations where the teacher might encourage peer support or shared knowledge evaluation.

Collaboration Within the MiGen System and the Metafora Platform

To illustrate what patterns of collaboration are possible with the MiGen and the Metafora systems, we analyse in this section some selected data from several learning episodes with these systems. For analyses of the wider pedagogical use of

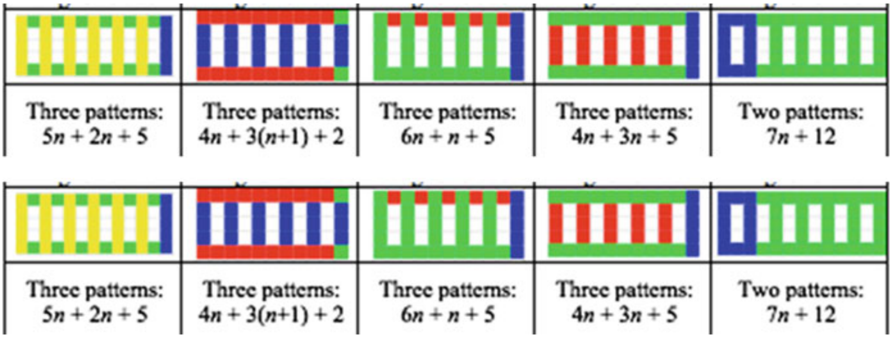

Fig. 3 Students' different TrainTrack models

eXpresser, we refer readers to Geraniou, Mavrikis, Kahn, Hoyles, and Noss (2009), Mavrikis, Noss, Hoyles, and Geraniou (2012) and Noss et al. (2012).

The first scenario is from the work of two 12 year-old students, Janet and Nancy[4] from a UK school. The students were part of a class of 22 Year seven students (aged 11–12) who participated in a series of lessons during which they were introduced to eXpresser through a number of introductory and practice tasks and solved a linear pattern generalisation task, namely TrainTrack. In this task the students were presented with the TrainTrack model (see, Fig. 1) animated in the *Activity Document*, a tool of eXpresser that presents the task-model, the task-questions and the task-goals and in which students can type their answers. The students were asked to construct the TrainTrack model in eXpresser using different patterns and combinations of patterns depending on their perceptions of the TrainTrack's geometrical structure and to derive a general rule for the number of square tiles needed for any Model Number.

At the end of the TrainTrack activity, and to prepare for the collaborative activity, students were asked to use the *Activity Document* tool to record some arguments that would support the correctness of their general rule. Students were then paired by the system's *Grouping Tool* based on the dissimilarity of their models and asked to work on a new collaborative activity that involved discussing the correctness and equivalence of their rules. This new activity was presented to them in a new eXpresser window which was automatically generated by the system and included the two students' models and rules, and also the following two questions in the Activity Document: *(1) Convince each other that your rules are correct, (2) Can you explain why the rules look different but are equivalent? Discuss and write down your explanations.*

The models and rules developed by the two students (Janet and Nancy) are presented in Fig. 2 in the form that these are represented in eXpresser. To prepare for the collaborative activity, Janet and Nancy were asked to type any arguments

[4] All names used for students are pseudonyms.

they had for the correctness of their rules (using the *Activity Document* tool). Their arguments were:

> Nancy: *"My rule is correct because each 'block' has 7 squares. So however many blocks there are, there are 7 squares for each one so you multiply the number of blocks by 7. But, at the end there is another block to finish the pattern off. In this block there are 5 squares so you add the number of squares (the blocks multiplied by 7) to the final block (the 5 squares). This rule should apply to this pattern each time."*

> Janet: *"I think my rule is correct, as it works every time and seems to make sense, as, because the number of red building blocks is unlocked, you can put any number in and it would work and is linked with the numbers of the blue and green building blocks".*

In the next lesson, Janet and Nancy were paired because they constructed the TrainTrack model in different ways (see, Fig. 2).

The two students worked together on the collaborative activity. They looked at each other's rules and compared them by interacting with each other's models in eXpresser (i.e. by changing the model number, animating the models, etc.). As a result they both stated that they understood each other's model:

> Nancy: *"I understand the rule so I don't see a reason why it shouldn't be correct"*

> Janet: *"Yeah, I understand yours too"*

In this way, both students were able to 'read' each other's rule and understand them. Yet it seems that the students viewed the 'correctness' of each other's models as so obvious and so 'understandable' that they failed to produce any justification during their collaboration. Even though they were prepared for this collaboration and had typed in their arguments during the previous lesson, the students failed to produce shared mathematically-valid arguments to justify the correctness of their rules. A possible reason for this is the limitation of the MiGen system in not drawing the students' attention to their written arguments.

After both of these students were convinced of the correctness of their rules, they continued by discussing their rules' possible equivalence. In this case, interacting with eXpresser and exploring each other's models acted as a catalyst to a constructive discussion. They benefitted from eXpresser's immediate feedback on their actions and were able to explore and validate their conjectures. After some debate, Nancy stated that:

> yeah, it's one red building block plus one blue building block so that would actually kind of make the . . .

and Janet interrupted to complete Nancy's chain of thought by saying

> yeah, it would make the same shape.

Nancy then added:

> because one red building block added to one blue building block

and Janet finished the argument:

> and that's the same as one of my green building blocks.

As a result, they reached agreement that their two seemingly different rules were in fact equivalent and they justified their conclusion by appealing to the structure of the models that they had constructed.

It is worth noting that there are many different ways of constructing the same model (see, Fig. 3) and pairing students with interestingly different constructions can lead to fruitful collaboration.

Collaboration in the context of the MiGen system entails students reading, deconstructing and matching their rule with their partner's by exploring, revisiting their actions, building on them and taking new actions using the tools available within the system. The eXpresser microworld provided the students with a visual means to express algebraic generalisation and through the manipulation of its entities, they were able to give meaning to algebraic concepts that are often elusive (such as constants, variables or the n-th term of a sequence). Such an expressive and exploratory tool proved to assist students in their development of complex mathematical ideas and this illustrates ways in which students can adopt an enquiry stance in making every effort to gain important mathematical skills (such as abstracting and generalising), as originally advocated by Papert (1980) and more recently by other researchers (e.g. Shaffer, 2007).

Although the potential of the MiGen system to support students' learning of algebraic generalisation, and of algebraic ways of thinking, was evident in the MiGen research (see, for example, Mavrikis et al., 2012; Noss et al., 2012), there was evidence of some inflexibility in terms of what collaborative actions the students could take (see, Geraniou et al., 2011). For example, the system is limited to groups of up to three students, they must work together on one machine, they have to store their shared answers on a local server and not on the web, all their collaboration is synchronous but offline, and while they can type their agreed answers they cannot easily post them for other students to see immediately. The latter action of sharing their final statements with fellow classmates could only be orchestrated by their teacher.

Taking into account the advances in digital media, and on the basis of relevant research on the affordances of new technology to support online collaboration (e.g. Stahl, 2006; Stahl, Zhou, Cakir, & Sarmiento-Klapper, 2011), the Metafora platform is innovative in integrating collaborative learning with microworlds that are extended for collaborative online use. A key technical and pedagogic innovation of the Metafora platform is that it gives students the opportunity to come together (not necessarily in the same time and space) using LASAD, an *argumentation tool*, to discuss the given challenge to solve, argue about their findings, and emerge with an agreed solution. In particular, the argumentation tool helps the students to organise their thoughts, discuss opinions, and display the relations between their arguments in graphical form. In this way, the students' discussions are structured and their learning scaffolded.

Taking further the example of students working on the TrainTrack task (the students who produced different models with equivalent rules and therefore were grouped together) we now demonstrate how LASAD, the argumentation tool, was used by the students to work on the collaborative activity of convincing each other of the correctness and possible equivalence of their algebraic rules. To do

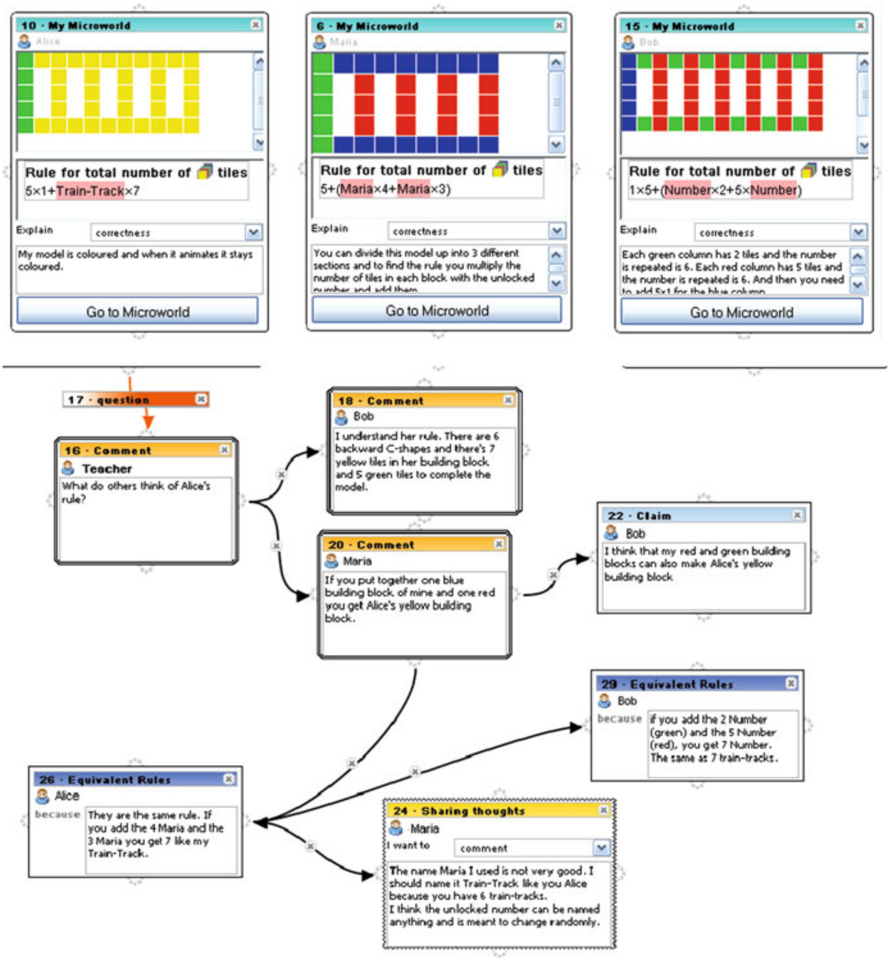

Fig. 4 Shared models, rules and arguments for correctness in LASAD

so, we analyse the collaborative process of three 12 year old students, Alice, Maria and Bob, while they interacted in the eXpresser mathematical microworld and simultaneously engaged in discussions and structured arguments using the argumentation tool LASAD.

After the students constructed their on-screen models and decided on the algebraic rules, they were directed not only to share their models and rules but also to prepare for the collaborative activity by stating arguments for their rule's correctness. All this is captured in Fig. 4.

From Fig. 4, we can see that Alice relied on the visual feedback from the microworld to validate her rule's correctness. Since her model remained coloured for different values of the Model Number (or, in other words, the unlocked number that she named as "Train-Track"), she was convinced that her rule was correct; providing further justification seemed unnecessary to her. In contrast, both the other two students, Maria and Bob, derived arguments for their rules' correctness based on the structure of their constructed models. As such, they both deconstructed their rules and models by matching each term in the rule to the corresponding component of the model. Throughout their preparation for the collaborative activity, all three students interacted with their models in eXpresser, explored their patterns' properties, and conjectured why their rules were correct. The combination of eXpresser and the argumentation tool LASAD provided them with the opportunity not only to reflect on their interactions within an exploratory learning environment, but also the opportunity to develop strategies to justify the correctness and equivalence of their rules. Additionally, the argumentation tool allowed them to share their way of thinking with each other and prompted their reflective thinking in terms of a valid argumentation and a mathematically-correct justification.

At this point the teacher intervened by prompting Maria and Bob to comment on Alice's rule. This triggered the students' reflective thinking and their discussion commenced. Maria and Bob tried to make sense of Alice's rule and compared it to theirs. They continued to think structurally as they focused on matching their building blocks to that of Alice's and recognised that their two building blocks formed Alice's yellow building block.

The students' discussion continued naturally until the issue arose of the equivalence of their rules. Alice immediately claimed that their rules were in fact the same rule. She supported her claim by explaining that if you add the terms "4 Marias and 3 Marias" in Maria's rule, you'll get the seven train-tracks she has. She recognised the unlocked number in Maria's rule and ignored the different name by focusing on the mathematical operations that would help her justify the rules' equivalence. Bob followed a similar approach. Maria, on the other hand, noticed that the main difference between their rules was in the name they chose for the unlocked number or, in other words, the variable. This triggered a conversation on the use of a meaningful name, like 'Train-Track', for the variable instead of 'Maria'. Their discussion revealed an appreciation of the notion of algebraic variable and what it represents in their model.

Throughout their discussions, the students used the language of the argumentation tool LASAD. As presented in Fig. 4, Bob and Maria, for example, added supportive comments about the correctness of Alice's rule and linked it to the teacher's prompt question; Bob made a claim reflecting on Maria's comment; Alice and Bob gave reasons for their rules being equivalent; and Maria shared her thoughts on the name she gave to her unlocked number and her view on its meaning. Such features allow students to go through stages in their argumentation process and form mathematically-valid arguments gradually.

In the above example of a use of the argumentation tool LASAD and the Metafora platform, the pedagogical benefits of allowing the interchange between the individual eXpresser workspace and the discussion space, i.e. LASAD, are evident. The students' collaboration encouraged them to recognise their different approaches to solving the same task as well as justifying the correctness and equivalence of their rules. Their reflective comments convey a mutual willingness to support their knowledge development and reach a consensus in terms of their collaborative task as well as recognise mathematically-valid arguments.

Student Collaboration and the Teacher

As mentioned earlier, research has documented the benefits of collaboration towards students' knowledge development (e.g. Cobb et al., 1997; Leonard, 2001). In presenting two different patterns of students' collaboration, we show students not only benefitting from being supported by the tool eXpresser (that provided them with visual feedback on their actions, and allowed them to share their solutions and their thoughts) but also another tool, LASAD, used in parallel to eXpresser, that provided a visual way of structuring their collaboration that scaffolded their knowledge development and mathematical thinking. As students' collaboration progresses from groupwork on paper, to groupwork with the assistance of digital tools (e.g. eXpresser and MiGen), to groupwork within a collaborative platform (e.g. LASAD and Metafora), attention needs to be paid to the integration of such tools in the mathematics classroom.

Through the cases of the MiGen and the Metafora projects (and their systems), and in parallel to the development of tools that support students' different collaboration patterns, we know that there is a need to encompass tools to support the teacher and move one step closer to successful integration of digital tools into the classroom. Both the MiGen and the Metafora projects foresaw the challenges for teachers in the digital era and aimed to provide assistance to teachers through the production of appropriate tools; ones that are able to provide feedback and draw the teacher's attention to prominent information regarding students' individual and collaborative work. Even though teacher support is not the focus of this section, we know that enabling the teacher to intervene when necessary is important in fostering students' collaboration. For example, when the teacher views a group's unproductive discussion, they could intervene in the argumentation space and remind students of their task (such a case was demonstrated in Fig. 4), give them hints to promote reflection on previous work, or extend their discussions to what they have learnt about collaborative learning. Our argument is that environments such as those created in the MiGen and the Metafora projects can offer the groundwork for the integration of digital technologies in the classroom by creating a collaborative workspace that can offer support to both students and teachers.

Cyberlearning: From Web 2.0 to Web 3.0

Having examined patterns of collaboration using current technologies, we now turn toward the next step in Web evolution that is the transition from the *Web* *of documents* to the *Web of data* (Berners-Lee et al., 2001). No doubt Web 2.0 technologies are enabling unprecedented growth in the volume of online content by enabling users not solely to be consumers of web content but also, if they chose, to be content producers. At the same time, the participation of people has been increasing significantly in intensity. The growth of online social networks in recent years has been phenomenal, giving rise to increased interaction among people on the Web. This includes the mechanisms of crowdsourcing that have evolved from content contribution on YouTube and collaborative knowledge construction on Wikipedia to the contribution of data and of applications that combine published data. This new era of Web 3.0 is not only that of the Web of data but also of the Web of online social networks.

This new stage of Web evolution provides significant opportunities for learning by leveraging the increasing amount of data that is getting published on the Web and by exploiting the connections that people form as part of their participation in online social networks. However, from a technological viewpoint, coping with the increased volume of content, data and people presents certain challenges when it comes of developing applications for learning. Certain questions arise, such as the following:

- How can one efficiently discover the most relevant content and data for learning on a Web that keeps increasing in size?
- How can one find the right people with whom one can collaborate and learn?
- How can one efficiently combine information that potentially comes from different data sources in order to provide new insights and new knowledge in a formal or informal learning context?
- What are the processes that transform online data to information and to knowledge and how can these processes be supported?
- What are the affordances of existing and emerging online social networks for learning on the Web?

The research community, and corresponding parts of the industry, have invested, and are continuing to invest, in technologies and operating standards in order to respond to these questions. In the emergent Web of data (or Web 3.0), a number of technologies for linked (open) data are available that enable both the publication of data on the Web in inter-operable formats (an example being RDF[5]) and the query and combination of those data (see, Bizer, 2009). The linked data movement has demonstrated on many occasions how these technologies are efficient enough to support 'crowdsourcing', not only of content production (as in classic Web 2.0 services) but also of linked data in a number of areas including e-government (see, Shadbolt, O'Hara, Salvadores, & Alani, 2011).

Regarding existing content, annotation (such as rating a Web resource) has always been central to efficient content discovery and aggregation. With the Web, the annotation process involves providing data about online content that will

[5]Resource Description Framework http://www.w3.org/RDF/

describe what the content is about; this data might be contained within this content (in other words, hidden inside the source of a Web page) or in a separate document. Annotation can also be used to describe people and their learning background and objectives. In Web 2.0 applications, the process of annotation is often supported by simple tags that are searchable by users. Support for more advanced searching often requires more elaborate annotation where the tags are not just keywords but are concepts and relationships (with such concepts and relationships being rigorously described in an ontology).

In Web 3.0, semantic technologies are central in the discovery of data, content or people, and in the combination with Web resources, the provision of innovative applications. Semantic technologies make use of ontologies and annotations in order to support searching and matching as well as drawing conclusions based on available metadata. The vision of a Web in which content (documents or data) is described using ontologies can enable the realisation of a plethora of advanced applications, with such developments being part of the Semantic Web vision leading to Web 3.0 (Berners-Lee et al., 2001; Hendler, 2009).

The significance of semantic technologies for learning is being researched widely. A recent survey of the value of semantic technologies for Higher Education in the UK found that there is increasing adoption of semantic technologies in this sector of education (see, Tiropanis et al., 2009a). The most significant value of semantic technologies was identified as their support for well-formed metadata, something which can enable efficient resource annotation and discovery. In addition, semantic technologies were found valuable in providing inter-operability and support for data integration. Finally, the potential for improved data analysis and reasoning was another significant benefit. A classification of the tools and services on which the value of semantic technology was surveyed produced the following categories:

- Collaborative authoring and annotation tools (including semantic wikis and argumentation tools). Semantic technologies can help with the forming of collaboration groups based on the similarity among individuals and with efficiently discovering relevant resources or arguments. Such technologies also support argumentation and visualisation of arguments to enable critical thinking in that semantic technologies can help the learner to navigate to arguments online or to seek patterns relevant patterns of argumentation. In addition, semantic technologies can provide for precise representation of shared knowledge and recommendation of related content and people for collaborative activities related to learning.
- Searching and matching tools for discovering relevant content and individuals related to learning activities. Semantic technologies can enable searches across repositories and enable more efficient question and answer systems. At the same time, they can provide for better matching among people for learning activities (i.e. group formation). Learners can be grouped according to their background, the skills that they need to develop and their learning objectives.

- Repositories and VLEs that provide access to their data using semantic technologies. These enable interoperability and provides for aggregation of the information that they hold. This supports learners (and learning solution developers) in discovering and processing learning content.
- Infrastructural tools and services that can assist in publishing, accessing and integrating data sources inside or across organisations in interoperable semantic formats. The growing number of such utilities illustrates the value of semantic technologies and the trend to develop software to support their development.

Given these technological responses to the issues of the Web of data and the Semantic Web, the former can be seen as a first step to reaching the latter. In terms of technologies, RDF is the common approach for both (i) publishing data in structured interoperable formats and (ii) publishing metadata about online content and describing ontologies. Existing content available in Web repositories that is collaboratively produced and/or annotated (tagged) could become available in semantic formats such as RDF and there is evidence that this process is well underway and is already bearing fruit (Tiropanis, Davis, Millard, & Weal, 2009b).

Opportunities presented by semantic technologies and Web 3.0 for education include not only the development of semantic applications for use in education but also the administration of educational programmes and educational institutions in general. As an example, the Higher Education sector is beginning to report benefits of using semantic technologies in terms of improving the visibility of degree programmes, helping with curriculum design, and supporting student recruitment and retention (Tiropanis et al., 2009a, 2009b). For more on semantic technologies and Web 3.0 for education, see, Bittencourt, Isotani, Costa, and Mizoguchi (2008) and Devedzic (2006).

The Likely Impact of Web 3.0 on Systems Such as MiGen and Metafora

Systems that make use of collaborative learning such as the MiGen system and Metafora can benefit from Web 3.0 technologies by enhancing their learning applications and improving the administration of the educational programmes that they support. Indicative enhancements might include:

- Enhancement of self-learning by proposing problems with patterns that the learner has not successfully solved. By the use of ontologies and annotation of the database of problems that can be presented to students, it is possible to propose problems that are better targeted to the students' past performance and learning objectives. If a student is having difficulties with problems of a specific type it will be possible to propose similar problems on a more appropriate level of difficulty.

- Enhancement of critical thinking by progressively offering 'hints' to problem solutions based on argumentation recorded during previous collaborative problem solving sessions. By annotating the database of problems and recording previous answers it will be possible to provide students with alternative ways to solve these problems. Further, it will be possible to recommend additional problems that required different argumentation patterns.
- Improvement of the collaborative learning experience by forming groups of learners based on their background, performance and goals. Providing semantic descriptions of learners, their achievements, their background and their learning objectives can provide for applications that will propose appropriate study groups with criteria that will be deemed to be pedagogically appropriate by the teacher. It will even be possible to form groups of learners across schools where, if desirable, the chances of achieving an increased mix of students could be higher.
- Improvement of exploratory learning by pointing learners to relevant content on the Web with historical information behind problems and additional visualisation of solutions. The potential of Web 3.0 technologies for content annotation, data interoperability and reasoning can enable the discovery of relevant content on a global scale. It could make it possible to recommend resources that will enable students to prepare to solve specific problems with which they appear to be having difficulties.

One of the most important strengths of Web 3.0 technologies is that they enable access to well-formed data in open repositories and that they can foster crowdsourcing for the development of additional components. In systems such as MiGen and Metafora, Web 3.0 technologies could enable the deployment of an ecosystem of problem resources and applications to foster further innovation for learning.

Conclusion

In the 30 years since the launch of the IBM model 510 PC, and particularly in the 20 years since the launch of the World Wide Web, networked computer-based tools have increasingly provided ways for users, be they learners or teachers, to collaborate in tackling visual representations of mathematics, both algebraic and geometric. As illustrated in this chapter, for learners these ways of collaborating can occur while they are tackling mathematical problems. Other research, for example, Lavicza, Hohenwarter, Jones, Lu, and Dawes (2010), is illustrating how, for teachers such collaborations can involve them supporting the development and take-up of technologies, not only through designing and sharing teaching ideas and resources, but also through designing and providing professional development workshops for other teachers of mathematics. The growth and increasingly-widespread availability of Web 2.0 technologies (such as blogs, wikis, social bookmarking, etc.) is meaning that users are enabled to discover information on a large scale, have ways of supporting collaborative knowledge construction amongst a large number of people, and benefit from easier and more efficient matching of individuals for learning purposes.

In reporting aspects of the MiGen project and the Metafora project, we focused on two different patterns of collaboration. In doing so, we showed students not only benefitting from using eXpresser, a tool that provided them with visual feedback on their actions and allowed them to share their solutions and their thoughts, but also LASAD, a tool that is used in parallel to eXpresser and that provides a visual means to help structure their collaboration and scaffold their knowledge development and mathematical thinking. The challenges in taking forward the findings of the MiGen project into the Metafora project are how to enable groupwork within a collaborative web-enabled digital platform, how the integration of such tools into the mathematics classroom can occur, and what additional digital tools might be provided to support the teacher to intervene when necessary to foster students' collaboration and learning of mathematics.

The likely impact of Web 3.0 on systems such as MiGen/Metafora is in terms of enhancing their learning applications and ways of administrating the educational programmes that they support (for example, from progressively offering 'hints' to problem solutions based on argumentation recorded during previous collaborative sessions through to suggesting groups of learners based on their background, previous performance and current learning goals).

Recently, Maddux and Johnson (2011a) have suggested that for emerging technologies to succeed in education, they must first become popular in society at large. In other words, successful technological innovations "succeed in education only if they have attained a significant degree of cultural momentum" (Maddux & Johnson, 2011b, p. 87). This notion has some similarities with the use that Jones (2011) makes of the term *canalization* to capture the phenomenon that, despite the widely-acknowledged potential of digital technologies, the integration into education has progressed more slowly than has been predicted. This could be because, according to Jones (ibid, p. 44), "the 'normal' pathway of educational change over time is one that innovative technology … *on its own* may not perturb enough to cause a major change". Given the findings of Tiropanis et al. (2009a, 2009b), it could be that, for the moment, it is the Higher Education sector where Web 3.0 will first begin to make an impact on educational practice. Tracking the impact of Web 3.0 on the teaching and learning of mathematics at the school level is going to be an exciting endeavour.

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Collaborative Mathematics Learning in Online Environments

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Abstract In formal education, learning mathematics is typically done by receiving direct instruction within the confines of a classroom. From first grade through graduate school, students are expected to learn mathematics primarily by being taught by instructors with previous knowledge of the subject. Research mathematicians, on the other hand, must rely on other methods; the mathematics they are trying to understand may not, as yet, be known to anyone else. Hence, they learn primarily through experimentation, self-directed study, and collaboration with peers. In recent years, these methods have been expanded to use modern tools and ideas. Research mathematicians initiated several successful large-scale online collaboration projects, such as the Polymath project and the MathOverflow website. In this chapter, we discuss these two projects, along with various other examples of online collaborative learning of mathematics. Our primary motivation is captured in the following question: *why aren't we all learning math this way?* While a complete answer is beyond the scope of this work, we hope to at least stimulate a debate among a wide audience. The major part of our discussion is thus informal; we defer the contextualization of these examples within modern education research until the end of the chapter.

Keywords Collaborative online learning of mathematics • Mathematics – Stack Exchange • MathOverflow website • Networked learning • Online collaboration at the research level • Polymath projects

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Introduction

> We sometimes talk as if "original research" were a peculiar prerogative of scientists or at least of advanced students. But all thinking is research, and all research is native, original, with him who carries it on, even if everybody else in the world already is sure of what he is still looking for. It also follows the all thinking involves a risk. Certainty cannot be guaranteed in advance. (Dewey, 1916/2010; p. 102)

In the modern world, almost all formal education takes place in the confines of a classroom environment. The classroom consists of learners led by a teacher who is charged with implementing a predetermined plan for what is to be learned and how it is to be learned. Since the teacher is vested with a significant amount of authority, major deviations from this plan are rare. The primary role of the teacher is to deliver knowledge, and the primary role of the learners is to absorb this knowledge. This is Freire's "banking" concept of education – "an act of depositing, in which the students are the depositories and the teacher is the depositor" (Freire, 1970/2006, p. 72). This "traditional/classroom model" is still pervasive and extends well beyond the boundaries of schools and universities. Governments use it to certify workers, and corporations use it to instruct employees on everything from work etiquette to trade secrets. Even home schooling typically replaces traditional in-school practices by emulating the teaching model found in schools, with the parent taking the place of the teacher (Lois, 2006). One consequence of the pervasiveness of the traditional model is that, broadly speaking, modern technology is often applied to education and learning as a means of "enhancing" the standard school classroom (e.g., with instruction tools such as interactive whiteboards and various courseware). In spite of its widespread use and its many successes, there are plenty of reasons to suspect that the classroom model is not the optimal learning environment for every student (Cheung & Slavin, 2011; Schank, 2011; Serow & Callingham, 2011; Widaman & Kagan, 1987). Instead of simply supplementing this model, modern technology can also enable us to explore entirely different models of learning. In this chapter, our focus will be on learning that involves a high degree of collaboration and cooperation performed in an online environment.[1] Our particular focus will be on collaborative online learning of mathematics. The major portion of our chapter will be spent on several detailed examples. Our aim is to discuss these examples in an informal and accessible manner and stimulate a debate around the question: *why aren't we all learning math this way?* In section "Analysis and Contextualization", we contextualize these examples within the framework of recent education research.

"Collaborative" and "Cooperative" Online Learning. Perhaps the most commonly encountered example of collaborative online learning is a Web forum run by a community of users with a shared interest (e.g., gardening, sailing, car maintenance, gaming, coding, etc.). The primary function of such a forum is to facilitate the

[1]For now, we will consider both "collaborative" and "cooperative" online learning, without worrying too much about the differences between the two; we will come back to this issue and discuss these differences briefly at the end of the chapter.

sharing of knowledge and skills related to this shared interest. These forums are quite numerous, and at least one exists for every imaginable topic or hobby. The communities that maintain them vary in size, level of activity, geographic reach, age range, etc. The quality of the learning that takes place in these communities is hard to quantify; however, discussions containing carefully written instructions and thorough explanations from such forums are often among the top Google results for hobby-related queries.[2] While the specific software used in a Web forum may vary, the basic format remains the same: when accessing the site, one sees a list of user-initiated topics, ordered according to the last time they were discussed; clicking on a topic opens up the relevant thread of discussion. This format is not entirely neutral, and has a tendency to promote certain kinds of discussions. For instance, a single thread can meander through many unrelated topics over months or even years. More recent tools (such as the Stack Exchange software) promote different kinds of interactions. In the Stack Exchange format, a discussion is initiated by posing a question, and is complete when the questioner accepts an answer given by another member of the community. Both questions and answers are collaboratively edited, and are rated according to the number of up/down votes they receive from members. The Stack Exchange format is employed in two of the examples discussed in detail later in the chapter, namely the "MathOverflow" (2011) and "math.stackexchange" (2011) websites.

The Classroom Model vs. a Collaborative Online Environment. Despite its ubiquity, the classroom model has not escaped criticism. Let us mention three of these criticisms in no particular order, and then discuss how some of the aforementioned collaborative online environments might do better. First, the classroom model is autocratic rather than democratic;[3] in particular, ideas can only be questioned or pursued within the narrow bounds of the syllabus and course goals. Second, the success of the model depends crucially on the qualities and abilities of a single individual: the teacher. Third, learner motivation in a classroom can be a significant problem (Hidi & Harackiewicz, 2000), as any high school teacher can attest.

The first criticism above does not apply to collaborative online learning communities almost by definition. The focus of the community is typically dictated by the desires of the individuals. The topics are chosen by a sort-of popular vote, where members vote for a topic simply by discussing it. The collaborative model also seems less susceptible to the second criticism. The learning process of an entire classroom can fail simply by the failure of one specific individual. Collaborative online settings are more resistant to failure, and can withstand the complete removal of many individuals before the learning process of the rest of the community suffers.[4] The third criticism is perhaps the most difficult one to overcome for any

[2]It would be an interesting study indeed, to examine effects of the collaboratively-edited (via search-engines) results from such communities versus ones provided by "authoritative sources."

[3]We are compelled to note that "democratic" in this context certainly does not mean that one should decide the veracity of mathematical propositions by vote.

[4]Of course, there are technical exceptions, such as the loss of a person who is running the server/software; in such a case, a strong community would presumably simply reform elsewhere on the Web.

model of learning or education. Nonetheless, our expectation is that motivation should be significantly higher among collaborative learners, especially when the choice of subject matter and learning methods are left entirely up to them (e.g., Engel, 2011).

In his introduction to the Polymath project, open-science researcher Michael Nielsen points out that open, large-scale collaboration has already been taking place for years in the open source coding community ("Polymath," 2011). An essential difference is that the primary goal of open source projects is to produce a particular outcome (i.e., a piece of software that performs a specific function) whereas the primary goal of a collaborative online learning project might be more accurately described as producing understanding or catalyzing learning. Nonetheless, at least some of the same principles of large-scale technology-assisted open collaboration that work for software engineering, should also work for learning mathematics.

Even if one accepts that collaborative online learning methods are superior, supplanting the classroom model as one of the most basic and established building blocks of all modern education is still largely a theoretical exercise. However, the availability and flexibility of the Web has made it possible for learners to experiment with other models. Given the preponderance of the activity-oriented collaborative Web forum, it is reasonable to speculate that the collaborative online model has already emerged as a victor in this setting; at some point in the future, it may even be adopted by educational institutions themselves. One should also keep in mind that a significant portion of the world population is simply not reached by these institutions; in these cases, collaborative online learning may be an important alternative (recognizing the fact that a large part of the world population does not yet have access to Internet). After all, it is reasonable to suspect that Internet access and basic computer technology has a wider geographic reach than physical schools with trained teachers (Warschauer & Matuchniak, 2010). Indeed, most of our examples of collaborative online learning of mathematics are open, in the sense that anyone with an Internet connection can participate. Finally, we remark that even where schools are available, some segments of the population choose to educate themselves and their children at home or in small communities organized around friends and family. It should not be taken as given that simply translating the school model to these situations is the best course of action.

The next few sections of the chapter will concentrate on specific examples of collaborative online learning of mathematics, ordered according to the reverse of the standard progression of schooling. We will begin with examples of such models as they are used at the highest level of academic research, continue to undergraduate mathematics, and end with elementary school student-teachers. While the complexity of subject matter obviously increases as one progresses through the standard course of schooling in mathematics, our concern here is with the teaching/learning methodology rather than the specifics of the subject matter. Given that these are our priorities, there are a number of reasons to begin with the academic research level.

Collaborative Online Learning of Research-Level Mathematics

Why Talk About the Research Level?

In this section, we will discuss collaborative online learning of mathematics at the research level. For our purposes, "the research level" refers to those areas of mathematics which are at or near the boundary of current human knowledge. This is a broad landscape covering everything from minor incremental advances in well-established areas to celebrated resolutions of fundamental conjectures. Given the colloquial meaning of "learning," it may seem strange that we are devoting a significant amount of attention to research-level mathematics. In fact, as we now argue, the decision to do so is quite natural.

We first point out that, in this context, "doing research" is simply a form of learning in which the learned matter may not have been known to anyone else. In fact, the lack of an outside authority which already knows the answers makes self-directed or collaborative learning at the research level in some sense more genuine. Second, the important complicating factor of learners' motivation is not a significant issue. It is reasonable to assume that most research-level mathematicians are self-motivated and driven by an enjoyment of the learning itself; this is in sharp contrast to, for example, the many high school students who view learning mathematics as an uncomfortable means to some unrelated but desired end (e.g., employment or parental approval) (Hannula, 2002; Schoenfeld, 1989). Last, as we will discuss in later sections, the use of technology for collaborative online learning appears to be fairly widespread and sophisticated at the research level. Collaboration is ubiquitous in academic research.[5] In this age, this typically means technology-assisted collaboration, even though the sophistication level of the technology might be quite low (e.g., e-mail). Academic researchers in general and research-level mathematicians in particular also seem genuinely interested in optimizing the productiveness of research through collaborative online learning. This is clearly demonstrated by the success of the Polymath projects and the MathOverflow website. Both of these are examples of open, large-scale self-directed online collaborations; we will discuss them in detail below.

It is an interesting problem to consider the extent to which these tools and methods could translate to collaborative online learning of mathematics at undergraduate and K-12 levels. Some examples (such as math.stackexchange, discussed in section "Collaborative Online Learning: Undergraduate and School-Age Mathematics" below) already exist. It is not merely fanciful to imagine that online tools of the

[5] One should be careful here not to confuse single authorship of papers (which is quite common in mathematics) with working in complete isolation (which is exceptionally rare.)

future will enable young students from across the globe to collaboratively explore mathematics with the curiosity and drive of research mathematicians.[6]

Overview of Online Collaboration at the Research Level

While the overall historical trend in research-level mathematics has been towards more collaboration rather than less, the growth has been slow indeed. The first long-running American mathematics journal, the *Annals of Mathematics*, began publishing in 1884, over two centuries after the first scientific journals appeared in Europe. And yet, the first 89 articles of the *Annals of Mathematics* were all single-author papers. In ten recent issues of the *Annals*, which is to this day one of the most prestigious Mathematics journals in the world, single-author papers comprised 36% of all articles, and the average number of authors per article was only 1.5 (Sarvate, Wetzel, & Patterson, 2009). Still, some of the greatest advances in modern mathematics have been the result of collaborations. Terence Tao, a famous modern-day mathematician and Fields Medalist, is best known for a joint result with Ben Green showing that there are arbitrarily long arithmetic progressions of prime numbers. In a discussion on the blog of another Fields Medalist, Tim Gowers, Tao wrote:

> I can't speak for others, but as for my own research, at least half of my papers are joint with one or more authors, and amongst those papers that I consider among my best work, they are virtually all joint.

This same discussion, begun by Gowers with the provocative question "Is massively collaborative mathematics possible?", eventually spawned the first Polymath project, a fascinating example of online collaboration, and one that we will return to in a moment ("Gowers's Weblog," 2011).

The frequency of joint-paper authorship is not the sole indicator of the level of collaboration taking place in mathematics research. Scientists often exchange information and learn from each other informally, and these exchanges surely fit the definition of "collaboration" and "collaborative learning" even if the result is not a new journal paper. In fact, some of these low-level collaborations have impacted science in a more significant way than many published journal papers. Around 1912, while struggling with what later came to be the General Theory of Relativity, Einstein realized that in reality (as opposed to abstract Euclidean geometry) the angles of a triangle only add up to approximately 180°. He knew that this called for a new kind of geometry, but such ideas were outside his area of expertise. He contacted a mathematician friend, Marcel Grossman, who made the crucial observation that Einstein's equations would be best described in terms of Riemannian geometry ("The Future of Science," 2011).

[6]Some projects of this kind already exist, e.g., the Global Nomads Group and iEarn; neither of these focus specifically on mathematics.

As we observe the way in which research-level mathematics is done today, it becomes clear that online interactions are having a dramatic effect on both types of collaborations mentioned above: the joint-author papers such as those of Tao and Green, and informal collaborations such as the exchange between Einstein and Grossman. The Polymath project is a natural example of the first kind of collaboration in an open, online environment; it is the subject of section "The Polymath Projects" below. The MathOverflow website, on the other hand, is a natural example of the second kind of collaboration within an online format designed to maximize the potential of such exchanges; it is the subject of section "The MathOverflow Website" below. Before going on to these discussions, we will briefly mention some other instructive examples of open, online collaborative mathematics at the research level.

An essential aspect of doing science is peer review. The traditional format for peer review, whereby a small number of experts (typically two or three) secretly judge a paper for accuracy and importance is the universal standard even today. In some cases, the potential importance of a result has recently driven scientists to apply modern tools and methods to peer review. An interesting case occurred in August of 2010, when Vijay Deolalikar, an engineer at Hewlett-Packard, made an online posting with a proposed proof that P is not equal to NP. Roughly speaking, P is the set of computational problems whose solution can be found quickly, while NP is the set of computational problems for which a proposed solution can be checked quickly. Whether P is equal to NP is one of the most fundamental questions in mathematics and computer science; the solution to this problem would have wide-ranging practical implications. It is one of the problems for which the Clay Mathematics Institute offers a million-dollar prize. Given the stature of this problem, it is not surprising that scientists immediately began discussing Deolalikar's proposed proof on blogs and wikis. Within 1 week of the original posting, an organic, massively collaborative online peer review process was completed. The conclusion was that Deolalikar's proof had interesting new ideas, but was nonetheless fundamentally flawed. Dozens of scientists were involved in the analysis, with possibly hundreds more reading and checking the analysis itself. By comparison, the traditional peer review process can take many months and depends crucially on the expertise and inerrancy of as few as two individuals (Markoff, 2010).

Much of the review of Deolalikar's work was undertaken on the research blogs of mathematicians and computer scientists. Such blogs are fairly common today, and the exchanges that take place in their comments sections are fundamentally very similar to the collaboration between Einstein and Grossman. One of the first (and also most famous) scientific blogs is John Baez's "This Week's Finds in Mathematical Physics." As Baez writes in a recent article in the AMS about blogging:

> My introduction to blogging came in 1993 when I started an online column called "This Week's Finds in Mathematical Physics". The idea was to write summaries of papers I'd read and explain interesting ideas. I soon discovered that, when I made mistakes, readers would kindly correct them—and when I admitted I didn't understand things, experts would appear from nowhere and help me out. Other math bloggers report similar results.

This organic process observed by Baez (2010) is precisely what we mean by *open, collaborative online learning*. As the examples in the next two sections demonstrate, the online exchange of knowledge among mathematicians has become more sophisticated in the time since Baez began his blog.

The Polymath Projects

Terence Tao's (2011, July 15) blog "What's new" is an illustrative example of why scientific blogs should be taken quite seriously in any study of how modern science (and more generally, modern learning) is done. The quality of Tao's blog postings is at such a level that the AMS published the first year of his posts as a 300-page book, *Structure and Randomness: Pages from year one of a mathematical blog*, essentially verbatim (Tao, 2008). Tao's blog also played a significant role in the Polymath project, which was itself spawned in 2009 as a result of a long discussion on the blog of fellow Fields Medalist Tim Gowers.[7] The discussion that initiated the project began with a post of Gowers', titled "Is massively collaborative mathematics possible?" He proposed that the answer was "yes" and suggested some ground rules for running an open, collaborative research project on his blog ("Gowers's Weblog," 2011). A typical research project in mathematics is done largely in secret, with only one or two mathematicians participating; the final results are announced at the end, in the form of a paper containing highly perfected reasoning and proofs. In Gowers' proposed project, the research would take place in the open, and anyone could observe and participate simply by visiting Gowers' blog. The final results would still be written up as a paper, but a full record of the work would remain, in the form of the relevant blog posts and comments of the participants. After a lengthy discussion, with expressions of support from several other well-known mathematicians, the blog participants agreed on some basic rules of conduct and selected a goal for the project. They would attempt to find a combinatorial proof for the so-called "density Hales-Jewett theorem," or DHJ for short. While the theorem is simple to state informally, the existing proof relied on heavy-duty ergodic theory.

The success of the Polymath project was nothing short of astounding. In his summary of the project, Michael Nielsen (2010) writes:

> On March 10, Gowers announced that he was confident that the polymaths had found a new combinatorial proof of DHJ. Just 37 days had passed since the collaboration began, and 27 people had contributed approximately 800 mathematical comments, containing 170,000 words.

---

[7] In this context "polymath" really means "many mathematicians" rather than the usual definition, i.e., "a person of wide-ranging knowledge or learning." Rather than depending on the powers of a single rare and remarkable individual of the mold of Newton or Einstein, the Polymath projects depend on the combined strength of a number of more ordinary mathematicians.

At a presentation at the Institute for Advanced Study, Gowers outlined some of the benefits of this new way of doing mathematics. He estimated that the project solved the DHJ problem in "six weeks rather than several years." According to Gowers, other benefits included a full record of all of the work done as part of the collaboration, as well as unanticipated connections and different perspectives arising from the openness of the model (The Institute Letter, 2010). The result of the project will be at least two published papers, under the pseudonym "D.H.J. Polymath." Since this project, there have been five more major polymath projects on various topics. The goal of "Polymath4," for instance, is to resolve the following conjecture:

> There exists a deterministic algorithm which, when given an integer k, is guaranteed to find a prime of at least k digits in length of time polynomial in k.

These projects have been studied as a new way of doing mathematics (Sarvate et al., 2009). They raise interesting research questions for education researchers as well. One might try to understand, for instance, what kinds of mathematics problems and learning tasks (research or not) are well-suited to large-scale online collaboration, and what kinds are better suited to working in small groups or in isolation.

The particular structure of hiring and promotion in academia poses a significant barrier to the Polymath project and open science in general. Hiring, tenure and promotion decisions are based largely on the candidate's record of publication in reputable journals. Young mathematicians are acutely aware of the pressure to "publish or perish," as the saying goes, and this pressure does not leave much room for experimenting with new methods of doing science. The credit for authorship in the Polymath project, for instance, amounts to a link to the full record of the collaboration (i.e., the blog thread). The mathematics community at large has yet to develop a methodical way of assigning credit for such work and taking it into consideration for hiring, tenure and promotion. The obstacles are sometimes simply technical, such as when journals require copyright release forms from every author ("The Polymath Blog," March 9, 2011).

The MathOverflow Website

In October of 2009, Berkeley graduate students and postdocs Anton Geraschenko, David Brown, and Scott Morrison started a new website called MathOverflow. The site was announced on the Secret Blogging Seminar, a mathematics blog maintained by a small group of students at Berkeley (Geraschenko & Morrison, 2009). "MathOverflow" (2011) was based on the Stack Exchange software; the hope was that the site would enjoy the same success as Stackoverflow, which is also built on Stack Exchange software and is a very popular question-and-answer site for programmers. While it started as only an experiment, MathOverflow immediately attracted a large number of mathematicians who quickly began exchanging questions and answers on any and all research topics in mathematics. The site has been featured in many news articles, and many questions and answers have been cited in published journal papers ("Meta Mathoverflow," 2011). A sampling of topics from questions posted on August 31st, 2011 included elliptic curves, Riemannian geometry, group theory, algebraic geometry, partial differential equations, Banach spaces, cohomology, and many others. As of that date, over 22,500 questions had been posed. Joining the site as a member is free, and members can ask questions and provide answers, and assign up/down votes to the questions and answers of other members. The votes a member receives are added up, and higher scores allow access to more advanced features of the site, such as editing the posts of others. Among the highest scorers are several prominent mathematicians and Fields Medalists. Some of the mathematicians involved have expended a tremendous amount of effort participating in the site. The currently highest-rated member is Joel David Hamkins, a logician at City University of New York, with 29 questions, 506 answers, and over 2,700 up-votes. A crude first-order approximation indicates that somewhere between 5 and 16% of all actively working mathematicians contribute to MathOverflow ("MathOverflow Contributors," 2011). It bears mentioning that, even though the vast majority of the participants at MathOverflow are professors or postdocs, the sixth-highest-rated and eighth-highest-rated contributors are undergraduate students from MIT and Caltech, respectively.

Unlike its precursors (for example, the newsgroups *sci.math* and *sci.math.research*), MathOverflow is focused on a question-and-answer role. The result is a particular kind of online collaboration of learners (in this case, research mathematicians). A single question-and-answer thread on MathOverflow may not seem particularly collaborative; it may only involve two people, one with a question and another with the answer. However, from the point of view of an involved member, the site is a wealth of collaboration and cooperation. Given the sum total expertise of the mathematicians involved, the community has a tremendous capacity to educate the member and fill in gaps in their knowledge of mathematics. In exchange, the member may provide answers to the questions of others and lend his/her insight through commentary and collaborative editing. The result is that, undoubtedly, a significant amount of mathematics learning is taking place.

As the meta discussions on the site itself can attest, MathOverflow is not without problems. For instance, as one meta thread indicates, a significantly fewer-than-representative number of women use MathOverflow ("Meta Mathoverflow Discussions," 2011). While this is not due to any active discrimination on the part of site administrators, it is telling that the discussion itself had to be closed due to less than amicable exchanges between members.

Collaborative Online Learning: Undergraduate and School-Age Mathematics

The success of the open, collaborative mathematics done on the MathOverflow website and within the Polymath project motivates us to consider the extent to which similar efforts have been made in undergraduate and K-12 education. As we pointed out earlier, some of the most active participants in MathOverflow are advanced undergraduate students. In this section we will instead consider some examples of the online collaborative learning of subject matter which is typically taught at undergraduate and K-12 levels in schools in the United States.

Mathematics – Stack Exchange

"Mathematics – Stack Exchange" (2011) (or M.SE for short) is self-described as a "free, community driven and collaboratively edited site for learners studying mathematics as well as for professionals in related fields." Like MathOverflow, M.SE is based on the Stack Exchange software, and follows roughly the same rules of etiquette, moderation, and collaborative editing of content. Unlike MathOverflow, it focuses on questions in mathematics which are not at the research level; as such, questions about typical undergraduate-level mathematics are common. According to the administrators of the site, the focus is on improving understanding of mathematical concepts, exchanging hints for solving specific problems, and learning about the history and development of mathematics ("Math Stack Exchange Q&A," 2011). As an illustration of how the site functions, we will examine a specific Q&A exchange on M.SE in the next section.

An Excerpt of an Exchange on Mathematics – Stack Exchange

Two engineering students familiar with M.SE were asked by one of the authors of this paper to solve the following problem in a virtual environment:

> Prove that the line segment joining the centers of two concurrent circles of equal radius is perpendicular to the line segment joining the two intersection points of the circles.

In addition, students were required to record their online conversations and provide a detailed solution to the problem. Students used Skype to chat about the problem, sketched a drawing using GeoGebra and kept a record of their activities in Google Documents. To illustrate their initial thinking, here is a verbatim excerpt from their initial conversation via Skype:

> [7/7/2011 12:36:27PM] S1: After reading the problem I have a few ideas on how to solve the problem. I am making assumptions as it's been a long time since I have opened my math books. We should first draw a diagram (asymmetric) so that we don't arrive at conclusions very easily. What do you think?
>
> [7/7/2011 12:41:39PM] S2: sketching diagram would be helpful. Using properties of circle and chords we could get an idea regarding the problem.
>
> [7/7/2011 12:44:09PM] S1: Lets use one of the online collaborative tools and see if it useful as I have never used it before. I hope we can save and make changes to the diagram whenever possible.

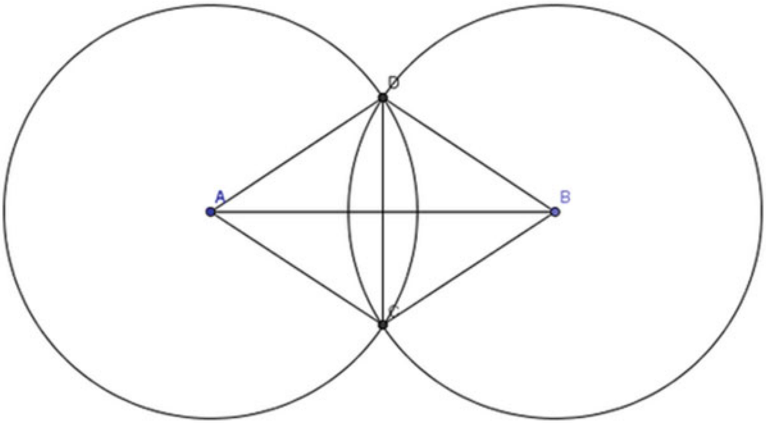

Fig. 1 A sketch of the problem made with "GeoGebra" (2011)

[7/7/2011 12:51:22PM] S2: i have'nt used online tools before. But i have an idea of google docs. I will try to draw a diagram by including my assumptions and share document with you. so that you can modify if you have any better ideas.

[7/7/2011 12:56:11PM] S1: ok. I will use different online tools and choose which tool is most user-friendly to work on.

S2 to S1: i found theorem proof for "line passing through the center of the circle bisect the chord."

[11:51:51AM] S2: will it be helpfull if l post the link to you.

[11:52:13] S2:

http://www.proofwiki.org/wiki/Perpendicular_Bisector_of_Chord_Passes_Through_Center

[11:56:53AM] S1: Yes, it works but it s a direct answer. If you look at the proof they have made assumptions based on a few definitions so we need to start at the basic level.

[11:59:56AM] S2: i will try. if i could get proof for statements in the proof.

[12:01:26PM] S1: They are bisecting AB at D. We have that construction in our proof Right?

[12:10:18PM] S2: yes. we have something similar. right now i have some assumptions through which we can draw a conclusion. i will modify the google document that we are using previously so that you can have visual of my idea (Fig. 1).

This conversation illustrates, in addition to use of Skype for communication, use of three additional online resources: (i) Google document to record the problem and solution attempts, (ii) GeoGebra (2011) to draw/construct an illustration supporting the students' understanding of the problem, and (iii) a wiki-proof – perpendicular bisector of chord passes through center – that the student S2 was thinking might be useful in the process. Eventually, the same student S2 posted a question to M.SE

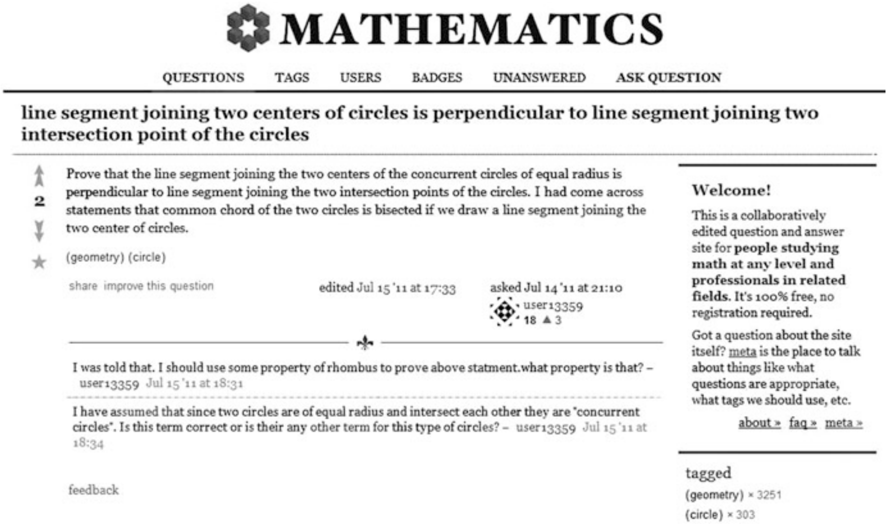

Fig. 2 Screen caption with the assigned problem posted to M.SE. (The Problem, 2011)

(The Problem, 2011) with an additional sentence: "I had come across statements that common chord of the two circles is bisected if we draw a line segment joining the two centers of circles."

Figure 2 is a capture of the question and the follow-up activity during the following 24 h by one of the engineering students trying to solve the problem. Within that short period of time, the problem was tagged under the categories *geometry* and *circle*, viewed 53 times, and given four answers. Notice that the student (as user13359) added two more comments after posting the original question, providing additional details about his thinking process.

Visible features of the M.SE are that viewers have an opportunity to (1) vote "yes" or "no" if "this answer was helpful," contributing to the reputation of the user that posted a particular answer, and (2) improve the answer.

Two M.SE users provided different ways of thinking about the problem. One wrote "This is true by symmetry. The figure with respect to reflection in the line joining the centers; if the line joining the two intersection points weren't perpendicular to that line, it would break the symmetry." The second user wrote,

> Let the points of interaction of the circles be A and B, and the centers be C and C'. The triangles ACB and AC'B are two isosceles triangles which share the same base. As you have said, the line joining C and C' bisects this base. But in an isosceles triangle, the main altitude is the same line as the main median. Hence the line CC' must be an altitude for each of the triangles ACB and AC'B, i.e., it is perpendicular to AB (The Problem, 2011).

A coordinate geometry solution was provided by another user within 24 h of posting the problem, Here's is a coordinate geometry proof: let the two circles be of the form

$$(x - h_1)^2 + (y - k_1)^2 = {r_1}^2,$$

$$(x - h_2)^2 + (y - k_2)^2 = {r_2}^2$$

I 'will be demonstrating a more general statement: the radical line of the two circles is perpendicular to the line through the two centers. In the special case of intersecting circles, the radical line is the line through the two intersection points.

It is trivial to write down the equation of the line joining the two centers:

$$(y - k_1) \div (x - h_1) = (k_2 - k_1) \div (h_2 - h_1)$$

To construct the equation of the radical line, we expand the Cartesian equations of the two circles:

$$x^2 - 2h_1 x + {h_1}^2 + y^2 - 2k_1 y + {k_1}^2 = {r_1}^2,\ x^2 - 2h_2 x + {h_2}^2 + y^2 - 2k_2 y + {k_2}^2 = {r_2}^2$$

And then subtract one from the other:

$$2\,(h_2 - h_1)\,x + {h_1}^2 - {h_2}^2 + 2\,(k_2 - k_1)\,y + {k_1}^2 - {k_2}^2 = {r_1}^2 - {r_2}^2$$

Whose slope is $(h_2 - h_1)/(k_2 - k_1)$, which when multiplied by the slope $(k_2 - k_1)/(h_2 - h_1)$ of the line joining the centers gives -1, thus showing the perpendicularity (The Problem, 2011).

What followed is an opinion and comment included with yet another way to think about the problem.

The argument by symmetry given by Joriki is in my opinion optimal.

If we want a maximally "high school" proof of the old-fashioned type, let the centers of the circles be C and C', and let the intersection points of the circles be A and B, as in the answer y Bruno Joyal. Let M be the point where lines AB and CC' meet.

Then triangles ACC' and BCC' are congruent, by what used to be called SSS.

It follows that $< \text{ACC'} = < \text{BCC'}$,

So triangles ACM and BCM are congruent, by what used to be called SAS.

Thus, $<$ CMA $=$ $<$CMB. But these angles add up to a "straight angle," so each is a right angle (The Problem, 2011).

The next day, two users exchanged comments on coordinate geometry versus "the classical route"

– Good demonstration of the "algorithmic" nature of coordinate geometry arguments. – André Nicolas
– Indeed, @André … On the other hand, there remains a sort of panache when one goes the classical route instead of the coordinate route … – J.M.

The follow-up reflective comment referred to a historical (not necessarily true) anecdote, "Euclid is said to have told a Ptolemy, who was asking for shortcuts to proofs, and/or Menaechmus told Alexander 'There is no royal road to geometry.' ... And the anecdote is unlikely to be true ... – André Nicolas". In response, J.M. reflects on a possibility of revising the proof at some later "... I could have made the centers lie on the horizontal axis, and the proof certainly becomes much tidier that way ... maybe I'll rewrite later."

Later, user6312 added a comment which included (a) an expansion of the problem, "... the result is true without the assumption of equal radius ...," and (b) a speculative assumption about information being provided to students in the class, "By a theorem which has presumably been proved in the course prior to this problem ...,"

> Curiously enough, we all seem to have missed the "equal radius" part of the statement of the problem, which as far as I can tell was there from the beginning! Of course, the result is true without the assumption of equal radius.
>
> But the following argument may have been the intended one. Since the radii are equal, the centers of the circles and their points of intersection form a rhombus. By a theorem which has presumably been proved in the course prior to this problem, the diagonals of a rhombus are perpendicular (and bisect each other). End of proof (The Problem, 2011).

The various users gave answers with very different approaches and styles, and then exchanged comments on these differences. The answers were well-thought-out, and were accompanied by LaTeX-coded equations and contextual commentary. The users also returned later to add comments and edit their responses. No additional interactions occurred between the student asking the question and the other users; the other users did interact with each other in a meaningful and socially friendly way. It should be mentioned that links to related problems are available on the right side of the web page (Mathematics – Stack Exchange, 2011), providing opportunities for additional exploration for self-regulated learners. We remark that, if one were to access this question at a later time, the total record of the collaboration would be available, providing a number of different approaches with complete proofs and useful commentaries.

Relevance of exploration in selecting representations toward students' conceptual understanding of mathematical ideas is recognized as a tool for meaningful learning by many researchers (e.g., Alagic, 2003; Greeno & Hall, 1997). In the example provided here, students were able to access different ways in which other users are thinking about the given problem. The same mathematical problem may be captured with different representations and different modes of the same representation. Multiple representations have been linked with greater flexibility in student thinking and better transfer of learning (Perkins & Salomon, 1992; Zheng, 2008). A student S2 reflected on his learning, "... it would be better understood and remembered for a long time when we try to learn things by comparing with other things that are having some similar properties."

The above exchange and accompanying reflection on M.SE raises many questions. For instance, what are the effects of integrating these kinds of interactions in a traditional classroom? How different is this learning from traditional mathematics education? How is the availability of websites such as M.SE already changing traditional education?

A Global Learning Project

One of the authors of this chapter co-facilitated a global learning project titled "From Kansas to Queensland" (Alagic, Gibson, & Doyle, 2004; Watters, Rogers, Gibson, Alagic, & Haack, 2004). The project connected preservice elementary teachers enrolled in an Australian science methods course with a similar group in the United States studying an integrated mathematics and science methods course. The goal of the project was two-fold: to develop greater understanding of practices in teaching elementary students, and to develop a global perspective on teaching and learning science.

The 2-year project involved approximately 60 participants each year. The participants consisted of a group of students from a major metropolitan university in Queensland, Australia and a similar group of US preservice elementary teachers from a university in Kansas, USA. Initially, the project faced many challenges, e.g., the timing of the respective courses. Many of these initial technological and organizational issues were resolved in the second year of the project, when most of the exchanges between the participants were facilitated by the discussion forums within the same courseware used by both universities. The observed exchanges between participants reflected Salmon's (2003) progressive stages of online learning, beginning with *access and motivation*, continuing to *on-line socialization* and *information exchange* and finally to *knowledge construction* and *development*. Eventually, the participants developed a level of autonomy that enabled them to engage in spontaneous communication and knowledge exchange/building (Alagic et al., 2004; Watters et al., 2004).

As part of the global learning project, students were encouraged to engage in computer-mediated communication and to learn about mutual cultural practices and primary/elementary science education in both countries. The initial social interaction was facilitated by suggesting to students that they introduce themselves, develop a name for their group, and reflect on their prior experiences of learning science. Many shared negative experiences in their prior learning of science and mathematics, which in turn fostered some discussions about possible ways to provide positive experiences for their own students. They also discussed challenges inherent to constructivist approaches to teaching and different ways in which curricula are designed in Australia and the United States. The outcomes demonstrated that preservice elementary teachers, at both locations, benefited from the exchanges by achieving a greater understanding of the common problems confronting science education in both countries. Reflecting on the project experience, students expressed support for the global learning initiative; below are four examples of their comments:

– *Thinking about learning*: "...the main thing I think I have learned is to not be afraid to try something new or to meet someone new. I was very nervous about having an e-mate, but so far, it has turned out to be awesome, and I am glad that I was forced to do it. I probably would not have done it if I had a choice, so I am glad I was pushed to do so."

– In terms of *science/mathematics content*, the teaching of the metric system generated an interesting debate. Some of the US students demonstrated a misconception about the use of the metric system internationally. One Australian student described her understanding of the system as the "English Metric System." A Kansas student shared, "I have also learned that I have a lot to learn about science. In his last e-mail he [e-mate] was talking about doing a lesson about how many Joules are in peanuts, and I don't even know what a Joule is. I have a lot to do still before I become a teacher."
– Reflective of *cultural awareness*: "... I've learned what netball is and how you play it. My e-mate is really good at it and her sister also plays it. Her sister represents Australia doing it, which I thought was really cool." Other details about daily lives were evident in communication related to "mascots" at the US universities, a practice unknown in Australia.
– *Value for preservice teachers' future classrooms*: Students made comments about the transferability of their experience to their own future classrooms. Several students shared ideas about e-corresponding next year when they have their own classes, with an attempt to link their students.

Furthermore, it appeared that for some participants the opportunity to interact with a neutral person (i.e., a peer, not the instructor), from another country, provided an outside authority who validated content and methods selected by instructors in respective courses. Although one could argue this can be accomplished in face-to-face courses, notice that in a face-to-face course students are exposed to the same curriculum and guided by the same instructor. Only a small number of students indicated that they needed more time to develop a deeper connection and engage more actively in discussions (Alagic et al., 2004; Watters et al., 2004).

Virtual Math Teams

The following is an example of a collaborative online learning opportunity for K-12 students facilitated in a completely different manner from the above Global Learning Project. The Virtual Math Teams (VMT) were implemented and researched by a group of researchers since 2003 (Stahl, 2009). They were built on the Problem-of-the-Week service at "Math Forum" (2011), an online resource for improving math learning, teaching, and communication. The Math Forum has been active since 1992, and offers a number of online services, including Problem of the Week, Ask Dr. Math, Math Tools, and Teacher2Teacher. Students, who once worked by themselves Problem of the Week, had an opportunity to work on open-ended problems with a group of peers. The book Studying Virtual Math Teams (Stahl) reports on empirical studies pursued by the author and his research team at the Math Forum at Drexel University. Studies utilized chat interaction analysis on researching how students in small online groups collaborate in solving problems and develop understanding of mathematical ideas. Studies incorporate a description of the so-called Computer-Supported Collaborative Learning (CSCL) environment and the corresponding pedagogical orientation. For the interested reader, a significant collection of publications about VMT is referenced at Stahl's webpage.

One of the studies (Litz, 2007) investigated why some K-12 students showed resistance to using the VMT chat tools and why the number of participants in the individually oriented Problems of the Week was substantially higher than those of the VMT Chat. Some of the identified reasons included a lack of teacher encouragement, a lack of integration of VMT Chat in math classes, difficulties with using the computer environment itself, and a lack of advertising for the program.

Analysis and Contextualization

The examples in sections "Collaborative Online Learning of Research-Level Mathematics" and "Collaborative Online Learning: Undergraduate and School-Age Mathematics" are a somewhat eclectic collection of online collaborative mathematics learning opportunities, selected to illustrate only a few (among many) existing different models for online collaborative learning. Collaborative online learning at the research level is illustrated by the Polymath project (a large-scale open research collaboration) and the MathOverflow website (a Q&A site for research mathematicians). At the undergraduate level, the two examples are Mathematics – Stack Exchange (a Q&A site similar to MathOverflow, but for non-research questions) and the Global Learning Project (which connected two undergraduate classrooms from different parts of the world with an overarching collaborative learning goal). Finally, we briefly discussed Virtual Math Teams as an example of collaborative online learning of mathematics appropriate for K-12 students, and known to many K-12 teachers and their educators. We introduced in more detail what we currently consider to be less known collaborative online opportunities, and briefly mentioned those we deem better known to the general and scholarly audiences. To clarify the significance of these examples, we now situate them in the context of recent education research and briefly deliberate about various forms of online collaborative learning, including open, self-directed, and massively collaborative learning.

Computer Supported Collaborative Learning

Computer-Supported Collaborative Learning (CSCL) is a pedagogical methodology in which learning takes place via social, collaborative interactions facilitated using computers and the Internet. CSCL is closely related to collaborative learning and computer-supported cooperative work. The distinguishing characteristic of CSCL is that learning is accomplished by the sharing and construction of knowledge among participants through online communication (e.g., Koschmann, 1996; Stahl, Koschmann, & Suthers, 2006). In addition to studying successful CSCL stories such as those mentioned in sections "Collaborative Online Learning of Research-Level Mathematics" (Polymath and Overflow) and "Collaborative Online Learning: Undergraduate and School-Age Mathematics" (Global Learning and VMT), (see also, Hurme, Merenluoto, & Järvelä, 2009; Stahl, 2010), it is relevant to recognize the challenges of computer-supported collaborative learning in mathematics. In Roberts (2004) two such challenges are identified: first, traditional mathematics textbook problems do not necessarily lend themselves to collaborative engagement in knowledge development; second, currently available tools for representation are limited in their ability to support constructive knowledge-building activities. Proposed remedies are given in the form of (1) authentic mathematical problems that engage students in the collaborative construction and necessary revision of mathematical models, and (2) user-friendly representational tools that not only enable students to effectively represent mathematical problems but also to translate within and across representations (Nason & Woodruff, 2004; Roberts, 2004). In section "An Excerpt of an Exchange on Mathematics – Stack Exchange" we illustrated how engagement of M.SE users around one relatively simple geometry problem brings together a set of different solutions using available representational tools. Readers of this chapter are encouraged to explore more complex, rich and authentic mathematical problems as well as representational tools available at M.SE in order to notice how Roberts' 'remedies' play out in 2012 as compared to 2004 when the Roberts' research was published.

The Computer Moderated Communication Model

Salmon (2003) described the Computer Moderated Communication (CMC) model of online communication and learning through five stages of progression: access, socialization, information exchange, knowledge construction, and further development. In the access stage, students familiarize themselves with the technology tools to be used in online interacting. During the socialization stage, students are engaged in conversation about their related experiences and compare these experiences in order to understand each other's perspectives. The information exchange stage is reached when students begin demonstrating an ability to exchange perspectives and synthesize their knowledge; this provides conditions for the knowledge construction stage, where students might operate as more independent learners. In the fifth and final stage, that of further development, the students displayed a willingness to engage in (or even initiate) similar projects. Salmon suggested that the CMC model be introduced to students as a "new way of learning through networking" (Salmon, p. 116).

In the Global Learning Project (as described in section "A Global Learning Project" the exchanges between the members of the two student teams exhibited Salmon's progression through the various stages of CMC. The initial socialization stage was initiated by instructors, and the information exchange stage was somewhat guided by the fact that the students were participating in the project as part of a class. The commitment to engage in collaborative discussions varied among groups; some committed to exchanging ideas well beyond the 6-week period, and continued collaborating even after they were in their own classrooms with their own students (Watters, Rogers, Gibson, Alagic, & Haack, 2004). The CMC model provided a useful structure for studying the development of the interactions in this project and for analyzing the collected data. However, researchers (Watters et al.) observed that the final CMC stage (Stage 5) did not capture the most advanced student behaviors. A small group of participants emerged from Stage Four with significant contributions to knowledge construction, and subsequently explored options of implementing similar projects with their own students. This display of leadership led researchers to further analyze data for elements of self-regulated learning (Alagic et al., 2004). Authors concluded with the proposition that making students aware of the CMC model stages and related expectations could assist in facilitating a more explicit path toward the interplay among the four major areas of self-regulated learning: cognition, motivation, behavior and context (Snow, Corno, & Jackson, 1996).

Networked Learning, and Collaboration Versus Cooperation

At first, research and practice related to collaborative online learning environments have focused mainly on tightly connected groups (as in the Global Learning Project) facilitated by instructors willing to collaborate and co-teach. The concepts of communities of practice have broadened this concept to include groups of people sharing common personal or professional interests, with a goal of sharing knowledge and developing shared understanding. With the expansion of social networking, the structure of online groups now also includes loosely organized communities in which people do not necessarily collaborate or communicate directly (Dron & Anderson, 2007; Jones, Ferreday, & Hodgson, 2006; Siemens, 2005; Roschelle & Teasley, 1995).

Although we made no attempt to differentiate between "cooperative learning" and "collaborative learning" in the previous sections, some authors do make a distinction (Dalsgaard & Paulsen, 2009; Panitz, 2003; Rockwood, 1995). For example, Dalsgaard and Paulsen described collaborative learning as group-work within a learning community, done with the expectation that all members actively participate in accomplishing the given complex open-ended task. Three basic criteria are often considered for collaboration: equal participation, genuine interactions, and unified synthesis of work. Consequently, collaborative learning is seen as limiting individual flexibility, while cooperative learning fosters an affinity to a learning community while maintaining certain individual flexibility. According to Dalsgaard and Paulsen, "collaborative learning depends on *groups*, and cooperative learning takes place in social and/or professional *networks*. One may also add that the ties between people are much tighter in groups than in networks" (p. 1). Classifying the examples from sections "Collaborative Online Learning of Research-Level

Mathematics" and "Collaborative Online Learning: Undergraduate and School-Age Mathematics" within this strict distinction made by Dalsgaard and Paulsen appears difficult. The participants of the Polymath project certainly referred to their work as "collaborative" (Gowers's Weblog, 2011); however, there were no specific expectations of any individual, and the complete list of participants was not even known until after the project was completed. The MathOverflow and Mathematics – Stack Exchange websites are "collaboratively edited" according to their own descriptions, but there again anyone is welcome to join and leave as they please, and no individual is held responsible for anything at all. All of the work on these sites is done voluntarily; it is thus distributed in a highly asymmetric way, owing to the differing levels of motivation and commitment among the various participants.

Networked learning is about connecting and communicating among individuals and groups in ways that support one's learning. In the information and communication technology paradigm, networked learning refers to the use of information and communication technologies to foster necessary interactions and connections (e.g., Goodyear, Banks, Hodgson, & McConnell, 2004; Benkler, 2006). Salmon (2003) suggested that we think about individual learners as nodes on a given network. In formal educational settings, networked learning is achieved through formally facilitated cooperative or collaborative processes (e.g., Dalsgaard & Paulsen, 2009; Panitz, 2003).

Dalsgaard and Paulsen (2009) were seeking answers to the following question: "what is the potential of social networking within cooperative online education?" They defined *transparency* in cooperative learning as the learners' insights into each other's activities and resources, within the defined learning environment. Transparency is seen as a way of enabling learners to utilize each other's work while maintaining individual learning autonomy. In a classroom-oriented online setting, the teacher facilitates transparency by setting or negotiating the conditions of the learning environment. Dalsgaard and Paulsen contend that the pedagogical potential of networked learning is situated within transparency and the capacity to foster awareness related to information and resources among learners. "Informal" and "transparent" seem to be crucial descriptors of Stack Exchange and Math Overflow. An additional crucial descriptor is "self-regulated;" indeed, none of these communities would exist if the participants did not choose to participate (Molenaar, van Roda, Boxtel, & Sleegers, 2012).

Self-Regulated Learning

Informal learning is sometimes classified according to intentionality and awareness; within this classification, the three forms of informal learning usually considered are autonomous, incidental and socialized (Schugurensky, 2000). Most of the examples discussed in sections "Collaborative Online Learning of Research-Level Mathematics" and "Collaborative Online Learning: Undergraduate and School-Age Mathematics" (Stack Exchange and Math Overflow) qualify as examples of informal, self-regulated learning within the context of an online learning environment. Global Learning project and Virtual Math Teams required various levels of instructor/teacher facilitation and therefore, the extent to which learners are self regulated is questionable and requires further analysis which is not a goal in this chapter.

Self-regulation or self-direction in learning can be seen as a high degree of learner independence that includes factors such as commitment to the learning task and critical reflection (Cannatella, 2000). One definition of self-regulated learning is "an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and the contextual features in the environment" (Wolters, Pintrich, & Karabenick, 2003, p. 4). The relations between individuals and the environment, and their resulting overall achievement are facilitated by these self-regulatory actions. Cognition, motivation, behavior and context are often considered major, though not distinct, areas of regulation. Models of self-regulation usually share four learner-directed phases: goal-setting, monitoring, control and regulation, and reflective processes (Snow et al., 1996). The development of lifelong learning is associated with certain criteria; among those are active learner involvement, a democratic learning approach, flexibility of time and amount of learning, provisions for individual learning differences, relevant motivating content, opportunities for collaboration, integration of knowledge from different fields, and an encouraging, risk-free atmosphere for learning (Knapper, 1988). To develop the qualities of long-term learner independence, opportunities for active involvement, relevance, and risk-taking must be provided at all stages of the educational process. Some learners require prolonged attention to these qualities, some less, or none (Alagic et al., 2004; Cannatella, 2000; Knapper, 1988).

Self-regulated learning is considered to contribute to cognitive self-engagement into higher-order thinking (Corno & Mandinach, 1983; Vygotsky, 1978; Winne & Hadwin, 2008) during collaborative learning that occurs when learners interact while seeking a solution to a task (Jonassen & Reeves, 1996). The purposeful planning and monitoring of cognitive and affective processes involved in successful completion of selected tasks is characterized by the use of metacognitive processes of planning and monitoring that for some learners seem to occur automatically. Gibbons (2004) explicated five stages/formats, each involving a set of tasks that can facilitate scaffolding of opportunities for self-regulated learning. These stages include: *Incidental self-direction; Independent thinking; Self-managed learning; Self-planned learning; and Self-directed learning*. Stages can be a guiding sequence for moving forward, or employed one at a time. The way in which the stages are implemented is highly dependent on the learners' active role in selecting, designing and implementing their own learning activities. It is not a goal here to study all the stages of self-regulation. However, there is a space for some illustration. If we subject "An Excerpt of an Exchange on Mathematics – Stack Exchange" to an analysis according to the mentioned stages, we may notice that the initial user showed some incidental self-direction by posting the problem and later adding some additional comments. Interaction among other users contributing to the conversation leads one of the users to promise improving his posted solution which at least illustrate intentional self-planned learning ("tiding").

Discussion

In the previous sections, we discussed several examples of collaborative online learning of mathematics. The examples varied in their essential nature; some involved small closed groups of K-12 peers (Virtual Math Teams) or preservice elementary teachers (Global Learning Project), and others were large-scale massive collaboration efforts open to anyone with an Internet connection, like Mathematics – Stack Exchange. Our hope is that the discussion of these examples, along with the contextual analysis of section "Analysis and Contextualization", will spur further discussion and research about the nature of both formal and informal learning of mathematics with modern online technologies. We have discussed many of the advantages and some of the challenges of collaborative online learning in its various incarnations. Still, it bears pointing out that, just like the classroom model, a collaborative online model has inherent limitations. For instance, most of the examples we cited (e.g., Polymath, MathOverflow, Mathematics – StackExchange) depend critically on self-motivation and self-regulation. The problem of developing such a degree of self-motivation and self-regulation in the first place is left unexamined. One should also exercise care in applying learning methods used by research mathematicians to school-age and undergraduate mathematics learning. In particular, learning mathematics without referring to outside authorities has strict limits. We would certainly not expect a group of young children to independently re-invent the entire field of mathematics without relying on any textbooks or outside instruction, even if they worked together with the most advanced collaboration tools available. We have only briefly touched upon the many challenges of integrating online learning with traditional/formal education (e.g., Virtual Math Teams). With these reservations in mind, we nonetheless see tremendous opportunities in collaborative online learning of mathematics for students at all ages and knowledge levels. In particular, we believe that online collaborative learning communities provide the greatest benefit to their participants when participation is self-regulated and open to anyone.

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The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team

Murat Perit Çakır and Gerry Stahl

Abstract Learning mathematics involves mastering specific forms of social practice. In this chapter, we describe socially situated, interactional processes involved with collaborative learning of mathematics online. We provide a group-cognitive account of mathematical understanding in an empirical case study of an online collaborative learning environment called Virtual Math Teams. The chapter looks closely at how an online small group of mathematics students coordinates their collaborative problem solving using chat, shared drawings and mathematics symbols. Our analysis highlights the methodic ways group members enact the affordances of their situation (a) to display their reasoning to each other by co-constructing shared mathematical artifacts and (b) to coordinate their actions across multiple interaction spaces to relate their narrative, graphical and symbolic contributions while they are working on open-ended mathematics problems. In particular, we identify key roles of referential and representational practices in the co-construction of deep mathematical understanding at the group level, which is achieved through methodic uses of the environment's features to coordinate narrative, graphical and symbolic resources.

Keywords Online mathematics • Collaborative learning • CSCL • Group cognition • Virtual math teams • Multiple representations • Discourse • Symbolic expression • Graphical reasoning • Mathematics education • Meaning making • Multimodal interaction space • Joint attention • Interaction analysis • Patterns • Indexical • Persistence • Presence • Artifact • Narrative • Reference • Sequential organization • Common ground • Practices • Affordance

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Mathematical Practices

Developing pedagogies and instructional tools to support learning mathematics with understanding is a major goal in Mathematics Education (CCSSI, 2011; NCTM, 2000). A common theme among various characterizations of mathematical understanding in the mathematics education literature involves *constructing relationships* among mathematical facts and procedures (Hiebert & Wearne, 1996). In particular, recognition of connections among *multiple realizations* of a mathematics concept encapsulated in various inscriptional forms is considered as evidence of deep understanding of that subject matter (Healy & Hoyles, 1999; Kaput, 1998; Sfard, 2008). For instance, the concept of function in the modern mathematics curriculum is introduced through its graphical, narrative, tabular and symbolic realizations. Hence, a deep understanding of the function concept is ascribed to a learner to the extent he/she can demonstrate how seemingly different graphical, narrative and symbolic forms are interrelated as realizations of each other in specific problem-solving circumstances that require the use of functions. On the other hand, students who demonstrate difficulties in realizing such connections are considered to perceive actions associated with distinct forms as isolated sets of skills, and hence are said to have a shallow understanding of the subject matter (Carpenter & Lehrer, 1999).

Reflecting on one's own actions and *communicating*/articulating mathematical rationale are considered as important activities through which students realize connections among seemingly isolated facts and procedures in mathematics education theory (Hiebert et al., 1996; Sfard, 2002). Such activities are claimed to help learners notice broader structural links among underlying concepts, reorganize their thoughts around these structures, and hence develop their understanding of mathematics (Carpenter & Lehrer, 1999; Skemp, 1976). Consequently, collaborative learning in peer-group settings is receiving increasing interest in mathematics education practice due to its potential for promoting student participation and creating a natural setting where students can explain their reasoning and benefit from each others' perspectives (Barron, 2003).

Representational capabilities offered by Information and Communication Technologies (ICT) provide important affordances for exploring connections among different realizations of mathematical objects. Dynamic geometry applications like Cabri, Geometer's Sketchpad, GeoGebra (Goldenberg & Cuoco, 1998); algebra applications such as Casyospee (Lagrange, 2005), or statistical modeling and exploratory data analysis tools like TinkerPlots (Konold, 2007) provide representational capabilities and virtual manipulatives that surpass what can be done with conventional methods of producing mathematical inscriptions in the classroom (Olive, 1998). In addition to this, widespread popularity of social networking and instant messaging technologies among the so-called Net Generation requires designers of educational technology to think about innovative ways for engaging the new generation of students with mathematical activity (Lenhart, Madden, Macgill, & Smith, 2007). Therefore, bringing the representational capabilities of existing mathematical packages together with communicational affordances of social-networking/messenger software can potentially support the kinds of interactions that foster deeper understanding of mathematics. Computer-Supported Collaborative Learning (CSCL) is a research paradigm in the field of Instructional Technology that investigates how such opportunities can be realized through carefully designed learning environments that support collective meaning-making practices in computer-mediated settings (Stahl, Koschmann, & Suthers, 2006).

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text-chat and a shared graphical workspace—have been widely employed in CSCL research and in commercial collaboration suites such as Elluminate and Blackboard-Wimba to support collaborative-learning activities of small groups online (Dillenbourg & Traum, 2006; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools not only brings various *affordances* (i.e., possibilities for and/or constraints on actions), but also carries important interactional consequences for the users (Çakır, Zemel, & Stahl, 2009; Dohn, 2009; Suthers, 2006). Providing access to a rich set of modalities for action allows users to demonstrate their reasoning in multiple semiotic forms. However, the achievement of connections that foster the kind of mathematical understanding desired by mathematics educators is conditioned upon team members' success in devising shared methods for coordinated use of these resources (Mühlpfordt & Stahl, 2007).

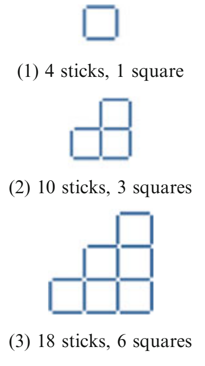
Although CSCL environments with multimodal interaction spaces offer rich possibilities for the creation, manipulation, and sharing of mathematical artifacts online, the *interactional organization of mathematical meaning-making activities* in such online environments is a relatively unexplored area in CSCL and in mathematics education. In an effort to address this gap, we have designed an online environment with multiple interaction spaces called Virtual Math Teams (VMT), which allows users to exchange textual postings as well as share graphical contributions online (Stahl, 2009). The VMT environment also provides additional resources, such as explicit referencing and special awareness markers, to help users coordinate their actions across multiple spaces. Of special interest to researchers, this environment includes a Replayer tool to replay a chat session as it unfolded in real time and inspect how students organize their joint activity to achieve the kinds of connections indicative of deep understanding of mathematics (Stahl, 2011).

In this chapter we focus on the interactional practices through which VMT participants achieve the kinds of connections across multiple semiotic modalities that are indicative of deep mathematical understanding. In particular, the chapter will look closely at how an online small group of mathematics students coordinated their collaborative problem solving using digital text, drawings and symbols. We take the mathematics-education practitioners' account of what constitutes deep learning of mathematics as a starting point, but instead of treating understanding as a mental state of the individual learner that is typically inferred by outcome measures, we argue that deep mathematical understanding can be located in the practices of collective multimodal reasoning displayed by groups of students through the sequential and spatial organization of their actions (Stahl, 2006). In an effort to study the practices of multimodal reasoning online, we employ an ethnomethodological case-study approach and investigate the methods through which small groups of students achieve *joint attention* to particular mathematical features of their representations in order to ground their co-construction of shared mathematical meaning (Sarmiento & Stahl, 2008; Stahl, Zhou, Çakır, & Sarmiento-Klapper, 2011). Our analysis of the excerpts presented below has identified key roles of *referential and representational practices* in the co-construction of deep mathematical understanding at the group level, which is elaborated further in the discussion section.

Data Collection and Methodology

The excerpts analyzed in this chapter are obtained from a problem-solving session of a team of three upper-middle-school students who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Singapore and Scotland to collaborate on an open-ended mathematics task on combinatorial patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat screen names and information that may have been communicated during the sessions. Each group participated in four sessions during a 2-week period, and each session lasted over an hour. Each session was moderated by a Math Forum member; the facilitators' task was to help the teams when they experienced technical difficulties, not to participate in the problem-solving work.

During their first session, all the teams were asked to work on a particular pattern of squares made up of sticks (see Fig. 1). For the remaining three sessions the teams



N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

1. Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?
2. Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki
3. Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Fig. 1 Task description for VMT Spring Fest 2006 were asked to come up with their own stick patterns, describe the patterns they observed as mathematical formulae, and share their observations with other teams through a wiki page.

This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods, such as the use of recursive functions and exploring the arithmetic properties of various number sequences. Moreover, the task had both algebraic and geometric aspects, which would potentially allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in a different kind of problem-solving activity as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional problem to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further (Moss & Beatty, 2006; Watson & Mason, 2005).

The VMT system that hosted these sessions has two main interactive components that conform to the typical layout of systems with dual-interaction spaces: a shared whiteboard that provides basic drawing features on the left and a chat window on the right. The online environment has features to help users relate the actions happening across dual-interaction spaces. One of the unique features of the VMT chat system is the *referencing* support mechanism (Mühlpfordt & Stahl, 2007), which allows users to visually connect their chat postings to previous postings or to objects on the whiteboard via arrows (e.g., Fig. 7 below illustrates a *message-to-message* reference, whereas Fig. 6 shows a *message-to-whiteboard* reference). The referential links attached to a message are displayed until a new message is posted. Messages including referential links are marked with an arrow icon in the chat window. A user can see where such a message is pointing at any time by clicking on it.

Studying the collective meaning-making practices enacted by the users of CSCL systems requires a close analysis of the process of collaboration itself (Koschmann, Stahl, & Zemel, 2007; Stahl et al., 2006). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the *small group* as the unit of analysis (Stahl, 2006), and we appropriate methods of Ethnomethodology (EM) (Garfinkel, 1967; Livingston, 1987) and Conversation Analysis (CA) (Sacks, 1962/1995; ten Have, 1999) to conduct case studies of online group interaction. Our work is informed by EM/CA studies of interaction mediated by online text-chat (Garcia & Jacobs, 1998; O'Neil & Martin, 2003). However, the availability of a shared drawing area and explicit support for deictic references in our online environment, as well as our focus on mathematical practice significantly differentiate our study from existing CA/EM studies of online text-chat.

The goal of ethnomethodological conversation analysis is to describe the commonsense understandings and procedures group members use to organize their conduct in particular interactional settings. Commonsense understandings and procedures are subjected to analytical scrutiny because they "enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings" (Goodwin & Heritage, 1990, p. 285). Group members' shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Since members enact these understandings and/or procedures in their visually displayed situated actions, researchers can discover them through detailed analysis of members' sequentially organized conduct (Schegloff & Sacks, 1973).

We conducted numerous VMT Project data sessions, where we subjected our analysis of VMT data to intersubjective agreement by conducting CA data sessions (Jordan & Henderson, 1995; ten Have, 1999). During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the timestamps of actions recorded in the log file. The temporal order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allowed us to study the sequential unfolding of events during the entire chat session. In short, the VMT environment provided us as researchers a perspicuous setting in which the mathematical meaning-making process is *made visible* as it is "observably and accountably embedded in collaborative activity" (Koschmann, 2001, p. 19).

Setting Up the Mathematical Analysis

In the following excerpts we will observe a team of three upper-middle-school students in action, who used "Qwertyuiop", "137" and "Jason" as login screen names. Prior to the session containing these excerpts, this team completed two chat sessions where they explored similar stick patterns together. In the current session, team members will be working on a new stick pattern that they co-constructed and named as the "hexagonal pattern", whose first three stages are illustrated in Fig. 2. Details of this co-construction process was analyzed and published elsewhere (Çakır, 2009; Çakır et al., 2009), so we will skip the part where the group constituted this pattern as a shared problem and figured out a method to count the number of *triangles* enclosed in its n-th stage. In the excerpts presented below, the team will be

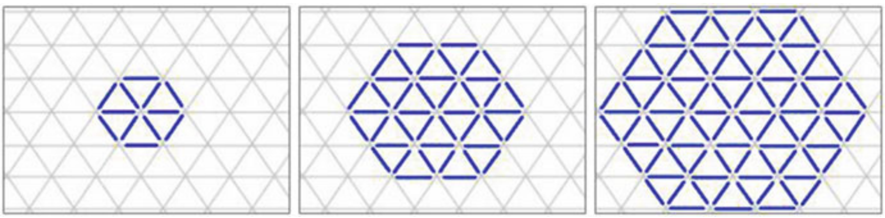


Fig. 2 Hexagonal stick pattern co-constructed by this team

working on devising a formula for characterizing the number of *sticks* that will be needed to construct the hexagonal pattern in general (i.e., in its n-th stage). Our main analytic goal is to identify the practices or *group methods* team members *enacted* to achieve a shared understanding of the problem at hand by co-constructing and acting on the mathematical artifacts in graphical, narrative and symbolic forms.

Excerpt 1: Constitution of a New Math Task

This excerpt illustrates a number of rich referencing methods: special terms, graphical practices, VMT tools, etc. Excerpt 1[1] opens with Qwertyuiop's announcement of "**an idea**"[2] in line 742. He suggests the team find the number of a set of objects he calls "**collinear sides**" and multiply that number by 3. The statement in parenthesis elaborates further that there are three such sets. The use of the term "**sides**" makes it evident that this statement is about the problem of finding the number of sticks to construct a given stage, rather than the problem of finding the number of triangles that make up a hexagon that has been recently discussed by the team.[3] Thus, Qwertyuiop seems to be proposing to his teammates a way to approach the problem of counting the number of sticks needed to construct the hexagonal shape in general.

A minute after this posting, 137 begins to type at 19:26:20. While the awareness marker continues to display that 137 is typing, he adds two green lines on the hexagon that intersect each other and two green arrows (see Fig. 3). The arrows are positioned outside the hexagon and their tips are mutually pointing at each other through a projected diagonal axis. Shortly after his last drawing move, 137 completes his typing action by posting the message "**as in those?**" in line 746, which is explicitly linked to Qwertyuiop's previous posting with a referential arrow. The plural[4] deictic term "**those**" in this posting instructs others to attend to some

[1]The referential links used by the students to connect their messages to previous messages are displayed in the right-most column in the excerpts. For instance, line 745 includes Message #742 in the right-most column. This indicates that message 745 was linked to 742 by its contributor (i.e. Nan in this case). References to whiteboard objects are also marked in this column (e.g. see Fig. 6). Whiteboard drawing actions are described in bold-italics to separate them from chat messages. Note that chat postings and whiteboard drawings often interleave each other.

[2]Phrases quoted from chat messages are printed in bold to highlight the terms used by the participants.

[3]There is a parallel conversation unfolding in chat at this moment between the facilitator (Nan) and Jason about an administrative matter. Lines 740, 743, 744, and 745 are omitted from the analysis to keep the focus on the math problem solving.

[4]137's referential work involves multiple objects in this instance. Although the referencing tool of VMT can be used to highlight more than one area on the whiteboard, this possibility was not mentioned during the tutorial and hence was not available to the users. Although the explicit referencing tool of the system seemed to be inadequate to fulfill this complicated referential move, 137 achieves a similar referential display by temporally coordinating his moves across both interaction spaces and by using the plural deictic term "those" to index his recent moves.

Excerpt 1 Constitution of a New Math Task

Chat index	Time start typing	Time of posting	Author	Content	Refers to
742	19:24:39	19:25:48	Qwertyuiop	an idea: Find the number of a certain set of colinear sides (there are 3 sets) and multiply the result by 3	
743	19:25:55	19:26:03	Jason	i did–apparently it didn't work for him	Message No. 740
744	19:26:05	19:26:13	Jason	or his internet could be down, as he's not even on IM right now	
745	19:26:10	19:26:13	Nan	i see. thanks!	Message No. 743
		19:26:23–19:26:33		***137 produces two green lines on the diagonals of the hexagon and two green arrows as displayed in Fig. 3***	
746	19:26:20	19:26:36	137	As in those?	Message No. 742
747	19:26:46	19:27:05	Qwertyuiop	no-in one triangle. I'll draw it . . .	Message No. 746
		19:27:10–19:28:08		***Qwertyuiop repositions some of the existing green lines on a particular section of the hexagon (see Fig. 4 below)***	
748	19:28:09	19:28:10	Qwertyuiop	Those	
		19:28:13–19:28:19		***137 makes the green lines thicker (see Fig. 4 below)***	
749		19:28:28	Qwertyuiop	find those, and then multiply by 3	
750	19:28:48	19:28:50	137	The rows?	
751	19:29:01	19:30:01	Qwertyuiop	The green lines are all colinear. There are 3 identical sets of colinear lines in that triangle. Find the number of sides in one set, then multiply by 3 for all the other sets.	
752	19:30:20	19:30:23	137	Ah. I see.	

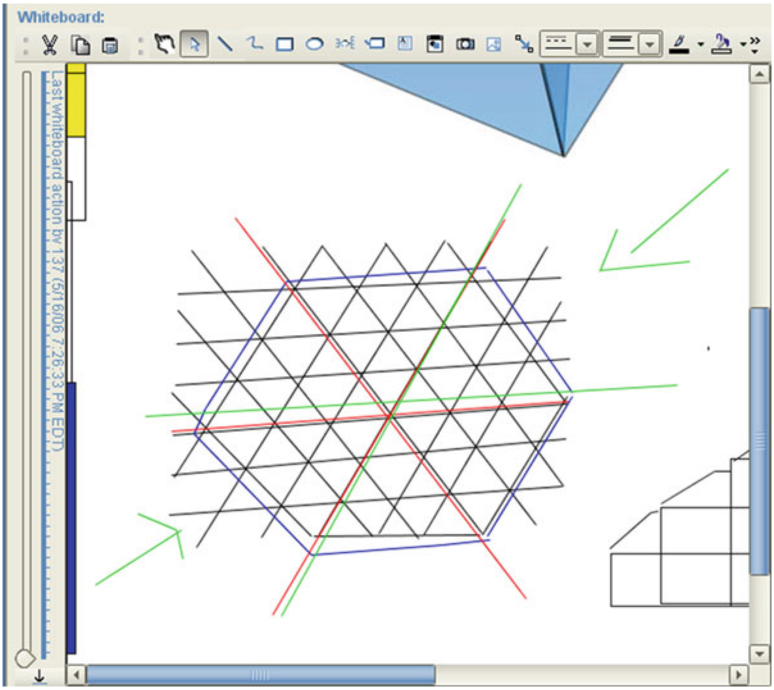


Fig. 3 Two arrows and two green lines produced by 137. One of the lines splits the hexagon into two halves horizontally, whereas the other line crosses the hexagon diagonally into two halves

objects beyond the chat statement itself, possibly located in the other interaction space. The way the drawing actions are embedded as part of the typing activity suggests that they are designed to be seen as part of a single turn or exposition. Hence, the deictic term "**those**" can be read as a reference to the objects pointed to by the recently added green arrows.[5] Moreover, the use of the term "**as**" and the referential link together suggest that these drawings are related to Qwertyuiop's proposal in line 746. Therefore, based on the evidence listed above, 137 proposes a provisional *graphical representation* of what was described in narrative form by Qwertyuiop earlier and calls for an assessment of its adequacy.

In line 747 Qwertyuiop posts a message linked to 137's message with the referential arrow, which indicates that he is responding to 137's recent proposal. The use of "**no**" at the beginning expresses disagreement and the following phrase "**in one triangle**" gives further specificity to where the relevant relationship should be located. The next sentence "**I will draw it** . . . " in the same posting informs other

[5] We have observed that students use "those" (or "that") in chat to reference items already existing in the whiteboard, but "these" (or "this") to reference items that they are about to add to the whiteboard.

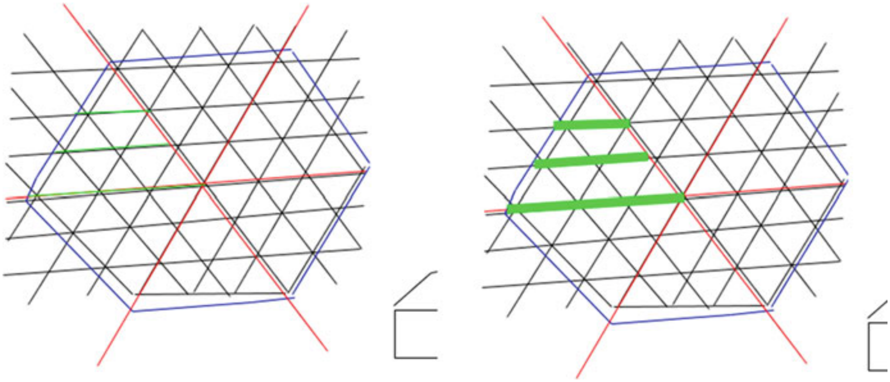


Fig. 4 Qwertyuiop repositions the green lines on the left. Shortly after, 137 increases their thickness

members that he will continue his elaboration on the whiteboard. The use of ellipsis "..." also marks the incomplete status of this posting, which informs others that his subsequent drawings should be seen as related to this thread.

Following this line, Qwertyuiop begins to *reposition* some of the green lines that 137 drew earlier. He forms three green horizontal lines within one of the six triangular partitions (see the snapshot on the left in Fig. 4). Then in line 748, he posts the deictic term "**those**" that can be read as a reference to the recently added lines. Immediately following Qwertyuiop's statement, 137 modifies the recently added lines by increasing their *thickness* (see the snapshot on the right in Fig. 4). These moves make the new lines more visible. In line 749, Qwertyuiop continues his exposition by stating that what has been marked (indexed by "**those**") is what needs to be found and then multiplied by 3.

137's posting "**the rows?**" follows shortly after in line 750. The term "**rows**" has been previously used by this team to describe a method to systematically count the triangles located in one of the six regions of the hexagonal array. By invoking this term here again, 137's posting proposes a relationship between what is highlighted on the drawing and a term the team has previously used to articulate a method of counting. The question mark appended invites others to make an assessment of the inferred relationship.

A minute after 137's question, Qwertyuiop posts a further elaboration. The first sentence states that the lines marked with green on the drawing are "**collinear**" to each other. The way he uses the term collinear here in relation to recently highlighted sticks indicates that this term is a reference to sticks that are aligned with respect to each other along a grid line. The second sentence asserts that there are "**3 identical sets of collinear lines**" (presumably located within the larger triangular partition, since the green lines are carefully placed in such a partition). Finally, the last sentence states that one needs to find the number of "**sides**" (i.e., sticks) in one set and multiply that number by "**3**" (to find the total number of sticks in one partition). Although Qwertyuiop does not explicitly state it here, the way he places the green lines indicates that he is oriented to one of the 6 larger partitions to perform the counting operation he has just described. Following Qwertyuiop's elaboration, 137 posts "Ah. I see." in line 752. This is a token of *cognitive change* (Heritage, 2002), where the person who made the utterance announces that she/he can see something he has not been able to see earlier. Yet, it is still ambiguous what is understood or seen since no display of understanding is produced by the recipients yet.

Excerpt 2: Co-construction of a Method for Counting Sticks

About 18 seconds after 137's last posting, Qwertyuiop begins typing, but he does not post anything in chat for a while. After 10 seconds elapsed since Qwertyuiop started typing, 137 begins to produce a drawing on the whiteboard. In about 10 s, 137 produces a smaller hexagonal shape with orange color on the triangular grid. The new elongated hexagonal shape is placed on the right side of the recently added green lines, possibly to avoid overlap (see Fig. 5). Once the hexagon is completed, 137 posts a chat message in line 753. The message starts with "**wait**"[6] which can be read as an attempt to suspend the ongoing activity. The remaining part of the message states that the aforementioned approach may not work for a case indexed by the deictic term "**that one**". Since 137 has just recently produced an addition to the shared drawing, his message can be read in reference to the orange hexagon. Moreover, since the referred case is part of a message designed to suspend ongoing activity for bringing a potential problem to others' attention, the recently produced drawing seems to be presented as a *counterexample* to the current approach for counting the sticks.

In the next line Jason posts the affirmative token "yes". Since it follows 137's remark sequentially, the affirmation can be read as a response to 137. Jason's following posting provides an account for the agreement by associating "**irregularity**" with an object indexed by the deictic term "**that**". When these two postings are read together in response to 137's message, the deictic term can be interpreted as a reference to the orange hexagon. In short, Jason seems to be stating that the strategy under consideration would not work for the *orange hexagon* because it is "**irregular**". In the meantime, 137 is still typing the statement that will appear in line 756, which asks whether the hexagon under consideration is still assumed to be regular. This question mitigates the prior problematization offered by the same author since it leaves the possibility that the proposed strategy by Qwertyuiop may still work for the regular case.

[6]The token "wait" is used frequently in math problem-solving chats to suspend ongoing activity of the group and solicit attention to something problematic for the participant who uttered it. This token may be used as a preface to request explanation (e.g., wait a minute, I am not following, catch me up) or to critique a result or an approach as exemplified in this excerpt.

Excerpt 2 Co-construction of a method for counting sticks

Chat index	Time start typing	Time of posting	Author	Content	Refers to
752	19:30:20	19:30:23	137	Ah. I see.	
		19:30:48–19:30:58		***137 drew an elongated hexagon in orange***	
753	19:31:00	19:31:07	137	Wait. Wouldn't that not work for that one?	
754	19:31:11	19:31:12	Jason	Yeah	
755	19:31:12	19:31:15	Jason	beacuse that's irregular	
756	19:31:09	19:31:17	137	Or are we still only talking regular ones?	
757	19:31:20	19:31:22	137	About	
758	19:30:38	19:31:24	Qwertyuiop	side length 1 = 1, side length 2 = 3, side length 3 = 6 . . .	
		19:31:45–19:32:15		***137 removes the orange hexagon***	
759	19:32:32	19:32:50	137	Shouldn't side length 2 be fore?	Message No. 758
760	19:32:52	19:32:53	137	*four	
761	19:33:06	19:33:10	Qwertyuiop	I count 3.	Message No. 759
762	19:33:20	19:33:25	137	Oh. Sry.	
763	19:33:24	19:33:30	Qwertyuiop	It's this triangle.	Reference to whiteboard (see Fig. 6)
764	19:33:44	19:33:45	137	We	
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet . . .	Message No. 758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10–19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position (see Fig. 7)***	
767	19:34:11	19:34:29	Qwertyuiop	no, 3 is only for side length 2.	Message No. 766

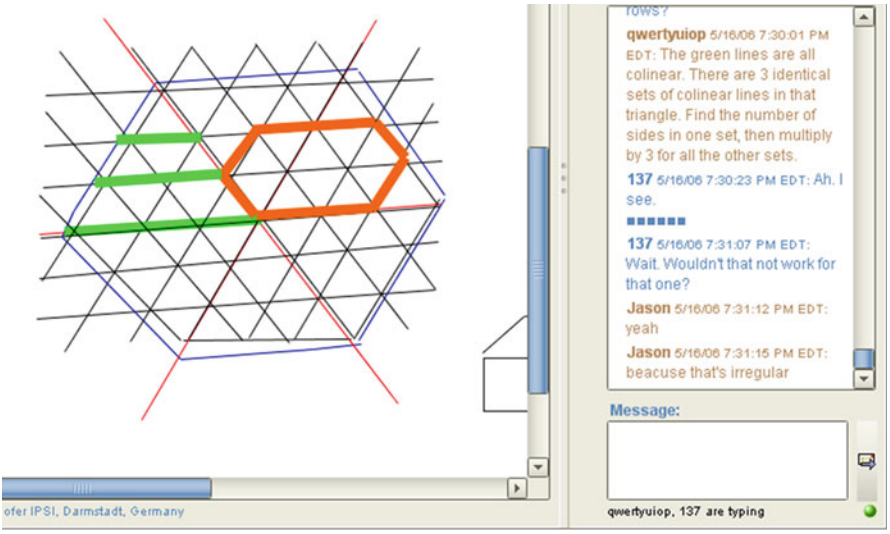


Fig. 5 137 adds an elongated hexagon in orange

In line 758, Qwertyuiop posts a chat message stating "**side length 1 = 1, side length 2 = 3, side length 3 = 6** . . . " It took about a minute for him to compose this message after he was first seen as typing at 19:30:38. The way the commas are used to separate the contents of the statement and the ellipsis placed at the end indicate that this posting should be read as an open-ended, ordered list. Within each list item the term "**side length**" is repeated. "Side length" was used by this team during a prior session as a way to refer to different stages of a growing stick-pattern. In the hexagonal case the pattern has 6 sides at its boundary and counting by side-length means figuring out how many sticks would be needed to construct a given side as the pattern grows step by step. Note that this method of indexing stages assumes a stick-pattern that grows symmetrically. So a side length equal to 1, 2 or 3 corresponds to the first, second or third stage of the hexagonal stick pattern, respectively. When the statement is read in isolation, it is not clear what the numbers on the right of the equals sign may mean, yet when this posting is read together with Qwertyuiop's previous posting where he described what needs to be found, these numbers seem to index the number of sticks within a set of collinear lines as the hexagonal array grows.

After Qwertyuiop's message, 137 removes the orange lines he drew earlier to produce an irregular hexagon. By erasing the irregular hexagon example, 137 seems to be taking Qwertyuiop's recent posting as a response to his earlier question posted in line 756, where he asked whether they were still considering regular hexagons or not. Although Qwertyuiop did not explicitly respond to this question, his message in line 758 (especially his use of the term "side length" which implicitly assumes such a regularity) seems to be seen as a continuation of the line of reasoning presented in his earlier postings. In other words, Qwertyuiop's sustained orientation to the symmetric case is taken as a response to the critique raised by 137.

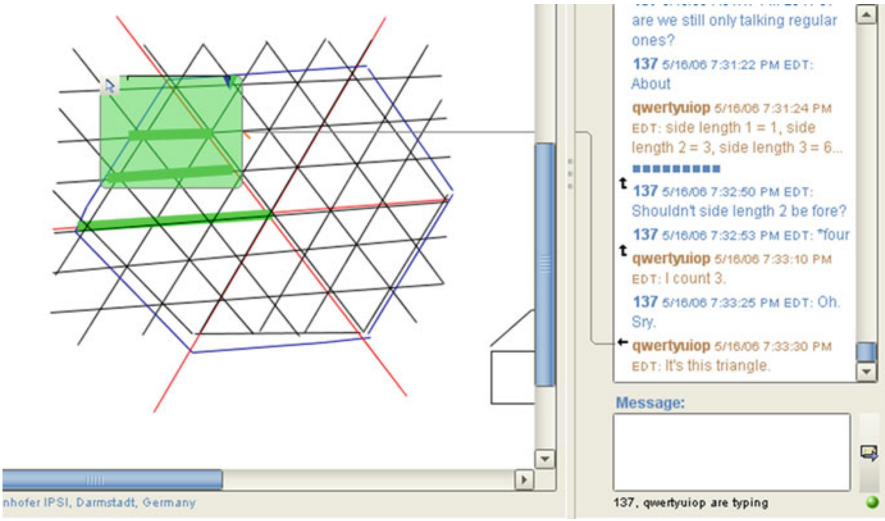


Fig. 6 Qwertyuiop points to the triangle which contains the sticks to be counted for the stage indexed by side length = 2. The *green lines* enclosed by the reference correspond to 1 + 2 = 3 sticks

In line 759, 137 posts a message explicitly linked to Qwertyuiop's most recent posting. It begins with the negative token "**Shouldn't**", which expresses disagreement. The subsequent "**side length 2**" indexes the problematic item and "**be fore**" offers a repair for that item. Moreover, the posting is phrased as a question to solicit a response from the intended recipient. 137's next posting in line 760 repairs his own statement with a repair notation peculiar to online chat environments. The asterisk at the beginning instructs readers to attend to the posting as a correction (usually to the most recent posting of the same author). In this case, due to its syntactic similarity to the word in the repair statement, "**fore**" seems to be the token that is supposed to be read as "**four**."

In his reply in line 761, Qwertyuiop insists that his counting yields "**three**" for the problematized case. In the next posting 137's "oh" marks the previous response as surprising or unexpected. The subsequent "**sry**"—short for "sorry"—can be read as backing down. In line 763, Qwertyuiop posts a message that states "**it's this triangle**" and explicitly points at a region on the shared drawing with the referencing tool. The explicit reference and the deictic terms again require the interlocutors to attend to something beyond the text involved in the posting. In short, the sequential unfolding of the recent postings suggests that this posting is designed to bring the relevant triangle in which the counting operation is done for the problematic case (indexed by side length 2) to other members' attention (see Fig. 6).

In line 765, Qwertyuiop posts another message explicitly pointing to his earlier proposal for the first few values he obtained through his method of counting, where he states that he has not been able to "**see a pattern yet**." Hence, this

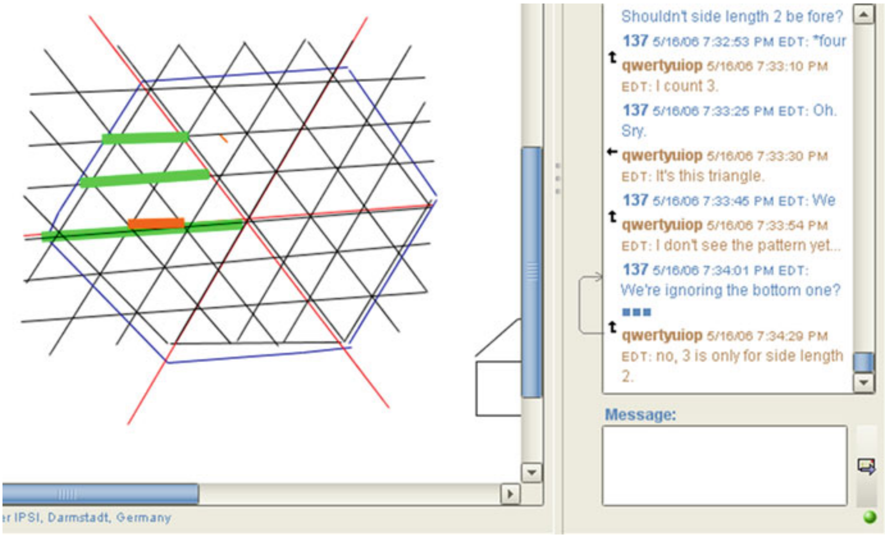


Fig. 7 137 adds a single stick in orange on the left of the line that horizontally splits the large hexagon

statement explicitly specifies "**the pattern**" as what is missing or needed in this circumstance. The message not only brings a *prospective indexical*[7] (Goodwin, 1996), "**the pattern,**" into the ongoing discussion as a problem-solving objective, but also invites other members of the team to join the search for that pattern.

In the next line, 137 posts a question that brings other members' attention to something potentially ignored so far. The term "**bottom one**" when used with "**ignore**" indexes something excluded or left out. Nine seconds after his posting, 137 performs some drawing work on the whiteboard. He moves the longest green line to the right first, then he adds a short line segment with orange color, and then he moves the same green line back to its original location (see Fig. 7). These moves make 137's orientation to a particular part of the drawing explicit. When read together with his previous question, the orange line could be seen as a graphical illustration for the left-out part previously referred as the "bottom one". When read as a response to Qwertyuiop's recent exposition in lines 761 and 763, the "bottom one" seems to be a reference to the part of the drawing that was not enclosed by Qwertyuiop's explicit reference.

The next posting by Qwertyuiop, which appears in line 767, is explicitly linked to 137's question in the previous line. The message begins with "no" which marks

[7]Goodwin (1996) proposes the term prospective indexicals for those terms whose sense is not yet available to the participants when it is uttered, but will be discovered subsequently as the interaction unfolds. Recipients need to attend to the subsequent events to see what constitutes a "pattern" in this circumstance.

the author's disagreement with the linked content, and the subsequent part of the message provides an account for the disagreement by stating that the value 3 is only relevant to the case indexed by "**side length 2**".

The sequence of exchanges between 137 and Qwertyuiop in this excerpt indicates that there is a misalignment within the group about the procedure used for counting the number of sticks. This misalignment is made evident through explicit problematizations and disagreements. The way the members make use of both spaces as they interact with each other makes it increasingly clear for them (a) where the relevant pieces indexed by the terms like "collinear" and "triangle" are located, and (b) how they are used in the counting process. Nevertheless, the misalignment between the counting procedures suggested in 137's and Qwertyuiop's contributions have not been resolved yet.

Excerpt 3: Collective Noticing of a Pattern of Growth

In line 768, 137 posts a message linked to Qwertyuiop's posting in line 765. The preface "And" and the explicit reference together differentiate this contribution from the ongoing discussion about a piece that was potentially excluded from the second stage. Note that Qwertyuiop's message in line 765 refers further back to an older posting where he proposed a sequence of numbers for the first three stages "side length $1 = 1$, side length $2 = 3$, side length $3 = 6\ldots$" When 137's message is read in relation to these two prior messages, the phrase "**they are all**" seems to be a reference to this sequence of numbers. Therefore, the message can be read as an uptake of the issue of finding a pattern that fits this sequence. Moreover, by proposing the term "**triangular numbers**" as a possible characterization for the sequence, 137 offers further specificity to the prospective indexical, the "**pattern**", which was initially brought up by Qwertyuiop.

Following his proposal, 137 changes the color of the longest green line segment at the bottom to red and then to green again. In the meantime Qwertyuiop is typing what will appear in line 769, which can be read as a question soliciting further elaboration of the newly contributed term "triangular numbers." 137 continues to act on the whiteboard and he adds a red hexagon to the shared drawing (see Fig. 8). Since the hexagon is located on the section referenced by Qwertyuiop several times earlier and shares an edge with the recently problematized orange section, this drawing action can be treated as a move related to the discussion of the ignored piece.

Jason joins the discussion thread about triangular numbers by offering a list of numbers in line 770. The term "**like**" is used here again to relate a mathematical term to what it may be indexing. This posting alone can be read as an assertion, but the question mark Jason posts immediately after in the next line mitigates it to a statement soliciting others' assessment. At roughly the same time, 137 posts a substantially longer sequence of numbers, and immediately after Qwertyuiop points out the difference between 137's sequence and what Jason offered as a list of

Excerpt 3 Collective noticing of a pattern of growth

Chat index	Time start typing	Time of posting	Author	Content	Refers to
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet . . .	Message No. 758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10-19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position***	
767	19:34:11	19:34:29	Qwertyuiop	No, 3 is only for side length 2.	Message No. 766
768	19:34:36	19:34:52	137	And I think the'y;re all triangular numbers.	Message No. 765
		19:35:03–19:35:16		***137's changes the color of the longest green line to red, and then to green again***	
769	19:35:06	19:35:17	Qwertyuiop	"triangular numbers"?	Message No. 768
		19:35:27–19:35:36		***137's draws a red hexagon on the diagram (Fig. 8)***	
770	19:35:28	19:35:37	Jason	You mean like 1, 3, 7, . . .	
771	19:35:39	19:35:39	Jason	?	
772	19:35:48	19:35:59	137	Like 1,3,6,10,15,21,28.	Message No. 770
773	19:35:51	19:36:02	Qwertyuiop	The sequence is 1, 3, 6 . . .	Message No. 770
774	19:36:02	19:36:30	137	Numbers that can be expressed as n(n+1)/2, where n is an integer.	
775	19:36:44	19:36:45	Qwertyuiop	Ah	
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message No. 766

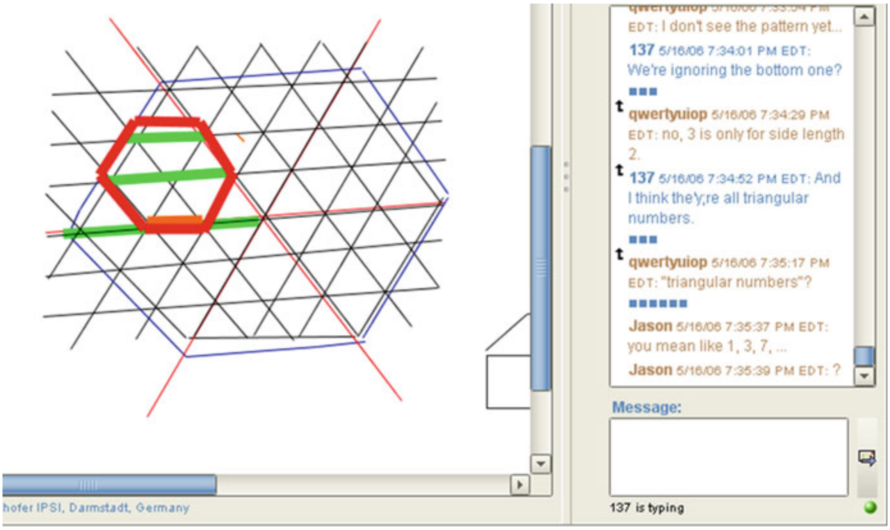


Fig. 8 137 adds a *red hexagon* inside the partition the team has been oriented to

triangular numbers. In line 774, 137 elaborates his definition further by offering an algebraic characterization of triangular numbers as integers that can be expressed with the formula "**n(n + 1)/2**".

In short, the sequence resulting from Qwertyuiop's counting work based on his notion of "**collinearity**" has led the team to notice a relationship between that sequence and a mathematical object called "**triangular numbers**". The latter symbolic definition offered by 137 for triangular numbers in response to the ongoing search for a pattern has established a relationship between geometrically motivated counting work and an algebraic/symbolic representation stated in generic form as **n(n + 1)/2**.

Excerpt 4: Resolution of Referential Ambiguity via Visual Proof

In line 776, 137 posts a message which is explicitly linked to his prior message in line 766 where he mentioned a potentially ignored piece indexed by the phrase "**the bottom one**". The use of "**So**" at the beginning can be read as an attempt to differentiate this message from the recently unfolding discussion about triangular numbers. The subsequent part of the message brings other team members' attention to a potentially ignored piece indexed by the phrase "**the bottom orange line**". 137 used the phrase "**the bottom one**" earlier, but this time he makes use of *color referencing* as an additional resource to provide further specificity to what he is referencing. At this moment a *red hexagon* and a *short orange segment* are visible on the shared drawing space, which are layered on top of the triangular grid (see

Excerpt 4 Resolution of referential ambiguity via visual proof

Chat index	Time start typing	Time of posting	Author	Content	Refers to
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message No. 766
777	19:37:32	19:37:36	Qwertyuiop	"green"?	Message No. 776
778	19:37:44	19:37:48	137	THe short orange segment.	
		19:37:59–19:38:02		***137 changes the color of the green lines enclosed by the red hexagon to blue (see Fig. 9)***	
779	19:37:49	19:38:05	137	PArallel to the blue lines.	
780	19:37:58	19:38:05	Qwertyuiop	I don't think so . . .	
781	19:38:20	19:38:26	137	Wait, we are counting sticks right now, right?	Message No. 780
782	19:38:35	19:38:48	Qwertyuiop	yes-one of the colinear ets of sticks	
783	19:38:55	19:39:08	Qwertyuiop	oops-"sets" not "ets"	
784	19:39:22	19:39:42	137	So we are trying to find the total number of sticks in a given regular hexagon?	Message No. 782
785	19:39:50	19:40:18	Qwertyuiop	not yet-we are finding one of the three sets, then multiplying by 3	Message No. 784
786	19:40:25	19:40:40	Qwertyuiop	that will give the number in the whol triangle	
787	19:40:34	19:40:51	137	Then shouldn't we also count the bottom line?	Message No. 785
788	19:40:52	19:41:01	Jason	are you taking into account the fact that some of the sticks will overlap	Message No. 786
789	19:41:25	19:41:41	137	Then number of sticks needed for the hexagon, right?	Message No. 786
790	19:41:16	19:42:22	Qwertyuiop	Yes. The blue and green/orange lines make up on of the three colinear sets of sides in the triangle. Each set is identical and doesn't overlap with the other sets.	Message No. 788
791	19:42:50	19:42:50	Jason	Ok	
792	19:43:03	19:43:11	Jason	this would be true for hexagons of any size right>	

(continued)

Excerpt 4 (continued)

Chat index	Time start typing	Time of posting	Author	Content	Refers to
793	19:43:09	19:43:13	Qwertyuiop	triangle, so far	Message No. 789
794	19:43:25	19:43:25	137	Oh.	
795	19:43:25	19:43:26	Qwertyuiop	this one	Reference to whiteboard (see Fig. 10)
796	19:43:42	19:43:52	137	Yes, but they will overlap . . .	
797	19:43:59	19:44:13	137	Eventually when you multiply by 6 to get it for the whole figure.	
798	19:44:01	19:44:30	Qwertyuiop	no, the sets are not collinear with eachother. I'll draw it . . .	Message No. 796
		19:44:35–19:44:56		***Qwertyuiop moves the small hexagon in red and blue lines out of the grid (see Fig. 11)***	
799		19:44:59	137		Message No. 798
		19:44:59-19:45:17		***Qwertyuiop repositions and resizes the red lines on the grid***	
		19:45:20		***Qwertyuiop continues adjusting the red lines***	
		19:45:23–19:45:37		***Qwertyuiop continues adjusting the red lines***	
		19:45:41–19:46:16		***Qwertyuiop adds purple lines (see Fig. 12)***	
800	19:46:22	19:46:34	137	Oh. I see.	
801	19:46:22	19:46:52	Qwertyuiop	Those are the 3 sets. One is red, one is green, one is purple.	
		19:47:07–19:47:11	137	***137 starts to make green lines thicker***	
802	19:47:04	19:47:12	Jason	wait— i don't see the green/purple ones	
		19:47:17–19:47:33	137	***137 makes the purple lines thicker (see Fig. 13 below)***	
803	19:47:18	19:47:40	Qwertyuiop	so we find a function for that sequence and multiply by 3	Message No. 774

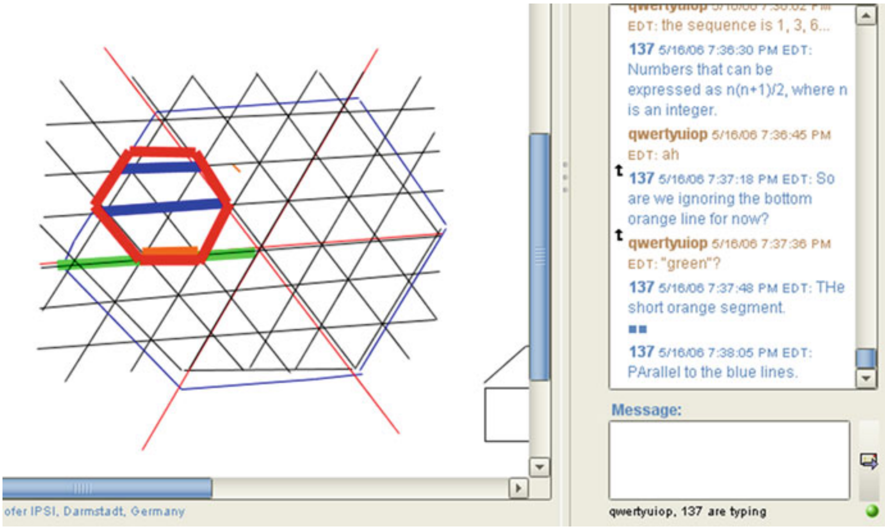


Fig. 9 137 changes the color of the *green lines* inside the *red hexagon* to *blue*

Fig. 8). The way 137 orients to the new state of the drawing indicates that his earlier drawing actions (marked in the prior excerpt before line 770) seem to be performed in preparation for this posting. Hence, this posting can be read as an attempt to *re-initiate a prior thread* about a potentially ignored piece in the counting work, which is distributed over both interaction spaces.

Qwertyuiop's message in the next line involves "**green**" in quotes, ends with a question mark, and is explicitly linked to 137's last message in line 776. The quotation marks seem to give significance to an object indexed by the color reference. Note that there are three green lines on the shared drawing at the moment (see Fig. 8). The use of the color reference and the explicit link suggest that this message is posted in response to 137's question in line 776. When it is read in this way, Qwertyuiop seems to be asking if the relevant line located at the bottom should have been the green one instead.

Following Qwertyuiop's posting, 137 provides further specificity to the problematized object by first stating that it is "**the short orange segment**" in line 778. Next, 137 modifies the two green lines inside the red hexagon by changing their color to blue (see Fig. 9). Then, he posts another message in line 779 that refers to a particular location on the whiteboard that is "**parallel**" to the recently added "**blue lines**". Thus, 137's recent actions suggest that the object indexed by his phrase, "**short bottom orange line**" segment, is the one parallel to the blue lines.

In line 780, Qwertyuiop states his disagreement. Since the message appears shortly after 137's point that the orange segment is left out of the computation, Qwertyuiop seems to be disagreeing with the remark that there is a missing piece in the counting method. In the next line, 137 posts a question prefaced with "**wait**" that calls for suspending the ongoing activity and asks if one can still characterize what the team ("**we**") is currently doing as "**counting the sticks**". The posting is explicitly linked to Qwertyuiop's last message. By posting a question about the ongoing group process following a sustained disagreement with his peer, 137 is making it explicit that there is a misalignment within the team with respect to the task at hand. Hence, this exchange marks a breakdown in interaction that needs to be attended to before the team can proceed any further.

In the next line, Qwertyuiop takes up this question by providing his account of the ongoing process as counting "**one of the collinear sets of sticks**." Next, 137 posts another question explicitly linked to Qwertyuiop's answer, which gives further specificity to 137's earlier characterization of the counting work undertaken by the team (i.e., counting the sticks for the "**whole hexagon**"). Qwertyuiop's response to this question states that the focus is not on the whole hexagon yet, but on what he is referring to as "**one of the three sets**", which would then be followed by a multiplication by 3. In the next line Qwertyuiop continues his explanation that this will give them the number of sticks for "**the whole triangle**", which can be read as a reference to one of the six triangular partitions that altogether form the hexagon.

In line 787, 137 posts a message explicitly linked to the first part of Qwertyuiop's explanation. The posting is phrased as a question problematizing again that the bottom line should also be included in the counting operation described by Qwertyuiop. Next, Jason joins the discussion by posting a question linked to the latter half of Qwertyuiop's explanation in line 786, which asks him if he has taken into account "**the fact that some of the sticks will overlap**". The way Jason phrases his posting brings "**overlap**" as an issue that needs to be addressed by the counting method under discussion.

In line 789, 137 posts a chat message with a referential link to Qwertyuiop's last posting in line 786. This message seems to extend the order of computations described in Qwertyuiop's exposition by anticipating the next step of the computation, namely calculating the number of sticks needed for the hexagon once the step mentioned in 786 is achieved. In other words, 137 displays that he is able to follow the order of computations suggested by his peer to address the task at hand.

In line 788 Qwertyuiop responds to the overlapping sticks issue raised by Jason. He makes reference to the blue and green/orange lines to describe one of the three collinear sets of sides within the triangular partition (since the shared image has remained unchanged, this message can be read in reference to the state displayed in Fig. 9). He further asserts that each set is identical and does not overlap. In the next line Jason concurs, and then asks if this should hold for hexagons of any size.

Following Jason's messages, Qwertyuiop posts a message linked to 137's earlier question in line 789. Qwertyuiop stresses again that the focus has been on the "**triangle**" so far. His next posting in line 795 includes a referential arrow to the shared diagram and a deictic term "**this one**" that together provide further specificity to which part of the hexagon he was referring to with the indexical term "**triangle**" (see Fig. 10).

In lines 796 and 797, 137 first accepts what Qwertyuiop has asserted, but points to a potential issue that will be faced when the result will be multiplied by 6 to extend the counting operation to the whole hexagon. Before 137 posts his elaboration in line

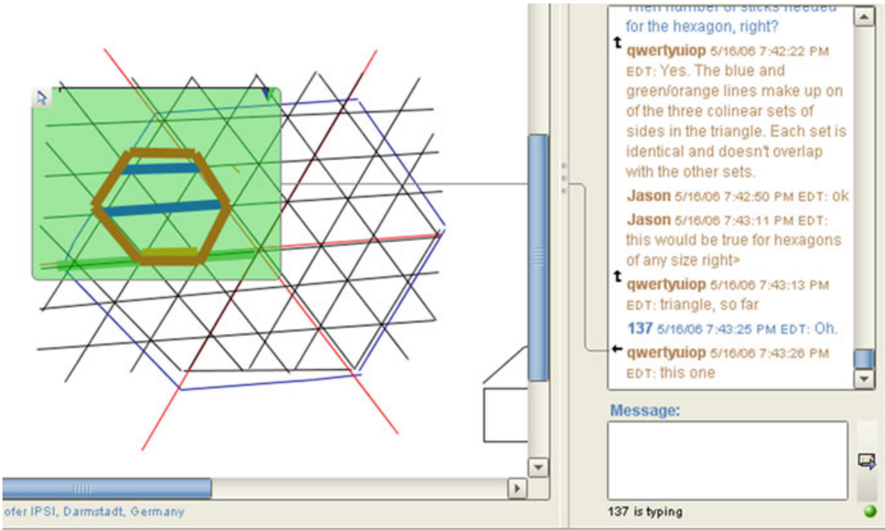


Fig. 10 Qwertyuiop highlights the triangle by using the referencing tool

797, Qwertyuiop begins typing a response to 137's first remark that appears in line 798. In that message Qwertyuiop expresses his disagreement and asserts that "**the sets are not collinear with each other**". Hence, this posting shows that Qwertyuiop has treated 137's use of the pronoun "**they**" in line 796 as a reference to the notion of collinear sets. In the latter part of his posting, Qwertyuiop announces that he will draw what he is talking about, so this section of the message projects that a related drawing action will follow his statement shortly.

Figures 11 and 12 display snapshots from Qwertyuiop's drawing actions following his last posting. First he moves the red and orange lines to the side, and then he repositions the red lines to highlight three segments that are parallel to each other. Next, he adds two green lines parallel to the remaining green line. Finally, he adds three purple lines to cover the remaining sticks in that triangular section. The green and purple lines are drawn with a thin brush (see Fig. 12).

Once the drawing reaches the stage in Fig. 12, 137 posts "**oh I see**" in line 800, which can be read in response to Qwertyuiop's recent drawing work. Qwertyuiop's graphical illustration seemed to have helped 137 to notice something he had not been able to see earlier. Next, Qwertyuiop posts a message that refers to the lines he has recently drawn with the plural deictic term "**those**". The message provides further specificity to the mathematics object "**3 sets**" by locating each set on the diagram through the use of color references "**red**", "**green**" and "**purple**". In other words, Qwertyuiop has provided a visual realization of the phrase "**3 sets of collinear sides**" he coined earlier, which has been treated as problematic by his teammates.

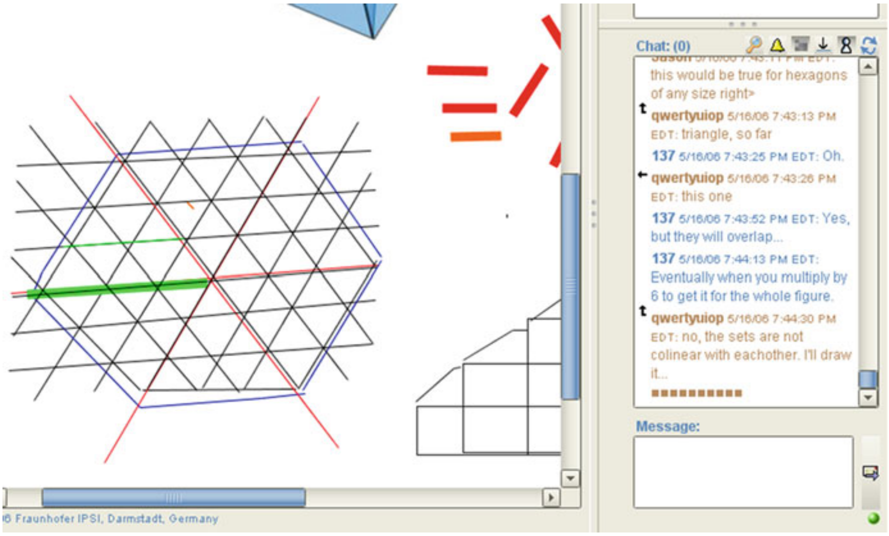


Fig. 11 Qwertyuiop moves the lines added by 137 away

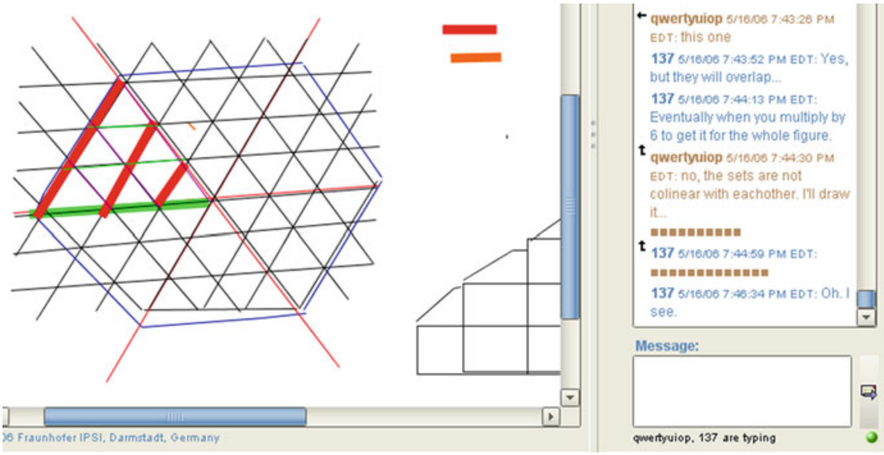


Fig. 12 Qwertyuiop repositions the *red lines* to mark a part of the larger triangle. Then he adds two horizontal lines in *green*, parallel to the existing *green line*. Finally, he adds three more lines in *purple*. Since Qwertyuiop uses a thinner brush to draw the *green* and *purple lines*, they are difficult to see

In line 802, Jason states that he cannot see the green/purple lines, which were marked with a thin brush by Qwertyuiop. In response 137 makes these new additions more visible by increasing their thickness (see Fig. 13). The final state of the diagram presents a *visual proof* that three sets of collinear lines marked with green, purple, and red do not indeed overlap with each other.

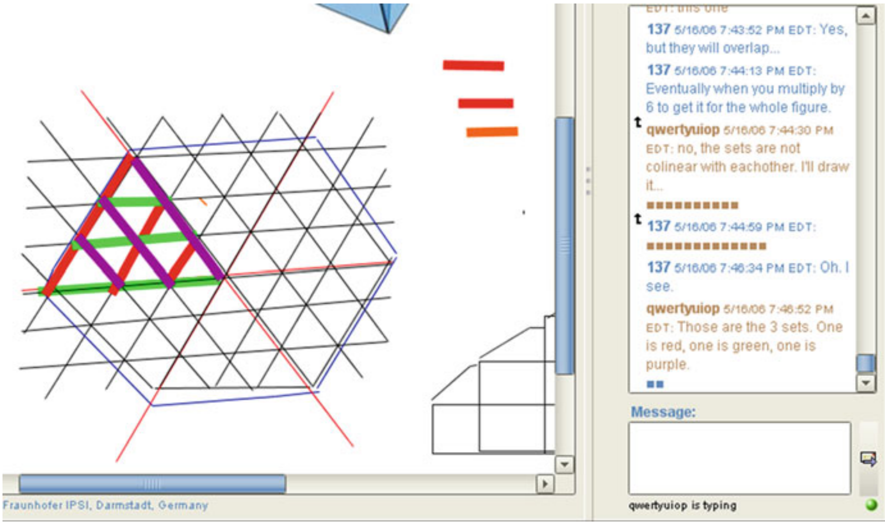


Fig. 13 137 increases the thickness of the newly added *green* and *purple lines*. The final state of the diagram presents a visual proof that three sets of collinear lines do not overlap with each other

In line 803, Qwertyuiop provides further specificity as to what needs to be found given the visual realization of the collinear sides recently produced on the whiteboard. His message is explicitly linked to an old message posted by 137 several lines ago (line 774 in Excerpt 3) that provides a formulaic realization for triangular numbers previously associated with the pattern of growth of collinear sides. Hence, Qwertyuiop's statement, "**find a function for that sequence and multiply by 3**", can be read as a proposal for a strategy to find the number of sticks required to build a triangular partition. In particular, Qwertyuiop is pointing (narratively) to a candidate (symbolic) algebraic realization of what he has just demonstrated with (graphical) visual resources on the whiteboard. This is the culmination of a subtle and complex collaborative process in which mathematical discourse, graphical reasoning and symbolic expression were tightly integrated by the group.

To sum up, in this episode the team has achieved a sense of *common ground* (Clark & Brennan, 1991), *intersubjectivity* (Stahl et al., 2011) or *indexical symmetry*[8] (Hanks, 1992, 2000) with respect to the term "**set of collinear sides**" and its projected application towards solving the task at hand. The challenges voiced by

[8]Hanks proposes the notion of indexical symmetry to characterize the degree to which the interactants share, or fail to share, a common framework relative to some field of interaction on which reference can be made. In particular, "...the more interactants share, the more congruent, reciprocal and transposable their perspectives, the more symmetric is the interactive field. The greater the differences that divide them, the more asymmetric the field." (Hanks, 2000, p. 8.). These excerpts show that mathematical terms are inherently indexical. Establishing a shared understanding of such indexical terms require collaborators to establish a reciprocity of perspectives towards the reasoning practices displayed/embodied in the organization of the texts and inscriptions in the shared scene (Zemel & Çakır, 2009).

137 and Jason through the course of the episode solicited further elaboration from Qwertyuiop regarding how **collinear sides** can be located in the shared diagram and how they can be used to devise a method to count the number of sticks. In particular, in this excerpt the team members worked out the overall organization of their joint problem-solving work by discussing what they are trying to find, how they should locate the objects relevant to the task, and how they should order some of the steps that have been proposed so far to arrive at a solution. For instance, Qwertyuiop's initial proposal including the indexical term "**collinear sets**" focuses on one of the triangular regions. Yet, the focus on a triangular region was left implicit, which seemed to have led 137 to treat Qwertyuiop's proposal as applied to the whole hexagon. Through their discussion across both interaction spaces the team has incrementally achieved a *shared understanding* in terms of how a triangular region is decomposed into three sets of collinear, non-overlapping sides, and how that can be used to systematically count the number of sticks in that region. The visual practices have been encapsulated in linguistic terms in ways that become shared within the small group through their interactions, which integrate graphical and narrative actions. The graphical moves are strategically motivated to decompose a complicated pattern into visually obvious sub-patterns, with an eye to subsequently constructing a symbolic representation of the pattern. The elaboration of a mathematical vocabulary allows the group to reference the elements of their analysis in order to establish a shared view of the graphical constructions, to make proposals about the patterns to each other and to index past established results.

Concluding the Mathematical Analysis

Excerpt 5: Re-initiating the Discussion of the Algebraic Formula

The group is now ready to return to the symbolic work. In line 818,[9] Qwertyuiop resumes the discussion about the shared task by proposing a formula "**f(n) = 2n − 1**" where he declares n to be the "**side length**" (see Excerpt 5). It is not evident from the text itself what the formula is standing for. Yet, the message is explicitly linked to an older posting (line 772) where 137 posted the statement "**Like 1,3,6,10,15,21,28**" as part of a prior discussion on triangular numbers (see Excerpt 3). Hence, when this message is read in reference to line 772, it can be treated as a proposal to generalize the values derived from Qwertyuiop's geometrically informed counting method with a formula stated in symbolic form.

137 rejects Qwertyuiop's proposal in line 819 and then makes a counter proposal in the next line. As we saw in Excerpt 3, the sequence of numbers resulting from Qwertyuiop's counting method was previously associated with a math artifact called triangular numbers by 137. The counter proposal includes the same expression 137

[9]A brief administrative episode including the facilitator took place between Excerpts 4 and 5, which is omitted in an effort to keep the focus of our analysis on problem solving.

Excerpt 5 Re-initiating the discussion of the algebraic formula

Chat index	Time start typing	Time of posting	Author	Content	Refers to
818	19:51:11	19:52:19	qwertyuiop	what about: f(n)=2n − 1 where n is side length	Message No. 772
		19:52:28	137	***137 changes the layout of the last straight line by making it a dashed line.***	
819	19:52:55	19:53:03	137	I don't think that works.	Message No. 818
820	19:53:07	19:53:18	137	Howbout just n(n+1)/2	
821	19:53:37	19:53:41	Jason	for # sticks?	
822	19:53:38	19:53:48	qwertyuiop	that's number of sides for one set	Message No. 820
823	19:53:50	19:53:51	qwertyuiop	?	
824	19:53:57	19:53:59	Jason	oh ok nvm	
825	19:54:26	19:54:29	137	Ya.	Message No. 822
826	19:54:36	19:54:58	qwertyuiop	then x3 is 3(n(n+1)/2)	Message No. 820
827	19:55:04	19:55:07	qwertyuiop	simplified to . . .	Message No. 826
828	19:55:11	19:55:37	qwertyuiop	(n(n+1)1.5	
829	19:55:34	19:55:44	137	On second thought, shouldn't we use n(n-1) for these:	Message No. 826
		19:55:50–19:55:55	137	***137 changes the color of two dashed lines into orange (see Fig. 13 below)***	
830	19:55:31	19:55:55	Nan	just a kind reminder: Jason mentioned that he needs to leave at 7p central time sharp	

provided earlier when he gave a definition of triangular numbers as "**integers that can be represented as n(n + 1)/2**" (see line 774). Jason joins the discussion in line 821 by asking if the proposed formula is for the number ("**#**") of sticks. Although Jason does not specify which object (e.g., the whole hexagon) he is associating the formula with, his posting can be read as an attempt to solicit further elaboration with regards to what the recently proposed formulas are about.

Qwertyuiop's posting in the next line states that the object indexed by the deictic term "that" corresponds to the "number of sides for one set". Note that Qwertyuiop's message is explicitly linked to 137's counterproposal in line 820, so the deictic term "**that**" can be read as a reference to the expression "**n(n + 1)/2**" included in 137's posting. Moreover, the message sequentially follows Jason's question. Hence, Qwertyuiop seems to be responding to Jason's query by pointing out which object the recently proposed formulas are about. The question mark Qwertyuiop posts in the next line mitigates his previous statement into a question. This can be read as a move to solicit the remaining member's (i.e. 137) assessment of the association Qwertyuiop has just offered. By making his reading of 137's formula explicit, Qwertyuiop also indicates that he concurs with the alternative expression proposed by his peer. Jason's next posting in line 824 indicates that he is now following his peers' reasoning, which comes just before 137's confirmation linked to Qwertyuiop's claim in 822. Therefore, at this point it seems to be evident for all members in the group that the algebraic expression n(n + 1)/2 is associated with one of the "collinear sets of sticks" within a triangular section.

In line 826, Qwertyuiop posts a message linked back to 137's proposal in 820. The use of "**then**" at the beginning suggests that this message is a consequence or follow up of the message he is referring to. "**x3**" can be read as a reference to multiplication by 3, where the remaining part of the message provides the expression yielded by this operation. In other words, Qwertyuiop seems to be proposing the next step in the computation, given the expression for the number of sticks for a single "**set**". In the next two lines he further simplifies this expression by evaluating 3/2 to 1.5.

In line 829, 137 posts a message phrased as a question. The posting begins with "on second thought" which indicates that the author is about to change a position he took prior with respect to the matter at hand. The rest of the statement is phrased as a question and it is addressed to the whole team as indicated by the use of the first person plural pronoun "we". The question part associates the expression "**n(n − 1)**" with the deictic term "**these**" which is yet to be specified.[10] The posting ends with "**:**" which projects that more content will likely follow this message subsequently. Next, 137 begins to act on the whiteboard by changing the color of two horizontal lines from green to orange (see Fig. 14). The temporal unfolding of these actions suggests that the sticks highlighted in orange are somehow associated with the

[10]See footnote to line 746 on the use of "these" and "those". The consistency of the usage of these terms for forward and backward references from the narrative chat to the graphical whiteboard suggests an established syntax of the relationships bridging those interaction spaces within the temporal structure of the multi-modal discourse.

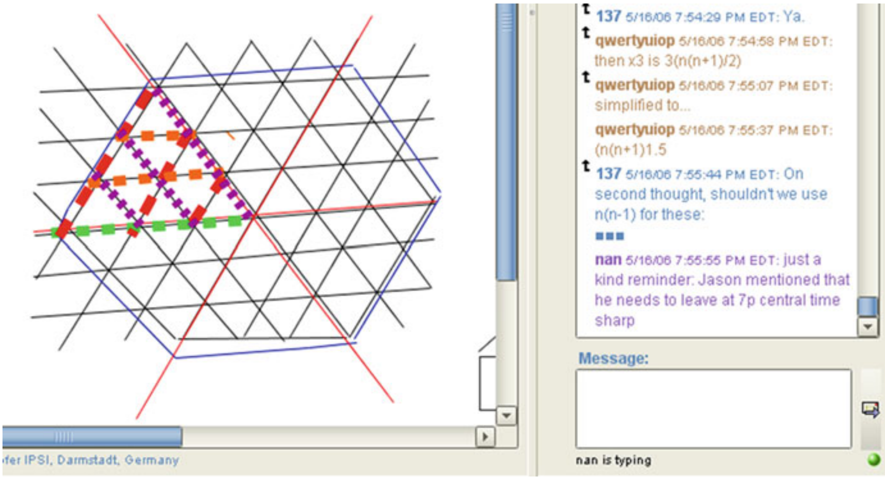


Fig. 14 137 highlights two horizontal lines in *orange* following his proposal at 7:55:44 (line 829)

expression $n(n-1)$. In other words, 137's recent actions can be seen as a move for adjusting the index values in the generalized formula.

In this episode, the team achieves an important transition from a geometrically motivated counting procedure applied on "**one of the collinear sets**" to a symbolic formula generalizing the procedure to a set of any given sidelength. The generality is achieved through one member's noticing that the sequence of numbers derived from the counting procedure corresponds to "**triangular numbers**", which seems to be a familiar concept at least for the member who proposed it. The formula that was provided as part of the definition of triangular numbers is then applied to the relevant portion of the pattern at hand to achieve the transition from geometric to algebraic mode of reasoning, mediated by the narrative concept of "triangular numbers".

Excerpt 6: Co-reflection on What the Team has Achieved So Far

At the end of Excerpt 5 an administrative discussion was initiated by the facilitator about Jason's departure from the chat session.[11] Some of this exchange is left out since it involved a brief chat about the schedule of the next session. However, while Jason was saying farewell to his peers, an exchange related to the task at hand occurred which is captured in Excerpt 6. This episode begins with Qwertyuiop's attempt to reinitiate the problem-solving work by making a reference to an older message posted in line 829 by 137. Following 137's acknowledgement in line 842,

[11]The session was scheduled to end at 7 p.m., yet the students were allowed to continue if they wished to do so. In this case Jason informed the facilitators in advance that he had to leave at 7 p.m. Central (the log is displayed in US Eastern time).

Excerpt 6 Co-reflection on what the team has achieved so far

Chat index	Time start typing	Time of posting	Author	Content	Refers to
841	19:58:23	19:58:25	qwertyuiop	Back to this?	Message No. 829
842	19:58:32	19:58:34	137	Ya	
843	19:58:39	19:58:49	qwertyuiop	why not n(n − 1)?	Message No. 829
844	19:58:39	19:58:50	Jason	you guys pretty much have the formula for this hexagon problem . . .	
845	19:58:57	19:59:28	qwertyuiop	We almost have it for the triangle. I don't know about the hexagon.	Message No. 844
846	19:59:35	19:59:50	Jason	well that's just multiplied by a certain number for a hexagon, provided that it is regular	Message No. 845
847	19:59:58	20:00:14	qwertyuiop	but the sides of the triangles making up the hexagon overlap	Message No. 846
848	19:59:52	20:00:18	Jason	well i have to leave now; sorry for not participating as much as i wanted to, it's a pretty busy night for me with school and extracurricular stuff	

Qwertyuiop posts a question linked to line 829 which indicates that he is oriented to the expression 137 proposed in that message.

About a second later, Jason posts a message stating that the formula for the hexagon problem is pretty much done. Jason's use of the phrase "you guys" ascribes this achievement to the remaining members of the team. In line 845, Qwertyuiop posts a message explicitly linked to Jason's last comment. The first sentence "**We almost have it for the triangle**" provides an alternative account of what has been achieved so far. In his second sentence, Qwertyuiop declares that he does not know about the hexagon yet. Hence, these postings make it evident how Qwertyuiop is treating what the team has accomplished so far.

In line 846, Jason posts a message linked to Qwertyuiop's latest remark. In his response Jason states that getting the formula for the hexagon requires a simple multiplicative step provided that the hexagon is regular. Qwertyuiop's response (as indicated by the referential arrow) follows next, where he brings in how the issue of overlap will play out when they move from the large triangles to the whole hexagon. This is followed by Jason's exiting remark where he apologizes for not being able to participate as much as he wanted.

In this excerpt, team members explicitly commented on how they characterize their collective achievement. In other words, these postings can be read as a joint reflection on what has been done so far. Another interesting aspect of this short exchange is the apparent shift in the positions with respect to the issue of overlapping sticks in the counting procedure. Jason was the person who raised the issue of overlap for the first time in Excerpt 3, yet his most recent characterization of the team's work seems to dismiss overlap as a relevant matter. Surprisingly, Qwertyuiop, who was the person previously critiqued by Jason for possibly ignoring the issue of overlapping sticks, explains now why it is a relevant matter that needs to be attended to, before the number of sticks in one triangle is multiplied by a certain number as Jason suggested in 846. In Excerpt 3, Qwertyuiop argued that overlaps would not be an issue in his counting work, but that assertion seems to be applied only to the triangular section he was oriented to at that time. His most recent posting displays his awareness with regards to when the overlapping sticks will become an issue, i.e. when they move from the triangular partition to the whole hexagon. These remarks also specify what has not been accomplished yet, and hence suggest the team to find a way to address overlaps as an issue to consider next.

Excerpt 7: Overcoming the Problem of Overlapping Sticks

Excerpt 7 follows Jason's departure.[12] In line 854, 137 re-initiates the problem-solving work by proposing to multiply by 3 what is indexed by "**the orange**". Figure 15 shows the state of the shared drawing at the moment, where there are

[12]The facilitator opens the possibility to end the session in line 855. The facilitator takes the sustained orientation of the remaining team members to the problem as an affirmative answer and lets the team continue their work.

Excerpt 7 Overcoming the problem of overlapping sticks

Chat index	Time start typing	Time of posting	Author	Content	Refers to
853		20:01:07		*Jason leaves the room*	
854	20:01:19	20:01:31	137	Anyways, if we multiply the orange by 3, we get the:	
855	20:01:14	20:01:34	Nan	do two of you want to continue working for a bit or stop here?	
		20:01:42–20:01:48	137	***137 begins to add blue lines on top of the triangular grid***	
856	20:01:40	20:01:44	Nan	i guess that's the answer	Message No. 854
857	20:01:47	20:01:48	Nan	go ahead	
		20:01:49–20:01:53	137	***137 continues to add blue lines. The resulting shape is displayed in Fig.*** 15	
858	20:01:57	20:02:14	137	So then we add 12n for:	
859	20:01:28	20:02:15	qwertyuiop	actually, this doesn't complicate it that much. The overlaps can be accounted for with "-6n"	Message No. 847
		20:02:32–20:02:52	137	***137 adds pink contours to the shared drawing, The resulting shape is displayed in Fig.*** 16	
860	20:02:54	20:02:55	137	Oh.	Message No. 859
861	20:02:56	20:03:07	137	I like addition more than subtraction.	

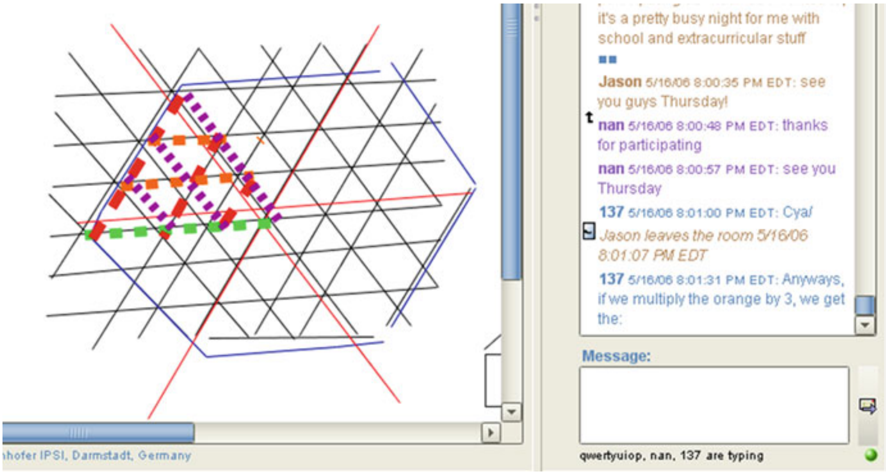


Fig. 15 The state of the whiteboard when 137 began his exposition at 8:01:31 (line 854)

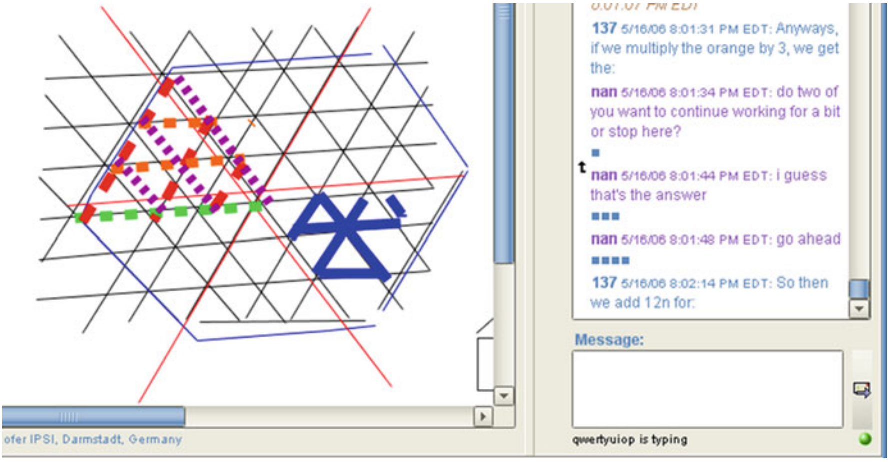


Fig. 16 137's drawing that followed his posting at 8:01:31 (i.e. line 854). The *triangles* added in *blue* follow the chat posting that proposes the multiplication of what is marked with *orange* by 3

two dashed orange lines covering a portion of the hexagon. The remaining part of the message announces the outcome of the suggested operation, but no result is provided yet. The message ends with a colon "**:**" indicating that more content is about to follow subsequently. Next, 137 performs a series of drawing actions where he highlights a set of sticks on the triangular grid with blue lines (see Fig. 16). These actions are done within a section of the shared drawing that has been empty. Based on the way these actions sequentially unfold and the way the drawing was set up in chat, one can read these actions as the ***visual outcome*** of the operation described in

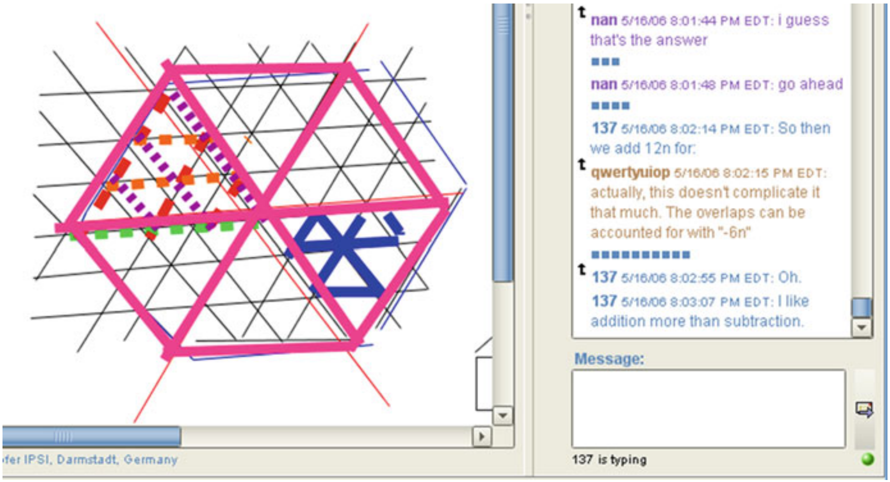


Fig. 17 137's posting "So then we add 12n for:" is followed by his drawing work where he adds the *pink lines*. Again the temporal sequencing suggests that the *pink lines* show visually which sticks will be covered when the proposed computation is performed (i.e., "adding 12n")

text in line 854. In short, multiplying the number of orange dashed lines by 3 seems to yield the number of sticks highlighted in blue, which is an elaborate mathematical move spanning across textual and graphical modalities.

137 posts another message in line 858 which announces adding "**12n**" as the next step in his ongoing exposition. The message ends with "**for:**" which is consistent with his prior use of the colon to project that more elaboration will follow, possibly in the other interaction space. Next, 137 begins to add pink lines to the shared drawing, which covers the boundaries and the diagonals of the hexagonal array (see Fig. 17). The sequential continuity of 137's actions suggests that the lines marked with pink provide a geometric ***realization*** of what is indexed by the symbolic expression "**12n**" on the particular instance represented by the shared drawing.

While 137 was composing his message, Qwertyuiop was busy typing the message that will appear in line 859. The message appears 1 s after 137's posting and just before he begins adding the pink lines. Hence, the temporal unfolding of actions suggests that these two messages were produced in parallel. In this posting Qwertyuiop makes a reference to an older message where he mentioned the problem of overlapping sticks among the six triangular regions. The current message announces that this may not be a big complication. The next sentence in the same post states that the overlaps can be accounted for with the expression "**–6n**". 137's response (as suggested by his use of the explicit reference) to Qwertyuiop's proposal comes after he is done with marking the pink lines on the whiteboard. The "**oh**" in line 861 makes 137's noticing of Qwertyuiop's proposal. In his next posting, 137 states that he prefers addition rather than subtraction. The contrast made between addition and subtraction suggests that 137 is treating his and Qwertyuiop's methods as distinct but related approaches to the task at hand.

What 137 is referring to as an "**additive**" approach can be observed through his prior actions distributed across both interaction spaces. 137's approach begins with a method to cover a specific portion of one of the six partitions of the hexagon. This is referred as "**multiplying the orange by three**" and the outcome of this operation is marked in blue. In other words, the orange lines seem to be used as a way to index a single side of a total of $1+2=3$ triangles (or $n(n-1)/2$ in general) inside one of the six partitions. Hence, multiplying this value by 3 covers the three blue triangles enclosed in a partition. Moreover, none of these triangles share a stick with the diagonals and the boundary of the hexagon, so the sticks highlighted in pink are added to cover the missing sticks. In short, 137's reasoning for the additive approach is evidenced in his drawing actions as well as in the way he coordinated his chat postings with the drawings.

The other approach referred to as "**subtraction**" by 137 has been discussed by the team for a while. This approach starts with counting the sticks for one of the six partitions of the hexagon. A partition is further split into "**3 collinear sets**" of sticks that do not "**overlap**" with each other. The number of sticks covered by a single set turned out to be equivalent to a "**triangular number**". Nevertheless, since this approach covers all the sticks forming a partition and partitions share a boundary with their neighbors, when this value is multiplied by 6 to cover the whole hexagon, the sticks at the boundaries (i.e., at the diagonals) would be counted twice. This is referred to by the team as the **overlap** problem. Qwertyuiop's latest proposal provides the expression that needs to be subtracted from the general formula to make sure all sticks at the internal boundaries are counted exactly once. In contrast, the additive approach does not need subtraction since it splits the shape in such a way that each stick is counted exactly once.

The main point we would like to make about this excerpt is that 137's approach takes the previously demonstrated approaches and their critiques as resources. He offers a new approach informed by previous discussion in an effort to address the practical issues witnessed (e.g., overlaps, adjusting the index in the expression for triangular numbers, etc.). Hence, 137's additive approach is firmly situated within the ongoing discussion. In other words, 137's reasoning has been *socially shaped*; it is not a pure cognitive accomplishment of an individual mind working in isolation from others.

Excerpt 8: Derivation of the Formula for the Number of Sticks

Excerpt 8 immediately follows the prior one. It begins with Qwertyuiop's question addressed to 137, which asks if he could see why subtracting 6n would work. In the meantime, 137 seems to be busy typing the message that will appear in line 864. The use of "**So**" suggests that this message is stated as a consequence of what has been discussed so far. The colon is followed by the formula "**9n(n + 1)–6n**", which involves the term "**–6n**" in it. By using the term "**–6n**", 137 makes his orientation to Qwertyuiop's proposal explicit. Moreover, the sequential build up suggests that the

Excerpt 8 Derivation of the formula for the number of sticks

Chat index	Time start typing	Time of posting	Author	Content	Refers to
862	20:03:11	20:03:16	Qwertyuiop	do you see why that works	Message No. 859
863	20:03:18	20:03:18	Qwertyuiop	?	
864	20:03:12	20:03:29	137	So: 9n(n+1)−6n.	
865	20:03:41	20:03:45	Qwertyuiop	9, not 3?	
866	20:04:13	20:04:14	137	?	Message No. 865
867	20:04:18	20:04:35	Qwertyuiop	you have "9n(n . . . "	
868	20:04:37	20:04:47	Qwertyuiop	not "3n(n . . . "?	
869	20:04:51	20:05:00	137	But we need to multiply by 6 then divide by 2	Message No. 868
870	20:05:10	20:05:22	Qwertyuiop	x6 and/2 for what?	Message No. 869
871	20:05:44	20:05:47	137	FOr each triangle	
872	20:05:48	20:06:02	137	and /2 because it's part of the equation.	
873	20:06:03	20:06:06	137	of n(n+1)/2	
874	20:05:36	20:06:20	Qwertyuiop	it's x3 for the 3 colinear sets, then x6 for 6 triangles in a hexagon . . . where's the 9 and 2?	
875	20:06:28	20:06:28	Qwertyuiop	Oh	Message No. 872
876	20:06:35	20:06:38	137	So 18/2.	
877	20:06:42	20:06:50	137	A.K.A. 9	
878	20:06:48	20:07:08	Qwertyuiop	(n(n+1)/2)x3x6	Message No. 873
879	20:07:14	20:07:15	137	Yeah.	
880	20:07:20	20:07:27	Qwertyuiop	Which can be simplified . . .	
881	20:07:42	20:07:46	137	To 9n(n+1)	Message No. 880
882	20:08:01	20:08:04	Qwertyuiop	that's it?	Message No. 881
883	20:08:10	20:08:12	137	−6n.	
884	20:08:17	20:08:24	137	So 9n(n+1)-6n	
885	20:08:20	20:08:34	Qwertyuiop	i'll put it with the other formulas . . .	

proposed expression stands for the formula for the number of sticks for the hexagonal array. In these ways—through the details of its contextual situating—the symbolic expression is tied to the on-going discourse, including the graphical features.

Qwertyuiop's next posting in line 864 problematizes the appearance of **9** in the proposed formula and asks if **3** should have appeared there instead. Next, 137 posts a question mark linked to Qwertyuiop's question, which can be read as a request for more elaboration. Qwertyuiop elaborates in the next two lines by posting the part of the formula that is problematic for him and then by suggesting a repair for that part. His elaboration ends with a question mark that can be seen as an attempt to solicit his peer's feedback. 137's reply in line 869 states that the steps of the computation should also include multiplication by 6 and division by 2. In response Qwertyuiop asks for what part of the pattern those operations need to be done. 137's reply spans three lines, where he first states "**for each triangle**" and then mentions that "**/2**" comes from the equation $n(n+1)/2$. Hence the sequential organization of these messages suggest that 137 associates multiplication by 6 with the triangles (i.e., the larger triangular partitions) and "**/2**" with the equation for triangular numbers.

In the meantime, Qwertyuiop has been typing what will appear in line 874. The first sentence associates each multiplication operation with a specific section of the hexagonal pattern, namely "**x3**" for the 3 "**collinear sets**" within a triangular partition and "**x6**" for the six triangular partitions of the hexagon. The next sentence in that posting problematizes again the appearance of 9 and 2 in the steps of the calculation. Eight seconds later, Qwertyuiop posts "**oh**" in response to 137's remark about the equation in line 872, which indicates that the referenced message has led him to notice something new. This is followed by 137's demonstration of the derivation of 9 from the numbers previously mentioned. Meanwhile, Qwertyuiop is composing an expression that brings all the items they have just talked about together in symbolic form, which appears in line 878 in response to line 873 where 137 reminded him about the equation $n(n+1)/2$. 137 expresses his agreement in the next line. Next, they simplify the expression and add"**−6n**" to derive the final formula for the number of sticks.

In short, the episode following 137's proposal shows that Qwertyuiop had trouble understanding how 137 derived the formula he reported in line 864. 137 seems to have gone ahead with putting together all the different pieces of the problem that have been discussed so far to produce the final formula. Note that the additive approach 137 was describing earlier included a step summarizing the pink boundary as 12n, which also includes the diagonals causing the overlap issue. The commonality between the two lines of reasoning may have informed 137's quick recognition of the algebraic implication of Qwertyuiop's subtraction move as an alternative to his approach.

Qwertyuiop's problematizations of some of the terms that appear in the proposed formula have led 137 to reveal more details of his algebraic derivation. This exchange has revealed how each algebraic move is based on the corresponding concept the team had developed earlier (e.g., $n(n+1)/2$ sticks to cover a collinear set, multiply by 3 to cover three collinear sets making up a triangular partition, multiply by 6 to cover the hexagon, subtract 6n to remove those sticks at the internal boundaries that are counted twice). 137's contributions in this and the previous excerpts demonstrate that he can competently associate the narrative descriptions and visual representations with symbolic formulas. Qwertyuiop's initial trouble and its resolution in the last excerpt provided us further evidence with regards to how participants made use of the narrative/geometric resources to co-construct a generalized symbolic formula addressing the problem at hand. In short, the team members complemented each other's skills as they incorporated geometric and algebraic insights proposed by different members into a solution for the task at hand during the course of their one hour long chat session.

Discussion

In this section we discuss the findings of our case study regarding the affordances of a multimodal CSCL environment for joint mathematical meaning making online and the interactional organization of mathematics discourse.

Visibility of the Production Process

Our first observation is related to the mathematical affordances of the drawing area. As we have seen in Excerpts 1, 2, 4, and 7, the construction of most shared diagrams includes multiple steps (e.g., addition of several lines). Moreover, the object-oriented design of the whiteboard allows users to re-organize its content by adding new objects and by moving, annotating, deleting and reproducing existing ones. Hence, the sequencing of drawing actions that produce and/or modify these diagrams is available for other members to observe. In other words, the whiteboard affords an *animated evolution* of the shared space, which makes the *reasoning process visually* manifest in drawing actions available for other members to observe. For instance, the sequence of drawing actions that led to the drawing displayed in Fig. 13 (Excerpt 4) allowed the team members to locate what was indexed by the term "set of 3 collinear sides." The drawing also served as a visual proof for the argument that those three sets do not share any sticks (i.e., they do not overlap). Finally, Figs. 16 and 17 show cases where a textually described algebraic operation was subsequently animated on the whiteboard. Such *demonstrable tweaks* make the mathematical details of the construction work visible and relevant to observers, and hence serve as a vital resource for joint mathematical sense making.

Persistent Presence of Contributions

In the VMT online environment, contributions have a *persistent* presence that allows participants to revisit a prior posting or reorganize a shared drawing to orient themselves to shared artifacts in new ways. One important consequence of persistence is illustrated by Qwertyuiop in Excerpts 4 and 5 (lines 803 and 818) and by 137 in Excerpt 3 (line 776), where they used the explicit referencing tool to point to a previous chat posting in an effort to re-initiate a past topic or thread. When combined with the referential arrows, the persistent availability of the chat messages affords re-initiation of past conversations and the management of multiple threads (e.g., the discussion on a missing stick and the formula for triangular numbers that unfolded in parallel in Excerpts 2 and 3 illustrates how users manage multiple threads).

One important consequence of quasi-synchronous interactions mediated by a persistent display of text messages is that participants are not subjected to the same set of physical constraints underlying the turn-taking apparatus associated with talk in face-to-face settings. In natural conversations, speakers take turns due to the practical intelligibility issues involved with overlapping speech. In contrast, the persistent availability of the text messages affords simultaneous production of contributions, and hence provides more possibilities for participation. This may introduce intelligibility issues referred to as chat confusion (Fuks, Pimentel, & de Lucena, 2006) or phantom adjacency pairs (Garcia & Jacobs, 1998), when simultaneously produced messages can be mistakenly treated in relation to each other. However, as we have seen in the excerpts analyzed above, participants routinely provide enough specificity to their contributions (e.g., by using the referential tool or specific tokens) and orient to the temporal/linear order in which messages appear on the screen to avoid such issues of intelligibility. Finally, when coupled with resources such as the explicit referencing tool and repetition of specific terms (e.g., "sidelength"), the persistency of chat messages also allows participants to make a previous discussion relevant to the current discussion. For instance, in line 818 in Excerpt 5, Qwertyuiop re-oriented the current discussion to the issue of devising a formula for the sequence of numbers that was stated back in line 772 by using the explicit referencing tool. Likewise, in line 841 in Excerpt 6 Qwertyuiop proposed that the team re-initiate a discussion on a point stated 13 lines above with his message "go back to this" coupled with an explicit referential link.

The possibility of engaging activities across multiple threads spanning both chat and whiteboard spaces is an important affordance of online environments like VMT due to the opportunities it brings in for more people to contribute to the ongoing discussion. For instance, in Excerpt 4 we have seen that 137 was engaged in two simultaneous threads where (a) he drew a line segment that was potentially ignored by the method of computation described by Qwertyuiop, and (b) he contributed to the simultaneously unfolding discussion about characterizing the pattern implicated by the numbers offered by Qwertyuiop as triangular numbers. Although the management of multiple threads across spaces can create confusion, the resolution of ambiguities and the intertwining of perspectives can lead to germination/fertilization of mathematical ideas across threads. This point is well demonstrated by how the aforementioned threads led to Qwertyuiop's visual proof, which (a) located visually what the term "3 sets of collinear lines" meant, (b) established that the sets do not overlap with each other, and (c) highlighted the association between the cardinality of a single set and a triangular number.

Finally, there is a subtle but important difference between the chat and whiteboard features in terms of the degree of persistence of their contents. As a session progresses chat postings gradually scroll away, but whiteboard drawings stay on the whiteboard until they are erased. For instance, in all the excerpts we have seen above, the particular illustration of the hexagonal pattern continued to serve as an interactional resource as team members illustrated and offered different ideas. Several chat postings presume the availability of such a persistent resource on display so that others can make sense of the contribution (e.g., indexical terms such as “the orange”, “3 sets”, etc.). Such persistently available artifacts provided the background against which new contributions were interpreted and made sense of.

Methods for Referencing Relevant Artifacts in the Shared Visual Field

Bringing relevant mathematical artifacts to other members’ attention requires a coordinated sequence of actions performed in both the chat and whiteboard spaces (Stahl et al., 2011). In the excerpts above we have observed several referential methods enacted by participants to bring relevant graphical objects on the whiteboard to other group members’ attention. In Excerpt 1, 137 *marked the drawing* with a different color to identify what he thought collinear sides meant in reference to the shared drawing. Qwertyuiop also used the same approach when he highlighted the collinear sides in the shared drawing with different colors in Excerpts 1 and 3. *Color coding* was another method used by members to draw others’ attention to specific parts of the drawing (e.g., “the orange”, “the green times 3”). Finally, members used the explicit referencing tool to support their *textual descriptions.* For instance, Qwertyuiop used the explicit referencing tool in Excerpts 2 and 4 to direct his teammates’ attention to the relevant section of the hexagon where he was performing his counting work. In all these cases, chat messages included either an explicit reference or a deictic term such as “this”, “that”, or “the green”, which are designed to inform other members of the group that they need to attend to some features beyond the textual statement itself to make sense of the chat message.

These referential mechanisms play a key role in directing other members’ attention to features of the shared *visual field* in particular ways. This kind of deictic usage isolates components of the shared drawing and constitutes them as relevant objects to be attended to for the purposes at hand. Hence, such referential work establishes a fundamental *relationship between the narrative and mathematical terminology used in text chat and the animated graphical constructions produced on the whiteboard*. The shared sense of the textual terms and the inscriptions co-evolve through the referential linkages established as the interaction sequentially unfolds in the dual-interaction space.

Deictic uses of text messages and drawings presume the availability of a *shared indexical ground* (Hanks, 1992) where the referential action can be seen as the *figure* oriented towards some part of the shared *background*. In other words, referential moves are not performed in isolation; they rely on a part/whole relationship between the referential action (i.e., figure) and a shared visual ground. For example, the color markings of collinear lines in Excerpt 4 worked as a referential action, because they were performed on top of an existing graphical artifact, namely the triangular grid. Even the design of the explicit referential tool, which attaches a semi-transparent green rectangle to a chat message, reflects this visual relationship between the figure (i.e., the green rectangle) and the background, which guides other members' attention to a particular location in the shared visual field. As virtual teams collaboratively explore their problem and co-construct shared artifacts, they collectively constitute a shared problem space with increasing complexity (Sarmiento & Stahl, 2008). By enacting referential practices, participants isolate features of the shared scene, assign specific terminology to them, and guide other members' perception of the ongoing activity to achieve a shared mathematical vision.

Coordination of Whiteboard Visualizations and Chat Narratives

The previous section focused on single actions that refer to some feature of the shared scene for its intelligibility. We argued that such actions involve a part/whole relationship that presumes the availability of a shared visual ground for their mutual intelligibility. In addition to this, such actions are also embedded within broader sequences of actions that establish their relevance. In other words, messages that establish a referential link between narrative and graphical resources routinely respond to practical matters made relevant or projected by prior actions. Thus, such actions are also tied to the *context* set by the sequentially unfolding discussion.

When the scope of analysis is broadened to sequences of actions that include messages with referential links, one can observe an important affordance of online environments with multiple interaction spaces: Since one can contribute to only one of the interaction spaces at a time, a participant cannot narrate his/her whiteboard actions with simultaneous chat postings, as can be done with talk in a face-to-face setting. However, as we have observed in 137's performance in Excerpts 1 and 7, participants can achieve a similar interactional organization by temporally coordinating their actions in such a way that whiteboard actions can be seen as part of an exposition performed in chat.

For instance, in Excerpt 1, Qwertyuiop's drawing activity was prefaced by his chat posting "I'll draw it". The posting was in response to a recent graphical illustration proposed by 137. Hence, the pronoun "it" included in the preface was not pointing to an existing drawing or to a prior posting. Instead, it *projected* a subsequent action to be performed next by the same author. In contrast, prior to Qwertyuiop's actions in Excerpt 1, 137 produced his drawings before he was seen as typing by others. Although the sequence of the chat and whiteboard actions are the opposite in this case (i.e., the referential move was made after the drawing was finished), 137 achieves a similar temporal organization through his use of deictic terms (e.g., "those", "that", "it"), referential arrows, and tokens of similarity such as "like" and "as". Therefore, these instances suggest that, although they can be ordered in different ways, the sequential organization and temporal proximity of actions are consequential for the treatment of a set of drawing actions in relation to a narrative account produced in chat.

In face-to-face settings, locational deictic terms such as "this" and "those" are used to point out contextual elements beyond the lexical content of the uttered statement, and they are often accompanied by co-occurring pointing gestures and body movements displaying the speaker's orientation towards what is being referred to in the vicinity (Goodwin, 2000; Hanks, 1992). As demonstrated by the actual cases of use in the excerpts analyzed above, a similar organization presents an interactional challenge for the participants in an online setting with dual interaction spaces like VMT. However, as participants demonstrated in these excerpts, a functionally comparable interactional organization can be achieved online through the use of available features so that chat messages can be seen as related to shared drawings that are either on display ("those") or in production ("these"). The sequential organization of actions, explicit referencing, and the temporal proximity of actions across both spaces together guide other members' attention so that they can treat such discrete actions as a coherent whole addressed to a particular prior message or to a thread of discussion unfolding at that moment.

Another important aspect of such achievements from a mathematics education perspective is that it shows us how *saming*[13] (Sfard, 2008) among narrative and graphical accounts or realizations can be done as an interactional achievement across dual-interaction spaces. This phenomenon is demonstrated in various episodes such as (a) Qwertyuiop's demonstration of collinear set of lines on the shared diagram in Excerpt 4, and (b) 137's exposition in Excerpt 7, where he showed the geometric implication of his proposal in narrative form by performing a drawing immediately after his chat message. The referential links, the temporal proximity of actions, the awareness indicators for those actions, and the persistent availability of both prior messages and the recently added drawings all work together as a semiotic system that allows group members to make connections among different realizations of the mathematical artifacts that they have co-constructed. Therefore, referential practices across modalities are consequential for the collective achievement of deep understanding of mathematics, which is characterized in mathematics education theory as establishing relationships between different realizations of mathematical ideas encapsulated in graphical, narrative or symbolic forms.

Past and Future Relevancies Implied by Shared Mathematical Artifacts

The objects on the whiteboard and their visually shared production index a horizon of past and future activities. The indexical terms in many proposals made in

[13]Sfard (2008) describes saming as the process of "... assigning one signifier (giving one name) to a number of things previously not considered as being the same" (p. 302).

the analyzed excerpts (like “hexagonal array”, “collinear lines”, “rows”) not only rely on the availability of the whiteboard objects to propose a relevant activity to pursue next, but also *reflexively* modify their sense by using linguistic and semantic resources to label or gloss the whiteboard object and its production. This allows actors to orient in particular ways to the whiteboard objects and the procedures of their co-construction—providing a basis for subsequent coordinated joint activity.

This suggests that shared representations are not simply manifestations or externalizations of mental schemas as they are commonly treated in cognitive models of problem-solving processes. Instead, our case studies suggest that shared representations are used as resources to interactionally organize the ways actors participate in collaborative problem-solving activities. As we have seen in this case study, once produced as shared mathematical artifacts, drawings can be mobilized and acted upon as resources for collective reasoning as different members continue to engage with them. Shared meanings of those artifacts are contingently shaped by these engagements, which are performed against the background of a shared visual space including other artifacts and prior chat messages (i.e., against a shared *indexical ground*). This does not mean that the achievement of shared understanding implies that each member has to develop and maintain mental contents that are isomorphic to each other’s, which is often referred as registering shared facts to a “common ground” in psycholinguistics (Clark & Brennan, 1991). Instead, shared understanding is a practical achievement of participants that is made visible through their reciprocal engagements with shared mathematical artifacts.

The way team members oriented themselves to the shared drawing while they were exploring various properties of the hexagonal array showed that the drawings on the whiteboard have a figurative role in addition to their concrete appearance as illustrations of specific cases. In other words, the particular cases captured by concrete, tangible marks on the whiteboard are routinely used as resources to investigate and talk about the general properties of the mathematical artifacts indexed by them. For example, the particular drawing of the hexagonal pattern in the excerpts studied above was illustrating one particular stage (i.e., $n = 3$), yet it was treated in a generic way throughout the whole session as a resource to investigate the properties of the general pattern implied by the regularity/organization embodied in that shared artifact. Noticing of such organizational features motivated the joint development of counting practices, where relevant components of the pattern were first isolated and then systematically counted.

Another important aspect of the team’s achievement of general formulas, which summarize the number of sticks and triangles included in the n-th case respectively, is the way they transformed a particular way of counting the relevant objects in one of the partitions (i.e., a geometric observation) into an algebraic mode of investigation. For instance, once the team discovered that a particular alignment of sticks that they referred to as “collinear sides” corresponded to triangular numbers, they were able to summarize the sequence of numbers they devised into the algebraic formula $9n(n + 1)-6n$. The shift to this symbolic mode of engagement, which relied on the presence of shared drawings and prior narratives as resources, allowed the team to progress further in the task of generalizing the pattern of growth by invoking algebraic methods. In other words, the team co-constructed general symbolic formulas for their shared tasks by making *coordinated* use of multiple realizations (graphical and linguistic) of the mathematical artifact (the hexagonal array) distributed across the dual-interaction spaces.

Conclusion

Perhaps the most important contribution of online learning environments like VMT to research is that they make the collective mathematical meaning-making process visible to researchers through their logs. This allows us to explore the mechanisms through which participants co-construct mathematical artifacts in graphical, narrative and symbolic forms; and to study how they incrementally achieve a shared understanding of them. Careful analysis of team members' actions helps us identify important affordances (i.e., possibilities and limitations on actions) of digital environments for supporting collaborative discussion of mathematics online. Such an understanding is vital not only for informing the design of cyberlearning environments for mathematics, but for investigating the social-interactional nature of mathematical practice.

Our analysis reveals that group members display their reasoning by enacting *representational affordances* of online environments like VMT. The persistent nature of the contributions and the availability of their production/organization allow other participants to *witness the mathematical reasoning* embodied in those actions. Group members establish relevancies across graphical, narrative and symbolic realizations of mathematics artifacts by enacting the referential uses of the available system features. Verbal references, highlighting a drawing with different colors, and the explicit referencing feature of the system are used to establish such relationships between contributions. Through referential practices group members:

(a) isolate objects in the shared visual field,
(b) associate them with local terminology stated in chat, and
(c) establish sequential organization among actions performed in chat and whiteboard spaces, which can be expressed in algebraic symbolism.

These practices had an important role in the achievement of a joint problem space against which indexical terms, drawings and symbols made shared sense for the team members. Development of shared understanding in cyberlearning environments such as VMT heavily relies on such representational and referential uses of its features.

Finally, this case study also showed us how mathematics terminology comes into being in response to specific communicational needs. Mathematical discourse has a deeply indexical nature; mathematics terminology often encodes certain ways of thinking about mathematical objects. As we have seen in the excerpts above, terminology such as "sides", "collinear set of sides", etc., emerge from the need to talk about and direct others' attention to specific aspects of the task at hand. Such glosses, names or indexicals become meaningful *mathematical narrative artifacts* through the ways participants enact them by organizing the shared space in particular ways and/or referring to some part of a drawing or a previous chat posting. Once a shared sense of a term is established in interaction, subsequent uses of the term encode certain ways of constructing/grouping/organizing some items and begin to serve as a convenient way to refer to an overall strategy of looking at a problem in a particular way. The term may then lead to a symbolic expression, drawing upon associated practices of computation and manipulation.

In short,*mathematical understanding at the group level is achieved through the organization of representational and referential practices*. Persistent whiteboard objects and prior chat messages form a shared indexical ground for the group. A new contribution is shaped by the indexical ground (i.e., interpreted in relation to relevant features of the shared visual field and in response to prior actions); it reflexively shapes the indexical ground (i.e., gives further specificity to prior contents) and sets up relevant courses of action to be pursued next. Shared mathematical understanding is an observable process, a temporal course of work in the actual indexical detail of its practical actions, rather than a process hidden in the minds of the group members. Deep mathematical understanding can be located in the *practices of collective multimodal reasoning* displayed by teams of students through the sequential and spatial organization of their actions. Mathematical results are reached through a sequence of discourse interactions that build successively (Stahl, 2011). The discourse moves within the media of graphical constructions, narrative terminology and manipulable symbolisms, allowing progress to be made through visual means, counting skills, encapsulation of knowledge in words, and generalization in symbols.

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Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies

Beste Güçler, Stephen Hegedus, Ryan Robidoux, and Nicholas Jackiw

Abstract In this chapter, we examined the experiences of 10-year old (fourth grade) students in the United States involved in a dynamic multi-modal environment as they explored the characteristics of 3D geometrical shapes. The environment we developed provided visual and physical feedback to students through the PHANTOM Omni® haptic device. This dynamic multi-modal environment enabled semiotic mediation where meanings are generated and substantiated through social interaction as students worked in groups. Adhering to a socio-cultural theoretical perspective, we mainly focused on students' discourse when exploring the affordances of multi-modal technologies in their mathematical experiences. Our preliminary findings indicated that such technologies have the potential to present students with the opportunities to explore 3D objects through multiple perceptions, supporting meaningful discourse as students engage in mathematical activities such as exploring, conjecturing, negotiating meaning, and sense-making.

Keywords Multi-modal technologies • Mathematical discourse • Semiotic mediation • Dynamic geometry • Haptic technology • 3D geometry • Elementary grades

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Background and Rationale

Our chapter focuses on the use of new technologies to create multi-modal learning environments for young children, allowing them to interact with mathematical objects in a variety of ways. A multi-modal environment supports user interactions in more than one modality or communication channel (e.g., speech, gesture, writing) through perceptual, attentive or interactive interfaces. These forms of interactions have evolved in various research areas and applications including computer visualization, psychology and artificial intelligence with increasing use in education, particularly in early learning and developmental psychology. Currently, the most common modalities in multi-media are audio-visuals, but in recent years alternative input modes have become increasingly more available. Multi-touch technologies (e.g., tablet PCs, SMART Boards, iPads) have been rapidly evolving and these multi-modal environments, which combine visual feedback and kinesthetic input through gesture, are being integrated into mathematics classrooms. For example, Thompson Avant and Heller (2011) examined the effectiveness of using TouchMath—a multisensory program that uses key signature points on mathematical objects— with students with physical learning disabilities. Although there is some work about student perceptions in multi-modal environments and how such environments can support students with particular needs, we do not yet know enough about students' general experiences in multi-modal environments, especially with respect to mathematics. The innovative and exploratory work we present here is part of a project that investigates the affordances of multi-modal environments in the mathematical experiences of young learners.

For the purposes of this chapter, we focus on a particular multi-modal environment that combines physical force-feedback—experienced through a haptic device—with visual feedback, in the context of dynamic geometry. In our work, we aim to show that the integration of the two technologies—haptic and dynamic geometry—forms an entirely new experience, dynamic haptic geometry. The multi-modal environment we use combines highly interactive 3D visual images with haptic hardware devices that allow users to touch and feel attributes of images they see and manipulate on the screen.

The evolution of haptic technology and dynamic, interactive software in mathematics education have evolved independently over the past 15 years. Our work has taken some innovative first steps in integrating these two fields and investigating the impact on how students talk and make meaning about 3D mathematical shapes. In what follows, we first outline how these two separate fields have evolved and provide a rationale for our study of integrating the affordances of such hardware and mathematical software to create a multi-modal learning environment that builds on prior work. We then outline our theory of change, which adheres to a social-cultural theoretical perspective and takes the position that such technologies—and resulting discourse practices of learners—are semiotic mediators. The second half of our chapter outlines a series of activities whose design principles are a direct product of this theoretical perspective and prior work with examples of student discourse from our preliminary data analysis. We conclude with next steps in our research that focus on using other types of multi-modal technologies and the refinement of our framework to analyze the evolution of young learners' expression from informal to formal mathematical discourse.

Dynamic Geometry

Dynamic media, in general, and dynamic geometry technology, in particular, have significantly influenced mathematics education research. Kaput (1992) considered abstraction of invariance as a key aspect of mathematical thinking and noted that "to recognize invariance—to see what stays the same—one must have variation. *Dynamic media inherently make variation easier to achieve*" (p. 525, italics in original). Dynamic geometry is a form of mathematics visualization where geometric constructions (and other forms of mathematical illustrations such as diagrams, figures, and graphs) can be manipulated via a computer mouse. In dynamic geometry environments, the dynamic illustrations on the screen are just as rigid as—but never more than—the essential mathematical properties that define them, making the exploration of invariance among the illustrations possible.

Dynamic software environments have certain anatomical features including the ability for users to interact with virtual objects; manipulate figures or diagrams according to the features of their construction; and construct mathematical figures or diagrams through specific tools (Hegedus, 2005). These dynamic features contribute to the co-action between a user's utilization of a tool and the software environment's use of a tool, where the modes of feedback continually inform both the user and environment (Hegedus, 2005). An example of this co-action can be seen within a dynamic geometry environment where learners can generate and explore a vast number of continuously related examples of a single set of mathematical relationships through the manipulation of "hot-spots," which results in the capability to link—automatically or on demand—the user's actions with those of the environment (Kaput, 1992).

Over the past 15 years, dynamic geometry environments such as Geometer's Sketchpad® (Jackiw, 1991, 2009) and Cabri-géométre (Laborde & Straesser, 1990) have had remarkable impact on geometry classrooms. Sketchpad's construction paradigm has led beyond high-school geometry material to enable varieties of modeling practice (Jackiw & Sinclair, 2007) that address a broad spectrum of mathematical topics, ranging from applications in the primary grades to advanced mathematics. Such construction paradigms support students' exploration of mathematical constraints and invariance as well as the co-action and interactivity among the users and the environment. This, in turn, allows students to reason quickly from the specific to the general, from concrete to abstract, from example or illustration to concept and idea (for a survey see King & Schattschneider, 1997).

Haptic Technology

Haptic literally means "ability to touch" or "ability to lay hold of" (Revesz, 1950) and has evolved in a technological era to be an interface for users to virtually touch, push, or manipulate objects created and/or displayed in a digital visual environment (McLaughlin, Hespanha, & Sukhatme, 2002). Haptic technology has evolved over the past 10 years, particularly out of a focus on virtual reality in the 1990s, and has become more available in a variety of commercial and educational applications, including 3D design and modeling as well as medical, dental and industrial applications. In a meta-analysis of the use of haptic devices, Minogue and Jones (2006) found over 1,000 articles on haptics in the electronic databases ERIC and PsycINFO, but only 78 related haptics to learning or education. A large proportion of these were focused on "haptic perception"—a major field in psychology focused on haptic sense—and the second main set was focused on multi-modality.

A main constraint observed by Dede (1999) was the high cost, and inflexibility of haptic devices; for example, one could only experience a single haptic experience at a time. However, recent technological progress has eroded these barriers, making force-feedback devices more affordable and flexible in their application and use. One particular haptic device, the PHANTOM Omni®[1] (hereon referred to as the Omni) by Sensable Technologies, Inc., has achieved significant success in bringing its product to scale at an affordable price, enabling it to be used in schools. We use this device in the work we report here and explore its affordances in terms of meaningful mathematical experiences later in this chapter.

Merging Fields

In dynamic geometry, feedback is visual, and does not include the additional sensory information of forces or kinesthetics that are directly linked to the properties of the objects being manipulated on a screen. We now present how we rationalized the potential educational benefits of integrating the dynamic geometry and haptic technologies based upon prior work in various fields.

Historically, a multi-modal learning environment has been translated as a way to create multiple learning pathways for students to work within, and predominantly these were dominated by auditory and visual modalities. In fact, the audio/visual modality is still the predominant "multi-" media form. However, students can interpret visual, auditory, and physical feedback to gather information, while using their proprioceptive system to navigate and control objects in their synthetic environment (Dede, Salzman, Loftin, & Sprague, 1999). Multiple sensory representations can offer mutually reinforcing information that a user can collect to develop understanding of a mathematical or scientific model. In addition, physical

[1]The website for PHANTOM Omni® is http://www.sensable.com/haptic-phantom-omni.htm.

feedback has been said to be superior to vision in the perception of properties of texture and microspatial properties of pattern (Lederman, 1983; Zangaladze, Epstein, Grafton, & Sathian, 1999), while vision is more useful in the perception of macrogeometry—particularly shape and color (Sathian, Zangaladze, Hoffman, & Grafton, 1997; Verry, 1998). Leveraging these relative strengths, our work proposes that it is relevant to integrate haptic technology with dynamic geometry technology to offer multiple sources of information-feedback for students; it is not enough to offer a way for students to just see a mathematical object or a scientific model in a static way (Moreno-Armella, Hegedus, & Kaput, 2008); they must also engage with it dynamically and tactically.

Given the aforementioned advancements in haptic technologies and mathematical affordances of dynamic geometry environments, it is timely for us to integrate haptic technologies into a highly visual, mathematics learning environment. For our work, we developed a dynamic, multi-modal environment that incorporates visual and physical feedback to the students via the Omni. The environment simultaneously provides students with the two modes of feedback (based on the students' input and the dynamic environment's reaction to that input), which are defined programmatically. The physical feedback is generated by the Omni and can be in the form of variable resistance (i.e., force-feedback) or kinesthetic resistance (i.e., friction). The visual feedback is provided through a computer display, and consists of the dynamic, digital representations of mathematical objects in the environment (e.g., a cube). Students utilize the multi-modal feedback to manipulate digital objects and navigate through the environment.

In the next section, we discuss our approach and theoretical assumptions regarding our research in efforts to combine the worlds of haptic and dynamic geometry technology with the world of mathematics education research to create a form of dynamic haptic geometry.

Theoretical Perspectives

One of the main assumptions that guides our work is that although technology—in particular, the Omni—is an important mediator for learners, what makes technology significant for mathematics education is the *mathematics* learning it enables in the classroom. In other words, what distinguishes the utilization of multi-modal technologies in mathematics education from other contexts is whether they support meaningful mathematical exploration and discussion, which is what we attend to in this chapter.

Another critical assumption we make when exploring the affordances of multi-modal technologies in students' mathematical experiences is the role of student discourse. In our design of a learning environment integrating new technologies in mathematically relevant ways, we adhere to a socio-cultural perspective of learning and focus on the interaction of students in terms of mathematically-relevant discourse as mediated by the various tools and supports available to them. As a result, we use a formulation of discourse that does not only attend to students' utterances but also to their interactions and the mediators shaping those interactions.

We use *discourse* to mean "the different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). This conceptualization goes beyond the approaches to discourse that focus only on students' word use or speech (e.g., Edwards, 1993), and highlights the roles of mediators (e.g., symbols, graphs), and routines (e.g., gestures, participation patterns, forms of argumentation) when exploring students' mathematical communications. Although Sfard mainly focuses on *visual mediators* in her framework, when exploring students' discourse, we extend her consideration of *visual mediators* to *any mediator* that contributes to students' mathematical communications. More specifically, we consider the multi-modal environment we developed through the Omni to be both a visual and a physical mediator due to the multi-modal nature of the environment. We also consider students' social interactions with each other as another form of mediation that shapes students' mathematical discourse.

Drawing on cross-cultural studies, Vygotsky (1980) highlighted an analogy between tool-use and sign-use, suggesting how such activity structures the social environment of interaction and the very behavioral routines of members of that environment. While doing so, he emphasized how mediators of discourse shape and are shaped by social interaction. Holland, Skinner, Lachicotte, and Cain (2003) mentioned that "a typical mediating device is constructed by the assigning of meaning to an object or a behavior" (p. 36), which again underlines the social norms of construction in the process of mediation. Accordingly, for the purposes of our study, mediating devices are not only the Omni but also the discursive behaviors that young learners volunteer to make sense of the mathematical task, and their actions that contribute to mediation and communication. The assumption that students' actions or routines impact their discursive behaviors is also widely held in the mathematics education literature on dynamic geometry. For example, actions of pointing, clicking, grabbing and dragging parts of geometric constructions allow a form of mediation (Falcade, Laborde, & Mariotti, 2007; Kozulin, 1990; Mariotti, 2000; Pea, 1993) between the object and the user who is trying to make sense of, or induce some particular attribute of a diagram or prove some theorem. Such form of mediation is referred to as *semiotic mediation*, which corresponds to mediation through the use of sign systems and artifacts whose meanings are generated by social construction (Hasan, 1992; Vygotsky, 1980).

In summary, consistent with the arguments provided in this section, we focus on (a) students' word use; (b) Omni as a mediator; and (c) social interaction as a mediator when examining students' discourse in our multi-modal environment. While exploring these aspects of students' discourse, we also take into account students' actions (e.g., use of gestures and deixis) to characterize their routines that impact their word use as well as discursive mediation.

Our main assumptions—mathematics and discourse matter—regarding the use of the multi-modal technology are also reflected in our activity design process. To ensure that the designed activities are mathematically meaningful, we examined their curricular relevance and took into consideration the critical ideas related to measurement and geometry. To ensure that the designed activities reflect the social nature of learning and lead to rich mathematical discourse, we created open-ended tasks that lend themselves to innovative mathematical explorations originating from students' own discourse. In addition, we asked students to work in groups as they discussed, developed, and negotiated mathematical meanings through their interaction with the Omni as a mediator of their discourse.

In the next section, we outline the design of our multi-modal environment. Our design process consisted of three stages: design of activities; intervention; and redesign/revision. The design of the activities originated from (1) our theoretical assumptions about the nature of learning and the importance of technology and social-based mediation in students' mathematical experiences; (2) mathematical and technical affordances and limitations of our dynamic, multi-modal environment based on our prior work; and (3) students' motor, cognitive, and discursive skills. The intervention stage provided us opportunities to observe and assess the mathematical as well as technical affordances and limitations of our dynamic, multi-modal environment; students' mathematical discourse and interaction patterns; and the enacted utilization of the activities by students. Finally, the redesign stage originated from the feedback we received from the intervention stage with respect to the utilization of technology; and elicited mathematical discourse.

Multi-modal Environment Design

In this section, we first discuss the specific technology that is employed within our multi-modal environment. We then describe the activities we developed for students' mathematical investigations and their curricular relevance. This is followed by the description of the context of our study.

Technical Specifications

The technical design of our multi-modal environment was built upon our prior research on the affordances of haptic technology (Hegedus, Güçler, Robidoux, & Burke, 2011). Our environment was a multi-modal environment consisting of visual and physical feedback presented to students through two separate, mediating platforms: a computer monitor and the Omni. Using the software development kit provided by Sensable Technologies, Inc., we created applications that were operated by students through the two mediating platforms. Therefore, in this chapter, we will only discuss the technical details regarding students' interactions within these applications and not the specifics of developing the software.

The Omni is comprised of a swiveling, jointed arm connected to a stylus, and operated within three-dimensional space. Physical feedback was relayed to students via the device's jointed arm, in the form of resistance, magnetism, and/or friction. Students operated the Omni by moving and rotating the stylus, which

Fig. 1 (**a**) A student operates the PHANTOM Omni® haptic device. (**b**) Students' view of the multi-modal environment

was designed to correspond to a pointer in our environment's visual mode. The visual mode was presented to students via the computer monitor and displayed the pointer and all other visual elements within our applications with which the students interacted. Unlike the Omni, which the students manipulated via its stylus, the computer monitor was used solely for the visual representation of elements within our software (see Fig. 1a, b). Researchers at the Kaput Center developed all software elements of the dynamic, multi-modal environment.

Design of Mathematical Activities

Given the technical affordances of the Omni and our theoretical assumptions, we focused on two types of mathematical investigations of 3D shapes: classification of solids and planar intersections. Below, we describe these investigations and how we expected them to mediate students' mathematical discourse.

Classification of Solids

In this type of investigation, students were presented with a toolbar containing a variety of digital representations of 3D shapes and a pointer (which was in the form of a bug to make it visually appealing to students). The goal of this activity was for students to select and inspect the shapes as well as classify them with respect to their similarities and differences based on their visual and physical perceptions of the shapes' attributes. Students were able to use multiple forms of classification depending on their focus on micro-terms (e.g., edges, vertices); macro-terms that described the whole shape (e.g., triangular, circular, pointy); or any other classifier they considered as relevant (e.g., color, size of the shapes).

The dynamic aspects of this investigation could be seen within each of the modalities present within our multi-modal environment. Students could use the Omni device to rotate any of the 3D shapes 360° around any axis, allowing

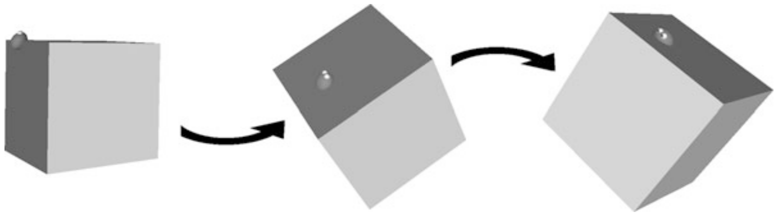

Fig. 2 A computer screen capture of a student's navigation of the bug along the faces of a cube within a multi-modal computer environment developed at the Kaput Center

them to visually inspect the mathematical properties of the shapes (e.g., number of faces, number of vertices). Students could also navigate the bug along the surface of any 3D shape and receive physical feedback from the environment via the Omni. For example, while moving along the surface of a cube, a student would feel a resistant force when they encountered an edge or vertex. Given that the activity required the students to classify 3D shapes, we expected them to hypothesize about the properties of the shapes through their visual and physical perceptions. In combining the dynamics of visually inspecting a 3D shape (i.e., 360° rotation) and physical feedback through the Omni, this investigation gave students an environment to test their hypotheses and group the shapes according to students' individual and collective perceptions and interactions. Figure 2 shows three chronological screenshots of a student's navigation of the bug along the faces of a cube, while also rotating the cube.

Planar Intersections

In this type of investigation, students were presented with a static 3D shape, a dynamic plane, and a pointer in the form of a bug. The goal of this activity was for students to explore the varied intersections of a 3D shape with a plane and describe the attributes of the intersections. There could be multiple types of descriptions depending on whether the students focused on and described each distinct intersection they created, or they focused on the generalizations pertaining to the attributes of the different sets of intersections.

This investigation contained dynamic elements within both the visual and physical feedback modalities of our environment. Students constructed planar intersections by rotating and translating the dynamic plane so that its position overlapped with the static 3D shape. Students could then visually inspect these intersections. Since the 3D shape was static, the intersection could only be viewed from one vantage point. Yet, students could gain physical feedback pertaining to the unseen portions of the intersections by navigating the bug around the intersection. For example, while navigating the bug along the boundaries of an intersection, students would feel a magnetic force—which restricted the bug's movements to the boundaries—and a resistant force when they encountered a vertex. As with

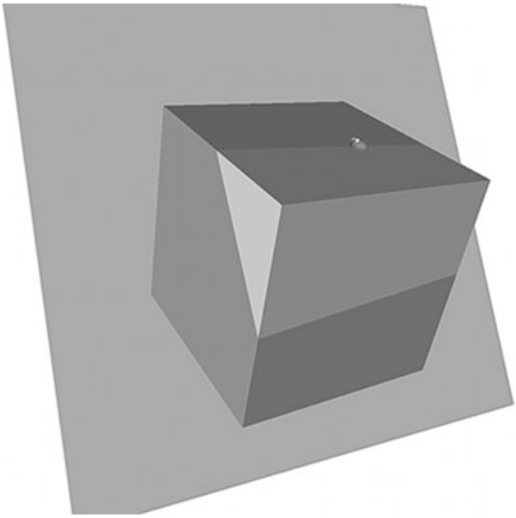

Fig. 3 A computer screen capture of a planar intersections activity within our multi-modal computer environment developed at the Kaput Center

the classification investigation, we expected students to hypothesize individually and collectively about the attributes of the intersections they constructed through their visual and physical perceptions. The dynamic, multi-modal features within this investigation provided students with an environment to test their hypotheses. Figure 3 provides a screenshot of this activity, where a student is navigating the bug along a planar intersection of a cube.

Curricular Relevance

The use of Omni in other fields and in our prior work indicated that it has potential affordances with respect to the visualization and feeling of 3D objects, so we designed our activities for the exploration of 3D solids and their attributes. Knowledge of geometric shapes and their characteristics is also a critical part of the geometry standards for elementary grade levels as highlighted by National Council of Teachers of Mathematics (NCTM) and Massachusetts Curriculum Framework for Mathematics that incorporates the Common Core Standards for Mathematics. Some general geometry standards for elementary level students are to "analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships" (NCTM, 2000, p. 41), and to "reason with shapes and their attributes" (Massachusetts Department of Education, 2011, p. 37). More specifically, by Grade 5, students are expected to have geometric skills such as:

- "Identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes" (NCTM, 2000, p. 164).

- "Classify two and three dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids" (NCTM, 2000, p. 164).
- "Make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions" (NCTM, 2000, p. 164).
- "Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size) ..." (Massachusetts Department of Education, 2011, p. 33).
- "Recognize and draw shapes having specific attributes such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes" (Massachusetts Department of Education, 2011, p. 37).

In this chapter, we focus on the discursive features that help us characterize students' experiences within our dynamic, multi-modal environment, and possible affordances of those discursive features to explore the mathematical affordances of such environments. Our examination of the curricular frameworks helped us situate our work in a curricular space but we also go beyond those frameworks since (a) our activities embody a combination of the aforementioned curricular standards rather than a single one, and (b) our activities also go beyond the elementary level standards to explore the span of mathematical discourse our dynamic, multi-modal environment supports. For example, students' knowledge of planar intersections is not required for elementary grades but their existing discourse does not necessarily hinder the exploration of planar intersections. To sum up, we designed an open exploration space to highlight its curricular relevance, which does not imply that our exploration space was bounded by the curriculum. We wanted to learn how students made sense of what they saw and felt collectively and also independently as they communicated and negotiated meanings about the features of 3D geometrical shapes. We wish to note that we do not use the curricular criteria mentioned above to assess student learning since our study utilizes a new approach and is mainly exploratory.

Context of the Study

This study is part of a 3-year project that examines the affordances of multi-modal technology in elementary and undergraduate classrooms, involving 182 students (150 elementary; 32 undergraduate-level) to date. In this chapter, we only focus on the part of the study about elementary-level students' use of the Omni, which took place in a suburban elementary school in the Southcoast region of Massachusetts. All seven fourth-grade teachers in the school agreed to select the students to be interviewed for our study. We interviewed four students from each of the seven classes over the course of 2 weeks. We interviewed students in groups because (a) given that we had only one operating Omni device, we wanted to give access to as many students as possible, and (b) we wanted to maintain students' chances for individual interaction with the device as well as their collective mathematical discussions with their peers. Although there was one device per group, every student used the Omni at some point during each activity. We conducted one semi-structured interview (lasting approximately 45 min) with each group as they worked on one or more of our activities. The interviews took place outside of the students' classrooms—in the school library—to avoid the interference of the teachers' ongoing classroom lessons.

In the next section, we discuss our data analysis and the common themes that emerged from students' interactions with the Omni and each other. The themes reported are elements of students' mathematical experience that we identified as important as they participated in the multi-modal environment. We do not make any claims about student learning at this stage, which may or may not have occurred during students' interactions with the multi-modal environment and each other. Instead, we explored whether students' discourse was mathematically rich and their experience, mathematically meaningful.

Data Analysis and Common Themes

Data gathered for the study included video recordings that captured students' individual and collective utterances as well as their actions with the Omni; video recordings of the computer monitor that captured students' manipulations of the 3D objects on the screen; and written work of the students. We formulated discourse using Sfard's (2008) work and transcribed the video-taped sessions both with respect to what the students said and what they did. We prepared the transcripts so that students' utterances and their actions were split into two distinct columns. The columns were enumerated to link the utterances with students' corresponding actions, which also enabled easy referencing during analysis. We investigated the general features of students' multi-modal experiences on their discourse as they associated mathematical meanings to their explorations.

Currently, we are still in the process of analyzing the data. Our goal is to develop a systematic way to investigate students' mathematical discourse in relation to their experience with multi-modal environments and each other. The analysis we present in this section is only at its preliminary stage and focuses on three important elements that impacted students' discourse: (1) students' word use, (2) Omni as a mediator of discourse, and (3) social interaction as a mediator of discourse. In what follows, we discuss these features that framed students' experiences in detail and provide examples from student data. All the student names mentioned in this section are pseudonyms.

Word Use

An important feature of students' discourse we explored was their word use. We were interested in whether, and how, students moved between non-scholastic (everyday and metaphorical word use) and scholastic (mathematical) word use. Although we are also interested in coherence of non-scholastic and scholastic word use, we are not making a claim about students' transition mechanisms from informal to formal mathematics in this chapter. Exploration of transition mechanisms in students' discourse would require a different implementation setting in which students are observed more than once since a single snapshot of their experiences would not suffice to capture "transitional" features, which occur across time. What we provide here is a descriptive report of the types of words that students used to make sense of their multi-modal experience.

During the activities, the students smoothly navigated between non-scholastic and mathematical word use. Students did not freeze or become silent even if the shapes or the goals of the activities were not apparent to them. There were instances in which they had difficulties naming the 3D shapes provided to them. However, not being able to name the object did not stop them from exploring the characteristics of the object. While doing so, they sometimes referred to metaphorical word use (e.g., saying "jelly bean" or "oval" for an ellipsoid; "tube" or "tunnel" for a cylinder; "diamond" or "kite" for a triangle-based pyramid) or used mathematical words for the 2D features of the 3D shapes[2] (e.g., saying "a triangle" for a triangle-based pyramid; "rectangle" for a rectangular prism; "circle" for a sphere). In other instances, they used mathematical language to describe the most basic features of the 3D shapes that were unfamiliar to them (e.g., identifying or counting the "edges," "corners," or "vertices" of the shapes through their visual and physical experiences).

The activities supported utilization of mathematical discourse even though students' words were not always mathematically accurate. Table 1 shows the family of words students commonly used as they explored 3D shapes, their characteristics, and their planar intersections.

The Omni as a Mediator of Discourse

Like other mediators, Omni has its affordances and constraints. Omni is somewhat clunky and fragile; it cannot be easily moved from one space to another. Omni also requires the designer of mathematical activities to have significant knowledge of the software environment. Despite these constraints, we found the Omni to be quite useful for mathematical experiences if it is used according to its affordances.

[2]Using 2D vocabulary to talk about 3D shapes was quite common in students' discourse.

Table 1 Examples of students' word use based on the types of their exploration

Type of exploration	Examples of words used
General features of the shapes	Edge, side, angle, vertices
Determining differential elements of the shapes	Smooth, flat, circular, rough
Positioning of the shapes	Inside, behind, top, bottom
Talking about 2D shapes or 2D features of 3D shapes	
Scholastic	Square, circle, trapezoid, parallelogram, pentagon, triangle, rectangle
Non-scholastic	Four sided-shape, circular shape
Talking about one-dimensional features of 3D shapes	Line, length, height, width
Talking about 3D shapes	
Scholastic	Rectangular prism, cube, sphere, cylinder, cone, pyramid
Non-scholastic	Kite, tunnel, jelly bean, ball, ice cream cone

A significant affordance of the Omni is its support for powerful 3D dynamic geometry activities, which we utilized specifically for the investigation of 3D shapes. The Omni allows for multi-modal and dynamic investigations of 3D shapes; enables three-dimensional manipulations of 3D objects; and allows a 360° view of those objects. Through force feedback, it makes possible the examination of depth and texture of 3D shapes helping to identify the characteristics of those shapes.

Another affordance of the Omni, which we observed during the implementation of our activities, is the level of student participation and engagement it supports. Students were immediately engaged in the activities designed by the Omni and they expressed their amazement frequently by means of gestures and words (e.g., "wow!," "this is cool," "I feel like touching the shape; so cool"). It may not be surprising for a new type of technology to create such an effect on students. What we want to particularly highlight is the continuous flow of hypothesizing, argumentation, and discussion that followed students' initial engagement. When working on the activities—which lasted about 45 min—the students were on task the entire time. In addition, they actively participated in the mathematical discourse the activities supported. We found the Omni useful for a meaningful mathematical experience in terms of "analyzing characteristics and properties of two- and three-dimensional geometric shapes and developing mathematical arguments about geometric relationships" (NCTM, 2000, p. 41) as described by the NCTM and other relevant curricular standards.

During the design and implementation of the mathematical activities, we paid attention to the impact of distracters in students' mathematical explorations and discourse. We wanted to minimize the visual and physical distracters as much as possible so that students focused more on the mathematical features of the shapes instead of color or size of the 3D shapes. This proved successful since none of the groups classified the shapes based on attributes that were mathematically irrelevant.

Table 2 A student's discussion of why he believes color is an irrelevant attribute when classifying 3D shapes

What is said	What is done
[1] Greg: there's one category that I know we're not gonna use, colors, because that would be the easiest one.	[1] All students giggle. They seem to agree with Greg.
[2] Interviewer 1: Well that's one way of categorizing it, but why are you not happy about it?	
[3] Greg: Because it doesn't really matter about the color. It matters about how it feels.	
[4] Interviewer 2: Greg, can you ... can you tell me a bit more about what you just said a moment ago, it doesn't matter about the color, it depends on how it feels? Can you explain to me more what you mean by that?	[4] Interviewer brings Greg's attention back to color.
[5] Greg: Like ... to me ... like it doesn't need to really be green, the square, it doesn't need to be green.	[5] He places the bug onto the front face of the cube, which is green.
[5a] It could be purple, blue, brown, red.	[5a] He moves and rotates the cube with the pointer.
[5b] But you can't have a circle that has edges.	[5b] He places the pointer on the surface of the sphere, which is white.
So, that's what I mean for how it feels.	
[5c] And you can't have a square that's round.	[5c] He places the bug on the front face of the cube, and makes a circular gesture with the pointer.
[5d] But you could have a circle that's green and a square that's white.	[5d] Places the bug on the surface of the sphere, which is white.

In fact, a student in one group explicitly conjectured that color should not be considered as a relevant attribute when classifying 3D shapes, as shown in Table 2.

Greg mentioned that color is irrelevant and considered his physical perception with respect to how the shapes felt more relevant when classifying 3D shapes (Table 2, [1], [3]). He later added that it did not matter whether the shape he investigated (a cube) was of a particular color (Table 2, [5a]). Instead, he focused on the attributes of the shape such as edges and roundness he could (or could not) feel through his experience with the Omni (Table 2, [5b], [5c]).

During the interviews, there were instances where students faced challenges related to cross-modality. In other words, sometimes what students saw visually was not compatible with what they felt physically. Interestingly—rather than constraining students—instances where students dealt with cross-modality often led to rich mathematical exploration and discourse as students examined individually the features of 3D shapes and their planar intersections and negotiated or justified their meanings with other students. Table 3 highlights one planar intersection episode in which the intersection of the 3D shape (a cube) with the plane formed

Table 3 Students examine a planar intersection and deal with the issue of cross-modality

What is said	What is done
[1] Duane: [Referring to the intersection] It looks like a pentagon.	[1] Points to the screen with his hand but does not use the Omni.
[2] James: Yeah, it kinda does. Yeah, it is a pentagon.	[2] Looks at the screen but doesn't use the Omni.
[3] Interviewer1: You said that really quick so why? … What made you think pentagon?	
[4] Duane: Yeah cuz it has five sides.	[4] Points a pencil towards the screen to count the sides a, b, c, d, e of the pentagon shown in Fig. 4b.
[5] Olga: Oh it … it … does feel like a pentagon.	[5] Uses Omni to trace the sides a, b, c (in Fig. 4b) but it's not clear what she considers as the pentagon.
[6] Jessica: A pentagon or a hexagon.	[6] Looks at the screen but doesn't use the Omni.
[7] Duane: No a hexagon has six sides.	[7] He turns to Jessica as he talks.
[8] Jessica: Oh, yeah.	
[9] Interviewer2: So Olga put the bug where the red [the plane] and the blue one [the cube] meet and move it along the path and let's see if it's a pentagon. How would you know if it's a pentagon?	[9] Verbally directs Olga to focus on the intersection, and to use the Omni to test the students' conjecture that the intersection is a pentagon.
[10] Olga: Cuz a pentagon has five sides.	[10] Looks at the interviewer.
[11] Interviewer 2: Okay, so do you feel five sides as you move along?	
[12] Olga: Uh huh … One … two, that's one side, three sides and four.	[12] Nods her head while tracing the sides a, b, c, f of the shape shown (in Fig. 4b).
[13] Jessica: Wait … one … two … three … four.	[13] Surprised, she traces in the air the same sides Olga traced with the Omni.
[14] Olga & Duane: Uh … that's one … two … three … four.	[14] Olga retraces the intersection a, b, c, f using the Omni, as she and Duane count the sides.
[15] Jessica: Because you don't … on the back you don't feel … you don't feel that curve. And you don't technically count that.	[15] Duane looks quite puzzled, while Jessica conjectures that the sides d and e cannot be felt while tracing the intersection with the Omni, and therefore shouldn't be counted.
[16] James: It's kinda like this shape …	[16] By repeatedly looking at the screen he begins to draw on a sheet of paper the sides of the intersection.
[17] Olga: It's just a straight side.	[17] Traces side f with the Omni, while Duane points to each side of the intersection a, b, c, f (in Fig. 4b) with his finger on the screen.
[18] Interviewer2: So how many sides did you end up counting?	[18] Students begin speaking simultaneously after this question.
[19] Olga & Jessica: One … two … three … four.	[19] Olga traces the sides of the intersection a, b, c, f (in Fig. 4b) using the Omni, as she and Jessica count the sides.

(continued)

Table 3 (continued)

What is said.	What is done.
[20] Duane: they're counting four.	[20] Refers to Olga and Jessica.
[21] Jessica: It's not a pentagon.	
[22] Olga: Oh no, actually four. It's not a pentagon.	
[23] James: Oh, it's a trapezoid.	[23] Finishes drawing the sides a, b, c, f (in Fig. 4b), looks at the drawing, and concludes it is a trapezoid.
[24] Duane: A trapezoid?	[24] Briefly glances at James' drawing, but looks confused.
[25] James: Yeah ... the trail that [the bug] is going on is a trapezoid.	[25] Points to the screen with his pencil.
[26] Duane: Oh yeah, I can see that too.	[26] Glances at James' drawing and then the screen, and agrees with James.
[27] Jessica: You see a pentagon but ... you feel a trapezoid.	

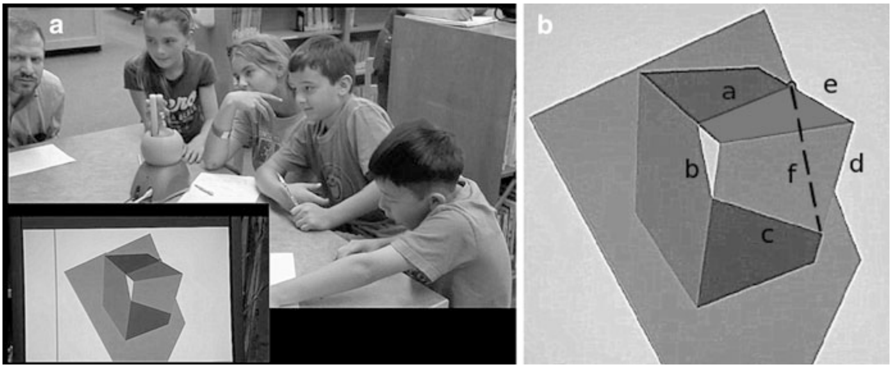

Fig. 4 (**a**) A group of students working on a planar intersection activity. (**b**) The planar intersection of a cube that students discuss in Table 3

a trapezoid (see Fig. 4a, b). The actual intersection consisted of sides *a*, *b*, *c*, and *f;* sides *d* and *e* were not part of the intersection but were among the sides of the pentagon (with sides *a*, *b*, *c*, *d*, *e*) that students saw on the computer screen (Fig. 4b).

Note that students identified the intersection as a pentagon, initially relying only on their visual perceptions (Table 3, [1–2], [6]). When one of the students used the Omni to trace the intersection and felt four sides instead of five (Table 3, [12]), students were faced with a cross-modality conflict resulting from the clash of their visual and physical experiences with the intersection. As students joined in the process of solving the conflict (Table 3, [13–27]), they eventually decided to rely on their physical experiences to conclude that the intersection was a trapezoid.

In our study, advantages of the Omni overpowered its limitations for the open-ended investigation of 3D shapes and their features. The distracters were minimal and generally supported further mathematizing as students continued to connect what they saw with what they felt physically.

Social Interaction as a Mediator of Discourse

One of the leading assumptions that guided our design is the social nature of the learning process as described in our theoretical perspectives section of this chapter. Consistent with this assumption, students worked in groups of four in our study. Each discussion drew on students' prior knowledge as well as cultural, everyday, metaphorical, or scholastic language they brought to the table. Students' socially mediated experience was also connected to the open-ended exploration space; presence and participation of the research team; and the Omni as a mediator of collectively developed mathematical reasoning.

The two types of activities we designed were open-ended in that initially students were only given prompts to classify a family of 3D objects or to talk about the intersection of a 3D object with a plane. Naming the objects, choosing the differential and similar features of the objects to focus on, and identifying the visual and physical prompts present in the investigation setting and associating them with a relevant feature of a 3D object were mostly up to the students. These are possible reasons why we observed continuous hypothesizing, conjecturing, argumentation, and negotiation in students' discourse as they worked on the activities.

The social nature of the activity setting was a crucial factor contributing to the continuity of students' discourse. There were instances in which one student could not find a mathematical name for the 3D shape but another student in the group did. In those cases, we observed that students using non-scholastic language to talk about the shapes could adopt other students' mathematical word use to refer to those shapes. For example, in Table 3 ([12–22]), students talked about the intersection being a four-sided shape but did not name the shape. Only after one of the students drew a picture of what he thought and proclaimed it to be a trapezoid (Table 3, [23]), did the other students start using the term (Table 3, [24], [27]). There were also instances in which students using inaccurate language to refer to a mathematical object were corrected by other students in the group. In Table 3 ([4–8]), Duane corrected Jessica's use of the term hexagon since the shape the students saw as the intersection had five sides whereas a hexagon has six sides.

The presence of the researcher(s) also contributed to the social interaction among students' and their mathematical discourse. For example, in Table 3 ([9]), the researcher directed the students' attention to the intersection of the two shapes and asked Olga to use the Omni after seeing that students focused on the pentagon-looking shape only through their visual perceptions. The researcher's mediation then supported students' interaction among each other and their mathematical discourse as they explored the issue of cross-modality.

Technology—the Omni—was also a factor in students' collective reasoning. For example, challenges students faced due to cross-modality resulted in rich discussions, hypothesis testing, and negotiations (see Table 3). We believe that the incompatibility of students' visual and physical perceptions had some roles in the refutation and negotiation process as students collectively and individually formed their discourse regarding the conflict. In this respect, the technology contributed to the continuation of the social and discursive argumentation space.

The Omni provided visual and physical feedback the students utilized as they explored the characteristics of 3D shapes. Students were engaged in mathematical thinking and generated plausible arguments about 3D shapes through their perceptions and interactions with each other. The learning environment we created through the use of a dynamic multi-modal technology has the potential to give students access to mathematical ideas that may otherwise be inaccessible to them. We have demonstrated this through our findings providing evidence that it is possible for students to generate mathematical discourse—though not necessarily formal—even when the content of the activities go beyond the requirements of their curriculum (see Table 3). We also have evidence that the Omni can be useful in facilitating students' perceptions so that they can identify and differentiate among the relevant and irrelevant features of 3D shapes (see Table 2). These findings suggest that our dynamic multi-modal environment has the potential to present students with meaningful opportunities to explore 3D objects through multiple perceptions, supporting mathematical discourse as students engage in mathematical activities such as exploring, conjecturing, negotiating meaning, and sense-making.

Discussion

Our study explored students' mathematical experiences in multi-modal environments with a particular focus on the Omni. Overall, we observed that the students had rich mathematical experiences as they interacted with the Omni. The experience was rich because it supported hypothesizing and testing those hypotheses as well as negotiating and creating mathematical meanings during social interactions. Our multi-modal environment enabled an open-ended exploration space that students found mathematically engaging. The open-ended exploration space supported students' word use both mathematically and metaphorically. In other words, the multi-modal environment and students' prior knowledge did not limit but supported their mathematical word use. The Omni and students' interactions with each other were among the factors that contributed to the semiotic mediation resulting in students' generation of scholastic and non-scholastic word use. Students used the Omni as a semiotic mediator when their visual perceptions contradicted their physical perceptions. Students also utilized social interaction as a form of mediation when they disagreed with each other; when they corrected each other's word use and adopted words used by peers; and when they individually and collectively hypothesized and tested their hypotheses. Our initial findings suggest that the dynamic, multi-modal exploration space we designed was compatible with our assumptions with respect to providing a rich experience in which individual and collective mathematical meanings are created or enhanced.

Although at its preliminary stage, our analysis provides some evidence that merging dynamic geometry environments with the haptic technology may be significant and relevant for mathematics education, which is an issue that is not addressed extensively by existing research on haptic technology. We propose that it is relevant to integrate haptic technology with dynamic geometry software to offer multiple sources of information-feedback for students; it is not enough to offer a way for students to just see a mathematical object or a scientific model in a static way; they must also engage with it dynamically, tactically, and naturally.

We have learned from our prior work that a key factor in the requirements for our instructional technology product is the core content within which it is used, in a sense redefining the roles of technology from being a prosthetic device for amplifying existing pedagogical practices to one which is a partner in the learning experience supporting the intentional constraints of the activity designer (Dede, 2007; Moreno-Armella et al., 2008). Such intentionality is deeply rooted in our objectives for mathematical learning and discovery and pedagogical strategy. This design philosophy allowed us to develop open-ended, multi-modal activities that did not integrate the technological affordances in just a merely entertaining way but a rather a sophisticated, rich mathematical way.

Our social-cultural perspective focused on how students interacted to make meaning of the complex mathematical tasks we have and the various roles of mediators within the learning environment. We examined student discourse to parse out how multi-modality offers multiple information feedback loops that can enhance meaning-making in mathematics and expose the challenges that students have in understanding 3D geometry.

Building on this work and the rich discursive practices used by the young learners in our present study, we aim to take two significant design trajectories forward:

1. Investigate the affordances of other types of haptic technologies in particular multi-touch (e.g., iPad).
2. Explore a wider variety of mathematical investigations that are categorized by mathematical routines (e.g., linking varying quantities to force feedback).

These two trajectories are deeply intersecting and work is already underway to investigate the impact of iPad technologies given its wide spread use and accessibility.

Another area we intend to focus on is the development of an analysis trajectory to explore more systematically the features of students' mathematical discourse as they interact with each other and the multi-modal environments. In each of these learning environments, our main aim is to observe changes in discourse patterns (e.g., non-scholastic to scholastic word use), as we believe this is a fundamental indicator of intellectual development and mathematical learning. We believe these new learning environments, designed with a deep focus on how the technology can affect access to mathematical ideas, will help the field focus on the centrality of the voice of the child in technology-enhanced mathematical investigations.

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Embodied Interaction as Designed Mediation of Conceptual Performance

Dragan Trninic and Dor Abrahamson

Abstract Can conceptual understanding emerge from embodied interaction? We believe the answer is affirmative, provided that individuals engaged in embodied-interaction activity enjoy structured opportunities to describe their physical actions using instruments, language, and forms pertaining to the targeted concept. In this chapter, we draw on existing literature on embodiment and artifacts to coin and elaborate on the construct of an *embodied artifact*—a cognitive product of rehearsed performance such as, for example, an arabesque penchée in dance or a flying sidekick in martial arts. We argue that embodied artifacts may encapsulate or "package" cultural knowledge for entry into disciplinary competence not only in explicitly embodied domains, such as dance or martial arts, but also implicitly embodied domains, such as mathematics. Furthermore, we offer that current motion-sensitive cyber-technologies may enable the engineering of precisely the type of learning environments capable of leveraging embodied artifacts as both means of learning *and* means for studying how learning occurs. We demonstrate one such environment, the Mathematical Imagery Trainer for Proportion (MIT–P), engineered in the context of a design-based research study investigating the mediated emergence of mathematical notions from embodied-interaction instructional activities. In particular, we discuss innovative features of the MIT–P in terms of the technological artifact as well as its user experience. We predict that embodied interaction will become a focus of design for and research on mathematical learning.

Keywords Embodied interaction • Sociocultural theory • Educational technology • Learning sciences • Mathematics • Proportion • Embodied artifact

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Introduction

Artifacts—cultural objects embedded in social practice—do not cease to fascinate scholars of human cognition and development. As design-based researchers of educational media, the pedagogical artifacts we investigate are historically young technologies. Nevertheless, we view these novel artifacts from the same theoretical perspectives as we would a seemingly humdrum manual tool. Specifically, we ask: What educational gains can such an artifact foster? What can it teach us about human learning?

Yet for the purposes of this particular chapter we are less interested in material artifacts such as a piano or an abacus; neither are we presently concerned with symbolic artifacts such as musical notes or numerals. We focus, instead, on *embodied artifacts*—the cognitive products of rehearsed performances or trained routines,[1] such as the capacity to play *Für Elise* or manipulate an abacus. As we shall argue, novel motion-sensitive cyber-technologies (e.g., Nintendo Wii) are uniquely geared both to craft and leverage embodied artifacts as means of fostering learning and, for researchers, opening a window into how learning occurs.

To illustrate and elaborate the construct of an embodied artifact, which will be central to our thesis, we begin by taking the readers on a guided tour of a few decidedly low-tech instantiations. For the sake of clarity, we initially focus on embodied artifacts within explicitly embodied domains. Later in the text, we will introduce a mathematical, technology-embedded embodied artifact.

To begin, imagine a first surfing lesson in Honolulu, Hawaii. Despite the endless crowds at the sun-drenched Waikiki beach, a neophyte surfer is eager to get in the water. Doing so immediately, however, is likely to invite disappointment. His inability to distinguish the many types of waves, crowding by dozens of other nearby surfers, neuromuscular fatigue from continuous paddling, an uneasy sense of unspoken social hierarchies among more experienced surfers, and a myriad other factors large and small all conspire to quickly dizzy and exhaust the novice. Yet the beachboys (surfing instructors) of Waikiki are famous for claiming they can make *anyone* ride a wave—at least, that is, for a second or two. How?

Before getting in the water, the beachboy will ask the first-timer to lay down upon the surfboard *on the sand*. There, the beginner is taught the elementary sequence of Stand Up (SU[2]) on the surfboard, roughly: (1) kneel; (2) one knee up; and (3) stand up. Only once the beachboy determines the neophyte is capable of executing this basic sequence with confidence does the surfer take to the water. There, the instructor will wait for an appropriate wave, a selection process beyond the novice's

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[1]We invite the reader to compare our "embodied artifact" with the construct of "organizational routines" (Feldman & Pentland, 2003). Though organizational routines share commonalities with embodied artifacts in terms of constituting structured procedures, our construct serves particular interests both in the *embodiment* of knowledge and in *learning* from an artifact-mediated perspective.

[2]By naming this sequence with a phrase commonly used in the context of this particular cultural practice, we are anticipating that it will be signified as a "chunked" performance.

capacity, then push his charge into said wave at the appropriate moment. At this point, all the neophyte must do is paddle hard into the wave—and (attempt to) execute SU. A complex activity is thus partitioned into: (1) select a wave; (2) approach a wave; and (3) SU. Hence the beachboys accomplish their claim of getting anyone to surf by performing (1) and (2) on behalf of their charge and having given the neophyte an embodied artifact, the elementary Stand Up sequence (3).

Note that the function of an embodied artifact is *modular*, in the sense that it can be taught and learned as a standalone sequence of operations, yet later it can be contextualized into a larger system as well as refined via analysis into component parts. For example, the learner becomes more adept at timing, instigating, and performing SU in respect to his distance to the wave (contextualization via integration—recall that the SU sequence was learned *on sand*); and learns the optimum placement for his knee during the kneeling portion of SU (refinement via analysis). We therefore arrive at disciplinary competence *by entering at the level of actions in the form of rehearsed performances*. In other words, embodied artifacts serve as entry into disciplinary engagement—as knowledge through practice (cf. Ericsson, 2002) and reflection (cf. Dewey, 1933; Schön, 1983). Importantly, the learner may rehearse operatory elements of this modular action (SU) independently of any larger activity system (surfing).

Because they are modular and thus portable, embodied artifacts tend to be adaptable in their application. Consider the Flying Sidekick (FS; see Fig. 1b), an aerial attack historically used to strike over ground fortifications (e.g., defensive spikes) and dismount fighters off of warhorses and other beasts of war. In modern times, neither mounted warriors nor spike-barricades pose a serious concern, yet FS continues its existence as more than a text-bound technique. The flying sidekick was practiced for centuries in martial arts halls concurrent to, yet independent of, its combative application: due to its modular nature, it survived the disappearance of its original context, mêlée warfare. Nowadays, FS continues its existence primarily as a test of a learner's discipline and body-mastery.

These two brief examples are meant to illustrate some of the variety of embodied artifacts. While embodied artifacts may work in tandem with other artifacts (as in the case of surfing, operating an abacus, or playing the piano), they may merely require space and gravity (such as dance, see Fig. 2a). So, what does this have to do with learning? The critical common thread is that *all embodied artifacts are rehearsed performances*, ready-to-hand cultural equipment created by "packaging" procedures for skillfully encountering particular situations in the world (cf. Rosenbaum, Kenny, & Derr, 1983, on motor learning via "chunking").[3] Indeed, as we have defined them, embodied artifacts, by mediating one's encounters with the world, constitute an integral part of cultural and individual development. First, humans embody cultural procedures through participating in social activities. Through observation, demonstration, imitation, and training, these cultural procedures become our resources

[3]Esther Gokhale (2008) argues that embodied artifacts, such as those found in traditional dances, serve to encapsulate and preserve traditional physiological knowledge, not unlike how a recipe may preserve traditional (tacit) nutritional knowledge.

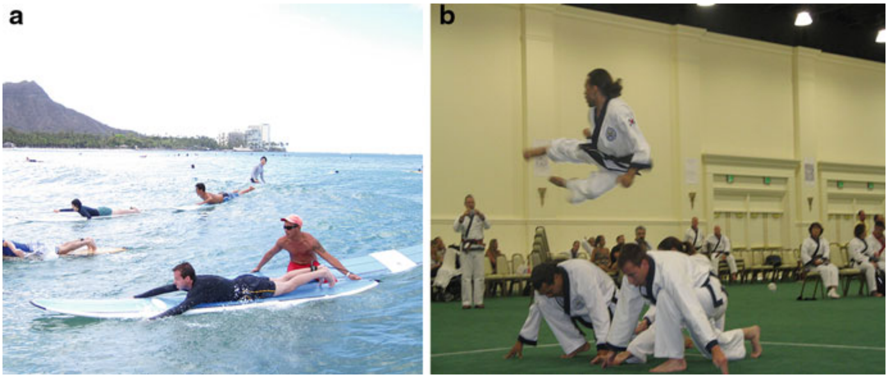

Fig. 1 Embodied artifacts in practice: (**a**) A novice surfer and his coach (*seated*); and (**b**) a Flying Kick demonstration by a Soo Bahk Do Master

Fig. 2 Embodied artifacts take many forms: (**a**) Traditional Cham dancers. (**b**) Mathematical Imagery Trainer (*MIT*) in use by two 10-year-old students, with the tutor (*center*) prompting and monitoring their problem solving

in the form of embodied artifacts. Therefore, through embodied artifacts we store cultural knowledge in the body, using the body as both the material for and means of encountering the world (cf. Dourish, 2001; Dreyfus & Dreyfus, 1999).

As learning scientists, we are interested in the role of embodied artifacts in the emergence of disciplinary competence, particularly disciplines traditionally viewed as "pure" in the sense of independence from the physical world, such as mathematics.[4] Our interest is twofold. First, as we elaborate in the next

[4] As the mathematician G. H. Hardy famously stated, not without pride: "I have never done anything 'useful'."

section, current empirically supported theories of mind suggest that embodiment—having and using a physical body in the world—is fundamentally linked to all reasoning, whether involving "pure" thought or getting one's hands dirty (literally or figuratively). Second, we hold that deliberate use of embodied artifacts in mathematics instruction may render hitherto undetectable learning processes open to both formative assessment in classrooms and empirical scrutiny in laboratories. The idea is simple: if students must perform physically in the service of doing mathematics, then such doing becomes publicly observable rather than hidden away "in their heads."

So, what does this have to do with technology? In addition to our practice as learning scientists, we are designers of pedagogical artifacts. As designers, we are interested in availing of novel technologies to engineer learning environments in which students appropriate embodied artifacts in pursuit of mathematical competence. We then observe students engaged with our design and, hopefully, we learn more about the process of learning (see Collins, 1992 on design-based research as educational science). So doing, in turn, we also learn more about designing learning environments. And on it goes. This chapter is, then, a design-meets-theory-meets-design piece on embodied artifacts and educational technologies.

We begin with observations about the pedagogical potential of embodied artifacts in light of increasingly ubiquitous motion-sensor technologies; these observations, in turn, form the theme of the following section, where we situate our study in the broader context of research on the role of embodiment in human learning and knowing. From the perspective of educational design, we consider the following question: How, if at all, may novel motion-sensor technologies be pedagogically utilized, particularly in light of recent advances indicating the fundamental role of embodiment?

Taking on this question, we present a proof-of-existence educational intervention that leverages cutting-edge technology, namely the Mathematical Imagery Trainer (hence, "MIT," see Fig. 2b). Working with the MIT for Proportion (MIT–P), students move their hands in an environment that changes its state in accord with the ratio of the hands' respective heights, effectively training an embodied artifact of moving the hands in parallel and at different rates, that is, proportionately to each other, with the distance between the hands increasing. Students then reflect on, analyze mathematically, and articulate this spatial–dynamical embodied artifact and then contextualize it as a particular case of proportionality.

Finally, we broaden our discussion to present a particular type of educational design, *embodied interaction*. This type of design, we argue, is ideally suited to foster embodied artifacts in a powerful way towards normative disciplinary competence and, furthermore, enables researchers a window into conceptual development. We then contextualize our arguments by presenting a case of embodied interaction design that suggests how mathematics education and embodied artifacts may be systemically linked in practice.

Theoretical Framework

The Rise of Embodied-Cognition Theory and Its Application to Mathematics Education

Can conceptual understanding emerge from embodied interaction? One answer is that we are physical beings living in a physical world; hence, attempts to understand the development of conceptual thought need look to physical, sensory interaction. Yet this answer appears naïve and, perhaps due precisely to its apparent simplicity, has been ignored by cognitive science throughout the last century. Traditional cognitivist views partitioned mundane interaction into three mutually exclusive constituent facets: perception, thought, and action (e.g., Fodor, 1975; Tulving, 1983). Thinking, or concepts, thus intervenes between perception and action and is characterized as distinct from those real-time embodied processes by token of being symbolic–propositional. Yet in alternative views discussed below, cognition is not secluded or elevated from perception and action but is rather embedded in, distributed across, and inseparable from these corporeal processes.

Embodiment studies rose fast in prominence towards the end of last century[5] through the converging efforts of numerous pioneers in fields as disparate as robotics, psychology, philosophy, and computer science (Brooks, 1991; Gibson, 1979; Varela, Thompson, & Rosch, 1991; Winograd & Flores, 1987). Though many of these perspectives initially emerged in opposition to then-prevalent symbolic architecture models of the mind, embodiment studies have, over the last few decades, burgeoned into a vast area of investigation in their own right—replete with a spectrum of proponents. Within this spectrum, we can roughly identify conservatives, who cautiously posit that reasoning may be connected with *some* aspects of non-corporeal cognition (e.g., Dove, 2009); moderates, who argue that physical action underpins or forms the substrate of cognition (Barsalou, 2010; Goldstone, Landy, & Son, 2010; e.g., Sheets-Johnstone, 1990); and radicals, who hold that cognition itself is merely another action (e.g., Melser, 2004). Indeed, the scope of embodiment studies has grown[6] to the point where scholars concern themselves defining what, exactly, it means to be "embodied" (Kiverstein & Clark, 2009).

In our current work we tend to hold with those who favor the middle ground, and we interpret available empirical evidence as indicating that physical action indeed undergirds thinking, including so-called "abstract" thinking (e.g., thinking about the word *antepenultimate*, or solving for x). We are therefore not concerned by the controversy over what role corporeality plays in thought: indeed, it gives us something

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[5]That said, these studies date back to American pragmatism in relatively recent times (see Chemero, 2009) and Buddhist psychologies many centuries before that (Varela, Thompson, & Rosch, 1991).

[6]It is telling that the most popular workshop at the CHI 2011 conference on Human-Computer Interaction was titled "Embodied Interaction"—and yet the idea of that very workshop was considered untenable in the previous years at the same venue.

to do. A fortiori, as interaction designers of mathematical learning we find ourselves in a unique position to contribute toward resolving this theoretical controversy.

Particularly relevant to our work, embodiment has been presented as a useful framework for theorizing processes inherent to "abstract" disciplinary mastery, including mathematics learning and reasoning (Abrahamson, 2009a; Campbell, 2003; Namirovsky, 2003; Roth & Thom, 2009). One consequence of this view is that observations, measurements, and analyses of physiological activities associated with brain and body behavior can provide insights into lived subjective experiences pertaining to cognition and learning in general, and mathematical thinking in particular. In a strong form, we conjecture that physical action is neither epiphenomenal nor merely supportive to "pure" mental activity. Rather, conceptual understanding—including reasoning about would-be "abstract" contents such as pure mathematics—emerges *through* and is phenomenologically *situated* and *embedded in* actual and simulated perceptuomotor interactions in the world.

Technology for Using the Body

Even as cognitive scientists recognize this essential role of the body, industry has made dramatic advances in engineering technological affordances for embodied interaction. At the time of this writing, Nintendo Wii and Playstation Move players worldwide are waving hand-held "wands" so as to remote-control virtual tennis rackets; iPhone owners are tilting their devices to navigate a virtual ball through a maze; and Xbox Kinect users are controlling video-game avatars with their bare hands—activities hitherto confined to the realms of futuristic fantasy, like flying cars. Moreover, innovative designers tuned to this progress are constantly devising ways of adapting commercial motion-sensor technology in the service of researchers and practitioners (Antle, Corness, & Droumeva, 2009; Lee, 2008). As such, media that only recently appeared as esoteric instructional equipment will imminently be at the fingertips of billions of potential learners. We are excited about the prospect of using these new media to create learning environments centered on embodied artifacts that may be rehearsed and consequently investigated via mathematics, allowing an embodied entry into this disciplinary domain. In the remainder of the chapter we document our attempts to utilize these capabilities and what we have learned doing so.

Learning as Performance: Appropriating Artifact-Bound Conceptual Systems

Our work in the Embodied Design Research Laboratory involves the design, testing, and refinement of pedagogical artifacts as well as the development of theoretical models of learning via interaction with said artifacts (Abrahamson, 2009b;

Abrahamson, Gutiérrez, Lee, Reinholz, & Trninic, 2011). The work we present here is subpart of Action-Before-Concept (ABC), a cluster of cross-disciplinary studies of performance in mathematics, music, climbing, and the martial arts centered around relations between procedural and conceptual knowledge. ABC, writ large, explores the relation between performance and knowledge. It is an inquiry into cultural precedence for pedagogical practice within explicitly embodied domains (e.g., martial arts), wherein procedures are initially learned on trust yet subsequently—only toward perfecting the procedures toward mastery and further dissemination—are interpreted by experts as embodying disciplinary knowledge. The results of these inquiries within *explicitly* embodied domains are then leveraged in investigations of, arguably, *implicitly* embodied domains, such as mathematics. In practical terms of design, much of our work consists of creating learning situations where (bi)manual performances culminate in the learner's guided reinvention of disciplinary knowledge (cf. Freudenthal, 1983). These performances take form as concerted dynamical coordination of embodied, material, and symbolic artifacts.

Thus we espouse a position that learning is the residual effect of engaging artifacts as means of accomplishing one's goals (cf. Salomon, Perkins, & Globerson, 1991; Vérillon & Rabardel, 1995). Yet against the backdrop view of learning as imitating, internalizing, and appropriating the elders' artifactual actions (e.g., Vygotsky, 1987), we foreground the pedagogical philosophy of learning-as-*discovering* these artifacts' horizons in the course of explorative problem solving and theory building (e.g., Karmiloff-Smith & Inhelder, 1975). The challenge for us as designers lies in taking this position on learning and making it a product, that is, designing a pedagogical artifact that encapsulates our theory of learning and respects current embodiment-informed theories of mind. Our response to this challenge is addressed in the following section.

Instructional and Experimental Design

Embodied-Interaction (EI) is a form of technology-supported training activity. By participating in EI activities, users encounter, discover, rehearse, and ultimately investigate embodied artifacts.

A general objective of EI design is for users to develop or enhance cognitive resources that presumably undergird specialized forms of human practice, such as proportional reasoning. As is true of all simulation-based training, EI is particularly powerful when everyday authentic opportunities to develop the targeted schemes are too infrequent, complex, expensive, or risky. Emblematic of EI activities, and what distinguishes EI from "hands on" educational activities in general, whether involving concrete or virtual objects, is that EI users' physical actions are intrinsic, and not just logistically instrumental, to obtaining information (Kirsh & Maglio, 1994). That is, the learner is to some degree physically immersed in the microworld, so that the embodied artifact—instantiated in finger, limb, torso, or even whole-body movements—emerges not only in the service of acting upon objects but rather the motions themselves become part of this learned cultural–perceptuomotor structure. EI is "hands in."

Before describing our design, it is useful to mention two related designs to illustrate the present scope of this emerging field to the reader. Antle et al. (2009) used EI to leverage participants' embodied metaphors of "Music is a physical body movement" as a means of developing fluency with music creation. Another EI design (Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011) used digital dance mats in design intended to improve kindergarteners' fluency with relative numerical magnitude. As EI technologies become increasingly ubiquitous, we anticipate an exponential growth in EI designs catering to various educational needs (see also Birchfield & Johnson-Glenberg, 2010).

Embodied Interaction Design: Mathematical Imagery Trainer

Our overarching design conjecture is that some mathematical concepts are difficult to learn because mundane life does not occasion opportunities to embody and rehearse their spatial–dynamical foundations. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—the misapplication of additive reasoning and procedures to multiplicative situations (see Lamon, 2007 for an overview)—indicate students' lack of appropriate dynamical imagery to ground proportion-related concepts (see also Pirie & Kieren, 1994).

In addition to theories of embodiment, this conjecture is grounded in our previous work. In particular, a pilot study by Fuson and Abrahamson (2005) suggested that children's cognitive difficulties in understanding proportional reasoning may be related to their difficulty in physically enacting proportion. Namely, when asked to enact scenarios involving proportional growth of plants—e.g., "If a rose grows twice as fast as a tulip, can you show me what that looks like with your hands?"—children manually demonstrated a "fixed difference" misconception; that is, they tended to raise their hands while keeping the distance between them fixed. The similarity between this physical "fixed difference" performance and aforementioned, presumably conceptual, "fixed difference" mathematical solution suggested a possible relation between the two.

Accordingly, we engineered an EI computer-supported inquiry activity for students to discover and rehearse the physical performance of a particular proportional transformation of our design. This activity, we reasoned, should train students' physical proportional skill via allowing them to experience "fixed difference" as contextually inappropriate. Let us step back and elaborate on this central design principle.

We define *conceptual performances* as embodied artifacts that are recognized by their enactors as physically inscribing essential semantics and syntax of corresponding mathematical content. For example, 'adding' gestures—such as bringing hands together as if amassing stuff into a single location—are well suited for signifying the arithmetic operation 'addition,' because the gestures objectify and manipulate

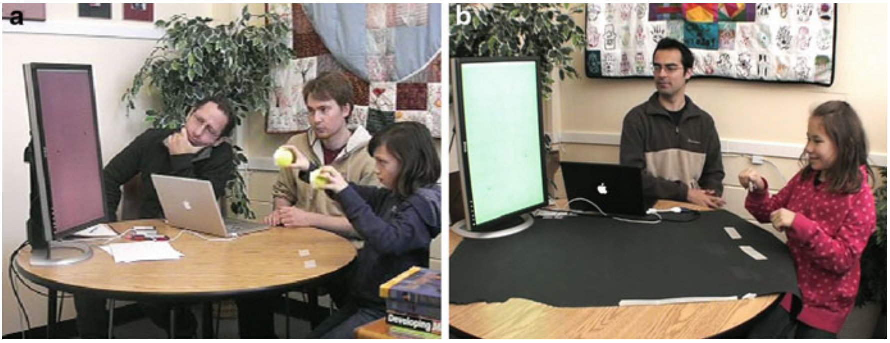

Fig. 3 MIT in action: (**a**) a student's "incorrect" performance (hand height locations do not match a 1:2 ratio as measured from the table) turns the screen *red*; (**b**) "correct" performance (*right hand* at approximately twice the height of the *left hand*, as measured from the table, forming a 1:2 ratio) keeps the screen *green*

imagined quantity sets. As such, the objective of embodied-interaction mathematics learning activities, per our framework, is to foster student development of customized embodied artifacts that subsequently—through symbolic instrumentation and regulating discursive interaction—emerge as conceptual performances. That is, embodied artifacts become conceptual performances once they have served as semiotic resources for discussing, and thus signifying, target curricular content. Our solution to this general design problem is the Mathematical Imagery Trainer (MIT),[7] which the following section further explains (see Fig. 3).

We wish to emphasize that we arrived at building the MIT–P technology only after having considered a variety of "low-tech" design solutions. We feel privileged to be designing in an age where we can expand on the vision of luminaries such as Froebel or Montessori by using available media to expand everyday experience.

Technical and Interface Properties

Our instructional design leverages the high-resolution infrared camera available in the inexpensive Nintendo Wii remote to perform motion tracking of students' hands. In our setup, an array of 84 infrared (940 nm) LEDs aligned with the camera provides the light source, and 3M 3000X high-gain reflective tape attached to tennis balls can be effectively tracked at distances as great as 12 ft. Later iterations used battery-powered, hand-held IR emitters that the students point directly at the Wii camera. The Wii remote is a standard Bluetooth device, with several open-source libraries available to access it through Java or .NET. Our

[7] See http://www.youtube.com/watch?v=n9xVC76PlWc for a video.

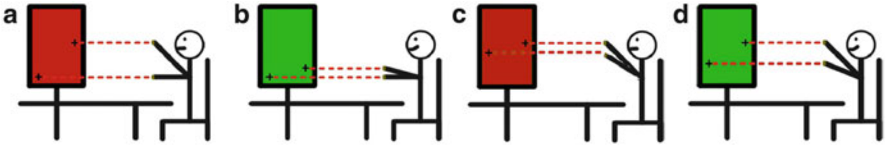

Fig. 4 The Mathematical Imagery Trainer for Proportion (*MIT-P*) set at a 1:2 ratio, so that the *right hand* needs to be twice as high along the monitor as the *left hand* in order to make the screen *green* (a "success"). Schema of student paradigmatic interaction sequence—while exploring, student: (**a**) positions hands "incorrectly" (*red feedback*); (**b**) stumbles on a "correct" position (*green*); (**c**) raises hands *maintaining constant distance between them* (*red*); and (**d**) corrects position (*green*). Compare (**b**) and (**d**) and note the embodied artifact constitutes different distances between the hands/cursors

accompanying software, called WiiKinemathics, is Java-based and presents students with a visual representation on a large display in the form of two crosshair symbols. Further details on technical (Howison, Trninic, Reinholz, & Abrahamson, 2011) and interface (Trninic, Gutiérrez, & Abrahamson, 2011) properties can be found elsewhere.

The orientation of the 22″ LED display (rotated 90 degrees and aligned to table height) and the responsiveness of the trackers are carefully calibrated so as to continuously position each tracker at a height that is near to the actual physical height of the students' hand above the desk. This feature is an attempt to enhance the embodied experience of the virtual, remote manipulation (Clinton, 2006).

In practice, the MIT measures the heights of the users' hands above the desk. When these heights (e.g., 10″ and 20″) match the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of this "mystery device" by proportionate increments the screen will remain green (e.g., raising by 5″ and 10″ to 15″ and 30″, thus maintaining a 1:2 ratio) but will otherwise turn red (e.g., raising by a equal increments of 5″ to 15″ and 25″). In other words, *the embodied artifact of the MIT–P activity is the continuous physical articulation of all the pairs effecting a green screen*. From this perspective, the initial purpose of the MIT–P is to train a particular proportion-relevant embodied artifact of Bimanual Proportional Transformation (BPT, see Fig. 4). As SU in surfing, BPT constitutes an activity whose meaning is situational.

Participants, Protocol, and Data Analysis

Participants included 22 students from a private K–8 suburban school in the greater San Francisco (33% on financial aid; 10% minority students). Care was taken to include students of both genders from low-, middle-, and high-achieving groups as ranked by their teachers. Students participated either individually or paired.

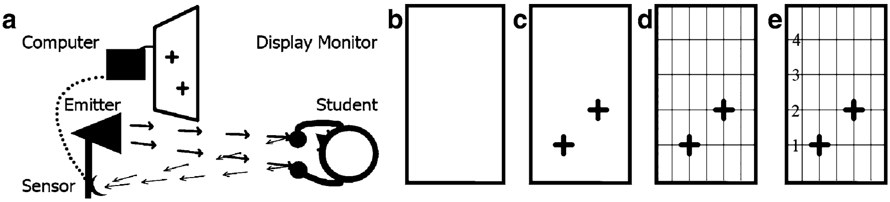

Fig. 5 The Mathematical Imagery Trainer: (**a**) overview of the system featuring an earlier MIT version, in which students held tennis balls with reflective tape. (**b**) 5b through 5e are schematic representations of different display configurations, beginning with (**b**) a blank screen, and then featuring a set of symbolical objects that are incrementally overlain onto the display: (**c**) crosshairs; (**d**) a grid; and (**e**) numerals along the *y*-axis of the grid (in the actual design, the flexible grid and corresponding numerals were initially set by default as ranging from 1 to 10)

Interviews took place in a quiet room within the school facility. Students participated either individually or paired with a classmate in semi-structured clinical interviews (duration: mean 70 min.; SD 20 min.). In addition to the interviewer, typically at least one observer was present, whose duty included taking written notes in real-time, crewing the video camera, and assisting in operating the technological system.

Study participants were initially tasked to move their hands so as to find a position that effects a green screen and, once they achieved this objective, to keep moving their hands yet maintain a continuously green screen. That is, the *participants* needed to discover a means of enacting a green-keeping performance that the *technology* interprets as a transformation of two values sustaining an invariant ratio, such as 1:2. In a sense, the MIT offers students a pre-numerical "What's-My-Rule?" mathematical game. The protocol included gradual layering of supplementary mathematical instruments onto this microworld, such as a Cartesian grid (see Fig. 5, below). Hence, once the proportional-transformation dynamical image is embodied, semiotic resources (mathematical instruments) and discursive support (the tutor) are present for it to be mathematically signified, elaborated, and analyzed.

The interview ended with an informal conversation, in which the interviewer explained the objectives of the study so as to help participants situate the activities within their school curriculum and everyday experiences. Finally, the interviewer answered any questions participants had, with the objective that they achieve closure and depart with a sense of achievement in this challenging task.

Our investigation of the empirical data—field notes and videography—was conducted post hoc in the leisure of the laboratory as collaborative, intensive micro-ethnographic analysis of participants' conceptual ontogenesis (Schoenfeld, Smith, & Arcavi, 1991). Microgenetic analysis is a research methodology, typically applied to video data, where study participants' presumed cognitive trajectories are interrogated and modeled via analyzing their moment-to-moment behaviors, essentially actions, interactions, and multimodal utterance. This methodology is emergent and iterative, in the sense that the researchers' insights from specific events inform successive waves of scrutinizing the entire data corpus. Importantly, microgenetic analysis enables us to maximize the theoretical significance of our work (see Yin, 2009 on analytic generalizability).

Findings

General Findings

We began the chapter by way of introducing the notion of embodied artifacts as well as their function in learning. We also mentioned the increasingly ubiquitous motion-sensor technologies that utilize users' bodily movements. Next, we explained our work at the intersection of these recent theoretical and technological advances, namely designing educational technologies that leverage embodied artifacts in the service of teaching the chronically challenging mathematical concept of proportion. Finally, we are in the position to summarily present some of our findings so as to provide evidence for the feasibility of this design-based research program. Presently we provide some general findings across all students and then focus on a case indicative of the struggles and insights encountered by them all.

Importantly, all students succeeded in devising and articulating strategies for making the screen green, and these initially qualitative strategies came to be aligned with the mathematical content of proportionality. This particular finding serves as a proof-of-existence supporting the conjecture that embodied artifacts such as Bimanual Proportional Transformation create pedagogical opportunities to support student learning of targeted mathematical concepts. Naturally, there existed minor variations in individual participants' initial interpretation of the task as well as consequent variation in their subsequent trajectory through the intervention protocol. However, the students progressed through similar problem-solving stages, with the more mathematically competent students generating more strategies and coordinating more among quantitative properties, relations, and patterns they noticed. We now elaborate on the learning trajectory.

Each student began either by working with only one hand at a time, waving both hands up and down in opposite directions, or lifting both hands up at the same pace, occasionally in abrupt gestures. They realized quickly (<1 min. on average) that the simultaneous actions of both hands are necessary to achieve green and, consequently, that the vertical distance between their hands was critical, although at first they viewed the distance between their hands as fixed. We found this default "fixed distance" approach of importance, as it arguably matches an enduring (mis)conception where students see 2/3 as "the same" as 4/5 (for both the numerator and denominator values respectively increased equally). Indeed, our hope was that by uncovering and addressing such conceptions physically, we could elicit and treat students' pre-numerical conceptual reasoning underlying their arithmetic competence.

The following sequence of insights into problem-solving the MIT-P compiles our observations based on real-time notes and close analysis of the video data from all study participants' interactions. Each step corresponds to students' "successful" or "correct" physical articulations with the MIT-P (that is, "making green") and consequent verbal articulations of what it is they are doing. The numerical example case will be a 2:3 ratio.[8]

Student discoveries:

(a) The actions of both hands are necessary to achieve green.
(b) Green is achieved by positioning the hands at particular stable locations.
(c) The critical quality for achieving green is a type of relation between the hands' relative positions.
(d) These positions can and should be reinterpreted as magnitudes—the distance between the objects or their respective heights above a common base line.
(e) The distance between the hands in correct (green) pairs is not constant—it will necessarily change between correct pairs.
(f) This distance should increase as the pair's height increases (and vice versa).
(g) Moving from one correct position to another can be achieved by increasing the hands' heights differentially, for example, for every 2 units the left hand rises, the right hand should rise 3 units (or the *distance* between the hands should grow by 1 unit from move to move)—a recursive rule for iterated transitions.
(h) The multiplicative relation within each pair—for example at 4 and 6 units the right hand is 1.5 times higher than the left hand—is also a constant across correct pairs.
(i) One and the same number pair (e.g., 2 and 3) expresses three aspects of the interaction: for example 2 and 3 units are the lowest correct integer pair of heights, raising the left hand by 2 units for every 3 raised by the right hand will result in another correct location, and 2/3 or 3/2 is the constant within-pair multiplicative relation.

In brief, students were given the initial opportunity to practice the embodied artifact BPT (Bimanual Proportional Transformation) *a*mathematically. Gradually the protocol encouraged integrating BPT within the broader world of proportional mathematics and providing mathematical tools for analyzing and expressing it in mathematically normative ways (see Fig. 5). As such, BPT gradually instantiated the practice of "proportional reasoning." Similar to the Waikiki surfer who embodied, utilized, contextualized, and refined SU, the students in our study integrated BPT into the broader world of proportional mathematics as well as analyzed its component parts and, so doing, displayed an emerging mastery of the mathematical concept of proportion. The following excerpt provides supporting evidence of this gradual emergence.

[8] Students initially worked with a 1:2 ratio, though the protocol included 1:3, 2:3 ratios and beyond. These more challenging scenarios were introduced only after a student displayed confidence with a 1:2 ratio.

Excerpts from an Empirical Study

Shani was a 5th-grade female student identified by her teachers as "low achieving." During the exploratory phase of the interview, as Shani attempted to discover a means of making the screen green (refer to earlier Fig. 5b, c), she stumbled upon the embodied artifact.

> Shani: [excitedly] Oh! Is it about the distance between these two [pointing to hand-held devices]?

Thus Shani, similar to all our participant students, noticed that an embedded property of the interaction, the distance between her hands, was associated with the desirable feedback. She articulated the "farther-up–more-apart" strategy, that is, the distance between the hands should increase with the hands' elevation in order to effect green (see Item f. in the list of discoveries, above). Once we overlaid the grid on the screen (see Fig. 5d), Shani discovered the "*a*-per-*b*" strategy, by which the hands rise at different yet constant intervals (see Item g.). When we next introduced the numerals (see Fig. 5d), Shani initially availed of them as mere location markers rather than quantitative indices. In particular, she used the numerals to recite the respective locations of her left and right hands, as she iteratively scaled the hands up the screen at 1-per-2 quotas: "One and two, two and four, three and six, and four and eight." Even though the "doubling" multiplicative relation within each of these number pairs is quite striking, Shani was oblivious to this relation. Indeed, it took a gentle suggestion by the interviewer.

> Interviewer: What else can you say about those numbers? One and two ...
> Shani: [continuing] One and two, then two and four, three and six. Hey wait. Um, oh, it's ... [fidgets, becomes animated] It's all doubles! The bottom number, like time ... times two is the top number. [motions at monitor] We had, like, one and two, then three and six, then, um, then four and eight, then five and ten.

Prior to the introduction of the grid, Shani's articulation of the embodied artifact was based on the qualitative relation of "farther up" and "more apart," yet once the grid and numerals were introduced, she instrumentalized them so as to analyze the embodied artifact BPT, rendering the description quantitative. Yet this was not a straightforward process—it is not the case that Shani noticed the green pairs and immediately saw them as proportionally related. Rather, her observation *emerged through interaction* with numerals, which she initially used merely to mark green locations.

Shani continued to discover new properties of the situation through appropriating symbolic artifacts as means of better enacting her strategy. In the following transcription, she responds to the interviewer's request to recount her recent findings, and in so doing she notices a relation between recursive (1-per-2) and multiplicative (double) strategies:

> Shani: Then ... this one [indicates right hand] is always going up by two, and this one [indicates left hand] is going up by one, which would mean that ... that, uhm, this one [right-hand side] is always double this [left-hand side].

Shortly after, Shani accomplished what we believe was an important shift from discrete proportion to continuous proportional reasoning.

> Shani: Wait a minute. A while ago you asked me, uhm, how many green there are. It could really be infinite. Like, because, if it is really all about the distance between them [the hands]—which is, like, I think it is, because it's getting darker depending on that—uhm, then it really doesn't matter where on the screen it is.

We would argue that this level of reasoning is surprisingly sophisticated for a fifth grader—particularly a student labeled by her teacher as "low achieving."

Eventually, Shani coordinated quantitative reasoning with a qualitative feel of "faster."

> Shani: So this one [indicates right controller] should be ... So my right hand should be moving faster. So that it can make ... be going up two spaces on the grid ... while the other one is only going up one.

Note how Shani's embodied experience with the green-making artifact supported her coordination between rate and speed, just as the embodied artifact supported her leap from discrete to continuous reasoning in the previous excerpt. Like the novice surfer, Shani used the embodied artifact as a means of gaining entry to a novel activity—in her case, proportion. Her actions, initially *a*mathematical, became mathematically meaningful, a *conceptual* performance.

Discussion

Epistemic, Cognitive, and Pedagogical Features of Embodied-Interaction Design

One of our design challenges rested on leading students via an embodied artifact towards a conceptual performance without explicit instruction. In the MIT–P activity, this is accomplished via the automated feedback "green," which is triggered whenever the user's bimanual action matches the ratio setting on the interviewer's console interface. The meaning of "green" evolved throughout the activity, and this evolution captures the process of embodying the dynamical artifact as well as integrating and refining it, as follows.

Green: (a) began as the *objective* of the "Make the screen green" task; (b) soon became *feedback* on the perceptuomotor activity, as the users attempted to complete this task objective, thus shaping the emerging embodied artifact; and finally (c) came to function as a *conceptual placeholder* by grouping a set of otherwise unrelated hand-location pairs sharing a common effect of "green." As such, "green" formed, sculpted, and refined the embodied artifact Bimanual Proportional Transformation (BPT), so that BPT—similar to ancient dance or martial arts forms perpetuated across generations—inherited, instantiated, and preserved a cultural practice. Ultimately, once users determined the activity's *mathematical* rule and recognized its power for anticipating, recording, and communicating BPT, this embodied artifact became situated within the larger practice of proportional reasoning.

Learning Is Where the Action Is, Then Down to Operations and Up to Activity

We offer a preliminary account for the emergence of conceptual knowledge from performance as seen in our data. We have found Leontiev's (1981) account of activity useful, and here we modify it to suit our needs. In brief, Leontiev proposes that social activity has a hierarchical structure with three distinct levels; the activity level, the action level, and the operation level. Activities consist of actions; actions, in turn, consist of operations. A typical example goes: building a house (activity), fixing the roof (action), and using a hammer (operation). While the levels are somewhat flexible, the basic message is that every activity consists of some number of actions; each action, in turn, consists of some number of operations.

Our current conjecture is that that *learning from others happens at the middle level of action in the form of embodied artifacts*. As an action becomes an embodied artifact via deliberate training, the learner may analyze her activity, moving "down" to the level of operations and refining those. Furthermore, through participation in discourse broadly construed and observing the embodied artifact in various contexts, the learner comes to understand the larger framework and how the activity integrates within it. That is, she moves "up" toward contextualization. For example, the students in our study practiced the embodied artifact BPT and then mathematically analyzed it by articulating its constituent physical operations with semiotic resources of the discipline. Even so doing, learners generate various observations connecting BPT to their existing knowledge (sometimes appropriately and sometimes not), and, in dialogue with the instructor and each other, come to see the activity and the various ways of mathematically treating it as "cases of" proportion. Thus the initially modular action becomes a conceptual performance.

Conclusion

Throughout this chapter we have been threading together two central themes. First is that movement matters. Physically interacting in a physical world is our mode of being and the roots of our thinking. This thread, then, dealt with the relation between performance and knowledge: namely, we interpreted existing embodiment studies as suggesting that conceptual understanding—including reasoning about would-be "abstract" contents such as pure mathematics—emerges through and is embedded in actual and simulated perceptuomotor interactions in the world. We introduced the construct of an embodied artifact as a means of articulating how cultural practices are “packaged” and “given” to learners, enabling their entry into the world of skillful action and, furthermore, disciplinary competence.

The second theme is that recent decades have witnessed advances not only in theoretical models of embodiment but also remote-interaction cyber-technologies, yet critical questions have remained unanswered regarding the interaction of the two. It is in embodied-interaction (EI) design that our two themes meet. We introduced EI as a form of physically immersed instrumented activity geared to augment everyday learning by crafting embodied artifacts targeted towards specific disciplinary practice, such as proportional reasoning. In pursuing these problems, our strategy has been to engage in conjecture-driven cycles of building, testing, and reflecting on these two themes. The current text aimed to share our conviction that EI offers unique affordances for teaching mathematical concepts via cultivating the conceptual performance of embodied artifacts.

To the extent that mathematics-education researchers and practitioners take seriously the grounded-cognition thesis, the community should pay far greater attention to the somatic substrate of subject matter. Students’ perceptuomotor manifestations as they engage in learning activities could be far more than mere support for, or communicative visualization of essentially abstract notions. On the contrary, notions become abstracted only through bodily incorporation. In fact, the grounded-cognition approach suggests that there need not be any tension at all between concrete and abstract ideas, because intrinsically embodied mathematical notions can transcend local contexts.

We anticipate that, when coupled with recent cyber-technological advances, EI stands to become a focus of design for and research on mathematical learning. As our work indicates, EI activities serve as highly useful empirical settings for research on the ontogenesis of mathematical concepts and, more generally, relations between performance and knowledge in mathematics education. These immersive activities create opportunities for design-based researchers to observe and help resolve tension between theoretical conceptualizations of: (a) unreflective orientation in a multimodal instrumented space, such as riding a bicycle or playing pong; and (b) reflective mastery over the symbol-based re-description of this acquired competence, such as in mathematical numerical forms.

We hope this line of investigation will contribute to developing a model of embodied mathematics instruction. Researchers could look to diverse cultural–historical forms of physical performance, such as music, dance, and the martial arts, as ethnographic entries into traditional and indigenous pedagogical acumen. The skills inherent to these cultural practices might, at first blush, be viewed as *a*conceptual and, as such, hardly bearing on mathematical reasoning and learning. Yet as recent theoretical and empirical work, including our own, suggests, our shared biology implies that even the most abstract of mathematical concepts may first be embodied, then verbally articulated, and finally reified in conventional semiotic forms. Such issues are more than academic, for all too often proverbial lines are drawn in the sand regarding the importance of “conceptual” knowledge versus “procedural” performance (e.g., see Schoenfeld, 2004 on “math wars”). Yet corporeal actions performed in the context of disciplinary activity constitute vital aspects of cognition and knowledge (cf. Alač & Hutchins, 2004; Kirsh, 2009, 2010), so that knowledge is developed, elaborated, and expressed as situated conceptual performance. In our future work, we will continue to investigate the embodiment of mathematical concepts through the reciprocal efforts of developing theories of embodied learning and designing educational technologies.

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Sensuous Cognition

Luis Radford

Abstract In the first part of this chapter I sketch an approach where human cognition is conceptualized in non-dualistic, non-representational, and non-computational terms. The basic idea is that cognition is a feature of living material bodies characterized by a capacity for *responsive sensation*. The sketched approach, which I term *sensuous cognition*, refers precisely to this view where sensation is considered to be the substrate of mind, and all psychic activity (cognitive, affective, volitional, etc.). I argue that, as far as humans are concerned, responsive sensation evolves—both at the phylogenetic and ontogenetic levels—interwoven with the material culture in which individuals live and growth. As a result, human cognition can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, transforming, and making sense of the world. In the second part of the chapter, I present classroom experimental data involving 7–8-year-old students dealing with pattern recognition. The classroom data allow me to illustrate the interplay of the various sensuous modalities in mathematical cognition. I end the chapter suggesting that a sensuous-based materialistic monistic view of cognition needs to attend not only to the plethora of sensorial modalities that teachers and students display while engaging in mathematical activities, but also to the manner in which sensorial modalities come to constitute more and more complex psychic *wholes* of sensorial and artifactual units.

Keywords Cognition • Sensation • Plasticity • Multimodality • Material culture • Gestures • Semiotic node • Pattern recognition

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Introduction

Mathematics has been enduringly renowned for its impressive use of symbols. Or at least this has been the case since the progressive invention of a succinct symbolism in the Renaissance by mathematicians such as Piero de la Francesca, Rafael Bombelli, and François Viète. Since then, mathematics has been conceived, implicitly or explicitly, as an essential activity mediated by written signs. However, mathematics can also be seen as an activity profoundly mediated by artifacts and signs other than those of the written register. This is certainly the case in classroom mathematics activity, where not only written signs, but also speech, gestures, body posture, kinesthetic actions, and artifacts mediate students' activities in a substantial manner. Figure 1 shows three Grade 11 students during a trigonometry lesson where they devised a formula to describe the position P(x(t), y(t)) of a train that moves at a constant speed along a circular route. The student to the left measures time with a chronometer; following the train, the student in the middle measures space with a pen; the student to the right coordinates the other two students' action and takes notes on a field sheet.

Now, what exact role do we ascribe to the artifacts to which student resort? What exact role do we ascribe to speech, body posture, gestures, and signs? Customarily, in traditional cognitive psychology, artifacts are considered convenient devices that reveal the functioning of the mind; they are hence thought to play a secondary role in cognition in general, and mathematical cognition in particular (see, e.g., Piaget's use of artifacts in his research). Body posture, gestures and other embodied signs have suffered the same fate. In some branches of traditional cognitive psychology, the mind is conceived of as a computational device; the research focus is on the

Fig. 1 Grade 11 students investigating the equation of the point P(x(t), y(t)) of a train that moves around a circle at constant speed

linguistic formats through which information is transmitted and decoded (e.g., declarative vs. procedural sentences). In other trends, when attention has been paid to embodiment and the sensorial realm—like in Piaget's epistemology—the body appears to play a role in the early stages of intellectual development only (the Piagetian sensory-motor stage), apparently disappearing in more advanced stages.

A new research trend, however, offers a different approach to the understanding of human cognition. In this trend our tactile-kinesthetic bodily experience of the world and our interaction with artifacts and material culture are considered as much more than merely auxiliary or secondary elements in our cognitive endeavours. For instance, Sheets-Johnstone (2009) argues that, as a result of our biological makeup, we are naturally equipped with a range of archetypal corporeal-kinetic forms and relations that constitute the basis on which we make our ways into the world. Echoing an increasing number of neuroscientists and linguists, Seitz (2000) contends that the basis of thought is to be found *in* the body. Yet, it is clear that there is not just one way in which to theorize the cognitive role of the body. As a result, it is not surprising to find a variety of perspectives on what has come to be termed "embodied cognition."[1] Thus, in her review of current perspectives, Wilson (2002, p. 626) highlights six claims to which, she argues, theorists of embodied cognition resort in their work:

1. cognition is situated;
2. cognition is time-pressured;
3. we off-load cognitive work onto the environment;
4. the environment is part of the cognitive system;
5. cognition is for action;
6. off-line cognition is body-based.

The most problematic of those claims, Wilson finds, is the fourth. Indeed, the fourth claim requires a completely different point of departure about our ordinary dualistic conceptualizations of thinking. As long as we keep a *dualistic* approach to mind, Wilson's fourth claim remains problematic. This is why it is not enough to merely put the body and material culture back into thought. What we need is a different starting point where thinking and environment are not conceptualized as separate entities.

Dwelling upon Vygotsky's and Leont'ev's work and enactivism (Maturana & Varela, 1998), in this chapter I elaborate on what I have previously termed *sensuous cognition* (Radford, 2009). Sensuous cognition refers to a non-dualistic, non-representational, and non-computational view of the mind. Starting from the premise that cognition and environment are intertwined entities, the basic idea is that cognition is a feature of living material bodies characterized by a capacity

[1] For some embodied perspectives in mathematics education, see the special isssue of *Educational Studies in Mathematics* edited by Edwards, Radford, and Arzarello (2009), and the special issue edited by Radford, Schubring, and Seeger (2011) in the same journal. See also Bautista and Roth (2011) and the seminal book of Lakoff and Núñez (2000).

for *responsive sensation*. In subsequent sections I argue that, as far as humans are concerned, responsive sensation evolves—both at the phylogenetic and ontogenetic levels—intertwined with the material culture in which individuals live and grow. As a result, cognition can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, transforming, and making sense of the world.

However, I should hasten to make clear that my interest is not purely cognitive. As a mathematics educator I am deeply interested in exploring how a non-dualistic view of mind translates into teaching and learning contexts. The chapter is divided into two parts. The first part is of a theoretical nature. The goal is to present a cogent sensuous-based monistic view of cognition. To reach this goal, I need to present in some detail the concept of sensuous cognition and to discuss some theoretical constructs, such as sensation—its plasticity and multimodal nature—as well as the entanglement of sensuous cognition and material culture. In the second part of the chapter, I present classroom experimental data involving 7–8-year-old students dealing with pattern recognition. The classroom data allows me to illustrate the interplay of the various sensuous modalities in mathematical cognition. I close the chapter by suggesting that a sensuous-based monistic view of cognition needs to attend not only to the plethora of sensorial modalities that teachers and students display while engaging in mathematical activities, but also to the manner in which sensorial modalities come to constitute more and more complex psychic *wholes* of sensorial and artifactual units.

Sensuous Cognition

The idea of sensuous cognition that I would like to advocate here rests on a non-dualistic view of the mind. In dualistic accounts, the mind is conceived of as operating through two distinctive planes, one internal and one external. The internal plane is usually considered to include consciousness, thought, ideas, intentions, etc., while the external plane refers to the material world—which includes concrete objects, our body, its movements, and so on. In opposition to this dualistic view, drawing on Vygotsky (1987–1999) and Leont'ev (1978, 2009), and Maturana and Varela (1998), I adopt a monistic position according to which mind is a property of matter. More specifically, mind is conceptualized as a feature of living material bodies characterized by a capacity for *responsive sensation*.

Sensation is a phylogenetically evolved feature of living organisms through which they *respond to, reflect* or *act* on their environment. Since the organism is itself a part of the material world, any reflection of reality is nothing other than a function of a material, corporeal organism (Leont'ev, 2009, p. 12) against a material milieu. As a result, reflection and action do not occur in two *separate* planes. They occur in the same plane—the plane of life.

Now, reflection, as understood here, cannot be considered a passive act of receiving sensorial impressions, as seventeenth and eighteenth century empiricists hold.

As its etymology suggests, in reflection the organism "bends back" something—the "reality" or the "environment," as the organism perceives or feels it. Reflection hence involves both (1) something that transcends the organism as such (something that, in order to differentiate it from the subject itself, we can call *objective*, namely *the object of reflection*), and (2) the *reflected object*, something that is *subjective* (in the sense that reflection depends on the specific organism reflecting the environment).[2]

It is worth noticing that, phylogenetically speaking, the relationship between the subjective world and objective reality is not absolute (Maturana & Varela, 1998); nor is it something that is given *a priori* (Leont'ev, 2009). As far as humans are concerned, this relationship is *dialectical*, where the objective world and its subjective reflection co-evolve. On the one hand, mind can only arise from the progressive complexity of processes of life; on the other hand, more complex conditions of life require organisms to have the capacity to reflect reality through more complex forms of sensation. This is why mind is not just something added to the organisms' vital functions: mind "arises in the course of [the organisms'] development and provides the basis for a qualitatively new, higher form of life—life linked with mind, with a capacity to reflect reality" (Leont'ev, p.18).

The historical origin of the printing press is a good example to illustrate these ideas. The apparition of the printing press can only be understood within the context of a complexification of previous forms of human labour, the unprecedented systematization and mechanization of actions in various spheres of life in the late Middle Ages, and the ensuing transformation of the human senses—e.g., mainly vision and tactility (McLuhan, 1962). Reciprocally, for such a complexification to occur, the capacity for psychic reflection of material reality was required. In short, the human mind, as a culturally and historically evolved form of sensation, and human consciousness—that is to say, the manner in which the individual's reality is *subjectively reflected*—can only be understood in light of the co-evolution of the nervous system, more evolved forms of sensation, and the concomitant complexity of social practices and material culture.

To sum up, instead of being purely "mental," reflection and its products remain, one way or another, intertwined with the environment that is been reflected and with the organism's capacities for sensation. Mind, in this context, is the ability of organisms to reflect, and act on, the reality around them. Thinking, memory, imagination and other cognitive functions are directly and indirectly related to a large range of sensorimotor functions expressed through the organism's movement, tactility, sound reception and production, perception, etc. What I term *sensuous cognition* refers precisely to this view where sensation is considered to be the substrate of mind, and of all psychic activity (cognitive, affective, volitional, etc.). In the next section I dwell in more detail on this point.

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[2]My use of the adjective *objective* does not refer to claims about truth. It is rather a claim about something that is distinct from the organism, something that *objects* the organism; in other words, the term *objective* refers to something that we can term *Otherness*.

The Plasticity of Human Sensation

A sensuous approach to mind cannot avoid noticing the specific *plastic* nature of human sensation alluded to in the previous section, and without which the co-evolution of life, social practices, material culture, and sensation would be meaningless. Indeed, as the German social theorist Arnold Gehlen (1988) argues, animals are endowed from birth with specific instincts and highly developed sensorial systems that make them fit to survive in specific environments. The human senses, by contrast, are highly unspecialized. Thus, the hearing, smelling, and perceptual sensitive organs of the deer trigger an alert signal when presented with recognized clues in the environment, clues that would remain beyond the range of human attention. Similarly, the acute visual perception of the eagle and the thermal sense of some predators surpass humans' sensorial acuity.

To cope with this *lack* of particular instinctual and environment-specific sensorial systems of animals, humans *develop* their highly unspecialized sensorial functions into complex forms that allow them to adapt to virtually any environment. Tactility, for instance, becomes a means to distinguish between temperature differences, soft and rough surfaces, and distances; in this way, tactility is transformed in "an intelligence in action" (Le Breton, 2007, p. 152). Through their hands humans grip things and explore and palpate the environment in movements that can become extremely specialized (Wilson, 1998). In short, the specific instincts and highly developed sensorial systems that we find in animals are compensated for in humans by the *plasticity* of their senses and the achievable levels of specialization that the senses can acquire.

Multimodal Sensuous Cognition

In the previous section we have mentioned that one chief characteristic of the human senses is constituted by their plasticity. Another central characteristic is the human senses' *interrelated* development and functioning. What this means is that the various senses develop in *integrative* manners and come to collaborate in ways that are truly specific to humans. The result is that the human mind and cognition are not merely sensuous but also *multimodal*. This idea is certainly a cornerstone in the new approaches to the human mind (Gogtay et al., 2004; Lewkowicz & Lickliter, 1994; Lickliter & Bahrick, 2000).

Indeed, the senses *collaborate* among themselves, allowing us to come up with a complex perception of reality. Touch and sight, for instance, collaborate with each other. Through a tactile experience, I can feel the weight of an orange; through a perceptual one, I can have a sense of its relative chromatic characteristics. Later, I can feel its porous skin even if it is out of my actual tactile reach. Touch and sight collaborate at close distance in their experience of the world. Sight and language, by contrast, collaborate at a long distance. Knowing hence is ensured through a multi-modal sensorial experience of the world. The sensorial modalities are *integrated* into a complex of properties that bring together different sensuous modal experiences (e.g., shape from perception and rigidity from tactility). Referring to vision as a complex modal experience, Varela says:

> vision is a patchwork of visual modalities, including at least form (shape, size, rigidity), surface properties (color, texture, specular reflectance, transparency), three-dimensional spatial relationships (relative positions, three dimensional orientation in space, distance) and three-dimension movement (trajectory, rotation). It has become evident that these different aspects of vision are emergent properties of concurrent subnetworks that make a visual percept coherent. (Varela, 1999, p. 48)

This multi-sensory characteristic of cognition is not specific to humans; it is shared by insects (Wessnitzer & Webb, 2006) and other primates as well. However, compared to the case of insects and other primates, human sensorial organs collaborate to a greater extent (Gómez, 2004; Köhler, 1951), so that what we perceive or touch is endowed with a variety of sensuous coordinated characteristics. For instance, the human hand does not only feel the trace of the object. We can say that the hand also "perceives its colour, its volume, its weight" (Le Breton, 2009, p. 151).

The Cultural Shaping of Senses

A third chief characteristic of the human senses is the manner in which they co-evolve with culture. Indeed, our senses are not merely part of our biological apparatus. The raw range of orienting-adjusting biological reactions we are born with is transformed into complex, historically constituted forms of sensing. The cultural nature of this transformation can be illustrated through the example of a child who was found in the woods of Aveyron, between Montpellier and Toulouse in France, in 1800. The young child

> who trotted and grunted like the beasts of the fields ... was apparently incapable of attention or even of elementary perceptions ... and spent his time apathetically rocking himself backwards and forwards like the animals at the zoo. (Humphrey in Itard, 1962, p. vi)

The so-called wild boy of Aveyron—or Victor as he was called later—was placed under the care of Dr. Jean-Marc-Gaspard Itard, who designed a series of exercises to teach Victor to speak and catch up with the development of his intellectual faculties. Thus, the boy started distinguishing incrementally between the sound of a bell and that of a drum, went on to discern among the tones of a wind instrument, and later distinguished between vowels. Itard comments:

> It was not without difficulty and much delay that I succeeded at last in giving him a distinct idea of the vowels. The first that he distinguished clearly was O, next the vowel A. The other three presented greater difficulty and for a long time he confused them. (Itard, 1962, p. 58)

The training of the sense of hearing was followed by the training of the sense of sight and touch. Although from a developmental viewpoint Victor was not able to catch up completely, the example shows clearly that without life in society the raw biological ensemble of orienting-adjusting reactions with which we are born remains undeveloped. As we live in society, interact with others, and participate in more or less specialized forms of training, the biological orienting-adjusting reactions undergo cultural transformation and are converted into complex historically constituted forms of sensing, leading to specific features of human development and the concomitant forms of cultural reflection. This is why in the process of development the child not only matures, but is also equipped with sophisticated ways of seeing, touching, hearing, tasting, and so on.

The ontogenetic process of the cultural transformation of the senses has been investigated in great detail in the past few years. To mention but one example, Zaporozhets (2002) reports research with 3- to 5-year old preschoolers who were learning to discriminate between variants of two geometric figures: triangles and quadrilaterals. In the beginning, the preschoolers were making a substantial number of errors. Then, they were invited to trace systematically with a finger the outline of the figure, paying attention to directional changes of the motions at angles, and accompanying the tactile exploration with side counting (one, two three . . .). The investigator reports that at this stage perception was accomplished through the tactile experience, while the eye performed an auxiliary role. "Later," Zaporozhets says, "the eye developed the ability to solve these types of perceptual tasks independently, consecutively tracing the outline of a figure, as it was earlier done by a touching hand" (2002, p. 31). During this process, the eye undergoes a transformative change: "initially, the eye motions have an extremely extensive nature, consecutively tracing the entire outline of the perceived figure and simulating its specifics in all details" (p. 32). In a subsequent stage, the eye's motions "gradually begin to decrease and to focus on the individual, most informative attributes of the object" (p. 32).

The Artifactual Dimension of Sensuous Cognition

A closer look at the previous examples shows that the new cultural forms of sensation are deeply interrelated with the use of *artifacts*. Indeed, in the first example, Itard makes recourse to two artifacts—a drum and a bell—and the artificial sounds that they produce. Victor learns to distinguish between them. Aural discrimination is consequently *shaped* by the cultural sounds that the ear meets. Itard also uses the cultural distinction between vowels of the human artifact par excellence—speech. In the second example, preschool children develop a mathematical form of perception that allows them to distinguish between cultural categories of geometrical figures. In doing so, the children have recourse to the material objects whose contours they cover with a finger while using numbers to count aloud. What these examples show is that our individual senses evolve intertwined not only one with the other senses (which is the claim I made in section "Multimodal Sensuous Cognition"), but also with the *materiality* of the objects in our surroundings. The materiality that shapes our senses is not, however, reduced to inert matter, but, as the examples show, matter already endowed with meaning (e.g., 'triangularity,' 'quadrilarity,' the bell's sounds, etc.).

It is this key role of artifacts in the constitution and evolution of forms of sensing and reflecting that Luria and Vygotsky underlined in their work. The use of artifacts, they contended, constitutes the first phase in cultural development (Luria & Vygotsky, 1998; Vygotsky & Luria, 1994). Such a phase marks the emergence of new forms of actions and reflection and the concomitant appearance of psychic functions. Vygotsky and Luria paid particular attention to the question of memory and argued that the construction of artificial signs, like a knot, transforms "natural" or eidetic memory (i.e., memory based on the recording of external impressions with great photographic precision). In some cultural formations knot-use appears as a material mechanism used to register events in information encoding systems, and gives rise to a new, cultural form of memory. Naturally, knots or writing are not the only historical artifacts at the base of the transformation of memory and other cognitive functions. Artifacts in general create *dispositions* through which to think, perceive, remember, etc. For instance, current electronic media and their forms of dynamic visualization are creating new *dispositions* through which to engage the world, much as pictorial representations and arithmetical calculations did in the Renaissance. In both cases, the senses are transformed to respond to new possibilities opened up by changes in material culture. For us mathematics educators, the challenge is to understand the sensuous possibilities of the new material culture so as to exploit it in design contexts as well as in teaching-learning cyber- and visually-based activities.

All in all, these examples amount to making a point about the embedded nature of artifacts in the evolution of our ways of sensing and reflecting. They stress the fundamentally cognitive role of artifacts and material culture in the ways we come to know. The claim that I am making, hence, goes beyond the conceptualization of artifacts as merely *mediators* of human thinking and experience, or as prostheses of the body. Artifacts do much more than mediate: they are a *constitutive part* of thinking and sensing. Behind this view lies, of course, the general concept of mind as a property of matter. This property expresses the enactive relationship between materiality and mind that inspired Vygotsky's, Luria's, and Leont'ev's work and that Bateson (1973) illustrates so nicely in his example of the blind person's stick. It is in this context that anthropologists Malafouris and Renfrew (2010) claim that we can speak of things as having a cognitive life. They say: "things have a cognitive life because minds have a material life" (p. 4).

Sensuous cognition is hence a perspective that highlights the role of sensation and materiality as the substrate of mind and of all psychic activity. But in contrast to other approaches where the focus remains on the individual's body, sensuous cognition offers a perspective where sensation and its cultural transformation in sensing forms of action and reflection are understood to be interwoven with cultural artifacts and materiality at large. Sensuous cognition calls into question the usual divide between mind and matter and casts in new terms the classical boundaries of mind.

A Classroom Example

As mentioned in the introduction, my interest is not purely cognitive. I am first of all interested in exploring how the idea of sensuous cognition translates into teaching-learning contexts. It is in this spirit that in this section I would like to discuss an example from a Grade 2 class (7–8-year-old students) involving the generalization of an elementary figural sequence. The example comes from the first of a series of lessons that were intended to introduce the students to a cultural-historical form of thinking that we recognize as algebraic. The first step is getting acquainted with what matters and what has to be attended to in the terms of a figural sequence. Figure 2 shows the terms of the sequence given to the students.

Mathematicians would attend without difficulty to those aspects of the terms that are relevant for the task at hand: they would, for instance, see the terms as divided into two rows and notice the immediate relationship between the number of the term and the number of squares in each one of the rows. The perception of those variational relationships usually moves so fast that mathematicians virtually do not even notice the complex work behind it. They would also extend without difficulty the noticed property of the rows to other terms that are not present in the perceptual field, like Term 100, and conclude that this term has100 + 101 squares, that is 2001 (see Fig. 3). Or even better, that the number of squares in any term, say Term n, is 2n + 1.

Yet, the novice eye does not necessarily see the sequence in this way. Figure 4 shows an example of how some Grade 2 students extend the sequence beyond the four given terms shown in Fig. 2.

The student focuses on the *numerosity* of the squares, leaving in the background the *spatiality* of the terms (Radford, 2011a). We cannot say, I think, that the student's answer shown in Fig. 4 is wrong. The answer makes sense for the student, even if it is probably true that by focusing on the numerosity of the terms of the sequence, it might be difficult later on to end up with a general formula like '2n + 1.' This is in fact what we have observed again and again in our research with older students (13–17-year-old students). In the latter case, the students tend to rely on trial-and-error

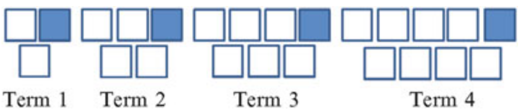

Fig. 2 The first terms of a sequence that 7–8-year-old students investigate in a Grade 2 class

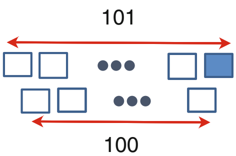

Fig. 3 A frequently reported quick imagination of Term 100 by the trained eye

Fig. 4 Terms 5 and 6 as drawn by a Grade 2 student

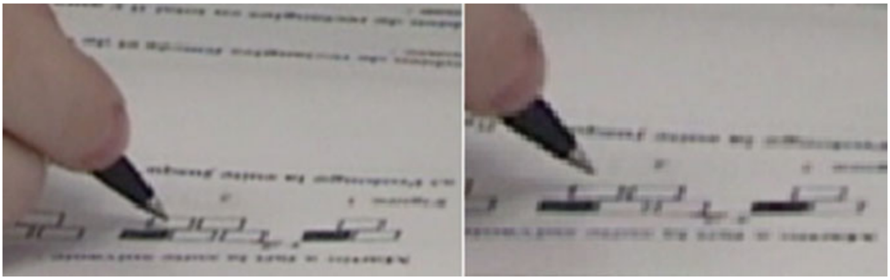

Fig. 5 A student pointing to the top row (*left*) and to the bottom top (*right*) of Term 2

methods that, as I have argued elsewhere, are not algebraic, but arithmetic in nature (Radford, 2008, 2010a, 2010b).

The issue is not that the students do not see the two rows of the terms. In Fig. 5, we see a Grade 2 student pointing with his pen to the top row, then to the bottom row, after moving the pen across the top and bottom rows to properly distinguish between them. However, when the student draws Term 5, the *spatial* dimension of the terms is relegated to a second plane and does not play an organizing role in the drawing of the term. He draws a *heap* of rectangles. The issue is rather about not realizing yet that the spatiality of the terms provides us with clues that are interesting from an algebraic viewpoint.

To become sensitive to the cultural-historical algebraic forms of perceiving terms in sequences like the one discussed here, students engage in processes that are far from mental. They engage with the task of exploring the sequence in a sensuous manner. I would like to illustrate this point by discussing the way in which the teacher and a group of students reflect on Term 8 of the sequence. As mentioned previously, the first question of the mathematical activity consisted in extending the terms of the sequence up to Term 6. Then, in questions 2 and 3, the students were asked to find out the number of squares in terms 12 and 25. In question 4, the students were given a term that looked like Term 8 of the sequence (see Fig. 6). They were told that this term was drawn by Monique (an imaginary Grade 2 student) and encouraged to discuss in small groups to decide whether or not Monique's term was Term 8. The trained eye would not have difficulties in noticing the missing white square on the top row. The untrained eye, by contrast, may be satisfied with the apparent spatial resemblance of these terms with the other terms of the sequence and might consequently fail to note the missing square.

Let me focus on the discussion that a group of students had with the teacher—a group formed by James, Sandra and Carla. When the teacher came to see their

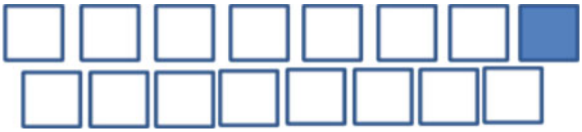

Fig. 6 The students were requested to discuss whether Monique's term is Term 8 of the given sequence

work the students had already worked together for about 32 min. They had finished drawing Terms 5 and 6, tried (unsuccessfully) to find the number of squares in Term 12 and 25, and answered the question about Term 8 (which they considered to be Term 8 of the sequence). The teacher engaged in collaborative actions to create the conditions of possibility for the students to perceive a general structure behind the sequence. She started by referring to the first four terms of the sequence that were drawn on the first page of the activity sheet (Monique's term, which the students examined previously, was drawn on the second page of the activity sheet and was hence not in the students' perceptual field in the first turns of the following episode):

1. Teacher: We will just look at the squares that are on the bottom (*while saying this, the teacher makes three consecutive sliding gestures, each one going from bottom row of Term 1 to bottom row of Term 4; Pics. 1–2 in Fig. 7 show the beginning and end of the first sliding gesture*). Only the ones on the bottom. Not the ones that are on the top. In Term 1 (*she points with her two index fingers to the bottom row of Term 1; see Pic. 3*), how many [squares] are there?
2. Students: 1!
3. Teacher: (*Pointing with her two-finger indexical gesture to the bottom row of Term 2*) Term 2?
4. Students: 2! (*James points to the bottom row of Term 2; see Pic. 4*).
5. Teacher: (*Pointing with her two-finger indexical gesture to the bottom row of Term 3*) Term 3?
6. Students: 3!
7. Teacher: (*Pointing with her two-finger indexical gesture to the bottom row of Term 4; see Pic. 5*) Term 4?
8. Students: 4!
9. Teacher: (*Making a short pause and breaking the rhythmic count of the previous terms*, *as if starting a new theme in the counting process*, *she moves the hand far away from Term 4 and points with a two-finger indexical gesture to the place where one would hypothetically expect to find Term 8; see Pic. 6*) How many squares would Term 8 have on the bottom?
10. Sandra: (*hesitantly, after a relatively long pause*) 4?

In Line 1, the teacher makes three sliding gestures to emphasize the fact that they will count the bottom row of the four given terms. The gestural dimension of the teacher-students' joint activity is somehow similar to the material-tactile experience of the students who, in the aforementioned experiment reported by Zaporozhets (2002), follow the contour of shapes with their fingers. Here, the material-tactile dimension is carried out instead with gestures through which the

Fig. 7 The teacher's and students' sensuous (perceptual, gestural, tactile, aural, vocal) engagement in the task

teacher suggests a cultural form of perceiving the terms of the sequence—one in which the mathematical ideas of variable, and the relationship between them, are emphasized.

Now, the teacher does not gesture silently. Gestures are coordinated with utterances. This is why it might be more useful to consider the teacher's utterance as a *multimodal utterance*: that is to say as a bodily expression that resorts to various sensorial channels and different semiotic registers. In this case, the teacher coordinates eye, hand, and speech through a series of organized simultaneous actions that orient the students' perception and emergent understanding of the target mathematical ideas. In our previous work we have termed *semiotic node* this complex coordination of various sensorial and semiotic registers (Radford, 2009). The investigation of semiotic nodes in classroom activity, we have suggested (Radford, Demers, Guzmán, & Cerulli, 2003), is a crucial point in understanding the students' learning processes. The concept of semiotic node rests indeed on the idea that the understanding of multi-modal action does not consist in making an inventory of material signs and sensorial channels at work in a certain context. From a methodological viewpoint, the problem is to understand how the diverse sensorial channels and semiotic signs (linguistic, written symbols, diagrams, etc.) are *related, coordinated,* and *subsumed* into a new thinking or psychic *unity* (Radford, 2011a, 2012). Such a methodological problem makes sense only against the background of a conception of the mind that overcomes the dualistic view of internal-external processes. In our case, the methodological problem makes sense against the background of a concept of the human mind as *sensuous* through and through.

Methodologically speaking, we still need to account for the manner in which the new ideational-material psychic activity comes into being. And in order to do so, we need to pay attention to the manner in which individuals engage in the task, and they position themselves towards each other. In the classroom passage under discussion we note the teacher's posture and other means to which she resorts to engage the students in the task, not only asking explicit questions but also opening up a space for an *ethical engagement* to occur. That is, she becomes a *presence* and a *call* to which students are invited to respond (Radford & Roth, 2011). The students *respond* to the teacher by perceptually and aurally following her hands' movement and speech along the material terms, and answering her sequence of questions. We cannot fail to notice the tremendous role that *rhythm* comes to play here. Rhythm appears through different sensuous modalities: it appears in the aural modality through the flow of speech (in the regularly time-governed occurrence of the words "Term 1?," "Term 2?," "Term 3?," "Term 4?"); it appears in the kinesthetic modality of hands' movement through the regularly spatial-governed occurrence of the two-finger pointing gesture; it appears at a kind of supra-level, where the aural and kinesthetic modalities are coordinated so that they occur in a synchronized manner (Radford, Bardini, & Sabena, 2007). It is easy to imagine how disastrous a mismatching between these two forms of rhythm would be—e.g., producing the pointing gestures faster or slower than the production of words. Indeed, it is rhythm that ensures an efficient link between the various sensorial modalities and material culture that paves the way for the students to become aware of the historical-cultural algebraic way of perceiving the given sequence. But, as mentioned previously, this link is not a mere connection between disparate and heterogenous sensuous-psychic elements. Rather, and this is the most important point, rhythm, psychologically speaking, is the token of the emergence of a new psychic unity: the unity of perception, gesture, symbol, and speech. Each one of these units has now been rearranged in a new plane of psychic activity where they do not operate in isolation, but along with the others as a *whole*.

From a semiotic viewpoint, let me note that rhythm is a sign, but of a very special sensuous sort. It does not point to an object as an indexical gesture or a linguistic term like 'this' does. That is, it does not have an existential relationship to its object. In other words, it is not an index. It is not a symbol either, in the Peircean sense of having an arbitrary relationship with its object (Peirce, 1931). Rhythm is an *icon* whose object is the embodied process that *incarnates* the target concept, in this case the relationship between mathematical variables. In our Grade 2 episode, rhythm appears as a complex icon embedded in various sensorial modalities (vocal, aural, kinesthetic, visual), wrapped in a composite supra coordination that emphasizes its object not by revealing it in an existential manner (as, for instance, when we point to a chair and say 'this chair') but by disclosing its *meaning*.

Yet, as Line 10 intimates, the passage from Term 4 to Term 8 was not successful. The objectification (that is, the becoming aware; see Radford, 2002, 2010b) of the algebraic manner in which sequences can be algebraically perceived has not yet occurred. The teacher hence decided to restart the process, with some important modifications, as we shall see.

As mentioned previously, Term 8 of the sequence was not materially drawn on the first page. Only the first four terms of the sequence were shown. In the previous excerpt, the teacher *pretends* that Term 8 is on the empty space of the sheet, somewhere to the right of Term 4. She points to the empty space, as she pointed to the other terms, to help the students imagine the term under consideration. During the second attempt, the teacher does not go from Term 4 to Term 8; this time she goes term after term until Term 8.

11. Teacher: We will do it again …
12. Teacher: (*Pointing to Term 1 with a two-finger indexical gesture*) Term 1, has how many?
13. Carla: (*Pointing with her pen to the bottom row*) 1, (*without talking to the teacher points to Term 2 with a two-finger indexical gesture; Carla points with her pen to the bottom row of Term 2)* 2, (again without talking *the teacher points to Term 3 with a two-finger indexical gesture; Carla points with her pen to the bottom row of Term* 3), 3, (*same as above*) 4, (*now moving to the hypothetical place of Term 5 would be expected to be and doing as above*) 5.
14. Teacher: Now it's Term 8! (*The teacher comes back to Term 1*. She points again *with a two-finger indexical gesture to the bottom row of Term 1*) Term 1, has how many [squares] on the bottom?
15. Students: 1.
16. Teacher: (*Pointing with a two-finger indexical gesture to the bottom row of Term 2)* Term 2?
17. Students: 2!
18. Teacher: (*Pointing with a two-finger indexical gesture to the bottom row of Term 3)* Term 3?
19. Students: 3!
20. Teacher: (*Pointing with a two-finger indexical gesture to the hypothetical place where bottom row of Term 4 would be)* Term 4?
21. Students: 4!
22. Teacher: (*Pointing as above*) Term 6?

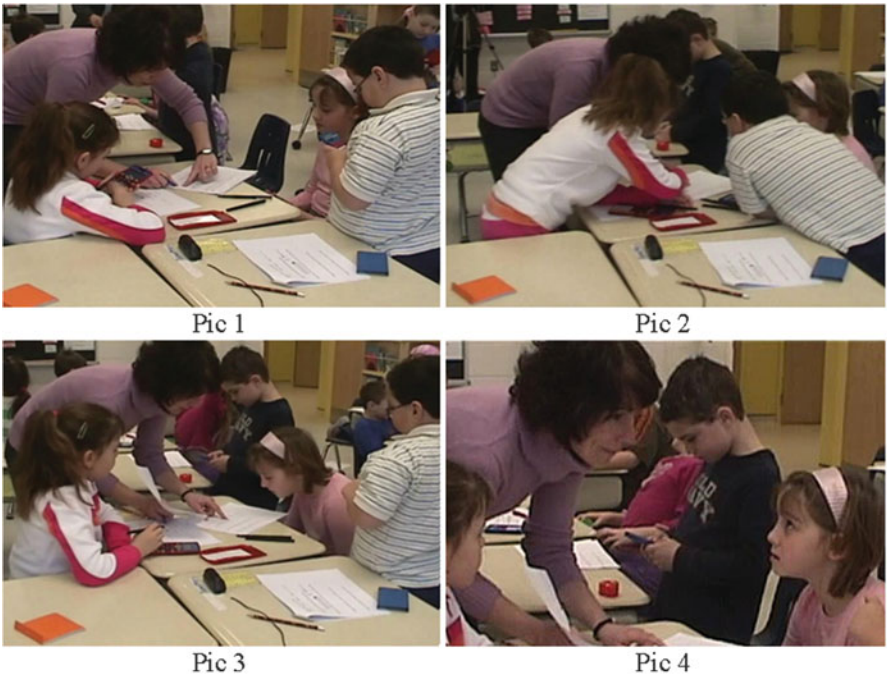

Fig. 8 Using the four given terms of the sequence, in Pics. 1 and 2, the teacher and the students count the *squares* on the *top row* of the visible Terms 1–4 and the imagined Terms 5–8. In Pics. 3 and 4 the students count the *squares* on the *top row* of Monique's term

23. Students: 6!
24. Teacher: (*Pointing as above*) Term 7?
25. Students: 7!
26. Teacher: (*Pointing as above*) Term 8?
27. Students: 8!
28. Sandra: There would be 8 on the bottom!

The teacher and the students counted together the squares on the bottom row of Monique's term and realized that the number was indeed 8. At this point the relationship between variables started becoming apparent for the students. The relationship started being objectified. The teacher then moved to a joint process of counting the squares on the top row:

29. Teacher: (*She turns the page and the students can see Monique's term*). Very, very good. Now, we will verify if Monique has the good amount [of squares] on top. We will just look at the top . . . (*like in the previous episode, she makes two sliding gestures, but this time pointing to the top row; see Fig.* 8, *Pic. 1*). Term 1 has how many?
30. Students: 2!

31. Teacher: Term 2?
32. Students: 3! ...
33. Teacher: Term 3?
34. Students: 4!
35. Teacher: Term 4?
36. Students: 5! (*see Pic. 2*)
37. Teacher: Term 6?
38. James: 7
39. Teacher: (*Repeating*) 7 ... Bravo! Term 8, will have how many?
40. Students: 9!
41. Teacher: Ok. Oh! Excellent. Are there 9 [squares] here (*pointing to Monique's term*)?
42. Sandra: Yes, there are 9.
43. Teacher: We will count it together.
44. Students: (The teacher points orderly and rhythmically to the terms one after the other, while the Sandra says) 1, 2, 3, 4, 5, 6, 7, 8 . . . !? (*long pause following a general surprise. See Pics. 3 and 4 in Fig. 8).*

The students were perplexed to see that contrary to what they believed, Monique's Term 8 did not fit into the sequence. Here the activity reached a tension. Picture 4 in Fig. 8 shows Sandra's surprise. The students and the teacher remained silent for 2.5 s, that is to say, for a lapse of time that was 21 times longer than the average elapsed time between uttered words that proceeded the moment of surprise (for details of this poetic moment in the teacher-students' objectification process, see Radford, 2010b).

Later on in the lesson the students were able to quickly answer questions about remote terms, such as term 12 and Term 25, which were not perceptually accessible. They refined the manner in which the terms of the sequence could be perceived. The number of squares on the bottom row was equated to the number of the term in the sequence, while the number of squares on the top row was equated to the number of the term plus one (identified as the dark square in the corner; see Radford, 2011b). Here is an excerpt from the dialogue of Sandra's group as they discuss without the teacher:

45. Sandra: (*Referring to Term 12*) 12 plus 12, plus 1.
46. Carla: (*Using a calculator*) 12 plus 12 ... plus 1 equal to ...
47. James: (*Interrupting*) 25.
48. Sandra: Yeah!
49. Carla: (*looking at the calculator*) 25!

At this point, the target cultural knowledge has been objectified and a new ideational-material psychic unity has been forged. The students no longer need to see the terms of the sequence or to touch them with their hands. What could only be made apparent through an intense interplay between material culture and the various sensorial modalities is now contracted, subsumed and reorganized in a new complex psychic unity where no reference is made to top or bottom rows (see lines 45–49).

Implications for Teaching and Learning in a Digital Era

As mentioned previously, the concept of sensuous cognition elaborated in this chapter is based on the idea that human sensation is consubstantial with material culture. This consubstantiality means, in particular, that our senses and the way we come to think about our world are shaped, oriented, and transformed by the ubiquitous presence of material culture in the activities we engage in. Our digital era is certainly provoking a transformation of the senses and our ways of thinking that we still need to better understand (Gee, 2003; Trend, 2010). Audiovisual forms of knowledge mediation are likely to lead to the creation of new dispositions or sensibilities in the ways we imagine, recall, reflect, visualize, and generalize. To give but one example, in her book *The skin of the film*, Canadian critic Laura Marks, shows that instead of being a flat screen, the skin of the film works rather as a "membrane that brings its audience into contact with the material forms of memory" (2000, p. 243) expressed through cultural forms of touching, smelling, and caressing, for example. The digital era and new material culture in general operates under new organizations of cultural sensoria with an increasing and definite "abstraction and symbolization of all sense modalities" (p. 244). Visual images and media are challenging the longstanding role that the written word has had for centuries in the Western world (see, e.g., Stephens, 1998). In a chapter entitled "The disappearing world," Trend summarizes the ongoing transformation as follows:

> The dawn of the digit era [in the 1990s] marked the biggest change in our relationship to language since the invention of the printing press, the rise of personal computing, networked communication, and other technologies at the turn of the millennium coincided with the death of Enlightenment thinking that separated the word from the image. ... [Written] Language is being taken over by images as experience itself becomes increasingly visual. (Trend, 2010, p. 31)

The implications for the teaching and learning of mathematics are evident. Hegedus and Moreno-Armella (2011), from the Kaput Research Center, are exploring the potentials of haptic devices, which work as semiotic mediators much as a dragged mouse does in dynamic geometry environments to mediate visual information. The haptic devices allow young students to sense mathematical structures, both physically and visually. Yet, we need to understand the reorganization of the senses and the creation of new sensibilities and forms of mathematical thinking that emerge from the contact and the engagement with digital environments and cyberlearning. Dynamic geometry, haptic devices and similar cultural artifacts provide the students with forms of exploration that are incommensurate with those offered by paper-and-pencil environments. The question concerns more than economy of time. These technological devices offer room for the creation of an experimental space that might require the appearance of new sensibilities and new embodied ways of thinking—dynamic new literate ways of scrutinizing, enquiring, looking into, and thinking about, mathematics, mathematical objects and their relationships.

Synthesis and Concluding Remarks

In the first part of this chapter, drawing on Vygotsky and his cultural-historical psychological school (Leont'ev, 1978, 2009; Luria, 1984; Luria & Vygotsky, 1998; Vygotsky, 1987–1999), as well as on the work of Maturana and Varela (1998), I sketched a theoretical approach to cognition that highlights the role of sensation as the substrate of mind and of all psychic activity. The role of sensation in our cognitive endeavours is not, however, something new. It has been a recurrent theme in philosophical inquiries since the pre-Socratics. Since Plato and in fact since the Eleatic thinkers sensation was nevertheless understood in negative terms—as something that hinders the road to knowledge (see e.g., Radford, Edwards, & Arzarello, 2009). This is the sense with which rationalist epistemologies of the seventeenth and eighteenth centuries up to the present have endowed sensation. Thus, to give but one example, for Kant, sensations such as colour, sound, heat, and smell, "are connected with the appearance only as effects accidentally added by the particular constitution of the sense organs" (Kant, 1787, p. 72), and as such they are unable to yield true knowledge. They do not constitute an objective determination of the object, as they pertain to the subjective dimension of the sensing subject. As a result, sensations are not, according to Kant, necessary conditions of the object's appearance and consequently are not a constitutive part of the process of knowing (for a detailed discussion, see Radford, 2011c). For some contemporary rationalists, sensation does play a cognitive and epistemic role. Yet, our sensing organs are considered as having little (if any) relation with culture and history. Their only history is the one of biology and natural development. Piaget's genetic epistemology is not, of course, the only example. By contrast, within the theoretical approach here sketched (which, I suggested, might be best captured by the term *sensuous cognition*) sensation is not merely part of our bodily and biological constitution. Sensation is rather conceived of as a culturally transformed sensing form of action and reflection interwoven with cultural artifacts (language, signs, diagrams, etc.) and material culture more generally. Sensuous cognition calls into question the usual divide between mind and matter and offers a perspective through which to cast the role of the body and artifacts in knowing processes.

In the second part of the article, I presented a short example that, I hope, gives an idea of the manner in which sensuous cognition may help us understand teaching-learning activity. Sensuous cognition, I argued, does not amount to claiming that our various senses come into play in classroom interaction. This is no more than a banal statement. The real question, I argued, is about understanding how, through classroom activity, our forms of sensing and reflecting are culturally transformed. The example discussed in the previous section intimates that as knowledge objectification proceeds, new ideational-material psychic unities emerge out of previous psychic formations. Within this context, the concept of semiotic node is a practical construct that may be useful in investigating this important aspect of teaching-learning and conceptual development. Thus, in the Grade 2 example we noticed how a complex psychic unity was revealed by a semiotic node constituted by the teacher's and the students' intercorporeal and material activity. This complex psychic unity gave rise to a new psychic formation where students were able to quickly tackle questions about remote terms (like Term 25). The complex and dynamic unity of perception, language, gestures, rhythm, diagrams revealed by the semiotic node yielded place to a new, more compact unity where language and cultural artifacts predominated (see lines 45–49). The issue, however, is not that knowledge has become disembodied. Behind language and cultural artifacts, there still resonates the complex bodily, material, and semiotic activity of the previous sensuous actions and forms of sensing. They have been *contracted* (Radford, 2010a) and *reorganized*, and will re-emerge if difficult questions arise (see Radford, 2011b). Previous psychic formations do not disappear. They are subsumed in the new ones and are reactivated if necessary, although not in an intact form: the eye, for instance, cannot regain the primary purity and naivety through which it saw the world before. The same could be argued of hearing and touching. Human sensuous cognition is not an additive formation. Its structure is rather made up of new formations subsuming the previous ones by relations of a dialectical nature. As French philosopher Jean Hyppolite (1961) notes in his essay on Hegel's system, a word, in its truly dialectical nature, signifies what is not there by signifying what is there, and signifies what is there by signifying what is not there.

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New Media and Online Mathematics Learning for Teachers

George Gadanidis and Immaculate Kizito Namukasa

Abstract In this chapter we offer a case study of an online Mathematics for Teachers course through the lens of four affordances of new media: democratization, multimodality, collaboration and performance. Mathematics, perhaps more so than other school subjects, has traditionally been a subject that people do not talk about outside of classroom settings. However, we demonstrate through the case of the Mathematics for Teachers course that this does not have to be the case. Mathematics, even mathematics that traditionally has been seen as abstract or inaccessible, can be talked about in ways that can engage not only adults but also young children. The affordances of new media can help us rethink and disrupt our existing views of mathematics (for teachers and for students) and of how it might be taught and learned, by (1) blurring teacher/student distinctions and crossing hierarchical curriculum boundaries; (2) communicating mathematics in multimodal ways; (3) seeing mathematics as a collaborative enterprise; and (4) helping us learn how to relate good math stories to classmates and family when asked "What did you do in math today?"

Online courses for teachers have gained popularity in a short period of time. For example, in just over a decade our Faculty of Education has grown from two online courses to over 150 online courses offered in its Continuing Teacher Education Program, several fully online and hybrid courses in its Preservice Teacher Education Program, a fully online cohort in its Masters of Education Program and several online courses offered in the rest of its graduate program. At the same time, the nature of our courses is changing due to the increased availability of multimodal, interactive and collaborative forms of communication. While our first courses were

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designed to be primarily text-based, they are now making use of new media forms of communication, such as online videos, interactive content, and collaborative tools.

One of our online courses that makes extensive use of new media is the Mathematics for Teachers course, which we have been offering for the last 4 years as an optional summer course (in August) to each incoming elementary (K-8) teacher candidate cohort. Teacher candidates pay for this course in addition to their tuition and it is listed on their university transcripts as a separate course. The course is advertised for teacher candidates who fear and/or dislike mathematics and as a form of "mathematics therapy" (Gadanidis & Namukasa, 2005). The course offers teacher candidates opportunities to experience the pleasure of doing mathematics.

In this chapter we examine the design of the Mathematics for Teachers course through the lens of four affordances of new media: democratization, multimodality, collaboration and performance. In the context of analyzing students' course discussions and shared assignments, we ask: in what ways are these affordances present in our course design?

Theoretical Perspective

It is important to note that through our theoretical perspective, we see new media not simply as a tool for pursuing and amplifying our educational purposes, but also as an actor that shapes our thinking and our purposes. As Borba and Villarreal (2005) suggest, humans-with-media form a collective where new media serves to disrupt and reorganize human thinking. Likewise, Levy (1993) sees technology not simply as a tool used by humans, but rather as an integral component of a *cognitive ecology* of the humans-with-technology *thinking collectives*. Levy (1998) claims "as humans we never think alone or without tools. Institutions, languages, sign systems, technologies of communication, representation, and recording all form our cognitive activities in a profound manner" (p. 121). Levy (1993) also suggests that technologies *condition* thinking, implying that the thinking collective of humans-with-media may persist even when the media is not present.

We are sympathetic to the view that the use of new media can help change how we think about mathematics education, and we have some anecdotal evidence from our personal experience. However, in this exploratory work we do not try to prove a causal link between the affordances of new media and the design of the Mathematics for Teachers course. Comparing the affordances of new media to the design features of our course, and thus better understanding how new media affordances may be seen to be manifested or mirrored in our pedagogical choices, nonetheless, may be a first step in investigating how thinking-with-new-media affects how educators and researchers think about mathematics education.

New Media Affordances

For the purpose of this chapter, *new media* refers to technology tools that are widely and publicly accessible, such as the YouTube, Google, Facebook, Ning, wikis and blogs. Below we describe four affordances of new media that we use in our analysis of the Mathematics for Teachers course.

Democratization

At the time of writing, the democratization affordance of new media is evident in the political upheavals in the Middle East, where hierarchical power relationships have been disrupted in part through the use of social media. In the last decade, authoritarian regimes that in the past have silenced political dissent are finding that new media tools have given people opportunities to express themselves through text, voice, and video (Al-Obaidi, 2003). In educational settings, new media also potentially disrupts existing power relationships in three important areas: (1) when/where learning takes place; (2) what curriculum and content is to be learned; and (3) who the teacher is. In a traditional classroom, what is available to learn is determined by "representatives" of hierarchical authority structures such as the teacher, the textbook and the mandated curriculum. Knowledge is classified by grade level and students do not have access to knowledge that is above their mandated grade. In contrast, a student in a new media setting can use Google or YouTube, for example, to search for information on a given topic they are studying or are curious about and freely access multimodal learning material that is not grade specific. Also in contrast to traditional education, where learning occurs in a classroom during a specified time period, new media learning resources are continuously available from any place with Internet access. Lastly, new media disrupts the labels of "teacher" and "student" as anyone with a camera and a YouTube account, for example, can teach about a topic of interest.

It needs to be cautioned, however, that the democratization affordances of new media do not provide a guarantee of democratization. Chester (2007) notes that there is a history of new communication technologies being subverted. In the classroom context, Cuban (1986), looking at 60 years of educational use of technology since 1920 notes that: classroom situations change minimally; when they do change, it is in non-standard ways; and, teachers and schools have a predominant effect on how technology is used in classrooms.

Multimodality

In contrast to the increasingly multimodal nature of the Web, many school experiences, especially in mathematics, continue to rely on discourses that are monomodal or bimodal (in cases where diagrams or graphs are employed). Kress and van Leeuwen (2001) suggest that in a digital environment "meaning is made in many different ways, always, in the many different modes and media which are co-present in a communicational ensemble" (p. 111). The shift from text-based communication to multimodal communication is not simply a quantitative change. It is not just a case of having more communication modes. It is a qualitative shift, analogous to the change that occurred when we moved from an oral to a print culture (Gadanidis, Hoogland, & Hughes, 2008). Print culture, for instance, supported the creation of fixed media and records.

Kaput (2002) also talked about the new technologies, used in the world of business but yet to be harnessed in the world of education, as connected and inexpensive technologies as opposed to isolated and expensive technologies. He also talked about "newly intimate connections among physical, linguistic, cognitive and symbolic experiences [of mathematics concepts]" that become possible with newer computer and WWW technologies (p. 91) as used in learning mathematics. "Revisiting the analogies with change made possible by the printing press ..." (p. 92) he talks about the "impact of the printing press on the democratization of literacy." Kaput hypothesizes a similar change in the twenty-first century "relative to the new representational infrastructures made possible by the computation mediums" (p. 92). He also stipulates that the "crossing between interfaces [on hand-held, networked devises used for learning] may help in exposing the mathematics structure" (p. 97).

Collaboration

Our digital age has been labelled as an information revolution (as contrasted with the industrial revolution). Schrage (2001) suggests that this label misses the essence of the paradigm shift.

> *In reality, viewing these technologies through the lens of "information" is dangerously myopic*. The value of the Internet and the ever-expanding World Wide Web does not live mostly in bits and bytes and bandwidth. To say that the Internet is about "information" is a bit like saying that "cooking" is about oven temperatures; it's technically accurate but fundamentally untrue. (p. 1, original emphasis)

Schrage suggests that a more appropriate label is *relationship revolution*.

> *The so-called "information revolution" itself is actually, and more accurately, a "relationship revolution."* Anyone trying to get a handle on the dazzling technologies of today and the impact they'll have tomorrow, would be well advised to re-orient their worldview around relationships. ... *When it comes to the impact of new media, the importance of information is subordinate to the importance of community*. The real value of a medium lies less in the information that it carries than in the communities it creates. (pp. 1–2; original emphasis)

Lankshear and Knobel (2006) suggest that the relatively recent "development and mass uptake of digital electronic technologies" represent changes on a "historical scale", which "have been accompanied by the emergence of different (new) ways of thinking about the world and responding to it." (pp. 29–30). These new ways of thinking can be characterized as "more 'participatory,' 'collaborative,' and 'distributed' and less 'published,' 'individuated,' and 'author-centric' ... also less 'expert-dominated.'" (Knobel & Lankshear, 2007, p. 9).

Performance

Hughes (2008) and Kress and van Leeuwen (2001), note that the multimodal nature of new media offers performative affordances. This is evident in the multimedia authoring tools used to create online content, such as Flash, which often use performance metaphors in their programming environment. For example, you program on what is referred to as the "stage", you use "scenes" to organise "actors" or "objects" and their relationships, and you control the performance using "scripts". The Web as a performative medium is evident in the popularity of portals like YouTube. Hughes suggests that the new media that is infusing the Web draws us into performative relationships with and representations of our "content". Gadanidis and Borba (2008) have explored digital performance in mathematics education settings, and the idea of students as *performance mathematicians.*

Research Setting and Participants

We study the design and implementation of a Mathematics for Teachers online course. This online course makes extensive use of new media. We have been offering it for the last 4 years as an optional summer course (in August) to each incoming elementary (K-8) teacher candidate cohort. Teacher candidates choose to enrol and pay for this course in addition to their tuition. Because the course is advertised for teacher candidates who fear and/or dislike mathematics several teacher candidates who enrol for it have specific needs for learning mathematics in different ways than those that likely turned them away from mathematics. The course, therefore, offers teacher candidates opportunities to experience the pleasure of doing mathematics.

In the summer of 2010, 37 elementary school teacher candidates attended the course. They consented to having their course participations used for research purposes. One of the online resources used in the course is the www.researchideas.ca website, where in collaboration with project schools we are creating multimodal and performative documentaries of how the "content" of our Mathematics-for-Teachers courses has been used in elementary school classrooms. In this paper we draw data from the Summer course online conversations.

Methods: Analyzing the Mathematics for Teachers Course

In the next section we analyze the design and implementation of the summer of 2010 Mathematics for Teachers course. We use a case study approach, with the course design, mathematics activities, and teacher candidate contributions to the online discussion constituting a single case. Case study method is suitable for collecting and re-telling in-depth stories of teaching and learning and for studying a 'bounded system' (that is, the thoughts and actions of participants of a particular education setting) so as to understand it as it functions under natural conditions (Stake, 2000a, 2000b; Yin, 2006). The analysis is qualitative in nature, in keeping with the established practice of in-depth studies of classroom-based learning and case studies in general (Stake, 2000a). Content analysis is employed to identify, code and organize patterns of discourse (Denzin & Lincoln, 2005; Patton, 2002) that fit the four new media affordances noted above: democratization, multimodality, collaboration and performance. The case study answers the question: in what ways are these affordances present in our course design? The style of writing of the case study is at once analytical (using the four new media affordances) and descriptive (offering a detailed account of the Mathematics for Teachers course experience).

Democratization

In what ways did the Mathematics for Teachers course reflect the democratization affordances of new media?

Where/When Learning Takes Place

Using an asynchronous online discussion platform, teacher candidates had access at any time and from any place with an Internet connection. Had we offered the course in a university classroom we would have limited the number of participants due to (a) increased cost of accommodation for out-of-area participants, (b) summer employment commitments, and/or (c) summer vacation plans. In fact, some of the teacher candidates participated in the course while working or vacationing outside of Ontario, and across various time zones.

What Is to Be Learned

The mathematics in our Mathematics for Teachers course disrupts common conceptions of content in two important ways. First, we don't distinguish between the mathematics that teachers study and the mathematics that students study (Gadanidis & Namukasa, 2007). That is, the activities we bring to our Mathematics for Teachers course are the same activities we use with students in elementary school classrooms. Second, the activities are designed to have a low mathematical floor, allowing for engagement with minimal prerequisite knowledge, and a high mathematical ceiling, allowing for extensions to more complex ideas (Gadanidis & Hughes, 2011). We elaborate on these ideas below.

Mathematics for All

Ball and Bass (2003), Ball (2002), and Davis and Simmt (2006) distinguish mathematics-for-teaching from the mathematics that students need to know. Ball, Bass, Sleep, and Thames (2005) give the example of the mathematical task of $307 - 168 = 139$ and state that "[t]o teach, being able to perform this calculation is necessary. But being able to carry out the procedure is not sufficient for teaching it." They identify four distinct domains of mathematical knowledge for teaching: (a) common content knowledge (calculating the answer to $307 - 168$), (b) specialized content knowledge (such as analyzing calculation errors), (c) knowledge of students and content (identifying student thinking that might have produced such errors), and (d) knowledge of teaching and content (recognizing which manipulative materials would best highlight place-value features of the algorithm). The suggestion is that the last three domains – knowledge of specialized content, of students and content, and of teaching and content – distinguish what teachers need to know from what students need to know. This work on mathematics-for-teaching offers a powerful unit of analysis of teacher education and development. However, the distinction that mathematics-for-teaching is different from students' mathematics may not fit well with research-based reform recommendations, where students are also expected to be engaged in the four mathematics-for-teaching domains identified by Ball et al.: performing calculations, analyzing errors, identifying thinking that might have produced such errors, and selecting tools for modelling mathematical ideas (e.g., Ontario Ministry of Education, 2005).

Begle (1979), Eisenberg (1977), Fennema and Franke (1992), General Accounting Office (1984), and Monk (1994) note that university mathematics courses may not offer the mathematics knowledge that teachers need for teaching. An emphasis on *more mathematics* may be inappropriate (Davis & Simmt, 2006). We agree that offering teachers more of the mathematics that turned them away from the subject is not helpful. But, that teachers need a totally different mathematics than what they experienced during their schooling does not necessarily imply that they need a different mathematics than their students. Rather, we think it implies that both teachers and students need to experience a *better* mathematics that is aligned to reform initiatives such as teaching through problem solving and focusing on understanding mathematical concepts (NCTM, 2000).

Low Floor, High Ceiling

How do we stretch mathematical concepts so that the key ideas are accessible across grades? This is not an easy task as traditionally our pedagogical thinking has focused in the opposite direction, by making mathematics content grade specific. Below we provide two examples of activities from the Mathematics for Teachers course that are designed to have a low mathematical floor, allowing engagement with minimal prerequisite knowledge, and a high mathematical ceiling, offering opportunities for extending ideas to more complex and more sophisticated representations.

Infinity and Limit

The Mathematics for Teachers course started with a module consisting of an activity on the themes of infinity and limit, which are typically covered in grade 11 through the topic of sequences and series and in grade 12 through Calculus. One goal of the activity is to help teacher candidates who fear mathematics realize that topics that are usually considered abstract and complex can be accessed in meaningful ways even by young children. In fact, a similar activity was completed by grades 2–3 students at a summer camp (as a context for exploring representations of fractions) and a grade 4 classroom (as an exploration of linear measurement). Videos of the students and one of the teachers were available online for teacher candidates to view.

All mathematics tasks were available at once, and teacher candidates were encouraged to complete the tasks in the sequence listed in order to achieve the best experience possible. After completing each task, they were asked to share and discuss their findings online. The tasks are carefully designed and sequenced to help teacher candidates experience mathematics ideas in new ways and in new light, and elicit mathematical surprise.

The first task of the module, specifically, asked teacher candidates to consider the following questions: (1) What do you know about infinity? (2) How big is it? (3) Can it fit in a gym? (3) Can you hold it in your hands? There was consensus among teacher candidates that infinity would not fit in a gym or in their hands. Three representative comments are listed below.

> By definition, infinity cannot be assigned a number value, and in math I believe it is usually associated with a symbol.

> It is all around us as “time” and “space”.

> Infinity is not something tangible, as it is a concept that is greater than any measurable space, like a gym. It cannot be held in your hand.

In the second task, teacher candidates accessed, printed and completed a handout consisting of ten 16×16 grids, with the first one shaded to represent the fraction 1/2. For each of the subsequent grids they were asked to shade in representations of the fractions 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512 and 1/1024, respectively. The last page had a 32×32 grid with the following instructions: “Suppose you repeated the above process forever. Now imagine taking all of the shaded pieces and joining them together to form one solid shape. What might the shape be and what would its size be? Draw the shape on the grid below.” (see Fig. 1)

Teacher candidates were then asked to consider the following “walk to the door” task.

- You are standing in a room, facing an open door, which is 1 m away from you.
- Is it possible to walk to the door, and actually go past it, and leave the room?
- You’re probably thinking this is a silly question, and that of course you can walk to the door, and even walk past it. After all, it’s open.
- But think of it this way: To get to the door, you first have to first pass half of the distance to the door (1/2 of a metre), then half the remaining distance to the door

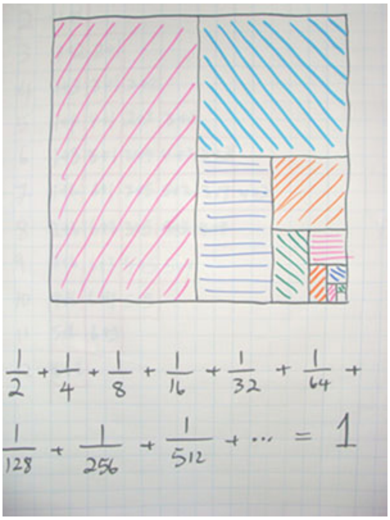

Fig. 1 The finite sum of an infinite number of fractions

(1/4 of a metre), then half of the remaining distance (1/8 of a metre), and so on and on and on ...
- So let me ask again, can you reach the door if you have to actually travel 1/2 of the distance, then 1/4, then 1/8, then 1/16 and so on?
- What does this tell us about the sum of 1/2, 1/4, 1/8, 1/16, and so on forever?
- Here's a song by grade 4 students on this theme ... http://www.edu.uwo.ca/mpc/mpf2010/mpf2010-131.html
- Teachers candidates also read a retelling of the story of Rapunzel, where Rapunzel's tower cell door is open but she does not escape because she is imagining passing through all the fractional distances discussed above.

Teacher candidates were also asked to consider the sum of the following infinite set of fractions: $1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128$ ... and to decide if there is a sum and what it might be. Then teacher candidates viewed the following online videos:

- A teacher talking about this activity (done with grade 2–3 students)

  http://www.edu.uwo.ca/mpc/camp2010/day4/infinity2.html

- Songs by the grade 2–3 students on the theme of infinity

  http://www.edu.uwo.ca/mpc/camp2010/day4/infinity1.html
  http://www.edu.uwo.ca/mpc/camp2010/day5/ms9.html
  http://www.edu.uwo.ca/mpc/camp2010/day5/ms10.html

Historically, mathematicians have struggled with the concepts of infinity and limit, as is evident by Zeno's paradoxes, which "have puzzled, challenged, influenced, inspired, infuriated, and amused philosophers, mathematicians, and

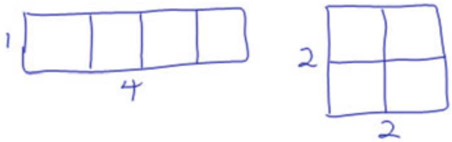

Fig. 2 Arrays for the number 4

physicists for over two millennia" (Tall & Tirosh, 2001; Wikipedia, 2011). The "walk to the door" problem posed in the Mathematics for Teachers course is a variation of Zeno's paradoxes. The traditional curriculum keeps the ideas of infinity, be it the limiting processes or finite values of limits, away from children, typically until they reach grades 11 and 12. Through the above activities, using familiar contexts of fractions and halving, teacher candidates came to realize that these ideas are in fact accessible to young children who are curious about ideas such as infinity even at kindergarten.

> I thought that this concept would be difficult to teach to anyone, let alone young students. After watching the videos, it became apparent that the students were in fact able to understand the idea of infinity and represent it in visual form.

> I was struck by the seeming simplicity of infinity (and its explanations) in all of this. Kids appeared to have little trouble grasping the fact that "infinity never stops" (to quote one of their catchy tunes).

> The exercise that the children did by creating the piece of paper to represent infinity was great. What I got out of the exercise and videos is that an abstract concept can go along way and be brought into easier terms for the students, causing them to express enthusiasm and creativity through a tangible means and at the same time using other media such as music and drama.

> Previous encounters with infinity usually involved an explanation that managed to complicate the whole thing unnecessarily. I am far less intimidated by infinity now.

Optimization

Another mathematics activity of the Mathematics for Teachers course involved an exploration of optimizing perimeter for a given rectangular area. Teacher candidates accessed, printed and completed a handout where they drew rectangular representations of the numbers 1 through 12. For example, two possible arrangements for the number 4 are shown in Fig. 2. Then they were given the following problem.

Imagine that you are planning for a mathematics party.

- You have 16 square tables that you can use to form a bigger rectangular table (like the rectangular arrays you created for the numbers 1–12, with no holes in the middle)
- What arrangement would you use so you can fit the most number of chairs around the big table?
- What arrangement would you use so you can fit the least number of chairs around the big table?

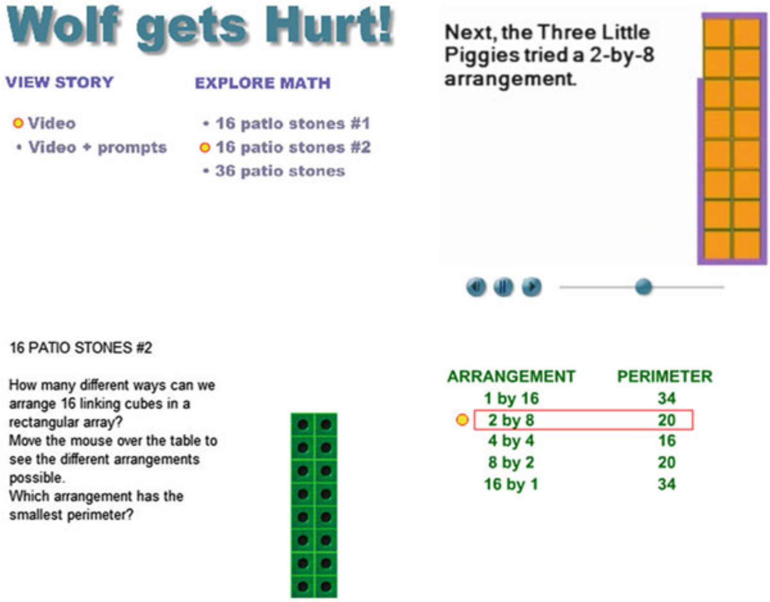

Fig. 3 The story "Wolf gets Hurt"

Use the grid in the handout to show the various table arrangements and the number of chairs that fit.

Then teacher candidates accessed an online story called "Wolf gets hurt" (see Fig. 3) that included a video reading and interactive mathematics content. In this retelling of the Three Little Pigs story, the Piggies capture Wolf and try to build the biggest rectangular pen possible with only 18 m of fence. They also accessed (1) an online video of a grade 2 teacher describing how she did this activity with her students (2) a video of the song that summarized the learning of the grade 2 students and (3) a stop motion animation video created by a teacher candidate who took the Mathematics for Teachers course in 2008:

- Teacher interview: http://www.edu.uwo.ca/mpc/bigideas/arrays
- Student song: http://www.edu.uwo.ca/mpc/bigideas/arrays/perimeter.html
- Animation: http://www.edu.uwo.ca/mpc/mpf2010/mpf2010-106.html

Optimization is a topic that is typically not taught until grades 9 and 10, when students first encounter quadratic functions. It is also part of the grade 12 Calculus curriculum. It can also potentially be found in grades 5 and 6, where students explore relationships between area and perimeter. However, when we have included optimization activities in workshops for grades 4–6 teachers, most teachers express surprise that the perimeter can change if the area does not, indicating that they have not addressed these area/perimeter relationships in their classrooms. Similarly, teacher candidates in the Mathematics for Teachers course were surprised by the optimization relationships between area and perimeter.

> I thought originally that I understood arrays, and immediately saw how it could be a lesson that might lead to factoring and such. Having to do the activity myself though, slowing down and taking the time, I found myself incredibly humbled.

> Terrific way to see how you can make the perimeter change.

> I learned that perimeter changes as shape changes, even though the area stays the same, and that there are fun ways to express this.

> I felt amazed at the patterns and relationships present in math - it really is cool that when area stays the same but changes shape, the perimeter changes.

They also realized that these concepts are accessible to young children, like the grade 2 students in one of the teacher interviews that they could view online, along with student samples of work.

> My most significant insight from Module 2 was the number of activities that were involved to teach one concept to children. When learning about arrays, the teacher had the children perform various activities. They read stories, learned about division with distributing their "cookies", created arrays and drew real life objects, used linking cubes and grid paper, and made up and performed songs. It seems like a lot of activities to learn one concept. However, as I watched the explanation of each activity and what the children learned, even I began to feel as if the concept was more ingrained in my memory. I also liked how the activities could be related to real life to make them more concrete and easily understood.

> I enjoyed looking at multiplication arrays and doing the Math Party. I had my 5 year old work along side me and he understood as well, and thought it was pretty neat to have a different number of chairs around the table with the different arrangements.

The preceding quote from one of the teacher candidates noted that the tables and chairs representation made the concepts of area and perimeter optimization accessible to her 5 year old child.

Who Is the Teacher

The role of the instructor in the Mathematics for Teachers course was to pose problems and scaffold and encourage teacher candidates to individually and collaboratively explore multiple solutions and representations. The instructor avoided giving answers or commenting on whether answers were correct. This helped create an atmosphere (discussed in greater depth in the section on collaboration) where teacher candidates relied on one another for learning.

> Sharing ideas with 30-some people in the manner we have also has confirmed for me that every class member will have their own take on the material and their own gift to offer to the discussion.

> I think it is SO important to work as a group and allow students to teach each other concepts as well, because who [is] better to teach something than someone who has JUST learned it! (Capitalization emphasis in original)

The use of video resources of classroom teachers discussing the activities of the course from the point of view of how they were implemented in their classrooms as well as videos of young students sharing their mathematics learning helped expand the scope of who the teacher was.

> The interview with Pam King and the song were helpful to watch. The song especially for me reinforced that 2×6 and 6×2 though mathematically the same, are two different ideas to children. 2 friends sharing 12 cookies, or 6 friends sharing 12 cookies - sooooo not the same thing to youngsters :)

> The children in all of the examples had a much better and more simpler means of understanding infinity and I LOVE that!

External resources including interviews with classroom teachers and performances by school children on the problem had a significant impact on the mathematical and pedagogical learning of teacher candidates. This helped to reinforce the idea that fellow teachers and even our students can be our “teachers”.

Multimodality

Teacher candidates noticed the multimodal nature of the course experiences and made ongoing comments about them. Figure 4 shows a screen shot of the interview with a Grade 2 teacher on the theme of optimization. Figure 5 shows the optimization stop-motion animation created by a teacher candidate in the 2008 Mathematics for Teachers course.

Following are representative comments about multimodality that arose from the infinity activity.

> All of this caused me to reflect on how I was taught in the lower grades as I recall far less visual aids and interesting stories. It impressed on me how much more effective teaching a subject could be if you present it in different formats (stories, videos, puzzles) rather explaining it and giving work to try out. This really has made me think about how teaching has evolved too.

> I found the structure of this exercise appealing because it (1) began with an intellectual challenge (i.e., the request that we interrogate our understanding of what “infinity” refers to); (2) invited us to engage in a series of hands-on paper-and-pen tasks; and, after encouraging us to contemplate the import of our doing so; (3) exposed us to a novel way in which an elementary school teacher has attempted to introduce the concept of infinity to her students and, simultaneously, elicit their creativity. Contemplating my reaction to these activities, I realized that what made the activity especially enjoyable was that it was not singular in its style of delivery and, instead, included activities that would appeal to different “types” of learners.

Teacher candidates made similar comments about the optimization activity.

> (I have developed an) awareness that one doesn’t have to explain a mathematical concept using only the lexicon of that discipline and with a singular reliance upon other mathematical concepts - one can/should exercise one’s creativity and, in doing so, encourage students to think holistically - to merge a mathematical concept with the concerns of other subjects - health studies, social studies, etc..

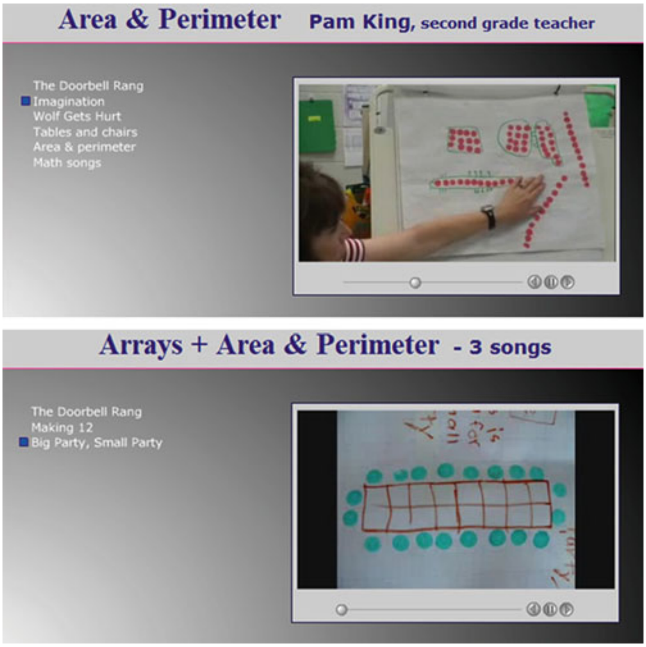

Fig. 4 Screenshots of teacher interview on optimization in Grade 2

Fig. 5 Screenshot of optimization stop-motion animation

After doing the assignment and watching the videos, I noticed that the visual representations made the concepts of area and perimeter more clear to me.

Treating math as a fun interplay between numbers (i.e., watching the buttons move around) and asking questions that require a little imagination from the learner seem to be very effective in making math a subject to enjoy.

> When I did the activities for this module and saw the video of the animation that visually demonstrated area and perimeter, I was surprised by how much these visual representations helped to clarify the concepts – even for me. It opened my mind to the possibility of using visual aids to help explain math, and has therefore changed my mind about taking a strictly auditory approach to the subject.

> I really enjoyed seeing the different ways that this lesson can be taught. I thought the button video was a really cool idea, having the different colour buttons and seeing that each time there are always 16 black buttons, (as the area doesn't change) but that different amounts of white buttons were needed as the perimeter does change.

> I have been wondering whether Pam King's methods may be too time consuming for a rigid classroom schedule. With more thought, I feel that developing a solid foundation in basic mathematical concepts is worth the time and effort Pam places in the planning and execution of her lesson. Really, she is able to kill two birds with one stone, as she incorporates music and English into her math lessons.

> For me I find visual examples to be very helpful, but I agree that all students learn differently, so it was neat to hear the songs and story which explained the lesson in a different way.

Teacher candidates also used multimodal ways to present their own ideas in the discussion, such as creating drawings and diagrams as attachments to their postings and authoring songs and stories that expressed their learning. One of them had this to say:

> I definitely enjoyed this course. It has been an eye opener because it was not at all what I was expecting. I love writing poetry, photography, music and art and now to have learned that I can use all of the above to teach math gave me insight to how I view math and how I have been afraid all these years.

For example, one of the teacher candidates wrote a song called "The infinity cake", shown in Fig. 6 and available as a musical performance at http://www.joyofx.com/music/m4t-2010-infinityCake.html.

Collaboration

Several teacher candidates commented that they had low expectations of the level of community that could be experienced in an online course.

> Admittedly, I was not expecting genuine community from an online course

> I had worried that taking a course that didn't involve in-class lectures would be intimidating and impersonal.

The course design attempted to create a sense of community and a collaborative spirit in the following ways. To reduce stress, especially since many elementary school teacher candidates fear or dislike mathematics, the course did not have any mathematics content tests and no final examination. In addition, to take the focus away from mathematics achievement, the course used a pass/fail assessment

Fig. 6 Lyrics of song "The infinity cake"

The Infinity Cake

I have the most delicious cake
I want the world to sample
I didn't have much time to bake
But I think just one
should be ample

All my friends had a laugh
They said it could not be done
But I'll just keep cutting
my cake in half
Until the whole world
has had some

I cut my cake in half for you
And gave a quarter to him
One eighth went to little Sue
And one sixteenth to Jim

The cake kept shrinking
My friends kept laughing
But I knew just what to do
I simply took what I had left
and cut it into two

At the end of the day
they all had a treat
Ever so small it may be
I smiled at my friends saying
"This was no feat"
Because I knew fractions
were the key

scheme. The standard for achieving a passing grade was high, based on a B or 75% level of performance on course assignments. However, none of the assignments required a mastery of any given set of mathematics content. Rather, the assessment expectation was that teacher candidates (a) explore a variety of ways of seeing or connecting mathematics concepts, (b) reflect deeply on mathematics teaching and learning, and (c) participate in course discussion on a regular basis. The comments made by the instructor focused on providing positive feedback and conceptual scaffolding when necessary. The course discussion was organized in smaller groups of 9–10 people in a discussion area, however teacher candidates had the freedom to read the discussions of other groups, and also comment on them. It was evident that teacher candidates read the discussions of other groups, as they regularly made explicit references to ideas discussed outside of their own discussion area. Part of the course discussion occurred in a wiki environment, where teacher candidates worked in small groups to collectively identify what they learned from certain activities and how they felt about the experience. The opportunity to express feelings, gave teacher candidates the opportunity to realize that they came to the course with a shared view of mathematics (fearing or disliking the subject) and shared similar experiences of initial apprehension and subsequent surprise and delight about the mathematics patterns they explored.

It is also important that the course focused on mathematics with (1) a low floor, allowing teacher candidates to engage with minimal mathematics knowledge, and (2) a high ceiling, offering opportunities for deep mathematical insight and consequently mathematics that is worth talking about. In traditional mathematics education, as will also be discussed in the next section on performance, when a student is asked "What did you do in math today?" the typical response is "Nothing" or "I don't know". Teacher candidates found the mathematics in this course interesting to talk about and in course discussions even related stories of their related mathematics conversations with family, friends and co-workers.

Below are representative comments about the community and collaborative nature of the Mathematics for Teachers course.

> This class, in the course of a single week, has fostered a greater sense of "community" and involvement among students than any of the classes that I took (including seminars with a very small group) during the course of my 4 year-undergraduate degree. It's really wonderful to read all the warm and supportive feedback that each submission has garnered (and I'm sure that others are equally appreciative, especially since so many of us admitted in our introductory remarks that we were white-knuckled re: teaching math). I have never taken an on-line course until now and it is awesome. I too feel the "community" atmosphere with everyone by such positive feedback, words of encouragement and creative energy from all.

> I'm enjoying learning because of the safe environment everyone here has helped to create. I so appreciate that.

> I have always been an individual who is nervous about participating in class and writing comments on discussion boards. This course has managed to help me break down my barriers and feel more confident in my ideas, comments, math ability, and myself overall.

> Hope that, in the future, I can encourage this kind of welcoming atmosphere in the classes that I teach.

In a hybrid course or in a fully online course involving students who met face to face before a sense community would be more expected in. In the four reflections above, it is surprising, that students in a fully online mathematics class experienced, in their own words, a sense of warm, supportive, positive, energizing, and safe community.

Performance

What makes for a good performance? Boorstin (1990) identifies three pleasures that we derive from performances such as movies: (1) the new and the surprising; (2) emotional moments; and (3) visceral sensations. It is interesting that Norman (2004) states that his principles for technological design "bear perfect correspondence" to the principles of what make movies work identified by Boorstin. These principles have been used in Canada and in Brazil to research how they might be used as a basis for pedagogical design in mathematics education and how they might help us see teachers and students as performance mathematicians (Gadanidis & Borba, 2008; Gadanidis & Hughes, 2011). Below we use Boorstin's principles of what makes movies work to examine the performative nature of experiences in the Mathematics for Teachers course.

New and Surprising

Good movies take you to a new world. The wide angle camera shot is a typical tool used in movies to give the audience of the new world in which the plot will unfold. Good movies also surprise you. When you watch a movie, you typically guess ahead; if your guesses are always correct, the movie becomes predictable and less interesting; however, if your guesses are incorrect then you experience the pleasure of surprise.

As two teacher candidates commented,

> After doing the activities and watching the videos I learned that it is possible to explain the concept of infinity to everybody. I guess I was initially thinking that it would be an impossible concept to explain clearly since it doesn't have a beginning or an end but now I see how it is possible.

> I really liked these activities, because they got me to think about infinity in a way I haven't before.

> It surprised me at how comfortable I felt and that it was a simple way to explain it to children.

The course intentionally sought to offer teacher candidates opportunities to see mathematics in new and wonderful ways. The design of low floor, high ceiling mathematics activities can be seen as being equivalent to the wide angle camera shot in movies. It helped teacher candidates see traditionally abstract concepts such as infinity and optimization as accessible.

> After watching the videos, I learned that *shock* [sic] you can hold infinity in your hands! It is surprising to see that there are ways of representing something so abstract and conveying these themes to such young students. (Asterisks in original)

> I felt amazed at the patterns and relationships present in math - it really is cool that when area stays the same but changes shape, the perimeter changes.

> Wow this activity completely changed how I looked at the concept of infinity.

Teacher candidates also experienced mathematical surprise, both from a mathematics and a pedagogical perspective.

Emotional Moments

Good movies help you experience emotional moments (vicariously through the actors). One of the opportunities that teacher candidates had to experience emotional moments in the course was by sharing how they felt during their mathematics activities of the course. One area of "feeling" that they commonly expressed was their unease with mathematics. This is not surprising as the course was advertised for teacher candidates who fear and/or dislike mathematics. Representative comments by the teacher candidates listed below indicate that in fact the majority fell in this category.

> I know when I received the email describing this course, I just kept saying "that's me ... that's me!" I immediately knew I had to take this course.

> I feel that others are in the same boat as I am when I read other comments about the "fear" of math.

> I think it seems to be a common thread that many of us found math "scary" or intimidating.

> I struggled so much with math throughout my schooling years but [the reason] wasn't because I wasn't good at it, I just needed more than one way of learning it. It led to many frustrating nights.

> I grew up hating math.

Some teacher candidates also expressed an apprehension about taking the course.

> What really surprised me in this course, is that there are a lot of people who felt the same way I did, very hesitant about starting this course at first, but felt it was important in order to be a future educator.

> When I signed up for this course I was uptight and scared because I thought that we were going to be going back to the old way of learning a lesson.

> As a student, I always was aware that math was a very important subject, but I was never taught it in a way where I actually enjoyed it.

> The methods that it was taught to me also did not help ... I was often told to "just do it".

As the course progressed, however, teacher candidates made repeated comments about the positive effect of course activities on their view of mathematics.

> At the beginning of each lesson so far I have felt a little anxious because I am realizing how much math I do need to brush up on. However, after completing each lesson I am feeling much more confident in my math abilities!

> When I signed up for this course I was very hesitant on taking it, but even more hesitant in taking a math course in the fall. You have dispelled my anxiety.

> This will make a huge difference in how I teach math!

> Math was never taught to me ever in such a creative, fun, interactive, and interpretive way!

Children's literature was used to either introduce or supplement all of the mathematics activities of the Mathematics for Teachers course. This created an opportunity for teacher candidates to experience mathematics through the characters involved and vicariously experience their emotional moments. Teacher candidates made overwhelmingly positive comments with respect to the use of children's literature.

> I really enjoyed the 3 little pigs video as a learner. I wanted to watch to the end.
>
> I felt at ease with this lesson. Incorporating the story of the three little pigs and wolf into learning arrays helped to keep me focused because I was also interested in the other (environmental) lesson I was learning with the wolf's story.
>
> I found the best activity for me was the wolf story. I suppose this is because I am such a reader and really connected to the visualizations.
>
> The use of stories and math is wonderful! A math lesson is taught but also as in Wolf Gets Hurt, an underlying message is relayed! Clever!
>
> I loved the moral of the story and I think the best thing a teacher can do is just create interest so the child wants to find out more for themselves.
>
> Wolf gets hurt: Having a dialogue was helpful because as the three little pigs verbalize their plan and rationale, its easier to understand the thought process that got them to the final solution. When the three little pigs try out a few ideas that do not work, it also demonstrates that it's ok to try out some ideas; they may not work the first time, but if you keep trying you will likely find the solution.
>
> The story was amazing it really helped bring it all together!

For these teacher candidates the stories were as source of motivation: "I wanted to watch to the end," "keep me focused," "really connected." Mathematics was cast as the "message," "the moral of the story", and "the thought processes" of the story characters.

Visceral Sensations

Good movies help you experience visceral sensations (such as beauty, fear, disgust, and lust). In the Mathematics for Teachers course, teacher candidates had numerous opportunities (through their hands-on explorations, discussions, interactive animations, and teacher interviews) to experience a sense of mathematics pattern and fit, which give a sense of mathematical beauty.

In movies, visceral sensations are enhanced by the movie soundtrack. Music was also present in the course, through songs from young students and from fellow classmates (see Fig. 7).

> That was amazing!! Definitely turns math into a more active and enjoyable subject, especially for students who are intimidated with numbers.
>
> Thank you so much!

Infinity in my Hand

Shade a square One-half
then a quarter
An eighth and a sixteenth
On and on forever
What's the sum of the parts?
My fractions get smaller
Yet still continue on and on
Can I hold it in my hand?

Infinity is pretty
intangible to me
Annoyingly elusive
and abstract
Something monstrous
or unreachable
I feel unsure, frustrated, overwhelmed
But I enjoy
that it makes me think
Think outside the box

An infinite number of fractions
Can fit in my hand
If I display them
As part of a whole

Cut an apple in half
Then the half in half
And the half in half in half
On and on and on
You get apple juice or mush
Eureka, it still fits in your hand

I am far less intimidated now
Infinity does not have to be large
It can be tiny and small
Wow, what a great exercise
I completely changed
How I look at infinity

It's kind of cool
A very odd pleasure
To physically do something that
Seems mathematically impossible
To hold infinity
in the palm of my hand

An infinite number of fractions
Can fit in my hand
If I display them
As part of a whole

Fig. 7 Lyrics of the song "Infinity in my hand"

I also really enjoyed the songs; I'm a big GreenDay fan, loved the melody choice ;)

I also learned that good math songs will really help when it's time to take a test! I can't get the "making 12" song out of my head now - haha.

In addition, the instructor composed songs that he used to summarize teacher candidate learning, where the songs used as much as possible actual comments made in course discussions.

Whooaa! This is real cool! all the songs act like a summation to the discussions.

What Did You Do in Math Today?

In this chapter we looked at online mathematics learning for teachers through the four affordances of new media: democratization, multimodality, collaboration and performance. All of these affordances enable us to better connect and communicate with one another in educational settings. It is interesting that mathematics, perhaps more so than other school subjects, has traditionally been a subject that people do not talk about outside of classroom settings. However, we have demonstrated through the case of the Mathematics for Teachers course that this does not have to be the case. Mathematics, even mathematics that traditionally has been seen as abstract or inaccessible, can be talked about in ways that can engage not only adults but also young children.

> After doing the activities and watching the videos I learned that it is possible to explain the concept of infinity to everybody.
>
> I guess I was initially thinking that it would be an impossible concept to explain clearly since it doesn't have a beginning or an end but now I see how it is possible.

In fact, many of the teacher candidates in the course did discuss the mathematics of the course with family, friends and colleagues.

> My son and I worked on some of the activities together, and math is becoming something fun for me.
>
> After debating with my husband about whether or not you can *actually* hold infinity, I am convinced that indeed you can.
>
> One of my kids walked by and asked if it was the bare naked ladies I was listening to. This has all been ridiculously fun to follow.

The affordances of new media do, in many ways, help us rethink and disrupt our existing views of mathematics (for teachers and for students) and of how it might be taught and learned, by (1) blurring teacher/student distinctions and crossing hierarchical curriculum boundaries; (2) communicating mathematics in multimodal ways; (3) seeing mathematics as a collaborative enterprise; and (4) helping us learn how to relate good math stories to classmates and family when asked "What did you do in math today?"

Looking Ahead

It is important to reiterate that our goal in this case study is not to prove a causal link between the affordances of new media and the design of the Mathematics for Teachers course. Rather, the affordances of new media are used as an analytical lens to better understand how new media affordances may be seen to be manifested or mirrored in our pedagogical choices and teacher candidate interactions in the course. At the same time, the case study is a glimpse of what-might-be in mathematics for teachers courses in a new media setting, especially in helping transform mathematics as a subject that can be discussed with family and friends as one might for a good movie or a favourite book. What needs to be noticed here is that such a transformation of mathematics can be facilitated through the affordances of new media. As such, our case study serves as an artefact for discussion and critique and as a starting point for designing similar or alternative course offerings in online settings. In our work, we continue to develop a capacity for democratization, multimodality, collaboration and performance in mathematics education, for teachers and for students. One example of this is the www.researchideas.ca website, where in collaboration with project schools we are creating multimodal and performative documentaries of how the "content" of our Mathematics-for-Teachers courses has been used in elementary school classrooms.

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Web-Based Video Clips: A Supplemental Resource for Supporting Pre-service Elementary Mathematics Teachers

Ann LeSage

Abstract Teacher understanding and confidence with rational numbers are important factors contributing to student success with this foundational concept. The challenge facing many Ontario elementary mathematics teacher educators is finding the time, within a 1-year teacher education program, to provide opportunities for elementary pre-service teachers to re-learn rational number concepts in ways they are required to teach. In an effort to address this challenge, web-based video clips were created as an accessible learning resource to support the needs of pre-service elementary teachers. This chapter describes how and why the videos were incorporated into the program and describes the reflections of elementary pre-service teachers after viewing selected videos. The reflections reveal the influence of web-based videos on pre-service teachers' perceived understanding of and confidence with rational numbers.

Keywords Elementary mathematics teacher education • Knowledge for teaching mathematics • Rational number understanding • Web-based videos • Teacher efficacy • Instructional design

Introduction

> Effective mathematics instruction is based on mathematical and pedagogical knowledge and understanding of students' mathematical development (Ontario Ministry of Education, 2011, p. 6).

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Inherent to this statement is the presumed breadth and depth of teachers' mathematical content and pedagogical content knowledge. Research highlights a significant relationship between student achievement and teachers' understanding of the mathematics they teach (Burton, Daane, & Giesen, 2008; Hill, Rowan, & Ball, 2005; Ma, 1999). More specifically, two seminal studies on the effect of teacher mathematical knowledge on student achievement conclude that the combined influence of teachers' mathematical content knowledge and pedagogical content knowledge more strongly correlate with student achievement than any other moderating factor, including socioeconomic and language status (Darling-Hammond, 2000; Hill et al., 2005).

Given this assertion, faculties of education are obliged to modify the structure and content of their programs to provide prospective elementary teachers opportunities to re-learn mathematics in ways they are required to teach. Regrettably, this mandate is particularly challenging for Ontario elementary mathematics teacher educators as instructional time devoted to mathematics methods courses is generally restricted to 36-h (18 h per semester). Consequently, with limited face-to-face instructional time, many pre-service teachers become frustrated and more anxious as they struggle to re-learn mathematics in new ways.

In an effort to address these issues, the author, an elementary mathematics teacher educator, designed a technology-enhanced elective course to support pre-service teachers' understanding of mathematics content and nurture their confidence as elementary mathematics teachers. The *Math4Teachers* course was introduced as an elective offered in the first semester of the program (9 weeks × 2 h/week). In addition to the technologies utilized during the face-to-face component of the course (e.g., interactive whiteboard, virtual manipulatives, interactive applets/software), web-based video clips were created as virtual resources to support learning beyond the physical classroom environment.

This chapter describes how and why web-based videos were integrated into the structure and content of a face-to-face elementary mathematics course. More specifically, the chapter summarizes the current literature on teaching and learning mathematics as well as integrating technology in teacher education; it describes how the web-based video clips (WBVCs) were incorporated into the course; conveys the reflections of pre-service teachers on the perceived impact of the videos on their understanding of and confidence with mathematics; and outlines some of the issues and implications of incorporating WBVCs into an elementary pre-service program.

Literature Review

Four bodies of literature were influential in guiding the development of the course and the WBVCs: knowledge for teaching mathematics, knowledge of rational numbers, web-based learning tools, and integrating digital video in teacher education. Analysis of the literature provides the theoretical lens for modifying the course structure and designing the web-based video clips (WBVCs).

Knowledge for Teaching Elementary Mathematics

Research highlights a direct correlation between student achievement in mathematics and their teachers' understanding of the mathematics content they teach (Burton et al., 2008; Hill et al., 2005; Ma, 1999; Schmidt, Houang, & Cogan, 2002). Although this connection seems self-evident, the paucity of empirical evidence on effective ways to develop knowledge for teaching mathematics has been highlighted in recent research (Berk & Hiebert, 2009; Burton et al., 2008; Hill & Ball, 2009; Kilpatrick, Swafford, & Findell, 2001). One explanation for the scarcity of this empirical research may be the lack of consensus on the nature and depth of knowledge required for teaching mathematics. Initially, Shulman (1986) described the knowledge required for teaching as the interconnection between subject-matter knowledge, pedagogical content knowledge and curricular knowledge. Yet, a decade later, Schifter (1998) bemoaned the absence of research on this topic. She urged researchers and mathematics teacher educators to pursue the question: "What kinds of understandings are required of teachers working to enact the new pedagogy?" (p. 57)

Since Schifter's (1998) call for research, the breadth and depth of teachers' mathematical knowledge has been explored largely by Ball and her colleagues (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2004; Hill et al., 2008; Hill & Ball, 2009; Hill, Schilling, & Ball, 2004; Hill et al., 2005). Ball and her colleagues have focused on a complex dimension of teacher knowledge, namely, *mathematical knowledge for teaching*. At the core of *mathematical knowledge for teaching* is a deep understanding of mathematics content. Specifically, *mathematical knowledge for teaching* assumes an understanding of "common" mathematics knowledge, which is the content knowledge "that any well-educated adult should have" (Ball et al., 2005, p. 22). Unfortunately, many elementary teachers lack this common knowledge and, therefore do not have the foundation to build their mathematical knowledge for teaching.

The collective effect of insufficient content knowledge and high levels of mathematics anxiety can overwhelm novice teachers as they begin their careers as elementary mathematics educators. Thus, the impetus for designing the WBVCs were two-fold: to foster an affinity for mathematics such that pre-service teachers can engage their students and get them excited about mathematics; and to contribute to the research on *how* to nurture pre-service elementary teachers' *mathematical knowledge for teaching,* and their efficacy as mathematics teachers and learners.

Knowledge of Rational Numbers

In decomposing the depth of content knowledge required for teaching elementary mathematics it becomes apparent that an understanding of rational numbers is central in the upper elementary curriculum (e.g. Grades 4–6). Gersten et al. (2009) acknowledge this content focus in their report on interventions that best support students struggling with mathematics. Specifically, the authors make five recommendations, of which one includes focusing "intensely on in-depth treatment of rational numbers in grades 4 through 8" (p. 18). Gersten et al. advocate focusing the curriculum content on "understanding the meaning of fractions, decimals, ratios, and percents, using visual representations, and solving problems with fractions, decimals, ratios, and percents" (p. 19).

Nurturing students' understanding of rational numbers is often deemed one of the most challenging aspects of teaching elementary mathematics (Gould, Outhred, & Mitchelmore, 2006; Li & Kulm, 2008). This pedagogical challenge is complicated by teachers' own conceptual misunderstandings of rational numbers (Hill et al., 2005; Jones Newton, 2009; Li & Kulm, 2008; Ma, 1999; McLeman & Cavell, 2009). Regrettably, teachers' misunderstandings inevitably lead to students' misunderstandings; which often follow children into adulthood (Lipkus, Samsa, & Rimer, 2001; Reyna & Brainerd, 2007) and further intensify should these adults pursue careers as elementary teachers (Ball et al., 2005; Ma, 1999; Menon, 2008; Yeping, 2008).

In spite of this pessimistic perspective, research also reveals that it may be possible to disrupt this cycle of conceptual misunderstanding by targeting teacher education. For example, Siegler et al. (2010) conducted an extensive review of research published over the past 20 years on the effects of instructional interventions on student understanding of rational numbers. The authors put forth five recommendations, of which one recommendation recognized the significant impact of teacher knowledge on student learning. Specifically, Siegler et al., believe it is critical for preservice teacher education and professional development programs to "place a high priority on improving teachers' understanding of fractions and of how to teach them" (p. 42).

As a mathematics teacher educator, I embrace this recommendation by providing opportunities for pre-service elementary teachers to re-form their mathematics content knowledge and re-learn mathematics in ways they are required to teach. To this end, the instructional design of the *Math4Teachers* elective course and WBVCs, adhere to research rooted in effective teaching strategies that support students struggling with mathematics and effective professional development models for teachers of mathematics. Specifically, research highlights positive student achievement outcomes for mathematics interventions which: (1) combine manipulatives and pictorial representations to model abstract concepts (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Siegler et al., 2010); (2) incorporate a mixed model of instruction which blends principles of explicit instruction including teacher modeling, guided practice, and corrective feedback (Baker, Gersten, & Lee, 2002; Flores & Kaylor, 2007; Gersten et al., 2009; Kroesbergen & Van Luit, 2003); and (3) feature ample time for discussion, including student-focused discussions which provide alternative solution strategies expressed in students' language (Grouws, 2004; Shellard, 2004).

A related body of research on effective professional development advocates providing teachers with opportunities to develop their pedagogical content knowledge and deepen their conceptual understanding of mathematics through actively engaging in the learning process (Hill, 2004; Manouchehri & Goodman, 2000; Ross, 1999; Spillane, 2000). The NCTM *Professional Teaching Standards* (1991) extends this recommendation stating that teacher education focus on "mathematical concepts and procedures and the connections among them; ... [as well as] multiple representations of mathematical concepts and procedures" (p. 132). Consequently, embedded in the *Math4Teachers* course design as well as the design of the WBVCs are opportunities for pre-service teachers to do similar tasks as their students (Saxe, Gearhart, & Nasir, 2001; Siegler et al., 2010), to explore multiple representations of concepts to discuss the nature of the mathematics and mathematics pedagogy, and to reflect on their learning experiences (Li & Kulm, 2008; Saxe et al., 2001; Tirosh, 2000).

Web-Based Learning Tools

Web-based learning tools (WBLTs), such as web-based video clips (WBVCs) evolved from a need for accessible, affordable and flexible learning via the Internet (Ally, 2004; Downes, 2004). WBLT are distinct from other digital resources in that "instructional design theory ... must play a large role in the application of WBLTs if they are to succeed in facilitating learning" (Wiley, 2000, p. 9). Kay and Knaack's (2005) review of the WBLT literature, categorized the many WBLT definitions as either technology-focused or learning-focused. They, in turn, defined WBLTs as "reusable, interactive web-based tools that support the learning of specific concepts by enhancing, amplifying, and guiding the cognitive processes of learners" (p. 231). The WBVCs discussed in this paper adhere to Kay and Knaack's definition of WBLTs.

Given the ubiquitous computing environment at this lap-top university, the development and use of WBLTs have been the focus of various research studies within our Faculty of Education (Kay & Kletskin, 2010; Kay & Knaack, 2009a, 2009b; Kay, Knaack, & Muirhead, 2009). Consequently, the design and implementation of the WBVCs discussed in this chapter are grounded in sound theory and practice both in the relevant literature and in other examples of implementation within the university. For example, the WBVC format and design were guided by Kay and Knaack's (2007) findings on conditions that most benefit student learning via technology, including student perceptions of the usefulness of the content, clear instructions, and visual appeal.

Thus, the WBVCs described in this chapter are brief (2–3 min), easy to navigate, have visual appeal and explore content considered useful to the end-user. More importantly, the WBVCs begin to address the challenge of limited face-to-face instructional time common in pre-service education programs.

Digital Videos and Teacher Education

In 2000, the National Council of Teachers of Mathematics (NCTM) established the Technology Principle as a component of their *Principles and Standards of School Mathematics* The Principle states that the "existence, versatility, and power of technology make it possible and necessary to re-examine what mathematics students should learn as well as how they can best learn it" (p. 25). Consequently, teachers need to re-examine how they might integrate technology into their existing mathematics curriculum to meet the diverse needs of their students. Similarly, teacher educators are obliged to prepare pre-service teachers for the task of guiding students in exploring mathematics supported by technology.

A variety of technologies can be useful for exploring elementary school mathematics, including: interactive whiteboards, interactive applets and software, classroom response systems, virtual manipulatives, dynamic geometry tools (i.e., *Geometer's Sketchpad*®), and exploratory data analysis software (i.e., *TinkerPlots*® & *Fathom*®). These technologies not only provide multiple representations of mathematical concepts; but they allow students to explore mathematics in more dynamic ways.

Niess and Walker (2010) advocate integrating digital videos as another viable technology tool. Specifically, they assert that digital videos provide "students with a different and often more engaging way for communicating what they know and understand" (p. 103). However, if teachers are to integrate digital videos into their teaching practices; they need opportunities to explore learning through this medium. Thus, the challenge for mathematics teacher educators is to restructure courses to prepared teachers to effectively incorporate "digital videos in ways that provide exciting, effective, and rigorous mathematics learning opportunities for K-12 students" (Niess & Walker, p. 104).

Kellogg and Kersaint (2004) advocate restructuring pre-service elementary mathematics methods courses to include videos which "help teachers examine mathematics teaching and learning" (p. 25). For that reason, they incorporated ready-made digital videos (i.e., accessible on the Internet) demonstrating reform-oriented mathematics teaching into their mathematics methods courses. Kellogg and Kersaint conclude that integrating these videos helped pre-service teachers "... appreciate alternatives to traditional methods for learning and presenting mathematics ideas" (p. 32).

Although an appreciation for alternative pedagogies is important, teachers can only provide "exciting, effective and rigorous mathematics learning opportunities for their students" (Niess & Walker, 2010, p. 104) if they possess a deep conceptual understanding of the mathematics they are expected to teach. Consequently, Gawlik (2009) recommends developing digital video tutorials to support elementary pre-service teachers' understanding of mathematical concepts. Specifically, she asserts that digital videos are a particularly valuable instructional tool for auditory or visual learners. Furthermore, Gawlik concludes that the length of the videos, the inclusion of step-by-step explanations and visual demonstrations were key components that impacted students' perceptions of the usefulness of the video tutorials.

Given the extant research on the positive effects of integrating digital videos into pre-service mathematics course, the videos described in this chapter extend the research by focusing on the particular mathematics concept of rational numbers. More specifically, the videos described in this chapter were designed to meet the needs of pre-service elementary teachers who lack the capacity to identify, understand and engage in the mathematics they are required to teach.

In the subsequent section I describe how current research informed the development of the video clips, and explain how the videos were incorporated into the pre-service mathematics program.

From Theory to Practice

In August 2009, a technology-enhanced elective course, titled *Math4Teachers,* was introduced to provide additional support for elementary pre-service teachers struggling with the mathematics they were expected to teach. However, as the course ended (November 2009), many of the teachers requested additional resources be provided that would support their continued development as elementary mathematics teachers. Consequently, the WBVCs evolved in an effort to provide this support in the most cost-effective manner.

Development of the WBVCs began in April 2010 with the intent of providing targeted instruction on specific mathematics concepts deemed problematic for elementary teachers. Previous research confirmed my experiences as an educator of elementary pre-service teachers; that strengthening teachers' conceptual understanding of rational numbers was critical (Hill et al., 2005; Jones Newton, 2009; Li & Kulm, 2008; Ma, 1999; McLeman & Cavell, 2009; Newton, 2008). Beyond the content focus, previous research on web-based learning objects (Kay & Knaack, 2007; Lim, Lee, & Richards, 2006; Wiley, 2000) and online videos in teacher education (Gawlik, 2009; Kellogg & Kersaint, 2004; Niess & Walker, 2010) guided the design of the video clips; while research on instructional practices supporting students struggling in mathematics guided the pedagogical focus (Baker et al., 2002; Butler et al. 2003; Gersten et al., 2004, 2009; Kroesbergen & Van Luit, 2003; Siegler et al., 2010). Consequently, each 2–4 min video explicitly demonstrates one aspect in an instructional sequence supporting the progressive development of rational number understanding.

Instructional Sequence = Developmental Continuum

The initial WBVC instructional sequence was modeled on three principal resources: (i) grade level curriculum expectations from the *Ontario Mathematics Curriculum, Grades 1–8* (Ontario Ministry of Education, 2005); (ii) research by Moss and Case (1999) on teaching rational numbers; and (iii) *Professional Resources and*

Instruction for Mathematics Educators, PRIME, an empirically validated teaching resource (Small, McDougall, Ross, & Ben Jaafar, 2006) which supports the progressive development of mathematical understanding.

In particular, the Canadian developed resource *PRIME*, details a continuum of developmental phases that students' progress through as they develop conceptual understandings in mathematics. *PRIME* also describes appropriate instructional strategies to support students' movement along the learning continuum (Small, 2005a, 2005b, 2005c). The earlier work of Canadian researchers, Moss and Case (1999), also highlighted the significance of exploring rational numbers in a specific teaching – learning sequence. Moss and Case advocate for teaching fractions and decimals using a lesson sequence which builds on students' understanding of benchmark percentages. Thus, their lesson sequence begins with visual representations of benchmark percentages, and then progresses to connecting percentage to decimal representations and finally connecting decimals to fractional representations.

In knowing that the WBVCs would not be viewed by pre-service teachers until they had completed at least 2 weeks of the *Math4Teachers* elective course, initial explorations of rational numbers began in a face-to-face environment. Thus, prior to viewing the WBVCs, pre-service teachers were introduced to benchmark numbers, including percentage values which could be represented using concrete or virtual manipulatives (e.g., 10×10 Geoboards, Base Ten Blocks, and money). By limiting exploration of percentages to only those that can be represented as terminating decimals (e.g., 25, 50, 90%, etc.), it ensures that each quantity can easily be modeled concretely. Consequently, the learner associates a visual model or concrete representation to an abstract concept. The face-to-face instructional sequence transitioned from representing benchmark percentages to concrete representations of decimals using the same virtual and concrete manipulatives (e.g., 10×10 Geoboards, Base Ten Blocks, and money).

Approximately half of the face-to-face instructional time ($\sim$10 h) was allocated to representing, comparing and decomposing decimals using manipulatives; and then connecting the concrete models to pictorial representations; and finally, to the abstract representation. The balance of the face-to-face instructional time was dedicated to the exploration of fractions. The subsequent fraction lessons progressed from representing tenths and hundredths using area models (e.g., Geoboards, Base Ten Blocks) to representing and comparing unit fractions and then simple fractions using area models (e.g., Geoboards, Tangrams, pattern blocks) and measured models (e.g., fraction strips, linking cubes, Cuisenaire Rods™). The face-to-face instructional sequence concluded with an introduction to addition and subtraction of fractions by extending the comparison of fractions lessons. For example, pre-service teachers were asked to use fraction strips to compare 1/2 and 1/3. Next, they were asked: "Which fraction is greater?" followed by "How much greater?" By creating a concrete or virtual model, the pre-service teachers were able to visualize the missing 1/6 fraction piece and then create an addition and subtraction sentence based on the concrete representation.

The face-to-face instructional sequence was used to create the parallel series of WBVCs. Consequently, the specific pedagogical sequence of the videos reinforces the progressive development of rational number understanding. Although the WBVCs' instructional sequence evolved from current research and my experiences teaching the concepts face-to-face; as the videos were created it became apparent that the sequence required refinement. Thus, multiple video clips were often necessary to demonstrate the micro-components within a single concept and ensure each video was within the 2–4 min time frame. Moreover, it was important to ". . . attend to potential cognitive overload caused by too much information being presented too quickly" (Bell & Bull, 2010, p. 2).

For example, the *Using Geoboards to Represent Decimals* video explores fraction and decimal representations to tenths and hundredths. However, upon creating the original version of this video clip, it became apparent that additional videos were required to introduce basic decimal vocabulary, and represent decimals to tenths and hundredths individually prior to representing them in a single WBVC. Additional video clips are currently under construction to address other conceptual gaps within the existing instructional sequence.

The next section summarizes the WBVCs pre-service teachers viewed for this research project, and illustrates a sample instruction sequence modeled on the video clips.

Development of the WBVCs

Development of the WBVCs began in April 2010 with the intent of providing targeted instruction on specific mathematics concepts deemed problematic for elementary teachers. The following WBVCs were viewed by pre-service teachers participating in this study:

- Decimal Vocabulary (4:55 min)
- Comparing Decimals (5:32 min)
- Exploring Tenths (3:24 min)
- Exploring Hundredth using Geoboards (2:55 min)
- Using Geoboards to Represent Decimals (3:28 min)
- Representing Fractions: Area Model (3:41 min)
- Representing Fractions: Measured Model (3:05 min)
- Representing Fractions: Set Model (3:26 min)
- Representing Mixed Fractions: Area Model (2:21 min)
- Representing Mixed Fractions: Measured Model (2:13 min)

The WBVCs posted online represent the first phase of a longer term research project (available at http://lesage.blogs.uoit.ca/?page_id=30). The clips are intentionally designed to build on the pre-service teachers' strengths, existing knowledge, and learning needs. The video clips allow for some user interaction by allowing the user to control the speed at which s/he views the demonstration (e.g., play, pause, stop, fast-forward and rewind). However, the long term goal is to expand the format of each WBVC to include: interactive tasks, extension problems, and an on-line self-assessment. That said, Bell and Bull (2010) acknowledge that "evidence is still evolving regarding the types of video and associated pedagogical methods that are most effective for teaching specific curricular topics" (p. 4).

Although the video clips to date include only instructor demonstration through guided discovery, the mathematics content within each clip is supported by current research on scaffolding rational number understanding (Moss & Case, 1999; Small, 2005a, 2005b, 2005c). The instructor guides the user through a series of tasks as she demonstrates the connection between the abstract concept (e.g., decimal numbers) and the concrete representation (e.g., Geoboard).

For illustrative purposes, the details of one WBVC are described here:

The WBVC on *Using Geoboards to Represent Decimals* is a 3:28 min video clip modelling how a 10×10 Geoboard can be used to represent decimal equivalents. The previous clips in this developmental video sequence demonstrate representing simpler decimal numbers, including decimals to the tenth and hundredth place value. Consequently, if the user views the clips in the sequence they are posted, the clips should extend the user's existing knowledge incrementally.

Three primary components of the instructional sequence demonstrated in this WBVC are highlighted in Table 1 including screen shots supplemented by a summary of the instructor's explanations.

Methodology

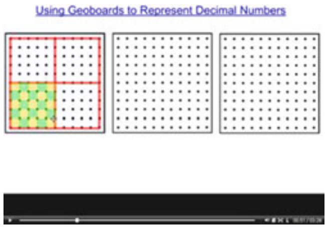
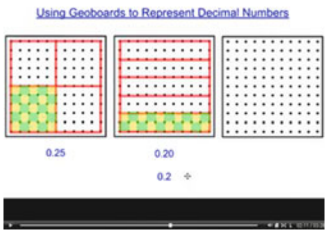
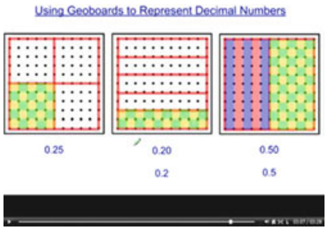
The goal of this qualitative study is to describe the experiences of elementary pre-service teachers in their journey toward understanding rational numbers assisted by online digital videos. The research participants are 40 elementary pre-service teachers who completed the *Math4Teachers* elective course from August to November 2010. The participants are a purposive sample of pre-service teachers who agreed to participate in the study.

The dual purpose of this study is to describe the experience of using WBVCs as a tool for developing rational number understanding; and to evaluate the efficacy of the WBVCs as established by pre-service teachers' perceptions of cognitive and affective gains.

Data Collection

"Qualitative research uses narrative, descriptive approaches to data collection to understand the way things are and what it means from the perspectives of the research participants," (Mills, 2003, p. 4). The qualitative data collection instruments include: narrative reflections from two homework assignments completed during the course; voluntary post-course feedback concerning the value of the WBVCs; and the official course evaluations completed at the end of the term.

Table 1 Sample WBVC instructional sequence

	**Step 1**: Divide the Geoboard into "4 equal pieces"; name the fractional piece: 1/4 **Step 2**: Scaffold content from previous clip. Prove 1 square represents 1/100 of the Geoboard **Step 3**: Count each coloured square by 1/100 to demonstrate 1/4 = 25/100 of the "whole" Geoboard **Step 4**: Include the decimal representation, stating that 0.25 is "*twenty-five hundredths*"
The instructor advises the viewer to periodically pause the clip to complete a specific task using the Geoboard paper provided in class	
	**Step 1**: Divide the Geoboard into "5 equal pieces"; name the fractional piece: 1/5 **Step 2**: Scaffold content from previous example in the same clip **Step 3**: Count pairs of coloured square by 2/100 demonstrating1/5 = 20/100 of the Geoboard **Step 4**: Include the decimal representation, stating that 0.20 is "*twenty hundredths*" 0.20 = 20/100 **Step 5**: Remind viewers that 0.2 is "*two-tenths*" which is the decimal equivalent of 0.20
	**Step 1**: The viewer is instructed to divide the Geoboard into "2 equal pieces"; and then record the decimal equivalents prior to viewing the demonstration **Step 2**: Demonstrate that 1/2 the Geoboard is covered by 50 small coloured squares or 50/100 **Step 3**: Include the decimal representation, stating that 0.50 is "*fifty hundredths*" **Step 4**: Demonstrate that 1/2 the Geoboard is also covered by 5 rectangular pieces or 5/10 **Step 5**: Include the decimal representation, stating that 0.5 is "*five tenths*" which is the decimal equivalent of 0.50 **Step 6**: Review decimal equivalents presented in the clip

The homework reflections required pre-service teachers to view the five *Representing Decimals* WBVCs (total viewing time of the 5 WBVCs = 20:14 min), and then provide a descriptive reflection of their *significant learning* and their *enduring questions*. At the end of the term, 40 of the 57 pre-service teachers voluntarily re-submit their reflective assignments and provided post-course feedback on the value of the WBVCs to be used for research purposes.

Data Analysis

Given the purpose of the study is to describe the phenomenon of using WBVCs as learning tools; qualitative content analysis was used to analyse the collected data. All data were transcribed into the qualitative data analysis program, ATLAS.ti™, which was used to process the data, create codes, and analyze and interpret codes by searching for common words, phrases, themes and patterns. ATLAS.ti™ provided an exploratory approach through which to build complex queries, and begin to develop a comprehensive understanding of the data.

Data analysis began by creating first level codes generated from extent literature on: knowledge for teaching mathematics; knowledge of rational numbers; and digital videos in teacher education. The first level codes were then entered into ATLAS.ti™. Next, all data were read repeatedly to achieve immersion and obtain a sense of the whole (Tesch, 1990). Each document in ATLAS.ti™ was then reviewed, both line by line and as paragraphs or chunks of information. As relevant data was encountered, the text was highlighted, creating a quotation to which a code was attached. Each quotation was automatically identified by ATLAS.ti™ and assigned a display name based on the document number, the location of the quotation within the document, the line numbers of the quotation and its first 20 letters. For example, "2:13 (100:101) *I did not know that we had to . . .* " identifies the quotation from Document #2, the 13th quotation within that document which is located from line 100 to line 101. A code is then attached to the quotation using either *open* coding (the researcher creates the code name) or *in-vivo* coding (the quotation text acts as the code name).

As this process continued, codes were added, merged, and removed as new insights emerged from the data (Miles & Huberman, 1994). Codes were then organized into categories based on how they were related "to one another in coherent, study-important ways" (Miles & Huberman, p. 62).

The selection of quotations and coding procedures denote the beginning of the interpretation phase. Through the coding process I developed initial interpretations and identified themes and patterns as they emerged from the data. Thus, as I began to analyze the data, I had created memos and anecdotal notes highlighting patterns that had become apparent during coding. These interpretations of the descriptive codes marked the initial stages in the development of pattern-based themes from the data.

Summary of Findings

It is through interpretation that raw data evolves into organized information allowing themes to emerge and inferences to develop. In this section, I present the themes that evolved from the data analysis and describe the 40 participants' experiences using web-based video clips with examples from their narratives to illustrate the responses of the group.

Based on the analyses of the participants' experiences viewing and reflecting upon the five *Representing Decimals* WBVCs, the following three shared themes emerged which describe the perceived impact of the videos on the participants' understanding of and confidence with mathematics:

- Development of rational number understanding + self-efficacy;
- Development of pedagogical content knowledge + teacher efficacy; and
- Considerations for instruction design of WBVCs.

Each of the three themes is described in detail as follows:

Development of Rational Number Understanding + Self-Efficacy

The majority (n = 34; 85%) of the pre-service elementary teachers indicated the WBVCs influenced their understanding of rational numbers. Specifically, 21 of the participants indicated that the WBVCs served as a "refresher" of previously "forgotten or never really understood" knowledge; while the remaining 13 participants indicated that the videos presented new information that they had not previously known or had misunderstood.

Within this theme, three sub-themes emerged highlighting specific domains participants identified as significant in supporting their understanding of rational numbers. These included: vocabulary of decimal numbers, comparing decimal quantities, and face value versus place value of decimal numbers.

Vocabulary of Decimal Numbers

The majority (n = 31; 78%) of the participants identified learning new vocabulary specific to decimal numbers as an outcome of viewing the 5 min "Decimal Vocabulary" video. In particular, participants misunderstood that the word "*and*" denotes a decimal; while the term "*point*" should not be used when naming a decimal (i.e., 3.2 should be read as "*three and two-tenths*" not "*three point two*"). Additionally, participants indicated that, prior to viewing the video, they did not realize that the place value of the digit furthest to the right dictates the name of a decimal number (i.e., 3.04 is read "*three and four hundredths*" because the place value of the 4 indicates hundredths).

Comparing Decimal Quantities

Although a half of the participants (n = 20) viewed the 5 min "Comparing Decimals" video; most of them (n = 14) focused their attention simply on the vocabulary of the decimal numbers as opposed to the concept of decimal equivalence. The principal focus of the video clip was to illustrate how one might think about the comparison between two equivalent decimal numbers: 0.2 and 0.20. However, most participants (n = 14) seemed to focus their attention on what they had just learned in the "Decimal Vocabulary" clip viewed previously.

As an example, one participant explained, "I would have pronounced these two numbers the same, not counting the zero. Now I have learned the proper way is pronouncing the second number as *twenty-hundredths*."

Although many of the participants acquired a rudimentary understanding of decimal comparisons, other participants (n = 6) were able to assimilate their previous knowledge of rational numbers to construct new understandings of the relationship between fractions, decimals and visual representations. As an example, one participant explained,

> [The video] helped me understand fractions as well as decimals. The part of clip on 0.20 and 0.2 being hundredths and tenths finally made sense to me. I always knew this, but I don't think that I really understood why.

Similarly, another participant highlighted a similar learning outcome based on the inclusion of a concrete model to represent the decimal numbers. She stated:

> When I saw the clip on this, it was as if a light bulb went on in my head! I always knew that 0.20 and 0.2 were the same, but it really makes a difference when you can see that 0.20 is twenty hundredths and 0.2 is two tenths on the Geoboard!

Face Value Versus Place Value of Decimal Numbers

Although only a few participants (n = 6; 15%) explicitly indicated that the videos helped them rectify their place value misunderstandings; this sub-theme is connected to the two previous sub-themes. For example, participants indicated that in viewing the "Comparing Decimals" video, they now understood that "it is the place value of the digits that matter not the number of digits." Similarly, another participant explained, "I thought 0.0948 was greater than 0.13. Unlike whole numbers, it is not how many digits you have (i.e., 948 versus 13), but their place value."

While another participant described her feelings of empowerment based on her newly-discovered understanding, stating:

> I could never picture a number written to three decimal places because I always said '*point four three two*' (0.432). I was hearing the 'hundred' and perhaps picturing the whole number; so it was difficult to visualize a hundredth decimal number. I was really confused about where the 'thousandths' came in. . . . [By watching the video, it] has now fused my imagined numeral with the verbal and visual representations.

Though the participants' descriptions of their place value learning experiences provide insight into the existing knowledge of pre-service elementary teachers; one participant's comment describes an undeveloped understanding of rational numbers. Specifically, the participant stated, "I now know that the first position behind the decimal is called 'tenths', I would have called it the 'ones' position."

Although the majority of the participants (n = 34; 85%) indicated that the WBVCs substantively influenced their understanding of rational numbers; only a few participants (n = 7; 18%) explicitly cited influences on their mathematics self-efficacy or improved confidence in their abilities to do mathematics. Participants offered comments concerning their improved mathematics self-efficacy, making statements such as; "After seeing these five short videos, I already feel more confident in my understanding of decimals . . . "

Finally, one participant's comment is worthy of sharing as it illustrates the potential impact of digital videos on pre-service teachers' confidence and self-efficacy:

> I am really surprised at how well I am grasping decimals. I remember this as one of my worst mathematics experiences; which usually ended in a lot of tears. But, watching the clips and using the manipulatives just made something click.

Development of Pedagogical Content Knowledge + Teacher Efficacy

Perhaps not surprising, all 40 of the study participants indicated that the WBVCs influenced their understanding of how to teach rational numbers. However, the purpose of the WBVCs was to provide explicit, teacher directed, just-in-time instruction on concepts deemed difficult for students struggling with mathematics. Thus, although the sequence of five *Representing Decimals* WBVCs is developmentally appropriate, illustrates decimal numbers using various manipulatives (e.g., Ten frames, Base-Ten Blocks, Geoboards), and demonstrates multiple representations of rational numbers (e.g., visual, verbal and symbolic); the WBVCs are not exemplary models of reform oriented mathematics teaching practices. Unfortunately, some participants (n = 8; 20%) extrapolated the explicit instruction model demonstrated in the videos as an example of exemplary classroom practice.

As an example, one participant concluded, "I will definitely be applying these video clips to teach my future students, as it made [the content] very clear and straightforward". Another participant commented, "I think that watching the videos was a good reminder to speak slowly when explaining a topic to children."

In spite of this unanticipated learning outcome, the majority of the pre-service teachers (n = 32; 80%) seemed to have generalized beyond the explicit instruction modeled in the WBVCs and gained new pedagogical insights into reform mathematics teaching practices. Specifically, participants emphasized the value of the following:

- encouraging multiple representations, including variability in the materials used to explore the same concept;
- focusing instruction on progressive development of concepts by encouraging concrete models, pictorial representations, verbal descriptions, and symbolic or numeric representations;
- providing students with sufficient time to explore concepts using concrete materials;
- utilizing diverse teaching strategies;
- incorporating technology (e.g., interactive whiteboard technology, virtual manipulatives, WBVCs); and
- supporting continued professional development and access to on-line resources (e.g., WBVCs).

As a final point, besides improved pedagogical content knowledge specific to teaching decimal numbers, all 40 pre-service elementary teachers indicated that the WBVCs influenced their teaching efficacy. The participants cited improved confidence in how to teach rational numbers and integrate reform teaching strategies into their classroom practice. As an example, one participant stated:

> I now feel more confident in introducing decimals while paving the way for the young learners to explore ways of interpreting and estimating decimals.

Considerations for the Instruction Design of WBVCs

In addition to the cognitive and affective outcomes described, the study participants emphasized the significant influence of the WBVC design on the quality of their learning experiences. Specifically, the participants highlighted the following instructional design components as contributing to their understanding of the concepts presented in the WBVC:

- integration of the WBVCs into the *Math4Teachers* face-to-face course design;
- careful sequencing of the content presented in each WBVC (e.g., each clip explores one component of a broader concept);
- clarity of the explanations including step-by-step explanations;
- combined use of visual models/virtual manipulatives + symbolic representation (numbers) + clear verbal explanations;
- abbreviated viewing time (e.g., each clip was less than 5 min in length);
- slower pace than actual classroom lessons;
- ability to control the pace of the learning (e.g., pause to take notes, rewind to review);
- the inclusion of practice questions; and
- the absence of judgement (e.g., pause or rewind the video as often as needed without the judgement of others).

Although the study participants deemed the WBVCs an effective tool for improving their understanding of decimal numbers; there was a general consensus that additional WBVCs are needed to support pre-service elementary teachers' understanding of and confidence with rational numbers. The participants offered the following 'next-step' suggestions which will be considered in subsequent stages of this research project:

- provide additional examples of each concept within each WBVC;
- include additional manipulatives for representing decimals (e.g., money, relational rods, fraction strips);
- create additional practice questions for each WBVC;
- include sample classroom vignettes for some of the WBVCs;
- discuss common misconceptions associated with teaching and learning rational numbers; and
- create WBVCs modelling <u>ineffective</u> teaching strategies for teaching rational numbers.

Issues and Implications

The WBVCs discussed in this chapter were developed to address challenging content areas; provide pre-service elementary teachers with accessible and flexible learning opportunities; and offer additional support for previewing or reviewing important concepts addressed during face-to-face instruction. These instructional design considerations allow pre-service teachers more control over their learning experiences. For elementary teachers with a history of negative mathematics experiences; being in control of mathematics is a novel yet welcome change. Thus, providing on-line resources which can be accessed in a "just-in-time" manner seems to be a promising strategy for supporting the individual learning needs of pre-service elementary teachers.

Although the WBVCs were positively received by the pre-service teachers in this study, the process of designing and creating this supplemental resource was an arduous task. When designing video clips to support pre-service mathematics teachers, careful consideration must be paid not only to the instructional design of the learning tool, but also to effectively supporting the development of pre-service teachers' pedagogical content knowledge and content knowledge of mathematics.

Consequently, video clips must model effective teaching strategies; such as, using concrete and pictorial representations to model rational numbers. However, similar to face-to-face instructional tasks, the video clip activities should also "help teachers understand how the representations relate to the concepts being taught" (Siegler et al., 2010, p. 44). Equally important is the careful sequencing of the video clip content. Conceptual understanding of mathematics is incremental and develops over a life-time of learning experiences. The specific sequence of the knowledge and skills developed while progressing along the mathematical learning continuum warrant further research and consideration from an adult learning/pre-service teacher education perspective.

The research described in this chapter marks a small step toward better understanding the specific program components and lesson sequence which may contribute to the development of pre-service elementary teachers' conceptual understanding of rational numbers. However, the ultimate goal for this research is to: (1) contribute to the collective body of research in determining the breadth and depth of mathematical knowledge necessary for teaching elementary mathematics; and (2) design effective, open-access learning tools, such as WBVCs, to support pre-service teachers in developing this knowledge.

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Visual Mathematics and Cyberlearning in View of Affordance and Activity Theories

Dragana Martinovic, Viktor Freiman, and Zekeriya Karadag

Abstract How do new digital tools and environments affect mathematics learning of students who belong to the Net Generation? In order to explore the complexity of the use of multimedia and Internet to make mathematics teaching and learning more visual and cyber-oriented, our chapter reflects on an add-on value of the new developments in research and practice as discussed by contributing authors of the book. This meta-conceptual analysis of the variety of perspectives on visual mathematics and cyberlearning presented in different chapters of the volume is conducted through the lenses of the Activity Theory and Affordance Theory thus allowing for comprehensive connections of affordances of computational tools to the new structures of activity system in the digital era that make mathematics learning collaborative and self-directed, and increase opportunities of democratization, emergence of mathematical discourse, and multimodalities of embodied interactions.

Keywords Activity Theory • Affordance Theory • Collaborative learning • Cyberlearning • Democratization of learning • Embodied interaction • Formalization of discourse • Mathematics education • Net Generation • Self-directed learning • Visual learning of mathematics

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Introduction

This chapter provides a global reflection on the various aspects of visual mathematics and cyberlearning emphasized by the contributing authors to this book. More precisely, we use *Activity Theory* and *Affordance Theory* as lenses to deepen our understanding of features and complexities of digital technology use in visual mathematics and cyberlearning. Further, possible impact of digital technology tools (Hoyles & Noss, 2009) on the mathematics teaching and learning in different contexts, advances in research and practice, as well as unresolved issues and prospective questions will be investigated.

The idea of using cross-theoretical analysis of teaching and learning mathematics with technology in order to build a meta-conceptual view of the work already accomplished in this area and to create a foundation for further steps is not new. For instance, Artigue, Cerulli, Haspekian, and Maracci (2009) used this method to address fragmentary character of existing theoretical approaches in the context of the Technology Enhanced Learning in Mathematics (TELMA) project thus establishing productive connections and complementarities. Based on the authors' remark of the limitation of such approach regarding making common frameworks easily accessible to researchers who do not share the same experience (Artigue et al., 2009), we begin this chapter with a clarification of the terms *visual mathematics* and *cyberlearning*.

Recent attention to cyberlearning has been fueled by the work of the National Science Foundation (NSF) Task Force and its subsequent call through its *Cyberlearning*: *Transforming Education* program (Montfort & Brown, 2012; NSF, 2011) for further research. Based on their meta-analysis of the existing literature, the Task Force defined cyberlearning as "…the use of networked computing and communications technologies to support learning" (Borgman, 2008, p. 5). Interestingly, although the prefix "cyber" has become associated with computer technology, the task force intended it to be used in its original sense: as a term "…built etymologically on the Greek term for 'steering'" (NSF, 2011, p. 11). In other words, although the focus of the NSF program was clearly on the networking technologies that define the so-called Information Age (e.g., cloud computing, social media), the report intentionally left the term "cyberlearning" open to refer to any form of future technology that mediates (i.e., "steers") the human interactions that are at the heart of education, including human–computer and human–human interactions in cyberspace. Thus, instead of attempting to name the newest technologically driven advances in education, the task force aimed to create a term that would encapsulate the way technology and education could be integrated, without specific reference to or limitation by any particular innovation or even era. It is this inclusiveness inherent in the term "cyberlearning" that also defines our approach to this field of study.

The use of the term 'visual mathematics' may refer to various concepts, like fractals, to the topics related to vector geometry (see, for example, http://vismath.tripod.com/) to the variety of representational tools, like diagrams, to aid the exploration and visualization of ideas (Howse & Stapleton, 2008), or visualization and image processing (Laidlaw & Vilanova, 2012). Another view shared by a number of researchers in mathematics education refers to the context of visual learning and/or visual approaches to learning mathematics (Duval, 1999; Laborde, 2001; Presmeg, 2006; Zimmermann & Cunningham, 1991).

In this sense, it is meaningful to consider visual mathematics as a way to denote the visual learning of mathematics, particularly in cyberlearning environments or with computational tools. In its turn, cyberlearning, as such, reflects dramatic changes of the ways people think, act, and learn in a digital era, especially those who grew up with technology and belong to so-called Net Generation. In the next section, we will examine these phenomena and their relationship to mathematics education.

The New Generation of Learners: Visual and Cyber-Oriented

Berk (2010) defines N-Gen'ers as individuals born between 1982 and 2003 (±2 years). The pervasiveness of digital media (e.g., the Internet, computers, cell phones) in the lives of these youth is confirmed by statistical data from numerous surveys (e.g., Ipsos-Reid, 2004; Junco & Mastrodicasa, 2007; Kaiser Family Foundation, 2010; Weiss, 2003). In fact, recent literature suggests that digital media and communication technologies have had a profound impact on the learning styles and behaviours of today's youth who prefer receiving information quickly, are adept at processing information rapidly, prefer multitasking and non-linear access to information, are kinesthetic, experiential, hands-on learners who must be engaged with first-person learning, games, simulations, and role-playing. They also rely heavily on communication technologies to access information and to carry out social and professional interactions (Martinovic, Freiman, & Karadag, 2011; Ministry of Child and Youth Services [MCYS], 2012; Pletka, 2007; Veen & Vrakking, 2006).

Other studies (Brown, 2005; Howe & Strauss, 2000; Oblinger & Oblinger, 2005) portray the N-Gen'er student as a strong visual but usually weaker textual learner, which may mean that N-Gen'ers' thinking and learning processes are primarily perceptual. According to a perceptual theory of knowledge, "perceptual experiences [that come in different sensory modalities based on audition, vision, taste, smell, touch, and movement] are directly stored in memory and can therefore form the basis for visual thinking" (Reed, 2010, p. 6).

What kind of mathematical activities are suitable for this type of learners? For instance, Rivera (2011) describes visual activities in mathematics as "informal and experimental," "intuitive," "experiential," and taking the form of "multiple paths," which coincides with how the literature describes the preferred learning activities of N-Gen'ers who (a) favor informal learning in exploring a concept or a process and therefore follow multiple paths in solving a problem, and (b) learn better when they are able to perform their exploration experientially and intuitively (e.g., Brown, 2005; Rivera, 2011; Windham, 2005).

The N-Gen'er does not limit his/her actions in cyberspace only to using information; as an experiential learner, s/he learns by exploring and by doing, thus taking opportunities to create new information (e.g., making movies in addition to downloading them) (Barone, 2005). This type of learner is an excellent collaborator and likes to work in groups (Gokhale, 2007; Veen & Vrakking, 2006), using a variety of technical skills and competencies to personalize the digital world for his/her needs. In view of such a learner's characteristics, Pletka (2007) emphasizes the benefits of providing multitasking, fast-paced, visually oriented environments in which the N-Gen'er student can randomly access information in associative contexts rather than in step-by-step, linear ways.

Virtual Tools to Support Cyberlearning

Here we briefly analyze some examples of Internet tools and environments used by N-Gen'ers, particularly those tools and environments referred to in earlier chapters in this book, to determine the kind of mathematical learning opportunities they might generate.

Solomon and Schrum (2007) refer to the year 2000 as a turning point in the evolution of the Internet with the development of a new Internet-based technology called Web 2.0. They begin their timeline with 2000, when the number of web sites stood at 20,000,000. The year 2001 was marked by the creation of Wikipedia (see http://en.wikipedia.org/wiki/Main_Page), the first online encyclopedia written by everyone who wanted to contribute to the creation of shared knowledge. In 2003, the site iTunes allowed for the creation and sharing of musical fragments. In 2004, the Internet bookstore Amazon.com allowed people to buy books entirely online. In 2005, the video-sharing site Youtube.com appeared, allowing for the production and sharing of short video sequences.

The result of this tremendous growth of the Internet-based educational resources and new socially oriented networked environments is that teachers have now a choice of multitude Web 2.0 tools—such as blogs, wikis, and other social software—to support the creation of ad hoc learning communities. Examples of these tools and learning opportunities they generate are provided by our authors:

- wikis that are suitable for the collective writing of mathematics texts and for sharing pictures and graphics (see the chapters in this book by Jones, Geraniou, & Tiropanis, Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"; and Gadanidis & Namukasa, Chapter "New Media and Online Mathematics Learning for Teachers")
- videocasting that allows for the creation and sharing of video sequences (some examples are given in this book in the chapter by LeSage, Chapter "Web-based video clips: A supplemental resource for supporting pre-service elementary mathematics teachers")

- math homework help sites that enable a student who is struggling with homework to ask others for help (see the Math Forum site at http://mathforum.org/; Martinovic, 2005, 2006, 2007; also in the chapters in this book by Jones et al., Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web"; Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team")
- blogs with their "journal" format that encourages students to keep a record of their mathematical thinking over time, thus facilitating critical feedback from teachers, peers, or a wider audience (see the chapters in this book by Jones et al., Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web"; Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; and Gadanidis & Namukasa, Chapter "New Media and Online Mathematics Learning for Teachers").

Overall, then, cyberlearning opportunities encompass: (a) making connections between individuals and groups working on challenging problems and engaging in questioning and finding solutions (Freiman & Lirette-Pitre, 2009; Renninger & Shumar, 2004); (b) helping students to gain a wide appreciation of mathematics (Jones & Simons, 1999); (c) supporting in-depth explorations of mathematics (Pallascio, 2003) that complement new opportunities for learning by means of tools that reinforce cognitive development (Depover, Karsenti, & Komis, 2007; Rotigel & Fello, 2004); and (d) collaborating online, which fosters initiative, creativity, critical thinking, and the generation of new knowledge (Palloff & Pratt, 2007).

Digital Tools to Support Visual Learning of Mathematics

The role of visual learning in mathematics has been explored by mathematicians, mathematics educators, and cognitive scientists (e.g., Arcavi, 2003; Barwise & Etchemendy, 1991; Giaquinto, 2007; Goldenberg, 1991; Ozdemir, Ayvaz-Reis, & Karadag, 2012; Reed, 2010; Rivera, 2011; Yerushalmy & Chazan, 2008; Zimmermann & Cunningham, 1991). There is consensus among these authors that visual learning in mathematics has been historically relatively less appreciated than the use of linguistic methods and algebraic approaches for teaching/learning mathematics, although these latter approaches may be overrated. Reed, for example, cautions that "language is a marvelous tool for communication, but it is greatly overrated as a tool for thought" (2010, p. 1). Zimmermann and Cunningham (1991) point out that visualization in both doing and learning mathematics contributes to the development of a deep and meaningful understanding of both mathematical ideas and the relationships among mathematical concepts. Moreover, problem-solving can be enhanced through visual exploration (Ozdemir et al. 2012).

Visualization as a cognitive tool to foster mathematical thinking and mathematical understanding is analyzed by Karadag and McDougall (2011) and Rivera

(2011). For Rivera, visual thinking in mathematics is not "static" as in "seeing images or pictures for the sake of having a visual or sense experience in order to make mathematics learning fun. It is, more importantly, a concept- or process-driven seeing with the mind's eye" (p. 36). Similarly, Karadag and McDougall (2011) put forward that

> [t]oday's students are more familiar with visual learning because they learn [to use] many new technologies such as computers, Internet, and cell phones visually. Therefore, it might be very challenging for them to learn [mathematics through] symbolic algebra first. Rather, they may better understand algebraic notations after they developed a visual understanding of mathematical concepts. (p. 179)

The contributors to this book approach visual learning in a number of ways. Gadanidis and Namukasa (Chapter "New Media and Online Mathematics Learning for Teachers") refer to the term "multimodality" for which the visual is but an instance. According to these authors, the multimodal nature of the Internet supports multiple modes of communication—speech, print, image, movement, gesture, and sound—thus bringing about a qualitative change in how we teach and learn mathematics and presenting a broad range of possibilities of how mathematics can be done. Taking another perspective, Radford (Chapter "Sensuous Cognition") prefers to use the term "sensuous" rather than a more restrictive term "visual cognition," and proposes that sensation is not only a biological way of understanding the world around us, but is culturally and historically shaped and inseparable from thinking.

For Jones et al. (Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web"), visual is a component of the micro-world in which students learn. Their students used visual means and pictorial symbols in a number of ways: (a) to visualize sub-tasks, stages of work, and required roles; (b) to discuss algebraic generalizations and to receive feedback from the computer environment (MiGen system); and (c) to visually compare their solutions during collaboration.

LeSage (Chapter "Web-based video clips: A supplemental resource for supporting pre-service elementary mathematics teachers") used online videos to provide elementary pre-service teachers with additional visual support in gaining a conceptual understanding of rational numbers. LeSage's means of instruction were especially beneficial for visual learners who could associate the visual model presented in a video with an abstract mathematical concept.

Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team") emphasize the visual nature of their virtual learning space. Their learners moved seamlessly between visual and linguistic practices, performed visual proofs, and used the visual display of the whiteboard to achieve a shared understanding of mathematics.

For Güçler, Hegedus, Robidoux, and Jackiw (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies"), the visual is an essential part of the multimodality of the environment. The authors wanted their learners to experience the mathematics of geometry in the same way as they experienced the 3D world around them naturally. They accomplished this by using haptic devices that allowed for the concretization of abstract geometry concepts and through which the visual once again became only one representation of geometry. Similar to Radford's position (Chapter "Sensuous Cognition"), the mathematics experience offered by Güçler et al. (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies") became sensuous, combining visual and kinesthetic experiences in doing geometry.

Trninic and Abrahamson's students (Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance") adapted their body movements to keep the computer screen green; to do so, both the student's hands had to be positioned properly to demonstrate a mathematical concept provided by the interviewer. This visual feedback from the computer helped students to "discover a means of enacting a green-keeping performance." With the gradual inclusion of symbols on the screen, such as crosshairs (to allow students to visualize the movement of their hands) and a grid (first unlabeled and then labeled with numerals), the students developed an embodied understanding of proportional mathematics and were able to move towards a more articulate and normative mathematics performance.

In summary, the authors of the chapters in this book have pushed the boundaries of our understanding of visual mathematics and visual learning by using existing technological means (e.g., videos) in novel ways or by developing new technological tools (e.g., online learning environments, haptic devices) for mathematics students.

Once the technological tools of different kind are applied, new questions arise regarding their potential to support mathematics learning which we address in the next section.

Digital Tools and How They "Steer" the Learning of Mathematics

Celia Hoyles and Richard Noss (2009) posit four types of computational tools, which have the potential to shape the formation of mathematical meanings in specific ways as described in the following list (see Fig. 1):

1. *Dynamic and interactive tools*, which put the locus of control in the learner and allow for the creation of accurate diagrams, which in turn help the learner both to notice important relationships and to make conjectures.
2. *Tools for outsourcing the processing power* (such as calculators and computer algebra software), which allow users to perform calculations that may be complicated, lengthy, or beyond their current level of skill.
3. *Representation tools*, which allow for previously hard-to-grasp insights to be made more obvious for the learners. Through the use of a specific semiotic system, mathematics terms and concepts may be represented in new ways (e.g., how the introduction of Arabic numerals to replace Roman numerals helped to turn multiplication into a basic skill). Similarly to the way the invention of the

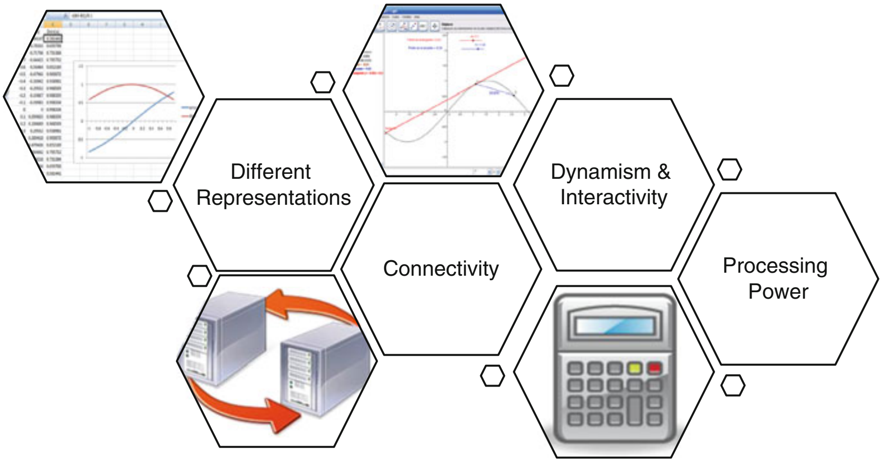

Fig. 1 Opportunities provided by the four types of computational tools, based on Hoyles and Noss (2009)

Cartesian system helped to establish connections between geometry and algebra, the new representations implemented here through computational tools allow for new understandings to emerge.

4. *Connectivity tools* (synchronous or asynchronous), which provide for remote communication and the exchange and/or collaborative development of mathematical ideas.

A computational tool may belong to more than one of the above categories and may be used for various, increasingly sophisticated purposes (e.g., a basic calculator can be used to help with calculations; a more advanced model may have a graphical display with different simultaneous representations; an even more advanced model may be networked with other calculators and used for exchange of data). Given both the complexities of learning/teaching situations and the rapid evolution in computational tools, these changes in technology and their effect on mathematics education are still to be grasped (Hoyles & Noss, 2009; Moreno-Armella, Hegedus, & Kaput, 2008). Through the contributions of our authors, this book seeks to raise questions about the potential of computational tools in relation to new ways of *learning* to create an outcome that is collaborative, self-directed, democratic, co-constructed, coordinated, multimodal, sensuous, and empowering.

In order to reflect globally on contributions regarding to our extended understanding on new ways of learning in both cyber- and visual aspects, we will use the lenses of two theories (see Fig. 2): *Activity Theory* (AcT) (Leont'ev, 1978) and *Affordance Theory* (AffT) (Gibson, 1979).

So as to increase our understanding of how two theories can work together to explain complexity of visual mathematics and cyberlearning, we shall begin with a brief overview of main ideas of each.

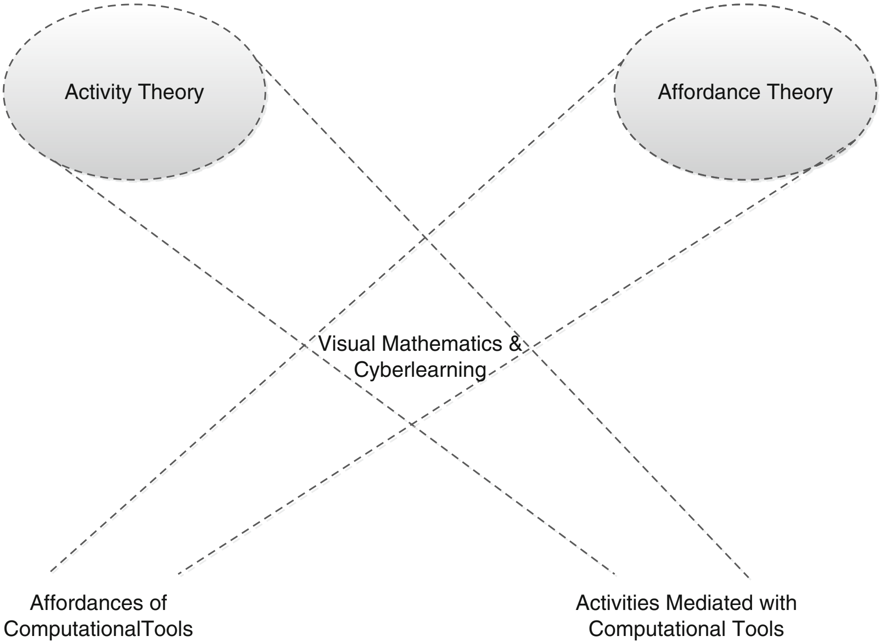

Fig. 2 Viewing the affordances *of* computational tools and the activities *with* computational tools in visual mathematics and cyberlearning through the lenses of AcT and AffT

Overview of the Activity Theory

The *Activity theory* (AcT) was developed in the 1920s—some 25 years before the first computers were created—by Soviet psychologists Vygotsky, Rubinshtein, Leont'ev, and others. Starting from the Vygotskian idea of a *mediated act* (i.e., a person's behavior in relation to his/her sociocultural environment), Leont'ev conceptualized activity as being composed of three different units of analysis: activity, actions, and operations. Then, in 1987, Engeström introduced the concept of an *activity system* that included the components of community, division of labor, and rules (Nunez, 2009).

Every activity has four major components: the *subject*, or the person doing the activity; the *object* on which the activity is performed; the *tool* used during the activity; and the activity's *goal/outcome*. According to Bedny and Karwowski (2007), the goal is the conscious, desired result of the subject's own actions or activity and is the cognitive component of the activity. An object may be physical, symbolic, or visual, and is modified and transformed during the activity. Depending on the character of the object, the performed actions could be practical/external or mental/internal; Leont'ev (1978) called the object the "true motive" (p. 62) of the activity.

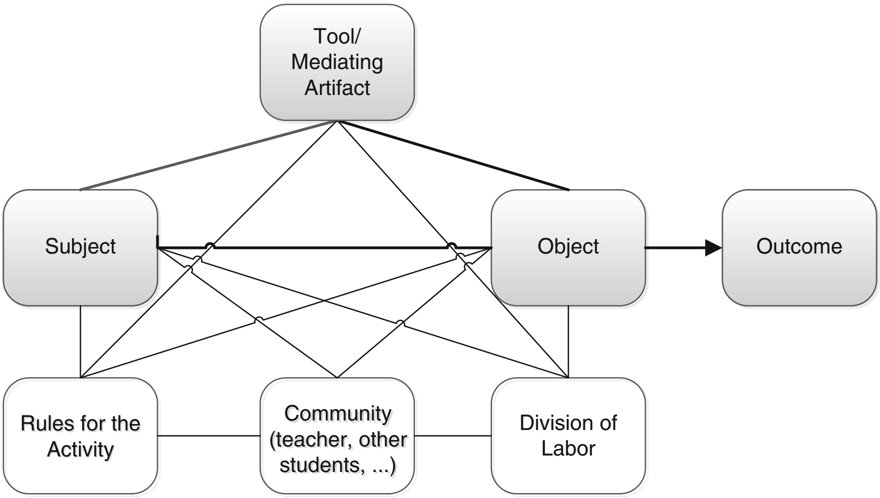

Fig. 3 Relationship between components during the mathematics learning activity

In class, the teacher usually sets up a task along with the conditions for the task. The student is then expected to perform actions (often modeled by the same teacher some time earlier in the process) that will ultimately lead to achieving the educational goal: namely, making a qualitative change in the student. From this perspective, the teacher is the facilitator of the activity and creates—together with the underlying pedagogy and class rules that s/he implements as well as his/her mathematics content knowledge and the mathematics content knowledge of the student—the conditions under which the activity is performed by the student (see Fig. 3). Note that the central triad of activity consists of three elements: a *Subject/Actor* (e.g., student), an *Object/Motive* (e.g., mathematical exercise, problem to solve) and a *Tool/Means/Mediating Artifact* (according to Bellamy, 1996; e.g., software, radio, pen and paper, language, computer). This triad is emphasized in the diagram, which also contains additional nodes, according to Engeström's (1987) model.

In Fig. 3 and in our interpretation, a Tool/Mediating Artifact (e.g., computing software) is a tool that the student uses to act upon the Object (e.g., worksheet, applet). The student can also use other mental tools to work on the object, which is why there is also a direct connection between the student (i.e., Subject) and the mathematical object (i.e., Object) in the diagram. The activity also takes place in the context of the classroom or other rules (e.g., rules of the mathematics task, mathematics discipline) with the teacher as facilitator of the activity and with other students (if they are involved in the activity) (i.e., Community). The teacher usually designs or suggests an activity with certain pedagogical goals in mind (e.g., to motivate students, to help them learn). The *student* (*Subject*) $\leftrightarrow$ *teacher* (*Community*) $\leftrightarrow$ *mathematical object* (*Object*) triangle represents didactical relationships in Brousseau's (1997) sense. Both learner and teacher know their roles in didactic situations: the learner, to attempt to perform the task; the teacher, to set the task and facilitate its realization. Other students participate in their own ways, depending on the activity. All the concepts in the diagram (except for the Outcome) are interconnected within a particular learning task. In the larger picture, all six nodes in the diagram present key elements of the activity, thereby representing a learning process and its consequent outcome (i.e., Outcome) that may be as general as "understanding the difference between a function and its derivative" or more task-related, as in "finding relations of a point on the graph of a function and the function."

The use of tools, including symbolic tools, is important in AcT (Bedny & Karwowski, 2007). Tools do not have to be physical objects. They can be symbols, signs, or images (e.g., according to Vygotsky (1962), language is a tool, too). Given that tools are the means for action and that they play a transformative role, computer software (e.g., spreadsheet and calculator programs) can also be considered a tool for learning, teaching, and doing mathematics.

Based on Engeström (1999), AcT went through three substantial changes: (a) from first being primarily concerned with the individual's activity as mediated by cultural artifacts (i.e., the subject/object/mediating-tool triangle), (b) to Engeström's (1987) model, which sought to present human activity in the context of the larger activity system (i.e., the original triangle extended to include the elements of community, rules, and division of labor), (c) to Engeström's model, which incorporates multiple activity systems and their connections, relationships, and evolution.

AcT is especially central for those who work on developing, investigating, and evaluating educational software because, as Jonassen and Rohrer-Murphy (1999) state: "A fundamental assumption of activity theory is that tools mediate or alter the nature of human activity and, when internalised, influence humans' mental development" (pp. 66–67). Thus, according to AcT, the alteration of the nature of human activity caused by computer use may translate into specific mental development of the user.

Actions in a digital environment depend on the student's knowledge and skill, the type of activity (i.e., individual or group, explorative or procedural), the software used, and many more factors (including technological and socio-cultural). In computer software, the environment for action is created with a purpose in mind by the software developer. To better understand what such an environment offers and how it can be acted upon, we now turn our attention to *affordance theory*.

Overview of the Affordance Theory

In 1966, American psychologist James Jerome Gibson developed the concept of *Affordance* in which he posited that the perception of environment inevitably leads to some course of action.

According to Gibson (1979), we perceive objects in terms of the possibilities for action they offer, or afford, us. Affordances, or clues in the environment that indicate possibilities for action, are perceived in a direct, immediate way with no sensory processing. While this view may be acceptable in the case of natural affordances (i.e., those that emerge in the natural environment), which became directly perceivable to humans through the evolution and adaptation of our species, this view has been criticized when applied to artificial environments (e.g., buttons on a keyboard for pushing, computer mouse for rolling, cursor for pointing, point for dragging on the screen, software package, computer).

Bærentsen and Trettvik (2002) argue that the affordances of artificial forms are nested in "the webs of social activities of praxis and ... these webs are in fact their objective basis" (p. 57). Affordances of such artificial forms are culturally and historically embedded in our present environment (e.g., the size, shape, placement, and features of the Windows ENTER and CTRL keys, which are common to all keyboards). This point can be easily illustrated by observing the convergence of the "feel and look" of software packages developed for different purposes and even by different companies. Hammond (2010) defines affordance as "the perception of a possibility of action (in the broad sense of thought as well as physical activity) provided by properties of, in this case, the computer plus software. These possibilities are shaped by past experience and context, may be conceptually sophisticated, and may need to be signposted by peers and teachers" (p. 216).

Walsh (2012, The affordance landscape: The spatial metaphors of evolution, personal communication) describes reciprocity between organisms and their affordances as "co-constituting and 'commingled'" (p. 15). He further specifies that a goal and a purpose are necessary conditions for an affordance to exist. In fact, "an affordance is an opportunity for, or an impediment to, the attainment of a goal" (p. 15), so some affordances may well be negative (Akhras & Self, 2002; Brown, Stillman, & Herbert, 2004; Gibson, 1979), such as the helpfulness of the teacher or a peer that precludes a student from constructing his/her own learning. Akhras and Self (2002), for example, propose that teachers need to be aware of the affordances of the environment and need to determine which affordances may support or prevent a student's learning in a particular situation.

Putting the Two Theories Together

AcT has been used in the analysis of human–computer interactions since the 1990s (Kaptelinin, 1996), while AffT helped to introduce the notion of affordance as "critical for building a science of educational technology" (de Vries, 2003, p. 170). However, although their terminology has become part of current educational jargon (e.g., affordance, activity, tool, division of labor, learning community), the use of these two theories in conjunction with each other is rare.

To rectify this oversight, we have looked for instances in this book where an explicit connection between these two theories and between these theories and the teaching/learning of mathematics can be made. Finding these connections is also important in terms of contributing to the theories themselves, which need to be constantly challenged in light of contextual changes brought about by the advent of new digital technologies. Or as Rückriem (2003) asks: “Is the current activity-oriented concept of mediation, with its notions of tool, symbol, and artifact, still sufficient for an adequate understanding of the societal and individual importance of digital technology? Is it adequate to deal with the epistemological quality of a ‘leading medium’ as if it were just a material object or a tool . . . ?” (pp. 88–89). He asks further: “Can activity theory and its methodology still be applied to a digitalized reality?” (p. 91).

Here, Rückriem refers specifically to the evolution of the World Wide Web (WWW) into Web 2.0, which in many ways removes the distinction between tool and medium (i.e., one uses the WWW, but is also immersed in it), as well as presenting an environment “in which the machines talk as much to each other as humans talk to machines or other humans” (2003, p. 91). Although the AffT was first posited when computers were already in use (i.e., in 1977), today’s more sophisticated and pervasive digital environment poses questions pertaining to AffT that remain to be answered, especially in terms of how to go beyond the perceptual notion of a tool, as initiated by Gibson (i.e., affordances of a tool are perceived directly), towards treating tools as social and cultural constructs (e.g., computers as tools for writing, drawing, calculating, etc., are perceived as such through social praxis; see Bærentsen & Trettvik, 2002; Hammond, 2010).

Our attempt to establish connections between two theories also fits in with the foundations for mathematics education in the twenty-first century suggested by Hegedus, Kaput, and Lesh (2007), which involve the symbiotic relationship between technology/mathematics education and the tools themselves. On one hand, there is the complexity of mathematics itself (e.g., the multiplicity of mathematical domains; the relationship between pure and applied mathematics; the epistemic dimensions within mathematics, such as the nature of mathematical objects and their relationships, the different kinds of representations and languages, and the forms and modes of justification and truth), which necessitates advances in both technology and education.

On the other hand, the advances in the development of the technological tools themselves provide opportunities for, and even require, new research into both mathematics and education (see, for example, Presmeg, 2006; Rivera, 2011). These advances in technology have, for example, added two major aspects to mathematics education: *representational* (e.g., dynamic geometry, programming languages, spreadsheets, new forms of computer–human interaction), and *communicational* (in terms of infrastructure: information systems, networks, computer devices; and in terms of visible interaction: Internet, WWW, electronic documents exchange, connectivity, social media) (Hegedus et al., 2007; see also Hoyles & Noss, 2009).

In other words, much as Rückriem (2003) does explicitly, Hegedus et al. (2007) implicitly raise issues in both AcT and AffT, such as the concept of a tool or a context. They note that the role of technology is gradually becoming “infrastructural,” such that “technology and tools co-constitute both the material upon which

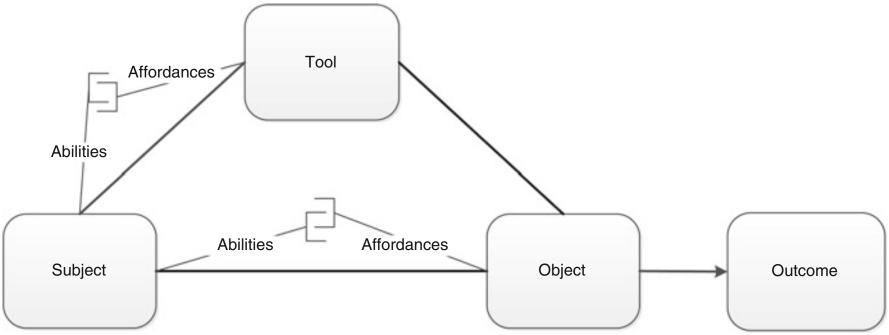

Fig. 4 Visualizing affordances in the context of activity

[individuals] operate and the conditions, particularly social conditions, within which such operations occur" (Hegedus et al., 2007, p. 173; see also Hoyles & Noss, 2009).

At the same time, the focus of education research is shifting to the "affordances of ubiquitous forms of technology in schools [e.g., representational and communicational] and its impact and co-evolution on new forms of teaching, improved learning, new curriculum and more effective problem solving skills" (Hegedus et al., 2007, p. 173). In our view, this sets the stage for using the two theories together.

There have been attempts to date to connect the AcT and AffT (e.g., Cram, Kuswara, & Richards, 2008). This seems to be quite natural: On one hand, AcT recognizes context/environment as a determining factor of any activity, while on the other hand, AffT is rooted in the interaction between the subject and its environment. Our interpretation of the connection between these two theories is portrayed in Fig. 4. Here, the living organisms (subjects) have abilities, while the objects (in the environment) have affordances.

Building on Greeno's (1994) notion of the interdependence between affordances and abilities, we conceptualize their relationship as a "handshake." In other words, a handshake is a prerequisite for an action by the subject to have a positive effect. For example, during explorative activities in mathematics software, a student needs to be able to use features of the software and to consider the objects created in/by the software as material/real. Thus, the student exploits the realness of the objects constructed in/by the software and uses this realness as a scaffold to gain an understanding of mathematics principles on the abstract level and/or to develop connections between different mathematics representations (e.g., visual and symbolic; geometric and algebraic).

The affordances of the tool need to match the abilities of the subject; the affordances of the object need to match the abilities of the subject. Cram et al. (2008) found that

> [b]oth activity theory and the concept of affordances are concerned with the way people interact with the world. However, while activity theory emphasizes the socially mediated aspect of group work, affordances emphasize how each individual within a group utilizes the environment to perform their contribution. A change in the form of activity is reflected by a change in which affordances are utilized. (p. 77)

Here, Cram et al. (2008) emphasize the social aspect of activity. At the same time, AffT needs to take the social component of the environment (e.g., in the case of group activity or when the teacher or an adult is present) into account, treating it as an affordance (Gibson, 1986) or a constraint (Greeno, 1998). Kennewell (2001), on his turn, assumes that "if students work collaboratively towards a single product, then their abilities may be considered jointly. When each student works towards an individual product, then the abilities of other students may be considered as affordances and constraints for the activity" (p.109). Kennewell uses as an example the research of Hoyles, Pozzi, and Healy (1994), in which it appeared that under the constraints of group collaboration, the affordances of peer discussion (i.e., the exchange of ideas and feedback) enhanced the learning.

Visual Mathematics and Cyberlearning in View of AcT and AffT

The work of Celia Hoyles and Richard Noss has been inspirational for this book. Its title, *Visual Mathematics and Cyberlearning*, touches on two new opportunities mentioned earlier: (a) making previously hard-to-grasp insights more obvious through the introduction of new semiotics systems, a concept closely related to the visual aspects of *different* mathematics *representations* in software (approached and discussed in various ways in chapters by Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"; Güçler et al., Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies"; Trninic & Abrahamson, Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance"; Radford, Chapter "Sensuous Cognition"; Gadanidis & Namukasa, Chapter "New Media and Online Mathematics Learning for Teachers"; and LeSage, Chapter "Web-based video clips: A supplemental resource for supporting pre-service elementary mathematics teachers"); and (b) establishing *connectivity*, which is the driving force of cyberlearning (Borgman, 2008; also in chapters by Jones et al., Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web"; Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"; Gadanidis & Namukasa, Chapter "New Media and Online Mathematics Learning for Teachers"; and LeSage, Chapter "Web-based video clips: A supplemental resource for supporting pre-service elementary mathematics teachers").

This book also contains references to the other two opportunities discussed in Hoyles and Noss (2009): *dynamism and interactivity* (in chapters by Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical

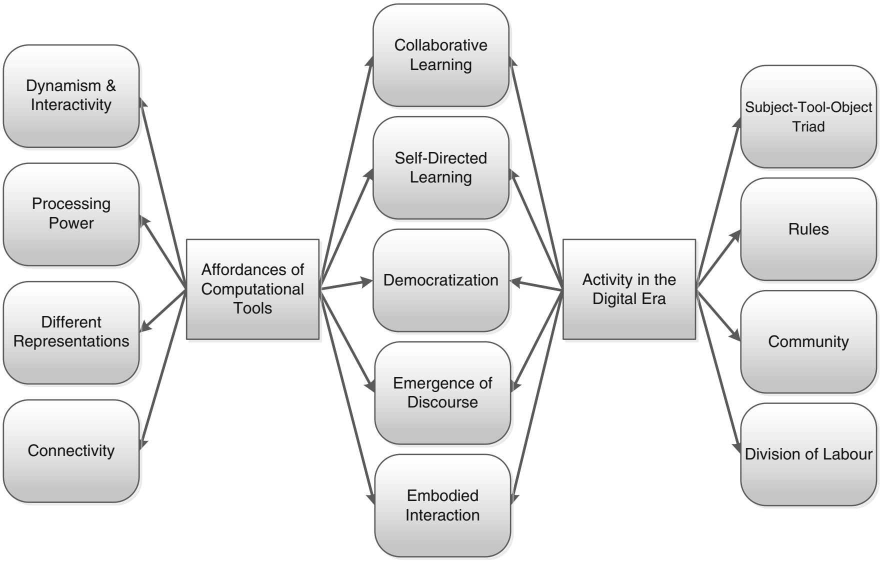

Fig. 5 Connecting the affordances with the activities described in *Visual Mathematics and Cyberlearning*: Introducing new research domains

Reasoning and Symbolic Expression by a Virtual Math Team"; Güçler et al., Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies"; Trninic & Abrahamson, Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance"), and *processing power* (in chapters by Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Çakır & Stahl, Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"; Güçler et al., Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies"). Such opportunities for action (see Fig. 5) also function as affordances in the examples provided by the authors of these chapters.

Which of these four affordances—that is, the potentials for action under given contextual constraints as well as the subject's ability and goal (Kennewell, 2001; Young, Depalma, & Garrett, 2002)—will be used in any given situation depends on the form the activity takes (Cram et al., 2008). In our view, the connection between these affordances (i.e., dynamism and interactivity, processing power, different representations, and connectivity) and the elements of the activity systems described in this book can be made in five domains: *collaborative learning, self-directed learning, democratization, formalization of discourse,* and *embodied interaction* (including coordinated action across multiple spaces).

Collaborative Learning

In their chapter, Gadanidis and Namukasa (Chapter "New Media and Online Mathematics Learning for Teachers") analyzed a teacher education course, Mathematics for Teachers, through the lens of affordances of new media. The affordance they explored is collaboration, which is also of interest among activity theorists. The teacher-candidates and the instructor divided the work involved in course activities. The instructor's role was to facilitate collaboration by providing positive feedback and by scaffolding ideas when needed.

Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"), for their part, demonstrate how their participants benefited from the affordances of an online collaborative learning environment called Virtual Math Teams (VMT). In fact, their demonstration provides a well-integrated example of how AcT and AffT come together because VMT provided participants with opportunities to perform activities in collaboration and to co-construct artifacts by utilizing the affordances of the environment.

Jones et al. (Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web") describe the features of eXpresser (a microworld "designed to provide students with a model for generalization") and the class activities designed to support collaborative learning. Their example illustrates how the affordance of the system, in this case the collaboration tool, needs to match the ability of the student to be able to benefit from it. A "handshake" between the affordance and the ability, however, did not appear to fully take place in their study, as the students "failed to produce shared mathematically valid arguments to justify the correctness of their rules." Jones et al. (Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web") posit the weak affordance of the system (i.e., collaboration) as a possible reason for this failure. Other analyses of different online learning environments (see Martinovic, 2007), instead, point to possible inherent rules of behavior in peer-to-peer online learning activity systems, where indirect language (e.g., avoidance of criticism) is used among peers both to save face either of other or of self (Goffman, 1967) and to keep the fellowship intact (Lim & Bowers, 1991).

Collaborative learning environments in Alagic and Alagic (Chapter "Collaborative Mathematics Learning in Online Environments"; e.g., the Polymath project and the MathOverflow website) are presented as activity systems resistant to failure. These massive, open-access online communities consist of volunteers who gather together to discuss mathematics based on their need and wish. Subgroups may instinctively gather together based on their interest, and individuals may join or leave without creating much disturbance in the community. Such distributed and flexible organizational structures are described by Alagic and Alagic as suitable for both research and the learning of mathematics.

Here we are reminded of the writing of Radford and Roth (2011) who introduce the concept of a *space of joint action*—a space in which collaborative activity happens where "the students . . . think, and act together" (p. 233, italics in the original). In Radford and Roth's example, this space was created in a classroom environment, but in our view it could be created in a cyberlearning environment, such as those described in this book. For collaborative learning to become *togetherness*, both the students and the teacher need to be committed to "a collectively motivated activity based on trust and responsibility" (Radford & Roth, 2011, p. 244). Togetherness is more than just working together on a problem; the object of the activity is changed as well, so that it is "reflected similarly in the consciousness of all participants" and "becomes a common object of activity because of togethering" (p. 242).

We are adding *togethering* to the list of collaborative activities described in this book; indeed, many of these activities deserve the question asked by Alagic and Alagic (Chapter "Collaborative Mathematics Learning in Online Environments"): "*Why aren't we all learning math this way*?" (italics in the original).

Self-Directed Learning

In this book, several authors address various opportunities for self-directed learning afforded by the new technologies. LeSage (Chapter "Web-based video clips: A supplemental resource for supporting pre-service elementary mathematics teachers") writes about pre-service teachers' use of online resources. Accessibility of online resources in a just-in-time manner is also recognized by other chapter contributors (e.g., Alagic & Alagic, Chapter "Collaborative Mathematics Learning in Online Environments"; Gadanidis & Namukasa, Chapter "New Media and Online Mathematics Learning for Teachers") as an important element of support for the individual's learning needs.

Namely, Alagic and Alagic describe various cyberlearning environments that provide self-regulated learners with opportunities for additional exploration, such as access to resources, records of discussions, complete proofs, and explanations. The authors see self-regulation or self-direction in learning as necessitating a high degree of learner independence, which includes the learner establishing and then following his/her own goals in conjunction with monitoring, regulating, and controlling his/her cognition, motivation, and behavior.

The question is how to develop such a learning style in school children. This is a valid question, especially since the literature suggests that children engage in informal self-initiated and self-directed learning more at home than at school. In homes with Internet access, computer use is often child-directed (i.e., with minimal parental involvement), with time for exploration and incidental learning. Compare this to school-based computer use, which is teacher-directed and focused on purposeful learning (Johnson & Puplampu, 2008; Kerawalla & Crook, 2002). At home, moreover, the child's skills in using the computer are recognized and appreciated, while in schools, this is rarely the case. Schools may need to embrace some of the typically out-of-school activities to increase engagement and achievement of children because, as Barron (2006) notes, out-of-school activities "allow for expertise development while simultaneously supporting aspects of identity development such as a sense of belonging in a community, feelings of competence, and interest development. The breadth and qualities of these activities are significant developmentally, as are the roles and relationships that emerge across contexts" (p. 194).

This situation is also recognized by a number of AcT theorists. Davydov and Markova (1983) criticize school as a place where genuine educational activity does not happen. This is because schools are not places where students formulate and accept the goals of actions in which they engage. The authors go on to explain: "In the course of development of educational activity, it is necessary to ascertain and create conditions that will enable activity to acquire personal meaning, to become a source [both] of the person's self-development and [of the] comprehensive development of his [sic] personality, [as well as] a condition for his [sic] entry into social practice" (Davydov & Markova, Theoretical Sources and Stages in the Development of the Concept). The authors distinguish here between formal (i.e., in-school) and informal (e.g., in-play) learning. In both instances, knowledge and abilities may be assimilated, but in the first case, assimilation is a goal, while in the second, assimilation is a by-product.

More recent critics of formal learning claim that it is antithetical to developing self-regulatory characteristics such as persistence, determination, use of multiple strategies, willingness to try and be wrong, confidence, and total lack of fear of the technology (Mason, 2004).

Instead of looking at learning as polarized between formal and informal, Barron (2006) writes about the "learning landscape," acknowledging that there are a number of contexts in which learning happens. Anyone can develop an interest and pursue it, if given the time, resources, and freedom to do so. For a person who has developed an interest in something, different contexts (formal or informal) do not present barriers to learning; on the contrary, they allow for the transfer and cross-fertilization of knowledge. Some examples of technology-mediated development of self-directed interests include searching for information on the Internet, creating materials using multimedia, following podcasts and YouTube videos, exploring new media, and taking a role of mentor in social media.

We argue that opportunities created by digital technologies blur the lines that exist between informal and formal learning. There are ample opportunities for a person to access online resources to satisfy his/her need or interest to learn. The policy framework for Youth Development in Ontario (MCYS, 2012) describes adolescence (13–17 years of age) as a period in life when (a) emotional self-regulation matures, (b) self-concept becomes more complex and situation-dependent, (c) self-appraisal and self-efficacy skills improve, and (d) youth become more self-sufficient in making decisions about their relationships and activities. Adolescence may be an appropriate time to develop new interests through the use of technology because, as Berk (2006) suggests: "An imagined future self helps motivate learning, as does the simple pleasure of creating" (p. 220). Engaging young people with mathematics through technology and describing opportunities for them to take part in society may spark an interest in mathematics that they will continue to nurture both formally and informally.

Democratization

In this book, both Alagic and Alagic (Chapter "Collaborative Mathematics Learning in Online Environments"), and Gadanidis and Namukasa (Chapter "New Media and Online Mathematics Learning for Teachers") caution that the current classroom model may be more autocratic than democratic, particularly when ideas can be questioned or pursued only within the narrow bounds of (a) the classroom walls and practices, and (b) the course syllabus and learning goals (Gadanidis and Namukasa refer to this as a "traditional" classroom, while Alagic and Alagic consider this even more generally as a "classroom model"). For these authors, new computer technologies are deemed disruptive in the sense that they allow students to escape from authoritarian education (i.e., an education system that is prescribed, controlled, fixed, and limited). New technologies provide opportunities for democratization; however, democratization will happen only in an activity system that supports it. Jones (2011), for example, cautions that "innovative technology … *on its own* may not perturb enough to cause a major change" (p. 44, italics in the original) in education.

Jones et al. in this book (Chapter "Patterns of Collaboration: Towards Learning Mathematics in the Era of the Semantic Web") predict that real changes and a real move towards democratization may start first in the higher education sector and then trickle down to other school levels. The experiences of Gadanidis and Namukasa (Chapter "New Media and Online Mathematics Learning for Teachers") in the teacher education program seem to confirm this prediction. Their Mathematics for Teachers course reflects the democratization affordance of the new media, which provide an asynchronous online discussion platform to teacher-candidates, providing access at any time and from any place (with an Internet connection), flexibility in what is to be learned and how, and distributed teaching within the community. Thus, once these teacher-candidates have their own students, they may be more willing to incorporate the democratic affordance of computer networks in their classes.

Another example of democratization—in this case where students took over the role of teacher—can be found in the chapter by Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"). After an open-ended problem in mathematics was given to the students, they reached an agreement on which interesting and worth-exploring direction to pursue, and then continued working as a group with minimal supervision from the tutor.

The chapters by Alagic and Alagic (Chapter "Collaborative Mathematics Learning in Online Environments"), and by Gadanidis and Namukasa (Chapter "New

Media and Online Mathematics Learning for Teachers") remind us that democratization does not necessarily mean lowering the standards of generated content, replacing professionalism with amateurishness, and expecting that novices can by themselves discover mathematics. Their cautionary remarks join the voices of others who believe educators should ensure that we do not enter the era of information without knowledge and computer networks without community (Martinovic & Magliaro, 2007; Noveck, 2000). There is much evidence that computer networks as part of the ecology of the Web have in fact a multifaceted nature, in that they provide for democratization on one hand, but also for globalization, surveillance, and control of communication and information on the other.

Emergence of Discourse

The chapter by Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team") analyzes how the VMT group members used representational affordances in an online environment to show, among other things, how mathematics terminology comes into being in response to the specific communications needs of the speakers. The participants in their study used verbal references or highlighted drawings with different colors to (a) isolate objects on the whiteboard, (b) associate these objects with communal chat terminology, and (c) use the verbal and nonverbal references to the objects to develop a shared understanding.

This multi-step process demonstrated how mathematical discourse can encode certain ways of thinking about mathematical objects—the students' new-found mathematical terms first emerging from the need to talk, then becoming meaningful mathematical artifacts through the ways that the participants enacted them, and finally reaching the stage of symbolic expressions.

In their analysis of the mathematical discourse of young learners involved in multimodal mathematical investigations (e.g., through the use of various communication channels—speech, gesture, writing), Güçler et al. in their chapter (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies") refer to Sfard's (2008) concept of discourse as evolving from being simply words used in speech to involving different (visual) mediators (e.g., symbols, graphs) and routines (e.g., gestures, participation patterns, forms of argumentation) when exploring how students communicate mathematical concepts. Using a combination of visual and haptic tools as well as social interactions to shape their students' discourse, the authors demonstrated how such tools could enhance meaning-making in mathematics in more dynamic, tactile, and natural ways. Through observing the changes in their students' discourse patterns (i.e., from non-scholastic "everyday language" to scholastic terminology), the authors could point at the changes in how the students discussed the concepts as indicators of intellectual development and mathematical learning.

Formalization of discourse does not happen automatically and without contradictions, but that is to be expected in any activity system. In fact, contradiction is the "motive force of change and development" (Engeström, 1999, p. 9), and the transitions and reorganizations within and between activity systems is part of these systems' evolution. The students of Güçler et al. (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies") "used the [haptic device, the PHANTOM Omni®] as a semiotic mediator when their visual perceptions contradicted their physical perceptions. Students also utilized social interaction as a form of mediation when they disagreed with each other; when they corrected each other's word use and adopted words used by peers; and when they individually and collectively hypothesized and tested their hypotheses." Thus, this "dynamic, multimodal exploration space" afforded the development of both personal and collective mathematical meanings.

Embodied Interaction

Understanding mathematical embodiment is becoming recently an important issue in mathematics education which requires building new paradigms in order to conceptualize the relationship between gesture and diagram. De Frietas and Sinclair (2012) suggest that "such an approach might open up new ways of conceptualizing the very idea of mathematical embodiment" (p. 134). In our volume, the topic is explored under the view of embodied interactions.

According to Trninic and Abrahamson (Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance"), embodied interaction is a particular type of pedagogical design that fosters learner development of embodied artifacts—that is, cognitive products targeted towards specific mathematical skills, such as proportional reasoning. The authors analyzed the pedagogical potential of the ubiquitous motion-sensor technologies (e.g., Mathematical Imagery Trainer for Proportion) to create learning environments in which students could use body movements to learn about proportions.

Trninic and Abrahamson (Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance") challenge the outdated view that conceptual learning is achieved essentially via developing fluency in a discipline's semiotic system centered on manipulating symbols in paper media. Instead, they foreground the critical role of subjective meaning in this acculturation process, and they offer that these meanings are generated through engaging in embodied interactions (EI) and then experiencing guided reflection on these interactions.

In his chapter, Radford refers to a person's capacity for *responsive sensation* (italics in the original) to represent the idea of cognition as a feature of *living material bodies* (italics by the authors). Here, Radford (Chapter "Sensuous Cognition") makes an important reference to Leont'ev (2009) by discussing how complex forms of sensations arising "from [the] progressive complexity of [the] processes of life" can develop the human mind's capacity to reflect reality. According to Radford (Chapter "Sensuous Cognition"), "our tactile–kinesthetic bodily experience of the world and our interaction with artifacts and material culture are considered as much more than merely auxiliary or secondary elements in our cognitive endeavors." In agreement with Güçler et al. (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies"), Radford (Chapter "Sensuous Cognition") argues that technological devices, such as dynamic geometry software and haptic devices, "offer room for the creation of an experimental space that might require the appearance of new sensibilities and new embodied ways of thinking—dynamic new literate ways of scrutinizing, enquiring, looking into, and thinking about, mathematics, mathematical objects and their relationships."

Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team"), for their part, analyze mathematical reasoning as the enactment of representational *affordances* (italics by the authors) as demonstrated in the interactions among their VMT participants. The authors reflect on the challenges of interaction in an online setting with a dual space for action, such as a whiteboard and a chat window, and compare this online interaction to face-to-face interaction where pointing gestures and body movements can clarify what the speaker is referring to. The authors found, however, that their students overcame the limitations of online environment and achieved functionally comparable interactions online through their use of available features (e.g., narrative, graphical, and symbolic). Çakır and Stahl's analysis seems to confirm that their students demonstrated "dynamic new literate ways" (see also the Chapter "Sensuous Cognition", by Radford, in this book) in using dynamic mathematics software in the whiteboard space and making references in the threaded discussion taking place simultaneously.

Embodied interaction calls for *coordinated action across multiple spaces*, and the chapters by Güçler et al. (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies") and by Çakır and Stahl (Chapter "The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team") both illustrate how learners can accomplish this. For example, Çakır and Stahl's participants performed their activities through verbal, symbolic, and graphical representations. They switched from one space of representation to another, as needed, and did so quite intuitively. This dynamic switch between representations as well as the ability to coordinate actions in one space with the intent of generating a specific response in another could be interpreted as a high level of mathematical thinking. Kaput (1992) argues that acting in any specific representation (notation) system evokes a particular cognition. When working across different representations, learners engage their corresponding cognitions as well as a "translation" mechanism that helps in switching from one notation system to another. Mathematics software may perform such a translation automatically and with a certain accuracy, which allows for the learning objective to be changed from obtaining mastery in one representation (notation) system to understanding the connections between different systems. This is especially important for learning mathematics, as mathematics concepts are usually complex and require multiple representations.

Similarly, the participants in the study by Güçler et al. (Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The Case of Haptic Technologies") switched between their physical manipulation with the haptic device and the visual representation of 3D objects produced on the computer screen. Moving between these physical and visual modalities allowed the students to learn to classify solids and to describe attributes of the planar intersections of the 3D object on the screen.

The research shared by these authors as well as by the other contributors to this book can guide the reader to follow how AffT and AcT have the potential to explore and explain the coordinated actions of participants across the multiple spaces of cyberlearning environments.

Conclusions

This book presents a diverse look at embodiment through technology: first, with haptic devices and body movement; second, within face-to-face interaction; and third, as reasoning embodied in interaction.

In this chapter, we have summarized the current understanding of how today's students can benefit from learning mathematics in conjunction with various digital technologies. Our intent was not to give an in-depth analysis of each author's contribution to this volume, but rather to review their chapters reflectively to uncover and discuss common findings and unresolved issues through the perspective of activity theory and affordance theory, two theories frequently used in modern mathematics education research in general and in teaching and learning in digital environments in particular.

From a global perspective, the opportunities for learning, teaching, and doing mathematics with computer-based technology are becoming more prevalent, and activity systems where this happens are becoming more complex. In addressing both the socially mediated aspects of learning and the ways in which individuals utilize the learning environment, we have identified various formal and informal opportunities for learning. In addition, by identifying the different roles that technology can play in learning/teaching/doing mathematics, we have also addressed issues around the superficial or inadequate use of technology. The emerging focus on embodiment, for example, may alleviate two fears: first, that technology might eliminate hand and body motions that may be necessary for cognitive development; and second, that human perceptual intelligence may degenerate in the technological era (Knipp, 2003). From the examples provided in this book, it is evident that today's technologies, in particular the haptic ones (e.g., the PHANTOM Omni® haptic device in Güçler et al., Chapter "Investigating the Mathematical Discourse of Young Learners Involved in Multi-Modal Mathematical Investigations: The

Case of Haptic Technologies"; the Mathematical Imagery Trainer for Proportion in Trninic & Abrahamson, Chapter "Embodied Interaction as Designed Mediation of Conceptual Performance") can deal with these issues.

Bypassing the limiting role that digital technology may be given in schools, such as being used to primarily grab the students' attention, to motivate them, or to reward their good behavior, we looked at its educational potential for becoming "a catalyst for changing pedagogy" (Zevenbergen & Lerman, 2007, p. 854). By investigating recent developments in mathematics education in the digital world, we have come to share the view that technologies, activities, artifacts, and environments do not exist independently of their use (LeBaron, 2002), but are constituted within practice in more collaborative and productive ways through the organized actions of their users.

Based on the literature and on our personal experience, it seems that active participation in virtual mathematical opportunities may help not only to preserve students' natural motivation and the interest they have in the world around them, but also turn such interest into meaningful mathematics learning, full of opportunities for enrichment and collaboration, and thus supporting the emergence of a new learning culture. However, more research is needed to better understand the outcomes of cyberlearning from (meta-)cognitive, affective, and social perspectives (Freiman, 2008). Moving away from the often too idealistic and unsubstantiated glorification of the positive changes brought about by technology and exploring instead the realities of young peoples' lives and educational settings would better inform effective policymaking and practice (Selwyn, 2009).

This goal is also evident in the recently released request for proposals in the NSF's Cyberlearning: Transforming Education program, which "seeks to integrate advances in technology with advances in *what is known about how people learn* to better understand how people learn with technology ... through individual use and/or through collaborations mediated by technology" (NSF Request for Proposals, 2011, p. 12; italics by the authors). The NSF anticipates that this research will produce new forms of educational practice as well as have positive effects on the productivity of the workforce and on the engagement of citizens in lifelong learning both in and out of school.

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