Murari L. Gambhir

Stability Analysis and Design of Structures

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With 159 Figures

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Preface

The stability considerations are extremely important and inevitable in the design of many engineering structures compassing aeronautical engineering, civil engineering, mechanical engineering, naval architecture and applied mechanics wherein a designer is confronted by numerous stability problems. Most of the national standards have based their design codal provisions on the stability criteria, especially in design of steel structures. In view of making the engineers appreciative of limitations associated with many structural design codal provisions, most of the engineering colleges and universities offer the course on the subject as a part of curriculum.

A number of books on the subject are available in the market, which had been written before mid-eighties and treated the problems normally encountered in engineering mainly by classical techniques. In view of rapid advancements and improvements in the methods of analysis and in the computing environment, stiffness methods supported by numerical techniques are being extensively applied to relatively complex real-life problems. The later approach is emphasized in the present book.

The text is specially designed to cater to the classroom or self-study needs of students at advanced undergraduate and graduate level in structural engineering, applied mechanics, aeronautical engineering, mechanical engineering and naval architecture. Although the special problems pertaining to these disciplines differ philosophically but analytical and design principles discussed in the text are generally applicable to all of them. The emphasis is on fundamental theory rather than specific applications.

The text addresses to the stability of key structural elements: rigid-body assemblage, column, beam-column, beam, rigid frame, thin plate, arch, ring and shell. The text begins with introduction to general basic principles of mechanics. This is followed by a detailed discussion on stability analysis of rigid-body assemblage, column, beam-column, beam, rigid frame, plates, arch and shell arranged in different chapters from 1 to 9. In Chap. 10, the elastic theories of buckling have been extended to the inelastic range. Where as in Chap. 11 on the design for structural stability, the American national standard, Australian standard AS: 1250–1981, British code BS: 5940-1985 (Part-I) and Indian code of practice IS: 800-1984 have been compared for the provisions related to stability considerations and number of design illustrations have also been given. Each chapter contains numerous worked-out problems to clarify the discussion of practical applications that will facilitate comprehension of basic principles from the field of stability theory. Wherever possible alternate approaches to the solution of important problems have been given. Tables and formulae are devised in the form suitable for the use in the design office. Thus the book would also prove useful to the practicing engineers engaged in actual design. In addition exercise problems designed to support and extend the treatment are given at the end of each chapter. For more important ones answers have also been given. The illustration problems have been treated by the practical methods, which are best suited. There is conscious effort to present results in non-dimensional form to render the subject matter independent of system of units. These non-dimensional parameters facilitate the application of results to different materials and structural configurations encountered in practice. A large amount of practical data in tabular form and simplified formulae are given to make them suitable for the use in the design of various components.

It is the opinion of the author that the undergraduate students should study first six chapters as a part of their required program of study. The remaining chapters can be studied at the graduate level. To make the fundamentals of stability analysis more understandable and meaningful, this text should be used at the level when the student has attained the basic knowledge of statics, solid mechanics or strength of materials and calculus. Only a minimum knowledge of calculus, Fourier series and Bessel functions is assumed on the part of reader. However, for reference necessary background information needed to deal with problems involving differential equations and Bessel functions is given in the appendix. The subject matter and its presentation sequence has been class tested over the past two decades. In the process students have made valuable suggestions for which author is grateful.

The author wishes to express his sincere gratitude to the authors of various books on the subject who have been an inspiration to developing this text. The author thanks all those who have assisted in various ways in preparation of this text. Particularly, he wishes to acknowledge the assistance rendered by Dr. Puneet Gambhir, Er. Mohit Gambhir and Er. Neha Gambhir in preparation of manuscript. The author is extremely grateful to his wife Ms Saroj Gambhir for the patience she has shown while he was busy completing this job. The assistance and advice received from Dr. Thomas Ditzinger and Ms. Gaby Maas, the Editor, of Springer-Verlag is gratefully acknowledged. The author welcomes suggestions from the readers for improvement in the subject matter in any manner.

Patiala, India
May 12, 2004

M. L. Gambhir

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About the Author

M.L. GAMBHIR is currently Director of Rayat Institute of Engineering and Information Technology, Railmajra, Punjab, India. Previously he was Professor & Head of Civil Engineering Department, and Dean Planning & Resource Generation at the Thapar Institute of Engineering & Technology, Patiala. He obtained his Bachelor's and Master's degrees from University of Roorkee (presently Indian Institute of Technology, Roorkee), and his Ph.D. from Queen's University, Kingston, Canada.

His major research interests have been in the areas of structural Engineering particularly in structural failures and rehabilitation of structures, structural reliability; structural stability and dynamics; and High Performance Concrete. He has supervised 40 Masters and 6 Doctoral theses. He has wide experience in structural design of diverse types of structures in reinforced concrete and structural steel. Dr. Gambhir has published over sixty technical papers in reputed journals and has authored: Concrete Technology, Tata McGraw-Hill Publishing Company, 3rd Edition; and Reinforced Concrete Design, Macmillan (I) Ltd., Ist Edition.

He has been recipient of several awards including Agra University Bursary, National Scholarship, Research Fellowship, Roorkee University (now IIT Roorkee) Gold Medal, Canadian Commonwealth Scholarship and merit scholarships. He is a member of Indian Society for Technical Education and the Indian Society for Earthquake Technology. He has been member of numerous committees.

1 Introduction

A structure is meant to withstand or resist loads with a small and definite deformation. In structural analysis problems, the aim is to determine a configuration of loaded system, which satisfies the conditions of equilibrium, compatibility and force-displacement relations of the material. For a structure to be satisfactory, it is necessary to examine whether the equilibrium configuration so determined is stable. In a practical sense, an equilibrium state of a structure or a system is said to be in a stable condition, if a disturbance due to accidental forces, shocks, vibrations, eccentricities, imperfections, inhomogeneities or irregularities do not cause the system to depart excessively from that state. The usual test is to impart a small disturbance to the existing state of the system, if the system returns back to its original undisturbed state when the cause of disturbance is removed, the system is said to be stable.

There are two types of failures associated with a structure namely material failure and form or configuration failure. In the former, the stresses exceed the permissible values which may result in the formation of cracks. In the later case, even though the stresses are within permissible range, the structure is unable to maintain its designed configuration under the external disturbances (or applied loads which could be tensile and/or compressive). The loss of stability due to tensile loads falls in the broad category of material instability, whereas the stability loss under compressive load is usually termed *structural* or *geometrical instability* commonly known as *buckling*.

A buckling failure is potentially very dangerous and may trigger the collapse of many types of engineering structures. It may take the form of instability of the structure as a whole or the localized buckling of an individual member or a part there of, which may or may not precipitate the failure of the entire structure. It is to be emphasized that the load at which instability occurs depends upon the stiffness of the structure or portion there of, rather than on the strength of material.

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1.1 Definitions of Stability

As discussed in the previous section, buckling is a phenomenon encountered in engineering structures under predominantly compressive forces. The requirement that a body should be in equilibrium seems insufficient even from purely practical point of view. For a sound structure, it is desirable that it is in stable state of equilibrium. The stable state of equilibrium is defined as the ability of the structure to remain in position and support the given load, even if forced slightly out of its position by a disturbance. The question of stability can be posed in three different ways. The first way of posing the stability question is: if there is a possibility of existence of another adjacent configuration beside straight configuration for which the structure can assume equilibrium for $P > P_{cr}$.

There are indeed two possible equilibrium states, the straight and the bent one. For illustration consider an initially straight vertical flag-post column of uniform cross-section subjected to a concentrated force (load) P acting along its centroidal axis. As the load P is continuously increased from zero to a particular critical value of the load P_{cr} for which the straight member sustains the load in the laterally bent configuration as shown in Fig. 1.1a.

At this value of load P called *critical* or *buckling load*, the member either remains in straight position or in the laterally deflected configuration. Below this critical value of the load the member will be straight and above it will be in bent position. Thus at critical value P_{cr} two adjacent equilibrium positions are possible for the same external force-called condition of *bifurcation* or *branching*. Moreover, the configurations of deformation for these two cases are totally different. Buckling, a condition of bifurcation, constitutes one of the ways in which the structural member becomes unstable. The load deformation curves P versus Δ, and P versus y are shown

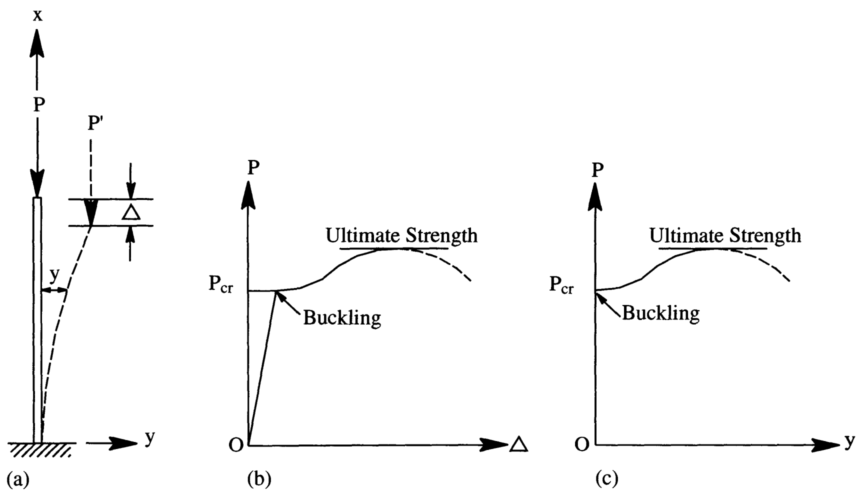

Fig. 1.1a–c. Load-deformation behaviour of cantilever subjected to axial compression. **a** Laterally deflected shape, **b** P-Δ curve, **c** p-y curve

in Figs. 1.1b and 1.1c., respectively. Normally the buckling does not necessarily correspond to the ultimate load carrying capacity of the member, it may indicate considerable strength above the buckling load, though deformations associated with increased load may be appreciable. As P is increased a stage is reached beyond which further increase is impossible and the member continues to deflect progressively. Such a situation also describes a condition of general structural instability. Thus it may be stated that the *instability occurs when two or more adjacent equilibrium positions correspond to fundamentally different deformation modes.*

The second way of posing question for stability investigation is that: if the column is given slight disturbance or perturbation causing it to vibrate, would the amplitude of vibration diminish or increase with the passage of time. This definition of stability is much more powerful than the preceding one since it puts the problem in wider context of dynamics. For $P > P_{cr}$, the system is *dynamically unstable.*

The third way to pose the question is: if there is a value of P for which the total potential energy of the system ceases to be minimum. This criterion is restricted to *conservative systems.*

The above three criteria are termed Euler's statical (non-trivial equilibrium state) criterion, Liapunov's dynamical criterion and potential energy stability criterion, respectively. It will be seen that for a continuous and conservative elastic system all these criteria are completely equivalent and within linearised analysis lead to an eigenvalue determinant from which the eigenvalue of critical or buckling load is retrieved.

In the above flag-post type vertical column illustration the effect of only axial force has been considered. However, in practice the members are normally subjected to lateral forces along with axial forces. To illustrate the influence of bending forces on the axial deformations consider a simply supported beam subjected to a single lateral load Q and compressive force P ($< P_{cr}$) as shown in Fig. 1.2a. The moment produces deflections which in turn cause additional moments along the member due to increased eccentricity of load P resulting in still more deflections. Finally a stable situation is reached where the deflections correspond to the bending moments due to lateral and axial loads. It should be noted that the iterative process just described actually need not be carried out to obtain a solution. The influence of axial force on bending moment can be incorporated directly into differential equation

$$EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -M_x = -(M_0 + Py) \tag{1.1}$$

where M_0 is the moment due to lateral forces, end moments etc. and the term Py takes into account the added influence of axial force and deflection. If the axial force and lateral loads increase proportionately and that the member deflects laterally in the plane of applied loads, the two load-deformation curves will be as shown in the Figs. 1.2a and 1.2b for P verses Δ, and P verses y, respectively. Here y corresponds to lateral deflection at the point of application of lateral load. Unlike in flag-post type column both axial and lateral deflections are observed from the outset of the load application.

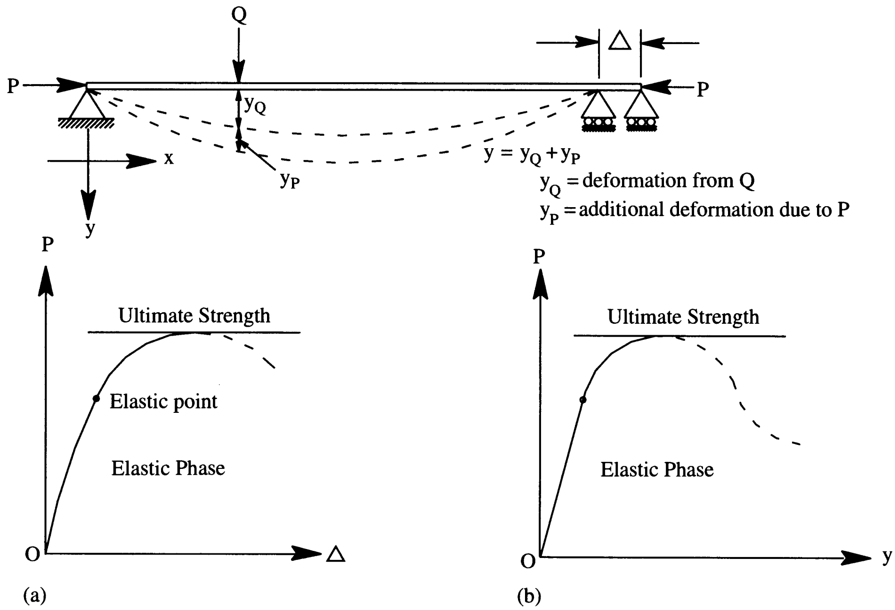

Fig. 1.2a,b. Laterally loaded beam subjected to axial thrust. **a** P-Δ diagram, **b** P-y diagram

The concept of stability, and its counterpart instability, is often explained by analogy to the behaviour of a rigid ball of some weight placed in position at different points on a surface shown in Fig. 1.3 with zero curvature normal to the plane of figure. The ball is assumed to be in equilibrium at the points of zero slopes as indicated. However, the response of the ball to a slight disturbance or perturbation from these positions is quite different. At position I of the ball positive work will be required, and the ball returns to its original position upon removal of the disturbance. This case corresponds to the points on the ascendancy sides of load-deformation curves shown in Figs. 1.1b, 1.1c, 1.2a and 1.2b. This equilibrium position is *stable*. Case II,

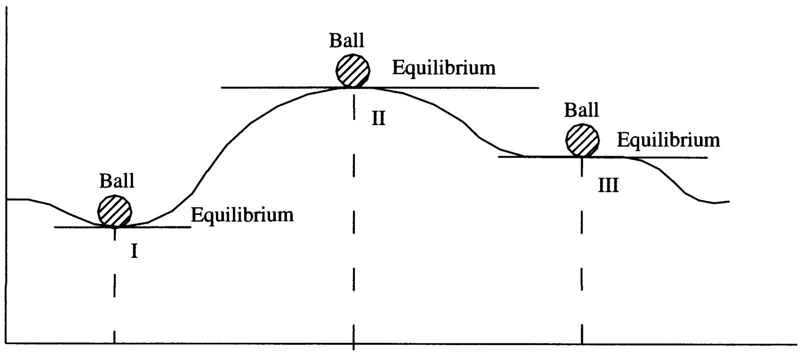

Fig. 1.3. Ball analogy for the bifurcation diagrams

on the other hand represents a state of *instability* or *unstable equilibrium* since the disturbance will result in the giving up of energy and ball will progressively move. This corresponds to the points on descendancy portions of the load deformation curves shown in Figs. 1.1b, 1.1c, 1.2a and 1.2b. In case III, the ball neither returns to its original position nor continues to move on removal of disturbance. This state is termed *neutral equilibrium condition.* This condition of neutral equilibrium is frequently stated as the *necessary condition for structural stability.* This would be the case when structural members buckle or when they reach their maximum load-carrying capacity. Buckling can occur in both the elastic and inelastic ranges of material behaviour. For real materials ultimate carrying capacity is realized in the inelastic range.

1.2 Structural Instability

The loss of structural stability is termed instability, which takes place in different ways depending on the material properties, structural configuration and loading conditions. The loss of stability in terms of structural behaviour can be expressed by the load-deformation relationship. For continuous conservative elastic systems stability is classified into three types of branching or bifurcations with distinct initial post-buckling behaviour:

1. A symmetric bent upward post-buckling curve. This bifurcation is stable and almost unaffected by imperfections,
2. A symmetric bent downward curve which is unstable and imperfection sensitive, and
3. An asymmetric post-buckling curve with a slope at the bifurcation point which is extremely sensitive even to a very small initial imperfection.

The first type shown in Fig. 1.4a is called symmetric stable branching, because for the loads above the critical point, the characteristic deflection can be increased only by increasing load P. On the other hand the second type Fig. 1.4b is termed

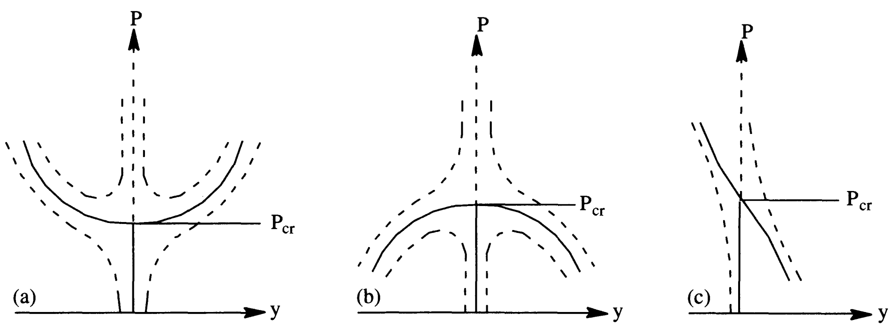

Fig. 1.4a–c. Points of static branching or bifurcation and points of imperfection. **a** Stable symmetric (positive curvature), **b** unstable symmetric (negative curvature), **c** unstable asymmetric (positive-negative curvature); —— perfect systems, – – – asymptotic imperfect systems

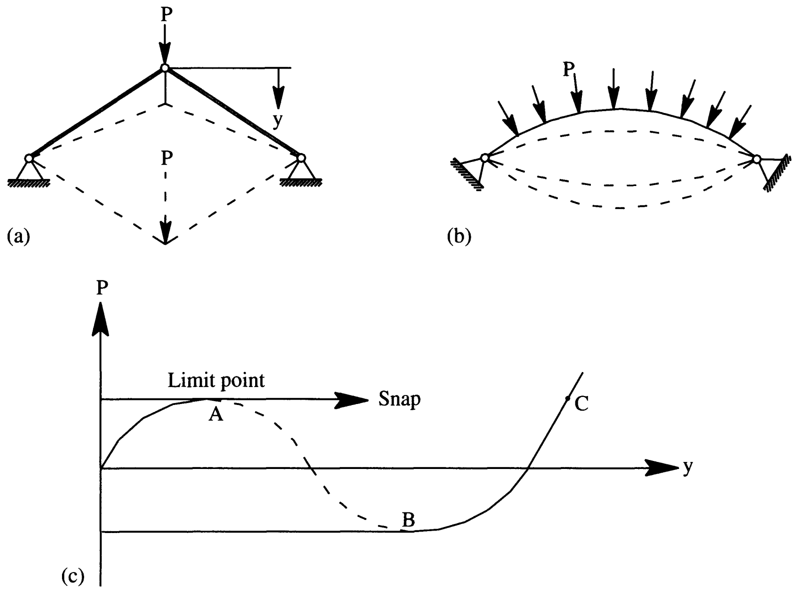

Fig. 1.5a–c. Von Mises truss and an arch under external pressure (snap through buckling). **a** Mises truss, **b** arch, **c** P-y diagram; —— stable, - - - - unstable

unstable because the deflection increases, if the system is perturbed above the critical point P_{cr} without increasing the load. In fact the load is to be reduced if an increase in deflections y is to be attained.

The third type of branching point as shown in Fig. 1.4c is unstable for deflection to the right hand side and stable for the left hand side. However, since the deflection can occur in either direction, the point is regarded unstable for practical purposes. The unavoidable imperfections such as crookedness of central axis makes the behaviour of imperfect system (shown by dotted lines) asymptotic as regard to the ideal system. The point at which the imperfect system turns, i. e. the maximum of load-deformation curve, is termed *limit point instability* and is similar to the snap-through buckling of the *mises truss* or the flat arch under external pressure shown in Figs. 1.5a and 1.5b, respectively. P-y relationship for mises truss is shown in Fig. 1.5c. If the load P is increasing monotonically and reaches the value P_{cr}, the point jumps to another branch of the curve corresponding to a new geometrical form of the truss as shown in the Fig. 1.5c. In these types of structures, the loss of stability consists in sudden transition to a non-proximate form of equilibrium.

A mass less cantilever column subjected to a *follower* force P where the direction of force P follows that of tangent at the free end i.e. the load is tangential load. The stability of such a system, in general, can not be determined by non-trivial equilibrium state approach, suggesting that buckling is impossible under such loading. The preceding conclusion is correct only if the structure would be truly mass less, which is of course, impossible in practice. A dynamical analysis is made possible by the inclusion of, say, a small mass at the tip of the structure. The analysis shows

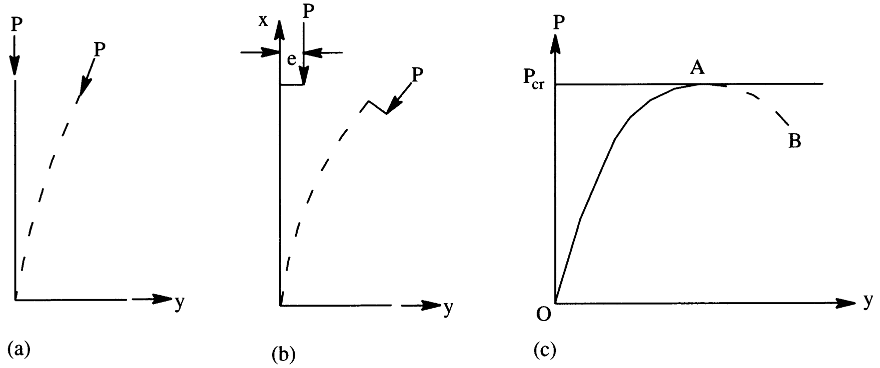

Fig. 1.6a–c. Cantilevers subjected to follower and eccentric loads. **a** Follower or tangential load, **b** eccentric load, **c** P-y relationship for eccentric load; —— stable, – – – unstable

that the column is stable up to the load level $P = P_{cr}$ and loss of stability occurs for $P > P_{cr}$ and column passes from state of rest to a state of motion.

Consider the case of a uniform cantilever column subjected to a compressive force P acting at an eccentricity e as shown in Fig. 1.6b. This case is equivalent to an imperfection due to crookedness in the centroidal axis. The load-deflection curve consists of an ascending branch and a descending branch with a definite apex which defines the maximum load carrying capacity of the member as shown in Fig 1.6c. Under monotonic loading ascending branch corresponds to a stable equilibrium state and descending to an unstable equilibrium state. As the load approaches, P_{cr} unlimited progressing growth of displacement occurs. The loss of stability of such imperfect systems is due to transition to *non-equilibrium states*. Consider a thin-walled cylindrical shell subjected to axial compression. In the load-volumetric strain diagram shown in Fig. 1.7b, the line OA represents the primary equilibrium path of unbuckled configuration of the shell where as the line BC represents the secondary equilibrium path of buckled non-cylindrical configuration of the shell. In such structures a finite disturbance during the application of the load can force the structure to pass from *primary equilibrium configuration* to a *secondary equilibrium configuration* even before the classical critical load is reached. This is due to diamond-shaped local buckling. In such structures the loss of stiffness after local buckling is so large that the buckled configuration can be maintained by returning to an earlier level of loading.

It must be noted that in each of types of loss of stability, a change in the geometry or configuration results from either due to introduction of additional new forces or due to the change in the nature of forces that existed in the un-deformed structure. In terms of new forces that appear during the loss of structural stability a further classification of instability can be provided as follows:

1. Flexural buckling,
2. Torsional buckling,
3. Torsional-flexural buckling, and
4. Snap-through buckling.

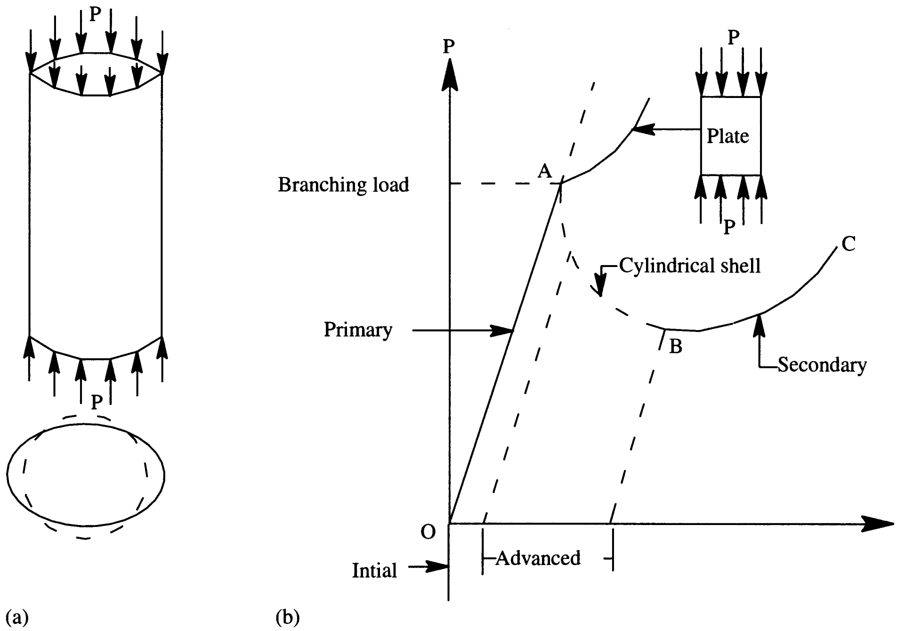

Fig. 1.7a,b. Post-buckling behaviour of a thin walled cylindrical shell. **a** Cylindrical shell, **b** post buckling behaviour; —— stable, – – – unstable

These instability modes may occur independently or in combinations. In order to explain persistent large discrepancy between theoretical and experimental results of eigenvalue buckling load, a study of post-buckling behaviour was suggested by Koiter. Koiter was the first to realize the immense importance of post-buckling in connection with imperfections as well as mode coupling phenomenon which combines two harmless stable buckling modes to a catastrophic highly unstable one.

The explanation of above referred discrepancy lies not in linear eigenvalue analysis but in post-buckling behaviour. The difference in the post-buckling behaviour lies in the linearizing the analysis by ignoring non-linear terms (Fig. 1.7). An adequate initial post-buckling analysis must in general consider up to fourth-order terms in the energy functional while the buckling load depends only on second-order terms, the slope of the post-buckling curve depends only on second- and third-order terms. Finally, the initial curvature depends only on second-, third- and fourth order terms unless the functional is symmetric in which case only fourth order terms have an effect on the initial curvature.

1.3 Methods for Stability Ananlysis

Stability analysis consists in determining the mode of loss of structural stability and corresponding load called critical load. The structure remains at rest before and after buckling except in the cases where loss of stability is due to transition from the state of rest to a state of motion called kinetic or dynamical instability. Four distinctly different classical methods available for the solution of buckling problems are:

1. Non-trivial equilibrium state approach,
2. Work approach,
3. Energy approach, and
4. Kinetic or dynamical approach.

The first of these, the so called statical equilibrium approach requires a second infinitesimally near equilibrium position which will sustain the load. In the other words it consists in determining the values of load for which a perfect system admits two or more different but adjacent equilibrium states. By different equilibrium states it is meant that the response of the structure is such that equilibrium can be maintained with different deformation patterns. The condition of infinitesimally close or adjacent equilibrium configuration renders the slope of deflection curve to be very small compared to unity. This enables the expression for curvature of deflection curve to be linearized. The method then requires the solution of governing differential equations subject to some prescribed boundary conditions. It leads to an eigenvalue problem. For a multi-degree-of-freedom system, the equations of equilibrium are expressed in matrix form. The determinant of coefficients of unknown displacements is termed *stability determinant*. For n degree-of -freedom system, the size of stability determinant would be $(n \times n)$ which according to the rules of linear algebra must vanish if the system of governing equations of equilibrium should have a non-trivial solution. It should be noted that the stability determinant is identical to the so-called Hessian of energy functional.

The second method known as work approach requires that zero force (load or moment) causes the system to remain in the deformed position.

The energy approaches (virtual work, minimum total potential energy or stationary potential energy, minimum complementary energy) as defined in Chap. 2, can also be used to establish neutrality of given equilibrium state. The method based on the principle of minimum potential energy may be stated as: a conservative (holonomic) system is in a configuration of stable equilibrium if, only if, the value of total potential energy Π, is a relative minimum i. e. $\partial\Pi/\partial y = 0$ (relative with respect to its immediate neighbourhood). Thus for stability Π must be a minimum i. e. $\partial^2\Pi/\partial y^2 > 0$ and for instability Π must be maximum i. e. $\partial^2\Pi/\partial y^2 < 0$. The critical state is thus given by vanishing of second variation i. e. $\partial^2\Pi/\partial y^2 = 0$. It is interesting to note that within linearized buckling instability analysis using energy formulation, the above criterion reduces to: $\Pi = 0$, $\delta\Pi = 0$, $\delta^2\Pi = 0$. It is stressed, however, that this equivalence is only true for the linear eigen-value analysis.

In the kinetic or dynamic approach, the equations of motions are formulated and the load is established which results in deformation with zero frequency of vibration ($\lambda = 0$). Thus the method consists in obtaining the so-called frequency equation. The frequency λ has both positive and negative real parts. If λ has a positive real part, the displacement increases as time t tends to infinity and structure is regarded unstable. On the other hand if λ has a negative real part, the displacement vanishes as time t

tends to infinity and structure must be regarded stable. Therefore, the critical state is represented by vanishing of real part of λ. In a case of multi-degree-of-freedom-system, the frequency equation is obtainable from frequency determinant which is the condition of non-trivial solution of equations of motion exactly as in the criterion of non-trivial equilibrium. On the other hand, the condition that λ should have zero real part is identical to the condition that the decrement should vanish. The decrement can be written in a determinant form. Thus in a dynamical investigation critical state is marked by vanishing of two determinants, the frequency determinant and so-called Burschart determinant. It should be noted in passing that the dynamical criterion of buckling is frequently used for experimental investigation of critical loads. This method is more general in the sense that the other approaches based on static concept are special cases of this approach when inertia forces are neglected. Moreover, since the dynamical method takes into account the inertia forces in its formulation, the mass distribution of the elastic system becomes as important as elastic stiffness of the system. The response of the system therefore becomes a function of both the space and the time coordinates.

From the foregoing discussion it is evident, that the dynamical criterion for buckling instability is the most general one. The energy method is restricted to conservative systems and the equilibrium method is limited to buckling to an adjacent equilibrium state and will thus fail in general to detect dynamical buckling. For a conservative elastic system, all the three approaches are equivalent as far as determination of critical load is concerned. All the three approaches lead to a *stability* or *frequency determinant*, the vanishing of which leads to an equation for determining critical parameters i. e. marginal stability.

In additional to classical approaches several approximate methods have been developed to predict the load carrying capacity in very specific cases.

1.4 Summary

A structure is said to have a branching critical buckling load P_{cr}, if for a loading $P > P_{\mathrm{cr}}$, it has more than one equilibrium state. In case of an Euler strut, for instance, these would be initial straight form and the slightly bent configuration. For a loading $P < P_{\mathrm{cr}}$, the structure is said to be stable while for $P > P_{\mathrm{cr}}$, the structure is unstable, There are three stability criteria associated with three methods of solution. The first is non-trivial equilibrium state criterion which is based on equilibrium method. The second is the dynamical criterion of stability which is based on vibrational analysis. Finally, the potential energy criterion states that an equilibrium state given by $\delta\Pi = 0$ is stable if the total potential energy is minimum i. e. the second variation $\delta^2\Pi = 0$. The state is unstable if $\delta^2\Pi < 0$. Consequently, the critical state is gives by $\delta^2\Pi = 0$. In the buckling analysis energy method plays an important role.

To explain discrepancy between theoretical and experimental results three types of bifurcation with distinct post-buckling behaviour have been outlined. For post-buckling analysis higher order terms in the energy functional must be considered.

2 Basic Principles

2.1 Introduction

In this chapter, the basic principles required to analyze the structural stability problems are discussed. Emphasis is laid on energy methods. In the beginning of the chapter, the idealization of the structures, equilibrium equations and rigid body diagrams have been described. The subject matter on energy principles starts with the definition of mechanical work for external and internal forces of an elastic system and establishes relationship between the two.

2.2 Idealization of Structures

The primary objective of structural analysis is to determine the reactions, internal forces and deformation at any point of given structure caused by applied loads and forces. To obtain this objective it becomes necessary to idealize a structure in a simplified form emendable to analysis procedures. The members are normally represented by their centroidal axes. This naturally does not consider the dimensions of the members or depth of joints and hence there may be considerable differences between clear spans and centre-to-centre spans ordinarily used in the analysis. These differences are ignored unless cross-sectional dimensions of the members are sufficiently large to influence the results. The supports and connections are also represented in a simplified form as illustrated in Fig. 2.1.

2.3 Equations of Equilibrium

For a stationary structure or a body acted upon by a system of forces which include external loads, reactions and gravity forces caused by the mass of the elements, the conditions of equilibrium are normally established with reference to a coordinate system X, Y and Z. It is also convenient to replace all the forces by their components along the chosen reference axes. The condition of equilibrium in X-direction

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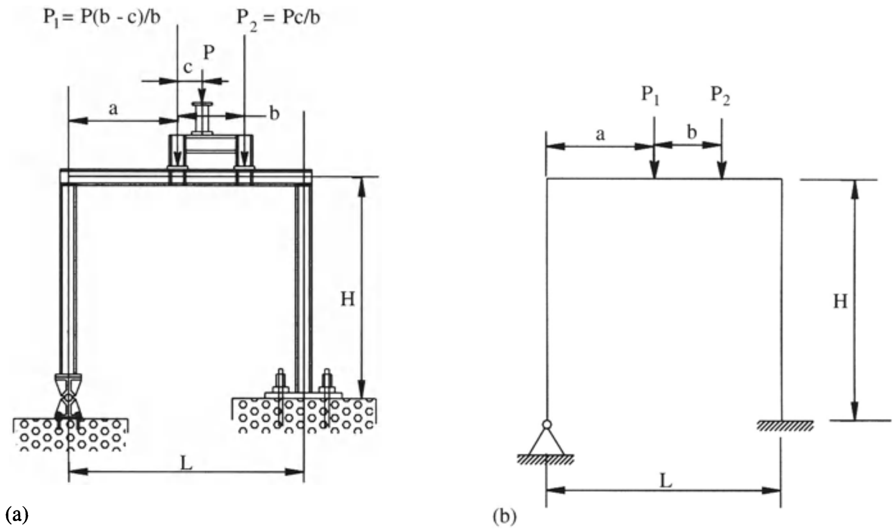


Fig. 2.1a,b. Idealization of structure. **a** Actual structure, **b** idealized structure

expresses the fact that there is no net unbalanced force to move the body in that direction. Thus for static equilibrium, the algebraic sum of all the forces along the co-ordinate axis X must be zero. Mathematically it can be expressed as $\sum F_x = 0$. Similar conditions hold good along co-ordinate axes Y and Z. Three additional conditions of equilibrium state that the structure or element does not spin or rotate about any of the three axes due to unbalanced moments. The satisfaction of three force conditions and three moment conditions establishes that the structure is in equilibrium or stationary condition. The six equilibrium conditions can be expressed as

- Translational equilibrium

$$\sum F_x = 0\,, \quad \sum F_y = 0 \quad \text{and} \quad \sum F_z = 0 \tag{2.1}$$

- Rotational equilibrium

$$\sum M_x = 0\,, \quad \sum M_y = 0 \quad \text{and} \quad \sum M_z = 0 \tag{2.2}$$

In the vector form they can be expressed as

$$F_{\mathrm{R}} = F_x\, i + F_y\, j + F_z\, k = 0 \quad \text{and} \quad M_{\mathrm{R}} = M_x\, i + M_y\, j + M_z\, k = 0 \tag{2.3}$$

For a planar structure lying in XY plane there is no force acting in Z-direction or any moment about X- and Y-directions (axes). The moment M_z represents moment about an axis perpendicular to XY plane. Thus for a planar structure the equilibrium conditions are:

$$\sum F_x = 0\,, \quad \sum F_y = 0 \quad \text{and} \quad \sum M_z = 0 \tag{2.4}$$

The major application of equilibrium analysis is in the evaluation of reactions and internal forces by representing a structure by a series of free body diagrams.

2.4 Free-Body Diagrams

The analysis of all the structures is based on the concept that any part or the structure is in equilibrium along with the structure as a whole. This concept is used to determine the internal forces in a structure by drawing free-body diagrams for the parts of the structure. The free-body diagrams are useful tools in structural analysis. These are obtained by cutting the structure hypothetically or disengaging some connections and supports. In constructing a free-body diagram, the correct depiction of all the possible forces in the structure at the cuts and disengaged connections by appropriate force vectors is of extreme importance. At this stage the correct direction of the internal forces is not known. Once the values of these quantities are ascertained by statics, the proper direction (sense) of each force component can be established. All the external forces acting on the body in its original state must also be depicted on the diagram. This procedure can be applied to each of the free-body diagrams into which the structure has been discretised or broken down. However, in dealing with the forces acting on the free bodies, the internal forces common to two free bodies are double action forces denoted as equal but appositely directed force vectors. It should be realized that the internal forces are the resultants of internal stresses which are decomposed into components, normal to cross-section, termed normal (axial) force N and tangent to cross-section shear force Q. In addition there are stress couples which are termed bending moment M. To illustrate the discretization of a structure into a number of free-bodies or elements consider the structure shown in Fig. 2.2a. The

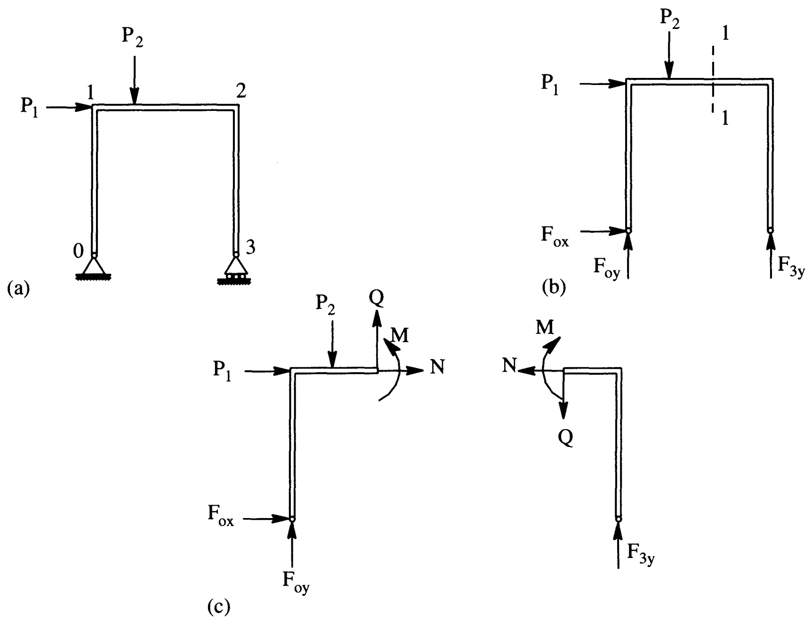


Fig. 2.2a–c. Free-body diagrams of the entire and discretized structure. **a** Structure, **b** free-body diagram of entire structure, **c** free bodies of individual parts

free-body diagrams of the entire structure and of the parts are shown in Figs. 2.2b and 2.2c, respectively.

To illustrate the application of equilibrium condition considers the loaded beam shown in Fig. 2.3a. The free body diagram of the entire beam released from the supports is shown in Fig. 2.3b. There are four unknown reaction components F_{ox}, F_{oy}, F_{2y} and M_o acting on the free-body diagram of the entire beam. The free-body diagrams of two parts disengaged at the hinge are shown in Fig. 2.3c. Three equilibrium conditions for this planar beam along with fourth structural condition that moment at the hinge 1 is zero, can be used to compute unknown reactions. The equilibrium condition

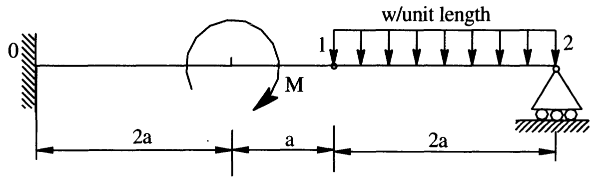
$$\sum F_x = F_{ox} = 0 \quad \text{gives} \quad F_{ox} = 0$$

Summation of moments at the hinge point 1, $\sum M_{1z}$

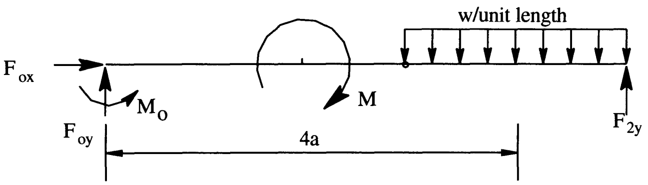
$$F_{2y}(2a) - 2wa(a) = 0 \quad \text{gives} \quad F_{2y} = wa$$

Summing up vertical forces

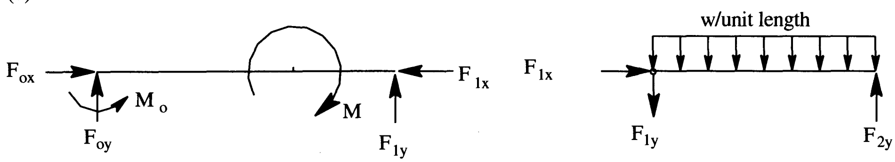
$$\sum F_y = F_{0y} + F_{2y} - 2wa = 0$$



(a)



(b)



(c)

Fig. 2.3a–c. Free-body diagrams of the entire and discretized structure. **a** The beam and the loading, **b** free-body diagram of entire beam, **c** free-body diagrams of two parts separated by the hinge

Therefore

$$F_{0y} + F_{2y} = 2wa \quad \text{or} \quad F_{0y} = wa$$

$$\sum M_{0z} = M_0 - M - 2wa(2a + a + a) + F_{2y}(5a) = 0$$

$$\text{or} \quad M_0 = M + 8wa^2 - 5F_{2y}a$$

substituting the value of F_{2y} we obtain

$$M_0 = M + 8wa^2 - 5(wa).a = (M + 3wa^2)$$

The positive sign indicates that the directions of reactions assumed are the correct directions.

2.5 Work of Externally Applied Forces

Consider a force F moving through a very small but finite distance δx along its direction of action. The force will not change in magnitude appreciably during this small movement δx and the elementary work is defined as

$$\delta W_{\mathrm{e}} = F(\delta x) \tag{2.5}$$

If the force has moved a total distance L the work done could be calculated by dividing the distance L into a number of arbitrary small distances $\delta x_i = L/n$ and the work would be approximated

$$W_{\mathrm{e}} = \sum_i^n \delta W_{\mathrm{e}i} = \sum_i^n F_i \delta x_i \tag{2.6}$$

To be able to calculate exact value of W_{e}, the number of parts (n) must be infinitely large. Thus in the limit $\delta(\,)$ tends to $d(\,)$, the summation ($\sum$) tends to integral ($\int$) and F_i renders a continuous function of x. Thus the expression for the work done by $F(x)$ is

$$W_{\mathrm{e}} = \int_0^L F(x)\,\mathrm{d}x \tag{2.7}$$

In structural analysis problems F varies during displacement or deformation, e.g. consider the very important case of linear relationship between the load F and the displacement y i.e. F is linear function of y

$$F = cy$$

where c is a constant. Substituting this into (2.7)

$$W_{\mathrm{e}} = \int_0^y (cy)\,\mathrm{d}y = \frac{1}{2}cy^2$$

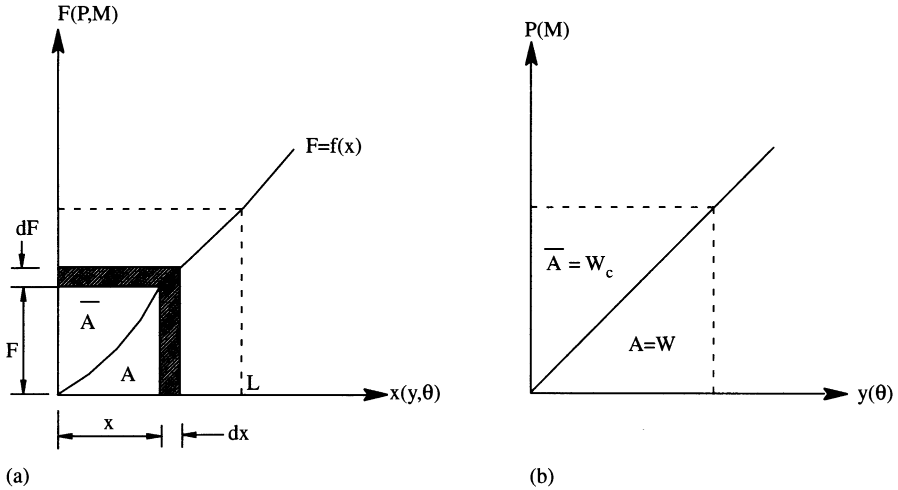


Fig. 2.4a,b. Geometric representation of work w.r.t. load-deformation diagram. **a** Generalised force-displacement curve, **b** load-deformation-work diagram

where y denotes the displacement or deflection of a point, and since $F = cy$, then

$$W_e = \tfrac{1}{2}Fy$$

Here the loading F represents a single force, P or moment M, and the deformation x represents the corresponding displacement and rotation, respectively. Thus F and x are frequently referred to as *generalized force* and *generalized displacement.* The curve $F = f(x)$ would be in general some non-linear continuous curve as shown in Fig. 2.4a. The work done expression $\int F(x)\,\mathrm{d}x$ represents the area under the curve. In the other words, the work can be interpreted geometrically as area under load-deflection diagram. If the curve $F = f(x)$ becomes straight line i. e. there exist a linear relationship between force and deformation, the area renders a triangle as shown in Fig. 2.4b and we obtain

$$W_e = \tfrac{1}{2}Py \quad \text{or} \quad \tfrac{1}{2}M\theta \tag{2.8}$$

The area $\bar{A}$ which together with A forms rectangle is obviously numerically identical to A in the case of linear force-deflection relationship. The area $\bar{A}$ is termed *complementary work* W_c. Thus the complementary work is defined as

$$W_c = \int y\,\mathrm{d}P \quad \text{or} \quad \int \theta\,\mathrm{d}M \tag{2.9}$$

Equation (2.8) implies that translational and rotational work must be equivalent. To illustrate this consider the hinged arm 0-1 as shown in Fig. 2.5 with a force applied at the free end 1 which would move by a distance dy. The work done by the force is

$$W_e = \int P\,\mathrm{d}y = \int PL\left(\frac{\mathrm{d}y}{L}\right)$$

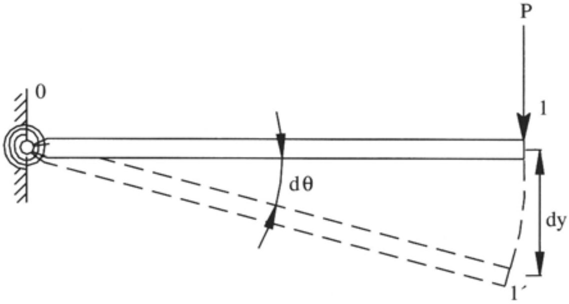


Fig. 2.5. Equivalence of translational and rotational works

Here PL is the bending moment M acting on the arm and $(\mathrm{d}y/L)$ is the angle of rotation $\mathrm{d}\theta$. Thus the work done can be expressed as

$$W_e = \int M\,\mathrm{d}\theta$$

2.5.1 Eigenwork and Displacement Works

Eigenwork is defined as the work done by a force moving along the deformation caused by it. As an example consider the structure shown in Fig. 2.6a. The deflection under distributed load w is $y(x)$, at the point of concentrated load P is y_1 and at the point of moment M is θ_2 ($y'_{x=a_2}$, where a prime ($'$) denotes differentiation with respect to x). The eigenwork of the loading in this case, with an assumed linear force-displacement relationship is

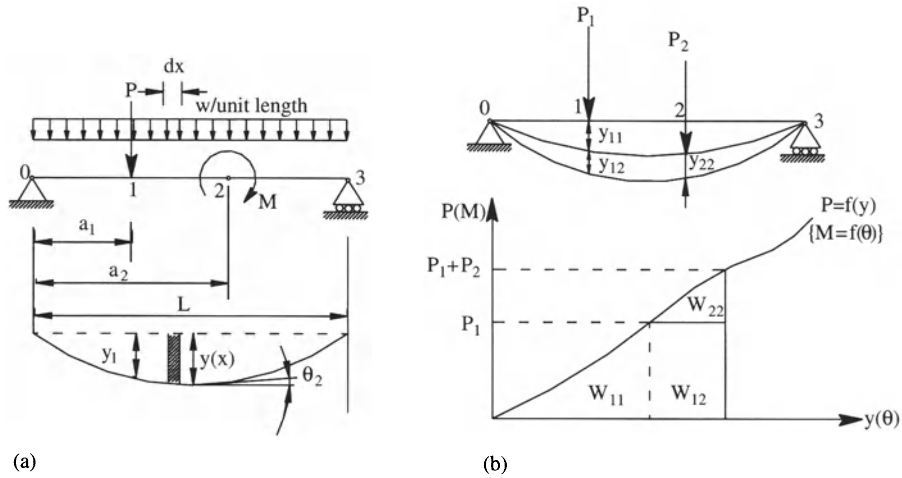


Fig. 2.6a,b. Eigenwork and displacement work. **a** Loaded beam and its deflection, **b** geometrical interpretation of eigenwork and displacement work

$$W_{\text{eig}} = \frac{1}{2}Py_1 + \frac{1}{2}\int_0^L w\,y(x)\,\mathrm{d}x + \frac{1}{2}M\theta_2 \tag{2.10}$$

In contrast to eigenwork, *the displacement work is the work done by a force P_1 along the displacement y_2 caused by another force P_2*. Thus force P_1 does not vary during deformation and displacement work is consequently

$$W_{\text{dis}} = \int P_1\,\mathrm{d}y_2 = P_1 y_2 \tag{2.11}$$

As an example consider the beam shown in Fig. 2.6b carrying multiple loads. In the double subscripted quantities term y_{ij} represents a displacement at the point i due to force P_j acting at the point j. Consider the force P_1 acting at point 1 which produces eigenwork $W_{11} = (P_1 y_{11})/2$. Now suppose that another load P_2 is applied to the beam at point 2. This load will cause additional displacements y_{22} and y_{12} at the points 2 and 1, respectively. Thus the eigenwork of load P_2, $W_{22} = (P_2 y_{22})/2$.

The eigenwork of the external forces (loading) of the system is

$$W_{\text{eig}} = \sum_{i=1}^{2} W_{ii} = W_{11} + W_{22} = \frac{1}{2}P_1\,y_{11} + \frac{1}{2}P_2\,y_{22}$$

while the displacement work is

$$W_{\text{dis}} = \sum_{i,j}^{2} W_{ij} = W_{12} = P_1 y_{12} \tag{2.12}$$

The displacement work wherein the load remains constant, could be interpreted as *virtual work* which *is the product of a constant load and an imaginary very small displacement (virtual displacement)*. This concept of virtual work will be discussed later in this chapter.

2.5.2 Linear Springs

There are two kinds of springs normally encountered in the idealized structures: a normal force or extensional spring and a moment or rotational spring. A spring is said to be linear when load-deformation relationship of the spring is linear.

Normal Force Spring

It is capable of carrying a normal force only, i. e. it has no bending, torsional and shear stiffness. The elongation $\varDelta$ of an ideal elastic spring subjected to a normal force P is given by

$$\varDelta = P/k_n \tag{2.13}$$

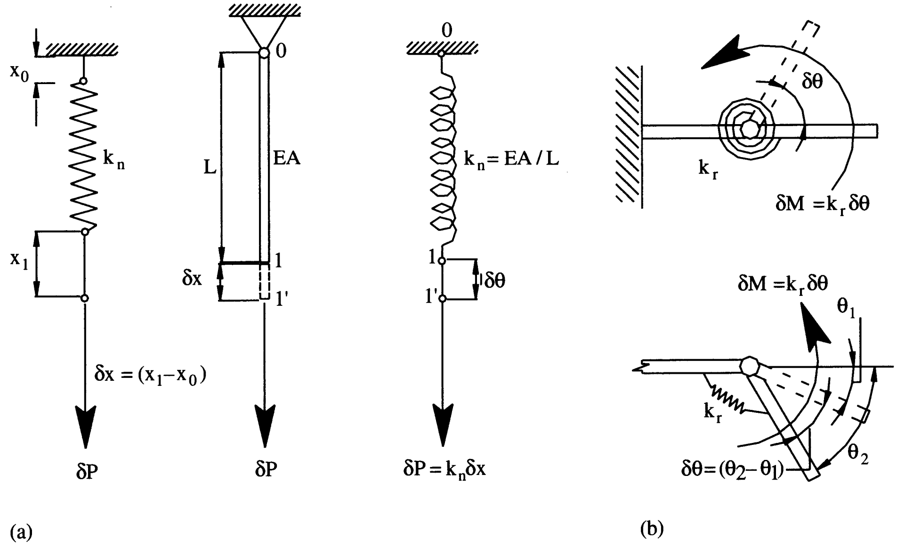


Fig. 2.7a,b. Concept of linear normal force and rotational springs. **a** Normal force spring, **b** moment or rotational spring

where k_n is spring constant termed spring stiffness (i.e. force required for unit deformation).

The inverse $(1/k_n)$ is described as the *flexibility of the spring*. In the structural analysis problems an ideal linear spring is represented symbolically as shown in Fig. 2.7b. An elastic bar of length L, cross-sectional area A and modulus of elasticity E carrying axial force δP, shown in Fig. 2.7b can also be modeled as a normal force spring. The total elongation Δ of the member following Hooke's law is given by:

$$\Delta = \varepsilon L = \left(\frac{\sigma}{E}\right) L = \left(\frac{P}{A}\frac{1}{E}\right) L = \frac{PL}{AE} = \frac{P}{(EA/L)}$$

Comparing this expression with the law of linear spring given by (2.13), the spring constant k_n is given by

$$k_n = (EA/L) \tag{2.14}$$

Work of internal force of the spring which is stored as the energy due to elastic deformation δx can be computed as follows. The internal force N produced in the spring due to δP is

$$N = \delta P = k_n \delta x$$

The eigenwork of the internal forces is thus

$$\begin{aligned} -W_{ii} &= \int N \,\mathrm{d}x = \int k_n \delta x \,\mathrm{d}x = \frac{1}{2} k_n (\delta x)^2 \\ &= \frac{1}{2} \text{ (spring constant) (elongation)}^2 \end{aligned}$$

The strain energy U of the spring is thus

$$U = \frac{1}{2} k_n (\delta x)^2 \tag{2.15}$$

Moment or Rotational Springs

A rotational spring is an idealized structure that is capable of resisting a rotation but does not have an axial stiffness. For an ideal linear rotational spring, the moment is directly proportional to the deformation which in this case is the angle of rotation θ. The law of a linear rotational spring is thus

$$M = k_{\mathrm{r}} \theta$$

where k_{r} is the moment stiffness i.e. the moment required for unit rotation and its inverse $(1/k_{\mathrm{r}})$ is the spring flexibility. An ideal rotational spring is symbolically represented as in Fig. 2.7b. The eigenwork of the internal forces in the moment spring is

$$\begin{aligned} -W_{\mathrm{in}} = W_{ii} &= \int \delta M \, \mathrm{d}\theta = \int (k_{\mathrm{r}} \delta\theta) \, \mathrm{d}\theta = \frac{1}{2} k_{\mathrm{r}} (\delta\theta)^2 \\ &= 1/2 \, (\text{spring constant}) \, (\text{angle of rotation})^2 \end{aligned} \tag{2.16}$$

In general, however, both ends of the spring will move, the relative rotation of the spring would be the net difference between the end rotations. The spring law should be stated more precisely as

$$\delta M = k_{\mathrm{r}} (\theta_2 - \theta_1) = k_{\mathrm{r}} \delta\theta$$

and the strain energy is thus

$$U = k_{\mathrm{r}} (\delta\theta)^2 / 2 \, . \tag{2.17}$$

2.5.3 Virtual Work

Virtual Work and Complementary Virtual Work

The equilibrium condition states that for a body to be in equilibrium, the sum of all the forces acting on the body must be zero ($\sum P_i = 0$). Suppose now that a rigid particle acted upon by several forces P_i has moved an arbitrary small distance $\delta\Delta$ which is compatible with the constraints on the particle. Then the work done by this force system would be given by the vector equation

$$W_{\mathrm{v}} = (P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + \ldots) \delta\Delta = \sum P_i \delta\Delta \tag{2.18}$$

which is nothing more than multiplying an equilibrium conditions by $\delta\Delta$. This formulation of *equilibrium conditions* has several computational advantages, e.g. the reactions of fixed supports drop out from the equations because they do not work.

In the preceding calculation, the virtual work has been defined as the product of a real force moving through a virtual displacement. In this form the principle is referred to more accurately as *principle of virtual displacement*. However, the principle of virtual work can be stated in another form known as *principle of complementary virtual work*. Here, the virtual work is defined as the product of virtual force moving through a real displacement. This principle thus states that if the system is in equilibrium, then the sum of all virtual complementary works is zero for *compatibility* or *geometric continuity* (e. g. zero slope at point of fixation in case of fixed support, and members meeting at a rigid joint have same absolute rotation). Thus, the principle of virtual work (or virtual displacement principle) assumes compatibility and leads to equations of equilibrium while the principle of complementary virtual work assumes equilibrium and leads to equations of compatibility.

To illustrate the dual character of these two fundamental principles of mechanics consider the model shown in Fig. 2.8.The model is discrete frame structure consisting of rigid-bars supported by rotational springs capable of activating reaction moments at the supports.

(1) Principle of virtual work

Give the system a virtual rotation $\delta\theta$ as shown in Fig. 2.8a. The virtual works of various forces are

$$\delta W_{\text{ex}} = P(\delta\Delta)$$
$$-\delta W_{\text{in}} = M_1\delta\theta_1 + M_2\delta\theta_2$$

For compatibility $\delta\theta_1 = \delta\theta_2 = \delta\theta = \delta\Delta/h$, then

$$-\delta W_{\text{in}} = M_1\left(\frac{\delta\Delta}{h}\right) + M_2\left(\frac{\delta\Delta}{h}\right) = \frac{1}{h}(M_1 + M_2)\delta\Delta$$

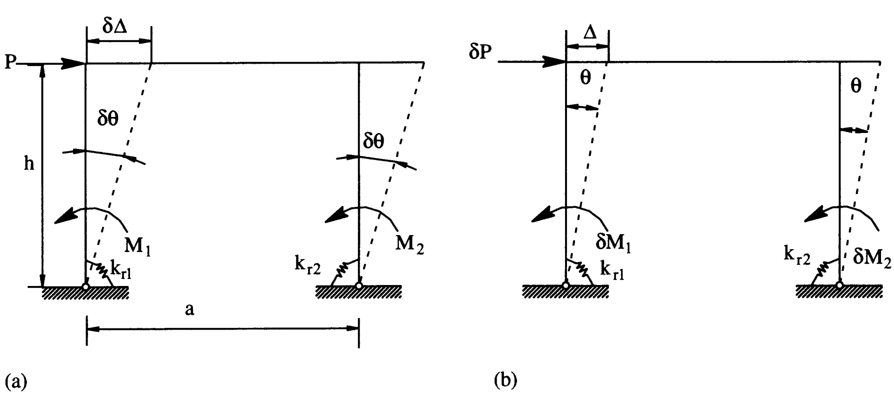


Fig. 2.8a,b. Concept of virtual displacements and virtual forces. **a** Virtual displacements, **b** virtual forces

From the principle of virtual work

$$\delta W_{ex} + \delta W_{in} = \left[P - \frac{1}{h}(M_1 + M_2) \right] \delta \Delta = 0$$

Since $\delta \Delta$ is arbitrary but non-zero, then

$$Ph = M_1 + M_2$$

which is equilibrium condition. Noting that $M_i = k_i \theta_i$

$$\theta = (Ph)/(k_{r1} + k_{r2})$$

(2) Principle of complementary virtual work

Apply on the system a virtual force δP. The virtual work done by various forces are

$$\begin{aligned} \delta W_{c,ex} &= \Delta(\delta P) \\ -\delta W_{c,in} &= \theta(\delta M_1) + \theta(\delta M_2) = \theta(\delta M_1 + \delta M_2) \end{aligned}$$

For moment equilibrium:

$$\delta M_1 + \delta M_2 = (\delta P)h \ .$$

Therefore

$$-\delta W_{c,in} = \theta(\delta P)h$$

From the principle of complementary work

$$\delta W_{c,ex} + \delta W_{c,in} = (\Delta - h\theta)\delta P = 0$$

Since δP is arbitrary but non-zero,

$$\Delta = h\theta$$

which is the compatibility condition.

2.5.4 The Principle of Superposition of Mechanical Work

An important property of linear deformation is the validity of principle of superposition which means that: if a force F_1 produce a deformation r_1 and F_2 produces another deformation r_2, then deformation due to F $(= F_1 + F_2)$ is r $(= r_1 + r_2)$. However, as far as mechanical work is concerned the principle of superposition can be applied to the displacement work component but it is not valid for eigenwork component. It can be noticed that in the case of eigenwork relationship between work W_{ii} and displacement δ_{ii} is parabolic and therefore principle of superposition does not hold good, while in case of displacement work, the relationship is linear and the principle of superposition is thus valid.

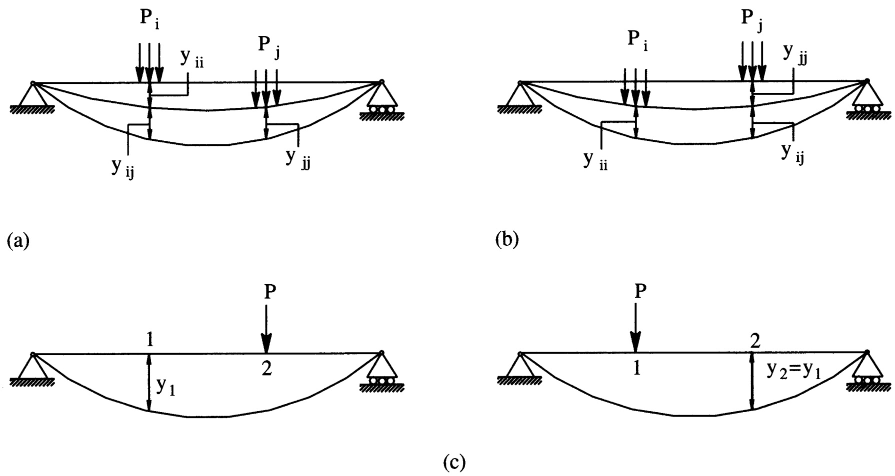


Fig. 2.9a–c. Basic features of Betti's and Maxwell theorems. **a** Loading sequence – case I, **b** loading sequence – case II, **c** Maxwell theorem

Theorems of Betti and Maxwell

Consider the beam shown in Fig. 2.9a subjected to load systems P_i and P_j at points i and j, respectively. Suppose the load system P_i is applied first and then subsequently the load system P_j. The work done by the forces is

$$W_I = W_{ii} + (W_{jj} + W_{ij}) \tag{2.19}$$

where W_{ii} and W_{jj} are eigenworks of P_i and P_j, respectively, and W_{ij} is the displacement work of P_i due to P_j. Now let the order of loading be reversed by bringing the load P_j first and then the load P_i as shown in the Fig. 2.9b. The work done by the forces in the second case is

$$W_{II} = W_{jj} + (W_{ii} + W_{ji}) \tag{2.20}$$

where W_{ji} is the displacement work done by P_j due to P_i. Since the total work done is independent of sequence of loading, W_I must be equal to W_{II}.

Thus

$$W_{ii} + W_{jj} + W_{ij} = W_{jj} + W_{ii} + W_{ji}$$

or

$$W_{ij} = W_{ji} \quad \text{or} \quad \sum P_i y_{ij} = \sum P_j y_{ji} \tag{2.21}$$

This theorem is known as *Betti's theorem* and may be stated: *for a linearly elastic structure, the work done by a set of external forces P_i acting through the displacements y_{ij} produced by another set of force P_j is equal to the work done by the later set of external forces P_j acting through the displacements y_{ji} produced by force P_i.*

Consider both the load systems P_i and P_j to be consisting of single load P (having the same magnitude but not necessarily in the same direction), then

$$Py_{ij} = Py_{ji} \quad \text{or} \quad y_{ij} = y_{ji} \tag{2.22}$$

This is known as *Maxwell's theorem of reciprocal deflection* and states that: the deflection of point j due to force P at point i is numerically equal to the deflection of point i due to force P applied at point j. It should be noted that deflections are measured in the direction of the forces. Here force means a generalized force (including moment).

2.5.5 Non-Linearities

While computing the work, it is essential that distinction be made between linear and non-linear force-deformation relationships. In a statical structural system, there are three main types of non-linearities: physical, geometrical and loading configuration non-linearities.

Physical Non-Linearity

This type of non-linearity is due to the physical properties of material used in the structure. All materials exhibit non-linearities to different degrees.

Geometrical Non-Linearity

This non-linearity is associated with the change in the geometry during deformation. To illustrate this type of non-linearity consider the model shown in Fig. 2.10. The

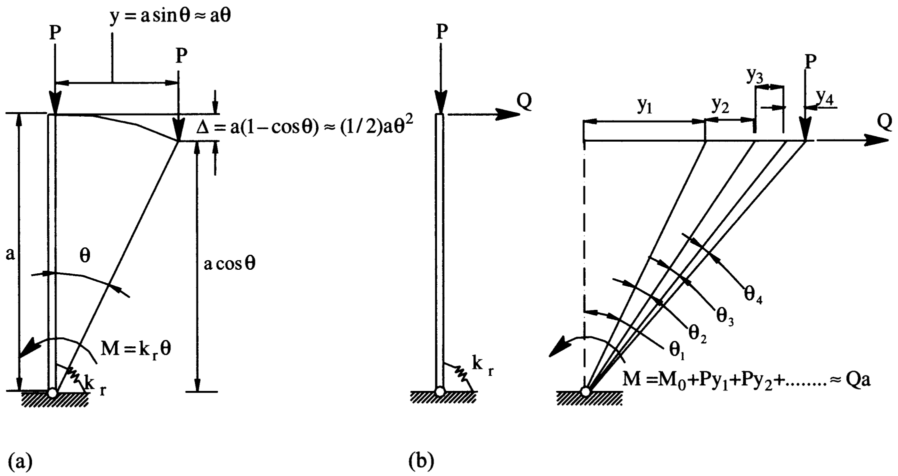


Fig. 2.10a,b. Geometric and loading configuration nonlinearities. **a** Geometric nonlinearity, **b** loading configuration nonlinearity

exact value of maximum bending is given by

$$M_{\max} = Pa\sin\theta = Pa\left(\theta - \frac{\theta^3}{3!} + - \cdots\right) \tag{2.23}$$

using Taylor series expansion.

If the expression is linearized i.e. all non-linear terms in θ are ignored, the equation reduces to

$$M_{\max} = Pa\theta$$

This is valid only when deflection is small. Large errors would result from using this simplification, if the structures are very flexible and tend to display large deformation. The classical theory based on small deflections ignores this type of non-linearity.

Loading Configuration Non-Linearity

This type of non-linearity is due to the effect of applied axial force on the deformed structure. Once a deformation has occurred, however small, the axial force will add to the bending moments and consequently to the deflection and so on. To illustrate this type of non-linearity consider the model shown in the Fig. 2.10b. In the initial equilibrium state, the maximum bending moment is $M_o = Qa$. This value of moment is based on ignoring the change in the length of lever arm from a to $a\cos\theta$. But due to the presence of axial force, the bending moment will be amplified to

$$M = M_o + Py_1 + Py_2 + \ldots \tag{2.24}$$

where y_1 is the deflection due to M_o, y_2 is that due to secondary moments Py_1 and so on. Thus, the presence of axial load introduces non-linearity to the system. In the stability analysis of structures under static loading normally the equilibrium equations are written with respect to deflected configuration of the system. Such an analysis is termed linearized theory analysis and leads to eigenvalue problems. In the absence of lateral force Q, the equilibrium equation of deflected system shown in Fig. 2.10b is

$$Pa\sin\theta = k_r\theta \quad \text{or} \quad P = \frac{k_r\theta}{a\sin\theta} \tag{2.25}$$

For small deflection theory, $\sin\theta$ is linearized to θ and P becomes independent of θ, i.e.

$$P = (k_r/a)\ .$$

2.6 Work of Internal Forces: Strain Energy

One-dimensional continuous elastic bodies will be discussed in detail in the following sections. Internal forces are the resultants of internal stresses which are resolved into component normal to cross-section, termed normal force N, and another tangential to the cross-section, termed shear force Q. In addition there are stress couples which are termed bending moment M. In case of 3-D or space structures there is another kind of internal force, called torsion moment M_t. All these internal forces are double action forces. They are different from the fixation or reactions at a support.

Ignoring, out of plane and shear deformations, a one-dimensional elastic body or say a rod can be imagined to be consisting of a number of discrete rigid elements of length Δx connected by perfectly elastic hinges, which are basically frictionless hinges with rotational spring devices simulating the bending flexibility of the rod. Such a discrete elements chain is shown in the Fig. 2.11a.

The strain energy of bending of such a chain model is defined as the sum of the energy stored in all the elastic hinges. Denoting the elastic hinge constant k_r. the strain energy stored in one spring (hinge) shown in Fig. 2.11c is

$$\begin{aligned}\delta U_r &= \frac{1}{2}\,\text{(moment in the spring) (rotation at the hinge)} = \frac{1}{2}(\delta M)(\delta\theta)\\ &= \frac{1}{2}(k_r\delta\theta)(\delta\theta) = \frac{1}{2}k_r(\delta\theta)^2 = -\delta W_i \end{aligned} \tag{2.26}$$

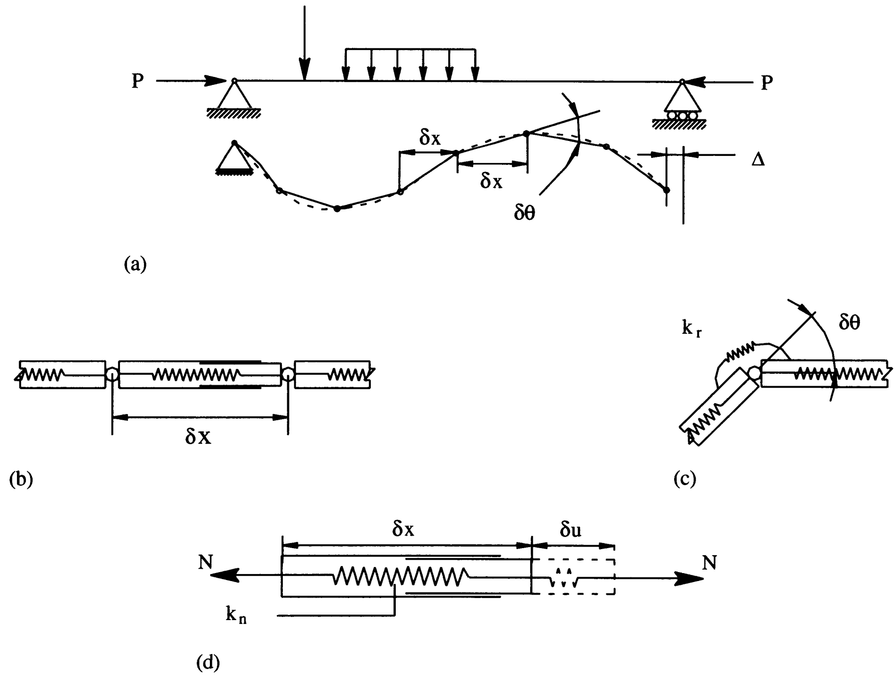


Fig. 2.11a–d. Deflection curve modeled by discrete rigid elements chain with bending and extensional flexibilities. **a** Discrete element chain, **b** extension flexibility, **c** elastic rotational hinge, **d** extension of an element

where $\delta\theta$ is the change in the angle between two adjacent elements. The total energy of bending for a chain of n elastic hinges is thus given by

$$U_r = \sum \delta U_r = \sum_{i=1}^{n} \frac{1}{2} k_r (\delta\theta_i)^2 = -W_{ir} \tag{2.27}$$

In order to include extensibility of middle axis of the rod imagine the small elements of length δx to be rigid but extensible. The extensibility is incorporated by internal extensional springs of spring constant k_n as shown in Fig. 2.11b by telescopic arrangement. If elongation of each element is δu then the energy of internal forces due to stretching is given by

$$\delta U_n = \frac{1}{2}(\delta N)(\delta u) = \frac{1}{2}(k_n \delta u)(\delta u) = \frac{1}{2} k_n (\delta u)^2$$

The stretching energy of the system is thus

$$U_n = \sum \delta U_n = \sum_{i=1}^{n} \frac{1}{2} k_n (\delta u_i)^2 = -W_{in} \tag{2.28}$$

Using the principle of superposition, the total strain energy U_i of the discrete mechanical model is therefore

$$U_i = \sum_{i=1}^{n} \left[\frac{1}{2} k_r (\delta\theta_i)^2 + \frac{1}{2} k_n (\delta u_i)^2 \right]$$

Since, $\delta\theta_i = (\delta M_i / k_r)$ and $\delta u_i = (\delta N_i / k_n)$, U_i can be expressed as

$$U_i = \sum_{i=1}^{n} \left[\frac{1}{2} \frac{(\delta M_i)^2}{k_r} + \frac{1}{2} \frac{(\delta N_i)^2}{k_n} \right] = -W_i \tag{2.29}$$

The above expression can be used to derive the corresponding expression for a continuous one-dimensional elastic structure. Noting the relationship from strength of material

$$\begin{aligned} M &= EI\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = EI\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = EI \lim \left(\frac{\delta\theta}{\delta x}\right) \\ &\approx EI\left(\frac{\delta\theta}{\delta x}\right) = \frac{EI}{\delta x}\delta\theta \end{aligned} \tag{2.30}$$

where $\delta\theta$ is the change in angle θ of the tangent to the line of deflection. Comparing the formula with the law of linear moment springs $M = k_r(\delta\theta)$, it is seen that $k_r \cong EI/(\delta x)$. Substituting this value in (2.27)

$$\begin{aligned} U_r &= \sum_{i=1}^{n} \frac{1}{2} k_r (\delta\theta_i)^2 = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{EI}{\delta x_i}\right)(\delta\theta_i)^2 \\ &= \sum_{i=1}^{n} \frac{EI}{2} (\delta\theta_i)^2 \delta x_i \end{aligned} \tag{2.31}$$

For a continuous body $n \to \infty$, summation $\sum$ can be replaced by $\int$ and difference expression by differential expression. Thus

$$U_\mathrm{r} = \int \frac{1}{2} EI(\theta')^2 \,\mathrm{d}x \tag{2.32}$$

Using the geometrically linear approximation $\theta' = y''$ where y is deflection, we obtain

$$U_\mathrm{r} = \int \frac{1}{2} EI(y'')^2 \,\mathrm{d}x = \int \frac{1}{2} EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \tag{2.33}$$

substituting the relationship $M = EIy''$, the relationship of (2.33) reduces to

$$U_\mathrm{rc} = \int \frac{1}{2} \left(\frac{M^2}{EI}\right) \mathrm{d}x$$

The subscript c, indicates that it is complementary strain energy connected with complementary work. In a similar manner

$$N = EA\epsilon = EAu' \cong EA(\delta u/\delta x)$$

Here EA is the axial stiffness, ϵ is the axial strain and u is the axial displacement component. Therefore elongational energy is

$$U_\mathrm{n} = \int \frac{1}{2} EA\epsilon^2 \,\mathrm{d}x = \int \frac{1}{2} EAu'^2 \,\mathrm{d}x = \int \frac{N^2}{2EA} \,\mathrm{d}x = U_\mathrm{n,c} \tag{2.34}$$

The total strain energy of an initially straight rod which is equivalent to the work of internal forces is thus

$$U_\mathrm{i} = U_\mathrm{r} + U_\mathrm{n} = \int \frac{1}{2} \left[EI(y'')^2 + EA(u')^2\right] \mathrm{d}x = -(W_\mathrm{ir} + W_\mathrm{in}) \tag{2.35}$$

A small change in U_i defined as the first variation δU_i is

$$\begin{aligned} \delta U_\mathrm{i} &= \int \frac{1}{2} \left[EI(2y'')\delta y'' + EA(2u')\delta u'\right] \mathrm{d}x \\ &= \int (EIy''\delta y'' + EAu'\delta u') \,\mathrm{d}x \end{aligned} \tag{2.36}$$

Since $EIy'' = M$ and $EAu' = N$, the expression for δU reduces to

$$\begin{aligned} \delta U_\mathrm{i} &= \int (M\delta y'' + N\delta u') \,\mathrm{d}x = \int (M\delta\theta' + N\delta\epsilon) \,\mathrm{d}x \\ &= \text{Virtual work of internal forces} = W_\mathrm{i} \end{aligned} \tag{2.37}$$

Thus small change in U defined as first variation δU is nothing else but the virtual work of internal forces.

2.7 The Work Equation

According to the *law of conservation of energy*, there is no gain or loss of energy in a conservative system during deformation under external loads. It means that the change in energy of applied load (i.e. external work done) is equal to the increase in stored or strain energy (i.e. the work done by the internal forces). This statement that the work of internal forces is equal to the work of external forces is valid for both eigenwork and displacement work. In a perfectly elastic closed system the work done by a load (stored as strain energy) will be released back completely when the load is removed in the absence of thermal dissipation or frictional or damping losses. Systems not following these conditions are commonly referred to as non-conservative systems. Thus we have

$$-W_{\text{in}} = W_{\text{ex}} \tag{2.38}$$

This work balance equation is equivalent to equilibrium conditions as has been shown earlier in Sect. 2.5.3. Using the principle of virtual work which is an axiom of equilibrium.

$$\begin{aligned} \delta W_{\text{v}} &= \delta W_{\text{ex}} + \delta W_{\text{in}} = 0 \quad \text{or} \\ \delta W_{\text{v}} - \delta W_{\text{in}} &= \delta W_{\text{ex}} \end{aligned} \tag{2.39}$$

which means that change in the energy of applied loads is equal to the increase in the stored energy. The work balance equation (2.38) can be used directly in solving problems in structural analysis. For illustration consider the cantilever, and simply supported beams of bending stiffness EI loaded by a concentrated load acting at the free end, and at a distance a from support O, respectively, as shown in Fig. 2.12. It is required to determine the deflections at the loaded points.

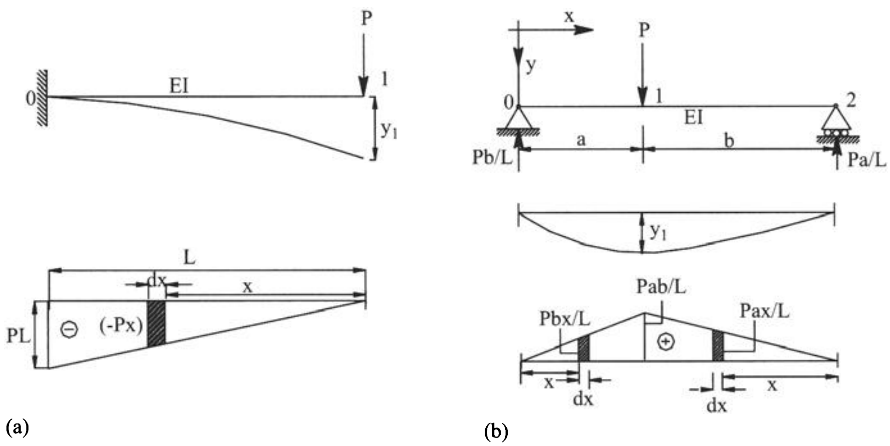


Fig. 2.12a,b. Cantilever and simply supported beams with b.m. diagrams. **a** Cantilever and its b.m. diagram, **b** simply supported beam and b.m. diagram

(a) Deflection of cantilever beam

Assuming linear small deflection theory, external work is

$$W_{ex} = \tfrac{1}{2} P y_1$$

where y_1 is the deflection of the beam. The internal work can be easily determined in terms of bending moment. Thus

$$-W_{in} = \int_0^L \frac{M^2}{2EI} dx = \int_0^L \frac{(-Px)^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

From work equation $-W_{in} = W_{ex}$

$$\frac{1}{2} P y_1 = \frac{P^2 L^3}{6EI},$$

we find that

$$y_1 = \frac{PL^3}{3EI} \tag{2.40}$$

(b) Deflection of simply supported beam

External work assuming linear small deflection theory is

$$W_{ex} = \tfrac{1}{2} P y_1$$

where y_1 deflection of beam at the load point. The internal work can be computed in two parts 0-1 and 1-2, since the bending moment diagram is discontinuous at the load point 1. Thus

$$-W_{in} = \int_0^L \frac{M^2}{2EI} dx = \int_0^a \frac{1}{2EI} \left(\frac{Pb}{L} x\right)^2 dx + \int_0^b \frac{1}{2EI} \left(\frac{Pa}{L} x\right)^2 dx$$

Thus

$$-W_{in} = \frac{P^2 b^2 a^3}{6EIL^2} + \frac{P^2 a^2 b^3}{6EIL^2} = \frac{P^2 a^2 b^2}{6EIL^2}(a+b) = \frac{P^2 a^2 b^2}{6EIL}$$

It should be noted that for the computation of internal work in the part 1-2 of the beam, the point 2 has been taken as the origin for simplification. Finally from work equation

$$\frac{1}{2} P y_1 = \frac{p^2 a^2 b^2}{6EIL} \quad \text{or} \quad y_1 = \left(\frac{Pa^2 b^2}{3EIL}\right) \tag{2.41}$$

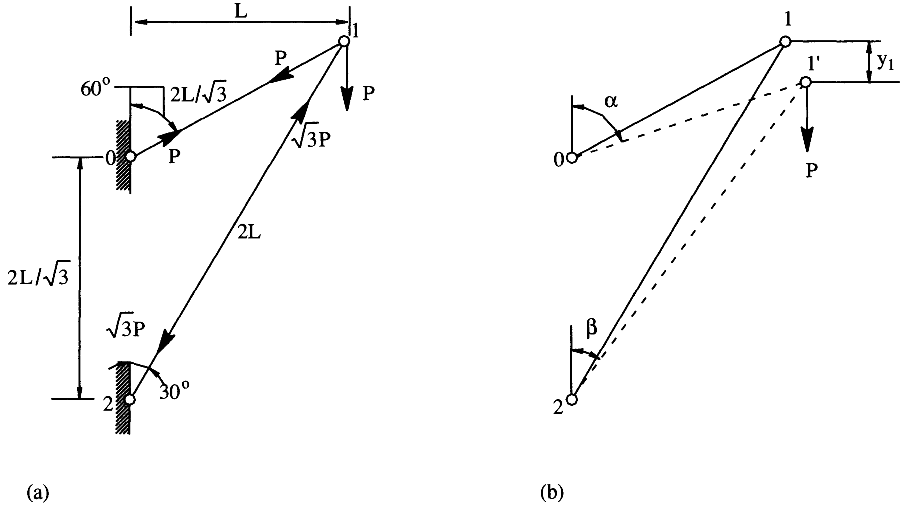


Fig. 2.13a,b. The truss and its deflection. **a** Pin-joint truss, **b** exaggered deflection of the truss

For multiple load system above method becomes complex. An alternate method using a trial shape function is suitable. The work equation is equally applicable to structures carrying axial loads. For illustration consider the truss shown in Fig. 2.13a. It is required to find the deflection of point 1.

An exaggerated deflection of the truss is shown in Fig. 2.13b for clarity. An assumption of small deflection theory for load-deflection relationship implies $\alpha \cong 60°$ and $\beta \cong 30°$. The external work done is

$$W_{ex} = \tfrac{1}{2} P y_1$$

The strain energy stored in the members of axial stiffness EA is given by

$$-W_{in} = U = \sum_i \frac{1}{2} \frac{F_i^2 L_i}{(AE)_i}$$

$$= \frac{1}{2AE}\left[P^2\left(\frac{2L}{\sqrt{3}}\right) + \left(\sqrt{3}P\right)^2 (2L)\right] = \frac{P^2 L}{AE}\left(\frac{1}{\sqrt{3}} + 3\right)$$

equating external work done to internal energy stored

$$\frac{1}{2} P y_1 = \frac{P^2 L}{AE}\left(\frac{1}{\sqrt{3}} + 3\right)$$

Therefore

$$y_1 = \left(\frac{2}{\sqrt{3}} + 6\right)\left(\frac{PL}{AE}\right)$$

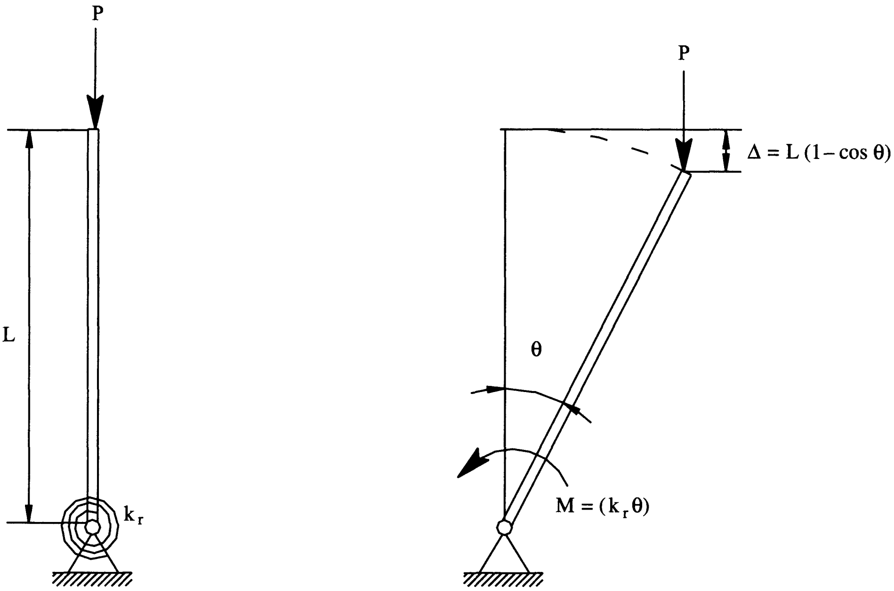


Fig. 2.14. Rigid column supported by a rotational spring

To illustrate the application of work balance equation to determine critical load consider a rigid (stiff) column hinged at the bottom and supported at the hinge by a rotational (moment) spring of stiffness k_r as shown in Fig. 2.14. When the P attains critical value P_{cr} (buckling load), the system moves from unbuckled to a buckled state. However, load remains constant. The work done by the load is thus

$$W_{ex} = P\Delta \tag{2.42}$$

where Δ is the descent or vertical movement of the load and can be easily found from the geometry of deformation as

$$\Delta = L(1 - \cos\theta) = L\left[1 - \left(1 - \frac{1}{2}\theta^2 + \ldots\right)\right] \cong L\left(\frac{1}{2}\theta^2\right) \tag{2.43}$$

In the Taylor series expansion of $\cos\theta$ only first two terms have been retained. Thus

$$W_{ex} = P\left(\frac{1}{2}L\theta^2\right)$$

The internal work, on the other hand, is equal to the internal work of rotational spring

$$-W_{in} = \frac{1}{2}k_r\theta^2$$

Equating both the works we get

$$\frac{1}{2}PL^2\theta = \frac{1}{2}k_r\theta^2 \quad \text{i.e.} \quad P_{cr} = (k_r/L) \tag{2.44}$$

As long as $P < (k_r/L)$ the system is stable and any sideway perturbation will cause the system to spring back to unbuckled state (after vibrating for sometime). However, for $P > (k_r/L)$, any perturbation will make the system to move from a vertical unbuckled state to a no return position.

For most of the beams and frames, axial and shear deformations can be ignored in determining deflection of central axis. On the other hand large axial force deformations in arches, suspension bridges and trusses have to be taken into account. Similarly, for deep beams shear deformation can reach large values. To determine the order of magnitude of contribution of shear force in the deflection, consider the maximum deflection due to load P applied at the free end of elastic cantilever of rectangular cross-section. The cross-section and magnitude of modulus of elasticity are assumed to be constant along the entire length of cantilever of Fig. 2.12a. The eigenwork of internal forces namely bending moment and shear force is

$$W_{\text{in}} = \int_0^L \frac{M_x^2\,\mathrm{d}x}{2EI} + K\int_0^L \frac{Q_x^2}{2GA}\mathrm{d}x \tag{2.45}$$

where K is a constant depending upon the shape of cross-section termed shape factor. The value of K for a rectangular cross-section is 1.2. Thus

$$-W_{\text{in}} = \int_0^L \frac{(-Px)^2\,\mathrm{d}x}{2EI} + 1.2\int_0^L \frac{P^2\,\mathrm{d}x}{2GA} = \frac{P^2L^3}{6EI} + \frac{6}{10}\frac{P^2L}{GA}$$

The external eigenwork done during the displacement $y_{\max}$ is

$$W_{\text{ex}} = Py_{\max}/2$$

Equating external work to the eigenwork of internal forces i.e. $W_{\text{ex}} = -W_{\text{in}}$

$$\begin{aligned} y_{\max} &= \left(\frac{PL^3}{3EI}\right) + \left(\frac{6PL}{5GA}\right) = y_{\text{bend}} + y_{\text{shear}} \\ &= \frac{PL^3}{3EI}\left[1 + \frac{3E}{10G}\left(\frac{h^2}{L^2}\right)\right] \end{aligned} \tag{2.46}$$

where h is the height of the cross-section. The relationship between material constants E and G is given by

$$\frac{E}{G} = 2(1 + \nu)$$

Taking $\nu = 0.2$ a typical value for concrete, the ratio E/G becomes 2.4. The total deflection

$$y_{\max} = \left[1 + 0.72(h^2/L^2)\right]y_{\text{bend}} \tag{2.47}$$

For a very short or deep beam say $L/h = 1$, the total deflection is 1.72 times that due to bending alone. Hence shear deformation are important. On the other hand for long or slender typical beam with $L/h = 15$, the deflection due to shear is 0.32 per cent. However, it should be realized that this is not always the case. For an I-beam with strong flanges and very thin web the shape factor can become up to ten times larger than rectangular section and shear force Q could not be ignored.

In the situations where the loading configuration is difficult to treat using standard methods of structural analysis or where the system is highly statically indeterminate due to complicated support and boundary conditions, the work method is viable alternative method to the classical integration of differential equation.

Differential Equations

The differential equation of a problem can be obtained by using standard equilibrium procedure. Consider an element dx isolated from the structure and carrying constant distributed load $w(x)$ within the element. Due to this loading force components, N, Q and M will generally vary as shown in the Fig 2.15b. The equilibrium conditions are

$$\sum F_x = 0\,, \quad N - (N + \mathrm{d}N) = 0$$

dividing by dx we obtain

$$\mathrm{d}N/\mathrm{d}x = N' = 0$$

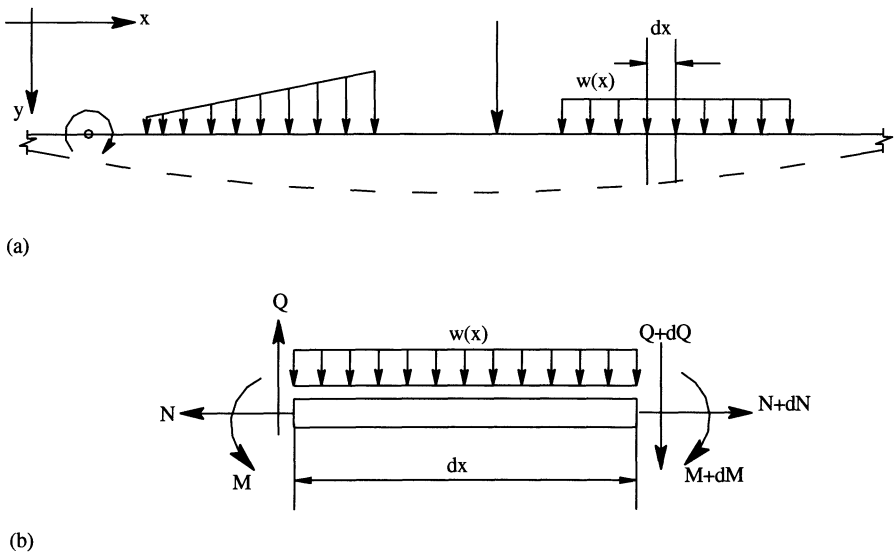


Fig. 2.15a,b. Forces acting on an isolated element. **a** Loaded structure, **b** internal forces acting on an element

i. e. N is constant

$$\sum F_y = 0\,, \quad Q - (Q + \mathrm{d}Q) - w(x)\,\mathrm{d}x = 0 \tag{2.48}$$

dividing by dx we obtain

$$\mathrm{d}Q/\mathrm{d}x + w(x) = Q' + w(x) = 0 \quad \text{or} \quad Q' = -w(x) \tag{2.49}$$

This means that the rate of change of Q with respect to x is equal to the negative of loading $w(x)$. The condition $\sum M = 0$ about the left end of element gives

$$-M + \left[(Q + \mathrm{d}Q)\,\mathrm{d}x\right] + \left[w(x)\,\mathrm{d}x\right]\left(\frac{\mathrm{d}x}{2}\right) + (M + \mathrm{d}M) = 0$$

On simplification and dividing by dx we obtain

$$Q + w(x)\frac{\mathrm{d}x}{2} + \frac{\mathrm{d}M}{\mathrm{d}x} = 0 \tag{2.50}$$

In the limit dx, the second term vanishes

$$Q + M' = 0 \quad \text{or} \quad Q = -M' \tag{2.51}$$

This means that the rate of change of M with respect to x is equal to the shear force Q.

Differentiation of (2.51) and substitution in (2.49) gives

$$M'' = w(x) \tag{2.52}$$

This means that differentiating bending moment twice is equal to the loading. In other words, integration of loading as a function of x twice, gives the bending moment. However, for determining the constants of integration two moment boundary conditions are required which may not be obvious. It is advantageous to express the differential equation in terms of lateral displacement $y(x)$. This can be accomplished by using standard linear relationship

$$M = (EI/R) \cong EIy'' \tag{2.53}$$

Thus the differential equation of elastic beam in the displacement form is

$$EIy'''' - w(x) = 0 \tag{2.54}$$

This is an ordinary linear differential equation of fourth-order and therefore requires determination of four constants of integration from the four displacement boundary conditions.

A shape functions of $y(x)$ which looks as close as possible to the expected one can be assumed. Such a assumed function is termed *trial or test function.* This trial function should in general approximate the deflection curve as far as possible but must satisfy all the boundary conditions related to deflection and slopes (y and y'). The internal work in terms of differential equation is given by

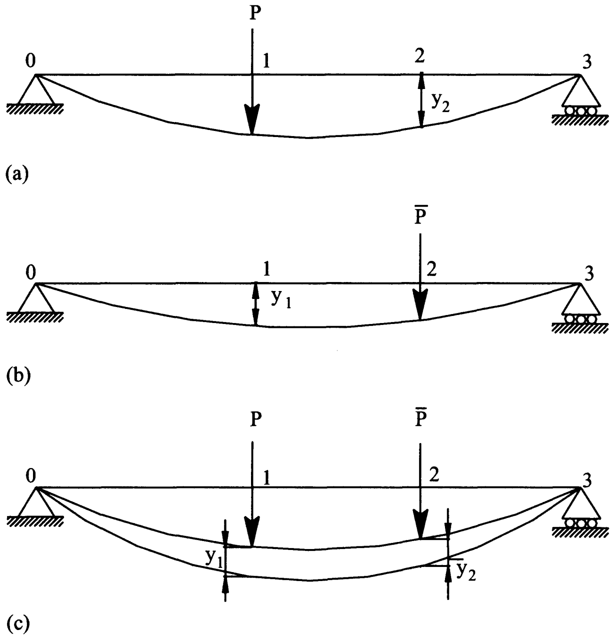


Fig. 2.16a–c. Virtual work method for calculation of displacement. **a** Displaced structure under load, **b** displacement due to virtual load $\bar{P}$, **c** displacements due to loads P and $\bar{P}$

$$-W_{\text{in}} = \int_0^L \frac{1}{2} EI(y'')^2 \mathrm{d}x \tag{2.55}$$

In this case trial function could be a second-order parabola or even simply a half sine wave. However, trial function should adapt to the boundary conditions of the problem. The work-balance approach used in the foregoing illustrations is limited to the determination of deflection caused by a single force at the point of application. In case of multiple loads more than one value of deflection will appear in the expression for external eigenwork, the resulting equations can not be solved. Due to these limitations, the method of real work is not widely used for deflection analysis. The principle of virtual (displacement) work may be utilized to solve this problem. This method is one of the most important methods used to calculate displacement of elastic structures and also forms the basis of force or flexibility method for the analysis of statically indeterminate systems. For illustration consider the simply supported beam shown in Fig. 2.16a. The beam supports load P at point 1. It is required to determine vertical displacement at point 2. Let the load P be removed temporarily from the beam and virtual load $\bar{P}$ of arbitrary magnitude be applied at point 2 in the direction of desired deflection as shown in Fig. 2.16b.

Now load the beam with the real load P, producing additional displacements y_1 and y_2 under loads P and $\bar{P}$, respectively, as shown in Fig. 2.16c. From the principle of energy balance that the external virtual work is equal to internal virtual work we have

$$\bar{W}_{ex} = -\bar{W}_{in}\,, \quad \bar{W}_{ex} = \bar{P}y_2$$

(i. e. product of virtual force $\bar{P}$ and real displacement).

The internal work is due to moment $\bar{M}$ caused by $\bar{P}$ acting through bending deformation $d\theta$ produced by real load P. The deformation $d\theta$ is defined by $M\mathrm{d}x/EI$, where M is due to real load P. Thus

$$-\bar{W}_{in} = \int_0^L \bar{M}(\mathrm{d}\theta) = \int_0^L \bar{M}\left(\frac{M\,\mathrm{d}x}{EI}\right)$$

Equating $\bar{W}_{ex}$ and $-\bar{W}_{in}$

$$\bar{P}.y_2 = \int_0^L \frac{\bar{M}M}{EI}\mathrm{d}x \quad \text{or} \quad y_2 = \frac{1}{\bar{P}}\int_0^L \frac{\bar{M}M}{EI}\mathrm{d}x$$

Since $\bar{P}$ is arbitrary and moment $\bar{M}$ is a linear function of $\bar{P}$, $\bar{P}$ can be replaced by a unit load and $\bar{M}$ by m,

$$y_2 = \int_0^L \left(\frac{mM}{EI}\right)\mathrm{d}x \tag{2.56}$$

where m is moment caused by unit load applied in place of $\bar{P}$. Thus, an application of unit external virtual force directly gives the desired displacement. This unit external force can be in the form of either a force or a moment depending upon the type of displacement to be determined.

The value of integral could be easily evaluated by numerical integration in tabular form. However, since m is always a linear quantity, the above integral reduces to following simple expression

$$y = \int \frac{mM}{EI}\mathrm{d}x = X\int\left(\frac{M}{EI}\right)\mathrm{d}x = X \quad \text{(Area of } M/EI \text{ diagram)} \tag{2.57}$$

where X is the value (ordinate) of m diagram at the location of centre of gravity of corresponding smooth (continuous) part of (M/EI) diagram as shown in Fig. 2.17. The method called area centre-of-gravity method is due to Otto Mohr, and is applicable to structures made up of straight members. For a rigid frame consisting of a number of discrete elements or members

$$y = \frac{A_1X_1}{E_1I_1} + \frac{A_2X_2}{E_2I_2} + \cdots = \sum_i \left(\frac{AX}{EI}\right)_i \tag{2.58}$$

By an analogous treatment it can be shown that deflections due to axial forces, shear forces and torsional moments are

$$y_n = \int \frac{nN}{EA}\mathrm{d}x\,, \quad y_q = \int \frac{qQ}{GA}\mathrm{d}x \quad \text{and} \quad y_{tor} = \int \frac{tT}{GI_p}\mathrm{d}x \tag{2.59}$$

respectively, where G is modulus of shear and I_p is polar moment of inertia.

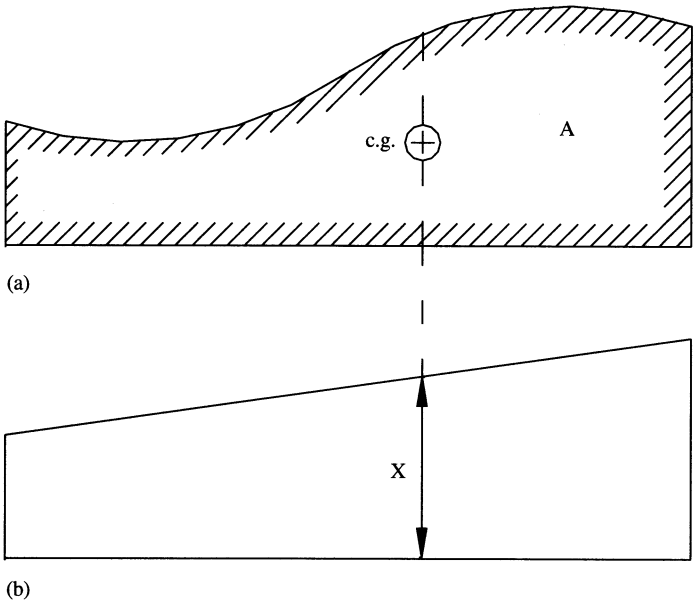


Fig. 2.17a,b. Computation of deflection by area centre of gravity method. **a** M/EI diagram, **b** m diagram

The virtual work unit load method can also be used for determining the deflection of a truss. Since bending moments in a pin-jointed truss are by definition all zero, the virtual work formula in this case is

$$-\mathrm{d}W_{\mathrm{in}} = \int \frac{nN}{EA}\mathrm{d}x$$

Noting that N and n are always constant over the whole length, the internal virtual work of a discrete member of length L is

$$-\mathrm{d}\bar{W}_{\mathrm{in}} = \frac{nN}{EA}\int_0^L \mathrm{d}x = \frac{nNL}{AE}$$

For a complete truss containing m members, the virtual work expression becomes

$$-\bar{W}_{\mathrm{in}} = \sum_{i=1}^{m}\left(\frac{nNL}{AE}\right)_i$$

Equating external virtual work $\bar{W}_{\mathrm{ex}}$ $(= 1.y)$ to internal virtual work $-\bar{W}_{\mathrm{in}}$

$$y = \sum_{i=1}^{m}\left(\frac{nNL}{AE}\right)_i \tag{2.60}$$

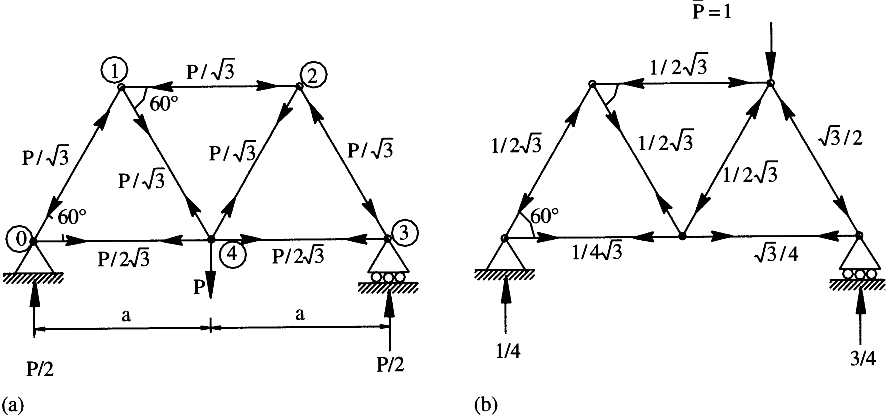


Fig. 2.18a,b. Pin jointed truss with real and virtual forces. **a** Pin jointed truss and real forces, **b** virtual forces

The following example illustrates the procedure for evaluating deflections of the truss joints. Consider the pin jointed truss shown in Fig. 2.18a and let it is desired to determine vertical component of deflection at the joint 2. AE is same for all the members. The real and virtual forces in the members of truss are shown in the Figs. 2.18a and 2.18b, respectively. The vertical displacement is

$$
\begin{aligned}
y_2 &= \frac{a}{AE}\sum_{i=1}^{m}(Nn)_i \\
&= \frac{a}{AE}\left[\left(\frac{P}{\sqrt{3}}\cdot\frac{1}{2\sqrt{3}}\right)+\left(\frac{P}{2\sqrt{3}}\cdot\frac{1}{4\sqrt{3}}\right)+\left(\frac{P}{\sqrt{3}}\cdot\frac{1}{2\sqrt{3}}\right)+\left(\frac{P}{\sqrt{3}}\cdot\frac{1}{2\sqrt{3}}\right)\right. \\
&\quad \left.+\left(\frac{P}{\sqrt{3}}\cdot\frac{1}{2\sqrt{3}}\right)+\left(\frac{P}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{4}\right)+\left(\frac{P}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}\right)\right]=\frac{32Pa}{24AE}
\end{aligned}
$$

2.8 Energy Theorems of Elastic Systems

In this section some of the important theorems of elastic systems will be discussed. As mentioned in the earlier section eigenwork could be used in conjunction with the work equation to determine the deflection at the point of application of a load. However, eigenwork could be generalized to calculate deflections. Consider for example a simple beam shown in Fig. 2.19a subjected to a system of loads P (P_1, P_2, $P_3, \ldots, P_n$) applied gradually. The beam undergoes deformation and strain energy, which is a function of external loads and is equal to external work done, is stored in the system, therefore,

$$W_{ex} = \phi(P_1, P_2, P_3, \ldots, P_n) = U$$

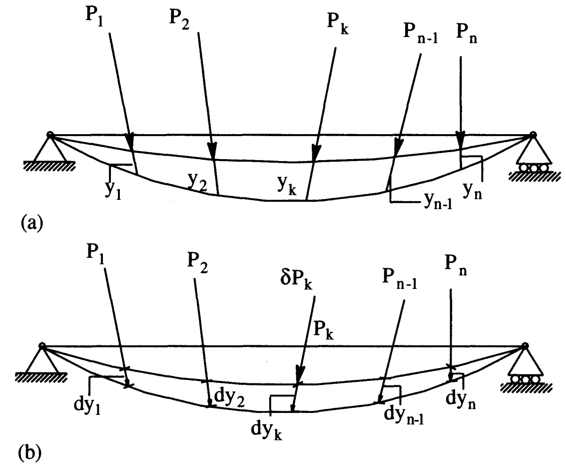


Fig. 2.19. **a** Deflections due to the load system, **b** additional deflections due to δP_k

Now, if any one of the loads, say P_k is increased by a differential amount δP_k, the strain energy of the system will change by an amount $(\partial U/\partial P_k)\delta P_k$. The expression for total strain energy reduces to

$$U_{\mathrm{t}} = U + \left(\frac{\partial U}{\partial P_k}\right)\delta P_k = W_{\mathrm{ex,t}} = U + \sum_{i=1}^{n} P_i \,\mathrm{d}y_i + \frac{1}{2}\delta P_k \,\mathrm{d}y_k$$

Neglecting the last term as being the product of two differential quantities, we have

$$U_{\mathrm{t}} = U + \sum_{i=1}^{n} P_i \mathrm{d}y_i \tag{2.61}$$

If the sequence of loading is reversed i.e. δP_k is applied first and then the system of loads P as shown in Fig. 2.19b the total work done is

$$W_{\mathrm{ex,t}} = U_{\mathrm{t}} = U + \left(\frac{1}{2}\delta P_k \mathrm{d}y_k\right) + \delta P_k y_k$$

The term $(\delta P_k \,\mathrm{d}y_k/2)$ being of second order can be neglected. Consequently expression of work done reduces to

$$U_{\mathrm{t}} = U + \delta P_k y_k \tag{2.62}$$

Since the order of application of loads is immaterial, the total work done or total internal strain energy in both the loading sequences must be equal. Therefore

$$U + \frac{\partial U}{\partial P_k}\delta P_k = U + \delta P_k y_k \quad \text{or}$$

$$\frac{\partial U}{\partial P_k} = y_k \quad \text{or} \quad y_k = \frac{\partial U}{\partial P_k} = \frac{\partial (W_{\mathrm{ex}})}{\partial P_k} = \frac{\partial(-W_{\mathrm{in}})}{\partial P_k} \tag{2.63}$$

The above expression can be generalized to

$$y_k = \frac{\partial(-W_{\text{in}})}{\partial P_k} \quad \text{and} \quad \theta_k = \frac{\partial(-W_{\text{in}})}{\partial M_k} \tag{2.64}$$

where, W_{in} is the internal work of the system. In the other words, for a linearly elastic structure, partial derivative of the total strain energy with respect to a typical load P_k gives the deflection due to this load in its direction. This is *Cotterill-Castigliano's second theorem.*

Now, if one of the displacements, say y_k, is changed by an infinitesimal amount $\mathrm{d}y_k$ while all other displacements are kept unchanged, the corresponding change in the strain energy would be $(\partial U/\partial y_k)\mathrm{d}y_k$. During such a change the force P_k is the only one which does work amounting to $P_k\,\mathrm{d}y_k$. Equating the change in the internal energy to the additional work done

$$\left(\frac{\partial U}{\partial y_k}\right)\mathrm{d}y_k = P_k\,\mathrm{d}y_k \quad \text{i.e.} \quad \frac{\partial U}{\partial y_k} = P_k \tag{2.65}$$

This is known as *Cotterill-Castigliano's first theorem*. It states that the partial derivative of strain energy of a system with respect to any one of the displacements at a certain point gives singular force at the same point. It should be noted that this theorem do not place any restriction on the relationship between deformation and force being linear.

To apply Cotterill-Castigliano's second theorem for determining deflections, the strain energy must be expressed in terms of external loads. Consider, for example, the flexural system where the internal strain energy is due to bending. The expression for deflection is

$$y_k = \frac{\partial U}{\partial P_k} = \frac{\partial}{\partial P_k}\int (M_x)^2/(2EI)\,\mathrm{d}x$$

It is much easier to first differentiate the quantity under integral sign and then evaluate the integral i.e.

$$y_k = \int_0^L \left(\frac{M_x}{EI}\right)\left(\frac{\partial M_x}{\partial P_k}\right)\mathrm{d}x \tag{2.66}$$

Similar expression can be developed for trusses where internal energy is due to axial strains. The expression for deflection of a truss point is

$$y_k = \sum\left(\frac{P_x}{EA}\right)\left(\frac{\partial P_x}{\partial P_k}\right)L_x \tag{2.67}$$

It must be noted that if a deflection component is required at a point where no action is applied or if an action exists at that point but not in the direction of desired deflection, then an imaginary action is assumed until the partial derivative for the total strain energy has been computed. In the resulting expression, the imaginary action is then reduced to zero.

The Cotterill-Castigliano's second theorem can be advantageously used for the calculation of redundant actions in statically indeterminate systems, The procedure consists in making the system statically determined by removing the redundant supports and replacing them with unknown redundant actions. Thus, the system under the given loading would have deflections at the location and in the direction of support reactions. The work is expressed in terms of external known loads and unknown redundant actions. The partial derivatives with respect to redundant actions (R_i) give deflection at their location and directions. However, the deflections are suppressed by the support that makes the system statically indeterminate in the first place. Thus

$$-\frac{\partial W_{\text{in}}}{\partial R_i} = 0, \quad i = 1, 2, \ldots, n \tag{2.68}$$

where n is the number of redundant actions. Suppose R_i is equal to unity, then we would have a unit deflection y_i. Thus

$$y_{Ri} = R_i y_i$$

Since y_R must be equal to the prescribed deflection y_{pi}, thus the net deflection

$$y_i = R_i y_i - y_{pi}$$

However, from Cotterill-Castigliano's second theorem

$$\left(-\frac{\partial W_{\text{in}}}{\partial R_i}\right) = R_i y_i - y_{pi}$$

Differentiating the equation with respect to R_i, we obtain

$$-\frac{\partial^2 W_{\text{in}}}{\partial R_i^2} = y_i > 0 \tag{2.69}$$

that is y_i must be positive. Thus (2.68) is the necessary condition that the $-W_{\text{in}}$ is an extremum. While (2.69) says that extremum is minimum. Thus *statically indeterminate redundancy takes a value that makes the work of internal forces $-W_{in}$ a minimum i. e.*

$$\delta(-W_{\text{in}}) = 0 \quad \text{and} \quad \delta^2(-W_{\text{in}}) > 0$$

The Cotterill-Castigliano's first and second theorems lead to the formulation of stiffness and flexibility methods.

2.9 Potential Energy

The word potential means ability or capability of achieving a particular goal. The ability of the load to do work is termed load potential which is very similar to the position energy of the load. In the structural mechanics the potential energy is always referred to an arbitrary datum. To establish basic concept, consider the surface profile shown in Fig. 2.20 which represents potential variation in terms of energy hills and valleys. It is evident that a particle or a ball on the surface could not be in equilibrium except at the top of a hill, at the bottom of a valley or at a point of inflection (or flat surface). As explained in Chap. 1, these are the points of *local maximum, minimum* or *minimax* on the energy surface where tangents are horizontal or have zero slope i.e. $\partial V/\partial y = O$. Mathematically, these points are termed *stationary points* (called equilibrium points in structural mechanics). At an equilibrium position corresponding to the minimum energy point 2, an infinitesimal displacement or perturbation of the ball requires positive energy which raises the energy level of the load, $V(y)$. This can be represented as

$$V(y_2 - \delta y) > V(y_2) < V(y_2 + \delta y) \tag{2.70}$$

On removal of perturbational force the ball roll backs to its position of minimum potential energy. This is termed as *stable equilibrium position.* Thus for a stable equilibrium which has minimum potential energy it must be ensured in addition to $\partial V/\partial y = 0$ that $\partial^2 V/\partial y^2 > 0$. Therefore, the principle of stationary potential energy may be stated as: *if a system is in static equilibrium, the total potential energy of the system has a stationary value.*

The positions corresponding to maximum and inflection points on the energy surface indicate *unstable* and *neutral equilibrium* conditions, respectively. In case of ball resting on a point of maximum potential energy, a perturbation makes the ball to roll down to lower energy levels. This is termed *unstable equilibrium* given by:

$$V(y_1 - \Delta y) < V(y_1) > V(y_1 + \Delta y)$$

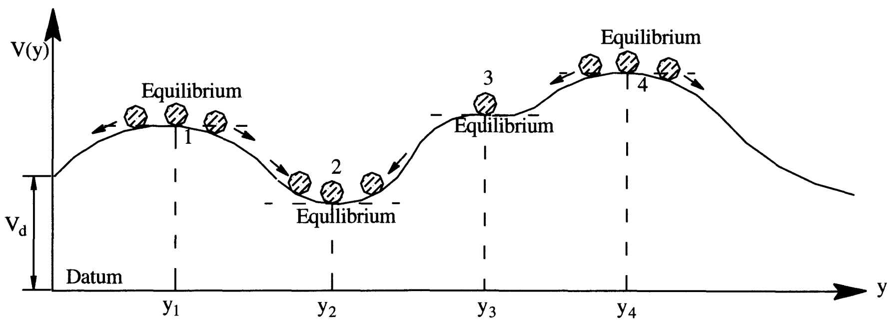


Fig. 2.20. Potential energy profile

Mathematically this condition is represented by

$$\partial^2 V/\partial y^2 < O \tag{2.71}$$

On the other hand at the point of inflection (i. e. at a flat surface), a perturbation makes the ball to stay back in the new position. There is no change in the energy level. This state is termed *neutral equilibrium* given by:

$$V(y_3 - \Delta y) < V(y_3) < V(y_3 + \Delta y)$$

Mathematically such a condition can be represented by

$$\partial^2 V/\partial y^2 = O \tag{2.72}$$

It is clear from Fig. 2.20, that the constant V_d could not effect the equilibrium on the surface since the relative heights are relevant and it does not matter if the surface is raised or lowered uniformly or the datum is changed arbitrarily.

2.9.1 Total Potential Energy of a Deformable Body

As explained in the preceding section, the potential of a rigid system is a function of loads and displacements which on extremizing gives state equation which may be equilibrium equation, equation of motion or any other governing equation. The total potential energy of a deformable body comprises of two components namely:

1. Potential energy of external load or force system i. e. load potential, and
2. Potential ability of internal forces to do work.

Potential of External Forces

As has been discussed earlier in Sect. 2.5, that when the point of application of a force acting on a system moves it does work equal to the product of the force and the linear displacement of the point of application in the direction of the force. Here the words force and displacement have been used in generalized sense. This potential of loads for doing work is termed load potential or potential energy of the external load system. This quantity has been previously defined as external work, $W_\mathrm{ex} = P\Delta$.

Timoshenko has defined the potential of a system in a deformed configuration as the work done by acting forces in moving from this configuration to some reference configuration. For static problems, it is convenient to take the shape of unloaded structure as the reference configuration. Thus the potential energy of the load system is the work done by all the acting forces when the structure is moved from its deflected loaded configuration along with the loads to its unloaded position. This process is known as *backing up process*. Due to negative work done by the loads during backing up process the potential energy of external loads is negative. For example for an elastic structure carrying a number of singular loads $P_1, P_2, P_3, \ldots, P_n$, the potential energy due to loads is: $W_\mathrm{ex} = -\sum_{i=1}^{n} P_i y_i$. This expression of load potential can be interpreted as *virtual displacement work*. However, it should be noted that during loading process (forward process) the loads are gradually increased from zero to their final values and the eigenwork done is $\frac{1}{2}\sum P_i y_i$. Thus in the forward process load should have full value to start with. This is not necessary in Timoshenko's concept of potential energy.

In case of vibration problems the mean position about which the mass oscillates is taken to be the reference position.

Potential of Internal Forces

As discussed earlier in the chapter, the internal work which is stored as strain energy of the system is always positive quantity. Therefore energy of internal forces is equal to the strain energy U. The total potential energy of a deformable structure designated by Π and is defined as the difference between strain energy U i.e. elastic work of internal forces $(-W_{\mathrm{in}})$ and the potential of external forces W_{ex} $(= V)$. Therefore

$$\Pi = U - (-W_{\mathrm{ex}}) = U + V \tag{2.73}$$

Thus the total potential energy functional of a system is computed as a function of displacements and deformations.

2.9.2 Principle of Stationary Potential Energy

The potential of the external forces which is defined as the work of external forces due to the displacement of the structure can be interpreted as a virtual work if the deflection is very small. The virtual work in turn can be written as an elementary work $P\,\mathrm{d}y = \mathrm{d}W_{\mathrm{ex}}$. Similarly the virtual work of the internal forces can be interpreted as an elementary work of the internal forces, $\mathrm{d}U = -\mathrm{d}W_{\mathrm{in}}$. The elementary potential function is thus given by

$$\mathrm{d}\Pi = \mathrm{d}U - \mathrm{d}V = -\mathrm{d}W_{\mathrm{in}} - \mathrm{d}W_{\mathrm{ex}}$$

However, for equilibrium $-\mathrm{d}W_{\mathrm{in}} = \mathrm{d}W_{\mathrm{ex}}$. Thus for equilibrium $\mathrm{d}\Pi = 0$. Since the loading is kept constant, the displacements (y) and strains are the only variables; the principle can be stated in the following form

$$\frac{\partial \Pi}{\partial y} = 0 \tag{2.74}$$

The elementary potential $\mathrm{d}\Pi$ could also be expressed in the form

$$\mathrm{d}\Pi = \frac{\partial \Pi}{\partial y}\mathrm{d}y = -\frac{\partial}{\partial y}(W_{\mathrm{in}})\,\mathrm{d}y - \frac{\partial}{\partial y}(W_{\mathrm{ex}})\,\mathrm{d}y = 0$$

and since dy is very small but not zero, $\frac{\partial \Pi}{\partial y} = 0$ for equilibrium. Thus the principle of virtual displacement can be stated that *a deformable system is in equilibrium only* *if the first variation of the total potential energy of the system is zero for every virtual displacement consistent with constraints.*

The variational principle of minimum total potential energy as derived from virtual work can be stated that: for equilibrium the first derivative or more generally the first variation vanishes ($\delta\Pi = 0$). Here only the displacement field is subjected to variation. In case of discrete systems represented by generalized co-ordinates y_i, the principle reduces to a simple form $\frac{\partial\Pi}{\partial y_i} = 0$ for equilibrium. This forms a set of algebraic equations of equilibrium. The principle $\delta\Pi = 0$ is a necessary and sufficient condition for equilibrium. Also in analogy with differential calculus, this stationary point can be shown to be minimum (which implies stability) if the second variation is larger than zero. Thus for stability we must have

$$\delta^2\Pi > 0 \tag{2.75}$$

It must be noted that within linear theory of small displacements, stationary point of Π is always minimum.

2.9.3 Applications of Total Potential Energy Principles

The *Cotterill-Castigliano's* theorems can be derived from the principles of total potential energy. The principle of stationary total potential energy can be applied to the determination of deflection of structures, and buckling of struts of different boundary conditions. It is extensively used in the generation of differential equation i. e. the *Euler-Lagrange* equation of the problems. The variational principle can be used directly to solve the problems. The classical form of the direct method is the *Rayleigh-Ritz* procedure wherein a trial shape function termed Rayleigh-Ritz function satisfying at least all the geometric boundary conditions is used in computation of Π.

Deflection Problem

For illustration consider a 3-hinged bars system shown in Fig. 2.21. The bars have same cross-sectional area. For writing the expression for total potential energy of the system the first step is to write the strain energy of stretching of the members due to deflection of the point of application of the load. The axial deformations of various members are computed in terms of vertical and horizontal components of the deflection y_v and y_h of load point as shown in Figs. 2.21b and 2.21c, respectively. The deformations taking the elongation to be positive are

$$\begin{aligned}
\Delta_{10} &= (y_v \cos 30 + y_h \cos 60) = \frac{1}{2}(\sqrt{3}y_v + y_h) \\
\Delta_{20} &= y_v \\
\Delta_{30} &= (y_v \cos 60 - y_h \cos 30) = \frac{1}{2}(y_v - \sqrt{3}y_h)
\end{aligned}$$

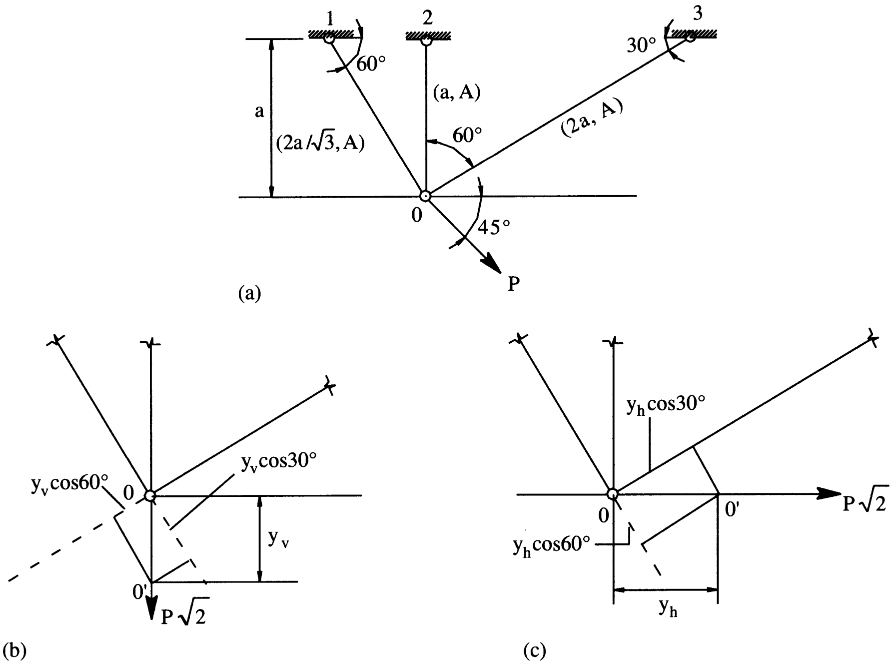


Fig. 2.21a–c. Deformation in the 3-hinged bar system. **a** 3-bar truss, **b** stretching due to y_v, **c** stretching due to y_h

The elastic strain energy U is

$$\begin{aligned} U &= \sum \frac{A_i E}{2L_i} \Delta_i^2 \\ &= \frac{AE}{2a}\left[\frac{\sqrt{3}}{2}(\sqrt{3}y_v + y_h)^2 \frac{1}{4} + (y_v)^2 + \frac{1}{2}(y_v - \sqrt{3}v_h)^2 \frac{1}{4}\right] \\ &= \frac{AE}{16a}\left[\sqrt{3}(3y_v^2 + y_h^2 + 2\sqrt{3}y_v y_h) + 8y_v^2 + (y_v^2 + 3y_h^2 - 2\sqrt{3}y_v y_h)\right] \\ &= \frac{AE}{16a}\left[(9 + 3\sqrt{3})y_v^2 + (3 + \sqrt{3})y_h^2 + (6 - 2\sqrt{3})y_v y_h\right] \end{aligned}$$

and the load potential,

$$V = -\left(\frac{P}{\sqrt{2}}y_v + \frac{P}{\sqrt{2}}y_h\right)$$

Therefore, the total potential energy of the system is

$$\begin{aligned} \Pi &= U + V \\ &= \left(\frac{AE}{16a}\right)\left[(9 + 3\sqrt{3})y_v^2 + (3 + \sqrt{3})y_h^2 + (6 - 2\sqrt{3})y_v y_u\right] - \frac{P}{\sqrt{2}}(y_v + y_h) \end{aligned}$$

Taking the variations of Π with respect to y_v and y_h

$$\delta\Pi = \frac{\partial \Pi}{\partial y_v} \cdot \delta y_v = 0 \quad \text{and} \quad \delta\Pi = \frac{\partial \Pi}{\partial y_h} \cdot \delta y_h = 0$$

On performing the above operations and noting that δy_v and δy_h are arbitrary or virtual displacements, following equations are obtained

$$\left(\frac{AE}{16a}\right)\left[2(9+3\sqrt{3})y_v + (6-2\sqrt{3})y_h\right] - \frac{P}{\sqrt{2}} = 0$$

$$\left(\frac{AE}{16a}\right)\left[2(3+\sqrt{3})y_h + (6-2\sqrt{3})y_v\right] - \frac{P}{\sqrt{2}} = 0$$

These equations can be arranged in the matrix form as

$$\left(\frac{AE}{8a}\right)\begin{bmatrix} 3(3+\sqrt{3}) & (3-\sqrt{3}) \\ (3-\sqrt{3}) & (3+\sqrt{3}) \end{bmatrix}\begin{Bmatrix} y_v \\ y_h \end{Bmatrix} = \frac{P}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

This is well known structure equilibrium equation encountered in the stiffness method. It should be noted that same result is obtained by using Cotterill-Castigliano's first theorem. On solving the above equations

$$\begin{Bmatrix} y_v \\ y_h \end{Bmatrix} = \frac{Pa}{AE}\left(\frac{\sqrt{2}}{(3+\sqrt{3})}\right)\begin{Bmatrix} 1 \\ (2+\sqrt{3}) \end{Bmatrix}$$

Buckling Problem

Consider the structure shown in Fig. 2.22 which carries an axial load P. The critical load of this so-called strut is $P_{cr} = \pi^2 EI/L^2$. To compute the value of critical load using total potential energy method, Π can be expressed as

$$\Pi = U + V = \int_0^L \frac{1}{2}EI(y'')^2 \mathrm{d}x - P\Delta$$

where the end shortening Δ can be expressed in terms of lateral deflection y by noting that

$$\Delta \cong \frac{1}{2}\int_0^L (y')^2 \mathrm{d}x$$

Thus,

$$\Pi = \int_0^L \frac{1}{2}EI(y'')^2 \mathrm{d}x - P\int_0^L \frac{1}{2}(y')^2 \mathrm{d}x$$

Assuming trial function $y = a\sin(\pi x/L)$. Substituting this value in Π, integrating and then differentiating with respect to the only coordinate a and equating to zero gives

$$P_{cr} = \pi^2 EI/L^2$$

This is the exact solution just because the trial function happens to be the exact solution of the corresponding differential equation.

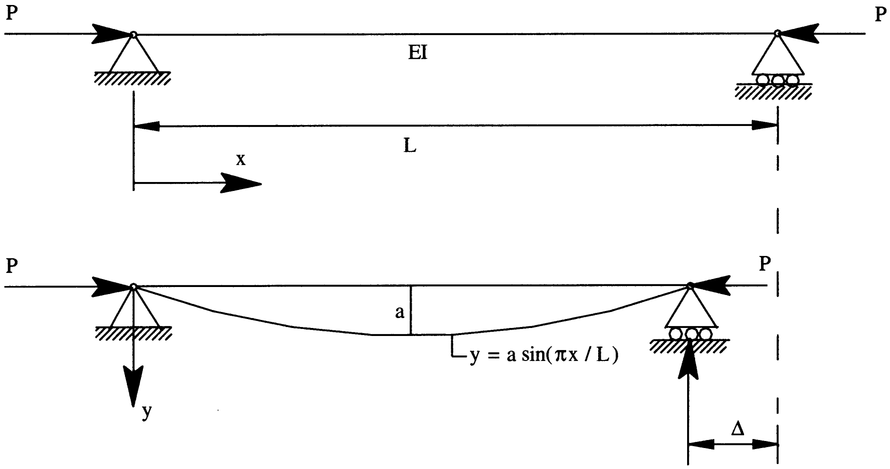


Fig. 2.22. Buckling of strut carrying axial load

2.10 Methods of Solution

Sometimes, the structural systems are not amenable to exact solution or are difficult to analyze, we take recourse to approximate solution using *stationary potential energy* procedures. As mentioned earlier, the total potential energy of a system can be expressed as the function of its joint displacements. In case the joint displacements which are also known as degrees-of-freedom are too many in number, it is possible to define true deflected shape by an approximate profile called *shape function* This may contain one or more undetermined parameters. An assumed shape function must satisfy all the *geometric or kinematic boundary conditions*. Such shape functions are called kinematically admissible shape functions. The method using assumed displacement function is also termed *trial* or *coordinate function* method. The geometrical compatibility or boundary conditions associated with assumed displacement function are the deflection (y) and slope (y'). Other boundary conditions, the so-called dynamical or force boundary conditions, associated with the bending moment and shear, and thus indirectly with y'' and y''' are optional.

2.10.1 Method of Trial Functions

The method of trial function which is coherently connected to the *minimum potential energy method* is described in details in this section. As pointed out earlier, the minimum or stationary potential energy theory is based on the *principle of virtual displacements* and has all the advantages and limitations of that principle. It replaces the equations of equilibrium but it does not guarantee geometrical compatibility of assumed deflected shape with prescribed geometrical conditions of the system. This is taken care of by using kinematically admissible functions.

The total potential energy associated with flexural action can be defined as

$$\Pi = W_{\text{in}} - W_{\text{ex}} = \frac{1}{2}\int_0^L EI(y'')^2 \mathrm{d}x - \frac{1}{2}P\int_0^L (y')^2\, dx \tag{2.76}$$

Here I, the moment of inertia of the member section has been assumed to be variable. If the trial function is assumed to be represented by

$$y \cong \tilde{y} = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 + \cdots + a_n\varphi_n \tag{2.77}$$

where $\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n$ are admissible functions of x, and a_i are the free coefficients representing the degrees-of-freedom or generalized co-ordinates which enable the function to take the best shape to extremize the potential energy. Substitution of (2.77) into (2.76) yields

$$\Pi = F_1(a_1, a_2, a_3, \ldots, a_n) + P.F_2(a_1, a_2, a_3, \ldots, a_n) \tag{2.78}$$

The functions F_1 and F_2 are quadratic functions of independent arbitrary free coefficients a_i. This means that principle of the minimum total potential energy is satisfied by minimizing a function rather than the integral of a function. This offers considerable simplification, and becomes nothing more than an ordinary maximum minimum problem with respect to independent variables $a_1, a_2, a_3, \ldots, a_n$, that is

$$\frac{\partial \Pi}{\partial a_i} = \frac{\partial}{\partial a_i}(W_{\text{in}} - W_{\text{ex}}) = 0 \quad \text{where } i = 1, 2, 3, \ldots, n \tag{2.79}$$

Since the first derivatives of a quadratic function are linear functions, (2.79), will represent a set of linear, homogeneous equations in terms of unknown independent variables $a_1, a_2, a_3, \ldots, a_n$ as follows.

$$\begin{aligned} \varphi_{11}a_1 + \varphi_{12}a_2 + \varphi_{13}a_3 + \cdots \varphi_{1n}a_n &= 0 \\ \varphi_{21}a_1 + \varphi_{22}a_2 + \varphi_{23}a_3 + \cdots \varphi_{2n}a_n &= 0 \\ &\vdots \\ \varphi_{n1}a_1 + \varphi_{n2}a_2 + \varphi_{n3}a_3 + \cdots \varphi_{nn}a_n &= 0 \end{aligned} \tag{2.80}$$

The terms φ's contain P, EI and L as well as numbers. For a non-trivial solutions, the determinant of coefficient must vanish, i. e.,

$$\begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi_{n1} & \varphi_{n2} & \varphi_{n3} & \cdots & \varphi_{nn} \end{vmatrix} = 0 \tag{2.81}$$

The expansion of determinant results in an n^{th} order polynomial equation which is solved for the smallest value. This method of solution is generally known as Rayleigh-Ritz or Ritz method.

Determination of Trial Function

A sound judgment, experience and a feel of physical behaviour of the structure facilitates determination of a trial function, which may be a polynomial expansion or a Fourier expansion displacement function.

(a) Polynomial trial-function

To find polynomial trial function the following procedure may be adopted.

1. Assume a polynomial trial function of an order one higher than the number of geometric boundary conditions that must be satisfied, e. g., if m is the number of boundary conditions then a polynomial of the order n $(= m + 1)$ need be assumed.
2. Using m boundary conditions, n constants can be expressed in terms of the $(n - m)$ constants.

For illustration consider the case of fixed-simply supported strut shown in Fig. 2.23a. The geometrical boundary conditions of structure

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(L) = 0 \tag{2.82}$$

thus there are three boundary conditions, therefore a polynomial chosen to approximate the buckled form should have four constants at least. The polynomial is thus

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{2.83}$$

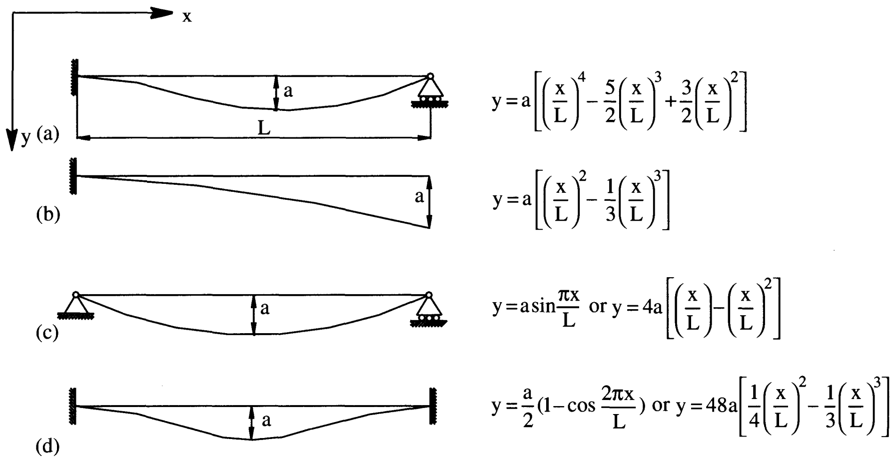


Fig. 2.23a–d. Deflected shapes of beams with different support conditions and corresponding trial functions

The conditions $y(o) = 0$, and $y'(o) = 0$ give $a_0 = a_1 = 0$. Finally, the condition $y(L) = 0$ gives $a_2 = -a_3 L$. The one-degree-of-freedom trial function obtained is

$$y = -a_3 L x^2 + a_3 x^3 = a_3(x^3 - Lx^2) \tag{2.84}$$

Setting for convenience, $a_3 = a$, the trial function can be written as

$$y = aL^3\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right] \tag{2.85}$$

If the dynamic boundary condition that the moment at the hinged end is zero, i.e., $M(L) = EIy''(L) = 0$ or $y''(L) = 0$ is also to be satisfied, there are four boundary conditions and the polynomial must therefore have five constants

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{2.86}$$

satisfying the four boundary conditions,

$$\begin{aligned} y(o) = y'(o) = 0 \quad & \text{give } a_0 = a_1 = 0 \\ y(L) = 0: \quad & a_2 L^2 + a_3 L^3 + a_4 L^4 = 0\,, \\ y''(L) = 0: \quad & 2a_2 + 6a_3 L + 12 a_4 L^2 = 0 \end{aligned}$$

Setting for convenience $a_4 L^4 = a$ the solution of two equation gives

$$a_2 = (3/2)(a/L^2) \quad \text{and} \quad a_3 = (-5/2)(a/L^3)$$

The one-degree-of-freedom trial function obtained is thus

$$y = a\left[\left(\frac{x}{L}\right)^4 - \frac{5}{2}\left(\frac{x}{L}\right)^3 + \frac{3}{2}\left(\frac{x}{L}\right)^2\right] \tag{2.87}$$

For a fixed-free strut shown in Fig. 2.23b, the deflected configuration can be represented by the polynomial

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{2.88}$$

The geometric boundary conditions $y(o) = y'(o) = 0$ give $a_0 = a_1 = 0$, and the deflection trial function reduces to a two-degree-of-freedom function.

$$y = a_2 x^2 + a_3 x^3$$

If the dynamic boundary (moment) condition $y''(L) = 0$ is also satisfied the trial function reduces to a one degree-of-freedom function

$$y = a\left[\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^3\right] \quad (a_2 L^2 = a) \tag{2.89}$$

For the fixed-fixed strut shown in Fig. 2.23d due to symmetry the geometric boundary conditions are: $y(0) = 0$, $y'(0) = 0$ and $y'(L/2) = 0$. The trial function obtained holds good for both the half portions with origin taken at the ends.

$$y = 48a\left[\frac{1}{4}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^3\right] \tag{2.90}$$

Similarly for a pin-ended (or simply supported) symmetrical strut the one-degree-of-freedom trial function is assumed to be $y = a_0 + a_1x + a_2x^2$. From boundary conditions $a_0 = 0$ and $a_1 + a_2L = 0$, the equation reduces to

$$y = 4a\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \tag{2.91}$$

However, it should be noted that the dynamic boundary conditions at the two ends are not satisfied.

Suitable trial functions can also be obtained by using Fourier expansion instead of polynomial expansion of exact function of y.

(b) Fourier trial-functions

The deflected configuration can be approximated by sine or cosine series. For example the one-degree-of-freedom trial function for a simply supported strut may be assumed to be

$$y = a\sin\left(\frac{\pi x}{L}\right) \tag{2.92}$$

which satisfies both the geometric and dynamic boundary conditions. Therefore this trial function is superior to the polynomial function given by (2.91). The trial function for a fixed-fixed strut may be assumed as

$$y = \frac{a}{2}\left(1 - \cos\frac{2\pi x}{L}\right) \tag{2.93}$$

For a fixed-free strut one degree-of-freedom trial function with co-ordinate system passing through the free end of the deflected strut can be assumed as:

$$y = a\cos\left(\frac{\pi x}{2L}\right) \tag{2.94}$$

When solving buckling problems by Rayleigh-Ritz method, it is useful to consider the trial function in the following general form:

$$y(x) = \sum_{n=0}^{\infty} a_n f(x) x^n \tag{2.95}$$

where $f(x)$ is a specifically chosen function which satisfies the geometric boundary conditions and $\sum a_n x^n$ is a power series

End conditions	*Function, $f(x)$*	
Fixed-free strut	$f(x) = x^2$	
Hinged-hinged strut	$f(x) = x(x - L)$	
Fixed-hinged strut	$f(x) = x^2(x - L)$	
Fixed-fixed strut	$f(x) = x^2(x - L)^2$	(2.96)

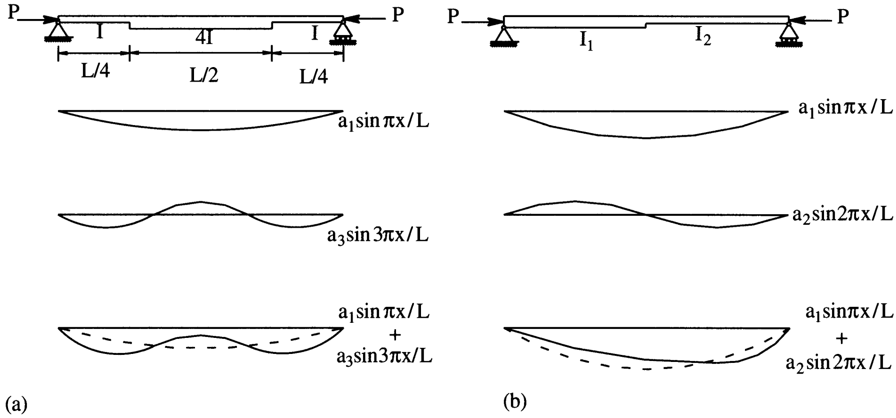


Fig. 2.24a,b. Trial functions for simply supported stepped beams. **a** Beam stepped at the centre, **b** beam stepped at the end

It must be noted that the polynomial trial functions given by (2.87), (2.89), (2.90) and (2.91) are simplest possible forms which satisfy (2.95).

Consider the case of a simply supported strut with variable moment of inertia as shown in Fig. 2.24. Because of discontinuity in the moment of inertia the potential energy must be written in the form

$$\Pi = \text{strain energy} - \text{load potential} = U - V$$

$$= \int_0^{L/4} \frac{EI}{2}(y'')^2 \mathrm{d}x + \int_{L/4}^{3L/4} \frac{4EI}{2}(y'')^2 \mathrm{d}x + \int_{3L/4}^{L} \frac{EI}{2}(y'')^2 \mathrm{d}x - \int_0^L \frac{P}{2}(y')^2 \mathrm{d}x$$

The trial function suitable for a simply supported symmetrical strut is given by $y = a_1 \sin(\pi x/L)$ which satisfies all the boundary condition of the problem. However, due to higher moment of inertia at the central portion the deflected curve is flatter than that for a uniform moment of inertia case, therefore another term $a_3 \sin(3\pi x/L)$ with three half sine waves satisfying all the geometric boundary conditions may be added to the above trial function. This inclusion of additional term will make the deflected shape flatter in the central portion. Thus a two-degree-of-freedom trial function recommended for this case is

$$y = a_1 \sin(\pi x/L) + a_3 \sin(3\pi x/L)$$

Substituting y, $(y')^2$ and $(y'')^2$ in the strain energy and load potential expressions

$$V = \frac{P}{2}\frac{\pi^2}{L^2}\int_0^L \left[a_1 \cos\left(\frac{\pi x}{L}\right) + 3a_3 \cos\left(\frac{3\pi x}{L}\right)\right]^2 \mathrm{d}x = \frac{P\pi^2}{4L}(a_1^2 + 9a_3^2)$$

$$U = \frac{EI\pi^4}{L^3}(0.864a_1^2 - 4.320a_1a_3 + 44.064a_3^2)$$

The equilibrium conditions are given by: $\partial \Pi / \partial a_1 = 0$ and $\partial \Pi / \partial a_3 = 0$

$$\left(1.728 - \frac{PL^2}{2\pi^2 EI}\right) a_1 - 4.320 a_3 = 0 \quad \text{and}$$

$$-4.320 a_1 + \left(88.128 - \frac{9PL^2}{2\pi^2 EI}\right) a_3 = 0$$

For non-trivial solution of these linear homogeneous equations, the determinant of coefficients should vanish i. e.

$$\begin{vmatrix} (1.728 - \alpha) & -4.320 \\ -4.320 & (88.128 - 9\alpha) \end{vmatrix} = 0$$

where

$$\alpha = \frac{PL^2}{2\pi^2 EI} \quad \text{or} \quad \alpha^2 - 11.520\alpha + 14.847 = 0$$

The smallest root $\alpha = 1.4786$ gives critical load $P_{cr} = 2.957\pi^2 EI/L^2$ which is about 13.7 per cent larger than the exact value of $2.600\pi^2 EI/L^2$.

If only the first term of the trial-function is adopted i. e. $a_3 = 0$, the total potential energy expression reduces to

$$\Pi = \frac{EI\pi^4}{L^3}(0.864 a_1^2) - \frac{P\pi^2}{4L}(a_1^2)$$

The equilibrium condition $\partial \Pi / \partial a_1 = 0$ gives

$$\left(\frac{1.728 EI\pi^4}{L^3} - \frac{P\pi^2}{2L}\right) a_1 = 0$$

For non-trivial solution

$$\left(\frac{1.728 EI\pi^4}{L^3} - \frac{P\pi^2}{2L}\right) = 0.$$

Thus,

$$P_{cr} = 3.456\pi^2 EI/L^2$$

This is far less accurate having an error of 32.9 per cent compared to the exact value. For asymmetrically stepped beam of Fig. 2.24b, a two-degree-of-freedom trial function shown in the figure will give better results.

Different Versions of the Rayleigh-Ritz Method

As explained earlier for stationary potential $\partial\Pi/\partial a = 0$ and the method is referred to as *equilibrium method.* However, for Π to be maximum representing unstable equilibrium $\partial^2\Pi/\partial a^2 < 0$, and to be minimum representing stable equilibrium $\partial^2\Pi/\partial a^2 > 0$, the critical situation must be given by $\partial^2\Pi/\partial a^2 = 0$ which gives the critical or buckling load and method is referred to as *stability method.* On the other hand the work-equation interpreted as equating the work of internal forces with that of external forces gives $\Pi = 0$. From (2.76)

$$P = \frac{\left[EI\int_0^L (y'')^2 \mathrm{d}x\right]}{\left[\int_0^L (y')^2 \mathrm{d}x\right]} \tag{2.97}$$

where y is the trial function. This expression is known as *Rayleigh quotient.* Another version of Rayleigh quotient due to Timoshenko is obtained by substituting the relationships

$$EI(y'')^2 = M^2/EI = P^2y^2/EI \tag{2.98}$$

in the Rayleigh quotient as

$$P = \frac{\int_0^L (P^2y^2/EI)\,\mathrm{d}x}{\int_0^L (y')^2 \mathrm{d}x} \quad \text{or} \quad P = \frac{\left[\int_0^L EI(y')^2 \mathrm{d}x\right]}{\left[\int_0^L y^2 \mathrm{d}x\right]} \tag{2.99}$$

It should be noted that the order of differentiation in this quotient is lower than that in the Rayleigh quotient. Since the error in using an approximate expression is increased in the process of differentiation, the Timoshenko quotient yields more accurate results compared with that of Rayeigh quotient. The Rayleigh-Ritz method could also be used in conjunction with the principle of minimum complementary energy $\partial\Pi_c = 0$. The complementary energy function of the problem is

$$\Pi_c = \int_0^L \left[\frac{M^2}{2EI} - \frac{P(y')^2}{2}\right] \mathrm{d}x = \int_0^L \left[\frac{M^2}{2EI} - \frac{(M')^2}{2P}\right] \mathrm{d}x \tag{2.100}$$

(since $Py = M$ and hence $y' = M'/P$ as P will remain constant and only M is subjected to variations). In terms of complementary work principle setting $\Pi_c = 0$, a quotient termed as complementary energy quotient is obtained

$$P = \frac{\int EI(M')^2 \mathrm{d}x}{\int M^2 \mathrm{d}x} \tag{2.101}$$

On replacing M by Py the quotient reduces to Timoshenko quotient. Observing the relationship $M = EIy''$ from which we obtain

$$y' = \int \frac{M}{EI} \mathrm{d}x + c \tag{2.102}$$

where c is an integration constant determined by geometric boundary conditions. Substituting this in the expression of Π_c given in (2.100).

$$\Pi_c = \int_0^L \left[\frac{M^2}{2EI} - \frac{P}{2} \left(\frac{M}{EI} \mathrm{d}x + c \right)^2 \right] \mathrm{d}x \tag{2.103}$$

Setting $\Pi_c = 0$, a new significant version of complementary energy quotient is obtained

$$P = \left[\int_0^L (M^2/EI) \,\mathrm{d}x \right] \Bigg/ \left\{ \int_0^L [(M/EI) \,\mathrm{d}x + c]^2 \mathrm{d}x \right\} \tag{2.104}$$

Since this quotient involves no differentiation at all, it yields more accurate results than that obtained by using any other quotient. Trial functions satisfying moment boundary conditions can lead to better approximate solutions. Sometime these trial functions are more convenient to use than *displacement trial functions.*

The fourth-order governing differential equation can also be obtained from the total potential energy functional which is generally a function of y'', y', y and x, i.e. $\Pi = \int_O^L \tilde{\Pi}(y'', y', y, x)\mathrm{d}x$. As explained earlier, the functional Π is stationary if $\delta\Pi = 0$ which is necessary and sufficient condition for equilibrium. For the stationary point to be minimum the second variation should be greater than zero i.e. $\delta^2\Pi \geq 0$. However, within the linear theory of small displacement, the stationary point is always minimum.

$$\begin{aligned} \delta\Pi &= \delta \int_0^L \tilde{\Pi}(y'', y', y, x) \,\mathrm{d}x = \int \delta\tilde{\Pi}\mathrm{d}x \\ &= \int_0^L \left(\frac{\partial\tilde{\Pi}}{\partial y''}\delta y'' + \frac{\partial\tilde{\Pi}}{\partial y'}\delta y' + \frac{\partial\tilde{\Pi}}{\partial y}\delta y \right) \mathrm{d}x \end{aligned} \tag{2.105}$$

Rewriting the terms containing $\delta y''$ and $\delta y'$ into the forms containing only δy using the integration by parts twice for the first and once for second term

$$\int_0^L \left(\frac{\partial\tilde{\Pi}}{\partial y''} \right) \delta y'' = \left[\left(\frac{\partial\tilde{\Pi}}{\partial y''} \right) \delta y' - \left(\frac{\partial\tilde{\Pi}}{\partial y''} \right)' \delta y \right]_0^L + \int_0^L \left(\frac{\partial\tilde{\Pi}}{\partial y''} \right)'' \delta y$$

$$\int_0^L \left(\frac{\partial\tilde{\Pi}}{\partial y'} \right) \delta y' = \left[\left(\frac{\partial\tilde{\Pi}}{\partial y'} \right) \delta y \right]_0^L - \int_0^L \left(\frac{\partial\tilde{\Pi}}{\partial y'} \right)' \delta y$$

Thus the variation of functional $\delta\Pi$ can be expressed as

$$\delta\Pi = \int_0^L \left[\left(\frac{\partial\tilde{\Pi}}{\partial y''}\right)'' - \left(\frac{\partial\tilde{\Pi}}{\partial y'}\right)' + \left(\frac{\partial\tilde{\Pi}}{\partial y}\right) \right] \delta y \, \mathrm{d}x$$
$$+ \left[\left(\frac{\partial\tilde{\Pi}}{\partial y''}\right) \delta y' - \left(\frac{\partial\tilde{\Pi}}{\partial y''}\right)' \delta y \right]_0^L + \left[\left(\frac{\partial\tilde{\Pi}}{\partial y'}\right) \delta y \right]_0^L \tag{2.106}$$

The first terms is the governing equation of the problem and the last two terms are boundary conditions. For $\delta\Pi$ to vanish the governing expressions and expressions for boundary conditions must vanish. Thus the governing differential equation is given by

$$\int_0^L \left[\left(\frac{\partial\tilde{\Pi}}{\partial y''}\right)'' - \left(\frac{\partial\tilde{\Pi}}{\partial y'}\right)' + \left(\frac{\partial\tilde{\Pi}}{\partial y}\right) \right] \delta y \, \mathrm{d}x = 0 \tag{2.107}$$

Since δy is arbitrarily small but non-zero, the differential equation reduces to

$$\left(\frac{\partial\tilde{\Pi}}{\partial y''}\right)'' - \left(\frac{\partial\tilde{\Pi}}{\partial y'}\right)' + \left(\frac{\partial\tilde{\Pi}}{\partial y}\right) = 0 \tag{2.108}$$

This is also called Euler-Lagrange equation. The general form of this equation is

$$(-1)^n \sum \frac{\partial^{(n)}}{\partial x^n} \left(\frac{\partial\tilde{\Pi}}{\partial y^{(n)}}\right) = 0 \tag{2.109}$$

The total potential energy functional of a flexural system is given by

$$\Pi = \int_0^L \left[\frac{EI}{2}(y'')^2 - \frac{P}{2}(y')^2 \right] \mathrm{d}x \tag{2.110}$$

Applying the criterion $\delta\Pi = 0$ through (2.107):

$$\int_0^L \left[\frac{EI}{2}(-1)^2 \frac{\partial^2}{\partial x^2} \left(\frac{\partial y''^2}{\partial y''}\right) - \frac{P}{2}(-1)\frac{\partial}{\partial x}\left(\frac{\partial y'^2}{\partial y'}\right) \right] dx = 0 \tag{2.111}$$

i.e.

$$\int_0^L (EIy'''' + Py'')\, \mathrm{d}x = 0$$

Thus fourth-order governing differential equation can be obtained by *Langrangian multiplier method*, and by equilibrium method. The most important application is in establishing an approximate solution. This is achieved by assuming an expression for the elastic curve of deflection

$$y = a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots$$

where y_i are admissible functions which can be admitted in making Π stationary and make the boundary terms vanish as discussed earlier. These terms are called geometrical boundary conditions. The independent coefficients a_i are then determined in such a way that potential functional Π is rendered stationary. The substitution of y in the functional Π reduces it into simple quadratic function with independent variables, so called generalized coordinates a_i. For equilibrium we must have

$$\frac{\partial \Pi}{\partial a_1} = \frac{\partial \Pi}{\partial a_2} = \frac{\partial \Pi}{\partial a_3} = \cdots = \frac{\partial \Pi}{\partial a_i} = 0 \tag{2.112}$$

The equations $\partial \Pi / \partial a_i$ are a linear system of algebraic equations in a. This method is called *Rayleigh-Ritz* method, which is discussed later in this chapter. A different version of this method based on above differential equation called Galerkin method will be described subsequently. As an application of the method of analysis using function in conjunction with *Hellinger-Reissner* functional Π_R. Consider fixed-fixed elastic strut. To account for the symmetry of the problem consider the region $0 \leq \Pi \leq L/2$ and functional reduces to

$$\Pi_R = 2\int_0^{L/2} \left[\left(\frac{M^2}{2EI} - My''\right) + \frac{P}{2}(y')^2\right] \mathrm{d}x$$

$$= \int_0^{L/2} \frac{M^2}{EI}\mathrm{d}x - 2\int_0^{L/2} My''\mathrm{d}x + P\int (y')^2 \mathrm{d}x$$

Consider two different trial functions for M and y as follows

$$M = a_1 + a_2 x \quad \text{and} \quad y = a_3\left(x^2 - \frac{L^2}{4}\right)^2$$

where a_1, a_2 and a_3 are independent coefficients. The derivatives of y are

$$y' = 4a_3 x\left(x^2 - \frac{L^2}{4}\right) \quad \text{and} \quad y'' = 4a_3\left(3x^2 - \frac{L^2}{4}\right)$$

Substitution in various terms of the functional and integration gives

$$\int_0^{L/2} \frac{M^2}{EI}\mathrm{d}x = \frac{1}{24EI}[12a_1^2 L + 6a_1 a_2 L^2 + a_2^2 L^3]$$

$$2\int_0^{L/2} My''\mathrm{d}x = \frac{1}{8}(a_2 a_3 L^4)$$

$$P\int (y')^2 \mathrm{d}x = \frac{P}{105}(a_3^2 L^7)$$

The function is thus reduced to a quadratic function of a_1, a_2 and a_3

$$\Pi_R = \frac{1}{24EI}(12a_1^2 L + 6a_1 a_2 L^2 + a_2^2 L^3) - \frac{1}{8}(a_2 a_3 L^4) + \frac{P}{105}(a_3^2 L^7)$$

The application of extremizing conditions gives

$$\frac{\delta \Pi_R}{\delta a_1} = \frac{1}{24EI}(24La_1 + 6L^2 a_2) = 0 \qquad \text{or} \quad 4a_1 + La_2 = 0$$

$$\frac{\delta \Pi_R}{\delta a_2} = \frac{1}{24EI}(6L^2 a_1 + 2L^3 a_2) - \frac{L^4}{8}a_3 = 0 \quad \text{or} \quad 6a_1 + 2La_2 - 3EIL^2 a_3 = 0$$

$$\frac{\delta \Pi_R}{\delta a_3} = -\frac{L^4}{8}a_2 + \frac{2PL^7}{105}a_3 = 0 \qquad \text{or} \quad -105a_2 + 16PL^3 a_3 = 0$$

These equations have a non-trivial buckled form solution, if and only if, the determinant of coefficients a_1, a_2 and a_3 i. e. stability determinant vanishes.

$$\begin{vmatrix} 4 & L & 0 \\ 6 & 2L & -3EIL^2 \\ 0 & -105 & 16PL^2 \end{vmatrix} = 32\,PL^4 - 1260\,EIL^2 = 0$$

The critical load is

$$P_{\text{cr}} = 39.375\,EI/L^2$$

The exact value is 39.48 EI/L^2. It should be noted that Reissner's principle is a very powerful method for obtaining accurate approximate solution. However it should be noted that in contrast to Raleigh's principle Hellinger-Reissner principle gives lower bound solution i. e.

$$P_{\text{cr}} \le P_{\text{cr,exact}}$$

2.10.2 Galerkin Method

This method is also based on the assumption of trial functions and gives identical solution for a conservative system as given by Ritz method, if same trial function is used. However the trial function in case of Galerkin method must satisfy both the kinematic (related to the geometry of the system) and the dynamical (i. e. related to the moment and shear force) boundary conditions. The solution procedure consists in formulating the Galerkin equations of problem. This can be done by writing the governing differential equation $EIy'''' + Py'' = 0$ in terms of the trial function, multiplied by its variation and then integrating over the domain of the independent variable. The equilibrium equations are then found from extremizing the galerkin equations with respect to the coefficients of the trial functions. Next step is to find the stability determinant and the eigenvalue equation of the buckling load. As an illustration consider the case of a fixed-hinged strut using a single degree of freedom trial function satisfying both the geometrical and dynamical boundary condition as

$$y = a_1(3L^2x^2 - 5Lx^3 + 2x^4) = a_1 y_1$$

The Galerkin equation is obtained by replacing y by y_1 in the governing differential equation and multiplying the resulting equation by y_1 and integrating. Thus

$$G = \int_0^L a_1(EIy_1'''' + Py_1'')y_1 \, dx = 0 \tag{2.113}$$

The various derivatives of trial function are

$$y_1'' = 6L^2 - 30Lx + 24x^2 \quad \text{and} \quad y_1'''' = 48$$

The integration of various terms of Galerkin equation is

$$EI \int_0^L 48(3L^2x^2 - 5Lx^3 + 2x^4) \, dx = \left(\frac{36EI}{5}\right) L^5$$

$$P \int_0^L (6L^2 - 30Lx + 24x^2)(3L^2x^2 - 5Lx^3 + 2x^4) \, dx = -\left(\frac{12P}{35}\right) L^7$$

Thus the Galerkin equation reduce to

$$a_1 \left(\frac{36EI}{5} - \frac{12PL^2}{35}\right) L^5 = 0$$

From which critical load is given by $P_{cr} = 21\,EI/L^2$. The exact solution is $20.19\,EI/L^2$. The error is only 4.0 per cent. Same procedure can be followed while using higher degree of freedom trial function. For example for the trial function having two-degree-of-freedom i.e. $a_1y_1 + a_2y_2$, the Galerkin equations are:

$$G_1 = \int_0^L \left[EI(a_1y_1'''' + a_2y_2'''') + P(a_1y_1'' + a_2y_2'')\right] y_1 \, dx = 0$$

$$G_2 = \int_0^L \left[EI(a_1y_1'''' + a_2y_2'''') + P(a_1y_1'' + a_2y_2'')\right] y_2 \, dx = 0$$

These are two linear algebraic homogeneous equations. The most of the solution then follows the standard stability investigation procedure which consists in finding stability determinant and eigenvalue equation of the buckling loads. The smallest root will correspond to the smallest critical load. The Galerkin method is also applicable to non-conservative system where no potential energy in classical since exists. It should be noted that the terms in the Galerkin equations have dimensions of work. Thus the differential equation is force equilibrium condition for an assumed displacement trial function and moment equilibrium condition in case of an assumed moment or rotational function.

2.10.3 Finite Difference Method

As discussed earlier the critical equilibrium of flexural or bending elements is expressed in terms of second-order or fourth-order linear differential equations. In some cases it is difficult if not impossible to perform formal integration to define deflected shape of the member. This is especially true when the cross-section of the member varies along its length. In such case a procedure called *finite difference or collocation method* can be used to convert the governing differential equations into a set of linear simultaneous algebraic equations i.e. the differential equations are replaced by appropriate difference equations. The method consists in application of governing differential equation in finite difference form at pre-selected locations along the member. This method presumes that within a given interval the function representing deflected shape can be expressed by a polynomial of order n. In this section an extremely short account of finite difference method is given to enable the reader to solve simple problems.

Consider the deflection curve modelled by a function $y = f(x)$ divided into a set of n equal divisions, with ordinate at a point x_i denoted by y_i as shown in Fig. 2.25 being represented by a polynomial of order n, i.e.,

$$y = f(x) \cong a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \tag{2.114}$$

A second-order approximation passing through three points, for example, would be

$$y = a_2 x^2 + a_1 x + a_0$$

Using local coordinates system shown in the Fig. 2.25 and assuming expansion about the reference point i, the coefficients of the polynomial are

$$\begin{aligned} a_0 &= y_i \\ a_1 &= \frac{1}{2h}(-y_{i-1} + y_{i+1}) \\ a_2 &= \frac{1}{2h^2}(y_{i-1} - 2y_i + y_{i+1}) \end{aligned}$$

An approximation to first and second derivatives of y with respect to x are

$$\frac{dy}{dx} = y' \cong 2a_2 x + a_1$$

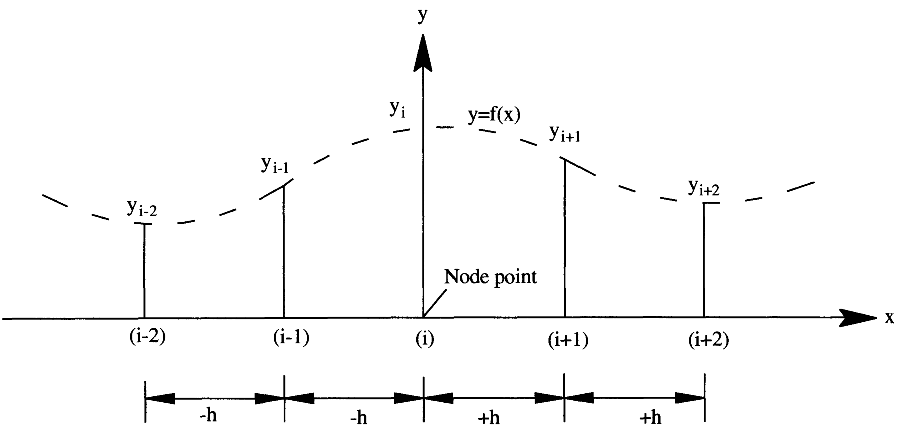


Fig. 2.25. Deflection curve divided into a set of equal increments

and

$$\frac{d^2y}{dx^2} = y'' \cong 2a_2$$

At the node point i, the values of derivatives are

$$y_i' \cong a_1 = \frac{1}{2h}(-y_{i-1} + y_{i+1})$$

and

$$y_i'' \cong 2a_2 = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \tag{2.115}$$

These expressions are referred to as first difference and second difference, respectively. This type of finite difference expression that uses the point $i-1$ and $i+1$ is called central finite difference. The other finite difference expressions using points i, $i+1, i+2$; and $i, i-1, i-2$ are known as *forward finite difference* and *backward finite difference,* respectively. It should be noted that these finite difference expressions are inadequate for the fourth-order differential equation. For the fourth-order case it is necessary to presume a fourth-order polynomial at the outset

$$y \cong a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{2.116}$$

requiring five reference points for its evaluation. Assuming these to be symmetrically placed about the central node point i, the resulting difference expressions for various derivatives can be obtained as

$$\begin{aligned}
y &= y_i \\
y_i' &= \left(\frac{dy}{dx}\right)_i \cong \frac{1}{12h}(y_{i-2} - 8y_{i-1} + 8y_{i+1} - y_{i+2}) \\
y_i'' &= \left(\frac{d^2y}{dx^2}\right)_i \cong \frac{1}{12h^2}(-y_{i-2} + 16y_{i-1} - 30y_i + 16y_{i+1} - y_{i+2}) \\
y_i''' &= \left(\frac{d^3y}{dx^3}\right)_i \cong \frac{1}{12h^3}(-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}) \\
y_i'''' &= \left(\frac{d^4y}{dx^4}\right)_i \cong \frac{1}{h^4}(y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2})
\end{aligned} \tag{2.117}$$

The finite difference expressions for the fourth-order differential equation can also be obtained from the first and second difference expressions of second-order differential equation.

$$\begin{aligned}
y_i''' &= (y_i'')' = \frac{1}{h^2}(y_{i-1}' - 2y_i' + y_{i+1}') \\
&= \frac{1}{h^2}\left[\frac{1}{2h}(-y_{i-2} + y_i) - 2 \cdot \frac{1}{2h}(-y_{i-1} + y_{i+1}) + \frac{1}{2h}(-y_i + y_{i+2})\right] \\
&= \frac{1}{2h^3}[-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}]
\end{aligned}$$

Similarly,

$$
\begin{aligned}
y_i'''' &= (y_i'')'' = \frac{1}{h^2}(y_{i-1}'' - 2y_i'' + y_{i+1}'') \\
&= \frac{1}{h^2}\left[\frac{1}{h^2}(y_{i-2} - 2y_{i-1} + y_i) - 2\cdot\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \right. \\
&\quad \left. + \frac{1}{h^2}(y_i - 2y_{i+1} + y_{i+2})\right] \\
&= \frac{1}{h^4}(y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2})
\end{aligned}
$$

However, it should be noted that depending on the problem, the boundary conditions also may need to be expressed in their expanded fourth-order form.

To demonstrate the methods consider the simply supported strut of uniform cross section as shown in the Fig. 2.26a. The governing differential equation for a simply supported strut is

$$y'' + (P/EI)y = 0 \quad \text{for the range } 0 \le x \le L$$

The displacement boundary conditions are: at $x = 0$, $y = 0$ and $x = L$, $y = 0$. The differences equation corresponding to this differential equation is obtained by substituting the value of derivatives. Figure 2.26a illustrates an example where the domain is subdivided into three equal increments of size $h = L/3$ (i.e. four nodes

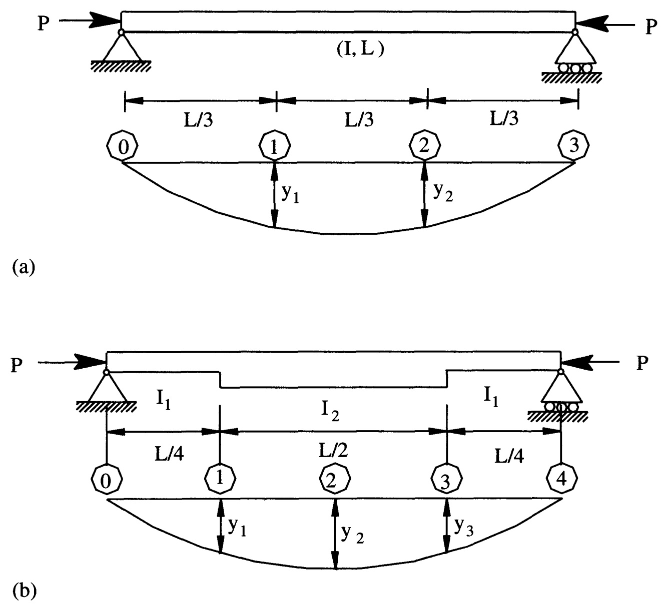


Fig. 2.26a,b. Nodes sub-dividing the deflection curves into equal increments. **a** Uniform simply supported strut, **b** stepped simply supported strut

including the ends) are used. The unknown values of y at each of the interior node points are y_1 and y_2 as indicated. For a node i the differential equation becomes.

$$y_i'' + \left(\frac{P}{EI}\right) y_i = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \left(\frac{P}{EI}\right) y_i = 0$$

or

$$y_{i-1} - 2y_i + y_{i+1} + \alpha^2 y_i = 0$$

where $\alpha^2 = (Ph^2/EI)$. The application of this difference equation to each of the interior nodes 1 and 2 furnish the following linear simultaneous equations in terms of unknown displacement values y_1 and y_2

$$\text{at node 1:} \qquad (0 - 2y_1 + y_2) + \alpha^2 y_1 = 0$$
$$\text{at node 2:} \qquad (y_1 - 2y_2 + 0) + \alpha^2 y_2 = 0$$

It should be noted that due to symmetry $y_2 = y_1$ and only one equation need to be considered. For a non-trivial solution

$$\alpha^2 \left(= \frac{Ph^2}{EI}\right) = 1.0 \quad \text{or} \quad P_{cr} = \frac{EI}{h^2} = \frac{9EI}{L^2}$$

If the simply supported strut of the above problem is subdivided into four and five equal increments, the values of the critical load P_{cr} will be $9.38\,EI/L^2$ and $9.55\,EI/L^2$, respectively. Thus it would be seen that greater accuracy is achieved by assuming more interior node points.

To illustrate the application of the method to the cases where cross-section of strut is variable, consider the simply supported stepped strut shown in Fig. 2.26b. In such cases at the nodes 1 and 3 where the cross-section changes, an average value of M/EI can be adopted as an approximation. Average value of M/EI at nodes 1 and 3 is:

$$\frac{M}{EI} = \frac{M}{E}\left[\frac{1}{I_1} + \frac{1}{I_2}\right]\frac{1}{2} = \frac{M}{EI_a} \quad \text{where} \quad I_a = \left(\frac{2I_1 I_2}{I_1 + I_2}\right)$$

Application of difference equation to the interior nodes

$$\text{at node 1:} \qquad (0 - 2y_1 + y_2) + (\alpha_a)^2 y_1 = 0$$
$$\text{at node 2:} \qquad (y_1 - 2y_2 + y_3) + (\alpha_2)^2 y_2 = 0$$

or

$$(2y_1 - 2y_2) + (\alpha_2)^2 y_2 = 0 \quad \text{since} \quad y_3 = y_1$$

Rearrangement of these equations gives

$$(\alpha_a^2 - 2)y_1 + y_2 = 0$$
$$2y_1 + (\alpha_2^2 - 2)y_2 = 0$$

For a nontrivial solution, the determinant of the system of equations must vanish i. e.

$$\begin{vmatrix} (\alpha_a^2 - 2) & 1 \\ 2 & (\alpha_2^2 - 2) \end{vmatrix} = 4 - 2(\alpha_a^2 + \alpha_2^2) + (\alpha_2^2)(\alpha_a^2) - 2 = 0$$

As a typical case consider $I_1 = I$ and $I_2 = 4I$ i. e.

$$I_a = \frac{2I_1 I_2}{I_1 + I_2} = \frac{2(I)(4I)}{I + 4I} = \frac{8I}{5}$$

Therefore,

$$\alpha_a^2 = \frac{Ph^2}{EI_a} = \frac{5Ph^2}{8EI}, \quad \alpha_2^2 = \frac{Ph^2}{EI_2} = \frac{Ph^2}{4EI} \quad \text{and} \quad h = \frac{L}{4}$$

Substituting the values of α_a^2 and α_2^2

$$2 - 2\left(\frac{5}{8} + \frac{1}{4}\right)\left(\frac{Ph^2}{EI}\right) + \frac{5}{8} \cdot \frac{1}{4}\left(\frac{Ph^2}{EI}\right)^2 = 0$$

$$\left(\frac{Ph^2}{EI}\right)^2 - \left(\frac{56}{5}\right)\left(\frac{Ph^2}{EI}\right) + \left(\frac{64}{5}\right) = 0$$

The smallest root of quadratic equation is $Ph^2/EI = 1.2919$. Thus the smallest critical load for the stepped strut is

$$P_{cr} = \frac{1.2919\,EI}{h^2} = \frac{20.67\,EI}{L^2}$$

which is 19.5 per cent less then the exact solution $P_{cr} = 25.66\,EI/L^2$.

2.10.4 Numerical Integration

Sometimes the function to be integrated may vary in a complex manner such that classical integration is not suitable. For example in the beams with sudden change in cross section (stepped beams) or where the cross section is continuously variable, the moment of inertia I can not be expressed as a simple function of x, the distance along the beam. In such situations even the finite difference method discussed in the previous section is not convenient due to reduced accuracy of the estimation. A numerical integration technique provides a powerful method to handle all the cases of loads and cross-section variations. Of all the methods available for numerical integration, the Newmark's method (1943) appears to be most convenient for the purpose.

The method consists in selecting specific points along the length of the members, known as node points, and then relating the loads, moments and deflection to these points. The domain of the member is thus subdivided into a number of increments or segments, and the position of the nodes and thus the segment lengths are selected to suit the problem so that solution thus obtained will be either exact or accurate enough. The position of the node points have to be chosen in such a manner that the important features of both loading and cross-section variation are properly accounted for. All the calculations deal with the discrete values of loading, moment, curvature, slope and deflection at the node points only, from these discrete values the corresponding diagrams can be drawn. The method described is a *forward integration* procedure, the integration being carried forward in a step-by-step manner from one node point to next.

Computation of Deflection

The integration to obtain deflection starts with the curvature M/EI produced at any point in a member. For small values of $\mathrm{d}y/\mathrm{d}x$ ($= y'$, the slope of the member due to bending), the curvature can be taken as $\mathrm{d}^2y/\mathrm{d}x^2$ ($= y''$). Thus,

$$\text{curvature,}\quad y'' = M/EI \tag{2.118a}$$

$$\text{slope,}\quad y' = \int \frac{M}{EI}\mathrm{d}x + A_1 \tag{2.118b}$$

$$\text{deflection,}\quad y = \int y'\mathrm{d}x = \iint \frac{M}{EI}\mathrm{d}x\,\mathrm{d}x + A_1x + A_2 \tag{2.118c}$$

The constants of integration A_1 and A_2 can be obtained from the boundary conditions for slope and deflection. A_1 is the value of slope at $x = 0$, and if this is unknown and is guessed incorrectly, the effect of this error will appear in y curve in terms of A_1x. Thus correction to y is linear. A_2 is the value of deflection at $x = 0$.

The first step of the procedure for computing deflections from a known or trial curvature distribution, is to evaluate concentrated values for curvature at N discrete node points $x_i = x_1, x_2, \ldots, x_n$ at an interval Δx. The slope is then computed by numerical integration of the curvature

$$y'_i = \int y''\,\mathrm{d}x = \sum_{k=1}^{i} y''_k \Delta x \tag{2.119}$$

In this integration the slope at the starting end is assumed to be zero i.e. $y'_0 = 0$, although this may not be the true boundary condition. The deflection y_c can now be computed by numerical integration of the slope y' as follows:

$$y_{ci} = \int y'\,\mathrm{d}x = \sum_{k=1}^{i} y'_k \Delta x \tag{2.120}$$

with initial boundary condition of $y_o = 0$.

If the computed deflected shape y_d does not conform to the end boundary condition, a linear correction is applied to the deflection values to make them conform to the true deflection condition. The correct deflection (Fig. 2.28f) is given by

$$y_i = y_{ci} - \frac{i}{N} y_{cN} \tag{2.121}$$

Boundary or End Conditions

In computation of deflection from discrete curvature values two known end conditions are required relating to slope and deflection. These may be either one slope or one deflection conditions-and in most cases these will be known locations of zero slope and deflection-or two deflection conditions, often two known locations of zero deflection. These known boundary conditions are essential for integration from curvature to slope and from slope to deflection. They are related to constants of integration and generally come directly from end conditions of the beam or the way the beam is supported. In case of two known deflection conditions, the computation of slope from curvature is not direct due to the lack of a slope condition, it require the computation of trial slope value and a subsequent linear correction to deflection. This can be achieved by assuming a value for slope at some suitable point as a basis for completing the calculations for slope and deflection, and then to apply a simple linear correction to the deflection values to make them conform to the two known deflection conditions. Any small error in the trial value of slope results in a constant error in the slope line and this in turn produces a linear error in the deflection line as can be seen from constants of integration of differential equations of slope and deflection.

Curvature Diagram

The curvature or M/EI diagram is a necessary bridge between calculation of moments and of deflections. The plot of the curve showing variation in M/EI is important if the cross section i.e. I of the beam is variable. The process of computation of moments normally provides values at the nodes. The shape of curvature (or moment) diagram passing through these points is important in the sense that it controls the concentration of curvature values at the node points. According to the shape of curvature y'' (i.e. M/EI) diagram various concentration formulae will be used to lump y'' at the node points. To ensure correctness of this step, moment diagram should be sketched through the spot values of M, the M/EI diagram should also be sketched in the cases where there is variation in I value.

Beams of Variable Cross-Section

The beams with variable I are very common in structural design. A variable I means that the shape of curvature diagram is different from that of moment diagram. However, as long as the general form of M/EI diagram is known and its values are available at every node point, the variation of I does not pose any difficulty in this numerical integration procedure. The accuracy of the numerical solution depends on how well the spot values of M/EI at the nodes define the total M/EI curve i.e. on the accuracy of the concentration formula. The discontinuity in y'' often lowers the overall accuracy of solution

Concentration of y'' Values at the Nodes

A continuously varying curvature (y'') curve can be concentrated at the node points for the purpose of numerical integration for calculating slopes (y') and displacements (y). A function or a curve in general can be defined in terms of power series

$$y'' = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k = \sum_{k=0}^{\infty} a_k x^k \tag{2.122}$$

where a_0, a_1, a_2, etc. are constants that specify the amount of x^0, x^1, x^2, etc. in the function. a_0 is always the spot value of function at the origin of curve where $x = 0$, other constants have to be determined from the known nodal values of $f(x)$.

The procedure for concentrating y'' curve at the node points in the form of Y'' values i. e. angle changes between adjacent rigid chords, depends on the variation of y'' curve. A linear or trapezoidal variation is defined by first two terms of the series and a parabolic variation is defined by first three terms. In case of trapezoidal variation the curve varies linearly between two known points whereas in case of parabolic variation the curve passes through three known points usually equally spaced along x-axis.

Consider the curve $f(x)$ shown in Fig. 2.27 passing through the values f_{i-1}, f_i and f_{i+1} at three equally spaced nodes $i-1$, i and $i+1$, respectively, with origin being taken at node i and hence at nodes $i-1$, and $i+1$, $x = -h$ and $x = +h$,

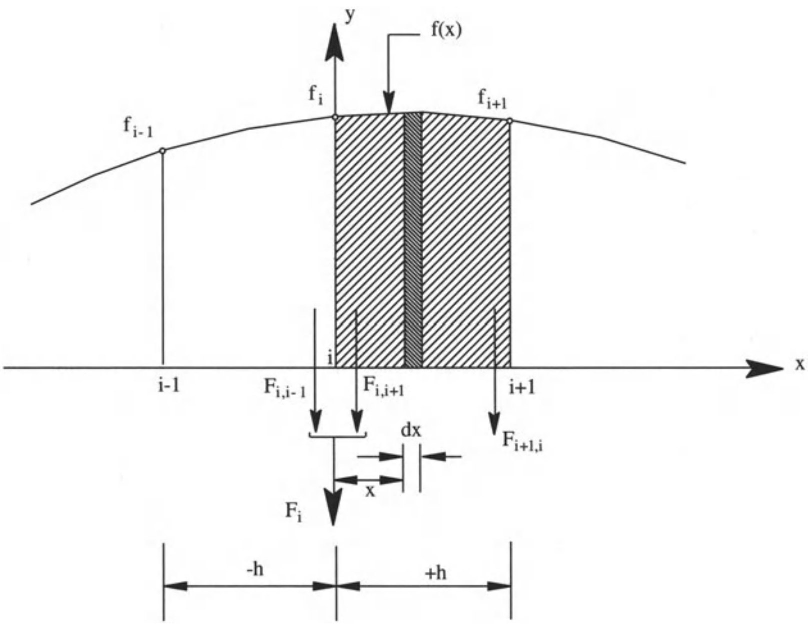


Fig. 2.27. Concentration of curvature values f $(= y'')$

respectively. At any point x the value of function is f, thus the total change over a short length $\mathrm{d}x$ at x is $f\mathrm{d}x$. The effect due to this change at the nodes i and $i+1$ are $f\mathrm{d}x(h-x)/h$ and $f\mathrm{d}x(x/h)$, respectively, such that c.g. of shaded area under the curve over chord i, $i+1$ remains unchanged. Thus the concentrations at the nodes i and $i+1$ are

$$F_{i,i+1} = \frac{1}{h}\int_0^h f(x)(h-x)\,\mathrm{d}x \tag{2.123}$$

and

$$F_{i+1,i} = \frac{1}{h}\int_0^h f(x)x\,\mathrm{d}x \tag{2.124}$$

Similarly,

$$F_{i,i-1} = \int_0^h f(x)(h-x)\,\mathrm{d}x \tag{2.125}$$

Total concentration at the node i is

$$F_i = F_{i,i+1} + F_{i,i-1}$$

These general expressions can be used for any form of curve.

(i) Linear or trapezoidal variation

In this case the general equation for f retains first two terms only

$$f(x) = a_0 + a_1 x \tag{2.126}$$

Applying the known node conditions

$$\text{at } x=0 \qquad f_i = a_0 + a_1(0)$$

$$\text{at } x=h \qquad f_{i+1} = a_0 + a_1(h)$$

From which, $a_0 = f_i$ and $a_1 = (f_{i+1} - f_i)/h$. The equation for $f(x)$ reduces to

$$f(x) = f_i + \left[(f_{i+1} - f_i)x/h\right] \tag{2.127}$$

Substituting this in (2.123) and (2.124)

$$F_{i,i+1} = \frac{1}{h}\int_o^h \left[f_i + \left(\frac{f_{i+1} - f_i}{h}\right)x\right](h-x)\,\mathrm{d}x$$

$$= h(2f_i + f_{i+1})/6 \tag{2.128}$$

and

$$F_{i+1,i} = \frac{1}{h}\int\limits_{o}^{h}\left[f_i + \frac{(f_{i+1} - f_i)}{h}x\right]x\,\mathrm{d}x$$

$$= h(2f_{i+1} + f_i)/6 \qquad (2.129)$$

In the particular case where function $f(x)$ also has trapezoidal shape between nodes $i-1$ and i with chord $i-1, i$ being of same length as $i, i+1$ and f_i being common to both the chords.

$$F_{i,i-1} = h(2f_i + f_{i-1})/6 \qquad (2.130)$$

Thus the total concentration at the node i is

$$F_i = F_{i,i+1} + F_{i,i-1} = h(f_{i-1} + 4f_i + f_{i+1})/6 \qquad (2.131)$$

(ii) Parabolic variation

In this case the variation of function $f(x)$ is parabolic passing through three known points f_{i-1}, f_i and f_{i+1}. The nodes $i-1$, i and $i+1$ are equally spaced. The power expression for $f(x)$ retains only three terms

$$f(x) = a_0 + a_1x + a_2x^2 \qquad (2.132)$$

Substituting the known values at the node points

$$x = -h \qquad f_{i-1} = a_0 + a_1(-h) + a_2(-h)^2$$

$$x = 0 \qquad f_i = a_0 + a_1(0) + a_2(0)^2$$

$$x = h \qquad f_{i+1} = a_0 + a_1(h) + a_2(h)^2$$

The values of constants are obtained as

$$a_0 = f_i\ , \quad a_1 = (f_{i+1} - f_{i-1})/(2h)$$

and

$$a_2 = (f_{i-1} - 2f_i + f_{i+1})/(2h^2)$$

Therefore, the function given by (2.132) can be rewritten as

$$f(x) = f_i + \left[(f_{i+1} - f_{i-1})/(2h)\right]x + \left[(f_{i-1} - 2f_i + f_{i+1})/(2h^2)\right]x^2 \qquad (2.133)$$

Substituting this expression for $f(x)$ in (2.123) and (2.124)

$$F_{i,i+1} = h(3f_{i+1} + 10f_i - f_{i-1})/24 \qquad (2.134)$$

$$F_{i+1,i} = h(7f_{i+1} + 6f_i - f_{i-1})/24 \qquad (2.135)$$

The equivalent expression for $F_{i,i-1}$ can be written as

$$F_{i,i-1} = h(3f_{i-1} + 10f_i - f_{i+1})/24 \qquad (2.136)$$

The total central concentration at the node i is give

$$F_i = F_{i,i-1} + F_{i,i+1} = h(f_{i-1} + 10f_i + f_{i+1})/12 \qquad (2.137)$$

Computation of Y'' Values at the Discontinuity

If the slope discontinuity in M/EI curve is caused by a point load W_i acting at i and beam cross-section is constant, the problem can be treated by adding a term $W_i h$ to the concentration Y_i'' to account for the slope discontinuity.

$$Y_i'' = \frac{h}{12} EI(M_{i-1} + 10M_i + M_{i+1} + W_i h) \tag{2.138}$$

where M_{i-1}, M_i and M_{i+1} are total computed moments at the nodes $i-1$, i and $i+1$, respectively. However, it should be noted that this approach is applicable only if I is constant. If I is variable and there are both distributed and point load systems acting, the problem can be handled by keeping two effects separate up to Y'' curve for best accuracy.

The method provides an iterative procedure starting from a assumed deflected shape or a trial function. The correct deflected shape is obtained by following the steps outlined in Fig. 2.28:

1. Assume a suitable buckling mode or a trial function y_a. The nearer this is to the true mode, the less is work of computation.
2. Using the trial function assumed in the step 1, compute bending moment distribution $M(x) = \overline{M}(x) + Py_a(x)$ in which $\overline{M}(x)$ represents primary bending moment in the straight beam-strut and Py_a is the disturbing moment due to P acting on $y_a(x)$.
3. From the moment distribution obtained in the step 2 compute curvature distribution $y''(x) = M(x)/EI$. For a strut with constant cross-section the shape of $y''(x)$ curve is same as $M(x)$ curve.
4. Discretize the continuous curvature distribution curve by subdividing the structure into N equal chords and compute nodal curvature values y_k''. Obtain the angle change at nodes given by Y'' by concentrating the y'' curve at the nodes by using appropriate formula (trapezoidal or parabolic) based on the shape of y'' curve between the nodes under consideration.
5. From Y'' values compute the derived deflection y_d by integrating twice by Newmark's method. The derived y_d values provide a better approximation to the true mode than y_a.
6. Repeat the steps 1 to 5 using y_d in place of y_a to derive a new set of y_d values until it converges to true buckling mode, i. e., $y_a = y_d$ at all the node points to the required accuracy. This condition indicates that the system is in equilibrium.
7. Compare y_d with y_a at the end of final cycle with fully converged results. The ratio of nodal values of two curves is unity for all the nodes for the stable configuration. Obtain the value of critical load P_{cr} from this comparison.

Special Features

1. The higher the number of nodes chosen (i. e. the larger number of chords used) the more accurate is the solution for P_{cr}.

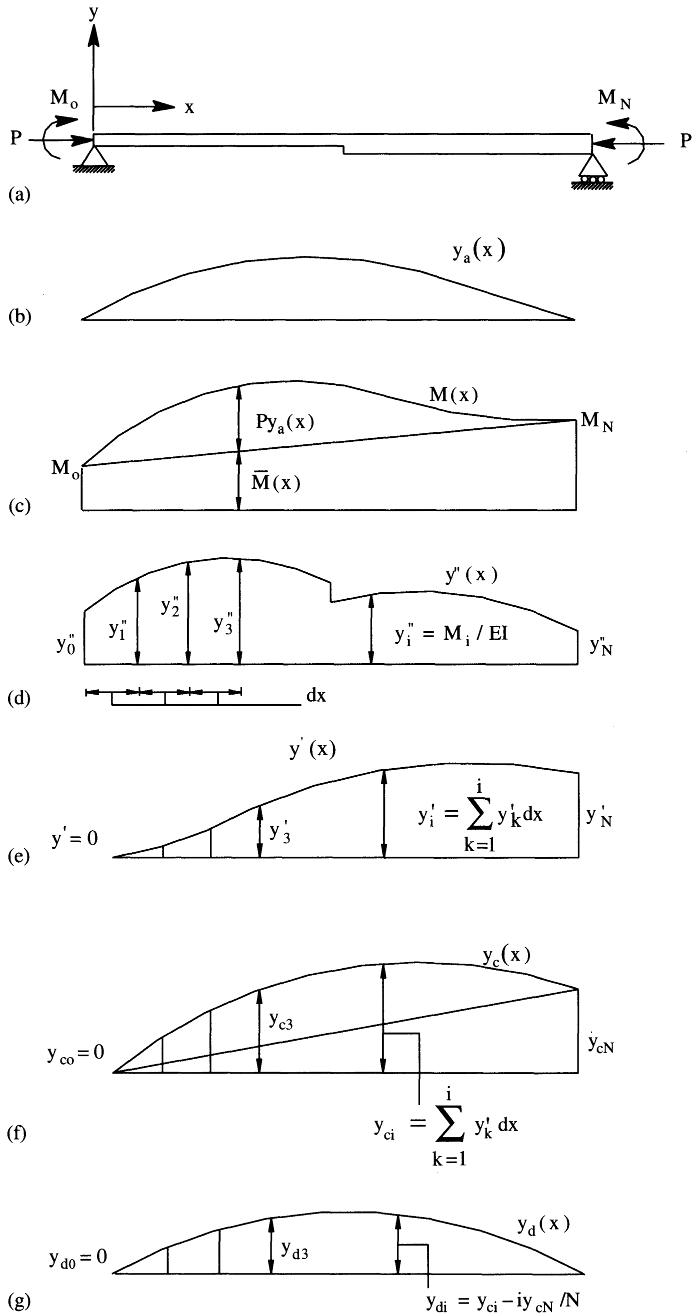


Fig. 2.28a–g. Computation procedure by Newmark's integration method. **a** Structure, **b** assumed buckling mode or trial function, **c** bending moment distribution $M(x)$, **d** curvature distribution **e** slope computation, **f** computated deflections, **g** corrected deflections i. e. derived deflections

2. The type of result expected depends upon the number of nodes chosen or the number of degrees-of-freedom given to the system. In the numerical solution, the degree-of-freedoms of the structure are restricted to make it buckle in one of a definite number of modes. Therefore one has to be certain that the first buckling mode of the original structure is closely represented by one of the possible modes of the discretized structure.
3. For a given discretization scheme i. e. a given number of nodes, the value of P_{cr} for the first mode will be more accurate than that of second mode. This is because, the higher the buckling mode the more complex is the deflected shape and consequently a shorter chord length is required to model it accurately.

This iterative procedure known as a method of successive approximations is similar to stodola's method of finding fundamental frequency of beams.

To illustrate the procedure consider the simply supported stepped strut shown in Fig. 2.29. The critical value of the load that will cause buckling, P_{cr} can be computed by using the following steps:

1. Assume trial displacement function or buckling mode y_a. Since the strut is simply supported at the ends, a buckling mode of the type $y_a = ax(L-x)/L^2$ satisfying boundary condition will be most suitable. The initial values at salient points are taken as 0.0, 5.5, 9.0, 10.0, 9.0, 5.5 and 0.0.
2. Draw the moment diagram $M(x) = -Py_a$.
3. Calculate M/EI values along the length of the strut. This is an important step as it indicates the correct way to apply concentration formulae.
4. Select the positions of node points to follow the important features of M/EI curve. In this particular problem, node points are required at the points of change

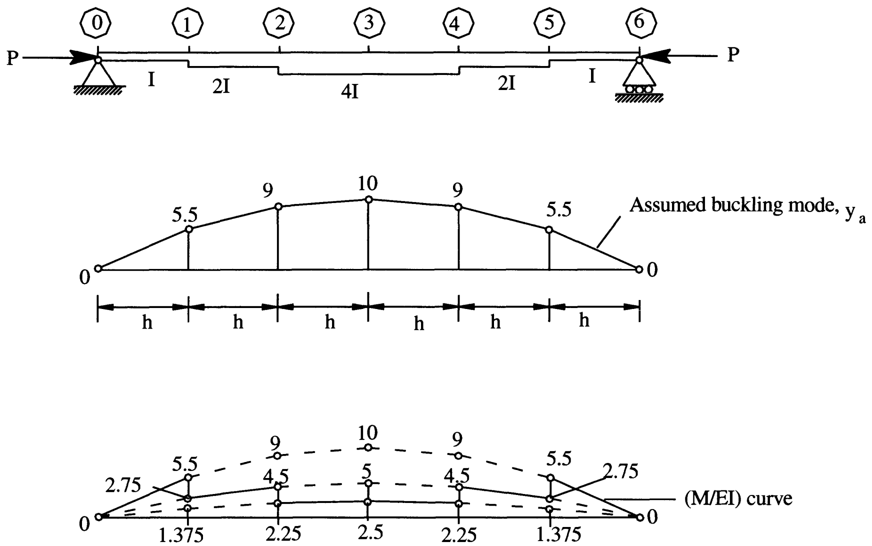


Fig. 2.29. Assumed buckling mode, M/EI curve of stepped strut

of cross-section and one at the mid length. A scheme with $N = 6$ adequately represents the important features of the problem and makes equal chord technique possible. Calculate values of y'' curve at the node points.

5. Compute Y'' values. In order to use parabolic distribution formulae at the discontinuities the curves are imagined to continue beyond discontinuity. The extrapolated portions of the curves are shown by dotted lines in Fig. 2.29 and the extrapolated values are known as fictitious values. Due to discontinuity formulae giving Y'' at each side of the nodes are used. Thus

$$Y''_{10} = \frac{h}{24}[3(0) + 10(-5.5) - 1(-9)]\frac{aP}{EI} = -46\left(\frac{aPh}{24EI}\right)$$

$$Y''_{12} = \frac{h}{24}[3(-4.5) + 10(-2.75) - 1(0)]\frac{aP}{EI} = -41\left(\frac{aPh}{24EI}\right)$$

$$Y''_{21} = \frac{h}{24}[3(-2.75) + 10(-4.5) - 1(-5)]\frac{aP}{EI} = -48.25\left(\frac{aPh}{24EI}\right)$$

$$Y''_{23} = \frac{h}{24}[3(-2.5) + 10(-2.25) - 1(-1.375)]\frac{aP}{EI} = -28.625\left(\frac{aPh}{24EI}\right)$$

$$Y''_{3} = \frac{2h}{24}[(-2.25) + 10(-2.5) + (-2.25)]\frac{aP}{EI} = -59\left(\frac{aPh}{24EI}\right)$$

Integrate Y'' to compute chord slopes Y' for which one slope value is required to be known. By symmetry $y'_3 = 0$, hence calculations are carried through from right to left. Integrate y' to calculate y from left where $y_a = 0$.

6. Compare y_d with y_a. The simplest way to accomplish this is to scale down the y_d values so that the value at node 4 is as in y_a. These form the new y_a values for the next cycle comprising of steps (2) to (5). When y_d and y_a become close as they do in the second cycle the iteration may be stopped.
7. Obtain the value of critical load P_{cr} by comparing y_d with y_a.

At buckling:

$$\frac{407.191aPh^2}{24\,EI} = 10a$$

Therefore,

$$P_{cr} = \frac{0.5894\,EI}{h^2} = \frac{2.15\pi^2 EI}{L^2} \quad (\text{as } h = L/6)$$

The computations are given in Table 2.1.

At buckling: $(40.646)(aPh^2)/(24EI) = a$ or

$$P_{cr} = 21.256\,EI/L^2 = 2.154\pi^2 EI/L^2 \quad (\text{as } h = L/6)$$

where $(\pi^2 EI/L^2)$ is buckling load for a simply supported strut of uniform cross-section.

The values of P_{cr} obtained by using $\sum y_a \sum y_d / \sum y_d^2$ is not necessarily an overestimate, except in the particular case – where EI is constant when it becomes identical with Rayleigh's estimate.

Table 2.1. Deflection computations by numerical integration technique

Node	0		1		2		3	4	Multiple factor
Iteration #1									
Assumed deflection, y_a	0.00		5.6		8.9		10.0	8.9	a
Moment due to P, $M(y)$	0.00		−5.6		−8.9		−10.0	−8.9	aP
Curvature, $y'' = (M/EI)$	0.00		−5.6 −2.80		−4.45 −2.225		−2.50	−2.225	$(aP)/(EI)$
Change in slope, Y''			−47.10 − 41.35 −88.45		−47.9 − 28.35 −76.25		−58.90		$(aPh)/(24EI)$
Slope, $y' = \sum Y''$		223.60		135.15		58.90	0		$(aPh)/(24EI)$
Deflection, $y = \sum y'$	0.00		223.60		358.75		417.65		$(aPh^2)/(24EI)$
Derived deflection, y_d	0.00		5.35		8.59		10.00	8.59	$(41.765)(aPh^2)/(24EI)$
Ratio y_a/y_d			1.04673		1.03609		1.000	–	
From minimum and maximum values of (y_a/y_d): $20.687(EI/L^2) < P_{cr} < 21.654(EI/L^2)$									
From $\sum y_a/\sum y_d$ (= 1.02957): $P_{cr} = 21.299(EI/L^2)$									
From $\sum y_a \sum y_d/\sum y_d^2$ (= 1.02625): $P_{cr} = 21.230(EI/L^2)$									
Iteration #2									
y_a	0.00		5.35		8.59		10.0	8.59	a
y''	0.00		−5.35 −2.675		−4.295 −2.1475		−2.50	−2.1475	$(aP)/(EI)$
Y''			−44.91 − 39.635 −84.545		−45.975 − 27.6375 −73.6125		−58.59		$(aPh)/(24EI)$
y'		216.7475		132.2025		58.59	0		$(aPh)/(24EI)$
y	0.00		216.7475		348.9500		407.5400		$(aPh^2)/(24EI)$
y_d	0.00		5.32		8.56		10.00		$(40.754)(aPh^2)/(24EI)$
Iteration #3									
y_a	0.00		5.32		8.56		10.0	8.56	a
y''	0.00		−5.32 −2.66		−4.28 −2.14		−2.50	−2.14	$(aP)/(EI)$
Y''			−44.64 − 39.44 −84.08		−45.78 − 27.57 −73.35		−58.56		$(aPh)/(24EI)$
y'		215.990		131.910		58.56	0		$(aPh)/(24EI)$
y	0.00		215.99		347.90		406.46		$(aPh^2)/(24EI)$
y_d	0.00		5.314		8.559		10.00		$(40.646)(aPh^2)/(24EI)$

Important Notes

1. For a converged solution the derived values y_d will be of exactly the same shape as y_a and will equal y_a at all the points.
2. If the assumed value y_a happens to be the true value, the solution will be found in one cycle, no convergence being necessary.
3. If the value of P_{cr} is calculated before the numerical procedure has converged completely i. e. iteration is stopped before $y_a/y_d = 1$ at all the nodes (i. e. before the governing equation of equilibrium is fully satisfied at all the nodes) a set of differing values of P_{cr} would be obtained by equating y_a to y_d at each different node in turn. For example in the above illustration the second iteration has been started with y_a being 5.33, 8.59, 10.00, 8.59 and 5.33 and has produced y_d as 5.32, 8.56, 10.00, 8.56 and 5.32 $(40.719\,aPh^2/24EI)$ after two iterations. If we were to compute P_{cr} by making $y_a = y_d$ at the nodes 1, 2, and 3 in turn this would have given $P_{cr} = 21.25, 21.29$, and $21.22\,(EI/L^2)$. The maximum value comes from the situation at node 2. The value $21.29\,(EI/L^2)$ is upper bound to the fully converged value of P_{cr} and the minimum $21.22\,(EI/L^2)$ is lower bound. Thus in any stability computation before complete convergence is attained, the values of P_{cr} obtained from y_a/y_d ratio across the structure bound the true value of P_{cr}.
4. The P_{cr} values computed from $y_a/y_d = 1$ at each node before complete convergence is achieved, can be used to provide good estimate of true buckling load. The following are commonly used methods of averaging.

(a) Weighted average of P_{cr} value from $\dfrac{\sum y_a}{\sum y_d} = 1$

$$P_{cr} = 21.26\,(EI/L^2)$$

(b) An average value of P_{cr} (based on least square solution) from $\dfrac{\sum y_a y_d}{\sum y_d^2} = 1$

$$\frac{\sum\limits_x y_a y_d}{\sum\limits_x y_d^2}\left(\frac{24}{40.719}\frac{EI}{Ph^2}\right) = \left(\frac{303.772}{303.152}\right)\left(\frac{24}{40.719}\frac{EI}{Ph^2}\right) = 1$$

or

$$P_{cr} = 21.262\,(EI/L^2)$$

This value is the same as that obtained by Rayleigh's method when EI is constant. The value obtained is always higher than the converged value. The later method of averaging is much more accurate than the first one.

Forced Convergence

In the procedure using successive approximation, there is a simple relationship between errors in the assumed y_a and derived y_d values which can be expressed as

$$\frac{\text{error in } y_a}{\text{error in } y_d} = \text{constant} \tag{2.139}$$

If $y_{a,1}$, $y_{a,2}$ and $y_{a,3}$ are three successive approximations for y_e in the iteration procedure where $y_{a,2}$ is derived from $y_{a,1}$ and $y_{a,3}$ from $y_{a,2}$. Thus the errors in these values being $y_e - y_{a,1}$, $y_e - y_{a,2}$ and $y_e - y_{a,3}$, respectively. From (2.139)

$$\frac{y_e - y_{a,1}}{y_e - y_{a,2}} = \frac{y_e - y_{a,2}}{y_e - y_{a,3}}$$

$$y_e = \frac{(y_{a,1})(y_{a,3}) - (y_{a,2})^2}{y_{a,1} - 2y_{a,2} + y_{a,3}} \tag{2.140}$$

In case $y_{a,1}$, $y_{a,2}$ and $y_{a,3}$ have close values, the value y_e obtained may be doubtful, but the problem can be circumvented by using the following procedure. If ε_1 is the error in $y_{a,1}$, then

$$y_e = \varepsilon_1 + y_{a,1} = \frac{y_{a,1}y_{a,3} - y_{a,2}^2}{y_{a,1} - 2y_{a,2} + y_{a,3}}$$

This reduces to

$$\varepsilon_1 = -\left[\frac{(y_{a,1} - y_{a,2})^2}{y_{a,1} - 2y_{a,2} + y_{a,3}}\right] \tag{2.141}$$

Equation (2.141) gives better results than (2.140) when successive values of y_a are close together. This process is called Aitken's procedure. In the problem illustrated in Table 2.1 the three successive values of y_a at the nodes 1 and 2 are 5.6, 5.35 and 5.32, and 8.9, 8.59 and 8.56, respectively. Using these values in (2.140):

at node 1: $y_e = [(5.6)(5.32) - (5.35)^2]/[5.6 - 2(5.35) + 5.32] = 5.3159$

at node 2: $y_e = [(8.9)(8.56) - (8.59)^2]/[8.9 - 2(8.59) + 8.56] = 8.5568$

2.11 Orthogonality of Buckling Modes

If y_m and y_n are two different modes in a buckling problem, and M_m and M_n are the moments throughout the structure corresponding to these modes, then according to the principle of orthogonality of buckling modes

$$\int_O^L M_m \left(\frac{M_n}{EI}\right) \mathrm{d}x = 0 \tag{2.142}$$

or in numerical integration solution $\sum_x M_m(M_n/EI) = 0$.

Here the integration is with respect to x over the length L of the structure or the summation is over all the node points. In the cases where EI is constant the moments M_m and M_n will be orthogonal, and in case of pin-ended strut (where $M = -Py$), the modes y_m and y_n will be orthogonal i. e.

$$\int_{O}^{L} y_m y_n \, \mathrm{d}x = \sum_{x} y_m y_n = 0 \tag{2.143}$$

In the relationship of (2.142) M_m can be considered as a generalized loading in the mode m, and $M_n/EI = y_n''$ which is curvature in mode n, can be considered as the corresponding generalized displacement in mode n. Equation (2.142) thus can be interpreted that the work done by load M_m as it is displaced through y_n'' is zero.

The orthogonality relationship in the bucking problem can be derived from the general equation of stability written as

$$M'' + \frac{P}{EI} M = 0 \tag{2.144}$$

where M is understood $M(x)$. Consider two buckling modes y_m and y_n with associated moment and load values being (M_m, P_m) and (M_n, P_n), respectively. Therefore

$$M_m'' + \frac{P_m}{EI} M_m = 0 \tag{2.145}$$

and

$$M_n'' + \frac{P_n}{EI} M_n = 0 \tag{2.146}$$

Multiplying (2.145) and (2.146) by M_n and M_m, respectively.

$$M_m'' M_n + \frac{P_m}{EI} M_m M_n = 0 \tag{2.147}$$

$$M_n'' M_m + \frac{P_n}{EI} M_n M_m = 0 \tag{2.148}$$

However, $M_m'' M_n = (\mathrm{d}/\mathrm{d}x)(M_m' M_n) - M_m' M_n'$ and $M_n'' M_m = (\mathrm{d}/\mathrm{d}x)(M_n' M_m) - M_n' M_m'$. Subtracting (2.148) from (2.147)

$$\frac{\mathrm{d}}{\mathrm{d}x}(M_m' M_n - M_n' M_m) = (P_n - P_m)\frac{M_m M_n}{EI} \tag{2.149}$$

Integrating (2.149) over the length of the strut between limits $x = 0$ and L

$$[M_m' M_n - M_n' M_m]_0^L = (P_n - P_m) \int_0^L \frac{M_m M_n}{EI} \mathrm{d}x \tag{2.150}$$

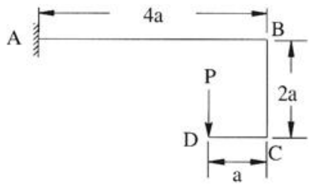
It should be noted that M or M' is always zero at an end. Hence left hand side of equation must be zero. Further, as $P_m \neq P_n$

$$\int_0^L \frac{M_m M_n}{EI} \mathrm{d}x = 0$$

This is the *orthogonality relationship* of buckling modes.

2.12 Problems

Problem 2.1. The rigid cantilever frame shown in Fig. P.2.1 has constant flexural rigidity (EI) and carries a concentrated load P at the free end. Using different versions of energy approach determine displacements at the points B, C and D and slope at the point D.

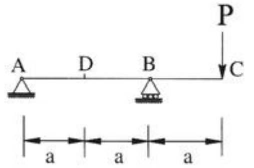


P.2.1

[Ans. $\delta_{Bv} = 40Pa^3/3EI(\downarrow)$, $\delta_{Bh} = 0$; $\delta_{Cv} = \frac{40Pa^3}{3EI}(\downarrow)$, $\delta_{Ch} = \frac{6Pa^3}{EI}(\leftarrow)$;

$$\delta_{Dv} = \frac{35Pa^3}{3EI}, \quad \delta_{Dh} = \frac{6Pa^3}{EI}(\leftarrow) \quad \textit{and} \quad \theta_D = \frac{3Pa^2}{2EI}(\textit{anticlockwise})]$$

Problem 2.2. Use Cotterill-Castigliano's theorem to compute vertical deflection at the point D of the beam shown in Fig. P.2.2.



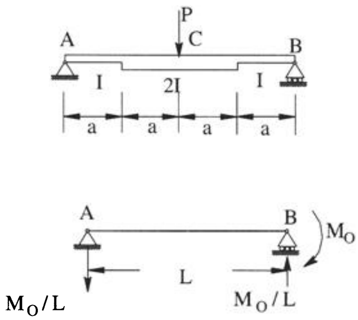
P.2.2

[Hint: Apply fictitious load W at the point D. The deflection at the point D is:

$$\delta_D = \partial U/\partial W = \int \left[M_x(\partial M_x/\partial W)\,\mathrm{d}x\right]/EI = Pa^3/4EI(\uparrow)]$$

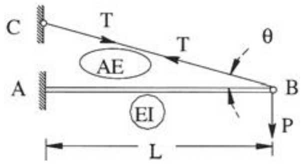
Problem 2.3. A simply supported beam AB shown in Fig. P.2.3 having moment of intertia of $2I$ at the central half portion and I for the remaining, is subjected to a concentrated load P at the centre. Determine central deflection and end slopes.
[Ans. $\delta_C = 3Pa^3/4EI$, $\theta_B = \theta_A = 5Pa^2/8EI$*]*

Problem 2.4. A simply supported beam shown in Fig. P.2.4 is subjected to an end moment M_0 at the end B. Determine the end slopes and maximum deflection.
[Ans. $\theta_A = M_oL/6EI$ *and* $\theta_B = M_oL/3EI$. *The deflection at distance* x *from the end is given by:* $\delta_x = M_oL(x - x^3/L^2)/6EI$. *Maximum deflection occurs at* $x = L/\sqrt{3}$ *and its magnitude is:* $\delta_{\max} = M_oL^2/(9\sqrt{3}EI)$*]*



P.2.4

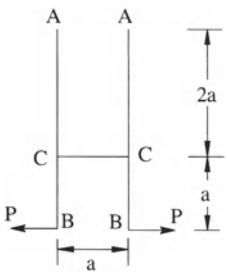
Problem 2.5. The free end of cantilever beam shown in Fig. P.2.5 is supported by an inclined tie rod. The cross-sectional area of tie rod is A and the flexural rigidity of cantilever beam is EI. Use strain energy method to determine vertical displacement at the joint B and tension in the tie rod due to concentrated load P acting at B. E is same for the beam and tie rod.



P.2.5

[Ans. $U = \{(AE \sin^2\theta \cos\theta/2L) + (8EI/2L^3)\}\Delta^2$, *where* Δ *is downward displacement at the point B.* $\Delta = P/\{(AE \sin^2\theta \cos\theta/L) + (3EI/L^3)\}$*]*

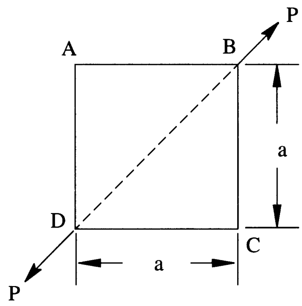
Problem 2.6. In the rigid frame shown in Fig. P.2.6 determine the distance by which the points A move closer under the action of force P acting at the points B. EI is constant throughout.



P.2.6

[Ans. $\Delta_A = 2Pa^3/EI$.*]*

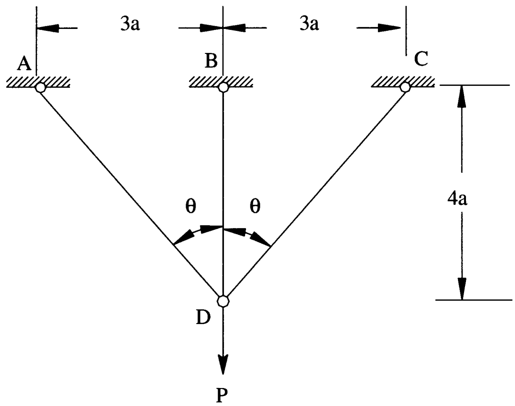
Problem 2.7. The square rigid frame with uniform cross-section shown in Fig. P.2.7 is subjected to diagonally opposite forces P at the points B and D. Ignoring axial deformations determine the distance by which the points A and C move closer.



P.2.7

[Hint: Due to symmetry only half the frame ABD carrying load $P/2$ at the ends B and D need be considered with roller supports such that movement is allowed only along the diagonal BD i. e. presume the reference coordinate system along the diagonals. $\Delta_{AC} = Pa^3/24EI$.]

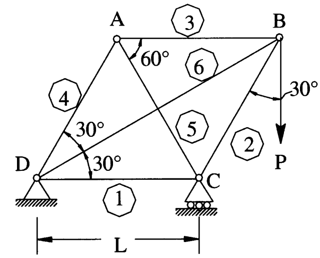
Problem 2.8. The three-bar pin-jointed frame in a vertical plane shown in Fig. P.2.8 is subjected to a vertical load P at the common point D. Use energy method to analyse the frame. The bars are of constant cross-section.



P.2.8

[Hint: Consider BD as the redundant member with force T_1. The forces in the other bars are T_2, and $\cos\theta = 4/5$, $\sum V_D = T_1 + 2T_2\cos\theta = P$ and $U = \frac{5a}{AE}\left\{\frac{2}{5}T_1^2 + \frac{25}{64}(P - T_1)^2\right\}$. Therefore, from the theorem of least work: $T_1 = 125\,P/253$.]

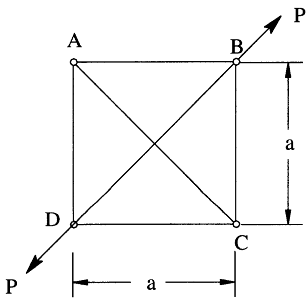
Problem 2.9. While fabricating the pin-jointed plane frame shown in Fig. P.2.9, it was found that member AC is fabricated Δ too short. Determine the forces in the members after assembly. AE is same for all members.



P.2.9

[Ans. $F_6 = \frac{3}{5+3\sqrt{3}}\left(\frac{\Delta AE}{L}\right)$ *(tension), and* $F_1 = F_2 = F_3 = F_4 = F_5 = F_6/\sqrt{3}$ *(compression)]*

Problem 2.10. All members of the truss shown in Fig. P.2.10 are of same cross-section and material. Compute force in each of the members due to opposite forces P acting at A and C.



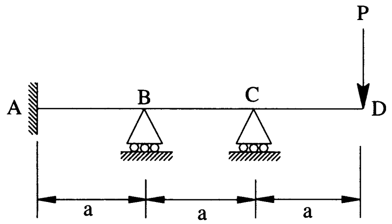
P.2.10

[Hint: Consider only half the frame ABC as in problem 2.7.]

Problem 2.11. Analyze the continuous beam ABCD shown in Fig. P.2.11. The beam is fixed at the end A and supported at B and C, and free at the end D. The beam carries a concentrated load P at the free end.

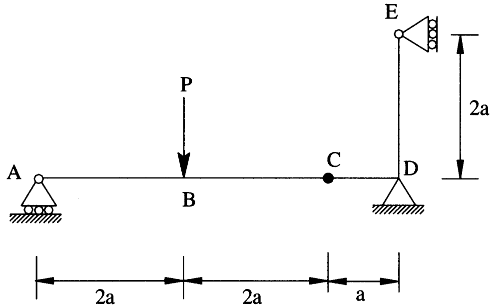
$$\left[\text{Ans: } R_A = \frac{3P}{7}(\uparrow)\ ;\ R_B = \frac{12P}{7}(\downarrow)\ ;\ R_C = \frac{16P}{7}(\uparrow) \text{ and}\right.$$

$$\left. M_{AB} = -\frac{Pa}{7}\ ,\ M_{BA} = M_{BC} = \frac{2Pa}{7} \text{ and} M_{CB} = M_{CD} = -Pa\right]$$



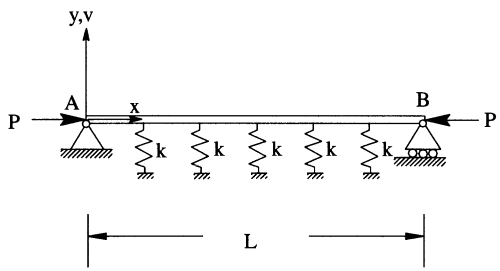
P.2.11

Problem 2.12. Analyze the rigid frame shown in the Fig. P.2.12. At the point C a frictionless hinge is provided. EI is constant throughout. Also calculate deflection at the point C due to the concentrated load acting at the point B.



P.2.12

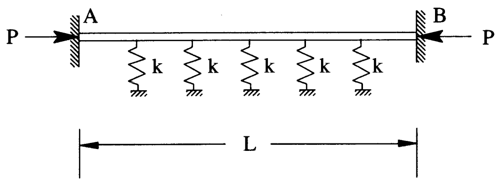
Problem 2.13. The governing differential equation of a hinged-hinged compression member supported along its entire length by an elastic medium applying a force k per unit length per unit deflection as shown in the Fig. P.2.13 is given by: $EI(\mathrm{d}^4v/\mathrm{d}x^4) + kv + P(\mathrm{d}^2v/\mathrm{d}x^2) = 0$. Determine the critical load using the finite difference technique when: (i) the member is divided into two segments, and (ii) the member is divided into three segments.



P.2.13

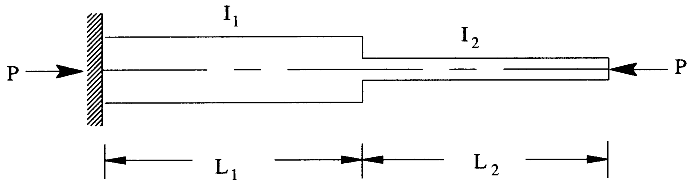
$$\left[Ans\text{: } P_{\mathrm{cr}} = \left(\frac{8EI}{L^2} + \frac{kL^2}{4}\right) \text{ with two segments}\right]$$

Problem 2.14. Solve the problem 2.13 when the ends of the compression member are fixed-fixed instead of hinged-hinged as shown in the Fig. P.2.14.



P.2.14

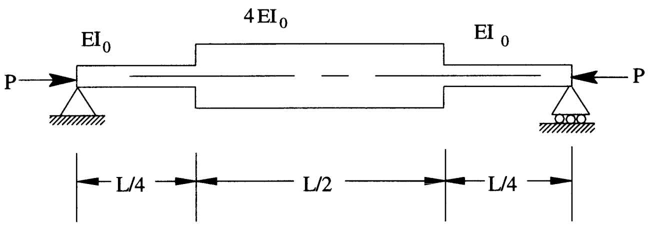
Problem 2.15. Analyze the stepped compression member shown in Fig. P.2.15 by using energy approach.



P.2.15

$$\left[\textit{Hint: Use the shape function: } v(x) = A\left(1 - \cos\frac{\pi x}{2L}\right) \right]$$

Problem 2.16. Analyse the stepped simply-supported compression member shown in Fig. P.2.16 by: (i) Rayleigh-Ritz method, and (ii) Galerkin technique.



P.2.16

3

Rigid-Body Assemblages

3.1 Introduction

This section deals with the class of structures consisting of rigid-body-assemblages wherein the elastic deformations are limited entirely to localized spring elements. In these systems, the rigid bodies are constrained by the support hinges so that only one type of displacement is possible. For the systems discussed here the formulation of the stability problem differs from the classical Euler formulation due to its basically discrete nature.

These systems can further be classified into single, two or multi-degree-freedom systems. The degrees of freedom are generally referred to *generalized coordinates* which represent the number of independent coordinates (displacements or rotations) which must be known in order to define the position (configuration) of the system. The word independent signifies that any of the generalized coordinates can be varied freely while others remain unchanged.

3.2 Methods of Analysis

Analytical approaches to stability analysis described here are based on static concept since the structure remains at rest before and after buckling. The methods are based on the investigation of the system close to its position of equilibrium and are applicable only if the external forces have a potential i. e. they are conservative. The aim is to predict the mode of loss of stability and corresponding load under which the structure gets into a critical state. The approaches discussed are:

1. Equilibrium approach, and
2. Energy approach.

3.2.1 Equilibrium Approach

This technique deals with the equilibrium configuration of the idealized perfect system and is characterized by the fact that there exist discrete values of the load at which additional equilibrium configurations (modes) appear in the neighbourhood of trivial solution (initial equilibrium position). In the other words the method consists in predicting the values of the loads for which a perfect system admits additional but adjacent (close) equilibrium states with different deformation patterns called *modes*. The assumption of an equilibrium configuration close to the initial one enables to consider the slopes of deflected elements as small compared to unity.

3.2.2 Energy Approach

This technique is based on the principle of *minimum potential energy* which states that a conservative system is in a configuration of stable equilibrium, if and only if, the value of potential energy is relative minimum (relative with respect to its immediate neighbourhood). A mechanical system is said to be conservative, if the virtual work W $(= W_{in} + W_{ex})$ vanishes for a virtual displacement that carries the system completely around any closed path. Here, W_{ex} and W_{in} are parts of virtual work performed by internal and external forces, respectively, during virtual displacement. Thus conservative system is in equilibrium when energy stored is equal to the work done by external loads. This criterion enables to predict the critical load at which response of the system ceases to be in stable equilibrium. The virtual displacement referred here is an *admissible displacement configuration* satisfying *geometric* or *force boundary conditions*.

When system deforms, the load point approaches the reference point and there is loss of potential energy. At the same time restraining springs develop or store elastic energy. The external virtual work done is given by

$$\delta \mathrm{W}_{ex} = -P\Delta \tag{3.1}$$

where Δ represents virtual displacement of the point of application of external force projected along its line of action. If the strain energy due to internal work is represented by δU, the principle of virtual work can be expressed as

$$\delta U = \delta W_{ex} \quad \text{or} \quad \delta U - \delta W_{ex} = 0 \tag{3.2}$$

Normally, the increment in external work due to virtual displacement is represented as a change in potential energy as

$$\delta V = -\delta W_{ex}$$

Thus (3.2) can be written as

$$\delta(U + V) = \delta(\Pi) = 0 \tag{3.3}$$

Hence, $U + V = \Pi = \text{const.}$

The quantity Π $(= U + V)$ is referred to as *total potential* or simply potential of the system. Thus, if a system is in static equilibrium, the total potential energy of the system has stationary value i.e. its first variation is zero $(\delta\Pi = 0)$. For small values of P, Π is positive in any non-trivial admissible configuration. For sufficiently large value of P, Π is $-ve$ making equilibrium configuration unstable. Thus the instability problem reduces to determination of value of load for which total potential of a perfect system ceases to be positive definite. For this case mathematically,

$$\delta^2 \Pi > 0 \tag{3.4}$$

for neutral equilibrium $\delta^2 \Pi = 0$ and for unstable equilibrium $\delta^2 \Pi < 0$.

The aim of above two approaches is to predict the smallest load for which non-trivial equilibrium exists. For the types of systems considered here, the energy approach is equivalent to the equilibrium method. One of the major advantages of the energy approach is that in its formulation, definition of coordinate system and sign convention is deemed unnecessary. Only expressions for strain energy and work done by external loads are needed.

For illustration of above principles consider the rigid-bar system constrained or supported by a linear and a rotational spring as shown in Fig. 3.1. The structure is a single-degree-of-freedom system since only one displacement (i.e. rotation, θ) is required to be known to define its deflected position or configuration. Both equilibrium, and energy approaches can be used to predict the critical load. The system is in equilibrium in undisturbed position under load P. To test its stability, the system is displaced by a small rotation θ.

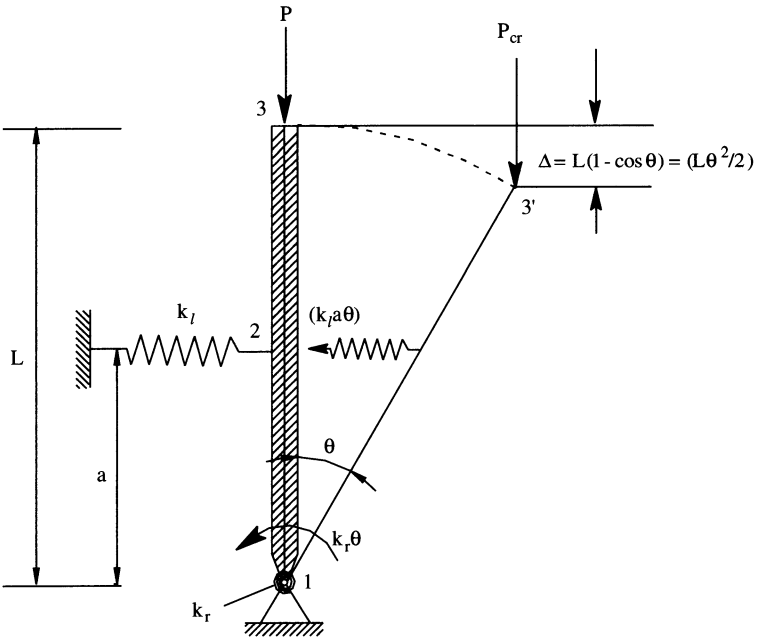


Fig. 3.1. Rigid-bar system with linear and rotational springs

(a) Equilibrium approach

In the displaced position, the restoring force is provided by linear and rotational springs. The destabilizing force producing displacement of the system is due to axial load P. If $P < P_{\text{cr}}$, the system will spring back to its original position, i.e., the restoring moment caused by spring forces is greater than the destabilizing or overturning moment produced by external load. On the other hand, if the destabilizing moment is greater than restoring moment, the equilibrium will be disturbed and the system will collapse or fall down.

At critical load condition, $P = P_{\text{cr}}$ i.e. at neutral equilibrium both the moments balance, each other. At this stage disturbance will make the system merely stay in that displaced position. The generalized coordinate in this case may be taken as rotation, θ. All other displacements can be computed in terms of θ. For equilibrium balance the two moments about an axis passing through the point 1.

$$\text{Destabilizing moment} = \text{Restoring moment}$$

$$P_{\text{cr}} \cdot L\theta = [k_\ell(a\theta)]\, a + k_r\theta$$

Therefore,

$$P_{\text{cr}} = \left(k_\ell a^2 + k_r\right)/L \tag{3.5}$$

(b) Energy approach

The various forms of energy approach are:

1. Principle of virtual displacements
2. Law of conservation of energy, and
3. Principle of stationary potential energy.

According to the principle of virtual displacement, if a system which is in equilibrium under the action of set of forces is subjected to *virtual displacement* i.e. *any displacement compatible with the system constraints*, the total work done by the forces will be zero. This method thus consists of first identifying all the forces acting on the system, and then imposing a small virtual displacement, corresponding to each degree-of-freedom and equating the work done to zero. Let the system shown in Fig. 3.1 is given small virtual displacement θ from the equilibrium position, then

$$\delta W_{\text{ex}} = \delta U$$

where $\delta W_{\text{ex}} = P\Delta$, in which Δ is descent or vertical movement of load P due to rotation θ, given by

$$\Delta = L - L\cos\theta = L - L\left(1 - \frac{\theta^2}{2} + \ldots\right) \approx \frac{L\theta^2}{2} \tag{3.6}$$

Thus,

$$\delta W_{\text{ex}} = P\left(\frac{L\theta^2}{2}\right) \tag{3.7}$$

and

$$\delta U = \frac{1}{2}k_r\theta^2 + \frac{1}{2}k_\ell a^2\theta^2$$

Therefore,

$$\frac{1}{2}P_{\mathrm{cr}}L\theta^2 = \frac{1}{2}k_r\theta^2 + \frac{1}{2}k_\ell a^2\theta^2$$
$$\mathrm{P}_{\mathrm{cr}} = \left(k_r + k_\ell a^2\right)/L$$

The buckling load can also be obtained by the application of law of conservation of energy. According to this law the work done by external load W_{ex} is equal to the internal energy W_{in} developed in the system. In the present case

$$W_{\mathrm{ex}} = P\Delta = PL\theta^2/2$$
$$W_{\mathrm{in}} = \frac{1}{2}[k_r\theta^2 + k_\ell a^2\theta^2]$$

Thus, at critical load condition

$$\frac{1}{2}P_{\mathrm{cr}}L\theta^2 = \frac{1}{2}[k_r\theta^2 + k_\ell a^2\theta^2] \quad \text{giving} \quad P_{\mathrm{cr}} = (k_r + k_\ell a^2)/L$$

The third approach of variation of energy namely the principle of stationary potential is a versatile technique. According to this principle, if the system is in static equilibrium, the potential energy Π of the system has a stationary value. Therefore for stationary potential energy

$$\delta\Pi = 0$$

In the rigid-bar system of Fig. 3.1, due to rigid body displacement the bar does not suffer any deformation, the restoring force is provided by springs and hence elastic energy is stored only in the springs.

The potential of the system is given by

$$\Pi = V + U$$

The potential energy due to external load can be expressed as

$$V = -P\Delta = -P(L/2)\theta^2$$

The strain energy stored in the springs is given by

$$U = \frac{1}{2}k_r\theta^2 + \frac{1}{2}k_\ell(a\theta)^2$$

Thus the total potential energy of the system is

$$\Pi = -(PL\theta^2/2) + \frac{1}{2}[k_r\theta^2 + k_\ell(a^2\theta^2)]$$

For stationary energy

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta}\delta\theta = \left\{-PL\theta + \frac{1}{2}[k_r(2\theta) + k_\ell a^2(2\theta)]\right\}\delta\theta = 0$$

Since $\delta\theta$ is any arbitrary virtual displacement, for non-trivial admissible configuration,

$$-PL\theta + (k_r + k_\ell a^2)\theta = 0$$

or

$$P_{\mathrm{cr}} = [k_r + k_\ell a^2]/L$$

In the following sections single- and multi-degree-of-freedom systems are discussed.

3.3 Single-Degree-of-Freedom Rigid-Bar Assemblages

A single-degree-of-freedom system or a structure which can be adequately idealized by a single-degree-of-freedom system can be analysed by equilibrium and energy approaches.

Example 3.1. In the two bar linkage with top end guided to move freely up and down shown in Fig. 3.2a, the movements at the joints 1 and 2 are constrained by a linear spring of stiffness k_ℓ and rotational springs of stiffness k_{r1} and k_{r2} as shown in the figure. The springs are un-stretched when linkage is vertical. Predict the maximum load P_{cr} for a stable equilibrium.

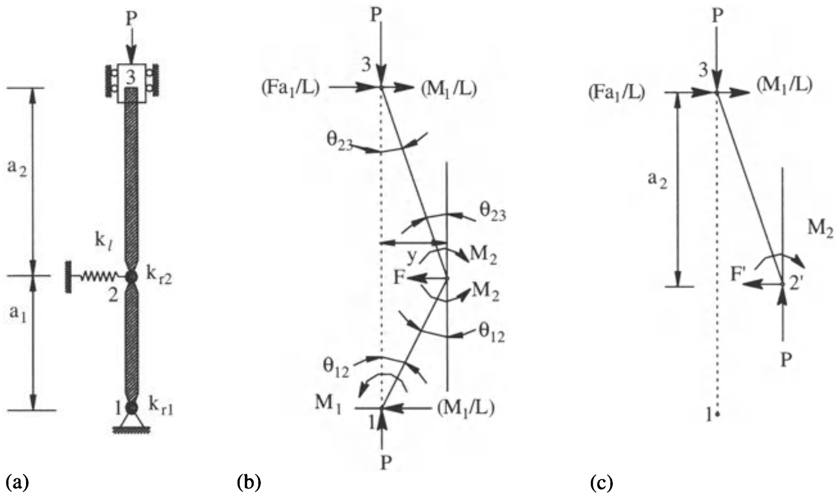


Fig. 3.2a–c. A SDOF rigid-bar assemblage with rotational and linear springs. **a** Two-rigid-bar system, **b** displaced configuration, **c** free-body diagram of rigid-bar 2-3

The generalized coordinate of the system is taken to be the displacement y of the joint 2. Since the linkage bars are rigid, they do not undergo any deformation and hence no energy of any kind is stored when a vanishingly small displacement is imposed on the system as shown in the Fig. 3.2b. However, potential energy is stored in the linear and rotational springs due to stretching and relative rotations of the springs.

(a) Equilibrium approach

Consider moment equilibrium of the forces shown in the free body diagram of Fig. 3.2c about an axis passing through the joint 2.

$$Py - (Fa_1a_2/L) - (M_1a_2/L) - M_2 = 0$$

where $F = k_\ell y; M_1 = k_{r1}(y/a_1); M_2 = k_{r2}(\theta_2) = k_{r2}(\theta_{12}+\theta_{23}) = k_{r2}y(1/a_1+1/a_2)$ and $L = a_1 + a_2$. On substitution the equilibrium equation reduces to:

$$Py = (k_\ell y)\left(\frac{a_1a_2}{a_1+a_2}\right) + \left(k_{r1}\frac{y}{a_1}\right)\left(\frac{a_2}{a_1+a_2}\right) + \left[k_{r2}\frac{y(a_1+a_2)}{a_1a_2}\right]$$

or

$$P_{\text{cr}} = \left\{k_\ell(a_1a_2) + k_{r1}(a_2/a_1) + k_{r2}(a_1+a_2)^2/(a_1a_2)\right\}/(a_1+a_2)$$

for typical case with $a_1 = a_2 = a$

$$P_{\text{cr}} = (k_\ell a^2 + k_{r1} + 4k_{r2})/(2a)$$

(b) Energy approach

Principle of stationary potential energy

The values of V and U with respect to displaced equilibrium configuration can be computed as follows. The potential energy, V associated with the descent or vertical displacement Δ [$= (a_1\theta_{12}^2 + a_2\theta_{23}^2)/2$] of the load towards base position is given by

$$V = P(-\Delta) = -P\frac{y^2}{2}\left[\frac{1}{a_1} + \frac{1}{a_2}\right]$$

The potential energy associated with the deformations of linear and rotational springs is given by

$$U = \frac{1}{2}\left\{k_\ell y^2 + k_{r1}\left(\frac{y}{a_1}\right)^2 + k_{r2}\left[y\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\right]^2\right\}$$

Potential,

$$\Pi = U + V = \frac{1}{2}\left\{k_\ell y^2 + k_{r1}\left(\frac{y}{a_1}\right)^2 + k_{r2}\left[y\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\right]^2 - Py^2\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\right\}$$

For potential to be stationary,

$$\delta \Pi = \frac{\partial \Pi}{\partial y} \delta y = 0$$

$$\delta \Pi = \frac{1}{2} \left\{ 2k_\ell y + k_{r1} \left(\frac{2y}{a_1^2} \right) + k_{r2} \left[2y \left(\frac{1}{a_1} + \frac{1}{a_2} \right)^2 \right] - 2Py \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \right\} \delta y = 0$$

For non-trivial solution $\delta y \neq 0$, and hence

$$k_\ell y + k_{r1} \left(\frac{y}{a_1^2} \right) + k_{r2} y \left(\frac{1}{a_1} + \frac{1}{a_2} \right)^2 - Py \left(\frac{1}{a_1} + \frac{1}{a_2} \right) = 0$$

or

$$\begin{aligned} P_{\text{cr}} &= k_\ell \left(\frac{a_1 a_2}{a_1 + a_2} \right) + k_{r1} \left[\left(\frac{a_2}{a_1 + a_2} \right) \frac{1}{a_1} \right] + k_{r2} \left(\frac{1}{a_1} + \frac{1}{a_2} \right)^2 \\ &= \left[k_\ell (a_1 a_2) + k_{r1} (a_2/a_1) + k_{r2} \frac{(a_1 + a_2)^2}{a_1 a_2} \right] \frac{1}{(a_1 + a_2)} \end{aligned}$$

Principle of virtual displacements

Let the system be given a virtual displacement y from the equilibrium position. The virtual work performed by the conservative load P as it moves (descends) through a distance Δ produced by virtual displacement is given by

$$\delta W_{\text{ex}} = P(\Delta) = P \frac{1}{2} \left(a_1 \theta_{12}^2 + a_2 \theta_{23}^2 \right) = P \frac{y^2}{2} \left[\frac{1}{a_1} + \frac{1}{a_2} \right]$$

The virtual work done by linear and rotational springs is

$$\delta U = \frac{1}{2} \left\{ k_\ell y^2 + k_{r1} \left(\frac{y}{a_1} \right)^2 + k_{r2} \left[y \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \right]^2 \right\}$$

From the principle of virtual displacements

$$\begin{aligned} \delta W_{\text{ex}} &= \delta U \\ P_{\text{cr}} &= \left[k_\ell + k_{r1} \left(\frac{1}{a_1} \right)^2 + k_{r2} \left(\frac{a_1 + a_2}{a_1 a_2} \right)^2 \right] \cdot \left(\frac{a_1 a_2}{a_1 + a_2} \right) \\ &= \left[k_\ell (a_1 a_2) + k_{r1} \left(\frac{a_2}{a_1} \right) + k_{r2} \frac{(a_1 + a_2)^2}{a_1 a_2} \right] \frac{1}{(a_1 + a_2)} \end{aligned}$$

It can be seen that energy approach is advantageous in the sense that in its formulation, definition of coordinate system and sign convention is not required.

Example 3.2. The rectangular rigid-bar-assemblage shown in Fig. 3.3 consists of three rigid-bars interconnected by frictionless hinges. The displacement at the joint 3 and rotation at the joint 4 are resisted by linear and rotational springs, respectively. The critical load for the systems is to be predicted.

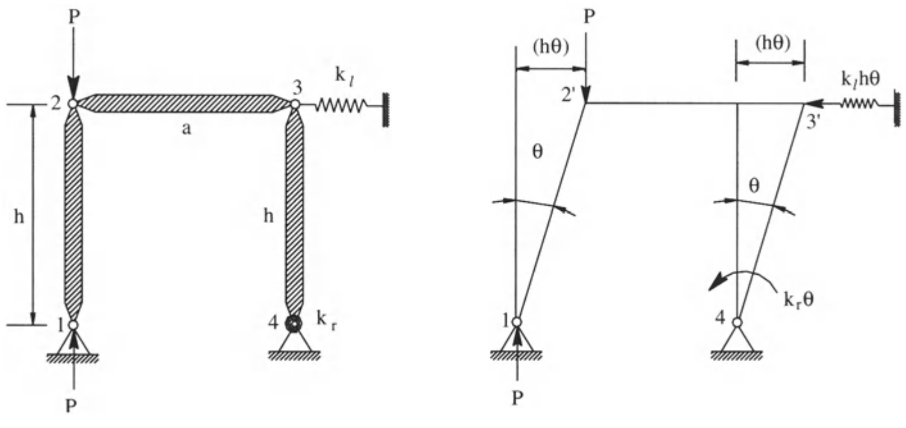


Fig. 3.3a,b. SDOF 3-rigid-bar assemblage with rotational and linear springs. **a** Rigid-bar assemblage, **b** displaced configuration

The generalized coordinate of this system is taken to be rotational angle θ. All other displacements are computed in terms of this generalized coordinate. The potential energy associated with the deformation of the linear and rotational springs and due to displacement (descent), Δ of load P towards the base position, is given by

$$\Pi = V + U = P\frac{1}{2}\left(-h\theta^2\right) + \frac{1}{2}\left[k_\ell\,(h\theta)^2 + k_r\,(\theta)^2\right] \quad \text{as} \quad \Delta = \left(\frac{1}{2}\mathrm{h}\theta^2\right)$$

For stationary potential energy

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta}\delta\theta = \left[-Ph\theta + k_\ell h^2\theta + k_r\theta\right]\delta\theta = 0$$

For non-trivial solution

$$-Ph + k_\ell h^2 + k_r = 0 \quad \text{or} \quad P_{\text{cr}} = (k_\ell h) + \left(\frac{k_r}{h}\right)$$

The result can also be obtained by equilibrium approach. For example consider moment equilibrium about an axis passing through point 1 of the system.

$$\text{Disturbing moment} = \text{restoring moment due to spring actions}$$
$$P(h\theta) = k_\ell(h\theta)h + k_r\theta$$

giving,

$$P_{\text{cr}} = (k_\ell h) + \left(\frac{k_r}{h}\right)$$

3.3.1 Modeling of Elastically Deformable Elements by Equivalent Springs

Sometimes it is convenient and conceptually simple to replace a flexible or elastically deformable member by equivalent linear or rotational spring. The stiffness constant of such a linear or rotational spring is defined as a force required for a unit deformation and thus equivalent to the inverse of displacement or rotation due to unit load or unit moment as the case may be. For illustration consider the simply supported beam shown in Fig. 3.4a carrying a concentrated load P at the mid-span point. For small deflection within elastic range, the displacement is proportional to the corresponding load. This can be expressed as $P = ky$ which is identical in form to the law of a linear normal force spring. If P is made unity

$$1 = ky \quad \text{or} \quad k = (1/y) = \left(48EI/L^3\right)$$

The beam clearly behaves as a spring support for the load P as shown in Fig. 3.4b. The beam can thus be replaced by an equivalent spring with stiffness $k = (1/y)$, where y is the deflection of the structure due to unit load acting at the same point where the equivalent spring effect of the beam is to be determined. Similarly the horizontal member of the structure shown in Fig. 3.4c can be simulated by computing rotation θ_1 due to unit moment and its inverse $(1/\theta_1)$ is the required value of the spring constant $k_r = 3EI/L$. The following three examples illustrate the modelling or idealization of flexible members by linear and rotational springs.

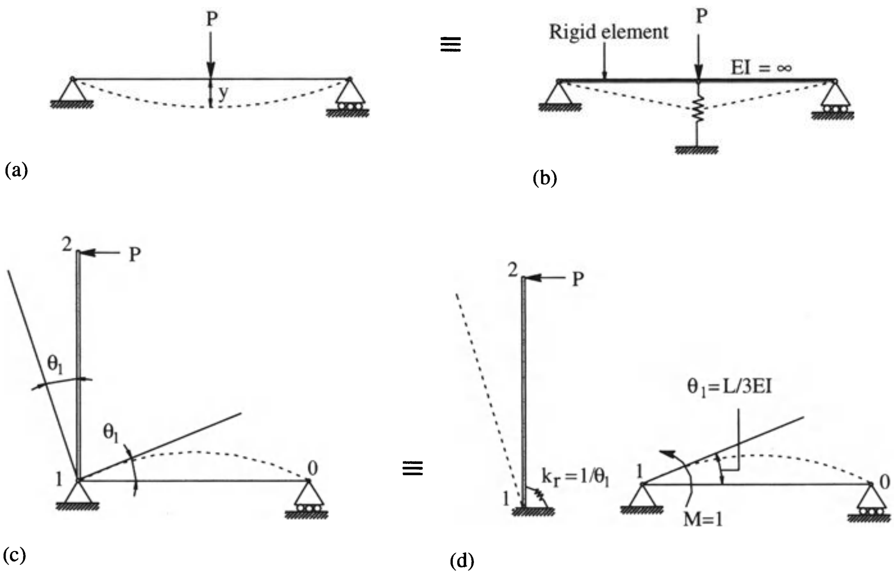


Fig. 3.4a–d. Modelling of elastically deformable elements by equivalent springs. **a** Continuous beam, **b** discrete model, **c** structure, **d** simulated model

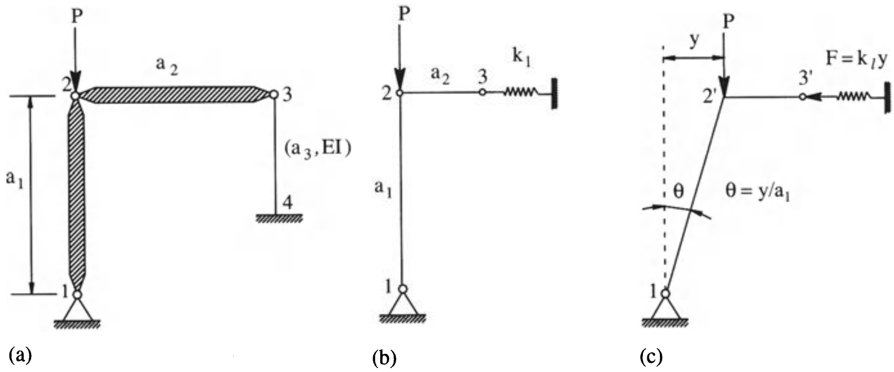


Fig. 3.5a–c. SDOF rigid and flexible-bar assemblage. **a** Rigid and flexible-bars system, **b** idealized rigid-bar system, **c** deflected position

Example 3.3. The rectangular rigid and flexible bars assemblage shown in Fig. 3.5a consists of two rigid bars 1-2 and 2-3, and one flexible member 3-4 with a given EI value, interconnected by frictionless hinges. It is required to predict the critical load for the system.

During deformation the flexible member 3-4 essentially behaves as a cantilever which is a modelled as a linear normal force spring with a spring constant $k_\ell = (3EI/a_3^3)$. The idealized model is shown in Fig. 3.5b. The generalized coordinate of the model is taken as displacement y at the joint 2. The potential energy associated with the descent, Δ of load P and deformation of linear spring is given by:

$$\Pi = V + U = P\left(\frac{y^2}{2a_1}\right) + \frac{1}{2}k_\ell y^2 \qquad \left[\text{since } \Delta = \left(\frac{a_1}{2}\right)\theta^2 = \frac{a_1}{2}\left(\frac{y}{a_1}\right)^2\right]$$

For stationary potential energy

$$\delta\Pi = \frac{\partial\Pi}{\partial y}\delta y = \left(-\frac{Py}{a_1} + k_\ell y\right)\delta y = 0$$

For non-trivial solution,

$$-\frac{Py}{a_1} + k_\ell y = 0$$

Thus,

$$P_{\text{cr}} = k_\ell a_1 = \left(3EIa_1/a_3^3\right)$$

The equilibrium approach is equally applicable. Consider moment equilibrium about the hinge 1.

$$P(y) = (k_\ell y)(a_1)$$

or

$$P_{\rm cr} = k_\ell a_1 = \left(3EIa_1/a_3^3\right)$$

Example 3.4. The rectangular rigid and flexible bars-assemblage shown in Fig. 3.6 consists of three rigid-bars 1-2, 2-3 and 3-4 and one flexible axial member 3-5 interconnected by frictionless hinges. The end 4 of bar 3-4 is rigidly connected to a flexible flexural member 4-6 as shown in the figure. It is desired to compute the critical load for the system.

The flexible hinged member 3-5 is modelled by a linear normal force spring of stiffness (EA/a_3) and the member 4-6 is modelled by a rotational spring of stiffness $(4EI/a_4)$. The idealized rigid-bars system with concentrated spring actions is shown in Fig. 3.6b. This reduced system is same as that given in Fig. 3.3. The generalized coordinate of this system is taken to be rotation θ. The potential energy of the system is given by

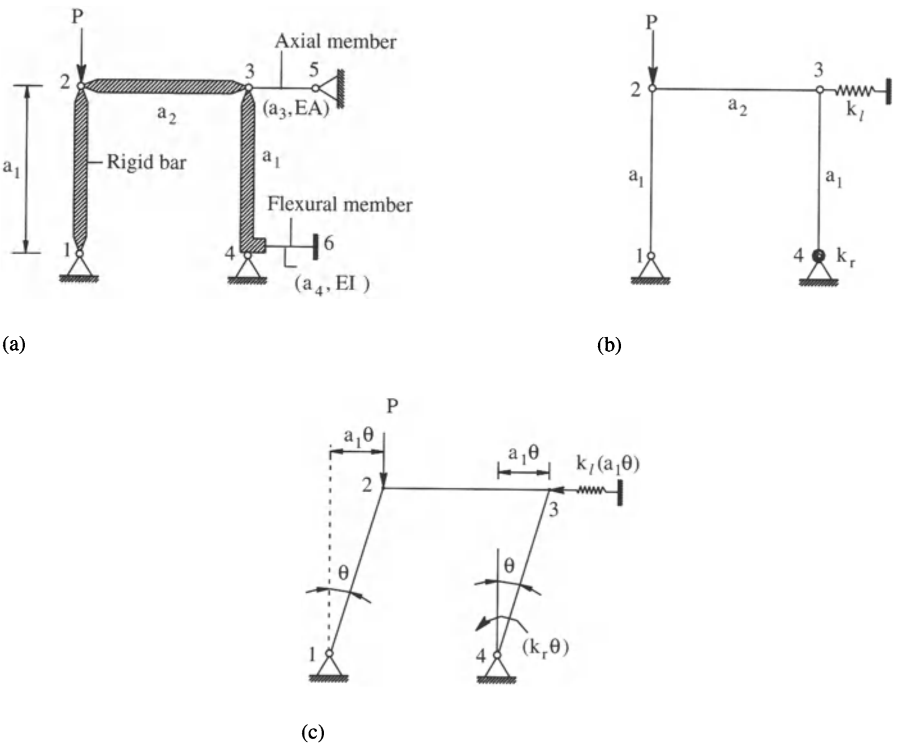


Fig. 3.6a–c. SDOF rigid-bar assemblage with flexural and axial action members. **a** System with rigid and flexible bars, **b** idealized rigid-bar system, **c** displaced configuration

$$\Pi = V + U = \frac{1}{2}\left(-Pa_1\theta^2\right) + \frac{1}{2}\left[k_\ell\,(a_1\theta)^2 + k_r(\theta)^2\right]$$

for stationary potential energy

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta}\delta\theta = \left[(-Pa_1\theta) + k_\ell a_1^2\theta + k_r\theta\right]\delta\theta = 0$$

Therefore for non-trivial solution

$$-Pa_1 + k_\ell a_1^2 + k_r = 0$$

and

$$P_{\mathrm{cr}} = \left(k_\ell a_1 + \frac{k_r}{a_1}\right) = \left(\frac{EA}{a_3}\right)a_1 + \left(\frac{4EI}{a_4}\right)\frac{1}{a_1}$$

Example 3.5. A crane consisting of a rigid bar 1-2 of length L hinged at 1 is supported by an elastic cable 2-3 as shown in Fig. 3.7a. In the unloaded condition the bar is inclined at an angle of 60° from the horizontal. If the load P is increased gradually at what angle will the system become unstable.

The flexible cable or member 2-3 in this case can be modelled by a linear spring of stiffness $k_\ell = (EA/a)$ as shown in Fig. 3.7b. The displaced equilibrium configuration is given in the Fig. 3.7b. In the displaced position spring takes an inclined position. However, for small displacement it may be assumed to act horizontally.

$$\text{Extension of the spring} = L\cos\theta - L\cos 60^\circ = L\left(\cos\theta - \frac{1}{2}\right)$$

$$\text{vertical descent of load } P,\ \Delta = L\sin 60^\circ - L\sin\theta = L\left(\frac{\sqrt{3}}{2} - \sin\theta\right)$$

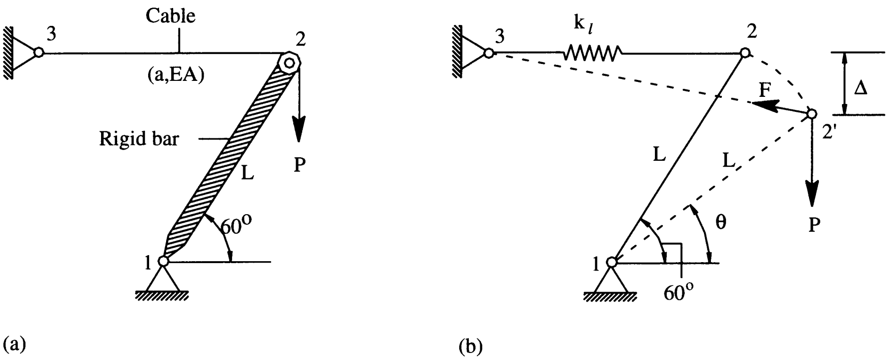


Fig. 3.7a,b. SDOF rigid-bar and cable assemblage. **a** Rigid-bar cable system, **b** idealized system with displaced configuration

The values of V and U based on the equilibrium configuration are given by

$$V = -P\Delta = PL\left(\sin\theta - \sqrt{3}/2\right)$$

$$U = \frac{1}{2}k_\ell\left[L\left(\cos\theta - \frac{1}{2}\right)\right]^2$$

Total potential of the system

$$\Pi = V + U = PL\left(\sin\theta - \sqrt{3}/2\right) + \frac{1}{2}k_\ell L^2\left(\cos\theta - \frac{1}{2}\right)^2$$

From the principle of stationary potential energy

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta}\delta\theta = \left[PL\cos\theta + k_\ell L^2\left(\cos\theta - \frac{1}{2}\right)(-\sin\theta)\right]\delta\theta = 0$$

Since $\delta\theta$ is an arbitrarily small virtual displacement

$$PL\cos\theta - k_\ell L^2\left(\cos\theta - \frac{1}{2}\right)\sin\theta = 0$$

Therefore,

$$P = k_\ell L\left(\cos\theta - \frac{1}{2}\right)\tan\theta = \left(\frac{EA}{a}\right)L\left(\cos\theta - \frac{1}{2}\right)\tan\theta$$

For a given load value of critical angle can be computed. This analysis can also be accomplished by equilibrium approach. Consider moment equilibrium about an axis passing through the hinge 1

$$PL\cos\theta = k_\ell L\left[\left(\cos\theta - \frac{1}{2}\right)\right]L\sin\theta$$

Giving,

$$P = k_\ell L\left(\cos\theta - \frac{1}{2}\right)\tan\theta$$

Thus, the value of P is dependent on θ. For critical value of P

$$\frac{\mathrm{d}P}{\mathrm{d}\theta} = k_\ell L\left(\cos\theta - \frac{1}{2}\sec^2\theta\right) = 0$$

or

$$\cos^3\theta = \frac{1}{2} \quad \text{i.e.} \quad \theta_{\mathrm{cr}} = 37.467^\circ$$

Therefore,

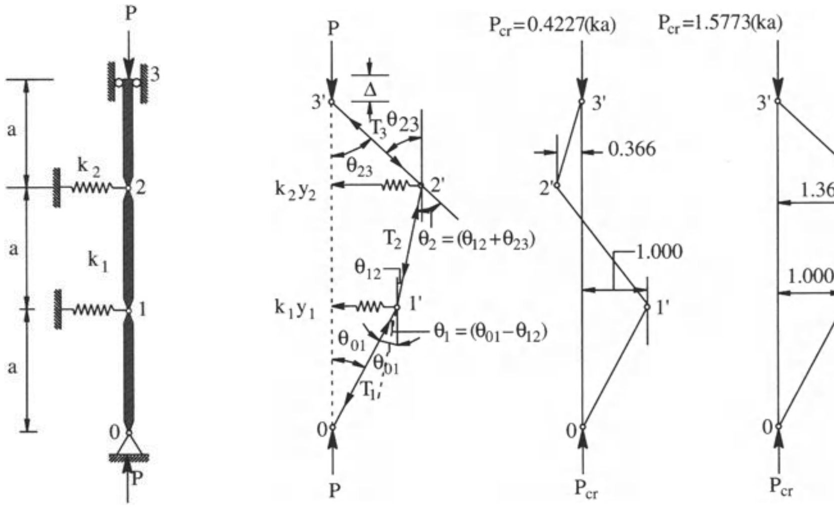
$$P_{\mathrm{cr}} = 0.2251 k_\ell L$$

It should be noted that the assumption that the spring force is horizontal is not valid and the analysis gives only a rough estimate of P_{cr}.

3.4 Two-Degree-of-Freedom Systems

The Examples 3.6 and 3.7 illustrate the application of foregoing principles to the two-degree-of-freedom rigid-bar systems with concentrated linear and rotational springs.

Example 3.6. In the rigid-bar-assemblage shown in Fig. 3.8, three uniform rigid bars of length a are interconnected by hinges at points 1 and 2 and their lateral displacements are resisted by linear springs located at each hinge with values as indicated. The assemblage is guided to move vertically by rollers at point 3 and hinged at point O. It is desired to compute critical load of the assemblage.



(a) (b) (c)

Fig. 3.8a–c. 2-DOF rigid-body assemblage with linear springs. **a** Rigid-bar assemblage, **b** deflected shape, **c** buckling modes

Unlike the rigid-bar systems discussed earlier, this system has two degrees-of-freedom since two generalized coordinates y_1 and y_2 are required to define its displaced configuration. Depending upon the magnitudes and signs of these ordinates the system has two buckling modes, and each mode has a corresponding critical load value. The system can be analysed by equilibrium and by principle of stationary potential energy approaches.

(a) Equilibrium approach

The angles of inclination of various rigid-bars are expressed in terms of displacement coordinates y_1 and y_2 as follows

$$\tan\theta_{01} = \frac{y_1}{a}, \quad \tan\theta_{12} = \frac{y_2 - y_1}{a} \quad \text{and} \quad \tan\theta_{23} = \frac{y_2}{a} \tag{a}$$

The axial forces transferred through the bars are computed by considering equilibrium in vertical direction. Equilibrium at

joint-0 $T_1 \cos\theta_{01} = P$ i.e. $T_1 = (P/\cos\theta_{01})$
joint-1 $T_1 \cos\theta_{01} - T_2 \cos\theta_{12} = 0$ i.e. $T_2 = T_1\,(\cos\theta_{01}/\cos\theta_{12}) = (P/\cos\theta_{12})$
joint-3 $T_3 \cos\theta_{23} = P$ i.e. $T_3 = (P/\cos\theta_{23})$ (b)

Now consider equilibrium in the horizontal direction
at joint-1:

$$T_1 \sin\theta_{01} - T_2 \sin\theta_{12} = k_1 y_1$$
$$P\tan\theta_{01} - P\tan\theta_{12} = k_1 y_1$$

Thus,

$$P\left(\frac{y_1}{a}\right) - P\left(\frac{y_2 - y_1}{a}\right) = k_1 y_1 \quad \text{or} \quad (2P - k_1 a)y_1 - Py_2 = 0 \tag{c}$$

at joint-2: $P\tan\theta_{12} + P\tan\theta_{23} = k_2 y_2$

$$P\left(\frac{y_2 - y_1}{a}\right) + P\left(\frac{y_2}{a}\right) = k_2 y_2 \quad \text{or} \quad -Py_1 + (2P - k_2 a)y_2 = 0 \tag{d}$$

The linear homogeneous equations (c) and (d) are expressed as

$$\begin{bmatrix} (2P - k_1 a) & -P \\ -P & (2P - k_2 a) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{e}$$

for non-trivial solution i.e. non-vanishing values y_1 and y_2

$$\begin{vmatrix} (2P - k_1 a) & -P \\ -P & (2P - k_2 a) \end{vmatrix} = 0$$

Therefore,

$$4P^2 - 2(k_1 + k_2)aP + k_1 k_2 a^2 - P^2 = 0$$
$$3P^2 - 2a(k_1 + k_2)P + k_1 k_2 a^2 = 0$$

giving

$$P_{\mathrm{cr}} = \frac{a}{3}\left\{(k_1 + k_2) \pm \left[(k_1 + k_2)^2 - 3k_1 k_2\right]^{1/2}\right\} \tag{f}$$

for computation of buckling modes consider either of the two equations (c) and (d)

$$\frac{y_2}{y_1} = \frac{2P - k_1 a}{P} = \left(2 - \frac{k_1 a}{P}\right)$$

Since the normalized modes have relative displacement values take $y_1 = 1$ and thus

$$y_2 = [2 - (k_1 a/P)] \tag{g}$$

As a typical case take $k_1 = k$ and $k_2 = 2k$. From equation (f)

$$P_{cr1} = 0.4227ka \quad \text{and} \quad P_{cr2} = 1.5773ka$$

The corresponding buckling modes or deformation configurations from equation (g) are

$$\begin{aligned} P_{cr1} &= 0.4227ka, \quad y_1 = 1.0 \quad \text{and} \quad y_2 = -0.366 \\ P_{cr2} &= 1.5773ka, \quad y_1 = 1.0 \quad \text{and} \quad y_2 = 1.366 \end{aligned}$$

These modes are shown in Fig. 3.8c. Since the load will attain lower value first, P_{cr1} gives critical load value. The readers will appreciate that this treatment is similar to computation of eigenvalues.

(b) Stationary potential energy approach

For small values of displacements, $\tan\theta_{01} \cong \theta_{01}$, $\tan\theta_{12} = \theta_{12}$ and $\tan\theta_{23} = \theta_{23}$. Due to rotation the downward movements of various bars are

$$\begin{aligned} \Delta_{01} &= \left(a\theta_{01}^2/2\right) = \left(y_1^2/2a\right) \\ \Delta_{12} &= \left(a\theta_{12}^2/2\right) = (y_2 - y_1)^2/2a \quad \text{and} \\ \Delta_{23} &= \left(a\theta_{23}^2/2\right) = \left(y_2^2/2a\right) \end{aligned}$$

The total descent or movement of load point is

$$\begin{aligned} \Delta = \Delta_{01} + \Delta_{12} + \Delta_{23} &= \frac{1}{2}\left(\theta_{01}^2 + \theta_{12}^2 + \theta_{23}^2\right) a \\ &= \frac{1}{2a}\left[y_1^2 + (y_2 - y_1)^2 + y_2^2\right] \end{aligned} \tag{h}$$

The potential energy associated with the movement of the load P is $V = -P\Delta$, and the strain energy stored in the linear springs is given by

$$U = (k_1 y_1^2 + k_2 y_2^2)/2\,.$$

Thus potential energy is given by

$$\Pi = (V + U) = \left\{-\frac{P}{2a}\left[y_1^2 + (y_2 - y_1)^2 + y_2^2\right] + \frac{1}{2}\left(k_1 y_1^2 + k_2 y_2^2\right)\right\} \tag{i}$$

For stationary potential energy

$$\delta\Pi = \frac{\partial\Pi}{\partial y_1}\cdot\delta y_1 = \left\{-\frac{P}{2a}\left[2y_1 - 2(y_2 - y_1)\right] + k_1 y_1\right\}\delta y_1 = 0$$

and

$$\frac{\partial \Pi}{\partial y_2}\delta y_2 = \left\{-\frac{P}{2a}\left[2(y_2 - y_1) + 2y_2\right] + k_2 y_2\right\}\delta y_2 = 0$$

$$-(2P - k_1 a)y_1 + Py_2 = 0$$

$$Py_1 - (2P - k_2 a)y_2 = 0 \qquad \text{(j)}$$

These equations are same as those obtained by equilibrium approach.

Example 3.7. Three uniform rigid-bars of lengths a_1, a_2 and a_3 are hinged together at the points 1 and 2 as shown in Fig. 3.9a. The top end of the assemblage is guided to move vertically up and down. Concentrated moment resisting elastic springs are attached to adjoining members at the points 1 and 2 with stiffness as indicated in the figure. It is desired to predict critical load for this assemblage.

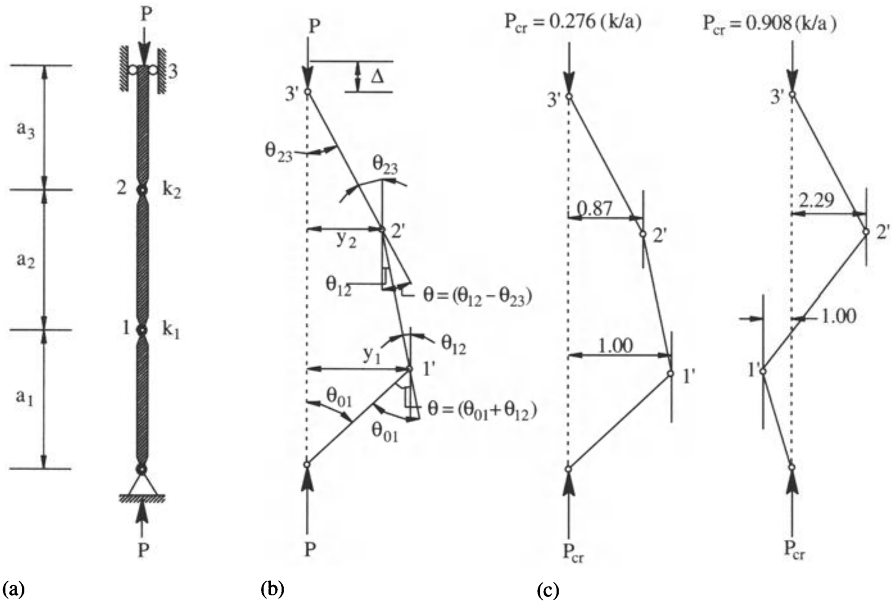


Fig. 3.9a-c. A 2-DOF rigid-body assemblage with rotational springs. **a** Rigid-bar assemblage, **b** deflected configuration, **c** buckling modes

The generalized co-ordinates for this system are taken to be the displacements of hinge points y_1 and y_2 as shown in the figure. Alternatively, the generalized co-ordinates may be taken to be the rotations of the rigid bars. However, in this example the former system is adopted, further it will be assumed that the displacements are small so that small deflection theory is valid.

(a) Equilibrium approach

For moment equilibrium at spring joints 1 and 2, the external moment should balance the internal moment.

At joint-1: $$Py_1 = (k_1\theta_1) = k_1(\theta_{01} + \theta_{12}) = k_1\left[\frac{y_1}{a_1} + \frac{y_1 - y_2}{a_2}\right]$$

$$\left(P - \frac{k_1}{a_1} - \frac{k_1}{a_2}\right) y_1 + \left(\frac{k_1}{a_2}\right) y_2 = 0 \tag{a}$$

At joint-2: $$Py_2 = k_2\theta_2 = k_2(\theta_{23} - \theta_{12}) = k_2\left[\frac{y_2}{a_3} - \frac{y_1 - y_2}{a_2}\right]$$

$$\left(\frac{k_2}{a_2}\right) y_1 + \left(P - \frac{k_2}{a_3} - \frac{k_2}{a_2}\right) y_2 = 0 \tag{b}$$

From equations (a) and (b), for non vanishing values of y_1 and y_2

$$\begin{vmatrix} (P - \frac{k_1}{a_1} - \frac{k_1}{a_2}) & \frac{k_1}{a_2} \\ \frac{k_2}{a_2} & (P - \frac{k_2}{a_2} - \frac{k_2}{a_3}) \end{vmatrix} = 0 \tag{c}$$

From equation (a)

$$\frac{y_2}{y_1} = -\left(P - \frac{k_1}{a_1} - \frac{k_1}{a_2}\right) \Big/ \left(\frac{k_1}{a_2}\right) \tag{d}$$

As a typical case let $a_1 = 4a$, $a_2 = 5a$, $a_3 = 6a$, $k_1 = k$ and $k_2 = 2k$, substitute these values in (c)

$$\begin{vmatrix} (P - \frac{k}{4a} - \frac{k}{5a}) & \frac{k}{5a} \\ \frac{2k}{5a} & (P - \frac{2k}{5a} - \frac{2k}{6a}) \end{vmatrix} = P^2 - \left(\frac{71k}{60a}\right) P - \left(\frac{k^2}{4a^2}\right) \tag{e}$$

The solution of quadratic equation (e) gives two values for P_{cr} corresponding to the first and second buckling modes

$$P = 0.276(k/a) \quad \text{and} \quad 0.908(k/a)$$

The lower value of P i. e. $0.276(k/a)$ gives the critical load. The two buckling modes are obtained by substituting these two critical loads separately into equation (d).

First mode $P_{cr1} = 0.276(k/a)$ $y_2/y_1 = 0.87/1.0$
Second mode $P_{cr2} = 0.908(k/a)$ $y_2/y_1 = -2.29/1.0$

The modes are shown in Fig. 3.9c.

(b) Stationary potential energy approach

The downward movement or descent of the point 3 is

$$\begin{aligned}\Delta &= \Delta_{01} + \Delta_{12} + \Delta_{23} = \frac{1}{2}\left(a_1\theta_{01}^2 + a_2\theta_{12}^2 + a_3\theta_{23}^2\right) \\ &= \frac{1}{2}\left[a_1\left(\frac{y_1}{a_1}\right)^2 + a_2\left(\frac{y_1 - y_2}{a_2}\right)^2 + a_3\left(\frac{y_2}{a_3}\right)^2\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{a_1} + \frac{1}{a_2}\right)y_1^2 + \left(\frac{1}{a_2} + \frac{1}{a_3}\right)y_2^2 - \frac{2y_1y_2}{a_2}\right] \qquad \text{(f)}\end{aligned}$$

The potential energy of axial force, $V = -P\Delta$.

Relative rotations of the bars at joints 1 and 2 are given by

$$\theta_1 = \theta_{01} + \theta_{12} = \left(\frac{y_1}{a_1} + \frac{y_1 - y_2}{a_2}\right) = \left[\left(\frac{1}{a_1} + \frac{1}{a_2}\right)y_1 - \left(\frac{1}{a_2}\right)y_2\right]$$

$$\theta_2 = \theta_{23} - \theta_{12} = \left(\frac{y_2}{a_3} - \frac{y_1 - y_2}{a_2}\right) = \left[-\left(\frac{1}{a_2}\right)y_1 + \left(\frac{1}{a_2} + \frac{1}{a_3}\right)y_2\right]$$

The potential energy of the spring is

$$\begin{aligned}U &= \frac{1}{2}(k_1\theta_1^2) + \frac{1}{2}(k_2\theta_2^2) \\ &= \frac{k_1}{2}\left[y_1\left(\frac{1}{a_1} + \frac{1}{a_2}\right) - \frac{y_2}{a_2}\right]^2 + \frac{k_2}{2}\left[y_2\left(\frac{1}{a_2} + \frac{1}{a_3}\right) - \frac{y_1}{a_2}\right]^2 \qquad \text{(g)}\end{aligned}$$

Total potential energy of the spring and axial force, $\Pi = V + U$

$$\begin{aligned}\Pi = \frac{1}{2}\Bigg\{&\left[k_1\left(\frac{1}{a_1} + \frac{1}{a_2}\right)^2 + k_2\left(\frac{1}{a_2^2}\right) - P\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\right]y_1^2 \\ &+ \left[k_1\left(\frac{1}{a_2^2}\right) + k_2\left(\frac{1}{a_2} + \frac{1}{a_3}\right)^2 - P\left(\frac{1}{a_2} + \frac{1}{a_3}\right)\right]y_2^2 \\ &+ \left[-2k_1\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\left(\frac{1}{a_2}\right) - 2k_2\left(\frac{1}{a_2} + \frac{1}{a_3}\right)\left(\frac{1}{a_2}\right) + \frac{2P}{a_2}\right]y_1y_2\Bigg\} \qquad \text{(h)}\end{aligned}$$

For a typical case, let $a_1 = a_2 = a_3 = a$. From stationary potential energy procedure

$$\left[\frac{1}{a^2}(4k_1 + k_2) - \frac{2P}{a}\right]y_1 + \left[\frac{1}{2a^2}(-4k_1 - 4k_2) + \frac{P}{a}\right]y_2 = 0$$

$$\left[\frac{1}{2a^2}(-4k_1 - 4k_2) + \frac{P}{a}\right]y_1 + \left[\frac{1}{a^2}(k_1 + 4k_2) - \frac{2P}{a}\right]y_2 = 0 \qquad \text{(i)}$$

A non-trivial solution of equation (i) is possible only when assemblage buckles under the action of axial force P, and this is indicated when determinant of the coefficients matrix equals zero. The expansion of determinant and rearrangement of terms gives:

$$(Pa)^2 - 2(k_1 + k_2)(Pa) + 3k_1k_2 = 0$$

$$P_{cr} = \frac{1}{a}\left[(k_1 + k_2) \pm \sqrt{k_1^2 + k_2^2 - k_1k_2}\right] \tag{j}$$

Thus equation (j) gives two values of P_{cr} corresponding to first and second buckling modes. The two mode shapes are found by substituting these two critical loads into either of equations (i) and solving for one of the generalized coordinator in terms of the other. The reader may note that in this particular example the equilibrium approach is much simpler.

The procedures illustrated in Examples 3.6 and 3.7 are equally applicable to higher degrees-of-freedom systems. However, the governing equations become progressively complicated with the increase in the degrees-of-freedoms.

3.5 Discrete Element Method

The method is similar to finite element method which idealizes a continuous structure by a finite degrees-of-freedom discrete model i. e. the method approximates the structure as a chain made up of rigid straight bars connected together by frictionless hinges. The bending rigidity in the model is accomplished by the provision of rotational springs at the hinges.

The major difference in continuous and discrete models is that the deformation of the continuous system is described by differential geometry while that of discrete system by elementary geometry i. e. deformation is given by straight lines and all relationships are obtained from elementary geometry which lead to algebraic equations. In the other words an approximation of a continuous deflection curve by a polygon of straight lines, results in approximating the differential equation of equilibrium by several algebraic equations which are easily solved. Thus this method is amenable to matrix formulation and the method has great potential in solving statically determinate structural problems.

To illustrate the basic idea of the discrete element method considers a simply supported beam shown in Fig. 3.10 wherein the deflection curve is approximated by two straight lines. These straight lines of deflection correspond to fictitious rigid bars assemblage consisting of two rigid elements connected at an elastic hinge or frictionless hinge with linear elastic rotational spring 1. The concept is thus equivalent to replacing the original continuous beam by a fictitious discrete elements beam and thus localizing the rotations at discrete nodal points. The elastic constants of these fictitious springs can be obtained from the standard relation

$$M = EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = EI\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = EI\frac{\mathrm{d}\theta}{\mathrm{d}x} = EI\theta' \tag{3.8}$$

Thus

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{M}{EI}$$

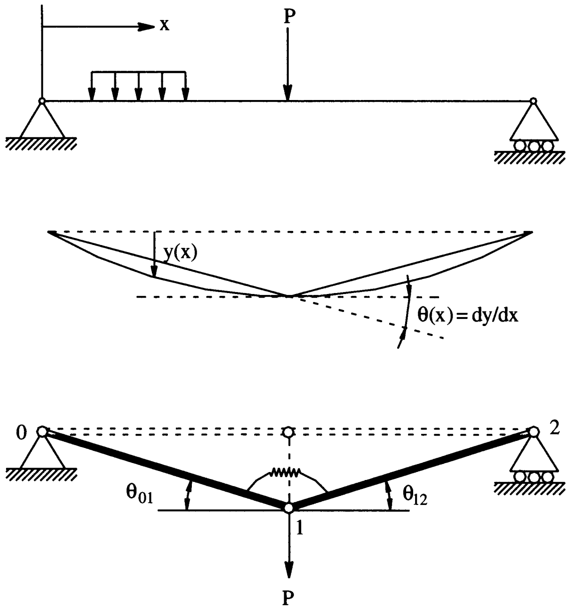


Fig. 3.10. Approximation of deflected curve by two straight lines

which means

$$\lim_{\Delta x \to 0} \frac{\Delta\theta}{\Delta x} = \frac{M}{EI}$$

If the limit to differential quotient is omitted we obtain

$$\frac{\Delta\theta}{\Delta x} \cong \frac{M}{EI} \quad \text{or} \quad M \cong \frac{\Delta\theta}{\Delta x} EI = \left(\frac{EI}{\Delta x}\right)\Delta\theta \tag{3.9}$$

Thus the bending moment at a hinge in a discrete element model is given by

$$M \cong k(\Delta\theta)$$

where $\Delta\theta$ is the change in slope at the hinge. For the bending moment to be same in both the discrete and continuous models

$$M = EI\theta' \cong EI\left(\frac{\Delta\theta}{\Delta x}\right) = k(\Delta\theta) \tag{3.10}$$

Thus we obtain $k = (EI/\Delta x)$. In this expression $\Delta x = L/n$, where n is the number of identical elements. The required expression for the constant of the fictitious spring is

$$k = nEI/L \tag{3.11}$$

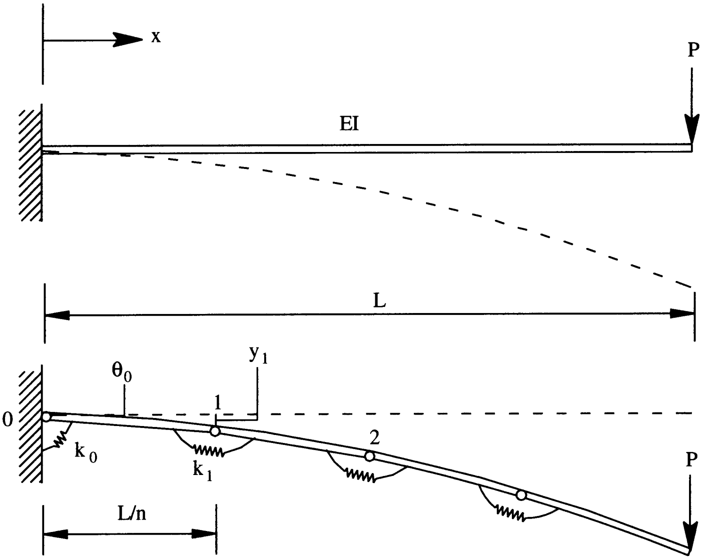


Fig. 3.11. A deflected cantilever approximated by n discrete elements

However, it is recognized that in general the fixed end of a structural system is relatively more stiff. Thus a provision of a spring of greater stiffness than that of common one $k = (nEI/L)$ at the fixed end of discrete element model will lead to more accurate results with fewer elements. To determine such a fixed end spring stiffness consider a cantilever of length L subjected to a concentrated load at the free end. Let the cantilever be divided into n equal discrete elements of length (L/n) as shown in Fig. 3.11. The deflection at a node distant (L/n) from the fixed end is

$$y_1 = \frac{P}{6EI}\left(\frac{L}{n}\right)^2\left(3L - \frac{L}{n}\right) \qquad \left[\text{since} \quad y(x) = \frac{Px^2(3L-x)}{6EI}\right] \tag{3.12}$$

Application of moment equilibrium equation $\Sigma M_0 = 0$ at the node 0 gives

$$k_0\theta_0 = PL \quad \text{where} \quad \theta_0 = \frac{ny_1}{L}$$

and substituting for θ_0 and y_1 gives

$$k_0 = \frac{6n^2}{3n-1}\left(\frac{EI}{L}\right) \tag{3.13}$$

Example 3.8. The deflection curve of the simply supported strut shown in Fig. 3.12 is approximated by three equal straight lines i. e. the system is divided into three rigid elements which are connected at elastic hinges with the spring stiffness k. Due to symmetry the system can be dealt as a single-degree-of-freedom system. From the geometry of deformation curve shown in the figure

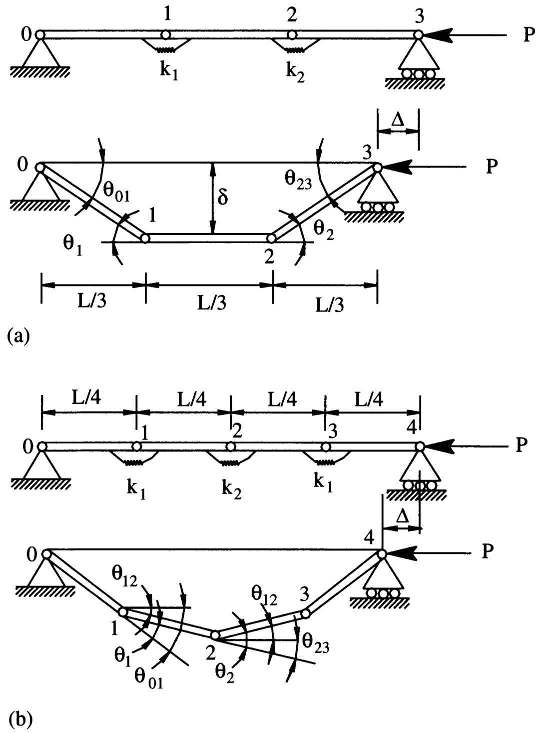


Fig. 3.12a,b. Three and four-element models of simply supported strut of Example 3.8. **a** Three-element model, **b** four-element model

$$\theta_1 = \frac{3\delta}{L}, \quad k_1 = k_2 = k = \frac{3EI}{L},$$

$$\theta_1 = \theta_{01} \quad \text{and} \quad \Delta = 2\left(\frac{L}{3}\right)(1 - \cos\theta_{01})$$

The total potential energy in this case is

$$\begin{aligned} \Pi &= 2\left(\frac{1}{2}\right)k\theta_{01}^2 - 2P\left(\frac{L}{3}\right)(1 - \cos\theta_{01}) \\ &= k\theta_{01}^2 - (2PL/3)(1 - \cos\theta_{01}) \end{aligned}$$

The equilibrium or stationary potential energy equation $\partial\Pi/\partial\theta_{01} = 0$ gives

$$P_{\text{cr}} = \frac{3k\theta_{01}}{L\sin\theta_{01}} = \frac{9EI\theta_{01}}{L^2\sin\theta_{01}}$$

Expanding $(\sin\theta_{01})^{-1}$ and retaining up to second-order terms only, the initial post buckling equation reduces to

$$P_{\text{cr}} = \frac{9EI}{L^2}\left(1 + \frac{1}{6}\theta_{01}^2\right)$$

where $\theta_{01} = 3\delta/L$ and δ is the maximum deflection, thus the buckling equation becomes

$$P_{\text{cr}} = \frac{9EI}{L^2} + (13.5)\frac{EI}{L^4}\delta^2$$

The asymptotically exact equation is

$$P_{\text{cr}} = \frac{\pi^2 EI}{L^2} + (12.17)\frac{EI}{L^4}\delta^2$$

For a mere single degree-of-freedom discrete element analysis the result is remarkably accurate both for buckling load ($P_{\text{cr1}} = 9EI/L^2$) and initial post buckling curvature ($P_{\text{cr2}} = 13.5EI/L^4$). The result will converge to the exact value as the number of elements increase. As an illustration consider the structure to be approximated by four element model as shown in Fig. 3.12b. Following the geometry of deformation shown in the figure, the total potential energy of this approximation which is essentially a two degree-of-freedom system due to symmetry is

$$\Pi = 2\left(\frac{1}{2}k_1\theta_1^2\right) + \frac{1}{2}k_2(\theta_2)^2 - \frac{1}{2}P\left(\frac{L}{4}\right)\left(\theta_{01}^2 + \theta_{12}^2\right)(2)$$

noting that

$$\theta_1 = \theta_{01} - \theta_{12} \quad \text{and} \quad \theta_2 = \theta_{12} + \theta_{23} = 2\theta_{12}.$$

Thus,

$$\Pi = k_1(\theta_{01} - \theta_{12})^2 + 2k_2\theta_{12}^2 - [PL(\theta_{01}^2 + \theta_{12}^2)/4]$$

The equilibrium equations from stationary principle are given by

$$\frac{\partial \Pi}{\partial \theta_{01}} = 2k_1(\theta_{01} - \theta_{12}) - \left[\frac{PL}{4}(2\theta_{01})\right] = 0$$

or

$$\left(2k_1 - \frac{PL}{2}\right)\theta_{01} - 2k_1\theta_{12} = 0,$$

and

$$\frac{\partial \Pi}{\partial \theta_{12}} = -2k_1(\theta_{01} - \theta_{12}) + 4k_2\theta_{12} - \frac{PL}{4}(2\theta_{12}) = 0$$

or

$$-2k_1\theta_{01} + \left(2k_1 + 4k_2 - \frac{PL}{2}\right)\theta_{12} = 0$$

for a non-trivial solution

$$\begin{vmatrix} (2k_1 - \alpha) & -2k_1 \\ -2k_1 & (2k_1 + 4k_2 - \alpha) \end{vmatrix} = 0, \quad \text{where} \quad \alpha = PL/2$$

$$\alpha^2 - 4(k_1 + k_2)\alpha + 8k_1k_2 = 0$$

In the present case $k_1 = k_2 = k_3 = k = (4EI/L)$, consequently

$$\alpha^2 - 8k\alpha + 8k^2 = 0$$

which gives

$$\alpha = \frac{PL}{2} = 4k \pm \sqrt{8}k = \left(4 \pm \sqrt{8}\right)\left(\frac{4EI}{L}\right)$$

The buckling load with four element approximation is given by

$$P_{\text{cr}} = (4 - \sqrt{8})(8EI/L^2) = (9.37EI/L^2)$$

This indicates significant improvement over the value obtained with three element approximation.

Example 3.9. To study the convergence of buckling load to its exact value consider a cantilever strut or a column which is fixed at the base and free at the top and subjected to an axial load in un-deformed equilibrium position. This is a standard case and the exact value of buckling load is $\left(\pi^2 EI/4L^2\right)$ or $2.467EI/L^2$. Consider two cases using one and two rigid element discretizations, respectively.

Case I: One-element model, Fig. 3.13b

The potential energy for computation of buckling load of this model with single element is

$$\Pi = \frac{1}{2}k_0\theta_{01}^2 - P\left(\frac{1}{2}\right)L\theta_{01}^2$$

The equilibrium equation from stationary energy principle is

$$\frac{\partial \Pi}{\partial \theta_{01}} = k_0\theta_{01} - PL\theta_{01} = 0$$

from which critical load is found to be

$$P_{\text{cr}} = k_0/L$$

where $k_0 = 3EI/L$ (a case of higher stiffness for spring at the fixed end).

Thus, $P_{\text{cr}} = 3EI/L^2$. As a first approximation the error is tolerable.

Case II: Two-element model, Fig. 3.13c

In this case

$$k_0 = \frac{24EI}{5L} = \frac{4.8EI}{L} \quad \text{and} \quad k = \frac{2EI}{L}$$

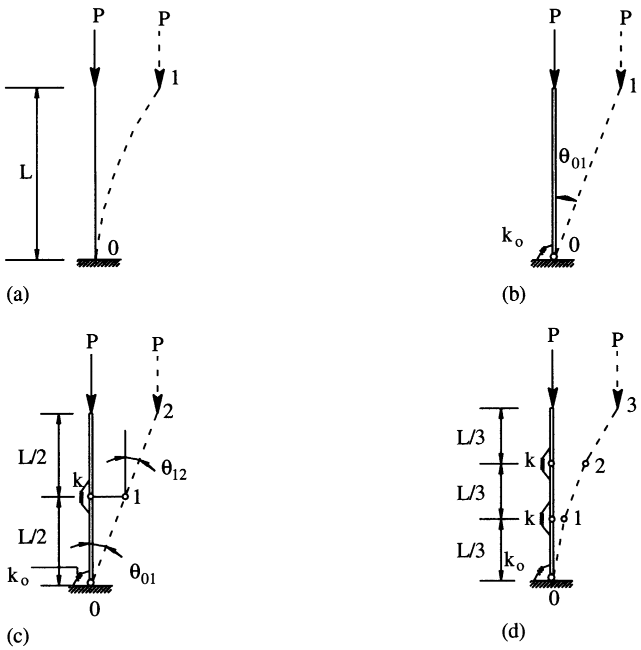


Fig. 3.13a–d. One, two and three-element models for a cantilever strut of Example 3.9. **a** Continuous strut, **b** one-element model, **c** two-element model, **d** three-element model

Thus the potential energy of the system is given by

$$\Pi = \frac{1}{2}k_0\theta_{01}^2 + \frac{1}{2}k(\theta_{12} - \theta_{01})^2 - \frac{1}{2}P\left(\frac{L}{2}\right)\left(\theta_{01}^2 + \theta_{12}^2\right)$$

The equilibrium conditions $\partial\Pi/\partial\theta_{01}$ and $\partial\Pi/\partial\theta_{12}$ lead to two linear homogeneous algebraic equations in θ_{01} and θ_{12}. These equations have non-trivial solution, if the determinant of coefficients of θ_{01} and θ_{12} vanishes. Expanding determinant and solution of resulting quadratic equation gives:

$$P_{cr} = \frac{1}{L}\left[(k_0 + 2k) - \left(k_0^2 + 4k^2\right)^{\frac{1}{2}}\right]$$

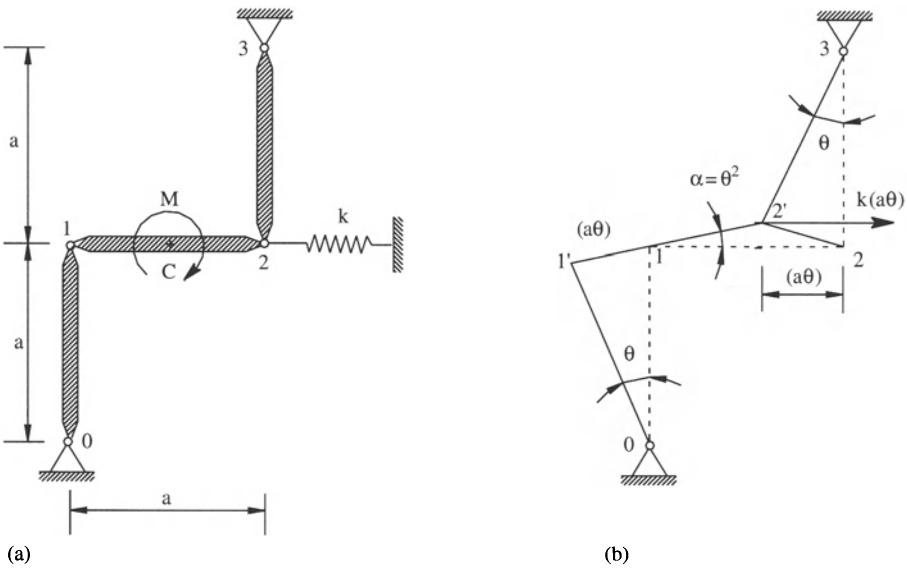
Substituting the values of k_0 and k

$$P_{cr} = 2.552EI/L^2$$

The value is much more closer to the exact value. A model with three-element will give still better result. It is evident that as the number of elements increase the result converges to the exact value.

3.6 Problems

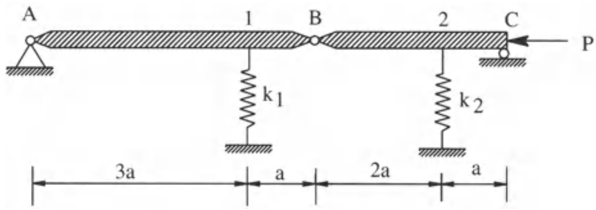
Problem 3.1. Three identical rigid bars of length a are hinged together at the joints 1 and 2, and are supported by hinges at the points 0 and 3 as shown in Fig. P.3.1. The assemblage is stabilized by a linear spring of stiffness k. A moment M is applied at the mid-point of bar 1-2. Predict the critical value of moment M which will make the system elastically unstable.



P.3.1

[Hint: Rotate the bar 0-1 by a small angle about the point o.
$W_{ex} = M\alpha = M\theta^2$ *and* $W_{in} = \frac{1}{2}(ka\theta)(a\theta) = k(a\theta)^2/2$*. Equating* W_{ex} *to* W_{in} *will give* $M_{cr} = ka^2/2$*]*

Problem 3.2. Two rigid bars AB and BC are hinged together at the joint B. The end A is hinged and end C is supported on a roller. Two linear springs of constant



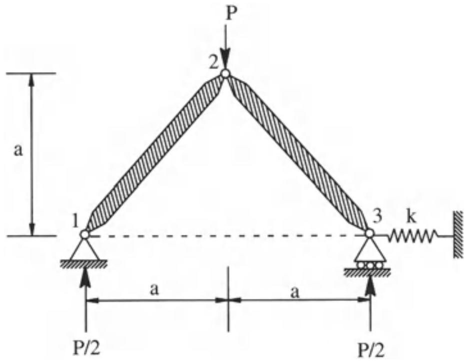
P.3.2

k_1 and k_2 are attached to the points 1 and 2, respectively, as shown in Fig. P.3.2. Determine the critical value of the load P.

[Hint: The displacement of hinge B may be taken as the generalized co-ordinate, and all other displacements are expressed in terms of it. $P_{cr} = (27k_1a/28) + (4k_2a/21)$*]*

Problem 3.3. Two identical rigid-bars 1-2 and 2-3 are hinged together at joint 2 as shown in Fig. P.3.3 and supported by a hinge at a point 1 and a roller at point 3. The movement at the roller end is resisted by a linear spring of constant k. Predict the load at which the assemblage becomes unstable. Also calculate the angle θ which the bar makes with the horizontal.

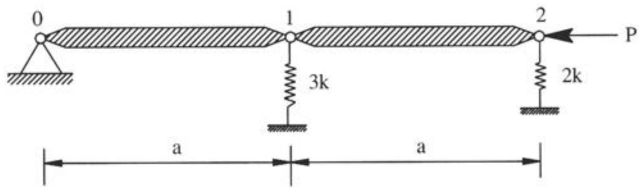
[Ans. $P_{cr} = (3\sqrt{2}/4)ka$ *and* $\cos^3\theta = (1/\sqrt{2})$*]*



P.3.3

Problem 3.4. Two uniform rigid-bars of length a are hinged together at joint 1 and are supported by a hinge at point 0. The displacements at points 1 and 2 are resisted by elastic springs having spring constants $3k$ and $2k$, respectively, as shown in Fig. P.3.4. A constant axial force P acts at the point 2. Determine the critical value of load P that will hold the assemblage in equilibrium in displaced position.

[Ans. $P_{cr} = 1.0(ka)$ *and* $6.0(ka)$ *and corresponding mode shapes can be obtained from:* $(y_2/y_1) = P/(P - 2ka)$*.]*



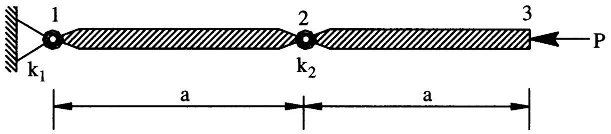
P.3.4

Problem 3.5. Two uniform rigid-bars of length a, 1-2 and 2-3 are hinged together at joint 2 and supported by a hinge at point 1. Concentrated moment resisting elastic springs of stiffnesses k_1 and k_2 are attached at the hinges 1 and 2, respectively. A constant axial force act at the free end 3 as shown in Fig. P.3.5. Determine the critical load and corresponding buckling modes.

[Hint: The potential energy of the assemblage is:

$$\Pi = \frac{1}{2}k_1\theta_1^2 + \frac{1}{2}k_2(\theta_1 - \theta_2)^2 - \frac{1}{2}Pa\left(\theta_1^2 + \theta_2^2\right),$$

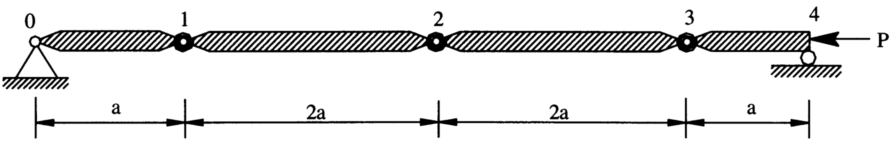
$$\textit{and}\ \ P_{cr} = \frac{1}{2a}\left[(k_1 + 2k_2) \pm \left(k_1^2 + 4k_2^2\right)^{1/2}\right]$$



P.3.5

Problem 3.6. Four uniform rigid-bars of lengths a, $2a$, $2a$ and a are hinged together at joints 1, 2 and 3 as shown in Fig. P.3.6 and are supported by a hinge at point 0 and roller at point 4. Concentrated moment resisting elastic springs of stiffness k are attached to adjoining bars at hinges 1, 2 and 3. A constant axial force acts at the roller end 4. Using small deflection theory determines the critical loads and corresponding buckling modes.

[Ans. $P_{cr} = 0.5(k/a)$, $1.0(k/a)$ *and* $4.0(k/a)$ *with corresponding buckling modes as* $(1, 0, -1)$, $(1, 2, 1)$ *and* $(1, -1, 1)$*]*



P.3.6

Problem 3.7. For calculating the critical load, a pin-ended Euler column of length $6a$ and stiffness EI is assumed to be divided into three equal segments of length $2a$ each, with flexural stiffness of each segment $k = EI/2a$ being concentrated at the centre of the segment by a spring-hinge. The hinges are imagined to be connected by rigid-bars. Determine the critical load and the percentage error introduced is this idealization.

[Hint: Refer to Problem 3.6, $P_{cr} = EI/(4a^2)$, Euler's load $= \pi^2 EI/(36a^2)$, percentage error $= 8.81$]

Problem 3.8. A column fixed at the base and hinged at the top carries an axial load P in the un-deformed equilibrium position. Determine the buckling load by considering one-, two- and three-rigid element discretizations. Also determine the percentage error when exact buckling load is $(20.19EI/L^2)$.

Problem 3.9. A cantilever column with fixed base and free top is discretized with three-rigid elements. The flexural stiffness of these elements is assumed to be lumped or concentrated at the interconnecting hinges. Determine critical loads and the percentage error introduced in the idealization.

Problem 3.10. A fixed–fixed strut subjected to an axial load P is discretized with two-and three-rigid elements. The flexural stiffness of these discrete elements is considered to be concentrated at the interconnecting hinges. Predict the buckling load for the models and percentage of error introduced in each of the models when exact value of buckling load is $4\pi^2 EI/L^2$.

4

Buckling of Axially Loaded Members (Columns)

4.1 Introduction

The classical critical load theory of perfect axial members assumes that the member in question is initially straight, slender, of solid cross section with flexural stiffness rigidity EI being constant throughout its length and subjected to an axial compressive force applied along the *centroidal axis* of the member. Moreover, it is presumed that the material of the member is homogeneous, isotropic and perfectly elastic. The assumption of *small deflection theory* of bending also holds good for the critical load theory.

The critical value of the axial thrust for a centrally loaded member is generally expressed in terms of that for an *idealized column* which is hinged at both the ends and subjected to an axial compressive force. This column is known as *Euler column* with the critical value of axial thrust being called as Euler buckling load which is denoted as P_e.

4.2 Buckling Loads for Members with Different End Conditions

The buckling loads can be derived directly from the governing differential equations obtained by considering the state of equilibrium of the member in its bend form caused by a disturbance. In view of small deflection theory being used, the *moment curvature relation* becomes linear and can be expressed as

$$M = -EI\left(\frac{d^2y}{dx^2}\right) \tag{4.1}$$

where M is bending moment and I is second moment of area. For a deformed shape to be in equilibrium, the internal resisting moment must balance the external disturbing moment Py. Hence

$$Py = -EI\left(\frac{d^2y}{dx^2}\right)$$

M. L. Gambhir, *Stability Analysis and Design of Structures*

or

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0 \tag{4.2}$$

where $\alpha^2 = P/(EI)$. Equation (4.2) is a linear homogeneous differential equation with constant coefficients. It should be noted that since the buckling will occur in the plane of minimum bending rigidity, minimum value of EI is to be used in (4.2). This is second-order formulation of the problem. The general solution is

$$y(x) = A\sin\alpha x + B\cos\alpha x \tag{4.3}$$

The arbitrary constants of integration A and B are evaluated from the prescribed boundary conditions associated with the end supports. The application of the method to the cases with standard boundary conditions is illustrated in the following section.

4.2.1 Hinged-Hinged Strut

Consider a strut hinged at both the ends as shown in Fig. 4.1. The boundary conditions, would be $y(0) = y(L) = 0$. The first condition gives $B = 0$ and in order to satisfy the second boundary condition

$$A\sin\alpha L = 0 \tag{4.4}$$

If A is set equal to zero, then $y(x) = 0$ everywhere along the length, meaning that the initial straight configuration of the strut is the only equilibrium state under the

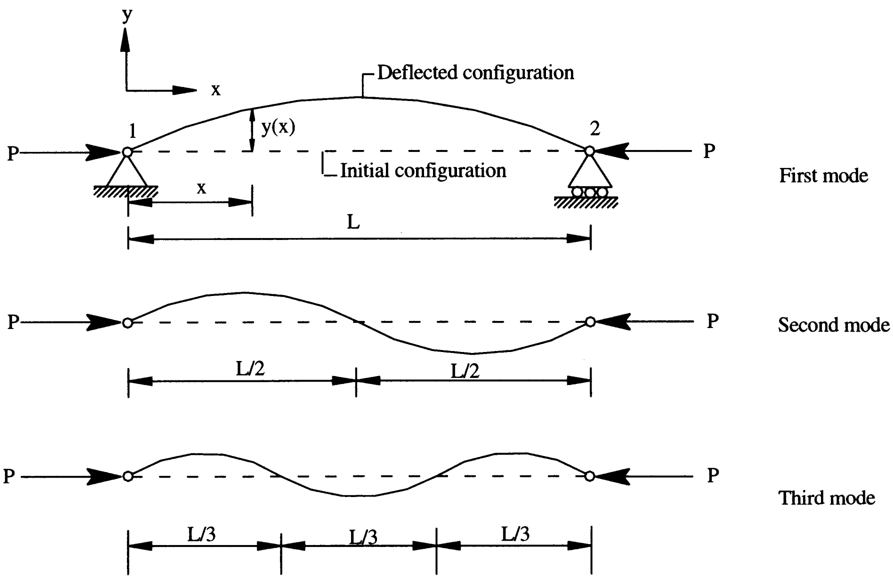


Fig. 4.1. Buckling modes of hinged-hinged strut

force P and no bend equilibrium state is available. This is a trivial solution. Thus for non-trivial solution second term must vanish i. e. $\sin \alpha L = 0$, for which it is necessary that

$$\alpha L = n\pi \quad n = 1, 2, 3 \ldots \tag{4.5}$$

Since $\alpha^2 = P/(EI)$, (4.5) can be written as

$$P_{\mathrm{cr}} = \frac{n^2 \pi^2 EI}{L^2} \tag{4.6}$$

The corresponding deflected shape is given by

$$y = A \sin \frac{n\pi x}{L} \tag{4.7}$$

The smallest P_{cr} value corresponds to the case where $n = 1$. Thus,

$$P_{\mathrm{cr}} = P_{\mathrm{e}} = \frac{\pi^2 EI}{L^2} \tag{4.8}$$

This smallest load P_{e} at which the strut ceases to be in a stable equilibrium is known as *Euler load*. The corresponding bent configuration called *buckled mode shape* is given by $y = A \sin(\pi x/L)$ which is shown in Fig. 4.1a. For $n = 2, 3, \ldots$ higher values of critical load are obtained, the corresponding buckled modes of the strut are defined by (4.7) and shown in Figs. 4.1b and 4.1c.

When the force P is different from the values defined by (4.6), then $A = 0$ i.e. only trivial straight form of strut is available, but when force P takes on any of the values defined by (4.6), the relation $A \sin \alpha L = 0$ is satisfied both with $A = 0$ and $A \neq 0$. It means that at these values both straight and non-trivial bent equilibrium states are possible. Hence these values are sometimes known as *bifurcation loads*.

The above procedure involving homogeneous differential equation of equilibrium along with homogeneous boundary conditions forms a class of problems known as *eigenvalue problems*.

4.2.2 Fixed-Free Cantilever Strut

Consider the cantilever strut shown in Fig. 4.2 acted upon by a compressive force P at its free-end. The external bending moment at any cross-section in the bent configuration is

$$EI \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right) = -M = P(y_m - y)$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha^2 y = \alpha^2 y_m \tag{4.9}$$

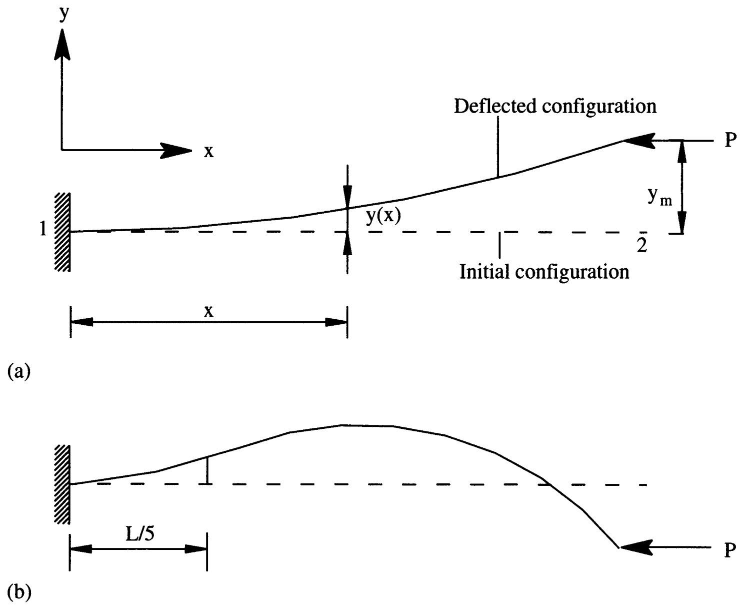


Fig. 4.2a,b. Buckling modes of fixed-free strut. **a** First mode, **b** second mode

As explained in Appendix Appendix C the general solution to (4.9) is

$$y(x) = A \sin \alpha x + B \cos \alpha x + y_m$$

where y_m is the unknown deflection at the free end. The integration constants are determined from the prescribed boundary conditions, namely $y(0) = y'(0) = 0$ at the fixed end. From the first of these $B = -y_m$ and from the second $A = 0$. Thus the bent configuration of the strut is given by

$$y(x) = y_m(1 - \cos \alpha x) \tag{4.10}$$

The boundary condition at the free end, i.e. $y(L) = y_m$ gives

$$y_m = y_m(1 - \cos \alpha L) \quad \text{or} \quad y_m \cos \alpha L = 0 \tag{4.11}$$

The solution requires either $y_m = 0$ or $\cos \alpha L = 0$. The solution $y_m = 0$ represents the initial straight form of the strut. Thus to ensure a non-trivial solution, $\cos \alpha L = 0$ for which it is necessary that

$$\alpha L = (2n - 1)\pi/2 \quad n = 1, 2, 3, \ldots$$

or

$$P_{\text{cr}} = \frac{(2n - 1)^2 \pi^2 EI}{4L^2} \tag{4.12}$$

The smallest value of P_{cr} corresponds to $n = 1$, thus:

$$P_{cr,1} = P_{cr} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2 EI}{(2L)^2} \tag{4.13}$$

The corresponding buckled mode shape is defined by $y(x) = y_m[1 - \cos\left(\frac{\pi x}{2L}\right)]$. For $n = 2, 3, \ldots$ higher values of critical loads obtained are $\frac{9\pi^2 EI}{4L^2}$, $\frac{25\pi^2 EI}{4L^2}$, $\ldots$ and the corresponding buckled modes are: $y_2 = y_m\left[1 - \cos\left(\frac{3\pi x}{L}\right)\right]$, $y_3 = y_m\left[1 - \cos\left(\frac{5\pi x}{L}\right)\right]$. The first two buckled mode shapes are shown in Fig. 4.2a,b.

4.2.3 Fixed-Hinged Strut

The governing differential equation of equilibrium for the fixed-hinged strut shown in Fig. 4.3 is

$$EI\left(\frac{d^2y}{dx^2}\right) = -M = -Q(L - x) - Py$$

where Q is shear force in the member. Differentiating this equation twice with respect to x

$$\frac{d^4y}{dx^4} + \alpha^2\left(\frac{d^2y}{dx^2}\right) = 0 \tag{4.14}$$

This is a fourth-order governing homogeneous differential equation for a general bending problem. As described in Appendix Appendix C, the general solution to (4.14) is

$$y = A\sin\alpha x + B\cos\alpha x + C(x/L) + D \tag{4.15}$$

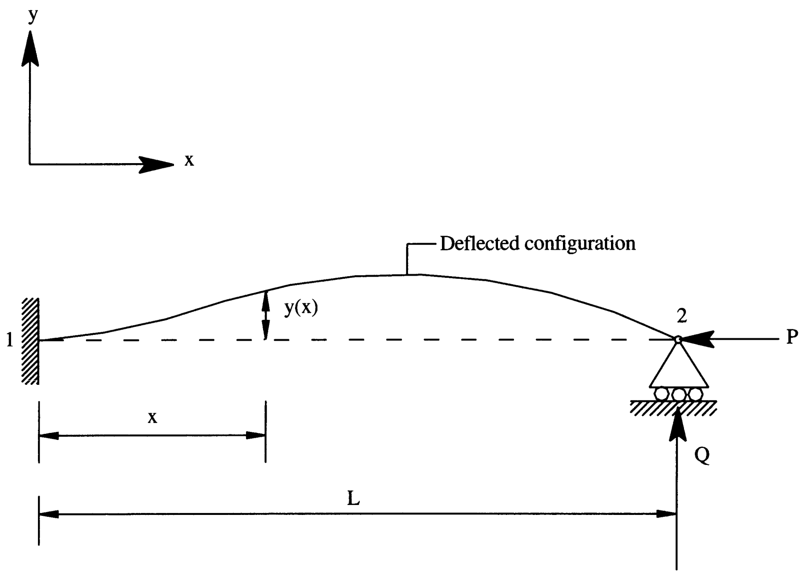


Fig. 4.3. Buckling mode of a fixed-hinged strut

The boundary conditions to be satisfied are

$$y(0) = y'(0) = y(L) = y''(L) = 0$$

These requirements lead to four linear homogeneous algebraic equations in terms of constants A, B, C and D as follows

$$\begin{aligned}
(0)A + \quad (1.0)B + \quad (0)C + (1.0)D &= 0\\
(\alpha L)A + \quad (0)B + (1.0)C + \quad (0)D &= 0\\
(\sin\alpha L)A + (\cos\alpha L)B + (1.0)C + (1.0)D &= 0\\
(\sin\alpha L)A + (\cos\alpha L)B + \quad (0)C + \quad (0)D &= 0
\end{aligned}$$

If $A \equiv B \equiv C \equiv D \equiv 0$, the member will remain in the initial straight configuration for all values of P which is a trivial solution. For non-trivial solution, the determinant of coefficients must vanish, i. e.,

$$\begin{vmatrix} 0.0 & 1.0 & 0.0 & 1.0 \\ \alpha L & 0.0 & 1.0 & 0.0 \\ \sin\alpha L & \cos\alpha L & 1.0 & 1.0 \\ \sin\alpha L & \cos\alpha L & 0.0 & 0.0 \end{vmatrix} = 0$$

or

$$\tan\alpha L = \alpha L \tag{4.16}$$

The solution to transcendental equation (4.16) can be obtained either numerically or graphically. The smallest root of (4.16) as obtained by trial and modification is 4.493.

Therefore, $\alpha L = 4.493$ or

$$P_{\mathrm{cr},1} = \frac{20.19EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2} \tag{4.17}$$

and

$$y(x) = A\sin\left(\frac{4.493x}{L}\right) - \left[\frac{x}{L} + \cos\left(\frac{4.493x}{L}\right) - 1\right]$$

4.2.4 Fixed-Fixed Strut

The fixed-fixed strut shown in Fig. 4.4 has four geometric boundary conditions hence the fourth-order governing differential equation given by (4.14) is required. In this type of strut both the ends of member are fixed against bending rotations and lateral translations, the boundary conditions to be satisfied are

$$y(0) = y'(0) = y(L) = y'(L) = 0 \tag{4.18}$$

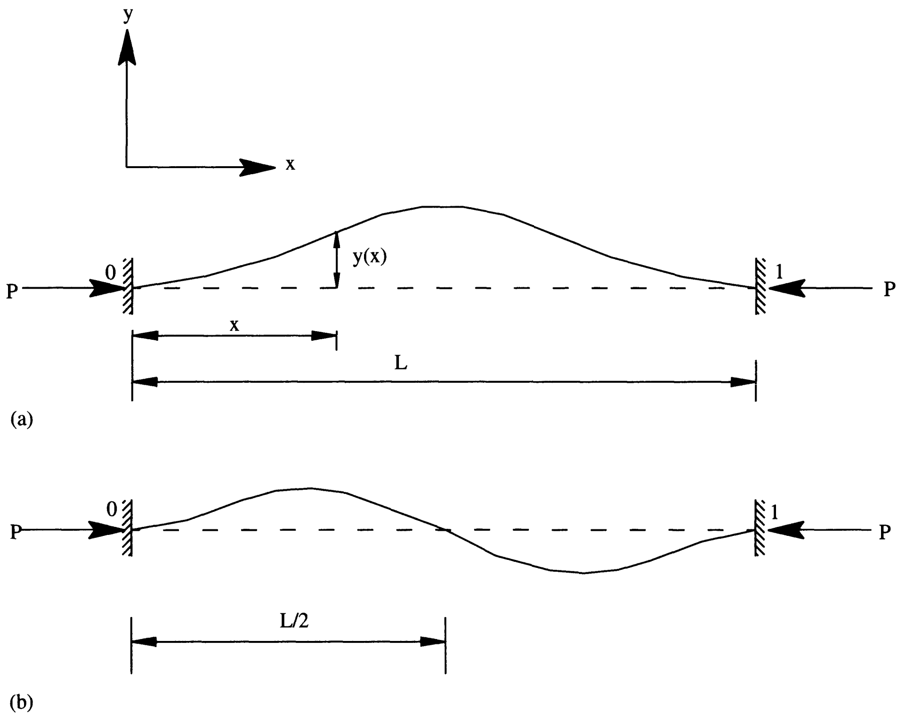


Fig. 4.4a,b. Buckling of fixed-fixed strut. **a** Symmetric mode, **b** antisymmetric mode

The solution to governing differential (4.14) is given by (4.15) and must satisfy boundary conditions given by (4.18). The stability condition or characteristic equation of this case is given by

$$\begin{vmatrix} 0.0 & 1.0 & 0.0 & 1.0 \\ \sin\alpha L & \cos\alpha L & 1.0 & 1.0 \\ \alpha L & 0.0 & 1.0 & 0.0 \\ \alpha L\cos\alpha L & -\alpha L\sin\alpha L & 1.0 & 0.0 \end{vmatrix} = 0$$

or

$$(\alpha L)\sin\alpha L + 2(\cos\alpha L - 1) = 0 \tag{4.19}$$

This equation can be simplified to a form

$$\sin\left(\frac{\alpha L}{2}\right)\left[\left(\frac{\alpha L}{2}\right)\cos\left(\frac{\alpha L}{2}\right) - \sin\left(\frac{\alpha L}{2}\right)\right] = 0 \tag{4.20}$$

For the solution of this equation, either $\sin\left(\frac{\alpha L}{2}\right) = 0$ or $\left(\frac{\alpha L}{2}\right)\cos\left(\frac{\alpha L}{2}\right) - \sin\left(\frac{\alpha L}{2}\right) = 0$.

(i) When $\sin\left(\frac{\alpha L}{2}\right) = 0$, then $\left(\frac{\alpha L}{2}\right) = n\pi, n = 1, 2, 3, \ldots$ Therefore,

$$P_{\mathrm{cr}} = \frac{4n^2\pi^2 EI}{L^2} \tag{4.21}$$

The corresponding mode shapes are given by

$$y(x) = B\left[\cos\left(\frac{2n\pi x}{L}\right) - 1\right] \tag{4.22}$$

The first or minimum critical load for $n = 1$, is:

$$P_{\mathrm{cr},1} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2} \tag{4.23}$$

and corresponding mode shape is $y(x) = B\left[\cos\left(\frac{2\pi x}{L}\right) - 1\right]$ and is shown in Fig. 4.4a.

(ii) If $\left(\frac{\alpha L}{2}\right)\cos\left(\frac{\alpha L}{2}\right) - \sin\left(\frac{\alpha L}{2}\right) = 0$

$$\left(\frac{\alpha L}{2}\right) = \tan\left(\frac{\alpha L}{2}\right)$$

The lowest root of the stability or transcendental equation is given by

$$\left(\frac{\alpha L}{2}\right) = 4.493 \quad \text{or} \quad P_{\mathrm{cr},1} = \frac{80.75EI}{L^2} = \frac{\pi^2 EI}{(0.35)^2} \tag{4.24}$$

The critical load given by (4.23) is lower than that given by (4.24). The value given by (4.24) corresponds to the first antisymmetric buckling mode as shown in the Fig. 4.4b.

4.2.5 Struts with Elastic Supports

The procedure described above is equally applicable to the struts with elastic supports. To illustrate this generality of procedure, consider the problem of buckling of fixed-partially restrained strut shown in Fig. 4.5 where the free end of the member is free to rotate but constrained against lateral deflection by a spring of stiffness k_n. As the boundary condition at the partially restrained end of the member involves shear which is a third-derivative consideration, the fourth-order differential equation must be used to solve the problem, i.e.

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \alpha^2\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = 0 \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The boundary conditions are

$$y(0) = y'(0) = y''(L) = 0, \tag{4.25a}$$

and the boundary condition at the restrained end stipulates that the shear developed in the strut at $x = L$ is resisted by the force in the spring due to lateral deflection, that is,

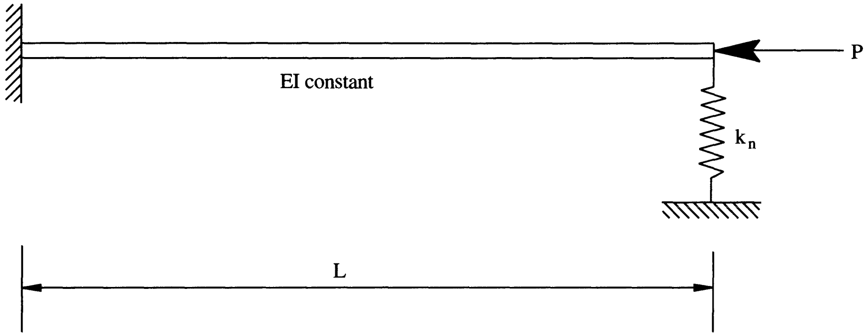


Fig. 4.5. Fixed-partially laterally restrained strut

$$EI\left(\frac{d^3y}{dx^3}\right) + P\frac{dy}{dx} = k_n y$$

or

$$(y''' + \alpha^2 y' - \gamma y)_{x=L} = 0 \quad \text{where} \quad \gamma = \frac{k_n}{EI} \tag{4.25b}$$

The substitution of the general solution i.e.

$$y = A\sin\alpha x + B\cos\alpha x + C\left(\frac{x}{L}\right) + D \tag{4.26}$$

into the boundary stipulations yields a set of four simultaneous homogeneous, linear equations expressed in the matrix form as

$$\begin{bmatrix} 0.0 & 1.0 & 0.0 & 1.0 \\ \alpha L & 0.0 & 1.0 & 0.0 \\ -\alpha^2 \sin\alpha L & -\alpha^2 \cos\alpha L & 0.0 & 0.0 \\ -\gamma L \sin\alpha L & -\gamma L\cos\alpha L & (\alpha^2 - \gamma L) & (-\gamma L) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{4.27}$$

For a solution other than trivial one $A = B = C = D = 0$, the determinant of coefficients must vanish. This condition yields following stability condition or characteristic equation.

$$\tan\alpha L = \alpha L - \left(\frac{\alpha^3}{\gamma}\right)$$

$$\tan\alpha L = \alpha L - (\alpha L)^3\left[\frac{EI}{k_n L^3}\right] \tag{4.28}$$

where

$$\alpha L = \sqrt{\frac{PL^2}{EI}} = \pi\sqrt{\frac{P_{\mathrm{cr}}}{P_{\mathrm{e}}}} \tag{4.29}$$

As a typical case consider $k_n = \frac{12EI}{L^3}$ and the characteristic equation reduces to

$$\tan\alpha L = \alpha L - \frac{(\alpha L)^3}{12}$$

The smallest root obtained by trial and modification is $\alpha L = 4.911304$ and corresponding critical load

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{(0.64L)^2} \tag{4.30}$$

If k_n is infinitely large the buckling problem reduces to that of a fixed-hinged strut with characteristic equation taking the form: $\tan\alpha L = \alpha L$, which is same as given by (4.16).

4.2.6 Framed Columns

The column members of a frame are typical examples of *elastically restrained columns* wherein elastic restraints are provided by connecting beams. The *flexural or rotational stiffness* k_{r} of a beam is $4EI/L$ if its far end is fixed and $3EI/L$ when the far end is hinged. The *axial or extensional stiffness*, k_n $(= EA/L)$ of the beam member is taken to be infinitely large for simplification. For illustration consider a column hinged at the lower end and connected to (i.e. elastically restrained by) a beam at the upper end as shown in Fig. 4.6. The boundary conditions of the idealized column shown in Fig. 4.6b for the end A are: $y(0) = y''(0) = 0$. At the end B, they are $y(L) = 0$ and $-EIy''(L) = k_{\mathrm{r}}y'(L)$ or $y''(L) + \gamma y'(L) = 0$ where k_{r} is the rotational spring constant associated with the beam BC, and $\gamma = \frac{k_{\mathrm{r}}}{EI}$. As discussed earlier, the rotational stiffness k_{r} is obtained by treating the beam BC hinged at B and fixed at the end C and is given by:

$$k_{\mathrm{r}} = \frac{4EI_1}{L_1}.$$

For this problem with four prescribed boundary conditions, the governing differential equation is

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \alpha^2\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = 0 \tag{4.31}$$

which has general solution, $y(x) = A\sin\alpha x + B\cos\alpha x + C\left(\frac{x}{L}\right) + D$. On substituting this solution into the boundary stipulations, following simultaneous homogeneous and linear equations are obtained

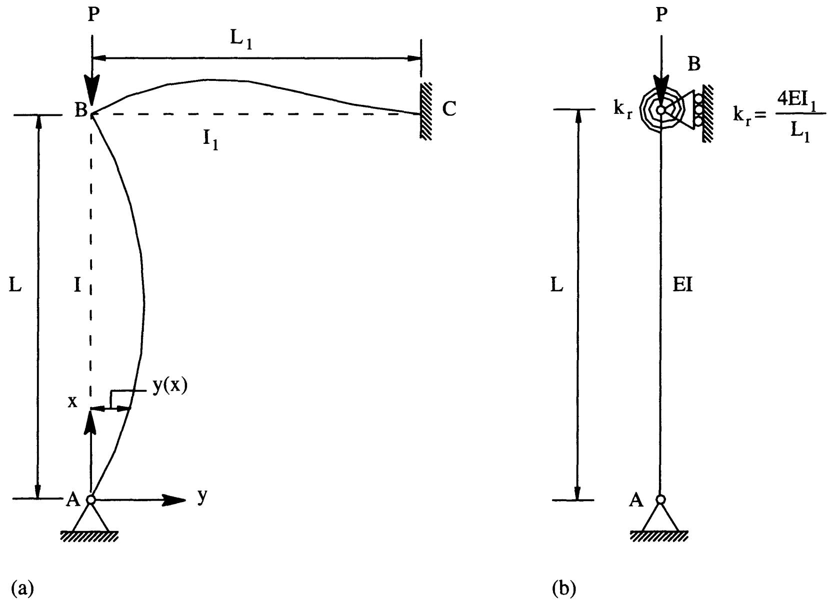


Fig. 4.6a,b. Buckling of framed column with hinged base. **a** Framed column, **b** idealized column with elastic support

$$
\begin{array}{llll}
 & B & & D = 0 \\
 & -B & & = 0 \\
A \sin \alpha L & B \cos \alpha L & C & D = 0 \\
A(-\alpha^2 \sin \alpha L + \alpha \gamma \cos \alpha L) & B(-\alpha^2 \cos \alpha L - \alpha \gamma \sin \alpha L) & C\left(\frac{\gamma}{L}\right) & = 0
\end{array}
\tag{4.32}
$$

From first three equations, $B = D = 0$, $C = -A \sin \alpha L$. Substituting these values in the fourth equation yields

$$A\left[-\alpha^2 \sin \alpha L + \alpha \gamma \cos \alpha L - \left(\frac{\gamma}{L}\right) \sin \alpha L\right] = 0$$

For the non-trivial solution $A \neq 0$, the characteristic equation is

$$\tan \alpha L = \frac{\alpha \gamma}{\alpha^2 + (\gamma/L)} = \frac{\alpha L}{(\alpha^2 L^2/\gamma L) + 1}$$

or

$$\cot \alpha L = \frac{\alpha L}{\gamma L} + \frac{1}{\alpha L} \tag{4.33}$$

This transcendental equation can be solved to obtain the smallest value of the root to compute critical load. In a typical case where $I_1 = I$ and $L_1 = L$, (4.33) reduces to

$$\tan \alpha L = \frac{4 \alpha L}{(\alpha L)^2 + 4} \tag{4.34}$$

The smallest root is: $\alpha L = 3.83$ and the corresponding critical load P_{cr} is given by

$$P_{cr} = \frac{(3.83)^2 EI}{L^2} = \frac{\pi^2 EI}{(0.82L)^2} \tag{4.35}$$

If I_1 is infinitely large, (4.33) reduces to $\tan \alpha L = \alpha L$ and smallest root obtained by trial and modification is $\alpha L = 4.493$ and corresponding critical load is:

$$P_{cr} = \alpha^2 EI = \frac{(4.493)^2 EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2} \tag{4.36}$$

The *effective length* of the compression member AB increases from $0.7L$ to $0.82L$ due to reduction in the rigidity of the beam. If BC is long with small I_1, then the restraint B will tend to vanish and the member AB will revert to a hinged-hinged column.

In another variation of the foregoing problem consider the case when member AB is in the same line as BC, making the structure a *two-span continuous strut* shown in Fig. 4.7a wherein the span AB is subjected to an axial force such that the axial force in the span BC is presumed to be zero. The procedure and results of the foregoing problem are also applicable to this case.

The column members of a symmetrical portal frame can also be modelled as *elastically restrained columns*, the elastic restraint being provided by connecting beam. The *flexural* or *rotational stiffness* of the beam depends upon the buckling mode considered for the analysis. In most of the practical cases, the axial or extensional stiffness (EA/L) of the beam member is taken to be infinitely large.

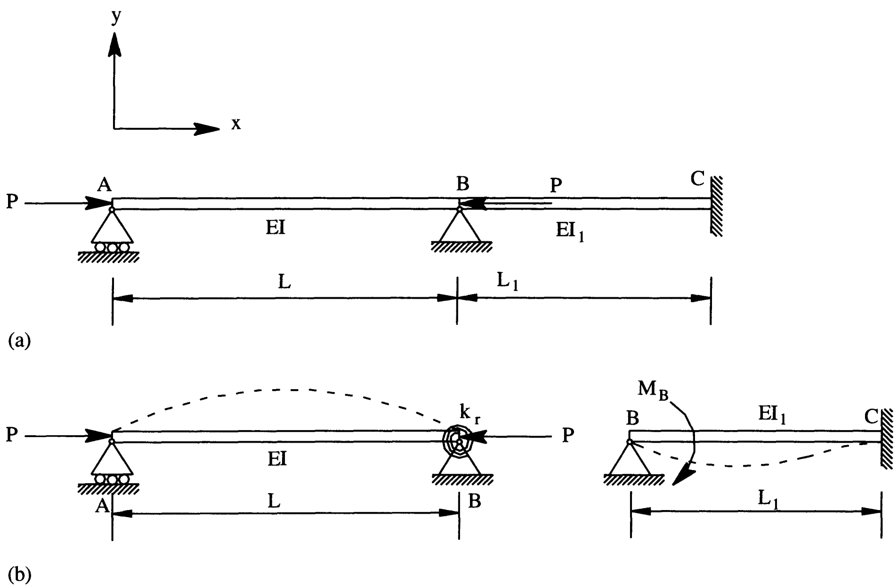


Fig. 4.7a,b. Two-span continuous strut. **a** Two-span continuous strut mode, **b** elastically supported strut model

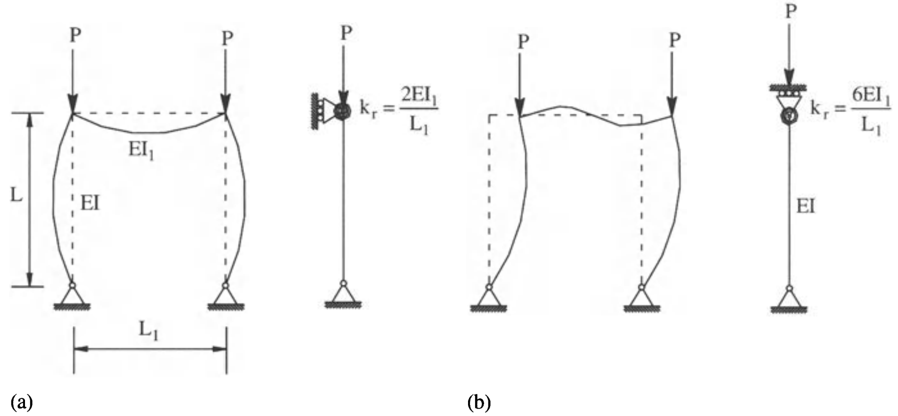


Fig. 4.8a,b. Portal frame with column hinged at the base. **a** Symmetric mode, **b** antisymmetric mode

(a) Portal frame with columns hinged at the base

Consider the symmetrical portal frame with columns hinged at the base as shown in Fig. 4.8. The portal may buckle either in a symmetric mode without sidesway or in an antisymmetric mode with side sway. The buckling of the frame can be viewed as buckling of column members with rotational restraint provided by horizontal beam member. The elastically restrained column models for the symmetric and antisymmetric buckling modes are shown in Fig. 4.8a and b, respectively.

(i) Symmetric buckling mode

The boundary conditions are:

$$y(0) = y''(0) = y(L) = 0$$

and

$$EIy''(L) + k_r y'(L) = 0 \quad \text{or} \quad y''(L) + \gamma y'(L) = 0 \tag{4.37}$$

where $\gamma = \frac{k_r}{EI}$ and k_r is the *rotational spring constant* (stiffness) associated with the connecting horizontal beam. For these four prescribed boundary conditions, the fourth-order governing equation must be used

$$\frac{d^4 y}{dx^4} + \alpha^2 \left(\frac{d^2 y}{dx^2} \right) = 0$$

with its general solution as: $y = A \sin \alpha x + B \cos \alpha x + C\left(\frac{x}{L}\right) + D$. The characteristic equation is given by (4.33) or

$$\cot \alpha L = \left(\frac{\alpha L}{\gamma L} \right) + \left(\frac{1}{\alpha L} \right) \tag{4.38}$$

The smallest root of this equation represents the first critical load for buckling of the portal frame. For the symmetrical buckling mode, the portal buckles in a manner shown in Fig. 4.8a such that side sway is prevented. Thus $k_r = \frac{2EI_1}{L_1}$.

$$\gamma L = \frac{k_r L}{EI} = \frac{2EI_1 L}{L_1 EI} = 2\left[\left(\frac{I_1}{I}\right)\left(\frac{L}{L_1}\right)\right].$$

For a typical portal frame with $I_1 = I$ and $L_1 = L$, $\gamma L = 2$ and the characteristics equation reduces to

$$\cot \alpha L = \left(\frac{\alpha L}{2}\right) + \left(\frac{1}{\alpha L}\right) \tag{4.39}$$

and the smallest root $\alpha L = 3.59$. Therefore

$$P_{cr} = \frac{(3.59)^2 EI}{L^2} = \frac{12.9EI}{L^2} = \frac{\pi^2 EI}{(0.875L)^2} \tag{4.40}$$

(ii) Antisymmetrical buckling mode

In this case the side sway is permitted and the frame is assumed to buckle in an antisymmetric mode shown in Fig. 4.8b. Here $k_r \doteq \frac{6EI_1}{L_1}$ and where

$$\gamma L = \frac{k_r L}{EI} = \frac{6EI_1 L}{L_1 EI} = 6\left[\left(\frac{I_1}{I}\right)\left(\frac{L}{L_1}\right)\right]$$

the boundary conditions are:

$$\begin{aligned} y(0) = y''(0) &= 0 \\ y''(L) + \gamma y'(L) &= 0 \\ \text{and} \quad y'''(L) + \alpha^2 y'(L) &= 0 \end{aligned} \tag{4.41}$$

Using fourth-order governing differential equation with its general solution being substituted in the boundary conditions yields

$$\alpha L \tan \alpha L = \gamma L \tag{4.42}$$

For the typical case $\frac{I_1}{L_1} = \frac{I}{L}$ reducing γL to 6 and the characteristic equation reduces to

$$\tan \alpha L = \frac{6}{\alpha L} \tag{4.43}$$

The smallest root obtained by trial and modification is $\alpha L = 1.35$. Therefore,

$$P_{cr} = \frac{(1.35)^2 EI}{L^2} = \frac{1.823EI}{L^2} = \frac{\pi^2 EI}{(2.327L)^2} \tag{4.44}$$

It must be noted that antisymmetric buckling is associated with the lower value of critical load.

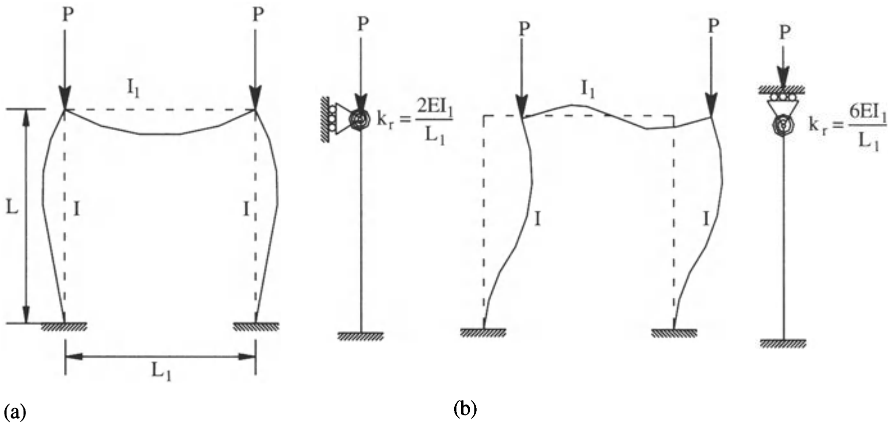


Fig. 4.9a,b. Portal frame with column fixed at the base. **a** Symmetric mode, **b** antisymmetric mode

(b) Portal frame with columns fixed at the base

Consider the portal frame shown in the Fig. 4.9. As in the case of portal with hinged columns, it may buckle either in a symmetric mode or in an antisymmetric mode.

(i) Symmetric buckling mode

The boundary conditions are:

$$y(0) = y'(0) = y(L) = 0 \quad \text{and} \quad y''(L) + \gamma y'(L) = 0 \tag{4.45}$$

Substituting the general solution of fourth-order differential equation in the boundary stipulations yields four linear algebraic homogeneous equations

$$\begin{array}{llll} & B & & D = 0 \\ \alpha A & & C\left(\frac{1}{L}\right) & \;\;= 0 \\ A \sin\alpha L & B\cos\alpha L & C & D = 0 \\ A(\alpha\gamma\cos\alpha L - \alpha^2 \sin\alpha L) & -B(\alpha\gamma\sin\alpha L + \alpha^2\cos\alpha L) & C\left(\frac{\gamma}{L}\right) & \;\;= 0 \end{array} \tag{4.46}$$

From first three boundary conditions

$$A(\sin\alpha L - \alpha L) + B(\cos\alpha L - 1) = 0$$

From second and fourth boundary conditions

$$A(\alpha\gamma\cos\alpha L - \alpha^2\sin\alpha L - \gamma\alpha) - B(\alpha\gamma\sin\alpha L + \alpha^2\cos\alpha L) = 0$$

For non-trivial solution ($A \neq 0$ and $B \neq 0$), the characteristic equation is

$$\alpha L(1 - \gamma L)\sin\alpha L - (2\gamma L + \alpha^2 L^2)\cos\alpha L + 2\gamma L = 0 \tag{4.47}$$

which can be solved for the smallest root of αL. Here

$$\gamma L = \frac{k_r L}{EI} = 2\left[\left(\frac{I_1}{I}\right)\left(\frac{L}{L_1}\right)\right] \quad \text{as} \quad k_r = \frac{2EI_1}{L_1}$$

For a typical case of portal frame with $\frac{I_1}{L_1} = \frac{I}{L}$, $\gamma L = 2$. The characteristics equation reduces to

$$\alpha L \sin \alpha L + [4 + (\alpha L)^2] \cos \alpha L = 4 \tag{4.48}$$

The smallest root obtained by trial and modification method is $\alpha L = 5.018186$. Therefore,

$$P_{cr} = \frac{(5.018186)^2 EI}{L^2} = \frac{25.182EI}{L^2} = \frac{\pi^2 EI}{(0.626L)^2} \tag{4.49}$$

It should be noted that if k_r i.e. γL tends to zero, this elastically restrained column reduces to a fixed-hinged column with $P_{cr} = \frac{\pi^2 EI}{(0.7L)^2}$ and on the other hand if k_r is infinitely large ($1/\gamma L = 0$), the column becomes fixed-fixed column with $P_{cr} = \frac{\pi^2 EI}{(0.5L)^2}$.

(ii) Antisymmetrical buckling mode

In this case column buckles with a side sway as shown in the Fig. 4.9b. The boundary conditions are:

$$y(0) = y'(0) = 0$$
$$y''(L) + \gamma y'(L) = 0$$
$$\text{and } y'''(L) + \alpha^2 y'(L) = 0$$

where

$$\gamma = \frac{k_r}{EI} = \frac{(6EI_1/L_1)}{EI} = \frac{6(I_1/L_1)}{I} \quad \text{or} \quad \gamma L = \frac{6(I_1/L_1)}{(I/L)} \tag{4.50}$$

Substituting the general solution of fourth-order differential equation namely, $y(x) = A \sin \alpha x + B \cos \alpha x + C(x/L) + D$ into the boundary conditions, following characteristic equation is obtained for non-trivial solution

$$\tan \alpha L = -\left(\frac{\alpha}{\gamma}\right) = -\left(\frac{\alpha L}{\gamma L}\right) \tag{4.51}$$

The behaviour of this elastically restrained column lies between that of fixed-free and fixed-hinged columns. Thus the smallest root of characteristic equation lies between $(\pi/2)$ and π depending on the value of γL. Typically consider $I_1/L_1 = I/L$ i.e. $\gamma L = 6$.

The characteristic equation reduces to tan $\alpha L = -(\alpha L/6)$ and the smallest root is given by $\alpha L = 2.7165$. Thus

$$P_{cr} = \frac{(2.7165)^2 EI}{L^2} = \frac{7.3794EI}{L^2} = \frac{\pi^2 EI}{(1.156L)^2} \tag{4.52}$$

4.3 Concept of Effective Length

In each of the above illustrations, the critical load has been expressed in the form $P_{cr} = \pi^2 EI/(KL)^2$ where K is termed *effective length factor*. This form of representation enables to express the *critical buckling load* in terms of *Euler load* of a hypothetical pin-ended member of length KL. Thus effective length factors could be obtained from the expression

$$K = \sqrt{\frac{P_e}{P_{cr}}} \tag{4.53}$$

The K values of simple cases are given in Table 4.1. When a *compression member* is an integral part of a structure, its ends are connected to the other members. The connected members provide rotational as well as translational restraint. To determine the buckling load of a particular compression member in a given structure, the engineers generally use their experience and judgment to estimate the effective length factor K for a given design situation as regard to the members immediately connected to the compression member in question. In most of the cases the effective length KL is actually the distance between the *points of contra flexure*.

The corresponding critical stress is given by

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(KL/r)^2} \quad \text{as} \quad I = Ar^2 \tag{4.54}$$

where A is the cross-sectional area and r is radius of gyration about an axis of the cross-section which governs buckling. The ratio KL/r is referred to as the effective *slenderness ratio* of the strut.

The classical procedure for computation of critical buckling load is equally applicable to the columns connected to the *rigid* or *flexible links* or having *internal hinges* as illustrated in Example 4.1.

Example 4.1. Structural members AB fixed at the base A and subjected to an axial force through a rigid link bar BC as shown in Fig. 4.10. The link bar is connected to the member by a hinge at B. Determine the critical buckling load of the system.

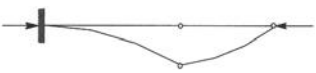
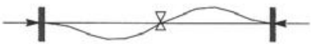
The deflected configuration and free-body diagrams are shown in Fig. 4.10b,c, respectively. The member AB is subjected to an axial force P as well as lateral force $M_A/(L_1 + L_2)$. From the moment equilibrium of the link BC

$$Py_m = \left[\frac{M_A}{(L_1 + L_2)}\right](L_2)$$

or

$$y_m = \frac{M_A L_2}{(L_1 + L_2)P}$$

Table 4.1. Effective length factors

Strut type	$P_{cr} = \pi^2 EI/(KL)^2$ Boundary conditions			K	Remarks
1. Rigid boundaries					
(a) Standard cases					
	fixed		free	2.00	
	hinged		slide	2.00	
	hinged		hinged	1.00	
	fixed		slide	1.00	
	fixed		hinged	0.70	
	fixed		fixed	0.50	
(b) Special cases					
	fixed	internal hinge	hinged	1.3495	
	hinged	continous	hinged	0.50	
	fixed	continous	fixed	0.33	
2. Elastic boundaries					
	hinged		hinged with elastic spring	0.82 0.875	$k_r = 4EI/L$ $k_r = 2EI/L$
	fixed		hinged with elastic spring	0.626	$k_r = 2EI/L$
	fixed		free with translational spring	0.64	$k_n = 2EI/L^3$

Equating the *internal resisting moment* to the *external disturbing moment* in the cantilever flexural member AB

$$EI\left(\frac{d^2y}{dx^2}\right) = -M = -\left[-P(y_m - y) - \frac{M_A x}{(L_1 + L_2)}\right]$$

where y_m is the maximum lateral deflection at the top end B of the cantilever AB. Therefore,

$$\frac{d^2y}{dx^2} + \alpha^2 y = \alpha^2 y_m + \frac{M_A x \alpha^2}{(L_1 + L_2)P} = \alpha^2\left[1 + \frac{x}{L_2}\right] y_m \tag{4.55}$$

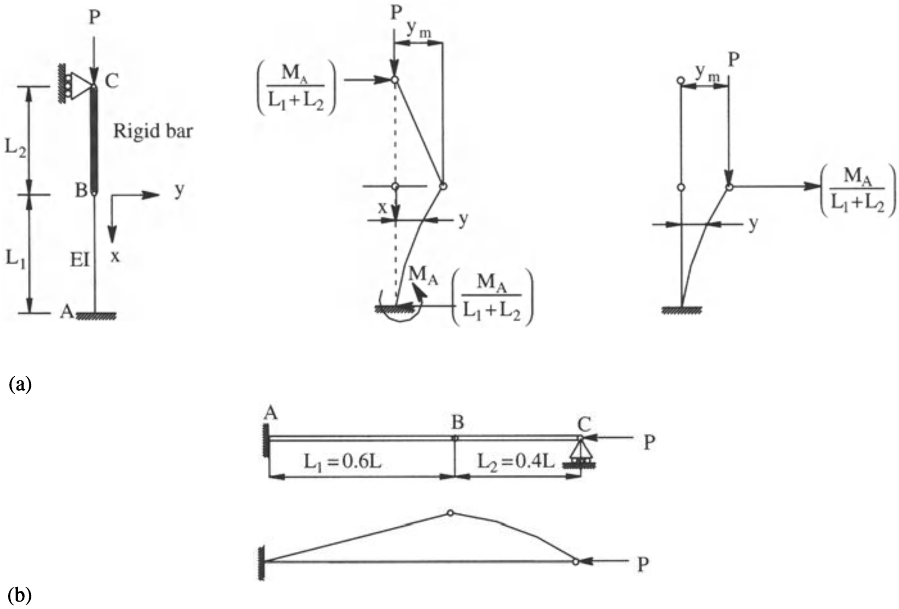


Fig. 4.10a,b. Column with internal hinge. **a** Column link system; *from left to right*: structure, deflected configuration, idealized column, **b** fixed hinged column with interior hinge

The boundary conditions to be satisfied are

$$y(0) = y_m\,, \quad y(L_1) = 0 \quad \text{and} \quad y'(L_1) = 0 \tag{4.56}$$

The general solution to the second order governing differential equation is

$$y = A\sin\alpha x + B\cos\alpha x + \left(\frac{x}{L_2}\right) y_m + y_m$$

Substituting the general solution into the prescribed boundary conditions yields:

$$B + y_m = y_m \quad \text{i.e. } B = 0$$

$$A\sin\alpha L_1 + B\cos\alpha L_1 + y_m\left(\frac{L_1}{L_2}\right) + y_m = 0$$

i.e.

$$A = \left[\frac{L_1 + L_2}{L_2}\right]\left[\frac{y_m}{\sin\alpha L_1}\right]$$

and

$$A\alpha\cos\alpha L_1 - B\alpha\sin\alpha L_1 + \frac{y_m}{L_2} = 0$$

or

$$\left[\frac{L_1 + L_2}{L_2}\right]\left[\frac{y_m \alpha}{\sin \alpha L_1}\right] \cos \alpha L_1 + \frac{y_m}{L_2} = 0$$

Therefore,

$$\tan \alpha L_1 = \alpha(L_1 + L_2).$$

As a typical case assume $L_1 = L_2$ $(= 0.5L)$ and characteristic equation reduces to

$$\tan(\alpha L_1) = 2(\alpha L_1)$$

The lowest root of this transcendental equation obtained by trial and modification procedure is given by: $\alpha L_1 = 1.164$. Hence,

$$P_{\mathrm{cr}} = \frac{(1.164)^2 EI}{L_1^2} = \frac{5.4196 EI}{L^2} = \frac{\pi^2 EI}{(1.3495L)^2} \tag{4.57}$$

If the rigid link bar BC is replaced by a flexible member of same cross-section as the member AB, the structure reduces to a strut with *internal hinge* as shown in the Fig. 4.10b. In this case the buckling failure may occur in two different modes. The first is the buckling of length BC as an Euler strut with buckling load of $\pi^2 EI/(L_2)^2$. The second mode of failure is the buckling of complete structure with point B moving laterally and the length BC acts as a link transmitting load from C to B. Thus it is again a case of buckling of the cantilever AB due to force becoming inclined as B deflects laterally. The lower of the two critical load values will provide the solution.

As a typical case consider $L_1 = 0.6L$ and $L_2 = 0.4L$. The critical load for the failure of this component.

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L_2^2} = \frac{\pi^2 EI}{(0.4L)^2} = \frac{61.685 EI}{L^2}$$

For the failure of entire structure, the characteristic equation $\tan \alpha L_1 = \alpha(L_1 + L_2)$ reduces to

$$\tan \alpha L_1 = \alpha L_1 \left[1 + \frac{L_2}{L_1}\right] = 1.667 \alpha L_1 \tag{4.58}$$

By trial and modification, the smallest root is given by

$$\alpha L_1 = 1.0526$$

Therefore,

$$P_{\mathrm{cr}} = \frac{(1.0526)^2 EI}{L_1^2} = \frac{1.10797 EI}{(0.6L)^2} = \frac{3.078 EI}{L^2} \tag{4.59}$$

Thus, the critical load for this fixed-hinged strut with an internal hinge is

$$P_{\mathrm{cr}} = \frac{3.078 EI}{L^2} = \frac{\pi^2 EI}{(1.791L)^2}$$

4.4 Approximate Techniques

The method based on integration of classical differential equation used in the preceding sections is suitable for the struts of uniform cross-section with relatively simple boundary conditions. The procedure becomes complicated when the member cross-section varies along its length or when the boundary conditions are complex. In such a situation, approximate techniques, discussed in Chap. 2, based on *work equation*, and *total potential energy* in conjunction with the *trial displacement functions* prove to be extremely useful tools of analysis. The energy approach has inherent ability to converge to the exact solution. On the other hand *Newmarks numerical integration technique* is very powerful method giving accurate results.

The *Rayleigh's quotient method* which is another form of energy method is frequently used in determining the elastic flexural buckling load. As explained in Chap. 2, the accuracy of the solution largely depends upon the accuracy of assumed displacement trial function. Consider the buckling of a pin-ended strut of length L with boundary conditions as $y(0) = y(L) = 0$. In Sect. 4.2.1, a *single sine wave* was selected as the trial function which predicted exact critical load $P_{\mathrm{cr}} = \pi^2 EI/L^2$. Let us consider a multi-degree-of-freedom trial function for deflected configuration

$$y(x) = a_1 \sin\left(\frac{\pi x}{L}\right) + a_3 \sin\left(\frac{3\pi x}{L}\right) + \ldots = \sum_n a_{2n-1} \sin\left[(2n-1)\frac{\pi x}{L}\right] \quad (4.60)$$

where $n = 1, 2, \ldots$. This equation satisfies all the *geometric boundary conditions*. The first and second derivatives are

$$y' = \sum_n a_{2n-1}\left[\frac{(2n-1)\pi}{L}\right]\cos\left[(2n-1)\frac{\pi x}{L}\right] \quad (4.61)$$

$$y'' = -\sum_n a_{2n-1}\left[\frac{(2n-1)\pi}{L}\right]^2 \sin\left[(2n-1)\frac{\pi x}{L}\right] \quad (4.62)$$

The integrals of the component of total potential energy can be expressed as

$$\begin{aligned}
\frac{P}{2}\int_0^L (y')^2 \mathrm{d}x &= \frac{P\pi^2}{2L^2}\sum_n a_{2n-1}^2 (2n-1)^2 \int_0^L \cos^2\left[\frac{(2n-1)\pi x}{L}\right]\mathrm{d}x \\
&= \frac{P\pi^2}{2L^2}\sum_n a_{2n-1}^2 (2n-1)^2 \left(\frac{1}{2}\right)\left[x + \frac{L}{2\pi(2n-1)}\sin\frac{2\pi(2n-1)x}{L}\right]_0^L \\
&= \frac{P\pi^2}{2L^2}\sum_n a_{2n-1}^2 (2n-1)^2 \left(\frac{L}{2}\right) = \frac{P\pi^2}{4L}\sum_n a_{2n-1}^2 (2n-1)^2
\end{aligned}$$

For a member of constant cross-section throughout its length

$$\frac{EI}{2}\int_0^L (y'')^2 \mathrm{d}x = \frac{EI\pi^4}{2L^4}\int_0^L a_{2n-1}^2 (2n-1)^4 \int_0^L \sin^2\left[\frac{(2n-1)\pi x}{L}\right]\mathrm{d}x$$

$$= \frac{EI\pi^4}{2L^4}\sum_n a_{2n-1}^2 (2n-1)^4 \left(\frac{1}{2}\right)\left[x - \frac{L}{2\pi(2n-1)}\sin\frac{2\pi(2n-1)x}{L}\right]_0^L$$

$$= \frac{EI\pi^4}{4L^3}\sum_n a_{2n-1}^2 (2n-1)^4$$

Considering only three terms in the series, the total potential is given by:

$$\Pi = U + V = \frac{\pi^4 EI}{4L^3}\left(a_1^2 + 3^4 a_3^2 + 5^4 a_5^2\right) - \frac{P\pi^2}{4L}\left(a_1^2 + 3^2 a_3^2 + 5^2 a_5^2\right) \tag{4.63}$$

$$= a_1^2\left(\frac{\pi^4 EI}{4L^3} - \frac{P\pi^2}{4L}\right) + a_3^2\left(\frac{3^4\pi^4 EI}{4L^3} - \frac{3^2 P\pi^2}{4L}\right) + a_5^2\left(\frac{5^4\pi^4 EI}{4L^3} - \frac{5^2 P\pi^2}{4L}\right)$$

Differentiating the total potential energy with respect to each of the unknowns, a_1, a_3 and a_5, and equating the resulting expressions to zero yields following three independent equations.

$$\frac{\partial \Pi}{\partial a_1} = 2a_1\left(\frac{\pi^4 EI}{4L^3} - \frac{P\pi^2}{4L}\right) = 0 \quad \text{i.e.} \quad P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2}$$

$$\frac{\partial \Pi}{\partial a_3} = 2a_3\left(\frac{3^4\pi^4 EI}{4L^3} - \frac{3^2 P\pi^2}{4L}\right) = 0 \quad \text{i.e.} \quad P_{\mathrm{cr}} = \frac{3^2\pi^2 EI}{L^2}$$

$$\frac{\partial \Pi}{\partial a_5} = 2a_5\left(\frac{5^4\pi^4 EI}{4L^3} - \frac{5^2 P\pi^2}{4L}\right) = 0 \quad \text{i.e.} \quad P_{\mathrm{cr}} = \frac{5^2\pi^2 EI}{L^2} \tag{4.64}$$

The smallest critical load at buckling is given by $P_{\mathrm{cr}} = \pi^2 EI/L^2$. This is infact the exact solution to the problem because the first term of assumed sine-series corresponds identically to the true deflected shape of the member at the buckling. Moreover, it should be noted that the partial differentiation with respect to a_1, a_3 and a_5 did not result in a set of simultaneous algebraic equations. Rather, each differentiation resulted in separate equation containing single unknown a_1 or a_3 or a_5, and a parenthesized containing P_{cr}. This was because the chosen sets of functions are *orthogonal* over the interval of integration. Due to the *property of orthogonality* the *Fourier series* are frequently used for the *Ritz solution*.

In the foregoing treatment it has been assumed that the moment of inertia of the cross-section is constant along the length of the member. However, if moment of inertia varies along the length of the member, the total potential energy associated with a flexural member can be defined as

$$\Pi = \frac{1}{2}E\int_0^L I(y'')^2 \mathrm{d}x - \frac{1}{2}P\int_0^L (y')^2 \mathrm{d}x \tag{4.65}$$

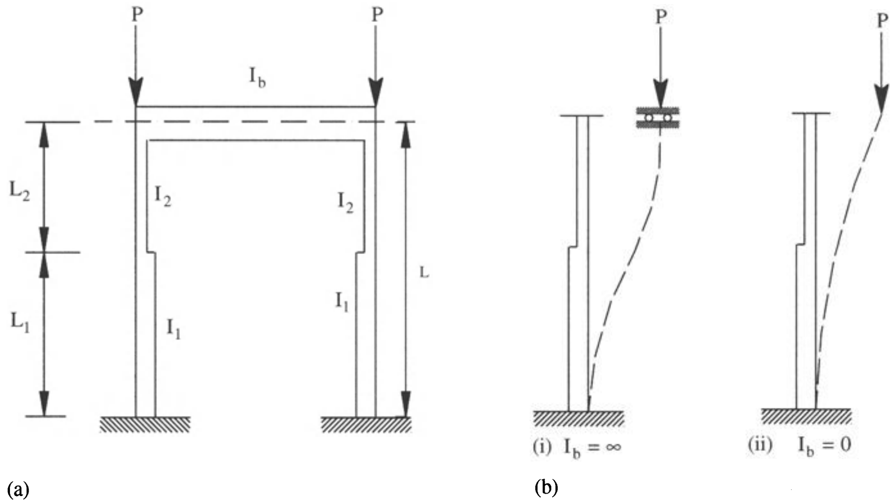


Fig. 4.11a,b. Frame with columns having variable cross-section. **a** Frame, **b** idealized columns

Example 4.2. Estimate the critical load P_{cr} that will cause buckling of the complete portal shown in Fig. 4.11. The columns are identical and change in cross-section from I_1 to I_2 at a height L_1 from the base. The horizontal beam is presumed to be rigid, preventing rotation at the top of columns but does not restrain the structure against side sway.

The idealized column is shown in Fig. 4.11b. The critical load corresponding to the sway buckling mode will be the smallest. The boundary conditions are

$$y(0) = y'(0) = y'(L) = 0 \quad \text{and} \quad y(L) = a \tag{4.66}$$

A trial function satisfying these *geometric* or *kinematic boundary condition* is given by

$$y(x) = \frac{a}{2}\left(1 - \cos\frac{\pi x}{L}\right) \tag{4.67}$$

Substituting this trial function in the expression for the total potential energy given by (4.65), The internal work done or strain energy term is:

$$\begin{aligned} W_{\text{in}} = U &= \frac{E}{2}\int_L \left(\frac{a\pi^2}{2L^2}\right)^2 I\cos^2\left(\frac{\pi x}{L}\right)\mathrm{d}x \\ &= \left(\frac{Ea^2\pi^4}{8L^4}\right)\int_L I\cos^2\left(\frac{\pi x}{L}\right)\mathrm{d}x \\ &= \left(\frac{Ea^2\pi^4}{16L^4}\right)\int_L I\left(1 + \cos\frac{2\pi x}{L}\right)\mathrm{d}x \end{aligned}$$

$$= \left(\frac{Ea^2\pi^4}{16L^4}\right)\left[\int_0^{L_1} I\left(1+\cos\frac{2\pi x}{L}\right)\mathrm{d}x + \int_{L_1}^{L} I\left(1+\cos\frac{2\pi x}{L}\right)\mathrm{d}x\right]$$

$$= \left(\frac{Ea^2\pi^4}{16L^4}\right)\left[I_1\left(x+\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_0^{L_1} + I_2\left(x+\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_{L_1}^{L}\right]$$

$$= \left(\frac{Ea^2\pi^4}{16L^4}\right)\left[I_1\left(L_1+\frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right)\right.$$
$$\left. + I_2\left((L-L_1)+\frac{L}{2\pi}\left\{0-\sin\frac{2\pi L_1}{L}\right\}\right)\right]$$

$$= \left(\frac{EI_1a^2\pi^4}{16L^4}\right)\left[L_1+\left(\frac{I_2}{I_1}\right)L_2+\left(\frac{L}{2\pi}\right)\left(1-\frac{I_2}{I_1}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right]$$

$$= \left(\frac{EI_1a^2\pi^4}{16L^3}\right)\left[\left(\frac{L_1}{L}\right)+\left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L}\right)\right.$$
$$\left. +\left(\frac{1}{2\pi}\right)\left(1-\frac{I_2}{I_1}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right] \tag{4.68}$$

The load potential or external work done is given by:

$$-W_{\text{ex}} = V = \frac{1}{2}\int_L P(y')^2\mathrm{d}x = \frac{P}{2}\int_L \left[\left(\frac{a}{2}\right)\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right)\right]^2\mathrm{d}x$$

$$= \left[\left(\frac{Pa^2\pi^2}{8L^2}\right)\int_L \frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right)\mathrm{d}x\right]$$

$$= \left(\frac{Pa^2\pi^2}{16L^2}\right)\left[x-\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_L$$

$$= \left(\frac{Pa^2\pi^2}{16L^2}\right)\left\{\left[x-\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L_1} + \left[x-\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_{L_1}^{L}\right\}$$

$$= \left(\frac{Pa^2\pi^2}{16L^2}\right)\left[L_1-\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right.$$
$$\left. +(L-L_1)+\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right]$$

$$= \left(\frac{Pa^2\pi^2 L}{16L^2}\right) = \left(\frac{Pa^2\pi^2}{16L}\right) \tag{4.69}$$

From work equation: $-W_{\text{ex}} = W_{\text{in}}$

$$\frac{Pa^2\pi^2}{16L} = \frac{EI_1a^2\pi^4}{16L^3}\left[\left(\frac{L_1}{L}\right)+\left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L}\right)+\left(\frac{1}{2\pi}\right)\left(1-\frac{I_2}{I_1}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right]$$

Therefore,

$$P_{cr} = \frac{\pi^2 EI_1}{L^2}\left[\left(\frac{L_1}{L}\right) + \left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L}\right) + \left(\frac{1}{2\pi}\right)\left(1 - \frac{I_2}{I_1}\right)\sin\left(\frac{2\pi L_1}{L}\right)\right] \tag{4.70}$$

As a typical case consider $L_2 = L_1 = \frac{L}{2}$ and $I_2 = \frac{I_1}{2} = I$.

$$\begin{aligned} P_{cr} &= \frac{\pi^2 E(2I)}{L^2}\left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2\pi}\frac{1}{2}\sin\pi\right) \\ &= \frac{1.5\pi^2 EI}{L^2} = \frac{14.8EI}{L^2} = \frac{\pi^2 EI}{(0.816L)^2} \end{aligned} \tag{4.71}$$

If the top ends of the columns are free to rotate and sway, the geometric or kinematic boundary conditions are:

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(L) = a$$

The trial displacement function satisfying these boundary conditions may be taken as

$$y(x) = a\left(1 - \cos\frac{\pi x}{2L}\right) \tag{4.72}$$

Substituting this trial function in the expressions for strain energy and external work

$$\begin{aligned} W_{in} = U &= \int_L \left[\frac{1}{2}EI\left(y''\right)^2\right]\mathrm{d}x \\ &= \frac{E}{2}\left(\frac{a\pi^2}{4L^2}\right)^2 \int_L I\left(\cos\frac{\pi x}{2L}\right)^2 \mathrm{d}x \\ &= \left(\frac{Ea^2\pi^4}{64L^4}\right)\int_L I\left(1 + \cos\frac{\pi x}{L}\right)\mathrm{d}x \\ &= \left(\frac{Ea^2\pi^4}{64L^4}\right)\left[I\left(x + \frac{L}{\pi}\sin\frac{\pi x}{L}\right)\right]_0^L \end{aligned}$$

Integrating over the column height

$$\begin{aligned} W_{in} &= \left(\frac{EI_1 a^2\pi^4}{64L^4}\right)\left[L_1 + \left(\frac{I_2}{I_1}\right)L_2 + \left(\frac{L}{\pi}\right)\left(1 - \frac{I_2}{I_1}\right)\sin\left(\frac{\pi L_1}{L}\right)\right] \\ &= \left(\frac{EI_1 a^2\pi^4}{64L^3}\right)\left[\frac{L_1}{L} + \left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L}\right) + \left(\frac{1}{\pi}\right)\left(1 - \frac{I_2}{I_1}\right)\sin\left(\frac{\pi L_1}{L}\right)\right] \end{aligned} \tag{4.73}$$

The external work done is given by

$$-W_{ex} = \frac{P}{2}\int_0^L \left(\frac{a\pi}{2L}\right)^2 \sin^2\left(\frac{\pi x}{2L}\right)\mathrm{d}x = \frac{Pa^2\pi^2}{16L^2}(L) \tag{4.74}$$

At critical load from work equation $-W_{ex} = W_{in}$

$$P_{cr} = \left(\frac{EI_1\pi^2}{4L^2}\right)\left[\frac{L_1}{L} + \left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L}\right) + \left(\frac{1}{\pi}\right)\left(1 - \frac{I_2}{I_1}\right)\sin\left(\frac{\pi L_1}{L}\right)\right] \tag{4.75}$$

Again as a typical case consider $L_2 = L_1 = \frac{L}{2}$ and $I_2 = \frac{I_1}{2} = I$.

$$P_{cr} = \left(\frac{\pi^2 EI}{2L^2}\right)\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{2\pi}\right] = \frac{(4.4865)EI}{L^2} = \frac{\pi^2 EI}{(1.483L)^2} \tag{4.76}$$

However, to study the effect of more terms in the representation of displacement, consider

$$\begin{aligned} y(x) &= a_1\left(1 - \cos\frac{\pi x}{2L}\right) + a_3\left(1 - \cos\frac{3\pi x}{2L}\right) + \ldots \\ &= \sum_n a_{2n-1}\left[1 - \cos\frac{(2n-1)\pi x}{2L}\right] \end{aligned} \tag{4.77}$$

$$y' = \sum_n a_{2n-1}\left[\frac{(2n-1)\pi}{2L}\right]\sin\frac{(2n-1)}{2L}\pi x$$

$$\text{and}\quad y'' = \sum_n a_{2n-1}\left[\frac{(2n-1)\pi}{2L}\right]^2\cos\frac{(2n-1)}{2L}\pi x$$

where $n = 1, 2, 3, \ldots$.

$$\begin{aligned} -W_{ex} &= \frac{P}{2}\int_0^L (y')^2 \mathrm{d}x = \frac{P\pi^2}{8L^2}\sum_n a_{2n-1}^2(2n-1)^2\int \sin^2\left(\frac{2n-1}{2L}\right)\pi x\,\mathrm{d}x \\ &= \frac{P\pi^2}{16L}\sum_n (2n-1)^2 a_{2n-1}^2 \end{aligned}$$

$$\begin{aligned} W_{in} &= \frac{1}{2}\int_0^L EI(y'')^2\mathrm{d}x \\ &= \frac{EI_1}{2}\left(\frac{\pi}{2L}\right)^4\sum_n (2n-1)^4 a_{2n-1}^2\left[\int_0^{L_1}\cos^2\frac{(2n-1)\pi x}{2L}\mathrm{d}x\right. \\ &\qquad\left. + \left(\frac{I_2}{I_1}\right)\int_{L_1}^{L}\cos^2\frac{(2n-1)\pi x}{2L}\mathrm{d}x\right] \end{aligned}$$

$$= \frac{EI_1\pi^4}{32L^4}\sum_n (2n-1)^4 a_{2n-1}^2 \left[\int_0^{L_1} \frac{1}{2}\left(1+\cos\frac{(2n-1)\pi x}{L}\right)\mathrm{d}x \right.$$

$$\left. + \left(\frac{I_2}{I_1}\right)\int_{L_1}^{L} \frac{1}{2}\left(1+\cos\frac{(2n-1)\pi x}{L}\right)\mathrm{d}x \right]$$

$$= \frac{EI_1\pi^4}{64L^4}\sum_n (2n-1)^4 a_{2n-1}^2 \left\{ \left[\left(L_1 + \frac{L}{(2n-1)\pi}\sin\frac{(2n-1)\pi L_1}{L}\right)\right]\right.$$

$$\left. + \left(\frac{I_2}{I_1}\right)\left[L_2 + \frac{L}{(2n-1)\pi}\left(\sin\frac{(2n-1)\pi L}{L} - \sin\frac{(2n-1)\pi L_1}{L}\right)\right]\right\}$$

$$= \frac{EI_1\pi^4}{64L^4}\sum_n (2n-1)^4 a_{2n-1}^2 \left[L_1 + \left(\frac{I_2}{I_1}\right)L_2\right.$$

$$\left. + \frac{L}{(2n-1)\pi}\left(1-\frac{I_2}{I_1}\right)\sin\frac{(2n-1)}{L}\pi L_1\right]$$

Assuming only two terms in the series, the total potential energy of the strut is

$$\Pi = W_{\text{in}} + W_{\text{ex}} = a_1^2\left[\frac{P\pi^2}{16L} - \frac{EI_1\pi^4}{64L^4}\left\{L_1 + \left(\frac{I_2}{I_1}\right)L_2 + \frac{L}{\pi}\left(1-\frac{I_2}{I_1}\right)\sin\frac{\pi L_1}{L}\right\}\right]$$

$$+ a_3^2\left[\frac{9P\pi^2}{16L} - \frac{81EI_1\pi^4}{64L^4}\left\{L_1 + \left(\frac{I_2}{I_1}\right)L_2 + \frac{L}{3\pi}\left(1-\frac{I_2}{I_1}\right)\sin\frac{3\pi L_1}{L}\right\}\right]$$

Differentiating the total potential energy with respect to each of the unknowns, a_1 and a_3, and equating resulting expression to zero yields:

$$\frac{\partial\Pi}{\partial a_1} = 0 \text{ gives } P_{\text{cr}} = \frac{EI_1\pi^2}{4L^2}\left[\left(\frac{L_1}{L}\right) + \left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L_1}\right) + \frac{1}{\pi}\left(1-\frac{I_2}{I_1}\right)\sin\pi\left(\frac{L_1}{L}\right)\right]$$

$$\frac{\partial\Pi}{\partial a_3} = 0 \text{ gives } P_{\text{cr}} = \frac{9EI_1\pi^2}{4L^2}\left[\left(\frac{L_1}{L}\right) + \left(\frac{I_2}{I_1}\right)\left(\frac{L_2}{L_1}\right) + \frac{1}{3\pi}\left(1-\frac{I_2}{I_1}\right)\sin 3\pi\left(\frac{L_1}{L}\right)\right]$$

For the case of uniform strut with $L_1 = L$, $L_2 = 0$ and $I_1 = I$, above expressions reduce to

$$P_{\text{cr}} = \frac{EI\pi^2}{4L^2} \quad \text{and} \quad P_{\text{cr}} = \frac{9EI\pi^2}{4L^2} \tag{4.78}$$

$P_{\text{cr}} = (\pi^2 EI/4L^2)$ being the smallest value gives the solution to the problem. For the typical case $L_1 = L_2 = (L/2)$ and $I_2 = (I_1/2) = I$, the critical load values reduce to

$$P_{\text{cr}} = \left(\frac{3}{4} + \frac{1}{2\pi}\right)\left[\frac{\pi^2 E(2I)}{(2L)^2}\right] = \frac{4.4865EI}{L^2} = \frac{\pi^2 EI}{(1.483L)^2}$$

$$P_{\text{cr}} = \left(\frac{3}{4} + \frac{1}{6\pi}\right)\left[\frac{9\pi^2 E(2I)}{(2L)^2}\right] = \frac{35.6661EI}{L^2} = \frac{\pi^2 EI}{(0.526L)^2} \tag{4.79}$$

The assumed deflected configuration or the trial function can also be of polynomial form as explained in Chap. 2. The following example will illustrate the use of polynomial trial functions.

Example 4.3. Estimate the critical value of the load P acting along the centroidal axis of the tapered strut shown in Fig. 4.12 that will cause its buckling. The strut is clamped at the end A and pinned to a roller support at B. The moment of inertia of cross-section of the strut reduces linearly from I_1 at the fixed end to I_0 at the hinged end. The variation in the moment of inertia of the cross-section may be defined by the relationship

$$I(x) = I_0\left[\beta - (\beta - 1)\left(\frac{x}{L}\right)\right] \quad \text{where } \beta = \frac{I_1}{I_0}$$

The parameter β is a measure of magnitude of the taper of the member. $\beta = 1$ represents a prismatic member. For the fixed-hinged supported case, the single-degree-displacement trial function is given by

$$y(x) = a\left[\frac{3}{2}\left(\frac{x}{L}\right)^2 - \frac{5}{2}\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right]$$

To facilitate computations a non-dimensional variable ξ $(= x/L)$ is introduced such that $0 \leq \xi \leq 1$. Thus the dimensional variables reduce to

$$I(\xi) = I_0[\beta - (\beta - 1)\xi] \tag{4.80}$$

and

$$y(\xi) = a\left[\frac{3}{2}\xi^2 - \frac{5}{2}\xi^3 + \xi^4\right]$$

The first and second derivatives of $y(\xi)$ with respect to ξ are

$$y' = \left(\frac{a}{L}\right)\left[3\xi - \frac{15}{2}\xi^2 + 4\xi^3\right]$$

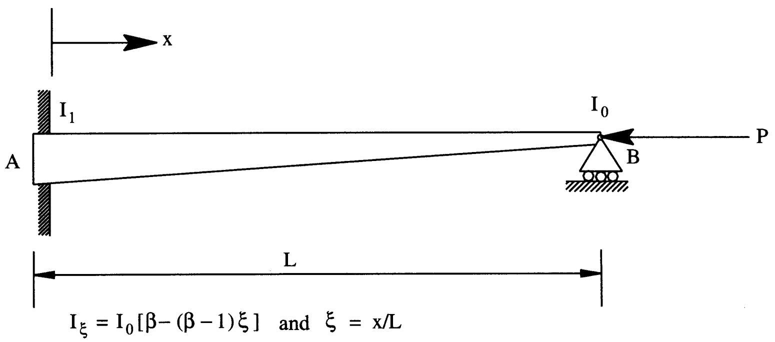


Fig. 4.12. Buckling of tapered strut

and

$$y'' = \left(\frac{a}{L^2}\right)\left[3 - 15\xi + 12\xi^2\right]$$

The total potential associated with flexural deformations in this case can be defined as

$$\Pi = W_{\text{in}} - W_{\text{ex}} = \frac{E}{2}\int_0^1 I(y'')^2 L\mathrm{d}\xi - \frac{P}{2}\int_0^1 (y')^2 L\mathrm{d}\xi = 0 \tag{4.81}$$

Substituting y' and y'' in the expression for Π

$$= \left(\frac{a^2 EI_0}{2L^4}\right)\int_0^L [\beta - (\beta - 1)\xi](3 - 15\xi + 12\xi^2)^2 L\mathrm{d}\xi$$

$$-\left(\frac{Pa^2}{2L^2}\right)\int_0^L \left[3\xi - \frac{15}{2}\xi^2 + 4\xi^3\right]^2 L\mathrm{d}\xi$$

or

$$\left(\frac{a^2 EI_0}{2L^3}\right)(1.05\beta + 0.75) - \frac{a^2 P}{2L}(0.0867) = 0$$

Therefore,

$$P_{\text{cr}} = 11.534(1.05\beta + 0.75)\frac{EI_0}{L^2} = \frac{\pi^2 EI_0}{(KL)^2} \tag{4.82}$$

where

$$K^2 = \pi^2/[11.534 \times (1.05\beta + 0.75)]$$

As a typical case consider $I_1 = 5I_0$ i.e. $\beta = 5$. The critical load at buckling becomes

$$P_{\text{cr}} = \frac{69.20 EI_0}{L^2} = \frac{\pi^2 EI_0}{(0.378L)^2}$$

For a prismatic member with $I_1 = I_0$ i.e. $\beta = 1$

$$P_{\text{cr}} = (11.534 \times 1.8)\frac{EI_0}{L^2} = \frac{20.76 EI_0}{L^2} = \frac{\pi^2 EI_0}{(0.6895L)^2}$$

The exact solution for this case is $(20.14EI_0/L^2)$. As discussed earlier in Chap. 2, it is advantageous to assume *displacement function* in the general form.

$$y(x) = \sum_{n=0}^{\infty} a_n f(x) x^n \tag{4.83}$$

where $f(x)$ is a *special function* satisfying the prescribed geometrical boundary conditions and $\sum a_n x^n$ is a *power series*. For the present case of fixed-hinged strut choosing $f(x) = x^2(x - L)$ with $n = 1$, the trial function becomes

$$y(x) = a_0 x^2(x - L) + a_1 x^3(x - L)$$

In terms of non-dimensional variable ξ $(= x/L)$, the trial function in the range $0 \le \xi \le 1$ can be defined as

$$y(\xi) = a_0(\xi^3 - \xi^2) + a_1(\xi^4 - \xi^3) \tag{4.84}$$

The first and second derivatives with respect to ξ are

$$y'(\xi) = \left(\frac{1}{L}\right)[a_0(3\xi^2 - 2\xi) + a_1(4\xi^3 - 3\xi^2)]$$

$$y''(\xi) = \left(\frac{1}{L^2}\right)[a_0(6\xi - 2) + a_1(12\xi^2 - 6\xi)]$$

Substituting the values of y' and y'' in various terms of the total potential expression, namely

$$\left(\frac{E}{2}\right)\int_0^L I(x)[y''(x)]^2 \mathrm{d}x = \left(\frac{E}{2}\right)\int_0^1 I(\xi)[y''(\xi)]^2(L\mathrm{d}\xi)$$

$$= \left(\frac{EI_0}{2L^4}\right)\int_0^1 [\beta - (\beta - 1)\xi][a_0(6\xi - 2) + a_1(12\xi^2 - 6\xi)]^2 L\mathrm{d}\xi$$

$$= \left(\frac{EI_0}{2L^3}\right)\int_0^1 [\beta - (\beta - 1)\xi][a_0^2(36\xi^2 + 4 - 24\xi) + a_1^2(144\xi^4 + 36\xi^2 - 144\xi^3) + 2a_0a_1(72\xi^3 - 60\xi^2 + 12\xi)]\,\mathrm{d}\xi$$

$$= \left(\frac{EI_0}{2L^3}\right)[a_0^2(\beta + 3) + a_1^2(0.6\beta + 4.2) + 2a_0a_1(0.6\beta + 3.4)]$$

$$\frac{P}{2}\int_0^L (y')^2 \mathrm{d}x = \frac{P}{2}\int_0^1 [y'(\xi)]^2(L\mathrm{d}\xi)$$

$$= \left(\frac{P}{2L}\right)\int_0^1 [a_0(3\xi^2 - 2\xi) + a_1(4\xi^3 - 3\xi^2)]^2 \mathrm{d}\xi$$

$$= \left(\frac{P}{2L}\right)\int_0^1 [a_0^2(9\xi^4 + 4\xi^2 - 12\xi^3) + a_1^2(16\xi^6 + 9\xi^4 - 24\xi^5) + 2a_0a_1(12\xi^5 - 17\xi^4 + 6\xi^3)]$$

$$= \left(\frac{P}{2L}\right)\left[a_0^2\left(\frac{2}{15}\right) + a_1^2\left(\frac{3}{35}\right) + 2a_0a_1\left(\frac{1}{10}\right)\right]$$

Therefore, total potential energy is given by

$$\Pi = \left(\frac{EI_0}{2L^3}\right)\left\{a_0^2\left[(\beta+3)-\left(\frac{2}{15}\right)\alpha^2\right]+a_1^2\left[(0.6\beta+4.2)-\left(\frac{3}{35}\right)\alpha^2\right]\right.$$
$$\left.+2a_0a_1\left[(0.6\beta+3.4)-\left(\frac{1}{10}\right)\alpha^2\right]\right\}$$

where $\alpha^2 = PL^2/(EI_0)$. The conditions of equilibrium are given by $\partial\Pi/\partial a_0 = 0$ and $\partial\Pi/\partial a_1 = 0$. For a non-trivial solution i.e. a buckled form solution, the determinant of the coefficients of a_0 and a_1 vanishes. This *stability determinant* provides characteristic equation. The characteristic equation is given by:

$$[7.5(\beta+3)-\alpha^2][7(\beta+7)-\alpha^2]-\left(\frac{7}{8}\right)[(6\beta+34)-\alpha^2]^2 = 0 \tag{4.85}$$

For a prismatic member $\beta = 1.0$ and the characteristic equation reduces to

$$[30-\alpha^2][56-\alpha^2]-\left(\frac{7}{8}\right)[40-\alpha^2]^2 = 0$$

or

$$\alpha^4 - 128\alpha^2 + 2240 = 0$$

The smallest root giving critical load is 20.92. Therefore,

$$\alpha^2 = 20.92 \quad \text{or} \quad P_{\text{cr}} = 20.92\left(\frac{EI_0}{L^2}\right)$$

For the typical case of tapered column with $\beta = 5$, the characteristic equation becomes

$$[60-\alpha^2][84-\alpha^2]-\left(\frac{7}{8}\right)[64-\alpha^2]^2 = 0$$
$$\alpha^4 - 256\alpha^2 + 11648 = 0$$

The smallest root which corresponds to the critical load is $\alpha^2 = 59.181$. Therefore,

$$P_{\text{cr}} = 59.181\left(\frac{EI_0}{L^2}\right) \tag{4.86}$$

It should be noted that for a prismatic member the one-degree-of-freedom trial function gives better results than two-degree-of-freedom trial function, because the former satisfied both geometrical and dynamical boundary condition. However, for tapered strut, later provided much better results.

The *Newmark's numerical integration* technique as described in Chap. 2 is extremely useful tool for analysis of variety of strut problems. The following examples will illustrate the versatility of the method. The example is a variation of Euler strut illustrating the effect of end conditions on the solution.

Example 4.4. A simply supported strut (Euler strut) has an overhang beyond the roller support as shown in Fig. 4.13. An axial force is applied at the unsupported end C. The load is free to deflect vertically with C while its line of action remains horizontal.

A buckled configuration is shown in Fig. 4.13c. It should be noted that any configuration involving a displacement of P will produce support (vertical) reactions which are to be taken into account while calculating moment M. To compute these support reactions, equate moments at the pinned support to zero i. e. $\sum M_0 = 0$.

$$\sum M_0 = -R_3(3h) + Pa_1 = 0 \quad \text{i.e.} \quad R_3 = \frac{Pa_1}{3h}$$

and

$$R_0 = -R_3 = -\frac{Pa_1}{3h}$$

For convenience take $a_1 = 3a$ at C. This will reduce values of reactions at the support nodes 0 and 3 to convenient values of Pa/h and $(-Pa/h)$, respectively.

The assumed deflection values of y_a may be obtained by tracing the buckled configuration on a graph paper with the help of a flexible elastic strip. It should be noted that the deflected shape is similar to one obtained by application of a concentrated

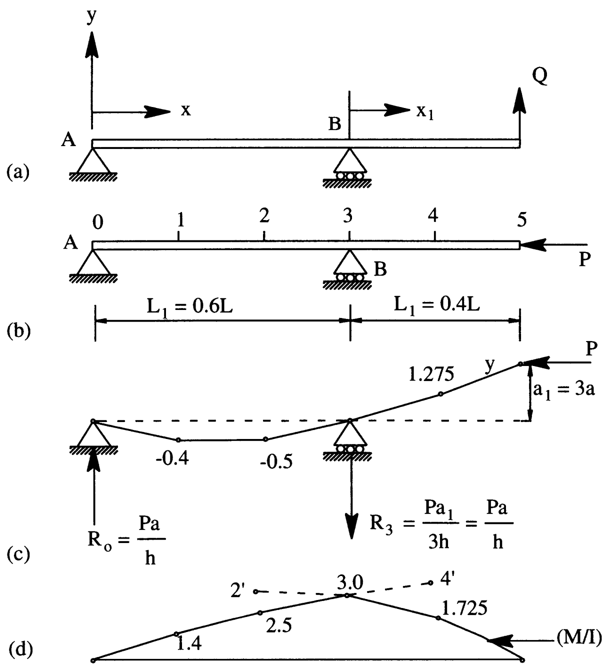


Fig. 4.13a–d. Buckling of strut with overhang. **a** Coordinate system and loads to obtain deflected configuration, **b** discretized strut, **c** buckled configuration, **d** moment diagram

vertical force at the overhang end C of the strut. The deflection due to a force Q acting at C are given by

$$y_x = \left(\frac{Q}{6EI}\right)\left(\frac{L_2}{L_1}\right)x(L_1^2 - x^2) \quad \text{for the range} \quad 0 \leq x \leq L_1$$

$$= \left(\frac{Q}{6EI}\right)x_1(2L_1L_2 + 3L_2x_1 - x_1^2) \quad \text{for the range} \quad 0 \leq x_1 \leq L_2$$

Consider a typical case with $L_1 = 0.6L$ and $L_2 = 0.4L$. The deflection are $y_0 = 0$; $y_1 = 0.064$, $y_2 = 0.080$, $y_3 = 0$, $y_4 = -0.204$ and $y_5 = -0.480$ $(\times 2QL^3/18EI)$. Since we are interested in the relative values, the displacement coordinates are scaled such that $y_5 = 3.00a$.

Expressing the value of displacements in the form $y_i = \xi_i a$. The coefficients ξ_i are: 0.000; -0.400; $= -0.500$; 0.000; 1.275 and 3.000. For the portions AB and BC the moments are given by $(i - \xi_i)$ and $(\xi_5 - \xi_i)$, respectively. For an improved accuracy at the node 3, the two M/I curves meeting at this node are extrapolated to the fictious points $4'$ and $2'$, respectively. For extrapolation the effect of vertical reaction at the enode 3 is ignored. The iterative procedure for computation of buckling load is given in Table 4.2.

At buckling:

$$3.000a = 138.284\frac{aPh^2}{12EI}$$

Therefore,

$$P_{\text{cr}} = \frac{6.5083EI}{L^2} = \frac{\pi^2 EI}{(1.231L)^2}$$

As another variation of Euler strut consider the vertical displacement of the free end C being restrained by a roller support such that the end is free to move horizontally. Thus the overhang strut reduces to a two-span continuous strut which is a single-degree indeterminate structure. The analysis of this strut is given in Example 4.5.

Example 4.5. Estimate the critical value of axial load P which will cause the two-span continuous strut of constant cross-section shown in Fig. 4.14 to buckle. As a typical case take $L_1 = 0.6L$ and $L_2 = 0.4L$.

The two-span continuous strut has a single-degree-of-indeterminacy. Consider R_3 to be the redundant action. The analysis of basic structure obtained by ignoring R_3 will result in a displacement at the support B i.e. node 3. This will violate the prescribed boundary condition of zero displacement at the node 3; hence a correction in displacement need be applied in each cycle of numerical integration. The value of R_3 should be adjusted such that $y_3 = 0$. Using triangular moment diagram (due to R_3) of arbitrary value, the form of correction y_c to be applied to y to account for the redundant force is computed. This can be achieved by means of *thrust line* concept.

Table 4.2. Computation of buckling load for a strut with an overhang

Node #	0	1	2	3	4	5	Multiplier
1st Cycle							
$y_a = \xi_a$	0	−0.400	−0.500	0.000	1.275	3.000	a
$\frac{M}{EI} = y''$	0	1.400	2.500 (3.500)	3.000 → ← 3.000	(2.725) 1.725	0.000	aP
Y''	–	16.50	29.40	$\frac{17.387+16.038}{33.425}$	20.25	–	$aPh/(12EI)$
y'	$(-20.80)^a$	−4.300	25.100	58.525	78.775		-do-
y	0	−20.800	−25.100	0.000	58.525	137.300	$aPh^2/(12EI)$
y_d	0	−0.454	−0.549	0.000	1.279	3.000	$(45.767)aPh^2/(12EI)$
2nd Cycle							
y_a	0	−0.454	−0.549	0.000	1.279	3.000	a
y''	0	1.454	2.549 (3.549)	3.000 → ← 3.000	(2.721) 1.721	0.000	aP
Y''	–	17.189	29.944	$\frac{17.463+15.807}{33.270}$	20.21	–	$aPh/(12EI)$
y'	$(-21.441)^a$	−4.252	25.692	58.962	79.172		-do-
y	0	−21.441	−25.693	0.000	58.962	138.134	$aPh^2/(12EI)$
y_d	0	−0.466	−0.558	0.000	1.281	3.000	$(46.045)aPh^2/(12EI)$
3rd Cycle							
y_a	0	−0.466	−0.558	0.000	1.281	3.000	a
y''	0	1.466	2.558 (3.558)	3.000 → ← 3.000	(2.719) 1.719	0.000	aP
Y''	–	17.218	30.046	$\frac{17.477+15.799}{33.277}$	20.19	–	$aPh/(12EI)$
y'	$(-21.494)^a$	−4.276	25.770	59.047	79.237		-do-
y	0	−21.494	−25.770	0.000	59.047	138.284	$aPh^2/(12EI)$
y_d	0	−0.466	−0.559	0.000	1.281	3.000	$(46.095)aPh^2/(12EI)$

[a] If the slope of the chord 0-1 is assumed to be $(-\theta_0)$, then the slopes of chords 1-2 and 2-3 are $(-\theta_0 + 16.50)$ and $(-\theta_0 + 16.50 + 29.40)$, respectively. Starting with 0.000 displacement at the node 0, the displacement at the node 3, $y_3 = [0.000 + (-\theta_0)h + (-\theta_0 + 16.50)h + (-\theta_0 + 16.50 + 29.50)h = \{-3\theta_0 + 2(16.50) + 29.40\}h$. However, the prescribed deflection at node 3 is zero, thus $\theta_0 = -20.80$.

The thrust line is known to pass through the supports A and C (i.e. has zero ordinate values) and must pass through the point of contra flexure. A computationally convenient comparable value say $0.3a$ is selected at the point 3 as shown in Fig. 4.14. The moment at any point in the strut is then P times the offset distance between the deflected configurations of the strut y_a and the thrust line. The thrust lines may be extended to the points $2'$ and $4'$ for calculating fictitious moments for better accuracy.

At buckling:

$$-1.00a = (-8.25)\frac{Pah^2}{12EI}$$

Therefore,

$$P_{cr} = \frac{36.364EI}{L^2} = \frac{\pi^2 EI}{(0.521L)^2}$$

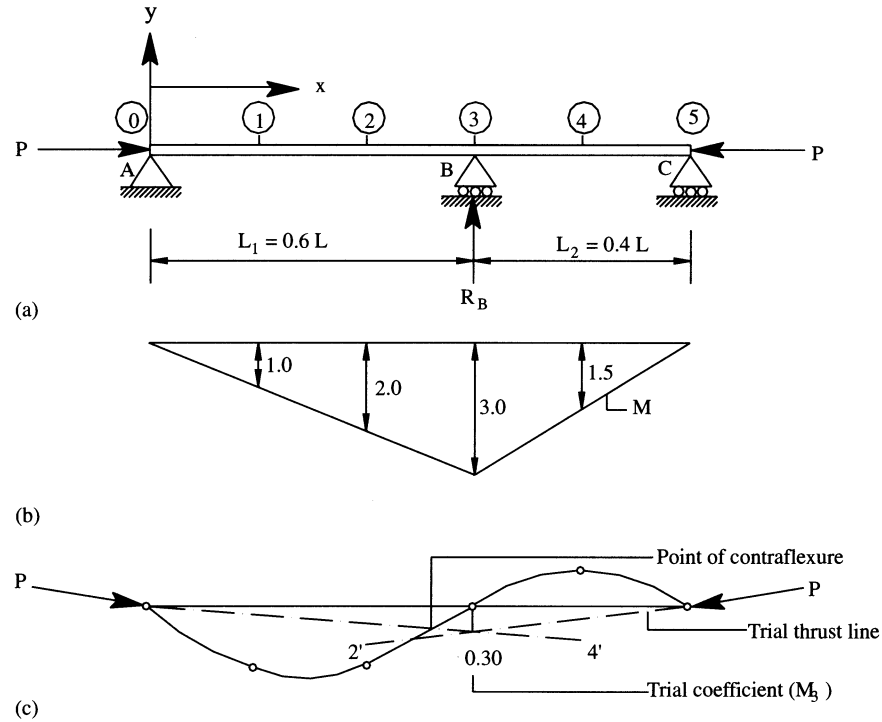


Fig. 4.14a–c. Buckling of a continuous two-span strut. **a** Strut with interior support, **b** moment due to arbitrary value of R_B, **c** buckled configuration

The analysis procedure as usual starts with the assumption of buckling configuration y_a with a minimum number of *points of contra flexure*. The displacements in the two spans are of opposite signs. The estimated values of displacements in assumed configuration as obtained from a sketch drawn on a graph paper with the help of flexible elastic strip are 0.00; -1.00; -0.95; 0.00; 0.42 and 0.00. The computations are shown in Table 4.3.

The application of numerical technique to a stepped strut is illustrated in Example 4.2. The procedure is equally convenient for strut with continuously variable cross-section.

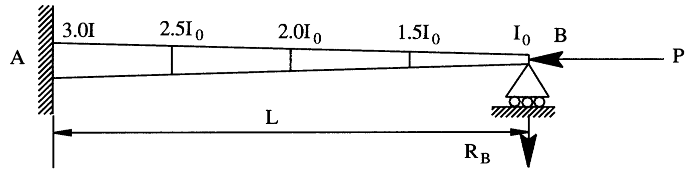
Example 4.6. Estimate the critical value of axial load P that will cause the propped cantilever of continuously variable section shown in Fig. 4.15 to buckle.

As in the previous example the strut is a first-degree redundant structure. The reaction R_B at the support B can conveniently be chosen as the redundant quantity. With origin at B the moment M at a section is given by: $M = -Py + R_Bx$. The term R_Bx is the effect of redundant and is represented in Fig. 4.15b by a triangular moment diagram. The first term $-Py$ is exactly the same as in Euler strut problem. The analysis of basic structure neglecting the second term will result in a displacement at

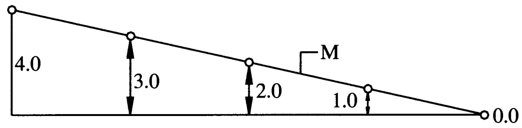
Table 4.3. Computations of buckling load for a continuous two-span strut

Node #	0		1		2		3		4		5	Multiplier/ remarks
I. Correction for arbitrary value of redundant, R_3												
M	0		−1.0		−2.0		−3.0		−1.5		0	Fig. 4.14b
Y''	–		−6.0		−12.0		$\frac{-8.0-7.5}{-15.5}$		−9.0		–	Trapezoidal formula
y'		(8.0)		2.0		−10.0		−25.5		−34.5		
y_c	0		8.0		10.0		0.0		−25.5		−60.0	
II. Computation Table												
1st Cycle												
y_a	0.0		−1.00		−0.95		0.00		0.42		0.00	a
$\frac{M}{EI}=y''$	0.0		0.90		0.75 (0.50)		−0.30 → ← −0.30		(−0.82) −0.57		0.00	Pa/EI
Y''	–		9.75		8.10		$\frac{0.04-2.61}{-2.57}$		−6.00		–	$Pah/(12EI)$
y'		(−9.20)[a]		0.55		8.65		6.08		0.08		-do-
y	0.0		−9.20		−8.65		0.00		6.08		6.16	$Pah^2/(12EI)$
y_c	0.0		0.82		1.03		0.00		−2.62		−6.16	-do-
y_d	0.0		−8.38		−7.62		0.00		3.46		0.000	$Pah^2/(12EI)$
$y_a = y_d$	0.0		−1.00		−0.91		0.00		0.41		0.000	$(8.38)Pah^2/(12EI)$
2nd Cycle												
Y''	0.00		0.90		0.71 (0.46)		−0.30 → ← −0.30		(−0.81) −0.56		0.00	Pa/EI
Y''	–		9.71		7.70		$\frac{0.03-2.57}{-2.54}$		−5.90		–	$Pah/(12EI)$
y'		(−9.04)[a]		0.67		8.37		5.83		−0.07		-do-
y	0.0		−9.04		−8.37		0.00		5.83		5.76	$Pah^2/(12EI)$
y_c	0.0		0.77		0.96		0.00		−2.45		−5.76	-do-
y_d	0.0		−8.27		−7.41		0.00		3.38		0.000	$Pah^2/(12EI)$
$y_a = y_d$	0.0		−1.00		−0.90		0.00		0.41		0.000	$(8.27)Pah^2/(12EI)$
3rd Cycle												
Y''	0.00		0.90		0.70 (0.45)		−0.30 → ← −0.30		(−0.81) −0.56		0.00	Pa/EI
Y''	–		9.70		7.60		$\frac{0.04-2.56}{-2.52}$		−5.90		–	$Pah/(12EI)$
y'		(−9.00)[a]		0.70		8.30		5.78		−0.12		-do-
y	0.0		−9.00		−8.30		0.00		5.78		5.66	$Pah^2/(12EI)$
y_c	0.0		0.75		0.94		0.00		−2.41		−5.66	-do-
y_d	0.0		−8.25		−7.36		0.00		3.37		0.000	$Pah^2/(12EI)$
$y_a = y_d$	0.0		−1.00		−0.89		0.00		0.41		0.000	$(8.25)Pah^2/(12EI)$

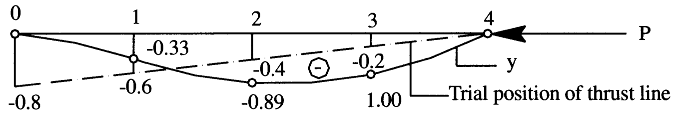
[a] See note at the end of Table 4.2



(a)



(b)



(c)

Fig. 4.15a–c. Buckling of propped cantilever of continuously variable section. **a** Propped cantilever of variable cross section, **b** moment diagram due to arbitrary redundant, **c** buckling configuration

the support B. The effect of second term represented by a triangular moment diagram of arbitrary value is added in such a way that all boundary conditions are satisfied. By neglecting $R_B x$ completely and taking $M = -Py_\mathrm{a}$ produces convergence problem. It should be noted that the moment diagram has a point of contra flexure where Py_a and $R_B x$ components of M_a cross over. Making a guess for this position of C will make the M values quite accurate resulting in a much faster convergence. The line joining points of zero moment namely BC is termed *thrust line*. The moment at any point in the strut is then P times the offset distance between deflected configuration of the strut and the thrust line. To illustrate the efficiency of the procedure, assume that the deflected shape is given by trial function $y(x) = a\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$ which is applicable to a strut of uniform cross-section and corresponding trial value of thrust line ordinate is $0.8a$. The relative values of y_a are: 0.00; -0.33; -0.89; -1.00 and 0.00. The computations are given in Table 4.4.

At buckling:

$$-1.000a = -4.919\frac{aPh^2}{12EI_0}$$

Therefore,

$$P_\mathrm{cr} = \frac{39.032EI_0}{L^2}$$

Table 4.4. Computations of buckling load of a redundant strut

Node #	0		1		2		3		4	Multiplier/remarks
I. Correction for arbitrary value of redundant, R_B										
M	4.000		3.000		2.000		1.000		0.000	
$M/EY = y''$	1.333		1.200		1.000		0.667		0.000	
Y''	7.767		14.333		11.867		7.670		–	Parabolic variation
y'		7.767		22.100		33.967		41.637		
y_c	0.000		7.767		29.867		63.834		105.471	
II. Computation Table										
1st Cycle										
y_a	0.000		−0.330		−0.890		−1.000		0.000	a
M	−0.800		−0.270		0.490		0.800		0.000	aP
y''	−0.267		−0.108		0.245		0.533		0.000	aP/EI_0
Y''	−1.136		−1.102		2.875		5.575		–	$aPh/(12EI_0)$
y'		−1.136		−2.238		0.637		6.212		–do–
y	0.000		−1.136		−3.374		−2.737		3.475	$Pah^2/(12EI_0)$
y_c	0.000		−0.256		−0.984		−2.103		−3.475	–do–
y'_d	0.000		−1.392		−4.358		−4.840		0.000	aPh^2 $/(12EI_0)$
2nd Cycle										
$y_a = y_d$	0.000		−0.288		−0.900		−1.000		0.000	$a = (4.840)aPh^2$ $/(12EI)$
M	−0.800		−0.312		0.500		0.800		0.000	aP
y''	−0.267		−0.125		0.250		0.533		0.000	aP/EI_0
Y''	−1.185		−1.267		2.908		5.580		–	$aPh/(12EI_0)$
y'		−1.185		−2.452		−0.465		6.036		–do–
y	0.000		−1.185		−3.637		−3.181		2.855	$Pah^2/(12EI_0)$
y_c	0.000		−0.210		−0.808		−1.728		−2.855	–do–
y'_d	0.000		−1.395		−4.445		−4.909		0.000	$aPh^2/(12EI_0)$
$y_a = y_d$	0.000		−0.284		−0.905		−1.000		0.000	$a = (4.909)aPh^2$ $/(12EI)$
3rd Cycle										
M	−0.800		−0.316		0.505		0.800		0.000	aP
y''	−0.267		−0.126		0.253		0.533		0.000	aP/EI_0
Y''	−1.186		−1.274		2.937		5.583		–	$aPh/(12EI_0)$
y'		−1.186		−2.460		0.477		6.06		–do–
y	0.000		−1.186		−3.646		−3.169		2.891	$Pah^2/(12EI_0)$
y_c	0.000		−0.213		−0.819		−1.750		−2.891	–do–
y'_d	0.000		−1.399		−4.465		−4.919		0.000	$aPh^2/(12EI_0)$
y_d	0.000		−0.284		−0.908		−1.000		0.000	$a = (4.919)aPh^2$ $/(12EI_0)$

4.5 Large Deflection Theory

The classical expressions for linear theory based on small deflection used in the preceding sections are not applicable to the geometrically non-linear problems involving large deflections. In classical linear theory, the presumption that the deflections are small renders the moment-curvature relation linear. They are based on un-deformed form of the structure. Moreover, the magnitude of deflections at the *post-buckling stage* remains undetermined. To determine these deflections accurate expression for the curvature of buckled strut is required. Consider an element of strut shown in Fig. 4.16.

$$\sin\theta = \frac{\mathrm{d}y}{\mathrm{d}s} \approx \frac{\mathrm{d}y}{\mathrm{d}x} = y'$$

Thus the slope $\theta = (\sin^{-1} y')$. Since the *curvature* is the rate of change of slope

$$\frac{1}{\bar{R}} = \theta' = \frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} y') = \frac{y''}{[1-(y')^2]^{1/2}}$$

Noting that the radius of curvature of an element before deflection is infinity i.e. curvature, $\frac{1}{R} = \frac{1}{\infty} = 0$. Hence change in curvature

$$\frac{1}{\bar{R}} - \frac{1}{R} = \frac{y''}{[1-(y')^2]^{\frac{1}{2}}} = y''[1-(y')^2]^{-\frac{1}{2}} \tag{4.87}$$

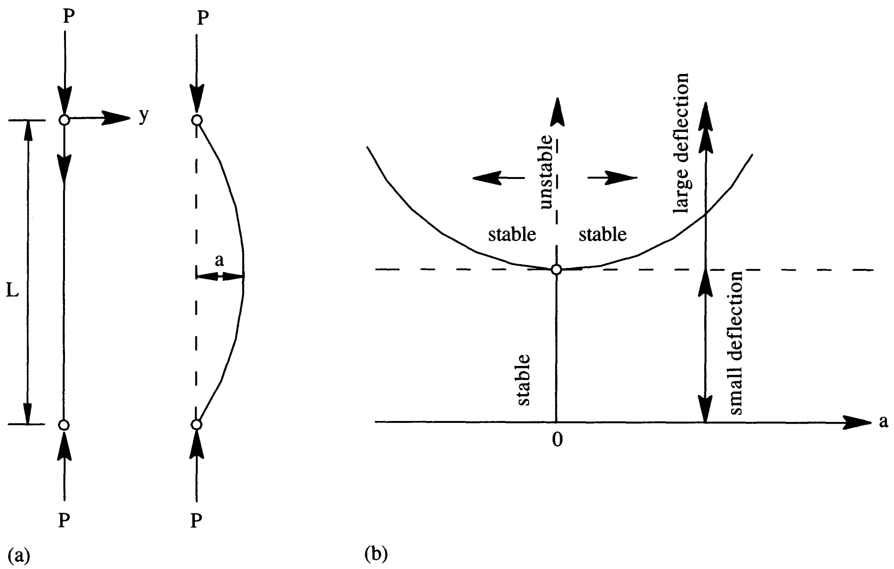


Fig. 4.16a,b. Equilibrium position of Euler strut and its bifurcation diagram. **a** Equilibrium position, **b** bifurcation diagram

A discerning reader will notice that this expression is different from that encountered in theory of elasticity namely, $1/R = y''/[1-(y')^2]^{3/2}$. The difference in the two expressions is due to the difference in the co-ordinate system adopted. In the former the coordinate axis is along the deflected shape while in the later the co-ordinate axis is along the un-deflected strut.

The displacement or movement of the load Δ can be expressed from geometrical considerations as

$$(\mathrm{d}s)^2 = (\mathrm{d}x - \mathrm{d}u)^2 + (\mathrm{d}y)^2$$

Divide by $\mathrm{d}x$:

$$\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 = \left(\frac{\mathrm{d}x - \mathrm{d}u}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

Noting that $\mathrm{d}s \approx \mathrm{d}x$

$$1 = (1-u')^2 + (y')^2 \quad \text{or} \quad (1-u') = [1-(y')^2]^{1/2}$$

Thus

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 - [1-(y')^2]^{1/2} \quad \text{or} \quad \mathrm{d}u = \{1 - [1-(y')^2]^{1/2}\}\,\mathrm{d}x$$

Integrating both sides

$$\int_0^L \mathrm{d}u = \Delta = \int_0^L [1-(1-y'^2)^{1/2}]\,\mathrm{d}x \tag{4.88}$$

The total potential energy of the strut $\Pi = (U - V)$ is given by:

$$\Pi = \int_0^L \left\{\left(\tfrac{1}{2}\right) EI(y'')^2(1-y'^2)^{-1} - P[1-(1-y'^2)^{1/2}]\right\} \mathrm{d}x$$

$$= \int_0^L \left\{\left(\tfrac{1}{2}\right) EI(y'')^2(1+y'^2+\ldots) - P\left[\left(\tfrac{1}{2}\right) y'^2 - \left(\tfrac{1}{8}\right) y'^4 + \ldots\right]\right\} \mathrm{d}x$$

Retaining terms upto fourth-order only:

$$\Pi = \int_0^L \left\{\left(\tfrac{1}{2}\right) EI(y'')^2 - \left(\tfrac{1}{2}\right) Py'^2\right\} \mathrm{d}x + \int_0^L \left[\left(\tfrac{1}{2}\right) EIy'^2y''^2 + \left(\tfrac{1}{8}\right) Py'^4\right] \mathrm{d}x \tag{4.89}$$

It must be noted that there are no constant terms. The first term is quadratic form of *classical eigenvalue problem* of a buckled strut and corresponds to the linearized differential equation of Euler strut: $y'''' + \alpha^2 y'' = 0$.

To illustrate the significance of the results consider a perfect simply supported strut under the action of an axial load P. The equilibrium configuration is assumed to be given by a trial function:

$$y(x) = a \sin\left(\frac{\pi x}{L}\right)$$

Substituting $y(x)$ in the expression for Π given by (4.89).

$$\Pi = \frac{1}{2}\left(\frac{\pi^2}{L^2}\right)\left(\frac{L}{2}\right)a^2\left[\left(\frac{EI\pi^2}{L^2}\right) - P\right] + \frac{1}{2}\left(\frac{\pi^4}{L^4}\right)\left(\frac{L}{8}\right)a^4\left[\left(\frac{EI\pi^2}{L^2}\right) - \left(\frac{3P}{4}\right)\right]$$

From

$$\frac{\partial \Pi}{\partial a} = 0 \tag{4.90}$$

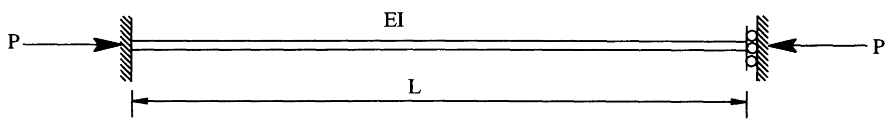
$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2} + \left(\frac{1}{8}\right)\left(\frac{\pi^4 EI}{L^4}\right)a^2 + \ldots \tag{4.91}$$

where terms up to a^2 are considered.

This equation is normally referred to as *initial post-buckling equation.* A plot of P versus mid-point displacement a is shown in the Fig. 4.16b. The curve starts at a constant first term ($\pi^2 EI/L^2$, the Euler critical load of a hinged-hinged strut). The curve has a horizontal tangent at P_{cr}. The local curvature of P_{cr} is equal to the coefficient of $a^2/2$. The critical load obtained is exact one because the trial function used happens to be the exact solution of the linearized eigen-value differential equation $y'' + \alpha^2 y = 0$.

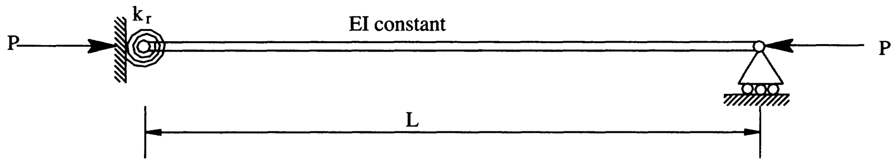
4.6 Problems

Problem 4.1. Compute the critical load for the uniform strut clamped at one end and constrained at the other end such that it is free to move or slide laterally without rotation as shown in Fig. P.4.1.



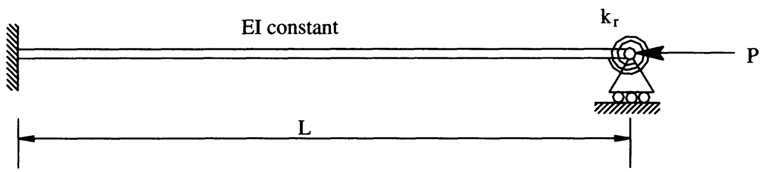
P.4.1

Problem 4.2. A strut of uniform cross-section is hinged at one end and restrained at the other by a rotation spring of stiffness k_{r} as shown in Fig. P.4.2. Determine the flexural buckling load of the strut as a function of L, EI and k_{r}. Describe graphically the variation of effective length factor K as a function of ratio k_{r}/EI.



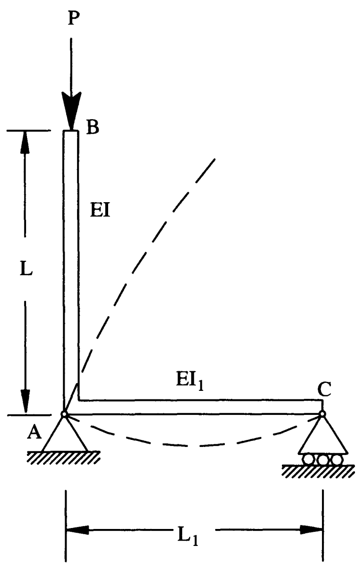
P.4.2

Problem 4.3. A strut of uniform cross-section is fixed at one end and at the other end it is restrained against bending rotation by a spring of stiffness k_r as shown in Fig. P.4.3. Determine the flexural buckling load of the strut as a function of L, EI and k_r. Describe graphically the variation of effective length factor K as a function of the ratio k_r/EI.



P.4.3

Problem 4.4. In a column and beam structure shown in Fig. P.4.4. The upper end of the column is free and the lower end is rigidly connected to a horizontal beam.

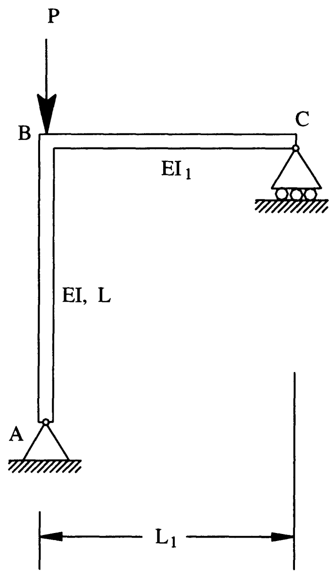


P.4.4

The column carries an axial thrust P. Determine the flexural buckling load of the structure.

[Hint: $k_r = 3EI_1/L_1$. The characteristic equation is $\alpha L \tan \alpha L = k_r L/(EI)$.]

Problem 4.5. In the column and beam structure of Fig. P.4.5, the lower end A of the column is hinged and the upper end is rigidly connected to a horizontal member BC with the end C: (i) roller supported, (ii) fixed and (iii) hinged. The column carries axial thrust P. Determine the flexural buckling load of the structure.



P.4.5

[Ans. The characteristic equations are:

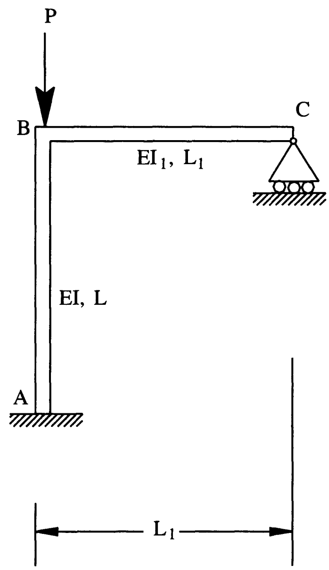
$$(i)\ \alpha L \tan(\alpha L) = 3\left(\frac{I_1}{L_1}\right)\left(\frac{L}{I}\right),$$

$$(ii)\ \tan \alpha L = \frac{3\alpha L}{(\alpha L)^2 + 3} \text{ and}$$

$$(iii)\ \tan \alpha L = \frac{4\alpha L}{(\alpha L)^2 + 4}.]$$

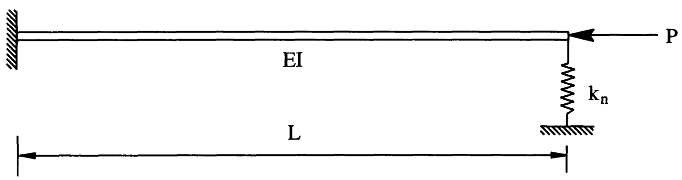
Problem 4.6. In a rigidly connected column and beam system shown in Fig. P.4.6, the base of the column is fixed and the far end of the beam is: (i) roller supported and (ii) fixed. The column is subjected to an axial force P. Determine the critical value of the load.

[Ans. Characteristic equations is: $(\alpha L)\cot(\alpha L) = -3\left(\frac{I_1}{L_1}\right)\left(\frac{L}{I}\right)$.]



P.4.6

Problem 4.7. A strut of constant cross-section is fixed at one end, and at the other end it is free to rotate but is constrained against translation by a spring of stiffness k_n as shown in Fig. P.4.7. Determine flexural buckling load of the strut as a function of L, EI and k_n. Describe graphically the variation of effective length K as a function of the ratio $(k_n L^3/EI)$.



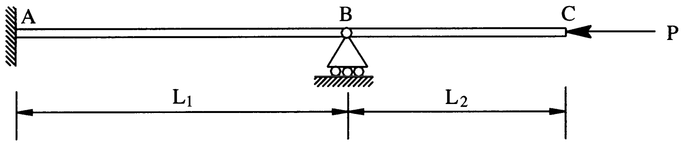
P.4.7

[Hint: $K = \pi/(\alpha L)$. The characteristic equation is:

$$\frac{k_n L^3}{EI} = \left[\frac{(\alpha L)^3}{\alpha L - \tan \alpha L}\right].$$

Assume values of ratio $k_n L^3/(EI)$ as 0.0, 5.0, 10.0, 15.0, etc. and calculate corresponding αL and hence K. Plot K versus $k_n L^3/(EI)$.]

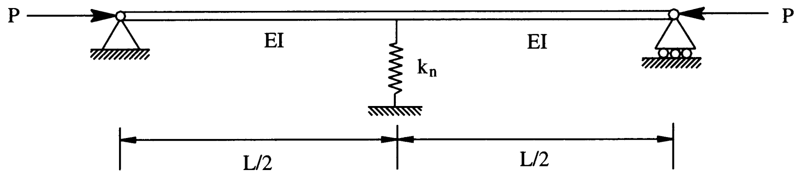
Problem 4.8. A continuous strut of constant cross-section and of length L $(= L_1 + L_2)$ is subjected to an axial load P as shown in Fig. P.4.8. If the strut is fixed at A and constrained against lateral deflection at a point B distant L_1 from A, determine
(a) the effective length factor of segment AB as function of the ratio L_1/L_2, and
(b) the effective length factor of segment BC as function of the ratio L_1/L_2.



P.4.8

Problem 4.9. If the strut of Problem 4.8 were hinged at the end A, what would be the corresponding effective length factors as functions of L_1/L_2?

Problem 4.10. The lateral displacement at the mid-point of a hinged-hinged strut is restrained by a spring of stiffness k_n attached at the point as shown in Fig. P.4.10. Determine the buckling load. What would be the critical load for the second mode of buckling?



P.4.10

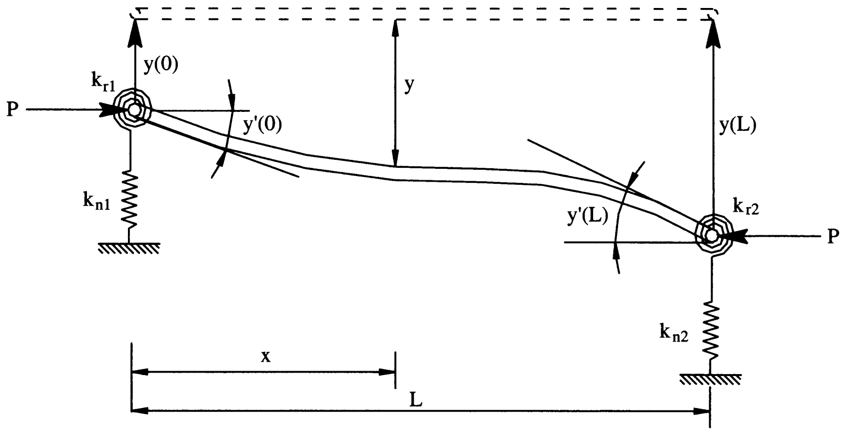
Problem 4.11. An axially loaded strut is supported by two translation and two rotation springs as shown in Fig. P.4.11. The boundary conditions of the strut are:

$$\begin{array}{llll} \text{At} & x = 0: \; y''(0) = \gamma_{r1} y'(0) & \text{and} & y'''(0) + \alpha^2 y'(0) = \gamma_{n1} y(0) \\ \text{At} & x = L: \; y''(L) = \gamma_{r2} y'(L) & \text{and} & y'''(L) + \alpha^2 y'(L) = \gamma_{n2} y(L) \end{array}$$

where $\gamma_{r1} = k_{r1}/(EI)$, $\gamma_{r2} = k_{r2}/(EI)$, $\gamma_{n1} = k_{n1}/(EI)$ and $\gamma_{n2} = k_{n2}/(EI)$. Determine the critical load of this elastically supported strut.

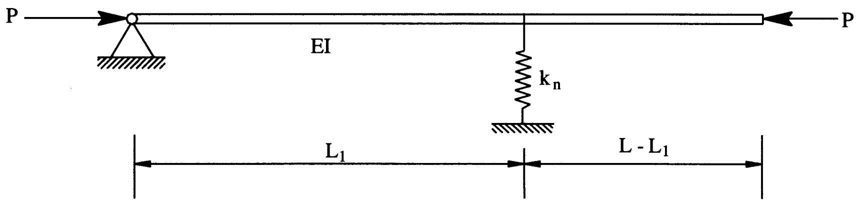
[Hint: Substitute the general solution to the governing differential equation i.e. $y(x) = A \sin \alpha x + B \cos \alpha x + C\left(\frac{x}{L}\right) + D$ in each of the boundary conditions. Equate determinant of coefficients to zero. That is,

$$\begin{vmatrix} 0 & -\gamma_{n1} L & \alpha^2 & -\gamma_{n1} L \\ -\alpha \gamma_{r1} & \alpha^2 & -\gamma_{r1}/L & 0 \\ (-\gamma_{n2} L \sin \alpha L) & (-\gamma_{n2} L \cos \alpha L) & (\alpha^2 - \gamma_{n2} L) & (-\gamma_{n2} L) \\ -(\alpha \gamma_{r2} \cos \alpha L + \alpha^2 \sin \alpha L) & -(\alpha \gamma_{r2} \sin \alpha L - \alpha^2 \cos \alpha L) & -\gamma_{n2} L & 0 \end{vmatrix} = 0\,]$$



P.4.11

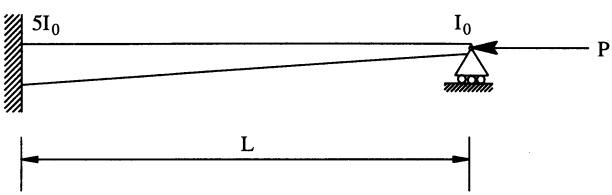
Problem 4.12. A strut of uniform cross section is hinged at one end and free at the other. The lateral displacement at a point distant L_1 from hinged end is restrained by an elastic spring of stiffness k_n as shown in Fig. P.4.12. Determine the critical load of the strut.



P.4.12

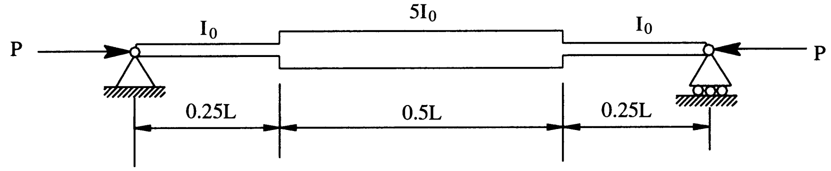
Problem 4.13. A tapered propped cantilever strut shown in Fig. P.4.13 is subjected to an axial load P. The moment of inertia of the cross-section varies linearly from I_0 at the propped hinged end to $5I_0$ at the fixed end. Use Newmark's numerical integration technique with four divisions to estimate the critical load P_{cr}.

[Ans. $P_{cr} = 54.18EI_0/L^2$]



P.4.13

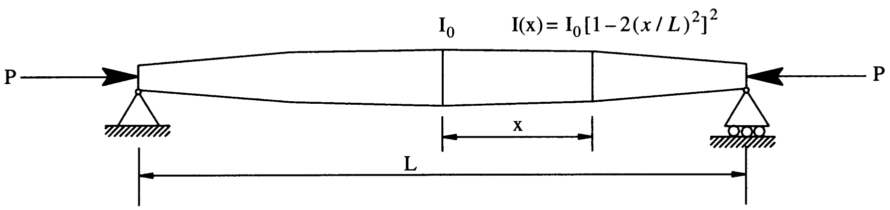
Problem 4.14. Use Newmark's numerical integration technique with four divisions to estimate the first critical value of the thrust P that will cause the buckling of the stepped strut shown in Fig. P.4.14.



P.4.14

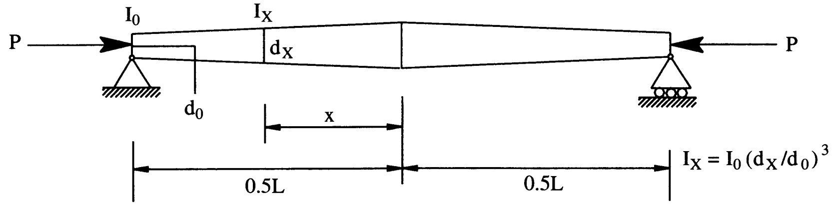
[Ans. $P_{\text{cr}} = 33.45EI_0/L^2$*]*

Problem 4.15. Estimate the critical load that will cause the buckling of the simply supported tapered strut of continuously variable cross-section as shown in Fig. P.4.15. The strut has maximum moment of inertia of I_0 at the mid-point. The moment of inertia at distance x from the mid-point is given by: $I(x) = I_0\left[1 - 2\left(\frac{x}{L}\right)^2\right]^2$.



P.4.15

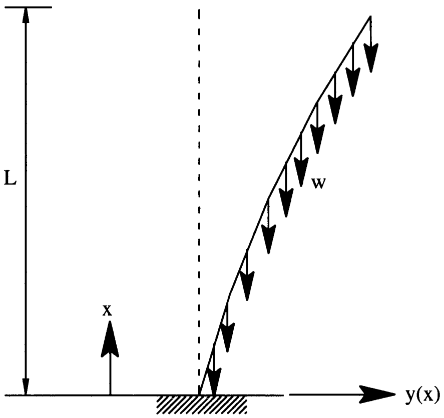
Problem 4.16. A simply supported strut of uniformly varying rectangular cross-section of constant width is subjected to a compressive load P as shown in Fig. P.4.16. Estimate the first critical load of the strut.



P.4.16

[Ans. $P_{\text{cr}} = 6.46EI_0/L^2$*]*

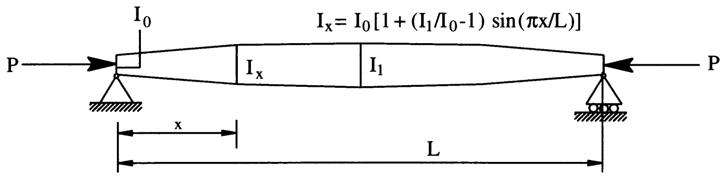
Problem 4.17. Estimate the first critical weight per unit length of vertical unsupported mast of constant cross-section fixed at the base that will cause the mast to buckle under its own weight. Use numerical integration with four divisions.



P.4.17

[Ans. $P_{cr} = 7.84EI_0/L^2$*]*

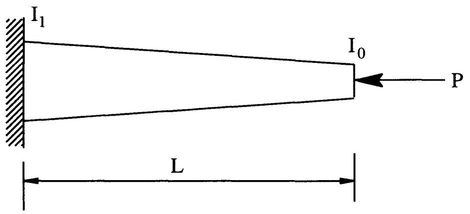
Problem 4.18. Use Rayleigh-Ritz technique to estimate the critical load for a non-uniform simply supported strut shown in Fig. P.4.18. The variation of moment of inertia may be assumed to be: $I(x) = I_0\left[1 + \left(\frac{I_1}{I_0}\right)\sin\left(\frac{\pi x}{L}\right)\right]$. As a typical case estimate P_{cr} for the strut with $\frac{I_1}{I_0} = 1$.



P.4.18

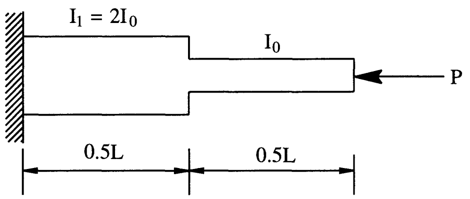
Problem 4.19. The linearly tapered cantilever strut shown in Fig. P.4.19 is subjected to an axial thrust along its centroidal axis. The moment of inertia of the strut is given by $I(x) = I_0[1 + \beta x]$, where I_0 is moment of inertia at $x = 0$ and β is a measure of magnitude of the taper of strut and is defined by: $\beta = [(I_1/I_0) - 1]/L$. For a prismatic member $\beta = 0$.

[Ans. For $\beta = 2$, $P_{cr} = 2.12EI_0/L^2$.*]*



P.4.19

Problem 4.20. The cantilever strut of stepped cross-section shown in Fig. P.4.20 is subjected to an axial thrust. Determine the critical load which will cause the cantilever to buckle. Use appropriate trial displacement function and
(a) Rayleigh-Ritz method
(b) Energy method
(c) Galerkin's Technique, and
(d) Newmark's numerical integration technique.



P.4.20

[Ans. $P_{\rm cr} = 4.1EI_0/L^2$ (approximate). Different procedures will provide critical load values to different degrees of accuracy.]

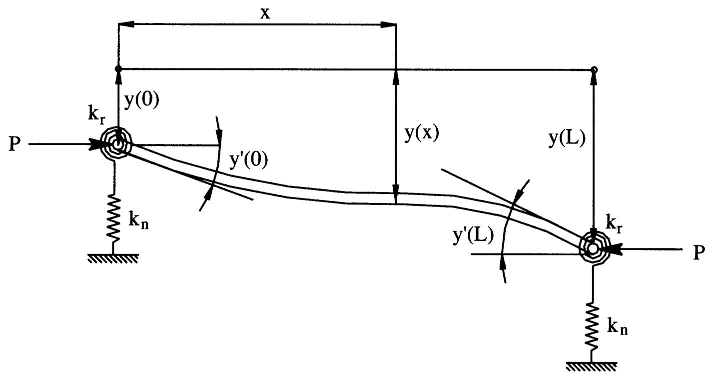
Problem 4.21. A strut is supported by a translation and a rotation springs at each of its ends as shown in Fig. P.4.21.The stiffness of translation and rotation springs is k_n and $k_{\rm r}$, respectively. The boundary conditions for the strut are:

Shear force, $Q = k_n y$ and moment, $M = k_{\rm r} y'$ at $x = 0$ and $x = L$. Obtain the equation of equilibrium for the strut from the variational principle of stationary potential energy:

$$\delta\Pi(y) = \delta\left[\frac{1}{2}\int_0^L [EI(y'')^2 - P(y')^2]\mathrm{d}x - \left(\frac{1}{2}k_n y^2 - \frac{1}{2}k_{\rm r}(y')^2\right)_0^L\right] = 0$$

[Hint:

$$\delta\Pi(y) = \int_0^L [\delta y'' EIy'' - \delta y' Py']\mathrm{d}x - [\delta y k_n y - \delta y' k_{\rm r}(y')]_0^L$$



P.4.21

Integrate the first part twice:

$$\int_0^L \delta y[(EIy'')'' + Py'']\mathrm{d}x - [\delta y\{(EIy''')' + Py'\}]_0^L$$
$$+ [\delta y' EIy'']_0^L - [\delta y k_n y - \delta y' k_r y']_0^L .$$

Since at the boundaries $x = 0$ *and* $x = L$, $Q = k_n y = -\{(EIy''')' + Py'\}$ *and* $M = k_r y' = -EIy''$. *Thus all boundary values* $[\ ]_0^L$ *vanish. Thus*

$$\delta\Pi(y) = \int_0^L \delta y[(EIy'')'' + Py'']\mathrm{d}x .$$

From the stationary condition $\delta\Pi(y) = 0$, *the equilibrium equation obtained is:* $(EIy'')'' + Py'' = 0$.*]*

Problem 4.22. Show that a cantilever strut is in a stable equilibrium at bifurcation point of $P_e = \pi^2 EI/(2L)^2$.

[Hint: Assume the deflection shape function, $y(x) = a\left[1 - \cos\frac{\pi x}{2L}\right]$.*]*

Problem 4.23. Solve Problem 4.17 by Rayleigh-Ritz method.

[Hint: As a trial function take two-degree-of-freedom function,

$$y(x) = a_1\left[1 - \cos\frac{\pi x}{L}\right] + a_2\left[1 - \cos\frac{3\pi x}{L}\right]$$

and substitute this in the energy functional

$$\Pi = \int_0^L \left[\frac{1}{2}EI(y'')^2 - \frac{1}{2}w(l-x)(y')^2\right] \mathrm{d}x$$

and integrate. Obtain equations of equilibrium $\frac{\partial \Pi}{\partial a_1} = \frac{\partial \Pi}{\partial a_2} = 0$ *and corresponding stability determinant. Solve the characteristic equation for smallest root,* $P_{cr} = 6.87EI/L^3$.*]*

5

Stability Analysis of Beam-Columns

5.1 Introduction

The primary objective of this chapter is to develop methods for predicting the deformation response of individual slender members or simple frames composed of such members subjected simultaneously to axial force and bending moment. Such structural members are termed *beam-columns*. In this chapter we are mainly concerned with lateral deformations i.e. deformations perpendicular to the longitudinal axis of the member. The analysis procedures are based upon the solution of appropriate differential equations.

It is recognised that the influence of axial force on bending deformations is one of the most important aspects of the structural analysis and design. The lateral loads and/or end moments cause deflections which are further amplified by axial compression causing moment, Py along the member. These additional deflections add significantly to the moments, which may result still further deflections. Finally, a stable situation is reached where deflections correspond to the bending moments due to both lateral load and Py. Because of this interaction between the axial force and the moments, the general superposition procedures are inappropriate. However, as the bending moment approaches zero, the member tends to become axially loaded strut, a problem that has been treated in details in Chap. 4. On the other hand, if the axial force vanishes, the problem reduces to that of a beam.

5.2 Derivation of Basic Equations

The iterative process described above actually need not be carried out to obtain a solution. The influence of axial force on the bending moment can be incorporated directly into the governing differential equation:

$$EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = -M_x = -[M_0(x) + Py(x)] \tag{5.1}$$

where $M_0(x)$ is the moment due to lateral forces, end moments, or from a known eccentricity of axial force at one or both ends and $Py(x)$ takes into account the added influence of the axial force and deflection. The moment $M_0(x)$ may vary along the length of the member. The moment-equilibrium equation (5.1) can be expressed in the standard form for the case when EI is constant.

$$\left(\frac{d^2y}{dx^2}\right) + \frac{Py}{EI} = -\frac{M_0(x)}{EI} \quad \text{or} \quad \left(\frac{d^2y}{dx^2}\right) + \alpha^2 y = -\frac{M_0(x)}{EI} \tag{5.2}$$

where $\alpha^2 = \frac{P}{EI}$. As described in Chap. 2, the *shear force equilibrium* expression of beam-column elements can be obtained by differentiating the *moment-equilibrium* relation given by (5.1) with respect to x, i. e.

$$Q(x) = \frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) + P\left(\frac{dy}{dx}\right).$$

If EI is constant

$$\left(\frac{d^3y}{dx^3}\right) + \alpha^2\left(\frac{dy}{dx}\right) = \frac{Q(x)}{EI} \tag{5.3}$$

Similarly, a second differentiation of (5.1) yields the equilibrium equation for lateral loads, i. e.

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\left(\frac{d^2y}{dx^2}\right) = w(x)$$

For the case when EI is constant.

$$\left(\frac{d^4y}{dx^4}\right) + \alpha^2\left(\frac{d^2y}{dx^2}\right) = \frac{w(x)}{EI} \tag{5.4}$$

where $w(x)$ is the intensity of load at a point on the element.

5.3 Analysis of Beam-Columns

Beam-column being the basic component of a rigid frame will be treated first, and then the analysis will be extended to the rigid frame. If EI is constant, the general solution of (5.4) has the form

$$y(x) = A\sin\alpha x + B\cos\alpha x + Cx + D + f(x) \tag{5.5}$$

where $\alpha^2 = \frac{P}{EI}$ and $f(x)$ is a particular solution of lateral load $w(x)$. The integration constants A, B, C and D are to be determined from the *prescribed boundary conditions*. The boundary conditions of a beam-column with uniform cross-section encountered in practice are:

$$
\begin{aligned}
&\text{deflection,} && y(x) = A\sin\alpha x + B\cos\alpha x + Cx + D + f(x) \\
&\text{slope,} && y' = \alpha(A\cos\alpha x - B\sin\alpha x) + C + f'(x) \\
&\text{moment,} && M = -EIy'' = EI\alpha^2(A\sin\alpha x + B\cos\alpha x) - EI\,f''(x) \\
& && \quad = P(A\sin\alpha x + B\cos\alpha x) - EI\,f''(x) \\
&\text{shear force,} && Q = -EI\,y''' = -\alpha P(A\cos\alpha x - B\sin\alpha x) + EI\,f'''(x)
\end{aligned} \tag{5.6}
$$

To obtain the general solution given by (5.5) it is required to find f(x) which depends on the lateral loading and to determine constants A, B, C, and D that satisfy the prescribed boundary conditions.

5.3.1 Beam-Column with Concentrated Loads

Consider the simply supported beam-column member of length L with constant EI subjected to a single lateral load W shown in Fig. 5.1. Because of the discontinuity at W, the problem is treated in two parts: one considering the beam to the left of W ($0 \leq x \leq L - z$); and other to the right ($L - z \leq x \leq L$). The moment equilibrium is defined by taking moment about an arbitrary point at a distance x from the left

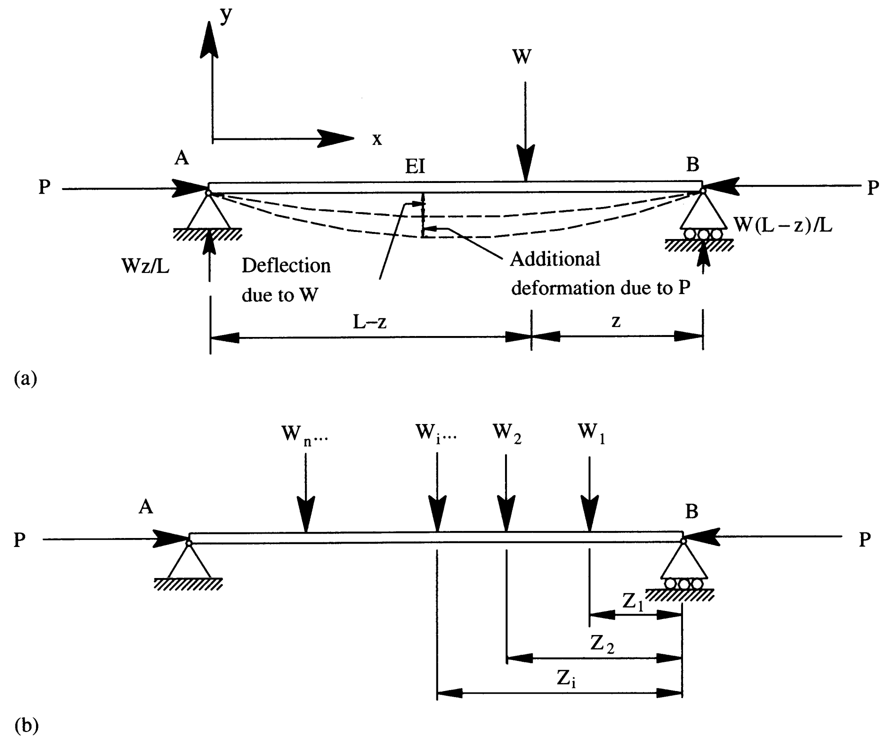


Fig. 5.1a,b. Simply supported strut with concentrated loads. **a** Equilibrium of a beam-column member, **b** beam-column with a number of concentrated loads

support of the member. Thus, for the left-hand part of the beam.

$$EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + Py(x) = -\left(\frac{Wz}{L}\right)x\,, \quad 0 \le x \le L - z$$

For the right-hand portion

$$EI\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + Py(x) = -\frac{W(L-z)(L-x)}{L}\,, \quad L - z \le x \le L$$

Substituting $\alpha^2 = \frac{P}{EI}$, the equations reduce to

$$y'' + \alpha^2 y = -\left(\frac{Wz}{EIL}\right)x$$
$$y'' + \alpha^2 y = -\left(\frac{W(L-z)}{EIL}\right)(L-x)$$

The solutions are:

$$y(x) = A\sin\alpha x + B\cos\alpha x - \left(\frac{Wz}{PL}\right)x\,, \qquad 0 \le x \le L - z$$
$$y(x) = C\sin\alpha x + D\cos\alpha x - \left[\frac{W(L-z)(L-x)}{PL}\right]\,, \quad L - z \le x \le L$$

The boundary conditions for evaluation of constants A, B, C and D are $y(0) = y(L) = 0$ giving $B = 0$ and $C = -D\cot\alpha L$ and at the point of application of load W the matching conditions are

$$y_{\text{left}}(L-z) = y_{\text{right}}(L-z) \quad \text{and} \quad y'_{\text{left}}(L-z) = y'_{\text{right}}(L-z)$$

The constants of integration are therefore

$$A = \frac{W\sin\alpha z}{P\alpha\sin\alpha L}\,; \quad B = 0$$
$$C = -\frac{W\sin\alpha(L-z)}{P\alpha\tan\alpha L} \quad \text{and} \quad D = \frac{W\sin\alpha(L-z)}{P\alpha}$$

Thus, the equations for the elastic curve are

$$y(x) = \left(\frac{W\sin\alpha z}{P\alpha\sin\alpha L}\right)\sin\alpha x - \left(\frac{Wz}{PL}\right)x\,, \quad 0 \le x \le (L-z)$$

$$y(x) = \left(\frac{W\sin\alpha(L-z)}{P\alpha\sin\alpha L}\right)\sin\alpha(L-x) - \left[\frac{W(L-z)(L-x)}{PL}\right]\,, \qquad (L-z) \le x \le L \tag{5.7}$$

From the differentiation of (5.7)

$$y' = \frac{W \sin \alpha z}{P \sin \alpha L} \cos \alpha x - \frac{Wz}{PL} \qquad 0 \le x \le (L-z)$$
$$y' = -\frac{W \sin \alpha (L-z)}{P \sin \alpha L} \cos \alpha (L-x) + \frac{W(L-z)}{PL} \qquad (L-z) \le x \le L \tag{5.8}$$

and

$$y'' = -\frac{W\alpha \sin \alpha z}{P \sin \alpha L} \sin \alpha x\,, \qquad 0 \le x \le (L-z)$$
$$y'' = -\frac{W\alpha \sin \alpha (L-z)}{P \sin \alpha L} \sin \alpha (L-x)\,, \quad (L-z) \le x \le L \tag{5.9}$$

If the load W is applied at mid-span i.e. $z = L/2$, the elastic curve is symmetrical and only one portion of the member need be considered. The mid-span i.e. $x = L/2$, deflection will be the maximum.

$$y(L/2) = y_{\max} = \frac{W}{2P\alpha}\left[\tan\left(\frac{\alpha L}{2}\right) - \left(\frac{\alpha L}{2}\right)\right]$$
$$= \frac{WL^3}{48\,EI}\left[\frac{3}{(\alpha L/2)^3}\right]\left[\tan\left(\frac{\alpha L}{2}\right) - \left(\frac{\alpha L}{2}\right)\right]$$

because $P = \alpha^2 EI$. Substituting $\psi = \dfrac{\alpha L}{2} = \dfrac{\pi}{2}\sqrt{\dfrac{P}{P_e}}$

$$y_{\max} = \frac{WL^3}{48EI}\eta(\psi) \quad \text{where} \quad \eta(\psi) = \frac{3(\tan\psi - \psi)}{\psi^3} \tag{5.10}$$

The parameter ψ depends on the ratio (P/P_e). It should be noted that the first term of the equation represents the deflection which is obtained by lateral load acting alone. The second term represents the influence of axial force P. For small values of P, the quantity ψ is also small and the factor $\eta(\psi)$ reduces to

$$\eta(\psi) = \frac{3}{\psi^3}\left[\left(\psi + \frac{\psi^3}{3} + \ldots\right) - \psi\right] \approx 1.0$$

On the other hand, if P approaches P_e, i.e., ψ tend to $\pi/2$, $\eta(\psi)$ becomes infinite and structure becomes instable. Thus when the axial compressive force approaches the limiting value P_e, even the smallest lateral load will produce considerable deflection. It should be noted that the deflection varies linearly with the lateral load W but not with the axial compression P.

The maximum slope (at the ends of the centrally loaded member) is

$$\theta_{\max} = \frac{WL^2}{16EI}\left[\frac{2(1-\cos\psi)}{\psi^2\cos\psi}\right]$$
$$= \frac{WL^2}{16EI}\varphi(\psi) \quad \text{where} \quad \varphi(\psi) = \left[\frac{2(1-\cos\psi)}{\psi^2\cos\psi}\right] \tag{5.11}$$

The maximum moment at the mid-span is given by

$$M_{\max} = EI\left(\frac{d^2y}{dx^2}\right)_{x=L/2} = \frac{WL}{4}\left(\frac{\tan\psi}{\psi}\right) \tag{5.12}$$

The proportionality of deflection to the lateral load enables to handle the case where the beam-column member is subjected to several concentrated lateral loads by using the principle of super-position. Consider the case of column-beam member subjected to n lateral loads $W_1, W_2, \ldots, W_n$ acting at distances $z_1, z_2, \ldots, z_n$, from the right support, respectively, as shown in Fig. 5.1b, where $z_1 < z_2, \ldots < z_n$. The deflection curve between the load W_m and W_{m+1} is obtained from (5.7).

$$\begin{aligned} y(x) = {} & \frac{\sin\alpha x}{P\alpha\sin\alpha L}\sum_{i=1}^{m} W_i\sin\alpha z_i - \frac{x}{PL}\sum_{i=1}^{m} W_i z_i \\ & + \frac{\sin\alpha(L-x)}{P\alpha\sin\alpha L}\sum_{i=m+1}^{n} W_i\sin\alpha(L-z_i) - \frac{(L-x)}{PL}\sum_{i=m+1}^{n} W_i(L-z_i) \end{aligned} \tag{5.13}$$

The expressions of the elastic curve of beam-column member subjected to concentrated load(s) given by (5.7) and (5.13) can conveniently be used to derive expressions for the other load cases.

5.3.2 Beam-Column with an Interior Moment

Consider the simply supported beam-column member AB of Fig. 5.2 subjected to an interior moment M_0 at distance z from the end B. The moment M_0 can be visualized as a couple of two equal and opposite loads of magnitude W $(= M_0/\delta z)$ acting at distances z and $z + \delta z$ from the end B as shown in the figure. The deflection curve for the portion to the left of the loads can be obtained from the (5.7).

$$y(x) = W[f(z+\delta z) - f(z)] = (W\delta z)\frac{f(z+\delta z) - f(z)}{\delta z}$$

where

$$f(z) = \left[\frac{\sin\alpha z}{P\alpha\sin\alpha L}\sin\alpha x - \frac{zx}{PL}\right]$$

In the limiting case when δz approaches 0, the second term represents the derivative of $f(z)$. The product $W\delta z$ remains finite and is equal to M_0. Consequently

$$y(x) = \frac{M_0}{P}\left[\left(\frac{\cos\alpha z}{\sin\alpha L}\right)\sin\alpha x - \frac{x}{L}\right], \quad 0 \le x \le (L-z) \tag{5.14a}$$

Similarly,

$$y(x) = -\frac{M_0}{P}\left[\left(\frac{\cos\alpha(L-z)}{\sin\alpha L}\right)\sin\alpha(L-x) - \frac{(L-x)}{L}\right], \quad (L-z) \le x \le L \tag{5.14b}$$

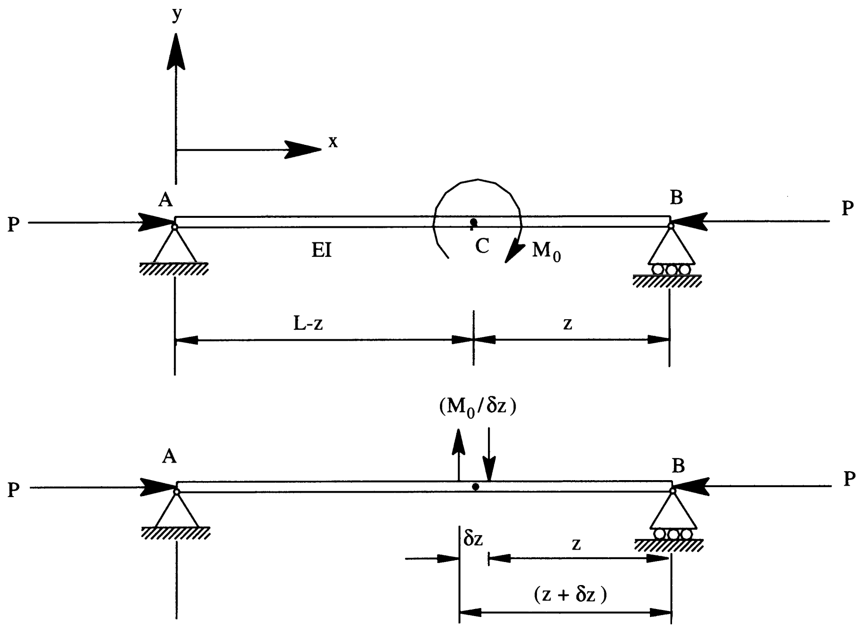


Fig. 5.2. Simply supported strut subjected to interior moment

In the typical case of a moment being acting at the mid-span of the member, the deflection curve is antisymmetric about the mid-span, and

$$y\left(\frac{L}{2}\right) = 0\,, \quad \text{and} \quad y'\left(\frac{L}{2}\right) = \left(\frac{M_0}{2P}\right)\left[\cot\left(\frac{\alpha L}{2}\right) - \frac{2}{(\alpha L)}\right] \tag{5.15}$$

5.3.3 Beam-Column Subjected to End Moments

The deflection curve for the simply supported beam-column member subjected to a moment M_B at the right end B as shown in Fig. 5.3a can be obtained from the expression of the interior moment case. To achieve this substitute $z = 0$ and $M_0 = M_B$ in (5.14a), i.e.

$$y = \frac{M_B}{P}\left[\left(\frac{\sin \alpha x}{\sin \alpha L}\right) - \frac{x}{L}\right] \tag{5.16}$$

Equation (5.16) can also be derived by using the fundamental governing differential equation:

$$EI\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\left(Py - \frac{M_B x}{L}\right) \quad \text{or} \quad y'' + \alpha^2 y = \frac{M_B x}{EIL}$$

where $\alpha^2 = P/EI$ and the boundary conditions are $y(0) = y(L) = 0$. The substitution of general solution $y = A \sin \alpha x + B \cos \alpha x + M_B x/PL$ in the boundary conditions

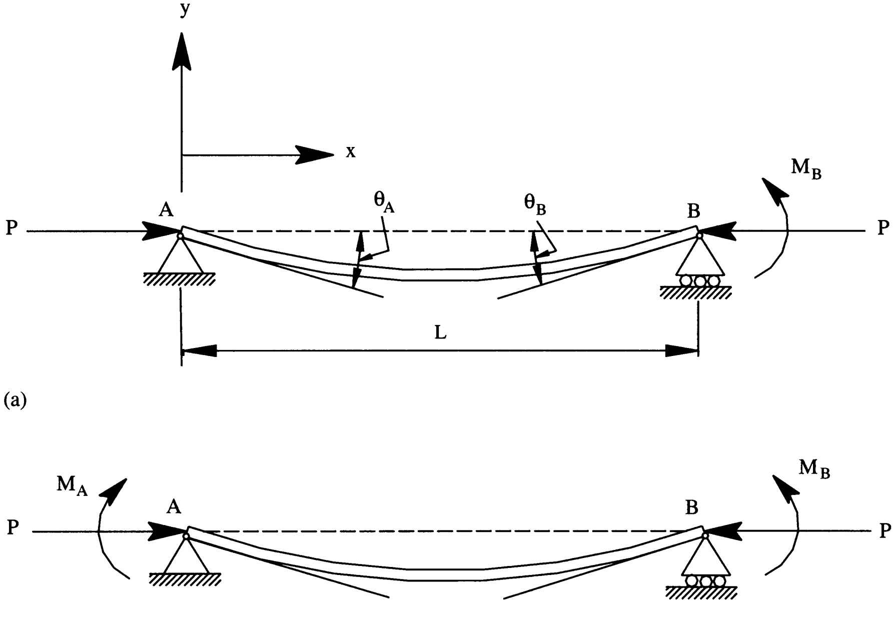


Fig. 5.3a,b. Beam-column subjected to end moments. **a** Beam-column with end-moment at one support, **b** beam-column with end-moments at both supports

yields the values of constants and hence

$$y(x) = \frac{M_B}{P}\left[\left(\frac{\sin\alpha x}{\sin\alpha L}\right) - \frac{x}{L}\right]$$

The end slopes θ_A and θ_B are given by

$$\theta_A = y'(0) = \frac{M_B}{P}\left[\frac{\alpha}{\sin\alpha L} - \frac{1}{L}\right] = \frac{M_B L}{6EI}\left[\frac{3}{\psi}\left(\frac{1}{\sin 2\psi} - \frac{1}{2\psi}\right)\right]$$
$$\theta_B = y'(L) = -\frac{M_B}{P}\left[\frac{\alpha\cos\alpha L}{\sin\alpha L} - \frac{1}{L}\right] = -\frac{M_B L}{3EI}\left[\frac{3}{2\psi}\left(\frac{1}{\tan 2\psi} - \frac{1}{2\psi}\right)\right] \quad (5.17)$$

It should be noted that the terms $\frac{M_B L}{6EI}$ and $\frac{M_B L}{3EI}$ are the angles produced by the moment M_B acting alone. In (5.17) these terms are multiplied by the trignometrical factors representing the influence of axial force P on the end rotations. Thus

$$\theta_A = \left(\frac{M_B L}{6EI}\right)\varphi_1(\psi) \quad \text{and} \quad \theta_B = \left(\frac{M_B L}{3EI}\right)\varphi_2(\psi)$$
$$\varphi_1(\psi) = \frac{3}{\psi}\left(\frac{1}{\sin 2\psi} - \frac{1}{2\psi}\right)$$
$$\varphi_2(\psi) = \frac{3}{2\psi}\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) \quad (5.18)$$

As P (i.e. ψ) approaches zero the factors $\varphi_1(\psi)$ and $\varphi_2(\psi)$ approach unity, and increase infinitely as ψ approaches $\pi/2$.

If the member is subjected two moments M_A and M_B at the ends A and B as shown in Fig. 5.3b, the elastic curve can be obtained by superposition

$$y(x) = \frac{M_A}{P}\left[\frac{\sin\alpha(L-x)}{\sin\alpha L} - \frac{(L-x)}{L}\right] + \left(\frac{M_B}{P}\right)\left[\frac{\sin\alpha x}{\sin\alpha L} - \frac{x}{L}\right] \tag{5.19}$$

The angles θ_A and θ_B are obtained by using (5.17) and (5.18)

$$\begin{aligned}\theta_A &= \left(\frac{M_A L}{3EI}\right)\varphi_1(\psi) + \left(\frac{M_B L}{6EI}\right)\varphi_2(\psi)\\ \theta_B &= \left(\frac{M_B L}{3EI}\right)\varphi_1(\psi) + \left(\frac{M_A L}{6EI}\right)\varphi_2(\psi)\end{aligned} \tag{5.20}$$

The end moments M_A and M_B in practice may appear as applied moments or as two eccentrically applied compressive axial loads P. Substituting $M_A = Pe_A$ and $M_B = Pe_B$ in (5.19), where e_A and e_B are, respectively, the eccentricities at the ends A and B. Thus

$$y(x) = e_A\left[\frac{\sin\alpha(L-x)}{\sin\alpha L} - \left(\frac{(L-x)}{L}\right)\right] + e_B\left[\frac{\sin\alpha x}{\sin\alpha L} - \left(\frac{x}{L}\right)\right] \tag{5.21}$$

To illustrate the effect of axial thrust on bending consider the case wherein the member is subjected to equal end moments to produce a single curvature type of deformation i.e. $M_A = -M_B = M_0$ as shown in Fig. 5.4a. The deflected shape is given by

$$\begin{aligned}y(x) &= \frac{M_0}{P\cos(\alpha L/2)}\left[\cos\left(\frac{\alpha L}{2} - \alpha x\right) - \cos\left(\frac{\alpha L}{2}\right)\right]\\ &= \left(\frac{M_0 L^2}{8EI}\right)\left(\frac{2}{\psi^2\cos\psi}\right)\left[\cos\left(\psi - \frac{2\psi x}{L}\right) - \cos\psi\right]\end{aligned} \tag{5.22}$$

The maximum deflection occurs at $x = L/2$ and is given by

$$y_{\max} = \left(\frac{M_0 L^2}{8EI}\right)\varphi(\psi) \tag{5.23}$$

where $\varphi(\psi)$ is the *multiplication* or *amplification factor*. The end slopes and maximum bending moment at the mid-span are:

$$\theta_A = \theta_B = y'(0) = \frac{M_0 L}{2EI}\left(\frac{\tan\psi}{\psi}\right) \tag{5.24}$$

and

$$M_{\max} = -EIy''(L/2) = M_0\sec\psi$$

or

$$\frac{M_{\max}}{M_0} = \sec\psi = \sec\left[\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right] \tag{5.25}$$

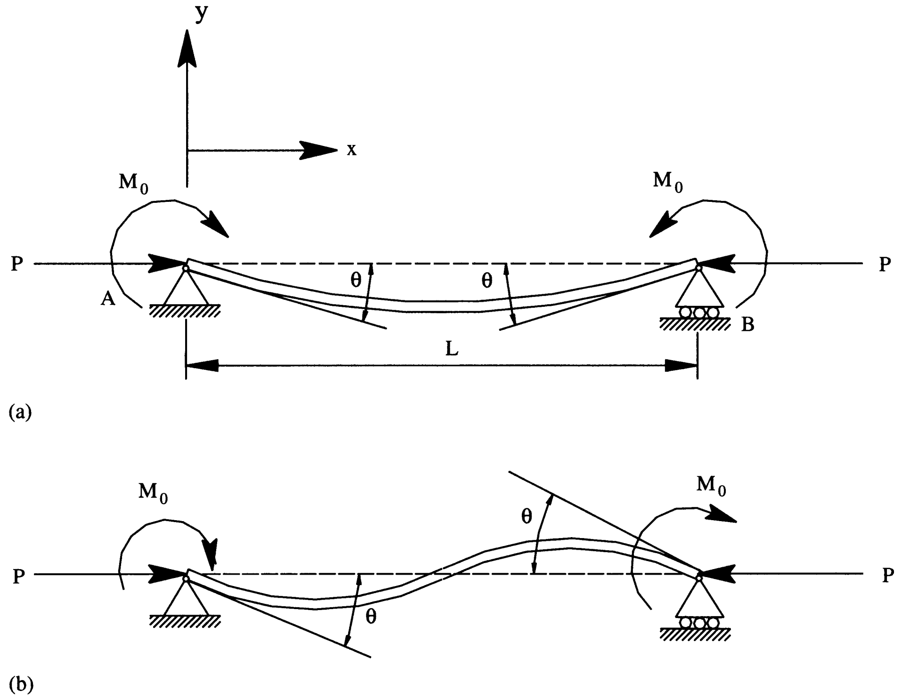


Fig. 5.4a,b. Beam-column subjected to end moments. **a** End-moments producing single curvature bending, **b** end-moments producing antisymmetric bending

where $P_e = \pi^2 EI/L^2$. For the case $P/P_e = 0.00$, $M_{max} = M_0$ i.e. the maximum moment is same as in the case of pure bending. The moment increases over that at zero thrust by 50 per cent for $P/P_e = 0.287$ and the increase is 200 per cent at $P/P_e = 0.614$. Finally at $P/P_e = 1.0$, the increase becomes infinite.

In case the end moments have same magnitude but opposite sense as shown in Fig. 5.4b, i.e. $M_A = M_B = M_0$, (5.19) yields

$$y(x) = \frac{M_0}{P}\left[\frac{\sin\alpha(L-x) - \sin\alpha x}{\sin\alpha L} + \frac{2x}{L} - 1\right] \tag{5.26}$$

When the axial force P is very small, $\alpha \approx 0$ and the sine term reduces to

$$\sin\alpha x = \alpha x - \frac{(\alpha x)^3}{6}$$

Thus (5.26) yields

$$y(x) = \frac{M_0}{6EIL}[x(x-L)(2x-L)]$$

which is identical to the solution of the beam subjected to end moment M_0. On the other hand when the axial load P approaches the critical value $P_e = \pi^2 EI/L^2$ or

$\alpha L = \pi$, the elastic curve becomes

$$y(x) = \left(\frac{M_0}{P}\right)\left[2\left(\frac{x}{L}\right) - 1 + \cos\left(\frac{\pi x}{L}\right)\right] \tag{5.27}$$

It should be noted that the elastic curve is always in two half waves no matter how small the end moment M_0 is.

5.3.4 Beam-Columns Subjected to Distributed Loads

Consider a beam-column member with axial force P subjected to a uniform load of intensity w over a portion or entire span as shown in Fig. 5.5.

(a) Uniformly distributed lateral load over a portion of the span

The load is applied over the portion extending from the point $x = a$ to $x = b$ from the right end as shown in Fig. 5.5a. The *uniformly distributed load* can be considered as a system of infinitely small concentrated forces and the *method of superposition* used in case of concentrated loads can be extended to the case of distributed load

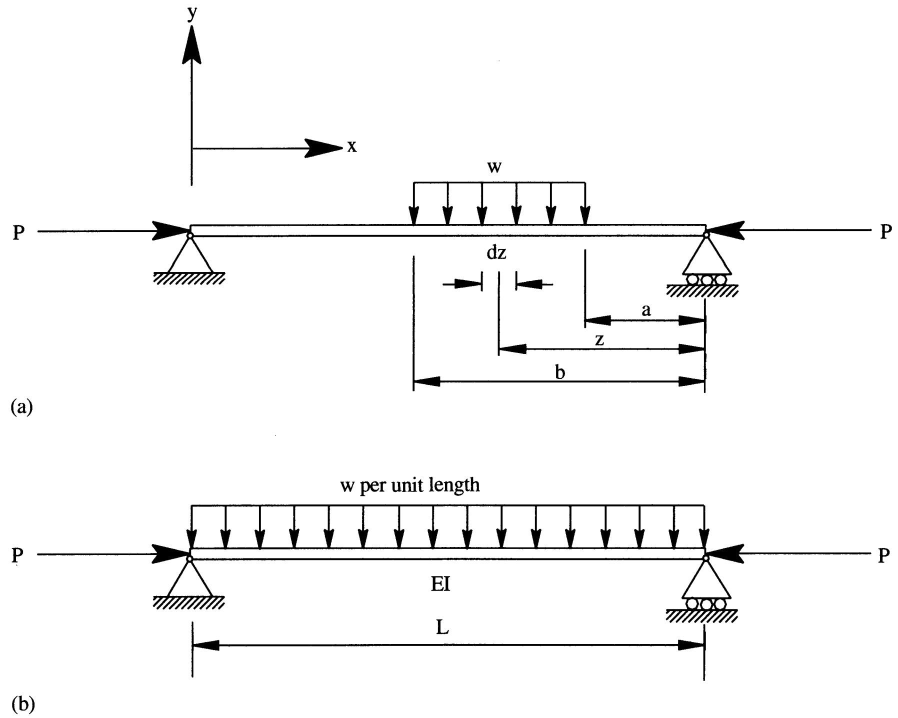


Fig. 5.5a,b. Beam-column carrying uniformly distributed load. **a** Uniformly distributed load over a portion, **b** uniformly distributed load over entire span

by replacing summation ($\sum$) by integration ($\int \mathrm{d}x$). Consider an infinitesimally small element of length $\mathrm{d}z$ of continuous load at distance z from the right hand support. The deflection produced by the elemental load $w\,\mathrm{d}z$ is obtained by treating it a concentrated load acting at distance z from the right-hand support. The deflection due to total load is then determined by integrating between the limits $z = a$ to $z = b$. Thus the deflection for the portion of the beam-column to the left of the load is given by (5.7).

$$
\begin{aligned}
y(x) &= \int_a^b \frac{(w\,\mathrm{d}z)\sin\alpha z}{P\alpha\sin\alpha L}\sin\alpha x - x\int_a^b \frac{wz\,\mathrm{d}z}{PL}\,, \quad 0 \le x \le (L-b) \\
&= \frac{w\sin\alpha x}{P\alpha^2\sin\alpha L}(\cos\alpha b - \cos\alpha a) - \frac{wx}{2PL}(b^2 - a^2)
\end{aligned}
\tag{5.28}
$$

For the portion to the right of load

$$
\begin{aligned}
y(x) &= \int_a^b \frac{(w\,\mathrm{d}z)\sin\alpha(L-x)}{P\alpha\sin\alpha L}\sin\alpha(L-z) - \int_a^b \frac{(w\,\mathrm{d}z)(L-z)(L-x)}{PL} \\
&= \frac{w\sin\alpha(L-x)}{P\alpha^2\sin\alpha L}[\cos\alpha(L-a) - \cos\alpha(L-b)] \\
&\quad - \frac{w(L-x)}{2PL}[2L(b-a) - (b^2 - a^2)]\,, \quad (L-a) \le x \le L
\end{aligned}
\tag{5.29}
$$

For the deflection at any point over the loaded portion

$$
\begin{aligned}
y(x) &= \int_a^{L-x} \frac{(w\,\mathrm{d}z)\sin\alpha z}{P\alpha\sin\alpha L}\sin\alpha x - \int_a^{L-x} \frac{x(w\,\mathrm{d}z)z}{PL} \\
&\quad + \int_{L-x}^b \frac{(w\,\mathrm{d}z)\sin\alpha(L-z)}{P\alpha\sin\alpha L}\sin\alpha(L-x) - \int_{L-x}^b \frac{(w\,\mathrm{d}z)(L-z)(L-x)}{PL}
\end{aligned}
\tag{5.30}
$$

This equation can be used to obtain deflections for the case when beam-column member carries uniformly distributed load over its entire length.

(b) Uniformly distributed load over the entire span

Consider the uniformly loaded simply supported beam-column member of length L with constant EI as shown in Fig. 5.5b. The governing differential equation is

$$
y'''' + \alpha^2 y'' = \frac{w}{EI}
\tag{5.31}
$$

The boundary conditions are: $y(0) = y''(0) = y(L) = y''(L) = 0$.

The particular solution is given by

$$f(x) = \frac{wx^2}{2P} = \frac{wx^2}{2\alpha^2 EI} \tag{5.32}$$

From the boundary conditions the constants of (5.5) are determined to be

$$A = \left(\frac{w}{\alpha^4 EI}\right)\left(\frac{1-\cos\alpha L}{\sin\alpha L}\right) = \left(\frac{w}{\alpha^4 EI}\right)\left[\tan\left(\frac{\alpha L}{2}\right)\right] = \left(\frac{w}{\alpha^2 P}\right)\left[\tan\left(\frac{\alpha L}{2}\right)\right]$$

$$B = -D = \frac{w}{\alpha^4 EI} = \frac{w}{\alpha^2 P}$$

$$C = -\frac{wL}{2\alpha^2 EI} = -\frac{wL}{2P}$$

The elastic deflection curve is therefore

$$\begin{aligned} y(x) &= \left(\frac{w}{\alpha^2 P}\right)\left[\tan\left(\frac{\alpha L}{2}\right)\sin\alpha x + \cos\alpha x - \frac{Lx\alpha^2}{2} - 1 + \frac{\alpha^2 x^2}{2}\right] \\ &= \left(\frac{w}{\alpha^2 P}\right)\left[\frac{\cos(\alpha L/2 - \alpha x)}{\cos(\alpha L/2)} - 1 - \frac{\alpha^2 x}{2}(L-x)\right] \end{aligned} \tag{5.33}$$

It is evident from the above expression that deflection varies linearly with the lateral load w but not with axial compression P. The maximum deflection at the mid-span of the member (i.e. $x = L/2$) is given by

$$\begin{aligned} y_{\max} &= \frac{w}{\alpha^2 P}\left[\sec\left(\frac{\alpha L}{2}\right) - 1 - \frac{(\alpha L)^2}{8}\right] \\ &= \frac{w}{2\alpha^4 EI}\left[2\sec\left(\frac{\alpha L}{2}\right) - 2 - \left(\frac{\alpha L}{2}\right)^2\right] \\ &= \frac{wL^4}{2(\alpha L)^4 EI}[2\sec\psi - 2 - \psi^2] \\ &= \frac{5wL^4}{384EI}\left[\frac{12(2\sec\psi - 2 - \psi^2)}{5\psi^4}\right] = \left(\frac{5wL^4}{384EI}\right)\eta(\psi) \end{aligned} \tag{5.34}$$

where

$$\psi = \frac{\alpha L}{2} = \frac{\pi}{2}\sqrt{\frac{P}{P_e}}\,.$$

Thus ψ depends upon the ratio $\frac{P}{P_e}$. The factor $\eta(\psi)$ which is function of P, EI and L is termed *amplification factor.* Thus the mid-span deflection is obtained by multiplying the pure bending deflection by the factor $\eta(\psi)$. When P approaches $P_e = (\pi^2 EI/L^2)$ or (αL) approaches π even smallest lateral load will produce considerable lateral deflection. For the mid-point deflection of a simply supported beam-column member under symmetrical loading condition, the amplification factor $\eta(\psi)$ can be approximated by the expression

$$\eta(\psi) \approx \frac{1}{(1 - P/P_e)}$$

provided that the ratio P/P_e is not large. For values of P/P_e less than 0.6, for example, the error in this approximate expression is less than 2 per cent. This approximate expression for estimating deflection at the centre of beam-column is frequently used in design computations.

5.3.5 Rotationally Restrained Beam-Columns

Consider the case of a beam-column in which lateral translation is prevented at both the ends and the end rotations are restrained by rotational springs as shown in Fig. 5.6. The problem is equivalent to the beam-column of Fig. 5.3b where $M_A = -k_{rA}\theta_A$ and $M_B = -k_{rB}\theta_B$. Substituting values of θ_A and θ_B in (5.20).

$$M_A\left[\frac{1}{k_{rA}} + \left(\frac{L}{3EI}\right)\varphi_1(\psi)\right] + M_B\left[\left(\frac{L}{6EI}\right)\varphi_2(\psi)\right] = 0 \quad \text{and}$$
$$M_A\left[\left(\frac{L}{6EI}\right)\varphi_2(\psi)\right] + M_B\left[\frac{1}{k_{rB}} + \left(\frac{L}{3EI}\right)\varphi_1(\psi)\right] = 0 \tag{5.35}$$

For non-trivial solution of these two homogeneous linear equations, the determinant of coefficients of M_A and M_B must vanish. The resulting stability condition or characteristic equation is

$$\left[\frac{1}{k_{rA}} + \left(\frac{L}{3EI}\right)\varphi_1(\psi)\right]\left[\frac{1}{k_{rB}} + \left(\frac{L}{3EI}\right)\varphi_1(\psi)\right] - \left[\left(\frac{L}{6EI}\right)\varphi_2(\psi)\right]^2 = 0 \tag{5.36}$$

As a typical case take $k_{rA} = k_{rB} = k_r$, (5.36) reduces to

$$\frac{1}{k_r} + \left(\frac{L}{3EI}\right)\varphi_1(\psi) \pm \left(\frac{L}{6EI}\right)\varphi_2(\psi) = 0 \tag{5.37}$$

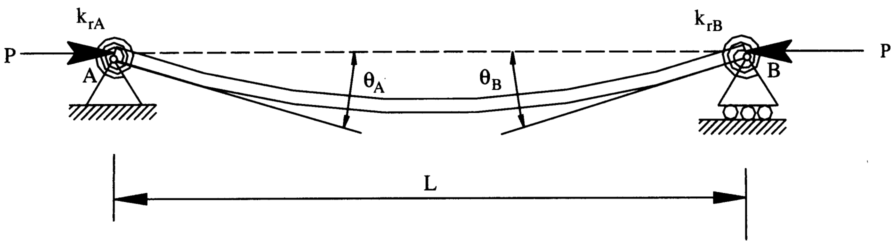


Fig. 5.6. Beam-column with elastically restrained leads

From first of (5.35):

$$M_B = -M_A \left[\frac{1}{k_r} + \left(\frac{L}{3EI}\right)\varphi_1(\psi)\right]\left[\left(\frac{L}{6EI}\right)\varphi_2(\psi)\right]^{-1}$$

The plus sign in (5.37) corresponds to the symmetric case ($M_A = -M_B$) and minus sign to asymmetric case ($M_A = M_B$). Substituting the values of $\varphi_1(\psi)$ and $\varphi_2(\psi)$ the symmetric case becomes

$$\tan\psi = -\left[\frac{2EI}{k_r L}\right]\psi \tag{5.38}$$

The value of ψ_{cr} lies between $\pi/2$ and π depending upon the value of k_r. When k_r approaches zero, $\psi_{cr} \Rightarrow \frac{\pi}{2}$ and $P_{cr} = \pi^2 EI/L^2$. For the antisymmetric case

$$\tan\psi = \frac{\psi}{1 + \left(\frac{2EI}{k_r L}\right)\psi^2} \tag{5.39}$$

The value of ψ_{cr} lies between π and 4.493. When $k_r \Rightarrow 0$, $\psi_{cr} \Rightarrow \pi$, and when $k_r \Rightarrow \infty$, $\psi_{cr} \Rightarrow 4.493$.

5.4 Beam-Column with Elastic Supports

5.4.1 Differential Equation Method

As has been discussed in earlier chapters that a structural member connected to an external spring system develops certain forces as it deflects and thus the free movement of member is restricted to greater or lesser extent. The forces developed in the springs depend on the amount of deformation produced in them due to deflection of the member. This results in a type of variable loading. The invariable (constant) lateral load on a strut causes initial eccentricity which introduces the stress problem, but it does not alter the P_e value of the system. Whereas in case of a spring, it results in a deflection dependent lateral load and a restoring force (as does the member) as deflection takes place. Thus the presence of spring affects the elastic stability value P_e and even small *spring stiffness* may cause a considerable increase in P_e.

There are two general types of translational elastic supports:

Translational springs

(i) Point elastic support, and
(ii) Distributed elastic support.

The former is represented by an individual spring, and it applies a force of value Q to the member as a point load; Q is dependent on the deflection y of the beam at that point. For a linear spring of stiffness k, the force Q is proportional to deflection y, i.e. $Q = -ky$. The negative sign is introduced as the direction of Q is opposite to that of the deflection.

On the other hand the *distributed elastic support* (medium) can be considered as a series of point supports very close together. If a force δQ acts over an element of length δx (direction x being normal to the direction of the springs), the intensity of distributed force in the elastic medium is

$$\frac{\delta Q}{\delta x} = \frac{\mathrm{d}Q}{\mathrm{d}x} = q = f(y) \tag{5.40}$$

For a *linear spring* action $q = -ky$, where k is the force per unit deflection per unit length of the elastic medium. k is frequently known as the *elastic modulus*. The stiffness of the spring affects the buckling load of the system considerably and it may cause a complete change in the *buckling mode*.

Consider the case of Euler strut with an added central spring of stiffness k as shown in Fig. 5.7a. The problem can be easily dealt with *potential energy approach.* The total potential energy of the system is given by

$$\Pi = \left[\int \frac{1}{2} EI(y'')^2\,\mathrm{d}x + \frac{1}{2}ky_c^2 - \int \frac{1}{2}P(y')^2\,\mathrm{d}x\right]$$

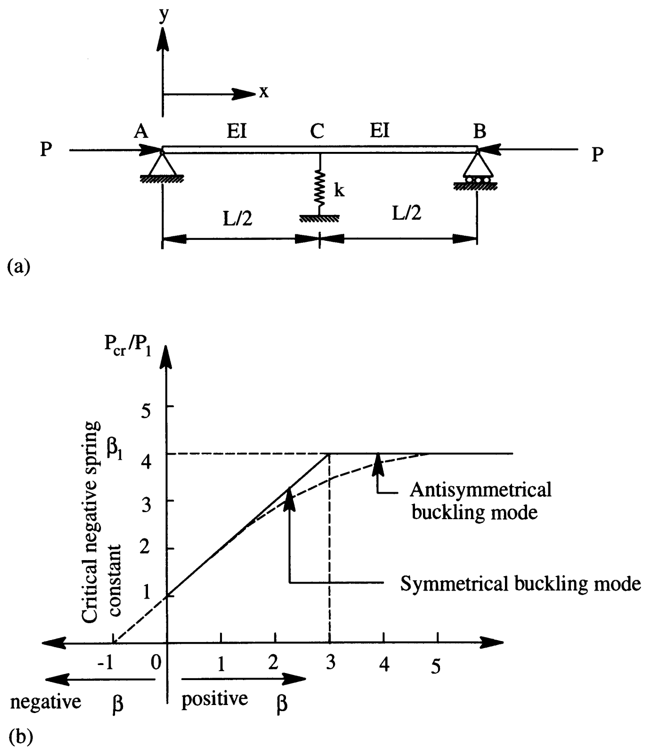


Fig. 5.7a,b. Buckling of simply supported strut with an elastic support at mid-point. **a** Simply supported strut with an elastic spring, **b** variation of P_{cr}/P_1 with β

where y_c is the deflection at the point of elastic support. Consider deflection *trial function* $y(x) = a\sin(\pi x/L)$, thus $y_c = a$ and

$$\frac{EI}{2}\int_0^L (y'')^2\,\mathrm{d}x = \frac{EI}{2}\int_0^L \left[\frac{a^2\pi^4}{L^4}\sin^2\left(\frac{\pi x}{L}\right)\right]\mathrm{d}x = \frac{EI\pi^4 a^2}{4L^3}$$

$$\frac{1}{2}ky_c^2 = \frac{1}{2}ka^2$$

$$\frac{P}{2}\int_0^L (y')^2\,\mathrm{d}x = \frac{P}{2}\int_0^L \left[\frac{a^2\pi^2}{L^2}\cos^2\left(\frac{\pi x}{L}\right)\right]\mathrm{d}x = \frac{P\pi^2 a^2}{4L}$$

Therefore,

$$\Pi = \left\{\frac{EI\pi^4}{4L^3} + \frac{1}{2}k - \left(\frac{P\pi^2}{4L}\right)\right\}a^2$$

For neutral equilibrium, $\delta\Pi = (\mathrm{d}\Pi/\mathrm{d}a)\delta a = 0$. Therefore,

$$\frac{EI\pi^4}{4L^3} + \frac{k}{2} - \frac{P\pi^2}{4L} = 0$$

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2}\left[1 + \left(\frac{L^2}{\pi^2 EI}\right)\left(\frac{2kL}{\pi^2}\right)\right]$$

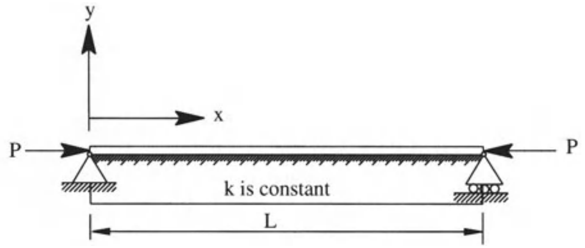
$$\text{or}\quad \frac{P_{\mathrm{cr}}}{P_{\mathrm{e},1}} = \left[1 + \left(\frac{2kL}{\pi^2 P_{\mathrm{e}}}\right)\right] = 1 + \beta \tag{5.41}$$

where $\beta = 2kL/\pi^2 P_{\mathrm{e}}$ is dimensionless quantity relating k to $P_{\mathrm{e},1}$ and L. When $\beta = 0$ i.e. spring is inoperative, $P_{\mathrm{cr}} = P_{\mathrm{e},1}$. With increasing positive values of β, P_{cr} increases linearly, and there comes a point at which the symmetrical buckling load equals the second critical load of the Euler strut problem, and the system could equally well buckle anti-symmetrically with the spring becoming inoperative. This would take place at $\beta = 3$ or $P_{\mathrm{cr}} = 4P_{\mathrm{e},1}$ as shown in Fig 5.7b. Beyond this point no increase will be obtained and the spring becomes unimportant.

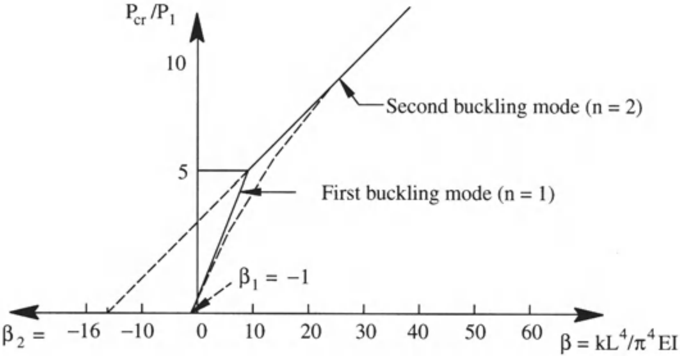
On the other hand with increasing negative values of β (a disturbing force) there comes a point at $\beta = -1$ where $P_{\mathrm{cr}} = 0$. The stiffness of the system reduces to zero. This value of β is called critical value β that alone will cause instability in the system. Thus the presence of springs may make the buckling mode higher than expected. In case of doubt about the true mode, both symmetrical and antisymmetrical modes will have to be studied.

This procedure is equally applicable to the strut with a larger number of elastic supports. Consider the problem of hinged-hinged strut of constant cross-section supported on an elastic medium of constant modulus k as shown in the Fig. 5.8a. The governing differential equation can be derived by following the standard procedure for the moment equilibrium of an element of the member

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + P\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) + ky = 0 \quad (\text{as } w = -ky)$$



(a)



(b)

Fig. 5.8a,b. Buckling of simply supported strut resting on an elastic medium. **a** Simply supported strut on an elastic medium of constant stiffness, **b** influence k on P_{cr} and buckling mode

or

$$EI\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)+P\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+ky=0 \quad \left(\text{as } M=EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

The four boundary conditions are: $y(0)=y(L)=y''(0)=y''(L)=0$. The general solution to this equation is: $y(x)=a_n\sin\frac{n\pi x}{L}$ where $y(x)$ represents the buckling mode and the value of n can be any integer as in the case of Euler strut problem. The total potential energy of the system is given by

$$\begin{aligned}\Pi &= \frac{1}{2}EI\int_0^L (y'')^2\,\mathrm{d}x+\frac{1}{2}k\int_0^L y^2\mathrm{d}x-\frac{P}{2}\int (y')^2\,\mathrm{d}x\\ &= \left[\frac{n^4\pi^4EI}{4L^3}+\frac{kL}{4}-\frac{Pn^2\pi^2}{4L}\right](a_n)^2\end{aligned}$$

For neutral equilibrium, $\delta\Pi=(\mathrm{d}\Pi/\mathrm{d}a_n)\delta a_n=0$. Since δa_n is arbitrary, for no-trivial solution $(\mathrm{d}\Pi/\mathrm{d}a_n)=0$. Therefore

$$P_{\mathrm{cr}}=\frac{\pi^2EI}{L^2}\left[n^2+\left(\frac{kL^4}{n^2\pi^4EI}\right)\right] \quad \text{or} \quad \frac{P_{\mathrm{cr}}}{P_{\mathrm{e},1}}=\left[n^2+\frac{\beta}{n^2}\right] \tag{5.42}$$

where $P_{e,1} = \pi^2 EI/L^2$, the Euler buckling load, and $\beta = kL^4/\pi^4 EI$, a dimensionless quantity relating k to the stiffness of the member and L. The different buckling modes are given by $n = 1, 2, 3$, etc. The relation is shown in the Fig. 5.8b. As β is increased from zero, the mode changes from single to two half-waves buckling at $\beta = 4$, and from two to three half-waves buckling at $\beta = 36$, etc. There is a series of negative values of β each of which produces a state of neutral equilibrium for $P = 0$. The first of these corresponding to single half-wave buckling mode occurs at $\beta_1 = -1$, the second at $\beta_2 = -16$ and so on.

5.4.2 Numerical Procedure

The complexities in the loading system, in the beam cross-section, and in the elastic supporting system can be conveniently handled by the numerical procedure discussed in Sect. 4.4. However, it should be appreciated that while dealing with struts or beam-columns with spring supports, the problem is somewhat different in that the member is subjected to two opposing forces, a *disturbing force* (due to axial and transverse loading) and a *restoring force* (due to the spring system). The beam is now not the only system resisting deflection, the springs also resist movement and in some cases they may even offer a stronger resistance than the beam.

In the problems where the resistance of the springs to deflection is small relative to that of beam, the convergence of the iteration procedure is very fast. On the other hand when the resistance of the springs to deflection is greater than that of the beam, as in the case of a flexible beam on a stiff foundation, the iteration procedure may diverge and the methods can become too unwieldy. In this type of problems it is often easier to use classical or a solution based on energy consideration.

The numerical solution for the stability problems with the added effect of the elastic supports as usual starts with a *trial deflection* y_a for the buckling mode. The values of the *derived deflection* y_d are made up of two components: (1) effect of spring loading k, and (2) the effect of axial loading P. If $y_{d,k}$ and $y_{d,p}$ are the derived deflection values due to k and P, respectively, the total derived deflection corresponding to the initial trial y_a is

$$y_d = y_{d,k} + y_{d,p} \tag{5.43}$$

where for compressive axial load $y_{d,p}$ is positive and $y_{d,k}$ is negative. It will be noticed that in the iteration cycle, starting from a factor 'a' in y_a, $y_{d,k}$ also has a single factor of 'a' and $y_{d,p}$ has a total factor of ($a\,Ph^2/EI$). Thus to use (5.43), a guessed value of P must be introduced in $y_{d,p}$ so that both $y_{d,p}$ and $y_{d,k}$ can be reduced to the same factor 'a'. The simplest way to do this is to satisfy the equation $y_a = y_{d,p} + y_{d,k}$ at the point of maximum displacement and to scale all the y_d values in the proportion. This procedure has been demonstrated in example 5.1.

Example 5.1. A simply supported beam of constant cross-section carries both lateral load and an axial compression of value $P = 0.4\pi^2 EI/L^2$ as shown in Fig. 5.9. Estimate the deflection at salient points.

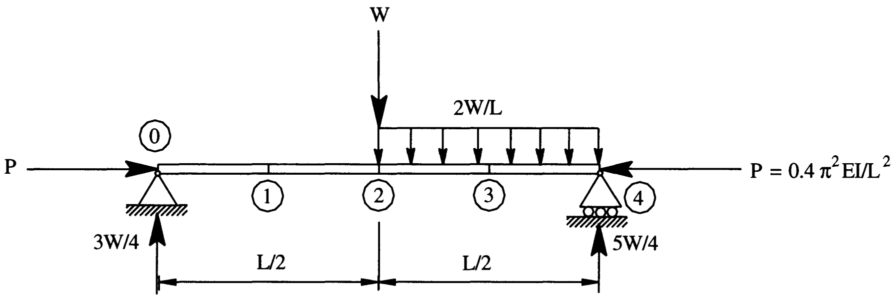


Fig. 5.9. Simply supported strut with lateral loads

This problem may be treated as a case of beam-column member subjected to a compression load at an *eccentricity* y_i. This *initial eccentricity* is due to lateral loading on the member and is represented by the deflection of the member due to lateral loading before axial forces are added. The compressive load P will tend to increase initial eccentricity, the total value of deflection approaching infinity as P approaches $P_{e,1}$, the first critical load in pure compression buckling. Thus P can not be greater than $P_{e,1}$.

In the numerical solution the first step is to calculate absolute values of eccentricity y_i at the node points. The next step is to assume trial values of additional deflection y_a due to the compressive load P. Using the total deflection $y = y_i + y_a$ compute moments through the member due to P, and hence determine derived deflection values y_d. As has been pointed out earlier, the closer the trial values y_a to the exact values of y_e, the lesser will be the computational effort. In most of the cases it is good to assume the trial values based on the relation

$$y_{a,1} = y_i \left[\frac{P_{e,1}}{P} - 1 \right]^{-1} \tag{5.44}$$

The numerical procedure is given in Table 5.1. In the first part, the deflections due to lateral loading have been computed in the absence of axial load P. This gives value of initial eccentricity y_i of P. In the second part the additional deflection y_a due to application of P are computed.

As $P_{e,1}$ for the simply supported beam-column member is Euler value $\pi^2 EI/L^2$, a good approximation is given by

$$y_{a,1} = y_i \left[\frac{P_{e,1}}{P} - 1 \right]^{-1} = y_i \left[\frac{1}{0.4} - 1 \right]^{-1} = 0.667 y_i$$

Therefore, $y = y_i + y_{a,1} = 1.667 y_i$

Table 5.1. Computation of deflection of beam-column subjected to lateral loads

Node No.	0		1		2		3		4	Multiplier
I. Computation of initial eccentricity (deflection), y_i due to lateral load										
Load, W			–		-1.25		-0.50		0	W
Shear, Q		0.75		0.75		-0.50		-1.00		W
$y'' = M/EI$	0		0.75		1.50		2.00		0	Wh/EI
Y''			9.00		$(7.50 + 8.25)$ 15.75		11.5			$Wh^2/12EI$
y'		(-17.5)[a]		-8.5		7.25		18.75		$Wh^3/12EI$
y_i	0		-17.50		-26.00		-18.75		0	$Wh^3/12EI = a$
II. Additional deflection due to axial load P										
First iteration										
$y = 1.667y_i$	0		-29.17		-43.33		-31.25		0	a
Y''			33.45		49.37		35.58			$10aPh/(12EI)$
y'		(-58.66)[a]		-25.02		24.29		59.63		$10aPh^2/(12EI)$
$\bar{y}_d$	0		-58.66		-83.88		-59.73		0	–do–
y_d	0		-12.06		-17.25		-12.28		a	
Second Iteration										
$y = y_i + y_d$	0		-29.56		-43.25		-31.03		0	a
Y''			33.88		49.31		35.35			$10aPh/(12EI)$
y'		(-58.90)[a]		-25.02		24.29		59.63		$10aPh^2/(12EI)$
$\bar{y}_d$	0		-58.90		-83.92		-59.63		0	–do–
y_d	0		-12.11		-17.26		-12.26			a
Third Iteration										
$y = y_i + y_d$	0		-29.61		-43.26		-31.01		0	a
Y''			33.94		49.32		35.34			$10aPh/(12EI)$
y'		(-58.95)[a]		-25.01		24.31		59.65		$10aPh^2/(12EI)$
$\bar{y}_d$	0		-58.95		-83.96		-59.65		0	–do–
y_d	0		-12.12		-17.26		-12.26			a

[a]Note: $y_{01} = (9 \times 3 + 15.75 \times 2 + 11.5)/4 = 17.5$, where 4 is the number of panels.

The elastic curve ordinate at the mid span

$$y = \frac{Wh^3}{12EI}\left(26 + \frac{839.6Ph^2}{12EI}\right)$$
$$= \frac{WL^3}{768EI}\left(26 + \frac{43.159P}{P_e}\right) = \frac{WL^3}{768EI}(26 + 17.26) = \frac{4WL^3}{71EI}$$

Example 5.2. A simply supported symmetrical strut of continuously variable cross-section $I_x = I_0\{1 - 4.0[(x/L)^2 - (x/L)^3]\}$ shown in Fig. 5.10 carries an axial compression of P. Estimate the first critical value of P that will cause its buckling.

If the strut has an initial eccentricity (at $P = 0$) measured as the off-set distance between the axis and the line joining end points i.e. the thrust line, estimate the de-

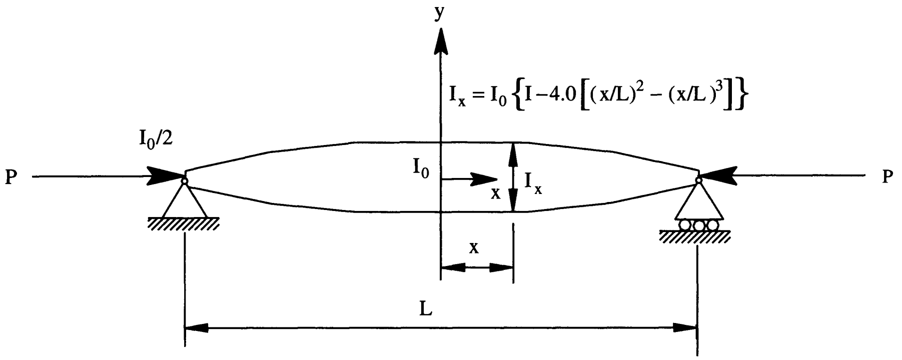


Fig. 5.10. Beam-column of variable cross section

flection due to an axial thrust of magnitude $P = 0.5\pi^2 EI/L^2$. The initial eccentricity is given by $y_i = a \sin \pi x/L$. The numerical integration for the buckling load is given in Table 5.2.

Table 5.2. Buckling load of a strut of a continously variable cross-section

Node No.	0		1		2		3	Multiplier and remarks
I_x	0.5		0.7		0.9		1.0	I_0
First Iteration								Half strut is considered
y_a	0.0		5.0		8.7		10.0	$a\ \ y(x) = a\sin(\pi x/L)$
$y'' = M/EI$	0.0		−7.1		−9.67		−10.0	Pa/EI_0
Y''			−80.67		−113.77		−119.34	$Pah/(12EI_0)$
y'		254.11		173.44		(59.67)	0.0	$Pah^2/(12EI_0)$
$\overline{y}_d$	0.0		254.11		427.55		487.22	–do–
y_d	0.0		5.22		8.77		10.0	$(48.72)[Pah^2/(12EI_0)]$
Second Iteration								
y_a	0.0		5.22		8.77		10.0	a
y''	0.0		−7.45		−9.74		−10.0	aP/EI_0
Y''			−84.24		−114.85		−119.48	$aPh/(12EI_0)$
y'		258.83		174.59		(59.74)	0.0	$Pah^2/(12EI_0)$
$\overline{y}_d$	0.0		258.83		433.42		493.16	–do–
y_d	0.0		5.25		8.79		10.0	$(49.32)[Pah^2/(12EI_0)]$
Third Iteration								
y_a	0.0		5.25		8.79		10.0	a
y''	0.0		−7.50		−9.77		−10.0	aP/EI_0
Y''			−84.77		−115.25		−119.54	$aPh/(12EI_0)$
y'		259.79		175.02		(59.77)	0.0	$aPh^2/(12EI_0)$
$\overline{y}_d$	0.0		259.79		434.81		494.58	–do–
y_d	0.0		5.25		8.79		10.0	$(49.46)[aPh^2/(12EI_0)]$

At buckling:

$$10.00a = 494.58\frac{aPh^2}{12EI_0}$$

Therefore,

$$P_{\mathrm{cr},1} = \frac{8.73EI_0}{L^2}$$

The estimation of deflection due to thrust of value $P = 0.5\pi^2 EI/L^2$ i.e. $P = 4.935EI/L^2$ is given in Table 5.3.

Table 5.3. Deflections of the eccentric strut of variable cross-section

Node No.	0		1		2		3	Multiplier and remarks
Initial eccentricity								Half strut is considered
y_i	0.00		0.50		0.87		1.0	a
I_x	0.50		0.70		0.90		1.0	I_0
First iteration								$Y_{a,1} = [(P_{\mathrm{cr},1}/P) - 1]^{-1} y_i = 1.30 y_i$
y_a	0.00		0.65		1.13		1.30	a
$y = y_i + y_a$	0.00		1.15		2.00		2.30	a
$y'' = M/EI$	0.00		−1.64		−2.22		−2.30	aP/EI_0
Y''			−18.62		−26.14		−27.44	$aPh/(12EI_0)$
y'		58.48		39.86		(13.72)	0.0	$Pah^2/(12EI_0)$
$\bar{y}_d$	0.00		58.48		98.34		112.06	$Pah^2/(12EI_0) = 0.01142a$
y_d	0.00		0.67		1.12		1.28	a
Second iteration								
y_a	0.00		0.67		1.12		1.28	a
$y = y_i + y_a$	0.00		1.17		1.99		2.28	a
y''	0.00		−1.67		−2.21		−2.28	aP/EI_0
Y''			−18.91		−26.05		−27.22	$aPh/(12EI_0)$
y'		58.57		39.66		(13.61)	0.0	$Pah^2/(12EI_0)$
$\bar{y}_d$	0.00		58.57		98.23		111.84	$Pah^2/(12EI_0) = 0.01142a$
y_d	0.00		0.67		1.12		1.28	a

The maximum deflection at the mid point is given by:

$$y = y_i + y_d = 2.28a$$

Example 5.3. A hinged-hinged strut of constant cross-section shown in Fig. 5.11 is supported by an elastic medium of constant stiffness $k = 50EI/L^4$. Estimate the first critical value of the axial thrust P.

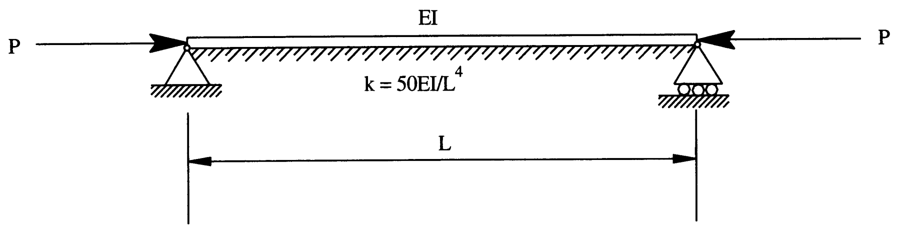


Fig. 5.11. Simply supported strut on elastic medium

Since the value of k is less than $-k_1 = \beta_1 \pi^4 EI/L^4$, the numerical solution is within the range of convergence. The numerical integration is given in the Table 5.4. Due to symmetry only half portion of the strut is considered for the first symmetrical mode.

Table 5.4. Buckling of hinged-hinged strut supported on uniform elastic medium

Node No.	0		1		2		3	Multiplier and remarks
First iteration								Consider $y(x) = a \sin \pi x/L$
y_a			5.0		8.67		10.0	a
I. Effect of k								
$q = -ky$	0		−5.0		−8.67		−10.0	ak
W	0		−58.67		−101.17		−117.34	$akh/12$
Q		218.51		159.84		(58.67)		-do-
M	0		218.51		378.35		437.02	$akh^2/(12)$
Y''			256.34		443.90		512.69	$10akh^3/(12)^2EI$
y'		−956.59		700.25		(−256.35)		-do-
$\bar{y}_d$	0		−956.59		1656.84		1913.19	$10akh^4/(12)^2EI$
$y_{d,k}$	0		−2.563		−4.444		−5.126	a
I. Effect of P								
M	0		−5.0		−8.67		−10.0	aP
Y''			−58.67		−101.17		−117.34	$aPh/12EI$
y'		218.51		159.84		58.67		-do-
$\bar{y}_d$	0		218.51		378.35		437.02	$aPh^2/12EI$
$y_{d,k}$	0		7.563		13.905		15.126	a
$y = y_{d,k} + y_{d,P}$			5.00		8.66		10.00	a

At buckling:

$$15.126a = 437.02aPh^2/12EI$$

Therefore, $P_{cr} = 14.952EI/L^2$

It should be noted that the solution has converged in one iteration because the trial displacement function happens to be the exact one.

Example 5.4. A rigid frame shown in Fig. 5.12 is composed of two identical members BA and BC each having uniform taper. They are rigidly connected at joint B and are hinged to rigid supports at A and C. The moment of inertia I vary from $4I_0$ at the rigid joint to I_0 at the hinged supports. The horizontal member BC carries a concentrated load W at the mid-span and the vertical member is subjected to an axial thrust P of magnitude $15EI_0/L^2$. Estimate the lateral deflections in the strut and draw bending moment diagram.

For analysis the horizontal member AB is treated as simply supported beam with moment M_0 at the point B. The vertical member reduces to a hinged-hinged strut with

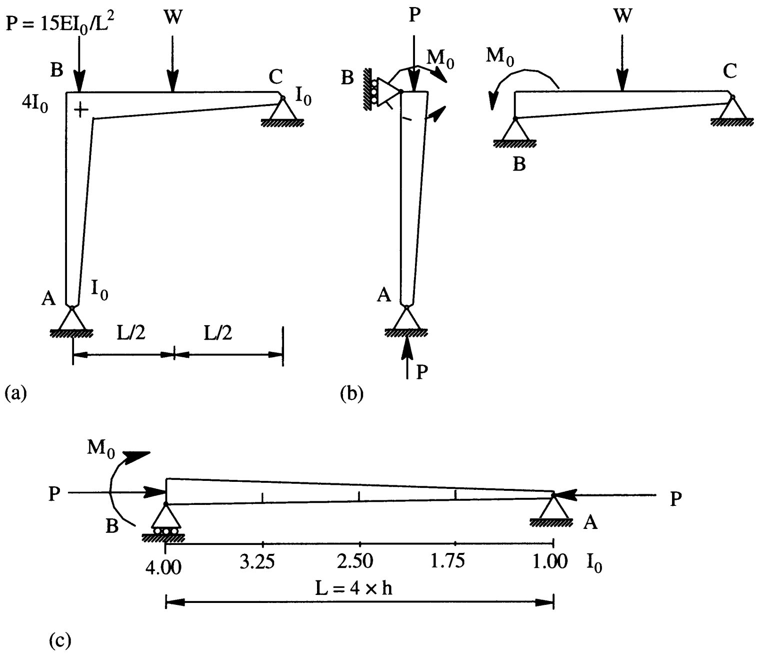


Fig. 5.12a–c. Buckling of frame with tapered members. **a** Frame with variable section, **b** free body diagrams **c** isolated beam-column

moment M_0 at the top end and carries an axial thrust P as shown in Fig. 5.12b. It is required to estimate the deflected position of the member AB under joint action of M_0 and P. The deflection in the member AB due to M_0 before P is applied provides the initial eccentricity to the load P. Since the two members are identical in all respects (including boundary conditions), hence the deflections in two due to M_0 are same. The calculations for displacement due to moment M_0 are given in Table 5.5.

Table 5.5. Calculations of deformation due to end moment M_0

Node No.	0 (B)	1	2	3	4 (C)	Multiplier
I	4.00	3.25	2.50	1.75	1.0	I_0
M	1.0	0.75	0.50	0.25	0.0	M_0
$y'' = M/EI$	0.2500	0.2308	0.2000	0.1429	0.0000	M_0/EI_0
Y''	1.4674	2.7580	2.3737	1.6290	–	$M_0h/(12EI_0)$
y'	−5.130 (−3.6626)[a]	−0.9046	1.4691	3.0981		–do–
y_d	0.0000	3.6626	−4.5672	−3.0981	0.0000	$M_0h^2/(12EI_0)$

$$y'_B = (-3.6626 - 1.4674)\frac{M_0h}{12EI_0} = -5.130\frac{M_0h}{12EI_0}$$

The stiffness;

$$K_{BC} = K_{BA} = \frac{M_0}{y'_B} = \frac{12EI_0}{5.130h} = \frac{9.357EI_0}{L}$$

Influence coefficient at node 2 for fixed-end-moment M_B

$$= -\frac{y_2}{y'_B} = -0.8903h\,.$$

The fixed end moment due to load W alone, $M_{BC} = -W(-0.8903h) = 0.2226WL$. The moment in the members BA and BC is $0.445WL$ each at the node B.

In the second part of analysis the effect of axial compression is taken into account. The initial eccentricity due to M_0 (produced by load W) before P is applied is taken from the first part of analysis. For a good trial approximation for the additional deflection due to P, the knowledge of the first critical load for the member BA under compression will be helpful. The value of first critical load may either be computed or assumed on the basis of average moment of inertia. For the present problem $P_{e,1}$ may be assumed to be

$$P_{e,1} = \frac{\pi^2 E(I_0 + 4I_0)/2}{L^2} = 24.67\left(\frac{EI_0}{L^2}\right)$$

The first trial approximation for y_a may be taken as

$$y_a = \left(\frac{P_{e,1}}{P} - 1\right)^{-1} y_i = 1.55y_i$$

The computation of additional displacement on account of axial thrust is given in Table 5.6.

Table 5.6. Calculation of additional deflection due to axial thrust

Node No.	0	1	2	3	4	Multiplier
I_x	4.00	3.25	2.50	1.75	1.00	I_0
y_i	0.000	−3.663	−4.567	−3.098	0.00	$M_0h^2/(12EI_0)$
y_a for first iteration	0.000	−5.678	−7.079	−4.802	0.00	$(y_{a,1} = 1.55y_i)$
Final Iteration						
y_a	0.000	−5.458	−8.320	−6.505		$M_0h^2/(12EI_0)$
$y = y_i + y_a$	0.000	−9.121	−12.887	−9.603		-do-
$y'' = Py/EI$	0.000	2.806	5.155	5.487	0.0	$M_0Ph^2/12(EI_0)^2$
Y''	5.840	33.215	59.843	60.029	–	$M_0Ph^3/12(EI_0)^2$
y'	(−69.84)*	−36.625	23.218	83.247		-do-
$\bar{y}_d$	0.000	−69.840	−106.465	−83.247	0.0	$M_0Ph^4/(12EI_0)^2$
y_d	0.000	−5.456	−8.318	−6.504	0.0	$M_0Ph^2/(12EI_0)^2$ $(Ph^2/12EI = 0.078125)$

$$\bar{y}'_{BA} = [-5.456 - (5.840/12)]\frac{M_0 h}{12EI_0} = -5.943\frac{M_0 h}{12EI_0}$$

Hence the total slope at B due to M_0 and P is given by

$$\bar{y}'_{BA} = [-5.943 - 5.130]\frac{M_0 h}{12EI_0} = -11.073\frac{M_0 h}{12EI_0}$$

Stiffness of member BA:

$$k_{BA} = \frac{M_B}{y'_{BA}} = 4.335\frac{EI_0}{L}$$

It should be noted that stiffness of the member BA reduces considerably due to the presence of axial thrust. However, the stiffness of the member BC remains unchanged as computed previously in the first part of analysis. The moment distribution factors at the joint B are:

$$D_{BC} = (9.357)/(9.357 + 4.335) = 0.6834$$

and

$$D_{BA} = (4.335)/(9.357 + 4.335) = 0.3166$$

From moment distribution

$$M_{BA} = 0.2226\,WL(0.3166) = 0.0705\,WL$$

Example 5.5. A section of a vertical wall AB constructed monolithic with a horizontal slab BC is shown in Fig. 5.13. The unit lengths of wall and slab have moments of inertia I and I_1, respectively, in transverse bending. The structure is idealized as being simply supported on continuous knife edges at B and C that can be assumed as acting at centre line of member. A continuous vertical knife edge load p per unit length (assumed concentrated) is applied at the centre of top of the wall. In addition to the axial thrust P, the wall is subjected to a horizontal shear force Q as shown in Fig. 5.13b.

The member AB is presumed to be an elastically restrained beam-column. The governing differential equation is

$$y'''' + k^2 y'' = 0$$

which has solution $y(x) = A\sin\alpha x + B\cos\alpha x + Cx + D$ with following boundary conditions.

$$y(0) = y''(L) = 0 \quad \text{and} \quad k_r y'(0) = (EI)y''(0)$$

In addition at the top end of the member AB i.e. at $x = L$, the end shear condition is:

$$Q + Py' + (EI)y''' = 0\,.$$

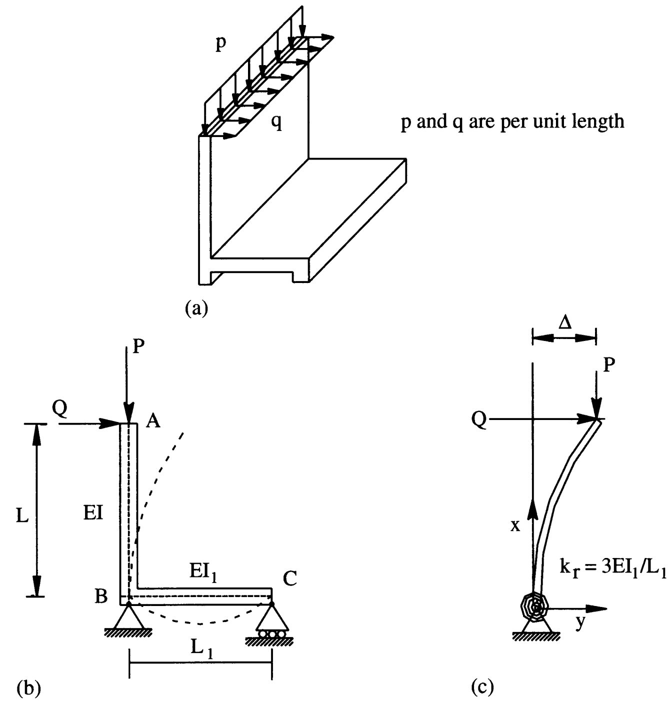


Fig. 5.13a–c. Wall-column monolithic with horizontal slab. **a** Structure, **b** idealized supports, **c** idealized wall-column

Substitution of the general solution in each of these boundary conditions yields the following set of simultaneous equations.

$$\begin{array}{lllll}
(0)A & + (1.0)B & + (0)C & + (1.0)D & = 0 \\
(\alpha\gamma)A & + (\alpha^2)B & + (\gamma)C & + (0)D & = 0 \\
(\sin\alpha L)A & + (\cos\alpha L)B & + (0)C & + (0)D & = 0 \\
(0)A & + (0)B & + (1.0)C & + (0)D & = -Q/P
\end{array}$$

where $\gamma = k_r/EI$. The spring constant due to bending of horizontal member $k_r = 3EI_1/L_1$. The moment M_B at the rigid joint B is of special interest

$$M_B = M(0) = (EI)y''(0) = -EI\alpha^2(B) = -P(B)$$

The constants can be computed as

$$C = -Q/P\,, \quad D = -B\,, \quad A = (-\cot\alpha L)B$$

and thus

$$B = -\left(\frac{Q}{P}\right)\left[\frac{\gamma\sin\alpha L}{\alpha(\gamma\cos\alpha L - \alpha\sin\alpha L)}\right]$$

Thus the moment at the rigid joint is given by

$$M_B = -\left(\frac{QL}{\alpha L}\right)\left[\frac{\sin\alpha L}{\cos\alpha L - (\alpha L/\gamma L)\sin\alpha L}\right] \quad \text{or}$$

$$\frac{M_B}{QL} = -\left[\frac{(\sin\alpha L)/\alpha L}{\cos\alpha L - (\alpha L/\gamma L)\sin\alpha L}\right]$$

where

$$\alpha L = \pi\sqrt{\frac{P}{P_{\mathrm{e}}}} \quad \text{and} \quad \gamma L = 3\left(\frac{I_1/L_1}{I/L}\right).$$

As a typical consider $I_1/L_1 = I/L$ and thus $\gamma L = 3$. For $P/P_{\mathrm{e}} = 0$, the problem reduces to one of pure bending with $M_B = QL$. At $P/P_{\mathrm{e}} = 0.1$, $M_B = 3.14\,QL$, which corresponds to an increase in moment over that at zero axial thrust of 214 per cent. From this example it is clear that axial force can have a profound effect upon maximum moment at the base of the column. At $P/P_{\mathrm{e}} = 1.0$, the axial load multiplying factor becomes infinitely large.

5.5 Strut with Initial Eccentricity

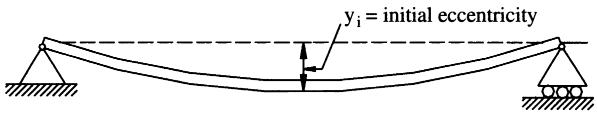
In the discussions so far the column has been assumed to be perfectly straight and the axial thrust is assumed to pass through the *centroidal axis*. However, in practice both the lack of straightness (i. e. imperfection of shape) and small eccentricity of load may be present in the structure. This type of problems can be easily handled by classical and numerical techniques. The numerical procedure is a powerful tool to estimate deflection. Irrespective of value of initial eccentricity y_{i} and value of end thrust, the numerical procedure converges to a set of values y_{e} which along with y_{i} will give the final equilibrium position of the structure.

For illustration consider the behaviour of hinged-hinged strut with initial eccentricity y_{i} measured from the position of thrust line as shown in Fig. 5.14a. Let the displacement of final deflected shape measured from the straight configuration is represented by $y(x)$. As in case of displacement functions, the initial eccentricity y_{i} can also be expressed in the form of a polynomial or trignometrical series. In the present case consider a trignometrical series

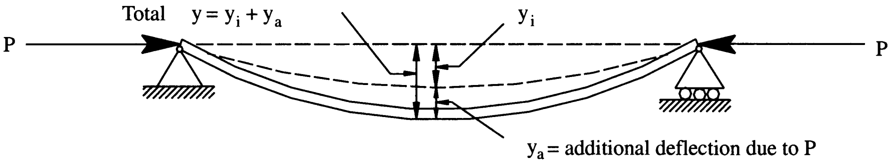
$$y_{\mathrm{i}} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{5.45}$$

Thus the internal resisting moment is

$$M = -EI\left[\frac{\mathrm{d}^2(y - y_i)}{\mathrm{d}x^2}\right] \tag{5.46}$$



(a)



(b)

Fig. 5.14a,b. Strut with initial eccentricity. **a** Initial eccentricity y_i at $P = 0$, **b** equilibrium configuration after application of P

which is balanced by external disturbing moment, i. e., $Py = M$. Therefore,

$$EI\left(\frac{d^2y}{dx^2}\right) + Py = EI\left(\frac{d^2y_i}{dx^2}\right) \tag{5.47}$$

Substituting for y_i from (5.45).

$$EI\left(\frac{d^2y}{dx^2}\right) + Py = -EI\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} \tag{5.48a}$$

$$y'' + \alpha^2 y = -\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} \tag{5.48b}$$

where $\alpha^2 = P/EI$.

The general solution to the governing equation is

$$y(x) = A\sin\alpha x + B\cos\alpha x + y_P$$

Let the particular solution y_P is given by

$$y_P = \sum_{n=1}^{\infty} Y_n \sin\frac{n\pi x}{L} \tag{5.49}$$

Substituting y_P from (5.49) in (5.48b)

$$-\sum_{n=1}^{\infty} Y_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} + \alpha^2\sum_{n=1}^{\infty} Y_n \sin\frac{n\pi x}{L} = -\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}$$

Consider the n^{th} term

$$Y_n = \frac{a_n}{1-(\alpha L/n\pi)^2} = \frac{a_n}{[1-(P/P_e)(1/n^2)]} \tag{5.50}$$

where $P_e = \frac{\pi^2 EI}{L^2}$. The total deflection can thus be written as

$$y = A\sin\alpha x + B\cos\alpha x + \sum\left[\frac{a_n}{[1-(P/P_e)(1/n^2)]}\right]\sin\frac{n\pi x}{L} \tag{5.51}$$

From the boundary conditions

$$y(0) = 0\,, \quad \text{gives} \quad B = 0 \quad \text{and} \quad y(L) = 0\,, \quad \text{yields} \quad A = 0$$

Thus,

$$y(x) = \sum\left[\frac{a_n}{[1-(P/P_e)(1/n^2)]}\right]\sin\left(\frac{n\pi x}{L}\right) \tag{5.52}$$

Considering a single term representation for elastic deflected curve, the mid-point deflection of the strut is given by

$$y(L/2) = a\left[1-\frac{P}{P_e}\right]^{-1} \tag{5.53}$$

As the thrust P approaches P_e the mid-span deflection tends to infinity. Sometimes this is known as *imperfection approach* to determination of critical load. According to this approach, the critical load of perfect column is defined as the load at which imperfect column develops an infinite displacement. It should be noted that the problem of an initially bent strut is not an *eigenvalue problem* since for every load there is a definite displacement.

5.6 Interaction Equation

In general the beam-columns are subjected to two basically different kinds of loading: axial thrust and bending moment. An expression relating these two, called *interaction equation*, gives a reasonable prediction of structural strength. It has been noticed in the preceding sections that the axial thrust significantly increases the primary moments, i. e. those which result from lateral loads and applied end moments etc. If limited *allowable normal stress* is the *design criterion*, the limited stress can be related to the axial thrust and bending moment by the equation.

$$\sigma_{\max} = \sigma_{\text{all}} = \frac{P}{A} + \frac{M_{\max}}{Z} \tag{5.54}$$

where A and Z are area and section modulus of the cross-section of the member, respectively. Since P is constant along the member, the stress depends on the location and magnitude of maximum bending moment. It is to be recognized that at a cut section of the member, the axial thrust acts through the centroid, and the bending moment is about an axis which passes through that point.

In case the allowable normal stress is the yield stress in the material, then

$$\frac{P}{A} + \frac{M_0(\zeta)}{Z} = \sigma_y \quad \text{or} \quad \frac{P}{P_y} + \frac{M_0(\zeta)}{M_y} = 1.0 \tag{5.55}$$

where $P_y = A\sigma_y$, the maximum yield value of the yield thrust when bending moment is zero; and $M_y = Z\sigma_y$, the initial yield value of the bending moment when axial thrust is zero.

Consider the case of a strut subjected to the equal and opposite compressive loads applied at eccentricities e and βe at the ends, A and B, respectively. The deflection curve for this case is from (5.21).

$$y(x) = e\left[\beta\left(\frac{\sin\alpha x}{\sin\alpha L} - \frac{x}{L}\right) + \frac{\sin\alpha(L-x)}{\sin\alpha L} - \left(\frac{L-x}{L}\right)\right] \tag{5.56}$$

The moment at a section is

$$M = -EIy'' = EI\,e\left(\frac{\alpha^2}{\sin\alpha L}\right)[\beta(\sin\alpha x) + \sin\alpha(L-x)] \tag{5.57}$$

The maximum moment occurs at the section where $\partial M/\partial x = 0$, i.e.

$$\beta\cos\alpha x - \cos\alpha(L-x) = 0$$

$$\text{or} \quad \beta = \frac{\cos\alpha(L-x)}{\cos\alpha x} \tag{5.58a}$$

Alternatively,

$$\cot\alpha x = \frac{\sin\alpha L}{\beta - \cos\alpha L} = \frac{\left(\sin\pi\sqrt{\frac{P}{P_e}}\right)}{\left(\beta - \cos\pi\sqrt{P/P_e}\right)} \tag{5.58b}$$

However, it must be understood that for a given value of β, if (5.58a) or (5.58b) predicts a location of maximum moment that is off the end of the member ($0 > x > L$), the maximum value occurs at the end.

As an illustration consider the case where $\beta = +1.0$, i.e. the strut is subjected to a compressive load applied at a eccentricity e at each end resulting in a single-curvature type primary bending moment. The maximum bending moment occurs at the mid-span ($x = L/2$) and its value is obtained from (5.57) as

$$\begin{aligned} M_{\max} &= \frac{(EI\alpha^2)e}{\sin\alpha L}\left[\sin\left(\frac{\alpha L}{2}\right) + \sin\left(\frac{\alpha L}{2}\right)\right] = \frac{Pe}{\sin\alpha L}\left[2\sin\left(\frac{\alpha L}{2}\right)\right] \\ &= \frac{Pe[2\sin(\alpha L/2)]}{2\sin(\alpha L/2)\cos(\alpha L/2)} = [Pe\sec(\alpha L/2)] = M_0(\zeta) \end{aligned} \tag{5.59}$$

From (5.55)

$$\frac{P}{P_y} + \frac{Pe\sec(\alpha L/2)}{(Ar^2/c)\sigma_y} = \frac{P}{P_y}\left[1 + \frac{ec}{r^2}\sec(\alpha L/2)\right] = 1.0$$

[as $M_y = Z.\sigma_y = (I/c)\sigma_y = (Ar^2/c)\sigma_y$]

$$\text{or} \quad \frac{P}{A} = \frac{\sigma_y}{\left[1 + \frac{ec}{r^2}\sec\frac{\alpha L}{2}\right]} = \frac{\sigma_y}{\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2\pi}\sqrt{\frac{P}{EA}}\right)\right]} \tag{5.60}$$

where r is the *radius of gyration* of the cross-section about the centroidal axis, and c is the distance from the centroid to the extreme fibre of the cross-section. The relation given by (5.60) is usually termed as *secant formula*. As mentioned earlier, the *amplification* or *magnification factor* ζ can be conservatively approximated as

$$\zeta_{\text{approx.}} = \frac{1}{\left(1 - \frac{P}{P_e}\right)} \tag{5.61}$$

It is to be recognized that the interaction relation given by (5.55) presumes initial yielding of extreme fibre. However, in practice the *allowable values* concept is more commonly used. Defining the desired allowable values as

$$P_a = C_p P_y = C_p(A\sigma_y) \quad \text{and} \quad M_a = C_b M_y = C_b(Z\sigma_y) \tag{5.62}$$

where C_p and C_b are prescribed factors to provide sufficient *margin of safety* against yielding, the modified interaction equation reduces to

$$\frac{P}{P_a} + \frac{M_0(\zeta)}{M_a} = 1.0 \tag{5.63}$$

Alternatively in terms of allowable stresses (5.63) can be expressed as

$$\frac{f_a}{F_a} + \frac{f_b(\zeta)}{F_b} = 1.0 \tag{5.64}$$

where f_a and f_b are actual compressive stress (P/A) and bending stress (M/Z), respectively, and F_a and F_b are corresponding *allowable stresses* when each one is acting independently.

It should be noted that the term (αL) appears in all of the beam-column problems. This term can conveniently be expressed in a meaningful non-dimensional form

$$\alpha L = L\sqrt{\frac{P}{EI}} = \pi\sqrt{\frac{P}{P_e}}$$

and

$$\frac{P}{P_e} = \left(\frac{P}{P_y}\right)\left(\frac{P_y}{P_e}\right) = \left(\frac{P}{P_y}\right)\left[\frac{A\sigma_y}{\pi^2 EI/L^2}\right] = \left(\frac{P}{P_y}\right)\left[\frac{(L/r)^2}{(\pi^2 E/\sigma_y)}\right] \tag{5.65}$$

where the ratio (L/r) is referred to as *slenderness ratio.* For smaller values of L/r the axial yield value governs. For larger values, the buckling occurs. For the known bending and axial forces the suitable cross-section for the beam-column can be determined by trial and modification.

Example 5.6. A wide flange rolled steel section (SC series) of length 3.75 m is to be selected to support an axial compressive force of 525 kN and a single-end bending moment ($\beta = 0$) of 5.0 kNm. Assume the effective length coefficient $K = 0.8$. The material has yield strength and modulus of elasticity of 250 MPa and 200 GPa, respectively. It is specified that the allowable axial stress (in absence of bending) should be $(0.55\, f_y)$, and the allowable stress in bending (in absence of axial force) should be $0.66\, f_y$, i.e.,

$$P_a = C_p A f_y = (0.55)(250)A$$
$$M_a = C_b Z f_y = (0.66)(250)Z$$

From the interaction equation

$$\frac{P}{P_a} + \frac{M(\zeta)}{M_a} = 1.0$$

where amplification factor ζ depends upon $P/P_{e,a}$

$$\frac{P}{P_{e,a}} = \frac{P}{0.55[\pi^2(200000)I/(0.8 \times 3750)^2]}$$
$$= \frac{(525 \times 10^3)}{0.55[\pi^2(200000)I/(0.8 \times 3750)^2]} = \frac{432205}{I}$$

The suitable section can be determined by trial and modification. Various trials are listed in Table 5.7. The terms in the last column of table are the sum of the two terms of the left-hand side of the interaction equation. The section whose value is closest to 1.0 (but slightly less) will be the most desirable section. For this example the most appropriate section in SC 150.

Table 5.7. Interaction values for the trial sections

Section	A	I_x	Z_x	$P/P_{e,a}$	ζ	$M_0(\zeta)/M_a$	P/P_a	(7) + (8)
(1)	(2)	(3) $\times 10^4$	(4) $\times 10^3$	(5)	(6)	(7)	(8)	(9)
SC140	4240	1470	211	0.296	1.421	0.204	0.901	1.105
SC150	4740	1970	259	0.221	1.284	0.152	0.805	**0.955**
SC160	5340	2420	303	0.180	1.219	0.122	0.715	0.837

Example 5.7. A structural aluminium round tube of 125 mm external diameter and 8 mm thickness is subjected to an eccentrically applied compressive axial load of 5 kN. At one end of the member the eccentricity is e while at the other end it is zero (i. e., $\beta = 0$). The yield strength of the material is 25 N/mm^2. Determine the limiting value of eccentricity that can be accepted over a length of 2.4 m with a load factor of safety of 1.9 against attainment of yield strength. Take $E = 6.0\,\text{kN/mm}^2$.

For the given section:

Area of cross-section,	$A = 2940.53\,\text{mm}^2$.
Second moment of area,	$I = 5055.14 \times 10^3\,\text{mm}^4$
Section modulus,	$Z = 80.88 \times 10^3\,\text{mm}^3$
Radius of gyration,	$r = 41.46\,\text{mm}$
Distance of extreme fibre,	$c = 62.50\,\text{mm}$

The form of interaction equation involving eccentricity is given by (5.60).

$$\frac{P}{P_y}\left[1+\left(\frac{ec}{r^2}\right)\sec\left(\frac{1}{2r}\sqrt{\frac{P}{EA}}\right)\right] = 1.0$$

Therefore,

$$e = \left(\frac{P_y}{P} - 1\right)\left(\frac{r^2}{c}\right)\cos\left(\frac{1}{2r}\sqrt{\frac{P}{EA}}\right)$$

To ensure the stipulated load *factor of safety*, the applied load of 5 kN will be increased to 9.5 kN and the analysis is carried out at that prorated value.

$$\left(\frac{1}{2r}\sqrt{\frac{P}{EA}}\right) = 0.00028 \quad \text{and} \quad \cos\left(\frac{1}{2r}\sqrt{\frac{P}{EA}}\right) \approx 1.0$$

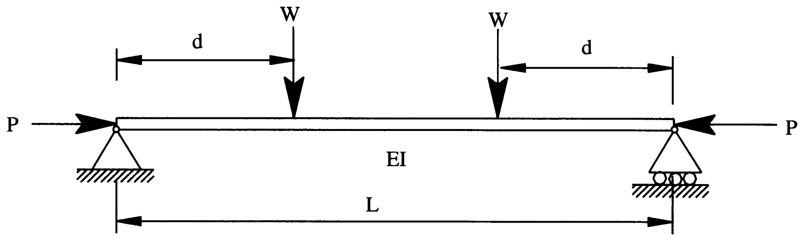
$$\frac{P_y}{P} = \frac{25 \times 2940.53}{9.5 \times 10^3} = 7.738$$

Thus,

$$e = (7.738 - 1)\left[\frac{(41.46)^2}{62.5}\right](1) = 185.32\,\text{mm} .$$

5.7 Problems

Problem 5.1. A beam-column is subjected to two loads of equal magnitude W acting at distance d from either support as shown in Fig. P.5.1. Determine the equation of the elastic curve and the end rotations.



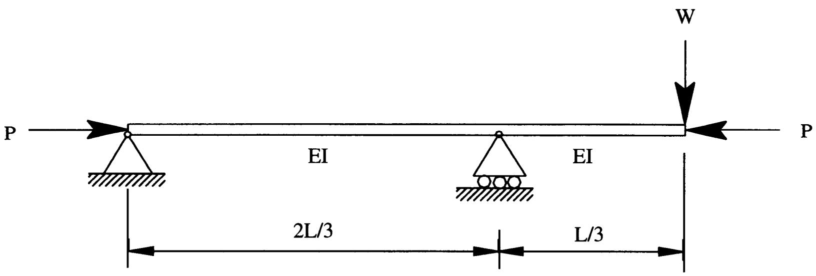
P.5.1

[Ans.

$$y(x) = \left(\frac{W}{P\alpha}\right) \sec\frac{\alpha L}{2} \cos\alpha L \left(\frac{1}{2} - \frac{d}{2}\right) \sin\alpha x - \alpha x \, , \quad 0 < x < d$$

$$\theta_A = \theta_B = y'(0) = \left(\frac{W}{P}\right) \sec\frac{\alpha L}{2} \cos\alpha L \left(\frac{1}{2} - \frac{d}{2}\right) - 1 \quad]$$

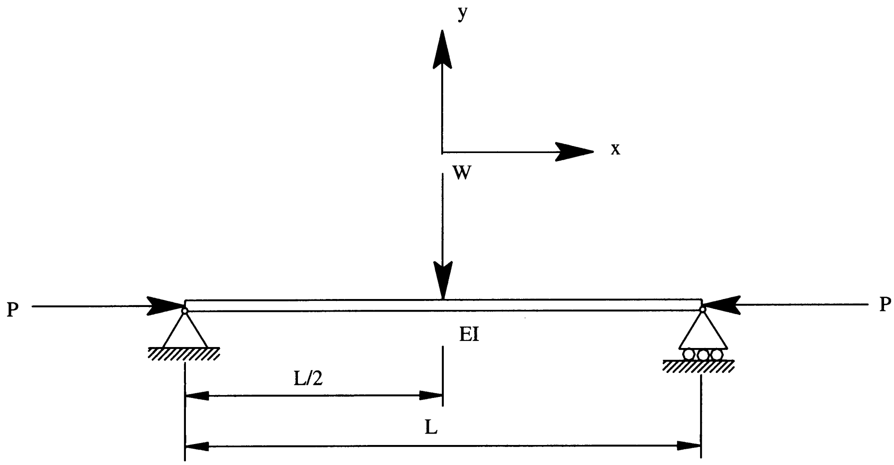
Problem 5.2. A simply supported beam with an over-hang is subjected to an axial compressive force P, and a lateral load W at its free end as shown in the Fig. P.5.2. Determine the equation of the elastic curve and rotations at the support.



P.5.2

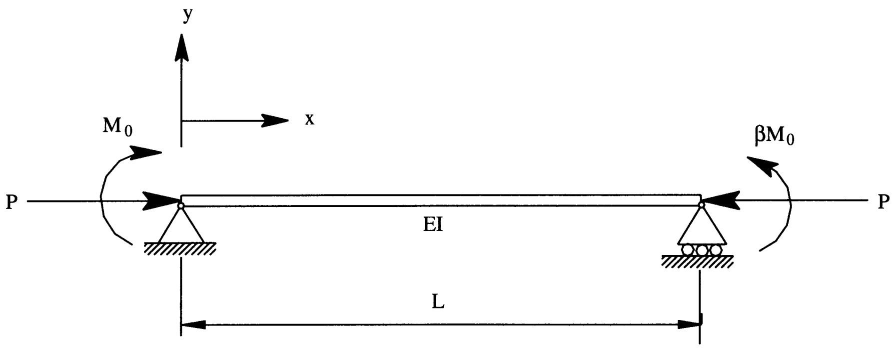
Problem 5.3. A cantilever beam of length L, bending stiffness EI, and cross-sectional area A is subjected to a lateral load$W$ at the free-end. In addition it carries (i) an axial thrust, and (ii) an axial tensile force. Derive the equations of elastic curve. If $y_{\max}$ is the deflection at the free end of the cantilever, plot the variation of $y_{\max}$ as a function of P/W.

Problem 5.4. A simply supported beam of length L and bending stiffness EI is subjected to an axial thrust P, and a concentrated load W at the mid-span as shown in Fig. P.5.4. Obtain approximations for the elastic curve using the series $y(x) = \sum a_n \cos(n\pi x/L)$, $n = 0, 1, 2, \ldots$ (origin at mid-span) with the principle of minimum potential energy.



P.5.4

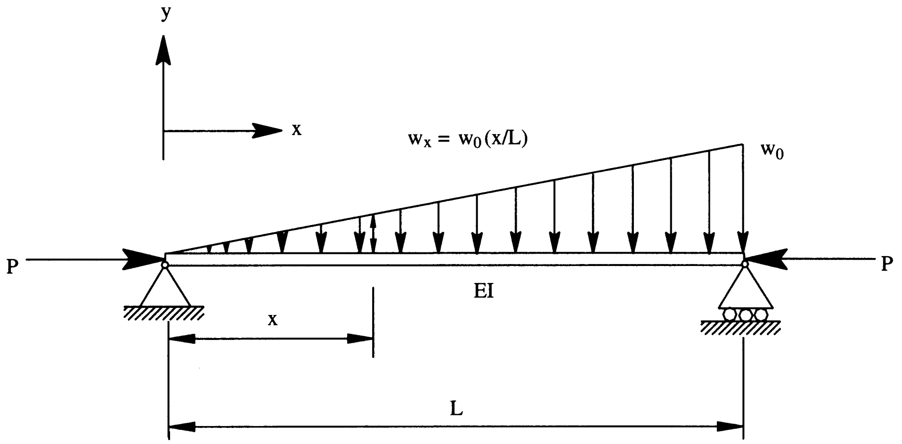
Problem 5.5. A simply supported beam of length L, bending stiffness EI is subjected to an axial thrust P, and two end moments M_0 and βM_0 at the left and right supports, respectively, as shown in Fig. P.5.5. Using the series $y(x) = \sum a_n \sin(n\pi x/L)$, $n = 1, 2, 3, \dots$ obtain approximations for the elastic curve based on principle of minimum potential energy.



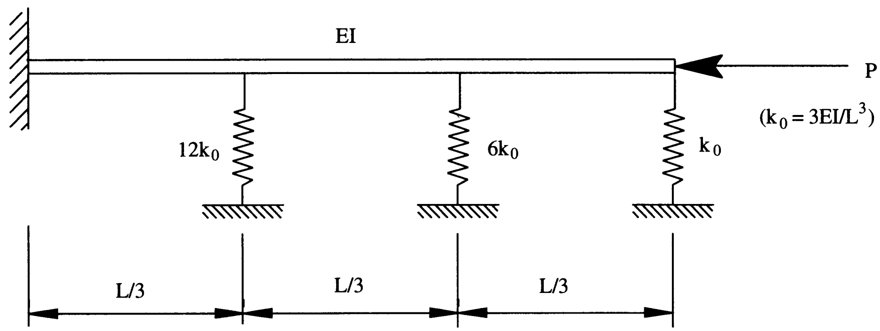
P.5.5

Problem 5.6. A simply supported beam of length L, bending stiffness EI is subjected to an axial thrust P, and a uniformly varying distributed load $w_x = w_0(x/L)$ as shown in Fig. P.5.6. Derive the expressions for elastic curve, maximum deflection and end slopes.

Problem 5.7. Estimate the first critical value of the axial thrust in the cantilever of constant cross-section with elastic supports as shown in Fig. P.5.7.



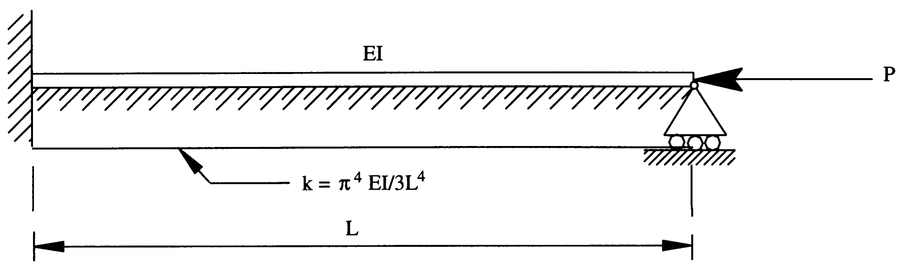
P.5.6



P.5.7

Problem 5.8. Estimate the first critical value of end thrust P in a fixed-hinged strut of constant cross-section supported by an elastic medium of constant stiffness, $k = \pi^4 EI/3L^4$ as shown in Fig. P.5.8.

[Hint: The problem is first degree redundant with critical negative modulus value of $-237.8\, EI/L^4$*].*



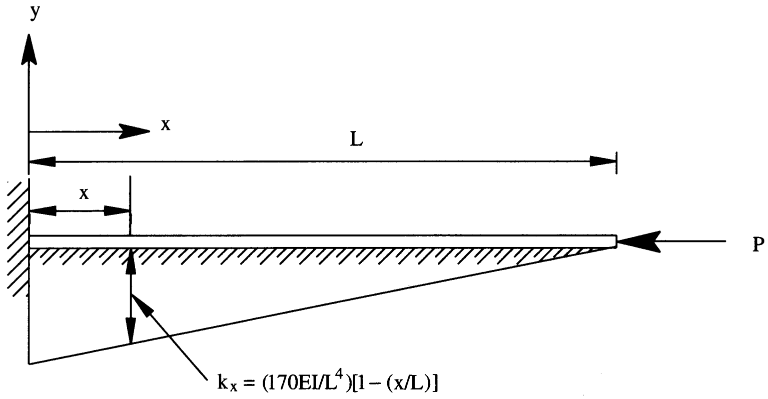
P.5.8

Problem 5.9. Calculate the first critical negative value of k for the beam of constant cross-section supported on a continuous elastic medium with end conditions as: (i) fixed-free, and (ii) fixed-fixed.

[Ans. (i) $k_1 = -123.6\, EI/L^4$, *and (ii)* $k_1 = -500.5\, EI/L^4$*].*

Problem 5.10. Estimate the first critical value of the axial load P acting on a cantilever of constant stiffness EI which is supported on an elastic medium whose modulus varies linearly from 0 at the free-end to a maximum of $170\, EI/L^4$ at the fixed end as shown in Fig. P.5.10. Use (i) Energy method, and (ii) Numerical integration for solution.

[Ans. $6.0\, EI/L^2$*].*

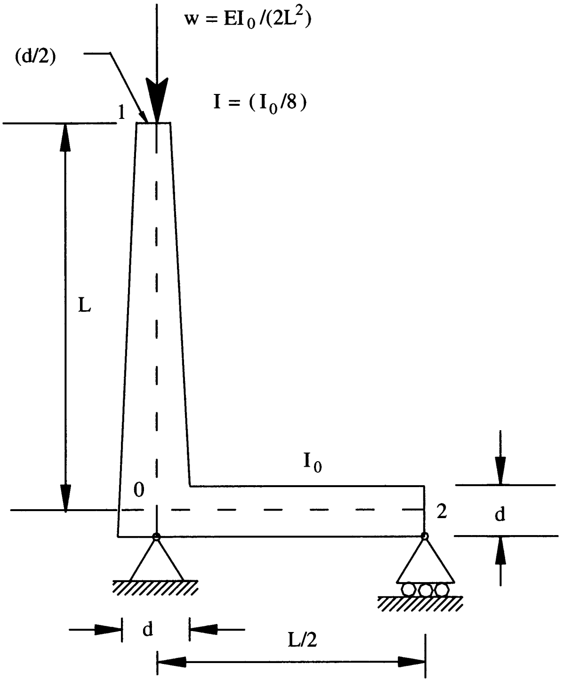


P.5.10

Problem 5.11. In the cross-section of part of idealized structure shown in Fig. P.5.11, the wall 0-1 is built monolithic with the horizontal slab of thickness d with moment of inertia of I_0. The thickness of wall is reduced uniformly from d at the junction with the horizontal slab to $d/2$ at the top with outside face being vertical. The structure is idealized as simply supported on the continuous knife edges at 0 and 2 and assumed to be acting at the centre line of the members. A continuous vertical knife edge load of value $w = EI_0/2L^2$ per unit length is applied at the centre of the top of the wall. Estimate the transverse deflections in the wall and draw the bending moment diagram.

[Hint: This is a case with initial eccentricity which vary from 0 at the top of the wall to $d/4$ *at the base. The first critical value of* $w_{cr,1} = 1.19\, EI_0/2L^2$ *and* $y_1 = 2.28a$*]*

Problem 5.12. A rigid frame shown in Fig. P.5.12 consists of two identical members 0-1 and 1-2, each having same uniform taper. The moment of inertia varies from I_0 at the hinged-end to $5I_0$ at the rigid-joint 1. Estimate the first critical value of the axial thrust P.



P.5.11

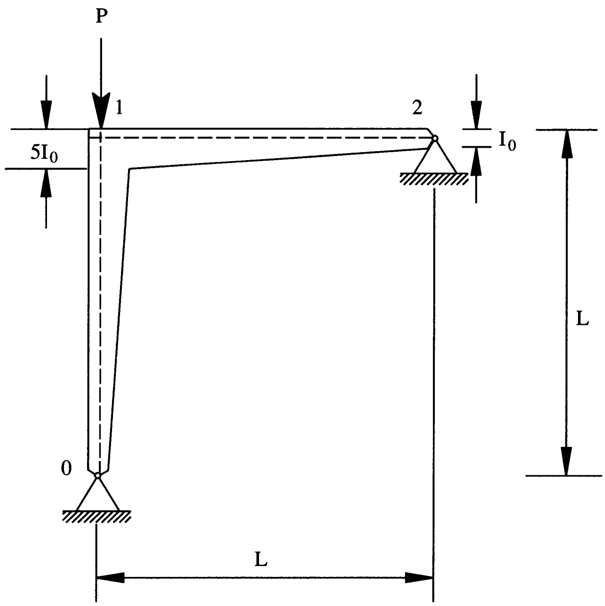
[Hint: Treat the tapered horizontal member 1-2 as simply supported with a moment M_0 acting at the joint 1].

Problem 5.13. A SC200 rolled steel beam-column of length 9 m, pin-connected at the ends is laterally supported against weaker direction of bending. In addition to an axial compressive force P applied through the centroid of the section at the ends, the member is subjected to an end-moment M_0 at one end of the member. Determine the value of M_0 for $P = 50\,\text{kN}$. Assume $E = 200\,\text{kN/mm}^2$, $f_y = 250\,\text{N/mm}^2$ and desired factor of safety of 1.05.

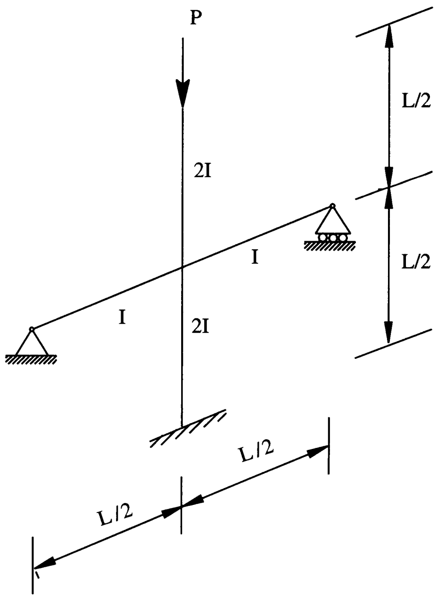
Problem 5.14. Design a beam-column member of length 3.75 m subjected to an axial force of 800 kN, and end moments $M_x = 2.5\,\text{kNm}$ and $M_y = 2.5\,\text{kNm}$. Assume effective length coefficient $K = 0.7$ and $F_y = 250\,\text{N/mm}^2$.

[Ans. SC200 section is adequate].

Problem 5.15. A column of length 3.5 m in a multistorey non-sway building frame is subjected to an axial force of 725 kN and a major axis moment M_x of 80 kNm at both the ends. At the top and bottom joints of the column $\sum k_c$ and $\sum k_b$ values are 6, 20 and 6, 18, respectively. If the section HB300 is readily available, check its adequacy for the present situation.



P.5.12



P.5.16

Problem 5.16. A vertical cantilever column of constant cross-section is strengthened by a horizontal beam at its mid-point as shown in the Fig. P.5.16. Estimate the first critical value of axial compression in the column for buckling of the column in the plane perpendicular to the one containing column and the beam, when the horizontal beam is (i) simply supported, and (ii) fixed at the ends.

[Hint: The beam will act as an elastic support of stiffness k at the mid-point of the column. k is equal to (i) $48\,EI/L^3$ *and (ii)* $192\,EI/L^3$*.]*

6

Stability Analysis of Frames

6.1 Introduction

In the previous chapters the stability of column, and beam-column was examined by treating them as independent or isolated members with appropriate boundary conditions. The simple frames have been treated as struts or beam-columns with elastically restrained ends wherein the effect of the connecting members has been modelled by end springs. However, in practice the columns, beams, and beam-columns are normally rigidly joined together to make skeletal structure called a frame in which the total structure is called upon to withstand the applied loads. In these rigid-jointed frames, the end conditions of a member and hence its effective length depends upon the relative stiffness of the members meeting at the ends and that of member itself. Moreover, in a frame the deflection even in a single member due to buckling causes distortion in all the members. Thus, the response of the frame needs be examined in its totality wherein actual buckling of total frame is considered. In this chapter the stability analysis of the frames using classical differential equation method, semi-geometrical method, matrix method and modified moment distribution method etc. has been described.

6.2 Classical Approach

In this section classical differential equation method has been used to obtain characteristic or stability equations for continuous columns, beam-columns, and frames. The solution to these equations yields the critical loads.

6.2.1 Continuous Columns and Beam-Columns

Continuous columns and beam-columns are the simplest forms of a rigid-jointed frame. For illustration consider two-span continuous column ABC shown in Fig. 6.1 which is statically indeterminate to the first degree. The bending moment at the

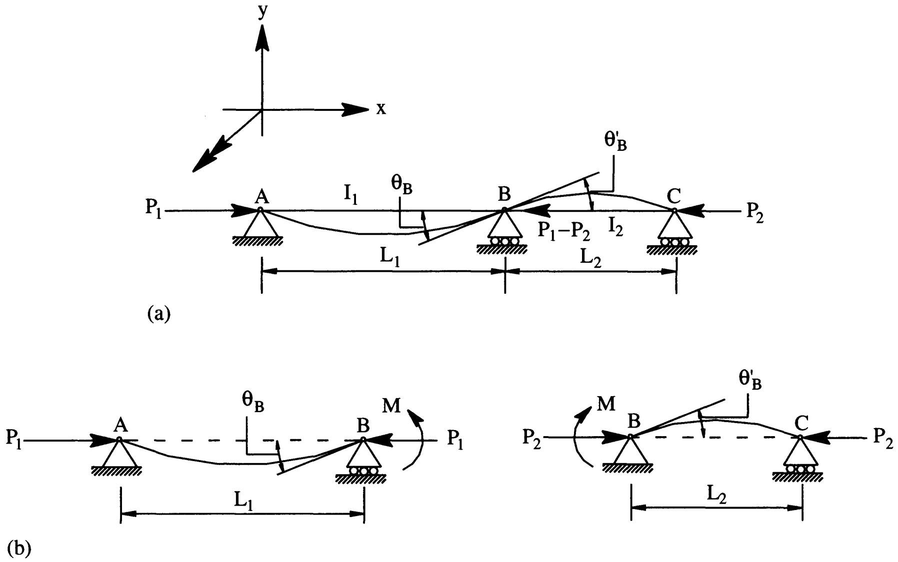


Fig. 6.1a,b. Two-span continuous beam-column, with both spans loaded. **a** Structure and its buckling mode, **b** both spans considered simply supported

rigid intermediate support is taken to be the redundant action. The sagging moment producing compression on the top is considered positive and the angle of rotation in the direction of positive moment is taken to be positive. The axial thrust and flexural rigidity remain constant within each span but are allowed to vary from span to span.

The *continuity* or *compatibility condition* to be satisfied by the moment M at the interior support B is

$$\theta_B = \theta'_B \quad \text{or} \quad \theta_B - \theta'_B = 0 \tag{6.1}$$

where θ_B and θ'_B are the angles of rotation at the support B obtained by treating each of the spans AB and BC to be simply supported beam-column with an end-moment as shown in Fig. 6.1b. The expressions for θ_B and θ'_B are given by (5.18).

$$\theta_B = \left(\frac{ML_1}{3EI_1}\right)\varphi(\psi_1)$$

where

$$\psi_1 = \left(\frac{\alpha_1 L_1}{2}\right) \quad \text{and} \quad \alpha_1 = \sqrt{\frac{P_1}{EI_1}}$$

Similarly

$$\theta'_B = -\left(\frac{ML_2}{3EI_2}\right)\varphi(\psi_2)$$

where

$$\psi_2 = \left(\frac{\alpha_2 L_2}{2}\right) \quad \text{and} \quad \alpha_2 = \sqrt{\frac{P_2}{EI_2}}$$

Substituting in (6.1)

$$\left(\frac{ML_1}{3EI_1}\right)\left[\varphi(\psi_1) + \left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right)\varphi(\psi_2)\right] = 0$$

Since $(ML_1/(3EI_1)) \neq 0$ the characteristic equation obtained is

$$\varphi(\psi_1) + \left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right)\varphi(\psi_2) = 0 \tag{6.2}$$

Let $P_2 = \lambda P_1$ where λ is a known constant. Then

$$\psi_2 = \gamma\psi_1 \quad \text{where} \quad \gamma = \left(\frac{L_2}{L_1}\right)\left[\lambda\left(\frac{I_1}{I_2}\right)\right]^{1/2}$$

Using these substitutions in (6.2), the characteristic equation reduces to

$$\varphi(\psi_1) + \left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right)\varphi(\gamma\psi_1) = 0 \tag{6.3}$$

The roots of this equation provide the critical loads for buckling by flexure of the two-span continuous beam-columns. As a typical case consider $L_2 = L_1 = L$, $I_2 = I_1 = I$ and $P_2 = P_1 = P$ i.e. $\lambda = 1$.

In this case the member will buckle as shown in Fig. 6.1a and the bending moment at the middle support will be zero and each of the spans can be treated as hinged-hinged strut. Therefore,

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L^2}$$

As a variation consider the case of a typical continuous beam with unequal spans of

$$L_1 = (2/3)L_2 = L\,, \quad I_2 = I_1 = I \quad \text{and} \quad \lambda = 1\,.$$

Therefore,

$$\psi_1 = \left(\frac{2}{3}\right)\psi_2 = \psi \quad \text{and} \quad \gamma = \left(\frac{3}{2}\right)$$

The Eqs. (6.3) and (5.18) give

$$\varphi(\psi) + \left(\frac{3}{2}\right)\varphi\left(\frac{3\psi}{2}\right) = 0$$

$$\left[\frac{3}{2\psi}\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) + \left(\frac{3}{2}\right)\left(\frac{3}{3\psi}\right)\left(\frac{1}{3\psi} - \frac{1}{\tan 3\psi}\right)\right] = 0$$

$$\left[\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) + \left(\frac{1}{3\psi} - \frac{1}{\tan 3\psi}\right)\right] = 0$$

Using trial and modification procedure the smallest root obtained is $2\psi = 2.427$. Therefore, $2\psi = \alpha L = 2.427$ or

$$P_{cr} = (5.89)\frac{EI}{L^2} = \frac{\pi^2 EI}{(1.294L)^2}$$

In another case consider the same two-span member with $P_2 = 0$ i.e. only the span AB is subjected to the axial force P_1. The rotations at the interior support B in this case are:

$$\theta_B = \left(\frac{ML_1}{3EI_1}\right)\varphi(\psi_1) \quad \text{and} \quad \theta'_B = -\left(\frac{ML_2}{3EI_2}\right)$$

Substituting these expressions in (6.1),

$$\left(\frac{ML_1}{3EI_1}\right)\left[\varphi(\psi_1) + \left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right)\right] = 0$$

Since $M \neq 0$, the characteristic equation becomes

$$\varphi(\psi_1) + \left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right) = 0 \tag{6.4}$$

Substituting (5.18) in (6.4)

$$2\psi_1 \cot 2\psi_1 = 1 + \left(\frac{1}{3}\right)\left(\frac{I_1}{I_2}\right)\left(\frac{L_2}{L_1}\right)(2\psi_1)^2 = 0 \tag{6.5}$$

The solution of (6.5) provides the critical load.

As a typical case consider $L_1 = 2L_2/3 = L$, and $I_2 = I_1 = I$. Equation (6.5) reduces to

$$2\psi_1 \cot 2\psi_1 = 1 + (1/2)(2\psi_1)^2 \tag{6.6}$$

Using trial and modification procedure the smallest root of (6.6) is given by

$$2\psi_1 = \alpha_1 L = 3.5909 \quad \text{where} \quad \alpha_1 = \sqrt{\frac{P_1}{EI_1}}$$

Therefore,

$$P_{cr} = (12.895)\frac{EI}{L^2} = \frac{\pi^2 EI}{(0.875L)^2}$$

It should be noted that the adjoining uncompressed member restrains the collapse of loaded span.

The procedure can be extended to a continuous beam-column having any number of spans. Consider a n-span continuous beam-column supported on $n + 1$ rigid supports $1, 2, 3, \ldots, n + 1$ with spans of length $L_1, L_2, L_3, \ldots, L_n$ and flexural

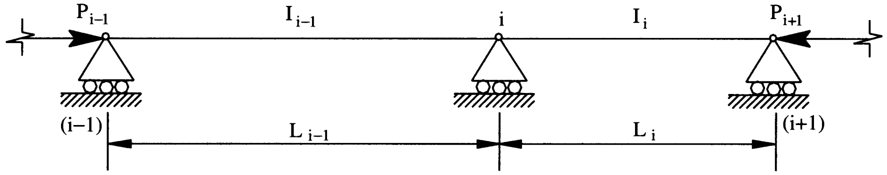


Fig. 6.2. Two consecutive spans of a continuous beam-column

rigidities of EI_1, EI_2, $EI_3, \ldots, EI_n$, respectively. The moments at the supports are denoted by $M_1, M_2, M_3, \ldots, M_{n+1}$. Consider two consecutive spans between supports $i-1$, i and $i+1$ as shown in Fig. 6.2. The continuity or compatibility condition at the intermediate support i requires the deflection curves of two spans to have the same tangent i.e. $\theta_i = \theta_i'$, where

$$\theta_i = \theta_{0i} + \left(\frac{M_{i-1}L_{i-1}}{6EI_{i-1}}\right)\varphi_2(\psi_{i-1}) + \left(\frac{M_i L_{i-1}}{3EI_{i-1}}\right)\varphi_1(\psi_{i-1})$$

$$\text{and}\quad \theta_i' = -\theta_{0i}' - \left(\frac{M_i L_i}{3EI_i}\right)\varphi_1(\psi_i) - \left(\frac{M_{i+1}L_i}{6EI_i}\right)\varphi_2(\psi_i)$$

Here, θ_{0i} and θ_{0i}' represent the rotations at the intermediate support i in the two adjacent spans due to lateral loads. The continuity condition $\theta_i = \theta_i'$, gives

$$\begin{aligned}
M_{i-1}\varphi_2(\psi_{i-1}) &+ 2M_i\left[\varphi_1(\psi_{i-1}) + \left(\frac{L_i}{L_{i-1}}\right)\left(\frac{I_{i-1}}{I_i}\right)\varphi_1(\psi_i)\right] \\
&+ M_{i+1}\left[\left(\frac{L_i}{L_{i-1}}\right)\left(\frac{I_{i-1}}{I_i}\right)\varphi_2(\psi_i)\right] \\
&= -\left(\frac{6EI_{i-1}}{L_{i-1}}\right)\left(\theta_{0i} + \theta_{0i}'\right)
\end{aligned} \tag{6.7}$$

Equation (6.7) is the general form of the *three-moment equation* at the interior support i. The moment quantities are positive when they cause compression at the top fibres of the beam-column. In applying the three-moment equation to a particular beam-column, the interior supports, such as 2, 3, 4, etc. are located successively and as many equations as the unknown redundant support moments are written. A simultaneous solution of the equations for the unknown moments yields the required result. The application of the method is illustrated in the following examples.

Example 6.1. A two-span continuous beam-column ABC of constant cross-section shown in Fig. 6.3 supports a uniformly distributed load of intensity w over the span BC. Estimate the moment at the support B, if the member is subjected to an axial thrust of magnitude $4EI/L^2$.

For this problem $M_{i-1} = M_{i+1} = 0$ and $M_i = M$. The quantities pertaining to the spans AB and BC are represented by the subscripts 1 and 2, respectively. Thus for the span AB,

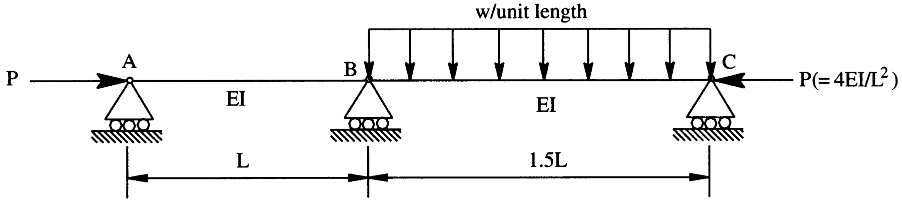


Fig. 6.3. Two-span continuous beam-column

$$\theta_{0B} = 0 \quad \text{and} \quad \psi_1 = \frac{\alpha L}{2} = \frac{\pi}{2}\sqrt{\frac{P}{P_e}} = \frac{\pi}{2}\left[\frac{(4EI/L^2)}{(\pi^2 EI/L^2)}\right]^{1/2} = 1.0$$

$$\varphi_1(\psi_1) = \left(\frac{3}{2\psi_1}\right)\left[\frac{1}{2\psi_1} - \frac{1}{\tan 2\psi_1}\right] = 1.4365$$

Similarly for the span BC

$$\psi_2 = \frac{\pi}{2}\left[\frac{(4EI/L^2)}{[\pi^2 EI/(1.5L)^2]}\right]^{1/2} = 1.5$$

$$\theta'_{OB} = \left[\frac{w(1.5L)^3}{24EI}\right]\left[\frac{3(\tan\psi_2 - \psi_2)}{\psi_2^3}\right] = \left[\frac{w(1.5L)^3}{24EI}\right](11.201) = 1.5751\left[\frac{wL^3}{EI}\right]$$

$$\varphi_1(\psi_2) = \left(\frac{3}{2\psi_2}\right)\left[\frac{1}{2\psi_2} - \frac{1}{\tan 2\psi_2}\right] = 7.3486$$

Substituting these values in (6.7)

$$2M\left[(1.4365) + \left(\frac{1.5L}{L}\right)\left(\frac{I}{I}\right)(7.3486)\right] = -\frac{6EI}{L}\left(\frac{1.5751wL^3}{EI}\right)$$

Therefore, $M = (0.3793)wL^2$

The value of support moment in the absence of axial thrust is $0.1125wL^2$. Thus the support moment increases by 237.16 per cent due to the presence of axial force.

Example 6.2. A two-span continuous beam-column is clamped at the end C and carries an axial thrust P as shown in Fig. 6.4. Determine P_{cr}, the first critical value of P that will cause the beam-column to buckle.

In this case there are two redundant moments M_B and M_C which require the application of three moment's equation at B and C.

$$2M_B\left[\varphi_1(\psi_1) + \left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\right\}\varphi_1(\psi_2)\right] + M_C\left[\left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\right\}\varphi_2(\psi_2)\right]$$
$$= -\frac{6EI_1}{L_1}\left(\theta_{OB} + \theta'_{OB}\right)$$

$$M_B\varphi_2(\psi_2) + 2M_C\varphi_1(\psi_2) = -\left(\frac{6EI_2}{L_2}\right)\theta_{OC}$$

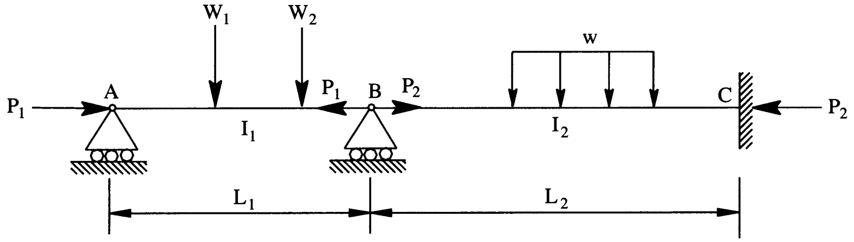


Fig. 6.4. Two-span continuous beam-column with clamped end

In the matrix form these equation can be expressed as

$$\begin{bmatrix} 2\left[\varphi_1(\psi_1)+\left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\varphi_1(\psi_2)\right\}\right] & \left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\varphi_2(\psi_2)\right\} \\ \varphi_2(\psi_2) & 2\,\varphi_1(\psi_2) \end{bmatrix} \begin{Bmatrix} M_B \\ M_C \end{Bmatrix} = -\begin{Bmatrix} \frac{6EI_1}{L_1}\left(\theta_{OB}+\theta'_{OB}\right) \\ \frac{6EI_2}{L_2}\theta_{OC} \end{Bmatrix} \tag{6.8}$$

At critical load the redundant moments approach infinity i.e. the determinant of matrix of coefficients on left hand side of (6.8) must vanish. Therefore,

$$\begin{vmatrix} 2\left[\varphi_1(\psi_1)+\left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\varphi_1(\psi_2)\right\}\right] & \left\{\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\varphi_2(\psi_2)\right\} \\ \varphi_2(\psi_2) & 2\,\varphi_1(\psi_2) \end{vmatrix} = 0 \tag{6.9}$$

The expansion of determinant gives the characteristic equation. It should be noted that the critical load is independent of lateral load acting on the beam-column. As a typical case consider

$$L_1 = 2L_2/3 = L\ ; \quad I_1 = I_2 = I \quad \text{and} \quad P_1 = P_2 = P$$

Therefore,

$$\left[\left(\frac{L_2}{L_1}\right)\left(\frac{I_1}{I_2}\right)\right] = 1.5\ ; \quad \psi_1 = \frac{2\psi_2}{3} = \psi \quad \text{since} \quad \alpha_1 = \alpha_2 = \sqrt{\frac{P}{EI}}$$

The characteristic equation obtained by the expansion of determinant is:

$$4[\varphi_1(\psi) + 1.5\varphi_1(1.5\psi)]\,\varphi_1(1.5\psi) - 1.5[\varphi_2(1.5\psi)]^2 = 0$$

Substituting (5.18) and by trial and modification the smallest root of the characteristic equation is $2\psi = 5.499$. Therefore,

$$2\psi = \alpha L = 5.499$$

Hence,
$$P_{cr} = (5.499)^2 \frac{EI}{L^2} = \frac{\pi^2 EI}{(0.571L)^2}$$

6.2.2 Rigid-Frames

The buckling of a rigid-jointed frame implies the buckling of its compression members. In simple cases, the beam-column members with end moments can be easily isolated. The results for the *beam-column* derived in Chap. 5 can be readily applied to determine the critical load at the buckling for this isolated framed member. For illustration consider the rigid frame $ABCD$ shown in Fig. 6.5a. The free-body diagram of the isolated member AB is shown in Fig. 6.5b. This member can be treated as a beam-column subjected to an end moment M_0 and axial thrust P at an eccentricity e as shown in Fig. 6.5b. Using results derived earlier in (5.17), the rotation θ_0 at the joint B is given by (6.9).

$$\theta_0 = \frac{e}{L}(\alpha L \operatorname{cosec} \alpha L - 1) + \frac{M_0}{PL}(\alpha L \cot \alpha L - 1) \tag{6.10}$$

For the beam element BC

$$M_0 = \frac{2EI_1}{L_1}\theta_0 \tag{6.11}$$

Eliminating M_0 from (6.10) and (6.11)

$$\theta_0 = \frac{(e/L)(\alpha L \operatorname{cosec} \alpha L - 1)}{(1/PL)(2EI_1/L_1)(1 - \alpha L \cot \alpha L) + 1} \tag{6.12}$$

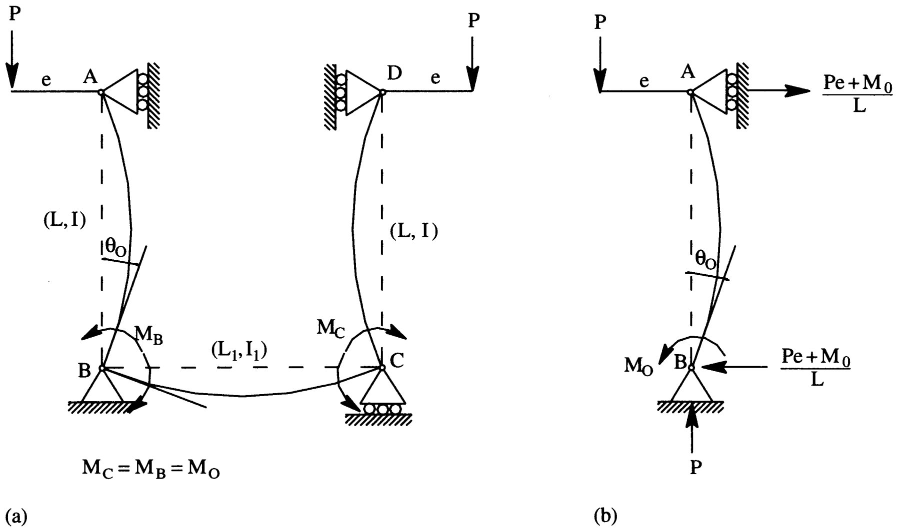


Fig. 6.5a,b. Rigid frame subjected to end moments and axial thrust. **a** Frame, **b** isolated beam-column

At the buckling of the frame rotation becomes very large i. e. it tends to infinity. This occurs when the denominator of (6.12) vanishes, that is

$$(\alpha L \cot \alpha L - 1) = \frac{PLL_1}{2EI_1} = \frac{1}{2}\left(\frac{P}{EI}\right)\left[\left(\frac{I}{I_1}\right)\left(\frac{L_1}{L}\right)\right]L^2$$
$$= \frac{1}{2}(\alpha L)^2\left[\left(\frac{I}{I_1}\right)\left(\frac{L_1}{L}\right)\right]$$

Thus

$$\alpha L \cot \alpha L = 1 + \frac{1}{2}(\alpha L)^2\left[\left(\frac{I}{I_1}\right)\left(\frac{L_1}{L}\right)\right] \tag{6.13}$$

For a typical case where $I/L = I_1/L_2$, (6.13) reduces to

$$\cot \alpha L = \frac{1}{\alpha L} + \frac{\alpha L}{2} \tag{6.14}$$

By trial and modification, the lowest root of transcendental equation is given by $\alpha L = 3.59$. Therefore,

$$P_{\mathrm{cr}} = (3.59)^2\frac{EI}{L^2} = \frac{\pi^2 EI}{(0.875L)^2}$$

In another variation of the above problem consider the symmetric closed frame shown in Fig. 6.6 wherein lateral joint movement is prevented. When the axial thrust attains the critical value, the columns AB and CD tend to deflect laterally as shown in the figure, resulting in bending of the beams AC and BD which in turn apply restraining moments at the column ends. Thus these compression members may be treated as columns with elastic restraints. The rotation at the ends of the columns are given by (5.20)

$$\theta_A = \frac{M_A L}{3EI}\varphi_1(\psi) + \frac{M_B L}{6EI}\varphi_2(\psi) \quad \text{and}$$
$$\theta_B = \frac{M_A L}{6EI}\varphi_2(\psi) + \frac{M_B L}{3EI}\varphi_1(\psi) \tag{6.15}$$

where

$$\varphi_1(\psi) = \frac{3}{2\psi}\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) \quad \text{and}$$
$$\varphi_2(\psi) = \frac{6}{2\psi}\left(\frac{1}{\sin 2\psi} - \frac{1}{2\psi}\right)$$

In which $2\psi = \alpha L = \pi\sqrt{P/P_{\mathrm{e}}}$ and $P_{\mathrm{e}} = \pi^2 EI/L^2$. In the present problem due to symmetry $M_B = -M_A = M_0$ and $\theta_B = -\theta_A = \theta_0$. For compatibility, the rotation θ_0 of the column must be the same as that of horizontal member which is given by

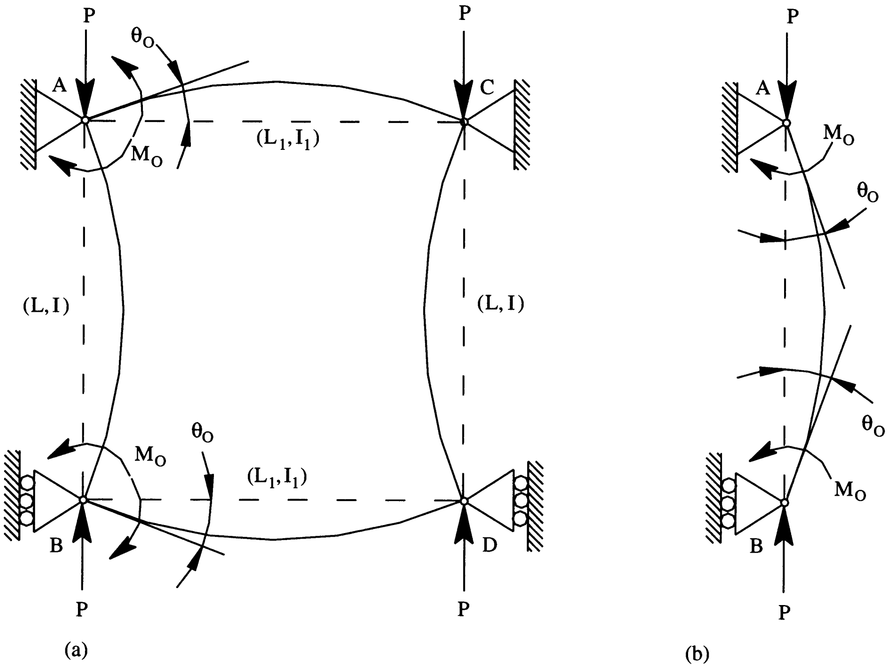


Fig. 6.6a,b. Non-sway buckling mode of closed frame. **a** Closed frame, **b** isolated beam column

$$\theta_0 = -\frac{M_0 L_1}{2EI_1}$$

Substituting this value of θ_0 in any one of (6.15) e.g.,

$$\theta_0 = -\frac{M_0 L_1}{2EI_1} = \frac{M_0 L}{6EI}[2\varphi_1(\psi) + \varphi_2(\psi)]$$

$$= \left(\frac{M_0 L}{6EI}\right)\left(\frac{6}{2\psi}\right)\left(\frac{1}{\sin 2\psi} - \frac{1}{\tan 2\psi}\right)$$

or

$$\frac{1}{2\psi}\left(\frac{1}{\sin 2\psi} - \frac{1}{\tan 2\psi}\right) = -\frac{1}{2}\left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right] \tag{6.16}$$

For a typical case where the members of the frame are identical i.e. $L_1 = L$ and $I_1 = I$, (6.16) reduce to:

$$\frac{1}{2\psi}\left(\frac{1}{\sin 2\psi} - \frac{1}{\tan 2\psi}\right) = -\frac{1}{2}$$

$$\text{or} \quad \tan\psi = -\psi \tag{6.17}$$

The lowest root of this transcendental equation is given by

$$\psi = \alpha L/2 = 2.02916$$

Therefore,

$$P_{\mathrm{cr}} = \frac{(4.0583)^2 EI}{L^2} = \frac{16.47 EI}{L^2} = \frac{\pi^2 EI}{(0.774L)^2}$$

In the preceding discussion the lateral joint displacement of the structures analysed has been prevented. The following example will illustrate the procedure to determine the buckling load of the frames undergoing lateral displacement or sway.

Example 6.3. The portal frame shown in Fig. 6.7 is subjected to axial load P. Determine the critical value of load P if the joints B and C are allowed to undergo lateral movement (sway). The deflected configuration is shown in the figure.

The governing differential equation for the vertical member can be written as

$$EI\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = P(\delta - y) - M$$

$$\text{or} \quad EI\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) + Py = P\delta - M \tag{6.18}$$

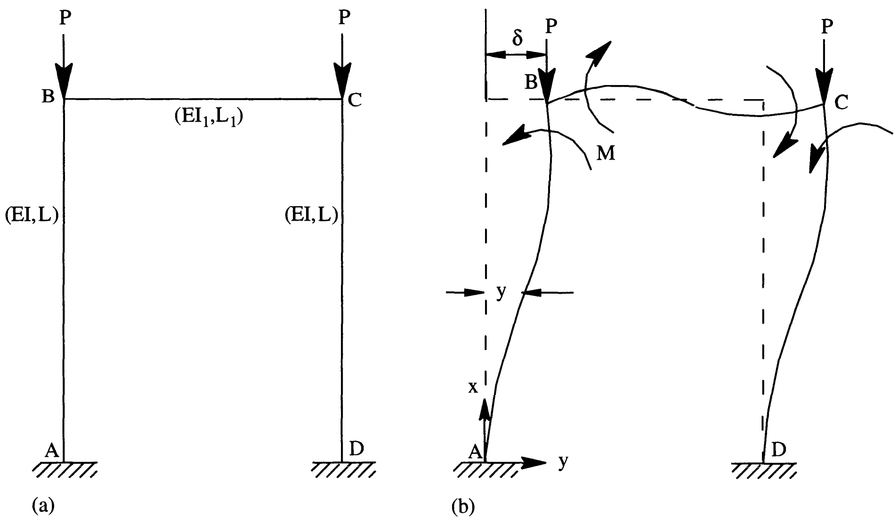


Fig. 6.7a,b. Buckling of a fixed base portal frame. **a** Symmetrical portal, **b** sway buckling mode

This is second-order non-homogeneous ordinary differential equation with constant coefficients. Its solution can be expressed as

$$y = A \sin \alpha x + B \cos \alpha x + \left(\delta - \frac{M}{P}\right) \tag{6.19}$$

where $\alpha^2 = (P/EI)$. The boundary conditions at $x = 0$ i.e. $y(0) = 0$ and $y'(0) = 0$ give

$$B = -\left[\delta - \frac{M}{P}\right] \quad \text{and} \quad A = 0$$

Thus the elastic curve is given by

$$y = \left[\delta - \frac{M}{P}\right](1 - \cos \alpha x) \tag{6.20}$$

The unknown δ and M can be evaluated by the additional boundary conditions at $x = L$

$$y(L) = \delta \quad \text{and} \quad y'(L) = \frac{ML_1}{6EI_1}$$

These conditions lead to

$$\delta \cos \alpha L + \frac{M}{P}(1 - \cos \alpha L) = 0$$

$$\text{and} \quad \delta \alpha \sin \alpha L - \frac{M}{P}\left(\alpha \sin \alpha L + \frac{L_1 P}{6EI_1}\right) = 0 \tag{6.21}$$

For non-trivial solution, the determinant of coefficients of δ and M/P must vanish. That is

$$\begin{vmatrix} \cos \alpha L & (1 - \cos \alpha L) \\ \alpha \sin \alpha L & -\left(\alpha \sin \alpha L + \dfrac{L_1 P}{6EI_1}\right) \end{vmatrix} = 0$$

$$\text{or} \quad \frac{\tan \alpha L}{\alpha L} = -\frac{1}{6}\left[\left(\frac{I}{I_1}\right)\left(\frac{L_1}{L}\right)\right] \tag{6.22}$$

For the given geometry, the transcendental equation can be solved for the critical load at buckling. For the typical case where $(L_1/I_1)/(L/I) = 1.0$, (6.22) reduces to

$$\frac{\tan \alpha L}{\alpha L} = -\frac{1}{6}$$

By trial and modification the smallest root obtained is $\alpha L = 5.53783$ and the first critical load is given by:

$$P_{cr} = \frac{(5.53783)^2 EI}{L^2} = \frac{30.668\, EI}{L^2} = \frac{\pi^2 EI}{(0.5673L)^2}$$

The following example will illustrate the application of the method to a hinged base portal frame.

Example 6.4. A symmetrical portal frame hinged at the base is subjected to an axial thrust P as shown in Fig. 6.8a. Determine the critical value of load P at the buckling of the frame.

In view of the symmetry of the structure and loading, the frame has been analysed for both the symmetric and antisymmetric buckling modes.

(i) Symmetric mode: In this mode, the joints B and C do not move but undergo the rotations. The rotations and hence the moments at these joints are equal in magnitude and cause compression at the top of the beam i.e. $M_B = -M_C = M$. As the ends A and D are hinged the support moments are zero.

Application of (6.7) to the members AB and BC yields

$$2M_B\left[\varphi_1(\psi) + \left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\varphi_1(\psi_1)\right] + M_C\left[\left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\varphi_2(\psi_1)\right] = 0$$

$$2M\varphi_1(\psi) + \left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right][2\varphi_1(\psi_1) + \varphi_2(\psi_1)]\ M = 0 \qquad (6.23)$$

Application of (6.7) to the members BC and CD results in the same (6.23) since the structure is symmetric. Further as $M \neq 0$, for non-trivial solution (6.23) reduces to

$$2\varphi_1(\psi) + \left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right][2\varphi_1(\psi_1) + \varphi_2(\psi_1)] = 0 \qquad (6.24)$$

As there is no axial load in the member BC, $\psi_1 = \frac{\pi}{2}\sqrt{P/P_e} = 0$ and hence $\varphi_1(\psi_1) = \varphi_2(\psi_1) = 1.00$, and (6.24) gives

$$\frac{3}{2\psi}\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) + \frac{3}{2}\left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right] = 0 \qquad (6.25)$$

For the given geometry of frame P_{cr} can be obtained from the transcendental equation (6.25). For the typical case where $[(L_1/L)(I/I_1)] = 1.0$, $2\psi = 2.59$. The critical

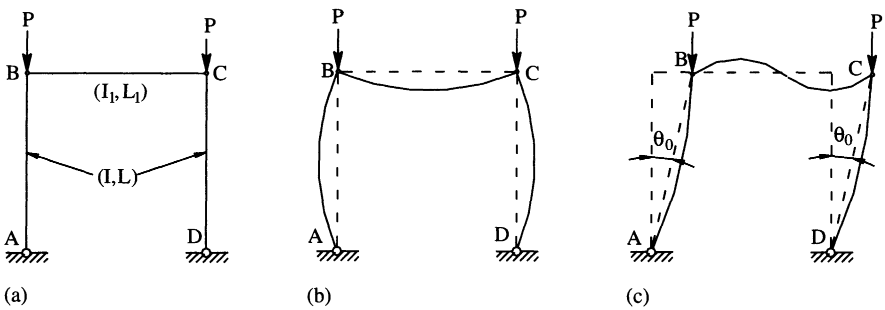


Fig. 6.8a–c. Buckling of a portal frame hinged at the supports. **a** Symmetrical portal frame, **b** symmetrical buckling mode, **c** anti-symmetrical buckling mode

load, P_{cr} is given by

$$P_{cr} = \frac{(2.59)^2 EI}{L^2} = \frac{\pi^2 EI}{(0.875L)^2}.$$

(ii) Antisymmetric Mode: In this mode, the joints B and C undergo lateral displacement and hence the vertical members AB and CD undergo rigid body rotation θ_0 as shown in Fig. 6.8c. Further, the moments at B and C are same, M. The application of (6.7) to the members AB and BC gives

$$2M_B\left[\varphi_1(\psi) + \left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\varphi_1(\psi_1)\right] + M_C\left[\left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\varphi_2(\psi_1)\right] = \left(\frac{6EI}{L}\right)\theta_0 \tag{6.26}$$

Consider the equilibrium of vertical member AB as a free body

$$M_B = PL\theta_0 \quad \text{or} \quad \theta_0 = (M_B/PL)$$

Therefore,

$$M\left[2\varphi_1(\psi) + \left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\{2\varphi_1(\psi_1) - \varphi_2(\psi_1)\} - \left(\frac{6EI}{PL^2}\right)\right] = 0$$

For non-trivial solution

$$2\varphi_1(\psi) + \left\{\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right\}\{2\varphi_1(\psi_1) - \varphi_2(\psi_1)\} - \left(\frac{6EI}{PL^2}\right) = 0 \tag{6.27}$$

Here

$$\frac{6EI}{PL^2} = \frac{6}{\pi^2}\left(\frac{(\pi^2 EI/L^2)}{P}\right) = \left(\frac{6}{\pi^2}\right)\left(\frac{P_e}{P}\right) = \frac{6}{(2\psi)^2} \quad \left(\text{since } 2\psi = \pi\sqrt{\frac{P}{P_e}}\right)$$

As there is no axial force in the member BC i.e. $\psi_1 = 0$ and hence $\varphi_1(\psi_1) = \varphi_2(\psi_1) = 1.0$. Equation (6.27) reduces to

$$\frac{6}{2\psi}\left(\frac{1}{2\psi} - \frac{1}{\tan 2\psi}\right) + \left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right] - \frac{6}{(2\psi)^2} = 0$$
$$\text{or} \quad \frac{\cot 2\psi}{2\psi} - \frac{1}{6}\left[\left(\frac{L_1}{L}\right)\left(\frac{I}{I_1}\right)\right] = 0 \tag{6.28}$$

This transcendental equation can be used to determine the critical load at the buckling. For a typical case when $[(L_1/L)(I/I_1)] = 1$, the lowest root of the equation is: $2\psi = 1.35$. Therefore

$$P_{cr} = \frac{(1.35)^2 EI}{L^2} = \frac{1.822EI}{L^2} = \frac{\pi^2 EI}{(2.327L)^2}$$

6.3 Semi-Geometrical Approach

This approach suggested by Haarman, uses the knowledge of buckled configuration of the frame structure made up of axial loaded straight members. The method is based on the observation that the elastic curve of an axially loaded, originally straight member can be described by a sine curve with respect of a rectangular coordinate system having origin at one flex point and one axis directed through the other flex point. Application of boundary or compatibility conditions will provide the critical value of the load that will cause the structure to buckle.

Consider the case of a fixed-hinged strut AB shown in Fig. 6.9. One flex point is at the hinged support A and the other C is in the column at distance kL (i. e. effective length of the strut). With point A as origin, direct the x-axis through the point C making an infinitesimally small angle θ with initial strut axis. x-axis makes an offset of δ at the fixed support. The y-axis is assumed to be normal to x-axis.

With the increase in value of axial load P, the strut starts deflecting and at buckling the resultant force on the strut P_{cr} becomes inclined since the line of action P must pass through the two flex points. For infinitesimal deformation $P \approx P_{\text{cr}}$, the equation for the elastic curve between the points A and C can be expressed as

$$y(x) = A \sin\left(\frac{\pi x}{kL}\right) \tag{6.29}$$

and hence

$$y'(x) = \left(\frac{A\pi}{kL}\right) \cos\left(\frac{\pi x}{kL}\right) \tag{6.30}$$

The geometric boundary conditions are

$$y(L) = -\delta = A \sin\left(\frac{\pi}{k}\right) \quad \text{i.e.} \quad A = -\frac{\delta}{\sin(\pi/k)} \tag{6.31}$$

$$y'(L) = -\theta = -\left(\frac{\delta}{L}\right) = \left(\frac{A\pi}{kL}\right) \cos\left(\frac{\pi}{k}\right) \tag{6.32}$$

Substituting (6.31) into (6.32)

$$\frac{\delta}{L} = -\left(\frac{A\pi}{kL}\right) \cos\left(\frac{\pi}{k}\right) = \frac{\delta}{\sin(\pi/k)} \left(\frac{\pi}{kL}\right) \cos\left(\frac{\pi}{k}\right)$$

$$\text{or} \quad \tan\left(\frac{\pi}{k}\right) = \frac{\pi}{k}$$

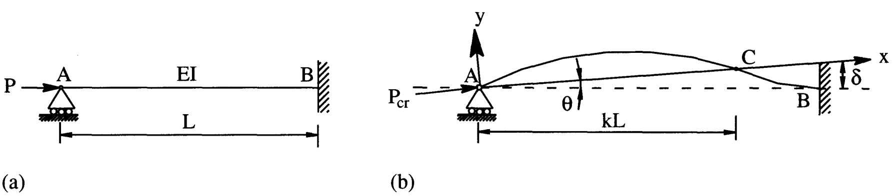


Fig. 6.9a,b. Buckling load by Haarman method. **a** Fixed-hinge strut, **b** buckling mode

By trial and modification $(\pi/k) = 4.4934$ or $k \approx 0.6992 = 0.7$. Therefore,

$$P_{cr} = \frac{\pi^2 EI}{(0.7L)^2} = \frac{20.191\, EI}{L^2}$$

The method is equally applicable to rigid frames.

Example 6.5. In the symmetrically loaded portal frame shown in Fig. 6.10, the columns have same area of cross-section which is different from that of beam. Determine the critical value of load P that will cause the frame to buckle. Use Haarman's semi geometrical method.

The frame can buckle in two different modes namely, symmetrical mode without side sway and antisymmetrical mode with side sway.

(i) Symmetrical Mode: The symmetrical buckling configuration is shown in Fig. 6.10a where B' and C' represent the flex points. The elastic curve AB' can be represented by

$$y(x) = A \sin\left(\frac{\pi x}{kL}\right) \quad \text{and} \quad y'(x) = \left(\frac{A\pi}{kL}\right) \cos\left(\frac{\pi x}{kL}\right) \tag{6.33}$$

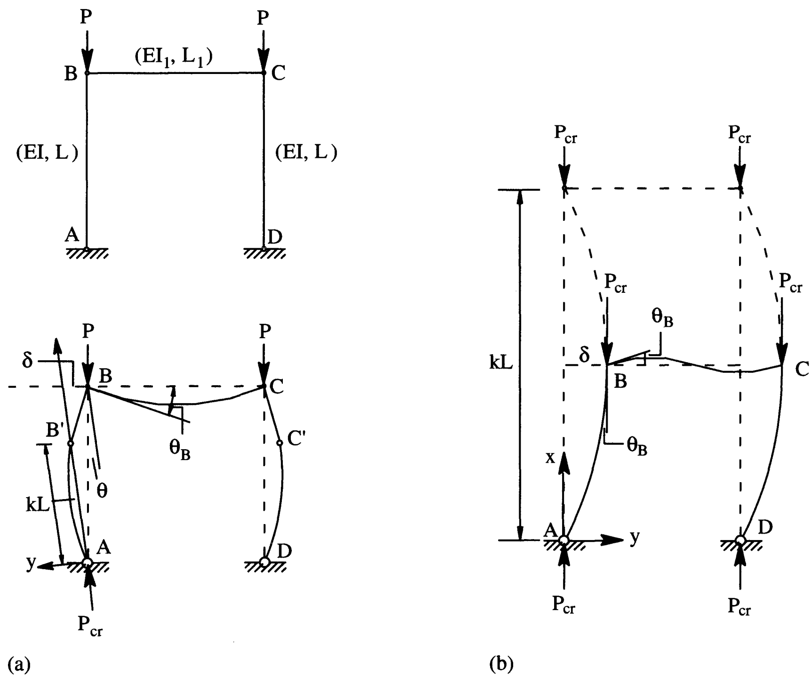


Fig. 6.10a,b. Buckling of a portal hinged at the support. **a** Buckling without sidesway, **b** buckling with sidesway

The boundary conditions are

$$y(L) = A\sin\left(\frac{\pi}{k}\right) = -\delta$$

$$\text{and}\quad y'(L) = -\theta - \theta_B = \frac{\pi A}{kL}\cos\left(\frac{\pi}{k}\right)$$

where

$$\theta = \frac{\delta}{L} = -\frac{A}{L}\sin\left(\frac{\pi}{k}\right) \quad \text{and thus}$$

$$\theta_B = -\theta - \frac{\pi A}{kL}\cos\left(\frac{\pi}{k}\right) = \frac{A}{L}\sin\left(\frac{\pi}{k}\right) - \frac{\pi A}{kL}\cos\left(\frac{\pi}{k}\right) \tag{6.34}$$

For the horizontal beam with two end moments ($M_C = -M_B$)

$$\theta_B = \frac{M_B L_1}{2EI_1} \quad \text{with} \quad M_B = P_{cr}\delta$$

Therefore,

$$\theta_B = \frac{P_{cr}\delta L_1}{2EI_1} = -\frac{P_{cr}L_1 A\sin(\pi/k)}{2EI_1} \tag{6.35}$$

Equating θ_B from (6.34) and (6.35),

$$-\frac{P_{cr}L_1 A\sin(\pi/k)}{2EI_1} = \frac{A}{L}\sin\left(\frac{\pi}{k}\right) - \frac{\pi A}{kL}\cos\left(\frac{\pi}{k}\right) \quad \text{or}$$

$$\left(\frac{\pi}{k}\right)\cot\left(\frac{\pi}{k}\right) = 1 + \frac{1}{2}\left(\frac{\pi^2}{k^2}\right)\left[\left(\frac{I}{L}\right)\left(\frac{L_1}{I_1}\right)\right] \quad \text{since} \quad P_{cr} = \frac{\pi^2 EI}{(kL)^2} \tag{6.36}$$

For a typical case with $I_1 = I$ and $L_1 = L$, i.e. $(IL_1/I_1L) = 1$, the characteristic equation reduces to

$$\left(\frac{\pi}{k}\right)\cot\left(\frac{\pi}{k}\right) = 1 + \frac{1}{2}\left(\frac{\pi}{k}\right)^2 \tag{6.37}$$

By trial and modification, $(\pi/k) = 3.591$, i.e. $k = 0.8749$ and

$$P_{cr} = \frac{\pi^2 EI}{(0.8749L)^2} = \frac{12.895EI}{L^2}\,.$$

(ii) Antisymmetrical Mode: The antisymmetrical buckling configuration is shown in Fig. 6.10b. For the beam with two end moments ($M_C = M_B$)

$$\theta_B = \frac{(M_B L_1)}{6EI_1} = \frac{(P_{cr}\delta)L_1}{6EI_1} = \left[\frac{\pi^2 EI}{(kL)^2}\right]\left[\frac{\delta L_1}{6EI_1}\right]$$

$$= \left(\frac{\pi}{k}\right)^2\left(\frac{IL_1}{I_1L}\right)\left(\frac{\delta}{6L}\right) \tag{6.38}$$

The equation to elastic curve is assumed to be

$$y = A\sin\left(\frac{\pi x}{kL}\right)$$

For the column the boundary conditions are

$$\begin{aligned} y(L) &= \delta = A\sin\left(\frac{\pi}{k}\right) \\ y'(L) &= \left(\frac{\pi A}{kL}\right)\cos\left(\frac{\pi}{k}\right) = \theta_B \end{aligned} \tag{6.39}$$

Equating θ_B from (6.38) and (6.39)

$$\frac{A\pi}{kL}\cos\left(\frac{\pi}{k}\right) = \left(\frac{\pi}{k}\right)^2\left(\frac{IL_1}{I_1L}\right)\left[\left(\frac{A}{6L}\right)\sin\left(\frac{\pi}{k}\right)\right]$$

$$\cot\left(\frac{\pi}{k}\right) = \left(\frac{\pi}{k}\right)\left(\frac{1}{6}\right)\left(\frac{IL_1}{I_1L}\right) \tag{6.40}$$

For the typical case $I_1/L_1 = I/L$ i.e. $IL_1/(I_1L) = 1$, the characteristic equation reduces to

$$\cot\left(\frac{\pi}{k}\right) = \left(\frac{1}{6}\right)\left(\frac{\pi}{k}\right) \tag{6.41}$$

By trial and modification, $\pi/k = 1.3495$ i.e. $k = 2.328$. Therefore,

$$P_{cr} = \frac{\pi^2 EI}{(2.328L)^2} = \frac{1.821EI}{L^2}$$

Example 6.6. In the closed-frame shown in Fig. 6.11, the vertical members have the same EI values and are subjected to equal axial loads. The EI values of horizontal members are different from columns as shown in the figure. Determine the critical value of load P that will cause the frame to buckle when it is restrained from undergoing any horizontal movement.

Due to the symmetry of the structure the beam end moments are equal and opposite i.e. $M_D = -M_A$ and $M_C = -M_B$. Therefore,

$$\theta_1 = \frac{P_{cr}\delta_1 L_1}{2EI_1} \quad \text{and} \quad \theta_2 = \frac{P_{cr}\delta_2 L_1}{2EI_1} \tag{6.42}$$

where

$$P_{cr} = \frac{\pi^2 EI}{(kL)^2} = \left(\frac{\pi}{k}\right)^2\left(\frac{EI}{L^2}\right)$$

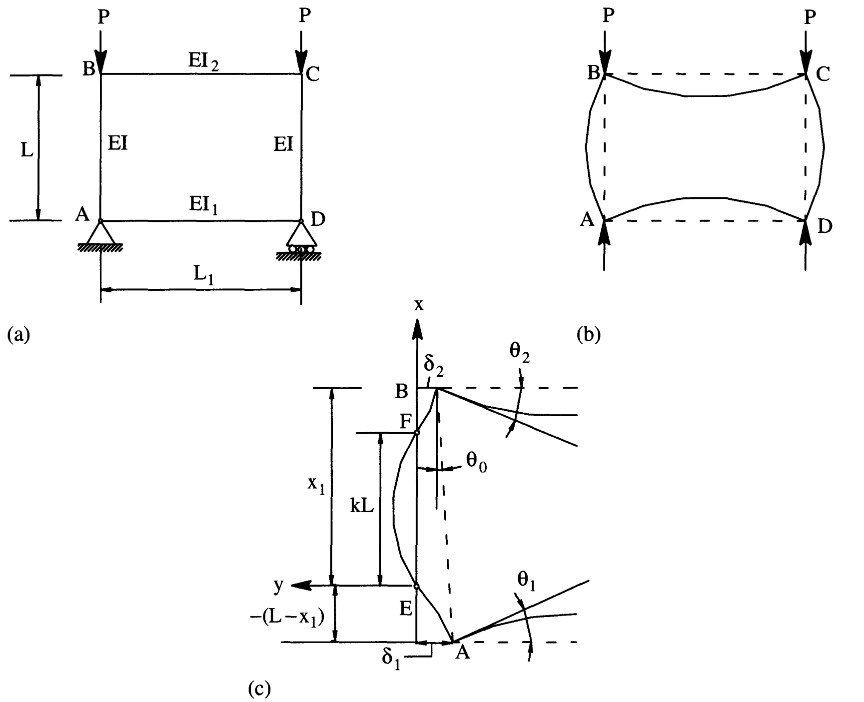


Fig. 6.11a–c. Nonsway buckling of a closed frame. **a** Closed frame, **b** nonsway buckling mode, **c** geometry of elastic curve

The equation of elastic curve between two flex points E and F can be expressed as

$$y = A \sin\left(\frac{\pi x}{kL}\right)$$

Therefore,

$$y' = \left(\frac{A\pi}{kL}\right) \cos\left(\frac{\pi x}{kL}\right) \tag{6.43}$$

Hence, the offsets at the level of column supports A and B are

$$\begin{aligned} \delta_2 &= -A \sin\left(\frac{\pi x_1}{kL}\right) \\ \delta_1 &= -A \sin\left(\frac{\pi(x_1 - L)}{kL}\right) \\ &= -A \left\{\sin\left(\frac{\pi x_1}{kL}\right) \cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi x_1}{kL}\right) \sin\left(\frac{\pi}{k}\right)\right\} \end{aligned} \tag{6.44}$$

For small deformation, $\theta_0 = (\delta_2 - \delta_1)/L$. Therefore,

$$\theta_0 = -\frac{A}{L}\left[\left\{1 - \cos\left(\frac{\pi}{k}\right)\right\} \sin\left(\frac{\pi x_1}{kL}\right) + \sin\left(\frac{\pi}{k}\right) \cos\left(\frac{\pi x_1}{kL}\right)\right] \tag{6.45}$$

The slopes of the beams at the joints A and B are

$$\theta_1 = y'(x_1 - L) + \theta_0 \quad \text{and} \quad \theta_2 = -y'(x_1) - \theta_0 \tag{6.46}$$

Substituting for $y'(x)$ and θ_0 from (6.43) and (6.45) into (6.46):

$$\begin{aligned}\theta_1 = \frac{A}{L}\Bigg[&\left\{\left(\frac{\pi}{k}\right)\sin\left(\frac{\pi}{k}\right) - 1 + \cos\left(\frac{\pi}{k}\right)\right\}\sin\left(\frac{\pi x_1}{kL}\right)\\ &+\left\{\left(\frac{\pi}{k}\right)\cos\left(\frac{\pi}{k}\right) - \sin\left(\frac{\pi}{k}\right)\right\}\cos\left(\frac{\pi x_1}{kL}\right)\Bigg]\end{aligned} \tag{6.47}$$

$$\theta_2 = \frac{A}{L}\left[\left\{1 - \cos\left(\frac{\pi}{k}\right)\right\}\sin\left(\frac{\pi x_1}{kL}\right) + \left\{\sin\left(\frac{\pi}{k}\right) - \left(\frac{\pi}{k}\right)\right\}\cos\left(\frac{\pi x_1}{kL}\right)\right] \tag{6.48}$$

Substituting δ_1 and δ_2 from (6.44) into (6.42)

$$\theta_1 = -\frac{A}{L}\left[\frac{1}{2}\left(\frac{\pi}{k}\right)^2\left(\frac{L_1 I}{L I_1}\right)\left\{\sin\left(\frac{\pi x_1}{kL}\right)\cos\left(\frac{\pi}{k}\right) - \cos\left(\frac{\pi x_1}{kL}\right)\sin\left(\frac{\pi}{k}\right)\right\}\right] \tag{6.49}$$

$$\theta_2 = -\frac{A}{L}\left[\frac{1}{2}\left(\frac{\pi}{k}\right)^2\left(\frac{L_1 I}{L I_2}\right)\sin\left(\frac{\pi x_1}{kL}\right)\right] \tag{6.50}$$

Eliminating θ_1 and θ_2 from (6.47) to (6.50) and for a non-trivial solution i. e.

$$\sin\left(\frac{\pi x_1}{kL}\right) = \cos\left(\frac{\pi x_1}{kL}\right) \neq 0$$

Following eigen value equation is obtained.

$$\begin{aligned}&\frac{1}{4}\left(\frac{L_1}{L}\right)^2\left(\frac{I^2}{I_1 I_2}\right)\left(\frac{\pi}{k}\right)^2 + I\left(\frac{L_1}{L}\right)\left[\frac{(I_1+I_2)}{I_1 I_2}\right]\left[1 - \left(\frac{\pi}{k}\right)\cot\left(\frac{\pi}{k}\right)\right]\\ &+\left[\tan\left(\frac{\pi}{2k}\right)\Big/\left(\frac{\pi}{2k}\right)\right] = 1\end{aligned} \tag{6.51}$$

As a typical case consider $L_1 = L$ and $I_2 = I_1 = I$, the characteristic equation reduces to

$$\frac{1}{4}\left(\frac{\pi}{k}\right)^2 + 2\left[1 - \left(\frac{\pi}{k}\right)\cot\left(\frac{\pi}{k}\right)\right] + \left[\tan\left(\frac{\pi}{2k}\right)\Big/\left(\frac{\pi}{2k}\right)\right] = 1 \tag{6.52}$$

By trial and modification

$$\left(\frac{\pi}{k}\right) = 4.2098 \quad \text{and hence} \quad k = 0.74626$$

Therefore,

$$P_{\text{cr}} = \frac{\pi^2 EI}{(0.74626L)^2} = \frac{17.722EI}{L^2}$$

6.4 Stiffness Method

The classical differential equation and semi-geometrical approaches discussed in the preceding sections are complex and can conveniently be used only for the analysis of simple frames. The more general basic *stiffness* (or *displacement*) approach which considers the equilibrium of forces or moments provides an extremely powerful tool for the stability analysis of framed structures. The stiffness formulation normally used in matrix form yields unknown nodal displacements which are frequently referred to as *kinematic redundant*. For the purpose of this section it is presumed that the reader has the basic knowledge of matrix stiffness or displacement method.

6.4.1 Criterion for Determination of Critical Load

Consider the rigid-jointed structure shown in Fig. 6.12a. The moment M_0 acting on the joint 0 causes the joint to rotate by an angle θ_0. The stiffness of the joint 0 is given by

$$k_0 = k_{01} + k_{02} + k_{03} \tag{6.53}$$

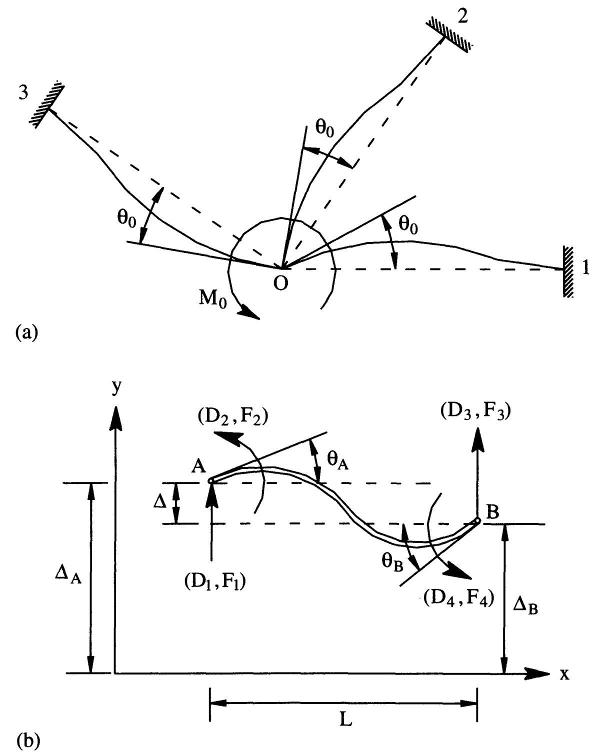


Fig. 6.12a,b. General displacements and signs convention. **a** Rotation at a rigid joint, **b** general displacement and forces

Therefore,

$$\theta_0 = M_0/k_0$$

In the above discussion it is presumed that there is no axial load present in the members of the structure. The effect of axial compression is to reduce the stiffness of the member. As the applied loads on the structure increase, the member forces increase and the overall resistance of the structure to any random disturbance decreases. At the critical load, the structure offers no resistance to the disturbance and the configuration of the structure is not unique i.e. any displaced position may be maintained without additional load. In the above referred rigid-jointed frame, if the loading is increased continuously, k_0 decreases and θ_0 continues to increase till at some multiple N_e of the working load (i.e. at critical load) the frame will collapse because of elastic instability at the joint O. At this stage the rotation θ_0 becomes infinite. This suggests a criterion for *elastic instability*, viz, that at the critical load displacements (rotation in this case) increase infinitely. For this condition to take place, the stiffness of the joint must reduce to a vanishingly small value.

For a structure with several rigid-joints, it is necessary to formulate the *stiffness matrix* of the entire structure. However, it should be noted that the structure stiffness matrix $[\bar{K}]$ for the structure in which members are subjected to axial load is different from the conventional stiffness matrix $[\bar{K}]$. The relationship of the externally applied loading to the displacements can be expressed as

$$\{F\} = [\bar{K}]\{D\}$$

$$\text{or} \quad \{D\} = [\bar{K}]^{-1}\{F\} = \left[\frac{\mathrm{adj}[\bar{K}]}{|\bar{K}|}\right]\{F\} \tag{6.54}$$

where

$$\begin{aligned} \{F\} &= \text{vector of joint loads} \\ \{D\} &= \text{joint displacements and} \\ [\bar{K}] &= \text{structure stiffness matrix.} \end{aligned}$$

It should be noted that in the solution for displacement $\{D\}$, the denominator will always be determinant of the stiffness matrix $[\bar{K}]$. For any displacement to become infinitely large, $|\bar{K}|$ must vanish and this condition means that every other displacement in the frame must also tend to infinity. Therefore, for elastic instability the condition is $|\bar{K}| = 0$ i.e. the stiffness matrix is *singular.* This equation usually referred to as characteristic equation may admit several different solutions of elastic instability *load factor* N_e, but smallest of these is of course, the value usually required. The higher eigen-values correspond to different types of external restraints acting on the structure and are therefore invalid unless these restraints can exist. Therefore, a solution should be checked to see if it implies any superfluous restraint.

As explained above the influence of compressive axial force is to reduce member's overall effective bending resistance and thereby to cause greater deformations. *Tensile* *forces* on the other hand reduce deformations. For a constant value of axial force P less than its critical value P_{cr}, the stiffness can be defined including the influence of axial thrust. Usually the expressions for the various bending *stiffness coefficients* are expressed as the product of stiffness with no axial thrust present times the *axial correction* or *magnification factors*. These correction or modification factors are function of the ratio P/P_e where P is the axial force in the member and P_e is the *Euler's buckling load* with both ends of the member being presumed pinned.

6.4.2 Stiffness Matrix Including Axial Force Effects

In the application of *stiffness matrix method* the real structure is modelled or replaced by a set of elements that are connected to one another at their node points. The load-deformation characteristics of the elements are pre-determined and described by *element* or *member stiffness matrix* $[k]$. As in the case of conventional analysis any element of the matrix, say k_{ij}, is defined as the force in *ith* direction due to unit displacement in *jth* direction – with all other displacements maintained at zero, i. e., the subscript i refers to the resulting or imposed force and the j to the deformation parameter. Thus the matrix equation that describes the equilibrium of an element AB shown in Fig. 6.12b is given by

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \quad \text{or} \quad [k]\{D\} = \{F\} \tag{6.55}$$

where the subscripts 1, 2, 3 and 4 are the directions shown in the Fig. 6.12b. $[k]$ is *local element stiffness matrix*, $\{D\}$ is *displacement vector* and $\{F\}$ the corresponding externally *applied force vector*.

Consider an axially loaded member AB of uniform cross-section of length L and having a bending rigidity EI as shown in the Fig. 6.13. For a member with no axial force, the *stiffness influence coefficients* are given in Fig. 6.13a:

For $D_1(=\Delta_A) = 1$

$$k_{11} = \frac{12EI}{L^3}, \quad k_{21} = \frac{6EI}{L^2}, \quad k_{31} = -\frac{12EI}{L^3}, \quad \text{and} \quad k_{41} = \frac{6EI}{L^2} \tag{6.56}$$

For $D_2(=\theta_A) = 1$

$$k_{12} = \frac{6EI}{L^2}, \quad k_{22} = \frac{4EI}{L}, \quad k_{32} = -\frac{6EI}{L^2} \quad \text{and} \quad k_{42} = \frac{2EI}{L} \tag{6.57}$$

For $D_3(=\Delta_B) = 1$

$$k_{13} = -\frac{12EI}{L^3}, \quad k_{23} = -\frac{6EI}{L^2}, \quad k_{33} = \frac{12EI}{L^3} \quad \text{and} \quad k_{43} = -\frac{6EI}{L^2} \tag{6.58}$$

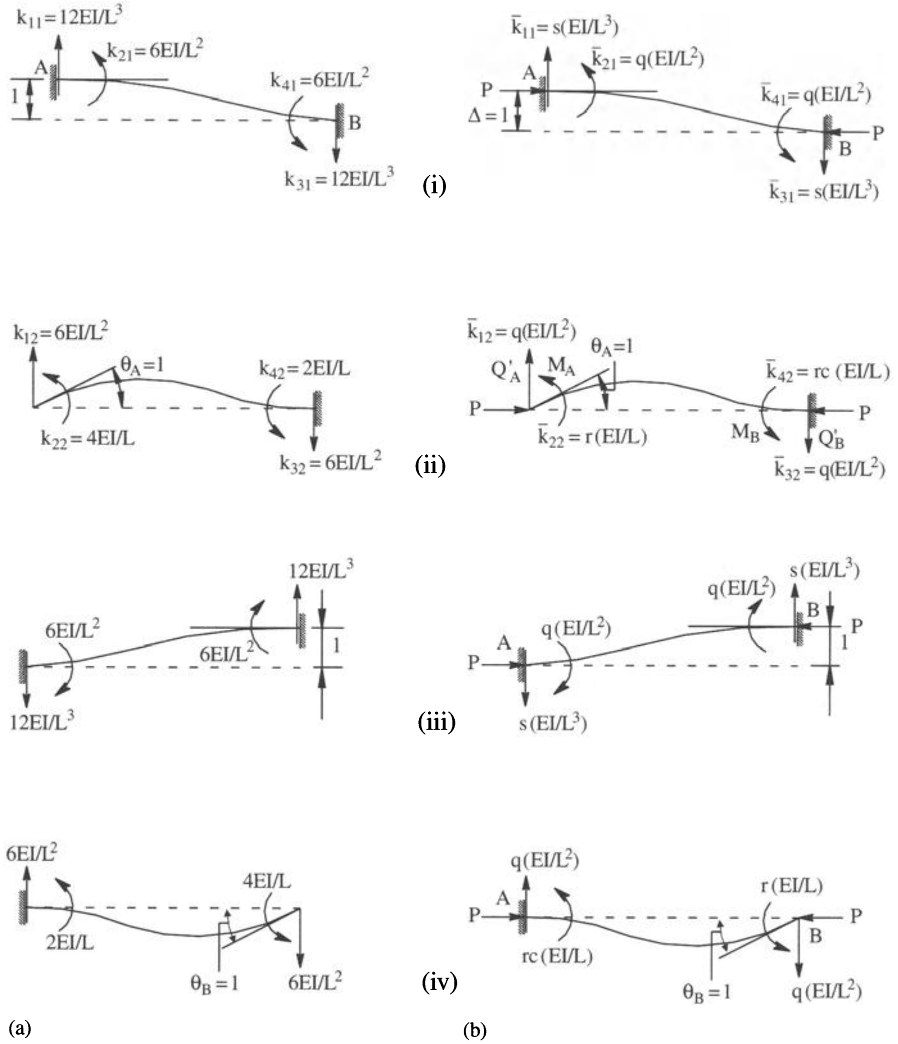


Fig. 6.13a,b. Stiffness influence coefficients. **a** Without axial load, **b** with axial load

For $D_4(=\theta_B) = 1$

$$k_{14} = \frac{6EI}{L^2}\,, \quad k_{24} = \frac{2EI}{L}\,, \quad k_{34} = -\frac{6EI}{L^2} \quad \text{and} \quad k_{44} = \frac{4EI}{L} \tag{6.59}$$

When axial force is also present, the *stiffness influence coefficients* as shown in Fig. 6.13b can be expressed as

$$\bar{k}_{11} = s\left(\frac{EI}{L^3}\right), \quad \bar{k}_{21} = q\left(\frac{EI}{L^2}\right), \quad \bar{k}_{31} = -s\left(\frac{EI}{L^3}\right) \quad \text{and} \quad \bar{k}_{41} = q\left(\frac{EI}{L^2}\right)$$

$$\vdots$$

$$\bar{k}_{13} = -s\left(\frac{EI}{L^3}\right), \quad \bar{k}_{23} = -q\left(\frac{EI}{L^2}\right), \bar{k}_{33} = s\left(\frac{EI}{L^3}\right) \quad \text{and} \quad \bar{k}_{43} = -q\left(\frac{EI}{L^2}\right)$$

$$\vdots \tag{6.60}$$

and so on. Here the correction or modification factors r, rc, q and s are functions of P, E, I and L, and are termed *stability coefficients*. The factors r and rc are termed *rotational coefficients*, and q and s are the *shear coefficients*. Thus the force-displacement relationship for an element in terms of stiffness matrix which is function of P/P_e is given by

$$\left(\frac{EI}{L}\right)\begin{bmatrix} s & q & -s & q \\ q & r & -q & rc \\ -s & -q & s & -q \\ q & rc & -q & r \end{bmatrix}\begin{Bmatrix} \Delta_A/L \\ \theta_A \\ \Delta_B/L \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} Q_A L \\ M_A \\ Q_B L \\ M_B \end{Bmatrix} \tag{6.61}$$

The size of stiffness matrix can be reduced by letting $Q_B = -Q_A = Q$ and combining the transverse displacements Δ_A and Δ_B into a relative term Δ ($= \Delta_A - \Delta_B$). Thus we obtain

$$\begin{Bmatrix} Q_A \\ M_A \\ M_B \end{Bmatrix} = \left(\frac{EI}{L}\right)\begin{bmatrix} s & q & q \\ q & r & rc \\ q & rc & r \end{bmatrix}\begin{Bmatrix} \Delta/L \\ \theta_A \\ \theta_B \end{Bmatrix} \tag{6.62}$$

The *elements of stiffness matrix* reduce to those of conventional stiffness matrix when $P = 0$. The *element stiffness matrices* can be assembled into a *structure stiffness matrix* $[\bar{K}]$ which can be used to determine the critical loading. The effects of elastic supports can be considered by treating the springs as members while formulating structure stiffness matrix. To illustrate the application of matrix approach to the stability analysis problems, consider the continuous, two-span strut AC of uniform EI shown in Fig. 6.14. The structure has three-degrees-of-freedom: one lateral displacement Δ_1 and two rotations θ_1 and θ_2. The strut is discretized into two elements AB and BC. The element stiffness matrices are obtained from the (6.61) for the relevant degrees of freedom.

Element #1

$$[\bar{k}]_1 = \left(\frac{EI}{L}\right)_1 \begin{array}{c} \begin{array}{cccc} \overbrace{\Delta_1/L \quad \theta_1}^{\text{Node 1}} & \overbrace{\Delta_2/L \quad \theta_2}^{\text{Node 2}} \end{array} \\ \left[\begin{array}{cc|cc} s & q & -s & q \\ q & r & -q & rc \\ \hline -s & -q & s & -q \\ q & rc & -q & r \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{c} \Delta_1/L \\ \theta_1 \end{array}\right\} 1 \\ \left.\begin{array}{c} \Delta_2/L \\ \theta_2 \end{array}\right\} 2 \end{array} \tag{6.63}$$

Element #2

$$[\bar{k}]_2 = \left(\frac{EI}{L}\right)_2 \begin{array}{c} \overbrace{\Delta_2/L \quad \theta_2}^{\text{Node 2}} \quad \overbrace{\Delta_3/L \quad \theta_3}^{\text{Node 3}} \\ \left[\begin{array}{cc|cc} s & q & -s & q \\ q & r & -q & rc \\ \hline -s & -q & s & -q \\ q & rc & -q & r \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l}\Delta_2/L\\ \theta_2\end{array}\right\} 2 \\ \left.\begin{array}{l}\Delta_3/L\\ \theta_3\end{array}\right\} 3\end{array} \tag{6.64}$$

Thus the structure stiffness matrix $[\bar{K}] = [\bar{k}]_1 + [\bar{k}]_2$ is given by

$$[\bar{K}] = \left(\frac{EI}{L}\right) \begin{array}{c} \overbrace{\Delta_1/L \quad \theta_1}^{\text{Node 1}} \quad \overbrace{\Delta_2/L \quad \theta_2}^{\text{Node 2}} \quad \overbrace{\Delta_3/L \quad \theta_3}^{\text{Node 3}} \\ \left[\begin{array}{cc|cc|cc} s & q & 0 & q & & \\ q & r & 0 & rc & & \\ \hline 0 & 0 & 0+0 & 0+0 & 0 & 0 \\ q & rc & 0+0 & r+r & 0 & 0 \\ \hline & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{array}\right] \end{array} = \left(\frac{EI}{L}\right) \begin{pmatrix} s & q & q \\ q & r & rc \\ q & rc & 2r \end{pmatrix} \tag{6.65}$$

For elastic instability:

$$|\bar{K}| = 0 \quad \text{i.e.} \quad q^2(2rc - 3r) + s[2r^2 - (rc)^2] = 0\,. \tag{6.66}$$

Use trial and modification procedure with values of the *stability functions obtained from table* given in Appendix A.1 for various values of parameters, $\rho = (P/P_e)$. For

$$\rho = 0.14: \quad [\bar{K}] = 5.0397$$

$$\rho = 0.16: \quad [\bar{K}] = -0.3543$$

By interpolation $[\bar{K}] = 0$, when $\rho = 0.1587$. Therefore,

$$P_{cr} = \frac{0.1587\pi^2 EI}{L^2} = \frac{1.5663EI}{L^2}\,.$$

Let us consider the case when the end C is hinged instead of being fixed. With this additional rotation the number of degrees-of-freedom increases to four. Following the above procedure, equilibrium equations of the structure in this case would be:

$$\left(\frac{EI}{L}\right) \begin{pmatrix} s & q & q & 0 \\ q & r & rc & 0 \\ q & rc & 2r & rc \\ 0 & 0 & rc & r \end{pmatrix} \begin{Bmatrix} \Delta/L \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} FL \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{6.67}$$

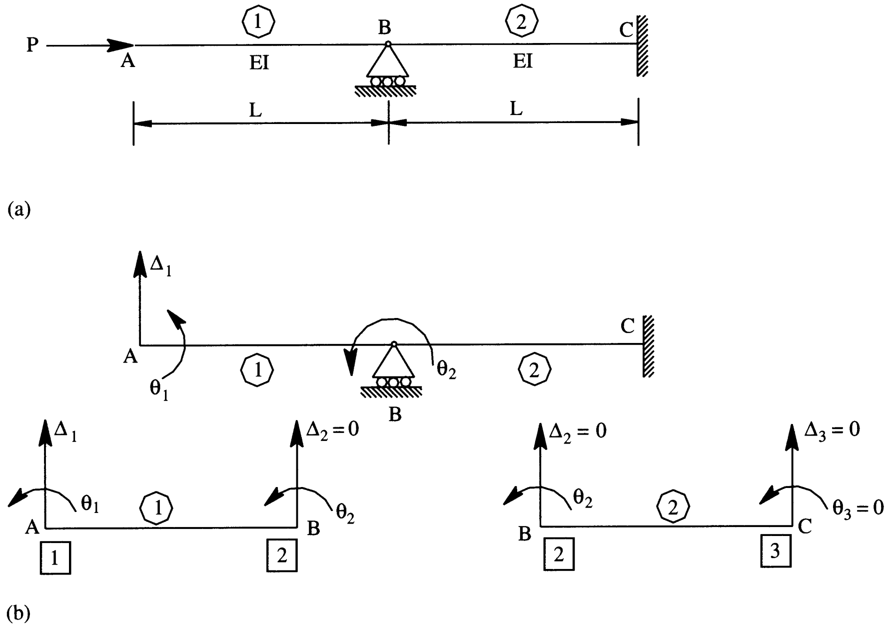


Fig. 6.14a,b. Formulation of structure stiffness matrix. **a** Continuous two-span strut, **b** degrees-of-freedom of the elements. (i) element number, [i] node number

Due to the presence of hinge at the node 3, $M_3 = 0$, hence from the fourth equilibrium equation: $rc\theta_2 + r\theta_3 = 0$ or $\theta_3 = -c\theta_2$. Substituting this value of θ_3 in third equilibrium equation, the force-displacement relation reduces to

$$\left(\frac{EI}{L}\right)\begin{bmatrix} s & q & q \\ q & r & rc \\ q & rc & 2r - rc^2 \end{bmatrix}\begin{Bmatrix} \Delta/L \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} FL \\ M_1 \\ M_2 \end{Bmatrix} \tag{6.68}$$

The quantity $2r - rc^2 = r + [r(1-c^2)] = r + r'$. Thus, for this case, elastic instability occurs when

$$|\bar{K}| = \begin{vmatrix} s & q & q \\ q & r & rc \\ q & rc & (r+r') \end{vmatrix} = q^2[2rc - (2r + r')] + s[r(r + r') - (rc)^2] = 0 \tag{6.69}$$

By trial and modification for

$$\rho = 0.12: \quad [\bar{K}] = 4.0863$$

$$\rho = 0.14: \quad [\bar{K}] = -0.5316$$

Hence by interpolation for $[\bar{K}] = 0$, $\rho = 0.1377$ and

$$P_{\text{cr}} = (0.1377)\frac{\pi^2 EI}{L^2} = \frac{1.359EI}{L^2}$$

It is evident that the matrix approach is a very powerful tool for analysis of structures when used in conjunction with computers. However, in the following sections a more direct approach is used.

6.5 Stability Functions

6.5.1 Member with No Lateral Displacement

For the structural element AB with the loading shown in the Fig. 6.13b-ii

$$\bar{k}_{11} = \frac{M_A}{\theta_A} \quad \text{and} \quad \bar{k}_{21} = \frac{M_B}{\theta_A} \tag{6.70}$$

The *carry-over* effect is defined by the relationship

$$c = \frac{M_B}{M_A} = \frac{\bar{k}_{21}}{\bar{k}_{11}} \tag{6.71}$$

In the absence of lateral loads along the element, the governing differential equation can be written as

$$\left(\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}\right) + \alpha^2 \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = 0 \tag{6.72}$$

where $\alpha^2 = \frac{P}{EI}$. The general solution to this fourth-order differential equation is

$$y = A \sin \alpha x + B \cos \alpha x + C \left(\frac{x}{L}\right) + D \tag{6.73}$$

For an imposed unit rotation at the end A (i.e. $\theta_A = 1$) while the end B is fixed against rotation, the boundary conditions to be satisfied at

$$\begin{aligned} x = 0: &\quad y(0) = 0 \quad \text{and} \quad y'(0) = 1\,, \\ x = L: &\quad y(L) = 0 \quad \text{and} \quad y'(L) = 0\,. \end{aligned} \tag{6.74}$$

On substitution of general solution in the boundary conditions the values of integration constants are obtained as follows:

$$y(0) = B + D = 0 \quad \text{i.e.} \quad D = -B$$

$$y'(0) = A\alpha + \frac{C}{L} = 1 \quad \text{i.e.} \quad \frac{C}{L} = 1 - A\alpha$$

$$\begin{aligned} y(L) &= A \sin \alpha L + B \cos \alpha L + C + D \\ &= A(\sin \alpha L - \alpha L) + B(\cos \alpha L - 1) + L = 0 \end{aligned} \tag{6.75}$$

$$\begin{aligned} y'(L) &= \alpha A \cos \alpha L - \alpha B \sin \alpha L + \frac{C}{L} \\ &= A\alpha(\cos \alpha L - 1) - B\alpha \sin \alpha L + 1 = 0 \end{aligned} \tag{6.76}$$

Solving (6.75) and (6.76) for A and B

$$A = \frac{1 - \alpha L \sin \alpha L - \cos \alpha L}{\alpha(2 - 2\cos \alpha L - \alpha L \sin \alpha L)} \tag{6.77}$$

$$B = \frac{\sin \alpha L - \alpha L \cos \alpha L}{\alpha(2 - 2\cos \alpha L - \alpha L \sin \alpha L)} \tag{6.78}$$

The end moments are given by

$$\begin{aligned} M_A &= -EIy''(0) = EI(\alpha^2 B) = \left(\frac{EI}{L}\right)\left[\frac{\alpha L(\sin \alpha L - \alpha L \cos \alpha L)}{2 - 2\cos \alpha L - \alpha L \sin \alpha L}\right] \\ &= r(\psi)\left(\frac{EI}{L}\right) \end{aligned} \tag{6.79}$$

$$\begin{aligned} M_B &= -EIy''(L) = EI\alpha^2(A \sin \alpha L + B \cos \alpha L) \\ &= \left(\frac{EI}{L}\right)\left[\frac{\alpha L(\alpha L - \sin \alpha L)}{2 - 2\cos \alpha L - \alpha L \sin \alpha L}\right] \\ &= rc(\psi)\left(\frac{EI}{L}\right) \end{aligned} \tag{6.80}$$

Since the forces M_A and M_B are due to unit rotation, they represent corresponding *stiffness influence coefficients*. Thus

$$\bar{k}_{11} = r\left(\frac{EI}{L}\right), \quad \bar{k}_{21} = rc\left(\frac{EI}{L}\right) \tag{6.81}$$

where

$$r = \left[\frac{\psi(S - \psi C)}{(2 - 2C - \psi S)}\right], \quad rc = \left[\frac{\psi(\psi - S)}{(2 - 2C - \psi S)}\right]$$

$$S = \sin \psi, \quad C = \cos \psi, \quad \psi = \alpha L = \pi\sqrt{\frac{P}{P_e}} = \pi\sqrt{\rho} \tag{6.82}$$

Therefore, the induced bending moments at the ends A and B of the element due to applied rotation θ_A at A are given by

$$M_A = r\left(\frac{EI}{L}\right)\theta_A \quad \text{and} \quad M_B = rc\left(\frac{EI}{L}\right)\theta_A \tag{6.83}$$

Thus *carry-over factor* is defined as

$$c = \frac{M_B}{M_A} = \frac{(\psi - S)}{(S - \psi C)} \tag{6.84}$$

The stiffness influence coefficient for an element AB hinged at the far end B can be obtained by applying a moment $-rc(EI/L)\theta_A$ at the end B, thereby reducing the net moment at B to zero i.e. reducing it to a hinged end-condition. This operation results in a carry-over moment of $c[-rc(EI/L)\theta_A]$ to end A. Thus the total moment at the end A becomes

$$\begin{aligned} M_A &= r\left(\frac{EI}{L}\right)\theta_A - rc^2\left(\frac{EI}{L}\right)\theta_A \\ &= r(1-c^2)\left(\frac{EI}{L}\right)\theta_A = r'\left(\frac{EI}{L}\right)\theta_A \end{aligned} \tag{6.85}$$

where $r' = r(1-c^2)$. The term r' represents *rotational stiffness influence coefficient* of a prismatic element when the far end is hinged.

The stiffness influence coefficients r, r' and rc reduce to 4, 3 and 2, respectively, when $P = 0$. This can be obtained by taking the limits $\psi \to 0$ using L'Hospital's rule four times and substituting $\psi = 0$.

The moment equilibrium of the element as a free body about the right hand end gives

$$\begin{aligned} &M_A + M_B - Q'_A L = 0 \\ \text{or}\quad &r\left(\frac{EI}{L}\right)\theta_A + rc\left(\frac{EI}{L}\right)\theta_A - Q'_A L = 0 \end{aligned} \tag{6.86}$$

Therefore, the *end shear* term can be defined from (6.86) as

$$Q'_A = (r + rc)\left(\frac{EI}{L^2}\right)\theta_A = q\left(\frac{EI}{L^2}\right)\theta_A \tag{6.87}$$

where

$$q = \left[\frac{\psi^2(1-C)}{(2-2C-\psi S)}\right].$$

For the case when $P = 0$, i.e. $\psi = 0$, $q(\psi) = 6$. It should be noted that the simplifications in the *stiffness* values applicable to the prismatic elements with no axial force are also applicable when these members constitute parts of the frame undergoing buckling. For example a symmetric element subjected to end moments which are equal in magnitude but opposite in sense i.e. $\theta_B = -\theta_A$ causing single curvature bending, the effective stiffness is $(2\,EI/L)$, whereas for the one with antisymmetric bending i.e. $\theta_B = \theta_A$, the modified stiffness is $(6\,EI/L)$. For the beam elements with moment applied at one end only i.e. $\theta_B = (\theta_A/2)$, the modified stiffness is $(3EI/L)$.

6.5.2 Member Subjected to a Relative End Displacement Δ

Consider the case of a member AB subjected to a relative displacement Δ at the ends while end rotations are prevented as shown in Fig. 6.13b-i. For an imposed unit end displacement $\Delta = 1$, the boundary conditions for determination of integration constants of the general solution of governing differential equation are

$$y(0) = 1\,, \quad y'(0) = y'(L) = 0 \quad \text{and} \quad y(L) = 0 \tag{6.88}$$

Substituting the general solution, $y = A\sin\alpha x + B\cos\alpha x + (Cx/L) + D$ into the boundary conditions given by (6.88):

$$\begin{aligned}
y(0) &= B + D = 1\\
y'(0) &= \alpha A + (C/L) = 0\\
y'(L) &= \alpha A\cos\alpha L - \alpha B\sin\alpha L + (C/L) = 0\\
y(L) &= A\sin\alpha L + B\cos\alpha L + C + D = 0
\end{aligned}$$

The values of the constants obtained are:

$$A = -\frac{\sin\alpha L}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)}, \quad B = -\frac{\cos\alpha L - 1}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)},$$

$$C = -\frac{\alpha L\sin\alpha L}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)} \quad \text{and}$$

$$D = \frac{1 - \cos\alpha L - \alpha L\sin\alpha L}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)} \tag{6.89}$$

The end moments are given by

$$\begin{aligned}
M_A &= -EIy''(0) = \alpha^2 B(EI)\\
&= -\left[\frac{(\alpha L)^2(\cos\alpha L - 1)}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)}\right]\left(\frac{EI}{L^2}\right) = q\left(\frac{EI}{L^2}\right)
\end{aligned} \tag{6.90}$$

$$\begin{aligned}
M_B &= -EIy''(L) = \alpha^2(A\sin\alpha L + B\cos\alpha L)(EI)\\
&- \left[\frac{(\alpha L)^2(1 - \cos\alpha L)}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)}\right]\left(\frac{EI}{L^2}\right) = -q\left(\frac{EI}{L^2}\right)
\end{aligned} \tag{6.91}$$

The end shear in the element is given by

$$\begin{aligned}
Q_A &= EIy'''(0) = -EI\alpha^3 A\\
&= \left[\frac{(\alpha L)^3\sin\alpha L}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)}\right]\left(\frac{EI}{L^3}\right) = s\left(\frac{EI}{L^3}\right)
\end{aligned} \tag{6.92}$$

$$\begin{aligned}
Q_B &= EIy'''(L) = -EI\alpha^3(A\cos\alpha L - B\sin\alpha L)\\
&= -\left[\frac{(\alpha L)^3\sin\alpha L}{(2 - 2\cos\alpha L - \alpha L\sin\alpha L)}\right]\left(\frac{EI}{L^3}\right) = -s\left(\frac{EI}{L^3}\right)
\end{aligned} \tag{6.93}$$

Since M_A, M_B, Q_A and Q_B are forces due to unit lateral displacement, they represent *stiffness influence coefficients*. Thus

$$\bar{k}_{13} = q\left(\frac{EI}{L^2}\right), \quad \bar{k}_{23} = q\left(\frac{EI}{L^2}\right), \quad \bar{k}_{33} = s\left(\frac{EI}{L^3}\right) \quad \text{and} \quad \bar{k}_{43} = -s\left(\frac{EI}{L^3}\right)$$

where

$$q = \left[\frac{\psi^2(1 - C)}{(2 - 2C - \psi S)}\right] \quad \text{and} \quad s = \left[\frac{\psi^3 S}{(2 - 2C - \psi S)}\right]$$

where

$$S = \sin\psi\ , \quad C = \cos\psi\ , \quad \psi = \alpha L = \pi\sqrt{\rho}\ .$$

The parameters r, rc, r', q and s which are functions of $\rho\ (= P/P_e)$ are termed *stability functions*. The selected values of these functions are tabulated in Appendix A.1. For intermediate values interpolation may be adopted.

The stability functions that have been developed for compressive forces can be readily modified for *axial tension.* This is accomplished by replacing P by $-P$ i.e. substitute αi $(= \alpha\sqrt{-1})$ for α and ψi $(= \psi\sqrt{-1})$ for ψ. Since $\sin(\mathrm{i}\psi) = \mathrm{i}\sinh\psi$ and $\cos(\mathrm{i}\psi) = \cosh\psi$, the functions become

$$r = \frac{\psi(\psi C - S)}{(2 - 2C + \psi S)}\ , \quad rc = \frac{\psi(S - \psi)}{(2 - 2C + \psi S)} \tag{6.94}$$

$$q = \left[\frac{\psi^2(C - 1)}{(2 - 2C + \psi S)}\right] \quad \text{and} \quad s = \left[\frac{\psi^3 S}{(2 - 2C + \psi S)}\right] \tag{6.95}$$

where $S = \sinh\psi$, $C = \cosh\psi$ and $\psi = \alpha L = \pi\sqrt{\rho}$. The selected values of the stability functions for axial tension are listed in Appendix A.3.

To illustrate the application of these functions in the elastic stability analysis of structure consider the symmetrical rigid-jointed plane frame subjected to loads P as shown in Fig. 6.15. It is required to estimate the critical values of load P to produce elastic instability of the frame. A possible buckled configuration of this two-degrees-of-freedom system is shown in the figure. The applied loads are directly transferred into the members 2-3 and 2′-3′ as axial compressive forces and hence their stiffness coefficients are expressed in terms of *rotational stability function r*. Since there is no axial force in the members 1-2, 2-2′ and 2′-1′, the usual stiffness and carry-over coefficients $(4EI/L)$ and $(2EI/L)$ are used. With these modifications the stiffness matrix $[\bar{K}]$ is formulated as in the case of conventional analysis $[\bar{K}]\{\Delta\} = \{F\}$

$$\begin{bmatrix} k_{22} & k_{22'} \\ k_{2'2} & k_{2'2'} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_{2'} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.96}$$

The stiffness matrix $[\bar{K}]$ is given by

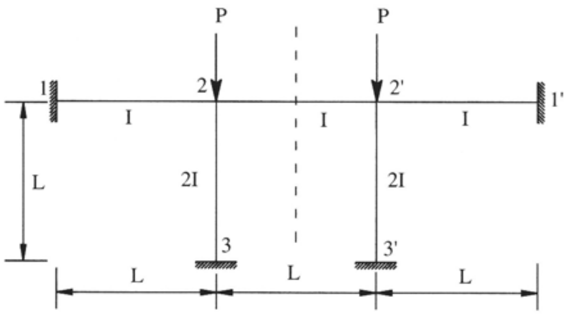
$$[\bar{K}] = \begin{bmatrix} \left(\dfrac{4EI}{L} + \dfrac{4EI}{L} + r\dfrac{E(2I)}{L}\right) & \left(\dfrac{2EI}{L}\right) \\ \left(\dfrac{2EI}{L}\right) & \left(\dfrac{4EI}{L} + \dfrac{4EI}{L} + r\dfrac{E(2I)}{L}\right) \end{bmatrix}$$

$$= \left(\frac{EI}{L}\right)\begin{bmatrix} (8 + 2r) & 2 \\ 2 & (8 + 2r) \end{bmatrix} \tag{6.97}$$

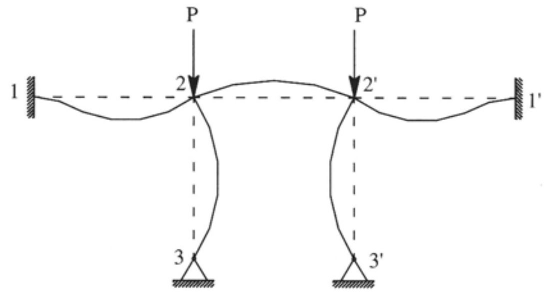
For elastic instability the determinant of stiffness matrix must vanish, that is,

$$|\bar{K}| = \begin{vmatrix} (8 + 2r) & 2 \\ 2 & (8 + 2r) \end{vmatrix} = 0 \quad \text{or} \quad (8 + 2r)^2 - (2)^2 = 0 \tag{6.98}$$

Therefore, $(6 + 2r)(10 + 2r) = 0$ giving $r = -3.0$ and -5.0



(a)



(b)

Fig. 6.15a,b. Buckling of symmetrical frame. **a** Frame fixed at the base, **b** frame hinged at the base

The lowest value of ρ $(= P/P_e)$ occurs for $r = -3.0$. Referring to the relevant stability functions table in Appendix A.1 for $r = -3.0$, $\rho = 2.730$. Therefore,

$$P_{cr} = \rho_{cr} P_e = \frac{2.730\pi^2 E(21)}{L^2} = \frac{5.46\pi^2 EI}{L^2} = \frac{53.888\, EI}{L^2}$$

The effective length L_{eff} of member 2-3 (or 2′-3′) is given by

$$\frac{\pi^2 E(2I)}{L_{eff}^2} = \rho_{cr}\left[\frac{\pi^2 E(2I)}{L^2}\right] \quad \text{i.e.} \quad L_{eff} = \frac{L}{\sqrt{\rho_{cr}}} = 0.605L$$

Identical results are obtained directly by considering stiffness of only one joint 2 or 2′ because the frame has geometric and loading symmetry $(\theta_{2'} = \theta_2)$. The stiffness of joint 2 is equal to the sum of stiffness of members 2-1, 2-2′ and 2-3. Thus

$$\begin{aligned}
\bar{k}_2 &= \bar{k}_{21} + \bar{k}_{22'} + \bar{k}_{23} \\
&= \frac{4EI}{L} + \frac{2EI}{L} + r\left[\frac{E(2I)}{L}\right] \quad (k_{22'} = 2EI/L \text{ due to symmetry}) \\
&= (6 + 2r)\left(\frac{EI}{L}\right)
\end{aligned}$$

For elastic instability $\bar{k}_2 = 0$, giving $r = -3.0$. From the stability functions table given in Appendix A.1 for $r = -3.0$, $\rho = 2.730$. Therefore, $P_{cr} = 53.888\, EI/L^2$.

As a variation consider the case when the joints of the frame 3 and 3′ are hinged as shown in Fig. 6.15b. This change makes the structure a four-degrees-of-freedom system. However, due to perfect symmetry in geometry and loading, only half the frame need be considered for stability analysis. The *rotational stability functions* for the members 2-3 and 2′-3′ with far ends hinged are represented by r' [$= (1 - c^2)r$]. Thus the stiffness of the joint 2 is given by

$$\begin{aligned}\bar{k}_2 &= \bar{k}_{21} + \bar{k}_{22'} + \bar{k}_{23} \\ &= \frac{4EI}{L} + \frac{2EI}{L} + r'\left[\frac{E(2I)}{L}\right] = (6 + 2r')\left(\frac{EI}{L}\right)\end{aligned}$$

The condition of elastic instability, $\bar{k}_2 = 0$ gives $r' = -3.0$. From the stability functions table given in Appendix A.1. For $r' = -3.0$, $\rho = 1.407$. Therefore,

$$P_{cr} = \frac{1.407\pi^2 E(2I)}{L^2} = \frac{27.773EI}{L^2}$$

Example 6.7. Estimate P_{cr}, the first critical value of the load P that will cause the rigid jointed frame shown in Fig. 6.16 to collapse under the following conditions: (i) load P is acting at the joint 1 only, (ii) each of the joints 1 and 2 carry load P. (iii) joints 1 and 2 carry loads P and $2P$, respectively, and (iv) beam member 1-2 only is subjected to compression. (EI/L) values are same for all the members. The horizontal displacement or sway is prevented.

Since the sway is prevented the system has two-degrees-of-freedom θ_1 and θ_2.

(i) In this case the member 1-3 alone is subjected to axial thrust P; hence its stiffness influence coefficients will be in terms of rotational stability function r. For members 1-2 and 2-4 with no axial force, the usual influence coefficient $(4EI/L)$ is used. Therefore, the *member terminal moments* are

$$M_{13} = r\left(\frac{EI}{L}\right)\theta_1\,, \quad M_{12} = 4\left(\frac{EI}{L}\right)\theta_1 + 2\left(\frac{EI}{L}\right)\theta_2$$

$$M_{21} = 4\left(\frac{EI}{L}\right)\theta_2 + 2\left(\frac{EI}{L}\right)\theta_1 \quad \text{and} \quad M_{24} = 4\left(\frac{EI}{L}\right)\theta_2\,.$$

For the equilibrium of joints 1 and 2

$$\begin{Bmatrix} M_1(= M_{12} + M_{13}) \\ M_2(= M_{21} + M_{24}) \end{Bmatrix} = \left(\frac{EI}{L}\right)\begin{bmatrix} (r+4) & 2 \\ 2 & (4+4) \end{bmatrix}\begin{Bmatrix}\theta_1 \\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix}$$

For elastic instability, the determinant of matrix $[\bar{K}]$ must vanish i.e.

$$|\bar{K}| = \begin{vmatrix} (r+4) & 2 \\ 2 & 8 \end{vmatrix} = 8r + 28 = 0$$

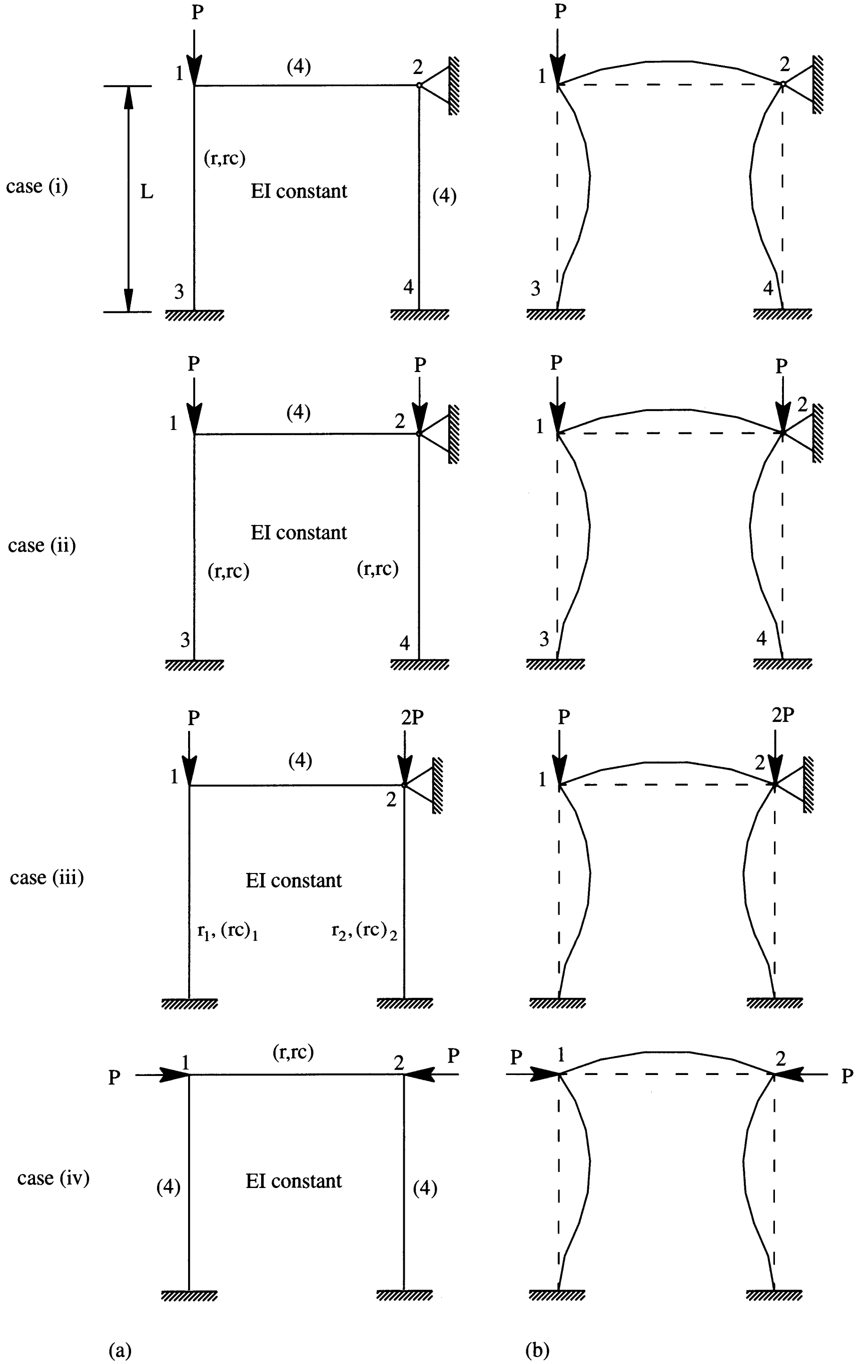


Fig. 6.16a,b. Buckling of a fixed base portal with different loading conditions. **a** Loading system, **b** buckling mode

Thus $r = -3.50$. From the stability function table of Appendix A.1 for $r = -3.5$, $\rho = 2.8079$. Therefore,

$$P_{\mathrm{cr}} = \frac{2.8079\pi^2 EI}{L^2} = \frac{27.71\,EI}{L^2}\,.$$

For determination of corresponding buckling mode substitute $r = -3.5$ in any of the equilibrium equations, e. g. consider the equation $(r+4)\theta_1 + 2\theta_2 = 0$. On substitution this equation reduces to $0.5\theta_1 + 2\theta_2 = 0$ i. e. $\theta_2 = -0.25\theta_1$. This buckling mode is shown in Fig. 6.16.

(ii) In this case both the columns carry axial thrust of magnitude P, hence their stiffness influence coefficients involve rotational stability function r. The modified stiffness matrix in this case would be

$$[\bar{K}] = \left(\frac{EI}{L}\right)\begin{bmatrix} (r+4) & 2 \\ 2 & (r+4) \end{bmatrix}$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix} (r+4) & 2 \\ 2 & (r+4) \end{vmatrix} = (r+2)(r+6) = 0$$

Therefore, $r = -2.0$ and -6.0. From the stability functions table in Appendix A.1 for

$$r = -2.0: \quad \rho = 2.551 \quad \text{and corresponding} \quad P_{\mathrm{cr}} = 25.177\,EI/L^2$$
$$r = -6.0: \quad \rho = 3.095 \quad \text{and} \quad P_{\mathrm{cr}} = 30.546\,EI/L^2.$$

The critical value of load at the buckling is given by smaller of these two values. As usual the buckling modes can be easily determined from any of the equilibrium equations. For the first buckling mode at $r = -2.0$, the equilibrium equation $(r+4)\theta_1 + 2\theta_2 = 0$ reduces to

$$2\theta_1 + 2\theta_2 = 0 \quad \text{i.e.} \quad \theta_2 = -\theta_1$$

i. e. the rotations at joints 1 and 2 are equal in magnitude but opposite in sense i. e. the buckling mode is symmetrical as shown in Fig. 6.16. Same result can be obtained by using the second equilibrium equation. For the second buckling mode at $r = -6.0$, the equilibrium equation reduces to

$$-2\theta_1 + 2\theta_2 = 0 \quad \text{i.e.} \quad \theta_2 = \theta_1$$

Thus, the second buckling mode has anti-symmetrical configuration. It should be noted that symmetric buckling mode gives lower value of critical load.

(iii) In contrast to the case (ii), the loads carried by two columns are different. Hence the rotational stability function r has different values for the two columns; say r_1 and r_2 and the critical load cannot be determined directly. In this case

$$\frac{\rho_1}{\rho_2} = \frac{(P/P_{\mathrm{e}})}{(2P/P_{\mathrm{e}})} = \frac{1}{2}$$

i.e. ρ_1 and ρ_2 are in the ratio of 1:2. The stiffness matrix in this case would be

$$[\bar{K}] = \left(\frac{EI}{L}\right)\begin{bmatrix}(r_1+4) & 2 \\ 2 & (r_2+4)\end{bmatrix}$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix}(r_1+4) & 2 \\ 2 & (r_2+4)\end{vmatrix} = (r_1+4)(r_2+4) - 4 = 0 \qquad \text{(a)}$$

The *load factor* N_e to cause the collapse can be determined by trial and modification. As a first trial assume $N_e = 1.0$, i.e. $\rho_1 = 1.0$ and $\rho_2 = 2.0$. The corresponding values of r as obtained from the stability functions table given in Appendix A.1 are $r_1 = 2.467$ and $r_2 = 0.143$, respectively. On substituting these values in equation (a), the value of determinant reduces to

$$|\bar{K}| = (2.467 + 4)(0.143 + 4) - 4 = 22.79$$

Assume $N_e = 1.48$ giving values of stability functions $r_1 = 1.502$ and $r_2 = -4.673$, the corresponding value of determinant is

$$|\bar{K}| = (1.502 + 4)(-4.673 + 4.0) = -3.70$$

Assume $N_e = 1.44$ with corresponding $r_1 = 1.591$ and $r_2 = -4.021$

$$|\bar{K}| = (1.591 + 4)(-4.021 + 4) = 0.117$$

By interpolation $N_e = 1.4412$. Therefore

$$\rho_1 = \frac{P}{P_e} = 1.4412$$

Thus,

$$P_{cr} = \frac{1.4412\pi^2 EI}{L^2} = \frac{14.224\,EI}{L^2}\,.$$

(iv) In this case the beam 1-2 alone is subjected to axial thrust P and the columns are free from axial compression. The instability condition is given by

$$|\bar{K}| = \begin{vmatrix}(4+r) & rc \\ rc & (4+r)\end{vmatrix} = (4+r)^2 - (rc)^2 = 0$$

Using stability functions table given in Appendix A.1, by trial and modification for

$$\rho = 2.12: \quad |\bar{K}| = (4 - 0.242)^2 - 13.987 = +0.1356$$

$$\rho = 2.16: \quad |\bar{K}| = (4 - 0.379)^2 - 14.582 = -1.4704$$

By interpolation for $|\bar{K}| = 0$, $\rho = 2.12338$. Therefore,

$$P_{cr} = \frac{2.12338\pi^2 EI}{L^2} = \frac{20.957 EI}{L^2}\,,$$

and the corresponding buckling mode is shown in Fig. 6.15b.

Example 6.8. A two span continuous strut of uniform cross section, shown in Fig. 6.17 is subjected to: (i) an axial force of P in the segment 1-2 with axial force in segment 2-3 presumed to be equal to zero, and (ii) an axial thrust P such that both the segments 1-2 and 2-3 carry axial compressive force P. Estimate the flexural buckling or critical load for the strut. EI is same for both the segments.

(i) In this case the segment 1-2 alone is subjected to an axial thrust P, hence its stiffness influence coefficients are in terms of r and rc. For the segment 2-3, the usual influence coefficients $(4\,EI/2L)$ and $(2\,EI/2L)$ are used. For this three-degrees-of freedom structure, the force-displacement equation $[K]\{D\} = \{F\}$ can be expressed as

$$\left(\frac{EI}{L}\right)\begin{bmatrix} r & rc & 0 \\ rc & (r+2) & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

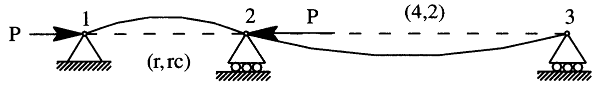
For elastic instability

$$|\bar{K}| = r[2(r+2) - 1] - rc(2rc) = 0$$

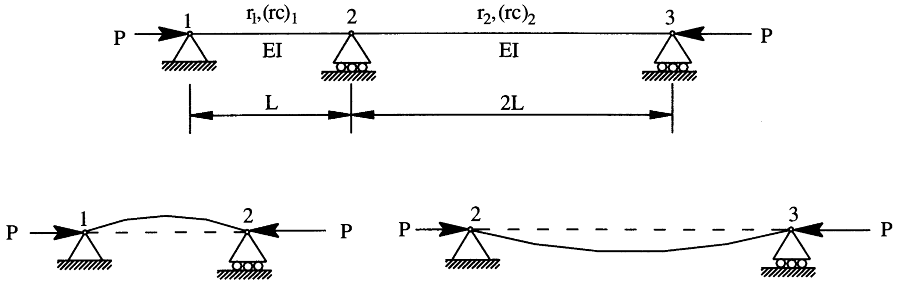
Using trial and modification procedure for

$$\rho = 1.24: \quad |\bar{K}| = (2.011)[2(2.011+2) - 1] - 2(6.977) = 0.1672$$

$$\rho = 1.28: \quad |\bar{K}| = (1.930)[2(1.930+2) - 1] - 2(7.150) = -1.0602$$



(a)



(b)

Fig. 6.17a,b. Buckling of a two segment continuous strut. **a** Strut subjected to axial load in one segment, **b** strut split into two parts

By interpolation for $|\bar{K}| = 0$, $\rho = 1.2454$. Therefore,

$$P_{cr} = \frac{1.2454\pi^2 EI}{L^2} .$$

(ii) In this case each of the segments 1-2 and 2-3 carry axial thrust P and hence stiffness influence coefficients are in terms of r and rc. The stiffness values for segments 1-2 and 2-3 have been identified by the subscripts 1 and 2, respectively.

Since the lengths of the segments are different

$$\rho_1 : \rho_2 = \frac{P}{(\pi^2 EI/L^2)} : \frac{P}{(\pi^2 EI/4L^2)} = 1 : 4 \tag{6.99}$$

Let $\rho_1 = \rho$ and hence $\rho_2 = 4\rho$. The stiffness matrix in this case becomes

$$[\bar{K}] = \left(\frac{EI}{L}\right) \begin{bmatrix} r_1 & (rc)_1 & 0 \\ (rc)_1 & (r_1 + 0.5r_2) & 0.5(rc)_2 \\ 0 & 0.5(rc)_2 & 0.5r_2 \end{bmatrix}$$

For elastic instability

$$|\bar{K}| = r_1[(r_1 + 0.5r_2)(0.5r_2) - 0.25(rc)_2^2] - (rc)_1[(rc)_1(0.5r_2)] = 0$$

By trial and modification procedure, using stability functions from the table given in Appendix A.1. For

$$\rho = 0.36: \quad r_1 = 3.502\,, \quad (rc)_1^2 = 4.549\,, \quad r_2 = 1.591\,, \quad (rc)_2^2 = 7.930$$
$$\text{and} \quad |\bar{K}| = 1.4107$$
$$\rho = 0.40: \quad r_1 = 3.444\,, \quad (rc)_1^2 = 4.621\,, \quad r_2 = 1.224\,, \quad (rc)_2^2 = 8.881$$
$$\text{and} \quad |\bar{K}| = -1.9256$$

By interpolation for $|\bar{K}| = 0$, $\rho = 0.3769$. Therefore,

$$P_{cr} = \frac{0.3769\pi^2 EI}{L^2} = \frac{3.72EI}{L^2} .$$

6.6 Frames with Sidesway

So far in this chapter structures with only rotational degrees-of-freedom have been discussed. Lateral displacement of an axially loaded structural member without joint rotations at the ends, and moments due to eccentric axial loads increase terminal moments and member rotations. These effects must be included in the moment sway-equation. The amplification effect could be covered by means of *magnification* or *modification factor* and the sway problem could be analyzed by any of the available methods.

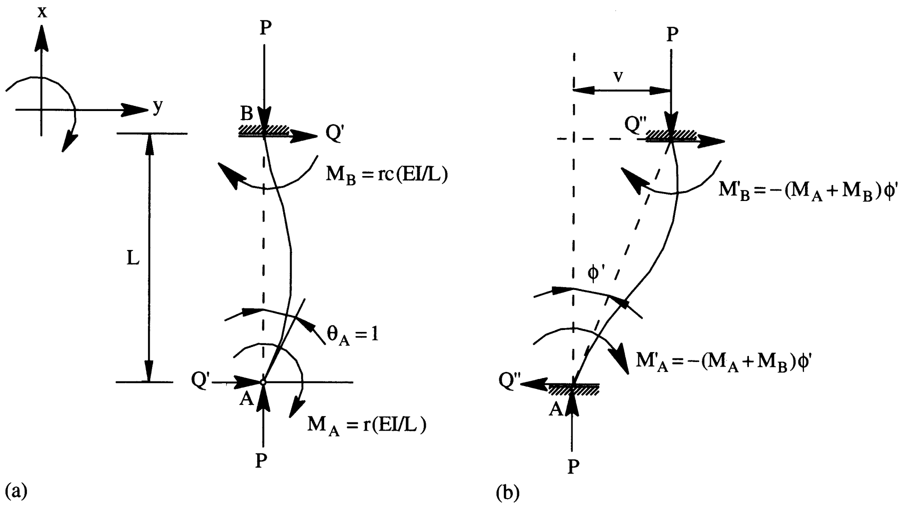


Fig. 6.18a,b. Behaviour of straight prismatic member. **a** Rotational stiffness, **b** sway or shear stiffness

Consider an axially loaded, straight, prismatic member AB subjected to an end rotation $\theta_A = 1$ as shown in Fig. 6.18a. The forces developed are shown in the figure. For static equilibrium take moment about bottom end.

$$M_A + M_B + Q'L = 0$$

$$\text{or} \quad Q' = -\frac{(M_A + M_B)}{L} = \frac{-r(1+c)(EI/L)}{L}$$

$$= -r(1+c)\left(\frac{EI}{L^2}\right) = -q\left(\frac{EI}{L^2}\right) \tag{6.100}$$

where $q = r(1 + c)$ which is generally referred to as *shear stiffness stability factor*.

If the column is restrained against additional rotation and the end B is allowed to sway by an amount v, the sway angle is given by $\phi' = v/L$. The restraining moments M'_A and M'_B at the ends A and B are both equal to $-(M_A + M_B)\phi'$. For static equilibrium

$$M'_A + M'_B + Q''L + Pv = 0$$

$$\text{or} \quad -2(M_A + M_B)\phi' + Q''L + Pv = 0 \tag{6.101}$$

Defining the sway angle when P is absent in the above equation (but not from its effect on M_A and M_B) by ϕ, the equation reduces to

$$-2(M_A + M_B)\phi + Q''L = 0 \tag{6.102}$$

Subtracting (6.102) from (6.101)

$$-2(M_A + M_B)(\phi' - \phi) + PL\phi' = 0 \quad (\text{since } v = L\phi') \tag{6.103}$$

Expressing the load P in terms of Euler's load P_e $(= \pi^2 EI/L^2)$

$$P = \rho P_e = \frac{\rho\pi^2 EI}{L^2}$$

Thus, (6.103) can be expressed as

$$-2r(1+c)\left(\frac{EI}{L}\right)(\phi' - \phi) + \rho\left[\frac{\pi^2 EI}{L^2}\right]\phi' L = 0$$

$$\text{or} \quad \phi' - \phi = \frac{\rho\pi^2\phi'}{2r(1+c)} \tag{6.104}$$

Therefore,

$$\frac{\phi'}{\phi} = m = \frac{1}{\left[1 - \frac{\rho\pi^2}{2r(1+c)}\right]} \tag{6.105}$$

The term m is defined as sway magnification factor. When P is absent and the joints do not rotate, the equilibrium equation (6.102) gives

$$M_A = M_B = -r(1+c)\left(\frac{EI}{L}\right)\phi = -\left(\frac{Q'L}{2}\right) \tag{6.106}$$

With the effect of axial load taken into account

$$M'_A = M'_B = -r(1+c)\left(\frac{EI}{L}\right)\phi' = -s\left(\frac{Q''L}{2}\right) \tag{6.107}$$

The above expressions can be used to determine the rotational stiffness at one end of an axially loaded column when the other end is allowed to sway. If the shear force is maintained zero, the corresponding rotational stiffness will automatically take into account the effect of side sway. From (6.100) and (6.101).

$$Q' = -\frac{M_A + M_B}{L}$$

$$\text{and} \quad Q'' = [2(M_A + M_B)\phi' - P(L\phi')]/L$$

If the total shear is to vanish $Q' + Q'' = 0$ i. e.

$$2(M_A + M_B)\phi' - PL\phi' = M_A + M_B$$

where

$$M_A = r\left(\frac{EI}{L}\right), \quad M_B = rc\left(\frac{EI}{L}\right) \quad \text{and} \quad P = \frac{\rho\pi^2 EI}{L}.$$

Therefore,

$$\phi' = \frac{1}{2\left[1 - \frac{\rho\pi^2}{2r(1+c)}\right]} = \frac{m}{2} \tag{6.108}$$

Thus the final moments are:

$$M'_A = r\left(\frac{EI}{L}\right) - r(1+c)\left(\frac{EI}{L}\right)\left(\frac{m}{2}\right)$$
$$= r\left[1-(1+c)\frac{m}{2}\right]\left(\frac{EI}{L}\right) = t\left(\frac{EI}{L}\right) \tag{6.109}$$
$$M'_B = rc\left(\frac{EI}{L}\right) - r(1+c)\left(\frac{EI}{L}\right)\left(\frac{m}{2}\right)$$
$$= r\left[c-(1+c)\frac{m}{2}\right]\left(\frac{EI}{L}\right) = t'\left(\frac{EI}{L}\right) \tag{6.110}$$

The terms t and t' are *rotational stiffness factors* for axially loaded compression member undergoing transverse relative displacement. The negative sign is used with t' because M_B is usually negative for varying ρ values, and hence positive values are tabulated. For $\rho = 0$, $t = +1$ and $-t' = -1$ and thus $M_B/M_A = -t'/t = -1$ and the ratio is recognized as a *carry-over-factor* used previously.

For the case when the member carries axial tension ($\rho = -\rho$)

$$m = \frac{1}{1+\left[\frac{\rho\pi^2}{2r(1+c)}\right]} \tag{6.111}$$

The modified value of m is used in (6.109) and (6.110) to compute t and t', respectively. However, it shall be noted that parameters r and rc in this case should correspond to member carrying axial tension. These values are also tabulated in Appendix A.2.

To demonstrate the effectiveness of the procedure developed above considers the continuous strut shown in Fig. 6.14 which has been previously analyzed by matrix stiffness approach. The terminal moments in this case are

$$M_{12} = t\left(\frac{EI}{L}\right)\theta_1 - t'\left(\frac{EI}{L}\right)\theta_2\,, \quad M_{21} = -t'\left(\frac{EI}{L}\right)\theta_1 + t\left(\frac{EI}{L}\right)\theta_2$$
$$\text{and} \quad M_{23} = r\left(\frac{EI}{L}\right)\theta_2$$

For the equilibrium of joints 1 and 2

$$M_1 = M_{12} = t\left(\frac{EI}{L}\right)\theta_1 - t'\left(\frac{EI}{L}\right)\theta_2 = 0$$
$$M_2 = M_{21} + M_{23} = -t'\left(\frac{EI}{L}\right)\theta_1 + (t+r)\left(\frac{EI}{L}\right)\theta_2 = 0$$
$$\text{i.e.} \quad \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \left(\frac{EI}{L}\right)\begin{bmatrix} t & -t' \\ -t' & (t+r) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

For elastic-instability,

$$|\bar{K}| = \begin{vmatrix} t & -t' \\ -t' & (t+r) \end{vmatrix} = t(t+r) - (t')^2 = 0$$

Using trial and modification procedure. For

$$\rho = 0.14: \quad |\bar{K}| = 0.4882$$

$$\rho = 0.16: \quad |\bar{K}| = -0.0338$$

By interpolation for $|\bar{K}| = 0$, $\rho = 0.1587$. Hence,

$$P_{\text{cr}} = \frac{0.1587\pi^2 EI}{L^2}.$$

For further illustration, consider the hinged base portal frame shown in the Fig. 6.19 with EI being same for all the members. An estimate of the critical value of load P at which the frame will buckle is required for the conditions: (i) when the frame is restrained from side sway movement at the beam level, and (ii) when restraint is removed to allow the frame to sway.

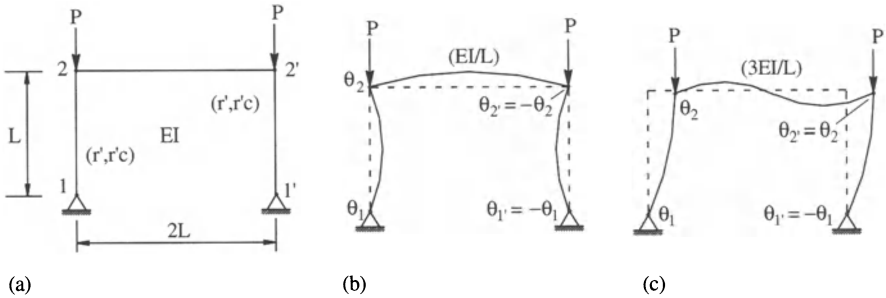


Fig. 6.19a–c. Buckling of hinged base portal frame. **a** Hinged based portal, **b** non-sway symmetrical mode, **c** sway antisymmetrical mode

Case-I: Due to perfect symmetry in loading and geometry, the symmetrical buckling mode will govern the critical value of the load. Thus $\theta_2 = \theta$ and $\theta_{2'} = -\theta$. The beam 2-2′ does not carry axial force and its effective stiffness is: $2(EI/2L) = (EI/L)$. The members 2-1 and 2′-1′ are hinged at the joints 1, and 1′, respectively, and hence their stiffness is given by $r'(EI/L)$ where $r' = r(1 - c^2)$. The terminal moments at the joint 2 can be written as

$$M_{21} = r'\left(\frac{EI}{L}\right)\theta \quad \text{and} \quad M_{22'} = \left(\frac{EI}{L}\right)\theta$$

Therefore,

$$M_2 = M_{21} + M_{22'} = (r' + 1)\left(\frac{EI}{L}\right)\theta$$

For elastic instability the stiffness of joint 2 must vanish, that is $r' + 1 = 0$ or $r' = -1$. From the stability functions table given in Appendix A.1 for $r' = -1$, $\rho = 1.1748$. Therefore,

$$P_{\text{cr}} = \frac{1.1748\pi^2 EI}{L^2} = \frac{11.5948EI}{L^2} \,.$$

Alternatively, the terminal moments at the nodes 2 and 1 are

$$\begin{aligned}
M_{21} &= r\left(\frac{EI}{L}\right)\theta_2 + rc\left(\frac{EI}{L}\right)\theta_1 \\
M_{12} &= r\left(\frac{EI}{L}\right)\theta_1 + rc\left(\frac{EI}{L}\right)\theta_2 \quad \text{and} \\
M_{22'} &= 2\left(\frac{EI}{2L}\right)\theta_2 = \left(\frac{EI}{L}\right)\theta_2
\end{aligned}$$

Thus for the equilibrium of joints 2 and 1

$$\begin{Bmatrix} M_2\ (= M_{21} + M_{22'}) \\ M_1\ (= M_{12}) \end{Bmatrix} = \left(\frac{EI}{L}\right)\begin{bmatrix} (r+1) & rc \\ rc & r \end{bmatrix}\begin{Bmatrix} \theta_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For elastic instability $|\bar{K}| = 0$. Therefore,

$$\begin{aligned}
\begin{vmatrix} (r+1) & rc \\ rc & r \end{vmatrix} &= (r+1)r - (rc)^2 = r[r(1-c^2)+1] \\
&= r(r'+1) = 0
\end{aligned}$$

where $r' = r(1-c^2)$. For non-trivial solution $r' + 1 = 0$. This result is the same as obtained directly.

Case-II: In this case there are four rotations and one translation resulting in a five-degrees-of-freedom system. The elastic instability condition will require an expansion of (5×5) order determinant. The solution will be cumbersome. However, the existence of perfect symmetry in loading and geometry and the fact that anti-symmetrical collapse mode occurs earlier than symmetrical collapse mode if there is no restraint against it, can be exploited for simplifying the computations. Due to the absence of lateral or side loading each column develops zero shears. With a sidesway moment, the joint rotations are equal in magnitude on two sides of the axis of symmetry. Thus using symmetry and no shear condition, the terminal moments are

$$\begin{aligned}
M_{21} &= t\left(\frac{EI}{L}\right)\theta_2 - t'\left(\frac{EI}{L}\right)\theta_1 \\
M_{12} &= t\left(\frac{EI}{L}\right)\theta_1 - t'\left(\frac{EI}{L}\right)\theta_2
\end{aligned}$$

The beam moment allowing for anti-symmetrical deformations is

$$M_{22'} = 6\left(\frac{EI}{2L}\right)\theta_2 = \left(\frac{3\,EI}{L}\right)\theta_2$$

Thus for equilibrium at the joints 2 and 1

$$(t+3)\left(\frac{EI}{L}\right)\theta_2 - t'\left(\frac{EI}{L}\right)\theta_1 = 0$$
$$-t'\left(\frac{EI}{L}\right)\theta_2 + t\left(\frac{EI}{L}\right)\theta_1 = 0$$

Therefore,

$$\begin{bmatrix} t+3 & -t' \\ -t' & t \end{bmatrix}\begin{Bmatrix} \theta_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Thus for elastic instability

$$|\bar{K}| = \begin{vmatrix} t+3 & -t' \\ -t' & t \end{vmatrix} = (t+3)t - (t')^2 = 0$$

Using trial and modification procedure with value of t and t' from the stability functions table given in Appendix A.2. For

$$\rho = 0.10: \quad |\bar{K}| = (0.647+3)(0.647) - (1.186)^2 = 0.953$$
$$\rho = 0.12: \quad |\bar{K}| = (0.570+3)(0.570) - (1.229)^2 = 0.524$$
$$\rho = 0.14: \quad |\bar{K}| = (0.491+3)(0.491) - (1.274)^2 = 0.091$$

By linear extrapolation for $|\bar{K}| = 0$, $\rho = 0.1442$. Therefore

$$P_{cr} = \frac{0.1442\pi^2 EI}{L^2} = \frac{1.423EI}{L^2}\,.$$

The following examples will illustrate the application of method to various types of rigid frames.

6.6.1 Single-Bay Multi-Storey Frames

Example 6.9. A symmetrical two-storey one-bay frame with (EI/L) values being equal for all the members shown in Fig. 6.20 is subjected to: (i) load P at the top of each column, (ii) load P at the top and $2P$ at the lower beam level in each column. Estimate the critical value of the load that will cause the frame to buckle.

Case-I: Frame subjected to loads P only at the top of columns:

(a) Non-sway symmetrical buckling mode

Because of perfect symmetry in loading and geometry, only half frame need be considered with $\theta_{i'} = -\theta_i$. The terminal moments for various members are the following.

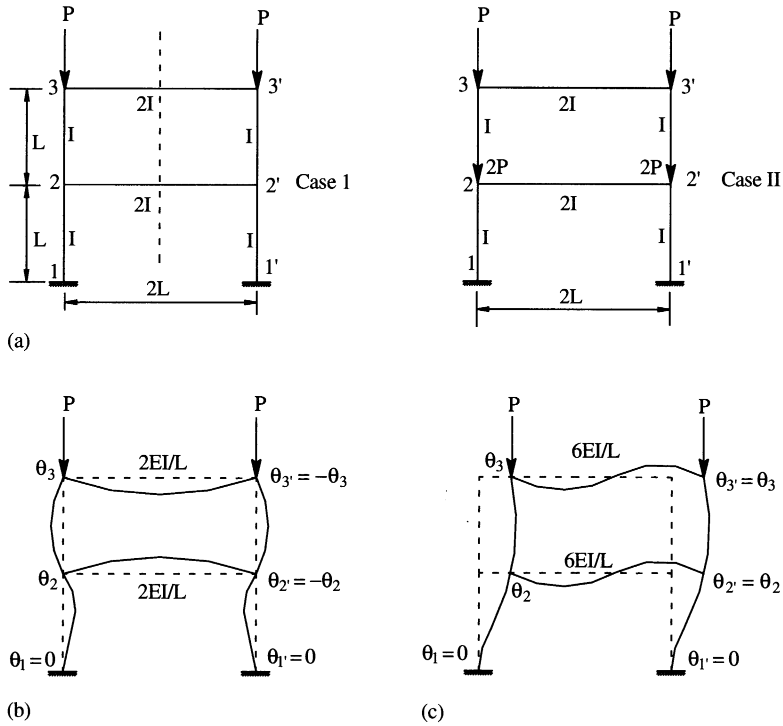


Fig. 6.20a–c. Buckling of symmetrical two storey one-bay frame. **a** Different loading cases, **b** non-sway symmetrical mode, **c** sway antisymmetrical mode

Columns:

$$M_{32} = r\left(\frac{EI}{L}\right)\theta_3 + rc\left(\frac{EI}{L}\right)\theta_2$$

$$M_{23} = r\left(\frac{EI}{L}\right)\theta_2 + rc\left(\frac{EI}{L}\right)\theta_3$$

$$M_{21} = r\left(\frac{EI}{L}\right)\theta_2$$

Beams:

$$M_{33'} = 2\left(\frac{EI}{L}\right)\theta_3 \quad \text{and} \quad M_{22'} = 2\left(\frac{EI}{L}\right)\theta_2$$

For the static equilibrium of joints 2 and 3

$$M_2 = M_{21} + M_{23} + M_{22'} = 2(r+1)\left(\frac{EI}{L}\right)\theta_2 + rc\left(\frac{EI}{L}\right)\theta_3$$

$$M_3 = M_{32} + M_{33'} = rc\left(\frac{EI}{L}\right)\theta_2 + (r+2)\theta_3$$

$$\text{or} \quad \begin{bmatrix} 2(r+1) & rc \\ rc & r+2 \end{bmatrix}\begin{Bmatrix}\theta_2 \\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix}$$

For elastic instability $|\bar{K}| = 0$ that is

$$|\bar{K}| = \begin{vmatrix} 2(r+1) & rc \\ rc & (r+2) \end{vmatrix} = 2(r+1)(r+2) - (rc)^2 = 0$$

Using trial and modification procedure with values of stability functions r and rc from the table given in Appendix A.1. Taking

$$\begin{aligned}
\rho = 1.72: \quad & r = 0.927 \quad \text{and} \quad (rc)^2 = 9.739\,, \\
& |\bar{K}| = 2(1.927)(2.927) - 9.739 = 1.542 \\
\rho = 1.76: \quad & r = 0.823 \quad \text{and} \quad (rc)^2 = 10.059\,, \\
& |\bar{K}| = 2(1.823)(2.823) - 10.059 = 0.234
\end{aligned}$$

By extrapolation for $|\bar{K}| = 0$, $\rho_{cr} = 1.7672$. Therefore,

$$P_{cr} = \frac{\rho_{cr}\pi^2 EI}{L^2} = \frac{1.7672\pi^2 EI}{L^2} = \frac{17.44\, EI}{L^2}$$

(b) Sway buckling mode

In this case due to antisymmetry $\theta_{i'} = \theta_i$ and the beam moments are $(6\,EI/L)\theta_i$. The terminal moments are given by the following.

Column moments:

$$\begin{aligned}
M_{32} &= t\left(\frac{EI}{L}\right)\theta_3 - t'\left(\frac{EI}{L}\right)\theta_2 \\
M_{23} &= t\left(\frac{EI}{L}\right)\theta_2 - t'\left(\frac{EI}{L}\right)\theta_3 \\
M_{21} &= t\left(\frac{EI}{L}\right)\theta_2
\end{aligned}$$

Beam moments: $M_{33'} = 6\left(\frac{EI}{L}\right)\theta_3$ and $M_{22'} = 6\left(\frac{EI}{L}\right)\theta_2$

Therefore, for elastic instability

$$|\bar{K}| = \begin{vmatrix} (2t+6) & -t' \\ -t' & t+6 \end{vmatrix} = (2t+6)(t+6) - (t')^2 = 0$$

Using trial and modification procedure for

$$\begin{aligned}
\rho = 0.50: \quad & t = -1.691 \quad \text{and} \quad t' = 2.792\,, \quad \text{thus} \\
& |\bar{K}| = [2(-1.691) + 6](-1.691 + 6) - (2.792)^2 = 3.4857
\end{aligned}$$

$$\rho = 0.54: \quad t = -2.099 \quad \text{and} \quad t' = 3.120\,, \quad \text{therefore}$$
$$|\bar{K}| = (-4.198 + 6) + (-2.099 + 6) - (3.120)^2 = -2.7048$$
$$\rho = 0.52: \quad t = -1.887 \quad \text{and} \quad t' = 2.949\,, \quad \text{thus}$$
$$|\bar{K}| = (-3.774 + 6)(-1.887 + 6) - (2.949)^2 = 0.459$$

For $|\bar{K}| = 0$, by interpolation $\rho_{\mathrm{cr}} = 0.5229$ Therefore,

$$P_{\mathrm{cr}} = \rho_{\mathrm{cr}} P_{\mathrm{e}} = \frac{0.5229\pi^2 EI}{L^2} = \frac{5.161\,EI}{L^2}\,.$$

(Since $\rho = 1.00$ for fixed-fixed column and 0.0625 for fixed-free column, this is value of $\rho = 0.5229$ is quite reasonable).

Case-II: In this case the axial loads carried by the columns of top and bottom storey are different. Axial force in members 3-2 and 2-1 are P and $3P$, respectively. Since (EI/L) and hence P_{e} is same for all the members, the values of ρ for the columns are proportional to axial load carried by them, i. e. $\rho_{23} = \rho$ and $\rho_{12} = 3\rho$.

(a) Symmetrical or non-sway buckling mode

The terminal moments are the following

Columns:

$$M_{21} = r_{12}\left(\frac{EI}{L}\right)\theta_2$$
$$M_{23} = r_{23}\left(\frac{EI}{L}\right)\theta_2 + (rc)_{23}\left(\frac{EI}{L}\right)\theta_3$$
$$M_{32} = r_{23}\left(\frac{EI}{L}\right)\theta_3 + (rc)_{23}\left(\frac{EI}{L}\right)\theta_2$$

Beams:

$$M_{22'} = 2\left[\frac{E(2I)}{2L}\right]\theta_2 = \left(\frac{2\,EI}{L}\right)\theta_2$$
$$M_{33'} = \left(\frac{2EI}{L}\right)\theta_3$$

For the equilibrium of joints 2 and 3

$$M_2 = (r_{12} + r_{23} + 2)\left(\frac{EI}{L}\right)\theta_2 + (rc)_{23}\left(\frac{EI}{L}\right)\theta_3 = 0$$
$$M_3 = (rc)_{23}\left(\frac{EI}{L}\right)\theta_2 + (r_{23} + 2)\left(\frac{EI}{L}\right)\theta_3 = 0$$

Thus for elastic instability

$$|\bar{K}| = \begin{vmatrix} (r_{12}+r_{23}+2) & (rc)_{23} \\ (rc)_{23} & (r_{23}+2) \end{vmatrix} = 0$$

$$\text{or} \quad (r_{12}+r_{23}+2)(r_{23}+2) - [(rc)_{23}]^2 = 0$$

Trial and modification procedure is used to compute the critical value of the loads.

For $\rho = 0.80$, $\rho_{23} = 0.80$ and $\rho_{12} = 2.40$. Substituting the values of stability functions r and rc from the stability functions table given in Appendix A.1.

$$|\bar{K}| = [(-1.301 + 2.816 + 2)(2.816 + 2) - 5.502] = 11.426$$

For

$$\rho = 1.00: \quad \rho_{23} = 1.00 \quad \text{and} \quad \rho_{12} = 3.00$$

$$|\bar{K}| = [(-5.032 + 2.467 + 2)(2.467 + 2) - (6.088)] = -8.612$$

For $|\bar{K}| = 0$, by interpolation

$$\rho = \frac{(-8.612)(0.80) - (11.426)(1.0)}{(-8.612) - (11.426)} = 0.914$$

For

$$\rho = 0.92: \quad \rho_{23} = 0.92 \quad \text{and} \quad \rho = 2.76$$

$$|\bar{K}| = [(-3.180 + 2.610 + 2)(2.610 + 2) - (5.839)] = 0.7533$$

By interpolation for $|\bar{K}| = 0$

$$\rho = \frac{(-8.612)(0.92) - (0.7533)(1.0)}{(-8.612) - (0.7533)} = 0.9263$$

Therefore,

$$P_{\text{cr}} = \frac{0.9263\pi^2 EI}{L^2} = \frac{9.14\, EI}{L^2}$$

(b) Antisymmetrical sway buckling mode ($\theta_{i'} = \theta_i$)

For this case the terminal beam moments are

$$M_{22'} = 6\left(\frac{EI}{L}\right)\theta_2 \quad \text{and} \quad M_{33'} = 6\left(\frac{EI}{L}\right)\theta_3$$

The corresponding equilibrium equations can be expressed as

$$\left(\frac{EI}{L}\right)\begin{bmatrix} (t_{12}+t_{23}+6) & -t'_{23} \\ -t'_{23} & (t_{23}+6) \end{bmatrix}\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix} (t_{12} + t_{23} + 6) & -t'_{23} \\ -t'_{23} & (t_{23} + 6) \end{vmatrix} = 0$$
$$= (t_{12} + t_{23} + 6)(t_{23} + 6) - (t'_{23})^2 = 0.$$

In this case also consider $\rho_{23} = \rho$ and $\rho_{12} = 3\rho$.

Trial and modification procedure is used to estimate the critical value of ρ. For

$\rho = 0.20$: $\rho_{23} = 0.20$ and $\rho_{12} = 0.60$

$$|\bar{K}| = (-2.842 + 0.235 + 6)(0.235 + 6) - (1.425)^2 = 19.125$$

$\rho = 0.24$: $\rho_{23} = 0.24$ and $\rho_{12} = 0.72$

$$|\bar{K}| = (-5.173 + 0.049 + 6)(0.049 + 6) - (1.540)^2 = 2.927$$

$\rho = 0.26$: $\rho_{23} = 0.26$ and $\rho_{12} = 0.78$

$$|\bar{K}| = (-7.217 - 0.050 + 6)(-0.050 + 6) - (1.603)^2 = -10.108$$

By interpolation for $|\bar{K}| = 0$

$$\rho = \frac{(-10.108)(0.24) - (2.927)(0.26)}{(-10.108) - (2.927)} = 0.2445$$

The values obtained for the sway case are lower than that for the non-sway case and hence are critical. Thus the critical value of P to produce elastic instability of the frame is

$$P_{\text{cr}} = \frac{0.244\pi^2 EI}{L^2} = \frac{2.413\,EI}{L^2}.$$

Example 6.10. A two-storey single-bay frame shown in Fig. 6.21 is subjected to loads P_1 at the top of the columns and P_2 at the mid-height as shown in the figure. If the magnitude of the load P_1 is equal to 0.4 P_e, estimate the value of P_2 that will cause the frame to buckle where $P_e = (\pi^2 EI/L^2)$ and EI is same for all the members.

The frame has perfect symmetry in geometry and loading. The buckling mode may be either symmetrical or antisymmetrical. In the symmetrical non-sway case shown in Fig. 6.21b the moment equilibrium equations for the joints 1 and 2 can be written as

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \left(\frac{EI}{L}\right) \begin{bmatrix} (r_{10} + r_{12} + 4) & (rc)_{12} \\ (rc)_{12} & (r_{12} + 4) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix} (r_{10} + r_{12} + 4) & (rc)_{12} \\ (rc)_{12} & (r_{12} + 4) \end{vmatrix} = (r_{10} + r_{12} + 4)(r_{12} + 4) - [(rc)_{12}]^2 = 0$$

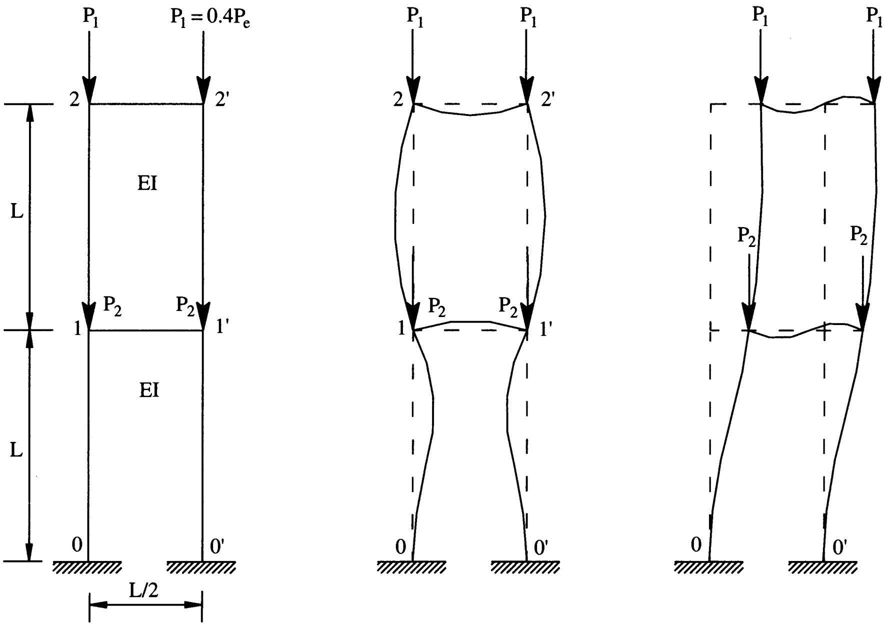


Fig. 6.21a–c. Buckling of two storey single bay frame. **a** Two-storey frame, **b** non-sway mode, **c** sway mode

For the member

$$1\text{-}2: \quad \rho_{12} = \frac{P_1}{P_e} = 0.4 \,, \quad r_{12} = 3.444 \quad \text{and} \quad (rc)^2 = 4.621$$

$$1\text{-}0: \quad \rho_{10} = \frac{P_1 + P_2}{P_e} = 0.40 + \rho \quad \text{where} \quad \rho = \frac{P_2}{P_e}$$

Therefore, the characteristic equation reduces to

$$(r_{10} + 3.444 + 4)(3.444 + 4) - 4.621 = 0 \quad \text{or} \quad r_{10} = -6.823$$

From the stability functions table given in Appendix A.1 for $r_{10} = -6.823$, $\rho_{10} = 3.1642$. Therefore,

$$\rho = 3.1642 - 0.40 = 2.7642 = \frac{P_2}{P_e}$$

$$P_2 = 2.7642\, P_e$$

For the sway antisymmetric buckling configuration (mode) shown in Fig. 6.21c, the corresponding instability equation reduces to

$$|\bar{K}| = (t_{10} + t_{12} + 12)(t_{12} + 12) - (t'_{12})^2 = 0 \,,$$

$$\text{since} \quad M_{22'} = M_{11'} = \frac{12EI}{L}\theta_i$$

For the member 1-2 with $\rho_{12} = 0.4$, $t_{12} = -0.878$ and $t'_{12} = 2.172$. Therefore,

$$(t_{10} - 0.878 + 12)(-0.878 + 12) - (2.172)^2 = 0$$

i.e. $t_{10} = -10.698$ and corresponding value of ρ_{10} from the stability functions table is 0.8396. Thus, $\rho = 0.8396 - 0.40 = 0.4396$ and hence $P_2 = 0.4396\, P_e$.

As a variation in the problem suppose that the force $P_2 = 0.40\, P_e$. It is required to estimate the values of force P_1 which will cause the frame to buckle. In this case

$$\rho_{12} = \frac{P_1}{P_e} \quad \text{and} \quad \rho_{10} = \frac{P_1 + P_2}{P_e} = (\rho_{12} + 0.40)$$

For the symmetrical buckling mode, for instability

$$|\bar{K}| = (r_{10} + r_{12} + 4)(r_{12} + 4) - [(rc)_{12}]^2 = 0$$

Adopt trial and modification procedure using stability functions from table given in Appendix A.1. For

$$\rho_{12} = 1.00: \quad |\bar{K}| = (1.678 + 2.467 + 4)(2.467 + 4) - 6.088 = 46.59$$

$$\rho_{12} = 1.80: \quad |\bar{K}| = (-0.519 + 0.717 + 4)(0.717 + 4) - 10.397 = 9.40$$

$$\rho_{12} = 2.00: \quad |\bar{K}| = (-1.301 + 0.143 + 4)(0.143 + 4) - 12.424 = -0.6496$$

$$\rho_{12} = 1.96: \quad |\bar{K}| = (-1.133 + 0.264 + 4)(0.264 + 4) - 11.967 = 1.3836$$

Therefore by interpolation for $|\bar{K}| = 0$, $\rho_{12} = 1.9872$ and thus

$$P_1 = 1.9872\, P_e = \frac{1.9872\pi^2 EI}{L^2}$$

For antisymmetric buckling configuration

$$|\bar{K}| = (t_{10} + t_{12} + 12)(t_{12} + 12) - (t'_{12})^2 = 0$$

Using stability functions values from the table given in Appendix A.2. For

$$\rho_{12} = 0.20: \quad |\bar{K}| = (-2.842 + 0.235 + 12)(0.235 + 12) - (1.425)^2 = 112.89$$

$$\rho_{12} = 0.40: \quad |\bar{K}| = (-8.159 - 0.878 + 12)(-0.878 + 12) - (2.172)^2 = 28.237$$

$$\rho_{12} = 0.44: \quad |\bar{K}| = (-10.725 - 1.174 + 12)(-1.174 + 12) - (2.392)^2 = -4.628$$

$$\rho_{12} = 0.42: \quad |\bar{K}| = (-9.303 - 1.022 + 12)(-1.022 + 12) - (2.278)^2 = 13.199$$

By interpolation for $|\bar{K}| = 0$, $\rho_{12} = 0.4348$ and thus

$$P_1 = 0.4348\, P_e\,.$$

6.6.2 Multi-Bay Rigid Frames

In case of multi-bay frames, the rotation degrees-of-freedom increase with the number of bays, thereby increasing the size of stiffness matrix. However, certain frames can be subdivided into a number of similar one-bay frames. This subdivision called the *principle of multiples* is based on an application of super positioning to structural properties. The super positioning depends upon the (EI/L) pattern for the entire frame being such that it breaks down into a number of patterns $\beta_1 R$, $\beta_2 R$ etc. for separate one-bay frames, where β_1, β_2, etc. are constants for individual subsidiary frames. The total loading P is also divided into $(P\beta_i / \sum \beta_i)$ components, so that each frame carries load proportional to its overall stiffness coefficient β. In such a frame satisfying the principle of multiples, all the joints at any particular beam level rotate by the same amount and the column sways are also, of course, identical in each storey. Therefore, the exact analysis of any of the subsidiary one-bay frames will lead directly to the exact analysis of the entire frame. For illustration consider the two-bay building frame shown in Fig. 6.22b along with its subsidiary one-bay frames. The two one-bay frames clearly add up to the original frame, since deformations are identical in each of the two subsidiary frames. Therefore, only one frame need be analysed. The moments and forces occurring in the actual frame are obtained by direct addition in the common interior columns.

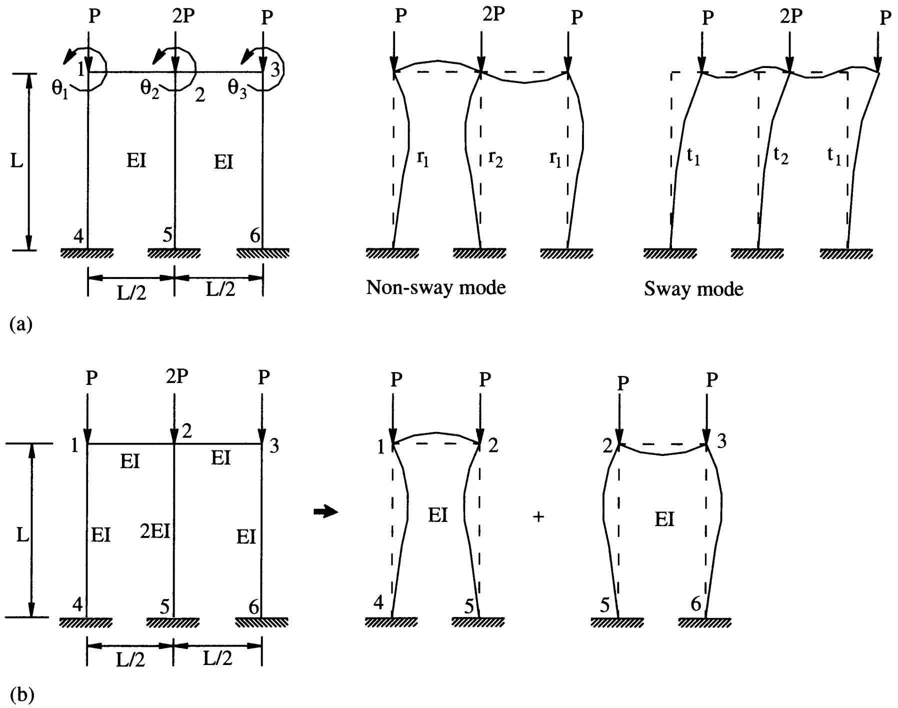


Fig. 6.22. Buckling of single storey two-bay frame. **a** Frame with EI values same for all the members, **b** frame with EI of interior column being twice of the exterior

Example 6.11. A single-storey two-bay (i.e. three columns) symmetrical frame is loaded symmetrically as shown in the Fig. 6.22. Estimate the first critical value of the load P that will cause the frame to buckle under following conditions: (i) EI values are same for all the members, and (ii) EI value for the interior column is twice that of a exterior column.

If the frame is prevented from the side sway there are three rotational degrees-of-freedom θ_1, θ_2 and θ_3. Here, the critical load has been determined without making use of symmetry of the system.

Case-I

$$\rho_{14} = \rho_{36} = \frac{P}{(\pi^2 EI/L^2)} = \rho \quad \text{and} \quad \rho_{25} = \frac{2P}{(\pi^2 EI/L^2)} = 2\rho$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix} (r_1+8) & 4 & 0 \\ 4 & (r_2+16) & 4 \\ 0 & 4 & (r_1+8) \end{vmatrix} = (r_1+8)[(r_2+16)(r_1+8) - 32]$$

$$(r_1+8)^2[(r_2+16) - 32/(r_1+8)] = 0$$

Two of the roots of the stability eigen-equation are apparent i.e. $r_1 = -8$ (i.e. $\rho = 3.2476$). For the third root $(r_2 + 16) - 32/(r_1 + 8) = 0$. Using trial and modification procedure, $\rho = 1.727$. Therefore,

$$P_{\text{cr}} = \frac{1.727\pi^2 EI}{L^2} = \frac{17.045\, EI}{L^2}$$

If the frame is allowed to sway the corresponding eigen-equation can be obtained by replacing r by t. For $t_1 = -8.00$, $\rho = 1.099$ and for $(t_2 + 16) - 32/(t_1 + 8) = 0$, $\rho = 0.4252$. Therefore,

$$P_{\text{cr}} = \frac{0.4252\pi^2 EI}{L^2} = \frac{4.197\, EI}{L^2}$$

Case-II

$$\rho_{14} = \rho_{36} = \frac{P}{(\pi^2 EI/L^2)} = \rho \quad \text{and} \quad \rho_{25} = \frac{2P}{(\pi^2 E(2I)/L^2)} = \rho\,.$$

Thus $r_2 = r_1 = r$ and for elastic instability

$$|\bar{K}| = \begin{vmatrix} (r+8) & 4 & 0 \\ 4 & (2r+16) & 4 \\ 0 & 4 & (r+8) \end{vmatrix} = 2(r+8)(r^2+16r+48)$$

$$= 2(r+8)(r+4)(r+12) = 0$$

Therefore, $r = -4.0$, -8.0 and -12.0.

Alternatively, as the frame satisfies the criterion of multiplies, it can be split into two single bay frames shown in Fig. 6.22b. Analysis of any one of these *subsidiary frames* leads to the analysis of entire frame. For elastic instability of the frame shown in Fig. 6.21b.

$$|\bar{K}| = \begin{vmatrix} r+8 & 4 \\ 4 & (r+8) \end{vmatrix} = (r+4)(r+12) = 0$$

Thus $r = -4.0$ and -12.0. The critical load P_{cr} is governed by the lowest value $r = -4.0$. From the stability functions table given in Appendix A.1 for $r = -4.0$, $\rho = 2.877$. The critical value of the load is

$$P_{cr} = \frac{2.877\pi^2 EI}{L^2} = \frac{28.396\, EI}{L^2} \, .$$

For the *sway buckling mode* with anti-symmetric configuration the force-displacement equation $|\bar{K}|\{\Delta\} = \{F\}$ is given by

$$\left(\frac{EI}{L}\right)\begin{bmatrix} t_1+8 & 4 \\ 4 & t_1+8 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

For anti-symmetric mode $\theta_2 = \theta_1$ and the above equations reduce to $(EI/L)(t_1 + 12)\theta_1 = F_1$. For elastic instability $t_1 + 12 = 0$ i.e. $t_1 = -12.0$. From the stability functions table for $t_1 = -12.0$, $\rho = 0.854$. Therefore,

$$P_{cr} = \frac{0.854\pi^2 EI}{L^2} = \frac{8.429\, EI}{L^2}$$

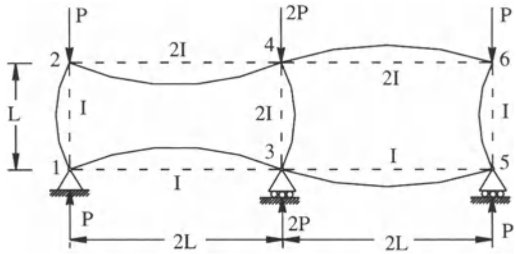
Example 6.12. The vertical members (columns) of a multi-bay closed framed structure are subjected to compressive loads as shown in Fig. 6.23. Estimate the critical values of the load that will cause the frame to buckle.

The critical loads predicted by non-sway symmetrical and anti-symmetrical buckling modes are larger than those of sway modes, thus only sway modes will be considered for the analysis.

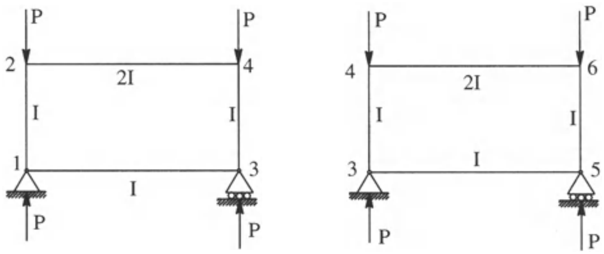
The frame has seven degrees-of-freedom (one sway and six rotations) and hence will involve operations with (7×7) determinant. However, a close scrutiny reveals that the frame satisfies the criterion for multiples and hence can be split into two single-bay closed frames. Consider the frame of Fig. 6.23b, the terminal moments are:

$$M_{13} = \left[\frac{6EI}{2L}\right]\theta_1 = \left(\frac{3EI}{L}\right)\theta_1 \quad \text{and} \quad M_{24} = \left[\frac{6E(2I)}{2L}\right]\theta_2 = \left(\frac{6EI}{L}\right)\theta_2$$

(Due to antisymmetry of the mode, $\theta_3 = \theta_1$ and $\theta_4 = \theta_2$).



(a)



(b)

Fig. 6.23a,b. Buckling of frame satisfying multiples criterion. **a** Multi-bay closed frame structure, **b** two single-bay subsidiary frames

Using rotational stiffness without shear coefficients, the terminal moments in the columns are:

$$M_{12} = t_{12}\left(\frac{EI}{L}\right)\theta_1 - t'_{12}\left(\frac{EI}{L}\right)\theta_2$$

$$\text{and} \quad M_{21} = t_{12}\left(\frac{EI}{L}\right)\theta_2 - t'_{12}\left(\frac{EI}{L}\right)\theta_1$$

Thus for equilibrium of joints 1 and 2

$$\begin{pmatrix} M_1 = M_{12} + M_{13} \\ M_2 = M_{21} + M_{24} \end{pmatrix} = \left(\frac{EI}{L}\right)\begin{bmatrix} (t_{12}+3) & -t'_{12} \\ -t'_{12} & (t_{12}+6) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

For elastic instability

$$|\bar{K}| = \begin{vmatrix} (t_{12}+3) & -t'_{12} \\ -t'_{12} & (t_{12}+6) \end{vmatrix} = (t_{12}+3)(t_{12}+6) - (t'_{12})^2 = 0$$

Using trial and modification method with stiffness values from the stability functions table. For

$$\rho = 0.50: \quad t_{12} = -1.691\,, \quad t'_{12} = 2.792 \quad \text{and} \quad |\bar{K}| = -2.15$$

$$\rho = 0.48: \quad t_{12} = -1.508\,, \quad t'_{12} = 2.648 \quad \text{and} \quad |\bar{K}| = -0.3098$$

$$\rho = 0.46: \quad t_{12} = -1.336\,, \quad t'_{12} = 2.515 \quad \text{and} \quad |\bar{K}| = 1.4357$$

By interpolation for $|\bar{K}| = 0$, $\rho = 0.4764$. Thus the critical value of the load P is:

$$P_{cr} = \frac{0.4764\pi^2 EI}{L^2} = \frac{4.7018EI}{L^2}$$

6.6.3 Substitute Frame Method

The special methods taking advantage of symmetrical or anti-symmetrical structural actions are dependent on both geometry and loading being symmetrical. The *substitute frame method* consists in replacing the actual frame with geometrical symmetry but unsymmetrical in its stiffness values by a symmetrical one-bay frame. The *flexural rigidity* of the columns of the substitute frame is taken to be the average of the flexural rigidities of the columns of the original frame. The flexural rigidity of the beam of the substitute frame is taken to be the sum of flexural rigidities of the beams in the original frame. The columns of the substitute frame are subjected to the loads, the magnitude of which is average of loads acting on the original frame. The substitute frame can now be analysed by any of the special methods to obtain a rough estimate of the buckling load.

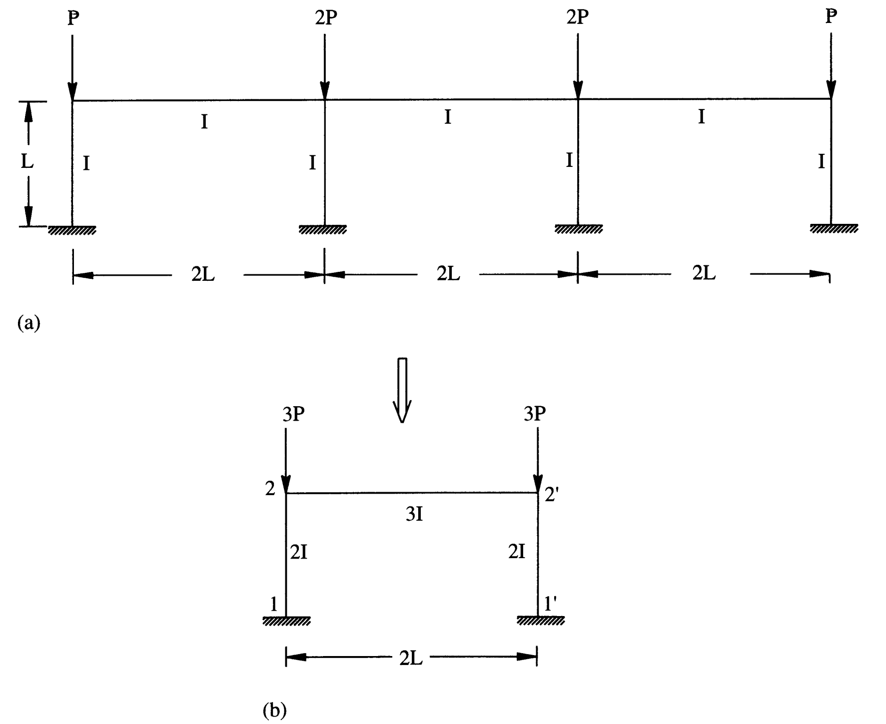


Fig. 6.24a,b. Substitute frame for a multibay frame. **a** Single-storey three-bay frame, **b** substitute frame

For illustration consider the single-storey multi-bay frame shown in the Fig. 6.24a. EI values are same for all the members. This frame does not satisfy the *principle of multiples*. The corresponding substitute frame is shown in Fig. 6.24b. The EI values of various components of substitute frame are

$$\textbf{Beam:} \qquad (EI_\mathrm{b})_\mathrm{s} = \sum EI_\mathrm{b} = 3EI$$

$$\textbf{Columns:} \quad (EI_\mathrm{c})_\mathrm{s} = \frac{1}{2}\sum EI_\mathrm{c} = \frac{1}{2}E(I + I + I + I) = 2EI$$

Load on each column of the substitute frame P_s is given by

$$P_\mathrm{s} = \frac{1}{2}\sum P = \frac{1}{2}(P + 2P + 2P + P) = 3P$$

Thus the degrees-of-freedom reduce from 5 (four rotational and one sway) to 3 (two rotations and one sway). The problem can further be simplified by invoking the perfect symmetry of substitute frame in geometry and loading. As usual the critical load is governed by the sway or anti-symmetric buckling mode.

In the sway mode, the terminal moments are

$$M_{21} = t\left(\frac{2EI}{L}\right)\theta_2$$

$$M_{22'} = \left[\frac{6E(3I)}{2L}\right]\theta_2 = \left(\frac{9EI}{L}\right)\theta_2$$

For the equilibrium of joint 2:

$$M_2 = M_{21} + M_{22'} = (2t + 9)\left(\frac{EI}{L}\right)\theta_2 = 0$$

Therefore, for elastic instability

$$(2t + 9) = 0 \quad \text{or} \quad t = -4.50$$

From the stability functions table given in Appendix A.1, for $t = -4.50$, $\rho_{\mathrm{s.cr}} = 0.6927$. The ρ_s value for each column of substitute frame is approximately given by

$$\rho_\mathrm{s} = \frac{\sum P}{\sum P_\mathrm{e}} = \frac{6P}{4P_\mathrm{e}} = 1.5\,\rho \quad \text{where} \quad \rho = \frac{P}{P_\mathrm{e}}$$

Therefore,

$$P_\mathrm{cr} = \left(\frac{\rho_{\mathrm{s,cr}}}{1.5}\right)\left(\frac{\pi^2 EI}{L^2}\right) = \left(\frac{0.6927}{1.5}\right)\left(\frac{\pi^2 EI}{L^2}\right) = \frac{4.558\,EI}{L^2}$$

As a variation considers the case when EI values of interior columns is twice that of exterior columns as shown in Fig. 6.25. In this case, the frame satisfies the principle of multiples and hence can be split into three symmetrical subsidiary frames.

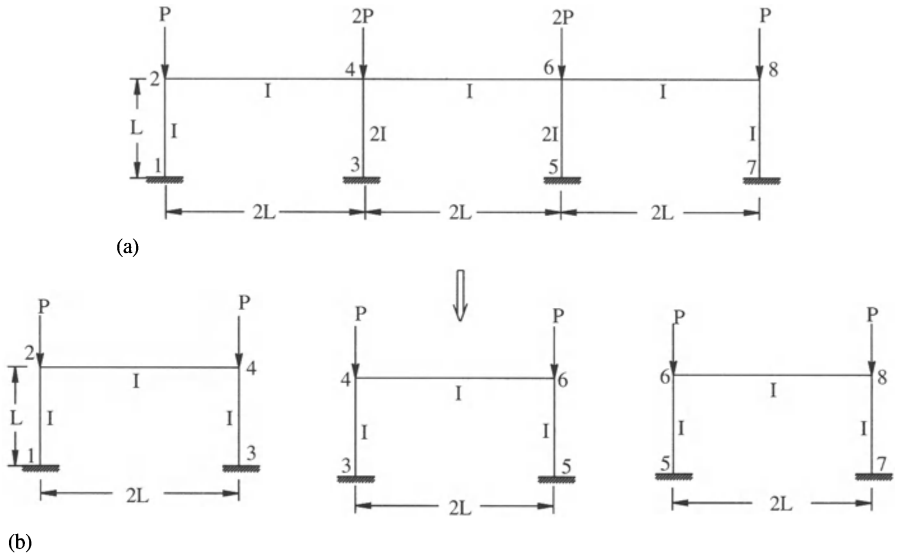


Fig. 6.25a,b. Splitting of multibay frame into single-bay subsidiary frames. **a** Multi-bay frame, **b** subsidiary frame

Each of the subsidiary frames is perfectly symmetric in geometry and loading with EI values being same for all the members. The critical load can be determined by the consideration of any one of the subsidiary frames. For a sway or anti-symmetric buckling mode

$$t + 3 = 0 \quad \text{or} \quad t = -3.00$$

From the stability functions table for $t = -3.00$, $\rho_{cr} = 0.610$, and hence

$$P_{cr} = \frac{0.61\pi^2 EI}{L^2} = \frac{6.02EI}{L^2}$$

If the principle of multiples is not invoked, and frame is analysed by using the substitute frame method

$$(EI_b)_s = \sum (EI)_b = 3\,EI$$

$$(EI_c)_s = \frac{1}{2}\sum EI_c = \frac{1}{2}E(I + 2I + 2I + I) = 3\,EI$$

and

$$P_s = 3P\,.$$

The terminal moments are

$$M_{21} = t\left(\frac{3\,EI}{L}\right) \quad \text{and} \quad M_{22'} = \left[\frac{6E(3I)}{2L}\right] = \left(\frac{9EI}{L}\right)$$

Thus, for elastic instability

$$3t + 9 = 0 \quad \text{or} \quad t = -3.00$$

From the stability functions table for $t = -3.00$, $\rho_{s,cr} = 0.61$. The ρ value for the substitute frame columns is given by

$$\rho_s = \frac{\sum P}{\sum P_e} = \frac{6P}{(1+2+2+1)P_e} = \rho \quad \text{where} \quad \rho = \frac{P}{P_e}$$

Therefore,

$$P_{cr} = \rho_{s,cr}\left(\frac{\pi^2 EI}{L^2}\right) = \frac{0.61\pi^2 EI}{L^2} = \frac{6.02\,EI}{L^2}$$

It should be noted that for the frames satisfying the principle of multiples both the methods give identical results.

6.7 Rigidly Connected Trusses

For *triangular subset trusses* (as opposed to *vierendeel truss* which are rectangular in form) the relative displacements of the ends of the members other than those due to axial shortening are zero. Thus in the stability analysis Δ will be zero. The following examples will illustrate the application of stability functions to this class of structures.

Example 6.13. A two-bar rigidly connected truss shown in Fig. 6.26 supports a load P. Estimate the value of P to produce elastic instability of truss.

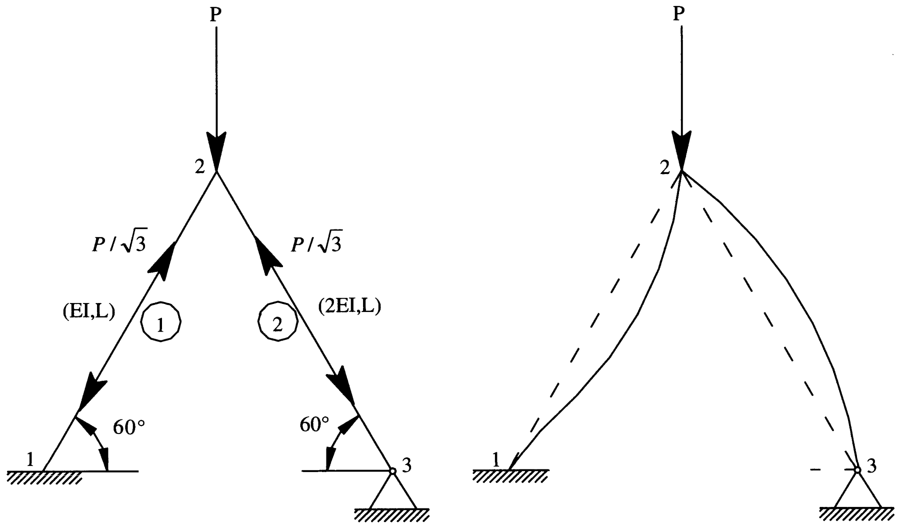


Fig. 6.26. Two-bar rigidly connected truss

The primary forces in the members are obtained by assuming the joint 2 to be hinged. These forces will be altered slightly by the moment induced by linear joint displacements. However, this effect is neglected. Ignoring the linear displacement of the joint 2, the only displacements to be considered are the rotation θ_2 and θ_3 at the joints 2 and 3, respectively.

The force-displacement relationship $[F] = |\bar{K}|[\Delta]$ is given by

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \left(\frac{EI}{L}\right) \begin{bmatrix} (r_1 + 2r_2) & 2(rc)_2 \\ 2(rc)_2 & 2r_2 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix}$$

For elastic instability $|\bar{K}| = 0$ i.e.

$$\begin{vmatrix} (r_1 + 2r_2) & 2(rc)_2 \\ 2(rc)_2 & 2r_2 \end{vmatrix} = r_1 + 2r_2[1 - (c_2)^2] = r_1 + 2r_2' = 0$$

This result can also be obtained directly by considering the stiffness of joint 2 with member 2-3 being hinged at the far end.

The primary forces in the members are:

$$P_1 = P_2 = (P/\sqrt{3})\ .$$

Thus ρ values for the members are given by

$$\rho_1 = \frac{P/\sqrt{3}}{\pi^2 EI/L^2} \quad \text{and} \quad \rho_2 = \frac{P/\sqrt{3}}{\pi^2 (2\,EI)/L^2}$$

or $\rho_1 = 2\rho$ and $\rho_2 = \rho$. The value of ρ_{cr} to cause collapse must, however, be determined by trial and modification.

From the stability functions table, by interpolation $\rho = 1.0086$. The critical value of the load is

$$P_{\text{cr}} = \sqrt{3}P_{2,\text{cr}} = \sqrt{3}\left[\frac{1.0086\pi^2(2EI)}{L^2}\right] = \frac{34.483\,EI}{L^2}$$

Example 6.14. A three-bar rigidly connected truss is subjected to a load P applied symmetrically as shown in Fig. 6.27. EI and L values are same for all the members. Estimate the critical value of loads P that will cause the frame to buckle.

The primary forces in the members are computed by assuming the joints to be pinned. The force in each member is equal to $(P/\sqrt{3})$. Therefore, ρ and rotational stiffness values r and rc are same for all members. The system has three degrees-of-freedom θ_1, θ_2 and θ_3. For the equilibrium of joints

$$\begin{Bmatrix} M_1\ (= M_{12} + M_{13}) \\ M_2\ (= M_{21} + M_{23}) \\ M_3\ (= M_{31} + M_{32}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \left(\frac{EI}{L}\right) \begin{bmatrix} 2r & rc & rc \\ rc & 2r & rc \\ rc & rc & 2r \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

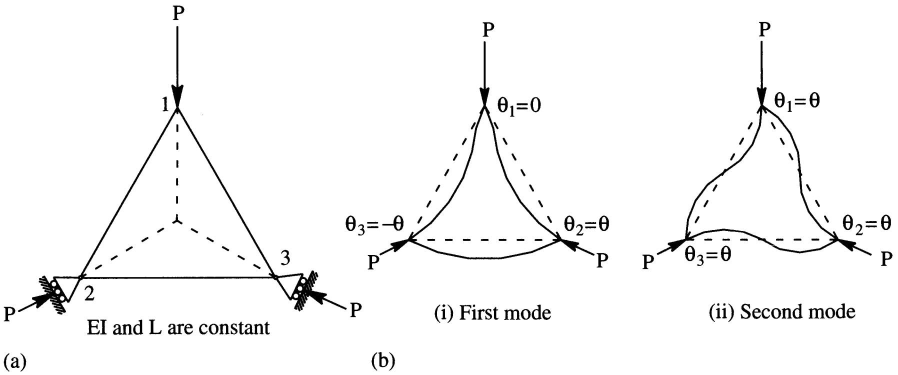


Fig. 6.27a,b. Buckling of three-bar rigidly connected truss. **a** Three-bar truss, **b** buckling modes

For elastic instability $|\bar{K}| = 0$, i.e.

$$\begin{vmatrix} 2r & rc & rc \\ rc & 2r & rc \\ rc & rc & 2r \end{vmatrix} = 2r^3(1+c)(2-c)^2 = 0$$

The first two roots of the eigen-value equation are given by: $(2-c)^2 = 0$ i.e. $c = 2.00$. This case corresponds to $\theta_1 = 0$ and $\theta_3 = -\theta_2 = \theta$. From the stability functions table for $c = 2.00$, $\rho = 1.5077$.

The third root of eigen-value equation is given by: $1 + c = 0$ or $c = -1.00$. This case corresponds to $\theta_1 = \theta_2 = \theta_3 = \theta$. From the stability functions table for $c = -1.00$, $\rho = 4.00$.

Therefore, the critical value of the load P is controlled by $\rho = 1.5077$ and hence

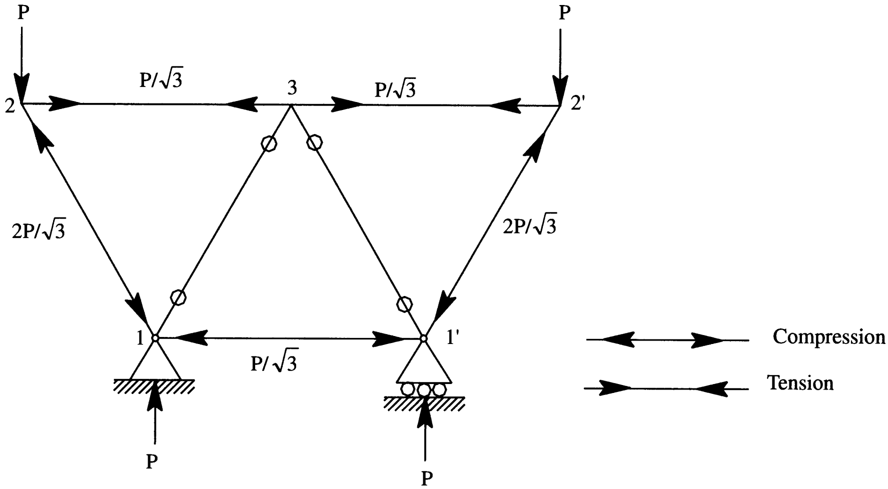
$$P_{\text{cr}} = \sqrt{3}\left(\frac{1.5077\pi^2 EI}{L^2}\right) = \frac{25.77\,EI}{L^2}$$

Example 6.15. Estimate the critical load factor N_e against elastic instability of rigidly connected truss shown in Fig. 6.28. EI and L are same for all members. The value of P_e for each member is $\sqrt{3}P$.

Treating the truss to be pin-jointed, the primary axial forces obtained in various members are shown in Fig. 6.28. Since the P_e is same for all the members, the ρ_{ij} $(= P_{ij}/P_{e,ij})$ values are proportional to the axial forces in the members i.e.,

$$\rho_{12} = 2\rho\,, \quad \rho_{13} = (0)\rho\,, \quad \rho_{11'} = \rho \quad \text{and} \quad \rho_{23} = -\rho$$

where $\rho = 1/3$. Disregarding the linear displacements of the joints, only rotational displacements at the joints 1 and 2 need be considered. Due to symmetry $\theta_{1'} = -\theta_1$, $\theta_{2'} = -\theta_2$ and $\theta_3 = 0$.



EI and L are the same for all the members

Fig. 6.28. Buckling of rigidly connected symmetrical truss

The terminal moments in the rigidly connected members meeting at joint 1 are:

$$M_{12} = r_{12}\left(\frac{EI}{L}\right)\theta_1 - (rc)_{12}\left(\frac{EI}{L}\right)\theta_2$$

$$M_{13} = r_{13}\left(\frac{EI}{L}\right)\theta_1 - (rc)_{13}\left(\frac{EI}{L}\right)\theta_3$$

$$M_{11'} = r_{11'}\left(\frac{EI}{L}\right)\theta_1 - (rc)_{11'}\left(\frac{EI}{L}\right)\theta_{1'}$$

For moment equilibrium of joint 1

$$\begin{aligned} M_1 &= M_{12} + M_{13} + M_{11'} \\ &= [r_{12} + r_{13} + r_{11'} - (rc)_{11'}]\left(\frac{EI}{L}\right)\theta_1 - (rc)_{12}\left(\frac{EI}{L}\right)\theta_2 = 0 \end{aligned}$$

(since $\theta_3 = 0$ and $\theta_{1'} = \theta_1$)

The terminal moments in the members meeting at joint 2 are:

$$M_{21} = r_{12}\left(\frac{EI}{L}\right)\theta_2 - (rc)_{12}\left(\frac{EI}{L}\right)\theta_1$$

$$M_{23} = r_{23}\left(\frac{EI}{L}\right)\theta_2 - (rc)_{23}\left(\frac{EI}{L}\right)\theta_3$$

Therefore, for the moment equilibrium of joint 2

$$M_2 = M_{21} + M_{23} = -(rc)_{12}\left(\frac{EI}{L}\right)\theta_1 + (r_{12} + r_{23})\left(\frac{EI}{L}\right)\theta_2 = 0$$

(since $\theta_3 = 0$)

For elastic instability

$$|\bar{K}| = \begin{vmatrix} r_{12} + r_{13} + r_{11'} - (rc)_{11'} & -(rc)_{12} \\ -(rc)_{12} & (r_{12} + r_{23}) \end{vmatrix} = 0$$

$$\text{or} \quad |\bar{K}| = [r_{12} + r_{13} + r_{11'} - (rc)_{11'}](r_{12} + r_{23}) - [(rc)_{12}]^2 = 0$$

The value of ρ to cause collapse is determined by trial and modification procedure. Let $\rho = 1.00$, therefore, the required stability coefficients for various members are:

$$\begin{aligned} &\rho_{12} = 2.00: && r_{12} = 0.143\,, && [(rc)_{12}]^2 = 12.424 \\ &\rho_{13} = 0.00: && r_{13} = 4.000 \\ &\rho_{11'} = 1.00: && r_{11'} = 2.467\,, && (rc)_{11'} = 2.467 \\ &\rho_{23} = -1.00: && r_{23} = 5.175 \end{aligned}$$

and

$$|\bar{K}| = (0.143 + 4.000 + 2.467 - 2.467)(0.143 + 5.175) - 12.424 = +9.608$$

For $\rho = 1.10$

$$\begin{aligned} &\rho_{12} = 2.20: && r_{12} = -0.519\,, && [(rc)_{12}]^2 = 15.219 \\ &\rho_{13} = 0.00: && r_{13} = 4.000 \\ &\rho_{11'} = 1.10: && r_{11'} = 2.282\,, && (rc)_{11'} = 2.535 \\ &\rho_{23} = -1.10: && r_{23} = 5.278 \end{aligned}$$

and

$$|\bar{K}| = (-0.519 + 4.000 + 2.282 - 2.535)(-0.519 + 5.278) - 15.219 = +0.143$$

For $\rho = 1.12$

$$\begin{aligned} &\rho_{12} = 2.24: && r_{12} = -0.665\,, && [(rc)_{12}]^2 = 15.904 \\ &\rho_{13} = 0.00: && r_{13} = 4.000 \\ &\rho_{11'} = 1.12: && r_{11'} = 2.245\,, && (rc)_{11'} = 2.550 \\ &\rho_{23} = -1.12: && r_{23} = 5.361 \end{aligned}$$

and

$$|\bar{K}| = (-0.665 + 4.000 + 2.245 - 2.550)(-0.665 + 5.361) - 15.904 = -1.675$$

By interpolation for $|\bar{K}| = 0$, $\rho_{cr} = 1.10157$. The load factor at collapse,

$$N_e = \frac{\rho_{cr}}{\rho} = \frac{1.10157}{(1/3)} = 3.3047\,.$$

It should be noted that the member 1-2 in compression would be weakest if all joints were pinned. If all joints were in this condition $N_e = 1.5$ would produce collapse of the member 1-2, and hence of entire structure. On the other hand if the truss is rigid-jointed (fixed), $N_e = 6.0$ will cause collapse. The N_e obtained lies between 1.5 and 6.0 and hence is reasonable.

6.8 Moment Distribution Method

In the preceding sections the *stiffness influence coefficients* of members subjected to axial loads have been used in expressing terminal moments as *superposition* of end moments caused by actual, unknown rotations and displacements. The equilibrium of joints in terms of terminal moments in the members meeting at the respective joints provided a set of simultaneous equations with displacements as unknowns. The matrix of coefficients of displacements furnished the *structure stiffness matrix* $|\bar{K}|$ which was used in computation of buckling load. A discerning reader will note that this is nothing but the *slope-displacement method* if the terminal moment due to external loads on the member with ends presumed to be restrained were superimposed on the terminal moments due to end displacements. For example for a prismatic member AB of length L with constant EI. The terminal moments are:

$$M_{AB} = r\left(\frac{EI}{L}\right)\theta_A + rc\left(\frac{EI}{L}\right)\theta_B - q\left(\frac{EI}{L}\right)\left(\frac{\Delta}{L}\right) + M_{fAB}$$
$$M_{BA} = rc\left(\frac{EI}{L}\right)\theta_A + r\left(\frac{EI}{L}\right)\theta_B - q\left(\frac{EI}{L}\right)\left(\frac{\Delta}{L}\right) + M_{fBA} \qquad (6.112)$$

where Δ is relative transverse displacement at the ends of the members, M_{fAB} and M_{fBA} are *fixed end moments* due to transverse loads.

As in conventional analysis, the iterative *moment distribution method* can be conveniently used for obtaining member end or terminal moments without actually solving any equation. However, in addition to stiffness influence coefficients, the procedure requires evaluation of fixed-end moments caused by transverse loads acting on members subjected to axial forces. These moments are function of both lateral and axial load. For a fixed-ended *beam-column i-j* of length L shown in Fig. 6.29 the fixed-end moments for a uniformly distributed load w over the entire span can be obtained as follows.

Using second-order formulation of the problem, the governing equation is given by:

$$EI\frac{d^2y}{dx^2} = -M_x = M_i - \frac{wLx}{2} + \frac{wx^2}{2} - Py \qquad (6.113)$$

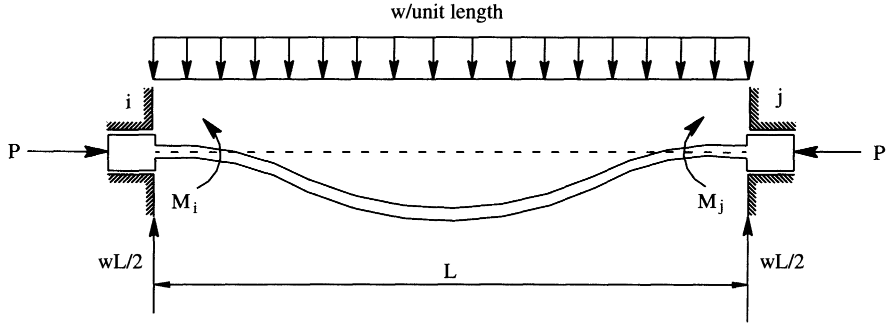


Fig. 6.29. Fixed-end moments due to uniformly distributed loads

where M_x is bending moment at a point at distance x from i. Thus

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{M_i}{EI} + \frac{w}{2EI}(x^2 - xL) \tag{6.114}$$

where $\alpha^2 = (P/EI)$. The general solution to (6.114) is

$$y = A\sin\alpha x + B\cos\alpha x + \frac{w}{2P}(x^2 - xL) + \left(\frac{M_i}{P} - \frac{w}{P\alpha^2}\right) \tag{6.115}$$

The boundary conditions are

$$y(0) = 0\,, \quad \text{and} \quad y'(L/2) = 0 \quad \text{(due to symmetry)}$$

$$y(0) = B + \left(\frac{M_i}{P} - \frac{w}{P\alpha^2}\right) = 0 \quad \text{or} \quad B = \frac{1}{P\alpha^2}(-\alpha^2 M_i + w)$$

$$y' = \alpha A\cos\alpha x - \alpha B\sin\alpha x + \frac{w}{2P}(2x - L)$$

$$y'\left(\frac{L}{2}\right) = \alpha A\cos\left(\frac{\alpha L}{2}\right) - \alpha B\sin\left(\frac{\alpha L}{2}\right) = 0$$

$$\text{or} \quad A = B\tan\left(\frac{\alpha L}{2}\right) = \frac{1}{P\alpha^2}(-\alpha^2 M_i + w)\tan\left(\frac{\alpha L}{2}\right)$$

The fixed end moment M_{fij} is given by the condition $y'(0) = 0$

$$y'(0) = \alpha A - \frac{wL}{2P} = \frac{\alpha}{P\alpha^2}(-\alpha^2 M_i + w)\tan\frac{\alpha L}{2} - \frac{wL}{2P} = 0$$

Therefore,

$$\begin{aligned} M_i = M_{fij} &= -wL^2\left[\frac{\tan(\alpha L/2) - (\alpha L/2)}{(\alpha L)^2\tan(\alpha L/2)}\right] \\ &= -wL^2\left[\frac{\tan\psi - \psi}{4\psi^2\tan\psi}\right] = -wL^2(m_{fw}) \end{aligned} \tag{6.116}$$

The quantity within the parentheses is termed *magnification factor* m_{fw}. Following the above procedure, the *fixed-end moments* for a concentrated load W acting at the mid-span of a beam-column is obtained as

$$M_{fij} = -WL\left[\frac{1 - \cos\psi}{8\psi\sin\psi}\right] = -WL(m_{fc}) \tag{6.117}$$

where

$$\psi = \left(\frac{\alpha L}{2}\right) = \frac{\pi}{2}\sqrt{\frac{P}{P_e}} = \frac{\pi}{2}\sqrt{\rho}\,.$$

For the member subjected to *axial tension* the corresponding expressions for the magnification factors are given by

$$m_{\mathrm{fw}} = \frac{\psi - \tanh\psi}{4\psi^2 \tanh\psi} \tag{6.118}$$

and

$$m_{\mathrm{fc}} = \frac{\cosh\psi - 1}{8\psi \sinh\psi} \tag{6.119}$$

For the selected values of ρ, these functions are tabulated in Appendix A.3.

The moment distribution method can be effectively used in determining the *load factor* N_{e} against elastic instability collapse of the entire structure. The method consists in applying an external moment at one of the joints, and balancing all other joints of the structure except the one to which moment is applied. If the *carry over of moments* back to this joint exceeds the originally applied moment, it is apparent that moments can not converge. On the other hand, when the collapse load factor N_{e} has been chosen correctly, the sum of the moments carried back to the joint considered will just be equal to the moment originally applied there i. e. the moment at joint will vanish. In the other words, it consists in finding the N_{e} value to make any particular joint have zero *rotational stiffness*. The following example will illustrate the principle.

Example 6.16. A pin-based single-span rigid portal frame shown in Fig. 6.30 is subjected to two vertical concentrated forces P, each acting directly over columns 1-2 and 1′-2′. Determine the critical value of load P that will cause the frame to collapse. EI is constant throughout.

(1) For the symmetrical buckling mode

$$\bar{k}_{21} = r\left(\frac{EI}{L}\right), \quad \bar{k}_{12} = rc\left(\frac{EI}{L}\right)$$

with carry over factor c.

$$\bar{k}_{22'} = r_{22'}(1 - c_{22'})\left(\frac{EI}{L}\right)_{22'} = \frac{2EI}{0.6L} = 3.333\left(\frac{EI}{L}\right)$$

The relative primary moments induced in the members due to unit rotation applied at the joint 2 are

$$M^0_{21} = r\left(\frac{EI}{L}\right) \quad \text{and} \quad M^0_{22'} = 3.3333\left(\frac{EI}{L}\right)$$

There is no carry over to, and carry back from joint 2′ as the modified stiffness of the member 2-2′ has been used. The moment distribution procedure is shown in the Fig. 6.30a.

Since for elastic instability, the total moment at the joint 2 must vanish i. e. sum of applied moment and the moments carried back to the joint must equal zero.

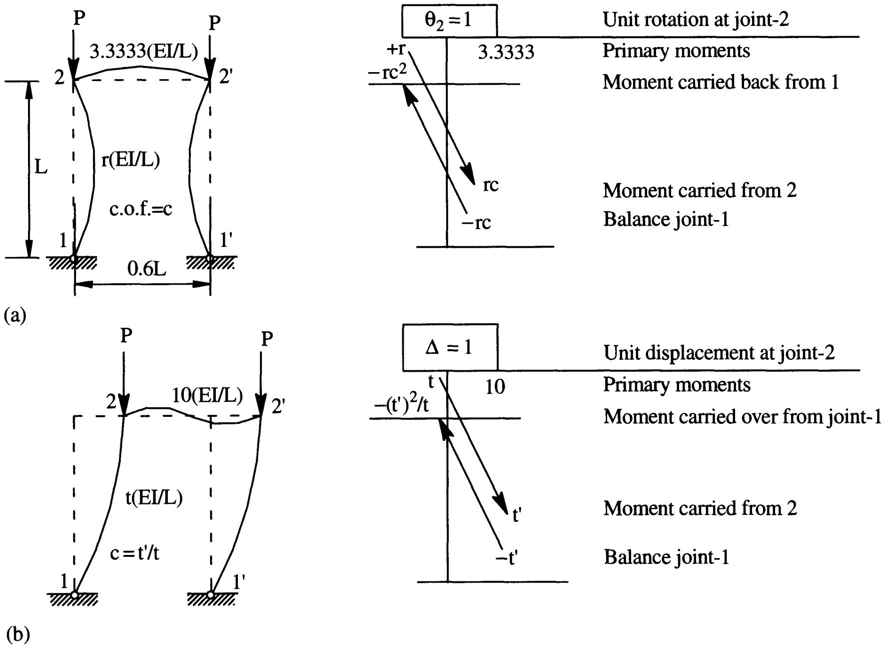


Fig. 6.30a,b. Calculation of terminal moments taking into account the effect of axial forces by moment distribution. **a** Symmetrical buckling mode, **b** antisymmetrical buckling mode

$$M_2 = [(r - rc^2) + 3.3333]\left(\frac{EI}{L}\right)$$

$$= [r(1 - c^2) + 3.3333]\left(\frac{EI}{L}\right) = [r' + 3.3333]\left(\frac{EI}{L}\right) = 0$$

i.e. $r' = -3.3333$. From the stability functions table given in Appendix A.1. For $r' = -3.3333$, $\rho = 1.4351$ and hence

$$P_{\text{cr}} = \frac{1.4351\pi^2 EI}{L^2} = \frac{14.1639\, EI}{L^2}\,.$$

(2) For anti-symmetric buckling mode

$$\bar{k}_{21} = t\left(\frac{EI}{L}\right), \quad \bar{k}_{12} = t'\left(\frac{EI}{L}\right) \quad \text{and c.o.f.,} \quad c = t'/t$$

$$\bar{k}_{22'} = r_{22'}(1 + c'_{22})\left(\frac{EI}{L}\right)_{22'} = \frac{6EI}{0.6L} = 10\left(\frac{EI}{L}\right)$$

There is no carry over to, and carry back from the joint $2'$ as the modified stiffness of the member $22'$ has been used. The relative primary moments induced in the members due to unit displacement ($\Delta = 1$ i. e. the rotation of $1/L$) applied at the joint 2 are:

$$M^0_{21} = t\left(\frac{EI}{L}\right) \quad \text{and} \quad M^0_{22'} = 10\left(\frac{EI}{L}\right)$$

The moment distribution procedure is shown in Fig. 6.30b. For elastic instability the total moment at the joint 2 must vanish i. e.

$$\left[t - \frac{(t')^2}{t}\right]\left(\frac{EI}{L}\right) + 10\left(\frac{EI}{L}\right) = 0 \quad \text{or} \quad f(\rho) = t(t+10) - (t')^2 = 0$$

Using trial and modification procedure with stiffness values from the stability functions table, for $f(\rho) = 0$, $\rho = 0.2068$. Therefore,

$$P_{\text{cr}} = \frac{0.2068\pi^2 EI}{L^2} = \frac{2.041 EI}{L^2} \,.$$

It should be noted that vanishing of total moment at a joint makes that particular joint to have *zero rotational stiffness*. The following example will illustrate the application of moment distribution method in computation of terminal moments in the frames where the members also carry axial forces.

Example 6.17. A symmetrical portal frame hinged at the base is subjected to a load system shown in Fig. 6.31. EI is constant for all the members. Determine the variation in the values of terminal moments when: (i) axial forces are taken into account, and (ii) axial forces are ignored.

The unknown axial force induced in the beam is expected to be small and hence no essential error is introduced by assuming $\rho_{22'} = 0$. In view of the symmetry of the system, the stiffness of the beam $k_{22'} = 2EI/(0.6L) = 3.3333(EI/L)$. The basic or fixed end moments in the beam are

$$M_{\text{f}22'} = -\frac{w(0.6L)^2}{12} = -(30.0 \times 10^{-3})wL^2 = -M_{\text{f}2'2}$$

The axial forces in the columns $= [w(0.6L)/2] + P$

To determine the variation in terminal moments let us consider the loading stage when $\rho = 0.20$.

Case-I: When axial force is taken into account. From the stability functions table for $\rho = 0.20$

$$\bar{k}_{21} = r'\left(\frac{EI}{L}\right) = 2.5808\left(\frac{EI}{L}\right)$$

$$\bar{k}_{22'} = 3.3333\left(\frac{EI}{L}\right)$$

The terminal moments obtained by moment distribution are shown in the Fig. 6.31b.

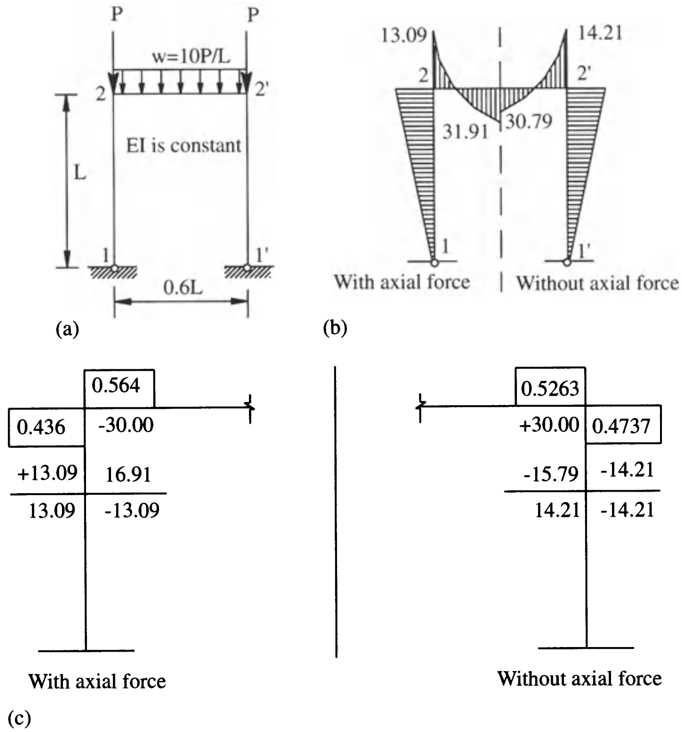


Fig. 6.31a–c. Calculation of terminal moments taking into account the effect of axial forces by moment distribution. **a** Structure, **b** bending moment diagrams ($\times 10^{-3} wL^2$), **c** moment distribution procedure

Case-II: When axial force is in the column is ignored. For this case $\rho_{21} = 0$ and

$$\bar{k}_{21} = 3.0000 \left(\frac{EI}{L}\right) \quad \text{and} \quad \bar{k}_{22'} = 3.3333 \left(\frac{EI}{L}\right)$$

The moments obtained by moment distribution are given in the Fig. 6.31b.

The difference in terminal moments in two cases is

Columns: $$\frac{(14.21 - 13.09)}{13.09} \times 100 = 8.556 \text{ per cent}$$

i. e. the column moment increases by 8.556 per cent when axial force is neglected.

The corresponding value for the mid-span moment in the beam is

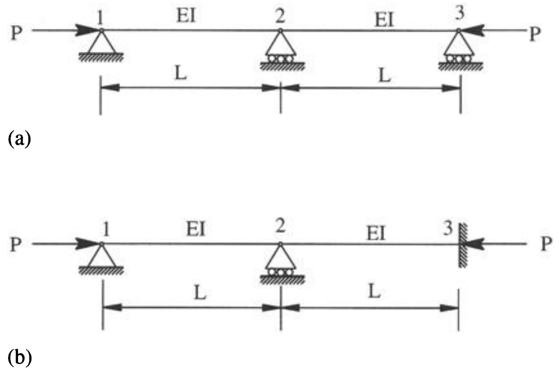
Beams: $$\frac{(30.79 - 31.91)}{31.91} \times 100 = -3.51 \text{ per cent}$$

i. e. the mid-span moment in the beam decreases by 3.51 per cent when axial force in the column is ignored.

6.9 Problems

Problem 6.1. A continuous beam 1-2-3 of uniform cross-section has two segments 1-2 and 2-3 of equal length L as shown in Fig. P.6.1a. The strut is subjected to an axial thrust P acting: (i) in the segment 1-2 and (ii) in the entire length 1-3. Estimate the critical value of load P that will cause the strut to buckle.

Also estimate the critical value of load P when the joint 3 is fixed against both rotation and translation as shown in the Fig. P.6.1b. Draw the buckling modes.



P.6.1

[Ans. (a) (i) Eigen-equation is: $r(r+3) - (rc)^2 = 0$ *or* $r(r'+3) = 0$, $\rho_{cr} = 1.408$, 2.047;

(ii) Eigen-equation is: $r^2(r') = 0$, $\rho_{cr} = 1.00$, 2.047.

(b) (i) Eigen-equation is: $r(r'+4) = 0$, $\rho_{cr} = 1.4853$ *and* 2.047;

(ii) Eigen-equation is: $r(r'+r) = 0$, $\rho_{cr} = 1.3143$ *and* 2.047*]*

Problem 6.2. A single column rigidly connected frame shown in Fig P.6.2 is fixed against both rotation and translation at the joints 2, 3 and 3′. The member 1-2 of the frame is subjected to axial compression P. Determine the critical value of load P that will cause the frame to buckle. Also calculate the critical load when: (i) joint 2 is released against rotation, (ii) joint 3′ only is released against rotation (iii) joints 3 and 3′ both are released against rotations, and (iv) joints 2, 3 and 3′ are released against rotations.

[Ans. (i) $r = -8.0$ *and* $\rho_{cr} = 3.2476$;

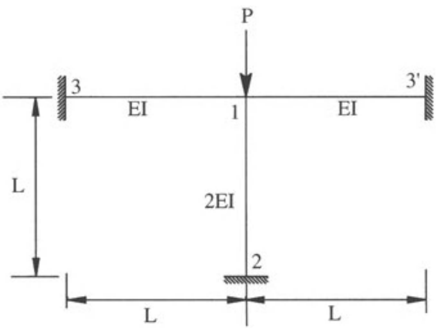
(ii) $r' = -8.00$ *and* $\rho_{cr} = 1.6748$;

(iii) $r = -7.00$ *and* $\rho_{cr} = 3.1776$;

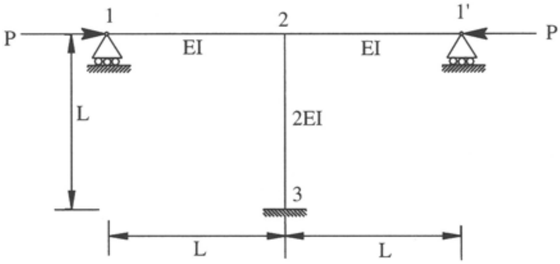
(iv) $r = -6.00$ *and* $\rho_{cr} = 3.095$, *and*

(v) $r' = -6.00$ *and* $\rho_{cr} = 1.5984$*]*

Problem 6.3. The member 1-2-1′ of the rigidly connected single column frame shown in Fig. P.6.3 is subjected to an axial thrust P. Determine the critical value of load P that will make the frame unstable when: (i) column base joint 3 is fixed, and (ii) when the joint 3 is hinged.



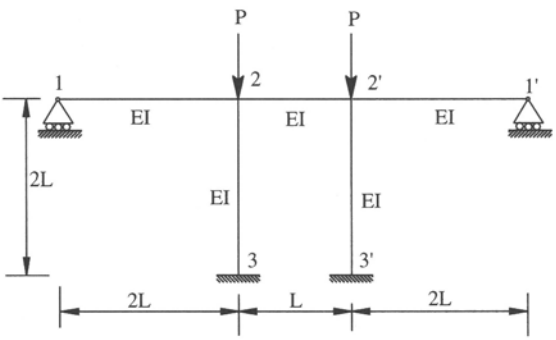
P.6.2



P.6.3

[Ans. (i) $r' = -4.0$ *and* $\rho_{cr} = 1.4853$,
(ii) $r' = -3.0$ *and* $\rho_{cr} = 1.4066$*]*

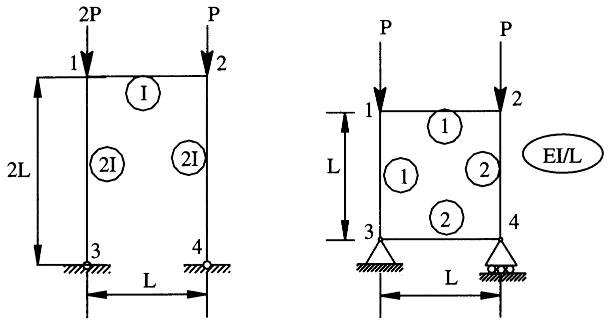
Problem 6.4. A two-column symmetrical system shown in Fig. P.6.4 is symmetrically loaded. EI values are same for all the members. Determine the critical value of load P that will make the system unstable when: (i) columns are fixed against rotation and translation at the base, and (ii) columns are hinged at the base.



P.6.4

[Ans. (i) for symmetrical buckling mode: $r = -7.00$ and $\rho_{cr} = 3.1776$ and for antisymmetric buckling mode: $t = -15.00$ and $\rho_{cr} = 0.880$;
(ii) for symmetrical mode: $r' = -7.00$ and $\rho_{cr} = 1.640$, for anti-symmetrical mode: $[t(t+15) - (t')^2] = 0$ and $\rho_{cr} = 0.2198$]

Problem 6.5. The single-bay, single-storey symmetrical frame shown in Fig. P.6.5a is subjected to unsymmetrical loading. Determine the critical value of the load P.



(a) (b)

P.6.5 **a** Symmetrical frame with unsymmetrical load, **b** Unsymmetrical frame with symmetrical load

Problem 6.6. The unsymmetrical single-bay, single-storey closed frame shown in Fig. P.6.5b is subjected to symmetrical loading. Determine the critical value of load P that will cause the frame to buckle.

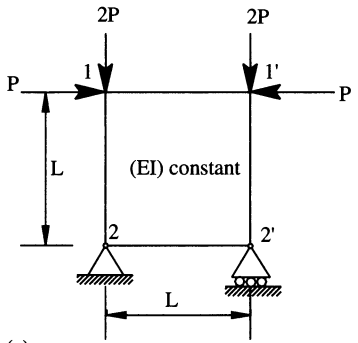
Problem 6.7. In the symmetrical closed frame shown in Fig. P.6.6a, each of the columns 1-2 and 1′-2′ carries an axial thrust of $2P$. In addition the horizontal member 1-1′ is also subjected to an axial load P. Determine the load factor, N_e, at which the frame will collapse.

Problem 6.8. The symmetrical closed frame shown in Fig. P.6.6b is symmetrically loaded. Determine the critical value of load at collapse.

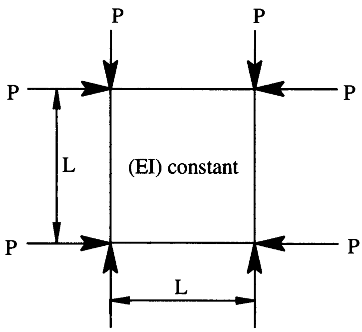
Problem 6.9. The members of multi-bay frame shown in the Fig. P.6.7 have (EI/L) values given in the circles. Determine the critical value of load P at which the frame will collapse.

[Hint: Use principle of multiples to split the frame into single-bay frames.]

Problem 6.10. Determine the load factor N_e at which the two-bay, two-storey frame shown in Fig. P.6.8a will buckle when: (i) $\beta = 2$ and $P_2 = 2P_1 = 2P$, (ii) when $\beta = 1$ and $P_2 = P_1 = P$.
[Hint: Use principle of multiples to split the frame into single-bay frames]

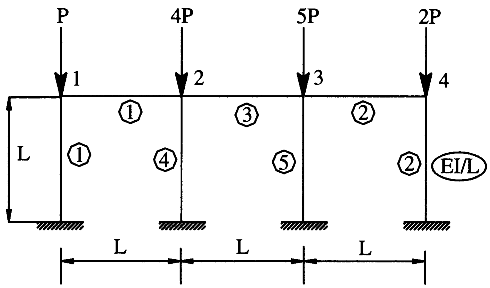


(a)

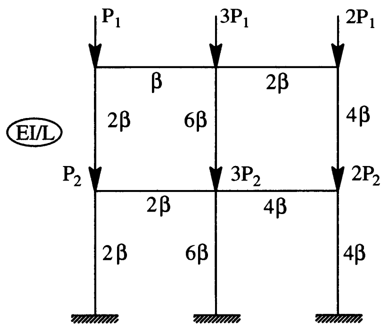


(b)

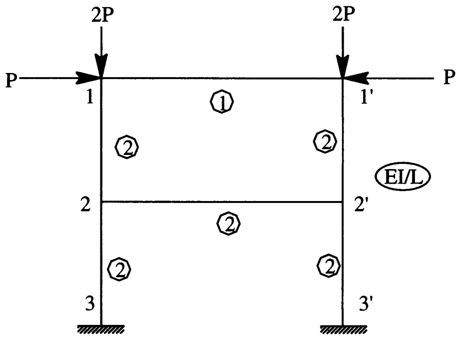
P.6.6



P.6.7



(a)

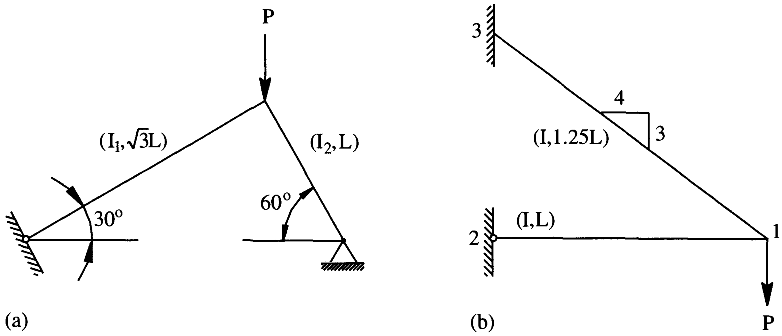


(b)

P.6.8

Problem 6.11. The single-bay, two-storey symmetrical frame shown in Fig. P.6.8b is symmetrically loaded. The frame is hinged at the base points 3 and 3′. Determine the critical value of load P which will cause the frame to buckle.

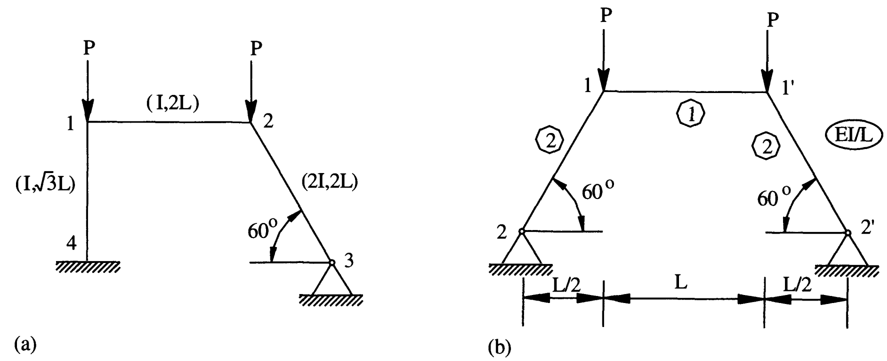
Problem 6.12. Determine the critical load for the rigidly connected two-bar structure shown in Fig. P.6.9a when: (i) $I_1 = \sqrt{3}I_2$; (ii) I_2 is very small compared to I_1, and (iii) I_1 is very small as compared to I_2.



P.6.9

Problem 6.13. The joints 2 and 3 of the cantilever bracket frame shown in Fig. P.6.9b are hinged and fixed, respectively, and the joint 1 is rigid. The members of the frame are of uniform cross-section. Determine the critical value of load P that will cause the frame to buckle.

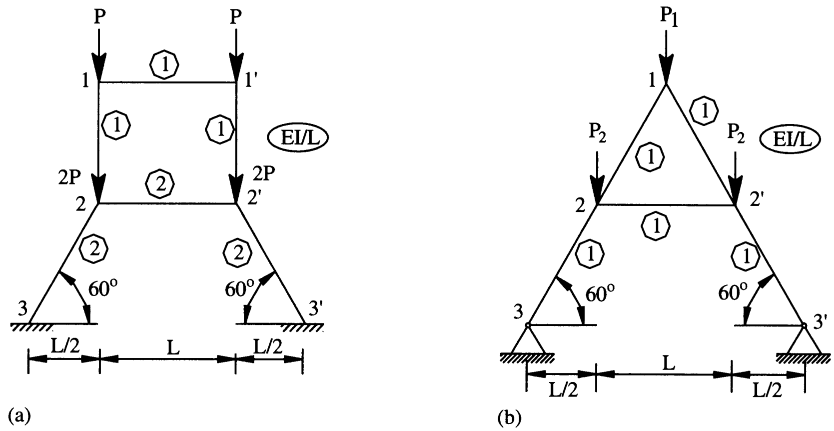
Problem 6.14. The column 2-3 of the portal frame shown in Fig. P.6.10a is inclined at an angle of 60° from the horizontal. Determine the critical load when the portal is subjected to the given load system.



P.6.10

Problem 6.15. The symmetrical single-bay portal frame shown in Fig. P.6.10b has inclined columns which are hinged at the base. Determine the critical value of the load which will cause the frame to collapse.

Problem 6.16. In the single-bay, two-storey symmetrical frame, the bottom storey has inclined columns as shown in Fig. P.6.11a. Determine the load factor N_e at collapse.



P.6.11

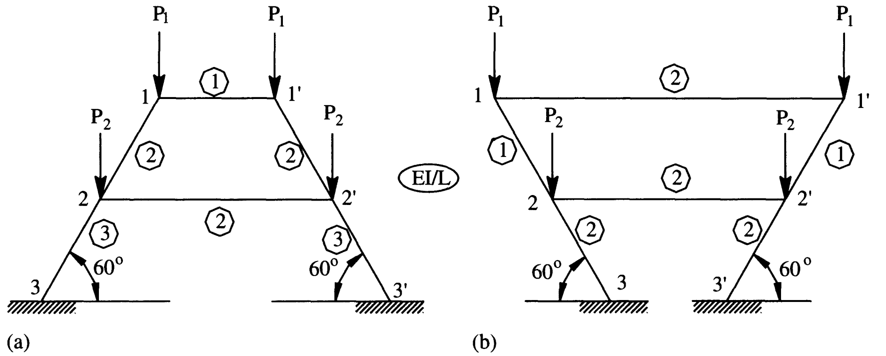
Problem 6.17. The symmetrical A-frame mast shown in Fig. P.6.11b is subjected to a load system shown in the figure. Determine the critical value of load P that will cause the mast to collapse, when $P_2 = P_1 = P$.

Problem 6.18. The wide base single-bay, two-storey symmetrical frame shown in Fig. P.6.12a has inclined columns and is subjected to a symmetrical loading shown in the figure. Determine load factor N_e against collapse when $P_2 = 2P_1 = 2P$.

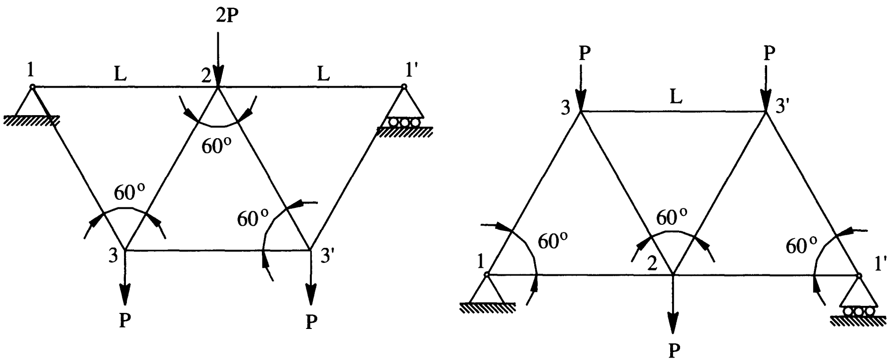
Problem 6.19. The narrow base, single-bay, two-storey symmetrical frame shown in Fig. P.6.12b has inclined columns. Determine the critical value of P when $P_2 = P_1 = P$.

Problem 6.20. The members of two-panel rigid-jointed truss shown in Fig. P.6.13a have constant EI and L throughout. The design is such that P_e $(= \pi^2 EI/L^2)$ of each member equals $3\sqrt{3}P$. Determine the load factor N_e against collapse of entire truss.

Problem 6.21. The members of rigid-jointed Warren truss shown in Fig. P.6.13b have constant EI and L throughout. The design of the frame is such that P_e $(= \pi^2 EI/L^2)$ of each member equals $2\sqrt{3}P$. Find the load factor N_e at which entire frame will collapse.

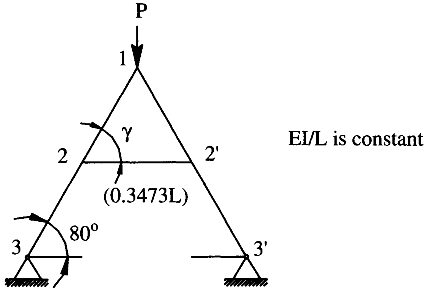


P.6.12



P.6.13

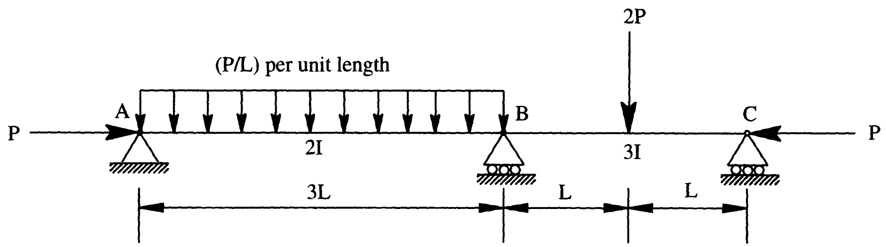
Problem 6.22. The symmetrical A-frame mast shown in Fig. P.6.14 is subjected to a vertical load P at the apex. Determine the critical value of load P that will result in the collapse of the mast, when it is: (i) hinged at the base points 3 and $3'$; and (ii) fixed at the points 3 and $3'$. EI is constant throughout.



P.6.14

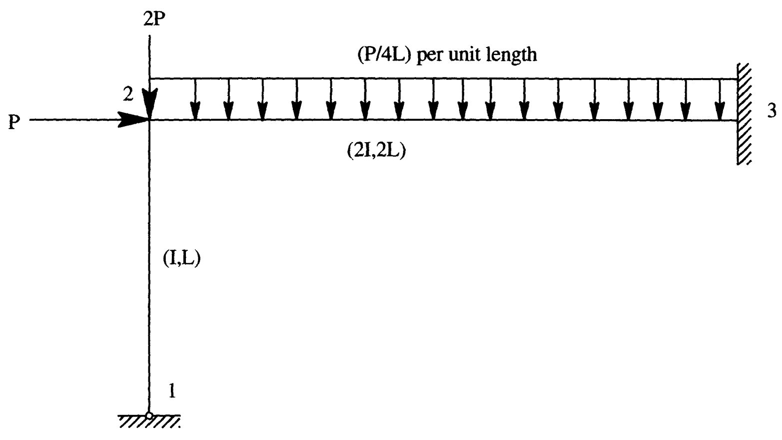
[Ans. $P_{cr} = 2(\rho_{cr} P_e) \sin \gamma$. *In the symmetrical non-sway mode, the joint 1 does not rotate.*
(a) $r + r' - 5.7587 = 0$ *and* $\rho_{cr} = 0.3465$, $P_{cr} = 2(0.3465\pi^2 EI/L^2) \sin 80°$,
(b) $r = -2.87935$ *or* $\rho_{cr} = 2.7021$ *and so on.*
In the anti-symmetrical or sway-mode assume the member 2-2′ to be displaced in such a way that upper part of the frame is rotated about the point 1 through unit angle. Therefore,
(a) $t(2t + 17.276) - (t')^2 = 0$ *or* $\rho_{cr} = 0.2237$
(b) $t = -8.638$ *and* $\rho_{cr} = 0.8084$ *and so on.]*

Problem 6.23. Analyze the continuous beam-column shown in Fig. P.6.15 by: (i) moment distribution method; and (ii) stiffness matrix method when: (a) the axial load P is equal to zero, and (b) $P = 0.4\, P_e$.



P.6.15

Problem 6.24. Analyze the rigid frame shown in Fig. P.6.16 by: (i) moment distribution; and (ii) by stiffness matrix method when the load factor $N_e = 2.0$.



P.6.16

7

Buckling of Members Having Open Sections

7.1 Introduction

Many flexural members are braced by other elements of the structures in such a manner that they are constrained to deflect only in the plane of applied transverse loads, e.g. slab-beam floor systems are extremely rigid in their own plane and the beams can deflect only in a plane perpendicular to the slab. The horizontal and rotational displacements are prevented by the floor system. On the other hand there are numerous instances where the members have no lateral support or bracings over their lengths and members can buckle in lateral direction under transverse loads. Similarly open column sections having only one or no axis of symmetry e.g. a channel section, and T-section or an angle section when subjected to axial compression; simultaneously undergo lateral displacement and rotation. This type of failure occurs because of low torsional rigidity of such sections. Further, in such sections, the critical load lies between the critical load for the *torsional mode* and that of *pure flexural mode*. A pure flexural mode exists when the *centroidal axis* coincides with *shear centre axis*. Therefore, a member subjected to an axial compressive force can also undergo lateral buckling.

7.2 Torsional Buckling

7.2.1 Member Subjected to Torque

When slender members are subjected to moments about their longitudinal axis, torsional shear stresses develop. In circular cross-sections the shearing stresses at every point in the plane of cross-section act in the direction perpendicular to a radius vector. On the other hand in *non-circular forms*, the shearing stress has components both perpendiculars to radius vector and in the direction of radius vector. This extra shearing force results in a shearing strain both within the plane of cross-section and normal to it. Since, the shearing force components vary from point to point, the cross-section does not remain flat and undergoes out of plane distortion which is called *warping*. The warping is predominant in the members having thin walled open cross-sections such as an I shape or a channel or an angle. In the cases where warping displacements are restrained, very high normal stresses called *warping normal stresses* develop. However, it is to be understood that the warping normal stresses σ_w are self equilibrating i. e. their integrated effect over the cross section in zero. There are two approaches namely equilibrium approach, and energy approach which are commonly used for the analysis of such sections.

I. Equilibrium approach

(a) Rectangular sections

The stresses developed in a non-circular cross-section subjected to a torque can be divided in two categories. The first of these referred to as *St. Venant* or *uniform torsional stresses* and the second as *non-uniform* or *warping stresses*. If the warping is unrestrained, then the applied torque is resisted completely by St. Venant shear stresses. On the other hand if the member is restrained from warping, then the torque is partly resisted by St. Venant shearing stresses and partly by the stresses produced due to constraint on warping. The St. Venant shearing stress in a rectangular cross section which is parallel to the edges is normally represented in the same form as used for a circular shaft i. e.

$$M_{x,s} = GJ(\mathrm{d}\beta/\mathrm{d}x) \tag{7.1}$$

where $M_{x,s}$, J and β are torsional moment, the *St . Venant torsional constant* and total angle of twist, respectively. G is the *shear modulus of rigidity* and X-axis is along the centroidal axis of the member.

The corresponding maximum shearing stress is given by

$$\tau_{sv,\max} = \frac{M_{x,s}}{J}t = Gt\frac{\mathrm{d}\beta}{\mathrm{d}x} \tag{7.2}$$

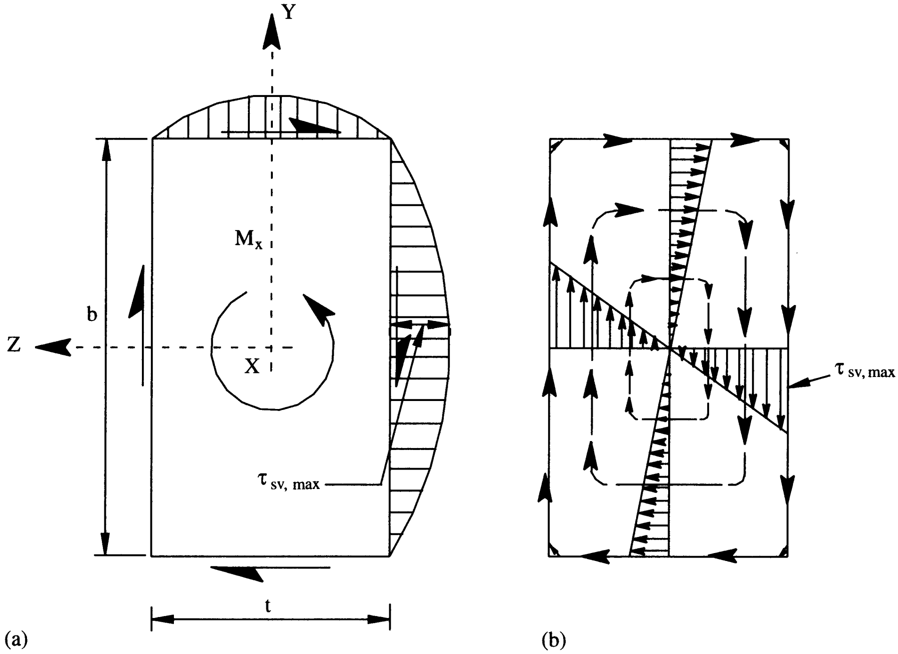
where t is thickness (smaller dimension) of the rectangular cross-section. It should be noted from Fig. 7.1b that in Y- and Z-coordinate directions, the shearing stresses are parallel to the outside surface of the members and vary from zero at the centre of the member to a maximum at the surface. $\tau_{sv,\max}$ occurs at the centre of longer side as shown in Fig. 7.1a. The shearing stresses due to St. Venant type torque at the four corners of the cross-section equal zero.

For a rectangular cross-section where aspect ratio (b/t) is large, the *torsional constant* J can be approximated as

$$J \approx \frac{1}{3}bt^3 \tag{7.3}$$

For open cross-sections consisting of several thin plate elements rigidly attached to one another to form a thin walled shape as shown in Fig. 7.1c, J is taken as

$$J = \sum_{i=1}^{n} J_i = \sum_{i=1}^{n} \frac{1}{3}b_i t_i^3$$



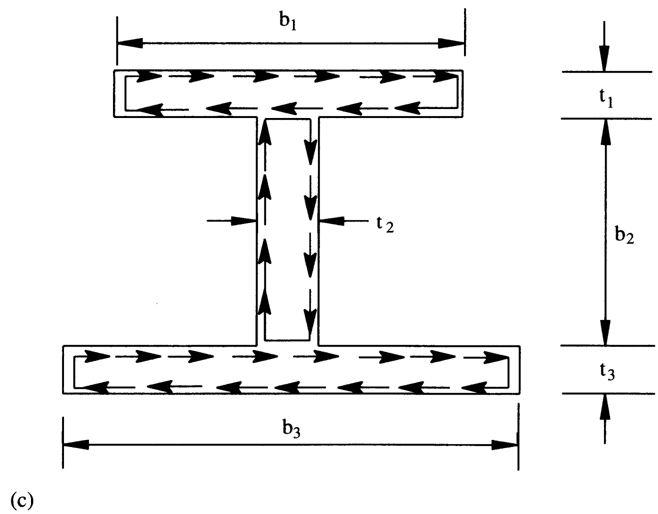


Fig. 7.1a–c. St. Venant shearing stresses in a rectangular- and I-cross sections due to torsional moment

where i refers to a typical of n connected plate elements. The values of J for some typical sections are given in Table 7.1. However, in practice J is modified by a factor α as

$$J = \alpha \sum_{i=1}^{n} J_i = \alpha \sum_{i=1}^{n} \frac{1}{3} b_i t_i^3 \tag{7.4}$$

where $\alpha = 1.3$ for I-sections, 1.12 for channels, 1.00 for angles, and 1.5 for welded beams with stiffening ribs welded to the flange. For riveted beams $\alpha = 0.50$.

For the I-section shown in Fig. 7.1c, J is given by:

$$J = \frac{1.3}{3}(b_1 t_1^3 + b_2 t_2^3 + b_3 t_3^3)$$

Consider three cases of I-shaped wide flange cross-section loaded as shown in Fig. 7.2. In the first case the torque is applied at the ends and rotation about X-axis is allowed i. e. the member is not restrained against *warping displacements* and hence no *warping stresses* will develop and the flanges will remain straight. This is a case of uniform torsion inducing only St. Venant stresses. In practice this case arises when a simply supported member is twisted at its ends by other members.

On the other hand in the case (b) shown in Fig. 7.2b the member is not free to rotate about the X-axis at the ends and hence the flanges do not remain straight i. e. warping stresses develop. However, due to symmetry all warping displacements are eliminated at mid point. In this case contribution of uniform torsional strength is maximum at the ends and decreases towards the centre. While warping strength is maximum at the centre and decreases towards the ends.

The beam in case (c) represents one-half the beam of case (b). In general, the applied torques are resisted by the sum of uniform (St. Venant) torsional resistance and the warping resistance i. e.

$$M_x = M_{x,\mathrm{s}} + M_{x,\mathrm{w}} \tag{7.5}$$

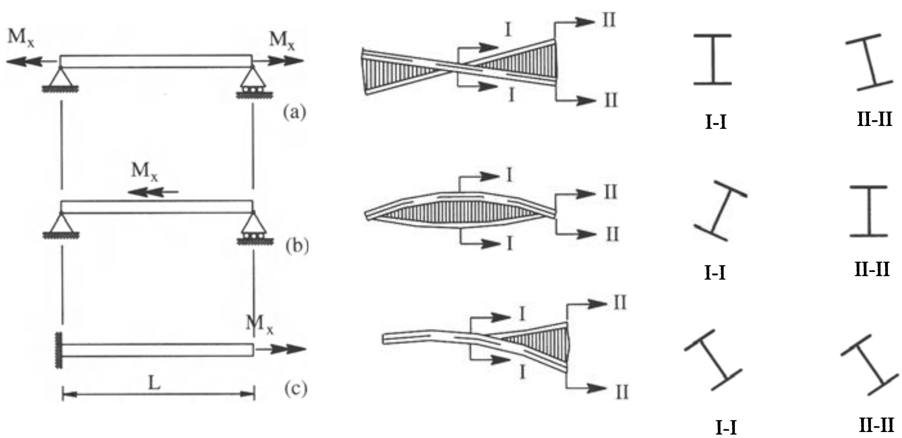


Fig. 7.2. Rotations of wide-flange member due to torsional moment

Table 7.1. Properties of some typical sections

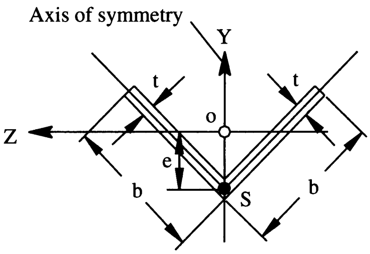
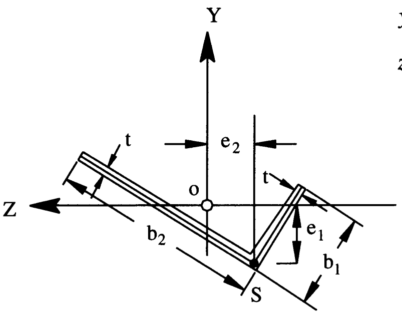
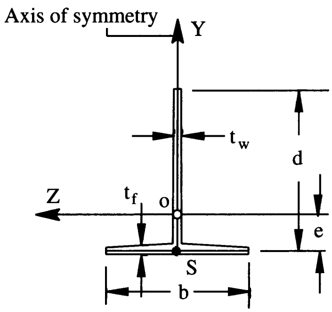
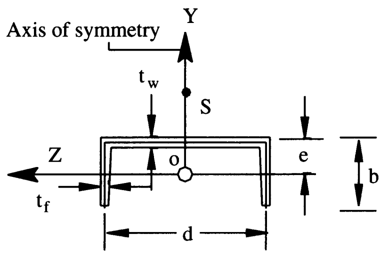
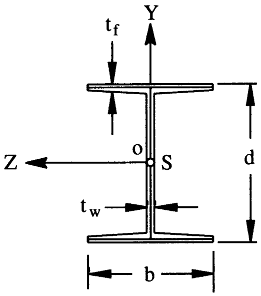
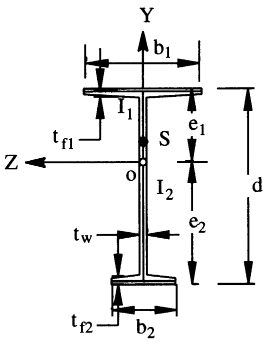
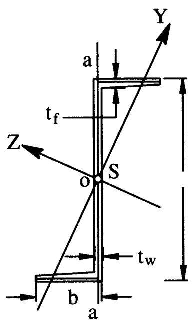
Shape of cross section	Location of shear centre, S	$J=\sum_{i=1}^{n} J_i$	I_w
	$y_0=-e$ $z_0=0$	$J=J_1+J_2$ $J_1=\frac{1}{3}bt^3$ $J_2=\frac{1}{3}bt^3$	$\frac{b^3t^3}{18}=\frac{A^3}{144}$
	$y_0=-e_1$ $z_0=-e_2$	$J=J_1+J_2$ $J_1=\frac{1}{3}b_1t^3$ $J_2=\frac{1}{3}b_2t^3$	$\frac{t^3}{36}(b_1^3+b_2^3)$
	$y_0=-e$ $z_0=0$	$J=J_1+J_2$ $J_1=\frac{1}{3}dt_w^3$ $J_2=\frac{1}{3}bt_f^3$	$\frac{t_w^3d^3}{36}+\frac{t_f^3b^3}{144}$
	$y_0=e\left(1+\frac{d^2A}{4I_y}\right)$ $z_0=0$	$J=2J_1+J_2$ $J_1=\frac{1}{3}bt_f^3$ $J_2=\frac{1}{3}dt_w^3$	$\frac{d^2}{4}\Big[I_z+e^2A$ $\times\left(1-\frac{d^2A}{4I_y}\right)\Big]$

Table 7.1. (*continued*)

Shape of cross section	Location of shear centre, S	$J = \sum_{i=1}^{n} J_i$	I_w
	$y_0 = z_0 = 0$	$J = 2J_1 + J_2$ $J_1 = \frac{1}{3}bt_f^3$ $J_2 = \frac{1}{3}dt_w^3$	$\frac{d^2}{4} I_y$
	$y_0 = \frac{e_2 I_2 - e_1 I_1}{I_1 + I_2}$ [a] $z_0 = 0$	$J = J_1 + J_2 + J_3$ $J_1 = \frac{1}{3}b_1 t_{f1}^3$ $J_2 = \frac{1}{3}b_2 t_{f2}^3$ $J_3 = \frac{1}{3}dt_w^3$	$d^2 \frac{I_1 I_2}{I_1 + I_2}$
	$y_0 = z_0 = 0$	$J = 2J_1 + J_2$ $J_1 = \frac{1}{3}bt_f^3$ $J_2 = \frac{1}{3}dt_w^3$	$\frac{d^2}{4} I_a$ [b]

[a] I_1 and I_2 are the moments of inertia of the top and the bottom flanges, respectively, with respect to the Y-axis

[b] I_a is the moment of inertia of the cross-section with respect to the centerline a-a of the web

Their relative magnitudes depend on the geometry of the cross-section and the ratio of elastic modulus E to shear modulus G, i.e. E/G.

(b) Warping of I and wide flange sections

In the following treatment it is assumed that the twist is small and the relative geometry of the cross-section does not change as member rotates. The warping effects can be described by considering lateral bending of members due to twisting. Consider a I or wide flange section subjected to *twisting moment* M_x as shown in Fig. 7.3a. Since

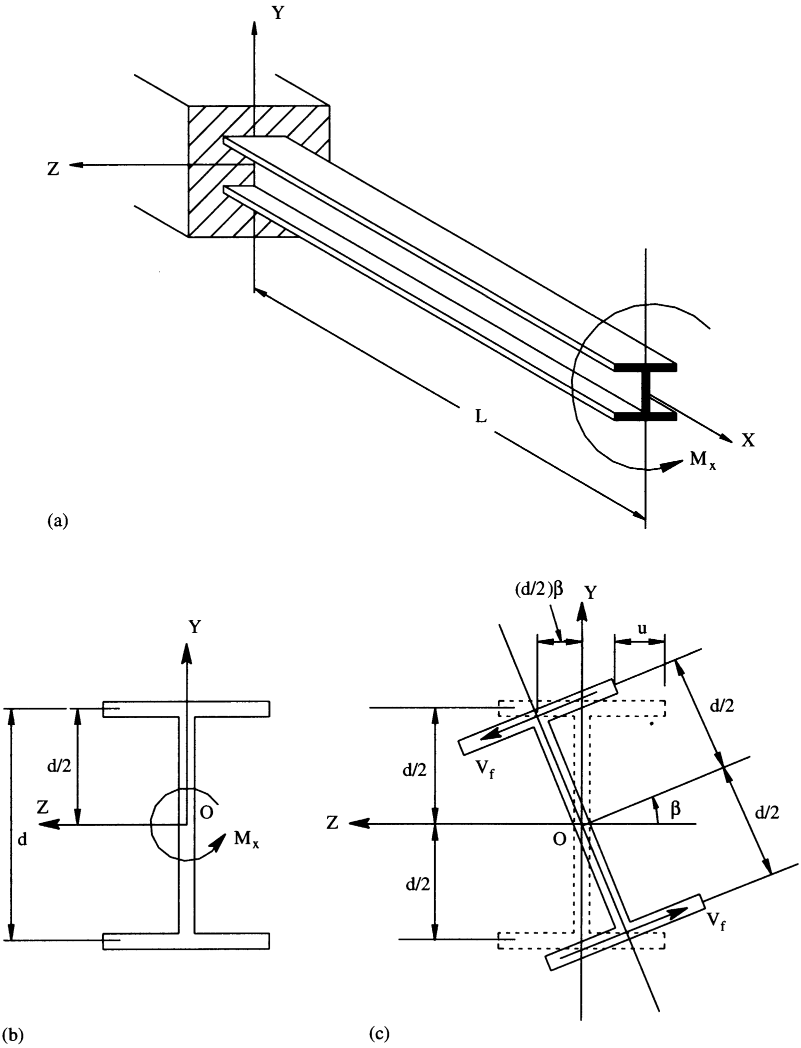


Fig. 7.3a–c. Warping of wide flange section due to twisting moment. **a** Beam subjected to twisting moment, **b** typical cross-section, **c** rotated cross-section

the section is not allowed to rotate at the support (i.e. warping deformations are constrained) it results in a differential lateral displacement of the top and bottom flanges, one bending to right and one to left as shown in Fig. 7.3c. This differential translation of the flanges accounts for primary warping and the rotation corresponds to St. Venant effect. The shear forces V_f in the flanges form a couple referred to as *warping torsion*, M_w $(= V_f d)$. The moment M_f induced in the flange about Y-axis due to lateral (horizontal) deformation u is given by

$$M_f = -EI_f \frac{d^2 u}{dx^2} \tag{7.6}$$

where I_f is the second moment of area of flange plus half the web section, about Y- or vertical axis.

For small angle of rotation β, displacement u can be expressed as: $u = \beta(d/2)$. Therefore, from (7.6)

$$M_f = (-EI_f)\left(\frac{d}{2}\right)\left(\frac{d^2\beta}{dx^2}\right) \tag{7.7}$$

The shear force in the flange is given by:

$$V_f = \frac{dM_f}{dx} = -(EI_f)\left(\frac{d}{2}\right)\left(\frac{d^3\beta}{dx^3}\right) \tag{7.8}$$

where d is the depth of cross-section centre to centre of flanges. The warping resistance of the cross-section resulting from the couple formed by two equal and opposite flange shear forces V_f is given by

$$M_{x,w} = -(EI_f)\left(\frac{d^2}{2}\right)\left(\frac{d^3\beta}{dx^3}\right)$$

For an I-shaped cross section, I_f may be approximated by $I_y/2$ of the total cross-section. Therefore,

$$M_{x,w} = -\left(EI_y\right)\left(\frac{d^2}{4}\right)\left(\frac{d^3\beta}{dx^3}\right) = -(EI_w)\frac{d^3\beta}{dx^3} \tag{7.9}$$

where I_w is referred to as *warping constant* of the cross-section. Warping constants for some of the typical sections are given in Table 7.1. Equation (7.9), though derived specifically for I or wide flange section, is also valid for thin walled open cross-sections with approximate value of I_w. Substitution from (7.1) and (7.9) into (7.5) gives twisting equilibrium equation

$$M_x = M_{x,s} + M_{x,w} = GJ\frac{d\beta}{dx} - EI_w\frac{d^3\beta}{dx^3} \tag{7.10}$$

Defining distributed variable torque as

$$m_x = \frac{\mathrm{d}M_x}{\mathrm{d}x} = GJ\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - EI_\mathrm{w}\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} \quad \text{or}$$

$$\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} - \left(\frac{GJ}{EI_\mathrm{w}}\right)\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} = -\frac{m_x}{EI_\mathrm{w}} \tag{7.11}$$

Thus the stresses produced in thin walled type open cross-sections subjected to torque are

(a) St. Venant shearing stress,

$$\tau_{\mathrm{sv,max}} = Gt(\mathrm{d}\beta/\mathrm{d}x) \tag{7.12}$$

and occurs on the surface at the mid point of thickest part.

(b) Warping normal stress,

$$\sigma_{\mathrm{w,max}} = \frac{M_\mathrm{f}b}{2(I_y/2)} = \frac{M_\mathrm{f}b}{I_y} = -\frac{Edb}{4}\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} \tag{7.13}$$

where b is the flange width.

(c) Warping shearing stress,

$$\tau_\mathrm{w} = \frac{V_\mathrm{f}(A\bar{y})}{I_\mathrm{f}t_\mathrm{f}} = -\frac{Ed(A\bar{y})}{2t_\mathrm{f}}\left(\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3}\right)$$

At the centre of flange $A\bar{y}$ is maximum and is given by $A\bar{y} = b^2t_\mathrm{f}/8$. Hence

$$\tau_{\mathrm{w,max}} = -\left(\frac{Edb^2}{16}\right)\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3} \tag{7.14}$$

7.2.2 Member Subjected to Axial Force

Consider a column with double symmetric cross-section subjected to an axial load P. Such a section where *shear centre* coincides with the *centre of gravity* can have pure flexural modes in X-Y and X-Z planes depending upon the axis about which second moment of area is minimum. It can also have a pure *torsional mode* simultaneously. In a buckled condition, the axial load P acts on a slightly rotated cross-section. A fibre element of length $\mathrm{d}x$ and cross-sectional area $t\mathrm{d}r$ located at a distance r from the axis undergoes a transverse or lateral displacement ξ $(= \beta r)$ as a result of rotation β (without translation or distortion). This element can be treated as column under axial load $\mathrm{d}P$ $(= \sigma t\mathrm{d}r)$ which has undergone a lateral displacement, ξ. This deformed elemental column is equivalent to a beam element subjected to a transverse fictitious load of magnitude $q(x)$ given by

$$q(x) = \frac{\mathrm{d}^2M_\xi}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathrm{d}P\xi) = \mathrm{d}P\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}$$

or

$$q(x) = \sigma(t\,\mathrm{d}r)\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2}$$

Substituting for ξ

$$q(x) = \sigma t r\,\mathrm{d}r\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} \tag{7.15}$$

where β is a function of x alone.

Since the load $q(x)$ is acting at distance r from the axis of column it causes a moment $q(x)r$ about Z-axis. This moment is very small if the width of the flange is small. Summing up the entire moment over the cross-section at distance x along the column axis.

$$m_x\,\mathrm{d}x = \sigma\,\mathrm{d}x\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}\int_A tr^2\,\mathrm{d}r = \sigma I_o\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2}\,\mathrm{d}x$$

where I_o is the *polar moment of inertia* of the cross-section about the shear centre. Thus the torque generated per unit length can be written as:

$$m_x = \sigma I_o\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} = \frac{P}{A}I_o\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} = Pr_o^2\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} \tag{7.16}$$

where $\sigma = P/A$ with A being the area of cross-section. Using (7.11) the torsional behaviour of a uniform cross-section member subjected to axial compressive load P is governed by differential equation:

$$\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} + \frac{Pr_o^2 - GJ}{EI_\mathrm{w}}\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} = 0 \tag{7.17}$$

The polar radius of gyration, r_o of the section about the centroidal axis is given by

$$r_o^2 = \frac{I_p}{A} = \frac{I_y + I_z}{A}$$

In contrast to doubly symmetrical cross-sections, in singly- or un-symmetrical section the centroid of the cross-section does not coincide with the *shear (elastic) centre*. If the distances of shear centre from the centroid are given by (y_o and z_o) the radius of gyration is given by

$$r_o^2 = \frac{I_p}{A} = \frac{I_y + I_z}{A} + y_o^2 + z_o^2$$

Equation (7.17) can be expressed in the standard form

$$\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} + \alpha^2\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} = 0 \quad \text{or} \quad \beta'''' + \alpha^2\beta'' = 0 \tag{7.18}$$

where $\alpha^2 = (Pr_o^2 - GJ)/EI_\mathrm{w}$. Equation (7.18) has a general solution of the form (Appendix Appendix C):

$$\beta = A\sin\alpha x + B\cos\alpha x + C\left(\frac{x}{L}\right) + D \tag{7.19}$$

Case I: Consider a column with simple end supports. The boundary conditions to be satisfied are

(i) at $x = 0$: $\beta(0) = \beta''(0) = 0$.

This condition provides $D = B = 0$, and the condition

(ii) at $x = L$: $\beta(L) = \beta''(L) = 0$

gives $C = 0$ and $A \sin \alpha L = 0$ (7.20)

Thus for a simply supported column with doubly-symmetric cross-section, the torsional buckling load is given by

$$P_{x,\mathrm{cr}} = \left(\frac{\pi^2 E I_\mathrm{w}}{L^2} + GJ\right) \frac{A}{(I_y + I_z)} \qquad (n = 1) \tag{7.21}$$

As noted earlier, in addition to torsional mode, there are two pure flexural modes. The buckling load will be the minimum of three loads corresponding to these modes. The other two buckling loads due to flexure are:

$$P_{y,\mathrm{cr}} = \pi^2 E I_y / L^2 \quad \text{and} \quad P_{z,\mathrm{cr}} = \pi^2 E I_z / L^2$$

It should be noted that these buckling modes are uncoupled.

Case II: Consider a column having built-in ends such that rotation and warping are prevented. The boundary conditions to be satisfied are:

$$\beta(0) = \beta'(0) = \beta(L) = \beta'(L) = 0$$

Incorporating these boundary conditions in the general solution given by (7.19) and vanishing the determinant of the coefficients of constants A, B, C and D for non-trivial solution provides characteristic equation

$$2(\cos \alpha L - 1) + \alpha L \sin \alpha L = 0 \quad \text{or} \quad \sin \frac{\alpha L}{2}\left(\frac{\alpha L}{2} \cos \frac{\alpha L}{2} - \sin \frac{\alpha L}{2}\right) = 0 \tag{7.22}$$

This characteristic equation provides two solutions:

$$\text{(i)} \ \sin \frac{\alpha L}{2} = 0 \text{ giving } P_{x,\mathrm{cr}} = \left(GJ + \frac{4\pi^2 E I_\mathrm{w}}{L^2}\right) \frac{A}{I_p} \quad (\text{for} \quad n = 1) \tag{7.23}$$

$$\text{(ii)} \ \tan \frac{\alpha L}{2} = \frac{\alpha L}{2} \text{ with lowest root } \frac{\alpha L}{2} = 4.4928,$$

and therefore

$$P_{x,\mathrm{cr}} = \left(GJ + \frac{80.75 E I_\mathrm{w}}{L^2}\right) \frac{A}{I_p} \tag{7.24}$$

This load is higher than the former and corresponds to antisymmetric mode of buckling.

II. Energy Solutions

Internal Potential Energy

As discussed in Chap. 2, the total potential Π of a member consists of two parts: strain energy U of the deformed member and potential energy V of the external loads. In the case under consideration, U can be divided into parts: U_1 due to longitudinal direct stresses and U_2 due to shear stresses. The strain energy due to direct stresses is given by

$$U_1 = \frac{1}{2}\int \left[EI_y(u'')^2 + EI_z(v'')^2 + EA\epsilon^2\right] \mathrm{d}x \tag{7.25}$$

The strain energy due to St. Venant shear stress is given by

$$\begin{aligned} U_{2,1} &= \frac{1}{2}\int M_x \frac{\mathrm{d}\beta}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{2}\int \left(-GJ\frac{\mathrm{d}\beta}{\mathrm{d}x}\right)\frac{\mathrm{d}\beta}{\mathrm{d}x}\,\mathrm{d}x \\ &= -\frac{1}{2}\int GJ\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}\right)^2 \mathrm{d}x = -\frac{1}{2}\int GJ(\beta')^2\,\mathrm{d}x \end{aligned} \tag{7.26}$$

The component of strain energy due to warping stresses can be expressed as

$$U_{2,2} = \frac{1}{2}\int EI_{\mathrm{w}}(\beta'')^2\,\mathrm{d}x \tag{7.27}$$

Thus, the total strain energy in a member is given by

$$\begin{aligned} U &= U_1 + U_2 = U_1 + (U_{2,1} + U_{2,2}) \\ &= \frac{1}{2}\int_0^L \left[EI_y(u'')^2 + EI_z(v'')^2 + EI_{\mathrm{w}}(\beta'')^2 - GJ(\beta')^2 + EA\epsilon^2\right] \mathrm{d}x \end{aligned} \tag{7.28}$$

The evaluation of the term $\int EA\epsilon^2\,\mathrm{d}x$ can be avoided, if the potential energy is reckoned from the value (zero) for the fully compressed but un-deflected (pre-buckled) state of column and potential energy of external loads is determined on the same basis. As the potential energy is zero for the straight column carrying its critical compressive load, potential energy in *post-buckled state* is expressed by first four terms of (7.28)

$$U = \frac{1}{2}\int_0^L \left[EI_y(u'')^2 + EI_z(v'')^2 + EI_{\mathrm{w}}(\beta'')^2 - GJ(\beta')^2\right] \mathrm{d}x \tag{7.29}$$

External Potential Energy

As explained earlier in Chap. 2, the potential energy of external loads, V is equal to the sum of negative products of external forces and the displacements of their points of application in the direction of forces. Reckoning V from zero for the fully loaded but un-deflected column, V will therefore represent change in potential due to lateral bending and twisting only. For an axially or centrally loaded column shown in Fig. 7.4 the compressive stress, $\sigma_x = P/A$ is uniformly distributed on the end surfaces. As the member buckles the stress σ_x changes to $\sigma_x + \mathrm{d}\sigma_x$. The work done by $\mathrm{d}\sigma_x$ may be neglected in comparison to work done by the stresses σ_x.

For computation of work done by σ_x consider a fibre column of area $\mathrm{d}A$ shown in Fig. 7.4b carrying a load $\sigma_x\,\mathrm{d}A$ at each end. The change in potential energy is $\mathrm{d}V = -\sigma_x\,\mathrm{d}A\Delta$, where Δ is relative displacement of the top w.r.t. bottom of the column

$$\Delta = \Delta_\mathrm{c} + \Delta_\mathrm{a}$$

where Δ_c and Δ_a are the contributions of curvature of fibre and the change in axial stress $\mathrm{d}\sigma_x$ in the post-buckled stage. Thus

$$\mathrm{d}W_\mathrm{e} = -\sigma_x\,\mathrm{d}A\Delta = -\sigma_x\,\mathrm{d}A\left[\Delta_\mathrm{c} + \frac{1}{E}\int\limits_0^L \mathrm{d}\sigma_x\,\mathrm{d}x\right]$$

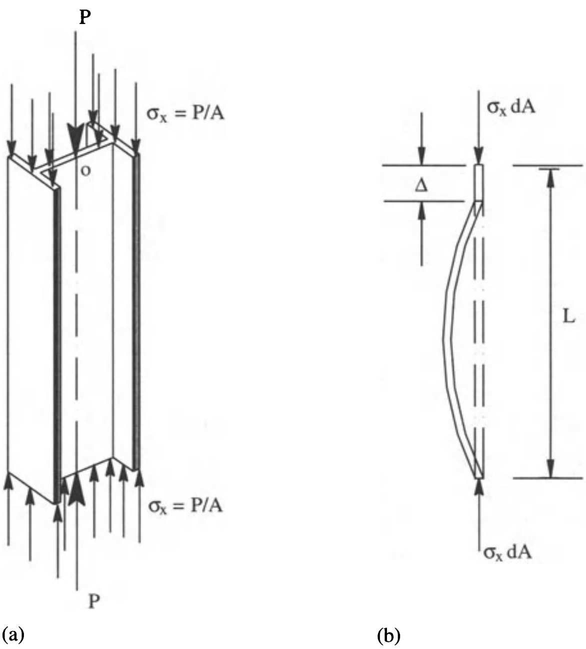


Fig. 7.4a,b. Column subjected to uniformly distributed compressive stress, $\sigma_x = P/A$. **a** Actual column under axial load, **b** fibre column

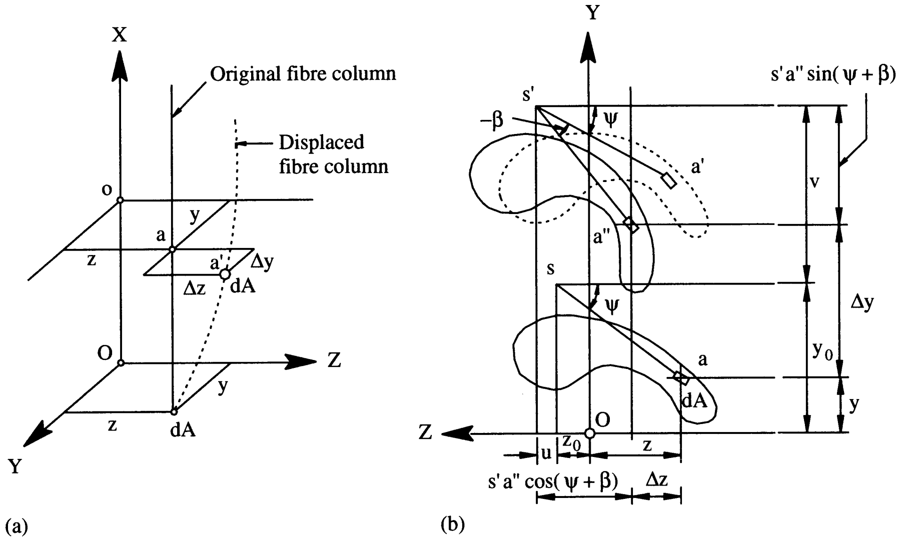


Fig. 7.5a,b. Movements Δy and Δz of fibre column of area dA. **a** Displacements of fibre column, **b** displacement of the cross-section

For the entire column cross-section

$$W_e = -\sigma_x \int_A \Delta_c \, dA - \frac{\sigma_x}{E} \int_A \int_0^L d\sigma_x \, dx \, dA$$

$$= -\sigma_x \int_A \Delta_c \, dA - \frac{\sigma_x}{E} \int_0^L \left[\int_A d\sigma_x \, dA \right] dx \tag{7.30}$$

The integral in the parentheses in (7.30) is the component of resultant of additional stress $d\sigma_x$ in X-direction, which must vanish because the external load does not change. Therefore,

$$W_e = -\sigma_x \int_A \Delta_c \, dA \tag{7.31}$$

Taking the coordinate axes X, Y and Z to pass through the centroid of the cross-section with the location of shear centre, s, being represented by y_0 and z_0 and the displacements by v, u and β as shown in Fig. 7.5. Consider a fibre column of area dA at the point (y, z) before deformation. Owing to deformation the new position of the fibre column is given by $(y + \Delta y, z + \Delta z)$, where Δy and Δz are functions of x.

For small deformation: $\cos\beta = 1$ and $\sin\beta = \beta$. Therefore Δy and Δz are given by:

$$\Delta y = v - (z_0 - z)\sin\beta = v - (z_0 - z)\beta, \quad \text{and}$$
$$\Delta z = u + (y_0 - y)\sin\beta = u + (y_0 - y)\beta \tag{7.32}$$

The movement of the load in X-direction, Δ_c for bi-planar deformation can be obtained by extending the uni-planar deformation given by (3.6):

$$\Delta_c = \frac{1}{2}\int_0^L \left[\left(\frac{d\Delta y}{dx}\right)^2 + \left(\frac{d\Delta z}{dx}\right)^2\right] dx \tag{7.33}$$

Substituting for Δy and Δz from (7.32) into (7.33):

$$\Delta_c = \frac{1}{2}\int_0^L \Big\{\left[(u')^2 + (v')^2\right] + 2y_0 u'\beta' - 2yu'\beta' - 2z_0 v'\beta' + 2zv'\beta' + \left[(y_0 - y)^2 + (z_0 - z)^2\right](\beta')^2\Big\} dx \tag{7.34}$$

Using the geometrical relations:

$$\int_A dA = A, \int_A y\,dA = 0, \int_A z\,dA = 0 \quad \text{and} \quad \int_A \left[(y_0 - y)^2 + (z_0 - z)^2\right] dA = I_p$$

the potential energy of external loads can be expressed as

$$V = -W_e = \sigma_x \int_A \Delta_c\,dA = \frac{1}{2}\int_0^L \Big\{\sigma_x A\left[(u')^2 + (v')^2\right] + 2\sigma_x A y_0 u'\beta' - 2\sigma_x A z_0 v'\beta' + \sigma_x I_p(\beta')^2\Big\} dx \tag{7.35}$$

where I_p is the *polar moment of inertia* of the cross-section with respect to the shear centre. The complete expression for potential Π $(= U + V)$ of beam is sum of (7.29) and (7.35)

$$\Pi = \frac{1}{2}\int_0^L \Big\{EI_y(u'')^2 + EI_z(v'')^2 + EI_w(\beta'')^2 - GJ(\beta')^2 + P\left[(u')^2 + (v')^2\right] + 2Py_0u'\beta' - 2Pz_0v'\beta' + \frac{P}{A}I_p(\beta')^2\Big\} dx \tag{7.36}$$

Governing Differential Equations of Buckling

The *total potential* of member Π is a function of three variables u, v and β. From *stationary potential principle*, Π will be stationary if the following Eulerian equations are satisfied

$$EI_y u'''' + Pu'' + Py_0\beta'' = 0 \tag{7.37a}$$

$$EI_z v'''' + Pv'' - Pz_0\beta'' = 0 \tag{7.37b}$$

$$Py_0u'' - Pz_0v'' + EI_w\beta'''' + \left(P\frac{I_p}{A} - GJ\right)\beta'' = 0 \tag{7.37c}$$

Equations 7.37 are in their most general form and are applicable to any type of boundary conditions which can be expressed without contradiction in terms of variables u, v, β and their derivatives. Depending upon the properties of cross-section the results can be simplified considerably.

1. Column having cross-section for which shear centre coincides with the centroid

For the cross-section having two axes of symmetry, the shear centre lies at the centroid i.e. $z_0 = y_0 = 0$ and the Eulerian governing buckling equations reduce to

$$EI_y u'''' + Pu'' = 0 \tag{7.38a}$$

$$EI_z v'''' + Pv'' = 0 \tag{7.38b}$$

$$EI_w \beta'''' + \left(P\frac{I_p}{A} - GJ\right)\beta'' = 0$$

or

$$EI_w \beta'''' + \left(\sigma_x I_p - GJ\right)\beta'' = 0 \tag{7.38c}$$

It should be noted that these buckling equations are uncoupled. The first two equations are identical in form with the differential equations for buckling of columns subjected to bending moment about Y- and Z-axes, respectively. The third equation describes buckling of column by twisting.

Example 7.1. A straight I-section column with the end cross-sections prevented from twisting but the flanges at the ends are free to rotate in their own planes, is subjected to a twisting moment about its centroidal axis, determine the value of the stress at which buckling takes place.

The governing differential equation for this case is:

$$EI_w \beta'''' + \left(\sigma_x I_p - GJ\right)\beta'' = 0 \tag{7.39}$$

Boundary Conditions. As the column is twisted about its centre line the displacement of the flanges in their own planes are: $u = \beta(d/2)$. Since the flanges are free to rotate at the ends, the bending moment in the flanges (i.e. curvature, u'') must be zero i.e. $\beta'' = 0$, and in addition as the ends of column are prevented from twisting, $\beta = 0$. Thus the four boundary conditions are:

$$\beta(0) = \beta''(0) = \beta(L) = \beta''(L) = 0$$

The displacement function satisfying these boundary conditions is given by:

$$\beta = A \sin\frac{n\pi x}{L} \tag{7.40}$$

where A is an arbitrary constant and n an integer. Substitution of β from (7.40) into (7.39) provides

$$EI_\mathrm{w}\left(A\frac{n^4\pi^4}{L^4}\sin\frac{n\pi x}{L}\right)+\left(\sigma_\beta I_\mathrm{p}-GJ\right)\left(-A\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}\right)=0$$

$$\left[EI_\mathrm{w}\left(\frac{n^2\pi^2}{L^2}\right)-\left(\sigma_\beta I_\mathrm{p}-GJ\right)\right]\left(\frac{n^2\pi^2}{L^2}A\sin\frac{n\pi x}{L}\right)=0$$

For a non-trivial solution

$$EI_\mathrm{w}\left(\frac{n^2\pi^2}{L^2}\right)-\left(\sigma_\beta I_\mathrm{p}-GJ\right)=0$$

$$\sigma_\beta=\frac{\pi^2E}{L^2}\left[n^2\left(\frac{I_\mathrm{w}}{I_\mathrm{p}}\right)+\frac{L^2}{\pi^2}\left(\frac{GJ}{EI_\mathrm{p}}\right)\right] \tag{7.41}$$

where I_w, I_p and J are the cross-sectional properties. The polar moment of inertia I_p is referred to the shear centre of cross-section. The lowest stress $\sigma_{\beta,\mathrm{cr}}$ at which buckling occurs corresponds to $n = 1$. Therefore,

$$\begin{aligned}\sigma_{\beta,\mathrm{cr}}A=P_{\beta,\mathrm{cr}}&=\frac{\pi^2EA}{L^2}\left[\left(\frac{I_\mathrm{w}}{I_\mathrm{p}}\right)+\frac{L^2}{\pi^2}\left(\frac{GJ}{EI_\mathrm{p}}\right)\right]\\&=\frac{\pi^2EA}{(L/r_\beta)^2}\end{aligned} \tag{7.42}$$

where r_β which has dimension of length is an *equivalent radius of gyration* and depends upon shape of the cross-section and length of the beam-column and is expressed as

$$r_\beta=\left[\left(\frac{I_\mathrm{w}}{I_\mathrm{p}}\right)+\frac{L^2}{\pi^2}\left(\frac{GJ}{EI_\mathrm{p}}\right)\right]^{1/2}$$

Therefore, a column will be torsionally unstable at a critical stress $\sigma_{\beta,\mathrm{cr}}$ which is equal to the critical stress for lateral buckling of an *equivalent column* having the slenderness ratio (L/r_β). The critical stress of the column will be lowest of the three stresses $\sigma_{y,\mathrm{cr}}$, $\sigma_{z,\mathrm{cr}}$ and $\sigma_{\beta,\mathrm{cr}}$, and it will correspond to the smallest radii of gyration r_y, r_z and r_β.

For the end conditions other than simply supported or hinged

$$\sigma_\mathrm{cr}=\frac{\pi^2E}{(KL/r_\beta)} \tag{7.43}$$

where KL is the *effective length* of the beam-column. In the case of fixed-ended column or fixed-free column, for torsional buckling the slenderness ratio considered is KL/r_β where r_β is given by

$$r_\beta=\left[\left(\frac{I_\mathrm{w}}{I_\mathrm{p}}\right)+\frac{K^2L^2}{\pi^2}\left(\frac{GJ}{EI_\mathrm{p}}\right)\right]^{1/2} \tag{7.44}$$

For rolled I-beam sections, $I_p = I_y + I_z$ and $I_w = [(d^2/4)I_y]$. The equivalent radius of gyration is given by

$$r_\beta = \left\{ \left(\frac{d^2}{4}\right) \frac{I_y}{I_y + I_z} + \frac{K^2L^2}{\pi^2} \left[\frac{GJ}{E(I_y + I_z)} \right] \right\}^{1/2} \tag{7.45}$$

For most of the standard steel beam sections r_β is larger than r_y and hence columns buckle laterally; only columns of short length with very wide flange have r_β values slightly smaller than r_y and such column will buckle torsionally.

2. Column with one axis of symmetry

If the cross-section has an axis of symmetry, say the Y-axis, then $z_0 = 0$ and (7.37) reduce to

$$EI_y u'''' + Pu'' + Py_0\beta'' = 0 \tag{7.46a}$$

$$EI_z v'''' + Pv'' = 0 \tag{7.46b}$$

$$Py_0 u'' + EI_w \beta'''' + \left(P\frac{I_p}{A} - GJ\right)\beta'' = 0 \tag{7.46c}$$

The second equation is uncoupled and is the usual differential equation of flexural buckling in Y-direction. For pin-ended column the critical stress is

$$\sigma_{y,\mathrm{cr}} = \frac{\pi^2 E}{(L/r_z)^2} \tag{7.47}$$

The first and third equations involve coupling between buckling in Z-direction and twisting. For a pin-ended column with twisting prevented, the boundary conditions are

$$u(0) = \beta(0) = u''(0) = \beta''(0) = 0$$
$$u(L) = \beta(L) = u''(L) = \beta''(L) = 0$$

The displacement functions satisfying these boundary conditions are given by

$$u = A\sin\frac{n\pi x}{L} \quad \text{and} \quad \beta = B\sin\frac{n\pi x}{L}$$

Substituting these deflection modes in the governing differential equations; two homogenous equations are obtained in terms of constants A and B. For non-trivial solution ($A \neq 0$, $B \neq 0$) the determinant of coefficients must vanish. The lowest critical load occurs for buckling in one half wave i.e. $n = 1$

$$\begin{vmatrix} (P_y - P_{\mathrm{cr}}) & -P_{\mathrm{cr}}y_0 \\ -P_{\mathrm{cr}}y_0 & \left[\frac{\pi^2 EI_w}{L^2} - \left(P_{\mathrm{cr}}\frac{I_p}{A} - GJ\right)\right] \end{vmatrix} = \begin{vmatrix} \left(\frac{r_y^2}{r_e^2} - 1\right) & -y_0 \\ -y_0 & -\frac{I_p}{A}\left[1 - \frac{r_\beta^2}{r_e^2}\right] \end{vmatrix} = 0$$

$$\left(P_{\mathrm{cr}} - P_y\right)\left(P_{\mathrm{cr}} - P_\beta\right) - P_{\mathrm{cr}}^2\left(\frac{Ay_0^2}{I_p}\right) = 0 \tag{7.48a}$$

or

$$\left(1 - \frac{r_y^2}{r_e^2}\right)\left(1 - \frac{r_\beta^2}{r_e^2}\right) - \frac{Ay_0^2}{I_p} = 0$$

where

$$r_\beta^2 = \left[\left(\frac{I_w}{I_p}\right) + \frac{L^2}{\pi^2}\left(\frac{GJ}{EI_p}\right)\right] \quad \text{and} \quad P_{cr} = \frac{\pi^2 EA}{(L/r_e)^2}, \tag{7.48b}$$

and r_e is an equivalent radius of gyration determined from the quadratic equation (7.48). Thus the critical load for buckling by torsion and flexure will therefore be the same as critical load in the ordinary column theory of an equivalent column having slenderness ratio L/r_e.

Equation (7.48) always has two positive roots of r_e, one of them is smaller than both r_y and r_β. Thus the column in question will buckle at a load corresponding to smaller of the values r_z and r_e. If r_z is smaller, the column will buckle in the Y-direction without twisting; if the root r_e is smaller than r_z, the column will deflect in the Z-direction and twist simultaneously. Since, r_e is always smaller than r_y, the critical load is smaller than that given by conventional column theory for buckling in the Z-direction.

3. Column with a cross-section with no axis of symmetry

This is the most general case with three coupled Eulerian governing equations given by (7.37). In this case both flexural and torsional displacements occur at the instant of buckling. For illustration consider a beam-column with simply supported end conditions. As in the previous cases the deformation functions may be assumed in the form:

$$u = A\sin(\pi x/L)\ , \quad v = B\sin(\pi x/L) \quad \text{and} \quad \beta = C\sin(\pi x/L)$$

For a non-trivial solution ($A \neq 0$, $B \neq 0$ and $C \neq 0$), vanishing the determinant of coefficients of constants A, B and C provides the characteristic equation as follows:

$$\begin{vmatrix} (P_z - P_{cr}) & 0 & P_{cr}z_0 \\ 0 & (P_y - P_{cr}) & -P_{cr}y_0 \\ P_{cr}z_0 & -P_{cr}y_0 & \left[\frac{\pi^2 EI_w}{L^2} - \left(P_{cr}\frac{I_p}{A} - GJ\right)\right] \end{vmatrix} = 0$$

$$(P_{cr} - P_z)\left(P_{cr} - P_y\right)\left(P_{cr} - P_\beta\right) \\ -P_{cr}^2\left(P_{cr} - P_y\right)\left(\frac{Az_0^2}{I_p}\right) - P_{cr}^2\left(P_{cr} - P_z\right)\left(\frac{Ay_0^2}{I_p}\right) = 0 \tag{7.49}$$

or $$\left(1 - \frac{r_y^2}{r_e^2}\right)\left(1 - \frac{r_z^2}{r_e^2}\right)\left(1 - \frac{r_\beta^2}{r_e^2}\right) - \left(1 - \frac{r_y^2}{r_e^2}\right)\left(\frac{Az_0^2}{I_p}\right) - \left(1 - \frac{r_z^2}{r_e^2}\right)\left(\frac{Ay_0^2}{I_p}\right) = 0$$

where $$r_\beta^2 = \left[\left(\frac{I_w}{I_p}\right) + \frac{L^2}{\pi^2}\left(\frac{GJ}{EI_p}\right)\right] \quad \text{and} \quad P_{cr} = \frac{\pi^2 EA}{(L/r_e)^2} \tag{7.50}$$

The equivalent radius of gyration, r_e can be obtained from the cubic equation (7.50). The smallest of the roots of (7.50) will provide critical buckling load for the column in question.

Example 7.2. A 7.0 m long member consisting of MB 400 @0.616 kN/m rolled steel section is to be used as a column. It is specified that all the end conditions of the member are of simple support type. Compute the buckling load of the column.

For the given cross-section of the column in which the shear centre coincides with the centroid, the flexural and torsional buckling modes would be uncoupled. The properties of the member are:
Length, $L = 700\,\text{cm}$; area, $A = 78.4\,\text{cm}^2$; depth, $d = 40\,\text{cm}$; flange width, $b = 14\,\text{cm}$; flange thickness, $t_f = 1.6\,\text{cm}$; web thickness $t_\omega = 0.89\,\text{cm}$; $I_y = 622\,\text{cm}^4$; $I_z = 20500\,\text{cm}^4$; $E = 20000\,\text{kN/cm}^2$; $G = (3/8)E$ and $I_p = I_y + I_z = 21122\,\text{cm}^4$.

$$J = \sum \frac{1}{3} b_i t_i^3 = \frac{2}{3} b t_f^3 + \frac{1}{3} d t_\omega^3 = \frac{2}{3} \times 14 \times 1.6^3 + \frac{1}{3} \times 40 \times 0.89^3 = 47.63\,\text{cm}^4$$

The three independent elastic buckling loads of the column are:

$$P_{y,\text{cr}} = \frac{\pi^2 E I_y}{L^2} = \frac{\pi^2 \times 20000 \times 622}{(700)^2} = 250.57\,\text{kN}$$

$$P_{z,\text{cr}} = \frac{\pi^2 E I_z}{L^2} = \frac{\pi^2 \times 20000 \times 20500}{(700)^2} = 8258.24\,\text{kN}$$

$$P_{\beta,\text{cr}} = \frac{\pi^2 E A}{(L/r_\beta)^2}$$

where,

$$r_\beta^2 = \left(\frac{d^2}{4}\right)\left(\frac{I_y}{I_p}\right) + \frac{L^2}{\pi^2}\left(\frac{GJ}{EI_p}\right)$$

$$= \left(\frac{40^2}{4}\right)\left(\frac{622}{21122}\right) + \left(\frac{700}{\pi}\right)^2\left(\frac{3}{8} \times \frac{47.63}{21122}\right) = 53.76\,\text{cm}^2$$

or

$$r_\beta = 7.332\,\text{cm}$$

Therefore,

$$P_{\beta,\text{cr}} = \frac{\pi^2 \times 20000 \times 78.4}{(700/7.332)^2} = 1697.90\,\text{kN}$$

The critical condition is that of flexural buckling about minor axis of the member at a value of axial thrust of 250.57 kN. The associated stress is 31.96 MPa ($= 250.57 \times 10^3/7840$), which is well within elastic limit. If the column in question had been supported in Z-direction such that bending about Y-Y axis was prevented, but twisting about longitudinal axis was allowed, then the column would buckle torsionally at the next higher critical value, $P_{\beta,\text{cr}} = 1697.90\,\text{kN}$ with critical stress of 216.57 MPa.

The foregoing procedures are equally applicable to clamped and free boundary conditions. In the following example application to clamped end conditions is illustrated.

Example 7.3. Determine the critical load at which a uniform column having a cross-section with one axis of symmetry (e.g. a channel section with symmetry about Z-axis) will buckle. The column has clamped end conditions. It is restrained from warping and is not allowed to rotate about its longitudinal X-axis.

As the section is symmetrical about Z-axis, $y_0 = 0$ and governing Eulerian equations (7.37), reduce to:

$$EI_y \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{7.51a}$$

$$EI_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - P(-z_0)\frac{\mathrm{d}^2 \beta}{\mathrm{d}x^2} = 0 \tag{7.51b}$$

$$EI_\mathrm{w} \frac{\mathrm{d}^4 \beta}{\mathrm{d}x^4} - \left(GJ - P\frac{I_\mathrm{p}}{A}\right)\frac{\mathrm{d}^2 \beta}{\mathrm{d}x^2} - P(-z_0)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0 \tag{7.51c}$$

The first of these equations contains the displacement u only and is the usual differential equation of flexural buckling about Y-axis. Therefore, for clamped columns the critical load is given by

$$P_{y,\mathrm{cr}} = \frac{4\pi^2 EI_y}{L^2} = \frac{\pi^2 EI_y}{(0.5L)^2}$$

The second and third equations contain both v and β but not u. This indicates that the equations are coupled, and that the buckling in Y-direction and twisting will occur simultaneously. Due to this coupling the equations need be solved simultaneously. However, since these equations are of even order with constant coefficients, a *variable separable type solution* can be used. This implies that the mode shapes v and β are not significantly affected by coupling because they are of second order, and hence can be neglected.

For the flexural mode the boundary conditions for clamped ends are

$$u(0) = u'(0) = u(L) = u'(L) = 0$$
$$v(0) = v'(0) = v(L) = v'(L) = 0$$

For the *torsional mode* the boundary conditions stipulate that the column is restrained from warping and can not rotate about X-axis at the ends. These can be expressed mathematically as

$$\beta(0) = \beta'(0) = \beta(L) = \beta'(L) = 0$$

The *shape* or *displacement functions* for u, v and β satisfying above boundary conditions may be taken as

$$u = A\left(1 - \cos\frac{2\pi x}{L}\right) \tag{7.52a}$$

$$v = B\left(1 - \cos\frac{2\pi x}{L}\right), \tag{7.52b}$$

$$\beta = C\left(1 - \cos\frac{2\pi x}{L}\right) \tag{7.52c}$$

Substituting (7.52a) in (7.51a)

$$\left[-EI_y\left(\frac{2\pi}{L}\right)^4 + P_{\mathrm{cr}}\left(\frac{2\pi}{L}\right)^2\right] A\cos\frac{2\pi x}{L} = 0$$

For non-trivial solution, $A\cos(2\pi/L) \neq 0$ and hence

$$P_{y,\mathrm{cr}} = \frac{4\pi^2 EI_y}{L^2} = \frac{\pi^2 EI_y}{(0.5L)^2}$$

This expression is identical to one written directly earlier.

Substituting (7.52b) and (7.52c) in (7.51b) and (7.51c)

$$\left[-EI_z\left(\frac{2\pi}{L}\right)^4 + P\left(\frac{2\pi}{L}\right)^2\right] B + Pz_0\left(\frac{2\pi}{L}\right)^2 C = 0$$

$$Pz_0\left(\frac{2\pi}{L}\right)^2 B + \left[-EI_{\mathrm{w}}\left(\frac{2\pi}{L}\right)^4 - \left(GJ - P\frac{I_{\mathrm{p}}}{A}\right)\left(\frac{2\pi}{L}\right)^2\right] C = 0$$

For non-trivial solution ($B \neq 0$, $C \neq 0$), the determinant of coefficients of B and C must vanish, that is

$$\begin{vmatrix} (P_{\mathrm{cr}} - P_z) & P_{\mathrm{cr}}z_0 \\ P_{\mathrm{cr}}z_0\left(\frac{A}{I_{\mathrm{p}}}\right) & (P_{\mathrm{cr}} - P_\beta) \end{vmatrix} = \left(1 - \frac{Az_0^2}{I_{\mathrm{p}}}\right) P_{\mathrm{cr}}^2 - (P_z + P_\beta)P_{\mathrm{cr}} + P_z P_\beta = 0 \tag{7.53}$$

where,

$$P_z = \frac{4\pi^2 EI_z}{L^2} = \frac{\pi^2 EI_z}{(0.5L)^2}$$

$$P_\beta = \frac{A}{I_{\mathrm{p}}}\left(\frac{4\pi^2}{L^2}EI_{\mathrm{w}} + GJ\right) = \frac{A}{I_{\mathrm{p}}}\left[\frac{\pi^2 EI_{\mathrm{w}}}{(0.5L)^2} + (GJ)\right]$$

These results can also be obtained from the case of simply supported boundary conditions by substituting KL for L where K is the *effective length factor*. For a given cross-section P_z and P_β are known quantities, the roots $P_{\mathrm{cr},1}$ and $P_{\mathrm{cr},2}$ of the quadratic characteristic equation can be computed. The first critical load will be given by smaller of $P_{\mathrm{cr},1}$ and $P_{y,\mathrm{cr}}$, where $P_{\mathrm{cr},1} < P_{\mathrm{cr},2}$. If $P_{y,\mathrm{cr}}$ is minimum, a pure flexural mode will control the critical load; on the other hand if $P_{\mathrm{cr},1}$ is minimum a *torsion-flexural* mode will provide the buckling load.

It should be noted that the problem of evaluating critical load is equivalent to that of finding the deflected configuration for which the system is in equilibrium. This can also be achieved by *stationary potential principle* using *Rayleigh–Ritz technique.* The procedure is illustrated in the following example.

Example 7.4. A uniform column having channel cross-section is clamped at one end and is completely free at the other. Estimate the critical load at which buckling will occur.

Since the boundary conditions at the ends of the column are not symmetric the analysis using Eulerian differential equation approach is difficult. The problem can be handled easily by stationary potential principle using Rayleigh–Ritz technique.

As the cross-section has axis of symmetry about Z-axis, $y_0 = 0$. The geometric boundary conditions at the column ends are:
For the clamped end:

$$u(0) = v(0) = \beta(0) = u'(0) = v'(0) = \beta'(0) = 0$$

For the free end

$$u''(L) = v''(L) = \beta''(L) = 0 \quad \text{and}$$
$$u'''(L) \neq 0, v'''(L) \neq 0, \beta'''(L) \neq 0$$

The *shape functions* satisfying the above boundary conditions may be assumed to be

$$u = Af(x); \;\; v = Bf(x) \quad \text{and} \quad \beta = Cf(x)$$

where,

$$f(x) = (3Lx^2 - x^3) \tag{7.54}$$

The potential functional Π given by (7.36) contains the first and second derivatives of shape functions and thus requires following integrations:

$$\int_0^L [f'(x)]^2 \mathrm{d}x = \int_0^L \left[6Lx - 3x^2\right]^2 \mathrm{d}x = \left(\frac{24}{5}\right) L^5$$
$$\int_0^L [f''(x)]^2 \mathrm{d}x = \int_0^L [6L - 6x]^2 \mathrm{d}x = 12L^3 \tag{7.55}$$

Substituting for the shape functions u, v and β from (7.54) into (7.36) and for stationary potential, the derivatives of Π w.r.t. A, B and C must vanish i.e.

$$\frac{\partial \Pi}{\partial A} = \left[\int_0^L EI_y (f'')^2 \mathrm{d}x - \int_0^L P(f')^2 \mathrm{d}x \right] A = 0$$

This equation represents an uncoupled flexural-mode. For non-trivial $A \neq 0$ solution

$$P_{y,\mathrm{cr}} = \frac{\int_0^L EI_y (f'')^2 \mathrm{d}x}{\int_0^L (f')^2 \mathrm{d}x} = (EI_y)(12L^3)\left(\frac{5}{24L^3}\right) = \frac{2.5EI_y}{L^2} = \frac{\pi^2 EI_y}{(1.98L)^2} = \frac{\pi^2 EI_y}{(KL)^2}$$

where $K = 1.98$. The exact value of K is 2.0. Vanishing other variations of Π

$$\frac{\partial \Pi}{\partial B} = \left[\int_0^L EI_z (f'')^2 \mathrm{d}x - \int P(f')^2 \mathrm{d}x \right] B + \left[\int_0^L Pz_0 (f')^2 \mathrm{d}x \right] C = 0$$

or

$$\left[\frac{2.5EI_z}{L^2} - P \right] B + (Pz_0)C = 0 \tag{7.56a}$$

$$\frac{\partial \Pi}{\partial C} = \left[\int_0^L Pz_0 (f')^2 \mathrm{d}x \right] B$$

$$+ \left[\int_0^L EI_\mathrm{w} (f'')^2 \mathrm{d}x + \int_0^L GJ(f')^2 \mathrm{d}x - \int_0^L \left(\frac{P}{A}\right) I_\mathrm{p} (f')^2 \mathrm{d}x \right] C = 0$$

or

$$(Pz_0)B + \frac{I_\mathrm{p}}{A}\left[\left(\frac{2.5EI_\mathrm{w}}{L^2} + GJ \right) \frac{A}{I_\mathrm{p}} - P \right] C = 0 \tag{7.56b}$$

For non-trivial ($B = C \neq 0$) solution of (7.56)

$$\begin{vmatrix} \left(\frac{2.5EI_z}{L^2} - P\right) & Pz_0 \\ Pz_0\left(\frac{A}{I_\mathrm{p}}\right) & \left[\left(\frac{2.5EI_\mathrm{w}}{L^2} + GJ\right)\frac{A}{I_\mathrm{p}} - P\right] \end{vmatrix} = 0$$

$$\left(1 - \frac{Az_0^2}{I_\mathrm{p}}\right) P^2 - (P_z + P_\beta)P + P_z P_\beta = 0$$

where

$$P_z = \frac{2.5EI_z}{L^2} = \frac{\pi^2 EI_z}{(1.98L)^2} \quad \text{and} \quad P_\beta = \left(\frac{A}{I_\mathrm{p}}\right)\left[\frac{\pi^2 EI_\mathrm{w}}{(1.98L)^2} + GJ\right] \tag{7.57}$$

The quadratic characteristic equation (7.57) can be solved for its roots $P_{\mathrm{cr},1}$ and $P_{\mathrm{cr},2}$. The minimum of three i.e. $P_{y,\mathrm{cr}}$; $P_{\mathrm{cr},1}$ and $P_{\mathrm{cr},2}$ will provide the critical load.

7.3 Lateral Buckling of Beams

In the preceding sections the discussion was mainly confined to the stability analysis of centrally and eccentrically loaded columns. In this section more complex buckling problems of *open thin walled sections* will be discussed. An I-beam supported at the ends and loaded longitudinally and transversely in the plane of web may buckle side ways if it is laterally unsupported at the supports. If the flexural rigidity of the beam in the plane of web is many fold its lateral stability, the beam may buckle and collapse long before bending stresses due to transverse load reach the yield point.

7.3.1 Torsional Buckling due to Flexure

Consider an I- or wide flange type simply supported beam subjected to planer moment as shown in Fig. 7.6a. The top flange of the beam is under uniform compression and would tend to buckle in weaker (i.e. downward) direction but the web prevents the same and hence the flange has tendency to buckle laterally (i.e. horizontally). On the other hand the bottom flange being in tension tends to remain straight. Thus the top flange bends farther than the bottom flange and in consequence the entire cross-section twists. This also holds good for planar rectangular beam. At the critical value of bending moment, $M_{\mathrm{oz,cr}}$, the member becomes unstable and warps i.e. undergoes rotation and lateral deflection which may cause collapse.

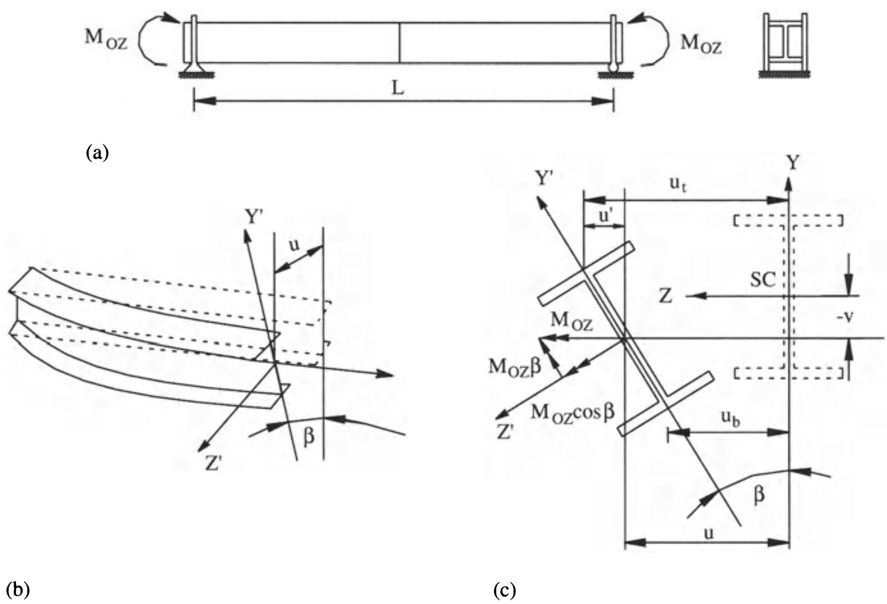


Fig. 7.6a–c. Lateral-torsional buckling of beam subjected to end moments. **a** Beam subjected to end moments, **b** deflected shape of beam, **c** deflections of cross-sections

In the following analysis it is presumed that the cross-section of the beam is constant and the fibre stresses due to external load do not exceed proportionality limit at the instant of buckling. Moreover, the distortion in the plane of cross-section is considered to be negligible such that section does not change shape. The displacements are also considered to be small such that secondary effects of displacements are ignored. Based on these assumptions, the lateral displacements of flanges are taken to be caused primarily by bending of flanges about Y-axis.

As defined in previous sections, Z- and Y-axes are principal axes of I-section which may have unequal flanges; the coordinates of the *shear centre* are generally represented by y_0 and z_0; v and u are the components of displacement of shear centre parallel to the rotated (variable or displaced) axes Y'- and Z'-; β is the angle of twist (rotation of axes) with respect to longitudinal axis X as shown in Fig. 7.6b. Normally the bending rigidity of I- or rectangular-beam cross-section about Z-axis (major axis) is quite large when compared to that about Y-axis. Thus deflection v in the plane of applied moment is small as compared to lateral displacement u and angle of twist, β. The buckling problem can be treated both by classical differential equation method and stationary potential energy principle.

Beams with Symmetric Cross-Section

For a doubly symmetric cross-section: $y_0 = z_0 = 0$

1. Differential equation solution

Three differential equations can be written by considering the problems of warping without translation, bending without warping, and that of constant torque using small displacement theory. Applicability of linear superposition of effects is presumed.

The lateral displacements of the top and bottom flanges u_t and u_b can be related to u and β which are the lateral displacement and angle of twist of shear centre with respect to longitudinal axis, respectively, of cross-section.

$$u_t = u + u' = u + \beta\left(\frac{d}{2}\right)$$

and

$$u_b = u - u' = u - \beta\left(\frac{d}{2}\right) \tag{7.58}$$

where u' is the displacement of center of flange with respect to the centroid of the cross-section and d is the depth. In the deflected configuration of beam the components of the applied moment M_{oz} with respect to displaced or variable axes X', Y' and Z' can be obtained from Fig. 7.6c.

$$M_{z'} = M_{\mathrm{oz}} \cos\beta = M_{\mathrm{oz}};$$

$$M_{y'} = M_{\mathrm{oz}} \sin\beta = \beta M_{\mathrm{oz}}$$

and

$$M_{x'} = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) M_{\mathrm{oz}} \tag{7.59}$$

Since $\sin\beta \approx \beta$ and $\cos\beta \approx 1.0$. The resistance of beam to these components can be expressed as:

$$M_{z'} = EI_z\frac{\mathrm{d}^2v}{\mathrm{d}x^2}, \quad M_{y'} = EI_y\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \tag{7.60}$$

$M_{x'}$ = torque developed by transverse shear in flanges i.e. warping torsion + St. Venant or pure torsional resistance = $M_\mathrm{w} + M_\mathrm{sv}$.

Transverse shear in the flange,

$$V_\mathrm{f} = \frac{\mathrm{d}M_\mathrm{f}}{\mathrm{d}x} = EI_\mathrm{f}\frac{\mathrm{d}^3u'}{\mathrm{d}x^3}$$

where M_f is the bending moment in the flange about Y-axis and I_f ($\approx I_y/2$) is the moment of inertia of the flange w.r.t. Y-axis. Thus

$$V_\mathrm{f} = EI_\mathrm{f}\left(\frac{d}{2} \times \frac{\mathrm{d}^3\beta}{\mathrm{d}x^3}\right)$$

Then torque developed is given by

$$M_\mathrm{w} = V_\mathrm{f}d = E\left(\frac{1}{2}I_\mathrm{f}d^2\right)\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3} = EI_\mathrm{w}\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3}$$

where,

$$I_\mathrm{w} = \frac{1}{2}I_\mathrm{f}d^2$$

Therefore,

$$M_{x'} = EI_\mathrm{w}\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3} - GJ\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}\right) \tag{7.61}$$

Thus the equilibrium equations are:

$$EI_z\frac{\mathrm{d}^2v}{\mathrm{d}x^2} - M_\mathrm{oz} = 0 \tag{7.62a}$$

$$EI_y\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \beta M_\mathrm{oz} = 0 \tag{7.62b}$$

$$EI_\mathrm{w}\left(\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3}\right) - GJ\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}\right) - M_\mathrm{oz}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0 \tag{7.62c}$$

The first equation describes bending about major axis of the cross-section and is independent of lateral and torsional displacements. The last two equations, however, are coupled and must be solved simultaneously. Eliminate ($\mathrm{d}^2u/\mathrm{d}x^2$) term from these two equations:

$$EI_\mathrm{w}\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} - GJ\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - \left(\frac{M_\mathrm{oz}^2}{EI_y}\right)\beta = 0 \tag{7.63a}$$

which is of the form: $\beta'''' - \lambda_1\beta'' - \lambda_2\beta = 0$ where

$$\lambda_1 = \frac{GJ}{EI_w} \quad \text{and} \quad \lambda_2 = \left(\frac{M_{oz}^2}{(EI_y)(EI_w)}\right) \tag{7.63b}$$

This is governing differential equation with general solution in the form

$$\beta = A\sinh\alpha_1 x + B\cosh\alpha_1 x + C\sin\alpha_2 x + D\cos\alpha_2 x \tag{7.64}$$

where

$$\alpha_1 = \left(\frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}\right)^{1/2} \quad \text{and} \quad \alpha_2 = \left(\frac{-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2}\right)^{1/2} \tag{7.65}$$

with boundary conditions for simply-supported ends

$$\beta(0) = \beta(L) = \beta''(0) = \beta''(L) = 0$$

Substituting the general solution into the boundary conditions and for non-trivial solution ($A \neq B \neq C \neq D \neq 0$) the determinant of coefficients of these constants must vanish i. e.

$$\left(\alpha_1^2 + \alpha_2^2\right)^2 \sinh\alpha_1 L \sin\alpha_2 L = 0$$

Since the term in the parentheses is sum of two positive numbers, hence can not be zero. On the other hand $sinh\alpha_1 L$ can be zero only when $(\alpha_1 L) = 0$, which is a trivial solution. Therefore, for a solution: $\sin\alpha_2 L = 0$.

$$\alpha_2 L = n\pi \quad \text{or} \quad \alpha_2^2 = \frac{n^2\pi^2}{L^2}$$

substituting the value of α_2 from (7.65).

$$\frac{-\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_2}}{2} = \frac{n^2\pi^2}{L^2}$$

or

$$\lambda_2 = \frac{1}{4}\left[\left(\frac{2n^2\pi^2}{L^2} + \lambda_1\right)^2 - \lambda_1^2\right] = \left(\frac{n^4\pi^4}{L^4} + \frac{n^2\pi^2}{L^2}\lambda_1\right)$$

where λ_1 and λ_2 are given by (7.63b). Therefore,

$$M_{oz,cr} = \frac{n\pi}{L}\sqrt{(EI_y)(GJ)\left[1 + \frac{n^2\pi^2 EI_w}{GJL^2}\right]}$$

For the smallest value of critical moment, $n = 1$ and hence

$$M_{\mathrm{oz,cr}} = \left(\frac{\pi}{L}\right)\sqrt{(EI_y)(GJ) + EI_y\left(\frac{\pi^2 EI_{\mathrm{w}}}{L^2}\right)} \tag{7.66}$$

The warping constant I_{w} is negligibly small for a rectangular cross-section and hence can be reasonably neglected i.e.

$$M_{\mathrm{oz,cr}} = \left(\frac{\pi}{L}\right)\sqrt{EI_y GJ} \tag{7.67}$$

2. Energy method

At buckling, the shortening and stretching of longitudinal fibres of the flange due to lateral displacement are

$$2\Delta_t = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}u_t}{\mathrm{d}x}\right)^2 \mathrm{d}x \quad \text{and} \quad 2\Delta_b = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}u_b}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{7.68}$$

The angle of rotation θ through which M_{oz} travels as obtained from Fig. 7.7 is:

$$\begin{aligned}\theta = \frac{\Delta_t - \Delta_b}{d} &= \frac{1}{4d}\int_0^L \left[\left(\frac{\mathrm{d}u_t}{\mathrm{d}x}\right)^2 - \left(\frac{\mathrm{d}u_b}{\mathrm{d}x}\right)^2\right] \mathrm{d}x \\ &= \frac{1}{4d}\int_0^L \left[4\frac{d}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}\right)\right] \mathrm{d}x = \frac{1}{2}\int_0^L \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}\beta}{\mathrm{d}x}\right) \mathrm{d}x \end{aligned} \tag{7.69}$$

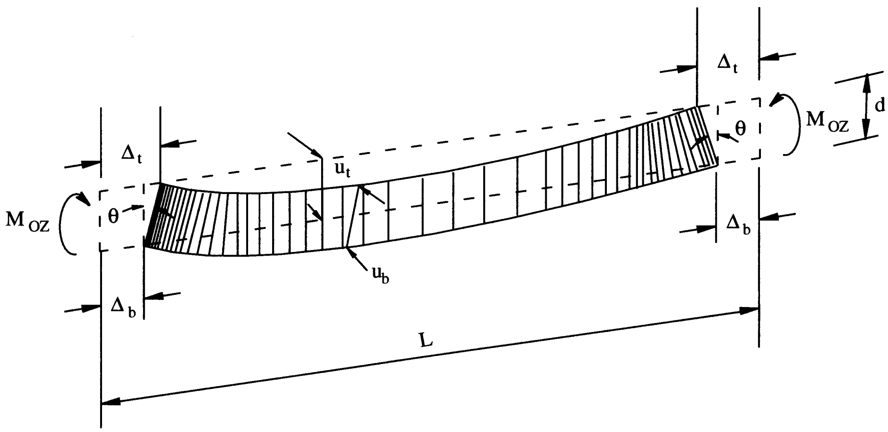


Fig. 7.7. Flexural member in laterally buckled mode

The external potential due to applied end moment is therefore,

$$W_e = V = M_{oz}(2\theta) = M_{oz}\int_0^L \left(\frac{du}{dx}\right)\left(\frac{d\beta}{dx}\right) dx \tag{7.70}$$

The internal potential (strain energy) due to combined lateral and torsional displacements are:

$$-W_i = U = \frac{1}{2}(EI_y)\int_0^L \left(\frac{d^2u}{dx^2}\right)^2 dx + \frac{1}{2}(GJ)\int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{1}{2}(EI_w)\int_0^L \left(\frac{d^2\beta}{dx^2}\right)^2 dx \tag{7.71}$$

Therefore, the total potential associated with buckled configuration is from (7.70) and (7.71)

$$\Pi = U + V = \frac{1}{2}(EI_y)\int_0^L (u'')^2 dx + \frac{1}{2}(GJ)\int_0^L (\beta')^2 dx + \frac{1}{2}(EI_w)\int_0^L (\beta'')^2 dx - M_{oz}\int_0^L (u')(\beta') dx \tag{7.72}$$

For illustration consider the case of a member with simply supported end conditions. Using Rayleigh–Ritz approach with the displacement function assumed as

$$u = A\sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \beta = B\sin\left(\frac{\pi x}{L}\right)$$

Noting that

$$\int_0^L \sin^2(\pi x/L)\, dx = \int_0^L \cos^2(\pi x/L)\, dx = (L/2)$$

The application of stationary potential principle gives

$$\frac{\partial \Pi}{\partial A} = \frac{1}{2}(EI_y)\left(\frac{\pi^4}{L^4}\right)(2A)\left(\frac{L}{2}\right) - M_{oz}(B)\left(\frac{\pi^2}{L^2}\right)\left(\frac{L}{2}\right) = 0$$

$$\frac{\partial \Pi}{\partial B} = \frac{1}{2}(GJ)\left(\frac{\pi^2}{L^2}\right)(2B)\left(\frac{L}{2}\right) + \frac{1}{2}(EI_w)\left(\frac{\pi^4}{L^4}\right)(2B)\left(\frac{L}{2}\right) - M_{oz}(A)\left(\frac{\pi^2}{L^2}\right)\left(\frac{L}{2}\right) = 0$$

For non-trivial ($A \neq 0$, $B \neq 0$) solution, the determinant of coefficients of A and B must vanish i. e.,

$$\begin{vmatrix} EI_y\left(\frac{\pi^2}{L^2}\right) & -M_{\text{oz}} \\ -M_{\text{oz}} & GJ + EI_{\text{w}}\left(\frac{\pi^2}{L^2}\right) \end{vmatrix} = 0$$

The expansion results in the following characteristic equation

$$(EI_{\text{w}})(EI_y)\left(\frac{\pi^4}{L^4}\right) + (EI_y)(GJ)\left(\frac{\pi^2}{L^2}\right) - (M_{\text{oz}})^2 = 0$$

Therefore,

$$M_{\text{oz,cr}} = \left(\frac{\pi}{L}\right)\sqrt{(EI_{\text{w}})(EI_y)\left(\frac{\pi^2}{L^2}\right) + (GJ)(EI_y)} \tag{7.73}$$

Equation (7.73) is identical to (7.66) obtained earlier by equilibrium approach. The first term under radical represents the contribution of flexural rigidity in the lateral plane and *warping torsional rigidity*. The second term represents the combined effect of lateral flexural rigidity and pure torsion. For a rectangular cross-section I_{w} is negligibly small, therefore

$$M_{\text{oz,cr}} = \left(\frac{\pi}{L}\right)\sqrt{(GJ)(EI_y)} \tag{7.74}$$

The critical moment obtained using either (7.73) or (7.74) corresponds to a member whose end supports are by definition, simply supported for both lateral bending and twisting. For the cases where supports are other than simple, the buckling load is greater, and critical moment can be expressed as

$$M_{\text{oz,cr}} = \left(\frac{\pi}{K_y L}\right)\sqrt{\left(\frac{\pi}{K_x L}\right)^2 (EI_{\text{w}})(EI_y) + (GJ)(EI_y)} \tag{7.75}$$

where K_x and K_y are *effective length factors* for twisting and lateral bending, respectively. For the members supported at both the ends since rotation β about longitudinal axis or warping is restrained ($u = u' = \beta = \beta'' = 0$) the effective length factor K_x for twisting mode or warping is 0.5. For cantilever K_x is unity. The effective length factor K_y for lateral bending of members with simple supports ($u = u' = \beta = \beta'' = 0$) at both ends and cantilever beams is taken as unity.

It is to be noted that the solution obtained above is valid for the buckling of member subjected to uniform moment. However, in practice, in most of the cases the members carry transverse loading producing varying moments. Typical loadings on the beams are the distributed loads, the concentrated loads, the unequal end moments, etc. The elastic lateral buckling strength of beams subjected to transverse loads can be defined as

Table 7.2. Lateral buckling coefficients for various loadings

Case	Type of beam and loading	Boundary conditions	K_y, K_x	C
I	Simply supported: Equal end moments (single curvature) $M_{x,\max} = M$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.00
II	Simply supported: Uniformly distributed load, w Maximum moment, $M_{x,\max} = wL^2/8$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.13 0.97
III	Simply supported: Concentrated load at the centre, W $M_{x,\max} = WL/4$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.35 1.07
IV	Simply supported: Moment at one end $M_{x,\max} = M$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.75
V	Simply supported: Equal end moments (double curvature) $M_{x,\max} = \pm M$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	2.56
VI	Clamped at both ends: Uniformly distributed load, w $M_{x,\max} = wL^2/12$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.30 0.86
VII	Clamped at both ends: Concentrated load at the mid-point, W $M_{x,\max} = WL/8$	$u = u'' = \beta = \beta'' = 0$ $u = u' = \beta = \beta' = 0$	$K_y = 1.00$ $K_x = 0.50$	1.70 1.04
VIII	Cantilever beam: Concentrated load W at the free end, $M_{x,\max} = WL$	$u = u' = \beta = \beta' = 0$	$K_y = K_x = 1.00$	1.30
IX	Cantilever beam: Uniformly distributed load, w $M_{x,\max} = wL^2/2$	$u = u' = \beta = \beta' = 0$	$K_y = K_x = 1.00$	2.05

$$M_{x,\max,\mathrm{cr}} = C\left(\frac{\pi}{K_y L}\right)\sqrt{\left(\frac{\pi}{K_x L}\right)^2 (EI_y)(EI_\mathrm{w}) + (EI_y GJ)} \tag{7.76}$$

where C is a modification factor, which accounts for different conditions of loading. The value of C for a number of loading conditions is given in Table 7.2. The corresponding critical stress can be obtained by dividing the critical maximum moment by the section modulus of cross-section about the stronger axis, Z.

The individual elements of the cross-section of plate girders may be so flexible that the members can not retain the cross-sectional form up to the calculated lateral buckling loads. In such cases the lateral buckling strength can be more reasonably predicted by assuming that the compression flanges act as column. The contribution of web portion is ignored. In such a case the critical value of end moment for lateral buckling will be

$$M_{\mathrm{oz,cr}} = P_{\mathrm{cr}} d = \frac{\pi^2 E(I_y/2)}{L^2} d \tag{7.77}$$

7.3.2 Torsional Buckling due to Flexure and Axial Force

For a beam-column with doubly symmetric cross-section subjected to an axial thrust P along with equal end-bending moments M_{oz} applied about the stronger axis, the differential equation of equilibrium may be written as

$$EI_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + Pv - M_{\mathrm{oz}} = 0 \tag{7.78a}$$

$$EI_y \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + Pu - M_{\mathrm{oz}}\beta = 0 \tag{7.78b}$$

$$EI_{\mathrm{w}} \frac{\mathrm{d}^3 \beta}{\mathrm{d}x} - \left(GJ - Pr_0^2\right) \frac{\mathrm{d}\beta}{\mathrm{d}x} - M_{\mathrm{oz}} \frac{\mathrm{d}u}{\mathrm{d}x} = 0 \tag{7.78c}$$

where r_0 is the polar radius of gyration given by: $r_0^2 = (I_y + I_z)/A$. Equation (7.78a) represents in-plane beam-column bending behaviour and (7.78b) and (7.78c) govern the lateral-torsional behaviour of the member. As in the previous cases, the buckled configuration of a hinged-end or simply supported beam-column can be assumed to be represented by

$$u = A\sin(\pi x/L) \quad \text{and} \quad \beta = B\sin(\pi x/L)$$

Substitution in (7.78b) and (7.78c) yields

$$\left[-\left(\frac{\pi^2 EI_y}{L^2} - P\right)A - (M_{\mathrm{oz}})B\right]\left(\frac{\pi^2}{L^2}\right)\sin\frac{\pi x}{L} = 0$$

$$\left[(-M_{\mathrm{oz}})A - \left(\frac{\pi^2 EI_{\mathrm{w}}}{L^2} + GJ - Pr_0^2\right)B\right]\left(\frac{\pi}{L}\right)\sin\frac{\pi x}{L} = 0$$

For a non-trivial solution the determinant of coefficients of A and B must vanish i. e.

$$\left(P_{\mathrm{ey}} - P\right)\left(r_0^2 P_{\mathrm{ex}} - Pr_0^2\right) = (M_{\mathrm{oz}})^2 \tag{7.79}$$

where

$$P_{\mathrm{ey}} = (\pi^2 EI_y/L^2) = \text{Euler's buckling load in the weaker direction,}$$

and

$$P_{\mathrm{ex}} = \frac{1}{r_0^2}\left(\frac{\pi^2 EI_{\mathrm{w}}}{L^2} + GJ\right) = \text{torsional buckling load under axial compression.}$$

Equation (7.79) can be rewritten as

$$M_{oz} = r_0\sqrt{(P_{ey} - P)(P_{ex} - P)} \tag{7.80}$$

Following special cases may arise.

(i) M_{oz} is equal to zero i. e. member is subjected to an axial load only

$$P_{cr,1} = \frac{\pi^2 EI_y}{L^2} \quad \text{and} \quad P_{cr,2} = \frac{1}{r_0^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right) \tag{7.81}$$

The lower value will govern the buckling.

(ii) P is equal to zero i. e. member is subjected to pure end moments

$$\begin{aligned} M_{oz,cr} &= r_o\sqrt{P_{ey}P_{ex}} \\ &= r_o\sqrt{EI_y\left(\frac{\pi^2}{L^2}\right)\frac{1}{r_o^2}\left(GJ + EI_w\frac{\pi^2}{L^2}\right)} \\ &= \left(\frac{\pi}{L}\right)\sqrt{(EI_y)(GJ) + \left(\frac{\pi^2}{L^2}\right)(EI_y)(EI_w)} \end{aligned} \tag{7.82}$$

The quantity under the radical in (7.80) must be positive for M_{oz} to have any realistic value i. e. P must be either greater than both P_{ey} and P_{ex} or less than both of them. That is *lateral-torsional buckling* load must be smaller than the individual buckling loads P_{ey} and P_{ex}. Equation (7.79) can also be written in non-dimensional form as:

$$\frac{M_{oz}^2}{r_o^2 P_{ey} P_{ex}} = \left(1 - \frac{P}{P_{ey}}\right)\left(1 - \frac{P}{P_{ey}}\frac{P_{ey}}{P_{ex}}\right) \tag{7.83}$$

7.4 Lateral Buckling of Beams with Transverse Loads

The forgoing procedures are quite general and can be conveniently applied to the lateral stability analysis of beams with transverse loads.

7.4.1 Lateral Buckling of a Cantilever Beam

Consider a cantilever beam of span length L subjected to a concentrated load at the centroid of the end cross-section as shown in Fig. 7.8. The critical load W_{cr}, at which the beam will buckle by lateral buckling (i. e. by warping), can be determined easily.

As explained earlier at a certain value of the load the planar mode of bending becomes unstable and the member buckles out of its plane accompanied by warping as shown in the figure. Consider a section at a distance x from the fixed end. The bending moments and torque on the deformed cross-section are given by (7.59):

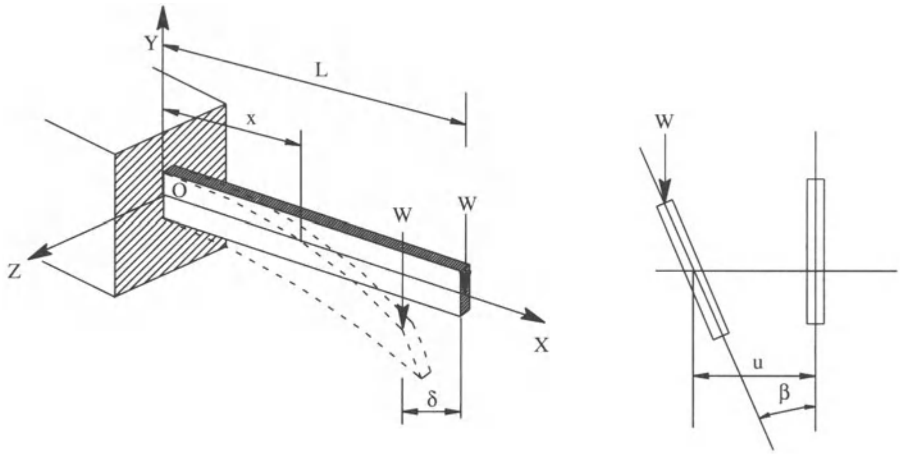


Fig. 7.8. Buckling of a cantilever beam loaded at its end

$$M_{y'} = -W(L-x)\beta \tag{7.84a}$$

$$M_{z'} = -W(L-x) \tag{7.84b}$$

and

$$M_{x'} = W(\delta - u) - W(L-x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \tag{7.84c}$$

where δ is lateral displacement at the free end of the cantilever. The equilibrium equations for lateral bending and warping are given by (7.60) and (7.61), respectively:

$$EI_y\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -W(L-x)\beta \tag{7.85}$$

and

$$EI_\mathrm{w}\frac{\mathrm{d}^3\beta}{\mathrm{d}x^3} - GJ\frac{\mathrm{d}\beta}{\mathrm{d}x} = W(\delta-u) - W(L-x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \tag{7.86}$$

Eliminating u from these equations by differentiating (7.86) with respect to x and substituting for $\mathrm{d}^2u/\mathrm{d}x^2$ from (7.85):

$$EI_\mathrm{w}\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} - GJ\frac{\mathrm{d}^2\beta}{\mathrm{d}x} = -W\frac{\mathrm{d}u}{\mathrm{d}x} - W\left[(L-x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{\mathrm{d}u}{\mathrm{d}x}\right] = \frac{W^2(L-x)^2}{EI_y}\beta$$

$$EI_\mathrm{w}\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} - GJ\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - \frac{W^2(L-x)^2}{EI_y}\beta = 0 \tag{7.87}$$

Introducing a new variable $s = L - x$, (7.87) takes the form:

$$\frac{\mathrm{d}^4\beta}{\mathrm{d}s^4} - \left(\frac{GI}{EI_\mathrm{w}}\right)\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} - \frac{W^2s^2\beta}{(EI_\mathrm{w})(EI_y)} = 0 \tag{7.88}$$

Table 7.3. Lateral buckling coefficients for various loadings

$L^2(GJ/EI_w)$	0.1	1	2	3	4	6	8	10	12	14	16	24	32	40
γ_1	44.3	15.7	12.2	10.7	9.76	8.68	8.03	7.58	7.20	6.96	6.73	6.19	5.87	5.64

Since the moment term Ws is not constant, simple exact solutions do not exist and approximate solution may be used in the form of infinite series. The critical load W_{cr} can be represented in the form

$$W_{cr} = \gamma_1 \left(\sqrt{GJEI_y}\right)/L^2 \tag{7.89}$$

where γ_1 is a dimensionless factor which depends upon the ratio $L^2(GJ/EI_w)$. As the ratio $L^2(GJ/EI_w)$ increases the factor γ_1 approaches the limiting value of 4.013 for the case when EI_w is vanishingly small. This case corresponds to a beam of narrow rectangular section. For large values of $L^2(GJ/EI_w)$, γ_1 can be approximated from

$$\gamma_1 = 4.013/\left[1 - \sqrt{(EI_w/L^2GJ)}\right]^2 \tag{7.90}$$

The values as computed by Timoshenko are given in Table 7.3.

Once the value of W_{cr} is determined, the corresponding critical stress is given by:

$$\sigma_{cr} = W_{cr}L/Z_z \tag{7.91}$$

It should be noted that this stress must be below the proportional limit of the material for (7.89) to be valid. Consider the case of a beam with narrow rectangular cross-section of width b and depth d where I_w is negligible. In this case (7.87) reduces to:

$$GJ\frac{d^2\beta}{dx} + \frac{W^2(L-x)^2}{EI_y}\beta = 0$$

or

$$\frac{d^2\beta}{dx^2} + \frac{W^2(L-x)^2\beta}{GJEI_y} = 0 \tag{7.92}$$

Introducing $L - x = s$, (7.92) transforms to:

$$\frac{d^2\beta}{ds^2} + \frac{W^2s^2\beta}{(GJ)(EI_y)} = 0$$

$$\frac{d^2\beta}{ds^2} + \left(\alpha_1^2 s^2\right)\beta = 0 \tag{7.93}$$

where $\alpha_1 = W/\sqrt{(GJ)\,(EI_y)}$. Equation (7.93) can be reduced to an equivalent form of *Bessel's differential equation* as explained in Appendix Appendix C. The general solution in terms of Bessel functions is:

$$\beta = \sqrt{s}\left[AJ_{1/4}\left(\frac{\alpha_1 s^2}{2}\right) + BJ_{-1/4}\left(\frac{\alpha_1 s^2}{2}\right)\right] \tag{7.94}$$

where $J_{1/4}$ and $J_{-1/4}$ represent *Bessel's functions* of the first kind of order 1/4 and −1/4, respectively. The arbitrary constants A and B are determined from the end conditions of the beam i. e.,

(i) at the built-in end, angle of twist is zero, i. e., at $x = 0$ or $s = L$;

$$\beta = 0 \tag{a}$$

(ii) at the free end, torque $M_{x'}$ is zero, i. e., at $x = L$ or $s = 0$; $\mathrm{d}\beta/\mathrm{d}x = -\mathrm{d}\beta/\mathrm{d}s = 0$

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \alpha_1 s^{3/2}\left[AJ_{-3/4}\left(\frac{\alpha_1 s^2}{2}\right) + BJ_{3/4}\left(\frac{\alpha_1 s^2}{2}\right)\right] \tag{b}$$

since at $s = 0$, $J_{-3/4}$ is infinite, the arbitrary constant $A = 0$ and from condition (a),

$$J_{-1/4}\left(\frac{\alpha_1 L^2}{2}\right) = 0 \tag{c}$$

From the table of zero *Bessel functions*, the lowest root of (c) is $(\alpha_1 L^2/2) = 2.006$, hence

$$W_{\mathrm{cr}} = \frac{4.012}{L^2}\sqrt{(GJ)(EI_y)} = \frac{4.012 EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.95}$$

It should be noted that (7.95) is valid within elastic region; beyond elastic region buckling occurs at a load which is smaller than that given by (7.95). The maximum bending stress is given by:

$$\sigma_{\mathrm{cr}} = \frac{M_{\max}}{Z_z} = \frac{W_{\mathrm{cr}}L}{(2I_z/d)} = \frac{4.012\sqrt{(GJ)(EI_y)}}{L(2I_z/d)} \tag{7.96}$$

For a narrow rectangular section of size $b \times d$ deep

$$I_y = b^3d/12; \quad I_z = bd^3/12 \quad \text{and} \quad J = b^3d/3$$

Taking $G = 0.4E$ (for steel), (7.96) reduces to:

$$\sigma_{\mathrm{cr}} = 2.537(b^2/dL)E \tag{7.97}$$

Therefore, for a material such as steel, buckling in the elastic region can occur only if the quantity b^2/dL is very small. Usually the lateral buckling is considered in the case of very narrow rectangular cross-section where b/d is a small quantity.

In the case of a uniform load of intensity q distributed along the length of the cantilever acting on its centroidal axis, critical value of q as obtained by Prandtl is given by:

$$q_{\mathrm{cr}}L = \frac{12.85 EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.98}$$

It should be noted that the critical value of the total uniformly distributed load is approximately 3.2 times the critical value of concentrated load acting at the free end.

7.4.2 Lateral Buckling of a Simply Supported Beam

Consider a simply supported beam subjected to a concentrated load W applied at the centroid of the mid-span cross-section as shown in Fig. 7.9. It is assumed that during deformation the ends of the beam can rotate freely with respect to the principal axes of inertia parallel to the Y- and Z-axes while rotation with respect to longitudinal X-axis is restrained. Thus lateral buckling is accompanied by twisting or warping of the beam.

With origin of axes at the mid-span cross-section, consider a section on the portion of the beam to the right of cross-section, at distance x from the origin. The external forces acting on this portion reduce to a single force $W/2$ due to reaction at the support. Comparing this case with that of cantilever subjected to a concentrated load at a free end, the equivalent moment at the section under consideration would be $(W/2)[(L/2) - x]$ instead of $W(L - x)$ for the cantilever beam. The governing equation (7.87) becomes

$$EI_{\mathrm{w}}\frac{\mathrm{d}^4\beta}{\mathrm{d}x^4} - GJ\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} - \frac{W^2\left[(L/2) - x\right]^2}{4EI_y}\beta = 0 \qquad (7.99)$$

As in the case of cantilever introducing a new variable $s = [(L/2) - x]$ (7.99) takes the form:

$$\frac{\mathrm{d}^4\beta}{\mathrm{d}s^4} - \left(\frac{GI}{EI_{\mathrm{w}}}\right)\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2} - \frac{W^2 s^2\beta}{4(EI_{\mathrm{w}})(EI_y)} = 0 \qquad (7.100)$$

Timoshenko had integrated this equation by the method of infinite series and using boundary conditions obtained the critical W_{cr} in the form

$$W_{\mathrm{cr}} = \gamma_2\left(\sqrt{GJEI_y}\right)/L^2 \qquad (7.101)$$

Typical values of the dimensionless buckling load factor (γ_2) are given in Table 7.4.

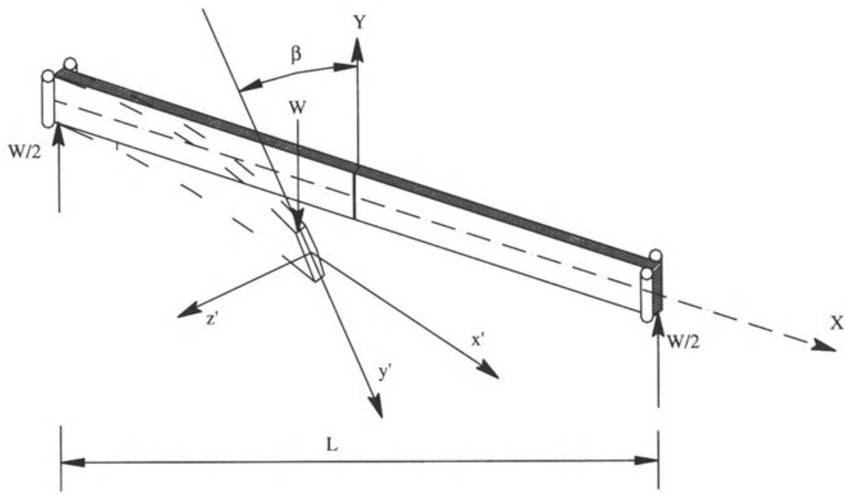


Fig. 7.9. Buckling of a beam with simple supports loaded at mid-span

Table 7.4. Values of the factor γ_2 for a simply supported I-beam with concentrated load at mid-span

$L^2(GJ/EI_w)$	0.4	4	8	16	24	32	48	64	80	96	160	240	320	400
γ_2	86.4	31.9	25.6	21.8	20.3	19.6	18.8	18.3	18.1	17.9	17.5	17.4	17.2	17.2

For a narrow rectangular cross-section, omitting the term containing warping rigidity EI_w in (7.99)

$$\frac{d^2\beta}{dx^2} + \frac{W^2\left[(L/2) - x\right]^2}{4(GJEI_y)}\beta = 0 \tag{7.102}$$

On substituting $L/2 - x = s$, (7.102) reduces to:

$$\frac{d^2\beta}{dx^2} + \frac{W^2 s^2 \beta}{4(GJEI_y)} = 0 \quad \text{or} \quad \frac{d^2\beta}{ds^2} + \left(\alpha_2^2 s^2\right)\beta = 0 \tag{7.103}$$

where $\alpha_2 = (W/2)/\sqrt{(GJ)(EI_y)}$.

The general solution in terms of Bessel functions is

$$\beta = \sqrt{s}\left[AJ_{1/4}\left(\frac{\alpha_2 s^2}{2}\right) + BJ_{-1/4}\left(\frac{\alpha_2 s^2}{2}\right)\right] \tag{7.104}$$

where $J_{1/4}$ and $J_{-1/4}$ represent Bessel functions of first kind of order $1/4$ and $-1/4$, respectively. For a beam with simple supports, the boundary conditions are:

(i) At the support angle of twist is zero i. e. at $x = L/2$ or $s = 0$, $\beta = 0$ giving B = 0
(ii) At the mid-span, the torque is zero i. e. at $x = 0$ or $s = L/2$, $d\beta/dx = -d\beta/ds = 0$ where

$$\frac{d\beta}{ds} = A\alpha_2 s^{3/2} J_{-3/4}\left(\frac{\alpha_2 s^2}{2}\right)$$

Therefore, to satisfy second condition:

$$J_{-3/4}\left(\frac{\alpha_2 s^2}{2}\right) = 0 \quad \text{at} \quad s = \frac{L}{2}$$

From the table of zeros of Bessel function of order $-3/4$:

$$\alpha_2 L^2/8 = 1.0585.$$

$$\left[(W_{cr}/2)\,/\sqrt{(GJ)(EI_y)}\right]\left(L^2/8\right) = 1.0585$$

Therefore,

$$W_{cr} = \frac{16.936\sqrt{(GI)(EI_y)}}{L^2} = \frac{16.936 EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.105}$$

If the simply supported beam carries a load uniformly distributed along the centroidal axis, the critical value of the load as obtained by Prandtl is given by:

$$q_{cr}L = \frac{28.3EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.106}$$

Thus the total uniformly distributed load is approximately 1.67 times the critical value of concentrated load acting at the mid-span.

The values of the critical loads can also be obtained from (7.76) by using the lateral buckling coefficients given in Table 7.2. For illustration again consider the case of cantilever beam of narrow rectangular cross-section subjected to a concentrated load at the free end (Case VIII in Table 7.2).

$$M_{x,\max,cr} = W_{cr}L = 1.3\left(\frac{\pi}{L}\right)\sqrt{(EI_y)(GJ)}$$

$$W_{cr} = \frac{1.3\pi}{L^2}\sqrt{(EI_y)(GJ)} = \frac{4.08EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.107}$$

The critical value of the load is fairly close to that given by (7.95). Similarly for a simply supported beam with a concentrated load at the centre

$$M_{x,\max,cr} = \frac{W_{cr}L}{4} = \frac{1.35\pi}{L}\sqrt{(EI_y)(GJ)}$$

$$W_{cr} = \frac{4 \times 1.35\pi}{L^2}(EI_y)\sqrt{\left(\frac{GJ}{EI_y}\right)}$$

$$= \frac{16.96EI_y}{L^2}\sqrt{\left(\frac{GJ}{EI_y}\right)} \tag{7.108}$$

The critical load given by (7.108) is in close agreement with that given by the (7.105).

7.5 Problems

Problem 7.1. Determine the critical load for a column of I-cross-section, if the column ends are simply supported such that the ends can rotate but cannot deflect about Y- and Z-axes. Further the ends are free to warp but cannot rotate about longitudinal X-axis.

Problem 7.2. Estimate the buckling load for the column of Problem 7.1 by using Rayleigh–Ritz technique with shape or displacement functions u, v and β as: $u = Af(x)$; $v = Bf(x)$, and $\beta = Cf(x)$, where

$$f(x) = \left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)\right]$$

Problem 7.3. A beam with clamped end conditions, such that the ends cannot rotate or deflect about Y- and Z-axes. Further the warping is constrained and the ends cannot rotate about X-axis. Using Rayleigh–Ritz method with displacement functions as

$$u = A\left(1 - \cos\frac{2\pi x}{L}\right) \quad \text{and} \quad \beta = B\left(1 - \cos\frac{2\pi x}{L}\right)$$

show that the characteristic equation for the lateral buckling due to uniformly applied moment M_{oz} is given by:

$$4EI_y\left(\frac{\pi^2}{L^2}\right)\left(GJ + 4EI_{\text{w}}\frac{\pi^2}{L^2}\right) - M_{\text{oz}}^2 = 0$$

Problem 7.4. Obtain the critical load for the beam of Problem 7.3 by considering the displacement functions as:

$$u = A\left(1 - \cos\frac{2\pi x}{L}\right); \quad v = B\left(1 - \cos\frac{2\pi x}{L}\right) \quad \text{and} \quad \beta = C\left(1 - \cos\frac{2\pi x}{L}\right)$$

Show that a better approximation to the exact solution can be obtained by considering two term representation for u, v and β as:

$$u = A_1\left(1 - \cos\frac{2\pi x}{L}\right) + A_2\left(1 - \cos\frac{4\pi x}{L}\right)$$
$$v = B_1\left(1 - \cos\frac{2\pi x}{L}\right) + B_2\left(1 - \cos\frac{4\pi x}{L}\right)$$
$$\beta = C_1\left(1 - \cos\frac{2\pi x}{L}\right) + C_2\left(1 - \cos\frac{4\pi x}{L}\right)$$

Problem 7.5. Derive relationship between the lateral buckling stress and aspect ratio d/b for a simply supported beam of rectangular cross-section of size $(b \times d)$.

What would be the aspect ratio when the critical buckling stress is limited to $2\sigma_y/3$? The effective span/depth ratio L/d can be assumed to be 25. The properties of the material of the beam are: $E = 2 \times 10^5$ MPa; $G = (3/8)E$ and $\sigma_y = 250$ MPa.

Problem 7.6. If the geometric properties of a built-up I-shaped cross-section are approximated by

$$I_z = \left[2bt_{\text{f}}\left(\frac{d}{2}\right)^2 + \frac{t_{\text{w}}d^3}{12}\right]; \quad I_y = 2\left(\frac{t_{\text{f}}b^3}{12}\right); \quad I_{\text{w}} = \left(\frac{1}{4}d^2I_y\right)$$

and

$$J = 2J_1 + J_2 = \frac{2}{3}bt_{\text{f}}^3 + \frac{1}{3}dt_{\text{w}}^3 \approx \frac{2}{3}bt_{\text{f}}^3$$

where b, d, t_{f} and t_{w} are flange width, depth of section, flange thickness and web thickness, respectively. Show that the critical stress corresponding to lateral buckling moment can be expressed as

$$\sigma_{\mathrm{cr}} = \sqrt{\left[\frac{k_1}{(Ld/bt_{\mathrm{f}})}\right]^2 + \left[\frac{k_2}{(L/r)}\right]^2}$$

where k_1 and k_2 are constants involving E and G; and r is radius of gyration about the Y-axis of compression flange of beam plus one-sixth of the web, and is given by:

$$r^2 = b^2/[12(1 + A_{\mathrm{w}}/A_{\mathrm{f}})^2]$$

where A_{f} is the area of the flange, $A_{\mathrm{w}} = \frac{1}{6}$ (area of web). Take $G = (3/8)E$.

Problem 7.7. Show that the energy due to axial constraint stress, σ developed when warping is restrained is given by:

$$V_{\mathrm{e}} = -\frac{1}{2}\int_0^L \int_A \sigma \left[\left\{\frac{\mathrm{d}}{\mathrm{d}x}[u + (y_0 - y)\beta]\right\}^2 + \left\{\frac{\mathrm{d}}{\mathrm{d}x}[v - (x_0 - x)\beta]\right\}^2\right] \mathrm{d}A\,\mathrm{d}x$$

Problem 7.8. A simply supported I-beam carries a uniformly distributed load of intensity w/unit length at a lateral eccentricity of e. Show that the angle of twist is given by:

$$\beta = \frac{we}{(GJ)}\left[\frac{\cosh\alpha L - 1}{\alpha^2 \sinh\alpha L}\sinh\alpha x - \frac{1}{\alpha^2}\cosh\alpha x + \frac{1}{\alpha^2} - \frac{Lx}{2} + \frac{x^2}{2}\right]$$

where $\alpha = (GJ)/(EI_{\mathrm{w}})$

Problem 7.9. A uniform straight member of length L with simple supports is subjected to an axial load P and end torque M_x. The cross-section of the member has same moment of inertia for all central axes. Show that critical combination of P and M_x is given by:

$$\frac{M_x^2}{4(EI)^2} \pm \frac{P}{EI} = \frac{\pi^2}{L^2}$$

where $-ve$ value of P (i.e. tension) indicates the load required to prevent buckling for $M_x > M_{x,\mathrm{cr}}$.

Problem 7.10. A thin circular tube of length L, external diameter D and thickness t is subjected to a torque, show that the critical stress is given by:

$$\sigma_{\mathrm{s,cr}} = \frac{\pi E D}{L(1 - \nu)}\left[1 - \left(\frac{t}{D}\right) + \frac{1}{3}\left(\frac{t}{D}\right)^2\right]$$

for helical buckling.

Problem 7.11. A uniform straight column of length L having channel cross-section has clamped end conditions at the base and simple support conditions at its top. Using Rayleigh–Ritz method estimate the critical load at which the buckling will occur. Assume displacement functions of the form: $u = Af(x)$, $v = Bf(x)$ and $\beta = Cf(x)$ where $f(x) = x^2(L - x)$.

Show that a better estimate can be obtained by selecting displacement functions of the form

$$f(x) = x^2(L - x)(3L - 2x) = (2x^4 - 5Lx^3 + 3L^2x^2)$$

8

Elastic Buckling of Thin Flat Plates

8.1 Introduction

In the preceding chapters, elastic buckling of structures composed of one-dimensional members has been discussed wherein deflections and bending moments are assumed to be the functions of a single independent variable. On the other hand, buckling of *plates* involves bending in two planes, and thus deflections and bending moments at a point become function of two independent variables. Consequently, the structural behaviour of plates is described by partial differential equations, whereas ordinary differential equations were adequate to describe the behaviour of columns. Further, the number of boundary conditions was four in the columns whereas in plates there are two boundary condition on each of its edges. Another basic difference between a column and a plate lies in their buckling behaviour. Once a column has buckled, it cannot resist any additional axial load i.e. critical load of a column is also its failure load. On the other hand, the plates which are invariably supported at edges or are interconnected to other plate elements continue to resist additional axial loads even after the loads reach their buckling values. This additional load is sometimes as high as 10–15 times the *initial elastic buckling load*. Thus, for a plate element the *post-buckling load* is much higher than the initial buckling load. This fact is largely exploited in the *minimum weight design* of the structures. The components of *open section columns* with wide flanges behave more like *plate elements*. The plates making up a column may undergo a form of *local failure,* thus necessitating the consideration of *instability of plate element*. In order to enhance buckling load of a plate sometimes *longitudinal* and *transverse stiffeners* are provided. The inherent discontinuities in these *stiffened structures* make their analysis complex.

In this chapter only *thin plates* have been considered for analysis. The plates are termed thin if their thickness is small as compared to the in-plane dimensions, and transverse shear deformations are negligible compared to bending deflections. In the following section idealization has been made to describe the two-dimensional plate behaviour by linear differential equations with constant coefficients.

M. L. Gambhir, *Stability Analysis and Design of Structures*

8.2 Governing Differential Equations of Bending

To determine the critical in-plane loading of a flat plate by the concept of equilibrium, it is essential to formulate equations of equilibrium of a plate element in a slightly displaced configuration. This plate element is acted upon by two sets of forces: *in-plane* or *membrane forces* balancing the externally applied loads and *shears* resulting from transverse bending of the plate. The derivation of the governing differential equation for a thin plate undergoing lateral or transverse displacements is based on the following assumptions:

1. The least lateral dimension of the plate is at least ten times the thickness i.e. it is a thin plate so that the effect of shear strains γ_{xz} and γ_{yz} are negligible and vertical plane of plate, which is perpendicular to the middle surface before bending remains perpendicular after bending.
2. The normal stress, σ_z and the corresponding strain ε_z are negligible for deflections less than the order of one hundredth of span length. The strains of middle surface are assumed to be negligibly small. Consequently the transverse deflection w at any point (x, y, z) is equal to the transverse deflection at the corresponding point (x, y, o) on the middle surface.
3. The material of the plate is homogeneous, isotropic and elastic.

The deflected configurations of a simply supported column and a rectangular plate are shown in Fig. 8.1. The stress resultants acting upon typical differential element of the plate are indicated on Fig. 8.2. The governing differential equation is obtained from the consideration of static equilibrium, namely.

$$\text{(i)}\ \sum F_x = 0\,; \quad \text{(ii)}\ \sum F_y = 0\,; \quad \text{(iii)}\ \sum F_z = 0\,;$$
$$\text{(iv)}\ \sum M^x = 0 \quad \text{and} \quad \sum M^y = 0$$

Equilibrium of in-plane forces in X-direction gives

$$\sum F_x = \left(p_x + \frac{\partial p_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y - p_x\,\mathrm{d}y + \left(p_{yx} + \frac{\partial p_{yx}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x - p_{yx}\,\mathrm{d}x = 0$$

$$\text{or} \quad \frac{\partial p_x}{\partial x} + \frac{\partial p_{yx}}{\partial y} = 0 \tag{8.1}$$

Similarly, the condition of equilibrium of in-plane forces in Y-direction $\sum F_y = 0$, results in

$$\frac{\partial p_y}{\partial y} + \frac{\partial p_{xy}}{\partial x} = 0 \tag{8.2}$$

Equilibrium of moments of in-plane forces about Z-axis passing through o', $\sum M^z = 0$ yields:

$$(p_{yx}\,\mathrm{d}x)\,\mathrm{d}y - (p_{xy}\,\mathrm{d}y)\,\mathrm{d}x = 0 \quad \text{i.e.} \quad p_{yx} = p_{xy} \tag{8.3}$$

(The second order quantities due to direct forces have been ignored).

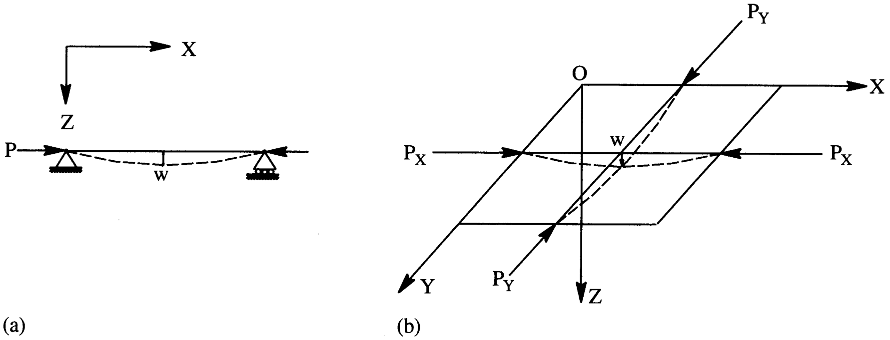


Fig. 8.1a,b. Deflected configurations of a column and plate. **a** Deflection of a column (one dimensional case), **b** Deflection of simply supported plate (two dimensional case)

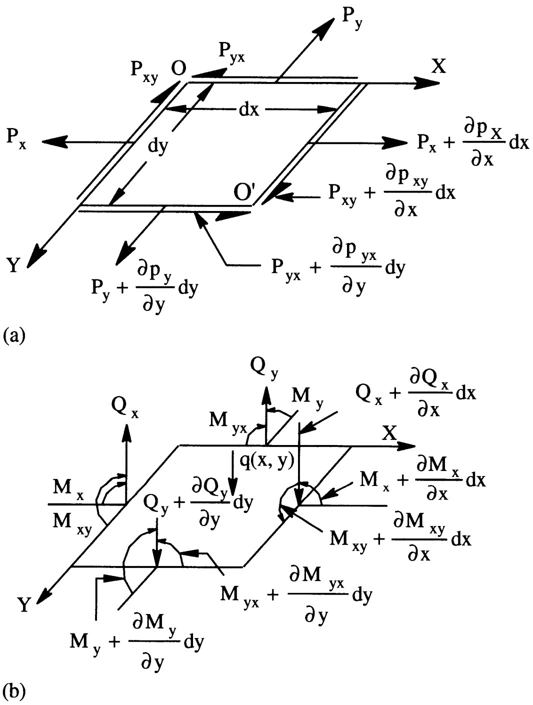


Fig. 8.2a,b. Element of the plate subjected to internal forces. **a** In-plane forces, **b** Shear forces and moments

Due to slight curvature in the elements due to transverse deflection, the in-plane forces p_x, p_y, p_{xy} and p_{yx} will have components along Z-axis. The slopes at the edges $x = 0$ and $x = dx$ are:

$$\left(\frac{\partial w}{\partial x}\right) \quad \text{and} \quad \frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) dx = \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx\right],$$

respectively. In the view of small deformation assumption:

$$\sin\left(\frac{\partial w}{\partial x}\right) \approx \frac{\partial w}{\partial x} \quad \text{and} \quad \cos\left(\frac{\partial w}{\partial x}\right) \approx 1$$

The resultant component of in-plane forces p_x and $[p_x + (\partial p_x/\partial x)\,\mathrm{d}x)]$ in the positive Z-direction is:

$$\left(p_x + \frac{\partial p_x}{\partial x}\,\mathrm{d}x\right)\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\right)\mathrm{d}y - p_x\left(\frac{\partial w}{\partial x}\right)\mathrm{d}y = \left(p_x\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\right)\mathrm{d}x\,\mathrm{d}y$$

(terms containing p_x only are retained).

Similarly the resultant component of forces p_y and $[p_y + (\partial p_y/\partial y)\,\mathrm{d}x]$ in the positive Z-direction is given by

$$\left(p_y\frac{\partial^2 w}{\partial y^2}\right)\mathrm{d}y\,\mathrm{d}x\,.$$

The resultant Z-component of in-plane forces p_{xy} and $[p_{xy} + (\partial p_{xy}/\partial x)\,\mathrm{d}x]$ is:

$$\left(p_{xy} + \frac{\partial p_{xy}}{\partial x}\,\mathrm{d}x\right)\left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\right)\mathrm{d}y - p_{xy}\left(\frac{\partial w}{\partial y}\right)\mathrm{d}y \approx \left(p_{xy}\frac{\partial^2 w}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y\right)$$

(retaining only p_{xy} terms). Similarly Z-component of p_{yx} and $[p_{yx} + (\partial p_{yx}/\partial y)\,\mathrm{d}y]$ is: $[p_{yx} + (\partial^2 w/\partial y \partial x)\,\mathrm{d}y\,\mathrm{d}x]$. Therefore, the resultant of all the in-plane forces in Z-direction is:

$$\left(p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)\mathrm{d}x\,\mathrm{d}y \quad (\text{since } p_{yx} = p_{xy}) \tag{8.4}$$

The component of shear forces along Z-direction is:

$$\left[\left(Q_x + \frac{\partial Q_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y - Q_x\,\mathrm{d}y\right] + \left[\left(Q_y + \frac{\partial Q_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x - Q_y\,\mathrm{d}x\right]$$
$$= \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \tag{8.5}$$

Equilibrium of forces along Z-direction i.e. $\sum F_z = 0$. From (8.4) and (8.5):

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \left(p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \tag{8.6}$$

Moment equilibrium of transverse forces

For equilibrium of moments about X-axis

$$M_y\,\mathrm{d}x - \left(M_y + \frac{\partial M_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x + \left(Q_y + \frac{\partial Q_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}y - Q_x\,\mathrm{d}y\left(\frac{\mathrm{d}y}{2}\right)$$
$$+ \left(Q_x + \frac{\partial Q_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\left(\frac{\mathrm{d}y}{2}\right) - M_{xy}\,\mathrm{d}y + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y = 0$$

Ignoring second order terms, the equation reduces to

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \tag{8.7}$$

For equilibrium of moments about Y-axis

$$- M_x \, \mathrm{d}y + \left(M_x + \frac{\partial M_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y - \left(Q_x + \frac{\partial Q_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}x + Q_y\,\mathrm{d}x\left(\frac{\mathrm{d}x}{2}\right)$$
$$- \left(Q_y + \frac{\partial Q_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x\left(\frac{\mathrm{d}x}{2}\right) - M_{yx}\,\mathrm{d}x + \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x = 0$$

Neglecting second order terms the equation reduces to:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \tag{8.8}$$

From (8.7) and (8.8)

$$\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial M_{xy}}{\partial x \partial y} \tag{8.9}$$

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} = \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} \quad \text{since } (M_{yx} = -M_{xy}) \tag{8.10}$$

Substituting $\partial Q_y/\partial y$ and $\partial Q_x/\partial x$ from (8.9) and (8.10) into (8.6)

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \left(p_x \frac{\partial^2 w}{\partial x^2} + p_y \frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \tag{8.11}$$

Equation (8.11) is the governing differential equation of *buckling of plate*. The moments M_x, M_y and M_{xy} can be expressed in terms of curvatures. Since a thin plate is essentially *two-dimensional*, the *constitutive laws* for an elastic plane-stress problem can be used. These are:

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\
\sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}
\end{aligned} \tag{8.12}$$

The strain-displacement relations for a linear problem are expressed as:

$$\varepsilon_x = \frac{\partial u}{\partial x}\,; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \text{and} \quad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{8.13}$$

where u and v are displacements along X- and Y-directions, respectively, at a distance z above the middle surface which remains unstrained during the transverse displacement, w, thus

$$u = -z\frac{\partial w}{\partial x} \quad \text{and} \quad v = -z\frac{\partial w}{\partial y} \tag{8.14}$$

Hence the strains ε_x, ε_y and γ_{xy} can be represented by

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \quad \text{and} \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{8.15}$$

Substituting the strains expressed in terms of w from (8.15) into (8.12)

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad \text{and}$$

$$\tau_{xy} = -\frac{Ez}{1+\nu} \cdot \frac{\partial^2 w}{\partial x \partial y} \tag{8.16}$$

The stress resultants M_x, M_y and M_{xy} are expressed as

$$M_x = \int_{-t/2}^{t/2} \sigma_x z \,\mathrm{d}z = -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{8.17a}$$

$$M_y = \int_{-t/2}^{t/2} \sigma_y z \,\mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{8.17b}$$

$$M_{yx} = -M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z \,\mathrm{d}z = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{8.17c}$$

$$D = \frac{Et^3}{[12(1-\nu^2)]} \tag{8.17d}$$

where D is the flexural rigidity per unit length of the plate. This is analogous to the bending stiffness EI of a beam. The rigidity of the plate is $1/(1-\nu^2)$ times that of a beam having the same width and depth as the plate. The plate is stiffer since each plate strip is restrained by the adjacent strips.

Substituting the values of M_x, M_y and M_{xy} from (8.17) into the general governing differential equation

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - \left(p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) = 0 \tag{8.18}$$

In terms of the operator, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, first term on the left hand side of (8.18) reduces to

$$D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = D\nabla^2\nabla^2 w = D\nabla^4 w$$

Thus (8.18) can be written as

$$D\nabla^4 w + p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x\partial y} = 0 \tag{8.19}$$

where p_x and p_y are compressive forces. It is interesting to note the similarity between (8.18) and fourth-order differential equation of the beam-column.

8.2.1 Boundary Conditions

The governing equation (8.18) or (8.19) is a fourth order partial differential equation in x and y, and thus for a unique solution it requires eight boundary conditions: four along X- and four along Y-edges. The commonly encountered boundary conditions for a typical boundary at $x = a$ are as follows:

(i) Simply supported or a hinged edge

This refers to an edge which is restrained against displacement but is free to rotate i. e. moment is zero i. e.

$$w(a, y) = 0 \quad \text{and}$$

$$M_x(a, y) = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0$$

Thus,

$$\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0 \tag{8.20}$$

Since, $\partial^2 w/\partial y^2 = 0$ for a supported edge, (8.20) can be written as:

$$\left[\nabla^2 w\right]_{x=a} = 0$$

(ii) Built-in or a clamped edge

This type of edge is restrained both against displacement and rotation i. e.

$$w(a, y) = 0 \quad \text{and} \quad \frac{\partial w(a, y)}{\partial x} = 0 \tag{8.21}$$

(iii) Free edge

This type of edge is characterized by zero moment and zero shear i. e.

$$M_x(a, y) = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{x=a} = 0 \quad \text{and}$$

$$Q_x(a, y) = D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a} = 0 \tag{8.22}$$

8.3 Energy Approach

8.3.1 Strain Energy of Plates

In a two dimensional plane-stress problem, the stress at a point can be expressed in terms of σ_x, σ_y and τ_{xy}, where σ_x and σ_y are normal stresses in the direction of the X- and Y-axes, respectively, and τ_{xy} is the shear stress in a section perpendicular to the plane of the plate cut parallel to the X- or Y-axis. The strain energy stored in a plate is given by:

$$-W_{\text{in}} = U = \frac{1}{2}\int_v \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right) \mathrm{d}v \tag{8.23}$$

Using strain-displacement and strain-curvature relations from (8.15) and (8.16), respectively, and carrying out the integration with respect to z over the total depth of the plate, (8.23) can be written as:

$$U = \frac{1}{2}\int_0^a\int_0^b D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 \right.$$
$$\left. - 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} \mathrm{d}x\,\mathrm{d}y$$

The term inside the square brackets is known as *Gaussian curvature* (G.C.):

$$\text{G.C.} = \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

If the displacement function $w(x, y)$ can be expressed in the form $w(x, y) = f(x).\phi(y)$ i. e. product of a function of x only and a function of y only with $w = 0$ at the boundaries, then integral of Gaussian curvature over entire plate surface equals zero. Under these conditions:

$$U = \frac{1}{2}D\iint_{\text{area}} \left(\nabla^2 w\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{8.24}$$

Thus, for the rectangular plates with simply supported or built-in edges the conditions; $w(x, y) = f(x).\,\phi(y)$ and $w = 0$ at boundaries hold and the strain energy can be determined from (8.24).

8.3.2 Potential Energy due to In-Plane Forces

To compute potential energy due to in-plane force component p_x, consider an elemental strip of width $\mathrm{d}y$ along X-direction. The load acting on this strip is $p_x\,\mathrm{d}y$ and potential energy of the strip due to this load is:

$$\mathrm{d}W_{\mathrm{ex}} = \mathrm{d}V_x = -\frac{1}{2}(p_x\,\mathrm{d}y)\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x$$

The potential energy of the entire plate can be obtained by integrating the elemental (strip) energy with respect to y. Thus

$$V_y = -\frac{1}{2}\int_0^a\int_0^b p_y \left(\frac{\partial w}{\partial y}\right)^2 \mathrm{d}y\,\mathrm{d}x$$

Similarly the potential energy due to p_y can be computed by considering an elemental strip of width $\mathrm{d}x$ in Y-direction. Thus

$$V_x = -\frac{1}{2}\int_0^b\int_0^a p_x \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

For computation of potential energy due to shearing forces p_{xy} and p_{yx} consider an element $\mathrm{d}x\,\mathrm{d}y$. This element is subjected to shear strain of $(\partial w/\partial y)(\partial w/\partial x)$ due to transverse displacement w.

Thus, the potential energy of the plate due to p_{xy} is

$$\begin{aligned}
V_{xy} &= -\frac{1}{2}\int_0^a\int_0^b \left[p_{xy}\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right) + p_{yx}\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right)\right] \mathrm{d}y\,\mathrm{d}x \\
&= -\frac{1}{2}\int_0^a\int_0^b 2p_{xy}\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right) \mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

Therefore, total work done or potential energy due to p_x, p_y and p_{xy} is

$$\begin{aligned}
W_{ex} &= V_e = V_x + V_y + V_{xy} \\
&= -\frac{1}{2}\int_0^a\int_0^b \left[p_x\left(\frac{\partial w}{\partial x}\right)^2 + p_y\left(\frac{\partial w}{\partial y}\right)^2 + 2p_{xy}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \mathrm{d}x\,\mathrm{d}y \quad (8.25)
\end{aligned}$$

The total potential of plate is given by: $\Pi = U + V_e$.

From stationary potential principle, for buckling

$$\delta\Pi = \delta(U + V_e) = 0 \quad (8.26)$$

8.4 Buckling Analysis of Rectangular Plates

The buckling analysis of thin rectangular plates can be accomplished by using either the governing differential equation or stationary potential principle. The most commonly used methods are:

1. Exact analysis in which a function representing deformed or buckled configuration of plate that satisfies governing and boundary conditions is known.
2. Analysis seeking variables type solution wherein w is expanded into *Fourier series* and the solution is obtained in series form.
3. Applications of principle of minimum potential energy using an assumed function for w which satisfies boundary conditions. When combined with *Rayleigh–Ritz method*, and *Galerkin's technique,* the method provides a powerful tool of analysis.
4. Approximate numerical techniques using: (i) Finite differences, and (ii) finite elements.

8.4.1 Governing Differential Equation Solution

I. Buckling of Plates Subjected to In-Plane Load in One-Direction

In this section a flat plate which is loaded on two simply supported edges b parallel to the Y-axis by a uniformly distributed load $p_x (= t\sigma_x)$ is considered. The edges parallel to X-axis (edges a) may be supported in different ways:

1. The plate is elastically restrained on both the edges a. This case includes as limiting cases simply supported and clamped edges.
2. One edge a is elastically restrained; the other is free. This case also includes the two limiting conditions in which the supported edge is either free to rotate or is clamped.

A solution to the partial differential equation (8.18) must satisfy the boundary conditions on all four edges. The conditions of simple support on the loaded edges b require:

$$w = 0 \quad \text{and} \quad \partial^2 w/\partial x^2 = 0 \quad \text{at the edges} \quad x = 0 \quad \text{and} \quad x = a. \tag{8.27}$$

The smallest critical load for a plate subjected to a compressive load p_x acting on the simply supported edges $x = 0$ and $x = a$ with edges $y = 0$ and $y = b$ being free, based on its analogy with column problem can be expressed as:

$$p_{x,\mathrm{cr}} = m^2\pi^2 D/a^2$$

such a plate is called *wide column*. The plate with other boundary conditions on the edges $y = 0$, b is expected to carry higher loads i.e. $p_{x,\mathrm{cr}} > m^2\pi^2 D/a^2$. Consequently, it can be expressed as:

$$p_{x,\mathrm{cr}} = k^2 \left(\frac{m\pi}{a}\right)^2 D \qquad k > 1 \tag{8.28}$$

Introducing the *aspect ratio*, $\mu = a/b$ into (8.28)

$$p_{x,\text{cr}} = \sigma_{x,\text{cr}} t = k^2 D\left(\frac{m\pi}{\mu b}\right)^2 = \left(\frac{km}{\mu}\right)^2 \frac{\pi^2 D}{b^2} \qquad (8.29)$$
$$= k_m^2\left(\frac{\pi^2 D}{b^2}\right)$$

where $k_m^2 = (km/\mu)^2$.

Substituting $D = Et^3/[12(1-\nu^2)]$ from (8.17d) into (8.29)

$$\sigma_{x,\text{cr}} = k_m^2 \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 \qquad (8.30)$$

It is convenient to select the buckled configuration $w(x, y)$ of the form:

$$w = f(y)\sin(m\pi x/a) \qquad m = 1, 2, 3\ldots \qquad (8.31)$$

satisfying the differential equation (8.18) and boundary conditions (8.27), where the function $f(y)$ is yet to be determined. The choice of function $f(y)$ will be governed by the boundary conditions on the unloaded edges. Introducing the assumed function $w(x, y)$ into the governing partial differential equation and canceling the term $D\sin(m\pi x/a)$ provides an ordinary differential equation of fourth order.

$$\frac{\mathrm{d}^4 f(y)}{\mathrm{d}y^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{\mathrm{d}^2 f(y)}{\mathrm{d}y^2} + \left[\left(\frac{m\pi}{a}\right)^4 - \frac{p_{x,\text{cr}}}{D}\left(\frac{m\pi}{a}\right)^2\right] f(y) = 0 \qquad (8.32)$$

In the above equation p_x is replaced by $p_{x,\text{cr}}$, the unknown *critical longitudinal uniformly distributed force* at which the plate buckles.

Substituting the value of $p_{x,\text{cr}}/D$ from (2.28) into (8.32), it assumes the form

$$\frac{\mathrm{d}^4 f(y)}{\mathrm{d}y^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{\mathrm{d}^2 f(y)}{\mathrm{d}y^2} + \left(\frac{m\pi}{a}\right)^4 (1-k^2) f(y) = 0 \qquad (8.33)$$

The general solution to the differential equation (8.33) is

$$f(y) = A\sinh\alpha y + B\cosh\alpha y + C\sin\beta y + D\cos\beta y \qquad (8.34)$$

where the parameters α and β are defined by:

$$\alpha^2 = \left(\frac{m\pi}{a}\right)^2 (k+1) \quad \text{and} \quad \beta^2 = \left(\frac{m\pi}{a}\right)^2 (k-1) \qquad (8.35\text{a})$$

Substituting $a = \mu b$, where μ is the aspect ratio, in (8.35a):

$$(\alpha b)^2 = \left(\frac{m\pi}{\mu}\right)^2 (k+1) \quad \text{and} \quad (\beta b)^2 = \left(\frac{m\pi}{\mu}\right)^2 (k-1) \qquad (8.35\text{b})$$

Thus the general solution to the partial differential equation takes the form:

$$w = (A \sinh \alpha y + B \cosh \alpha y + C \sin \beta y + D \cos \beta y) \sin(m\pi x/a) \qquad (8.36)$$

The constants A, B, C and D are determined from four boundary conditions at the unloaded edges a. Special of these boundary conditions will be considered in the following sections.

For analysis the plate buckling problems can be divided into two categories. First category deals with the plates having equal elastic restraints on both unloaded edges; this includes the cases where both the unloaded edges are either clamped or simply supported or free. In the second category unequal restraints exist on the unloaded edges of the plate; this includes the cases where two edges have different or *mixed boundary conditions.*

Case-I. Plate simply supported on loaded edges, and elastically restrained on unloaded edges.

For the type of problems involving plate elements with equal elastic restraints on both the unloaded edges, it is convenient to assume the origin of the co-ordinates X, Y at the mid point of the left edge as indicated in Fig. 8.3. Due to symmetry of boundaries, for smallest value of $p_{x,\mathrm{cr}}$, w is *symmetric function* of y and hence the terms $A \sinh \alpha y$ and $C \sin \beta y$ of solution (8.36) vanish. The solution reduces to

$$w = (B \cosh \alpha y + D \cos \beta y) \sin(m\pi x/a) \qquad (8.37)$$

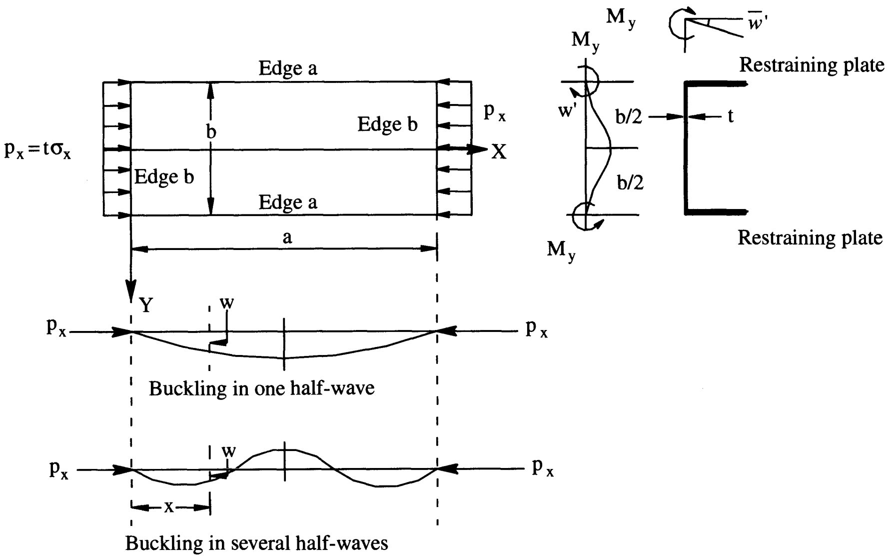


Fig. 8.3. Flat plate elastically restrained on unloaded edges a and simply supported on loaded edges b

The constants B and D can be determined from the boundary conditions at the unloaded edges, namely, the support conditions.

$$w\left(x, \pm\frac{b}{2}\right) = 0 \tag{8.38a}$$

i.e. the edges $y = \pm b/2$ remain straight when plate buckles. The other condition is that of continuity stipulating that the angle of rotation $w'(= \partial w/\mathrm{d}y)$ at the edge of buckling plate is equal to the angle of rotation of the adjoining restraining plate which is assumed to be rigidly connected. This can be expressed as

$$w' = \left(\frac{\partial w}{\partial y}\right)_{y=\pm\frac{b}{2}} = \overline{w}' \tag{8.38b}$$

The restraining moment M_y per unit length that occurs along the unloaded edges when the plate distorts is proportional to the angle $\overline{w}'$ and is given by:

$$M_y = -\overline{\eta}\,\overline{w}' \tag{8.39}$$

where $\overline{\eta}$ is *elastic stiffness factor* depending upon the properties of the restraining element. However, M_y is given by (8.17b) as

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]_{y=\pm\frac{b}{2}} = -D\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=\pm\frac{b}{2}} \tag{8.40}$$

since $\partial^2 w/\partial x^2 = 0$ for a supported edge. Substitution of (8.40) into (8.39) furnishes

$$\overline{w}' = \frac{D}{\overline{\eta}}\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=\pm\frac{b}{2}}$$

From (8.38b), the boundary conditions take the form:

$$\left[\frac{\partial w}{\partial y} + \frac{D}{\overline{\eta}}\frac{\partial^2 w}{\partial y^2}\right]_{y=\pm\frac{b}{2}} = 0 \tag{8.41a}$$

It should be noted that the quantity $D/\overline{\eta}$ has dimension of length. Defining a dimensionless parameter $2D/b\overline{\eta} = \eta$, which is constant along the edge and is a function of the dimensions of buckling and restraining plates. η which is referred to as *coefficient of restraint* can assume values from zero to infinity. When $\eta = 0$ (i.e. $\overline{\eta} = \infty$) it represents a completely fixed edge and when $\eta = \infty$ (i.e. $\overline{\eta} = 0$) the edge is free to rotate i.e. it is a simply supported edge.

Equation (8.41a) reduces to

$$\left[\frac{\partial w}{\partial y} + \frac{\eta b}{2}\frac{\partial^2 w}{\partial y^2}\right]_{y=\pm\frac{b}{2}} = 0 \tag{8.41b}$$

Substitution of solution (8.37) into boundary conditions given by (8.38a) and (8.41b) results in the following homogeneous linear equations:

$$B\cosh\frac{\alpha b}{2} + D\cos\frac{\beta b}{2} = 0$$

$$\left(B\alpha\sinh\frac{\alpha b}{2} - D\beta\sin\frac{\beta b}{2}\right) + \eta\frac{b}{2}\left(B\alpha^2\cosh\frac{\alpha b}{2} - D\beta^2\cos\frac{\beta b}{2}\right) = 0$$

For a non-trivial solution (i.e. $B = D \neq 0$), the determinant of coefficients of B and D must vanish. This leads to the following characteristic equation or stability condition.

$$\alpha\tanh\frac{\alpha b}{2} + \beta\tan\frac{\beta b}{2} + \eta\frac{b}{2}\left(\alpha^2 + \beta^2\right) = 0$$

or

$$\frac{\alpha b}{2}\tanh\frac{\alpha b}{2} + \frac{\beta b}{2}\tan\frac{\beta b}{2} + \eta\left[\left(\frac{\alpha b}{2}\right)^2 + \left(\frac{\beta b}{2}\right)^2\right] = 0 \tag{8.42}$$

Introducing $\alpha b/2$ and $\beta b/2$ from (8.35b) into (8.42)

$$(k+1)^{\frac{1}{2}}\tanh\left[\left(\frac{m\pi}{2\mu}\right)(k+1)^{\frac{1}{2}}\right] + (k-1)^{\frac{1}{2}}\tan\left[\left(\frac{m\pi}{2\mu}\right)(k-1)^{\frac{1}{2}}\right] + \eta\frac{m\pi k}{\mu} = 0 \tag{8.43}$$

This general transcendental equation defines a relationship between parameters k and m/μ and can be solved for k for a given value of m/μ. To illustrate the application of (8.43) following special cases have been considered.

(1) Plate simply supported along the unloaded edges

This condition is obtained by introducing $\eta = \infty$ into (8.43) and knowing that the function tanh attains values only between $+1$ and -1, (8.43) reduces to

$$\tan\left[\left(\frac{m\pi}{2\mu}\right)(k-1)^{\frac{1}{2}}\right] = -\infty$$

The smallest root satisfying this equation is given by:

$$\left(\frac{m\pi}{2\mu}\right)(k-1)^{\frac{1}{2}} = \frac{\pi}{2}$$

or

$$k^2 = \left[\left(\frac{\mu}{m}\right)^2 + 1\right]^2 = 1 + 2\left(\frac{\mu}{m}\right)^2 + \left(\frac{\mu}{m}\right)^4 \tag{8.44}$$

Substituting k^2 in (8.29) yields

$$p_{x,\text{cr}} = \left(\frac{\mu}{m} + \frac{m}{\mu}\right)^2 \frac{\pi^2 D}{b^2} = k_m^2(\mu)\frac{\pi^2 D}{b^2} \tag{8.45a}$$

or from (8.30)

$$\sigma_{x,\text{cr}} = \left(\frac{\mu}{m} + \frac{m}{\mu}\right)^2 \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 = k_m^2(\mu)\frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 \tag{8.45b}$$

where

$$k_m(\mu) = \left(\frac{\mu}{m} + \frac{m}{\mu}\right)$$

Only unknown now left in (8.45) is m, which indicates the number of half-waves in which plate buckles in the X-direction. The value of m that corresponds to the minimum value of $p_{x,\text{cr}}$ is given by:

$$\frac{\mathrm{d}p_{x,\text{cr}}}{\mathrm{d}m} = \frac{\pi^2 D}{b^2}\frac{\mathrm{d}k_m^2(\mu)}{\mathrm{d}m} = 0\,, \quad \text{i.e.} \quad \frac{\mathrm{d}k_m^2(\mu)}{\mathrm{d}m} = 0$$

$$\frac{\mathrm{d}k_m^2(\mu)}{\mathrm{d}m} = \left(\frac{m}{\mu} + \frac{\mu}{m}\right)\left(\frac{1}{\mu} - \frac{\mu}{m^2}\right) = m\left[\left(\frac{1}{\mu}\right)^2 - \left(\frac{\mu}{m^2}\right)^2\right] = 0$$

or

$$\frac{1}{\mu} = \frac{\mu}{m^2} \quad \text{i.e.} \quad m = \mu.$$

Thus, from (8.44)

$$k_m^2 = k^2 = \left[\left(\frac{\mu}{m}\right)^2 + 1\right]^2 = 4$$

Therefore,

$$p_{x,\text{cr}} = \frac{4\pi^2 D}{b^2} \quad \text{or} \quad \sigma_{\text{x,cr}} = \frac{\pi^2 E}{3(1-\nu^2)}\left(\frac{t}{b}\right)^2 \tag{8.46}$$

The critical load given by (8.45a) is valid only when μ is an integer i.e. plate buckles in μ square waves in X-direction. For small values of μ, i.e. for sufficiently short plates, buckling will occur in one *half-wave*. Above a certain value of μ two half-waves will be formed. For the limiting ratio at which there is transition from one state of equilibrium to another, i.e., when both the cases are equally possible at the same buckling stress $\sigma_{x,\text{cr}}$, (8.45b) will yield same value of $\sigma_{x,\text{cr}}$ whether m is 1 or 2. In general the limiting ratio $\overline{\mu}$ at which either m or $m+1$ half-waves can occur is given by:

$$k_m(\mu) = k_{m+1}(\mu)$$

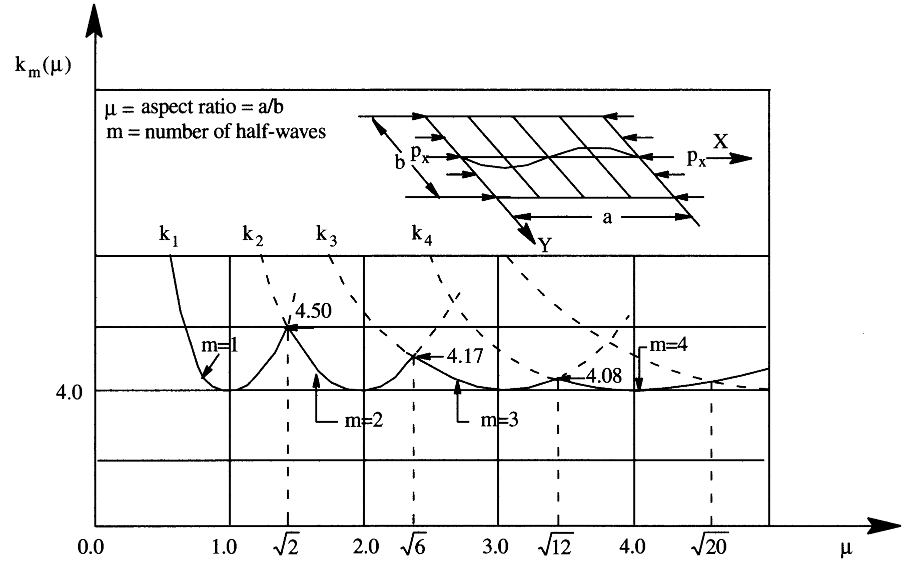


Fig. 8.4. Relationship between k_m and μ for a plate simply supported on all the edges

i.e.

$$\frac{\overline{\mu}}{m} + \frac{m}{\overline{\mu}} = \frac{\overline{\mu}}{m+1} + \frac{m+1}{\overline{\mu}}$$

or

$$\overline{\mu} = [m(m+1)]^{\frac{1}{2}} \quad \text{for} \quad m = 1, 2, 3, \ldots \quad \overline{\mu} = \sqrt{2}, \sqrt{6}, \sqrt{12}, \ldots \tag{8.47}$$

Thus the buckling occurs in one half-wave up to $\mu = 1.414$ (i.e. $a = 1.414b$), and from $\mu = 1.414$ to $\mu = 2.449$ in two half-waves and so on. For long plates the length of the half-waves approaches the width b. This dependence of k on aspect ratio μ is shown in Fig. 8.4 in the form of a sequence of curves which correspond to *harmonic number* of buckling modes in the direction of loading, $m = 1, 2, 3, \ldots$. It should be noted that curves for $m = 2, 3, \ldots$ can be readily drawn from the curve for $m = 1$ by multiplying the abscissa by 2, 3, etc. and keeping the ordinate unchanged.

For the general buckling condition of elastically restrained plate, Bleich had proposed an algebraic relationship between k^2 and μ/m, similar to the one given by (8.44) for the simply supported plate.

$$k_1^2 = 1 + p\left(\frac{\mu}{m}\right)^2 + q\left(\frac{\mu}{m}\right)^4 \tag{8.48}$$

where p and q are parameters depending on the coefficient of restraint η and were computed by Bleich for various values of η from the characteristic (8.43).

Substituting the expression (8.48) into (8.30) an equation for $\sigma_{x,\mathrm{cr}}$ which is valid for all possible values of elastic restraint, is obtained

$$\sigma_{x,\mathrm{cr}} = \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2\left[\left(\frac{m}{\mu}\right)^2 + p + q\left(\frac{\mu}{m}\right)^2\right]$$

Introducing the notation

$$k_m^2 = \left[\left(\frac{m}{\mu}\right)^2 + p + q\left(\frac{\mu}{m}\right)^2\right] \tag{8.49}$$

the equation for $\sigma_{x,\mathrm{cr}}$ assumes standard form

$$\sigma_{x,\mathrm{cr}} = \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 k_m^2 \tag{8.50}$$

where k_m is a non-dimensional parameter depending upon μ and boundary conditions at unloaded edges. As in the case of simply supported plate, the limiting aspect ratio $\overline{\mu}$ at which either m or $m+1$ half-waves can exist is given by

$$\overline{\mu} = \left(\frac{1}{q}\right)^{\frac{1}{4}} [(m)(m+1)]^{\frac{1}{2}} \tag{8.51}$$

The parameter q lies between 1 (for both unloaded edges simply supported) and 5 (for both edges clamped). For clamped edges $\overline{\mu} = 0.6687[m(m+1)]^{1/2}$. Thus, the length of half-waves is appreciably shortened by clamping. The value μ_0 for which $\sigma_{x,\mathrm{cr}}$ becomes minimum can be based on

$$\frac{\partial \sigma_{x,\mathrm{cr}}}{\partial \mu} = 0 \quad \text{i.e.} \quad \mu_o = m\left(\frac{1}{q}\right)^{\frac{1}{4}}$$

and the corresponding k_m^2 is given by:

$$k_m^2 = \left(p + 2\sqrt{q}\right) \tag{8.52}$$

The parameters p and q are given by:

$$\begin{aligned}
&\text{For } \eta = \infty: && p = 2 \quad \text{and} \quad q = 1 \\
&\text{For } \eta = 0: && p = 2.5 \quad \text{and} \quad q = 5.0 \\
&\text{For } \eta > 1.6: && p \approx 2.0 + \frac{0.047}{0.73+\eta} \quad \text{and} \quad q \approx 1.0 + \frac{0.70}{0.077+\eta}
\end{aligned} \tag{8.53}$$

The values of $\overline{\mu}$ for different number of half-waves that can exist for the case of plate with unloaded edges clamped ($q = 5$) are given by (8.51).

For the analysis of plates with unequal restraints on the unloaded edges, an approximate technique based on the method outlined above for equal restraints on unloaded edges can be used. The technique consists of first using the coefficient of restraint η_1 of one edge to find $k_{m,1}$ and then using the other value η_2 to find a plate coefficient $k_{m,2}$. The mean value $(k_{m,1} + k_{m,2})/2$ provides a fairly good estimation of the exact value of k_m and can be used to obtain critical stress of the plate under consideration.

Number of half-waves, m	Aspect ratio, μ
1	$\mu < 0.945$
2	$0.945 < \mu < 1.638$
3	$1.638 < \mu < 2.316$
4	$2.316 < \mu < 2.990$
5	$2.990 < \mu < 3.662$

(2) Plate clamped along the unloaded edges

This condition is attained by introducing $\eta = 0$ into (8.43). The resulting characteristic equation is:

$$(k+1)^{\frac{1}{2}} \tanh\left[\left(\frac{m\pi}{2\mu}\right)(k+1)^{\frac{1}{2}}\right] + (k-1)^{\frac{1}{2}} \tan\left[\left(\frac{m\pi}{2\mu}\right)(k-1)^{\frac{1}{2}}\right] = 0 \quad (8.54)$$

As discussed above for an aspect ratio μ up to 0.94 the minimum value of k and hence of axial load is obtained for $m = 1$ and for the ratios approximately between 0.9 to 1.6, $m = 2$ and so on. For illustration consider a square plate ($\mu = 1$) for which minimum critical stress will be given for $m = 2$. By trial and modification the least value of k_m satisfying (8.54) is 2.77332. Therefore, critical load is given by

$$\begin{aligned} p_{x,\text{cr}} &= \frac{k_m^2 \pi^2 D}{b^2} = \frac{(2.77332)^2 \pi^2 D}{b^2} \\ &= \frac{7.6913\pi^2 D}{b^2} \end{aligned} \quad (8.55)$$

The values of k_m^2 for various aspect ratios are tabulated in Table 8.1.

Case-II. Plate simply supported on the loaded edges, elastically restrained on one of the unloaded edges, and free at the other

The origin of co-ordinate axes in this case is taken to coincide with a corner of the plate such that X-axis is along supported edge as shown in Fig. 8.5. Since there is no symmetry with respect to X-axis, the general solution of governing differential equation given by (8.35) is to be used i. e.

$$w(x, y) = \sin\left(\frac{m\pi x}{a}\right)(A \sinh \alpha y + B \cosh \alpha y + C \sin \beta y + D \cos \beta y)$$

$$\text{where } \alpha = \left(\frac{m\pi}{a}\right)(k+1)^{\frac{1}{2}} \quad \text{and} \quad \beta = \left(\frac{m\pi}{a}\right)(k-1)^{\frac{1}{2}} \quad (8.56)$$

The constants A, B, C and D are determined from four boundary conditions at the restrained or supported edge $y = 0$ and free edge, $y = b$:
(i) The boundary conditions at the restrained edge are:

Table 8.1. Buckling Load Factor k_{m}^2 for Rectangular Plates in Compression. (Loaded edges $x = 0$ and $x = a$ are simple supports)

Aspect ratio, $\mu \downarrow$	Types of unloaded edges						
	Type-I	Type-II	Type-III		Type-IV	Type-V	
Boundary conditions of unloaded edges	Clamped –clamped	Clamped –hinged	Clamped–free		Simple support–simple support	Simple support–free a	
Poisson's ratio →			$\nu = 0.25$	$\nu = 0.30$		$\nu = 0.25$	$\nu = 0.30$
0.4000	9.4479	8.8619	6.7248	6.6608	8.4100	6.6148	6.5788
0.6000	7.0552	5.9177	3.3342	3.2841	5.1378	3.1921	3.1502
0.8000	7.3037	5.4099	2.1900	2.1437	4.2025	1.9894	1.9539
0.8910	7.7986[b]					1.4342	
1.0000	7.6913	5.7402	1.6983	1.6525	4.0000	1.2269	1.4016
1.1254		6.1862[b]				1.1334	1.1953
1.2000	7.0552	5.9177	1.4669	1.4205	4.1344	0.9524	1.1022
1.4000	7.0008	5.5118	1.3625	1.3151	4.4702	0.9424	0.9220
$\sqrt{2}$		5.4962			4.5000[b]		0.9120
1.5000	7.6912[b]					0.8353	
1.6000	7.3037	5.4099	1.3296	1.2811	4.2025		0.8052
1.6300			1.3290				
1.6451				1.2804		0.7551	
1.8000	7.0552	5.5049	1.3420	1.2926	4.0446	0.7109	0.7253
1.9487		5.6684[b]				0.6979	0.6811
2.0000	6.9716	5.6056	1.3862	1.3358	4.000		0.6681
2.2900	7.1578[b]						
2.3229				1.4947[b]		0.6236	
2.4000	7.0552	5.4099	1.5423	1.4908	4.1344	0.6169	0.5938
$\sqrt{6}$		5.4140			4.1667[b]	0.5830	0.5871
2.7556		5.5391[b]				0.5790	0.5531
2.8000	7.0008	5.5118	1.7662	1.3151			0.5491
2.9600	7.0835[b]					0.5630	
3.0000	7.0552	5.4312	1.9881	1.2912	4.000	0.5362	0.5331
$\sqrt{12}$		5.4547			4.0833[b]		0.5062
4.015				1.3381[b]			
∞	6.9716	5.4099	1.3290	1.2804	4.0000	0.4500	0.4250
Wave length, $\lambda = \mu_{cr}/m$	0.6667	0.8000	1.6300	1.6451	1.0000	[a]	[a]

[a] Value of m is always unity i.e. wave-length is always equal to the length of the plate.
[b] Indicates a point where slope discontinuity occurs due to change of the number of half-waves to the next higher one. Such points should not be used as intermediate points in interpolation.

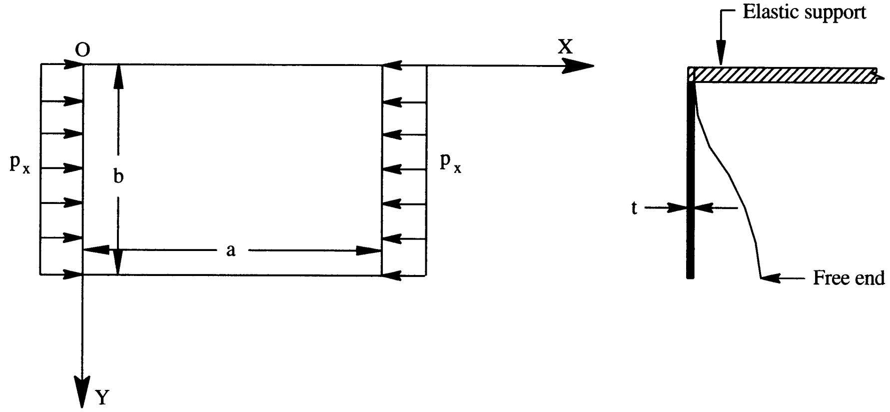


Fig. 8.5. One of the unloaded edge elastically restrained and other free

$$w = 0 \quad \text{and} \quad \left[\frac{\partial w}{\partial y} - \frac{\eta b}{2}\frac{\partial^2 w}{\partial y^2}\right]_{y=0} = 0 \qquad \text{(from case-I)} \tag{8.57}$$

(ii) The boundary conditions at the free edge are:

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]_{y=b} = 0$$

$$\text{Transverse shear,} \quad Q_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=b} = 0 \tag{8.58}$$

Substitution of the solution (8.56) in the boundary conditions given by (8.57) yields the relations

$$w = B + D = 0 \quad \text{or} \quad B = -D$$

$$A\alpha + C\beta - \eta\frac{b}{2}\left(B\alpha^2 - D\beta^2\right) = 0 \quad \text{or} \quad A = -C\left(\frac{\beta}{\alpha}\right) - D\eta\frac{b}{2}\left(\frac{\alpha^2+\beta^2}{\alpha}\right)$$

Therefore, solution (8.56) can be expressed as

$$w(x, y) = \sin\left(\frac{m\pi x}{a}\right)\left[\left(\sin\beta y - \frac{\beta}{\alpha}\sinh\alpha y\right)C + \left\{\cos\beta y - \cosh\alpha y - \frac{\eta b}{2}\left(\frac{\alpha^2+\beta^2}{\alpha}\right)\sinh\alpha y\right\}D\right] \tag{8.59}$$

Introducing this equation into two remaining boundary conditions given by (8.58)

$$\left(\bar{\beta}\sin\beta b + \frac{\bar{\alpha}\beta}{\alpha}\sinh\alpha b\right)C + \left(\bar{\beta}\cos\beta b + \bar{\alpha}\cosh\alpha b + \gamma\bar{\alpha}\sinh\alpha b\right)D = 0$$

$$-\left(\frac{\bar{\alpha}\beta}{\alpha}\cos\beta b + \frac{\bar{\beta}\beta}{\alpha}\cosh\alpha b\right)C + \left(\frac{\bar{\alpha}\beta}{\alpha}\sin\beta b - \bar{\beta}\sinh\alpha b - \gamma\bar{\beta}\cosh\alpha b\right)D = 0$$

where

$$\overline{\alpha} = \alpha^2 - \nu\left(\frac{m\pi}{a}\right)^2 = \beta^2 + (2-\nu)\left(\frac{m\pi}{a}\right)^2$$
$$\overline{\beta} = \beta^2 + \nu\left(\frac{m\pi}{a}\right)^2 = \alpha^2 - (2-\nu)\left(\frac{m\pi}{a}\right)^2$$
$$\gamma = \left(\frac{\eta b}{2}\right)\left(\frac{\alpha^2+\beta^2}{\alpha}\right) \tag{8.60}$$

For non-trivial solution, vanishing the determinant of coefficients of C and D, yields stability condition.

$$2\overline{\alpha}\overline{\beta} + \left(\overline{\alpha}^2 + \overline{\beta}^2\right)\cosh\alpha b\cos\beta b - \left[\left(\frac{\alpha}{\beta}\right)\overline{\beta}^2 - \left(\frac{\beta}{\alpha}\right)\overline{\alpha}^2\right]\sinh\alpha b\sin\beta b$$
$$+\gamma\left[\overline{\alpha}^2\sinh\alpha b\cos\beta b - \overline{\beta}^2\left(\frac{\alpha}{\beta}\right)\cosh\alpha b\sin\beta b\right] = 0 \tag{8.61}$$

The parameters αb, βb, $\overline{\alpha}$ and $\overline{\beta}$ can be expressed as :

$$\alpha b = \left(\frac{m\pi}{\mu}\right)(k+1)^{\frac{1}{2}}, \quad \beta b = \left(\frac{m\pi}{\mu}\right)(k-1)^{\frac{1}{2}}$$
$$\overline{\alpha} = \frac{1}{b^2}\left(\frac{m\pi}{\mu}\right)^2(k+1-\nu) \quad \text{and} \quad \overline{\beta} = \frac{1}{b^2}\left(\frac{m\pi}{\mu}\right)^2(k-1+\nu) \tag{8.62}$$

The critical stress $\sigma_{x,\mathrm{cr}}$ can be expressed in the form

$$\sigma_{x,\mathrm{cr}} = \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 k_m^2$$

where

$$k_m^2 = \left[\left(\frac{m}{\mu}\right)^2 + p + q\left(\frac{\mu}{m}\right)^2\right] \tag{8.63}$$

The parameters p and q are dependent upon coefficient of restraint η. As in the case-I, the limiting $\overline{\mu}$ at which either m or $m+1$ half-waves can exist; i. e.

$$\overline{\mu} = \left(\frac{1}{q}\right)^{\frac{1}{4}}[(m)(m+1)]^{\frac{1}{2}} \tag{8.64}$$

The ratio μ_o corresponding to minimum value of $\sigma_{x,\mathrm{cr}}$ is given by

$$\mu_0 = m\left(\frac{1}{q}\right)^{\frac{1}{4}} \tag{8.65}$$

and minimum

$$\sigma_{x,\mathrm{cr}} = \frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2(p + 2\sqrt{q}) \tag{8.66}$$

If one edge of the plate is simply supported or free to rotate, $q = 0$ and the plate will bulge in one half wave, regardless of its length. The following special cases arise.

1. The edge $y = 0$ is clamped i. e. $\eta = \gamma = 0$, and the edge $y = b$ is free.

The characteristic equation given by (8.61) reduces to;

$$2\overline{\alpha}\overline{\beta} + \left(\overline{\alpha}^2 + \overline{\beta}^2\right) \cosh \alpha b \cos \beta b - \left[\left(\frac{\alpha}{\beta}\right)\overline{\beta}^2 - \left(\frac{\beta}{\alpha}\right)\overline{\alpha}^2\right] \sinh \alpha b \sin \beta b = 0 \tag{8.67}$$

where $\overline{\alpha}$ and $\overline{\beta}$ are defined by (8.62).

Consider a square plate ($\mu = 1$). The least value of k_m satisfying the stability condition given by (8.67) is 1.28550. Therefore,

$$p_{x,\text{cr}} = \frac{(1.28550)^2\pi^2 D}{b^2} = \frac{1.6525\pi^2 D}{b^2}$$

For other aspect ratios the values of k_m^2 are given in the Table 8.1.

2. The edge $y = 0$ is simply supported or hinged (i. e. η or $\gamma = \infty$) and the edge $y = b$ is free.

The characteristic equation for this case is obtained from (8.61) by substituting $\gamma = \infty$. Before affecting this substitution divide the equation by γ. The characteristic equation reduces to

$$\overline{\alpha}^2 \sinh \alpha b \cos \beta b - \overline{\beta}^2 \left(\frac{\alpha}{\beta}\right) \cosh \alpha b \ \sin \beta b = 0$$

or

$$\tanh \alpha b \cot \beta b = \left(\frac{\overline{\beta}}{\overline{\alpha}}\right)^2 \left(\frac{\alpha}{\beta}\right) = \left[\frac{k - 1 + \nu}{k + 1 - \nu}\right]^2 \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}} \tag{8.68}$$

To illustrate application of this equation consider a square plate ($\mu = 1$) of a material having $\nu = 0.3$. For this problem the characteristic equation reduces to

$$\tanh \alpha b. \cot \beta b = \left[\frac{k - 0.7}{k + 0.7}\right]^2 \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}$$

Using trial and modification procedure, the least value of k_m satisfying the equation is 1.18389. Therefore, critical load is given by

$$p_{x,\text{cr}} = \frac{(1.18389)^2\pi^2 D}{b^2} = \frac{1.4016\pi^2 D}{b^2}$$

For various aspect ratios the values of k_m^2 are given in the Table 8.1.

3. The edge $y = 0$ is clamped and $y = b$ is simply supported or hinged

Substitution of general solution given by (8.36) in the boundary conditions at the unloaded edges $y = 0$ and $y = b$:

at $y = 0$: $\quad w = 0$ i.e. $w = B + D = 0$ or $D = -B$

and

$$\left(\frac{\partial w}{\partial y}\right)_{y=0} = \alpha A + \beta C = 0 \quad \text{i.e.} \quad C = -\left(\frac{\alpha}{\beta}\right) A$$

at $\quad y = b$: $\quad w = 0$ and $M_y = 0$

$$w = A \sinh \alpha b + B \cosh \alpha b - \left(\frac{\alpha}{\beta}\right) A \sin \beta b - B \cos \beta b = 0$$

and

$$M_y = \left(\frac{\partial^2 w}{\partial y^2}\right)_{y=b} = \alpha^2 A \sinh \alpha b + \alpha^2 B \cosh \alpha b + \alpha\beta A \sin \beta b + \beta^2 B \cos \beta b = 0$$

For non-trivial solution, the determinant of coefficients of A and B must vanish that is:

$$\begin{vmatrix} \sinh \alpha b - \left(\frac{\alpha}{\beta}\right) \sin \beta b & \cosh \alpha b - \cos \beta b \\ \alpha^2 \sinh \alpha b + \alpha\beta \sin \beta b & \alpha^2 \cosh \alpha b + \beta^2 \cos \beta b \end{vmatrix} = 0$$

The expansion of the determinant yields stability condition or characteristic equation

$$\tanh \alpha b - \left(\frac{\alpha}{\beta}\right) \tan \beta b = 0 \tag{8.69}$$

where αb and βb are defined in (8.35b) as

$$\alpha b = \left(\frac{m\pi}{\mu}\right)(k+1)^{\frac{1}{2}}; \quad \beta b = \left(\frac{m\pi}{\mu}\right)(k-1)^{\frac{1}{2}} \quad \text{and} \quad \left(\frac{\alpha}{\beta}\right) = \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}$$

For a square plate, the minimum value of k_m satisfying (8.69) is 2.39587. Therefore,

$$p_{x,\mathrm{cr}} = \frac{(2.39587)^2 \pi^2 D}{b^2} = \frac{5.7402\pi^2 D}{b^2}$$

For other aspect ratios μ, values of k_m^2 are given in Table 8.1.

The stability conditions for the type-I plates with symmetric boundary conditions can also be derived directly using the co-ordinate system of type-II problems. For illustration consider the plate with both the edges $y = 0$ and $y = b$ clamped. The boundary conditions $w = 0$ and $\partial w/\partial y = 0$ at the edges $y = 0, b$ when transformed in terms of general solution provide at $y = 0$:

$$B + D = 0 \quad \text{or} \quad D = -B \tag{a}$$

$$\alpha A + \beta C = 0 \quad \text{or} \quad C = -(\alpha/\beta)A \tag{b}$$

at $y = -b$:

$$A \sinh \alpha b + B \cosh \alpha b - (\alpha/\beta)A \sin \beta b - B \cos \beta b = 0 \tag{c}$$

$$\alpha A \cosh \alpha b + \alpha B \sinh \alpha b - \alpha A \cos \beta b + \beta B \sin \beta b = 0 \tag{d}$$

For non-trivial ($A = B \neq 0$) solution of simultaneous equations (c) and (d) the determinant of coefficients of A and B must vanish, i. e.

$$\begin{vmatrix} \sinh \alpha b - \left(\frac{\alpha}{\beta}\right) \sin \beta b & \cosh \alpha b - \cos\ \beta b \\ \alpha \cosh \alpha b - \alpha \cos \beta b & \alpha \sinh \alpha b + \beta \sin \beta b \end{vmatrix} = 0$$

The expansion of determinant leads to the stability condition or characteristic equation

$$\frac{1}{2}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)(\sinh \alpha b. \sin \beta b) = (1 - \cosh \alpha b. \cos \beta b)$$

$$\frac{1}{2}\left(\frac{\beta^2 - \alpha^2}{\alpha\beta}\right)(\sinh \alpha b. \sin \beta b) + (\cosh \alpha b. \cos \beta b) - 1 = 0$$

$$-(k^2 - 1)^{-1/2}(\sinh \alpha b. \sin \beta b) + (\cosh \alpha b. \cos \beta b) - 1 = 0 \tag{8.70}$$

where αb and βb are defined by (8.35b). Though (8.70) differs from (8.54) due to difference in the origin of co-ordinate system, but they provide same results. Consider a square plate i. e. $\mu = 1$. The least value of k_m satisfying the characteristic (8.70) is 2.77332. Therefore, the critical load is given by

$$p_{x,\text{cr}} = \frac{k_m^2 \pi^2 D}{a^2} = \frac{(2.77332)^2 \pi^2 D}{a^2} = \frac{7.691\pi^2 D}{a^2}$$

The above procedure can also be used for plates with unloaded edges $y = 0$ and $y = b$ simply supported or hinged. The boundary conditions $w = 0$ and $w'' = 0$ at the edges $y = 0, b$ when transformed in terms of general solutions provide

at $y = 0$:

$$B + D = 0 \quad \text{or} \quad D = -B$$

and

$$\alpha^2 B - \beta^2 D = 0 \quad \text{or} \quad D = (\alpha^2/\beta^2)B$$

These two conditions can be true only when $D = B = 0$

at $y = b$:

$$A \sinh \alpha b + C \sin \beta b = 0$$

$$\alpha^2 A \sinh \alpha b - \beta^2 C \sin \beta b = 0$$

For non-trivial ($A = C \neq 0$) solution

$$\begin{vmatrix} \sinh \alpha b & \sin \beta b \\ \alpha^2 \sinh \alpha b & -\beta^2 \sin \beta b \end{vmatrix} = 0$$
$$-(\alpha^2 + \beta^2) \sinh \alpha b. \sin \beta b = 0$$

The quantity in bracket is sum of two positive quantities and hence cannot be zero. On the other hand $\sinh \alpha$b is zero only at $\alpha b = 0$, this is a trivial solution. Hence the only feasible solution is

$$\sin \beta b = 0 \quad \text{or} \quad \beta b = n\pi$$

Therefore, from (8.35b):

$$(\beta b)^2 = \frac{m^2\pi^2}{\mu^2}(k-1) = n^2\pi^2$$

or

$$k = 1 + n^2\left(\frac{\mu^2}{m^2}\right) \tag{8.71}$$

For minimum value of k, integer $n = 1$. Thus from (8.29)

$$p_{x,\text{cr}} = k^2 D\left(\frac{m\pi}{\mu b}\right)^2 = \left[1 + \left(\frac{\mu}{m}\right)^2\right]^2 \left(\frac{m}{\mu}\right)^2 \left(\frac{\pi^2 D}{b^2}\right)$$
$$= \left[\left(\frac{m}{\mu}\right) + \left(\frac{\mu}{m}\right)\right]^2 \left(\frac{\pi^2 D}{b^2}\right) = k_m^2\left(\frac{\pi^2 D}{b^2}\right)$$

where $\mu = a/b$. This equation is identical to one derived earlier and given by (8.45a).

II. Buckling of plates subjected to in-plane loads in two directions

In this type of plate problems in addition to a uniform axial compressive force p_x acting along the edges $x = 0$, a the plate is subjected to a uniform compressive force p_y per unit length along the edges $y = 0$, b, i.e. the plates are subjected to in-plane loads in two direction as shown in Fig. 8.6.

Here, since $p_{xy} = 0$ the governing partial differential (8.18) reduces to:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} = 0 \tag{8.72}$$

Consider a rectangular plate simply supported along its four edges as shown in Fig. 8.6. The conditions of zero lateral deflection and moment at the edges implies.

$$w = \partial^2 w/\partial x^2 = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a$$
$$w = \partial^2 w/\partial y^2 = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b$$

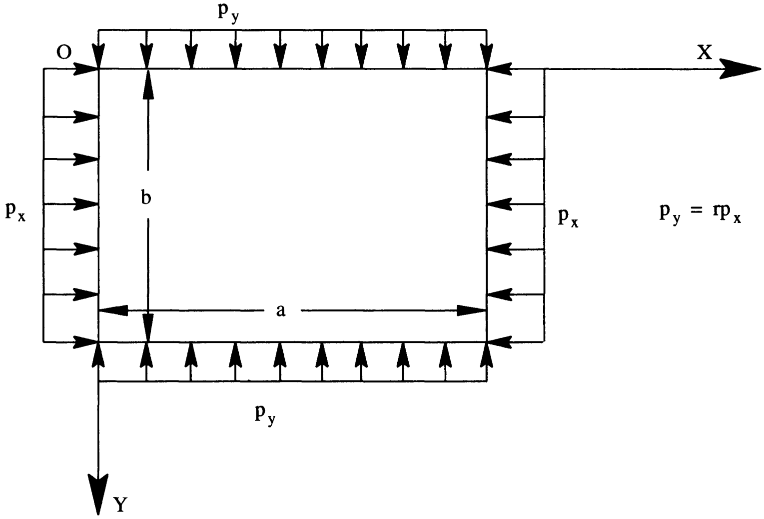


Fig. 8.6. Rectangular plate compressed in two directions

A typical series solution satisfying the above boundary condition is

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{8.73}$$

where m and n are integers. For $w(x, y)$ to be a buckled configuration, it should satisfy the governing differential equation, i. e.,

$$\sum_m \sum_n DA_{mn} \left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 - \frac{p_x}{D}\left(\frac{m\pi}{a}\right)^2 - \frac{p_y}{D}\left(\frac{n\pi}{b}\right)^2\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0$$

Such a sum in series form can vanish only when the coefficient of every term is zero. Thus,

$$A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{p_x}{D}\left(\frac{m\pi}{a}\right)^2 - \frac{p_y}{D}\left(\frac{n\pi}{a}\right)^2\right] = 0$$

Thus for a non-trivial solution

$$p_x\left(\frac{m}{a}\right)^2 + p_y\left(\frac{n}{b}\right)^2 = \pi^2 D\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \tag{8.74}$$

This stability or characteristic equation can lead to several cases of plate buckling problem. Some of them are given below:

1. p_x and p_y are proportional i.e. $p_x = f p_{\mathrm{cr}}$ and $p_y = g p_{\mathrm{cr}}$ where f and g are specified fractions or ratios and allow p_{cr} to be evaluated.
2. p_x (or p_y) has a fixed value and corresponding $p_{y,cr}$ (or $p_{x,\mathrm{cr}}$) need be computed.

For illustration consider the problem of rectangular plate where $p_y = r p_x$. The stability condition given by (8.74) reduces to

$$p_{x,\mathrm{cr}} = \frac{\pi^2 D\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}{\left[\left(\frac{m}{a}\right)^2 + r\left(\frac{n}{b}\right)^2\right]} = \left(\frac{\pi^2 D}{b^2}\right)\frac{\left[\frac{m}{\mu} + n^2\left(\frac{\mu}{m}\right)\right]^2}{\left[1 + n^2 r\left(\frac{\mu}{m}\right)^2\right]} \tag{8.75}$$

Equation (8.75) is sometimes referred to as *interaction equation.* For the given values of load ratio, r and aspect ratio, μ the values of m and n may be determined by trial and modification procedure or otherwise to obtain the smallest value of critical load.

As a typical case of buckling under bi-directional in-plane loading consider a square plate ($\mu = 1.0$) with $r = 1$ ($p_y = p_x$). Equation (8.75) reduces to

$$p_{x,\mathrm{cr}} = \left(\frac{\pi^2 D}{b^2}\right)(m^2 + n^2)$$

Obviously the lowest critical value is obtained for $m = n = 1$

$$p_{x,\mathrm{cr}} = \frac{2\pi^2 D}{b^2}$$

i.e., the critical load is just half of that for a square plate loaded only in one direction and buckling occurs with single half wave in each direction.

If $r = 0$, i.e. plate is subjected to the load p_x only, (8.75) reduces to

$$p_{x,\mathrm{cr}} = \left(\frac{\pi^2 D}{b^2}\right)\left[\frac{m}{\mu} + n^2\frac{\mu}{m}\right]^2$$

The minimum value of $p_{x,\mathrm{cr}}$ corresponds to $n = 1$ and hence

$$p_{x,\mathrm{cr}} = \left(\frac{\pi^2 D}{b^2}\right)\left[\frac{m}{\mu} + \frac{\mu}{m}\right]^2 \tag{8.76}$$

This expression is identical to that given by (8.45a).

For a square plate (8.75) reduces to

$$p_{x,\mathrm{cr}} = \left(\frac{\pi^2 D}{b^2}\right)\frac{(m^2 + n^2)^2}{(m^2 + r\,n^2)}$$

$p_{x,\mathrm{cr}}$ will again correspond to $m = n = 1$ so that

$$p_{x,\mathrm{cr}} = \left(\frac{4}{1 + r}\right)\left(\frac{\pi^2 D}{b^2}\right) \tag{8.77}$$

It should be noted that the buckling may occur even if one of the in-plane forces is tensile, although it retards the instability. The effect of unidirectional tension on retardation of buckling can be demonstrated by considering negative value of r in (8.77). In many practical cases of unidirectional compression, the boundaries in the other direction, parallel to the direction of loading are restrained in middle plane. As the plate deflects, these boundaries develop tensile forces in the middle plane which retard or delay the onset of instability or increase the critical load as given by (8.77).

8.4.2 Stationary Potential Principle

As is seen in the preceding sections the calculations for obtaining critical load and corresponding buckling mode become involved for the plates with different boundary conditions at the edges. In such situations approximate methods like *Ritz method*, *Galerkin's method* etc. prove to be useful. For illustration consider the case of a simply supported rectangular plate subjected to a uniform axial compressive force p_x per unit length along the edges $x = 0$ and $x = a$. Let the displaced configuration $w(x, y)$ is still given by (8.73) as:

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{8.78}$$

For this type of series displacement function with $w(x, y) = 0$ at all edges, the *Gaussian curvature* term of strain energy U reduces to zero, hence

$$U = \frac{D}{2}\iint\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{8.79}$$

Substituting various derivatives of $w(x, y)$ in (8.79) and after integrating, the strain energy of the plate is given by:

$$U = \frac{D}{2}\sum_m\sum_n A_{mn}^2\left[\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\right]^2\left(\frac{ab}{4}\right) \tag{8.80a}$$

Noting that $p_{xy} = p_y = 0$, the potential energy V_e is given by (8.25):

$$V_e = -\frac{1}{2}\int_0^a\int_0^b p_x\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}y\,\mathrm{d}x$$

$$= -\frac{1}{2}\sum_m\sum_n A_{mn}^2\left[\left(\frac{m\pi}{a}\right)^2 p_x\right]\left(\frac{ab}{4}\right) \tag{8.80b}$$

Thus the total potential of the plate is given by:

$$\Pi = U + V_e = \frac{1}{2}\sum_m\sum_n A_{mn}^2\left[\frac{\pi^4 abD}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{\pi^2 b}{4a}p_x m^2\right] \tag{8.80}$$

The stationary potential condition, $\partial \Pi/\partial A_{mn} = 0$ gives all the possible equilibrium configurations. For non-trivial $(A_{mn} \neq 0)$ solution:

$$p_{x,\mathrm{cr}} = \frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{n^2 a}{mb}\right)^2 = \frac{\pi^2 D}{b^2}\left(\frac{m}{\mu} + n^2\frac{\mu}{m}\right)^2 \tag{8.81}$$

For smallest $p_{x,\mathrm{cr}}$, $n = 1$. Hence,

$$p_{x,\mathrm{cr}} = \frac{\pi^2 D}{b^2}\left(\frac{m}{\mu} + \frac{\mu}{m}\right)^2$$

where $a/b = \mu$, represents the aspect ratio. This equation is identical to (8.45a).

Following examples will illustrate the application of stationary potential principle to the plates with different boundary conditions at the edges. The stability conditions of these plates have already been derived in the preceding sections by differential equation method.

Example 8.1. Consider a rectangular plate clamped at all the edges and subjected to a uniformly distributed load p_x at the edges $x = 0$ and $x = a$.

Displacement function satisfying boundary conditions at the clamped edges may be taken in the form:

$$w(x, y) = A\left(1 - \cos\frac{2\pi mx}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right) \tag{8.82}$$

For evaluation of U and V following derivatives are required:

$$\frac{\partial w}{\partial x} = \left(\frac{2\pi mA}{a}\right)\sin\left(\frac{2\pi mx}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{4\pi^2 m^2 A}{a^2}\right)\cos\left(\frac{2\pi mx}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{4\pi^2 A}{b^2}\left(1 - \cos\frac{2\pi mx}{a}\right)\cos\frac{2\pi y}{b}$$

Substituting these derivatives in the expressions for U and V given by (8.24) and (8.25), respectively, and on integration the potential Π is given by:

$$\begin{aligned}
\Pi = U + V &= \left(\frac{Dm^3 b}{2a^3}\right)\left(\frac{16\pi^4 A^2}{4}\right)\left[3 + 3\left(\frac{a^4}{m^4 b^4}\right) + 2\left(\frac{a^2}{m^2 b^2}\right)\right] \\
&\quad - \left(\frac{3\pi^2 mbA^2}{2a}\right)p_x \\
&= \left(\frac{2\pi^4 mDA^2}{ab}\right)\left[3\left(\frac{mb}{a}\right)^2 + 3\left(\frac{a}{mb}\right)^2 + 2\right] \\
&\quad - \left(\frac{3\pi^2 A^2}{2}\right)\left(\frac{mb}{a}\right)p_x
\end{aligned} \tag{8.83}$$

From stationary potential principle $\partial \Pi / \partial A = 0$, the stability condition is given by:

$$p_{x,\mathrm{cr}} = \frac{4\pi^2 D}{3b^2}\left[3\left(\frac{m}{\mu}\right)^2 + 2 + 3\left(\frac{\mu}{m}\right)^2\right] \tag{8.84}$$

where $\mu = a/b$. For smallest value of $p_{x,\mathrm{cr}}$

$$\frac{\partial p_{x,\mathrm{cr}}}{\partial m} = \frac{4\pi^2 D}{3b^2}\frac{\partial}{\partial m}\left[3\left(\frac{m}{\mu}\right)^2 + 2 + 3\left(\frac{\mu}{m}\right)^2\right] = 0 \tag{8.85}$$

$$\frac{6m}{\mu^2} - \frac{6\mu^2}{m^3} = 0 \quad \text{i.e.} \quad m = \mu$$

Thus the absolute smallest value of $p_{x,\mathrm{cr}}$ is $10.6667\pi^2 D/b^2$ which is independent of μ. For a square plate $\mu = 1$ and $m = 1, k_m^2 = 10.6667$. The exact solution obtained by Levy using an infinite series for $w(x, y)$ is $10.07\pi^2 D/b^2$. It should be noted that the stationary principle yields an *upper bound solution*. Further, the displacement function assumes that at the critical load, a plate buckles with one half-wave along Y-direction.

With one term the error is of the order 6 per cent. The error can be minimized by using more terms in the representation of $w(x, y)$, e. g. with two terms

$$\begin{aligned} w(x, y) = {} & A\left(1 - \cos\frac{2\pi m x}{a}\right)\left(1 - \cos\frac{2\pi y}{b}\right) \\ & + B\left(1 - \cos\frac{4\pi m x}{a}\right)\left(1 - \cos\frac{4\pi y}{b}\right) \end{aligned} \tag{8.86}$$

The values of $p_{x,\mathrm{cr}}$ for different aspect ratios μ with one term solution are given in the Table 8.2. This example clearly demonstrates the advantage of using energy method. If the edges $x = 0$ and $x = a$ are simply supported instead of being clamped, the displacement function satisfying boundary conditions at the edges may be assumed to be

$$w(x, y) = A \sin\frac{m\pi x}{a}\left(1 - \cos\frac{2\pi y}{b}\right)$$

This displacement function is based on the observation that, in general, at the critical load a plate buckles with one-half wave along Y-direction. The derivatives of w required for computation of potential of the plate $\Pi = U + V$ are:

$$\frac{\partial^2 w}{\partial x^2} = -\left(\frac{m\pi}{a}\right)^2 A \sin\frac{m\pi x}{a}\left(1 - \cos\frac{2\pi y}{b}\right)$$

$$\frac{\partial^2 w}{\partial y^2} = A \sin\frac{m\pi x}{a}\left[\left(\frac{2\pi}{b}\right)^2 \cos\frac{2\pi y}{b}\right]$$

Table 8.2. Buckling factor, k_m^2 for various aspect ratios, μ (Stationary Potential Principle)

$\mu = \frac{a}{b}$	All edges simply supported		Loaded edges: clamped Unloaded edges: Simply supported		Loaded edges: simply supported Unloaded Edges: clamped		All edges clamped	
	m	k_m^2	m	k_m^2	m	k_m^2	m	k_m^2
0.4	1	8.4100	1	26.1200	1	9.7700	1	28.3067
0.6	1	5.1378	1	12.3811	1	7.3644	1	15.2178
0.8	1	4.2025	1	7.7300	1	7.6425	1	11.4767
1.0	1	4.0000	1	5.7500	2	8.0000	1	10.6667
1.2	1	4.1344	1	4.8578	2	7.3644	1	11.2044
1.4	1	4.4702	1	4.5108	2	7.3208	1	12.5475
1.6	2	4.2025	1	4.4825	2	7.6425	2	11.4767
1.8	2	4.0446	1	4.6646	3	7.3644	2	10.8449
2.0	2	4.0000	1	5.0000	3	7.2870	2	10.6667
2.4	2	4.1344	2	4.8578	4	7.3644	2	11.2044
2.8	3	4.2191	2	4.5108	4	7.3208	3	10.7429
3.0	3	4.0000	2	4.4652	5	7.3644	3	10.6667
4.0	4	4.0000	3	4.5833	6	7.2870	4	10.6667
∞	4.0000		4.4641		7.2855		10.6667	

Substituting these derivatives in Eqs. (8.24) and (8.25) and on integration

$$U = \frac{D}{2}\frac{\pi^4 A^2 b}{a^3}\left[\frac{3m^4}{4} + \frac{4a^4}{b^4} + \frac{2m^2a^2}{b^2}\right]$$

and

$$V = -\frac{1}{2}(A^2 m^2 \pi^2)\left(\frac{p_x b}{a}\right)\left(\frac{3}{4}\right)$$

Equating variation $\partial \Pi/\partial A$ of potential to zero and for a non-trivial ($A \neq 0$) solution:

$$\begin{aligned} p_{x,\text{cr}} &= \frac{D\pi^2}{3m^2b^2}\left[3\left(\frac{b}{a}\right)^2 m^4 + 16\left(\frac{a}{b}\right)^2 + 8m^2\right] \\ &= \frac{D\pi^2}{3b^2}\left[3\left(\frac{m}{\mu}\right)^2 + 8 + 16\left(\frac{\mu}{m}\right)^2\right] \end{aligned} \tag{8.87}$$

For a given aspect ratio μ, the value of m at which $p_{x,\text{cr}}$ will be smallest is given by:

$$\frac{\partial p_{x,\text{cr}}}{\partial m} = \left(\frac{\pi^2 D}{3b^2}\right)\frac{\partial}{\partial m}\left[3\left(\frac{m}{\mu}\right)^2 + 8 + 16\left(\frac{\mu}{m}\right)^2\right] = 0$$

$$\frac{6m}{\mu^2} - \frac{32\mu^2}{m^3} = 0 \quad \text{i.e.,} \quad m = 1.5197\mu \tag{8.88}$$

Substituting this value of m in (8.87), the absolute smallest value of $p_{x,\text{cr}}$ obtained is $7.28547\pi^2 D/b^2$. For a square plate i. e. $\mu = 1$, $m = 2$ and $k_m^2 = 8.0000$. Thus

$$p_{x,\text{cr}} = 8.00\pi^2 D/b^2 \tag{8.89}$$

This is upper bound to its exact solution $7.6913(\pi^2 D/b^2)$.

For different aspect ratios, the k_m^2 values are given with Table 8.2. Like all other plate problems analyzed by stationary principle, the values in the Table 8.2 are upper bound to their respective exact values.

In the foregoing and other plate problems with symmetrical edge support conditions, it is convenient to define the origin at the center of the plate. For example the displacement functions for the symmetrical edge support conditions shown in Fig. 8.7 are:

1. All edges simply supported

$$w(x, y) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right) \tag{8.90}$$

2. Loaded edges simply supported, unloaded edges clamped

$$w(x, y) = A\cos\left(\frac{m\pi x}{a}\right)\left(1 + \cos\frac{2\pi y}{b}\right) \tag{8.91}$$

3. All edges clamped

$$w(x, y) = A\left(1 + \cos\frac{2m\pi x}{a}\right)\left(1 + \cos\frac{2\pi y}{b}\right) \tag{8.92}$$

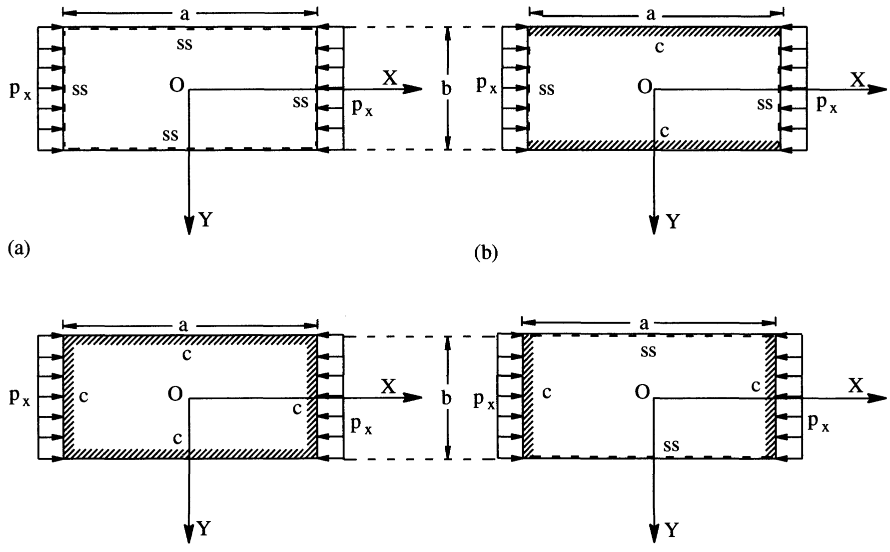


Fig. 8.7a–d. Types of symmetrical edge support conditions

4. Loaded edges clamped and unloaded edges simply supported

$$w(x, y) = A\left(1 + \cos\frac{2m\pi x}{a}\right)\left(\cos\frac{\pi y}{b}\right) \tag{8.93}$$

For illustration, consider the buckling problem of the plate clamped at the loaded edges and simply supported at the unloaded edges. The displacement function satisfying the boundary condition is given by (8.93).

Following the procedure adopted in the preceding plate buckling problems, the value of critical load is given by:

$$p_{x,\mathrm{cr}} = \left(\frac{\pi^2 D}{4b^2}\right)\left[16\left(\frac{m}{\mu}\right)^2 + 4 + 3\left(\frac{\mu}{m}\right)^2\right] = \frac{\pi^2 D}{4b^2}(k_m^2) \tag{8.94}$$

For the smallest value of $p_{x,\mathrm{cr}}$, m can be obtained from

$$\frac{\partial p_{x,\mathrm{cr}}}{\partial m} = \left(\frac{\pi^2 D}{4b^2}\right)\frac{\partial}{\partial m}\left[16\left(\frac{m}{\mu}\right)^2 + 4 + 3\left(\frac{\mu}{m}\right)^2\right]$$

$$\frac{32m}{\mu^2} - \frac{6\mu^2}{m^3} = 0 \quad \text{i.e.,} \quad m = 0.6580\mu \tag{8.95}$$

Substituting $m = 0.6580\mu$ in (8.94), the absolute minimum value of k_m^2 obtained is 4.4641. For a square plate, (i.e., $\mu = 1$) $m = 1$ gives $k_m^2 = 5.75$. This value again is upper bound to its exact value. For different aspect ratios values of k_m^2 are given in Table 8.2.

8.5 Buckling of Web Plates of Girders

Thin web plates of girders under the action of compressive stress during bending are susceptible to buckling. In general a web panel of girder is subjected to a uniformly distributed *shear forces* $\tau_{xy}t$ along all the four edges and in addition it is loaded on the edges $x = 0$ and $x = a$ by longitudinal forces $\sigma_x t$ linearly varying along these edges. In practice the shear stresses are parabolically distributed along the edges $x = 0$ and $x = a$ and also vary along $y = 0$ and $y = b$. Furthermore, σ_x also varies along the span of girder with bending moment. The stability analysis of *web plates* is considerably simplified by assuming a loading condition with average values of σ_x and τ_{xy}.

The web plates of the deep girders are in general too thin to develop a sufficiently high buckling strength for an economical design without a provision of stiffeners. Therefore, *longitudinal* and *transverse stiffeners* play an important role in the design of the web plates. In this section the stability analysis of both unstiffened and stiffened web plates under various loading and support conditions is discussed.

8.5.1 Buckling of Rectangular Plate in Shear

Consider a rectangular plate of length a, width b and thickness t, simply supported along the four edges, subjected to uniformly distributed *shear forces* $p_{xy}(\tau_{xy}t)$ along the edges as shown in Fig. 8.8. In the analysis *principle of stationary potential* is used. For this pure shear load problem, ($p_x = p_y = 0$) with all the edges assumed to be simple supports, the Ritz solution to boundary conditions i.e. displacement $w(x, y)$ can be represented by:

$$w(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij} \sin\left(\frac{i\pi x}{a}\right) \sin\left(\frac{j\pi y}{b}\right) \tag{8.96}$$

Substituting w(x, y) in the strain energy expression given by (8.24)

$$-W_{\text{in}} = U = \frac{1}{8}\pi^4 abD \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}^2 \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2 \tag{8.97a}$$

The potential energy, V of the uniformly distributed forces $p_{xy}(= \tau_{xy}t)$, is expressed from (8.25) as:

$$\begin{aligned} W_e = V &= -\int_0^a\int_0^b p_{xy} \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \mathrm{d}y\,\mathrm{d}x \\ &= -\left(\frac{\pi^2}{ab}\right)\int_0^a\int_0^b p_{xy} \\ &\times \sum_{\substack{i=1\\k=1}}^{m}\sum_{\substack{j=1\\ \ell=1}}^{n} A_{ij}A_{k\ell}\left(i\cos\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right)\left(\ell\sin\frac{k\pi x}{a}\cos\frac{\ell\pi y}{b}\right)\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

where i, k, and j, ℓ can assume values from $1, \ldots m$ and $l, \ldots n$, respectively. Noting that:

$$\begin{aligned} \int_0^a \cos\frac{i\pi x}{a}\sin\frac{k\pi x}{a}\mathrm{d}x &= 0 && \text{when } i+k \text{ is an even number} \\ &= \frac{2a}{\pi}\frac{i}{(i^2-k^2)} && \text{if } i+k \text{ is an odd number} \end{aligned}$$

Thus expression for V reduces to

$$V = -4p_{xy}\sum_i\sum_j\sum_k\sum_\ell A_{ij}A_{k\ell}\frac{ijk\ell}{(i^2-k^2)(\ell^2-j^2)} \tag{8.97b}$$

In which i, k and j, ℓ are integers such that both $(i+k)$ and $(j+\ell)$ are odd numbers. For illustration consider $m = n = 2$. Therefore i, j, k and ℓ can therefore have values

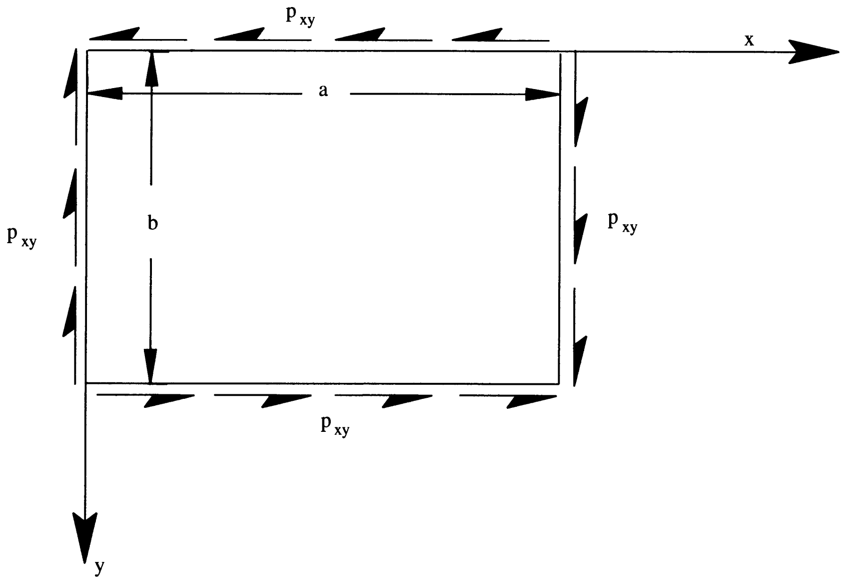


Fig. 8.8. Rectangular plate in pure shear

1 or 2 only. Equation (8.97) will thus have four terms in which combinations of values i, j, k and ℓ satisfy condition that $(i+k)$ and $(j+\ell)$ are odd, as (1,1,2,2), (1,2,2,1), (2,1,1,2) and (2,2,1,1). Therefore, potential function $\Pi(=U+V)$ is given by:

$$\Pi = \frac{\pi^4 D}{8\mu^3 b^2}\left[A_{11}^2(1+\mu^2)^2 + A_{12}^2(1+4\mu^2)^2 + A_{21}^2(4+\mu^2)^2 + A_{22}^2(4+4\mu^2)^2\right] - \frac{6}{9}p_{xy}(A_{12}A_{21} - A_{11}A_{22}) \tag{8.98}$$

where $\mu = a/b$. The application of stationary potential energy principle $\partial\Pi/\partial A_{11} = 0$, $\partial\Pi/\partial A_{12}$, $\partial\Pi/\partial A_{21}$ and $\partial\Pi/\partial A_{22}$ yields following set of four linear homogeneous algebraic equations:

$$\begin{bmatrix} \rho(1+\mu^2)^2 & 0 & 0 & p_{xy} \\ 0 & \rho(1+4\mu^2)^2 & -p_{xy} & 0 \\ 0 & -p_{xy} & \rho(4+\mu^2)^2 & 0 \\ p_{xy} & 0 & 0 & \rho(4+4\mu^2)^2 \end{bmatrix} \begin{Bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\rho = 9\pi^4 D/(128\mu^3 b^2)$. For non-trivial solution, the determinant of the coefficients A_{11}, A_{12}, A_{21} and A_{22} must vanish. The expansion of determinant leads to stability condition or characteristic equation. This equation of fourth degree in p_{xy} has four roots.

$$p_{xy,\mathrm{cr},1,2} = \pm \frac{9\pi^4 D}{128\mu^3 b^2}(1+\mu^2)(4+4\mu^2)$$

$$p_{xy,\mathrm{cr},3,4} = \pm \frac{9\pi^4 D}{128\mu^3 b^2}(1+4\mu^2)(4+\mu^2) \tag{8.99}$$

The lowest value of $p_{xy,\mathrm{cr}}$ is given by $p_{x,\mathrm{cr},1,2}$ i.e. by $p_{xy,\mathrm{cr},1}$ or $p_{xy,\mathrm{cr},2}$. This implies that instability of plate does not depend upon the sense of p_{xy}. The critical load thus is given by

$$p_{xy,\mathrm{cr}} = \frac{9\pi^4 D}{32\mu^3 b^2}(1+\mu^2)^2 = \frac{\pi^2 D}{b^2}\left[\frac{9\pi^2}{32}\frac{(1+\mu^2)^2}{\mu^3}\right] = \frac{\pi^2 D}{b^2}k_m^2 \tag{8.100}$$

where k_m^2 is the plate factor

$$k_m^2 = \frac{9\pi^2}{32}\frac{(1+\mu^2)^2}{\mu^3} \tag{8.101}$$

For a square plate $\mu = 1$, $p_{xy,\mathrm{cr}} = 11.1\pi^2 D/b^2$. This is upper bound to exact value $p_{xy,\mathrm{cr}} = 9.34\pi^2 D/b^2$. The computed value is 15.86 per cent higher than the exact value. Bleich has suggested a simple formula for design purposes

$$k^2 = 5.34 + \frac{4}{\mu^2} \quad \text{for} \quad \mu > 1 \tag{8.102}$$

For validity of this relation 'a' must always be selected as the larger of the dimensions. The values of factor k_m^2 for various aspect ratios are given in Table 8.3. This problem can also be analyzed by *Galerkin's method.* The buckled configuration satisfying boundary conditions may be taken as

$$w(x,y) = A_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + A_2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} = A_1 g_1 + A_2 g_2 \tag{8.103}$$

Table 8.3. Buckling coefficient k_m^2 for the plate under uniform shearing stress on all edges

Aspect ratio. $\mu = a/b$	Case-I: All edges clamped	Case-II : Two short edges simply supported and both long edges clamped	Case-III: All edges simply supported
1.0	14.710	12.280	9.338
1.5	11.500	11.120	7.070
2.0	10.340	10.210	6.590
2.5	9.820	9.810	6.066
3.0	9.620	9.610	5.890
∞	8.976	8.980	5.330

The Galerkin's equations are:

$$\int_0^a \int_0^b \left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{p_{xy}}{D} \frac{\partial^2 w}{\partial x \partial y} \right] g_i \, \mathrm{d}y \, \mathrm{d}x = 0 \qquad i = 1, 2 \tag{8.104}$$

The equation within brackets is the governing differential equation. Substituting $w(x, y)$ in (8.104) and simplification yields:

$$\begin{bmatrix} \frac{\pi^4}{a^2} & -\frac{32}{9}\frac{p_{xy}}{D} \\ -\frac{32}{9}\frac{p_{xy}}{D} & \frac{16\pi^4}{a^2} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.105}$$

For non-trivial solution vanishing of determinant of coefficients of A_1 and A_2 provides

$$p_{xy,\mathrm{cr}} = \frac{9\pi^4 D}{8b^2} = 11.10\frac{\pi^2 D}{b^2} \tag{8.106}$$

which is identical to the one obtained earlier by Ritz method.

For the plates clamped at all four edges, Bleich has suggested a parabolic formula for approximating k_m^2:

$$k_m^2 = 8.98 + \frac{5.60}{\mu^2} \tag{8.107}$$

The values of k_m^2 for plates with different boundary conditions for various μ are given in Table 8.3.

For the case of very long plate ($\mu = \infty$) with short edges simply supported or clamped; one long edge simply supported and other clamped, $k_m^2 = 6.628$.

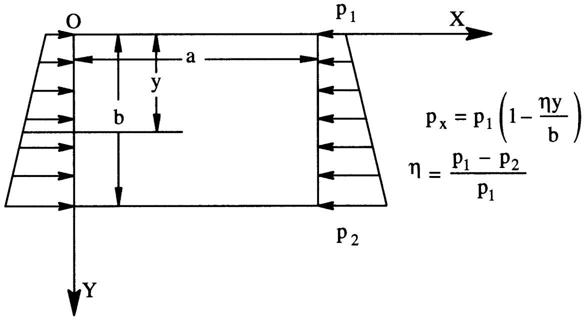
8.5.2 Buckling of Rectangular Plate due to Non-Uniform Longitudinal Stresses

Consider a simply supported rectangular plate subjected to varying in-plane axial load due to bending along two opposite edges. The magnitude of the force p_x at a distance y from the upper edge of plate as shown in Fig. 8.9 can be expressed by a linear relationship:

$$p_x = p_1\left(1 - \frac{\eta y}{b}\right) \tag{8.108}$$

where $\eta = (p_1 - p_2)/p_1$. The parameter $\eta = 0$ corresponds to uniformly distributed compressive load and $\eta = 2$ to pure bending, and $0 < \eta < 2$ indicates combined bending and compression.

As is seen earlier in the chapter that a plate in longitudinal compression buckles in X-direction in half-waves of equal length with straight nodal lines perpendicular to the X-axis. Thus each buckle represents a plate simply supported on its four edges and can as such be treated independent unit. The deflection $w(x, y)$ can therefore be assumed in the following series form:



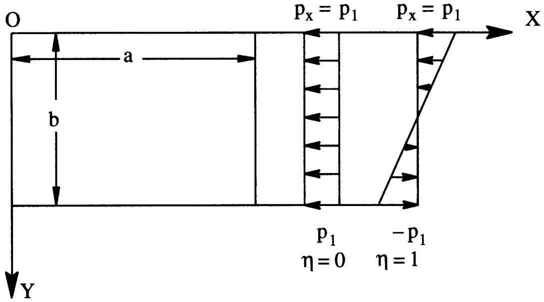


Fig. 8.9a,b. Non-uniform longitudinal stresses. **a** Linearly varying load, **b** special cases

$$w(x, y) = \sin\left(\frac{\pi x}{a}\right) \sum_{i=1}^{n} A_i \sin\left(\frac{i\pi y}{b}\right) \tag{8.109}$$

The strain energy U is obtained by substituting $w(x, y)$ from (8.109) into (8.24).

$$\begin{aligned} U &= \frac{\pi^4 Dab}{8} \sum_{i=1}^{n} A_i^2 \left(\frac{1}{a^2} + \frac{i^2}{b^2}\right)^2 \\ &= \frac{\pi^4 D}{8\mu^3 b^2} \sum_{i=1}^{n} A_i^2 \left(1 + i^2\mu^2\right)^2 \end{aligned} \tag{8.110}$$

The potential energy, V due to external forces is given by:

$$V = -\frac{p_1}{2} \int_0^a \int_0^b \left(1 - \eta\frac{y}{b}\right) \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}y\,\mathrm{d}x \tag{8.111a}$$

Noting that:

$$\left(\frac{\partial w}{\partial x}\right)^2 = \left(\frac{\pi^2}{a^2}\right) \cos^2\frac{\pi x}{a} \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j \sin\frac{i\pi y}{b} \sin\frac{j\pi y}{b}$$

Substituting this expression into (8.111a):

$$V = -\frac{p_1}{2}\frac{\pi^2}{a^2}\int_0^a\int_0^b\left(1-\eta\frac{y}{b}\right)\cos^2\frac{\pi x}{a}\sum_{i=1}^{n}\sum_{j=1}^{n}A_iA_j\sin\frac{i\pi y}{b}\sin\frac{j\pi y}{b}\mathrm{d}y\,\mathrm{d}x \tag{8.111b}$$

Making use of following integration formula

$$\int_0^b y\sin\frac{i\pi y}{b}\sin\frac{j\pi y}{b}\mathrm{d}y = \begin{cases} b^2/4 & \text{for } i=j \\ 0 & \text{for } i+j \text{ is an even number.} \\ -\left(\frac{4b^2}{\pi^2}\right)\frac{ij}{(i^2-j^2)^2} & \text{for } i+j \text{ is an odd number.}\end{cases}$$

Therefore,

$$V = -\frac{\pi^2 p_1}{8\mu}\sum_{i=1}^{n}A_i^2 + \frac{\eta\pi^2 p_1}{16\mu}\sum_{i=1}^{n}A_i^2 - \frac{2\eta p_1}{\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{ijA_iA_j}{(i^2-j^2)^2}\right] \tag{8.111}$$

where i assumes all values 1 to n, while j can have such values for which $(i+j)$ is an odd number.

The total potential $\Pi = U + V$ can then by obtained from (8.110) and (8.111). The stationary potential principle $\partial(U+V)/\partial A_i = 0$, $(i = 1, 2, \ldots, n)$ yields a system of n simultaneous homogenous algebraic equations:

$$\left[\left(1+i^2\mu^2\right)^2 - \frac{p_1 b^2\mu^2}{\pi^2 D}\left(1-\frac{\eta}{2}\right)\right]A_i - \frac{8\eta p_1 b^2\mu^2}{\pi^4 D}\sum_{j=1}^{n}\frac{ijA_j}{(i^2-j^2)^2} = 0$$

$$i = 1, 2, \ldots n \tag{8.112}$$

The summation $\sum_j$ is to extend only on those numbers j which satisfy the condition that $i+j$ is an odd number. A non-trivial solution exists only if the determinant of coefficients of A_i vanishes. The stability condition $\Delta = 0$, then can be used to compute the value of the buckling factor k_m^2. Timoshenko and Gere have computed k^2 values for various η and μ values.

For illustration consider $n = 2$, i.e. a two term solution:

$$\left[\left(1+\mu^2\right)^2 - \frac{p_1 b^2\mu^2}{\pi^2 D}\left(1-\frac{\eta}{2}\right)\right]A_1 - \frac{16\eta p_1 b^2\mu^2}{9\pi^4 D}A_2 = 0 \tag{a}$$

$$-\frac{16\eta p_1 b^2\mu^2}{9\pi^4 D}A_1 + \left[\left(1+4\mu^2\right)^2 - \frac{p_1 b^2\mu^2}{\pi^2 D}\left(1-\frac{\eta}{2}\right)\right]A_2 = 0 \tag{b}$$

For a square plate ($\mu = 1$) with $\eta = 2$ i.e. for pure bending case, Eqs. (a) and (b) reduce to:

$$4A_1 - \frac{32p_1b^2}{9\pi^4 D}A_2 = 0$$

$$-\frac{32p_1b^2}{9\pi^4 D}A_1 + 25A_2 = 0$$

Table 8.4. Buckling coefficient k_m^2 for the rectangular plate with all edges simply supported under linearly varying stress on two opposite edges. (Combined bending and compression)

Aspect ratio.	Coefficient, k_m^2					
$\mu = a/b$	$\eta = 2$ (pure bending)	$\eta = 4/3$	$\eta = 1.00$ (triangular load)	$\eta = 4/5$	$\eta = 2/3$	$\eta = 0$ (pure compression)
0.4	29.055	18.721	15.081	13.310	10.820	8.409
0.6	24.076	12.890	9.725	8.265	6.439	5.167
0.8	24.396	11.186	8.143	6.926	5.953	4.193
1.0	25.603	11.064	7.778	6.561	5.831	4.005
1.5	24.076	11.551	8.387	7.048	6.074	4.337

For a non-trivial ($A_1 = A_2 \neq 0$) solution

$$\left(\frac{32 p_1 b^2}{9\pi^4 D}\right)^2 - 100 = 0 \quad \text{i.e.} \quad p_{x,\text{cr}} = \frac{45}{16}\frac{\pi^4 D}{b^2} = 27.758\frac{\pi^2 D}{b^2} \tag{8.113}$$

This value of $k_m^2 = 27.758$ is upper bound to its exact value 25.6 and differ by 8.4 per cent. Timoshenko and Gere have shown that exact value is obtained by taking $n = 4$ in the series representation for the displacement $w(x, y)$.

It is seen that minimum value of k_m^2 occurs at $\mu = 2/3$ for $\eta = 2$ and at $\mu = 1$ when $\eta = 0$. A very long plate, therefore buckles in half-waves of length $\lambda = 2b/3$ in case of pure bending. The *wavelength* increases as η decreases and approaches limiting value $\lambda = b$ in case of uniform compressive load. The values of buckling factor k_m^2 for various values of aspect ratio μ for different values of η are given in the Table 8.4. For the pure bending case ($\eta = 2$), third approximation i. e. $n = 3$ is used while for others second approximation i. e. $n = 2$ is used for calculating $p_{x,\text{cr}}$.

8.5.3 Buckling of Stiffened Plates

There are numerous engineering applications where the thin plate elements are stiffened by means of *stiffeners* or *stringers* to prevent buckling at lower loads. This increase in critical load is due to the increase in flexural rigidity of the plates. For a rectangular plate of specified aspect ratio, the critical flexural stress is proportional to t^2/b^2. Thus the stability of such a plate can be improved either by increasing t or by decreasing b. It is economical to achieve this objective by introducing stiffeners in the longitudinal direction, thereby decreasing b. The stiffeners placed transversely are not very effective in increasing the flexural buckling strength unless they are closely spaced.

On the other hand critical stress in *shear* depends upon the ratio of width or smaller dimension of the plate (say b) to its thickness, t. The provision of transverse stiffener considerably reduces the width to thickness ratio, and the critical stress being inversely proportional to the square of this ratio, is substantially increased. Sufficiently rigid *transverse stiffeners* divide the plate into smaller panels which may be considered approximately as simply supported. There exists a limiting value I_0 of moment of inertia of stiffener which ensures straight nodal lines at these stiffeners. If I is smaller than I_o, the stiffener buckle and deflect together with the plate. With increasing flexural rigidity the buckling strength of stiffened plate increases until when $I = I_0$. A further increase of I does not add to the buckling strength of the plate. When reinforced by stiffeners having moment of inertia I_0, each plate panel can be considered as simply supported plate in shear and critical stress reaches the maximum possible value.

1. Longitudinally stiffened plates

In deep *plate girders* it is often economical to stiffen the *web plate* by longitudinal stiffeners at the locations where the longitudinal compressive stresses due to bending are high. If the stiffener is located at the longitudinal center line of web i. e. at neutral axis, it does not carry compressive force. Its effect is negligible for small aspect ratios μ $(= a/b)$, but becomes marked when $\mu = 2/3$. The increase of buckling strength amounts to only 50 per cent of the strength of the unstiffened plate. The stiffeners at the center line are therefore not very effective in improving the stability of web plates in case of pure bending.

A larger effect is obtained when a stiffener is placed between compression flange and the center line. The problem can be analyzed easily by stationary potential principle by extending the analysis for unstiffened plate discussed in the preceding section by including the bending energy U_s of the stiffener in the expression for the potential energy of the system.

Consider a simply supported rectangular plate of uniform thickness t, length a and width b stiffened by a typical ℓ^{th} *longitudinal stiffener* of cross-sectional area A_s and moment of inertia I_s, located at a distance y_ℓ from the edge $y = 0$. The buckled configuration satisfying all boundary condition may be represented by a double Fourier sine series of the form:

$$w(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{8.114}$$

The strain energies of the plate and the ℓ^{th} stiffener, U_p and $U_{s\ell}$, respectively, are given by

$$U_p = \frac{\pi^4 D}{8\mu a^2} \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 \left(i^2 + \mu^2 j^2\right)^2 \tag{a}$$

$$\begin{aligned} U_{sl} &= \frac{(E_s I_s)_\ell}{2} \int_0^a \left(\frac{\partial^2 w}{\partial x^2}\right)_\ell^2 \mathrm{d}x \\ &= \frac{(E_s I_s)_\ell}{4a^3} \pi^4 \sum_{i=1}^{m} i^4 \left(A_{i1} \sin \frac{\pi y_1}{b} + A_{i2} \sin \frac{2\pi y_1}{b} + ...\right)^2 \end{aligned} \tag{b}$$

The potential energies due to compressive force p_x acting on the plate and $p_{s\ell}$ on ℓ^{th} stringer are:

$$V_p = -\frac{p_x}{2}\left(\frac{ab}{4}\right)\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{i^2\pi^2}{a^2}A_{ij}^2 = -\frac{\pi^2}{8\mu}\sum_{j=1}^{n}\sum_{i=1}^{m}p_x A_{ij}^2 i^2$$

$$V_{sl} = -\frac{p_{s\ell}}{2}\int_0^a\left(\frac{\partial w}{\partial x}\right)_\ell^2 \mathrm{d}x$$

$$= -\frac{p_{s\ell}}{2}\frac{\pi^2}{a^2}\left(\frac{a}{2}\right)\sum_{i=1}^{m} i^2\left(A_{i1}\sin\frac{\pi y_1}{b} + A_{i2}\sin\frac{2\pi y_1}{b} + \ldots\right)^2$$

Defining following parameters

$$\gamma_\ell = \frac{(E_s I_s)_\ell}{Db} = \frac{12(1-\nu^2)\,(E_s I_s)_\ell}{t^3 b}$$
$$\delta_\ell = \frac{A_s}{bt} = \frac{p_{s\ell}}{bp_x} \tag{8.115}$$

Summing up over all stringers r in number. The stationary potential condition $\partial(U_p + U_{s\ell} + V_p + V_{s\ell})/\partial A_{in} = 0$ $(i = 1, 2, \ldots m)$ forms a system of m homogenous linear algebraic equations

$$\frac{\pi^2 D}{\mu^2 b^2}\left[A_{ij}(i^2 + \mu^2 j^2)^2 + 2\sum_{\ell=1}^{r}\gamma_\ell \sin\frac{j\pi y_\ell}{b}.i^4\sum_{k=1}^{n}A_{mk}\sin\frac{k\pi y_\ell}{b}\right]$$
$$-\left[p_{x,cr}\left(i^2 A_{ij} + 2\sum_{\ell=1}^{r}\delta_\ell \sin\frac{j\pi y_\ell}{b}.i^2\sum_{k=1}^{n}A_{mk}\sin\frac{k\pi y_\ell}{b}\right)\right] = 0 \tag{8.116}$$
$$(i = 1, 2, 3\ldots).$$

Equating to zero the determinant of system of equations provides characteristic equation. As discussed earlier minimum buckling load of plate is given by one-half wave along X-direction i.e., $i = m = 1$. On the other hand the values of n represent the number of half-waves along y-direction.

As a typical case consider a plate stiffened by one stiffener ($r = 1$) located at $y = b/2$, i.e., at the centre line. In this case for even values of $n(= 2, 4, 6, \ldots)$ i.e. antisymmetric configuration, the stiffener does not contribute to the plate buckling load since for such values of n, the location of stiffener corresponds to the nodal line, and it remains straight. Each half of the plate behaves as a plate of length a and width $b/2$, simply supported on all four edges. No bending moment is carried over from one half to the other half due to the presence of inflection point at the *nodal line*. The buckling load of stiffener in this case reaches its maximum value. The critical stress for antisymmetric buckling is independent of γ, but it is the critical stress of simply supported plate of width $b/2$. In the symmetric mode, i.e. for odd values of n, stiffeners deflects with the plate. For first approximation consider $j = n = 1$.

$$\frac{\pi^2 D}{\mu^2 b^2}\left[(1+\mu^2)^2+2\gamma_1\right]A_{11}-\left[p_{x,\mathrm{cr}}(1+2\delta_1)\right]A_{11}=0$$

Therefore,

$$k_m^2=\frac{\left[\left(1+\mu^2\right)^2+2\gamma_1\right]}{\left[\mu^2(1+2\delta_1)\right]}$$

where,

$$p_{x,\mathrm{cr}}=k_m^2\frac{\pi^2 D}{b^2} \tag{8.117}$$

For better approximation consider $j=n=1,3$ and following equations are obtained:

$$\left[(1+\mu^2)^2+2\gamma_1-(1+2\delta_1)\,(\mathrm{k}\mu)^2\right]A_{11}-\left[2\gamma_1-(2\delta_1)\,(\mathrm{k}\mu)^2\right]A_{13}=0 \qquad \text{(a)}$$

$$-\left[2\gamma_1-(2\delta_1)(k\mu)^2\right]A_{11}+\left[\left(1+9\mu^2\right)^2+2\gamma_1-(1+2\delta_1)(k\mu)^2\right]A_{13}=0 \qquad \text{(b)}$$

For definite values of γ_1, δ_1 and μ, the determinant of coefficients of A_{11} and A_{13} must vanish for non-trivial solution. The resulting characteristic equation will enable the computation of k_m^2. The procedure is quite general and is applicable to any number of stiffeners.

2. Transversely stiffened plates

As has been discussed earlier, the provision of transverse stiffeners, subdivide the plate into smaller panels and there exists an optimum value I_0 of moment of inertia of stiffeners which ensure straight nodal lines at these stiffeners, and panels may be considered approximately as simply supported in shear. Introducing ratio b/d, the maximum value of critical stress in the elastic range can be determined from *pure shear plate problems*. For $\mu=d/b\leq 1$,

$$\tau_{\mathrm{c}}=\frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 k_m^2$$

where

$$k_m^2=4.00+\frac{5.34}{(d/b)^3} \tag{8.118}$$

3. Both longitudinal and transverse stiffeners

If the stiffeners are closely spaced, this type of arrangement can be conveniently handled by *orthotropic plate theory*. For illustration consider a simply supported rectangular stiffened plate of size $a\times b$. The plate is stiffened with both longitudinal and transverse stiffeners. The stiffeners are of equal stiffness and are closely spaced.

Such a stiffened plate can be considered to be an orthotropic plate having two different flexural rigidities in two perpendicular directions. The moment curvature relations for such a plate are given by:

$$M_x = -\frac{EI_x}{1-\nu_x\nu_y}\left(\frac{\partial^2 w}{\partial x^2} + \nu_y\frac{\partial^2 w}{\partial y^2}\right) \tag{8.119a}$$

$$M_y = -\frac{EI_y}{1-\nu_x\nu_y}\left(\frac{\partial^2 w}{\partial y^2} + \nu_x\frac{\partial^2 w}{\partial x^2}\right) \tag{8.119b}$$

$$M_{xy} = 2GI_{xy}\left(\frac{\partial^2 w}{\partial x \partial y}\right) \tag{8.119c}$$

Substituting the expressions for M_x, M_y and M_{xy} given by (8.119) in the governing differential (8.11):

$$D_x\frac{\partial^2 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = p_x\frac{\partial^2 w}{\partial x^2} + p_y\frac{\partial^2 w}{\partial y^2} + 2p_{xy}\frac{\partial^2 w}{\partial x \partial y} \tag{8.120}$$

where,

$$D_x = EI_x/(1-\nu_x\nu_y)\ , \quad D_y = EI_y/(1-\nu_x\nu_y)\ , \quad D_{xy} = 2GI_{xy}\ ,$$
$$G = E/[2(1+\sqrt{\nu_x\nu_y})]\ , \quad D_1 = [E(\sqrt{\nu_x\nu_y})I_x]/(1-\nu_x\nu_y)$$
$$\text{and} \quad H = D_1 + 2D_{xy}\ .$$

If the plate is subjected to a uniformly distributed in-plane compressive load p_x along the edges $x = 0$ and $x = a$, the governing (8.120) reduces to:

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} + p_x\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.121a}$$

The procedure for obtaining the critical value of p_x is exactly similar to one described in the preceding sections for the isotropic plates. Considering the buckled configuration as

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\alpha_m x\ \sin\beta_n y \tag{8.121b}$$

where

$$\alpha_m = m\pi/a \qquad \text{and} \quad \beta_n = n\pi/b \tag{8.121c}$$

Substituting (8.121b) in the governing equation (8.121a)

$$D_x\alpha_m^4 + 2H\alpha_m^2\beta_n^2 + D_y\beta_m^4 - \alpha_m^2 p_x = 0$$

or

$$p_x = \frac{1}{\alpha_m^2}\left[D_x\alpha_m^4 + 2H\alpha_m^2\beta_n^2 + D_y\beta_m^4\right]$$

Introducing (8.121c)

$$p_x = \frac{\pi^2 a^2}{m^2}\left[\frac{m^4}{a^4}D_x + \frac{2m^2n^2}{a^2b^2}H + \frac{n^4}{b^4}D_y\right]$$

$$= \frac{\pi^2 a^2}{m^2} \times \frac{m^2}{a^2b^2}\left[\frac{m^2b^2}{a^2}D_x + 2n^2H + \frac{a^2n^4}{b^2m^2}D_y\right]$$

$$= \frac{\pi^2}{\mathrm{b}^2}\left[\left(\frac{m}{\mu}\right)^2 D_x + 2n^2H + \left(\frac{\mu}{m}\right)^2 n^4 D_y\right] \tag{8.121d}$$

where μ is aspect ratio. Multiplying both the numerator and denominator by $[D_xD_y]^{1/2}$

$$p_x = \frac{\pi^2\sqrt{D_xD_y}}{b^2}\left[\left(\frac{m}{\mu}\right)^2\sqrt{\frac{D_x}{D_y}} + 2n^2\frac{H}{\sqrt{D_xD_y}} + n^4\left(\frac{\mu}{m}\right)^2\sqrt{\frac{D_y}{D_x}}\right] \tag{8.121e}$$

It is obvious that for the value of $n = 1$, p_x will assume the minimum value, i.e., the plate will buckle along Y-direction as single half-wave. This makes (8.121d) and (8.121e) to assume the form:

$$p_x = \frac{\pi^2\sqrt{D_xD_y}}{b^2}\left[\left(\frac{m}{\mu}\right)^2\sqrt{\frac{D_x}{D_y}} + 2\frac{H}{\sqrt{D_xD_y}} + \left(\frac{\mu}{m}\right)^2\sqrt{\frac{D_y}{D_x}}\right] \tag{a}$$

$$= \left(\frac{\pi^2}{b^2}\right)\left[\left(\frac{m}{\mu}\right)^2 D_x + 2H + \left(\frac{\mu}{m}\right)^2 D_y\right] \tag{b}$$

$$= k_m^2\left(\frac{\pi^2}{b^2}\right) \tag{8.121f}$$

where

$$k_m^2 = \left[\left(\frac{m}{\mu}\right)^2 D_x + 2H + \left(\frac{\mu}{m}\right)^2 D_y\right]$$

The value of m i.e. the number of half-waves along X-direction can be obtained by minimising p_x with respect to m. For the case when the aspect ratio μ is an integer number

$$\frac{\partial p_x}{\partial m} = \frac{\partial k_m}{\partial m} = 0 \qquad \text{gives:} \quad \frac{2\mathrm{m}}{\mu^2}(D_x) = \frac{2\mu^2}{m^3}(D_y)$$

or

$$\mu^4 = m^4\frac{D_x}{D_y}$$

or

$$\mu_0 = m\left(\frac{D_x}{D_y}\right)^{1/4} \tag{8.121g}$$

with this μ_o, the critical compressive load $p_{x,\mathrm{cr}}$ can be obtained from (8.121e) as:

$$p_{x,\mathrm{cr}} = \frac{\pi^2\sqrt{D_x D_y}}{b^2}\left[\sqrt{\frac{D_y}{D_x}}\cdot\sqrt{\frac{D_x}{D_y}} + \frac{2H}{\sqrt{D_x D_y}} + \sqrt{\frac{D_x}{D_y}}\sqrt{\frac{D_y}{D_x}}\right]$$

or

$$p_{x,\mathrm{cr}} = \frac{2\pi^2\sqrt{D_x D_y}}{b^2}\left[1 + \frac{H}{\sqrt{D_x D_y}}\right] = \frac{2\pi^2}{b^2}\left(\sqrt{D_x D_y} + H\right) \tag{8.121h}$$

For an isotropic plate, $D_x = D_y = H = D$ and (8.121h) reduces to

$$p_{x,\mathrm{cr}} = \frac{4\pi^2 D}{b^2}$$

The limiting value of aspect ratio $\overline{\mu}$ at which either m or $m+1$ half-waves can exist is obtained as explained earlier

$$D_x\left(\frac{m}{\mu}\right)^2 + 2H + D_y\left(\frac{\mu}{m}\right)^2 = D_x\left(\frac{m+1}{\mu}\right)^2 + 2H + D_y\left(\frac{\mu}{m+1}\right)^2$$

i.e.

$$\mu^4 = m^2(m+1)^2\left(\frac{D_x}{D_y}\right)$$

or

$$\overline{\mu} = [m(m+1)]^{1/2}\left(\frac{D_x}{D_y}\right)^{1/4} \tag{8.121i}$$

The expression is similar to (8.51). The number m corresponding to a given value of the aspect ratio μ of the plate can be obtained on identical lines as done for isotropic case. For

$$\begin{aligned} 0 < \mu^2 < 2\left(\frac{D_x}{D_y}\right)^{1/2}, &\quad m = 1 \\ 2\left(\frac{D_x}{D_y}\right)^{1/2} < \mu^2 < 6\left(\frac{D_x}{D_y}\right)^{1/2}, &\quad m = 2 \\ 6\left(\frac{D_x}{D_y}\right)^{1/2} < \mu^2 < 12\left(\frac{D_x}{D_y}\right)^{1/2}, &\quad m = 3 \end{aligned} \tag{8.121j}$$

and so on. Alternatively, using trial and modification procedure the value of m is selected for a given aspect ratio μ for the minimum value of p_x. Following example will illustrate the above procedure.

Example 8.2. A rectangular flat plate (slab) of size $2400 \times 1000 \times 20$ mm thickness is stiffened in the X-direction by ribs of size 6×28 mm at 50 mm c/c and in Y-direction by ribs of size 7.5×26 mm at 80 mm c/c. The slab which is simply supported at all the edges is subjected to a uniformly distributed in-plane compressive load p_x in X-direction (i.e. at 1000 mm wide edges). Determine the critical value of p_x at which the plate will buckle, if both the plate and the ribs are made of same material having: $E = 2 \times 10^5$ MPa and $\nu = 0.3$.

The various parameters are: aspect ratio: $\mu = a/b = 2.4$ and hence $\mu^2 = 5.76$, stiffness coefficients:

$$D = \frac{(2 \times 10^5) \times 20^3}{12\left[1 - (0.3)^2\right]} = 1.4652 \times 10^8 \,\text{N mm}$$

$$D_x = \left[1.4652 \times 10^8 + \frac{2 \times 10^5}{50}\left(\frac{1}{12} \times 6 \times 28^3\right)\right] = 1.90424 \times 10^8 \,\text{N mm}$$

$$D_y = \left[1.4652 \times 10^8 + \frac{2 \times 10^5}{80}\left(\frac{1}{12} \times 7.5 \times 26^3\right)\right] = 1.73983 \times 10^8 \,\text{N mm}$$

Since the plate (presumed isotropic) is cross stiffened by two sets of equidistant stiffeners

$$H = D = 1.4652 \times 10^8 \,\text{N mm}$$

Therefore, $\sqrt{D_x/D_y} = 1.0462$ and $\sqrt{D_x D_y} = 1.8202 \times 10^8 \,\text{N mm}$. Since, $6\sqrt{D_x/D_y} (= 6.2772) > \mu^2$, the number of half-waves in X-direction $m = 2$. From first form of (8.121fa)

$$k_m^2 = \frac{4 \times 1.0462}{5.76} + \frac{2 \times 1.4652 \times 10^8}{1.8202 \times 10^8} + \frac{5.76}{4 \times 1.0462} = 3.7129$$

Hence, the critical load is:

$$p_{x,\text{cr}} = k_m^2 \frac{\pi^2\sqrt{D_x D_y}}{b^2} = \frac{3.7129\pi^2 \times 1.8202 \times 10^8}{(1000)^2} = 6.67 \times 10^3 \,\text{N/mm}$$

Alternatively, (8.121f) can be used to obtain minimum value of k_m^2 for the given aspect ratio by trial and modification

$$\begin{aligned} k_m^2 &= \frac{1.0462}{5.76} m^2 + \frac{5.76}{1.0462 m^2} + \frac{2 \times 1.4652}{1.8202} \\ &= 7.2972 \quad \text{for} \quad m = 1 \\ &= 3.7129 \quad \text{for} \quad m = 2 \text{ (minimum)} \\ &= 3.8564 \quad \text{for} \quad m = 3 \end{aligned}$$

Thus, k_m^2 becomes minimum for $m = 2$. This value is identical to one arrived at earlier.

The *orthotropic plate* which is subjected to uniformly distributed compressive load p_x on two simply supported edges 'b' parallel to Y-axis with the unloaded edges 'a' parallel to X-axis being supported in different ways, can be handled conveniently by selecting buckled configuration of the form given by (8.31) as: $w(x, y) = f(y) \sin \alpha_m x$ where $\alpha_m = m\pi/a$. Substitution of this equation into the governing differential equation (8.121a) provides an ordinary differential equation of fourth order.

$$D_y f''''(y) - 2\alpha_m^2 H f''(y) + \left[D_x\alpha_m^4 - p_{x,\text{cr}}\alpha_m^2\right] f(y) = 0 \tag{8.122a}$$

The general solution to the differential equation (8.122a) is

$$f(y) = A\sinh\alpha y + B\cosh\alpha y + C\sin\beta y + D\cos\beta y \tag{8.122b}$$

where α and β are defined by

$$\alpha^2, \beta^2 = \left[\left(\frac{H}{D_y}\right)^2\alpha_m^4 - \left(\frac{D_x}{D_y}\right)\alpha_m^4 + \left(\frac{p_{x,\text{cr}}}{D_y}\right)\alpha_m^2\right]^{1/2} \pm \left(\frac{H}{D_y}\right)\alpha_m^2 = 0 \tag{8.122c}$$

where $\alpha_m = (m\pi/a)$. For an isotropic plate $D_x = D_y = H = D$ and (8.122c) reduces to

$$\alpha^2, \beta^2 = \alpha_m\left(\sqrt{p_{x,\text{cr}}/D}\right) \pm \alpha_m^2$$

since $p_{x,\text{cr}}/D = k^2\alpha_m^2$ from (8.28):

$$\alpha^2, \beta^2 = \left(\frac{m\pi}{a}\right)^2 (k \pm 1) \tag{8.122d}$$

These values are identical to those given by (8.35a). The arbitrary constants A, B, C and D of (8.122b) can be evaluated from the prescribed boundary conditions at the edges 'a' i.e. at $y = 0$ and $y = b$. Satisfying two boundary conditions at each edge results in a set of four simultaneous equations. Vanishing the determinant of coefficients of constants A, B, C and D yields a transcendental equation which will finally lead to the critical load $p_{x,\text{cr}}$ for the orthotropic plate as in the case of isotropic plates.

For the orthotropic plates subjected to in-plane compressive loads p_x and p_y in the X- and Y-directions, respectively, the governing differential equation is given by:

$$D_x\frac{\partial^2 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} + p_x\left(\frac{\partial^2 w}{\partial x^2} + r\frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{8.123a}$$

where $r = p_y/p_x$. For illustration consider rectangular orthotropic plate simply supported along its four edges with assumed buckling configuration as

$$w(x, y) = A_{mn}\sin\alpha_m x\,\sin\beta_n y \tag{8.123b}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

Introducing (8.123b), (8.123a) reduces to

$$D_x\alpha_m^4 + 2H\alpha_m^2\beta_n^2 + D_y\beta_n^4 - p_x\left(\alpha_m^2 + r\beta_n^2\right) = 0$$

Substituting for α_m and β_n

$$p_x\left[\left(\frac{m}{a}\right)^2+r\left(\frac{n}{b}\right)^2\right]=\pi^2\left[D_x\left(\frac{m}{a}\right)^4+2H\left(\frac{mn}{ab}\right)^2+D_y\left(\frac{n}{b}\right)^4\right]$$

$$p_x\left(\frac{m}{a}\right)^2\left[1+rn^2\left(\frac{\mu}{m}\right)^2\right]=\left(\frac{\pi^2}{b^2}\right)\left(\frac{m^2}{a^2}\right)\left[\left(\frac{m^2}{\mu^2}\right)D_x+2Hn^2+n^4\left(\frac{\mu^2}{m^2}\right)D_y\right]$$

or

$$p_x\left[1+rn^2\left(\frac{\mu}{m}\right)^2\right]=\left(\frac{\pi^2}{b^2}\right)\sqrt{D_xD_y}$$

$$\times\left[\left(\frac{m}{\mu}\right)^2\sqrt{\frac{D_x}{D_y}}+\frac{2Hn^2}{\sqrt{D_xD_y}}+\left(\frac{\mu}{m}\right)^2n^4\sqrt{\frac{D_y}{D_x}}\right]$$

where $\mu = a/b$. For minimum critical load $n = 1$. Thus,

$$p_{x,\text{cr}}=\frac{\pi^2\sqrt{D_xD_y}}{b^2}\left[\frac{(m/\mu)^2\sqrt{D_x/D_y}+2H/\sqrt{D_xD_y}+(\mu/m)^2\sqrt{D_y/D_x}}{1+r(\mu/m)^2}\right] \tag{8.124}$$

For an isotropic plate $D_x = D_y = H = D$, (8.124) becomes

$$p_{x,\text{cr}}=\left(\frac{\pi^2D}{b^2}\right)\left[\left(\frac{m}{\mu}\right)+\left(\frac{\mu}{m}\right)\right]^2\Big/\left[1+r\left(\frac{\mu}{m}\right)^2\right] \tag{8.125}$$

This expression is identical to one given by (8.75) for $n = 1$.

8.6 Strength of Thin Plates in Compression

During the experimental determination of ultimate strength of plates, it had been observed that the flat plates do not fail when the computed *initial-buckling load* is reached. The stress across the width of the plate is nearly constant prior to buckling. The *post-buckling* loads are very much higher than the initial-buckling load and the stress distribution is no longer constant across the width. The end strips of the plate carry higher stresses than the middle portion as shown in Fig. 8.10, i.e. the side portions are stiffer than the middle portion and are capable of resisting additional stresses. However, the change in the stress in the middle portion before and after buckling is almost negligible. The *tensile stresses* developed in the transverse direction at *post-buckling* stage are responsible for the *post-buckling strength.* If the unloaded edges are not supported, then the post-buckling load is much smaller than that for plate with supported edges. Thus at the time of collapse the two strips adjacent to the supported edges are stressed to the yield point and carry total load where as the heavily distorted middle portion is considered to be (relatively) unstressed.

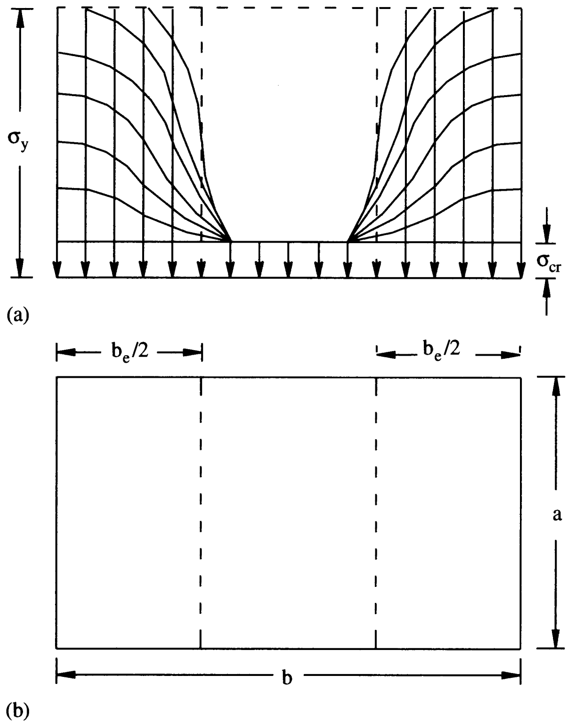


Fig. 8.10a,b. Schematic variation of stress across the width. **a** Stress distribution, **b** effective width

The *total effective width*, b_e of the load carrying strips as given by Karman is:

$$b_e = \frac{2\pi t}{\sqrt{12(1-\nu^2)}}\sqrt{\frac{E}{\sigma_y}} = Ct\sqrt{\frac{E}{\sigma_y}} \qquad (8.126)$$

where $C = 1.90$ (for $\nu = 0.3$) and σ_y is the yield strength. Hence the *ultimate strength* is given by:

$$\sigma_{ult} = \left(\frac{b_e}{b}\right)\sigma_y = C\left(\frac{t}{b}\right)\sqrt{E\sigma_y} \qquad (8.127)$$

However, later on it was noticed that coefficient C is not constant but is a function of a non-dimensional parameter $(t/b)\sqrt{E/\sigma_y}$ and decreases with increase in this parameter. The coefficient C approaches 1.90 for wide thin plates ($b/t > 100$) and the ultimate load is nearly independent of the width of the plate. For a stress levels' lower than the yield point stress σ_y the effective width can be expressed as:

$$b_e = Ct\sqrt{E/\sigma_y} \qquad (8.128)$$

where the value of coefficient C is given by:

$$C = 1.90\left[1 - 0.475\left(\frac{t}{b}\right)\sqrt{\frac{E}{\sigma_y}}\right] \tag{8.129}$$

For a plate supported on three edges and free on the fourth, $C = 0.60$ and the effective width, b_e can be approximated from the relation

$$b_e = 0.60t\sqrt{E/\sigma_y} \tag{8.130}$$

Example 8.3. An aluminum sheet panel is stiffened by two longitudinal stringers as shown in the Fig. 8.11. The panel is simply supported along the loaded edges and free at the side edges. Determine the compressive load carrying capacity of the panel for the following stipulations:

- plate size: $a = 300\,\text{mm}$, $b = 210\,\text{mm}$
- area of each stringer $= 80.00\,\text{mm}^2$
- thickness of sheet, $t = 1.50\,\text{mm}$

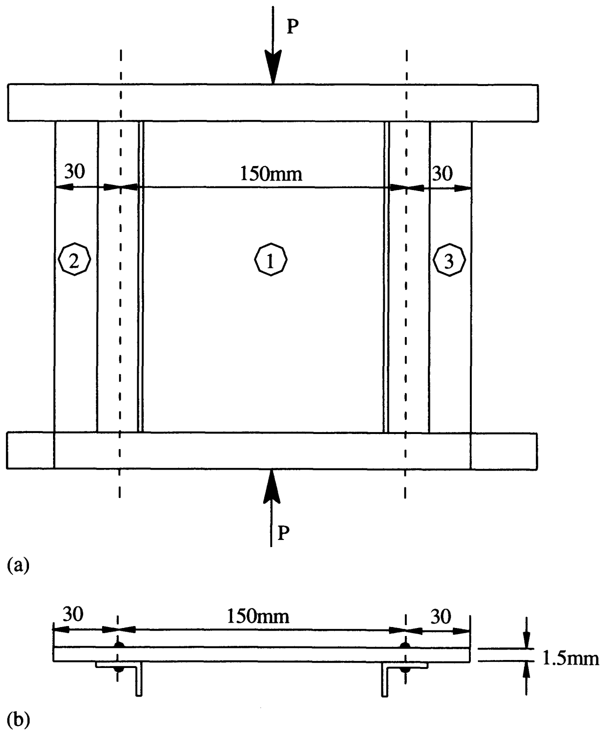


Fig. 8.11a,b. Stiffened-sheet panel. **a** Front elevation of sheet panel, **b** cross section of sheet panel

- elastic modulus, $E = 58.80\,\text{GPa}$
- yield strength of material, $\sigma_y = 58.84\,\text{MPa}$, and
- poisson's ratio, $\nu = 0.30$

The central panel 1 may be considered as simply supported at the rivet lines, i. e.; this panel is simply supported on all sides and the side panels 2 and 3 are simply supported on three sides and free at the fourth.

I. Initial critical stress:

Panel #1 ($\mu = 2.0$)

$$\sigma_{1,\text{cr}} = \frac{k^2\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 = \frac{4 \times \pi^2 \times 58.80 \times 10^3}{12(1-0.3^2)}\left(\frac{1.50}{150}\right)^2 = 21.257\,\text{MPa}$$

Panels #2 and 3 ($\mu = 10.0$)

$$\sigma_{2,\text{cr}} = \sigma_{3,\text{cr}} = \frac{0.425 \times \pi^2 \times 58.80 \times 10^3}{12(1-0.09)}\left(\frac{1.50}{30}\right)^2 = 56.436\,\text{MPa}$$

Thus the sheet between the stringers is first to buckle. Just prior to initial buckling the entire width of the plate is effective in supporting the load. Therefore, the load carried by entire cross-section is:

$$P = \sigma_{\text{cr}} A = 21.257 \times [(30 + 150 + 30) \times 1.5 + (2 \times 80)] = 10097.07\,\text{N}.$$

II. Post-buckling stage

Panel #1

$$C = 1.90 \times \left[1 - 0.475\left(\frac{1.5}{150}\right)\sqrt{\frac{58.80 \times 10^3}{58.84}}\right] = 1.615$$

Thus,

$$\text{b}_{1,\text{e}} = Ct\sqrt{E/\sigma_y} = 1.615 \times 1.5 \times \sqrt{58.80 \times 10^3/58.84} = 76.58\,\text{mm}$$

Panels #2 and 3:

$$C = 0.60$$

$$\text{b}_{2,3,\text{e}} = 0.60 \times 1.5 \times \sqrt{58.80 \times 10^3/58.84} = 28.45\,\text{mm}$$

Effective cross-sectional area of sheet = [76.58 + 2 (28.45)] $\times$ 1.5 = 200.22 mm^2.

Compressive load carrying capacity = σ_y A$_\text{e}$ = 58.84 $\times$ (200.22 + 2 $\times$ 80) = 21195.345 N. It should be noted that the post-buckling load is 2.099 times the initial buckling load.

8.7 Plates Under Longitudinal Compression and Normal Loading

The problem of plates subjected to longitudinal compression and carrying normal loads is encountered in the design of outer *hull plating of vessels*. In this section a simplified method of analysis is discussed. For the analysis of very thin plates occurring in aeronautical engineering more accurate solutions are required. In the absence of in-plane compression the plate can be analyzed by *classical plate theory*, which is valid up to deflections equal to one-half the plate thickness. But for larger deflections the classical theory estimates the deflections and stresses which are 10 per cent or more in excess of the actual values. In this section the well-known *linearized theory* which is linear in deflection as it is based on the presumption that the deflections are small enough to neglect their higher powers is considered. However, it should be remembered that as in the case of a beam in compression along with transverse loads, the deformations are not proportional to external loads and the principle of superposition is not valid.

The differential equation of the linearized theory can be obtained from the differential equation of elastic buckling given by (8.18) by adding the transverse load term p_o/D

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{p_x}{D}\frac{\partial^2 w}{\partial x^2} = \frac{p_o}{D}$$

$$\nabla^4 w + \left(\frac{p_x}{D}\right)\frac{\partial^2 w}{\partial x^2} = \frac{p_o}{D} \tag{8.131}$$

This equation is linear in w and its derivatives but contains a product of load and deflection which makes it nonlinear. Both differential equation and energy methods will be used to derive approximate solutions of this plate problem.

8.7.1 Governing Differential Equation Method

Consider the rectangular plate shown in the Fig. 8.12 subjected to transverse or normal pressure p_o and constant in-plane longitudinal load p_x in the X-direction. The origin of the reference coordinate system has been assumed to lie at the corner of the plate. Let the displacement function is still given by (8.73) as

$$w(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{8.132a}$$

where A_{ij} is *Fourier coefficient* for displacement in general harmonics i and j. Thus

$$\nabla^4 w = \pi^4 \sum\sum \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2 A_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{8.132b}$$

$$\frac{\partial^2 w}{\partial x^2} = -\pi^2 \sum\sum \left(\frac{i}{a}\right)^2 A_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{8.132c}$$

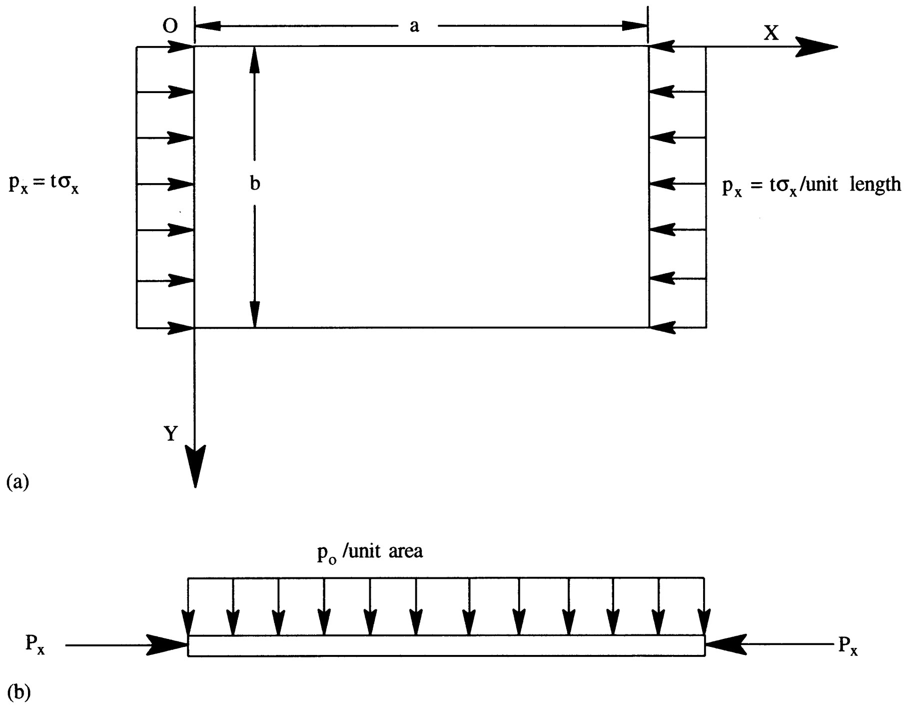


Fig. 8.12a,b. Plate panel subjected to compression and normal pressure. **a** Plan of the plate panel, **b** cross section

The uniform load can also be represented by *Fourier series* as:

$$p(x, y) = \sum \sum a_{ij} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{8.132d}$$

Substituting from (8.132b), (8.132c) and (8.132d) into the governing equation (8.131).

$$\sum \sum \left\{ A_{ij} \left[\pi^4 \left(\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right)^2 - \pi^2 \frac{p_x}{D} \left(\frac{i}{a}\right)^2 \right] - \frac{a_{ij}}{D} \right\} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} = 0$$

Since this equation must be valid for all values of x and y, it follows

$$A_{ij} = \frac{1}{\pi^4 D} \frac{a_{ij}}{\left[\left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 - \frac{p_x}{\pi^2 D} \left(\frac{i}{a}\right)^2 \right]} \tag{8.133}$$

The fourier coefficient a_{ij} for the applied transverse loading can be evaluated from:

$$a_{m'n'} = \frac{4}{ab} \int_0^a \int_0^b p(x, y) \sin \left(\frac{m'\pi x}{a} \right) \sin \left(\frac{n'\pi y}{b} \right) \mathrm{d}y \, \mathrm{d}x$$

For the uniformly distributed load $p(x, y) = p_o$

$$a_{m'n'} = \frac{4p_o}{ab}\int_0^a\int_0^b \sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)\mathrm{d}y\,\mathrm{d}x = \frac{16p_0}{\pi^2 m'n'}$$

where m' and n' are odd integers. $a_{m'n'} = 0$, if either m' or n' is even. Therefore, $a_{ij} = 16p_o/(ij\pi^2)$ and from (8.133)

$$\begin{aligned} A_{ij} &= \frac{16p_0}{\pi^6 D(ij)}\frac{1}{\left[\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 - \frac{p_x}{\pi^2 D}\left(\frac{i}{a}\right)^2\right]} && \text{for odd } i \text{ and } j \\ &= 0 && \text{for even } i \text{ and } j \end{aligned}$$

Therefore,

$$w(x, y) = \frac{16p_0}{\pi^6 D}\sum_{i=1,3,5.....}^{\infty}\sum_{j=1,3,5.....}^{\infty}\frac{\sin(i\pi x/a)\sin(j\pi y/b)}{\left[\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 - \frac{p_x}{\pi^2 D}\left(\frac{i}{a}\right)^2\right](ij)} \tag{8.134}$$

Thus under compressive force deflection is increased and if p_x is tensile deflection is reduced from no in-plane load case. Critical value is for specific harmonics $i = m$ and $j = n$ for which the denominator of (8.134) vanishes and deflection becomes infinite. Thus

$$\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{p_{x,\mathrm{cr}}}{\pi^2 D}\left(\frac{m}{a}\right)^2 = 0$$

or

$$p_{x,\mathrm{cr}} = \pi^2 D\left(\frac{a}{m}\right)^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^2 = \frac{\pi^2 D}{b^2}\left[m\left(\frac{b}{a}\right)+\frac{n^2}{m}\left(\frac{a}{b}\right)\right]^2 \tag{8.135}$$

It is obvious from (8.135) that $n = 1$ will give lowest $p_{x,\mathrm{cr}}$, so that

$$p_{x,\mathrm{cr}} = \frac{\pi^2 D}{b^2}\left[\frac{m}{\mu}+\frac{\mu}{m}\right]^2 = \frac{\pi^2 D}{b^2}k_m^2 \tag{8.136}$$

where $\mu(= a/b)$ is the aspect ratio and $k_m^2 = [(m/\mu) + (\mu/m)]^2$. $p_{x,\mathrm{cr}}$ is the elastic buckling load for uniaxially compressed simply supported rectangular plate and is identical to that obtained earlier in (8.45a) and (8.91). It should be noted that $p_{x,\mathrm{cr}}$ is unaffected by the presence of transverse load. Here, m represents the harmonic number of buckling modes in the direction of compressive loading. For the given harmonics

$$w(x, y) = \frac{16p_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi x/a)\sin(n\pi y/b)}{mn\left[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{p_x}{\pi^2 D}\left(\frac{m}{a}\right)^2\right]} \tag{8.137}$$

In the absence of in-plane compressive force, the deflection of the plate as obtained from (8.137) is given by:

$$w_0 = \frac{16p_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x/a)\sin(n\pi y/b)}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \tag{8.138}$$

Equation (8.137) can now be expressed as

$$w(x, y) = \frac{w_o}{1 - \left\{\frac{p_x}{\pi^2 D}\left(\frac{m}{a}\right)^2 \Big/ \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2\right\}} = \frac{w_o}{\left[1 - \frac{p_x b^2}{\pi^2 D k^2}\right]}$$

$$= \frac{w_o}{\left[1 - \left(p_x/p_{x,\text{cr}}\right)\right]} = \psi w_o \tag{8.139}$$

where $\psi = 1/[1 - (p_x/p_{x,\text{cr}})] = 1/[1 - (\sigma_x/\sigma_{x,\text{cr}})]$ is the *magnification factor*. It can also be written as:

$$\frac{1}{\psi} = 1 - \left\{\sigma_x \Big/ \left[\frac{k^2\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2\right]\right\} \tag{8.140}$$

The values of k^2 for different support conditions of the plate can be obtained from the Table 8.1.

8.7.2 Energy Approach

Consider the rectangular plate shown in Fig. 8.12 subjected to normal pressure p_o and longitudinal load p_x in X-direction. Reckoning the change of potential energy with reference to a state in which the plate carries the longitudinal load p_x but no normal load. If the normal load p_o is added to this reference state the plate will deflect, and the change in the strain energy of bending of plate will be given by (8.24)

$$W_{\text{in}} = U = \frac{D}{2} \iint \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 \mathrm{d}y\,\mathrm{d}x \tag{8.141}$$

The change in potential energy V of external loads p_o and p_x is

$$-W_{\text{ex}} = V = -\iint \left[p_o w + \frac{p_x}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \mathrm{d}y\,\mathrm{d}x \tag{8.142}$$

The potential $\Pi (= U + V)$ of the plate is therefore is given by:

$$\Pi = \iint \left[\frac{D}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - \frac{p_x}{2}\left(\frac{\partial w}{\partial x}\right)^2 - p_o w\right] \mathrm{d}y\,\mathrm{d}x \tag{8.143}$$

Using single term *Ritz-solution* for the deflection $w(x, y)$ of the plate

$$w(x, y) = A\varphi(x, y) \tag{8.144}$$

where A is an unknown constant and $\varphi(x, y)$ is a function of x and y satisfying the boundary conditions. From the *work-equation*: $W_{\text{in}} = W_{\text{ex}}$

$$A = \frac{p_o \iint \varphi(x, y)\mathrm{d}y\,\mathrm{d}x}{\frac{D}{2}\iint \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)^2 \mathrm{d}y\,\mathrm{d}x - \frac{p_x}{2}\iint \left(\frac{\partial\varphi}{\partial x}\right)^2 \mathrm{d}y\,\mathrm{d}x} \tag{8.145}$$

For the case of plate without p_x, representing w and A by w_o and A_o, respectively:

$$w_o = A_o\varphi(x, y) \tag{8.146}$$

$$A_0 = \frac{p_o \iint \varphi(x, y)\mathrm{d}y\,\mathrm{d}x}{\frac{D}{2}\iint \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)^2 \mathrm{d}y\,\mathrm{d}x} \tag{8.147}$$

Dividing (8.145) by (8.147) following relation is obtained

$$A = \psi A_o \tag{8.148}$$

where ψ is the magnification factor defined by:

$$\frac{1}{\psi} = 1 - \left(\frac{p_x}{D}\right)\frac{\iint \left(\frac{\partial\varphi}{\partial x}\right)^2 \mathrm{d}y\,\mathrm{d}x}{\iint \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right)^2 \mathrm{d}y\,\mathrm{d}x} \tag{8.149}$$

From (8.144), (8.146) and (8.148)

$$w(x, y) = \psi w_o \tag{8.150}$$

that is the deflection $w(x, y)$ of the plate is ψ times the deflection w_o of the plate subjected to normal loads alone. The bending stresses in plate being proportional to the second derivative of w_o, the magnification factor ψ can also be applied to these stresses. The maximum fibre stresses in X- and Y-directions of the plate are given by:

$$\sigma_{x,\max} = \sigma_x + \psi\sigma_{x,\mathrm{b}} \tag{8.151a}$$

$$\sigma_{y,\max} = \psi\sigma_{y,\mathrm{b}} \tag{8.151b}$$

where $\sigma_{x,\mathrm{b}}$ and $\sigma_{y,\mathrm{b}}$ are bending stresses in the plate subjected to normal loads alone, in X- and Y-directions, respectively. Although the above treatment is based on single term Ritz solution but still it gives good approximation within following limits.

1. The maximum deflection computed by (8.150) must be less than half the thickness of plate.
2. The longitudinal compressive stress, σ_x must be less than the proportional limit of the plate material, and
3. The longitudinal compressive stress, σ_x must be less than critical buckling stress $\sigma_{x,\mathrm{cr}}$ of the plate.

The magnification factor ψ given by (8.149) can be expressed in the general form given by (8.150).

Normal pressure causes a decrease in effective width at strains below the normal buckling strain and an increase in the effective width for strains somewhat greater than the normal buckling strains.

8.8 Problems

Problem 8.1. Using differential equation approach derive the characteristic equations or stability conditions for the rectangular plate of size $a \times b$ with the given support conditions assuming appropriate displacement functions. Considering the origin of the reference co-ordinate system to lie at the centre of the plate. The supports $x = \pm a/2$ are the loaded edges.

(a) All edges clamped

$$w(x, y) = \left(1 + \cos\frac{2m\pi x}{a}\right)\left(1 + \cos\frac{2n\pi y}{b}\right)$$

(b) All edges simply supported

$$w(x, y) = \left(\cos\frac{m\pi x}{a}\right)\left(\cos\frac{n\pi y}{b}\right)$$

(c) Loaded edges clamped and the longitudinal edges simply supported

$$w(x, y) = \left(1 + \cos\frac{2m\pi x}{a}\right)\cos\frac{n\pi y}{b}$$

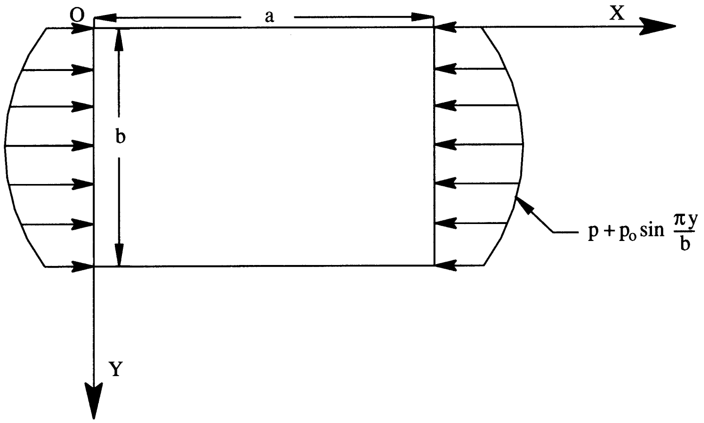
(d) Loaded edges simply supported and the longitudinal edges clamped

$$w(x, y) = \left(\cos\frac{m\pi x}{a}\right)\left(1 + \cos\frac{2n\pi y}{b}\right)$$

Problem 8.2. Use stationary potential principle to derive stability conditions for the plates of problem 8.1.

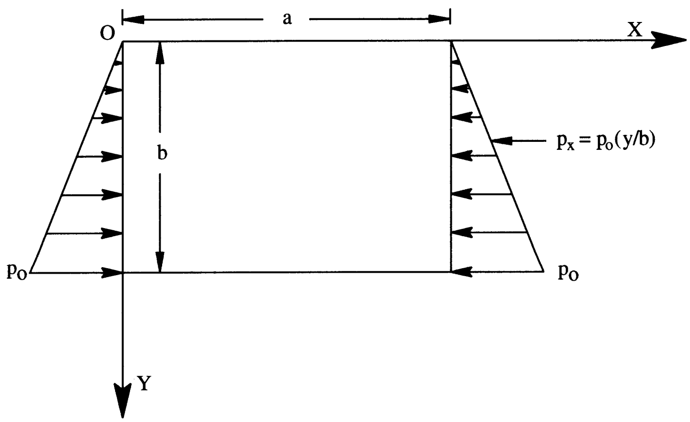
Problem 8.3. A simply supported rectangular plate is subjected to in-plane end loads along the edges $x = 0$ and $x = a$. Derive the characteristic equation for obtaining critical load when the end load p_x per unit length shown in Fig. P.8.3 is given by

$$p_x = p + p_o \sin(\pi y/b)$$



P.8.3 Rectangular plate subjected to sinusoidal loading

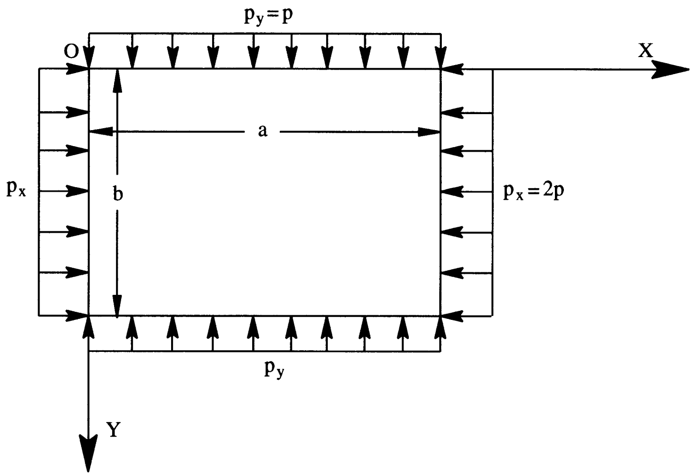
Problem 8.4. A simply supported rectangular plate is compressed along X-direction by a linearly varying load $p_x = p_o(y/b)$ as shown in the Fig. P.8.4. Use Rayleigh–Ritz method to determine the critical load.



P.8.4

Problem 8.5. A simply supported rectangular plate shown in Fig. P.8.5 is compressed in two perpendicular directions by uniformly distributed loads. Determine critical load by using: (1) Rayleigh–Ritz method, and (2) Galerkin's Technique.

Problem 8.6. Analyse the plate of problem 8.3 when the edges carrying compressive load are simply supported while the other two are clamped.



P.8.5

Problem 8.7. For analysis a bounded plate panel is modelled as a simply supported rectangular plate subjected to a varying in-plane axial load, $p_x(y) = p_o[1 - \alpha(y/b)]$ per unit length along its two opposite edges $x = 0$ and $x = a$. Determine the critical load for the panel.

[Hint: A simply supported rectangular plate subjected to in-plane compression buckles into half-waves in X-direction with nodal lines perpendicular to X-direction. Each sub-panel may be treated to buckle as a plate simply supported on all its four sides. Take the buckled configuration of the type:

$$w(x, y) = \sin\left(\frac{\pi y}{b}\right) \sum_{i=1}^{m} A_i \sin\left(\frac{i\pi x}{a}\right)$$

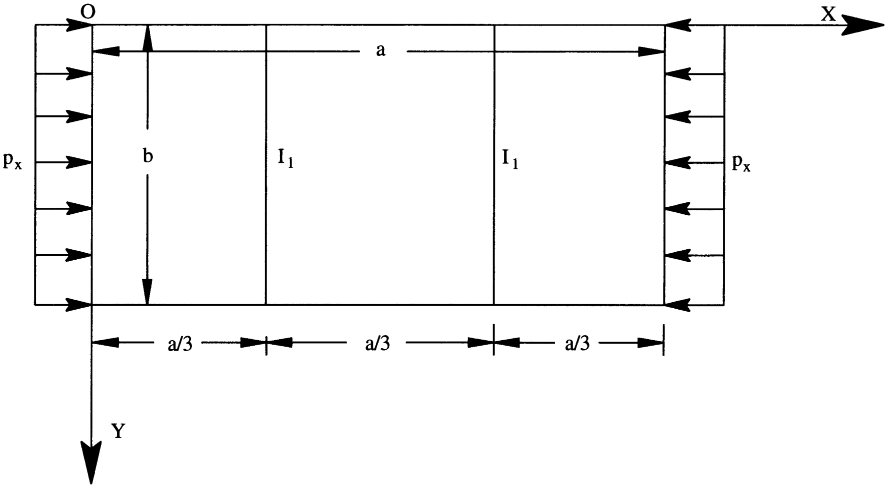
Use energy approach for its solution.

Ans. For a rectangular plate of aspect ratio $\mu = a/b$ and $m = 1$

$$p_{x,\text{cr}} = \pi^2 D\left(\mu + \frac{1}{\mu}\right)^2 / \left[b^2(1 - \alpha/2)\right] \quad]$$

Problem 8.8. A rectangular plate is stiffened by two transverse stiffeners at one-third points as shown in Fig. P.8.8. If the plate is uniformly compressed along the edges $x = 0$ and $x = a$, obtain the expression for critical load when: (i) all edges are simply supported, and (ii) all edges are clamped. Assume stiffeners to be of same material as the plate.

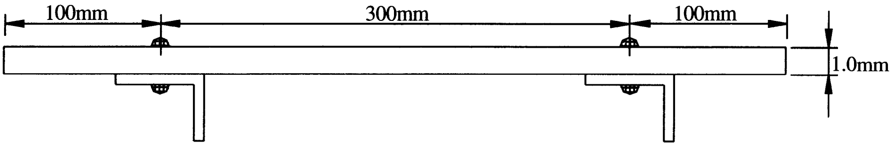
Problem 8.9. The skin sheet of an aircraft wing of gauge 1.0 mm is stiffened by stringers and ribs spaced 125 mm and 600 mm, respectively. Determine critical stress



P.8.8

if the sheet panel edges are assumed to be simply supported between stringers and ribs. Take $E = 70.63\,\text{GPa}$, $\sigma_y = 62.78\,\text{MPa}$ and $\nu = 0.28$.

Problem 8.10. A sheet-stringer panel shown in Fig. P.8.10 is subjected to uniform axial compression. Determine the total load carrying capacity P assuming the sheet to be simply supported at the loaded ends and along the rivet lines. Each stringer has an area of $140\,\text{mm}^2$. Assume $E = 70.63\,\text{GPa}$, $\sigma_y = 62.78\,\text{MPa}$ and $\nu = 0.27$.



P.8.10

Problem 8.11. A rectangular plate with short edges b simply supported is compressed by two equal and opposite forces P acting at the mid points of long edges a. Determine critical value of force P_{cr} at which the plate will buckle when: (i) the long edges are simply supported; and (ii) the long edges are clamped.

[Ans. (i) $1.273\pi^2 D/b^2$ *and (ii)* $2.546\pi^2 D/b^2$*, for* $a/b > 2$*]*

Problem 8.12. A rectangular plate with simply supported edges is subjected to a combination of pure shear with uniform longitudinal compression. Use energy method to determine the buckling load factor k^2 for computation of critical load.

Problem 8.13. Use: (i) differential equation method, and (ii) stationary potential principle, to derive stability conditions for a rectangular plate of problem 8.1 (a) if a polynomial of form:

$$w(x, y) = A\left(x^2 - \frac{a^2}{4}\right)^2 \left(y^2 - \frac{b^2}{4}\right)^2$$

is used to describe deflected configuration instead of double circular functions.

9

Stability Analysis of Arches, Rings and Shells

9.1 Introduction

Arches, rings and shells constitute a very important class of structures in themselves. An arch and a ring are usually considered to be the basic components of a more versatile shell structure. The classical stability analysis of these structures is cumbersome. In general they can be conveniently analysed by finite element method. For the cases where the structure axis follows the pressure curve, shear forces appear only at the stage of collapse and the solution can be obtained in a simple manner by using corresponding differential equations e. g. for a circular curve, this situation is realized for a uniform pressure normal to the axis i.e. radial pressure. The following analysis of a flat arch may serve as simple illustration.

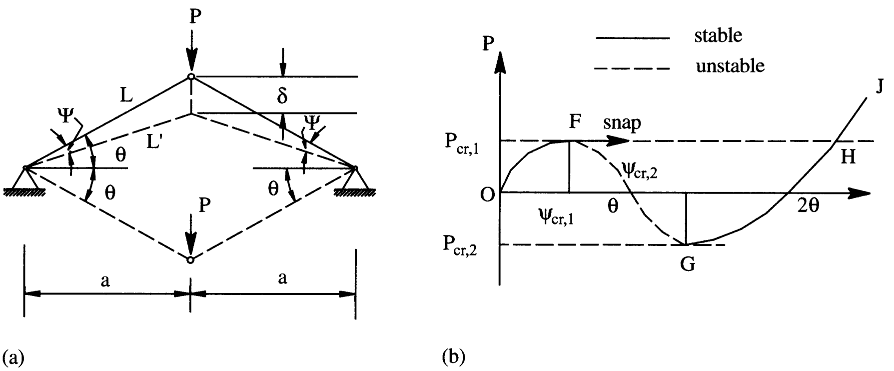


Fig. 9.1a,b. Buckling of flat-arches (snap-through). **a** Flat-arch, **b** load-deflection curve

9.2 Arches

9.2.1 Flat Arches

Consider the flat arch consisting of two axially deformable bars of stiffness k $(= AE/L)$, hinged together at the crown and as well as in the foundation as shown in Fig. 9.1a. There is no shear force and hence bending in the bars. The two bars are of equal length and the distance between the two supports of the arch is $2a$. The two bars initially make angle θ with the horizontal and under the action of the load P, this angle diminishes by an infinitesimal quantity ψ.

Considering only symmetrical deformations the system has only one degree of freedom, and the strain energy of the arch can be expressed as:

$$W_{\text{in}} = U = 2 \times \left[\frac{1}{2}k(L - L')^2\right] = 2 \times \left[\frac{1}{2}k\left(a\sec\theta - a\sec(\theta - \psi)\right)^2\right] \tag{9.1}$$

Using Taylor series expansion

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left[1 - \frac{\theta^2}{2} + \ldots\right]} \cong 1 + \frac{\theta^2}{2} \tag{9.2a}$$

Similarly,

$$\sec(\theta - \psi) \cong 1 + \frac{1}{2}(\theta - \psi)^2 \tag{9.2b}$$

Substituting (9.2) into (9.1)

$$U = \frac{1}{4}ka^2\left[\theta^2 - (\theta - \psi)^2\right]^2 = \frac{1}{4}ka^2\psi^2(2\theta - \psi)^2 \tag{9.3}$$

The applied load, P moves downward by δ, therefore

$$-W_{\text{ex}} = V = -P\delta = -P[a\tan\theta - a\tan(\theta - \psi)] \cong -Pa\psi \tag{9.4}$$

The total potential energy of system Π is therefore given by:

$$\Pi = U + V = \frac{1}{4}ka^2\psi^2(2\theta - \psi)^2 - Pa\psi \tag{9.5}$$

The equilibrium configuration is as usual given by a stationary value of Π. Thus from $\partial\Pi/\partial\psi = 0$.

$$\begin{aligned} P &= ka\psi(\psi^2 - 3\theta\psi + 2\theta^2) \\ &= ka\psi(\psi - \theta)(\psi - 2\theta) \end{aligned} \tag{9.6}$$

The load-deflection relation of the flat arch is shown in Fig. 9.1b. There exist three positions of equilibrium with $P = 0$, when $\psi = 0, \theta, 2\theta$. Whilst the first and third represent conditions of stable equilibrium with connecting bars being unstressed, the second one is the condition of unstable equilibrium represented by the configuration with connecting bars being aligned and compressed. The critical points can be determined from $\partial P/\partial\psi = 0$ which gives:

$$\psi^2 - 2\theta\psi + \frac{2}{3}\theta^2 = 0$$

and therefore,

$$\psi_c = \theta\left(1 \pm \frac{1}{3}\sqrt{3}\right) \tag{9.7}$$

Introduction of ψ_c from (9.7) into (9.6) yields critical values of load

$$\text{for}\quad \psi_{cr,1} = \left(1 - \frac{1}{3}\sqrt{3}\right)\theta, \qquad P_{cr,1} = \frac{2}{3\sqrt{3}}ka\theta^3$$

$$\text{for}\quad \psi_{cr,2} = \left(1 + \frac{1}{3}\sqrt{3}\right)\theta, \qquad P_{cr,2} = -\frac{2}{3\sqrt{3}}ka\theta^3 \tag{9.8}$$

These results could have arrived at directly from the energy stability criterion, that critical state is given by, $\partial^2\Pi/\partial\psi^2 = 0$.

Under slowly increasing load P, the portion OF of the load-deflection curve of Fig. 9.1b is traversed in a stable manner until stationery point F is reached. As the load P is further increased, there is abrupt jump on the stable branch HJ at the same load. This instability phenomenon which gives rise to a sudden change in configuration at constant load is termed *snap-through*. In the stable portion HJ the angle ψ is much greater and corresponds to a configuration of the system which is inverted with respect to initial one as shown by dotted line on Fig. 9.1a. The phenomenon of snap-through can also develop in more complex cases of flat arches and shells, so giving rise to sudden change of configuration.

9.2.2 Circular Arches

(a) Uniform radial pressure

As mentioned earlier in the preceding section for a circular arch subjected to a radial pressure, the shear forces appear only at the stage of collapse and solution can be obtained by corresponding differential equation. Consider a circular arch compressed by a radial uniformly distributed load of intensity p. At a certain value of this load the circular form of the arch becomes unstable and the arch buckles. Consider an isolated elementary segment of length ds from the buckled arch with its local radius of curvature, r which is assumed to differ only slightly from its initial radius of curvature R. The initial and deformed positions of the segment ds are shown in Fig. 9.2. The change in curvature of the element is related to the moment M by the well known expression.

$$\left(\frac{1}{r} - \frac{1}{R}\right) = -\frac{M}{EI} \tag{9.9}$$

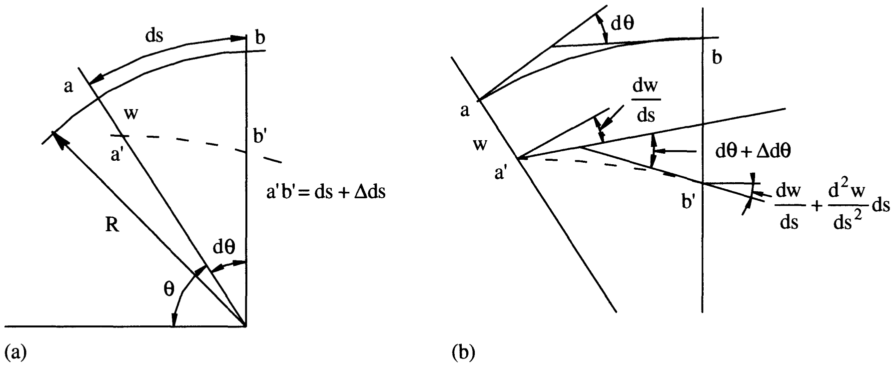


Fig. 9.2a,b. Initial and deformed geometry of an element. **a** An element of arch, **b** enlarged deformation

where M is the bending moment in the cross-section and EI is the flexural rigidity of the arch. M is assumed to be positive if it reduces the initial curvature of the arch. From the geometry of the element:

$$\mathrm{d}s = R\,\mathrm{d}\theta \quad \text{or} \quad \frac{1}{R} = \frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{9.10a}$$

The curvature of the deformed or the strained element is given by:

$$\frac{1}{r} = \frac{\mathrm{d}\theta + \Delta\mathrm{d}\theta}{\mathrm{d}s + \Delta\mathrm{d}s} \tag{9.10b}$$

where

$$\Delta\,\mathrm{d}\theta = \left(\frac{\mathrm{d}w}{\mathrm{d}s} + \frac{\mathrm{d}^2w}{\mathrm{d}s^2}\mathrm{d}s\right) - \frac{\mathrm{d}w}{\mathrm{d}s} = \frac{\mathrm{d}^2w}{\mathrm{d}s^2}\mathrm{d}s \tag{9.10c}$$

It follows from the similar sectors Fig. 9.2a

$$\frac{\mathrm{d}s}{R} = \frac{\mathrm{d}s + \Delta\mathrm{d}s}{R - w} \qquad \text{i.e.} \quad \mathrm{d}s + \Delta\mathrm{d}s = \mathrm{d}s\left(1 - \frac{w}{R}\right) \tag{9.10d}$$

Substituting (9.10c) and (9.10d) into (9.10b) we obtain:

$$\frac{1}{r} = \frac{1}{(1 - w/R)}\left[\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\mathrm{d}^2w}{\mathrm{d}s^2}\right]$$

or

$$\frac{(1 - w/R)}{r} = \left[\frac{1}{R} + \frac{\mathrm{d}^2w}{\mathrm{d}s^2}\right]$$

or

$$\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{\mathrm{d}^2w}{\mathrm{d}s^2} + \frac{w}{Rr} \tag{9.11}$$

With first degree of approximation $w/Rr \cong w/R^2$ and introducing moment-curvature relation from (9.9):

$$\frac{d^2w}{ds^2} + \frac{w}{R^2} = -\frac{M}{EI} \quad \text{or} \quad \frac{d^2w}{ds^2/R^2} + w = -\frac{MR^2}{EI}$$

Therefore,

$$\frac{d^2w}{d\theta^2} + w = -\frac{MR^2}{EI} \tag{9.12}$$

This is the governing equation of bending of a curved beam in polar coordinates.

Example 9.1. A two hinged high circular arch having central angle of 2ϕ is subjected to a uniform radial pressure of magnitude p per unit circumferential length as shown in Fig. 9.3a. Determine the critical load at which arch will buckle.

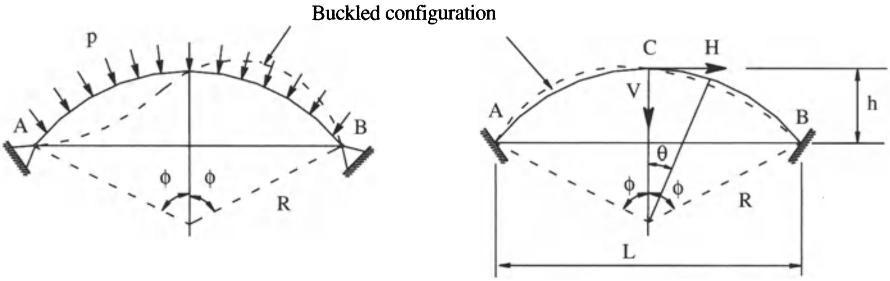


Fig. 9.3a,b. Circular two-hinged and fixed arches subjected to hydrostatic pressure. **a** Two-hinged arch, **b** fixed or hingeless arch

In this case the pressure curve coincides with the arch axis i.e. the bending moment caused by pressure on the arch axis is negligible. Thus, the normal pressure on each cross-section is $P = pR$. In the deformed configuration, the bending moment produced at a section is $M = Pw = pRw$ and the governing differential equation (9.12) reduces to:

$$\frac{d^2w}{d\theta^2} + \alpha^2 w = 0$$

where $\alpha^2 = 1 + (pR^3/EI)$. The general solution to the above differential equation is given by:

$$w = A \sin \alpha\theta + B \cos \alpha\theta$$

The application of boundary conditions that $w = 0$ at $\theta = 0$ and $\theta = \phi$ yield

$$B = 0 \quad \text{and} \quad A \sin \alpha\phi = 0$$

For non-trivial ($A \neq 0$) solution $\alpha\phi = \pi, 2\pi \ldots$ Thus the minimum critical load is given by:

$$\alpha^2 = 1 + \left(\frac{pR^3}{EI}\right) = \frac{\pi^2}{\phi^2}$$

whence,

$$p_{\text{cr}} = \frac{EI}{R^3}\left(\frac{\pi^2}{\phi^2} - 1\right) = (k^2 - 1)\frac{EI}{R^3} \tag{9.13}$$

where $k = \pi/\phi$. For a semi-circular arch $2\phi = \pi$ and

$$p_{\text{cr}} = 3EI/R^2 \tag{9.14}$$

If the above uniformly compressed arch is clamped instead of being hinged at the supports, due to inextensibility of the centre line of arch, it will buckle as shown in Fig. 9.3b by dotted line. The middle point C does not undergo any displacement after buckling and is acted upon by (horizontal) thrust H and radial (vertical) shear V. The bending moment at a section at an angle θ from the middle point is given by:

$$M = Hw - VR\sin\theta \tag{9.15a}$$

And the governing differential equation (9.12) becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + w = -\frac{R^2}{EI}(Hw - VR\sin\theta) \tag{9.15b}$$

where $H = pR$. In terms of parameter $\alpha^2 = 1 + (pR^3/EI)$, equation (b) reduces to

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + \alpha^2 w = \frac{VR^3\sin\theta}{EI} \tag{9.15c}$$

The general solution of the equation (c) is:

$$w = A\sin\alpha\theta + B\cos\alpha\theta + \frac{VR^3\sin\theta}{(\alpha^2 - 1)EI} \tag{9.15d}$$

There are three unknown quantities A, B and V, which can be determined by the boundary conditions.

(i) $w = \dfrac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} = 0$ at $\theta = 0$ (mid-point C)

giving $B = 0$

(ii) $w = \frac{\mathrm{d}w}{\mathrm{d}\theta} = 0$ at $\theta = \phi$ (support)

$$A\sin\alpha\phi + VR^3\sin\phi/[(\alpha^2 - 1)EI] = 0, \quad \text{and}$$
$$A\alpha\cos\alpha\phi + VR^3\cos\phi/[(\alpha^2 - 1)EI] = 0$$

For non-trivial ($A \neq 0$, $V \neq 0$) solution vanishing the determinant of A and V

$$\sin\alpha\phi\cos\phi - \alpha\cos\alpha\phi\sin\phi = 0$$

or

$$\alpha\tan\phi\cot\alpha\phi = 1 \tag{9.15e}$$

and the critical pressure $p_{\rm cr}$ is

$$p_{\rm cr} = \frac{EI}{R^3}(\alpha^2 - 1) \tag{9.15f}$$

The values α for various values angle ϕ are

ϕ (Degrees)	15	30	60	90	120	150	180
α	17.243	8.621	4.375	3.000	2.364	2.066	2.000

Thus value of $p_{\rm cr}$ is always greater than that obtained from (9.13). If the span L and height h of a circular arch is given the semi-central angle is given by:

$$\phi = \cot^{-1}\left(\frac{L}{4h} - \frac{h}{L}\right) \tag{9.15g}$$

Equations analogous to above arch equations can be obtained for bending of long cylindrical shells if the load does not change along the length of the shell. In such a case an elemental arch cut out of the shell by two cross-sections perpendicular to the length and unit distance apart is considered for analysis. The critical value of compressive force in such an arch (with rectangular cross-section $1 \times t$) is obtained by substituting $E/(1 - \nu^2)$ for E and $t^3/12$ for I. Thus,

$$p_{\rm cr} = \frac{Et^3}{12(1 - \upsilon^2)R^3}(\alpha^2 - 1) \tag{9.15h}$$

For practical use it is convenient to represent critical pressure as function of span L and rise h of the arch and the expression for $p_{\rm cr}$ takes the form:

$$p_{\rm cr} = \gamma EI/L^3 \tag{9.16a}$$

where the coefficient γ depends on the ratio h/L and the number of hinges, e.g. hinge less, one-hinged, two-hinged and three-hinged. For a two-hinged circular arch from (9.13)

$$\gamma = 8\left(\frac{\pi^2}{\phi^2} - 1\right)\sin^3\phi \tag{9.16b}$$

and for a circular arch built-in at supports

$$\gamma = 8(\alpha^2 - 1)\sin^3\phi \tag{9.16c}$$

Example 9.2. A uniformly compressed circular arch of span L and rise $h = L/5$ is: (i) hinged, and (ii) built-in at the springing. Determine the critical value of the uniform pressure at which the arch will buckle.

For the given arch geometry the rise to span ratio h/L is 0.20. Therefore, from (9.15),

$$\phi = \cot^{-1}\left(\frac{L}{4h} - \frac{h}{L}\right) = 43.603^\circ$$

(i) For two-hinged arch, from (9.16a) and (9.16b)

$$p_{\mathrm{cr}} = \gamma\frac{EI}{L^3} = 8\left[\left(\frac{\pi}{\phi}\right)^2 - 1\right]\sin^3\phi\left(\frac{EI}{L^3}\right) = 42.096\frac{EI}{L^3}$$

(ii) For built-in or hinge less circular arch from (9.16a) and (9.16c)

$$p_{\mathrm{cr}} = 8(\alpha^2 - 1)\sin^3\phi\left(\frac{EI}{L^3}\right)$$

where α is given by (9.15e); by trial and modification

$$\alpha = 5.96321 \quad \text{and hence}$$

$$p_{\mathrm{cr}} = 8\left[(5.96321)^2 - 1\right](\sin 43.603)^3\left(\frac{EI}{L^3}\right) = 90.69\left(\frac{EI}{L^3}\right)$$

In the preceding discussion of buckling of circular arches it was assumed that during bucking, the external forces remained normal to the buckled configuration as in the case of hydrostatic pressure. But in practice sometimes forces retain their initial directions during buckling. The slight changes in the direction of forces during buckling have only small influences on the values of critical pressure.

(b) Uniformly distributed load along the span

The bending moments introduced by uniformly distributed load in the three-hinged, two-hinged and fixed arches are given in Fig. 9.4. A largest moment is introduced in a three-hinged arch at one-fourth span. These arches behave as two crescent-shaped half arches. In the two-hinged arch moments are distributed rather uniformly as shown by curve 2 in Fig. 9.4, thus they constitute the most preferred type. On the other hand in the fixed arches the moment in the middle half of the span is minimal and high at the supports requiring strong supports and foundations, thus not used frequently.

The most efficient rise-to-span ratios, h/L is approximately 1/6 to 1/5. An increase in the rise, h leads to reduction in axial force and increase in the moment, and vice versa. As in case of radial pressure, the most probable form of buckling configuration of an arch in the vertical plane is S-shaped curve with an inflexion point at the axis near the middle of arch length as shown in Fig. 9.3. The critical thrust can be approximately determined from the Euler–Yasinki formula

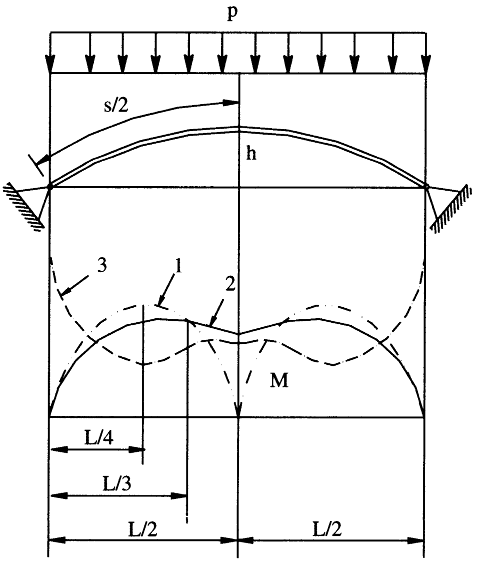


Fig. 9.4. Moments in the circular arches subjected to distributed load. 1 three-hinged arch, 2 two-hinged arch, 3 clamped arch

$$P_{\rm cr} = \frac{\pi^2 EI}{L_{\rm e}^2} = \frac{\pi^2 EI}{(KS/2)^2} \tag{9.16d}$$

where the effective length, $L_{\rm e}$ = half arch length ($S/2 = R\phi$) × effective length coefficient (K).

The radius, $R = \frac{h}{2} + \frac{L^2}{8h}$ and ϕ is given by (9.15g). The effective length coefficient K is given in Table 9.1. The moment of inertia I of the arch section is taken at one-fourth span length L. For the above formula to be valid, the ratio $P_{\rm cr}/P$ must lie in the range 1.2 to 1.3, where P is the thrust induced by design loads. In case of parabolic arch subjected to a load p uniformly distributed along the span, it will be subjected to an axial compression and there will be no bending in the arch. For symmetrical arches of uniform cross-section the critical value of load intensity can be expressed as $p_{\rm cr} = \gamma EI/L^3$. Here again γ depends on the ratio h/L. For flat parabolic arches ($h/L < 0.2$) the value γ differ only slightly from those for circular arches.

Table 9.1. Effective length coefficient, K

Type of arch	Rise -to-span ratio, h/L			
	1/20	1/5	1/3	1/2.5
Three-hinged	1.20	1.20	1.20	1.30
Two-hinged	1.00	1.10	1.20	1.30
Clamped	0.70	0.75	0.80	0.85

9.3 Stability of rings and tubes

Consider the problem of stability of a ring compressed by a radial uniformly distributed load of intensity p (hydrostatic pressure). As in the case of arch isolate an elementary segment of length ds from the buckled ring, as shown in Fig. 9.5b, with local radius of curvature r which is presumed to differ only slightly from the initial curvature, R. There are normal axial force and bending moment acting at a cross-section of the buckled ring.

In the prebuckled state P_o is the axial normal force at the cross-section and there are no shear force and bending moment. The equilibrium condition is:

$$P_o = pR \tag{9.17}$$

The forces acting at the buckled element are shown in Fig. 9.5(b). The equilibrium conditions are:
(i) In the direction of normal to the element (radial):

$$p\,\mathrm{d}s + \mathrm{d}Q - (P_o + P)\,\mathrm{d}s/r = 0$$

Substituting for P_o from (9.17):

$$p\left(\frac{1}{R} - \frac{1}{r}\right) + \frac{1}{R}\frac{\mathrm{d}Q}{\mathrm{d}s} - \frac{P}{rR} = 0 \tag{9.18}$$

Representing the change in curvature by β, the moment in the element is given by well known curvature-moment relation

$$M = EI\left(\frac{1}{R} - \frac{1}{r}\right) = EI\beta \tag{9.19}$$

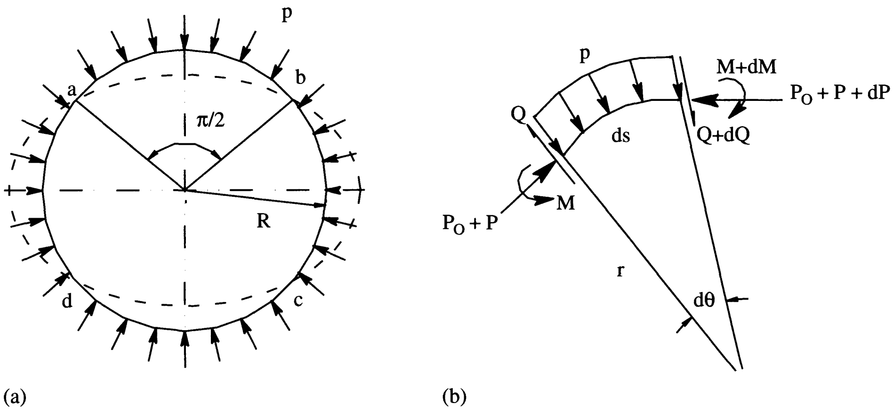


Fig. 9.5a,b. A ring compressed radially by external pressure. **a** Buckled ring, **b** forces acting on an element

where EI is rigidity of the ring. Further noting that $r \approx R$, (9.18) reduces to

$$-p\beta + \frac{1}{R}\frac{\mathrm{d}Q}{\mathrm{d}s} - \frac{P}{R^2} = 0 \tag{9.20a}$$

(ii) In the direction of tangent

$$\frac{\mathrm{d}P}{\mathrm{d}s} + \frac{Q}{R} = 0 \tag{9.20b}$$

(iii) Moment equilibrium

$$\frac{\mathrm{d}M}{\mathrm{d}s} + Q = 0 \tag{9.20c}$$

Eliminating P and Q from (9.20a), (9.20b) and (9.20c):

$$p\frac{\mathrm{d}\beta}{\mathrm{d}s} + \frac{1}{R}\frac{\mathrm{d}^3M}{\mathrm{d}s^3} + \frac{1}{R^3}\frac{\mathrm{d}M}{\mathrm{d}s} = 0 \tag{9.21}$$

On integration (9.21) reduces to

$$p\beta + \frac{1}{R}\frac{\mathrm{d}^2M}{\mathrm{d}s^2} + \frac{1}{R^3}M = C$$

Putting $M = EI\beta$

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}s^2/R^2} + \alpha^2\beta = C\left(\frac{R^3}{EI}\right)$$

$$\text{or} \quad \frac{\mathrm{d}^2\beta}{\mathrm{d}\theta^2} + \alpha^2\beta = C\left(\frac{R^3}{EI}\right) \tag{9.22}$$

where

$$\alpha^2 = 1 + \frac{pR^3}{EI} \tag{9.23}$$

The solution to the governing equation (9.22) is

$$\beta = A\sin\alpha\theta + B\cos\alpha\theta + C\left(\frac{R^3}{\alpha^2 EI}\right)$$

The critical load for the closed ring can best be determined from the condition of periodicity of the solution i.e. if the variable θ is increased by 2π, the function β remains unaltered. Hence α shall be changed by a multiple of 2π. Thus,

$$\alpha(\theta + 2\pi) - \alpha\theta = 2\pi n \qquad \text{or} \qquad 2\pi\alpha = 2\pi n$$

where n is an integer. Thus $\alpha = n$ or $\alpha^2 = n^2$. Therefore, from (9.23)

$$p_{\mathrm{cr}} = \frac{(n^2 - 1)EI}{R^3} \tag{9.24}$$

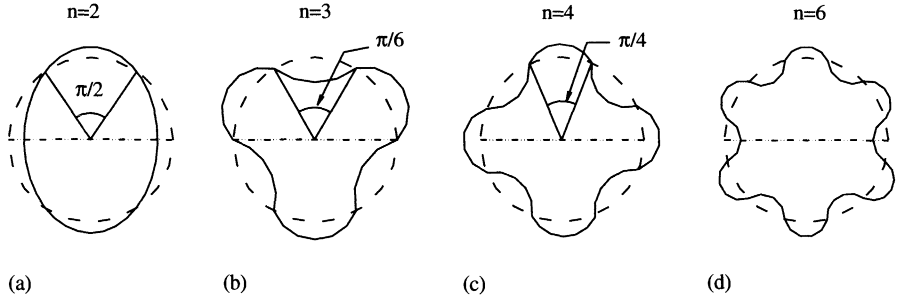


Fig. 9.6a–d. Buckled configurations (number of lobes, n) of closed ring under radial pressure. **a** 4 half-waves, **b** 6 half-waves, **c** 8 half-waves, **d** 12 half-waves

The smallest non-zero value of p_{cr} occurs when $n = 2$, i. e. β undergoes two complete periods of change while passing around the ring. The ring will buckle into two lobes or four half-waves assuming an ellipse like shape as shown in Fig. 9.6a. The corresponding value of critical pressure is:

$$p_{\mathrm{cr}} = \frac{3EI}{R^3} \tag{9.25a}$$

This result is same as the obtained for a semi-circular arch in example 9.1.

If the cross-section of the ring is $b \times t$ then $I = bt^3/12$ and (9.25a) reduces to

$$p_{\mathrm{cr}} = \frac{Eb}{4}\left(\frac{t}{R}\right)^3 = 2Eb\left(\frac{t}{d}\right)^3 \tag{9.25b}$$

where d is outer diameter of the ring. If the ring is stiffened by an even number $2n$ ($n > 2$) of equally spaced supports as shown in the Fig. 9.7a, the buckling occurs in the form of n lobes or $2n$ half-waves and value of p_{cr} is given by (9.24) for the given n.

It should be noted that in deformed configuration of the ring, the portion between two successive contra flexure points is subjected to direct compression just as is each section of straight slender column under an axial load. For example, consider the case when $n = 2$ wherein the ring buckles in two lobes, at the four contra flexure points a, b, c and d the bending moments are zero. If the curvature of an arch portion (say a–b) is neglected and Euler's equation is applied:

$$p_{a,\mathrm{cr}} = \frac{p_{\mathrm{cr}}d}{2} = \frac{\pi^2 EI}{(\pi d/4)^2}$$

or

$$p_{\mathrm{cr}} = \frac{32EI}{d^3} = \frac{4EI}{R^3} = \frac{8Eb}{3}\left(\frac{t}{d}\right)^3$$

where $I = bt^3/12$. This value is only approximate. The factor $8/3 (= 2.67)$ compares to 2.0 for the exact value given by (9.25b).

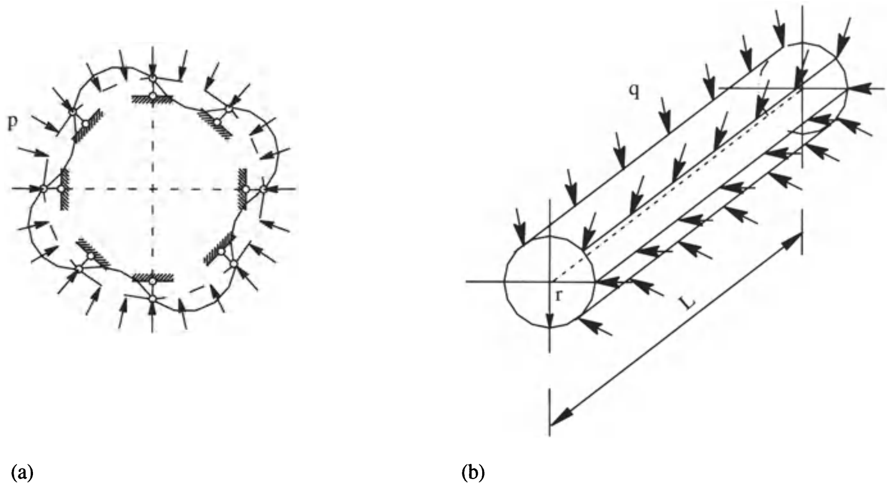


Fig. 9.7a,b Stiffened ring and a very long pipe under radial pressure. **a** Stiffened ring ($2n = 8$), **b** long pipe

The results obtained for a ring can readily be extended to the very long pipes subjected to external radial pressure q as shown in Fig. 9.7b.

As discussed in Chap. 8 that for a two-dimensional plate element bending in two perpendicular planes, the moment in a direction (say X-) is given by (8.17) as

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -\frac{EI_x}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

where $I_x = 1 \times t^3/12$. The terms $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ represent curvatures of the deflected plate in two right-angle transverse X-Z- and Y-Z-planes. For a long tube in longitudinal direction Z- (Y- in case of plate) $\partial^2 w/\partial z^2 = 0$ and expression for moment becomes

$$M_x = \frac{EI_x}{1-\nu^2}\frac{\partial^2 w}{\mathrm{d}x^2} = \left(\frac{EI_x}{1-\nu^2}\right)\frac{1}{R}$$

For the bending in one plane $M_x = EI_x/R$. Thus E in the expression for pressure in the ring is replaced by $E/(1-\nu^2)$ for pressure in the long tubes i.e. the flexural rigidity, EI of the ring is replaced by that of plate/shell, i.e. $EI = Et^3L/[12(1-\nu^2)]$.

Moreover, in case of tubes $p = qL$ thus,

$$q_{\text{cr}} = \frac{(n^2-1)Et^3}{12(1-\nu^2)R^3}$$

For smallest non-zero value of q_{cr}, $n = 2$ i.e.

$$q_{\text{cr, min}} = \frac{E}{4(1-\nu^2)}\left(\frac{t}{R}\right)^3 = \frac{2E}{1-\nu^2}\left(\frac{t}{d}\right)^3 \tag{9.26a}$$

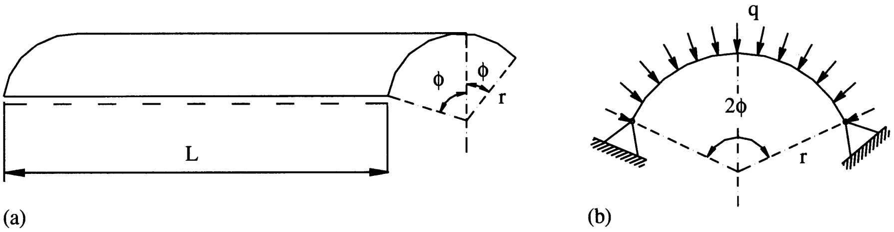


Fig. 9.8a,b. Cylindrical curved panel subjected to radial pressure. **a** Panel geometry, **b** cross-section

since $q = 2\sigma t/d$

$$\sigma_{cr} = \frac{E}{1-\nu^2}\left(\frac{t}{d}\right)^2 \tag{9.26b}$$

where t is the thickness of pipe wall and d is the external diameter of the pipe. Similarly the elastic buckling pressure for a long cylindrical curved panel of radius r and central angle 2ϕ, with curved edges free and straight edges simply supported (i.e. hinged) as shown in Fig. 9.8 is given by replacing EI by $Et^3/[12(1-\nu^2)]$ in (9.13). Thus,

$$p_{cr} = q_{cr}L = \frac{E\left(\frac{\pi^2}{\phi^2}-1\right)}{12(1-\nu^2)}\left(\frac{t}{r}\right)^3 \tag{9.27}$$

If the straight edges are clamped instead of being simple supports the corresponding buckling pressure can be obtained from (9.15f) as follows:

$$p_{cr} = \frac{E(\alpha^2-1)}{12(1-\nu^2)}\left(\frac{t}{r}\right)^3 \tag{9.28}$$

For various values of semi-central angle ϕ, α is given by (9.15e).

9.4 Elastic Instability of Thin Shells

In Chap. 8, it is seen that a thin plate resists loads by two dimensional bending and shear. On the other hand a shell is a three-dimensional structure whose basic resistance to loads is through in-plane or membrane forces i.e. tension and compression are predominant. A membrane resists the loads through in-plane tensile stresses but a thin shell must be capable of developing both tension and compression. However, the similarity between the behaviour of a shell and a membrane is not complete because of so called boundary disturbances which arise in the shells. These boundary disturbances give rise to bending moments and shears which are localized in the region immediately adjacent to the boundary. Moreover, in contrast to general instability problem wherein entire shell buckles as a beam-column, local instabilities are characterized by displacements of comparatively small wave-length. There are several possible approaches to the elastic instability analysis of shells. In one of the commonly used approaches, a general solution for the normal displacement due to transverse and in-plane loading is obtained, then the discrete values of P which cause the displacement to become excessively large are determined which are termed critical loads for a perfect shell. It should be realized that critical load P_{cr} is independent of transverse loading.

9.4.1 Governing Differential Equation

For simplicity of treatment the flat plate theory in Cartesian coordinates will be extended to the buckling analysis of thin shallow shells. A flat plate differential element can be considered to be a special case of differential shell element with zero curvatures in two perpendicular X-Z and Y-Z planes. However, the nature and direction of forces acting on an undeformed shell element are similar to that on a plate element in the deformed state.

Let the pre-buckled membrane state or in-plane forces in a shell be represented by P_{xo}, P_{yo} and P_{xyo}. In shells at transition point, the greatly increased normal displacement results in development of normal components of these in-planes forces or shear resultants. Such normal components may be treated as surface loading in n-direction. The determination of normal components of membrane state resultants for shallow shells can be based on the in-plane stress resultants of a plate.

With transverse shear forces and in-plane loads omitted, the normal force q_n can be expressed in cartesian coordinates from (8.4) as:

$$q_n = P_{xo}\frac{\partial^2 w}{\partial x^2} + P_{yo}\frac{\partial^2 w}{\partial y^2} + 2P_{xyo}\frac{\partial^2 w}{\partial x \partial y} \tag{9.29}$$

In case of shallow shells that cover rectangular plan areas, if rise is smaller than one-fifth of smaller side of rectangle, the assumption of small displacement theory that $(\partial w/\partial x)$ and $(\partial w/\partial y)$ may be neglected in comparison to unity can be invoked and the radii of curvatures can be expressed as:

$$\frac{1}{R_x} = -\frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad \frac{1}{R_y} = -\frac{\partial^2 w}{\partial y^2} \tag{9.30}$$

In terms of orthogonal curvilinear co-ordinates (ϕ, θ) shown in Fig. 9.9a the equilibrium equation for the shells can be expressed as:

$$D\nabla^4 w + \left(\frac{P_\phi}{R_\phi} + \frac{P_\theta}{R_\theta}\right) = q_n \tag{9.31}$$

where

$$\frac{P_\phi}{R_\phi} + \frac{P_\theta}{R_\theta} = \frac{\partial}{\partial \phi}\left(\frac{1}{R_\theta}\frac{\partial £}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{R_\phi}\frac{\partial £}{\partial \theta}\right) = \left(\frac{1}{R_\theta}\frac{\partial^2 £}{\partial \phi^2} + \frac{1}{R_\phi}\frac{\partial^2 £}{\partial \theta^2}\right) = \nabla_*^2(£) \tag{9.32}$$

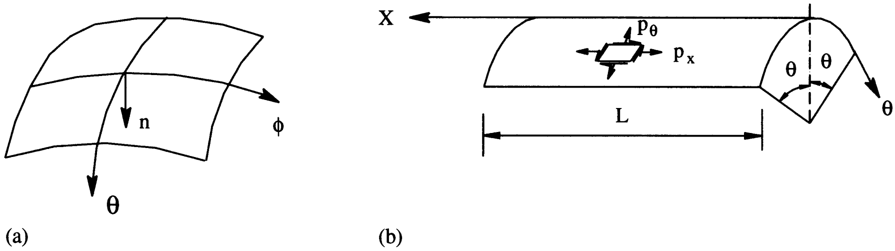


Fig. 9.9a,b. Co-ordinate systems for shells. **a** Orthogonal curvilinear co-ordinates, **b** co-ordinate system for cylindrical surfaces

where £ is a stress function. For a shallow shell described by cartesian co-ordinates R_ϕ and R_θ are approximated by average values of Radii of curvatures R_x and R_y, respectively, i.e. $R_\phi = R_x$ and $R_\theta = R_y$, and the cartesian coordinates equivalent of curvilinear coordinates are given by:

$$\frac{P_\phi}{R_\phi} + \frac{P_\theta}{R_\theta} = \frac{P_x}{R_x} + \frac{P_y}{R_y} \tag{9.33}$$

$$D\nabla^4 w + \nabla_*^2 \pounds = q_n \tag{9.34}$$

The operators $\nabla^2()$ and $\nabla_*^2()$ are given by

$$\nabla^2() = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{and}$$

$$\nabla_*^2() = \frac{1}{R_y}\frac{\partial^2}{\partial x^2} + \frac{1}{R_x}\frac{\partial^2}{\partial y^2} \tag{9.35}$$

The stress function £ representing the in-plane stress resultants, P_ϕ, P_θ and $D = Et^3/[12(1-\nu^2)]$ is related to w by

$$\frac{1}{Et}\nabla^4 \pounds - \nabla_*^2 w = 0 \tag{9.36}$$

Defining a potential function G related to w and £ by

$$w = \nabla^4 G \qquad \text{and} \qquad \pounds = Et\nabla_*^2 G \tag{9.37}$$

The governing differential equation (9.34), therefore, reduces to

$$D\nabla^4(\nabla^4 G) + \nabla_*^2(Et\nabla_*^2 G) = q_n$$

or

$$\nabla^8 G + \frac{12(1-\nu^2)}{t^2}\nabla_*^4 G = \frac{q_n}{D} \tag{9.38}$$

where

$$\frac{q_n}{D} = \frac{1}{D}\left[P_{xo}\frac{\partial^2}{\partial x^2}(\nabla^4 G) + P_{yo}\frac{\partial^2}{\partial y^2}(\nabla^4 G) + 2P_{xyo}\frac{\partial^2}{\partial x \partial y}(\nabla^4 G)\right] \tag{9.39}$$

This equation can conveniently be applied to the stability analysis of shallow shells. In the following example its application to a spherical shell has been illustrated.

Example 9.3. Determine the lowest buckling pressure for a pressurized spherical shell of radius r which is subjected to a uniform internal suction or external pressure, p.

The spherical shell is axisymmetrically loaded due to the uniform pressure p, i.e., $q_n = -p$. For a sphere of radius r,

$$\nabla_*^2() = \frac{1}{r}\nabla^2() \qquad \text{and} \quad P_\phi = P_\theta = -\frac{pr}{2} \tag{9.40}$$

Therefore, $P_{xo} = P_{yo} = -pr/2$ and $P_{xyo} = 0$. Thus, the right hand side of (9.38) becomes

$$\frac{q_n}{D} = -\frac{pr}{2D}\left[\frac{\partial^2}{\partial x^2}(\nabla^4 G) + \frac{\partial^2}{\partial y^2}(\nabla^4 G)\right] = -\frac{pr}{2D}(\nabla^6 G) \tag{9.41}$$

Therefore, in view of (9.40) and (9.41), (9.38) can be written as:

$$\nabla^8 G + \left[\frac{12(1-\nu^2)}{t^2r^2}\right]\nabla^4 G + \left(\frac{pr}{2D}\right)\nabla^6 G = 0 \tag{9.42}$$

Seeking the solution in the form $\nabla^2 G = \alpha G$, where α is an Eigen value related to p_{cr}. Equation (9.42) reduces to

$$\alpha^2\left[\alpha^2 + \left(\frac{p_{\text{cr}}r}{2D}\right)\alpha + \frac{12(1-\nu^2)}{t^2r^2}\right] = 0 \tag{9.43}$$

For non-trivial ($\alpha \neq 0$) solution

$$\alpha^2 + \left(\frac{p_{\text{cr}}r}{2D}\right)\alpha + \frac{12(1-\nu^2)}{t^2r^2} = 0$$

or

$$p_{\text{cr}} = -\left(\frac{2D}{r}\right)\left[\alpha + \frac{12(1-\nu^2)}{\alpha t^2r^2}\right]$$

For lowest value of p_{cr}:

$$\frac{\partial p_{\text{cr}}}{\partial \alpha} = 0 = 1 - \frac{12(1-\nu^2)}{\alpha^2t^2r^2}$$

or

$$\alpha = \pm\frac{2\sqrt{3(1-\nu^2)}}{tr} \tag{9.44}$$

The negative root of α results in a positive value of p_{cr} which has already been defined as an internal suction or external pressure. Thus critical pressure for a spherical shell is

$$p_{\text{cr}} = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{r}\right)^2 \qquad \text{(ideal case)} \tag{9.45}$$

The possible actual minimum is:

$$p_{\mathrm{cr},1} = \frac{0.365\, Et^2}{r^2}$$

This expression is valid for complete sphere as well as spherical shells that have ideal membrane boundary conditions. For determination of displacement or buckling configuration consider (9.37):

$$w = \nabla^4 G = \nabla^2(\alpha G) = \alpha \nabla^2 G$$

For the computed value of α, the potential function G in turn can be determined from:

$$\nabla^2 G - \alpha G = 0$$

Solution to this equation provides eigen-functions of the problem. Since the above analysis is based on a governing equation derived on the basis of simplest form of Cartesian coordinate system, the feasible solution to the equation is the one that satisfies *kinematic boundary conditions*, if the shell covers rectangular plan area. However, the solution is sufficient to estimate the buckling pressure. In this regard it should be noted that quite large *factor of safety* of the order 5 to 7 against elastic buckling are commonly specified for the design of thin shells.

Example 9.4. A cylindrical shell of radius R; length L and thickness t is subjected to a uniformly distributed axial (longitudinal) load p acting along its periphery as shown in Fig. 9.10. Determine critical value of load p at which shell will buckle.

In this particular case $P_{xo} = p$, $P_{yo} = P_{xyo} = 0$, $R_x = 0$, $R_y = R$ and $\nabla_*^2() = [\nabla^2()]/R$. Thus governing differential equation (9.38) and (9.39) reduces to

$$\nabla^8 G + \frac{12(1-\nu^2)}{t^2R^2}\nabla^4 G + \frac{p}{D}\nabla^6 G = 0$$

If the solution is assumed in the form $\nabla^2 G = \alpha G$ where α is an eigenvalue related to p $(= p_{\mathrm{cr}})$, the above equation reduces to:

$$\alpha^2\left[\alpha^2 + \left(\frac{p_{\mathrm{cr}}}{D}\right)\alpha + \frac{12(1-\nu^2)}{t^2R^2}\right] = 0$$

For a nontrivial solution $\alpha \neq 0$

$$\alpha^2 + \left(\frac{p_{\mathrm{cr}}}{D}\right)\alpha + \frac{12(1-\nu^2)}{t^2R^2} = 0$$

$$p_{\mathrm{cr}} = -D\left[\alpha + \frac{12(1-\nu^2)}{\alpha t^2R^2}\right]$$

For lowest value of p_{cr}:

$$\frac{\partial p_{\mathrm{cr}}}{\partial \alpha} = 1 - \frac{12(1-\nu^2)}{\alpha^2t^2R^2} = 0 \quad \text{or} \quad \alpha = \pm\frac{2\sqrt{3(1-\nu^2)}}{tR}$$

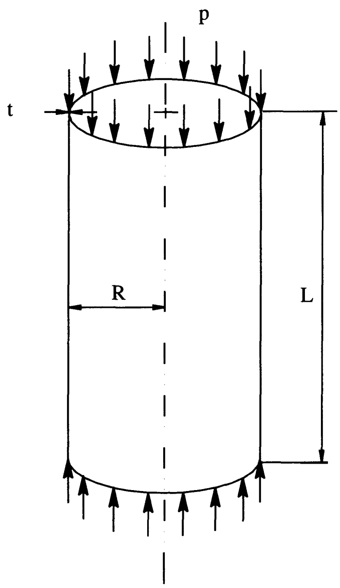


Fig. 9.10. Axisymmetrically loaded cylindrical shell

Using negative root of α as it results in a positive p_{cr}.

$$
\begin{aligned}
p_{\text{cr}} &= D\left[\frac{2\sqrt{3(1-\nu^2)}}{tR} + \frac{12(1-\nu^2)}{t^2R^2}\frac{tR}{2\sqrt{3(1-\nu^2)}}\right] \\
&= \frac{4D\sqrt{3(1-\nu^2)}}{tR} = \frac{4Et^3}{12(1-\nu^2)}\cdot\frac{\sqrt{3(1-\nu^2)}}{tR} = \frac{Et^2}{R\sqrt{3(1-\nu^2)}} \\
&= 0.605Et^2/R \quad \text{for} \quad \nu = 0.3
\end{aligned}
\tag{9.46}
$$

In case of axially compressed thin-walled cylindrical shells, the buckling deformations remain confined to small portion of the shell surface. This phenomenon is referred to as *localized buckling*. The critical value of corresponding load is: $p_{\text{cr}} = 0.323Et^2/R$.

This type of solutions may also be obtained for other geometries with some judicious geometric approximations.

For shorter cylinders with $(\pi R/L)^2 > (2R/t)\sqrt{3(1-\nu^2)}$, $p_{\text{cr}} = \pi^2Et^3/[12(1-\nu^2)L^2]$.

For long cylinders, $p_{\text{cr}} = (\pi^2ER^2t)/(2L^2)$.

9.4.2 Energy Approach

From the energy point of view, the transition between pre-buckled and post-buckled states may be represented by: (i) there is no bending prior to the onset of buckling, so

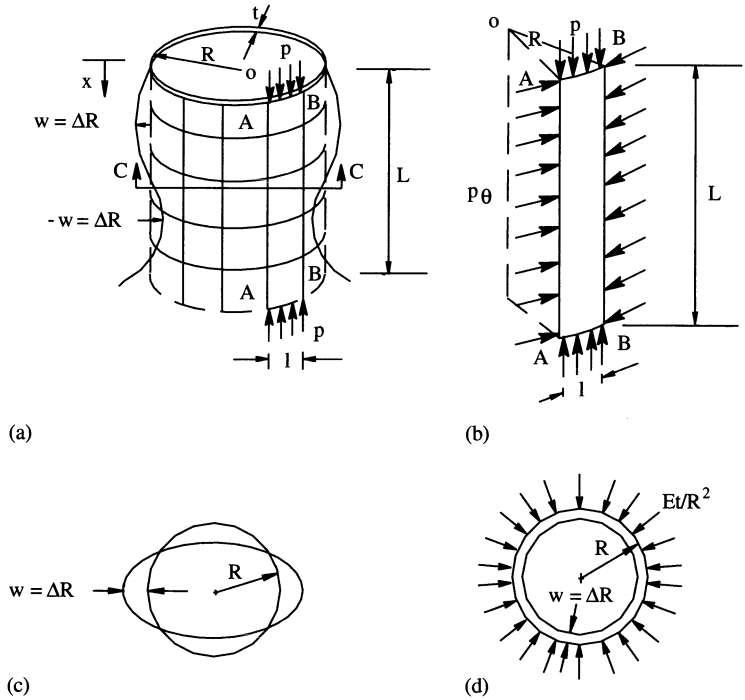


Fig. 9.11a–d. Deformation of an axisymmetrically loaded cylindrical shell. **a** Cylindrical shell under axial pressure, **b** beam element (A-B), **c** deformation of ring element, **d** ring element (C-C)

that total strain energy is due to in-plane stress resultants; (ii) at the onset of instability, there are additional contributions to the strain energy due to straining and bending of middle surface; and (iii) the increase in the strain energy as buckling occurs must be equal to the work done by the external loading and by components of in-plane forces that act through the normal displacements. This latter source of external work is analogous to the load components q_n used in the differential equation method and is absent during infinitesimal deformation.

Consider the case of axisymmetrical deformation of the cylindrical shell shown in Fig. 9.11 wherein the cylindrical surface is viewed as longitudinal beam strips resting on ring elements. For axisymmetrical (rotationally symmetric) deformation, the ring element can suffer uniform expansion or uniform contraction only. The longitudinal or beam elements, on the other hand, can bend freely but maintaining compatibility with uniform deformation of the ring elements. The deflection w of the beam elements is identical to the change in the radius of cylinder, ΔR. Consequently, the ring elements are viewed to act as an *elastic foundation* for the beam elements and hence longitudinal elements may be treated as beams on elastic foundation.

For a uniform contraction or expansion of a ring equal to $\Delta R = \pm w$ the circumferential strain is given by:

$$\varepsilon = \frac{2\pi(R - \Delta R) - 2\pi R}{2\pi R} = -\frac{\Delta R}{R} = \frac{w}{R} \tag{9.47a}$$

If the equivalent radial pressure on the ring is represented by p, then

$$\text{Hoop force,} \quad P = pR = E\varepsilon A = E\left(\frac{w}{R}\right)A$$

Therefore,

$$p = \left(\frac{EA}{R^2}\right)w = Cw \tag{9.47b}$$

where C is the elastic constant of the imaginary elastic foundation. For a ring element with a unit width, $A = (t)(1)$ where t is the thickness of the shell, $C = (Et/R^2)$. Each beam element can now be treated as a strut on an elastic foundation. To account for the continuity of beam elements, the bending stiffness EI of a unit beam element is replaced by $EI = Et^3/[12(1-\nu^2)]$. Using the preceding analogy, the total potential energy of the strut on an elastic foundation can be used for the analysis. However, in the following section total potential energy expression has been developed directly.

In this treatment the pre-buckled state is designated by the subscript o when the bending and normal load components are absent. For the analysis of cylindrical shells the axes X and θ are used for coordinate system (instead of ϕ and θ) as shown in Fig. 9.9b. The axial and corresponding circumferential strains just before buckling are:

$$\varepsilon_{xo} = -p_{\text{cr}}/Et \quad \text{and} \quad \varepsilon_{\theta 0} = -\nu\varepsilon_{xo} \tag{9.48}$$

At buckling the normal displacements w produce circumferential strains, as given by (9.47a) of magnitude $\varepsilon_{\theta 1} = w/R$. Total circumferential strain after the shell has buckled is therefore given by:

$$\varepsilon_\theta = \varepsilon_{\theta 0} + \varepsilon_{\theta 1} = \frac{w}{R} - \nu\varepsilon_{xo} \tag{9.49}$$

The longitudinal or meridional strain ε_x can be obtained as:

$$\varepsilon_x = \varepsilon_{xo} - \nu\varepsilon_{\theta 1} = \varepsilon_{xo} - \nu(w/R) \tag{9.50}$$

The change in curvature after buckling is:

$$\beta = \frac{1}{R_{x0}} - \frac{1}{R_{x1}} = \frac{1}{\infty} - \frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 w}{\partial x^2} \tag{9.51}$$

Since $P_{xo} = -p_{x,\text{cr}}$, $P_{yo} = P_{xyo} = 0$

$$q_n = -p_{x,\text{cr}}\frac{\partial^2 w}{\partial x^2} \tag{9.52}$$

The strain energy of the cylindrical shell under axisymmetrical loading is given by:

$$U_\varepsilon = \int_0^L \frac{1}{2}\left\{E(1)t(\varepsilon_x^2 + \varepsilon_\theta^2 + 2\nu\varepsilon_x\varepsilon_\theta) + D\beta^2\right\}(2\pi R)\,\mathrm{d}x \tag{9.53}$$

The change in the strain energy due to buckling is

$$\delta U = U_\varepsilon(\varepsilon_x, \varepsilon_\theta, \beta) - U_{\varepsilon 0}(\varepsilon_{xo}, \varepsilon_{\theta 0}, 0) \tag{9.54}$$

Substituting for ε_x, ε_θ and β, (9.54) reduces to

$$\delta U = \pi R \int_0^L \left\{ Et\left(\frac{w}{R}\right)^2 - 2\nu Et\varepsilon_{xo}\left(\frac{w}{R}\right) + D\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right\} \mathrm{d}x \tag{9.55}$$

Third and higher order terms have been ignored. The work done by external loading and normal components of in-plane stress resultants during buckling is given by:

$$\begin{aligned}
\delta V &= \delta V_\mathrm{e} + \delta V_\mathrm{n} \\
&= \int_0^L -p_\mathrm{cr}(\varepsilon_x - \varepsilon_{xo})(2\pi R\,\mathrm{d}x) + \frac{1}{2}\int_0^L (q_\mathrm{n} w)(2\pi R\,\mathrm{d}x) \\
&= 2\pi\nu p_\mathrm{cr}\int_0^L w\,\mathrm{d}x - \pi R p_\mathrm{cr}\int_0^L w\frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x \\
&= 2\pi p_\mathrm{cr}\int_0^L \left[\nu w - \frac{R}{2}w\frac{\partial^2 w}{\partial x^2}\right]\mathrm{d}x
\end{aligned} \tag{9.56}$$

From the work equation:

$$\delta V = \delta U$$

Consider one term Rayleigh–Ritz approximation for w as

$$w = A\sin\frac{m\pi x}{L} \tag{9.57}$$

Substituting (9.57) into (9.55) and (9.56)

$$\delta U = \pi R\left[\frac{EtLA^2}{2R^2} - \frac{2\nu Et\varepsilon_{xo}A}{R}\int_0^L \sin\frac{m\pi x}{L}\mathrm{d}x + \frac{Dm^4\pi^4A^2}{2L^3}\right] \tag{9.58a}$$

and

$$\delta \mathrm{V} = 2\pi \mathrm{p}_\mathrm{cr}\left[\nu A\int_0^L \sin\frac{m\pi x}{L}\mathrm{d}x + \frac{Rm^2\pi^2}{4L}A^2\right]. \tag{9.58b}$$

Noting that $\varepsilon_{xo} = -p_\mathrm{cr}/(Et)$ and equating δV and δU (retaining only quadratic terms in A which control post-buckling behaviour).

$$p_\mathrm{cr} = D\left[\frac{m^2\pi^2}{L^2} + \frac{Et}{R^2D}\left(\frac{L^2}{m^2\pi^2}\right)\right] \tag{9.59}$$

For the lowest value of p_{cr}, $\partial p_{\text{cr}}/\partial m = 0$, giving

$$\frac{m\pi}{L} = \left(\frac{Et}{R^2 D}\right)^{\frac{1}{4}} \tag{9.60}$$

Therefore,

$$p_{\text{cr},1} = 2D\left(\frac{Et}{R^2 D}\right)^{\frac{1}{2}} = \frac{Et^2}{R\sqrt{3(1-\nu^2)}} \tag{9.61a}$$

The result is identical to one obtained earlier by equilibrium method. The length of buckled half-waves is given by

$$\lambda = L/m = \pi\left(\frac{R^2 D}{Et}\right)^{1/4} = \frac{\pi\sqrt{Rt}}{\left[12(1-\nu^2)\right]^{1/4}} \cong 1.73\sqrt{Rt} \quad \text{for} \quad \nu = 0.30 \tag{9.61b}$$

However, it should be realized that the smallest critical pressure has been obtained by differentiating p_{cr} with respect to m which assumes p_{cr} to be a continuous function of m, which is only true for moderately long to long cylinders. The characteristic term $\sqrt{Rt}$ plays an important role in development of bending stresses near the boundaries. The results are accurate only for very long tubes or cylinders, but are applicable if the length is several times as large as $1.73\sqrt{Rt}$. Tests indicate an actual buckling strength of between 40 to 60 per cent of the theoretical value or $\sigma_{\text{cr}} = 0.30E(t/R)$ approximately.

9.5 Problems

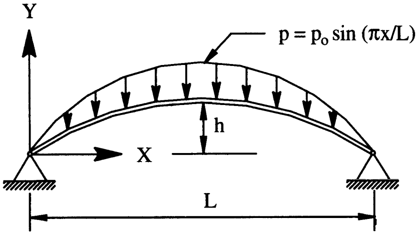
Problem 9.1. The arches and rings have curvilinear axes which are considered to be inextensible and non-deformable in shear. Show that the general governing equation for this type of structures is given by:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}\theta^4} + (1+\alpha^2)\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} + \alpha^2 w = C\left(\frac{R^5}{EI}\right)$$

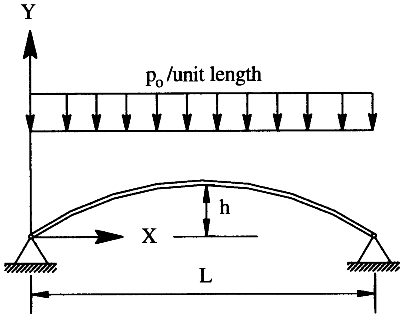
where various symbols have usual meaning.

[Hint: Substitute the expression representing change in curvature β into the governing equation in terms of β].

Problem 9.2. A shallow sinusoidal arch of rise h with simple supports (i. e. hinged at the ends) shown in Fig. P.9.2 is subjected to a sinusoidal loading represented by $p = p_o \sin(\pi x/L)$ per unit horizontal run (span), where L is the span of the arch. The origin is at the left end of the arch. Determine the critical value of the load to cause buckling of the arch in its own plane.



P.9.2



P.9.3

Problem 9.3. Determine the critical buckling load of the arch of Problem 9.2 when it is subjected to: (a) a uniform load of intensity p_o per unit horizontal run shown in Fig. P.9.3 (b) central concentrated load P.

Problem 9.4. If an axisymmetrically loaded cylindrical shell is modelled as longitudinal-strips supported on closely spaced ring elements which act an elastic foundation i.e. it is analogous to beams/struts on an elastic foundation, show that the total potential energy functional Π can be written as:

$$\Pi = \int_0^L \left\{ \frac{1}{2} \left[\frac{Et^3}{12(1-\nu^2)} \cdot \frac{w''^2}{1-w'^2} + \frac{Et}{R^2} w^2 - p(1-w'^2)^{\frac{1}{2}} \right] \right\} \mathrm{d}x$$

[Hint: In the expression of potential of a strut on elastic foundation given by

$$\Pi = \frac{1}{2} \int_0^L \left[EI \frac{w''^2}{1-w'^2} + kw^2 - p(1-w'^2)^{\frac{1}{2}} \right]$$

where k is the elastic constant of the imaginary elastic foundation, replace the bending stiffness EI of the unit beam element by $Et^3/[12(1-\nu^2)]$ to account for continuity and Poisson's ratio. The elastic constant can be shown to be $k = Et/R^2$].

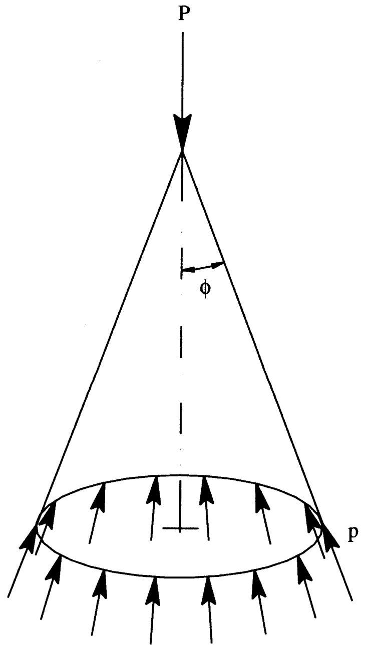
Problem 9.5. Using energy approach determine the buckling pressure for a spherical shell of radius r subjected to a uniform external radial pressure (hydrostatic pressure) p producing compressive stress $\sigma = pr/2t$ in the wall of shell of thickness t.

$$\left[\text{Ans.} \quad p_{\text{cr}} = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{r}\right)^2 \quad \text{or} \quad \sigma_{\text{cr}} = \frac{E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{r}\right)\right]$$

Problem 9.6. Derive an expression for the critical uniform internal suction for a closed cylindrical shell of length L and radius r.

Problem 9.7. Compute the value of critical density of material, γ (force per unit volume) at which a hemispherical shell of thickness t and radius r would buckle under self weight.

Problem 9.8. A high conical shell with semi-vertex angle ϕ as shown in Fig. P.9.8 is subjected to an axial vertex load P. Determine the buckling load assuming that there are only membrane stresses in the post-buckled shell.



P.9.8

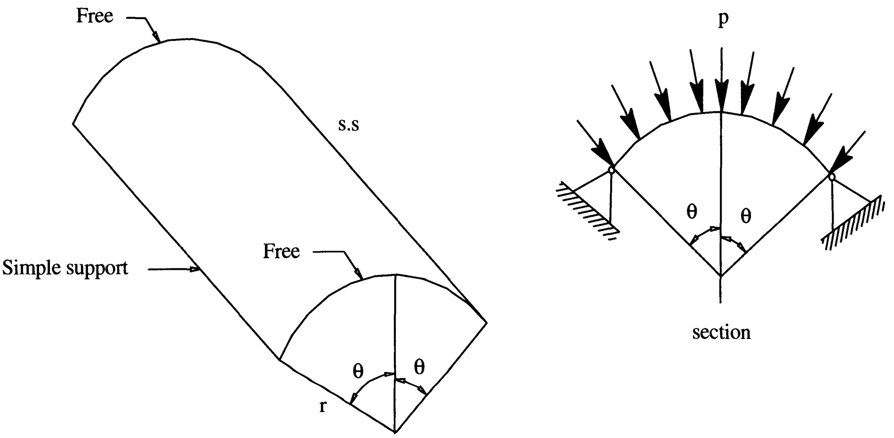
[Ans. $P_{cr} = 2\pi E t^2 \cos^2\phi/\sqrt{3(1-\nu^2)}$*]*

Problem 9.9. A very long tube of length L and radius r is subjected to a uniform lateral (radial) external pressure p. Compute the critical value of the pressure at which the tube will buckle.

$$\left[\text{Ans.}\quad P_{\text{cr}} = \frac{E}{4(1-\nu^2)}\left(\frac{t}{r}\right)^3\right]$$

Problem 9.10. A cylindrical curved panel of radius r, central angle 2θ, with curved edges free and straight edges simply supported or hinged shown in Fig. P.9.10 is subjected to a uniform radial pressure p. Determine the critical value of p at which elastic buckling will occur.

$$\left[\text{Ans.}\quad p_{\text{cr}} = \frac{Et^3\left(\frac{\pi^2}{\theta^2}-1\right)}{12r^3(1-\nu^2)}\right]$$



P.9.10

Problem 9.11. If the straight edges of the curved panel of Problem 9.10 are clamped, determine the external pressure p_{cr} at which elastic buckling will occur.

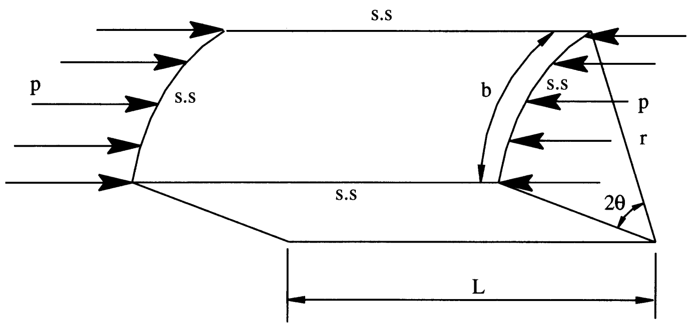
θ	15°	30°	60°	90°	120°	150°	180°
k	17.243	8.621	4.376	3.000	2.364	2.066	2.000

[Ans. $p_{cr} = \frac{Et^3\left(k^2-1\right)}{12r^3(1-\nu^2)}$, *where k is given by the equation* $k\tan\theta\cos k\theta = 1$ *and has the values given in the table above.]*

Problem 9.12. A cylindrical curved panel of length L, width as measured on the arc b and radius r shown in Fig. P.9.12 is subjected to a uniform longitudinal compression

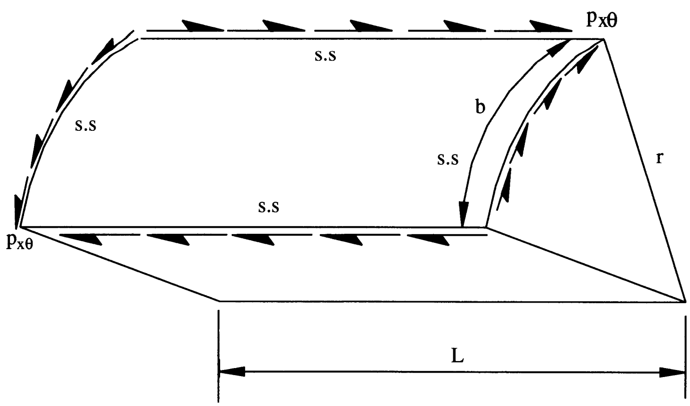
p on the curved edges. Determine the critical values of p_{cr} at which buckling will occur. All the edges may be assumed to be simply supported.

$$\left[\text{Ans.} \quad p_{cr} = \frac{E}{6(1-\nu^2)} \left\{ \left[12(1-\nu^2)\left(\frac{t}{r}\right)^2 + \left(\frac{\pi t}{b}\right)^4 \right]^{1/2} + \left(\frac{\pi t}{b}\right)^2 \right\} \right]$$



P.9.12

Problem 9.13. If the curved panel of the Problem 9.12 is subjected to a uniform shear $p_{x\theta}$ on all the four edges as shown in Fig. P.9.13. Compute the critical buckling shear stress, when (i) all edges are simply supported, and (ii) all edges are clamped.



P.9.13

[Ans: (i) $p_{x\theta,\mathrm{cr}} = 0.10E(t/r)+5.0E(t/b)^2$ *and* (ii) $p_{x\theta,\mathrm{cr}} = 0.10E(t/r)+7.5E(t/b)^2$]

Problem 9.14. A uniformly compressed circular arch of span L and rise $h = 3L/10$ is: (a) (i) hinged, and (ii) built-in, at the supports. Determine the critical value of the uniform radial pressure at which the arch will buckle. (b) What will be the values when the rise is increased to $4L/10$?

[Ans. (a) (i) $p_{\mathrm{cr}} = 40.9EI/L^3$*, (ii)* $p_{\mathrm{cr}} = 93.5EI/L^3$*; (b) (i)* $p_{\mathrm{cr}} = 32.8EI/L^3$ *and (ii)* $p_{\mathrm{cr}} = 80.8EI/L^3$*]*

Problem 9.15. If the arch of Problem 9.14 is provided with a hinge at the crown. Determine the values of p_{cr} for elastic buckling.

[Ans. (a) (i) $p_{\mathrm{cr}} = 34.9EI/L^3$*, and (ii)* $p_{\mathrm{cr}} = 52.0EI/L^3$*, (b) (i)* $p_{\mathrm{cr}} = 30.2EI/L^3$*, and (ii)* $p_{\mathrm{cr}} = 46.0EI/L^3$*].*

10

Inelastic Buckling of Structures

10.1 Introduction

In the elastic stability analysis discussed in the preceding chapters, the material of the structure is presumed to behave according to Hooke's Law i.e. the stress in the structure does not exceed the *initial yield stress* in compression and the member undergoes configuration or shape failure. For many real structures the elastic analysis results in flexural buckling load estimation that exceeds the one associated with the yield stress or *proportional limit stress* of the material. This is especially true for the relatively short or stocky compression members in the framed structures. For this category of members the prorated design stresses based on *safety factors*, generally, fall in the range of *plastic* or *inelastic behaviour*. For steel framed structures many real designs occur in that range and most of the concrete framed structure columns are short. In these shorter columns the elastic limit is exceeded before the inception of buckling, and the modulus of elasticity E, hitherto constant, becomes a function of critical stress $\sigma_{cr} = P_{cr}/A$.

10.2 Inelastic Buckling of Straight Columns

For an idealized axially loaded compression member with presumed pin-ended conditions, the flexural buckling load within elastic range of material behaviour is given by:

$$P_{cr} = P_e = \pi^2 EI/L^2 \tag{10.1}$$

In terms of stress this would be:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2} \tag{10.2}$$

where $r = \sqrt{I/A}$ is the radius of gyration of the cross-section, and (L/r) is the *slenderness ratio.*

M. L. Gambhir, *Stability Analysis and Design of Structures*

Equation (10.2) gives a relationship between critical stress at elastic buckling and the slenderness ratio (L/r) of the member. The equation is valid only as long as σ_{cr} does not exceed elastic limit σ_y of the material i.e. modulus E does not change its value before buckling occurs. This condition restricts the applicability of (10.2) to the so called elastic range of buckling and confines the validity of this equation to a slenderness ratio (L/r) above a certain limiting value which depends on the properties of the material. The condition for the applied load to reach Euler's load before axial stress exceeds the yield stress σ_y i.e. $\sigma_{cr} < \sigma_y$ is given by:

$$(L/r) > \pi\sqrt{E/\sigma_y} \tag{10.3}$$

and is shown in Fig. 10.1. A compression member satisfying this condition is termed *long* or *slender column* and can be analysed by elastic analysis. On the other hand the axial stress in the columns with L/r ratio less than that given by (10.3) called *short columns* will exceed the yield stress before the applied load reaches the critical load, P_{cr}. For example for a material with E value of 200×10^6 kN/m^2 and average stress σ_y of 27.0×10^4 kN/m^2, L/r ratio must exceed 86 for elastic limit not to be exceeded before inception of buckling. The shorter columns will not buckle in the elastic range but the material will yield first. The modulus of elasticity, which is constant in elastic range, now becomes a function of critical stress σ_{cr}. Hence, the results are not valid for short columns and the load carrying capacity of such columns must be determined by taking into account the inelastic behaviour of the material.

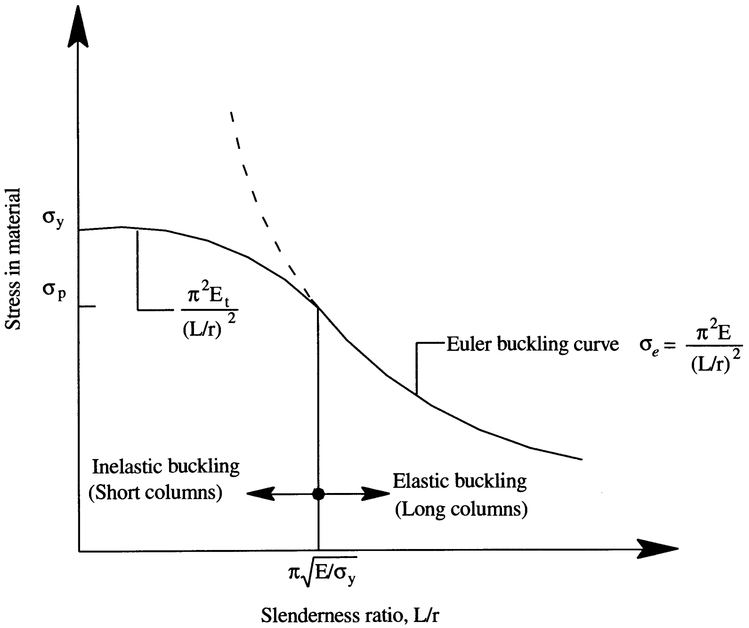


Fig. 10.1. Variation of σ vs. L/r for an ideal axially loaded compression member

It should be noted that nonlinear stress-strain relationship is not unique but differs from material to material. For these reasons, an idealization of the stress-strain relationship in the inelastic or plastic range is desirable in order to develop a reasonably simple inelastic theory.

10.2.1 Stress-Strain Relationship

An examination of the typical stress-strain relationship of a structural metal shown in Fig. 10.2, reveals that in the initial loading range from the origin O up to point A, the *proportional limit*, the material responds linearly to the imposed stresses i.e. the line O-A is straight and the modulus E has a unique value within this elastic range. From the point A to the point B, the *elastic limit*, the curve is not straight but the state is still elastic i.e. strain is *reversible*. Thus on unloading from the point B, the unloading path will follow B-A-O and no residual strain will remain. When the load is increased beyond the elastic limit point B, the strains increase at an ever increasing rate and have *irreversible strain component (plastic strain)*. In this range strains become nonlinear function of stress. At the point C, the *initial yield point*, the plastic or irreversible strain increases appreciably. The point D in the Fig. 10.2a is the *maximum stress point* or the *ultimate strength* of the material. At this state, the stress and strain distributions in the cross-section cease to be uniform and the local instability called *necking* occurs at the critical section. Finally, the specimen breaks at the point F. If the same test results are plotted in terms of true stress ($= P/A$) and true strain ($= \int dl/l$) then the stress-strain relation will follow the curve A-B-C-D-F′, as shown in Fig. 10.2a. The true stress continues to rise until fracture occurs although the load drops. This is due to necking of the critical section. The metal continues to work harden. In the stress-strain relation for the mild steel (low and medium carbon structural steels) on the portion of curve drawn by dotted line in Fig. 10.2b, two yield points C_u and C_l are observed. At the *upper yield point* C_u, the elastic behaviour breaks down in an unstable manner and at the *lower yield point*

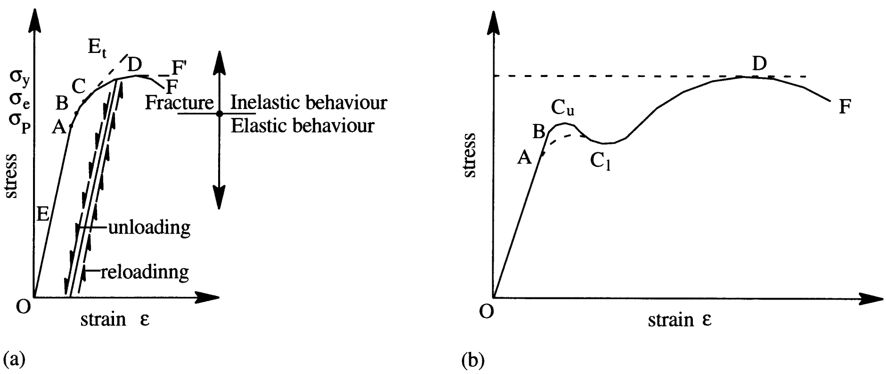


Fig. 10.2a,b. Stress-strain relationships for structural metals. **a** High carbon steel, **b** low and medium carbon steels

C_l appreciable plastic deformation occurs at the almost constant load. Beyond the lower yield point C_l *plastic flow* takes place i.e. strain increases while the stress remains constant. The upper yield point is not ordinarily observed. After large plastic strain, the stress starts to go up again which is known as *strain hardening* of the material.

As mentioned above beyond elastic limit the strain become function of stress and for many materials the yield point is not observed clearly, hence an operating level of strain is required to determine the modulus. In such cases, the 0.2 per cent *offset yield point* i.e. the point at which a *residual or permanent strain* of 0.002 is produced, is often used as the definition of initial yield value. The differences among the proportional limit σ_p, the elastic limit σ_e and initial yield point are usually small. Thus a linear elastic stress-strain relationship called *Hooke's law* is assumed up to the yield point for convenience,

$$\sigma = E\varepsilon \quad \text{for} \quad \sigma \leq \sigma_y$$

in which the proportionality constant E is *modulus of elasticity*. Typical values of E for steel and aluminum are $207 \times 10^6\,\mathrm{kN/m^2}$ and $73 \times 10^6\,\mathrm{kN/m^2}$, respectively.

Beyond initial yielding, the stress-strain relation is nonlinear and slope of stress-strain curve is not constant and depend upon the stress level. The slope of stress-strain curve $\mathrm{d}\sigma/\mathrm{d}\varepsilon$ is called *tangent modulus*, E_t which is generally less than *elastic modulus, E*, thus

$$E_t = \mathrm{d}\sigma/\mathrm{d}\varepsilon \leq E \quad \text{for} \quad \varepsilon \geq \varepsilon_y \tag{10.4}$$

This implies reduction in resistance of the material due to plasticity. In the plastic range, the *elastic (reversible)* and *plastic (irreversible) strains* exist simultaneously. Thus above the initial yielding, the slope E_t varies from E to a small value with increasing stress. Figure 10.3b is a plot of variation of E_t as a function of stress σ.

It is recalled that just prior to buckling an axially loaded member is subjected to a uniform distribution of the stress over the entire cross-section i.e. all longitudinal fibres of the member correspond to the same point on the stress-strain curve. Thus it seems reasonable to assume that in the elastic range of applied axial stresses, the member at buckling will respond as if it were composed of a material whose modulus of elasticity is tangent modulus, E_t at the stress level in question, i.e.,

$$\sigma_{cr,t} = \frac{\pi^2 E_t}{(L/r)^2} \tag{10.5}$$

However, as elastic deformation occurs, bending moments develop and compressive stresses increase on the concave side and decrease on the convex side of the member. For increasing values of compressive stress, the changes due to bending are related to the bending strains by tangent modulus E_t and for the segments of the cross-section where unloading (or decrease in stress) occurs, the elastic modulus E governs the

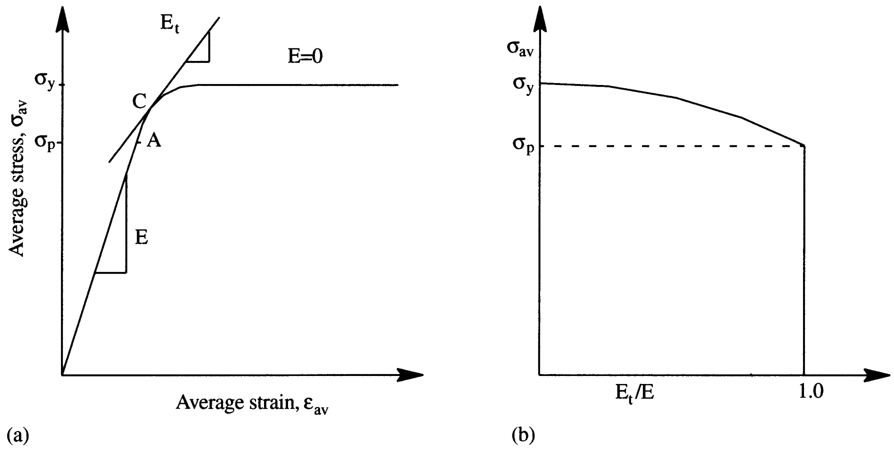


Fig. 10.3a,b. Buckling stress as a function of the tangent modulus

behaviour of each of the fibres. Thus the *effective bending stiffness* of the member which governs lateral bending deformations and therefore the buckling, is a function of both E_t and E. The effective bending stiffness of the member can be determined by the relationship

$$M = E_{\text{eff}} I\phi \quad \text{or} \quad M/\phi = E_r I \tag{10.6}$$

where E_r is the *reduced modulus* for the cross-section and I is the moment of inertia about the weaker axis of the section. Thus, there are four basic values of direct compressive stress σ in the longitudinal fibres of the member compressed by an axial load P. For an ideal pin-ended strut these are:

Euler stress,	$\sigma_{\text{cr,e}} = (\pi^2 E)/(L/r)^2$.
initial yield stress in compression,	$\sigma_{\text{cr,y}} = \sigma_y$
tangent modulus stress,	$\sigma_{\text{cr,t}} = (\pi^2 E_t)/(L/r)^2$
reduced modulus stress,	$\sigma_{\text{cr,r}} = (\pi^2 E_r)/(L/r)^2$

The variation of various buckling stresses with slenderness ratio is shown in Fig. 10.4. Hence, in the nonlinear region of stress-strain curve for the range $\sigma \leq \sigma_y$, $\sigma_{\text{cr,t}} < \sigma_{\text{cr,r}} < \sigma_{\text{cr,e}}$. At the proportionality limit ($\sigma = \sigma_y$), $\sigma_{\text{cr,t}} = \sigma_{\text{cr,e}}$. Therefore, there are four different stress ranges, and the buckling mechanism will depend upon the range which contains the yield stress.

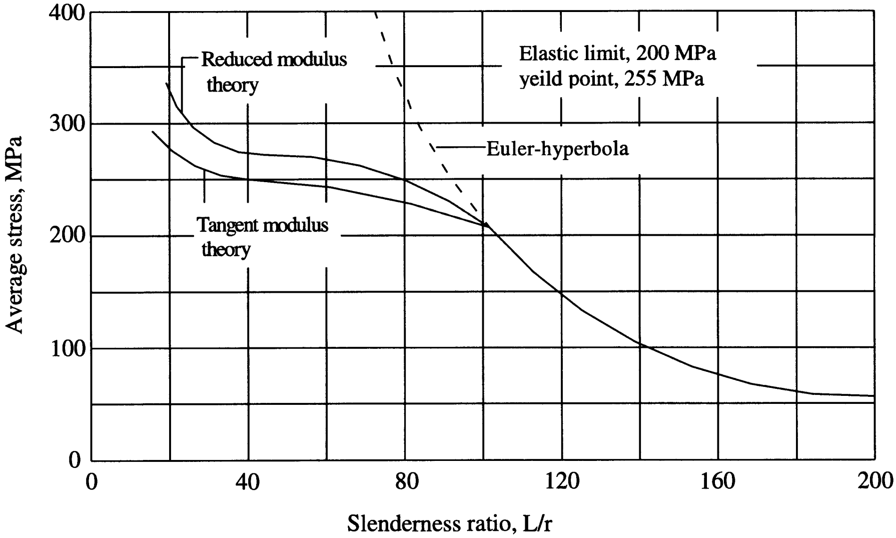


Fig. 10.4. Buckling stress as a function of the slenderness ratio

10.3 Theories of Inelastic Buckling

Based on the two different types of loading concepts discussed above there are two major theories in inelastic buckling, one is so called *tangent modulus theory* and the other is the *reduced modulus theory.* The reduced modulus theory assumed that strain reversal took place on the convex side and that such strain reversal relieved only elastic portion of the stress. The tangent modulus theory on the other hand assumes that no strain reversal takes place and that tangent modulus E_t applies over the whole cross-section.

10.3.1 Reduced Modulus Theory

The reduced modulus theory is based on the assumptions that: (i) displacements are small relative to the cross-sectional dimensions, (ii) plane sections remain plane and normal to the centre-line after bending, (iii) the relationship between stress and strain in any longitudinal fibre is given by the stress-strain curve for the material, and (iv) the plane of bending is a plane of symmetry of the column section.

Consider a perfectly straight member with symmetrical cross-section as shown in Fig. 10.5b subjected to an axial thrust P through the centroid causing a uniformly distributed normal stress, such that σ_1 $(= P/A)$ is greater than the proportionality limit. Consider the load to be further increased until member reaches the condition of unstable equilibrium. At this particular value of load, the member is assumed to buckle and therefore deflects laterally by an infinitesimal amount. In every cross-section there will be an axis NA perpendicular to the plane of bending where the stress σ developed prior to deflection remains unchanged. The bending due to deflection

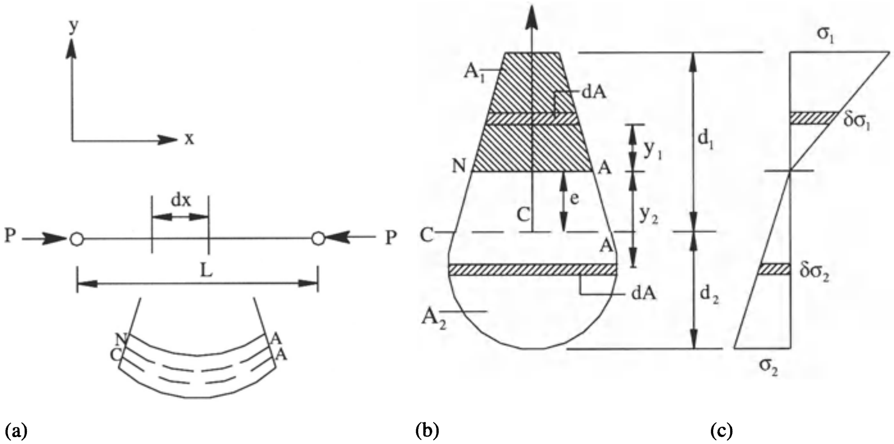


Fig. 10.5a–c. Stress distribution at the onset of bending-reduced modulus concept. **a** Straight and bent fibres, **b** cross section, **c** stress distribution

will increase the compressive stress on concave side (top in this case) and reduce it on the convex (bottom) side. The rate of increase on the concave side will be proportional to E_t, where $E_t(= d\sigma/d\varepsilon)$ is tangent modulus at the stress level σ_1. On the convex side the superposition of bending stresses relieves only the elastic portion of the strain, thus the law of proportionality of stress and strain with constant E applies. The stress diagrams are different on the two sides as shown in Fig. 10.5c.

If the curvature of centroidal axis lying at a distance d_1 from the most highly compressed fibre is represented by $1/R$, and $d\phi$ denotes the angle subtended by two originally parallel normal sections ds ($\approx dx$) apart, then $d\phi = ds/R$. From the standard pure bending relationship: $M/I = E/R = \sigma/y$,

$$\sigma = \frac{E}{R}y \tag{10.8}$$

The total compression added and relieved on two sides of the neutral axis NA are $\int_{A_1} \delta\sigma_1 \, dA$ and $\int_{A_2} \delta\sigma_2 \, dA$, respectively. Thus for P to remain unaltered by deformation

$$\int\limits_{A_1} \delta\sigma_1 \, dA - \int\limits_{A_2} \delta\sigma_2 \, dA = 0 \quad \text{or} \quad \frac{E_t}{R}\int\limits_{A_1} y_1 \, dA - \frac{E}{R}\int\limits_{A_2} y_2 \, dA = 0 \tag{10.9a}$$

or

$$E_t/E = \eta = \left[\int\limits_{A_2} y_2 \, dA\right]\left[\int\limits_{A_1} y_1 \, dA\right]^{-1} \tag{10.9b}$$

For the moment equilibrium on the cross-section

$$\int_{A_1} \delta\sigma_1(y_1+e)\,\mathrm{d}A + \int_{A_2} \delta\sigma_2(y_2-e)\,\mathrm{d}A = M = Py$$

or

$$\frac{E_\mathrm{t}}{R}\int_{A_1} y_1(y_1+e)\,\mathrm{d}A + \frac{E}{R}\int_{A_2} y_2(y_2-e)\,\mathrm{d}A = Py$$

or

$$\frac{E_\mathrm{t}}{R}\int_{A_1} y_1^2\,\mathrm{d}A + \frac{E}{R}\int_{A_2} y_2^2\,\mathrm{d}A + e\left[\frac{E_\mathrm{t}}{R}\int_{A_1} y_1\,\mathrm{d}A - \frac{E}{R}\int_{A_2} y_2\,\mathrm{d}A\right] = Py \quad (10.10)$$

In view of (10.9a), the second term on the left hand side vanishes. Hence

$$\frac{1}{R}(E_\mathrm{t}I_1 + EI_2) = Py \quad \text{or} \quad -\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(E_\mathrm{t}I_1 + EI_2) = Py \quad (10.11)$$

where I_1 and I_2 are second moment of cross-sectional areas separated by the NA. The effective bending stiffness of the member can be determined by:

$$M = -(E_\mathrm{eff}I)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad (10.12)$$

or

$$E_\mathrm{eff}I\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Py = 0$$

Comparing (10.11) and (10.12)

$$E_\mathrm{eff} = \frac{1}{I}(E_\mathrm{t}I_1 + EI_2) = \frac{E}{I}\left[\left(\frac{E_\mathrm{t}}{E}\right)I_1 + I_2\right] \quad (10.13)$$

$$\frac{E_\mathrm{eff}}{E} = \eta_r = \left(\frac{E_\mathrm{t}}{E}\frac{I_1}{I} + \frac{I_2}{I}\right) = \left(\eta\frac{I_1}{I} + \frac{I_2}{I}\right)$$

where the ratio $\eta_r = E_\mathrm{eff}/E$ and $\eta = E_\mathrm{t}/E$ is termed *modifying factor or plasticity reduction factor*. I is the total moment of inertia of the cross-section about the axis through centroid. E_eff is the *effective modulus* which is also known as *reduced modulus* E_r, or *double modulus* or the *Von Karman modulus*. In the elastic range $E_t = E$ and the formula yields $E_\mathrm{r} = E$ as can be expected. It should be noted that the value of E_r depends upon the shape of cross-section and properties of the materials. Like E, E_r is also independent of the abscissa. Equation (10.12) has the same form as differential equation for perfectly elastic column in the state of unstable equilibrium. Thus (10.12) is valid in the elastic as well as in the inelastic range. In the inelastic range E_r is variable and depends upon stress level σ $(= P/A)$ while in elastic range E_r becomes equal to E. Therefore, in the inelastic range, the bending rigidity is $E_\mathrm{r}I$

rather than EI. If the bending rigidity of a perfectly elastic column is replaced by reduced bending rigidity, then

$$P_{\mathrm{cr,r}} = \frac{\pi^2 E_r I}{L^2} \quad \text{and} \quad \sigma_{\mathrm{cr,r}} = \frac{\pi^2 E_r}{(L/r)^2} \tag{10.14}$$

Introducing the ratio $\eta_{\mathrm{r}} = E_{\mathrm{r}}/E$, the fundamental governing equation assumes the form

$$EI\eta_{\mathrm{r}} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Py = 0$$

where $\eta_{\mathrm{r}} = \left(\frac{E_{\mathrm{t}}}{E}\frac{I_1}{I} + \frac{I_2}{I}\right)$. From (10.14)

$$\sigma_{\mathrm{cr,r}} = \frac{\pi^2 E \eta_{\mathrm{r}}}{(L/r)^2} \tag{10.15}$$

The additional subscript r indicates that the values are calculated with respect to the reduced modulus. For $E_t/E < 1.0$, the bracketed term of (10.13) is always less than 1.0, since $I_1 + I_2 > I$ (as I_1 and I_2 refer to an axis NA which does not coincide with the centre of gravity), and therefore E_{r} is always less than E but greater than tangent modulus E_{t}. Thus the reduced modulus buckling load will always be greater than the tangent modulus buckling load. The actual magnitude of the increase depends on the stress-strain relationship, magnitude of average stress prior to buckling, and the cross-section of the member.

10.3.2 Tangent Modulus Theory

The tangent modulus theory is based on the presumption that there is no strain reversal when the compression member passes from a straight to a bent configuration and that the tangent modulus, E_{t} applies over the whole cross-section. In the other words it means that bending may proceed simultaneously with increase in the axial load. According to Stanley there is a continuous spectrum of deflected configurations corresponding to the values of axial load P between tangent modulus load $P_{\mathrm{cr,t}}$ and reduced modulus load $P_{\mathrm{cr,r}}$. The deflection y associated with a load has a definite value and increases from zero to infinity when P varies from $P_{\mathrm{cr,t}}$ to $P_{\mathrm{cr,r}}$. The modulus corresponding to the stage where there is no strain reversal is the local tangent modulus. Thus the differential equation of the deflected centre line takes the form.

$$E_{\mathrm{t}} I \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Py = 0$$

and upon introducing $E_t/E = \eta$, the buckling load $P_{\mathrm{cr,t}}$ and hence critical stress, therefore, is defined by:

$$\sigma_{\mathrm{cr,t}} = \frac{P_{\mathrm{cr,t}}}{A} = \frac{\pi^2 E \eta}{(L/r)^2} \tag{10.16}$$

where η and consequently $P_{cr,t}$ are not affected by the shape of the column cross-section and depend only on elastic-plastic properties of the material. For commonly used metals η lies between 0.80 and 0.95.

In general, in the inelastic range, the load corresponding to true buckling is somewhere between tangent modulus load $P_{cr,t}$ and reduced modulus load, $P_{cr,r}$. In the elastic range, the Euler solution governs. However, the experimentally determined values of critical load from carefully conducted tests by Shanley provided a much greater degree of correspondence with the *tangent modulus load* than did with *reduced modulus load.*

It is important to note the basic difference between elastic and inelastic buckling theories. For elastic buckling the bending stiffness of a member is constant and there is a unique buckling load associated with given bending stiffness and prescribed set of member geometrical parameters. On the other hand for inelastic buckling, the bending stiffness is reduced from that in elastic range, but the extent of this reduction depends upon the magnitude of applied load at the time of buckling. Thus for the prediction of *inelastic buckling load* at least one assumption has to be made in order to provide sufficient operating condition for the solution. Theoretically, an infinite number of inelastic buckling loads, all bounded by $P_{cr,t}$ and $P_{cr,r}$ can be determined depending upon the assumed strain distribution.

The difference between tangent modulus and reduced modulus buckling loads is relatively small for most practical cases. Moreover, the inherent imperfections in the member namely lack of straightness, applied thrust not being precisely located along the centroidal axis tend to reduce the critical load of the member. Thus a prediction of buckling loads based on tangent modulus theory provides an adequate basis for the development of working load formulae.

Equations (10.15) and (10.16) may be generalised to include effect of restraints at the ends of the member as:

$$\sigma_{cr} = \frac{\pi^2 E\eta}{(KL/r)^2} \tag{10.17}$$

where KL represents the effective length of the compression member. Following example will illustrate the dependence of E_r on the cross-section of columns.

Example 10.1. Determine the coefficient η_r required for the estimation of inelastic buckling load for an axially loaded column, when the cross-section is: (a) rectangle with width b and depth d, and (b) an I-section with negligibly small web thickness.

(a) *Rectangular cross-section* (shown in Fig. 10.6a). From (10.9b):

$$\frac{E_t}{E} = \eta = \frac{\int\limits_{A_2} y\,dA}{\int\limits_{A_1} y\,dA} = \frac{\int\limits_0^{y_2} y(b\,dy)}{\int\limits_0^{y_1} y(b\,dy)} = \frac{(1/2)by_2^2}{(1/2)by_1^2} = \left(\frac{y_2}{y_1}\right)^2 \tag{a}$$

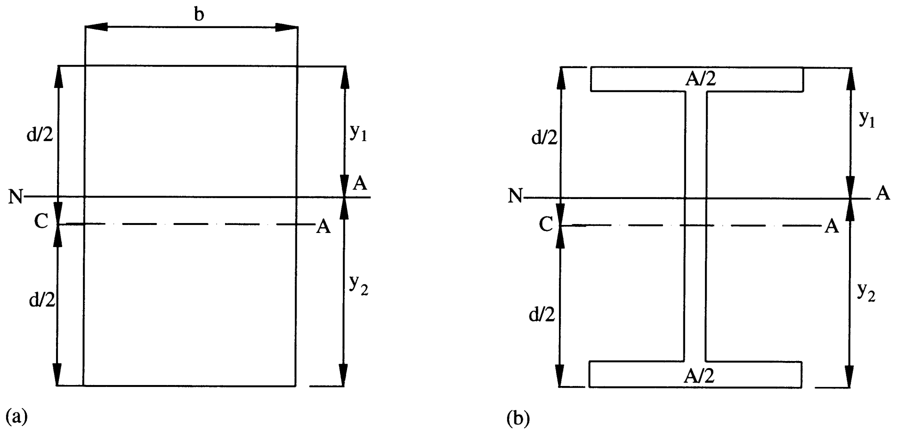


Fig. 10.6a,b. Reduced modulus for different cross sections. **a** Rectangular section, **b** I-section with thin web (idealized)

In addition

$$y_1 + y_2 = d \quad \text{or} \quad y_2 = d - y_1 \tag{b}$$

From equations (a) and (b)

$$y_1 = \frac{d}{1+\sqrt{\eta}} \quad \text{and} \quad y_2 = \frac{\sqrt{\eta}d}{1+\sqrt{\eta}} \tag{c}$$

From (10.13)

$$\eta_r = \left(\eta\frac{I_1}{I} + \frac{I_2}{I}\right)$$

where

$$I_1 = \frac{by_1^3}{3}, \quad I_2 = \frac{by_2^3}{3} \quad \text{and} \quad I = \frac{bd^3}{12} \tag{d}$$

Therefore,

$$\eta_r = 4\left[\eta\left(\frac{y_1}{d}\right)^3 + \left(\frac{y_2}{d}\right)^3\right] \tag{e}$$

Substituting from equation (c) into equation (e):

$$\eta_r = \frac{4\eta}{\left(1+\sqrt{\eta}\right)^2}$$

Substituting for η and η_r

$$E_r = \frac{4EE_t}{\left(\sqrt{E}+\sqrt{E_t}\right)^2} \tag{f}$$

(b) *I-Section with negligible web thickness.* In this case for simplicity half the cross-sectional area is assumed to be lumped at each flange and the area of the web is neglected as shown in Fig. 10.6b. From (10.9b)

$$\eta = \frac{(A/2)y_2}{(A/2)y_1} = \frac{y_2}{y_1} \tag{a}$$

In addition

$$y_1 + y_2 = d \tag{b}$$

From equations (a) and (b)

$$y_1 = \frac{d}{1+\eta} \quad \text{and} \quad y_2 = \frac{\eta d}{1+\eta} \tag{c}$$

From (10.13)

$$\eta_r = \left(\eta \frac{I_1}{I} + \frac{I_2}{I}\right)$$

where $I_1 = (A/2)y_1^2$, $I_2 = (A/2)y_2^2$ and $I = A(d/2)^2$. Therefore,

$$\eta_r = 2\left[\eta\left(\frac{y_1}{d}\right)^2 + \left(\frac{y_2}{d}\right)^2\right]$$

Substituting from equation (c)

$$\eta_r = 2\left[\frac{\eta}{(1+\eta)^2} + \frac{\eta^2}{(1+\eta)^2}\right] = \frac{2\eta}{1+\eta} \tag{d}$$

Substituting for η and η_r

$$E_r = \frac{2EE_t}{(E+E_t)} \tag{e}$$

10.4 Eccentrically Loaded Columns

In the eccentrically loaded column, the bending and direct stresses occur simultaneously from the start and grow together with increasing load P. Every cross-section will be subjected to stress $\sigma = \sigma_0 + \sigma_b$ where σ_0 represents the average stress $(= P/A)$, and σ_b denotes the stresses due to bending shown by shaded portion in Fig. 10.7d. Consider an initially straight column of rectangular cross-section of width b and depth d loaded by compressive load P acting at an eccentricity of e from the centroidal axis of the cross-section.

The conditions of equilibrium assume the form:

$$\int_{A_1} \sigma_b \, dA - \int_{A_2} \sigma_b \, dA = 0 \quad \text{or} \quad b\int_{y_2}^{y_1} \sigma_b \, dy = 0 \tag{10.18}$$

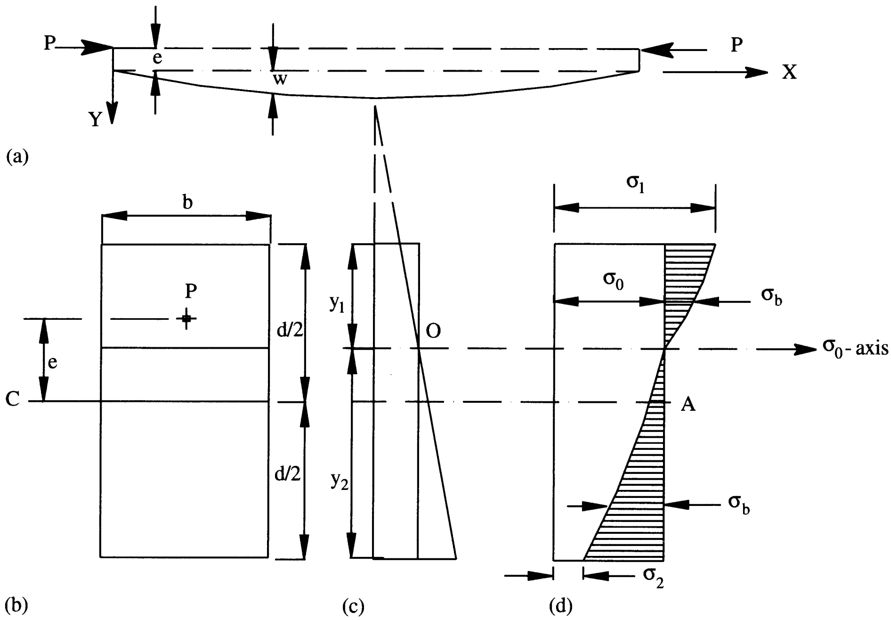


Fig. 10.7a–d. Stresses in eccentrically loaded column. **a** Eccentrically loaded member, **b** cross-section, **c** strain diagram, **d** stress diagram

and

$$b\int_{y_2}^{y_1} \sigma_b y\,dy = P(e + w) \tag{10.19}$$

where the deflection w refers to the centroidal axis of the column, and y_1 and y_2 are distances of extreme fibres from the σ_0-axis on concave and convex sides, respectively. In Fig. 10.7d, σ_1 and σ_2 represent extreme fibre stresses including bending. If ε_1 and ε_2 are strains corresponding to the stresses σ_1 and σ_2, respectively, then

$$\frac{1}{R} = \frac{\varepsilon_1 - \varepsilon_2}{d} \quad \text{or} \quad -\frac{d^2w}{dx^2} = \frac{\varepsilon_1 - \varepsilon_2}{d} \tag{10.20}$$

Defining modulus $\overline{E}$ as

$$\overline{E} = (\sigma_1 - \sigma_2)/(\varepsilon_1 - \varepsilon_2) \tag{10.21}$$

Equation (10.20) reduces to:

$$-\frac{d^2w}{dx^2} = \frac{\sigma_1 - \sigma_2}{\overline{E}d} \tag{10.22}$$

The stresses σ_1 and σ_2 are given by:

$$\sigma_1 = \frac{P}{A} + \frac{Md}{2I} \quad \text{and} \quad \sigma_2 = \frac{P}{A} - \frac{Md}{2I} \tag{10.23}$$

where the bending moment M at any section is equal to $P(e+w)$. Introducing (10.23), (10.22) reduces to

$$\frac{d^2w}{dx^2} = -\frac{M}{\overline{E}I} \tag{10.24}$$

If the shape of the deflected centre line is assumed to be:

$$w = w_m \sin\frac{\pi x}{L}$$

The curvature at the mid-length of the column may be expressed by

$$\frac{1}{R_m} = -\left[\frac{d^2w}{dx^2}\right]_{x=\frac{L}{2}} = \frac{\pi^2}{L^2}w_m \tag{10.25}$$

Substituting (10.25) into (10.24)

$$w_m = \frac{P(e+w_m)L^2}{\pi^2\overline{E}I}$$

or

$$w_m = \frac{PeL^2}{(\pi^2\overline{E}I - PL^2)} = \frac{e}{\left[\frac{\pi^2\overline{E}}{\sigma_0(L/r)^2} - 1\right]} \tag{10.26}$$

where r is radius of gyration of the cross-section given by $r = \sqrt{I/A}$.

The stresses in the extreme fibres (i. e. at $y = \pm d/2$) at the mid-length of column are given by (10.23):

$$\sigma_1 = \sigma_0\left[1 + \frac{ed}{2r^2}\left\{\frac{1}{1-\frac{\sigma_0(L/r)^2}{\pi^2\overline{E}}}\right\}\right] \tag{10.27a}$$

and

$$\sigma_2 = \sigma_0\left[1 - \frac{ed}{2r^2}\left\{\frac{1}{1-\frac{\sigma_0(L/r)^2}{\pi^2\overline{E}}}\right\}\right] \tag{10.27b}$$

In (10.27) $\overline{E}$ being the function of σ_1 and σ_2 (by definition) is unknown. If the stress-strain curve for the material of the column and σ_0 are given, an iterative procedure can be used to solve the problem. The procedure is based on assuming a value of $\overline{E}$ and computing σ_1 and σ_2 from (10.27) and hence ε_1 and ε_2 are estimated from the stress-strain curve. The new value of $\overline{E}$ is obtained from (10.21). The process is repeated until value of $\overline{E}$ is obtained to the desired accuracy. Once $\overline{E}$ is known, the deflection at the centre of column is obtained from (10.26).

As pointed out earlier the design of compression member for its particular loading and end conditions is based on the elastic-plastic properties of the material of the member. Therefore, it is logical and convenient to base the design in all cases of instability upon the ideal column curve representing the effect of elastic-plastic properties of the material on the strength of member controlled by slenderness ratio, L/r. An inspection of the relationship between critical stress σ_{cr} and slenderness ratio of an ideal column of structural steel reveals that the shape of the curve is controlled essentially by the modulus of elasticity E, which defines the Euler hyperbola, and proportional limit σ_p and yield point stress σ_y, marking the inelastic range. The curve of inelastic range derived from tangent modulus concept can be modelled by the quadratic parabola

$$\sigma_{cr} = \sigma_y - \frac{\sigma_y - \sigma_p}{(L/r)_p^2}\left(\frac{L}{r}\right)^2 \tag{10.28}$$

The coefficient of second term is defined by σ_p, σ_y and by the slenderness ratio $(L/r)_p$ which corresponds to the critical stress $\sigma_{cr} = \sigma_p$. This slenderness ratio is given by

$$\left(\frac{L}{r}\right)_p^2 = \frac{\pi^2 E}{\sigma_p} \tag{10.29}$$

Introduction of (10.29) into (10.28) results in

$$\sigma_{cr} = \sigma_y - \frac{\sigma_p(\sigma_y - \sigma_p)}{\pi^2 E}\left(\frac{L}{r}\right)^2 \tag{10.30}$$

Thus for a material with distinctly marked yield or inelastic zone with known E, σ_p and σ_y the column formula can easily be established. In practice the safe working stress of an ideal column can be obtained by dividing the *ultimate average stress* given by column formula σ_{cr} by a *factor of safety*, n which takes care of accidental imperfections, unintentional eccentricities of the axial load etc. which vary over a wide range. The ultimate carrying capacity of compression members forming integral parts of structures is affected by continuity conditions at the ends connected to adjoining members, eccentricity due to member end moments from the frame, eccentric transfer of compression load from the adjacent members. These uncertainties are taken care of by the factor of safety.

Equation (10.30) can be used to derive a simple analytical expression for the ratio η $(= E_t/E)$ which plays an important role in the analysis of various buckling problems. Rewriting (10.30) in the form

$$\sigma_{cr} = A - B\left(\frac{L}{r}\right)^2 \tag{10.31}$$

where A and B are constants depending upon the characteristics σ_p, σ_y and E of the material. The critical stress σ_{cr} can also be given by (10.16):

$$\sigma_{cr} = \frac{\pi^2 E\eta}{(L/r)^2}$$

Eliminating $(L/r)^2$ from these equations

$$\eta = \frac{\sigma_{cr}(A - \sigma_{cr})}{\pi^2 EB} \tag{10.32}$$

Replacing A by σ_y and $\pi^2 EB$ by $\sigma_p(\sigma_y - \sigma_p)$ (10.32) leads to:

$$\eta = \frac{(\sigma_y - \sigma_{cr})\sigma_{cr}}{(\sigma_y - \sigma_p)\sigma_p} \tag{10.33}$$

For a given material with known σ_p and σ_y, η values can be tabulated. To account for the variations in the characteristics of materials supplied under the stipulated specifications, probability based minimum values of σ_p and σ_y are introduced.

10.4.1 Analysis of Short Columns

The relationship between carrying capacity of column σ_{cr} and slenderness ratio (L/r) given in the preceding sections overestimates the buckling load for short columns. Based on large number of experimental investigations, a number of simple relationships for a wide variety of materials are now available. Of these, most commonly used empirical relationship is power law of the form

$$\sigma_{cr} = \sigma_{co} - \kappa(L_{eff}/r)^n \tag{10.34}$$

where σ_{co} is the critical stress intercept at $(L_{eff}/r) = 0$ and n is a parameter. Both the quantities depend upon the material properties and on the manufacturing or fabrication conditions of the member (residual stress etc.) The coefficient κ is defined by bifurcation slenderness ratio $(L_{eff}/r)_b$ at which bifurcation of equilibrium position occurs. This bifurcation is considered to be the criterion of instability. At this point critical stress as obtained form (10.34) and that from Euler curve should be identical. Therefore,

$$\sigma_{co} - \kappa(L_{eff}/r)_b^n = \frac{\pi^2 E}{(L_{eff}/r)_b^2} \tag{10.35}$$

and for tangential bifurcation equating the slopes of two curves

$$-\kappa n(L_{eff}/r)_b^{n-1} = -\frac{2\pi^2 E}{(L_{eff}/r)_b^3}$$

or

$$\kappa = \frac{2\pi^2 E}{n(L_{eff}/r)_b^{n+2}} \tag{10.36}$$

Introduction of (10.36) into (10.35) leads to

$$\sigma_{co} = \frac{\pi^2 E}{(L_{eff}/r)_b^2}\left[1 + \frac{2}{n}\right] \tag{10.37a}$$

or

$$(L_{eff}/r)_b^2 = \frac{\pi^2 E}{\sigma_{co}}\left[1 + \frac{2}{n}\right] \tag{10.37b}$$

Substitution for $(L_{\text{eff}}/r)_{\text{b}}$ in (10.36) yields

$$\kappa = \frac{2\pi^2 E}{n\pi^{n+2}\left[\frac{E}{\sigma_{\text{co}}}\left(1+\frac{2}{n}\right)\right]^{(n+2)/2}} \tag{10.38}$$

Therefore, the basic relationship given by (10.34) reduces to

$$\sigma_{\text{cr}} = \sigma_{\text{co}} - \frac{\frac{2E}{n\pi^n}\left(\frac{L_{\text{eff}}}{r}\right)^n}{\left[\frac{E}{\sigma_{\text{co}}}\left(1+\frac{2}{n}\right)\right]^{(n+2)/2}} \tag{10.39}$$

Introducing non-dimensional parameters

$$\beta = \sigma_{\text{cr}}/\sigma_{\text{co}} \quad \text{and} \quad \alpha = \frac{(L_{\text{eff}}/r)}{\pi\sqrt{E/\sigma_{\text{co}}}}$$

the basic column (10.39) reduces to

$$\beta = \left[1 - \left\{\frac{2}{n}\left(\frac{n+2}{n}\right)^{-(0.5n+1)}\right\}\alpha^n\right] = \left[1 - C\alpha^n\right] \tag{10.40}$$

where $C = \frac{2}{n}\left(\frac{n+2}{n}\right)^{-(0.5n+1)}$ and corresponding κ is given by

$$\kappa = C\sigma_{\text{co}}\left[\alpha/(L_{\text{eff}}/r)\right]^n = C\sigma_{\text{co}}/\left(\pi\sqrt{E/\sigma_{\text{co}}}\right)^n \tag{10.41}$$

For different values of n different relations can be obtained. The commonly used relations are:

1. Straight line relationship ($n = 1$)

$$\beta = 1 - 0.3849\alpha$$

and
$$\kappa = 0.1225\sqrt{\sigma_{\text{co}}^3/E} \tag{10.42}$$

2. Johnson's Parabolic formula ($n = 2$)

$$\beta = 1 - 0.2500\alpha^2$$

and
$$\kappa = 0.0253\sigma_{\text{co}}^2/E \tag{10.43}$$

3. Semi-cubic formula ($n = 1.5$)

$$\beta = 1 - 0.3027\alpha^{1.5}$$

$$\kappa = 0.0544\sqrt{\sigma_{\text{co}}^5/E^3} \tag{10.44}$$

The Euler curve is

$$\beta = 1.0/\alpha^2 \tag{10.45}$$

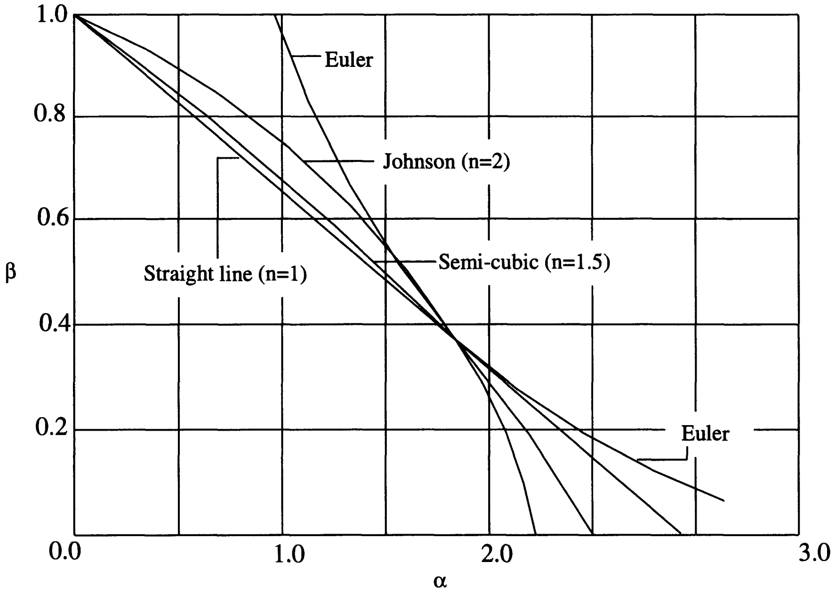


Fig. 10.8. Variation of β with α

These relations are shown in Fig. 10.8. The following example will illustrate the application of above relations.

Example 10.2. A column of hollow circular cross-section with centre line diameter of 30 mm and thickness of 1.2 mm supports an axial load over a length of 1120 mm with end fixity coefficient of 4. Estimate the critical stress at which it will buckle. For the material of the member $E = 70$ GPa and column curve is: $\sigma_{cr} = 0.25 - \kappa(L_{eff}/r)^{1.1}$.

For this problem $\sigma_{co} = 0.25$ GPa and for fixity coefficient of 4, the effective length of this column is

$$L_{eff} = L/\sqrt{4} = 560\,\text{mm}.$$

The radius of gyration is given by: $r = D/\sqrt{8}$ (for very thin tubes) or

$$\begin{aligned} r &= \sqrt{I/A} = \sqrt{(\pi \times 30^3 \times 1.2/16)/(\pi \times 30 \times 1.2/2)} \\ &= 10.607\,\text{mm} \end{aligned}$$

Slenderness ratio, $L_{eff}/r = 560/10.607 = 52.795$

From (10.40) for $n = 1.1$, $C = 0.3649$ and

$$\alpha = \frac{L_{eff}/r}{\pi\sqrt{E/\sigma_{co}}} = \frac{52.795}{\pi\sqrt{70/0.25}} = 1.0043$$

Thus, $\beta = 1.0 - 0.3649\alpha^{1.1} = 1.0 - 0.3649 \times (1.0043)^{1.1} = 0.6334$. Therefore, critical stress at bifurcation $\sigma_{cr} = \sigma_{co}\beta = 0.25 \times 0.6334 = 0.1583$ GPa.

10.5 Inelastic Buckling by Torsion and Flexure

If the stresses in a member subjected to torsion and flexure exceed the proportional limit at the instant of buckling, the modulus of elasticity E and modulus of rigidity G in any element of the member will change into E_t and G_t, respectively, where E_t and G_t are effective values according to the tangent-modulus theory. In Sect. 10.3.1, the tangent-modulus E_t is related to the elastic modulus E in the form $E_t = \eta E$ where η is dependent on the stress. On the other hand no information is available concerning tangent modulus of rigidity G_t. However, due to similarity of definitions of E_t ($= \mathrm{d}\sigma/\mathrm{d}\varepsilon$) and G_t ($= \mathrm{d}\tau/\mathrm{d}\gamma$), it is possible to introduce relation $G_t = \eta G$. This definition results in a smaller value being used than the actual G_t, which leads to lower critical stresses and is thus on safer side. With this substitution the expression for the strain energy, U given by (7.28) reduces to

$$U = \frac{1}{2}\int_0^L \left[\eta E I_y \left(u''\right)^2 + \eta E I_z \left(v''\right)^2 + \eta E I_w \left(\beta''\right)^2 + \eta G J \left(\beta'\right)^2\right]\mathrm{d}x \tag{10.46}$$

and governing Eulerian equation (7.37) reduce to

$$\begin{gathered}\eta E I_y u'''' + P u'' + P y_o \beta'' = 0\\ \eta E I_z v'''' + P v'' - P z_o \beta'' = 0, \quad \text{and}\\ P y_o u'' - P z_o (v'') + \eta E I_w \beta'''' + \left(P\frac{I_p}{A} - \eta G J\right)\beta'' = 0\end{gathered} \tag{10.47}$$

The critical loads can be determined from (10.47) and the new values of critical load or stress will be values for the elastic buckling multiplied by η. Thus the value of critical stress will be σ_{cr}/η where σ_{cr} can be determined in the same manner as in case elastic buckling. For example, consider the case of tension buckling due to flexure in a member subjected to a pure end moment, M_{oz}.

The critical value of moment $M_{oz,cr}$ for elastic torsional buckling is given by (7.66).

$$M_{oz,cr} = \left(\frac{\pi}{L}\right)\sqrt{(EI_y)(GJ) + EI_y\left(\pi^2 EI_w/L^2\right)}$$

Replacement of E and G by ηE and ηG yields

$$M_{oz,cr} = \eta\left(\frac{\pi}{L}\right)\sqrt{(EI_y)(GJ) + EI_y\left(\pi^2 EI_w/L^2\right)} \tag{10.48}$$

wherever the ratio (GJ/EI_w) appears it remains unchanged due to presumption $G\eta/E\eta = G/E$.

10.6 Lateral Buckling of Beams in the Inelastic Range

Unlike the problem of inelastic buckling of a column, due to different stresses in different elements of the beam, E_t and G_t are also variable making the problem complicated for a rational solution. However, replacement of E and G by $E_t = \eta E$ and $G_t = \eta G$ where η corresponds to the value applicable to the maximum compressive stress occurring anywhere in the beam, provides a lower limit for the critical load. Not only the critical stress obtained is a lower limit for actual critical stress, but the difference can not be very large because actual critical stress must necessarily be below the yield stress. The procedure, thus, will always result in conservative or safer values of the critical stress.

As mentioned in the preceding section, the information on tangent-modulus of rigidity G_t is lacking. However, the effective or reduced modulii E_r and G_r have been reported to satisfy the relation $E_r/E = G_r/G$.

10.7 Inelastic Buckling of Plates

The governing differential equation (8.18) of thin plates derived in Chap. 8 is valid within the range of proportionality limit. If the stresses exceed the elastic limit before buckling stress is reached, the governing differential equation need be modified. There are two schools of thought. The first one is based on two-modulus concept that in the direction in which the stress exceeds the proportional limit the tangent-modulus E_t will be effective while in the direction in which stress is within proportional limit E remains valid. In the other words anisotropic behaviour of plate is assumed when critical stress σ_{cr} lies above the proportional limit, i.e. stretching of the plate beyond proportional limit in one direction does not materially affect the elastic properties in the perpendicular direction. The other school of thought presumes the plastic deformation of the material to be isotropic i.e. stretching of plate beyond the elastic limit in one direction produces yielding in all directions, and recommends replacement of E by E_r, the reduced or effective modulus. However, the former approach has been reported to give results which are closer to the experimental results. In the following treatment former approximation is used.

As explained earlier in the Chap. 8 that the rigidity, D of the plate is $1/(1-\nu^2)$ times the stiffness EI of a beam having same width and thickness. The plate is stiffer since each plate strip is restrained by adjacent strips. In the governing differential equation (8.18), the first term within parentheses $D\,\partial^4 w/\partial x^4$ is analogous to the differential equation of elastic curve of a typical bent strip (bar) under axial load $p_x t$ i.e. $EI\mathrm{d}^4 w/\mathrm{d}x^4$. Since these strips of the plate are stressed by longitudinal force $p_x t$, the factor E_t $(= \eta E)$ must be substituted for E when σ_x exceeds the proportional limit and hence $D\eta$ must be substituted for D. Thus the first term in the parentheses will be $\eta D\,\partial^4 w/\partial x^4$. In the same manner, the third term may be taken as bending term arising from the bending of strips running parallel to Y-axis. In the absence of external in-plane load in that direction, the normal stresses in that direction are small and the third term $D\,\partial^4 w/\partial y^4$ will remain unaffected. The second term represents interaction or coupling between the bending behaviour in the two directions and give rise to twisting moment. Since, the elastic-plastic behaviour in the two directions are different the coefficient is expected to take mean value between 1 and η. For simplicity, $\sqrt{\eta}$ is taken to be coefficient for the second term. Thus (8.18) assumes the generalized form

$$D\left(\eta\frac{\partial^4 w}{\partial x^4}+2\sqrt{\eta}\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+p_x\frac{\partial^2 w}{\partial x^2}=0 \tag{10.49}$$

The boundary conditions for the differential equation (10.49) are obtained by modifying the expressions for the support moments M and shears Q by introducing η or $\sqrt{\eta}$ in a similar manner. The expressions for transverse shear forces and moment curvatures given by (8.7), (8.8) and (8.17) change to

$$\begin{aligned}
Q_x &= -D\frac{\partial}{\partial x}\left(\eta\frac{\partial^2 w}{\partial x^2}+\sqrt{\eta}\frac{\partial^2 w}{\partial y^2}\right)\\
Q_y &= -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2}+\sqrt{\eta}\frac{\partial^2 w}{\partial x^2}\right)\\
M_x &= -D\left(\eta\frac{\partial^2 w}{\partial x^2}+\nu\sqrt{\eta}\frac{\partial^2 w}{\partial y^2}\right)\\
M_y &= -D\left(\nu\sqrt{\eta}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \quad \text{and}\\
M_{xy} &= -M_{yx} = D\sqrt{\eta}(1-\nu)\frac{\partial^2 w}{\partial x\partial y}
\end{aligned} \tag{10.50}$$

10.7.1 Plates Subjected to Uniaxial Loading

The theoretical results derived in Chap. 8 for perfectly elastic plate could be applied in the inelastic range by replacing E by E_t ($=\eta E$) and D by D_t ($=\eta D$). The expression for critical load given by (8.29) reduces to

$$p_{x,\text{cr}}=\sigma_{x,\text{cr}}t=\left(\frac{k_1 m}{\mu}\right)^2\left(\frac{\pi^2\eta D}{b^2}\right)$$

or

$$\sigma_{x,\text{cr}}=\left(\frac{k_1 m}{\mu}\right)^2\left(\frac{\pi^2\eta D}{tb^2}\right)=\left(\frac{k_1 m}{\mu}\right)^2\left[\frac{\pi^2\eta E}{12(1-\nu^2)}\right]\left(\frac{t}{b}\right)^2 \tag{10.51}$$

The general solution to the modified differential equation (10.49) satisfying boundary conditions on all four edges can be obtained by using the procedure followed in Chap. 8. For a flat plate loaded on two simply-supported edges $x=0$ and $x=a$ by uniformly distributed compressive load, p_x and having general conditions at the other two edges, (8.33) gets modified for inelastic range to

$$\frac{\text{d}^4 f(y)}{\partial y^4}-2\sqrt{\eta}\left(\frac{m\pi}{a}\right)^2\frac{\text{d}^2 f(y)}{\text{d}y^2}+\eta\left(\frac{m\pi}{a}\right)^4(1-k_1^2)f(y)=0 \tag{10.52}$$

with the general solution

$$f(y)=A\sinh\alpha y+B\cosh\alpha y+C\sin\beta y+D\cos\beta y \tag{10.53}$$

where

$$(\alpha b)^2 = \left(\frac{m\pi}{\mu}\right)^2 \sqrt{\eta}(k_1 + 1) \quad \text{and} \quad (\beta b)^2 = \left(\frac{m\pi}{\mu}\right)^2 \sqrt{\eta}(k_1 - 1) \tag{10.54}$$

The constants A, B, C and D are determined such that boundary conditions at two edges $y = 0$ and $y = b$ are satisfied. The general stability condition given by transcendental equation (8.43) becomes

$$\begin{aligned}(k_1 + 1)^{1/2} \tanh &\left[\left(\frac{m\pi}{2\mu}\right)(\eta)^{1/4}(k_1 + 1)^{1/2}\right] \\ &+ (k_1 - 1)^{1/2} \tan\left[\left(\frac{m\pi}{2\mu}\right)(\eta)^{1/4}(k_1 - 1)^{1/2}\right] + \zeta\left(\frac{m\pi k_1}{2\mu}\right)(\eta)^{1/4} = 0\end{aligned} \tag{10.55}$$

Special Cases

(i) Plate simply supported along the unloaded edges

The expression (8.44) giving smallest root reduces to

$$k_1^2 = \left[\left(\frac{\mu}{m}\right)^2 \sqrt{\eta} + 1\right]^2 \tag{10.56}$$

The limiting ratio $\overline{\mu}$ at which either m or $m + 1$ half-waves can occur given by (8.47) becomes

$$\overline{\mu} = (\eta)^{1/4}[m(m + 1)]^{1/2} \tag{10.57}$$

For $m = 1, 2, 3, 4\ldots$; $\left\{\overline{\mu}/(\eta)^{1/4}\right\} = \sqrt{2}, \sqrt{6}, \sqrt{12}, \sqrt{20}\ldots$

In the elastic range when $\eta = 1$, the number of half-waves become independent of the nature of material. In the inelastic range $\eta < 1$, the waves become shorter.

(ii) Plate elastically supported along the unloaded edges

For elastically restrained edges the relation given by (8.49) can be expressed as

$$\left(\frac{mk_1}{\mu}\right)^2 = k^2 = \left(\frac{m}{\mu}\right)^2 \sqrt{\eta} + p + q\left(\frac{\mu}{m}\right)^2 \frac{1}{\sqrt{\eta}} \tag{10.58}$$

and equation for $\sigma_{x,\text{cr}}$ assumes the standard form

$$\sigma_{x,\text{cr}} = \frac{\pi^2 E k^2 \sqrt{\eta}}{12(1 - \nu^2)}\left(\frac{t}{b}\right)^2 \tag{10.59}$$

where k is a non-dimensional coefficient that depends on the μ, and on the boundary conditions at the unloaded edges. Equation (10.57) giving limiting value $\overline{\mu}$ assumes the form

$$\overline{\mu} = \left(\frac{\eta}{q}\right)^{1/4} [m(m + 1)]^{1/2} \tag{10.60}$$

As explained earlier in Chap. 8, q is reported to lie between 1 and 5, $q = 1$ for simply supported edges a and $q = 5$ for clamped edges a. Therefore,

$$\text{for simply-supported edges,} \quad \overline{\mu} = (\eta)^{1/4}\,[m(m+1)]^{1/2} \tag{10.61a}$$

$$\text{for clamped edges,} \quad \overline{\mu} = 0.6687(\eta)^{1/4}\,[m(m+1)]^{1/2} \tag{10.61b}$$

The value μ_o which makes $\sigma_{x,\mathrm{cr}}$ a minimum is given by

$$\frac{\partial \sigma_{x,\mathrm{cr}}}{\partial \mu} = 0 \quad \text{i.e.} \quad \mu_o = m\left(\frac{\eta}{q}\right)^{1/4} \tag{10.62}$$

and corresponding k^2 is given by

$$k^2 = \sqrt{\eta}(p + 2\sqrt{q}) \tag{10.63}$$

Here the expression $(p + 2\sqrt{q})$ itself becomes independent of η.

(iii) Asymmetrical elastic supports

(a) Elastically restrained at $y = 0$ and free at $y = b$

With the origin of coordinate axes taken to be coinciding with the corner of plate such that X-axis is along the supported edge, the stability condition for the inelastic range can be obtained from that of elastic range condition given by (8.61).

$$\begin{aligned} &2\overline{\alpha}\overline{\beta} + (\overline{\alpha}^2 + \overline{\beta}^2)\cosh \alpha b \cos \beta b - \left[\left(\frac{\alpha}{\beta}\right)\overline{\beta}^2 - \left(\frac{\beta}{\alpha}\right)\overline{\alpha}^2\right]\sinh \alpha b \sin \beta b \\ &\quad + \gamma\left[\overline{\alpha}^2 \sinh \alpha b \cos \beta b - \overline{\beta}^2\left(\frac{\alpha}{\beta}\right)\cosh \alpha b \sin \beta b\right] = 0 \end{aligned} \tag{10.64}$$

where $\alpha b = (m\pi/\mu)\,(\eta)^{1/4}\,(k_1 + 1)^{1/2}$ and $\beta b = (m\pi/\mu)\,(\eta)^{1/4}\,(k_1 - 1)^{1/2}$

$$\overline{\alpha} = \frac{1}{b^2}\left(\frac{m\pi}{\mu}\right)^2 (\eta)^{1/2}\,(k_1 + 1 - \nu) \quad \text{and} \quad \overline{\beta} = \frac{1}{b^2}\left(\frac{m\pi}{\mu}\right)^2 (\eta)^{1/2}\,(k_1 - 1 + \nu) \tag{10.65}$$

The critical stress is given by

$$\sigma_{x,\mathrm{cr}} = \frac{\pi^2 E k^2 \sqrt{\eta}}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 \tag{10.66}$$

where

$$k^2 = \left\{\left(\frac{m}{\mu}\right)^2 (\eta)^{1/2} + p + q\left(\frac{\mu}{m}\right)^2 \left(\frac{1}{\eta}\right)^{1/2}\right\}$$

The limiting value of aspect ratio $\overline{\mu}$ at which either m or $m+1$ half-waves can exist, i. e.

$$\overline{\mu} = \left(\frac{\eta}{q}\right)^{1/4} [(m+1)m]^{1/2} \tag{10.67}$$

For minimum value of $\sigma_{x,\mathrm{cr}}$

$$\mu_o = m\left(\frac{\eta}{q}\right)^{1/4} \tag{10.68}$$

and

$$k^2 = \left(p + 2\sqrt{q}\right) \tag{10.69}$$

The half-wave length is $\lambda = \beta b(\eta)^{\frac{1}{4}}$. Here k^2 is independent of η. Thus,

$$\min \sigma_{x,\mathrm{cr}} = \frac{\pi^2 E\sqrt{\eta}}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 \left(p + 2\sqrt{q}\right) .$$

(b) Plate hinged at $y = 0$ and free at $y = b$

It is evident from (10.68) that with decreasing q (i. e. elastic restraint) μ_o increases steadily. In the limiting case of a simply supported edge $y = 0$, $q = 0$, the ratio μ_o will be infinite indicating that plate buckles in one half-wave only. With increasing μ_o the critical stress min $\sigma_{x,\mathrm{cr}}$ decreases asymptotically to the value

$$\operatorname{Min} \sigma_{x,\mathrm{cr}} = \frac{\pi^2 E\sqrt{\eta}}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 p$$

This value is absolute minimum. If one edge ‘a’ of the plate is free to rotate the plate will buckle in one half-wave regardless of its length. If one edge is elastically built, the plate will buckle in several half-waves for a long plate as is indicated by (10.67).

10.7.2 Plate Subjected to In plane Biaxial Loading

For a plate subjected to in-plane biaxial loading, the stresses along both X- and Y-directions may exceed the elastic limit. In such a case, a reduced modulus of elasticity is effective in both the directions, therefore, buckling load or critical stress in the inelastic range can be obtained by replacing E by E_r ($\approx \eta E$) and D by D_r ($\approx \eta D$), in the values for elastic range.

Example 10.3. A rectangular plate of size $a \times b$ is simply supported at the loaded edges $x = 0$ and $x = a$, and is clamped at the unloaded edges $y = 0$ and $y = b$. Determine the critical load in the inelastic range if $\eta = 0.9$.

The stability condition for this case is given by (8.70) as

$$-(k_1^2 - 1)^{-1/2}\,(\sinh \alpha b.\sin \beta b) + (\cosh \alpha b.\cos \beta b) - 1 = 0 \tag{10.70}$$

where the parameters αb and βb are defined by (10.65) as

$$\alpha b = \left(\frac{m\pi}{\mu}\right)(\eta)^{1/4}(k_1 + 1)^{1/2} \quad \text{and}$$
$$\beta b = \left(\frac{m\pi}{\mu}\right)(\eta)^{1/4}(k_1 - 1)^{1/2} \tag{10.71}$$

The critical load $p_{x,\mathrm{cr}}$ is given by

$$p_{x,\mathrm{cr}} = \frac{\pi^2 D\sqrt{\eta}}{b^2} k_m^2$$

where

$$k_m^2 = \left(\frac{m k_1}{\mu}\right)^2 \tag{10.72}$$

the values k_m^2 for various aspect ratios are given in Table 10.1.

Table 10.1. Values of k_m^2 for various aspect ratios

Aspect ratio, μ	Plasticity reduction factor, η ($= E_t/E$)				
	1.00	0.95	0.90	0.85	0.80
0.4	9.4479	9.5549	9.6719	9.8007	9.9431
0.6	7.0552	7.2170	7.3949	7.5916	7.8105
0.8	7.3037	7.5414	7.8037	8.0947	8.4196
1.0*	7.6913	8.9396	9.3102	9.7221	10.1831

*At $\mu = 0.8910\,(\eta)^{1/4}$, the buckling mode changes from one half-wave to two-half waves. For $\mu = 1$ the values listed in the table are not reliable.

10.8 Inelastic Buckling of the Shells

When a shell is not very thin, the buckling can occur at a stress that is beyond the proportionality limit. The critical load in this case can be obtained from (9.38) by substituting in the expression for D, the tangent-modulus E_t instead of E as in the case of flat plates. For example for the spherical shells from (9.45)

$$p_\mathrm{cr} = \frac{2E_\mathrm{t}}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{r}\right)^2 \tag{10.73}$$

In the case of cylindrical shell from (9.46) the critical load in inelastic range is given by:

$$p_{\mathrm{cr}} = \frac{t^2\sqrt{EE_{\mathrm{t}}}}{R\sqrt{3(1-\nu^2)}} \tag{10.74}$$

In this case of buckling beyond the proportionality limit, the half-wave length, λ given by (9.61b) becomes

$$\lambda = \frac{L}{m} = \pi\left[\frac{R^2t^2}{12(1-\nu^2)}\right]^{1/4}\left[\frac{E_{\mathrm{t}}}{E}\right]^{1/4} \approx 1.73\sqrt{Rt(E_t/E)} \tag{10.75}$$

Thus the length of waves becomes shorter for buckling beyond the proportionality limit. If the mechanical properties of the material beyond the proportionality limit are the same in the axial and circumferential directions, (10.75) reduces to

$$p_{\mathrm{cr}} = \frac{E_{\mathrm{t}}t^2}{\left[R\sqrt{3(1-\nu^2)}\right]} \tag{10.76}$$

and $\lambda = 1.73\sqrt{Rt}$ (for $\nu = 0.30$). That is beyond the proportionality limit the wave-length remains unchanged.

If the material has yield point, the value of E_{t} is zero at this stress and p_{cr} becomes equal to zero. Since E_{t} is function of σ_{cr} and hence of p_{cr}. The solution can be obtained by trial and modification.

Example 10.4. A long cylindrical steel tube having outside diameter of 250 mm is required to withstand an external pressure $p = 20$ MPa. Compute the thickness t of tube so as to take twice the pressure at buckling. The stress and strain diagram of the metal of the tube is shown in Fig. 10.9. Take $\nu = 0.40$. The tube is expected to buckle in the inelastic range, therefore, replacing E by E_{t} the critical pressure expression given by (9.26) becomes:

$$p_{\mathrm{cr}} = \frac{E_{\mathrm{t}}}{4(1-\nu^2)}\left(\frac{t}{R}\right)^3$$

$$\sigma_{\mathrm{cr}} = \frac{p_{\mathrm{cr}}d}{2t} = \frac{E_{\mathrm{t}}}{(1-\nu^2)}\left(\frac{t}{d}\right)^2$$

Since E_{t} is function of σ_{cr} and hence of p_{cr}, the equation need be solved by trial and modification procedure. Therefore,

$$\sigma_{\mathrm{cr}} = \frac{E_{\mathrm{t}}}{(1-\nu^2)}\left(\frac{2\times p}{2\sigma_{\mathrm{cr}}}\right)^2 = \left(\frac{E_{\mathrm{t}}}{0.84}\right)\left(\frac{2\times 20}{2\sigma_{\mathrm{cr}}}\right)^2$$

Using σ and E_{t} relation shown in Fig 10.9 trial and modification procedure provides $E_{\mathrm{t}} = 3.45 \times 10^4$ MPa and $\sigma = \sigma_{\mathrm{cr}} = 2.55 \times 10^2$ MPa satisfy the equation. Thus the thickness of the tube is

$$t = \frac{pd}{2\sigma_{\mathrm{cr}}} = \frac{2\times 20\times 250}{2\times 2.55\times 10^2} = 19.60\,\mathrm{mm} \quad (\text{say } 20\,\mathrm{mm})$$

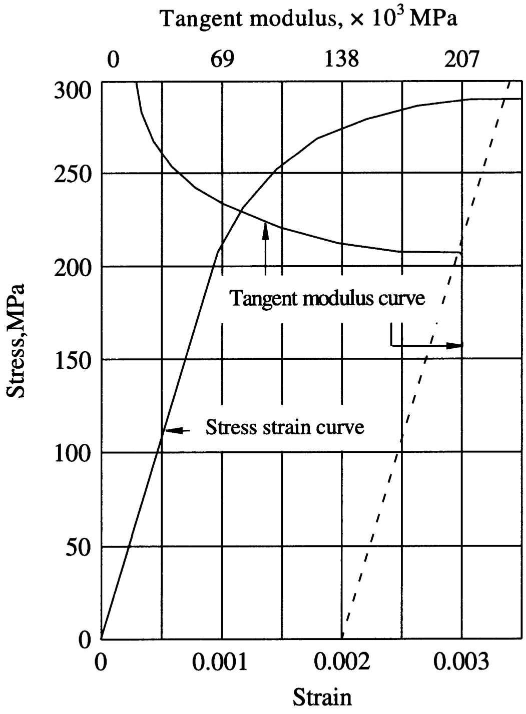
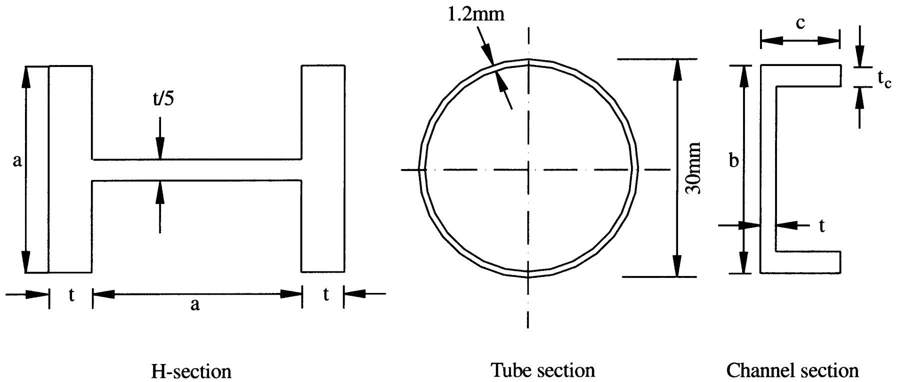


Fig. 10.9. Tangent modulus for a metal not having a yield point

10.9 Problems

Problem 10.1. Obtain the reduced modulus E_r for the sections shown in Fig. P.10.1.



P.10.1

Problem 10.2. Estimate the load carrying capacity of round alloy steel tube column of length 800 mm. The external diameter of the tube is 40 mm and thickness is 1.5 mm. The critical stress of the material σ_{co} is 1.05 GPa. The end fixity coefficient is 2.0.

[Hint: $L_{eff} = L/\sqrt{2}$.]

Problem 10.3. If the column curve for a material is represented by the equation: $\sigma_{cr} = \sigma_{co} - \kappa(L_{eff}/r)^{1.65}$. Determine the value of κ at the bifurcations point. The Euler curve is given by: $\sigma_{cr} = \pi^2 E/(L_{eff}/r)^2$. Further calculate the critical value of slenderness ratio (L_{eff}/r) and corresponding value of σ_{cr}.

Problem 10.4. A column of hollow circular cross-section of size 30 mm mean diameter and 1.2 mm thickness and of length 900 mm is subjected to an axial load. Obtain the buckling load by using straight line column formula with $\sigma_{co} = 400$ MPa, and $E = 65$ MPa. Take the end fixity coefficient as 1.5.

Problem 10.5. A rectangular plate is subjected to an in-plane compressive force p_x per unit length along the edges $x = 0$ and $x = a$. The edges $y = 0$ and $y = b$ are unloaded. If all the edges are simply supported show that the minimum value of critical load in the inelastic range is given by: $p_{x,cr} = 4\pi^2 D\sqrt{\eta}/b^2$.

Problem 10.6. A rectangular aluminum plate of size 1000×800 mm with thickness of 5 mm is simply supported on all the four edges. It is subjected to a uniform in plane compressive load in both the directions. Determine p_x in the inelastic range when $p_y = 0.8p_x$. The material properties are: $E = 60$ GPa, $\nu = 0.30$ and $\eta = 0.85$.

Problem 10.7. The channel section shown in Fig. P.10.1 is subjected to a uniform axial compressive stress. The loaded edges are assumed to be simply supported. The coefficient of restraint ζ is given by:

$$\zeta = \left(\frac{t}{t_c}\right)^3 \left(\frac{c}{b}\right) \frac{1}{\left[1 - 0.106(t^2c^2)/(t_c^2b^2)\right]}$$

where $b = 120$ mm; $c = 300$ mm; $t_c = 3$ mm and $t = 1.0$ mm. Determine the critical load in the inelastic range when $\eta = 0.90$.

Problem 10.8. Plot the interaction curve for the buckling of a rectangular plate of $\mu = 1.2$ in the inelastic range. The plate is simply supported on all the four edges and subjected to biaxial in plane compressive loads of magnitude of $p_y = 0.6p_x$.

Problem 10.9. Determine critical stresses in the inelastic range for the web plate of a girder subjected to pure shear stress τ_{xy}. The plate may be assumed to be simply supported on all the edges.

11

Structural Design For Stability Of Members

11.1 Introduction

In a narrow sense, *structural design* is an art that is concerned with determination of minimum cross-sectional geometries of structural members based upon the results of structural analysis using acceptable *performance criteria* such as allowable stress, ultimate strength, maximum deformation, stiffness or stability etc. For the stress or strength and deformation the design procedures are straight forward which ensure the realization of a particular desired state for given loading. It is very rare when stability is the controlling condition; therefore objective is to have more than a *predetermined reserve in capacity* to preclude possibility of instability at the given loading. In the other words a *margin of safety* has to be provided. When buckling is a controlling factor, the problem can be handled by adding supporting or bracing members or a sufficiently large cross-section can be selected thereby eliminating buckling as a real problem. The choice will be governed by economy and practicality of the solution.

The performance limitations for practical design situations are given in national and international *design codes* and specifications. However, these manuals of acceptable design practice vary from region to region and from country to country. Moreover, they are updated from time to time. Therefore, in this chapter only basic concepts and procedures for stability design of structural members is dealt with. The Indian code of practice IS: 800 has mainly drawn on other national standards e.g. American Institute of Steel Construction Specifications (AISCS), British Standards Institution (BS: 5940), Australian Standards Association (AS: 1250) etc. In the following discussion, the specifications are restricted to the clauses pertaining to the stability of structures.

11.2 Column Design Formula

Most of the national codal provisions are based on the following basic column formula.

$$F_e = \frac{\pi^2 E}{(KL/r)^2}$$

where K is the *effective length factor* and r is the *radius of gyration* of the cross-section. The parameter (KL/r) is termed *effective slenderness ratio* and is almost universally used for column strength formula. The values of factor K are given in Appendix B for both uniform and stepped columns.

AISCS

The specifications have based the *allowable axial compressive stress,* F_a upon the *limiting effective-slenderness ratio,* C_c corresponding to *Euler stress equal* to $0.5F_y$ i. e.

$$\frac{\pi^2 E}{(KL/r)^2} = 0.5F_y \quad \text{or} \quad \frac{KL}{r} = \left(\frac{2\pi^2 E}{F_y}\right)^{1/2} = C_c \tag{11.1}$$

(a) when $KL/r \leq C_c$

$$F_a = \frac{F_y}{FS}\left[1 - \frac{(KL/r)^2}{2C_c^2}\right] \tag{11.2}$$

where F_y is the *minimum guaranteed yield stress* in tension, KL/r is the effective slenderness ratio and FS is a *variable factor of safety* for columns buckling inelastically and is given by

$$FS = \frac{5}{3} + \frac{3}{8}\frac{(KL/r)}{C_c} - \frac{1}{8}\frac{(KL/r)^3}{C_c^3} \tag{11.3}$$

FS varies between 1.67 for $KL/r = 0$ and 1.917 $(= 23/12)$ for $KL/r = C_c$. This increase in FS of 15 per cent takes into account the following factors:

1. The increased sensitivity of long columns to variations in the effective length factor.
2. The practical difficulty in determining effective length factor.
3. Initial crookedness.

(b) when $200 > KL/r > C_c$

$$F_a = \frac{\pi^2 E}{(KL/r)^2}\frac{1}{FS} \tag{11.4}$$

where $FS = 23/12$
(c) when $200 > L/r > 120$, (K is taken as 1.0) the allowable compressive stress for bracing and secondary members is given by:

$$F_a = \frac{F_a \text{ given by (11.2) or (11.4)}}{1.6 - [(L/r)/200]} \tag{11.5}$$

BS: 449

The allowable stress is given by:

$$F_a = \frac{1}{2FS}\left[F_y + F_e(1+\eta) - \frac{1}{2FS}\sqrt{[(F_y + F_e(1+\eta)]^2 - 4F_eF_y}\right] \tag{11.6}$$

where $\eta = \delta_o B/(2r^2)$, dimensionless imperfection parameter

δ_o = mid-span deflection

B = width of the member and

F_e = Euler critical stress $= \pi^2 E/(L_e/r)^2$

For thick members, $F_a \rightarrow F_y$; for slender members, $F_a \rightarrow F_e$. Equation (11.6) is infact Perry's formula for average stress in a column at failure. Subsequent British modification use following expression for η:

$$\eta = 0.03(L_e/100r)^2 \quad \text{and} \quad FS = 1.7 \tag{11.7}$$

AS: 1250

The Australian code formula for allowable compressive stress is the same as that of BS: 449 but with $FS = 5/3$ instead of 1.7.

IS: 800

Based on experimental observations, the relation for prediction of failure load is derived from the following reciprocal formula.

$$\frac{1}{P_f^n} = \frac{1}{P_p^n} + \frac{1}{P_e^n} \tag{11.8}$$

where P_f, P_p and P_e are failure load, fully plastic load, and elastic critical load, respectively. The factor n is exponential coefficient. For low slenderness ratio L_e/r, $P_f \rightarrow P_p$ and for high L_e/r value, $P_f \rightarrow P_e$. In terms of corresponding stresses the formula can be expressed as

$$\frac{1}{(AF_f)^n} = \frac{1}{(AF_y)^n} + \frac{1}{(AF_{ec})^n}$$

or

$$\frac{1}{F_f^n} = \frac{1}{F_y^n} + \frac{1}{F_{ec}^n} \quad \text{or} \quad F_f^n = \frac{F_y^n F_{ec}^n}{F_{ec}^n + F_y^n}$$

Therefore,

$$F_f = \frac{F_y F_{ec}}{\left[F_{ec}^n + F_y^n\right]^{1/n}} = \frac{F_y}{\left[1 + (F_y/F_{ec})^n\right]^{1/n}} \tag{11.9}$$

The allowable stress F_{ac} is obtained by dividing the failure stress F_f by the factor of safety of 1.67. Hence

$$F_{ac} = \frac{F_f}{1.67} = \frac{0.6F_y}{\left[1 + \left(F_y/F_{ec}\right)^n\right]^{1/n}} \tag{11.10}$$

where F_y and F_{ec} [$= \pi^2 E/(L_e/r)^2$] are yield stress and elastic critical compressive stress, respectively. For very short columns allowable axial compressive stress is $F_{ac} = 0.6F_y$. The value of exponential coefficient n is determined by experimental results. The value of n lies in the range 1.0 to 3.0. IS: 800 adopts $n = 1.4$ and $E = 2 \times 10^5$ MPa for all Indian rolled sections. Some national standards use multiple column design curves by varying n for different sections, like open rolled sections, built-up sections, tubular sections etc.

When a column is likely to buckle about the weaker axis, it is laterally supported at certain intervals, so that the critical stress reaches yield stress and the column strength is not influenced by buckling, but depends on the yield strength. Whenever, a column fails in yielding, the allowable stress $F_{ac} = 0.60F_y$.

Example 11.1. Determine allowable axial load that a rolled steel MB 500 @ 86.9 kg/m section pin-ended column can sustain over an effective length of: (i) 1.75 m and (ii) 5.25 m. The maximum compressive residual stress F_r is $0.30F_y$. It is defined that the weak-axis flexural buckling will govern. Take $F_y = 250$ MPa and $E = 2.1 \times 10^5$ MPa. Desired factor of safety is 1.95.

For MB 500 section, $A = 11100\,\text{mm}^2$, $r_y = 35.2$ mm. The proportional limit stress, $F_p = F_y - F_r = 0.70F_y = 175$ MPa.

Case I: For $L_e = 1.75$ m, $L_e/r_y = 49.716$.

$$\text{Elastic buckling stress,} \quad F_{ec} = \frac{\pi^2 E}{(L_e/r_y)^2} = 838.55\,\text{MPa}.$$

which is considerably more than F_p. Therefore, the buckling will occur in the inelastic range. The allowable stress is given by (ref. 24 appendix-D)

$$\begin{aligned} F_a &= \frac{F_{ine,c}}{FS} = \frac{F_y}{FS}\left[1 - \left(\frac{F_r}{F_y}\right)\frac{F_p}{\pi^2 E}\left(\frac{L_e}{r_y}\right)^2\right] \\ &= \frac{250}{1.95}\left[1 - (0.30) \times \frac{0.7 \times 250}{\pi^2 \times 2.1 \times 10^5} \times (49.716)^2\right] = 120.18\,\text{MPa}. \end{aligned}$$

And allowable axial load would be

$$P_{all} = F_a A = 120.18 \times 11100(\times 10^{-3}) = 1334\,\text{kN}.$$

Case II: For $L_e = 5.25$, $L_e/r_y = 149.148$

$$\text{Elastic buckling stress,} \quad F_{ec} = \frac{\pi^2 E}{(L_e/r_y)^2} = 93.17\,\text{MPa} < 175\,\text{MPa}.$$

Therefore, the load carrying capacity is based upon elastic buckling, and the allowable load is

$$P_a = \frac{93.17}{1.95} \times 11100(\times 10^{-3}) = 530.36\,\text{kN}.$$

The allowable axial load can also be determined by the method proposed by AISCS.
From (11.1)

$$C_c = \left(\frac{2\pi^2 E}{F_y}\right)^{1/2} = 128.77$$

Case I: For KL/r_y (= 49.716) < C_c,

$$FS = \frac{5}{3} + \frac{3}{8}\left(\frac{49.716}{128.77}\right) - \frac{1}{8}\left(\frac{49.716}{128.77}\right)^3 = 1.8$$

Therefore, from (11.2)

$$F_a = \frac{F_y}{FS}\left[1 - \frac{(KL/r_y)^2}{2C_c^2}\right] = \frac{250}{1.8}\left[1 - \frac{(49.716)^2}{2 \times (128.77)^2}\right] = 128.537\,\text{MPa}.$$

F_a with a FS of 1.95 = 128.537 × 1.8/1.95 = 118.65 MPa.
Case II : For KL/r_y (= 149.148) > C_c, from (11.4)

$$F_a = \frac{\pi^2 E}{(KL/r_y)^2}\frac{1}{FS} = \left(\frac{12}{23}\right)\frac{\pi^2 \times 2.1 \times 10^5}{(149.148)^2} = 48.61\,\text{MPa}.$$

$$P_a = 48.61 \times 11100(\times 10^{-3}) = 539.58\,\text{kN}.$$

11.3 Local Plate Buckling of Structural Members

As discussed in the preceding section that when stability is a design consideration, the effective slenderness ratio is the most important parameter. The smaller the value of this ratio, the greater will be the load that the member can sustain. To achieve this objective, the cross-sectional shape is so selected that it provides largest radius of gyration about an axis perpendicular to the direction of anticipated buckling. The optimal form takes different shapes. The cross-sectional forms composed of thin elements provide cost effective solution. However, the load carrying capacity of the cross-section is noticeably affected by the *local stability* of its elements. The local stability depends on the slenderness ratio of flanges, webs or other elements. The slenderness ratio of these elements are determined by the ratio between their *characteristical dimension* (width of flange, depth of web, element width) and their thickness i.e. by b/t or h/t. Depending upon the loading, material properties and type of cross-section etc. of the element the slenderness ratio will reach a value

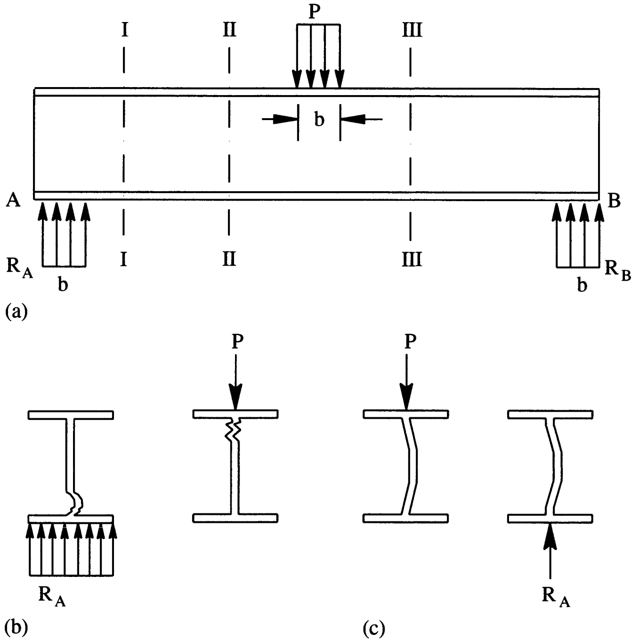


Fig. 11.1a–c. Local buckling of beam web. **a** Loaded beam, **b** crippling, **c** buckling

above which the cross-section fails to maintain its original form, and *local wrinkling* (*crippling*) or *buckling* of that individual element occurs much before the anticipated load is attained as shown in Fig. 11.1. In this section the objective is to develop criteria for prevention of local buckling of cross-sectional elements subjected to typical loading.

The commonly used sections are composed essentially of flat plate elements which are welded, glued or intermittently connected by rivets, bolts etc. The wide flange or similar type built-up cross-sections are commonly used in Civil Engineering constructions. These elements are subjected to direct compression, bending, shear or any combination thereof. In particular situations where transverse loads are applied directly to the cross-sectional elements, *stiffeners* or *cover plates* etc. are introduced to ensure that the cross-section maintains its form. Away from these points, the individual plate elements of the cross-section are subjected to in-plane forces, and it is this loading condition that is of major concern in proportioning the various elements which constitute the cross-section.

The *built-up column* buckling differs from the plate buckling described in Chap. 8 in several ways. Firstly buckling strength of an individual column element is greatly influenced by the edge conditions along the length of the element. Secondly a considerably larger post-buckling strength exists for plates than does for columns. This additional *post-buckling strength* is due to redistribution of stress at the critical load. For columns, the increase in the load carrying capacity beyond buckling load is negligibly small and hence *tangent modulus* load may be taken as criterion for design

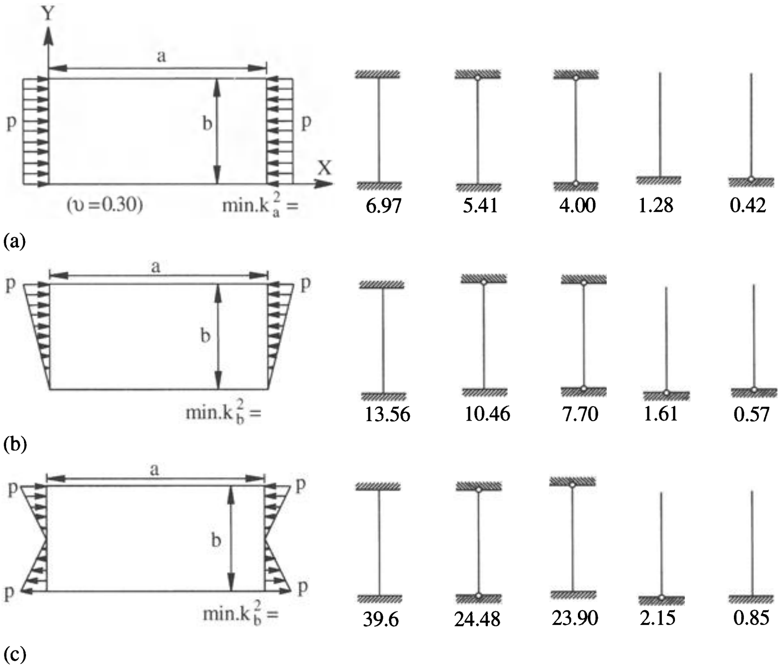


Fig. 11.2a–c. Minimum values of plate buckling coefficients, k^2 for various boundary conditions at the unloaded edges of the plate. **a** Uniform compression-pure axial force case, **b** non-uniform longitudinal compression – no tension (axial force and bending combined) **c** non-uniform longitudinal compression-pure bending case

purposes, while on the other hand post-buckling strength of plates must be taken into consideration for an efficient design. In the plate elements which have length, width and thickness, if the loading is in the direction of length (longer of the two plane dimensions), and length being at least several times the width, the buckling load is essentially independent of actual length, the buckling deformation will be of wave form. The elastic buckling of such a long element is primarily dependent on width-to-thickness ratio (b/t) and on the restraint conditions that exist along the longitudinal boundaries of the element. The critical buckling stress of flat plate elements in columns that are subjected to uniform compressive loads is given by (10.59)

$$\sigma_{\mathrm{a,cr}} = k^2 \frac{\pi^2 E\sqrt{\eta}}{12(1-\nu^2)(b/t)^2} \tag{11.11}$$

where $\eta = E_{\mathrm{t}}/E$, ν is possion's ratio, b and t are width and thickness of the plate, respectively; and k^2 is a factor depending upon the longitudinal boundary conditions shown in Fig. 11.2 and is given in Table 8.1. In a column composed of various connected flat plate elements, some elements may be more susceptible to buckling than the others. For such a case, the less critical parts provide edge constraint to the more flexible elements. An analysis based on the presumption that all elements reach simultaneously their buckling point and provides no bending resistance to their adjacent connected elements, gives a lower bound solution. Simply supported or hinged (or free edges where exist) would then be presumed.

To preclude the possibility of local buckling problem it is desirable to limit the value b/t. To this end many codes stipulate that the proportions of elements of a cross-section shall be such that an element is capable of reaching the yield stress of the material prior to attainment of the buckling load. It should be noted that such a requirement does not provide any additional safety factor, i. e. $SF = 1.0$. For such cases $F_y \le \sigma_{\text{cr}}$ or

$$\frac{b}{t} \le k\left[\frac{\pi^2 E}{12(1-\nu^2)F_y}\right]^{1/2}$$

or

$$\left(\frac{b}{t}\right)_{\max} \le Ck/\sqrt{F_y} \tag{11.12}$$

where C is a numerical coefficient depending upon material properties. For commonly used structural steel with $E = 2 \times 10^5$ MPa and $\nu = 0.3$, $C = 425.16$. For illustration consider the outstanding flange of I-section. If the outstanding flange or overhang is assumed to be hinged at its junction with the web and free at the other, the buckling coefficient k^2 obtained from Table 8.1 or Fig. 11.2a is 0.420.

Therefore, from (11.12):

$$b/t \le 17.426 \tag{11.13a}$$

On the other hand if the flange overhang is assumed to be clamped at the junction, k^2 from Table 8.1 or Fig. 11.2a is 1.2804 and the ratio b/t from (11.12) is:

$$b/t \le 30.427 \tag{11.13b}$$

In certain situations, some of the elements of cross-section are subjected to both compression and bending, e. g., web of an I-shaped section beam-column.

To account for the variation in the stress across the plate element following equation can be used:

$$\sigma_{\text{b,cr}} = k_{\text{b}}^2 \frac{\pi^2 E}{12(1-\nu^2)(b/t)^2} \tag{11.14}$$

The values of k_{b}^2, the buckling coefficient, are tabulated in Table 11.1. When a plate element is subjected to a uniformly applied shear along the four edges, as shown in Fig. 11.3, the critical buckling stress can be computed from the relationship

$$\sigma_{\text{s,cr}} = k_{\text{s}}^2 \frac{\pi^2 E}{12(1-\nu^2)(b/t)^2} \tag{11.15}$$

where k_{s}^2 is the shearing buckling coefficient given by:

(i) *For all edges simply supported case.*

$$\begin{aligned} k_{\text{s}}^2 &= \sqrt{3}\left[5.34 + (4.00/\mu^2)\right] \quad \text{for} \quad \mu \ge 1.0 \\ &= \sqrt{3}\left[4.00 + (5.34/\mu^2)\right] \quad \text{for} \quad \mu \le 1.0 \end{aligned} \tag{11.16}$$

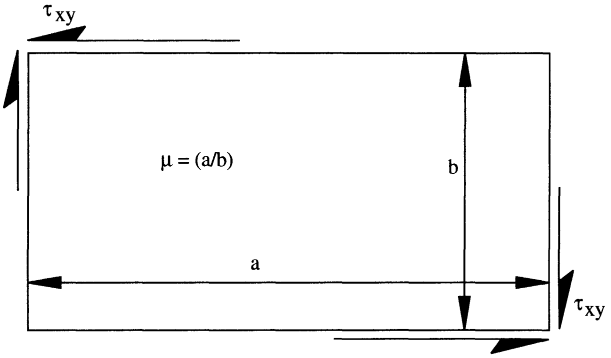


Fig. 11.3 Shearing buckling coefficients, k_s^2 for pure shear case

(ii) *For all edges fixed case.*

$$k_s^2 = \sqrt{3}\left[8.98 + (5.6/\mu^2)\right] \quad \text{for} \quad \mu \geq 1.0 \qquad (11.17)$$

where μ is the aspect ratio ($\mu = a/b$, $a \geq b$). The critical shear stress, $\tau_{cr} = \sigma_{s,cr}/\sqrt{3}$. Thus the web of a girder may buckle under vertical compressive stress, in pure flexure or in pure shear or combination thereof. Different codes require different b/t ratios for web and flanges. Some of the significant clauses are given below.

11.3.1 Average Shear Stress

AISCS

(i) *For stocky (stiffened or unstiffened) webs.* When

$$\frac{D}{t_w} \leq \frac{998}{\sqrt{F_y}} \qquad F_s = 0.40F_y \quad \text{(elastic)} \qquad (11.18)$$

The shear stress is based on overall depth, D. When

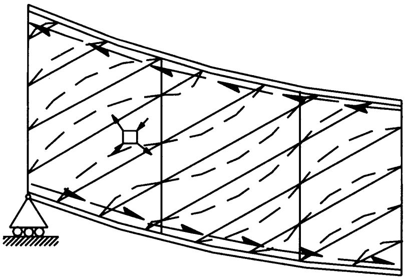
$$\frac{h}{t_w} > \frac{998}{\sqrt{F_y}} \qquad F_s = \frac{F_y}{2.89}C_v \leq 0.40F_y \quad \text{(elastic)} \qquad (11.19)$$

where C_v is defined by (11.23) and F_y is in MPa (N/mm^2). Here shear stress is on clear distance between flanges, h.

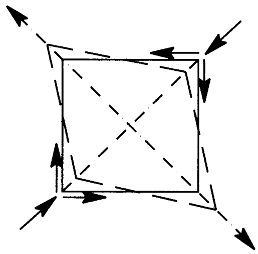
(ii) *For slender unstiffened webs.*

$$\text{For} \quad \frac{998}{\sqrt{F_y}} \leq \frac{D}{t_w} \leq \frac{1440}{\sqrt{F_y}} \qquad F_s = \frac{399\sqrt{F_y}}{(D/t_w)}$$

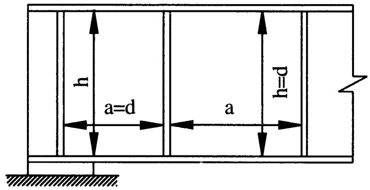
$$\text{For} \quad \frac{1440}{\sqrt{F_y}} \leq \frac{D}{t_w} \leq \frac{9600}{\sqrt{F_y(F_y+114)}} \qquad F_s = \frac{574000\sqrt{F_y}}{(D/t_w)^2} \qquad (11.20)$$



(a)



(b)



(c)

Fig. 11.4a–c. Loss of local stability of web due to shearing stresses. **a** Buckling of web, **b** distorsion of an element of web, **c** stiffening of web

(iii) *For slender stiffened webs.*

Near the support the web of a beam is subjected to the action of large shearing stresses resulting in a distortion along the lines of shortened diagonals of the web which are under compression as shown in Fig. 11.4b. Along the extended diagonals the web is stretched under the action of tensile stresses. Thus, under the action of compression the web may buckle, forming waves inclined at an angle of about 45* to the axis of beam web as shown in Fig. 11.4a. To prevent buckling of web, *vertical stiffeners (stiffening ribs)* are provided that intersect possible buckling waves as shown in Fig. 11.4c. This arrangement divides the web into rectangles bounded on four sides by the flanges and the stiffeners.

In this case shear force is resisted by the web as in the case of beam and is called *beam shear action*. Due to shear the web buckles in the direction perpendicular to the direction of principal compressive stress in the plane of web. After the web has buckled, a part of each web panel acts as a diagonal tension member and the *stiffeners* act as vertical compression members. This is called *tension field action*. Thus, both the beam shear action and tension field action contribute to shear strength of a stiffened slender web of the beam. The shear stress in the web is given by:

$$F_s = \frac{F_y}{FS\sqrt{3}}\left[C_v + \frac{(1 - C_v)}{1.15\sqrt{1 + (a/h)^2}}\right] \leq 0.40F_y \quad \text{for } a/h \leq 3 \text{ and } C_v \leq 1.0 \tag{11.21}$$

where a is the clear spacing between transverse or vertical stiffeners and h is the clear height between flanges. The first term of (11.21) represents the shear (V/ht) due to beam action which is limited by shear buckling whereas second term represents the shear due to tension field action which is limited by yielding due to combined stresses present in the *web-tension zone*. For the webs with widely spaced stiffeners i.e. $a/h > 3.0$, second term becomes insignificant and (11.21) reduces to:

$$F_s = \frac{F_y}{FS\sqrt{3}} C_v \leq 0.4F_y \quad \text{for} \quad a/h > 3.0 \quad \text{and} \quad C_v > 1.0 \tag{11.22}$$

For the factor of safety of 1.65 the term $FS\sqrt{3} = 2.89$. The factor C_v is computed as follows:

$$\begin{aligned} C_v &= \frac{F_{s,cr}}{F_{s,y}} = \frac{k_s^2 \pi^2 E}{(F_y/\sqrt{3}) \times 12(1-\nu^2)\,(h/t_w)^2} \\ &= \frac{313089 k_s^2}{\left[F_y (h/t_w)^2\right]} && \text{when } C_v \leq 0.80 \\ &= \sqrt{\frac{F_{s,cr}}{F_{s,y}}} = \frac{560 k_s}{\left[(h/t_w)\sqrt{F_y}\right]} && \text{when } C_v > 0.80 \end{aligned}$$

The above computations are based on $E = 2 \times 10^5$ MPa, and $\nu = 0.30$, and F_y is in MPa. The values adopted by AISCS (converted to SI units) are:

$$\begin{aligned} C_v &= \frac{310275\, k_v}{\left[F_y (h/t_w)^2\right]} && \text{when} \quad C_v \leq 0.80 \\ &= \frac{499}{(h/t_w)} \sqrt{\left(\frac{k_v}{F_y}\right)} && \text{when} \quad C_v > 0.80 \end{aligned} \tag{11.23}$$

Here $k_v = k_s^2$

This equation governs elastic behaviour. The following equation governs inelastic behaviour:

$$C_v = \frac{F_{s,cr}}{F_{s,y}} = \sqrt{\left(\frac{0.8}{F_y/\sqrt{3}}\right) \times \frac{\pi^2 E k_s^2}{12(1-\nu^2)(h/t_w)^2}}$$

or

$$C_v = 500 k_s / \left[(h/t_w)\sqrt{F_y}\right] \quad \text{when} \quad C_v > 0.8 \tag{11.24}$$

If simply supported edge conditions are assumed, the buckling coefficient k_s^2 is given by:

$$\begin{aligned} k_s^2 &= 4.00 + \left[5.34/(a/h)^2\right] && \text{when} \quad a/h \leq 1.0 \\ &= 5.34 + \left[4.00/(a/h)^2\right] && \text{when} \quad a/h > 1.0 \\ &= 5.34 && \text{when} \quad a/h > 3.0 \end{aligned} \tag{11.25}$$

where t_w is web thickness in m; a is clear distance between intermediate transverse stiffeners in m, h is the clear distance between flanges at the section under investigation in m and F_y is in MPa.

BS: 449

The shear stress in a stiffened slender web is

$$F_s = 0.4F_y\left[1.3 - \frac{(b/t)}{\{250 + (b/a)^2/2\}}\right] \le 0.40F_y \tag{11.26}$$

where a and b are clear panel dimensions with $a > b$. A load factor equal to 1.45 has been used in this equation.

AS: 1250

For stocky and slender unstiffened webs specifications are similar to those given by AISC specifications. For slender stiffened webs

$$F_s = 0.37F_y\left[1.3 - \frac{(b/t)\sqrt{F_y}}{4000\left\{1 + \frac{1}{2}(b/a)^2\right\}}\right] \le 0.37F_y \tag{11.27}$$

IS: 800

(a) Shear buckling of unstiffened beam webs

The critical shear stress is given by

$$\sigma_{s,cr} = k_s^2\pi^2E/\left[12(1 - \nu^2)(d/t)^2\right] \tag{11.28}$$

where

$$\begin{aligned} k_s^2 &= 4.00 + (5.34/\mu^2) \qquad \text{for} \quad \mu \le 1.0 \\ &= 5.34 + (4.00/\mu^2) \qquad \text{for} \quad \mu > 1.0 \end{aligned}$$

where $\mu(= a/b)$ is the aspect ratio. For $\mu = \infty$ and $\sigma_{s,cr} \le F_y/\sqrt{3}$, $k_s^2 = 5.34$. Therefore, from (11.28) for very long plate with $E = 2 \times 10^5$ MPa, $\nu = 0.3$ and $F_y = 250$ MPa

$$\frac{d}{t} \le \left[\frac{5.34 \times \pi^2 \times (2 \times 10^5) \times \sqrt{3}}{12(1 - 0.3^2) \times 250}\right]^{1/2} = 81.78 \approx 82 \tag{11.29}$$

If the d/t ratio of web is more than 82, buckling due to shear occurs. The code stipulates that vertical stiffeners should be provided when (d/t) exceeds 85.

In unstiffened web, allowable average shear stress

$$F_{s,a} = \left(\frac{F_y}{\sqrt{3}}\right)\frac{1}{FS} = \left(\frac{F_y}{\sqrt{3}}\right) \times \frac{1}{1.44} = 0.40F_y \tag{11.30}$$

where $F_y\sqrt{3}$ is yield stress in shear and factor of safety $FS = 1.44$.

(b) Shear stress in the stiffened webs

If the web is stiffened with vertical or intermediate stiffeners the allowable shear stress is governed by panel dimensions a and d, thickness t and grade of steel. Here a is the spacing of the stiffeners in the horizontal direction and d is the other dimension of the panel in the vertical direction i. e. clear distance from tension flange to horizontal stiffener or to the compression flange (if horizontal stiffener is not provided). Two cases arise for computing the maximum permissible or allowable average shear stress, $F_{\mathrm{s,a}}$.

(i) The spacing of vertical stiffeners a is less than d i. e. $a < d$

$$F_{\mathrm{s,a}} = 0.40F_y \left[1.3 - \frac{(a/t)\sqrt{F_y}}{4000\left\{1 + \frac{1}{2}\left(\frac{a}{d}\right)^2\right\}} \right] \leq 0.40F_y \tag{11.31}$$

(ii) The spacing a of vertical stiffeners is more than d i. e. $a > d$

$$F_{\mathrm{s,a}} = 0.40F_y \left[1.3 - \frac{(d/t)\sqrt{F_y}}{4000\left\{1 + \frac{1}{2}\left(\frac{d}{a}\right)^2\right\}} \right] \leq 0.40F_y \tag{11.32}$$

The spacing, a, should not be less than $d/3$ and greater than $1.5d$.

11.3.2 Flexural Buckling of Webs

The web of a beam being thin may buckle locally in the longitudinal direction due to bending compressive stress over a part of the depth of beam; it is called flexural or bending buckling. The web is strengthened by providing horizontal stiffeners. The bending critical stresses are given by:

$$\sigma_{\mathrm{b,cr}} = k_{\mathrm{b}}^2 \pi^2 E / \left[12(1 - \nu^2)(d/t)^2\right] \tag{11.33}$$

where k_{b}^2 is the plate buckling coefficient. k_{b}^2 for simply supported and clamped edge conditions of a flat plate under bending are 23.9 and 39.6, respectively, as is shown in Fig. 11.2c. Therefore,

(i) for simple supports with $\sigma_{\mathrm{b,cr}} \leq F_y (= 250\,\mathrm{MPa})$, $\quad d/t \leq 131.5$

(ii) for clamped supports with $\sigma_{\mathrm{b,cr}} \leq F_y (= 250\,\mathrm{MPa})$, $\quad d/t \leq 169.2$

However, web *flexure buckling* being localized in nature does not reduce the ultimate strength of the beam and hence a reduced factor of safety is used in arriving at d/t ratio.

According to IS: 800 for unstiffened webs d_1/t ratio is limited to 85, where d_1, is the clear distance between flanges or between inner toes of flange angles as appropriate.

For vertically stiffened webs d_2/t ratio is limited to 200, beyond which a *horizontal stiffener* has to be provided at $2/5^{\text{th}}$ the distance from the compression flange to the neutral axis. This happens to be the most efficient location for single horizontal stiffener. When d_2/t ratio exceeds 250 an additional horizontal stiffener is provided at the neutral axis. In any case d_2/t ratio should not exceed 400. Here d_2 is twice the clear distance from compression flange angles, or plate, or tongue plate to the neutral axis.

11.3.3 Built-up Sections

When the axial loads acting on columns are very large, it may not be possible to design a column with only rolled sections, it becomes necessary to use *built-up sections*. The various elements of the built-up section must be securely connected as shown in Fig. 11.5 so that they act together, rather than as individual components. The simplest built-up column type member is a rolled section with *additional plates* (called *cover plates*) attached to the flanges or two or more rolled sections at a distance apart are tied by *lacing* or *battens*. Various elements of a built-up column are so arranged that the moment of inertia about the minor axis is equal to that about the major axis making it equally strong about both the axes. To limit the lateral deflection i.e. to increase the lateral stiffness the size (lateral dimension of the column generally expressed in terms of its depth) is normally kept at 1/10 to 1/15 the height of the column for the yield stress of 250 MPa. Following example will illustrate the design procedure.

Example 11.2. A 3.8 m high column with both ends fixed is to support an axial load of 4200 kN. Design the column using SC series section with cover plates. The yield stress of structural steel is 250 MPa and $E = 2 \times 10^5$ MPa. Depth of column $\approx 3800/15 = 253.33$ mm.

Consider SC 250 @ 65.6 kg/m rolled steel section with cover plates. For SC 250 section.

$$A = 10.9 \times 10^3\,\text{mm}^2, \quad I_x = 125 \times 10^6\,\text{mm}^4, \quad I_y = 32.6 \times 10^6\,\text{mm}^4.$$
$$r_x = 107\,\text{mm}, \quad r_y = 54.6\,\text{mm} \quad \text{and} \quad K = 0.65$$

Therefore,

$$L_e/r_{\min} = 0.65 \times 3800/54.6 = 45.24$$
$$F_{ec} = \pi^2 E/(L_e/r_{\min})^2 = 964.46\,\text{MPa}$$
$$F_a = 0.6F_y/[1 + (F_y/F_{ec})^{1.4}]^{1/1.4} = 135.66\,\text{MPa}$$

Load carrying capacity of SC 250 section

$$= (10.9 \times 10^3)(135.66) \times 10^{-3} = 1478.7\,\text{kN}.$$

Balance to be carried by the cover plates

$$= 4200 - 1478.7 = 2721.3\,\text{kN}.$$

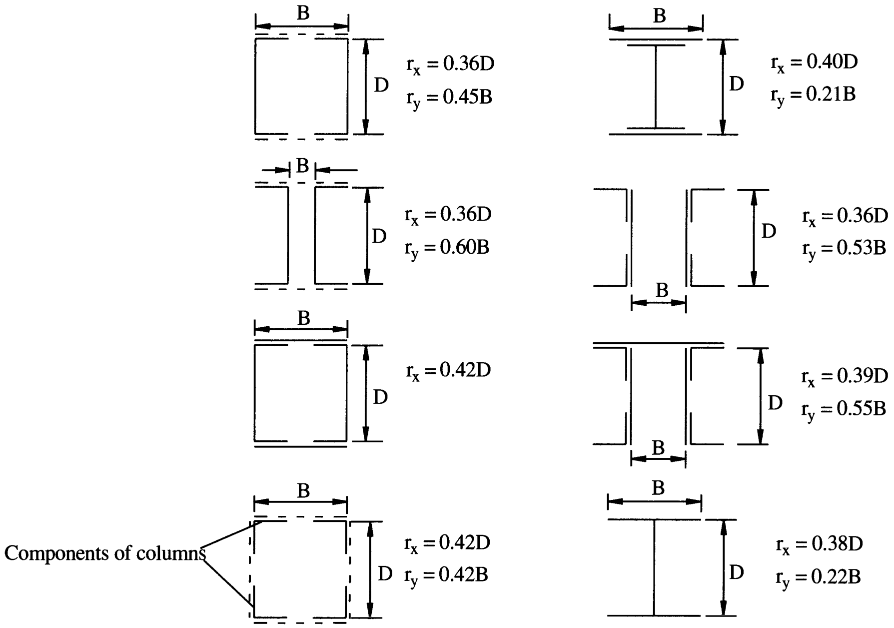


Fig. 11.5. Approximate radii of gyration of some of the commonly used built-up sections

The approximate radii of gyration for this arrangement of components are from Fig. 11.5: $r_x = 0.4D$ and $r_y = 0.21B$. For the case $r_x = r_y$, $B = 0.4D/0.21 = 0.4 \times 250/0.21 = 476.2$ mm (say 520 mm). Therefore,

$$KL/r = 0.65 \times 3800/(0.4 \times 250) = 24.7$$

and corresponding

$$F_{\mathrm{ec}} = \pi^2 \times (2 \times 10^5)/(24.7)^2 = 3235.46\,\mathrm{MPa}$$
$$F_{\mathrm{a}} = 0.6F_y/[1 + (F_y/F_{\mathrm{ec}})^{1.4}]^{1/1.4} = 147.10\,\mathrm{MPa}$$

Thickness of the plate, $t = (2721.3 \times 10^3)/(2 \times 147.10 \times 520) = 17.79$ mm.
Consider 520×18 mm cover plates.

- Total depth $= 250 + (2 \times 18) = 286$ mm. (OK)
- Overhang projection $= (520 - 250)/2 = 135$ mm
- Limiting overhang $= 16t = 16 \times 18 = 288\,\mathrm{mm} > 135$ mm. (OK)

$$I_x = 125 \times 10^6 + 2 \times 520 \times 18 \times (125 + 9)^2 = 461.14 \times 10^6\,\mathrm{mm}^4$$
$$A = 10.9 \times 10^3 + (2 \times 520 \times 18) = 29.62 \times 10^3\,\mathrm{mm}^2$$
$$r = [(461.14 \times 10^6)/(29.62 \times 10^3)]^{1/2} = 124.77\,\mathrm{mm}$$
$$KL/r = 0.65 \times 3800/124.77 = 19.8$$
$$F_{\mathrm{ec}} = \pi^2 \times (2 \times 10^5)/(19.8)^2 = 5035.0\,\mathrm{MPa}$$
$$F_{\mathrm{a}} = (0.6 \times 250)/[1 + (250/5035)^{1.4}]^{1/1.4} = 148.42\,\mathrm{MPa}$$

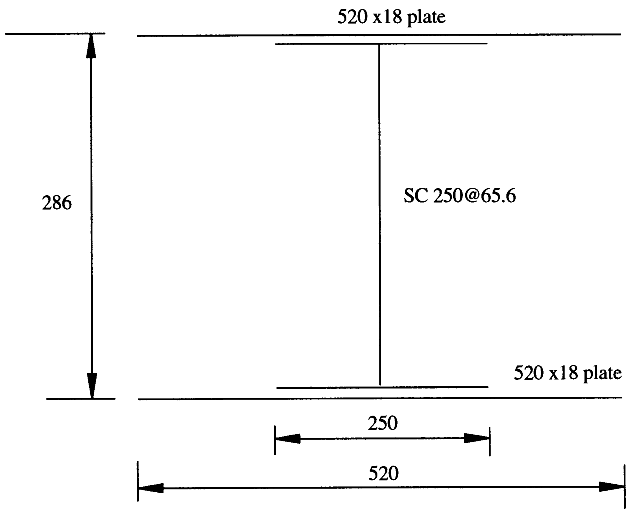


Fig. 11.6. Built-up columns cross-section for example 11.2

- Load carrying capacity of built-up column

$$P_u = (29.62 \times 10^3) \times 148.42 \times 10^{-3} = 4396.2\,\text{kN} > 4200\,\text{kN}.$$

Thus SC 250 rolled section with 2 – 520 × 18 mm cover plates as shown in Fig. 11.6 is adequate.

11.4 Beam Design Formula

11.4.1 Lateral Buckling of Beams

As discussed in Chap. 7, in a laterally unsupported beam where minor axis moment of inertia is less than the major axis moment of inertia, there is likelihood of occurrence of lateral buckling. The compression flange deflects normal to the plane of loading besides bending in the plane of loading. Thus the beam gets twisted while undergoing vertical displacement, which is called *flexural-torsional buckling.*

A simply supported beam having doubly symmetrical cross-section and an unsupported compression flange, when subjected to equal end moments about the major axis (i. e. moment diagram is rectangular consequently compression flange is under uniform compression) buckles laterally beside bending in the plane of loading when the moment reaches its critical value. The critical moment is given by (7.66) which can be expressed as:

$$M_{\text{cr}} = \frac{\pi}{L_e}\sqrt{(EI_yGJ)}\sqrt{1+\left(\frac{\pi^2 EI_w}{L_e^2 GJ}\right)}$$

or

$$M_{\text{cr}} = \frac{\pi^2\sqrt{EI_yEI_w}}{L_e^2}\sqrt{1+\left(\frac{L_e^2 GJ}{\pi^2 EI_w}\right)} \tag{11.34}$$

However, this critical moment is influenced by following factors:

1. *The moment gradient.* The critical moment given by (11.34) is applicable to uniform moment case when moment gradient is zero. Under other types of loads, beam buckles at a moment obtained from (11.34) multiplied by moment factor C which depends on the loading and boundary conditions as given in Table 7.2. By ignoring the effect of moment factor, a conservative design is obtained. IS: 800 adopts a value of unity for C.
2. *Load position.* The derivation of (11.34) assumes the transverse load to act at the centroid of the section which is true for doubly symmetrical section where centroid coincides with the *shear centre*. If the load P is acting at a point above or below the shear centre at distance e and if it is free to move sideways with the beam it exerts additional torque $Pe\phi$. This additional torque causes a decrease or increase in the resistance of the beam against lateral buckling, depending upon the type and position of the load; and torsional parameter $(\pi^2 EI_w/L_e^2 GJ)$. The bottom flange loading increases and top flange loading reduces the critical moment. These effects are taken care of by introducing factor C as given in (11.37).
3. *Adjacent spans.* The critical moment is influenced by division of span into segments due to lateral supports within the span or by existence of adjacent spans. The adjacent spans, increase the critical moment due to their restraining effect. Since the presence of adjacent spans has a beneficial effect on the critical moment, most codes consider the effect indirectly by increasing the effective length of compression flange.
4. *Inplane deflections.* In the derivation of (11.34), the effect of major axis deflection has not been considered. If this effect is included critical moment is obtained from the following equation:

$$M_{cr} = \frac{\pi}{L_e}\sqrt{\frac{EI_y GJ}{\tau}}\sqrt{1+\left(\frac{\pi^2 EI_w}{L_e^2 GJ}\right)} \tag{11.35}$$

or

$$M_{cr} = \left(\frac{\pi^2}{L_e^2}\right)\sqrt{\left(\frac{EI_y EI_w}{\tau}\right)}\sqrt{1+\frac{L_e^2 GJ}{\pi^2 EI_w}} \tag{11.36}$$

where $\tau = (1 - I_y/I_z)$. For a typical beam where I_y/I_z is 1/5, the effect is an increase of 10 per cent on moment. For the beams of MB series where $I_y \ll I_z$, the effect is negligible.

11.4.2 Effective Length of Compression Flange

In order to consider the effect of boundary conditions, load position and restraint provided at the ends, the length of unsupported portion of compression flange is replaced by effective length of compression flange, which is the product of unsupported length of compression flange and the effective length coefficient of the compressive flange. Thus, a general equation for critical stress $\sigma_{cr,c}$ in the compression flange of a doubly symmetric I-section subjected to major axis bending can be written in the following form :

$$\sigma_{\mathrm{cr,c}} = \frac{C\pi\sqrt{EI_y GJ}}{L_e Z_z}\sqrt{\left[1 + \frac{\pi^2 EI_w}{L_e^2 GJ}(C_1^2 + 1)\right]} + \frac{C_1 \pi}{L_e}\sqrt{\frac{EI_w}{GJ}} \tag{11.37}$$

where EI_y, EI_w and GJ are minor axis flexural rigidity, warping rigidity and torsional rigidity, respectively. Z_z is section modulus about Z-axis, L_e is effective length, C is coefficient which accounts for the beneficial effect of moment gradient along the beam axis; and C_1 is the coefficient which accounts for the toppling or stabilizing effects due to load acting at the top or bottom flange of the beam. The last term should be added when load is applied at the bottom flange and subtracted when it is applied at the top flange.

Example 11.3. A 30 m long WB 600 @ 145 kg/m beam is laterally unsupported over its entire length, determine the maximum normal stress in the beam corresponding to lateral torsional buckling for each of the following loading and support conditions:

(1) subjected to equal end moments M_o causing single curvature bending (uniform moment) with $v = \beta = v'' = \beta'' = 0$ at both the supports; and
(2) subjected to a concentrated load at the mid span with $v = \beta = v' = \beta'' = 0$ at both the supports.

Consider the plane of web of the member to be in the plane of the applied bending moments. Neglect the weight of the beam itself. Assume $F_y = 450\,\mathrm{MPa}$, $E = 2.7G = 2.1 \times 10^5\,\mathrm{MPa}$.

The cross-sectional properties for WB 600 are:

$$\begin{aligned} I_y &= 52.983 \times 10^6\,\mathrm{mm}^4; \\ I_w = I_y(D^2/4) &= 476.847 \times 10^{10}\,\mathrm{mm}^6; \\ Z_z &= 3.854 \times 10^6\,\mathrm{mm}^3; \\ J = 0.9BT^3 = 0.9 \times 250 \times (23.6)^3 &= 2.9575 \times 10^6\,\mathrm{mm}^4. \end{aligned}$$

Case I: For the uniform moment case, from (11.34)

$$M_{\mathrm{cr}} = \frac{\pi^2}{L^2}\sqrt{EI_y EI_w}\sqrt{1 + \left(\frac{L^2 GJ}{\pi^2 EI_w}\right)} = 171.48\,\mathrm{kNm}$$

The maximum normal stress which occurs at the extreme fibres is

$$f_b = M_{\mathrm{cr}}/Z_z = 44.495\,\mathrm{MPa}.$$

This is less than the yield stress value minus any compressive residual stress. Therefore, the member buckles in the elastic range.

Case II: For the concentrated load case, the buckling moment can be determined using (11.37) and the values given in Table 7.2. In this case $C_1 = 0$ and for

$$K_x = K_z = 1.00, \quad C = 1.70$$

Therefore,

$$M_{z,\mathrm{cr}} = 1.7 \times 171.48 = 291.516\,\mathrm{kNm}$$

Assuming elastic behaviour, the maximum normal stress is:

$$f_{x,\mathrm{b}} = 1.7 \times 44.495 = 75.64\,\mathrm{MPa}.$$

For a maximum compressive residual stress, $F_\mathrm{r} = 0.50F_y$, the proportional limit stress is:

$$F_\mathrm{p} = F_y - 0.50F_y = 0.50F_y = 225\,\mathrm{MPa}.$$

The cross-sectional still remains in the elastic range.

11.4.3 Codal Provisions

AISCS: The allowable stresses in bending F_b (in MPa) are:

1. For sections in tension, $F_\mathrm{b} = 0.60F_y$
2. For sections in compression,
   a) For $\sqrt{703 \times 10^3 C_\mathrm{b}/F_y} \leq L/r_\mathrm{T} \leq \sqrt{3516 \times 10^3 C_\mathrm{b}/F_y}$

$$F_\mathrm{b} = F_y\left[\frac{2}{3} - \frac{F_y(L/r_\mathrm{T})^2}{10550 \times 10^3 C_\mathrm{b}}\right] \leq 0.60F_y \tag{11.38}$$

   b) For $L/r_\mathrm{T} \geq \sqrt{3516 \times 10^3 C_\mathrm{b}/F_y}$

$$F_\mathrm{b} = 1172 \times 10^3 C_\mathrm{b}/(L/r_\mathrm{T})^2 \leq 0.60F_y \tag{11.39}$$

   c) When compression flange is solid and rectangular with its area not less than tension flange.

$$F_\mathrm{b} = 83 \times 10^3 C_\mathrm{b}/(LD/A_\mathrm{f}) \leq 0.60F_y \tag{11.40}$$

   where L, D and r_T are unbraced length of compression flange (m), overall depth of girder (m) and radius of gyration of section comprising the compression flange and one-third of compression web (m), respectively, A_f is the area of compression flange.

AISCS does not provide for any allowance for potentially dangerous condition of top flange loading. A loading condition other than pure bending is represented as an equivalent uniform moment, M_eq multiplied by a factor C_b i.e.

$$M_\mathrm{cr} = C_\mathrm{b} M_\mathrm{eq} \tag{11.41}$$

The factor C_b depends upon the type of loading and is approximated as:

$$C_\mathrm{b} = 1.75 + 1.05(M_1/M_2) + 0.3(M_1/M_2)^2 \not> 2.3 \tag{11.42}$$

where M_1 and M_2 are smaller and larger end moments, respectively. The ratio M_1/M_2 is considered positive when the beam bends in reverse curvature (i.e. M_1 and M_2 have the same sign).

BS: 449

(a) For beams having flanges of uniform cross-section throughout or each being of uniform cross-section but where $I_{yc} > I_{yt}$, $F_b \le 0.66F_y$ for parts in tension and compression. For a doubly symmetric I-beam subjected to pure bending (11.34) can be written in the form:

$$F_{cr} = \frac{M_{cr}}{Z_z} = \frac{\pi\sqrt{EI_yGJ}}{LZ_z}\left[\sqrt{\left\{1 + \frac{EI_y}{GJ}\left(\frac{\pi}{2}\frac{D}{L}\right)^2\right\}\left(\frac{I_z}{I_z - I_y}\right)}\right] \tag{11.43}$$

Since $I_w = I_y\,(D^2/4)$ for I-section. Further using following approximations for the geometric and material properties:

$$I_z = 1.1BTD^2/2\,, \quad Z_z = 1.1BTD\,, \quad I_y = B^3T/6\,, \quad J = 0.9BT^3\,.$$

$B = 4.2r_y$ and $E = 2.5G = 2 \times 10^5$ MPa, (11.43) reduces to:

$$F_{cr} = \left(\frac{1675}{L/r_y}\right)^2\left[1 + \frac{1}{20}\left\{\frac{(L/r_y)}{(D/T)}\right\}^2\right]^{1/2} \tag{11.44}$$

(b) For beams having unequal flanges

(i) For the section having flanges of equal moment of inertia i.e. $I_{ft} = I_{fc}$

$$F_{cr} = \left(\frac{1675}{L/r_y}\right)^2\left[1 + \frac{1}{20}\left\{\frac{(L/r_y)}{(D/T)}\right\}^2\right]^{1/2} \tag{11.45}$$

(ii) If $I_{fc} > I_{ft}$

$$F_{cr} = \left(\frac{1675}{L/r_y}\right)^2\left[\sqrt{\left\{1 + \frac{1}{20}\left(\frac{L}{r_y}\frac{T}{D}\right)^2\right\}} + k_2\right] \tag{11.46}$$

where $k_2 = 2m - 1$ for $m \le 0.5$, and $(2m - 1)/2$ for $m > 0.5$, in which m is the ratio of moment of inertia of compression flange to moment of inertia of whole section about Y-axis. Both are calculated at the point of maximum bending moment.

(iii) $I_{fc} < I_{ft}$

$$F_{cr} = \left(\frac{1675}{L/r_y}\right)^2\left[\sqrt{\left\{1 + \frac{1}{20}\left(\frac{L}{r_y}\frac{T}{D}\right)^2\right\}} + k_2\right]\left(\frac{y_c}{y_t}\right) \tag{11.47}$$

where L, D and T are effective length, overall depth of girder, and effective thickness of flange, respectively; y_c and y_t are distances of compression and tension fibres from the neutral axis, respectively. r_y is radius of gyration of whole section about Y-axis, at the point of maximum bending moment. The parameter k_2 allows for inequality of tension and compression flanges that depends upon factor m. The allowance for the increased danger of top flange loading is provided by calculating F_{cr} using 20 per cent increased length of compression flange.

AS: 1250

The relations between critical stress F_{cr} and allowable bending stress F_b in compression are:

$$F_b = F_{cr}\left[0.55 - 0.1\left(\frac{F_{cr}}{F_y}\right)\right] \quad \text{when} \quad F_{cr} \leq F_y, \tag{11.48}$$

$$F_b = F_{cr}\left[0.95 - 0.5\sqrt{\frac{F_y}{F_{cr}}}\right] \quad \text{when} \quad F_{cr} > F_y \tag{11.49}$$

The expressions for F_{cr} are the same as proposed by British code and are given by Eqns (11.45) and (11.46), except for the factor $(1675)^2$ which has been replaced by $(1627.88)^2 \approx 2650 \times 10^3$ i.e.

$$F_{cr} = k_1 \frac{2650 \times 10^3}{(L/r_y)^2}\sqrt{\left\{1 + \frac{1}{20}\left(\frac{L}{r_y}\frac{T}{D}\right)^2\right\}} \tag{11.50}$$

where $k_1 = 0.2 + 0.8\,\mathrm{N} \not< 0.25$. The coefficient k_1 allows for curtailment of thickness and/or breadth of flanges between points of effective lateral restraint and depend upon N, the ratio of the area of both the flanges at the point of minimum bending moment to the area of flanges at the point of maximum bending moment.

IS: 800

The critical stress for prismatic doubly symmetric sections given by following equation is modified to make it applicable to monosymmetric sections, angles, tees and non prismatic sections. The relation:

$$F_{cr} = \frac{2650 \times 10^3}{(L/r_y)^2}\left[1 + \frac{1}{20}\left(\frac{L/r_y}{D/T}\right)^2\right]^{1/2} \tag{11.51}$$

is replaced by:

$$F_{cr} = k_1(X + k_2 Y)\left(\frac{c_2}{c_1}\right)$$

where

$$Y = 2650 \times 10^3/(L/r_y)^2 \quad \text{and} \quad X = Y\left[1 + \frac{1}{20}\left(\frac{L/r_y}{D/T}\right)^2\right]^{1/2} \tag{11.52}$$

- k_1 = a coefficient depending on the ratio R_1 of total areas of both flanges at the points of least and maximum bending moments,
- k_2 = a coefficient which depends on the ratio R_2 of moment of inertia of compression flange alone to that of the sum of the moments of inertia of flanges about their own axes parallel to Y-axis,

- $c_1, c_2 =$ the lesser and larger distances of extreme fibres from the neutral axis of the section,
- $r_y =$ radius of gyration of section with respect to Y-axis.
- T = mean thickness of compression flange.
- $B, D =$ the breadth and overall depth of the cross-section, and
- $L =$ effective length of the compression flange.

The coefficients k_1 and k_2 are given in Table 11.1. The value of F_{cr} given by (11.52) is increased by 20 per cent if $T/t \leq 2$ and $d_1/t \leq 1344/\sqrt{F_y}$.

The quantities, T, t and d, represent mean thickness of flange, thickness of web and unsupported width of web (distance between fillet lines for rolled beams). In order to predict critical stress at both elastic and inelastic stages, empirical relations of the type used for columns are recommended:

$$F_{\mathrm{cr}} = F_y/[1 + (F_y/F_{\mathrm{cr}})^n]^{1/n} \tag{11.53}$$

The exponent n is taken as 1.4 for all Indian rolled sections. Using a factor of safety of 1.5 instead of 1.67 for columns, the allowable bending compressive stress F_{bc} is given by:

$$F_{\mathrm{bc}} = \frac{F_{\mathrm{cr}}}{1.5} = 0.66\frac{F_y}{\left[1 + (F_y/F_{\mathrm{cr}})^n\right]^{1/n}} \tag{11.54}$$

where F_y and F_{cr} are yield stress of the material and elastic bending critical stress. If the multiplying factor $[1/[1 + (F_y/F_{\mathrm{cr}})^n]^{1/n}]$ is less than one, lateral buckling occurs before material reaches the yield point. Thus,

$$\text{(i)}\quad F_{\mathrm{bc}} \leq 0.66F_y \qquad \text{for no buckling case} \tag{11.55}$$

$$\text{(ii)}\quad F_{\mathrm{bc}} \leq 0.66F_y/[1 + (F_y/F_{\mathrm{cr}})^n]^{1/n} \qquad \text{for buckling case} \tag{11.56}$$

The allowable bending stress is lesser of the two values for buckling and no buckling cases.

Table 11.1 Values of coefficients k_1 and k_2

k_1 values for the beams with curtailed flanges											
R_1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
k_1	1.0	1.0	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
k_2 values for the beams with unequal flanges											
R_2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
k_2	0.5	0.4	0.3	0.2	0.1	0.0	−0.2	−0.4	−0.6	−0.8	−1.0

where

$$R_1 = \frac{\text{sum of areas of both flanges at } M_{\min}}{\text{sum of areas of both flanges at } M_{\max}}$$

$$R_2 = \frac{I_{\mathrm{yc}}}{I_{\mathrm{yc}} + I_{\mathrm{yt}}}$$

$I_{\mathrm{yc}}, I_{\mathrm{yt}} = I_y$ of compression and tension flanges, respectively

Example 11.4. A built-up girder of an industrial building is composed of a MB 300 rolled steel section with MC 250 at the top flange and a cover plate of 220 × 16 mm at the bottom flange as shown in Fig. 11.7. The girder has a span of 5.25 m with an unsupported compression flange. The ends are simply supported with partial torsional restraint. Determine the allowable bending compressive stress and shear stress to be used in the design of girder.

The girder cross-section is shown in Fig. 11.7. The sectional properties from metal tables are:

MB 300	**MC 250**	**Plate: 220 × 16 mm**
$a = 5860\,\text{mm}^2$	$a = 3900\,\text{mm}^2$	$a = 3520\,\text{mm}^2$
$T = 13.1$ mm	$T = 14.1$ mm	
$t = 7.7$ mm	$t = 7.2$ mm	
$I_z = 89.90 \times 10^6\,\text{mm}^4$	$I_z = 38.80 \times 10^6\,\text{mm}^4$	
$I_y = 4.86 \times 10^6\,\text{mm}^4$	$I_y = 2.11 \times 10^6\,\text{mm}^4$	
$R_\text{b} = 14.0$ mm	$C_y = 23$ mm	

$$y_\text{c} = \frac{5860 \times 157.2 + 3900 \times 23 + 3520 \times (300 + 7.2 + 8)}{5860 + 3900 + 3520} = 159.67\,\text{mm} = c_1$$

$$y_\text{t} = 300 + 7.2 + 16 - 159.67 = 163.53\,\text{mm} = c_2$$

For the whole section

$$A = 13280\,\text{mm}^2$$

$$I_z = 89.90 \times 10^6 + 5860 \times (159.67 - 157.2)^2 + 2.11 \times 10^6$$
$$+ 3900 \times (159.67 - 23)^2 + 3520 \times (163.53 - 8)^2 = 250.04 \times 10^6\,\text{mm}^4.$$

$$I_y = 4.86 \times 10^6 + 38.80 \times 10^6 + 16 \times 220^3/12 = 57.86 \times 6\,\text{mm}^4.$$

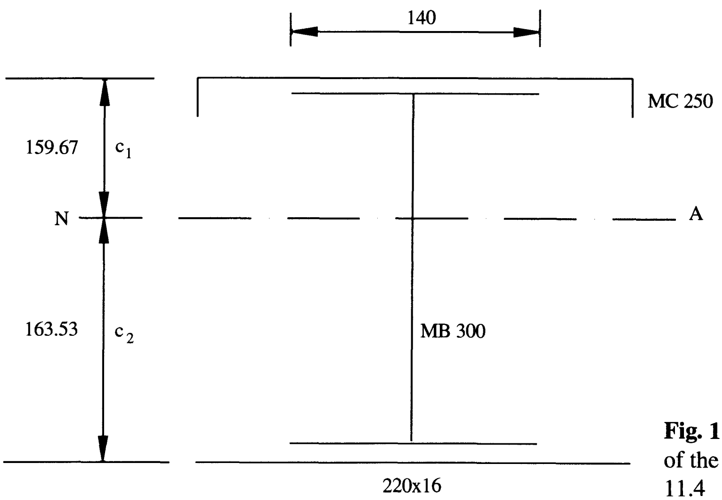


Fig. 11.7. Cross-section of the girder of example 11.4

Average flange thickness, $T = (250 \times 7.2 + 140 \times 13.1)/250 = 14.54\,\text{mm}$. Total depth, $D = 300 + 7.2 + 16 = 323.2\,\text{mm}$

$$D/T = 323.2/14.54 = 22.27$$

Effective length is increased by 20 per cent. Therefore,

$$L_e = 5250 \times 1.0 \times 1.2 = 6300\,\text{mm}.$$

$$r_y = \sqrt{I_y/A} = \sqrt{57.86 \times 10^6/13280} = 66.00\,\text{mm}$$

Therefore, $L_e/r_y = 6300/66 = 95.44$ and the ratio

$$(L_e/r_y)/(D/T) = 95.44/22.27 = 4.29$$

As the section is prismatic $k_1 = 1.0$. For the compression and tension flanges

$$I_{yc} = 38.80 \times 10^6 + (13.1 \times 140^3/12) = 41.80 \times 10^6\,\text{mm}^4.$$

$$I_{yt} = (16 \times 220^3/12) + (13.1 \times 140^3/12) = 17.19 \times 10^6\,\text{mm}^4.$$

$$R_2 = I_{yc}/(I_{yc} + I_{yt}) = 0.71$$

For $R_2 = 0.71, k_2 = 0.21$

Parameters for critical stress:

$$Y = 2650 \times 10^3/(95.44)^2 = 290.93\,\text{MPa}.$$

$$X = 290.93 \times [1 + 4.29^2/20]^{1/2} = 403.14\,\text{MPa}$$

Therefore,

$$\begin{aligned} F_{cr} &= k_1(X + k_2Y)(c_2/c_1) \\ &= 1.0 \times (403.14 + 0.21 \times 290.93) \times (163.53/159.67) = 475.46\,\text{MPa}. \end{aligned}$$

$$T/t = 14.54/7.7 = 1.89 < 2.0$$

$$d_1/t = [300 - 2(13.1 + 14.0)]/7.7 = 31.9 < 85$$

Hence, increase F_{cr} by 20 per cent. Therefore,

$$F_{cr} = 475.46 \times 1.2 = 570.55\,\text{MPa}.$$

For the Ratio, $F_y/F_{cr} = 250.0/570.55 = 0.438$ and $n = 1.4$

$$F_{bc} = 0.66 \times 250/[1 + (0.438)^{1.4}]^{1/1.4} = 135.69\,\text{MPa}.$$

Allowable shear stress, $F_s = 0.4 \times 250 = 100.00\,\text{MPa}$.

11.4.4 Bearing Compressive Stress

High *bearing compressive stress* due to application of excessive load over a small length of beam may result in *web crippling*. In practice the bearing lengths b provided are usually large enough to prevent web crippling. The bearing stress requirements of various codes are given below.

AISCS

Load dispersion through flanges has been taken to be 45° which provide a length of $(b+k)$ or $(b+2k)$ for the end reactions and interior loads, respectively. The critical area for the stress, which occurs at the toe of fillet is given by $(b+k)t_{\mathrm{w}}$ or $(b+2k)t_{\mathrm{w}}$, and the permissible bearing stress F_{p} is equal to:

1. For thick webs:

$$F_{\mathrm{p}} = 0.75F_y \tag{11.57}$$

2. For thin webs: (a) When compression flange is quite free to rotate about its longitudinal axis.

$$F_{\mathrm{p}} = \left[2 + 4\left(\frac{h}{a}\right)^2\right]\frac{69000}{(h/t)^2} \quad \text{(in MPa)} \tag{11.58}$$

(b) When the flange is restrained from rotating by rigid slab or some other means.

$$F_{\mathrm{p}} = \left[5.5 + 4\left(\frac{h}{a}\right)^2\right]\frac{69000}{(h/t)^2} \quad \text{(in MPa)} \tag{11.59}$$

BS: 449

(i) For thick webs load distribution is taken at 30° (from horizontal) through the flanges and bearing stress is equal to:

$$F_{\mathrm{p}} = 0.75F_y \tag{11.60}$$

(ii) For thin webs, the bearing stress is given by the axial load-carrying capacity of strut whose effective slenderness ratio and area are $(h\sqrt{3})/t$ and Bt, respectively, where B is the length of the stiff portion of the bearing plus additional length given by 30° dispersion as shown in Fig. 11.8.

$$\text{For concentrated load,} \quad B = b + 2\sqrt{3}k$$
$$\text{For support reaction,} \quad B = b + \sqrt{3}k$$

where k is the fillet depth or distance from the outer face of the flange to the web toe. h is clear depth of web between root fillets $(= D - 2k)$ and b is the bearing length of concentrated loads or reactions.

AS: 1250

The specifications are same as those given by the British code.

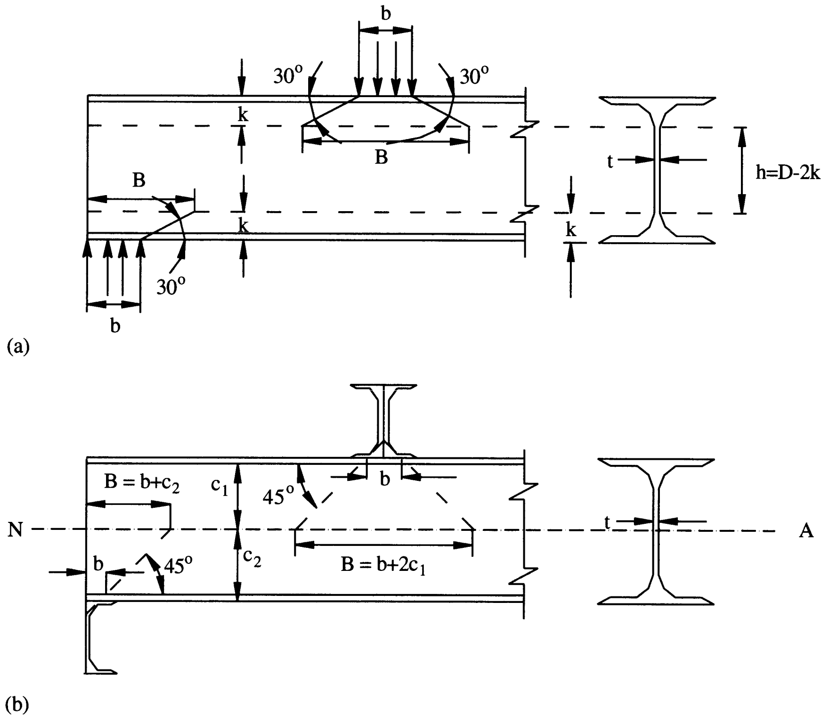


Fig. 11.8a,b. Dispersion through flanges and web for computing of bearing area. **a** Dispersion for web crippling, **b** dispersion for web buckling

IS: 800

The requirements are the same as those given by the British code.

11.5 Stiffeners

The web of a girder may buckle locally either under pure shear due to diagonal compression, or under flexure due to bending compressive stress, or under concentrated loads due to bearing compressive stress. This local buckling of web is prevented by providing *stiffeners*, called *intermediate vertical* (transverse) *stiffeners*, *horizontal stiffeners* and *bearing stiffeners*.

11.5.1 Vertical Stiffeners

The vertical stiffeners divide the web into panels and enhance the buckling strength under diagonal compression due to pure shear. The recommended spacing is from one-third to one and a half times the clear distance between flanges or from the tension flange (farthest flange) to the nearest horizontal stiffener, if they exist.

AISCS

According to these specifications intermediate stiffeners are required when the ratio (h/t_w) is greater than 260 and maximum web shear stress is greater than permitted by (11.22).

The moment of inertia of a pair of intermediate stiffeners, or a single intermediate stiffener I_s w.r.t. an axis in the plane of the web shall not be less than $(h/50)^4$ and the gross area of the stiffener is given by

$$A_s = \frac{1 - C_v}{2}\left[\frac{a}{h} - \frac{(a/h)^2}{\sqrt{1 + (a/h)^2}}\right] YDht \tag{11.61}$$

The spacing of the stiffeners, a is governed by the equation:

$$\frac{a}{h} \leq \left[\frac{260}{(h/t_w)}\right]^2 \quad \text{and} \quad 3.0 \tag{11.62}$$

where

Y = ratio of yield stress of web steel to yield stress of stiffener steel,
D = 1.0 for stiffeners provided in pairs,
= 1.8 for single angle stiffeners, and
= 2.4 for single plate stiffeners.

BS: 449

In order to ensure that the intermediate stiffeners effectively restrain the web plate and enable it to act as a rectangular plate supported at four edges. The minimum moment of inertia should be

$$I_s = 1.5\frac{h^3t^3}{a^2} \tag{11.63}$$

where I_s for a pair of stiffeners is about the centre of the web, and for single stiffener about the face of the web; a is the maximum permitted clear distance between vertical stiffeners, and h is unsupported web depth.

The requirements for the vertical stiffeners in AS: 1250 and IS: 800 are the same as given in the British code.

11.5.2 Horizontal Stiffeners

The horizontal stiffeners in single or in pairs are provided between vertical stiffeners to prevent local flexural buckling of the web.

AISCS

There is no provision for horizontal stiffener in the specifications. However, the United States Steel Handbook suggests provision of horizontal stiffeners in the compression zone of the web, preferably at $0.2h$ from the compression flange, to increase buckling strength.

BS: 449

For deep webs i.e. $200 < d_2/t < 250$, one horizontal stiffener is provided at 0.4 ($= 2/5$) times the distance from the compression flange to the neutral axis with $I_s \geq 4\,\mathrm{at}^3$, where d_2 is equal to twice the clear distance of compression flange or plate from neutral axis, a is actual distance between vertical stiffeners, and t is the minimum thickness of the web.

When further stiffening of the web is desired, another horizontal stiffener is placed at the neutral axis of the girder. This stiffener serves exclusively to reduce the web panel dimensions and carries no significant load. Therefore, smaller $I_s = ht^3$ is provided, where $h = d_2$ as defined above. The requirements for the horizontal stiffener in AS: 1250 and IS: 800 are the same as given by British code. According to IS: 800 the outstand of a stiffener from the web to the outer unstiffened edge of the stiffener should not be more than $256t/\sqrt{F_y}$ for rolled sections and $12t$ for flats where t is the thickness of the web or flat.

11.6 Beam-Column Design Formulae

In the preceding chapters various possible failure modes for the beam-columns have been discussed and procedures have been developed for determination of associated critical loads. In general, a beam-column is subjected to two different kinds of loading: axial thrust and bending moment. Based on the ratio P/P_u and M/M_u, a single interaction equation that provides reasonable prediction of structural strength is given by:

$$\frac{P}{P_u} + \frac{M}{M_u} = 0 \tag{11.64}$$

where P and P_u are applied axial load, and the axial force carrying capacity of the member, respectively, when axial force alone exists. M and M_u are maximum bending moment due to applied transverse loading or applied end moments acting in the plane of symmetry, and the bending moment carrying capacity, respectively, when moment alone exists. P_u and M_u may be either elastic or inelastic. P_u depends upon the slenderness ratio and for very short columns it approaches the yield load i.e. $P_u = \sigma_y A$. M_u depends upon the lateral support conditions, for a laterally supported member, M_u will be the maximum bending moment that the cross-section can sustain i.e. fully plastic moment.

To include the effect of secondary moment due to axial thrust i.e. thrust times the deflection, on maximum bending moment M, an amplification factor $1/(1 - P/P_e)$ is introduced, and interaction equation (11.64) reduces to

$$\frac{P}{P_u} + \frac{M}{M_u(1 - P/P_e)} \leq 1.0 \tag{11.65}$$

where P_e is the Euler buckling load of the member in the plane of applied bending moment. Equation (11.65) is valid for the cases when end moments are equal ($\beta = 1.0$) i.e. the moment gradient is zero. To account for the *moment gradient* due to different loading and support conditions, a modification factor to M_u, C_m which is less than unity but greater than 0.40 is introduced. The interaction equation changes to:

$$\frac{P}{P_u} + \frac{M}{(M_u/C_m)(1 - P/P_e)} \leq 1.0 \tag{11.66}$$

Equation (11.66) can be expanded to handle biaxial bending conditions.

$$\frac{P}{P_u} + \frac{M_x}{(M_{ux}/C_{mx})(1 - P/P_{ex})} + \frac{M_y}{(M_{uy}/C_{my})(1 - P/P_{ey})} \leq 1.0 \tag{11.67}$$

Here P_u is computed for the larger *effective slenderness ratio* of the member. The subscripts x and y refer to the two principal directions of bending. In terms of stresses

$$\frac{P}{A} = f_a, \quad \frac{P_u}{A(FS)} = F_a, \quad \frac{M}{Z} = f_{bc} \quad \text{and} \quad \frac{M_u}{Z(FS)} = F_{bc}$$

where A, FS and Z are area of cross-section, factor of safety and section modulus, and

- F_a = axial compressive stress that would be permitted if axial load alone existed,
- F_{bc} = compressive bending stress that would be permitted if bending moment alone existed,
- f_a = computed axial stress,
- f_{bc} = computed bending stress, and
- C_m = a reduction factor used to modify the amplification factor $1/[1 - f_a/(F'_e)]$

On substitution of values (11.67) reduce to

$$\frac{f_a}{F_a} + \frac{f_{bcx}}{F_{bcx}} \frac{C_{mx}}{\left[1 - f_a/(F'_{ex})\right]} + \frac{f_{bcy}}{F_{bcy}} \frac{C_{my}}{\left[1 - f_a/(F'_{ey})\right]} \leq 1.0 \tag{11.68}$$

where F_{ex} and F_{ey} are elastic critical stresses w.r.t. principal axes of column.

F_e and F'_e are given by:

$$F_e = \pi^2 E/(L_e/r)^2 \quad \text{and} \quad F'_e = F_e/FS \tag{11.69}$$

11.6.1 Codal Provisions

The approximate expression for combined stresses in a short beam-column subjected to an axial load and bending moments with respect to both the axes may be expressed as:

$$f_{\max} = \frac{P}{A} \pm \left(\frac{M_x c_1}{I_x}\right) \pm \left(\frac{M_y c_2}{I_y}\right) \tag{a}$$

The expression can be rewritten as:

$$f_{\max} = f_{\mathrm{a}} + f_{\mathrm{b}x} + f_{\mathrm{b}y} \tag{b}$$

with the negative signs neglected (i. e. considering the absolute value), dividing both sides by $f_{\max}$

$$1 = \frac{f_{\mathrm{a}}}{f_{\max}} + \frac{f_{\mathrm{b}x}}{f_{\max}} + \frac{f_{\mathrm{b}y}}{f_{\max}} \tag{c}$$

For design application the above expression can be improved by introducing the applicable *allowable stresses* in place of $f_{\max}$, the following interaction formula is obtained.

$$\frac{f_{\mathrm{a}}}{F_{\mathrm{a}}} + \frac{f_{\mathrm{b}x}}{F_{\mathrm{b}x}} + \frac{f_{\mathrm{b}y}}{F_{\mathrm{b}y}} \leq 1.0 \tag{d}$$

where the subscripts x and y indicate the axis of bending about which a particular stress applies. The interaction equations used in various national codes are described below.

AISCS

(a) Axial compression and biaxial bending

If $f_{\mathrm{a}}/F_{\mathrm{a}} < 0.15$, the effect of compression on bending is relatively small and linear interaction formula is recommended.

$$\frac{f_{\mathrm{a}}}{F_{\mathrm{a}}} + \frac{f_{\mathrm{bc}x}}{F_{\mathrm{bc}x}} + \frac{f_{\mathrm{bc}y}}{F_{\mathrm{bc}y}} \leq 1.0 \tag{11.70}$$

For $f_{\mathrm{a}}/F_{\mathrm{a}} \geq 0.15$, the secondary moment due to member deflection may be of a significant magnitude which may be taken care of by the *amplification factor*

$$1/\left\{1 - \left[f_{\mathrm{e}}/\left(F_{\mathrm{e}}'\right)\right]\right\}$$

where $F_{\mathrm{e}}' = F_{\mathrm{e}}/FS$ and $FS = 23/12$. Thus for the factor of safety $FS = (23/12)$ (11.68) is expressed as:

$$\frac{f_{\mathrm{a}}}{F_{\mathrm{a}}} + \frac{f_{\mathrm{bc}x}}{F_{\mathrm{bc}x}} \frac{C_{\mathrm{m}x}}{\left[1 - \frac{23 f_{\mathrm{a}}}{12 F_{\mathrm{e}x}}\right]} + \frac{f_{\mathrm{bc}y}}{F_{\mathrm{bc}y}} \frac{C_{\mathrm{m}y}}{\left[1 - \frac{23 f_{\mathrm{a}}}{12 F_{\mathrm{e}y}}\right]} \leq 1.0 \tag{11.71}$$

At the supports and points braced in the plane of bending.

$$\frac{f_a}{0.6F_y} + \frac{f_{bcx}}{F_{bcx}} + \frac{f_{bcy}}{F_{bcy}} \le 1.0 \tag{11.72}$$

These equations recommended for the design of beam-columns are essentially empirical. The factor of safety depends partly upon the relation selected to define allowable stresses F_a and F_b. The quantity F_a varies with slenderness ratio L_e/r and has a factor of safety varying from 1.67 to 1.92. On the other hand F_b normally corresponds to 1/1.67 of the plastic moment capacity of the section. Thus the real factor of safety for the beam-columns can not be determined from individual components. The reduction factor C_m is defined as:

(i) A beam-column in a frame where computed moments are maximum at the ends and joint translations are permitted: $C_{mx}, C_{my} = 0.85$.

(ii) A beam-column in a frame that is subjected to end moments with joint translations prevented and transverse loading being absent:

$$C_{mx}, C_{my} = 0.6 + 0.4\beta \ge 0.40$$

where β is the end moment ratio (smaller end moment to larger end moment) < 1.0 in the portion unbraced in the plane of bending. β is to be taken positive if the end moments tend to produce single curvature. β is negative if double curvature is induced.

(iii) A beam-column which is braced against joint translation and subjected to transverse loading:

$$C_m = 1 + \Psi f_a/F_e' \quad \text{and} \quad F_e' = F_e/FS \tag{11.73}$$

where value of Ψ depends upon the transverse loads and end restraints.

$$\Psi = (\pi^2 \delta_o E I_x / M_o L^2) - 1 \tag{11.74}$$

where δ_o and M_o are the maximum deflection and maximum bending moment between the supports, respectively, due to transverse loads.

(b) Axial tension and biaxial bending

$$\frac{f_a}{0.6F_y} + \frac{f_{bcx}}{F_{bcx}} + \frac{f_{bcy}}{F_{bcy}} \le 1.0 \tag{11.75}$$

where $F_{bcx} = F_{bcy} \le 0.66F_y$

BS: 449

(a) Axial tension and biaxial bending

$$\frac{f_a}{F_a} + \frac{f_{btx} + f_{bty}}{F_{bc}} \le 1.0 \tag{11.76}$$

(b) Axial compression and biaxial bending

$$\frac{f_a}{F_a} + \frac{f_{bcx} + f_{bcy}}{F_{bc}} \leq 1.0 \tag{11.77}$$

where F_b is the appropriate stress for the member subjected to bending. The code has not differentiated between F_{bx} and F_{by}, they may be different.

AS: 1250

The specifications are identical to those of AISCS except that a factor of safety equal to $1.67(1/FS = 0.6)$ has been used instead of $23/12$.

IS: 800

Specifications are the same as those of AS: 1250.

11.6.2 Design of a Beam-Column Member

As noted earlier indeterminate structural design is basically an iterative process. To start the process, based on the judgement or limiting simplifications a cross-section is selected for examination. The selected cross-section is examined for its adequacy to sustain the imposed loading safely. Based on this examination another lighter or heavier cross section (as situation warrants) is selected for examination. The process is repeated till a cross-section with just adequate margin of safety is obtained. To help in the above procedure an approximation to the area of cross-section of a beam-column can be arrived at from the interaction formula (11.68). Let in an extreme case $C_{mx}, C_{my} = 1.03$; Therefore,

$$\left(1 - \frac{f_a}{0.6F_e}\right) F_{bc} = \left(\frac{3}{4} \quad \text{to} \quad \frac{2}{3}\right) F_{bc} = F_a$$

and the interaction formula becomes

$$\frac{f_a}{F_a} + \frac{f_{bcx}}{F_a} + \frac{f_{bcy}}{F_a} = 1 \quad \text{i.e.} \quad f_a + f_{bcx} + f_{bcy} = F_a$$

or

$$\frac{P}{A} + \frac{M_x}{Z_x} + \frac{M_y}{Z_y} = F_a$$

or

$$P + M_x\frac{A}{Z_x} + M_y\frac{A}{Z_y} = F_aA = P_{eff} \tag{11.78}$$

where P and P_{eff} are actual and equivalent axial loads, respectively.

For Indian rolled steel sections in the MB series in the practical range of the sections: $A/Z_x = 0.01$ and $A/Z_y = 0.08$; and for SC series in practical range:

$A/Z_x = 0.015$ and $A/Z_y = 0.050$. The area of cross-section can be estimated by making use of appropriate values for A/Z_x and A/Z_y in (11.78)

$$A = \left[P + M_x \frac{A}{Z_x} + M_y \frac{A}{Z_y}\right] \Big/ F_a \tag{11.79}$$

Alternatively, a compressive allowable stress F_a is assumed based on the slenderness ratio of column, and the area required on the basis of compression only is determined. The area is multiplied by a factor of 2 or more to account for the effect of moments acting on the column. The procedure has been illustrated in the following example.

Example 11.5. Design a rolled steel SC series beam-column of 3.8 m length to resist an axial compressive load of 820 kN and biaxial moments M_x of 2.90 kNm and M_y of 1.95 kNm. The effective length factor for the column is 0.7. F_y for the column material is 250 MPa and $E = 2 \times 10^5$ MPa.

For SC Series section:

$$A/Z_x = 0.015; \quad A/Z_y = 0.050 \quad \text{and} \quad F_a = 132\,\text{MPa} \quad (\text{for assumed } L_e/r = 50).$$

Approximate area of the cross-section from (11.79)

$$\begin{aligned} A &= (820 \times 10^3 + 2.90 \times 10^6 \times 0.015 + 1.95 \times 10^6 \times 0.050)/132 \\ &= 7280\,\text{mm}^2. \end{aligned}$$

Consider SC 200 @ 60.3 kg/m rolled steel section. For this section from metal tables: $A = 7680\,\text{mm}^2$, $B = D = 200\,\text{mm}$, $I_x = 5530 \times 10^4\,\text{mm}^4$, $I_y = 1530 \times 10^4\,\text{mm}^4$, $r_x = 84.8\,\text{mm}$, $r_y = 44.6\,\text{mm}$, $Z_x = 553 \times 10^3\,\text{mm}^3$ and $Z_y = 153 \times 10^3\,\text{mm}^3$. For $L/B = 3800/200 = 19 < 23$ (for the case when weaker axis is unsupported)

$$F_{bc} = 0.66 F_y = 0.66 \times 250 = 165\,\text{MPa}.$$

The slenderness ratios are

$$\begin{aligned} L_e/r_x &= 0.7 \times 3800/84.8 = 31.37 \quad \text{and} \\ L_e/r_y &= 0.7 \times 3800/44.6 = 59.64 \end{aligned}$$

Therefore,

$$\begin{aligned} F_{ex} &= \pi^2 E/(L_e/r_x)^2 = 2005.86\,\text{MPa}. \\ F_{ey} &= \pi^2 E/(L_e/r_y)^2 = 554.95\,\text{MPa}. \end{aligned}$$

From (11.10) for $L_e/r = 59.64$

$$\begin{aligned} F_a &= 0.6 F_y / \left[1 + (F_y/F_{ec})^{1.4}\right]^{1/1.4} = \frac{0.6 \times 250}{\left[1 + (250/554.95)^{1.4}\right]^{1/1.4}} \\ &= 122.52\,\text{MPa} \end{aligned}$$

The individual simple stresses are:

$$f_{\mathrm{a}} = 820 \times 10^3/7680 = 106.77\,\mathrm{MPa}.$$
$$f_{\mathrm{bc}x} = 2.90 \times 10^6/(553 \times 10^3) = 5.24\,\mathrm{MPa}.$$
$$f_{\mathrm{bc}y} = 1.95 \times 10^6/(153 \times 10^3) = 10.13\,\mathrm{MPa}.$$

Moment amplifications factor due to axial thrust

$$1 - [106.77/(0.6 \times 2005.86)] = 0.9113$$
$$1 - [106.77/(0.6 \times 554.95)] = 0.6793$$

Substituting the value in the interaction formula

$$\frac{106.77}{122.52} + \frac{5.24}{0.9113 \times 165} + \frac{10.13}{0.6793 \times 165} = 0.997 < 1.0$$

The selected cross-section is adequate as it meets the specification interaction equation requirement.

Example 11.6. A typical vertical member of a rigid multistory sway frame of height 3.8 m is subjected to $P = 200\,\mathrm{kN}$, $M_x = 30\,\mathrm{kNm}$ and $M_y = 8\,\mathrm{kNm}$. At the top and bottom of the member, the $\sum k_{\mathrm{c}}$ and $\sum k_{\mathrm{b}}$ values are 10, 25 and 16, 25, respectively. HB 300 @ 588 N/m section is readily available. Check the adequacy of the section. F_y and E for the material of member are 250 MPa and 2×10^5 MPa, respectively.

For H 300 @ 588 N/m section: $A = 5880\,\mathrm{mm}^2$, $r_x = 130\,\mathrm{mm}$, $r_y = 54.1\,\mathrm{mm}$, $Z_x = 836 \times 10^3\,\mathrm{mm}^3$ and $Z_y = 175 \times 10^3\,\mathrm{mm}^3$.

For the given member of the sway frame, the member end distribution factors are:

$$\beta_1 = \frac{\sum k_{\mathrm{c}}}{\sum k_{\mathrm{c}} + (3/2)\sum k_{\mathrm{b}}} = \frac{10}{10 + (3/2) \times 25} = 0.211$$
$$\beta_2 = \frac{16}{16 + (3/2) \times 25} = 0.299$$

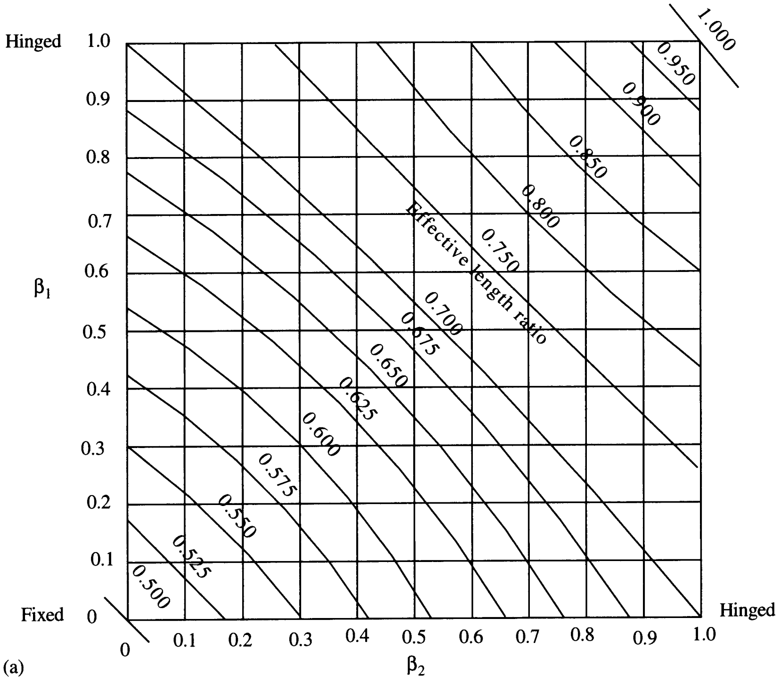
Since the beams bend antisymmetrical i.e. in double curvature during sway a factor 3/2 has been used. For the sway case with $\beta_1 = 0.211$ and $\beta_2 = 0.299$ the effective length ratio from the effective length charts given in Fig. 11.9b is: $K = 1.16 < 1.2$.

Therefore, $K = 1.2$. The slenderness ratios are

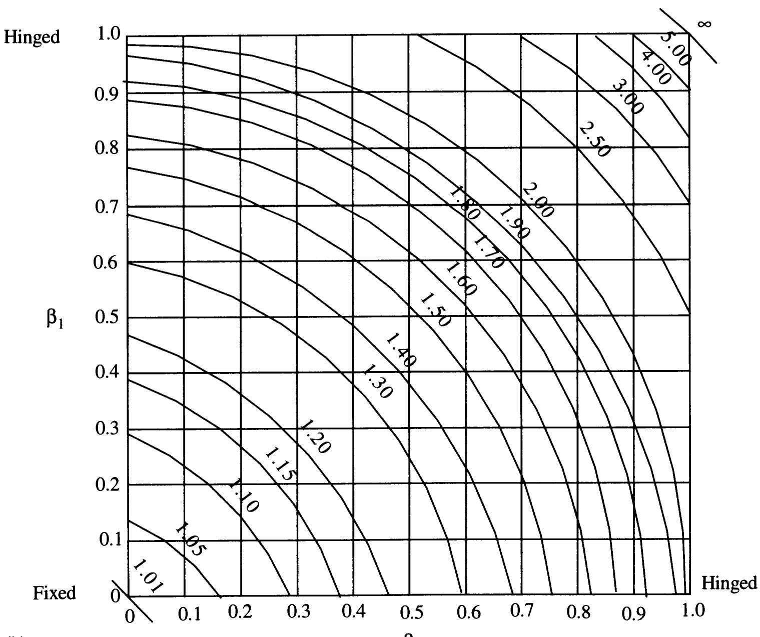
$$KL/r_x = 1.2 \times 3800/130 = 35.08$$
$$KL/r_y = 1.2 \times 3800/54.1 = 84.29$$

Therefore,

$$F_{\mathrm{ec}x} = \pi^2 E/(KL/r_x)^2 = 1604.02\,\mathrm{MPa}$$
$$F_{\mathrm{ec}y} = \pi^2 E/(KL/r_y)^2 = 277.83\,\mathrm{MPa}.$$



(a)



(b)

Fig. 11.9a,b. Effective length ratio curves for columns. **a** For columns in non-sway frame, **b** for columns in a sway frame

The permissible stresses are:

$$F_a = 0.6F_y/\left[1 + (F_y/F_{ec})^{1.4}\right]^{1/1.4} = 0.6 \times 250/\left[1 + (250/277.83)^{1.4}\right]^{1/1.4}$$
$$= 96.19\,\text{MPa}$$

$$f_{bcx} = F_{bcy} = 0.66F_y = 0.66 \times 250 = 165\,\text{MPa}.$$
$$f_a = (200 \times 10^3)/5880 = 34.01\,\text{MPa}.$$
$$f_{bcx} = (30 \times 10^6)/(836 \times 10^3) = 35.89\,\text{MPa}.$$
$$f_{bcy} = (8 \times 10^6)/(175 \times 10^3) = 45.71\,\text{MPa}.$$

The moment amplification factors due to axial thrust are:

$$1 - [34.01/(0.6 \times 1604.02)] = 0.9647$$
$$1 - [34.01/(0.6 \times 277.83)] = 0.7960$$

Substituting the values in the interaction formula

$$\frac{34.01}{96.19} + \frac{35.89}{(0.9647 \times 165)} + \frac{45.71}{(0.7960 \times 165)} = 0.927 < 1.0$$

Minimum depth required = 3800/15 = 253.33 mm < 300 mm (available). The available section is more than adequate.

11.7 Optimum Design

The cost of a structural element depends on its weight, which in turn is related to its cross-sectional area. The structural design procedure used in the preceding sections consists in selecting a cross-section and checking its adequacy to sustain applied loads. The process is repeated till a safe design is obtained. This procedure may not yield the most efficient cross-section. To illustrate the underlying principles consider the design of a beam of length L having rectangular cross section of size $t \times d$.

Most of the national codes take the lateral stability a design consideration. For the uniform moment case the critical moment is given by (11.35). The equation is rewritten here for the convenience.

$$M_{cr} = \frac{\pi}{L_e}\sqrt{\frac{(EI_yGJ)I_x}{(I_x - I_y)}}\left[\sqrt{1 + \left(\frac{\pi^2 EI_w}{L_e^2 GJ}\right)}\right] \tag{11.80}$$

For thin rectangular cross sections, the torsional parameter $(\pi^2 EI_w/L_e^2 GJ)$ being small can be ignored hence (11.80) reduces to:

$$M_{cr} = \frac{\pi}{L_e}\sqrt{\frac{(EI_yGJ)I_x}{(I_x - I_y)}} \tag{11.81}$$

It may be seen from the quantity under the radical that as I_y approaches I_x (i.e., for squarish cross-sections) the denominator becomes small so that critical moment becomes very large, a disadvantageous proposition.

Consider the beam to be subjected to an applied moment M which is constant over the length of the beam. The cross-section $t \times d$ of the beam should be such that the calculated stress in the beam does not exceed the allowable stress in the material F_b. Thus

$$F_b \geq \frac{M(d/2)}{td^3/12} \quad \text{or} \quad F_b \geq \frac{6M}{td^2} \quad \text{i.e.} \quad td \geq \frac{6M}{dF_b} \tag{11.82}$$

For the cross-sectional area td to be as small as possible, d should be as large as possible. Therefore, replace the inequality in (11.82) by the equality:

$$td = 6M/(dF_b) \tag{11.83a}$$

Therefore,

$$t/d = (6M)/(d^3 F_b) \tag{11.83b}$$

The maximum value of d is subject to a constraint based on (11.81). Thus using a factor of safety FS in this connection.

$$M(FS) \leq \left(\frac{\pi}{L_e}\right)\sqrt{\frac{(EI_y GJ)I_x}{I_x - I_y}} \tag{11.84}$$

For the given section:

$$I_x = td^3/12, \quad I_y = dt^3/12 \quad \text{and} \quad J = dt^3/3$$

Therefore, from (11.84)

$$M(FS) \leq \frac{\pi\sqrt{EG}}{6L_e}\left(\frac{t}{d}\right)\left[\frac{1}{1-(t/d)^2}\right]^{1/2}(td)^2 \tag{11.85}$$

Since it is desired to minimize td and maximize d, replace the inequality sign by equality sign. As the square of t/d is expected to be small compared to unity,

$$\frac{1}{1-(t/d)^2} = 1 + \left(\frac{t}{d}\right)^2 + \left(\frac{t}{d}\right)^4 + \ldots \approx \left[1 + \left(\frac{t}{d}\right)^2\right]$$

Therefore,

$$M(FS) = \frac{\pi\sqrt{EG}}{6L_e}\left(\frac{t}{d}\right)\left[1 + \left(\frac{t}{d}\right)^2\right]^{1/2}(td)^2 \tag{11.86}$$

Substituting from (11.83) into (11.86)

$$M(FS) = \frac{\pi\sqrt{EG}}{6L_e}\left(\frac{6M}{d^3F_b}\right)\left[1 + \left(\frac{6M}{d^3F_b}\right)^2\right]^{1/2}\left(\frac{6M}{dF_b}\right)^2 \tag{11.87}$$

Further neglecting $(t/d)^2$ in comparison to unity yields:

$$d = \left[\frac{\pi (6M)^2 (EG)^{1/2}}{(FS) L_e F_b^3} \right]^{1/5} \tag{11.88}$$

This equation can be used as a starting approximation for an iterative solution to the optimal depth of the cross-section. For illustration consider steel beam with $E = 2.5G = 2 \times 10^5$ MPa, $F_b = 200$ MPa and $FS = 1.67$. From (11.88).

$$d = \left[\frac{\pi (36) \left[(2 \times 10^5)(0.8 \times 10^5) \right]^{1/2}}{(1.67)(200)^3} \frac{M^2}{L_e} \right]^{1/5} = 1.0138 \left(\frac{M^2}{L_e} \right)^{1/5} \tag{11.89}$$

Equation (11.89) provides a reasonable approximation to the optimal depth of the cross-section. For the purpose of iteration (11.85) can be rewritten as:

$$d = \left\{ \frac{\pi (6M)^2}{(FS) L_e F_b^3} \left[\frac{EG}{1 - \left[(6M)/(d^3 F_b) \right]^2} \right]^{1/2} \right\}^{1/5} \tag{11.90}$$

Example 11.7. A two metre long rectangular beam is laterally unsupported over its entire length. Design the beam for lateral torsional buckling when it is subjected to a uniform moment of 15 kNm. The design stipulations are:

$$E = 2.5G = 2 \times 10^5 \text{ MPa}, \quad F_b = 200 \text{ MPa} \quad \text{and} \quad FS = 1.67.$$

With these stipulations the first approximation to the optimal depth of beam is given by (11.89)

$$d = 1.0138[(15 \times 10^6)^2/(2 \times 10^3)]^{1/5} = 162.27 \text{ mm}$$

For iteration (11.90) becomes

$$\begin{aligned} d &= \left\{ \frac{[\pi^2 (6M)^4 EG]/[(FS) L_e F_b^3]^2}{1 - [(6M)/(d^3 F_b)]^2} \right\}^{1/10} \\ &= \left\{ \frac{1.451165 \times 10^{22}}{1 - (2.025 \times 10^{11}/d^6)} \right\}^{1/10} \end{aligned} \tag{a}$$

Substituting the value of $d = 162.27$ mm in the right hand side of (a) results in a value of $d = 164.686$. The latest value of $d = 164.686$ when resubstituted in the iteration equation (a) gives $d = 164.670$. Further iteration does not improve the value of d; hence the optimum depth d of the cross-section is 164.670 mm. The corresponding thickness t of the cross section can be obtained from (11.83b):

$$t = [(6M)/(d^3 F_b)]d = (6M/d^2 F_b) = 16.60 \text{ mm}$$

Thus the optimum cross-section of the beam is 16.6 × 164.7 mm. It should be noted that (11.88) provides a reasonable first approximation to the optimal depth.

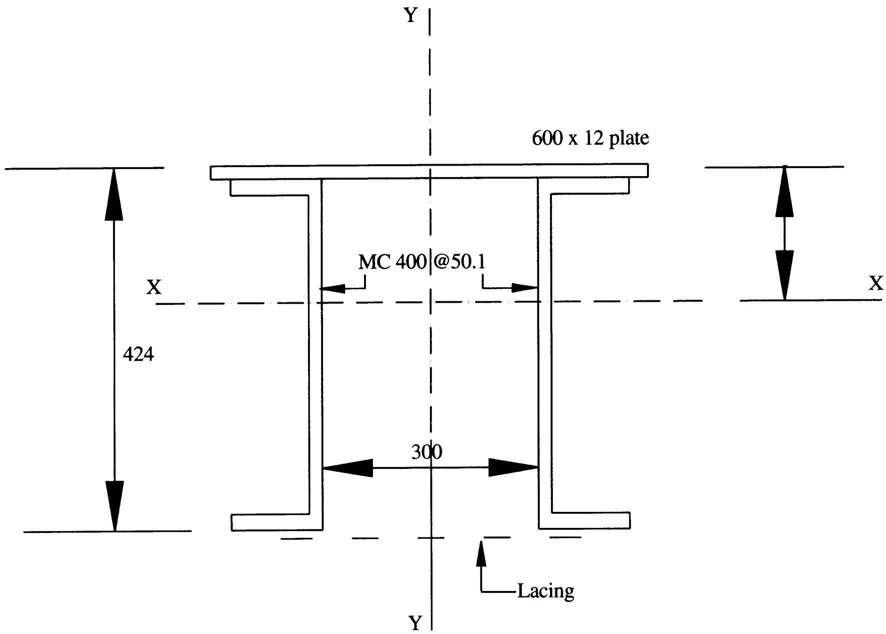
11.8 Problems

Problem 11.1. A centrally loaded column, is simply supported about strong axis at both ends, and fixed about weak axis with warping free at the top and restrained at the bottom. Determine the axial buckling load if the column cross-section is MB500 with slenderness ratio L_e/r_x: (i) 25 and (ii) 110. Take $E = 2.5G = 2 \times 10^5$ MPa.

[Hint: The boundary conditions are:
(i) at the top: $v = v'' = u = u' = \beta = \beta'' = 0$
(ii) at the bottom: $v = v'' = u = u' = \beta = \beta' = 0$, $K_x = 1.00$, $K_y = 0.5$ *and* $K_z = 0.70$.
Compute $P_{cr,x}$, $P_{cr,y}$ *and* $P_{cr,z}$. *Lowest value will give the critical load.]*

Problem 11.2. A typical welded built-up cross-section compression chord of a bridge is shown in Fig. P.11.2. Determine the axial thrust that the member can sustain if its length is 5.75 m, $K = 1.00$, $F_y = 250$ MPa and FS is 1.8. For the welded cross-section, maximum compression residual stress F_r may be presumed to be $0.50F_y$.

[Hint: Determine A, I_x *and* I_y *of the built-up cross-section and hence the least radius of gyration and governing slenderness ratio to compute* F_a *by AISCS method.]*



P.11.2

Problem 11.3. A beam-column member of length 3.8 m is subjected to an axial thrust of 750 kN and a moment of 5 kNm about the major axis. The weaker plane of the member is strengthened by bracing. Design the member, if its effective length coefficient is 0.70.

Problem 11.4. Design a member of length of 3.8 m if it is subjected to an axial load of 750 kN, a major axis moment of 5.0 kNm and a weaker axis moment of 2.0 kNm. The effective length coefficient is 1.2. The column is free to buckle in any plane.

Problem 11.5. A simply supported beam of span of 5.4 m carries a uniformly distributed load of 42 kN/m and two concentrated loads of 100 kN each at one-third points. Design the beam using an available rolled steel section MB400, if the beam is laterally supported throughout the span.

Problem 11.6. A 6.2 m long simply supported beam has lateral supports at the ends only. The ends of the beam are free to rotate at the bearings and are torsionally restrained. The beam section is composed of MB300 rolled steel section with a 220×16 mm plate attached to the top flange, determine the allowable bending stress.

Appendix A

Stability Functions

A.1 Stability Functions for Compression Members

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
0.00	4.0000	4.0000	0.5000	3.0000	6.0000	12.0000	1.0000	1.0000	−1.0000
0.02	3.9736	4.0265	0.5050	2.9603	5.9802	11.7631	1.0168	0.9333	−1.0337
0.04	3.9471	4.0535	0.5101	2.9201	5.9604	11.5260	1.0343	0.8648	−1.0690
0.06	3.9204	4.0808	0.5153	2.8795	5.9405	11.2889	1.0525	0.7943	−1.1060
0.08	3.8936	4.1086	0.5206	2.8384	5.9206	11.0516	1.0714	0.7218	−1.1448
0.10	3.8667	4.1369	0.5260	2.7968	5.9006	10.8142	1.0913	0.6471	−1.1856
0.12	3.8396	4.1656	0.5316	2.7547	5.8805	10.5767	1.1120	0.5701	−1.2285
0.14	3.8123	4.1947	0.5372	2.7120	5.8604	10.3391	1.1336	0.4905	−1.2737
0.16	3.7849	4.2244	0.5430	2.6688	5.8403	10.1014	1.1563	0.4083	−1.3213
0.18	3.7574	4.2545	0.5490	2.6251	5.8200	9.8636	1.1801	0.3233	−1.3715
0.20	3.7297	4.2851	0.5550	2.5808	5.7998	9.6256	1.2051	0.2351	−1.4245
0.22	3.7019	4.3162	0.5612	2.5359	5.7794	9.3875	1.2313	0.1438	−1.4805
0.24	3.6739	4.3479	0.5676	2.4904	5.7590	9.1493	1.2589	0.0489	−1.5398
0.26	3.6457	4.3801	0.5741	2.4443	5.7385	8.9110	1.2880	−0.0498	−1.6027
0.28	3.6174	4.4128	0.5807	2.3975	5.7180	8.6726	1.3186	−0.1527	−1.6694
0.30	3.5889	4.4460	0.5875	2.3500	5.6974	8.4340	1.3511	−0.2599	−1.7402
0.32	3.5602	4.4799	0.5945	2.3019	5.6768	8.1953	1.3854	−0.3720	−1.8157
0.34	3.5314	4.5143	0.6017	2.2531	5.6561	7.9565	1.4218	−0.4894	−1.8961
0.36	3.5024	4.5493	0.6090	2.2035	5.6353	7.7176	1.4604	−0.6125	−1.9820
0.38	3.4732	4.5849	0.6165	2.1532	5.6145	7.4785	1.5015	−0.7418	−2.0738
0.40	3.4439	4.6211	0.6242	2.1021	5.5936	7.2393	1.5453	−0.8781	−2.1723
0.42	3.4144	4.6580	0.6321	2.0502	5.5726	7.0000	1.5922	−1.0219	−2.2781
0.44	3.3847	4.6955	0.6402	1.9974	5.5516	6.7605	1.6423	−1.1741	−2.3919
0.46	3.3548	4.7337	0.6485	1.9438	5.5305	6.5210	1.6962	−1.3357	−2.5148
0.48	3.3247	4.7725	0.6571	1.8893	5.5093	6.2813	1.7542	−1.5076	−2.6477
0.50	3.2945	4.8121	0.6659	1.8338	5.4881	6.0414	1.8168	−1.6910	−2.7918
0.52	3.2640	4.8524	0.6749	1.7774	5.4668	5.8015	1.8846	−1.8875	−2.9487
0.54	3.2334	4.8934	0.6841	1.7200	5.4455	5.5614	1.9583	−2.0986	−3.1199
0.56	3.2025	4.9351	0.6937	1.6615	5.4240	5.3211	2.0387	−2.3264	−3.3075
0.58	3.1715	4.9776	0.7035	1.6020	5.4026	5.0807	2.1267	−2.5733	−3.5137
0.60	3.1403	5.0210	0.7136	1.5414	5.3810	4.8402	2.2234	−2.8419	−3.7414
0.62	3.1088	5.0651	0.7239	1.4795	5.3594	4.5996	2.3304	−3.1359	−3.9941
0.64	3.0771	5.1100	0.7346	1.4165	5.3377	4.3588	2.4491	−3.4592	−4.2758
0.66	3.0453	5.1558	0.7456	1.3522	5.3159	4.1179	2.5819	−3.8172	−4.5918
0.68	3.0132	5.2025	0.7570	1.2866	5.2941	3.8768	2.7311	−4.2162	−4.9485
0.70	2.9809	5.2500	0.7687	1.2197	5.2722	3.6356	2.9003	−4.6645	−5.3541

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
0.72	2.9484	5.2985	0.7807	1.1512	5.2502	3.3943	3.0935	−5.1725	−5.8190
0.74	2.9156	5.3479	0.7932	1.0814	5.2282	3.1528	3.3165	−5.7540	−6.3571
0.76	2.8826	5.3983	0.8060	1.0099	5.2060	2.9112	3.5766	−6.4273	−6.9865
0.78	2.8494	5.4497	0.8193	0.9368	5.1838	2.6694	3.8839	−7.2174	−7.7323
0.80	2.8159	5.5020	0.8330	0.8621	5.1616	2.4275	4.2526	−8.1592	−8.6295
0.82	2.7822	5.5555	0.8472	0.7855	5.1392	2.1854	4.7032	−9.3032	−9.7285
0.84	2.7483	5.6100	0.8618	0.7071	5.1168	1.9432	5.2664	−10.7253	−11.1051
0.86	2.7141	5.6655	0.8770	0.6267	5.0943	1.7008	5.9904	−12.5445	−12.8784
0.88	2.6797	5.7223	0.8927	0.5442	5.0718	1.4583	6.9556	−14.9591	−15.2467
0.90	2.6450	5.7801	0.9090	0.4596	5.0491	1.2157	8.3069	−18.3264	−18.5672
0.92	2.6100	5.8392	0.9258	0.3727	5.0264	0.9728	10.3336	−23.3606	−23.5541
0.94	2.5748	5.8995	0.9433	0.2835	5.0036	0.7299	13.7113	−31.7284	−31.8742
0.96	2.5392	5.9611	0.9615	0.1917	4.9808	0.4867	20.4663	−48.4298	−48.5275
0.98	2.5035	6.0239	0.9804	0.0972	4.9578	0.2434	40.7308	−98.4647	−98.5138
1.00	2.4674	6.0881	1.0000	0.0000	4.9348	0.0000	∞	∞	∞
1.02	2.4311	6.1536	1.0204	−0.1002	4.9117	−0.2436	−40.3255	101.4645	101.5141
1.04	2.3944	6.2206	1.0416	−0.2035	4.8885	−0.4874	−20.0610	51.4286	51.5283
1.06	2.3575	6.2889	1.0638	−0.3102	4.8652	−0.7313	−13.3060	34.7259	34.8762
1.08	2.3202	6.3588	1.0868	−0.4204	4.8419	−0.9754	−9.9283	26.3561	26.5576
1.10	2.2827	6.4302	1.1109	−0.5343	4.8185	−1.2196	−7.9016	21.3194	21.5725
1.12	2.2448	6.5032	1.1360	−0.6522	4.7950	−1.4640	−6.5503	17.9491	18.2544
1.14	2.2066	6.5778	1.1623	−0.7743	4.7714	−1.7086	−5.5851	15.5308	15.8889
1.16	2.1681	6.6541	1.1898	−0.9009	4.7477	−1.9534	−4.8610	13.7074	14.1189
1.18	2.1293	6.7321	1.2185	−1.0324	4.7239	−2.1983	−4.2978	12.2806	12.7459
1.20	2.0901	6.8119	1.2487	−1.1690	4.7001	−2.4434	−3.8472	11.1312	11.6510
1.22	2.0506	6.8935	1.2804	−1.3112	4.6761	−2.6886	−3.4785	10.1835	10.7584
1.24	2.0107	6.9770	1.3137	−1.4592	4.6521	−2.9341	−3.1711	9.3869	10.0176
1.26	1.9705	7.0625	1.3487	−1.6137	4.6280	−3.1797	−2.9110	8.7066	9.3936
1.28	1.9299	7.1499	1.3855	−1.7750	4.6038	−3.4255	−2.6880	8.1174	8.8615
1.30	1.8889	7.2394	1.4244	−1.9437	4.5795	−3.6714	−2.4947	7.6012	8.4029
1.32	1.8476	7.3311	1.4655	−2.1204	4.5552	−3.9176	−2.3255	7.1441	8.0041
1.34	1.8058	7.4249	1.5089	−2.3058	4.5307	−4.1639	−2.1762	6.7357	7.6547
1.36	1.7637	7.5210	1.5549	−2.5006	4.5061	−4.4104	−2.0434	6.3677	7.3464
1.38	1.7212	7.6195	1.6038	−2.7058	4.4815	−4.6571	−1.9246	6.0337	7.0729
1.40	1.6782	7.7203	1.6557	−2.9221	4.4568	−4.9039	−1.8176	5.7286	6.8289
1.42	1.6348	7.8237	1.7109	−3.1507	4.4319	−5.1510	−1.7208	5.4481	6.6103
1.44	1.5910	7.9296	1.7699	−3.3929	4.4070	−5.3982	−1.6328	5.1888	6.4138
1.46	1.5468	8.0383	1.8329	−3.6499	4.3820	−5.6457	−1.5523	4.9480	6.2363
1.48	1.5021	8.1496	1.9005	−3.9233	4.3569	−5.8933	−1.4786	4.7231	6.0758
1.50	1.4570	8.2638	1.9731	−4.2150	4.3317	−6.1411	−1.4107	4.5123	5.9301
1.52	1.4114	8.3810	2.0512	−4.5269	4.3064	−6.3891	−1.3480	4.3139	5.7975
1.54	1.3653	8.5012	2.1356	−4.8616	4.2809	−6.6373	−1.2900	4.1264	5.6768
1.56	1.3187	8.6246	2.2271	−5.2217	4.2554	−6.8857	−1.2360	3.9486	5.5667
1.58	1.2716	8.7512	2.3264	−5.6105	4.2298	−7.1343	−1.1858	3.7794	5.4661
1.60	1.2240	8.8813	2.4348	−6.0320	4.2041	−7.3831	−1.1389	3.6179	5.3741
1.62	1.1759	9.0148	2.5534	−6.4906	4.1783	−7.6321	−1.0949	3.4634	5.2900
1.64	1.1272	9.1519	2.6838	−6.9919	4.1524	−7.8813	−1.0537	3.3150	5.2130
1.66	1.0780	9.2928	2.8278	−7.5424	4.1264	−8.1307	−1.0150	3.1722	5.1426
1.68	1.0282	9.4376	2.9877	−8.1502	4.1003	−8.3803	−0.9786	3.0344	5.0782
1.70	0.9779	9.5864	3.1662	−8.8253	4.0741	−8.6302	−0.9442	2.9012	5.0195
1.72	0.9270	9.7394	3.3667	−9.5800	4.0478	−8.8802	−0.9116	2.7720	4.9659
1.74	0.8754	9.8968	3.5936	−10.4299	4.0213	−9.1305	−0.8809	2.6465	4.9170
1.76	0.8233	10.0587	3.8524	−11.3949	3.9948	−9.3809	−0.8517	2.5244	4.8727
1.78	0.7705	10.2252	4.1504	−12.5011	3.9681	−9.6316	−0.8240	2.4053	4.8325
1.80	0.7170	10.3966	4.4969	−13.7828	3.9414	−9.8825	−0.7977	2.2889	4.7963
1.82	0.6629	10.5731	4.9051	−15.2868	3.9145	−10.1336	−0.7726	2.1750	4.7638
1.84	0.6081	10.7548	5.3929	−17.0776	3.8876	−10.3850	−0.7487	2.0634	4.7347
1.86	0.5526	10.9419	5.9859	−19.2479	3.8605	−10.6365	−0.7259	1.9537	4.7090

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
1.88	0.4964	11.1347	6.7223	−21.9352	3.8333	−10.8883	−0.7041	1.8459	4.6864
1.90	0.4394	11.3335	7.6612	−25.3521	3.8059	−11.1404	−0.6833	1.7397	4.6668
1.92	0.3817	11.5383	8.8990	−29.8466	3.7785	−11.3926	−0.6633	1.6349	4.6500
1.94	0.3232	11.7496	10.6056	−36.0301	3.7510	−11.6451	−0.6442	1.5314	4.6360
1.96	0.2639	11.9675	13.1087	−45.0844	3.7233	−11.8978	−0.6259	1.4291	4.6246
1.98	0.2038	12.1923	17.1355	−59.6290	3.6955	−12.1508	−0.6083	1.3277	4.6157
2.00	0.1428	12.4244	24.6841	−86.8644	3.6676	−12.4040	−0.5914	1.2272	4.6093
2.02	0.0809	12.6640	43.9616	−156.3627	3.6396	−12.6574	−0.5751	1.1275	4.6052
2.04	0.0182	12.9114	197.3863	−709.2395	3.6115	−12.9111	−0.5594	1.0284	4.6034
2.06	−0.0455	13.1671	−79.8138	289.5707	3.5832	−13.1650	−0.5443	0.9298	4.6039
2.08	−0.1101	13.4313	−33.2921	121.9015	3.5548	−13.4192	−0.5298	0.8316	4.6066
2.10	−0.1757	13.7045	−21.0722	77.8328	3.5263	−13.6736	−0.5158	0.7337	4.6113
2.12	−0.2423	13.9870	−15.4361	57.4874	3.4976	−13.9283	−0.5022	0.6360	4.6182
2.14	−0.3099	14.2793	−12.1925	45.7629	3.4689	−14.1832	−0.4892	0.5385	4.6272
2.16	−0.3786	14.5818	−10.0850	38.1320	3.4400	−14.4384	−0.4765	0.4409	4.6382
2.18	−0.4485	14.8950	−8.6059	32.7650	3.4109	−14.6939	−0.4643	0.3433	4.6512
2.20	−0.5194	15.2194	−7.5107	28.7813	3.3818	−14.9496	−0.4524	0.2456	4.6662
2.22	−0.5916	15.5555	−6.6673	25.7044	3.3525	−15.2055	−0.4410	0.1476	4.6832
2.24	−0.6649	15.9039	−5.9978	23.2542	3.3231	−15.4618	−0.4298	0.0493	4.7022
2.26	−0.7395	16.2652	−5.4537	21.2552	3.2935	−15.7183	−0.4191	−0.0494	4.7231
2.28	−0.8154	16.6400	−5.0027	19.5917	3.2638	−15.9751	−0.4086	−0.1486	4.7460
2.30	−0.8926	17.0289	−4.6230	18.1845	3.2340	−16.2321	−0.3985	−0.2483	4.7709
2.32	−0.9713	17.4328	−4.2988	16.9775	3.2040	−16.4895	−0.3886	−0.3487	4.7978
2.34	−1.0513	17.8523	−4.0190	15.9298	3.1739	−16.7471	−0.3790	−0.4498	4.8267
2.36	−1.1328	18.2883	−3.7750	15.0109	3.1436	−17.0050	−0.3697	−0.5517	4.8576
2.38	−1.2159	18.7416	−3.5604	14.1977	3.1132	−17.2632	−0.3607	−0.6545	4.8906
2.40	−1.3006	19.2131	−3.3703	13.4723	3.0827	−17.5216	−0.3519	−0.7582	4.9256
2.42	−1.3869	19.7038	−3.2006	12.8204	3.0520	−17.7804	−0.3433	−0.8630	4.9628
2.44	−1.4749	20.2148	−3.0484	12.2310	3.0212	−18.0395	−0.3350	−0.9689	5.0021
2.46	−1.5647	20.7470	−2.9111	11.6949	2.9902	−18.2988	−0.3268	−1.0761	5.0435
2.48	−1.6563	21.3018	−2.7865	11.2047	2.9591	−18.5585	−0.3189	−1.1845	5.0872
2.50	−1.7499	21.8804	−2.6732	10.7543	2.9278	−18.8184	−0.3112	−1.2943	5.1332
2.52	−1.8454	22.4841	−2.5695	10.3386	2.8964	−19.0787	−0.3036	−1.4057	5.1814
2.54	−1.9430	23.1144	−2.4744	9.9534	2.8648	−19.3393	−0.2963	−1.5186	5.2321
2.56	−2.0427	23.7728	−2.3869	9.5952	2.8330	−19.6001	−0.2891	−1.6332	5.2852
2.58	−2.1447	24.4610	−2.3061	9.2607	2.8011	−19.8613	−0.2821	−1.7496	5.3409
2.60	−2.2490	25.1808	−2.2312	8.9475	2.7691	−20.1229	−0.2752	−1.8679	5.3991
2.62	−2.3557	25.9341	−2.1618	8.6533	2.7368	−20.3847	−0.2685	−1.9883	5.4600
2.64	−2.4650	26.7230	−2.0971	8.3761	2.7044	−20.6469	−0.2620	−2.1107	5.5237
2.66	−2.5769	27.5497	−2.0369	8.1142	2.6719	−20.9094	−0.2556	−2.2355	5.5902
2.68	−2.6915	28.4165	−1.9805	7.8662	2.6392	−21.1722	−0.2493	−2.3626	5.6597
2.70	−2.8091	29.3262	−1.9278	7.6308	2.6063	−21.4353	−0.2432	−2.4922	5.7323
2.72	−2.9296	30.2815	−1.8784	7.4067	2.5732	−21.6988	−0.2372	−2.6245	5.8080
2.74	−3.0533	31.2854	−1.8319	7.1931	2.5400	−21.9627	−0.2313	−2.7596	5.8871
2.76	−3.1803	32.3413	−1.7882	6.9889	2.5066	−22.2269	−0.2255	−2.8976	5.9696
2.78	−3.3108	33.4526	−1.7470	6.7934	2.4730	−22.4914	−0.2199	−3.0389	6.0558
2.80	−3.4449	34.6234	−1.7081	6.6059	2.4393	−22.7563	−0.2144	−3.1834	6.1456
2.82	−3.5828	35.8577	−1.6714	6.4257	2.4054	−23.0215	−0.2090	−3.3314	6.2394
2.84	−3.7246	37.1601	−1.6366	6.2522	2.3713	−23.2871	−0.2037	−3.4832	6.3374
2.86	−3.8707	38.5358	−1.6038	6.0849	2.3370	−23.5531	−0.1984	−3.6389	6.4396
2.88	−4.0213	39.9901	−1.5726	5.9234	2.3025	−23.8195	−0.1933	−3.7987	6.5463
2.90	−4.1765	41.5290	−1.5430	5.7671	2.2678	−24.0862	−0.1883	−3.9629	6.6578
2.92	−4.3366	43.1591	−1.5149	5.6158	2.2330	−24.3533	−0.1834	−4.1318	6.7743
2.94	−4.5019	44.8876	−1.4882	5.4690	2.1979	−24.6207	−0.1785	−4.3057	6.8960
2.96	−4.6727	46.7223	−1.4628	5.3264	2.1627	−24.8886	−0.1738	−4.4847	7.0233
2.98	−4.8492	48.6720	−1.4387	5.1878	2.1273	−25.1568	−0.1691	−4.6694	7.1564
3.00	−5.0320	50.7463	−1.4157	5.0528	2.0917	−25.4255	−0.1645	−4.8599	7.2957

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
3.02	−5.2212	52.9557	−1.3937	4.9212	2.0558	−25.6945	−0.1600	−5.0567	7.4416
3.04	−5.4174	55.3121	−1.3728	4.7927	2.0198	−25.9640	−0.1556	−5.2603	7.5943
3.06	−5.6209	57.8285	−1.3529	4.6672	1.9836	−26.2338	−0.1512	−5.4709	7.7545
3.08	−5.8323	60.5193	−1.3339	4.5444	1.9472	−26.5041	−0.1469	−5.6892	7.9225
3.10	−6.0519	63.4008	−1.3157	4.4242	1.9105	−26.7747	−0.1427	−5.9156	8.0988
3.12	−6.2805	66.4910	−1.2983	4.3063	1.8737	−27.0458	−0.1386	−6.1507	8.2840
3.14	−6.5186	69.8101	−1.2817	4.1907	1.8366	−27.3174	−0.1345	−6.3952	8.4787
3.16	−6.7669	73.3808	−1.2659	4.0771	1.7993	−27.5893	−0.1304	−6.6496	8.6836
3.18	−7.0262	77.2287	−1.2508	3.9654	1.7618	−27.8617	−0.1265	−6.9148	8.8994
3.20	−7.2971	81.3826	−1.2363	3.8556	1.7241	−28.1345	−0.1226	−7.1915	9.1269
3.22	−7.5807	85.8752	−1.2224	3.7474	1.6862	−28.4078	−0.1187	−7.4806	9.3670
3.24	−7.8779	90.7436	−1.2092	3.6407	1.6480	−28.6815	−0.1149	−7.7833	9.6206
3.26	−8.1899	96.0299	−1.1965	3.5355	1.6096	−28.9557	−0.1112	−8.1004	9.8890
3.28	−8.5178	101.7826	−1.1844	3.4317	1.5710	−29.2304	−0.1075	−8.4333	10.1732
3.30	−8.8629	108.0569	−1.1729	3.3291	1.5321	−29.5055	−0.1039	−8.7834	10.4746
3.32	−9.2269	114.9167	−1.1618	3.2276	1.4930	−29.7810	−0.1003	−9.1520	10.7948
3.34	−9.6114	122.4357	−1.1512	3.1272	1.4537	−30.0571	−0.0967	−9.5411	11.1354
3.36	−10.0183	130.6995	−1.1412	3.0278	1.4141	−30.3336	−0.0932	−9.9524	11.4983
3.38	−10.4497	139.8079	−1.1315	2.9293	1.3743	−30.6107	−0.0898	−10.3880	11.8857
3.40	−10.9082	149.8780	−1.1223	2.8316	1.3342	−30.8882	−0.0864	−10.8506	12.3001
3.42	−11.3965	161.0474	−1.1135	2.7347	1.2939	−31.1662	−0.0830	−11.3428	12.7442
3.44	−11.9178	173.4792	−1.1052	2.6385	1.2533	−31.4448	−0.0797	−11.8679	13.2211
3.46	−12.4757	187.3678	−1.0972	2.5428	1.2125	−31.7238	−0.0764	−12.4294	13.7346
3.48	−13.0745	202.9458	−1.0896	2.4478	1.1714	−32.0034	−0.0732	−13.0316	14.2888
3.50	−13.7190	220.4940	−1.0824	2.3532	1.1301	−32.2835	−0.0700	−13.6794	14.8886
3.52	−14.4149	240.3536	−1.0755	2.2591	1.0884	−32.5641	−0.0668	−14.3785	15.5397
3.54	−15.1689	262.9423	−1.0690	2.1653	1.0465	−32.8453	−0.0637	−15.1356	16.2488
3.56	−15.9890	288.7758	−1.0628	2.0719	1.0044	−33.1270	−0.0606	−15.9586	17.0239
3.58	−16.8845	318.4967	−1.0570	1.9787	0.9619	−33.4093	−0.0576	−16.8568	17.8742
3.60	−17.8668	352.9140	−1.0514	1.8857	0.9192	−33.6921	−0.0546	−17.8417	18.8111
3.62	−18.9494	393.0567	−1.0462	1.7930	0.8762	−33.9755	−0.0516	−18.9268	19.8483
3.64	−20.1492	440.2504	−1.0413	1.7003	0.8329	−34.2595	−0.0486	−20.1290	21.0024
3.66	−21.4868	496.2245	−1.0367	1.6077	0.7893	−34.5441	−0.0457	−21.4687	22.2941
3.68	−22.9879	563.2705	−1.0324	1.5151	0.7455	−34.8292	−0.0428	−22.9719	23.7493
3.70	−24.6852	644.4737	−1.0284	1.4225	0.7013	−35.1149	−0.0399	−24.6712	25.4005
3.72	−26.6208	744.0683	−1.0247	1.3298	0.6568	−35.4013	−0.0371	−26.6086	27.2898
3.74	−28.8496	867.9886	−1.0212	1.2370	0.6120	−35.6883	−0.0343	−28.8391	29.4721
3.76	−31.4449	1024.7578	−1.0180	1.1441	0.5669	−35.9758	−0.0315	−31.4360	32.0208
3.78	−34.5066	1226.9671	−1.0151	1.0509	0.5215	−36.2641	−0.0288	−34.4991	35.0356
3.80	−38.1745	1493.8426	−1.0125	0.9575	0.4758	−36.5529	−0.0260	−38.1683	38.6565
3.82	−42.6506	1855.9180	−1.0101	0.8638	0.4297	−36.8424	−0.0233	−42.6456	43.0854
3.84	−48.2381	2364.0441	−1.0079	0.7698	0.3834	−37.1326	−0.0206	−48.2341	48.6254
3.86	−55.4130	3108.0245	−1.0061	0.6754	0.3367	−37.4234	−0.0180	−55.4100	55.7527
3.88	−64.9691	4258.6957	−1.0045	0.5805	0.2896	−37.7149	−0.0154	−64.9669	65.2609
3.90	−78.3349	6174.3571	−1.0031	0.4852	0.2422	−38.0070	−0.0127	−78.3333	78.5786
3.92	−98.3675	9714.4644	−1.0020	0.3894	0.1945	−38.2999	−0.0102	−98.3665	98.5630
3.94	−131.7337	17392.3511	−1.0011	0.2930	0.1464	−38.5934	−0.0076	−131.7331	131.8806
3.96	−198.4334	39414.6837	−1.0005	0.1960	0.0980	−38.8877	−0.0050	−198.4331	198.5316
3.98	−398.4666	158814.7958	−1.0001	0.0983	0.0492	−39.1827	−0.0025	−398.4665	398.5158
4.00	0.0000	0.0000	−1.0000	0.0000	0.0000	−39.4784	0.0000	0.0000	0.0000
4.50	16.5908	322.3967	−1.0823	−2.8415	−1.3646	−47.1425	0.0579	16.6303	−17.9159
5.00	7.4983	111.7156	−1.4096	−7.4004	−3.0712	−55.4905	0.1107	7.6683	−10.3996
5.50	3.4609	76.8742	−2.5334	−18.7516	−5.3069	−64.8967	0.1636	3.8948	−8.3338
6.00	0.2974	76.1659	−29.3489	−255.8395	−8.4299	−76.0775	0.2216	1.2315	−7.7932
6.50	−3.1859	100.7494	3.1505	28.4371	−13.2233	−90.5991	0.2919	−1.2559	−8.1074
7.00	−8.3115	181.7543	1.6220	13.5564	−21.7931	−112.6735	0.3868	−4.0963	−9.2664
7.50	−19.3323	532.5795	1.1937	8.2164	−42.4100	−158.8420	0.5340	−8.0091	−11.7544
8.00	−85.6372	7760.1380	1.0287	4.9793	−173.7288	−426.4145	0.8148	−14.8570	−17.3114
8.50	54.6147	2850.6399	0.9776	2.4191	108.0060	132.1204	1.6350	−33.6783	−34.9016
9.00	19.2650	371.1392	1.0000	0.0000	48.0000	7.1736	−6.5491	145.4326	145.4326

A.2 Stability Functions for Tension Members

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
0.00	4.000	4.0000	0.5000	3.0000	6.0000	12.0000	1.0000	1.0000	−1.0000
0.04	4.0524	3.9482	0.4903	3.0781	6.0394	12.4735	0.9684	1.1283	−0.9371
0.08	4.1042	3.8979	0.4810	3.1545	6.0785	12.9466	0.9390	1.2503	−0.8796
0.12	4.1555	3.8491	0.4721	3.2293	6.1174	13.4192	0.9117	1.3668	−0.8268
0.16	4.2063	3.8018	0.4635	3.3025	6.1562	13.8915	0.8863	1.4782	−0.7784
0.20	4.2567	3.7559	0.4553	3.3743	6.1947	14.3633	0.8626	1.5850	−0.7337
0.24	4.3065	3.7113	0.4473	3.4447	6.2330	14.8346	0.8403	1.6876	−0.6924
0.28	4.3559	3.6680	0.4397	3.5138	6.2711	15.3056	0.8194	1.7865	−0.6542
0.32	4.4048	3.6259	0.4323	3.5816	6.3089	15.7762	0.7998	1.8818	−0.6188
0.36	4.4532	3.5850	0.4252	3.6482	6.3466	16.2463	0.7813	1.9739	−0.5859
0.40	4.5013	3.5452	0.4183	3.7137	6.3841	16.7161	0.7638	2.0631	−0.5553
0.44	4.5488	3.5065	0.4117	3.7780	6.4214	17.1854	0.7473	2.1495	−0.5268
0.48	4.5960	3.4689	0.4052	3.8412	6.4585	17.6544	0.7317	2.2333	−0.5002
0.52	4.6428	3.4323	0.3990	3.9035	6.4954	18.1230	0.7168	2.3148	−0.4754
0.56	4.6891	3.3967	0.3930	3.9648	6.5321	18.5912	0.7027	2.3940	−0.4521
0.60	4.7351	3.3620	0.3872	4.0251	6.5687	19.0591	0.6893	2.4712	−0.4303
0.64	4.7807	3.3282	0.3816	4.0845	6.6050	19.5266	0.6765	2.5465	−0.4099
0.68	4.8259	3.2953	0.3762	4.1430	6.6412	19.9937	0.6643	2.6199	−0.3907
0.72	4.8707	3.2633	0.3709	4.2007	6.6772	20.4604	0.6527	2.6916	−0.3726
0.76	4.9152	3.2321	0.3658	4.2576	6.7130	20.9268	0.6416	2.7618	−0.3556
0.80	4.9593	3.2017	0.3608	4.3137	6.7486	21.3929	0.6309	2.8304	−0.3396
0.84	5.0031	3.1720	0.3560	4.3690	6.7841	21.8586	0.6207	2.8975	−0.3245
0.88	5.0465	3.1431	0.3513	4.4237	6.8194	22.3240	0.6109	2.9634	−0.3103
0.92	5.0896	3.1148	0.3468	4.4776	6.8545	22.7890	0.6016	3.0279	−0.2968
0.96	5.1323	3.0873	0.3424	4.5308	6.8894	23.2537	0.5925	3.0912	−0.2841
1.00	5.1748	3.0605	0.3381	4.5834	6.9242	23.7180	0.5839	3.1533	−0.2720
1.04	5.2169	3.0342	0.3339	4.6353	6.9588	24.1820	0.5755	3.2144	−0.2606
1.08	5.2587	3.0086	0.3298	4.6866	6.9933	24.6457	0.5675	3.2744	−0.2498
1.12	5.3003	2.9836	0.3259	4.7373	7.0276	25.1091	0.5598	3.3334	−0.2396
1.16	5.3415	2.9592	0.3221	4.7875	7.0617	25.5722	0.5523	3.3914	−0.2298
1.20	5.3824	2.9354	0.3183	4.8370	7.0957	26.0349	0.5451	3.4485	−0.2206
1.24	5.4231	2.9121	0.3147	4.8861	7.1295	26.4974	0.5381	3.5047	−0.2118
1.28	5.4634	2.8893	0.3111	4.9346	7.1632	26.9595	0.5314	3.5601	−0.2035
1.32	5.5035	2.8671	0.3077	4.9825	7.1967	27.4213	0.5249	3.6147	−0.1955
1.36	5.5433	2.8453	0.3043	5.0300	7.2301	27.8829	0.5186	3.6685	−0.1880
1.40	5.5828	2.8240	0.3010	5.0770	7.2633	28.3441	0.5125	3.7216	−0.1808
1.44	5.6221	2.8032	0.2978	5.1235	7.2964	28.8051	0.5066	3.7739	−0.1739
1.48	5.6611	2.7829	0.2947	5.1696	7.3294	29.2657	0.5009	3.8256	−0.1674
1.52	5.6999	2.7630	0.2916	5.2152	7.3621	29.7261	0.4953	3.8766	−0.1611
1.56	5.7384	2.7435	0.2886	5.2603	7.3948	30.1862	0.4899	3.9269	−0.1552
1.60	5.7767	2.7244	0.2857	5.3051	7.4273	30.6460	0.4847	3.9766	−0.1495
1.64	5.8147	2.7058	0.2829	5.3494	7.4597	31.1055	0.4796	4.0258	−0.1440
1.68	5.8525	2.6875	0.2801	5.3933	7.4919	31.5647	0.4747	4.0743	−0.1388
1.72	5.8901	2.6696	0.2774	5.4369	7.5240	32.0237	0.4699	4.1223	−0.1339
1.76	5.9274	2.6521	0.2747	5.4800	7.5560	32.4824	0.4652	4.1698	−0.1291
1.80	5.9645	2.6349	0.2721	5.5228	7.5878	32.9409	0.4607	4.2167	−0.1246
1.84	6.0014	2.6181	0.2696	5.5652	7.6195	33.3991	0.4563	4.2632	−0.1202
1.88	6.0381	2.6016	0.2671	5.6072	7.6511	33.8570	0.4520	4.3091	−0.1160
1.92	6.0745	2.5855	0.2647	5.6489	7.6825	34.3146	0.4478	4.3546	−0.1120
1.96	6.1108	2.5697	0.2623	5.6903	7.7138	34.7720	0.4437	4.3996	−0.1082
2.00	6.1468	2.5542	0.2600	5.7313	7.7450	35.2292	0.4397	4.4441	−0.1045
2.04	6.1826	2.5390	0.2577	5.7720	7.7761	35.6861	0.4358	4.4882	−0.1010
2.08	6.2183	2.5240	0.2555	5.8124	7.8070	36.1428	0.4320	4.5319	−0.0976
2.12	6.2537	2.5094	0.2533	5.8524	7.8378	36.5992	0.4283	4.5752	−0.0944
2.16	6.2889	2.4951	0.2512	5.8922	7.8685	37.0553	0.4247	4.6181	−0.0913
2.20	6.3239	2.4810	0.2491	5.9316	7.8991	37.5113	0.4212	4.6606	−0.0883

ρ	Non-sway Frames						Sway Frames		
	r	$(rc)^2$	c	r'	q	s	m	t	t'
2.24	6.3588	2.4672	0.2470	5.9708	7.9295	37.9669	0.4177	4.7027	−0.0854
2.28	6.3934	2.4536	0.2450	6.0097	7.9598	38.4224	0.4143	4.7444	−0.0826
2.32	6.4279	2.4403	0.2430	6.0483	7.9901	38.8776	0.4110	4.7858	−0.0799
2.36	6.4622	2.4273	0.2411	6.0866	8.0202	39.3326	0.4078	4.8268	−0.0774
2.40	6.4963	2.4145	0.2392	6.1246	8.0501	39.7873	0.4047	4.8675	−0.0749
2.44	6.5302	2.4019	0.2373	6.1624	8.0800	40.2419	0.4016	4.9079	−0.0726
2.48	6.5640	2.3895	0.2355	6.1999	8.1098	40.6962	0.3986	4.9479	−0.0703
2.52	6.5975	2.3774	0.2337	6.2372	8.1394	41.1503	0.3956	4.9876	−0.0681
2.56	6.6310	2.3655	0.2319	6.2742	8.1690	41.6041	0.3927	5.0270	−0.0660
2.60	6.6642	2.3538	0.2302	6.3110	8.1984	42.0578	0.3899	5.0661	−0.0639
2.64	6.6973	2.3423	0.2285	6.3475	8.2277	42.5112	0.3871	5.1049	−0.0620
2.68	6.7302	2.3310	0.2269	6.3838	8.2569	42.9644	0.3844	5.1434	−0.0601
2.72	6.7629	2.3199	0.2252	6.4199	8.2860	43.4174	0.3817	5.1816	−0.0582
2.76	6.7955	2.3089	0.2236	6.4557	8.3150	43.8702	0.3791	5.2195	−0.0565
2.80	6.8280	2.2982	0.2220	6.4914	8.3439	44.3228	0.3765	5.2572	−0.0548
2.84	6.8602	2.2877	0.2205	6.5268	8.3727	44.7752	0.3740	5.2946	−0.0532
2.88	6.8924	2.2773	0.2189	6.5620	8.4014	45.2273	0.3715	5.3317	−0.0516
2.92	6.9243	2.2671	0.2174	6.5969	8.4300	45.6793	0.3691	5.3686	−0.0501
2.96	6.9562	2.2571	0.2160	6.6317	8.4585	46.1311	0.3667	5.4052	−0.0486
3.00	6.9878	2.2472	0.2145	6.6662	8.4869	46.5826	0.3644	5.4416	−0.0472
3.04	7.0194	2.2375	0.2131	6.7006	8.5152	47.0340	0.3621	5.4777	−0.0458
3.08	7.0508	2.2280	0.2117	6.7348	8.5434	47.4852	0.3598	5.5137	−0.0445
3.12	7.0820	2.2186	0.2103	6.7687	8.5715	47.9362	0.3576	5.5493	−0.0432
3.16	7.1131	2.2094	0.2090	6.8025	8.5995	48.3869	0.3554	5.5848	−0.0419
3.20	7.1441	2.2003	0.2076	6.8361	8.6274	48.8375	0.3533	5.6200	−0.0407
3.24	7.1749	2.1913	0.2063	6.8695	8.6552	49.2880	0.3512	5.6550	−0.0396
3.28	7.2056	2.1825	0.2050	6.9027	8.6829	49.7382	0.3491	5.6898	−0.0385
3.32	7.2362	2.1738	0.2038	6.9357	8.7106	50.1882	0.3471	5.7244	−0.0374
3.36	7.2666	2.1653	0.2025	6.9686	8.7381	50.6381	0.3451	5.7587	−0.0363
3.40	7.2969	2.1569	0.2013	7.0013	8.7655	51.0877	0.3432	5.7929	−0.0353
3.44	7.3271	2.1486	0.2001	7.0338	8.7929	51.5372	0.3412	5.8269	−0.0343
3.48	7.3571	2.1405	0.1989	7.0662	8.8202	51.9865	0.3393	5.8607	−0.0334
3.52	7.3870	2.1325	0.1977	7.0983	8.8473	52.4357	0.3375	5.8942	−0.0325
3.56	7.4168	2.1246	0.1965	7.1304	8.8744	52.8846	0.3356	5.9276	−0.0316
3.60	7.4465	2.1168	0.1954	7.1622	8.9014	53.3334	0.3338	5.9608	−0.0307
3.64	7.4760	2.1091	0.1943	7.1939	8.9283	53.7820	0.3320	5.9939	−0.0299
3.68	7.5055	2.1016	0.1932	7.2255	8.9552	54.2304	0.3303	6.0267	−0.0291
3.72	7.5348	2.0941	0.1921	7.2568	8.9819	54.6787	0.3285	6.0594	−0.0283
3.76	7.5640	2.0868	0.1910	7.2881	9.0085	55.1268	0.3268	6.0918	−0.0276
3.80	7.5930	2.0796	0.1899	7.3192	9.0351	55.5747	0.3252	6.1242	−0.0268
3.84	7.6220	2.0725	0.1889	7.3501	9.0616	56.0225	0.3235	6.1563	−0.0261
3.88	7.6509	2.0654	0.1878	7.3809	9.0880	56.4701	0.3219	6.1883	−0.0254
3.92	7.6796	2.0585	0.1868	7.4115	9.1143	56.9175	0.3203	6.2201	−0.0247
3.96	7.7082	2.0517	0.1858	7.4420	9.1406	57.3648	0.3187	6.2517	−0.0241
4.00	7.7367	2.0450	0.1848	7.4724	9.1668	57.8119	0.3171	6.2832	−0.0235
5.00	8.4169	1.9032	0.1639	8.1908	9.7965	68.9410	0.2842	7.0248	−0.0125
6.00	9.0436	1.7990	0.1483	8.8447	10.3849	79.9873	0.2597	7.6953	−0.0070
7.00	9.6272	1.7195	0.1362	9.4486	10.9385	90.9643	0.2405	8.3119	−0.0041
8.00	10.1754	1.6568	0.1265	10.0126	11.4626	101.8820	0.2250	8.8858	−0.0025
9.00	10.6937	1.6063	0.1185	10.5435	11.9611	112.7486	0.2122	9.4248	−0.0015

A.3 Stability Magnification Factors for Members with Lateral Load

ρ	Nature of Axial Force			
	Compression		Tension	
	m_{fw}	m_{fc}	m_{fw}	m_{fc}
0.00	0.0833	0.0625	0.0789	0.0625
0.04	0.0839	0.0630	0.0828	0.0620
0.08	0.0845	0.0635	0.0823	0.0615
0.12	0.0850	0.0641	0.0817	0.0610
0.16	0.0856	0.0646	0.0812	0.0605
0.20	0.0862	0.0652	0.0807	0.0601
0.24	0.0868	0.0658	0.0802	0.0596
0.28	0.0874	0.0664	0.0797	0.0591
0.32	0.0881	0.0670	0.0793	0.0587
0.36	0.0887	0.0676	0.0788	0.0583
0.40	0.0894	0.0682	0.0783	0.0578
0.44	0.0901	0.0688	0.0779	0.0574
0.48	0.0908	0.0695	0.0774	0.0570
0.52	0.0915	0.0702	0.0770	0.0566
0.56	0.0922	0.0709	0.0765	0.0562
0.60	0.0929	0.0716	0.0761	0.0558
0.64	0.0937	0.0723	0.0757	0.0554
0.68	0.0944	0.0730	0.0753	0.0550
0.72	0.0952	0.0738	0.0749	0.0546
0.76	0.0960	0.0745	0.0745	0.0543
0.80	0.0969	0.0753	0.0741	0.0539
0.84	0.0977	0.0761	0.0737	0.0536
0.88	0.0986	0.0770	0.0733	0.0532
0.92	0.0995	0.0778	0.0729	0.0529
0.96	0.1004	0.0787	0.0726	0.0525
1.00	0.1013	0.0796	0.0722	0.0522
1.04	0.1023	0.0805	0.0719	0.0519
1.08	0.1033	0.0814	0.0715	0.0515
1.12	0.1043	0.0824	0.0711	0.0512
1.16	0.1053	0.0834	0.0708	0.0509
1.20	0.1064	0.0844	0.0705	0.0506
1.24	0.1075	0.0855	0.0701	0.0503
1.28	0.1086	0.0866	0.0698	0.0500
1.32	0.1098	0.0877	0.0695	0.0497
1.36	0.1110	0.0889	0.0692	0.0494
1.40	0.1122	0.0900	0.0688	0.0491

ρ	Nature of Axial Force			
	Compression		Tension	
	m_{fw}	m_{fc}	m_{fw}	m_{fc}
1.44	0.1135	0.0913	0.0685	0.0488
1.48	0.1148	0.0925	0.0682	0.0486
1.52	0.1161	0.0938	0.0679	0.0483
1.56	0.1175	0.0952	0.0676	0.0480
1.60	0.1189	0.0966	0.0673	0.0477
1.64	0.1204	0.0980	0.0670	0.0475
1.68	0.1219	0.0995	0.0667	0.0472
1.72	0.1235	0.1011	0.0665	0.0470
1.76	0.1252	0.1026	0.0662	0.0467
1.80	0.1269	0.1043	0.0659	0.0465
1.84	0.1286	0.1060	0.0656	0.0462
1.88	0.1304	0.1078	0.0654	0.0460
1.92	0.1323	0.1096	0.0651	0.0457
1.96	0.1343	0.1116	0.0648	0.0455
2.00	0.1363	0.1136	0.0646	0.0453
2.04	0.1384	0.1156	0.0643	0.0450
2.08	0.1407	0.1178	0.0640	0.0448
2.12	0.1430	0.1200	0.0638	0.0446
2.16	0.1454	0.1224	0.0635	0.0444
2.20	0.1479	0.1249	0.0633	0.0441
2.24	0.1505	0.1274	0.0631	0.0439
2.28	0.1532	0.1301	0.0628	0.0437
2.32	0.1561	0.1329	0.0626	0.0435
2.36	0.1591	0.1359	0.0623	0.0433
2.40	0.1622	0.1390	0.0621	0.0431
2.44	0.1655	0.1422	0.0619	0.0429
2.48	0.1690	0.1456	0.0617	0.0427
2.52	0.1726	0.1493	0.0614	0.0425
2.56	0.1765	0.1531	0.0612	0.0423
2.60	0.1806	0.1571	0.0610	0.0421
2.64	0.1849	0.1614	0.0608	0.0419
2.68	0.1895	0.1659	0.0606	0.0417
2.72	0.1943	0.1707	0.0603	0.0415
2.76	0.1995	0.1758	0.0601	0.0413
2.80	0.2050	0.1813	0.0599	0.0412
2.84	0.2109	0.1871	0.0597	0.0410
2.88	0.2172	0.1933	0.0595	0.0408
2.92	0.2239	0.2001	0.0593	0.0406
2.96	0.2312	0.2073	0.0591	0.0404
3.00	0.2390	0.2151	0.0589	0.0403

ρ	Nature of Axial Force			
	Compression		Tension	
	m_{fw}	m_{fc}	m_{fw}	m_{fc}
3.04	0.2475	0.2235	0.0587	0.0401
3.08	0.2568	0.2327	0.0585	0.0399
3.12	0.2669	0.2427	0.0583	0.0398
3.16	0.2779	0.2537	0.0581	0.0396
3.20	0.2900	0.2658	0.0580	0.0394
3.24	0.3034	0.2791	0.0578	0.0393
3.28	0.3183	0.2940	0.0576	0.0391
3.32	0.3349	0.3105	0.0574	0.0390
3.36	0.3536	0.3291	0.0572	0.0388
3.40	0.3747	0.3503	0.0570	0.0386
3.44	0.3989	0.3744	0.0569	0.0385
3.48	0.4268	0.4022	0.0567	0.0383
3.52	0.4594	0.4347	0.0565	0.0382
3.56	0.4978	0.4731	0.0563	0.0380
3.60	0.5439	0.5192	0.0562	0.0379
3.64	0.6003	0.5755	0.0560	0.0377
3.68	0.6707	0.6458	0.0558	0.0376
3.72	0.7612	0.7363	0.0557	0.0375
3.76	0.8819	0.8570	0.0555	0.0373
3.80	1.0509	1.0258	0.0553	0.0372
3.84	1.3042	1.2791	0.0552	0.0370
3.88	1.7265	1.7013	0.0550	0.0369
3.92	2.5709	2.5457	0.0549	0.0368
3.96	5.1040	5.0787	0.0547	0.0366
4.00	∞	∞	0.0545	0.0365

Appendix B

Effective Length of Stepped and Multiple Level Load Columns

A column with steps i.e. having portions of different rigidity or a uniform column with loads at different levels is commonly encountered in practice. The stability analysis of such a column can be performed by means of differential equations, one for each segment of uniform rigidity with appropriate boundary conditions and *continuity or matching conditions* at the joints between the segments. These matching or *compatibility conditions* are the equality of displacements, slopes and curvatures etc. The steel stepped columns in the industrial buildings are normally fixed in their foundations by means of anchor bolts and hence may be treated as clamped at the foundation level. In single-span frames such a column may be considered as a separate column having free horizontal displacement at the top, i. e., shear force $Q = 0$. On the other hand when the frames cover two or more bays, the upper end of a column when determining the effective length can be considered as restrained, i. e., shear force at the upper end of the column $Q \neq 0$.

For illustration of the procedure for computation of effective length, consider a single-stepped column clamped at the bottom and free at the top as shown in Fig. B.1a. The governing differential equations for the two segments are:

In the upper segment: $M = P_1(\delta_1 - y_1)$ and the corresponding differential equation is:

$$EI_1 y_1'' + P_1 y_1 = P_1 \delta_1 \tag{a}$$

In the lower segment: $M = P_1(\delta_1 - y_2) + P_2(\delta_2 - y_2)$ and corresponding governing differential equation is:

$$EI_2 y_2'' + (P_1 + P_2) y_2 = P_1 \delta_1 + P_2 \delta_2 \tag{b}$$

The solution to (a) and (b) can be written in the form

$$y_1 = A \sin \alpha_1 x + B \cos \alpha_1 x + \delta_1$$
$$y_2 = C \sin \alpha_2 x + D \cos \alpha_2 x + [(P_1 \delta_1 + P_1 \delta_1)/(P_1 + P_2)]$$

where

$$\alpha_1 = \sqrt{P_1/(EI_1)} \quad \text{and} \quad \alpha_2 = \sqrt{(P_1 + P_2)/(EI_2)} \tag{c}$$

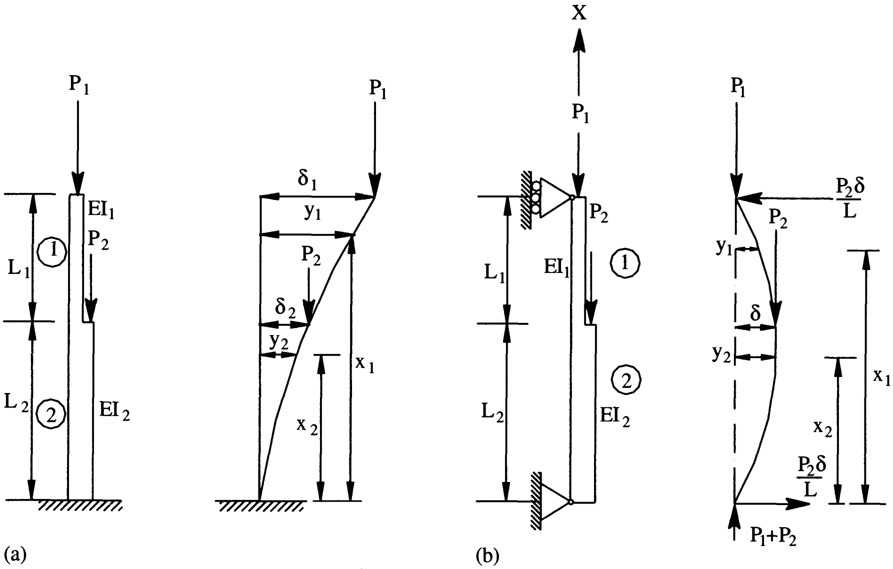


Fig. B.1a,b. Stepped columns with multiple level loads. **a** Fixed at the base and free at top, **b** simple supports

The values of the unknown coefficients A, B, C and D can be determined by the following boundary, and continuity conditions at the joint between the two segments

(i) Boundary conditions

At $x = 0$:

$$y_2' = 0 \quad \text{i.e.} \quad C\alpha_2 = 0$$

$x = (L_1 + L_2) = L$:

$$y_1 = \delta_1 \quad \text{i.e.} \quad A \sin \alpha_1 (L_1 + L_2) + B \cos \alpha_1 (L_1 + L_2) = 0$$

(ii) Matching or continuity conditions

At $x = L_2$:

$$y_1' = y_2' \quad \text{i.e.} \quad A\alpha_1 \cos \alpha_1 L_2 - B\alpha_1 \sin \alpha_1 L_2 = -D\alpha_2 \sin \alpha_2 L_2$$
$$y_1'' = y_2'' \quad \text{i.e.} \quad - A\alpha_1^2 \sin \alpha_1 L_2 - B\alpha_1^2 \cos \alpha_1 L_2 = -D\alpha_2^2 \cos \alpha_2 L_2$$

For non-trivial solution the determinant of coefficients of A, B and D must vanish i.e.

$$\begin{vmatrix} \sin \alpha_1 (L_1 + L_2) & \cos \alpha_1 (L_1 + L_2) & 0 \\ \cos \alpha_1 L_2 & -\sin \alpha_1 L_2 & (\alpha_2/\alpha_1) \sin \alpha_2 L_2 \\ -\sin \alpha_1 L_2 & -\cos \alpha_1 L_2 & (\alpha_2/\alpha_1)^2 \cos \alpha_2 L_2 \end{vmatrix} = 0$$

The expansion of determinant provides the stability or characteristic equation

$$\tan(\alpha_1 L_1)\tan(\alpha_2 L_2) = (\alpha_2/\alpha_1) \tag{d}$$

For illustration consider the typical case where

$$P_1 = P; \quad P_2 = 3P; \quad I_1 = I_2 = I \quad \text{and} \quad L_1 = L_2 = L/2.$$

This case corresponds to a uniform column, which is subjected to axial forces at different levels. Thus from (c) the stability parameters are

$$\alpha_1 = \sqrt{P/EI}; \quad \alpha_2 = \sqrt{(P+3P)/EI} = 2\sqrt{P/EI} = 2\alpha_1$$

The characteristic equation (d) reduces to:

$$\tan(\alpha_1 L/2)\tan(\alpha_1 L) = 2$$

By the method of trial and modification, $\alpha_1 L = 1.23096$, whence

$$P_{\text{cr}} = \frac{(1.23096)^2 EI}{L^2} = \frac{\pi^2 EI}{(2.552L)^2}$$

Thus, the effective length factor $K = 2.552$.

To study the effect of a step i.e. sudden change in the rigidity consider a typical single-stepped column with $P_1 = P$; $P_2 = 0$, $I_1 = I$, $I_2 = 2I$ and $L_1 = L_2 = L/2$. Therefore, from (c):

$$\alpha_1 = \sqrt{P/EI}, \quad \alpha_2 = \sqrt{P/2EI} = \alpha_1/\sqrt{2} \quad \text{and} \quad (\alpha_2/\alpha_1) = 1/\sqrt{2}$$

The characteristic equation (d) becomes

$$\tan\left(\frac{\alpha_1 L}{2}\right)\tan\left(\frac{\alpha_1 L}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

By trial and modification $\alpha_1 L = 1.6442$. Therefore,

$$P_{\text{cr}} = \frac{(1.6442)^2 EI}{L^2} = \frac{\pi^2 EI}{(1.9107L)^2}$$

The effective length factor K reduces to 1.9107.

In the above analysis of stepped column it is assumed that the column reaches its critical state upon a simultaneous and proportional increase of the loads in both the segments. The procedure is equally applicable to stepped-columns with other boundary conditions. For illustration consider the stepped column with simple supports as shown in Fig. B.1b. In this case horizontal reactions $Q = P_2\delta/L$ will be produced during buckling.

In the upper segment:

$$M = -P_1 y_1 - \frac{P_2\delta}{L}(L - x)$$

or

$$EI_1 y_1'' + P_1 y_1 = -\frac{P_2\delta}{L}(L - x) \tag{e}$$

In the lower segment:

$$M = -P_1 y_2 - \frac{P_2\delta}{L}(L - x) + P_2(\delta - y_2)$$

or

$$EI_2 y_2'' + (P_1 + P_2) y_2 = \frac{P_2\delta}{L}x \tag{f}$$

Defining parameters:

$$\alpha_1^2 = P_1/EI_1, \quad \alpha_2^2 = (P_1 + P_2)/EI_2, \quad \alpha_3^2 = P_2/EI_2 \quad \text{and} \quad \alpha_4^2 = P_2/EI_1 \tag{g}$$

The solution to the governing differential equations (e) and (f) can be written in the form

$$y_1 = A\sin\alpha_1 x + B\cos\alpha_1 x - \frac{\delta(L - x)}{L}\left(\frac{\alpha_4}{\alpha_1}\right)^2$$

and

$$y_2 = C\sin\alpha_2 x + D\cos\alpha_2 x + \frac{\delta x}{L}\left(\frac{\alpha_3}{\alpha_2}\right)^2$$

The arbitrary constants of integration A, B, C and D can be evaluated from boundary and continuity conditions.

$$\begin{aligned}
\text{At} \quad x = 0: &\quad y_2 = 0 \quad \text{giving} \quad D = 0 \\
x = L: &\quad y_1 = 0 \quad \text{i.e.} \quad A\sin\alpha_1 L + B\cos\alpha_1 L = 0 \\
x = L_2: &\quad y_1 = \delta \quad \text{i.e.} \quad A\sin\alpha_1 L_2 + B\cos\alpha_1 L_2 - \frac{\delta L_1}{L}\left(\frac{\alpha_4}{\alpha_1}\right)^2 = \delta \\
&\quad y_2 = \delta \quad \text{i.e.} \quad C\sin\alpha_2 L_2 + D\cos\alpha_2 L_2 + \frac{\delta L_2}{L}\left(\frac{\alpha_3}{\alpha_2}\right)^2 = \delta
\end{aligned}$$

From these conditions following values of A, B, C and D are obtained

$$A = \frac{\delta(\alpha_1^2 L + \alpha_4^2 L_1)}{\alpha_1^2 L(\sin\alpha_1 L_2 - \tan\alpha_1 L\cos\alpha_1 L_2)}, \quad B = -A\tan\alpha_1 L$$

$$C = \frac{\delta(\alpha_2^2 L - \alpha_3^2 L_2)}{\alpha_2^2 L\sin\alpha_2 L_2} \quad \text{and} \quad D = 0$$

Substituting these constants into continuity condition at $x = L_2$; $y_1' = y_2'$

$$\left(\frac{\alpha_4}{\alpha_1}\right)^2 - \frac{\alpha_1^2 L + \alpha_4^2 L_1}{\alpha_1\tan\alpha_1 L_1} = \left(\frac{\alpha_3}{\alpha_2}\right)^2 + \frac{\alpha_2^2 L - \alpha_3^2 L_2}{\alpha_2\tan\alpha_2 L_2} \tag{h}$$

This transcendental equation can be used to obtain the critical loads. As a typical case consider following illustration.

Example B.1. Consider a singly-stepped column having simple supports with:
(a) $P_1 = P$, $P_2 = 0$; $L_1 = L_2 = L/2$; $EI_1 = EI$ and $EI_2 = 4EI$; and
(b) $P_1 = P$, $P_2 = P$, $L_1 = L_2 = L/2$ and $EI_1 = EI_2 = EI$.

Case (a): Defining $P/(4EI) = \alpha^2$. Thus

$$\alpha_1^2 = \frac{P}{EI} = 4\alpha^2; \quad \alpha_2^2 = \frac{P}{4EI} = \alpha^2; \quad \alpha_3^2 = 0 \quad \text{and} \quad \alpha_4^2 = 0$$

On substituting these values the characteristic equation (h) reduces to:

$$-\alpha_1 \tan(\alpha_2 L_2) - \alpha_2 \tan(\alpha_1 L_1) = 0$$
$$-2\tan(\alpha L/2) - \tan(\alpha L) = 0$$

By trial and modification, the smallest root of this equation $\alpha L = 0.955317$. Therefore,

$$P_{\text{cr}} = \frac{(0.955317)^2(4EI)}{L^2} = \frac{\pi^2 EI}{(1.644L)^2}$$

The effective length factor, $K = 1.644$.

Case (b): Defining $P/EI = \alpha^2$. Thus

$$\alpha_1^2 = \frac{P}{EI} = \alpha^2; \quad \alpha_2^2 = \frac{2P}{EI} = 2\alpha^2; \quad \alpha_3^2 = \frac{P}{EI} = \alpha^2; \quad \alpha_4^2 = \frac{P}{EI} = \alpha^2;$$
$$(\alpha_4/\alpha_1)^2 = 1, \quad \text{and} \quad (\alpha_3/\alpha_2)^2 = \frac{1}{2}$$

On substitution the characteristic equation (h) reduces to:

$$3\alpha L\left[\frac{1}{\tan(\alpha L/2)} + \frac{1}{\sqrt{2}\tan(\alpha L/\sqrt{2})}\right] = 1$$

Using trial and modification procedure smallest $\alpha L = 2.55656$. Hence,

$$\alpha_2^2 = (P_1 + P_2)/EI = 2\alpha^2 = \frac{2 \times (2.55656)^2}{L^2}$$

Therefore,

$$(P_1 + P_2)_{\text{cr}} = \frac{13.0719EI}{L^2} = \frac{\pi^2 EI}{(0.8689L)^2}$$

The effective length coefficient, $K = 0.8689$.

In the case where $P_1 = 0$, the characteristic transcendental equation (h) is not applicable. In this case effective length factor can be determined by direct application of differential equations. For example consider the above case with $P_1 = 0$, i.e. a uniform hinged end column carries a longitudinal force at the mid-point. The differential equations for the two segments are:

$$y_1'' = -\frac{\delta P_2}{EI_1 L}(L - x) = -\delta\alpha_4^2\left(1 - \frac{x}{L}\right)$$

$$y_2'' + \alpha_2^2 y_2 = \frac{\delta P_2 x}{EI_2 L} = \delta\alpha_2^2\left(\frac{x}{L}\right)$$

The solutions to the equations are:

$$y_1 = -\delta\alpha_4^2\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + Ax + B$$

$$y_2 = C\sin\alpha_2 x + D\cos\alpha_2 x + \delta\left(\frac{x}{L}\right)$$

Using boundary and continuity conditions the arbitrary constants A, B, C and D can be determined and hence the critical load. The constants are

$$A = -\frac{\delta}{L_1} + \frac{\delta}{L_1}\alpha_4^2\left[\frac{L^2}{3} - \frac{L_2^2}{2} + \frac{L_2^3}{6L}\right]$$

$$B = \frac{\delta}{3}L^2\alpha_4^2 - AL$$

$$C = \delta L_1/(L\sin\alpha_2 L_2) \quad \text{and} \quad D = 0$$

The continuity condition $y_1' = y_2'$ at $x = L_2$ gives the characteristic equation or stability condition as:

$$\cot(\alpha L/2) = -\frac{36 - (\alpha L)^2}{6(\alpha L)} = \frac{(\alpha L/2)^2 - 9}{3(\alpha L/2)}$$

Using trial and modification method, the smallest root of this equation is $\alpha L/2 = 2.160201$. Therefore,

$$P_{cr} = \frac{(2.160201)^2 \times 4EI}{L^2} = \frac{18.6659EI}{L^2}$$

$$P_{cr} = \frac{\pi^2 EI}{(0.7272L)^2}$$

The effective length factor K is 0.7272. It will be found that energy approach for the stability analysis of stepped columns is equally effective.

The foregoing stability analysis for single-stepped columns subjected to two forces P_1 and P_2 provides stability condition or characteristic equation in terms of transcendental coefficients and two parameters. This analysis is not convenient for design purposes. The following simple procedure wherein the values of length coefficients are determined separately for the upper part of the column K_1 and lower part K_2, is extremely useful for design purposes. The method consists in performing

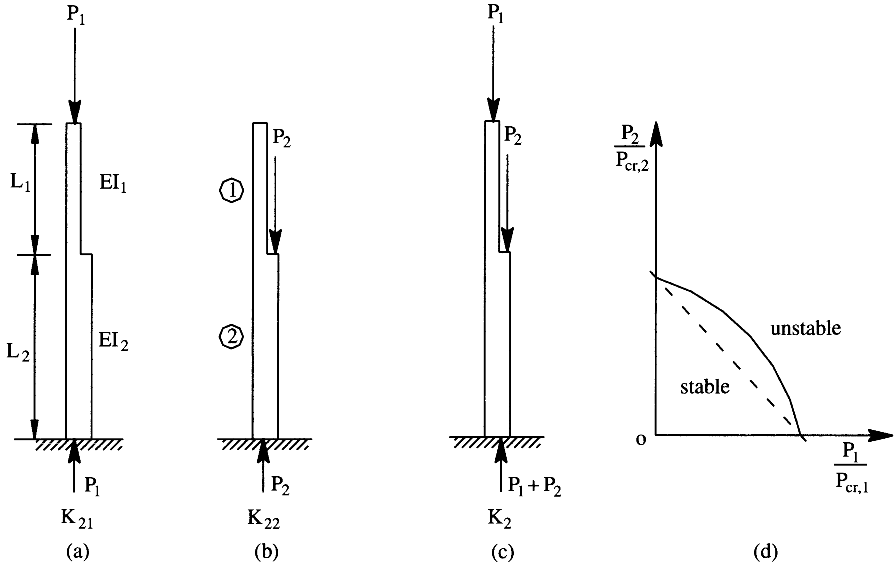


Fig. B.2a–d. Determination of coefficient K for single stepped column

the analysis twice, first under the action of force P_1 alone and determining critical force $P_{\mathrm{cr},1}$ and coefficient K_{21} for the lower part, and then under the action of force P_2 alone and determining $P_{\mathrm{cr},2}$ and coefficient K_{22} again for the lower part.

When two forces are applied simultaneously, then the sum of the two ratios between applied forces and critical ones will characterize an area divided into stable and unstable parts as shown in Fig. B.2d. Each of the ratios $P_1/P_{\mathrm{cr},1}$ and $P_2/P_{\mathrm{cr},2}$ is less than or equals unity. If the end points of the convex curve of the stability condition are connected by a straight line, it will provide a margin of stability (safety). This limiting straight line can be written in the form:

$$\frac{P_1}{P_{\mathrm{cr},1}} + \frac{P_2}{P_{\mathrm{cr},2}} = 1 \tag{i}$$

Assuming $P_{\mathrm{cr},1} = \pi^2 EI_2/(K_{21}L_2)^2$, $P_{\mathrm{cr},2} = \pi^2 EI_2/(K_{22}L_2)^2$ and $(P_1 + P_2)/P_1 = m$ or $P_1 = [P_2/(m-1)]$. Substituting in (i)

$$P_2\left[\frac{K_{21}^2}{(m-1)\pi^2 EI_2/L_2^2} + \frac{K_{22}^2}{\pi^2 EI_2/L_2^2}\right] = 1 \tag{j}$$

Noting that $P_1 + P_2 = mP_2/(m-1)$ or $P_2 = (P_1 + P_2)(m-1)/m$. Substituting this expression for P_2 in (j) and assuming that when both forces act simultaneously:

$$(P_1 + P_2)_{\mathrm{cr}} = \frac{\pi^2 EI_2}{(K_2 L_2)^2}$$

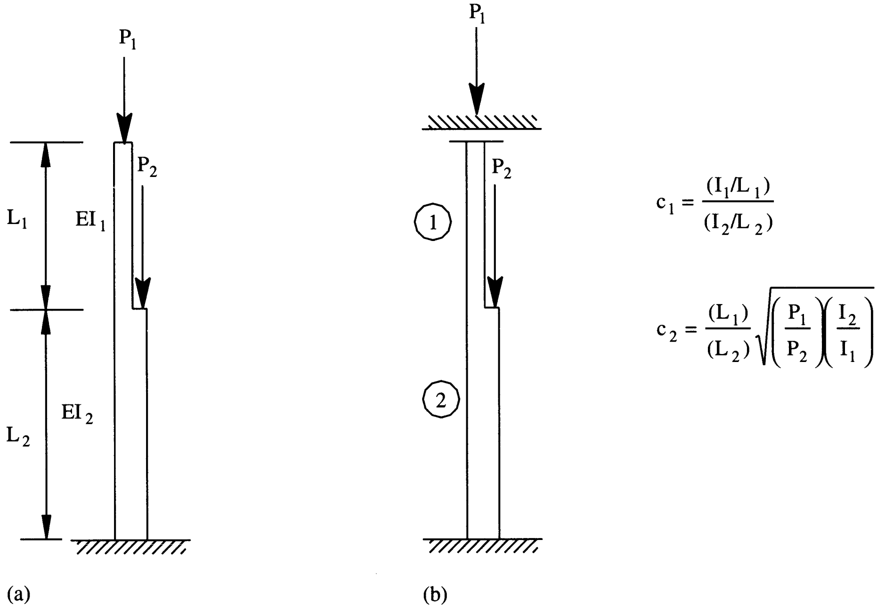


Fig. B.3a,b. Parameters for length coefficient for Tables B.1 and B.2. **a** Case I, **b** case II

Therefore,

$$K_2 = \sqrt{\frac{K_{22}^2(m-1) + K_{21}^2}{m}} \tag{k}$$

For the upper part of the columns $K_1 = K_2/c_1 \leq 3$.

Thus the coefficient K_2 is determined as a function of K_{21}, the length coefficient of lower part of column with $P_2 = 0$, and of K_{22}, the length coefficient of lower part with $P_1 = 0$. The values of K_{21} and K_{22} as function of the ratio $L_1/L_2 = n$ and $I_1/I_2 = \beta$ are given in the Tables B.1 and B.2 for the different values of parameters c_1 and c_2 which are defined as:

$$c_1 = \frac{I_1/L_1}{I_2/L_2} \quad \text{and} \quad c_2 = \frac{K_2}{K_1} = \sqrt{\frac{\left(P_1/P'_{\mathrm{cr},1}\right)}{\left(P_2/P'_{\mathrm{cr},2}\right)}} = \left(\frac{L_1}{L_2}\right)\sqrt{\left(\frac{P_1}{P_2}\right)\left(\frac{I_2}{I_1}\right)} \tag{l}$$

where L_1, I_1, P_1 = height, moment of inertia and longitudinal force for the upper part of the column. L_2, I_2, P_2 = above quantities for the lower part of the column.

Table B.1 is for hinged and B.2 is for fixed connection of the collar beam to the column. The procedure outlined above predicts conservative values within two to eight per cent of the exact values.

In the single-storey frames having single-stepped columns with the ratios $L_1/L_2 \leq 0.6$ and $P_2/P_1 \geq 3$, the values of the length coefficient K can be taken from Table B.3, which differ slightly from mean values. The use of effective length tables is illustrated in the following example.

Table B.1. Length coefficients K_2 for columns with top end free

	c_1										
c_2	0	0.2	0.4	0.6	0.8	1.0	1.4	1.8	2.5	10.0	20.0
0	2.0	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
0.2	2.0	2.02	2.04	2.06	2.07	2.09	2.12	2.15	2.21	2.76	3.38
0.4	2.0	2.08	2.13	2.21	2.28	2.35	2.48	2.60	2.80		
0.6	2.0	2.20	2.36	2.52	2.66	2.80	3.05	3.28			
0.8	2.0	2.42	2.70	2.96	3.17	3.36	3.74				
1.0	2.0	2.73	3.13	3.44	3.74	4.00					
1.5	3.0	3.77	4.35	4.86							
2.0	4.0	4.90	5.67								
2.5	5.0	6.08	7.00								
3.0	6.0	7.25									

Table B.2. Length coefficients K_2 for columns with top end fixed against rotation

	c_1										
c_2	0	0.2	0.4	0.6	0.8	1.0	1.4	1.8	2.5	10.0	20.0
0	2.00	1.86	1.76	1.67	1.60	1.55	1.46	1.40	1.32	1.10	1.05
0.2	2.00	1.87	1.76	1.68	1.62	1.56	1.48	1.41	1.33	1.11	
0.4	2.00	1.88	1.77	1.72	1.66	1.61	1.53	1.48	1.40		
0.8	2.00	1.94	1.90	1.87	1.85	1.82	1.79				
1.0	2.00	2.00	2.00	2.00	2.00	2.00					
1.5	2.00	2.25	2.38	2.48							
2.0	2.00	2.66	2.91								
2.5	2.50	3.17	3.50								
3.0	3.00	3.70	4.12								

Table B.3. Values of length coefficients K_2 for single-stepped columns of single storey industrial buildings.

Shear force at the top end	Constraint of upper end	For lower part, K_2 with		For upper part, K_1
		$0.3 \geq \frac{I_1}{I_2} \geq 0.1$	$0.1 \geq \frac{I_1}{I_4} \geq 0.05$	
Q = 0	Free end	2.5	3.0	3.0
	End fixed only against rotation	2.0	2.0	3.0
Q $\neq$ 0	Immovable pin-supported end	1.6	2.0	2.5
	Immovable end fixed against rotation	1.2	1.5	2.0

Example B.2. A stepped column of 21.0 m total height for a single-bay shop has crane runway part of height $L_2 = 15.5$ m and roof supporting part above the crane runway of height $L_1 = 5.5$ m. The effective axial loads in roof supporting part P_1 and in the crane runway (lower) part P_2 are 950 kN and 3050 kN, respectively. The column is clamped at foundation level and is pin connected to the collar at the top.

For the investigation of the possibility of using Table B.3 for finding the effective length coefficients, the parameters required are

$$\frac{L_1}{L_2} = \frac{5.5}{15.5} = 0.355 < 0.60 \quad \text{and}$$
$$\frac{P_2}{P_1} = \frac{3050}{950} = 3.211 > 3.00$$

Since these parameters satisfy the stipulations, Table B.3 can be used and the effective length coefficients are $K_1 = 3.0$ and $K_2 = 2.5$. The effective lengths for computing slenderness ratios of the upper and lower parts of the column are:

$$L_{e,1} = K_1 L_1 = 3.0 \times 5.5 = 16.5\,\text{m}$$
$$L_{e,2} = K_2 L_2 = 2.5 \times 15.5 = 38.75\,\text{m}.$$

A similar approximate solution can be obtained for double-stepped columns as shown in Fig. B.4. The tables drawn up for solving single-stepped columns are used with moments of inertia averaged over the length of the parts of the columns.

$$\frac{P_2}{P_1} = s_2; \quad \frac{P_3}{P_1} = s_3; \quad \frac{I_1}{I_3} = \beta_1; \quad \frac{I_2}{I_3} = \beta_2; \quad \frac{L_1}{L_3} = n_1; \quad \text{and} \quad \frac{L_2}{L_3} = n_2$$

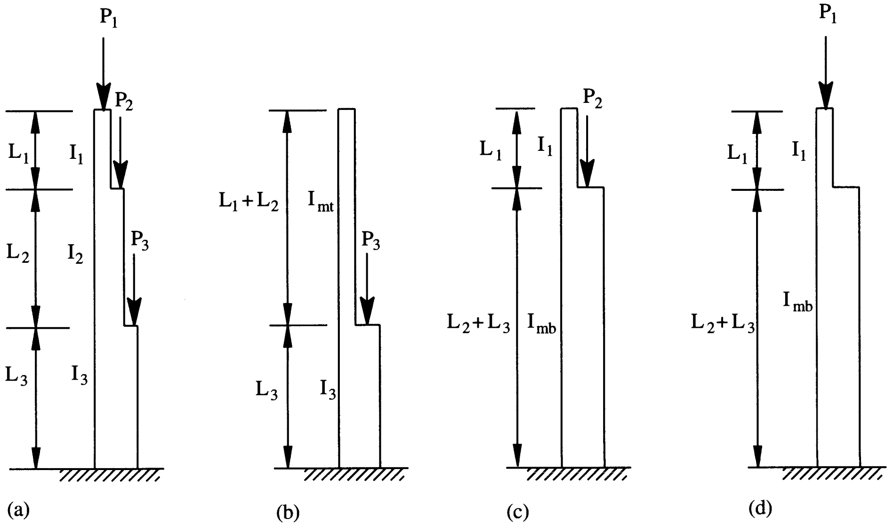


Fig. B.4a–d. Stages for determination of coefficients K for double stepped columns

Table B.4. Coefficients for the columns fixed at the bottom $\overline{K}_1$, $\overline{K}_2$ and $\overline{K}_3$.

Shear force at upper end	Upper end conditions	$\overline{K}_1$	$\overline{K}_2$	$\overline{K}_3$
$Q = 0$	Free end	$\overline{K}_1 = \overline{K}_2$, from Table B.1 with $c_2 = \frac{L_1}{L_2+L_3}\sqrt{\frac{I_{mb}}{I_1}}$	$\overline{K}_2 = 2$	$\overline{K}_3 = 2$
	End fixed only against rotation	$\overline{K}_1 = \overline{K}_3$, from Table B.2 with $c_2 = \frac{L_1}{L_2+L_3}\sqrt{\frac{I_{mb}}{I_1}}$	$\overline{K}_2 = K_3$, from Table B.2 with $c_2 = 0$	$\overline{K}_3 = K_3$, from Table B.2 with $c_2 = 0$

then the coefficient K_3 for the lower part will be

$$K_3 = \sqrt{\frac{s_3\overline{K}_3^2 + \left(s_2\overline{K}_2^2 + \overline{K}_1^2\right)(1+n_2)^2\frac{I_3}{I_{mb}}}{1+s_2+s_3}}$$

where $I_{mb} = (I_2L_2 + I_3L_3)/(L_2 + L_3)$ is the mean value of the moment of inertia for the part of the column $L_2 + L_3$. The coefficients $\overline{K}_1$, $\overline{K}_2$ and $\overline{K}_3$ are determined in the same way as for single-stepped columns according to the Fig. B.4b, c and d, using Table B.4.

For the part $L_1 + L_2$ the value of I_{mt} is found from the equation.

$$I_{mt} = \frac{I_1L_1 + I_2L_2}{L_1 + L_2}$$

The length coefficient for the middle part of the column length is determined from the equation

$$K_2 = \frac{K_3}{c_2} \quad \text{where} \quad c_2 = n_2\sqrt{\frac{P_1 + P_2}{(P_1 + P_2 + P_3)\,\beta_2}}$$

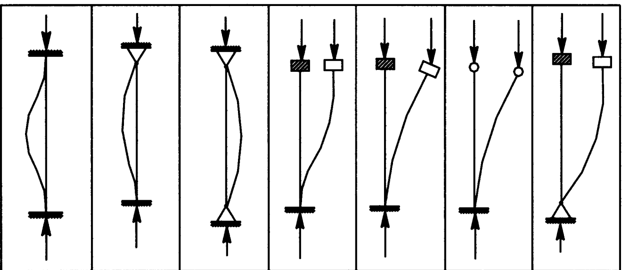
The length coefficient for the upper part of the column is determined from the expression

$$K_1 = \frac{K_3}{c_1} \leq 3$$

$$c_1 = n_1\sqrt{\frac{P_1}{(P_1 + P_2 + P_3)\,\beta_1}}$$

The effective length factors, K for uniform columns with different end conditions subjected to end compressive load are given in the Table B.5.

Table B.5. Effective length factors K for axially loaded columns with various idealized end conditions

Buckled shape of column							
Recommended value of k for idealised end conditions	0.65	0.80	1.0	1.2	1.5	2.10	2.0
End condition code		Rotation fixed			Translation fixed		
		Rotation free			Translation fixed		
		Rotation fixed			Translation free		
		Rotation partially fixed			Translation free		
		Rotation free			Translation free		

Appendix C

Mathematical Essentials

C.1 Linear Differential Equations

The general linear differential equation of n^{th} order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + \alpha_1 \frac{d^{n-1} y}{dx^{n-1}} + \alpha_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + \alpha_n y = X \tag{C.1}$$

where $\alpha_1, \alpha_2 \ldots \alpha_n$ are constants. In the symbolic form the equation can be written as:

$$(D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \ldots + \alpha_n)y = X \tag{C.2}$$

where the symbol D stands for the operation of differentiation and is treated much the same as an algebraic quantity i.e., it can be factorized by ordinary rules of algebra and the factors may be taken in any order. The solution of equation consists of two parts viz complementary function (C.F.) y_c and particular integral (P.I.) y_p. The complete solution is $y = y_c + y_p$.

1. Complementary Function

An auxiliary equation is obtained by equating the coefficient of differential equation in symbolic form to zero i.e.

$$D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \ldots + \alpha_n = 0 \tag{C.3}$$

Let $m_1, m_2, \ldots m_n$ be its roots. The complementary solution is given by:

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$$

Following cases may arise:

1. If some roots are equal. For example $m_3 = m_2 = m_1$, then C.F. is

$$y_c = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \ldots c_n e^{m_n x}$$

2. If one pair of roots is imaginary e.g. $m_1 = \beta + \mathrm{i}\gamma$, $m_2 = \beta - \mathrm{i}\gamma$

$$y_c = (c_1 \cos gx + c_2 \sin gx)\mathrm{e}^{bx} + c_3\mathrm{e}^{m_3 x} + \ldots + c_n \mathrm{e}^{m_n x}$$

3. If two pairs of roots are imaginary. For example $m_2 = m_1 = \beta + \mathrm{i}\gamma$ and $m_4 = m_3 = \beta - \mathrm{i}\gamma$, then

$$y_c = [(c_1 x + c_2)\cos gx + (c_3 x + c_4)\sin gx]\,\mathrm{e}^{bx} + \ldots + c_n \mathrm{e}^{m_n x}$$

2. Inverse Operator

$$\frac{1}{D}X = \int X\,\mathrm{d}x$$
$$\frac{1}{D-a}X = \mathrm{e}^{ax}\int X\mathrm{e}^{-ax}\,\mathrm{d}x$$

3. Particular Integral

For the linear differential equation expressed in symbolic form, the particular integral is given by:

$$y_p = \frac{1}{D^n + \alpha_1 D^{n-1} + \alpha_2 D^{n-2} + \ldots + \alpha_n}X = \frac{1}{f(D)}X \quad \text{or} \quad \frac{1}{\phi(D^2)}X$$

Depending upon the type of X, various forms of particular integral are:

1. where

$$X = \mathrm{e}^{ax}, \quad y_p = \frac{1}{f(D)}\mathrm{e}^{ax} = \frac{1}{f(a)}\mathrm{e}^{ax} \quad \text{provided} \quad f(a) \neq 0.$$
$$\text{If} \quad f(a) = 0, \quad \frac{1}{f(D)}\mathrm{e}^{ax} = x\frac{1}{f'(a)}\mathrm{e}^{ax} \quad \text{provided} \quad f'(a) \neq 0 \quad \text{etc.}$$

2. when $X = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b) \quad \text{provided} \quad \phi(-a^2) \neq 0$$

if $\phi(-a^2) = 0$,

$$\frac{1}{\phi(D^2)}\sin(ax+b) = x\frac{1}{\phi'(-a^2)}\sin(ax+b) \quad \text{provided} \quad \phi'(-a^2) \neq 0$$

3. when $X = x^m$, m being a positive integer

$$y_p = \frac{1}{f(D)}x^m = [f\,(D)]^{-1}\,x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending power of D by Binomial theorem as far as D^m and operate on x^m term by term.

4. when $x = e^{ax} g(x)$ where g is a function of x.

$$y_p = \frac{1}{f(D)} e^{ax} g(x) = e^{ax} \frac{1}{f(D+a)} g(x)$$

Evaluate $\dfrac{1}{f(D+a)} g(x)$ as in (1.), (2.) and (3.)

4. Various types of linear differential equations encountered in the stability analysis of structures

1. $\dfrac{d^2 y}{dx^2} + \alpha^2 y = 0$ i.e. $(D^2 + \alpha^2)y = 0$

   Its auxiliary equation is $D^2 + \alpha^2 = 0$, thus $D = \pm i\alpha$ The solution is:

$$\begin{aligned} y &= c_1 e^{i\alpha x} + c_2 e^{-i\alpha x} \\ &= c_1(\cos\alpha x + i\sin\alpha x) + c_2(\cos\alpha x - i\sin\alpha x) \\ &= (c_1 + c_2)\cos\alpha x + (ic_1 - ic_2)\sin\alpha x \\ &= A\sin\alpha x + B\cos\alpha x \end{aligned}$$

   where $A = i(c_1 - c_2)$ and $B = (c_1 + c_2)$, are arbitrary constants.

2. $\dfrac{d^2 y}{dx^2} + \alpha^2 y = kx$

$$\begin{aligned} y_c &= A\sin\alpha x + B\cos\alpha x \\ y_p &= \frac{1}{D^2 + \alpha^2} kx = \frac{1}{\alpha^2}\left[1 + \left(\frac{D}{\alpha}\right)^2\right]^{-1}(kx) \\ &= \frac{1}{\alpha^2}\left(1 - \frac{D^2}{\alpha^2}\right) kx = \frac{kx}{\alpha^2} \end{aligned}$$

   Hence, $y = y_c + y_p = A\sin\alpha x + B\cos\alpha x + \frac{kx}{\alpha^2}$

3. $\dfrac{d^3 y}{dx^3} + \alpha^2 \dfrac{dy}{dx} = 0$ or $(D^3 + \alpha^2 D) = 0$ giving $D = 0, +i\alpha, -i\alpha$.

   Therefore, $y = A\sin\alpha x + B\cos\alpha x + C$.

4. $\dfrac{d^4 y}{dx^4} + \alpha^2 \dfrac{d^2 y}{dx^2} = 0$ or $D^2(D^2 + \alpha^2) = 0$ giving $D = 0, 0, +i\alpha$ and $-i\alpha$

   Therefore, $y = A\sin\alpha x + B\cos\alpha x + Cx + D$.

5. $\dfrac{d^4 y}{dx^4} + \alpha^2 \dfrac{d^2 y}{dx^2} = k$

$$y_{\mathrm{p}} = \frac{1}{D^2(D^2+\alpha^2)}k = \frac{1}{D^2}\frac{1}{\alpha^2}\left(1+\frac{D^2}{\alpha^2}\right)^{-1}k = \frac{1}{D^2}\frac{1}{\alpha^2}\left(1-\frac{D^2}{\alpha^2}\right)k$$
$$= \frac{1}{D^2}\frac{k}{\alpha^2} = \frac{k}{\alpha^2}\left(\frac{x^2}{2}\right) = \frac{kx^2}{2\alpha^2}$$

Thus, $y = y_{\mathrm{c}} + y_{\mathrm{p}} = A\sin\alpha x + B\cos\alpha x + Cx + D + \dfrac{kx^2}{2\alpha^2}$

6. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - \alpha^2 y = 0$

   $y = y_{\mathrm{c}} = c_1\mathrm{e}^{\alpha x} + c_2\mathrm{e}^{-\alpha x} = A\sinh\alpha x + B\cosh\alpha x$

7. $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} - \alpha^4 y = 0$ or $(D^4-\alpha^4)y = 0$ giving $D = \pm\alpha, \pm i\alpha$

   Hence solution is $y = c_1\mathrm{e}^{\alpha x} + c_2\mathrm{e}^{-\alpha x} + c_3\mathrm{e}^{\mathrm{i}\alpha x} + c_4\mathrm{e}^{-\mathrm{i}\alpha x} = A\sinh\alpha x + B\cosh\alpha x + C\sin\alpha x + D\cos\alpha x$

8. $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} - \alpha^4 y = k$

$$y_{\mathrm{c}} = A\sinh\alpha x + B\cosh\alpha x + C\sin\alpha x + D\cos\alpha x$$
$$y_{\mathrm{p}} = \frac{1}{D^4-\alpha^4}k = k\frac{1}{D^4-\alpha^4}\mathrm{e}^{o.x} = -\frac{k}{\alpha^4}$$

   Thus, complete solution is: $y = A\sinh\alpha x + B\cosh\alpha x + C\sin\alpha x + D\cos\alpha x - \dfrac{k}{\alpha^4}$

C.2 Bessel Functions

The linear second order differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2-n^2)y = 0 \tag{C.4}$$

where n is a constant, is known as Bessel's differential equation. Since n appears only as n^2, n may be assumed to be either zero or a positive number without any loss of generality. Every value of the parameter n is associated with a pair of basic solutions of (C.4) called Bessel functions of order n. One of them which is finite at $x = 0$ is called Bessel function of the first kind and the other which has no finite limit (i. e. is unbounded) called Bessel function of second kind.

Thus the general solution of the (C.4) is given by:

$$y = AJ_n(x) + BY_n(x) \tag{C.5}$$

where A and B are arbitrary constants and functions $J_n(x)$ and $Y_n(x)$ are Bessel functions of first and second kinds of order n, respectively. The function $J_n(x)$ is defined by the infinite series:

Table C.1. Values of Bessel functions of first kind of the order of 0 and 1. (For more extensive tables see [1] in Appendix D)

x	$J_o(x)$	$J_1(x)$	X	$J_o(x)$	$J_1(x)$	x	$J_o(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3991	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

$$\begin{aligned}
J_n(x) &\equiv \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\Gamma(n+r+1)} \\
&= \frac{x^n}{2^n \Gamma(n+1)} \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{2r}}{r!\,[(n+1)\ldots(n+r)]} \\
&= \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \times 4 \times (2n+2)(2n+4)} \cdots \right]
\end{aligned} \tag{C.6}$$

where $\Gamma(n+r+1)$ represents a Gamma function. When n is integer $\Gamma(n+1) = n!$. On the other hand $Y_n(x)$ is defined as:

$$Y_n(x) = \lim_{r \to n} \left[\frac{J_r(x)\cos r\pi - J_{-r}(x)}{\sin r\pi} \right] \tag{C.7}$$

The function $J_{-n}(x)$ obtained from $J_n(x)$ by replacing n by $-n$ is a Bessel function of first kind of negative order n. If n is not an integer, the functions $J_n(x)$ and J_{-n} (x) constitute two linearly independent solutions of (C.4) and the solution is given by:

$$y = C_1 J_n(x) + C_2 J_{-n}(x), \quad n \neq 0, 1, 2, \ldots \tag{C.8}$$

where C_1 and C_2 are arbitrary constants.

If n is an integer i. e. $n = 0, 1, 2, 3 \ldots$, then

$$J_{-n}(x) = (-1)^n J_n(x) \tag{C.9}$$

and (C.8) reduces to

$$y = C_1 J_n(x) + C_2(-1)^n J_n(x) = [C_1 + C_2(-1)^n] J_n(x) = A J_n(x) \tag{C.10}$$

Therefore, $J_n(x)$ does not any more represent an independent solution of (C.4). In this case the other independent solution is taken to be $Y_n(x)$ which is Bessel function of order n of second kind and general solution to (C.4) is given by (C.5).

The functions $J_n(x)$, $J_{-n}(x)$ and $Y_n(x)$ have been tabulated and behave somewhat like trigonometric functions of damped amplitudes.

Special properties of some Bessel functions

1. If the independent variable x is changed to λx where λ is a constant, the general solution becomes

$$y = A J_n(\lambda x) + B Y_n(\lambda x) \tag{C.11}$$

2. If n is any real number then $J_n(x) = 0$ has infinite number of real roots. Difference between successive roots approach π as the roots increase in value. The roots of $J_n(x) = 0$ lie between those of $J_{n-1}(x) = 0$ and $J_{n+1}(x) = 0$. Similar remarks are applicable to $Y_n(x)$.
3. For the particular case $n = 0$ (i. e. Bessel equation of zero order)

$$y'' + \frac{1}{x} y' + y = 0 \tag{C.12}$$

Taking derivative of left hand side of (C.12)

$$\frac{d^2 y'}{dx^2} + \frac{1}{x}\frac{dy'}{dx} + \left(1 - \frac{1}{x^2}\right) y' = 0$$

where $y' = dy/dx$ and solution is:

$$y' = \frac{dy}{dx} = A J_1(x) + B Y_1(x) \tag{C.13}$$

Table C.2. Values of Bessel functions of second kind of the order 0 and 1.

x	$Y_o(x)$	$Y_1(x)$	X	$Y_o(x)$	$Y_1(x)$	x	$Y_o(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

Table C.3. Zeros of Bessel functions

(i) Typical Bessel functions of integer order

Zero No.	**1**	**2**	**3**	**4**	**5**
$J_o(x)$	2.40483	5.52008	8.65373	11.79153	14.93092
$J_1(x)$	3.83171	7.01559	10.17347	13.32369	16.47063
$J_2(x)$	5.13562	8.41724	11.61984	14.79595	17.95982
$J_3(x)$	6.38016	9.76102	13.01520	16.22347	19.40942

(ii) First zero of typical Bessel functions of fractional order

$J_{-1/4}(x)$	$J_{-1/3}(x)$	$J_{-3/4}(x)$
2.0063	1.8660	1.0585

4. The zero of J_n, J_{-n} and Y_n are approximately equal to the zeros of $\cos\phi$ and $\sin\phi$ where $\phi = (\pi/2) + m\pi$ for zeros of J_n, and $\phi = m\pi$ (including $m = 0$) for zero of Y_n.
5. Zeros of typical Bessel functions are given in the Table C.3.
6. Values of Bessel functions for various magnitudes of x:

$$\text{For small value } x: \lim_{x\to 0} J_n(x) = x^n/2^n\Gamma(n) \tag{C.14}$$

The value of $J_{-n}(x)$, where n is non-integer, tends to ∞ as $x \to 0$. Similarly $Y_n(x)$ also tends to ∞ as $x \to 0$ for all values of n and therefore in most of the problems of practical interest the solution $Y_n(x)$ may be ignored.

For large values of x:

$$\lim_{x\to\infty} J_n(x) = \frac{\cos(x - \pi/4 - n\pi/2)}{\sqrt{(\pi x/2)}} \tag{C.15}$$

$$\lim_{x\to\infty} Y_n(x) = \frac{\sin(x - \pi/4 - n\pi/2)}{\sqrt{(\pi x/2)}} \tag{C.16}$$

i. e. for large values of argument x, Bessel functions behave like trigonometrical functions of decreasing amplitude.

$$J_n(0) = 0 \quad \text{for} \quad n > 0 \quad \text{and} \quad J_o(0) = 0 \tag{C.17}$$

$$J_{1/2}(x) = \left(\sqrt{\frac{2}{\pi x}}\right) \sin x \quad \text{and} \quad J_{-1/2}(x) = \left(\sqrt{\frac{2}{\pi x}}\right) \cos x \tag{C.18}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x} J_o(x) = J_1(x) \quad \text{and} \quad -\frac{\mathrm{d}}{\mathrm{d}x} Y_o(x) = Y_1(x) \tag{C.19}$$

7. Recurrence relations:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{C.20}$$

Therefore, when n is half of an odd number e.g. $n = 1/2$

$$J_{3/2}(x) = \left(\sqrt{\frac{2}{\pi x}}\right) \left[\frac{\sin x}{x} - \cos x\right] \quad \text{and}$$

$$J_{-3/2}(x) = \left(\sqrt{\frac{2}{\pi x}}\right) \left[-\frac{\cos x}{x} - \sin x\right]$$

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) = x J_{n-1}(x) - n J_n(x) \tag{C.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[x^n J_n(x)\right] = x^n J_{n-1}(x) \tag{C.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[x^{-n} J_n(x)\right] = -x^n J_{n+1}(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \tag{C.23}$$

8. Equations with solutions in terms of Bessel Functions:
   A large number of differential equations which are apparently not similar to the standard form of Bessel's differential equation given by (C.4), can be transformed into standard form by proper substitution and solved in terms of Bessel functions. Introducing independent variable $x = \alpha s^\beta$ in (C.4)

$$s^2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + s \frac{\mathrm{d}y}{\mathrm{d}s} + \left(\beta^2 \alpha^2 s^{2\beta} - \beta^2 n^2\right) y = 0 \tag{C.24}$$

Also changing the dependent variable as $y = s^{-\gamma} \xi(s)$, the (C.24) becomes

$$\frac{\mathrm{d}^2 \xi}{\mathrm{d}s^2} + \left(\frac{1 - 2\gamma}{s}\right) \frac{\mathrm{d}\xi}{\mathrm{d}s} + \left\{\left(\beta \alpha s^{\beta-1}\right)^2 + \frac{\gamma^2 - \beta^2 n^2}{s^2}\right\} \xi(s) = 0 \tag{C.25}$$

The general solution of (C.4) given by $y = A J_n(x) + B Y_n(x)$ gets modified to

$$\xi(s) = s^\gamma \left[A J_n(\alpha s^\beta) + B Y_n(\alpha s^\beta)\right] \tag{C.26}$$

Equation (C.25) has four parameters α, β, γ and n. By comparing the given differential equation with (C.25), the parameters can be assigned appropriate values and solution in terms of Bessel functions can be obtained. This transformation enables substantial simplifications in the analysis.

Typical Differential Equations Reducible to Bessel Equation

1.
$$\frac{d^2y}{dx^2} - \left(\frac{2n-1}{x}\right)\frac{dy}{dx} + y = 0 \tag{C.27}$$

A comparison with (C.25) reveals that this equation is same as (C.25) for $\alpha = \beta = 1$ and $\gamma = n$. Hence general solution from (C.26) is:

$$y = x^n[AJ_n(x) + BY_n(x)] \cong x^n B_n(x) \tag{C.28}$$

where $B_n(x)$ is symbolic representation of $[AJ_n(x) + BY_n(x)]$.

2. $$x\frac{d^2y}{dx^2} + (1-n)\frac{dy}{dx} + \frac{1}{4}y = 0; \quad \alpha = 1; \quad \beta = \frac{1}{2}; \quad \gamma = n/2 \quad \text{and} \quad n = n \tag{C.29}$$

$$y = x^{n/2} B_n\left(\sqrt{x}\right) \tag{C.30}$$

3.
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + 4\left(x^2 - \frac{n^2}{x^2}\right)y = 0; \quad y = B_n(x^2) \tag{C.31}$$

4.
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{4}\left(\frac{1}{x} - \frac{n^2}{x^2}\right)y = 0; \quad y = B_n(\sqrt{x}) \tag{C.32}$$

5.
$$\frac{d^2y}{dx^2} + \left(\frac{a}{x}\right)\frac{dy}{dx} + by = 0 \tag{C.33}$$

$$\alpha = \sqrt{b}; \quad \beta = 1; \quad \gamma = (1-a)/2 \quad \text{and} \quad n = \gamma = (1-a)/2$$

$$y = x^\gamma\left[AJ_n\left(x\sqrt{b}\right) + BY_n\left(x\sqrt{b}\right)\right] = x^\gamma B_n\left(x\sqrt{b}\right) \tag{C.34}$$

where $n = \gamma = (1-a)/2$ and n is an integer.

6.
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{9x^2}\right)y = 0 \tag{C.35}$$

$$\alpha = 1, \quad \beta = 1, \quad \gamma = 0 \quad \text{and} \quad n = 1/3$$

$$y = AJ_{1/3}(x) + BJ_{-1/3}(x) \tag{C.36}$$

7.
$$\frac{d^2y}{dx^2} + \left(k^2x^2\right)y = 0 \tag{C.37}$$

$$\alpha = k/2; \quad \beta = 2; \quad \gamma = 1/2 \quad \text{and} \quad n = 1/4$$

Thus the solution is:

$$y = \sqrt{x}\left\{AJ_{1/4}(kx^2/2) + BJ_{-1/4}(kx^2/2)\right\} \tag{C.38}$$

8.
$$\frac{d^2y}{dx^2} + xy = 0 \tag{C.39}$$
$$\alpha = \frac{2}{3}; \quad \beta = \frac{3}{2}; \quad \gamma = \frac{1}{2} \quad \text{and} \quad n = \frac{1}{3}$$
$$y = \sqrt{x} B_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

9.
$$\frac{d^2y}{dx^2} + \frac{1}{\sqrt{x}}y = 0$$
$$y = \sqrt{x} B_{2/3}\left(\frac{4}{3}x^{3/4}\right) \tag{C.40}$$

10.
$$\frac{d^2y}{dx^2} + \left(k^2 n^2\, x^{2k-2}\right) y = 0; \quad y = \sqrt{x} B_{1/2k}\left(nx^k\right) \tag{C.41}$$

11.
$$x^2\frac{d^2y}{dx^2} + (2k+1)\,x\frac{dy}{dx} + \left(m^2x^{2r} + t^2\right) y = 0 \tag{C.42}$$
$$y = x^{-\gamma}\left[A J_{k/r}(mx^r/r) + B Y_{k/r}(mx^r/r)\right] \tag{C.43}$$

where $\gamma = \sqrt{k^2 - t^2}$

This equation can further be used in transforming a large number of equations in this form by assessing appropriate values to the constant k, m, r and t.

C.3 Fourier Series

1. Types of function

The function $f(x)$ is termed odd if $f(-x) = -f(x)$ e.g. $\sin x, \tan x, x, x^3 \ldots$ etc. are odd functions. Graphically an odd function is symmetrical about the origin. On the other hand if $f(-x) = f(x)$, the function is termed an even function, e.g. $\cos x, \sec x, x^2, x^4 \ldots$ etc. Even functions are symmetrical about Y-axis. An important property of these function is

$$\int_{-\ell}^{\ell} f(x)\,dx = 2\int_{o}^{\ell} f(x)\,dx \qquad \text{when } f(x) \text{ is an even function.}$$
$$= 0 \qquad \text{when } f(x) \text{ is an odd function.} \tag{C.44}$$

2. Fourier series expansion

The Fourier series expansion of a periodic function defined in the range $(-\ell,\ell)$ is expressed as:

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

where

$$a_o = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{d}x, \quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,\mathrm{d}x, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,\mathrm{d}x \tag{C.45}$$

Following cases arise:

Case I. When $f(x)$ is an even function expansion contains only cosine terms i.e.

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

where

$$a_o = \frac{2}{\ell}\int_{o}^{\ell} f(x)\,\mathrm{d}x \quad \text{and} \quad a_n = \frac{2}{\ell}\int_{o}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,\mathrm{d}x \tag{C.46}$$

Case II. When $f(x)$ is an odd function the expansion contain only sine terms i.e.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{C.47}$$

where

$$b_n = \frac{2}{\ell}\int_{o}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,\mathrm{d}x$$

3. Special properties

Square values of $f(x)$:

1. Full range Fourier series ($-\ell$ to ℓ)

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\right) \tag{C.48}$$

$$\int_{-\ell}^{\ell} [f(x)]^2\,\mathrm{d}x = \ell\left\{\frac{a_o^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right\} \tag{C.49}$$

$$\int_{o}^{2\ell} [f(x)]^2 \, dx = \ell \left\{ \frac{a_o^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) \right\} \quad \text{(C.50)}$$

2. Half range series of period 2ℓ in the range (0 to ℓ) for $f(x)$

(a) Cosine series

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} \quad \text{(C.51)}$$

$$\int_{o}^{\ell} [f(x)]^2 \, dx = \frac{\ell}{2} \left\{ \frac{a_o^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right\} \quad \text{(C.52)}$$

(b) Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} \quad \text{(C.53)}$$

$$\int_{o}^{\ell} [f(x)]^2 \, dx = \frac{\ell}{2} \left\{ \sum_{n=1}^{\infty} b_n^2 \right\} \quad \text{(C.54)}$$

4. Definite Integrals

$$\int_{o}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) dx = 0 \qquad (n \neq 0) \quad \text{(C.55)}$$

$$\int_{o}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) dx = 0 \qquad \text{when } n \text{ is an even number and } n \neq 0$$

$$= \frac{2\ell}{\pi} \qquad \text{when } n \text{ is an odd number} \quad \text{(C.56)}$$

$$\int_{o}^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = 0 \qquad (m \neq n)$$

$$= \frac{\ell}{2} \qquad (m = n) \quad \text{(C.57)}$$

$$\int_{o}^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = 0 \qquad \text{when } m \pm n \text{ is an even number}$$

$$= \frac{2m\ell}{\pi(m^2 - n^2)} \qquad \text{when } m \pm n \text{ is an odd number.}$$

$$\int_o^\ell \sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\mathrm{d}x = 0 \qquad (m \neq n) \tag{C.58}$$

$$= \frac{\ell}{2} \qquad (m = n) \tag{C.59}$$

$$\int_o^\ell x\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\mathrm{d}x = 0 \quad \text{when } m \neq n \text{ and } m \pm n \text{ is an even number}$$

$$= \left(\frac{4\ell^2}{\pi^2}\right)\frac{\mathrm{mn}}{(m^2 - n^2)^2} \quad \text{when } m \pm n \text{ is an odd number.}$$

$$= \frac{\ell^2}{4} \quad \text{when } m = n \tag{C.60}$$

These integrals demonstrate the orthogonality property that greatly facilitates the evaluation of Fourier coefficients.

Appendix D

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