Ravindra B. Bapat · Steve J. Kirkland
K. Manjunatha Prasad
Simo Puntanen *Editors*

Combinatorial Matrix Theory and Generalized Inverses of Matrices

Springer

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Editors

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Foreword

The success of a conference has to be evaluated by the quality of the scientific presentations and the ability of the organizers to publish the proceedings. If this is the yardstick used, I reckon the International Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices makes the grade easily.

I congratulate the Department of Statistics, Manipal University, on the successful conduct of the conference and the publication of the proceedings.

Manipal, India Dr. H. Vinod Bhat

Preface

The *International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices-2012* was organized by the Department of Statistics, Manipal University, Manipal, India. There were more than 100 registered participants for both of the workshop and conference and 22 invited speakers representing 12 nations across the globe. Two-day conference consisted of four plenary sessions. Invited papers and contributed papers were presented in three parallel sessions dedicated, respectively, to the branches of "Combinatorial Matrix Theory," "Generalized Inverse of Matrices," and "Matrix Methods for Statistics." The international workshop and conference on CMTGIM provided a delightful and fruitful opportunity for all Indian and foreign mathematicians and statisticians at all levels of experience to interact in person and exchange ideas.

"Combinatorial Matrix Theory and Generalized Inverses of Matrices" is being published in an attempt to document the recent developments in the areas discussed at the conference. This book contains about 18 research and expository articles from different speakers and their collaborators. Only the selected articles, after the refereeing process, were retained for the inclusion in this volume. A report on the workshop and conference is presented as an appendix, covering narrative photographs and messages from the Pro Vice-Chancellor, Joint Secretary, CSIR, and the Director General, CSO.

New Delhi, India — Ravindra B. Bapat
Maynooth, Ireland — Steve Kirkland
Manipal, India — K. Manjunatha Prasad
Tampere, Finland — Simo Puntanen

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We are grateful to all patrons of the conference and the Chairman of organizing committee Dr. H. Vinod Bhat, Pro Vice-Chancellor, MU, for their consistent support. Also, we thank Dr. N.S.K. Nair, the Head of the Department, and other faculty members of the Department of Statistics for their active involvement in the conduct of conference.

Our sincere thanks are also due to the National Board of Higher Mathematics (NBHM), the Central Statistics Office (CSO) Ministry of Statistics and Programme Implementation, the Council of Scientific and Industrial Research (CSIR), and the International Center for Theoretical Physics (ICTP) for their financial support.

Assistance of K.S. Mohana, Santhi Sheela and M. Vinay in compiling this volume was incredible, and we proudly applaud their effort.

Finally, we thank the team at Springer for their cooperation at all stages of publishing this volume.

September 2012

Ravindra B. Bapat
Steve Kirkland
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Skew Spectrum of the Cartesian Product of an Oriented Graph with an Oriented Hypercube

A. Anuradha and R. Balakrishnan

Abstract Let σ be an orientation of a simple graph H yielding an oriented graph H^σ. We define an orientation ψ to the Cartesian product $G = H \,\square\, Q_d$ of H with the hypercube Q_d by orienting the edges of G in a specific way. The skew adjacency matrices $S(G^\psi)$ obtained in this way for some special families of G answer some special cases of the Inverse Eigenvalue Problem. Further we present a new orientation ϕ to the hypercube Q_d for which the skew energy equals the energy of the underlying hypercube, distinct from the two orientations of hypercubes defined by Tian (Linear Algebra Appl. 435:2140–2149, 2011) and show how one of the two orientations of Q_d described by Tian is a special case of our method.

Keywords Oriented graph · Oriented hypercubes · Spectrum of graph · Skew spectrum

Mathematics Subject Classification (2010) 05C50

1 Introduction

Let $G = (V, E)$ be a finite simple undirected graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$ as its vertex set and E as its edge set. Let σ be any orientation of the edge set E yielding the oriented graph $G^\sigma = (V, \Gamma)$, where Γ is the arc set of G^σ. The arc $(v_i, v_j) \in \Gamma$ denotes the arc having head v_j and tail v_i. The adjacency matrix of G is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise. As the matrix A is real and symmetric, all its eigenvalues are real. The skew adjacency matrix of the oriented graph G^σ is the $n \times n$ matrix $S(G^\sigma) = (s_{ij})$, where $s_{ij} = 1 = -s_{ji}$ whenever $(v_i, v_j) \in \Gamma(G^\sigma)$ and $s_{ij} = 0$ otherwise. Clearly, $S(G^\sigma)$ is a skew symmetric matrix, and hence its eigenvalues are all

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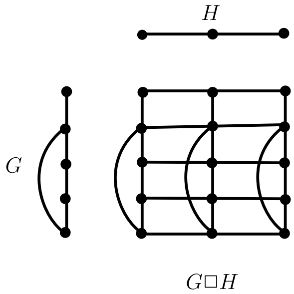


Fig. 1 Cartesian product of two graphs

pure imaginary. The spectrum of G, denoted by $\mathrm{Sp}(G)$, is the set of all eigenvalues of A. In a similar manner, the skew spectrum of the oriented graph G^σ is defined as the spectrum of the matrix $S(G^\sigma)$. For further properties on skew spectrum of oriented graphs, the reader may refer to Adiga [1] and Cavers [4].

The Cartesian product $G = G_1 \square G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has $V(G) = V(G_1) \times V(G_2)$ as its vertex set, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$ (see Fig. 1).

2 Cartesian Product of Oriented Graphs with Hypercubes

In this section, we define an orientation of the Cartesian product of a graph H with a hypercube and compute its skew spectrum.

Suppose that H^σ is an oriented graph of order n with H as its underlying graph. For $d \geq 1$, let $G_d = H \square Q_d \simeq Q_d \square H$, where $Q_d = K_2 \square K_2 \square \cdots \square K_2$ (d times), be the Cartesian product of the undirected graph H with the hypercube Q_d of dimension d. We now construct an oriented Cartesian product graph G_d^ψ with orientation ψ defined by the following algorithm.

Algorithm 1 The Cartesian product graph $G_d = Q_d \square H = K_2 \square (Q_{d-1} \square H) = K_2 \square G_{d-1}$ contains two copies of G_{d-1}. Set $G_0^\psi \simeq H^\sigma$.

1. For $k < d$, assume that G_i has been oriented to G_i^ψ, $i = 1, 2, \ldots, k$. The oriented graph G_{k+1}^ψ is formed as follows:
   (a) Take two copies of G_k^ψ. Assume that the vertex set of the first copy is $\{1, 2, \ldots, n \cdot 2^k\}$ and the corresponding vertex set of the second copy is $\{n \cdot 2^k + 1, n \cdot 2^k + 2, \ldots, n \cdot 2^{k+1}\}$.
   (b) Extend ψ to G_{k+1} by assigning an oriented arc $j \to n \cdot 2^k + j$ for $1 \leq j \leq n \cdot 2^k$ (see Fig. 2).
2. If $k + 1 = d$, stop; else $k \leftarrow k + 1$, return to Step 1.

If $H^\sigma \simeq P_3^\sigma$, an oriented path on three vertices (see Fig. 3(a)), the oriented graph G_2^ψ is given in Fig. 3. For $d \geq 1$, let S_d be the skew adjacency matrix of the Cartesian

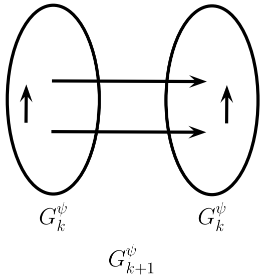


Fig. 2 Construction of G_{k+1}^ψ from two copies of G_k^ψ in Algorithm 1

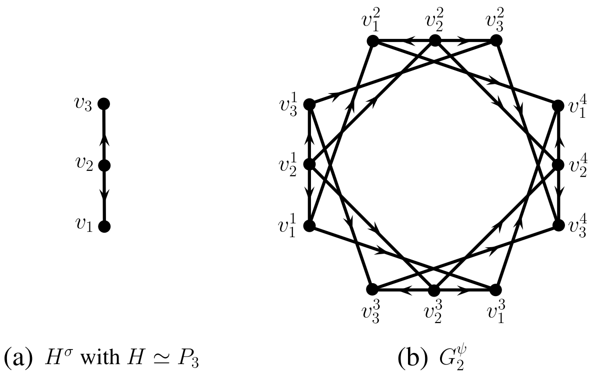


Fig. 3 Oriented graph $G_2^\psi = (P_3 \,\square\, Q_2)^\psi$ obtained by Algorithm 1

product graph G_d^ψ oriented as in Algorithm 1. Then the skew adjacency matrix S_{d+1} for the oriented graph G_{d+1}^ψ is given by

$$S_{d+1} = \begin{bmatrix} S_d & I \\ -I & S_d \end{bmatrix},$$

where I is the identity matrix of order 2^d.

Theorem 1 *For $d \geq 1$, the skew spectrum of the oriented graph G_{d+1}^ψ, oriented by Algorithm 1, is*

$$\mathrm{Sp}_S\big(G_{d+1}^\psi\big) = \big\{\mathbf{i}(\mu \pm 1)\colon\ \mathbf{i}\mu \in \mathrm{Sp}_S\big(G_d^\psi\big)\big\}.$$

Proof Given $d \geq 1$, consider $A_d = S_d^t S_d$, where S_d is the skew adjacency matrix of G_d^ψ, and A^t denotes the transpose of the matrix A. If $\mathbf{i}\mu$ is an eigenvalue of the skew symmetric matrix S_d, then μ^2 is an eigenvalue of the symmetric matrix A_d. Let X_d be an eigenvector corresponding to the eigenvalue μ^2 of the matrix A_d. Then

$$A_d X_d = \mu^2 X_d. \tag{1}$$

Suppose that

$$S_d X_d = Y_d. \tag{2}$$

Then since S_d is skew symmetric,

$$\begin{aligned} S_d Y_d &= S_d(S_d X_d) = -(-S_d)(S_d X_d) \\ &= -\left(S_d^t S_d\right) X_d = -A_d X_d \\ &= -\mu^2 X_d. \end{aligned} \tag{3}$$

Further, by Eqs. (3) and (2),

$$\begin{aligned} A_d Y_d &= \left(S_d^t S_d\right) Y_d = -\mu^2 S_d^t X_d = \mu^2 (S_d X_d) \\ &= \mu^2 Y_d. \end{aligned} \tag{4}$$

Consequently,

$$\begin{aligned} A_{d+1} &= S_{d+1}^t S_{d+1} \\ &= \begin{bmatrix} S_d & I \\ -I & S_d \end{bmatrix}^t \begin{bmatrix} S_d & I \\ -I & S_d \end{bmatrix} \\ &= \begin{bmatrix} S_d^t & -I \\ I & S_d^t \end{bmatrix} \begin{bmatrix} S_d & I \\ -I & S_d \end{bmatrix} \\ &= \begin{bmatrix} S_d^t S_d + I & S_d^t - S_d \\ S_d - S_d^t & I + S_d^t S_d \end{bmatrix} \\ &= \begin{bmatrix} A_d + I & -2S_d \\ 2S_d & A_d + I \end{bmatrix}, \quad \text{since } S_d^t = -S_d. \end{aligned}$$

Hence,

$$\begin{aligned} A_{d+1} \begin{bmatrix} Y_d \\ \mu X_d \end{bmatrix} &= \begin{bmatrix} A_d + I & -2S_d \\ 2S_d & A_d + I \end{bmatrix} \begin{bmatrix} Y_d \\ \mu X_d \end{bmatrix} \\ &= \begin{bmatrix} (A_d + I) Y_d - 2\mu S_d X_d \\ 2 S_d Y_d + \mu (A_d + I) X_d \end{bmatrix}, \\ &= \begin{bmatrix} (\mu^2 + 1) Y_d - 2\mu Y_d \\ -2\mu^2 X_d + \mu(\mu^2 + 1) X_d \end{bmatrix} \quad \text{by Eqs. (1)–(4)} \\ &= \begin{bmatrix} (\mu - 1)^2 Y_d \\ \mu(\mu - 1)^2 X_d \end{bmatrix} \\ &= (\mu - 1)^2 \begin{bmatrix} Y_d \\ \mu X_d \end{bmatrix}, \end{aligned}$$

which shows that $(\mu - 1)^2$ is an eigenvalue of A_{d+1} with the corresponding eigenvector $\left[\begin{smallmatrix} Y_d \\ \mu X_d \end{smallmatrix}\right]$. Similarly, $(\mu + 1)^2$ is an eigenvalue of A_{d+1} corresponding to the eigenvector $\left[\begin{smallmatrix} -Y_d \\ \mu X_d \end{smallmatrix}\right]$.

Hence, for each eigenvalue $\mathbf{i}\mu$ of S_d, there are two eigenvalues $\mathbf{i}(\mu + 1)$ and $\mathbf{i}(\mu - 1)$ for the matrix S_{d+1}, each with the same multiplicity as that of μ of S_d, and this accounts for all the eigenvalues of G_{d+1}^{ψ}. □

Lemma 1 (Balakrishnan [2]) *Let* $\mathrm{Sp}(G_1) = \{\lambda_1, \ldots, \lambda_n\}$ *and* $\mathrm{Sp}(G_2) = \{\mu_1, \ldots, \mu_t\}$ *be respectively the adjacency spectra of two graphs* G_1 *of order* n *and* G_2 *of order* t. *Then*

$$\mathrm{Sp}(G_1 \,\Box\, G_2) = \{\lambda_i + \mu_j \colon 1 \le i \le n,\ 1 \le j \le t\}.$$

Corollary 1 *Suppose that* σ *is an orientation of a bipartite graph* H *for which* $\mathrm{Sp}_S(H^\sigma) = \mathbf{i}\,\mathrm{Sp}(H)$. *Then, for each* $d \ge 1$, *the oriented Cartesian product graph* $G_d^\psi = (H \,\Box\, Q_d)^\psi$ *has the property that*

$$\mathrm{Sp}_S\big(G_d^\psi\big) = \mathbf{i}\,\mathrm{Sp}(G_d).$$

Proof For $d \ge 1$, the oriented Cartesian product graph G_{d+1}^ψ is isomorphic to $(G_d \,\Box\, K_2)^\psi$. Applying Theorem 1 and Lemma 1, the proof follows by induction on d. $\Box$

Corollary 2 (Tian [8]) *Consider the orientation* σ *for a complete graph* K_2, $V(K_2) = \{v_1, v_2\}$ *that orients* $v_1 \to v_2$. *Then for* $H^\sigma \simeq K_2^\sigma$, *the oriented Cartesian product graph* $G_d^\psi = (H \,\Box\, Q_d)^\psi$, $d \ge 1$, *constructed by Algorithm 1, has the property that*

$$\mathrm{Sp}_S\big(G_d^\psi\big) = \mathbf{i}\,\mathrm{Sp}(G_d).$$

Proof Proof is immediate from Corollary 1. $\Box$

It is easy to verify that when $H^\sigma \simeq K_2^\sigma$, the oriented graph $G_d^\psi = (H \,\Box\, Q_d)^\psi \simeq Q_{d+1}^\psi$ is one of the oriented hypercubes constructed by G.-X. Tian [8, Algorithm 2] for which $\mathrm{Sp}_S(G_d^\psi) = \mathbf{i}\,\mathrm{Sp}(G_d)$.

Theorem 2 *Let* $\mathrm{Sp}_S(H^\sigma) = \left(\begin{smallmatrix} \mathbf{i}\mu_1 & \mathbf{i}\mu_2 & \ldots & \mathbf{i}\mu_p \\ m_1 & m_2 & \ldots & m_p \end{smallmatrix}\right)$, *where* m_j, $1 \le j \le p$, *be the multiplicity of* $\mathbf{i}\mu_j$. *Then the skew spectrum of the oriented graph* $G_d^\psi = (H \,\Box\, Q_d)^\psi$ *defined by Algorithm 1 is given by*

$$\mathrm{Sp}_S\big(G_d^\psi\big) = \begin{pmatrix} \mathbf{i}(\mu_j + d) & \mathbf{i}(\mu_j + d - 2) & \ldots & \mathbf{i}(\mu_j + d - 2d) \\ \binom{d}{0} m_j & \binom{d}{1} m_j & \ldots & \binom{d}{d} m_j \end{pmatrix},$$
$$j = 1, 2, \ldots, p.$$

Proof By induction on d. The case $d = 1$ is a consequence of Theorem 1. Indeed, for each $j = 1, 2, \ldots, p$, both $\mathbf{i}(\mu_j + 1)$ and $\mathbf{i}(\mu_j - 1)$ are repeated $\binom{1}{0} m_j$ and $\binom{1}{1} m_j$ times, respectively, and this accounts for $2m_j$ eigenvalues of G_1^ψ. Assume that the result holds for $d - 1$, $d \ge 2$. The skew spectrum of the oriented graph G_{d-1}^ψ is then given by

$$\mathrm{Sp}_S\big(G_{d-1}^{\psi}\big) = \begin{pmatrix} \mathbf{i}(\mu_j + d - 1) & \mathbf{i}(\mu_j + (d-1) - 2) & \ldots & \mathbf{i}(\mu_j + (d-1) - 2(d-1)) \\ \binom{d-1}{0} m_j & \binom{d-1}{1} m_j & \ldots & \binom{d-1}{d-1} m_j \end{pmatrix},$$

where $j = 1, 2, \ldots, p$.

As $G_d^{\psi} = (G_{d-1} \,\square\, K_2)^{\psi}$, the eigenvalue $\mathbf{i}(\mu_j + d - 1)$ of G_{d-1}^{ψ} gives rise, again by Theorem 1, to the eigenvalue $\mathbf{i}(\mu_j + (d-1) + 1) = \mathbf{i}(\mu_j + d)$ of G_d^{ψ} repeated $\binom{d-1}{0} m_j = \binom{d}{0} m_j$ times. Further, $\mathbf{i}(\mu_j + (d-1) - 1)$ also gets repeated $\binom{d-1}{0} m_j$ times while the eigenvalue $\mathbf{i}(\mu_j + (d-1) - 2)$ of G_{d-1}^{ψ} gives rise to the eigenvalue $\mathbf{i}(\mu_j + (d-1) - 1)$ repeated $\binom{d-1}{1} m_j$ times. Thus, $\mathbf{i}(\mu_j + d - 2)$ is an eigenvalue of G_d^{ψ} with multiplicity $\binom{d-1}{0} m_j + \binom{d-1}{1} m_j = \binom{d}{1} m_j$. The computation of the general term is now clear, and the proof is complete. □

Note that as $\binom{d}{0} + \binom{d}{1} + \cdots + \binom{d}{d} = 2^d$, all the $2^d(m_1 + m_2 + \cdots + m_p)$ eigenvalues of the oriented graph G_d^{ψ} have been accounted for in the statement of Theorem 2.

Definition 1 Suppose that G^{σ} is an oriented graph with skew spectrum $\mathrm{Sp}_S(G^{\sigma}) = \begin{pmatrix} \mathbf{i}\mu_1 & \mathbf{i}\mu_2 & \ldots & \mathbf{i}\mu_p \\ m_1 & m_2 & \ldots & m_p \end{pmatrix}$. Then G^{σ} is said to be skew integral if each μ_j, $1 \le j \le p$, is an integer.

The following corollary constructs many skew integral oriented graphs from a given skew integral oriented graph.

Corollary 3 *Let H^{σ} be a skew integral oriented graph. Then for each $d = 1, 2, \ldots,$ the oriented Cartesian product graph $G_d^{\psi} = (H \,\square\, Q_d)^{\psi}$ obtained by applying Algorithm 1 to H^{σ} is also skew integral.*

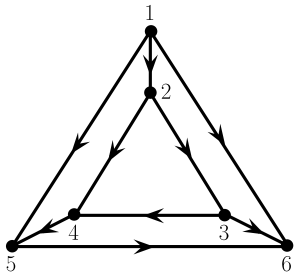
Proof Proof is immediate from Theorem 2. □

3 Structured Inverse Eigenvalue Problem (SIEP)

Given a certain spectral data, the objective of a Structured Inverse Eigenvalue Problem (SIEP) is to construct a matrix that maintains a certain specific structure and the given spectral property (see Chu [5]). Structured Inverse Eigenvalue Problems arise in a wide variety of fields: control design, system identification, principal component analysis, exploration and remote sensing, geophysics, molecular spectroscopy, particle physics, structure analysis, circuit theory, mechanical system simulation, and so on [5]. For further study on inverse eigenvalue problems, the reader may refer to the survey papers by Byrnes [3] and Chu [5]. The SIEP is defined as follows:

Structured Inverse Eigenvalue Problem (SIEP) Given scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ in a field $\mathbb{F}$, find a specially structured matrix A such that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ forms the set of eigenvalues of A.

Fig. 4 Oriented graph H^σ



In this section we construct skew symmetric matrices that solve the SIEP for some interesting special sets of complex numbers.

Theorem 3 *Let H^σ be the oriented graph given in Fig. 4. Let $\mathcal{G} = \{G_d^\psi : d = 1, 2, \ldots\}$ be the family of oriented graphs obtained by applying Algorithm 1 to H^σ. Then the oriented graph $G_d^\psi = (H \,\Box\, Q_d)^\psi$, $d = 0, 1, 2, \ldots$, of order $n_d = 6 \cdot 2^d$ is skew integral.*

Proof We first prove that the oriented graph $G_0^\psi = H^\sigma$ given in Fig. 4 is skew integral. The skew adjacency matrix of H^σ is

$$S(H^\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

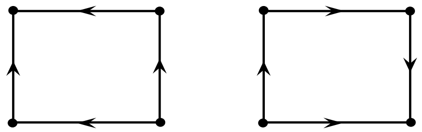
Consider the symmetric matrix $A = S(H^\sigma)^t S(H^\sigma)$,

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 \\ -1 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

Direct computations show that 1 is an eigenvalue of A with eigenvectors $[0\ 1\ -1\ 1\ 0\ 0]^t$ and $[1\ 0\ 0\ 0\ -1\ 1]^t$, and 4 is an eigenvalue of A with eigenvectors $[-1\ 0\ 0\ 0\ 0\ 1]^t$, $[1\ 0\ 0\ 0\ 1\ 0]^t$, $[0\ -1\ 0\ 1\ 0\ 0]^t$, and $[0\ 1\ 1\ 0\ 0\ 0]^t$. Thus, the skew spectrum of the matrix $S(H^\sigma)$ is given by

$$\mathrm{Sp}_S(H^\sigma) = \begin{pmatrix} 2\mathbf{i} & \mathbf{i} & -\mathbf{i} & -2\mathbf{i} \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

As H^σ is skew integral, Theorem 2 shows that $G_d^\psi = (H \,\Box\, Q_d)^\psi$ is also skew integral. $\square$



(a) An evenly oriented cycle (b) An oddly oriented cycle

Fig. 5 Cycle C_4 with two different orientations

Theorem 4 *Given a positive integer $k \geq 3$, there exists a skew symmetric matrix A such that the set $\{0, \pm\mathbf{i}, \pm 2\mathbf{i}, \dots, \pm k\mathbf{i}\}$ forms the set of all distinct eigenvalues of A.*

Proof Proof follows from Theorem 3. Given $k \geq 3$, the skew adjacency matrix of the oriented graph $G_{k-2}^{\psi} = (H \,\Box\, Q_{k-2})^{\psi}$ constructed by Algorithm 1, where H is the underlying graph of the oriented graph H^{σ} given in Fig. 4, is the required matrix. $\square$

Recall that a cycle in an oriented graph G^{σ} need not necessarily be a directed cycle. An even cycle C of G^{σ} is said to be *evenly oriented* (respectively *oddly oriented*) if the number of arcs of C in each direction is even (respectively odd) (see Hou [7]). See Fig. 5.

From Adiga [1] we know that the skew adjacency matrices of both the evenly oriented cycles C_4^{σ} have the same skew spectrum, namely,

$$\mathrm{Sp}_S(C_4^{\sigma}) = \begin{pmatrix} 2\mathbf{i} & 0 & -2\mathbf{i} \\ 1 & 2 & 1 \end{pmatrix}.$$

Applying Algorithm 1 to an evenly oriented cycle C_4^{σ}, we can construct oriented graphs $G_d^{\psi} = (C_4 \,\Box\, Q_d)^{\psi}$, $d = 1, 2, \dots$, with the following special property:

Theorem 5 *For any $d \in \mathbb{N}$, there exists a skew symmetric matrix A for which the set $\{\pm d\mathbf{i}, \pm(d-2)\mathbf{i}, \pm(d-4)\mathbf{i}, \dots, \pm(d-k)\mathbf{i}\}$, where $k = d$ or $d-1$ according as d is even or odd, forms the set of all distinct eigenvalues of A.*

Proof The proof is a consequence of the fact that the numbers given in the statement of the theorem constitute the set of eigenvalues of the skew adjacency matrix of the oriented graph $G_{d-2}^{\psi} = (C_4 \,\Box\, Q_{d-2})^{\psi}$, where C_4^{σ} is an evenly oriented 4-cycle. $\square$

4 Orientation of Hypercubes

The skew energy $\mathscr{E}_S(G^{\sigma})$ of an oriented graph G^{σ} was defined by Adiga et al. [1] as the sum of the absolute values of the eigenvalues of its skew adjacency matrix $S(G^{\sigma})$. This was a generalization of the concept of graph energy defined by Ivan Gutman for undirected graphs. Recall that the energy of a graph is the sum of the absolute values of its eigenvalues (see Gutman [6]). The reader may refer to [1] for the basic properties of skew energy of an oriented graph.

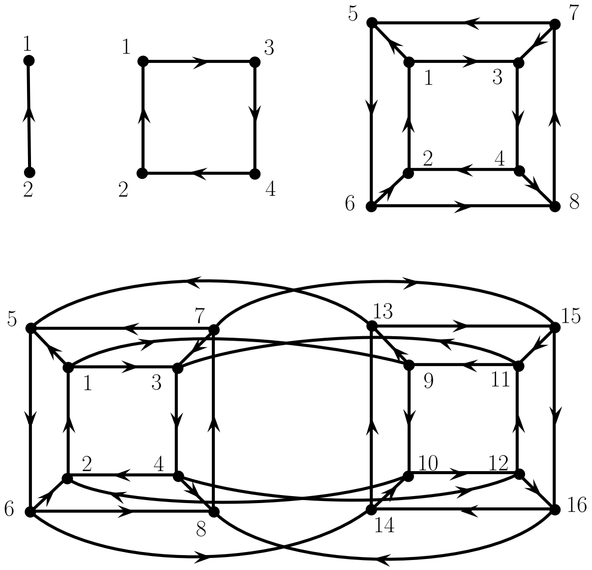


Fig. 6 Orientation ψ of hypercubes Q_1, Q_2, Q_3, and Q_4 as defined in Algorithm 2

Tian [8] has constructed two nonisomorphic orientations of the hypercube Q_d such that the first orientation yields the maximum possible skew energy among all d-regular graphs of order 2^d, namely, $2^d\sqrt{d}$ (see Adiga [1]), while in the second orientation, the skew energy equals the energy of the underlying undirected hypercube. We now give a new orientation ϕ to the edge set of Q_d using the following algorithm and show that for the resulting oriented hypercube Q_d^ϕ, $\mathscr{E}_S(Q_d^\phi) = \mathscr{E}(Q_d)$.

Algorithm 2 The hypercube $Q_d, d \geq 2$ can be constructed by taking two copies of Q_{d-1} and making the corresponding vertices in the two copies adjacent. Let $V(Q_d) = \{1, 2, \ldots, 2^d\}$ be the vertex set of Q_d.

1. For the hypercube $Q_1 = K_2$, set $(2, 1) \in \Gamma(Q_1^\phi)$.
2. Assume that for $i = 1, 2, \ldots, k\ (< d)$, Q_i has been oriented to Q_i^ϕ. For $i = k+1$, the digraph Q_{k+1}^ϕ is formed as follows:
   (a) Take two copies of Q_k^ϕ. Reverse the orientation of all the arcs in the second copy. Assume that the vertex set of the first copy is $\{1, 2, \ldots, 2^k\}$ and the corresponding vertex set of the second copy is $\{2^k + 1, 2^k + 2, \ldots, 2^{k+1}\}$.
   (b) Let $\{j_1, j_2, \ldots, j_{2^{k-1}}\}$, $\{j_{2^{k-1}+1}, j_{2^{k-1}+2}, \ldots, j_{2^k}\}$ be a bipartition of $\{1, 2, \ldots, 2^k\}$ in the first copy. Also let $\{j_{2^k+1}, j_{2^k+2}, \ldots, j_{2^k+2^{k-1}}\}$, $\{j_{2^k+2^{k-1}+1}, j_{2^k+2^{k-1}+2}, \ldots, j_{2^{k+1}}\}$ be the corresponding bipartition of $\{2^k + 1, 2^k + 2, \ldots, 2^{k+1}\}$ in the second copy. Extend ϕ to Q_{k+1} by assigning the orientations $j_r \to j_{2^k+r}$ for $1 \leq r \leq 2^{k-1}$ and $j_{2^k+r} \to j_r$ for $2^{k-1} + 1 \leq r \leq 2^k$ (see Fig. 6).
3. If $k + 1 = d$, stop; else take $k \leftarrow k + 1$, return to Step 2.

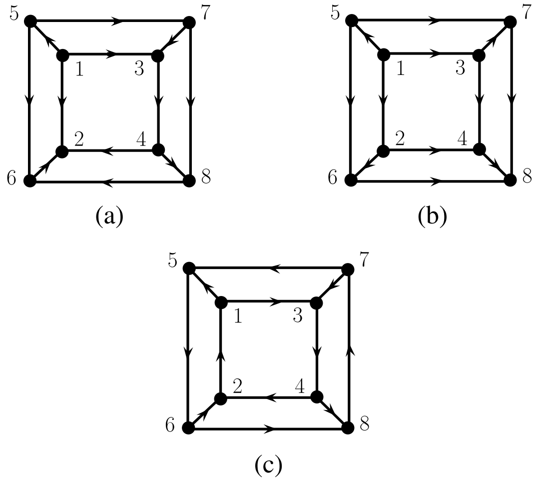


Fig. 7 (**a**)–(**b**) Two nonisomorphic orientations of Q_3 constructed by Tian [8]. (**c**) Oriented hypercube Q_3^ϕ defined in Algorithm 2

In the orientation defined by our Algorithm 2, for every vertex $u \in V(Q_d)$, $|\deg^+(u) - \deg^-(u)| \neq d$. However, in both the orientations defined by Tian [8], there is at least one vertex $v \in V(Q_d)$ such that $|\deg^+(v) - \deg^-(v)| = d$ (see Fig. 7), and hence both of these oriented hypercubes are non-Hamiltonian. The following result shows that the oriented hypercube Q_d^ϕ, $d \geq 2$, defined by our Algorithm 2 is Hamiltonian and hence is nonisomorphic to the two oriented hypercubes defined in [8].

Observation 1 *Algorithm 2 yields a directed Hamilton circuit in Q_d^ϕ.*

Proof Proof is by induction on d. Clearly, Q_2^ϕ is a directed circuit (see Fig. 6). Assume that Q_i^ϕ contains a directed Hamilton circuit for all $i \leq k$. Consider the hypercube Q_{k+1}^ϕ. By Algorithm 2, Q_{k+1}^ϕ contains two copies of Q_k^ϕ where the orientation of all the arcs is reversed in the second copy. Denote the second copy by $Q_k^{\phi'}$. By induction hypothesis, the hypercubes Q_k^ϕ and $Q_k^{\phi'}$ both contain directed Hamilton circuits C and C', respectively, each of length 2^k. By the construction defined in Algorithm 2, there exist at least an arc (u, v) in C and an arc (v', u') in C' such that (u, u') and (v', v) are arcs in Q_{k+1}^ϕ. Then the directed Hamilton paths $(v, \ldots, u)$ in C, $(u', \ldots, v')$ in C', together with the arcs (u, u') and (v', v), form a directed Hamilton circuit in Q_{k+1}^ϕ. □

5 The Skew Energy of the Oriented Hypercube Q_d^ϕ

The skew adjacency matrix of Q_d^ϕ, as defined by Algorithm 2 is

$$S(Q_d^\phi) = \begin{pmatrix} S(Q_{d-1}^\phi) & B_{d-1} \\ -B_{d-1} & -S(Q_{d-1}^\phi) \end{pmatrix}, \quad \text{where}$$
$$B_{d-1} = \operatorname{diag}(b_{1,2^{d-1}+1}, b_{2,2^{d-1}+2}, \ldots, b_{2^{d-1},2^d}),$$

where for $1 \le k \le 2^{d-1}$,

$$b_{k,2^{d-1}+k} = \begin{cases} 1 & \text{if } (k, 2^{d-1}+k) \in \Gamma(Q_d^\phi), \\ -1 & \text{if } (2^{d-1}+k, k) \in \Gamma(Q_d^\phi). \end{cases}$$

Theorem 6 *The skew energy of the hypercube Q_d^ϕ defined in Algorithm 2 is the same as the energy of the underlying hypercube Q_d.*

Proof Proof by induction on d. If $d = 1$, the skew adjacency matrix of Q_1^ϕ defined in Algorithm 2 is given by $S(Q_1^\phi) = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then for the orthogonal matrices $U_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $V_1 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, it is easy to verify that

$$U_1 S(Q_1^\phi) V_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A(Q_1),$$

where $A(Q_1)$ is the adjacency matrix of the underlying graph $Q_1 = K_2$.

If $d = 2$, for the orthogonal matrices $U_2 = \operatorname{diag}(1, -1, 1, -1)$ and $V_2 = \operatorname{diag}(-1, -1, 1, 1)$, $U_2 S(Q_2^\phi) V_2 = A(Q_2)$. The singular values of a real matrix are invariant under multiplication by an orthogonal matrix. Hence, if λ is any eigenvalue of $A(Q_2)$, then $\mathbf{i}\lambda$ is an eigenvalue of the skew symmetric matrix $S(Q_2^\phi)$. Thus, $\operatorname{Sp}_S(Q_2^\phi) = \mathbf{i}\operatorname{Sp}(Q_2)$, and the skew energy of the hypercube Q_2^ϕ defined by means of Algorithm 2 is the same as that of the underlying hypercube Q_2.

We now apply induction on d. Assume that there exist two diagonal matrices U_k, V_k with each diagonal element being either 1 or -1 such that $U_k S(Q_k^\phi) V_k = A(Q_k)$ for $k \le d-1$. Then $\operatorname{Sp}_S(Q_k^\phi) = \mathbf{i}\operatorname{Sp}(Q_k)$ and $\mathscr{E}_S(Q_k^\phi) = \mathscr{E}(Q_k)$ for $k \le d-1$. The skew adjacency matrix of Q_{k+1}^ϕ is

$$S(Q_{k+1}^\phi) = \begin{bmatrix} S(Q_k^\phi) & B_k \\ -B_k & -S(Q_k^\phi) \end{bmatrix},$$

where $B_k = \operatorname{diag}(b_{1,2^k+1}, b_{2,2^k+2}, \ldots, b_{2^k,2^{k+1}})$ is defined by

$$b_{l,2^k+l} = \begin{cases} 1 & \text{if } (l, 2^k+l) \in \Gamma(Q_k^\phi), \\ -1 & \text{if } (2^k+l, l) \in \Gamma(Q_{k+1}^\phi),\ 1 \le l \le 2^k. \end{cases}$$

Let $U_{k+1} = \left[\begin{smallmatrix} U_k & \mathbf{0} \\ \mathbf{0} & U_k \end{smallmatrix}\right]$ and $V_{k+1} = \left[\begin{smallmatrix} V_k & \mathbf{0} \\ \mathbf{0} & -V_k \end{smallmatrix}\right]$, where $\mathbf{0}$ denotes the zero matrix of order 2^k. Let $A(Q_k)$ be the adjacency matrix of Q_k. Now consider the product

$$U_{k+1} S(Q_{k+1}^\phi) V_{k+1} = \begin{bmatrix} U_k & \mathbf{0} \\ \mathbf{0} & U_k \end{bmatrix} \begin{bmatrix} S(Q_k^\phi) & B_k \\ -B_k & -S(Q_k^\phi) \end{bmatrix} \begin{bmatrix} V_k & \mathbf{0} \\ \mathbf{0} & -V_k \end{bmatrix}$$

$$= \begin{bmatrix} U_k S(Q_k^{\phi}) V_k & -U_k B_k V_k \\ -U_k B_k V_k & U_k S(Q_k^{\phi}) V_k \end{bmatrix}$$
$$= \begin{bmatrix} A(Q_k) & -U_k B_k V_k \\ -U_k B_k V_k & A(Q_k) \end{bmatrix},$$

by induction hypothesis. By construction of the diagonal matrices B_k, U_k, V_k, it is easy to verify that $-U_k B_k V_k = I_k$, the identity matrix of order k. (For example, $U_2 = \mathrm{diag}(1, -1, 1, -1)$, $B_2 = \mathrm{diag}(1, -1, -1, 1)$, and $V_2 = \mathrm{diag}(-1, -1, 1, 1)$.) Therefore,

$$U_{k+1} S\big(Q_{k+1}^{\phi}\big) V_{k+1} = \begin{bmatrix} A(Q_k) & I_k \\ I_k & A(Q_k) \end{bmatrix} = A(Q_{k+1}),$$

and hence, $\mathscr{E}_S(Q_{k+1}^{\phi}) = \mathscr{E}(Q_{k+1})$. □

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References

1. Adiga, C., Balakrishnan, R., Wasin, S.: The skew energy of a digraph. Linear Algebra Appl. **432**, 1825–1835 (2010)
2. Balakrishnan, R., Ranganathan, K.: A Textbook of Graph Theory, 2nd edn. Springer, New York (2012)
3. Byrnes, C.I.: Pole placement by output feedback. In: Three Decades of Mathematical Systems Theory. Lecture Notes in Control and Inform. Sci., vol. 135, pp. 31–78. Springer, New York (1989)
4. Cavers, M., Cioabă, S.M., Fallat, S., Gregory, D.A., Haemers, W.H., Kirkland, S.J., McDonald, J.J., Tsatsomeros, M.: Skew-adjacency matrices of graphs. Linear Algebra Appl. **436**, 4512–4529 (2012)
5. Chu, M.T.: Inverse eigenvalue problems. SIAM Rev. **40**, 1–39 (1998)
6. Gutman, I.: The energy of a graph: old and new results. In: Betten, A., Kohner, A., Laue, R., Wassermann, A. (eds.) Algebraic Combinatorics and Applications, pp. 196–211. Springer, Berlin (2001)
7. Hou, Y., Lei, T.: Characteristic polynomials of skew-adjacency matrices of oriented graphs. Electron. J. Comb. **18**, P156 (2011)
8. Tian, G.-X.: On the skew energy of orientations of hypercubes. Linear Algebra Appl. **435**, 2140–2149 (2011)

Notes on Explicit Block Diagonalization

Murali K. Srinivasan

Abstract In these expository notes we present a unified approach to explicit block diagonalization of the commutant of the symmetric group action on the Boolean algebra and of the nonbinary and q-analogs of this commutant.

Keywords Block diagonalization · Symmetric group action · q-Analogue

Mathematics Subject Classification (2010) 06E25

1 Introduction

We present a unified approach to explicit block diagonalization in three classical cases: the commutant of the symmetric group action on the Boolean algebra and the nonbinary and q-analogs of this commutant.

Let $B(n)$ denote the set of all subsets of $[n] = \{1, 2, \dots, n\}$, and, for a prime power q, let $B(q, n)$ denote the set of all subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$. Let $p \geq 2$, and let $A(p)$ denote the alphabet $\{L_0, L_1, \dots, L_{p-1}\}$ with p letters. Define $B_p(n) = \{(a_1, \dots, a_n) : a_i \in A(p)$ for all $i\}$, the set of all n-tuples of elements of $A(p)$ (we use $\{L_0, \dots, L_{p-1}\}$ rather than $\{0, \dots, p-1\}$ as the alphabet for later convenience and do not want to confuse the letter 0 with the vector 0).

Let S_n denote the symmetric group on n letters, and let $S_p(n)$ denote the wreath product $S_{p-1} \sim S_n$. The natural actions of S_n on $B(n)$, $S_p(n)$ on $B_p(n)$ (permute the n coordinates followed by independently permuting the nonzero letters $\{L_1, \dots, L_{p-1}\}$ at each of the n coordinates), and $GL(n, \mathbb{F}_q)$ on $B(q, n)$ have been classical objects of study. Recently, the problem of explicitly block diagonalizing the commutants of these actions has been extensively studied. In these expository notes we revisit these three results. Our main sources are the papers by Schrijver [17], Gijswijt, Schrijver and Tanaka [9], and Terwilliger [22].

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We emphasize that there are several other classical and recent references offering an alternative approach and different perspective on the topic of this paper. We mention Bachoc [2], Silberstein, Scarabotti and Tolli [3], Delsarte [4, 5], Dunkl [6, 7], Go [10], Eisfeld [8], Marco-Parcet [12–14], Tarnanen, Aaltonen and Goethals [21], and Vallentin [23].

In Sect. 2, we recall (without proof) a result of Terwilliger [22] on the singular values of the up operator on subspaces. In Srinivasan [18], the $q = 1$ case of this result, together with binomial inversion, was used to derive Schrijver's [17] explicit block diagonalization of the commutant of the S_n action on $B(n)$. In Sect. 3 we show that the general case of Terwilliger's result, together with q-binomial inversion, yields the explicit block diagonalization of the commutant of the $GL(n, \mathbb{F}_q)$ action on $B(q, n)$. In Sect. 4 we define the concept of upper Boolean decomposition and use it to reduce the explicit block diagonalization of the commutant of the $S_p(n)$ action on $B_p(n)$ to the binary case, i.e., the commutant of the S_n action on $B(n)$. The overall pattern of our proof is the same as in Gijswijt, Schrijver and Tanaka [9], but the concept of upper Boolean decomposition adds useful additional insight to the reduction from the nonbinary to the binary case.

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2 Singular Values

All undefined poset terminology is from Stanley [20]. Let P be a finite *graded poset* with *rank function* $r : P \to \mathbb{N} = \{0, 1, 2, \dots\}$. The *rank* of P is $r(P) = \max\{r(x) : x \in P\}$, and, for $i = 0, 1, \dots, r(P)$, P_i denotes the set of elements of P of rank i. For a subset $S \subseteq P$, we set $\text{rankset}(S) = \{r(x) : x \in S\}$.

For a finite set S, let $V(S)$ denote the complex vector space with S as basis. Let P be a graded poset with $n = r(P)$. Then we have $V(P) = V(P_0) \oplus V(P_1) \oplus \cdots \oplus V(P_n)$ (vector space direct sum). An element $v \in V(P)$ is *homogeneous* if $v \in V(P_i)$ for some i, and if $v \neq 0$, we extend the notion of rank to nonzero homogeneous elements by writing $r(v) = i$. Given an element $v \in V(P)$, we write

$$v = v_0 + \cdots + v_n, \quad v_i \in V(P_i),\ 0 \le i \le n.$$

We refer to the v_i as the *homogeneous components* of v. A subspace $W \subseteq V(P)$ is *homogeneous* if it contains the homogeneous components of each of its elements. For a homogeneous subspace $W \subseteq V(P)$, we set

$$\text{rankset}(W) = \big\{r(v) : v \text{ is a nonzero homogeneous element of } W\big\}.$$

The *up operator* $U : V(P) \to V(P)$ is defined, for $x \in P$, by $U(x) = \sum_y y$, where the sum is over all y covering x. Similarly, the *down operator* $D : V(P) \to$

$V(P)$ is defined, for $x \in P$, by $D(x) = \sum_y y$, where the sum is over all y covered by x.

Let $\langle , \rangle$ denote the standard inner product on $V(P)$, i.e., $\langle x, y\rangle = \delta(x, y)$ (Kronecker delta) for $x, y \in P$. The *length* $\sqrt{\langle v, v\rangle}$ of $v \in V(P)$ is denoted $\|v\|$. In this paper we study three graded posets. The *Boolean algebra* is the rank-n graded poset obtained by partially ordering $B(n)$ by inclusion. The rank of a subset is given by cardinality. The q-*analog* of the Boolean algebra is the rank-n graded poset obtained by partially ordering $B(q, n)$ by inclusion. The rank of a subspace is given by dimension. We recall that, for $0 \le k \le n$, the q-*binomial coefficient*

$$\binom{n}{k}_q = \frac{(1)_q(2)_q \cdots (n)_q}{(1)_q \cdots (k)_q(1)_q \cdots (n-k)_q},$$

where $(i)_q = 1 + q + q^2 + \cdots + q^{i-1}$, denotes the cardinality of $B(q, n)_k$.

Given $a = (a_1, \ldots, a_n) \in B_p(n)$, define the *support* of a by $S(a) = \{i \in \{1, \ldots, n\} : a_i \neq L_0\}$. For $b = (b_1, \ldots, b_n) \in B_p(n)$, define $a \le b$ if $S(a) \subseteq S(b)$ and $a_i = b_i$ for all $i \in S(a)$. It is easy to see that this makes $B_p(n)$ into a rank-n graded poset with rank of a given by $|S(a)|$. We call $B_p(n)$ the *nonbinary analog* of the Boolean algebra $B(n)$. Clearly, when $p = 2$, $B_p(n)$ is order isomorphic to $B(n)$.

We give $V(B(n))$, $V(B_p(n))$, and $V(B(q, n))$, the standard inner products. We use U to denote the up operator on all three of the posets $V(B(n))$, $V(B_p(n))$, and $V(B(q, n))$ and do not indicate the rank n (as in U_n, say) in the notation for U. The meaning of the symbol U is always clear from the context.

Let P be a graded poset. A *graded Jordan chain* in $V(P)$ is a sequence

$$s = (v_1, \ldots, v_h) \tag{1}$$

of nonzero homogeneous elements of $V(P)$ such that $U(v_{i-1}) = v_i$ for $i = 2, \ldots, h$ and $U(v_h) = 0$ (note that the elements of this sequence are linearly independent, being nonzero, and of different ranks). We say that s *starts* at rank $r(v_1)$ and *ends* at rank $r(v_h)$. A *graded Jordan basis* of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of graded Jordan chains in $V(P)$.

The graded Jordan chain (1) is said to be a *symmetric Jordan chain* (SJC) if the sum of the starting and ending ranks of s equals $r(P)$, i.e., $r(v_1) + r(v_h) = r(P)$ if $h \ge 2$ or $2r(v_1) = r(P)$ if $h = 1$. A *symmetric Jordan basis* (SJB) of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of symmetric Jordan chains in $V(P)$.

The graded Jordan chain (1) is said to be a *semisymmetric Jordan chain* (SSJC) if the sum of the starting and ending ranks of s is $\ge r(P)$. A *semisymmetric Jordan basis* (SSJB) of $V(P)$ is a basis of $V(P)$ consisting of a disjoint union of semisymmetric Jordan chains in $V(P)$. An SSJB is said to be *rank complete* if it contains graded Jordan chains starting at rank i and ending at rank j for all $0 \le i \le j \le r(P), i + j \ge r(P)$.

Suppose that we have an orthogonal graded Jordan basis O of $V(P)$. Normalize the vectors in O to get an orthonormal basis O'. Let $(v_1, \ldots, v_h)$ be a graded Jordan chain in O. Put $v'_u = \frac{v_u}{\|v_u\|}$ and $\alpha_u = \frac{\|v_{u+1}\|}{\|v_u\|}$, $1 \le u \le h$ (we take $v'_0 = v'_{h+1} = 0$). We have, for $1 \le u \le h$,

$$U\big(v'_u\big) = \frac{U(v_u)}{\|v_u\|} = \frac{v_{u+1}}{\|v_u\|} = \alpha_u v'_{u+1}. \tag{2}$$

Thus, the matrix of U with respect to O' is in block diagonal form, with a block corresponding to each (normalized) graded Jordan chain in O and with the block corresponding to $(v'_1, \ldots, v'_h)$ above being a lower triangular matrix with subdiagonal $(\alpha_1, \ldots, \alpha_{h-1})$ and 0s elsewhere.

Now note that the matrices, in the standard basis P, of U and D are real and transposes of each other. Since O' is orthonormal with respect to the standard inner product, it follows that the matrices of U and D, in the basis O', must be adjoints of each other. Thus, the matrix of D with respect to O' is in block diagonal form, with a block corresponding to each (normalized) graded Jordan chain in O and with the block corresponding to $(v'_1, \ldots, v'_h)$ above being an upper triangular matrix with superdiagonal $(\alpha_1, \ldots, \alpha_{h-1})$ and 0s elsewhere. Thus, for $0 \le u \le h-1$, we have

$$D\big(v'_{u+1}\big) = \alpha_u v'_u. \tag{3}$$

It follows that the subspace spanned by each graded Jordan chain in O is closed under U and D. We use (2) and (3) without explicit mention in a few places.

The following result is due to Terwilliger [22], whose proof is based on the results of Dunkl [7]. For a proof based on Proctor's [15] $\mathfrak{sl}(2, \mathbb{C})$ method, see Srinivasan [19].

Theorem 1 *There exists an SJB $J(q,n)$ of $V(B(q,n))$ such that*

(i) *The elements of $J(q,n)$ are orthogonal with respect to $\langle , \rangle$ (the standard inner product).*
(ii) (Singular values) *Let $0 \le k \le n/2$, and let $(x_k, \ldots, x_{n-k})$ be any SJC in $J(q,n)$ starting at rank k and ending at rank $n-k$. Then we have, for $k \le u < n-k$,*

$$\frac{\|x_{u+1}\|}{\|x_u\|} = \sqrt{q^k (u+1-k)_q (n-k-u)_q}. \tag{4}$$

Let $J'(q,n)$ denote the orthonormal basis of $V(B(q,n))$ obtained by normalizing $J(q,n)$.

Substituting $q = 1$ in Theorem 1, we get the following result.

Theorem 2 *There exists an SJB $J(n)$ of $V(B(n))$ such that*

(i) *The elements of $J(n)$ are orthogonal with respect to $\langle , \rangle$ (the standard inner product).*
(ii) (Singular values) *Let $0 \le k \le n/2$, and let $(x_k, \ldots, x_{n-k})$ be any SJC in $J(n)$ starting at rank k and ending at rank $n-k$. Then we have, for $k \le u < n-k$,*

$$\frac{\|x_{u+1}\|}{\|x_u\|} = \sqrt{(u+1-k)(n-k-u)}. \tag{5}$$

Let $J'(n)$ denote the orthonormal basis of $V(B(n))$ obtained by normalizing $J(n)$.

Theorem 2 was proved by Go [10] using the $\mathfrak{sl}(2,\mathbb{C})$ method. For an explicit construction of an orthogonal SJB $J(n)$, together with a representation theoretic interpretation, see Srinivasan [18]. It would be interesting to give an explicit construction of an orthogonal SJB $J(q,n)$ of $V(B(q,n))$.

3 q-Analog of $\mathrm{End}_{S_n}(V(B(n)))$

We represent elements of $\mathrm{End}(V(B(q,n)))$ (in the standard basis) as $B(q,n) \times B(q,n)$ matrices (we think of elements of $V(B(q,n))$ as column vectors with coordinates indexed by $B(q,n)$). For $X, Y \in B(q,n)$, the entry in row X, column Y of a matrix M will be denoted $M(X,Y)$. The matrix corresponding to $f \in \mathrm{End}(V(B(q,n)))$ is denoted M_f. We use similar notations for $B(q,n)_i \times B(q,n)_i$ matrices corresponding to elements of $\mathrm{End}(V(B(q,n)_i))$. The finite group $G(q,n) = GL(n,\mathbb{F}_q)$ has a rank and order-preserving action on $B(q,n)$. Set

$$\mathscr{A}(q,n) = \left\{M_f : f \in \mathrm{End}_{G(q,n)}\left(V\left(B(q,n)\right)\right)\right\},$$
$$\mathscr{B}(q,n,i) = \left\{M_f : f \in \mathrm{End}_{G(q,n)}\left(V\left(B(q,n)_i\right)\right)\right\}.$$

Thus, $\mathscr{A}(q,n)$ and $\mathscr{B}(q,n,i)$ are $*$-algebras of matrices.

Let $f : V(B(q,n)) \to V(B(q,n))$ be linear, and $g \in G(q,n)$. Then

$$f\left(g(Y)\right) = \sum_X M_f\left(X, g(Y)\right)X \quad \text{and} \quad g\left(f(Y)\right) = \sum_X M_f(X,Y)g(X).$$

It follows that f is $G(q,n)$-linear if and only if

$$M_f(X,Y) = M_f\left(g(X), g(Y)\right), \quad \text{for all } X, Y \in B(q,n), g \in G(q,n), \tag{6}$$

i.e., M_f is constant on the orbits of the action of $G(q,n)$ on $B(q,n) \times B(q,n)$.

Now it is easily seen that $(X,Y), (X',Y') \in B(q,n) \times B(q,n)$ are in the same $G(q,n)$-orbit if and only if

$$\begin{aligned} &\dim(X) = \dim\left(X'\right), \quad \dim(Y) = \dim\left(Y'\right), \quad \text{and} \\ &\dim(X \cap Y) = \dim\left(X' \cap Y'\right). \end{aligned} \tag{7}$$

For $0 \le i, j, t \le n$, let $M_{i,j}^t$ be the $B(q,n) \times B(q,n)$ matrix given by

$$M_{i,j}^t(X,Y) = \begin{cases} 1 & \text{if } \dim(X) = i, \dim(Y) = j, \dim(X \cap Y) = t, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\left\{M_{i,j}^t | i + j - t \le n, 0 \le t \le i, j\right\}$$

is a basis of $\mathscr{A}(q,n)$, and its cardinality is $\binom{n+3}{3}$.

Let $0 \le i \le n$. Consider the $G(q,n)$-action on $V(B(q,n)_i)$. Given $X, Y \in B(q,n)_i$, it follows from (7) that the pairs (X,Y) and (Y,X) are in the same orbit of the $G(q,n)$-action on $B(q,n)_i \times B(q,n)_i$. It thus follows from (6) that the algebra $\mathscr{B}(q,n,i)$ has a basis consisting of symmetric matrices and is hence commutative. Thus, $V(B(q,n)_i)$ is a multiplicity free $G(q,n)$-module, and the $*$-algebra $\mathscr{B}(q,n,i)$ can be diagonalized. We now consider the more general problem of block diagonalizing the $*$-algebra $\mathscr{A}(q,n)$.

Fix $i, j \in \{0,\dots,n\}$. Then we have

$$M_{i,t}^t M_{t,j}^t = \sum_{u=0}^{n} \binom{u}{t}_q M_{i,j}^u, \quad t = 0,\dots,n,$$

since the entry of the left-hand side in row X, column Y with $\dim(X)=i$, $\dim(Y)=j$ is equal to the number of common subspaces of X and Y of size t. Apply q-binomial inversion (see Exercise 2.47 in Aigner [1]) to get

$$M_{i,j}^t = \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q M_{i,u}^u M_{u,j}^u, \quad t = 0,\dots,n. \tag{8}$$

Before proceeding further, we observe that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k+1)^2 = \binom{n+3}{3} = \dim \mathscr{A}(q,n) \tag{9}$$

since both sides (of the first equality) are polynomials in r (treating the cases $n=2r$ and $n=2r+1$ separately) of degree 3 and agree for $r = 0, 1, 2, 3$.

For $i, j, k, t \in \{0,\dots,n\}$, define

$$\beta_{i,j,k}^{n,t}(q) = \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2} - ku} \binom{u}{t}_q \binom{n-2k}{u-k}_q \binom{n-k-u}{i-u}_q \binom{n-k-u}{j-u}_q.$$

For $0 \le k \le \lfloor n/2 \rfloor$ and $k \le i, j \le n-k$, define $E_{i,j,k}$ to be the $(n-2k+1) \times (n-2k+1)$ matrix, with rows and columns indexed by $\{k, k+1, \dots, n-k\}$, and with entry in row i and column j equal to 1 and all other entries 0. Let $\mathrm{Mat}(n \times n)$ denote the algebra of complex $n \times n$ matrices.

In the proof of the next result we will need the following alternate expression for the singular values:

$$\sqrt{q^k (u+1-k)_q (n-k-u)_q} = q^{\frac{k}{2}} (n-k-u)_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{u+1-k}_q^{-\frac{1}{2}}. \tag{10}$$

We now present a q-analog of the explicit block diagonalization of $\mathrm{End}_{S_n}(V(B(n)))$ given by Schrijver [17].

Theorem 3 *Let $J(q,n)$ be an orthogonal SJB of $V(B(q,n))$ satisfying the conditions of Theorem 1. Define a $B(q,n) \times J'(q,n)$ unitary matrix $N(n)$ as follows: for $v \in J'(q,n)$, the column of $N(n)$ indexed by v is the coordinate vector of v in the standard basis $B(q,n)$. Then*

(i) *$N(n)^*\mathscr{A}(q,n)N(n)$ consists of all $J'(q,n)\times J'(q,n)$ block diagonal matrices with a block corresponding to each (normalized) SJC in $J(q,n)$ and any two SJCs starting and ending at the same rank give rise to identical blocks. Thus, there are $\binom{n}{k}_q-\binom{n}{k-1}_q$ identical blocks of size $(n-2k+1)\times(n-2k+1)$, for $k=0,\dots,\lfloor n/2\rfloor$.*

(ii) *Conjugating by $N(n)$ and dropping duplicate blocks thus gives a positive semidefiniteness-preserving C^*-algebra isomorphism*

$$\Phi:\mathscr{A}(q,n)\cong\bigoplus_{k=0}^{\lfloor n/2\rfloor}\mathrm{Mat}\big((n-2k+1)\times(n-2k+1)\big).$$

It will be convenient to reindex the rows and columns of a block corresponding to a SJC starting at rank k and ending at rank $n-k$ by the set $\{k,k+1,\dots,n-k\}$. Let $i,j,t\in\{0,\dots,n\}$. Write

$$\Phi\big(M^t_{i,j}\big)=(N_0,\dots,N_{\lfloor n/2\rfloor}).$$

Then, for $0\le k\le\lfloor n/2\rfloor$,

$$N_k=\begin{cases} q^{\frac{k(i+j)}{2}}\binom{n-2k}{i-k}_q^{-\frac12}\binom{n-2k}{j-k}_q^{-\frac12}\beta^{n,t}_{i,j,k}(q)E_{i,j,k} & \text{if } k\le i,\ j\le n-k,\\ 0 & \text{otherwise.}\end{cases}$$

Proof (i) Let $i\ge u$, and let $Y\subseteq X$ with $X\in B(q,n)_i$ and $Y\in B(q,n)_u$. The number of chains of subspaces $X_u\subseteq X_{u+1}\subseteq\cdots\subseteq X_i$ with $X_u=Y$, $X_i=X$, and $\dim(X_l)=l$ for $u\le l\le i$ is clearly $(i-u)_q(i-u-1)_q\cdots(1)_q$. Thus, the action of $M^u_{i,u}$ on $V(B(q,n)_u)$ is $\frac{1}{(i-u)_q(i-u-1)_q\cdots(1)_q}$ times the action of U^{i-u} on $V(B(q,n)_u)$.

Now the subspace spanned by each SJC in $J(q,n)$ is closed under U and D. It thus follows by (8) that the subspace spanned by each SJC in $J(q,n)$ is closed under $\mathscr{A}(q,n)$. The result now follows from Theorem 1(ii) and the dimension count (9).

(ii) Fix $0\le k\le\lfloor n/2\rfloor$. If both i,j are not elements of $\{k,\dots,n-k\}$, then clearly $N_k=0$. So we may assume that $k\le i,\ j\le n-k$. Clearly, $N_k=\lambda E_{i,j,k}$ for some λ. We now find $\lambda=N_k(i,j)$.

Let $u\in\{0,\dots,n\}$. Write $\Phi(M^u_{i,u})=(A^u_0,\dots,A^u_{\lfloor n/2\rfloor})$. We claim that

$$A^u_k=\begin{cases} q^{\frac{k(i-u)}{2}}\binom{n-k-u}{i-u}_q\binom{n-2k}{u-k}_q^{\frac12}\binom{n-2k}{i-k}_q^{-\frac12}E_{i,u,k} & \text{if } k\le u\le n-k,\\ 0 & \text{otherwise.}\end{cases}$$

The otherwise part of the claim is clear. If $k\le u\le n-k$ and $i<u$, then we have $A^u_k=0$. This also follows from the right-hand side since the q-binomial coefficient $\binom{a}{b}_q$ is 0 for $b<0$. So we may assume that $k\le u\le n-k$ and $i\ge u$. Clearly, in this case we have $A^u_k=\alpha E_{i,u,k}$ for some α. We now determine $\alpha=A^u_k(i,u)$. We have, using Theorem 1(ii) and expression (10),

$$A^u_k(i,u)=\frac{\prod_{w=u}^{i-1}\{q^{\frac{k}{2}}(n-k-w)_q\binom{n-2k}{w-k}_q^{\frac12}\binom{n-2k}{w+1-k}_q^{-\frac12}\}}{(i-u)_q(i-u-1)_q\cdots(1)_q}$$

$$= q^{\frac{k(i-u)}{2}} \binom{n-k-u}{i-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}}.$$

Similarly, if we write $\Phi(M^u_{u,j}) = (B^u_0, \ldots, B^u_{\lfloor n/2 \rfloor})$, then we have

$$B^u_k = \begin{cases} q^{\frac{k(j-u)}{2}} \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} E_{u,j,k} & \text{if } k \leq u \leq n-k, \\ 0 & \text{otherwise.} \end{cases}$$

So, from (8) we have

$$N_k = \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k B^u_k = \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k B^u_k.$$

Thus,

$$\begin{aligned} N_k(i,j) &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q \left\{ \sum_{l=k}^{n-k} A^u_k(i,l) B^u_k(l,j) \right\} \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q A^u_k(i,u) B^u_k(u,j) \\ &= \sum_{u=k}^{n-k} (-1)^{u-t} q^{\binom{u-t}{2}} \binom{u}{t}_q q^{\frac{k(i-u)}{2}} \binom{n-k-u}{i-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} \\ &\quad \times q^{\frac{k(j-u)}{2}} \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q^{\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} \\ &= q^{\frac{k(i+j)}{2}} \binom{n-2k}{i-k}_q^{-\frac{1}{2}} \binom{n-2k}{j-k}_q^{-\frac{1}{2}} \left\{ \sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}-ku} \binom{u}{t} \right. \\ &\quad \left. \times \binom{n-k-u}{i-u}_q \binom{n-k-u}{j-u}_q \binom{n-2k}{u-k}_q \right\}, \end{aligned}$$

completing the proof. □

We now explicitly diagonalize $\mathscr{B}(q,n,i)$. Let $0 \leq i \leq n$. We set $i^- = \max\{0, 2i-n\}$ and $m(i) = \min\{i, n-i\}$ (note that i^- and $m(i)$ depend on both i and n. The n will always be clear from the context). It follows from (6) that $\mathscr{B}(q,n,i)$ has a basis consisting of $M^t_{i,i}$, for $i^- \leq t \leq i$ (here we think of $M^t_{i,i}$ as $B(q,n)_i \times B(q,n)_i$ matrices). The cardinality of this basis is $1+m(i)$. Since $\mathscr{B}(q,n,i)$ is commutative, it follows that $V(B(q,n)_i)$ is a canonical orthogonal direct sum of $1+m(i)$ common eigenspaces of the $M^t_{i,i}$, $i^- \leq t \leq i$ (these eigenspaces are the irreducible $G(q,n)$-submodules of $V(B(q,n)_i)$).

Let $J(q,n)$ be an orthogonal SJB of $V(B(q,n))$ satisfying the conditions of Theorem 1. For $k = 0, 1, \ldots, m(i)$, define

$$J(q,n,i,k) = \{v \in J(q,n) : r(v) = i, \text{ and the Jordan chain containing } v \\ \text{starts at rank } k\}.$$

Let $W(q,n,i,k)$ be the subspace spanned by $J(q,n,i,k)$ (note that this subspace is nonzero). We have an orthogonal direct sum decomposition

$$V\big(B(q,n)_i\big) = \bigoplus_{k=0}^{m(i)} W(q,n,i,k).$$

It now follows from Theorem 3 that the $W(q,n,i,k)$ are the common eigenspaces of the $M^t_{i,i}$. The following result is due to Delsarte [4].

Theorem 4 *Let $0 \le i \le n$. For $i^- \le t \le i$ and $0 \le k \le m(i)$, the eigenvalue of $M^t_{i,i}$ on $W(q,n,i,k)$ is*

$$\sum_{u=0}^{n} (-1)^{u-t} q^{\binom{u-t}{2}+k(i-u)} \binom{u}{t}_q \binom{n-k-u}{i-u}_q \binom{i-k}{i-u}_q .$$

Proof Follows from substituting $j = i$ in Theorem 3 and noting that

$$\binom{n-2k}{i-k}_q^{-1} \binom{n-2k}{u-k}_q \binom{n-k-u}{i-u}_q = \binom{i-k}{i-u}_q . \qquad \square$$

Set

$$\mathscr{A}(n) = \{M_f : f \in \mathrm{End}_{S_n}\big(V\big(B(n)\big)\big)\},$$

and for $i, j, k, t \in \{0, \dots, n\}$, define

$$\beta^{n,t}_{i,j,k} = \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t} \binom{n-2k}{u-k} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u}.$$

Substituting $q = 1$ in Theorem 3, we get the following result of Schrijver [17]. We shall use this result in the next section.

Theorem 5 *Let $J(n)$ be an orthogonal SJB of $V(B(n))$ satisfying the conditions of Theorem 2. Define a $B(n) \times J'(n)$ unitary matrix $N(n)$ as follows: for $v \in J'(n)$, the column of $N(n)$ indexed by v is the coordinate vector of v in the standard basis $B(n)$. Then*

(i) *$N(n)^* \mathscr{A}(n) N(n)$ consists of all $J'(n) \times J'(n)$ block diagonal matrices with a block corresponding to each (normalized) SJC in $J(n)$ and any two SJCs starting and ending at the same rank give rise to identical blocks. Thus, there are $\binom{n}{k} - \binom{n}{k-1}$ identical blocks of size $(n-2k+1) \times (n-2k+1)$, for $k = 0, \dots, \lfloor n/2 \rfloor$.*

(ii) *Conjugating by $N(n)$ and dropping duplicate blocks thus gives a positive semidefiniteness-preserving C^*-algebra isomorphism*

$$\Phi : \mathscr{A}(n) \cong \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathrm{Mat}\big((n-2k+1) \times (n-2k+1)\big).$$

It will be convenient to reindex the rows and columns of a block corresponding to a SJC starting at rank k and ending at rank $n-k$ *by the set* $\{k, k+1, \dots, n-k\}$. *Let* $i, j, t \in \{0, \dots, n\}$. *Write*

$$\Phi\left(M_{i,j}^t\right) = (N_0, \dots, N_{\lfloor n/2 \rfloor}).$$

Then, for $0 \le k \le \lfloor n/2 \rfloor$,

$$N_k = \begin{cases} \binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \beta_{i,j,k}^{n,t} E_{i,j,k} & \text{if } k \le i, j \le n-k, \\ 0 & \text{otherwise}. \end{cases}$$

4 Nonbinary Analog of $\mathrm{End}_{S_n}(V(B(n)))$

Let (V, f) be a pair consisting of a finite-dimensional inner product space V (over $\mathbb{C}$) and a linear operator f on V. Let (W, g) be another such pair. By an isomorphism of pairs (V, f) and (W, g) we mean a linear isometry (i.e, an inner product-preserving isomorphism) $\theta : V \to W$ such that $\theta(f(v)) = g(\theta(v))$, $v \in V$.

Consider the inner product space $V(B_p(n))$. An *upper Boolean subspace* of rank t is a homogeneous subspace $W \subseteq V(B_p(n))$ such that rankset$(W) = \{n-t, n-t+1, \dots, n\}$, W is closed under the up operator U, and there is an isomorphism of pairs $(V(B(t)), \sqrt{p-1}U) \cong (W, U)$ that sends homogeneous elements to homogeneous elements and increases rank by $n-t$.

Consider the following identity:

$$p^n = (p-2+2)^n = \sum_{l=0}^{n} \binom{n}{l} (p-2)^l 2^{n-l}. \tag{11}$$

We shall now give a linear algebraic interpretation to the identity above. For simplicity, we denote the inner product space $V(A(p))$, with $A(p)$ as an orthonormal basis, by $V(p)$. Make the tensor product

$$\bigotimes_{i=1}^{n} V(p) = V(p) \otimes \cdots \otimes V(p) \ (n \text{ factors})$$

into an inner product space by defining

$$\langle v_1 \otimes \cdots \otimes v_n, u_1 \otimes \cdots \otimes u_n \rangle = \langle v_1, u_1 \rangle \cdots \langle v_n, u_n \rangle. \tag{12}$$

There is an isometry

$$V\left(B_p(n)\right) \cong \bigotimes_{i=1}^{n} V(p) \tag{13}$$

given by $a = (a_1, \dots, a_n) \mapsto \bar{a} = a_1 \otimes \cdots \otimes a_n$, $a \in B_p(n)$. The rank function (on nonzero homogeneous elements) and the up and down operators, U and D, on $V(B_p(n))$ are transferred to $\bigotimes_{i=1}^{n} V(p)$ via the isomorphism above.

Fix a $(p-1)\times(p-1)$ unitary matrix $P=(m_{ij})$ with rows and columns indexed by $\{1,2,\ldots,p-1\}$ and with first row $\frac{1}{\sqrt{p-1}}(1,1,\ldots,1)$. For $i=1,\ldots,p-1$, define the vector $w_i\in V(p)$ by

$$w_i=\sum_{j=1}^{p-1} m_{ij}L_j. \tag{14}$$

Note that $w_1=\frac{1}{\sqrt{p-1}}(L_1+\cdots+L_{p-1})$ and that, for $i=2,\ldots,p-1$, the sum $\sum_{j=1}^{p-1} m_{ij}$ of the elements of row i of P is 0. Thus we have, in $V(p)$,

$$U(w_i)=D(w_i)=0,\quad i=2,\ldots,p-1, \tag{15}$$

$$U(w_1)=D(L_0)=0, \tag{16}$$

$$U(L_0)=\sqrt{p-1}w_1,\qquad D(w_1)=\sqrt{p-1}L_0. \tag{17}$$

Set

$$\mathscr{S}_p(n)=\big\{(A,f):A\subseteq[n],\ f:A\to\{2,\ldots,p-1\}\big\}, \tag{18}$$

$$\mathscr{K}_p(n)=\big\{(A,f,B):(A,f)\in\mathscr{S}_p(n),\ B\subseteq[n]-A\big\}. \tag{19}$$

Note that

$$\big|\mathscr{S}_p(n)\big|=\sum_{l=0}^{n}\binom{n}{l}(p-2)^l,\qquad \big|\mathscr{K}_p(n)\big|=\sum_{l=0}^{n}\binom{n}{l}(p-2)^l 2^{n-l}.$$

For $(A,f,B)\in\mathscr{K}_p(n)$, define a vector $v(A,f,B)=v_1\otimes\cdots\otimes v_n\in\bigotimes_{i=1}^n V(p)$ by

$$v_i=\begin{cases} w_{f(i)} & \text{if } i\in A,\\ w_1 & \text{if } i\in B,\\ L_0 & \text{if } i\in[n]-(A\cup B).\end{cases}$$

Note that $v(A,f,B)$ is a homogeneous vector in $\bigotimes_{i=1}^n V(p)$ of rank $|A|+|B|$. For $(A,f)\in\mathscr{S}_p(n)$, define $V_{(A,f)}$ to be the subspace of $\bigotimes_{i=1}^n V(p)$ spanned by the set $\{v(A,f,B):B\subseteq[n]-A\}$. Set $K_p(n)=\{v(A,f,B):(A,f,B)\in\mathscr{K}_p(n)\}$.

We have, using (15), (16), and (17), the following formula in $\bigotimes_{i=1}^n V(p)$:

$$U\big(v(A,f,B)\big)=\sqrt{p-1}\Big\{\sum_{B'} v\big(A,f,B'\big)\Big\}, \tag{20}$$

where the sum is over all $B'\subseteq([n]-A)$ covering B.

It follows from the unitariness of P and the inner product formula (12) that

$$\big\langle v(A,f,B),v\big(A',f',B'\big)\big\rangle=\delta\big((A,f,B),\big(A',f',B'\big)\big), \tag{21}$$

where $(A,f,B),(A',f',B')\in\mathscr{K}_p(n)$.

We can summarize the discussion above in the following result.

Theorem 6

(i) *$K_p(n)$ is an orthonormal basis of $\bigotimes_{i=1}^n V(p)$.*
(ii) *For $(A, f) \in \mathscr{S}_p(n)$, $V_{(A,f)}$ is an upper Boolean subspace of $\bigotimes_{i=1}^n V(p)$ of rank $n - |A|$ and with orthonormal basis $\{v(A, f, B) : B \subseteq [n] - A\}$.*
(iii) *We have the following orthogonal decomposition into upper Boolean subspaces*:

$$\bigotimes_{i=1}^{n} V(p) = \bigoplus_{(A,f)\in\mathscr{S}_p(n)} V_{(A,f)}, \tag{22}$$

with the right-hand side having $(p-2)^l\binom{n}{l}$ upper Boolean subspaces of rank $n-l$, for each $l = 0, 1, \ldots, n$.

Certain nonbinary problems can be reduced to the corresponding binary problems via the basis $K_p(n)$. We now consider two examples of this (Theorems 7 and 9 below).

For $0 \le k \le n$, note that $0 \le k^- \le k$ and $k \le n + k^- - k$. For an SSJC c in $V(B_p(n))$, starting at rank i and ending at rank j, we define the *offset* of c to be $i + j - n$. It is easy to see that if an SSJC starts at rank k, then its offset l satisfies $k^- \le l \le k$, and the chain ends at rank $n + l - k$. For $0 \le k \le n$ and $k^- \le l \le k$, set

$$\mu(n, k, l) = (p-2)^l \binom{n}{l}\left\{\binom{n-l}{k-l} - \binom{n-l}{k-l-1}\right\}.$$

The following result is due to Terwilliger [22].

Theorem 7 *There exists a rank complete SSJB $J_p(n)$ of $V(B_p(n))$ such that*

(i) *The elements of $J_p(n)$ are orthogonal with respect to $\langle,\rangle$ (the standard inner product).*
(ii) (Singular values) *Let $0 \le k \le n$, $k^- \le l \le k$, and let $(x_k, \ldots, x_{n+l-k})$ be any SSJC in $J_p(n)$ starting at rank k and having offset l. Then we have, for $k \le u < n + l - k$,*

$$\frac{\|x_{u+1}\|}{\|x_u\|} = \sqrt{(p-1)(u+1-k)(n+l-k-u)}. \tag{23}$$

(iii) *Let $0 \le k \le n$ and $k^- \le l \le k$. Then $J_p(n)$ contains $\mu(n, k, l)$ SSJCs starting at rank k and having offset l.*

Proof Let $V_{(A,f)}$, with $|A| = l$, be an upper Boolean subspace of rank $n - l$ in the decomposition (22). Let $\gamma : \{1, 2, \ldots, n-l\} \to [n] - A$ be the unique order-preserving bijection, i.e., $\gamma(i) = i$th smallest element of $[n] - A$. Denote by $\Gamma : V(B(n-l)) \to V_{(A,f)}$ the linear isometry given by $\Gamma(X) = v(A, f, \gamma(X))$, $X \in B(n-l)$.

Use Theorem 2 to get an orthogonal SJB $J(n-l)$ of $V(B(n-l))$ with respect to $\sqrt{p-1}U$ (rather than just U) and transfer it to $V_{(A,f)}$ via Γ. Each SJC in $J(n-l)$

will get transferred to a SSJC in $\bigotimes_{i=1}^{n} V(p)$ of offset l, and, using (5), we see that this SSJC will satisfy (23). The number of these SSJCs (in $V_{(A,f)}$) starting at rank k is $\binom{n-l}{k-l} - \binom{n-l}{k-l-1}$.

Doing this for every upper Boolean subspace in the decomposition (22), we get an orthogonal SSJB of $\bigotimes_{i=1}^{n} V(p)$. Transferring via the isometry (13), we get an orthogonal SSJB $J_p(n)$ of $V(B_p(n))$ satisfying (23). Since the number of upper Boolean subspaces in the decomposition (22) of rank $n-l$ is $(p-2)^l \binom{n}{l}$, Theorem 7 now follows. □

Denote by $J'_p(n)$ the orthonormal basis of $V(B_p(n))$ obtained by normalizing $J_p(n)$.

We represent elements of $\mathrm{End}(V(B_p(n)))$ (in the standard basis) as $B_p(n) \times B_p(n)$ matrices. Our notation for these matrices is similar to that used in the previous section. The group $S_p(n)$ has a rank- and order-preserving action on $B_p(n)$. Set

$$\mathscr{A}_p(n) = \left\{ M_f : f \in \mathrm{End}_{S_p(n)}\left(V\left(B_p(n)\right)\right) \right\},$$
$$\mathscr{B}_p(n,i) = \left\{ M_f : f \in \mathrm{End}_{S_p(n)}\left(V\left(B_p(n)_i\right)\right) \right\}.$$

Thus, $\mathscr{A}_p(n)$ and $\mathscr{B}_p(n,i)$ are $*$-algebras of matrices.

Let $f : V(B_p(n)) \to V(B_p(n))$ be linear, and $\pi \in S_p(n)$. Then f is $S_p(n)$-linear if and only if

$$M_f(a,b) = M_f\big(\pi(a), \pi(b)\big) \quad \text{for all } a, b \in B_p(n), \pi \in S_p(n), \tag{24}$$

i.e., M_f is constant on the orbits of the action of $S_p(n)$ on $B_p(n) \times B_p(n)$. Now it is easily seen that $(a,b), (c,d) \in B_p(n) \times B_p(n)$ are in the same $S_p(n)$-orbit if and only if

$$\begin{aligned} &\big|S(a)\big| = \big|S(c)\big|, \quad \big|S(b)\big| = \big|S(d)\big|, \quad \big|S(a) \cap S(b)\big| = \big|S(c) \cap S(d)\big|, \\ &\text{and} \quad \big|\big\{i \in S(a) \cap S(b) : a_i = b_i\big\}\big| = \big|\big\{i \in S(c) \cap S(d) : c_i = d_i\big\}\big|. \end{aligned} \tag{25}$$

For $0 \le i, j, t, s \le n$, let $M_{i,j}^{t,s}$ be the $B_p(n) \times B_p(n)$ matrix given by

$$M_{i,j}^{t,s}(a,b) = \begin{cases} 1 & \text{if } |S(a)| = i, |S(b)| = j, |S(a) \cap S(b)| = t, \\ & |\{i : a_i = b_i \neq L_0\}| = s, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\mathscr{I}_p(n) = \big\{(i,j,t,s) : 0 \le s \le t \le i, j, i+j-t \le n\big\}.$$

It follows from (24) and (25) that $\{M_{i,j}^{t,s} : (i,j,t,s) \in \mathscr{I}_p(n)\}$ is a basis of $\mathscr{A}_p(n)$. Note that

$$p \ge 3 \quad \text{implies} \quad \big|\mathscr{I}_p(n)\big| = \dim \mathscr{A}_p(n) = \binom{n+4}{4} \tag{26}$$

since $(i, j, t, s) \in \mathscr{I}_p(n)$ if and only if $(i-t)+(j-t)+(t-s)+s \le n$ and all four terms are nonnegative. When $p=2$, this basis becomes $\{M_{i,j}^{t,t} : (i, j, t, t) \in \mathscr{I}_2(n)\}$, and its cardinality is $\binom{n+3}{3}$.

Let $0 \le i \le n$. Consider the $S_p(n)$-action on $V(B_p(n)_i), 0 \le i \le n$. Given $a, b \in B_p(n)_i$, it follows from (25) that the pairs (a, b) and (b, a) are in the same orbit of the $S_p(n)$-action on $B_p(n)_i \times B_p(n)_i$. It thus follows from (24) that the algebra $\mathscr{B}_p(n, i)$ has a basis consisting of symmetric matrices and is hence commutative. Thus, $V(B_p(n)_i)$ is multiplicity-free as an $S_p(n)$-module, and the $*$-algebra $\mathscr{B}_p(n, i)$ can be diagonalized. We now consider the more general problem of block diagonalizing the $*$-algebra $\mathscr{A}_p(n)$.

Before proceeding, we observe that

$$\sum_{k=0}^{n}\sum_{l=k^-}^{k}(n+l-2k+1)^2 = \binom{n+4}{4} \tag{27}$$

since both sides are polynomials in r of degree 4 (treating the cases $n=2r$ and $n=2r+1$ separately) and agree for $r=0,1,2,3,4$.

Consider the linear operator on $V(B_p(n))$ whose matrix with respect to the standard basis $B_p(n)$ is $M_{i,j}^{t,s}$. Transfer this operator to $\bigotimes_{i=1}^n V(p)$ via the isomorphism (13) above and denote the resulting linear operator by $\mathscr{M}_{i,j}^{t,s}$. In Theorem 8 below we show that the action of $\mathscr{M}_{i,j}^{t,s}$ on the basis $K_p(n)$ mirrors the binary case.

Define linear operators $\mathscr{N}, \mathscr{Z}, \mathscr{R} : V(p) \to V(p)$ as follows:

- $\mathscr{Z}(L_0) = L_0$ and $\mathscr{Z}(L_i) = 0$ for $i = 1, \ldots, p-1$,
- $\mathscr{N}(L_0) = 0$ and $\mathscr{N}(L_i) = L_i$ for $i = 1, \ldots, p-1$,
- $\mathscr{R}(L_0) = 0$ and $\mathscr{R}(L_i) = (L_1 + \cdots + L_{p-1}) - L_i$ for $i = 1, \ldots, p-1$.

Note that

$$\mathscr{R}(w_1) = (p-2)w_1, \tag{28}$$

$$\mathscr{R}(w_i) = -w_i, \quad i = 2, \ldots, p-1, \tag{29}$$

where the second identity follows from the fact that the sum of the elements of row $i, i \ge 2$, of P is zero.

Let there be given a 5-tuple $\mathscr{X} = (S_U, S_D, S_{\mathscr{N}}, S_{\mathscr{Z}}, S_{\mathscr{R}})$ of pairwise disjoint subsets of $[n]$ with union $[n]$ (it is convenient to index the components of S in this fashion). Define the linear operator

$$F(\mathscr{X}) : \bigotimes_{i=1}^n V(p) \to \bigotimes_{i=1}^n V(p)$$

by $F(\mathscr{X}) = F_1 \otimes \cdots \otimes F_n$, where each F_i is U or D or $\mathscr{N}$ or $\mathscr{Z}$ or $\mathscr{R}$ according as $i \in S_U$ or S_D or $S_{\mathscr{N}}$ or $S_{\mathscr{Z}}$ or $S_{\mathscr{R}}$, respectively.

Let $b \in B_p(n)$. It follows from the definitions that

$$F(\mathscr{X})(\bar{b}) \ne 0 \quad \text{iff} \quad S_D \cup S_{\mathscr{N}} \cup S_{\mathscr{R}} = S(b), \quad S_U \cup S_{\mathscr{Z}} = [n] - S(b). \tag{30}$$

Given a 5-tuple $r=(r_1,r_2,r_3,r_4,r_5)$ of nonnegative integers with sum n, define $\Pi(r)$ to be the set of all 5-tuples $\mathscr{X}=(S_U,S_D,S_{\mathscr{N}},S_{\mathscr{Z}},S_{\mathscr{R}})$ of pairwise disjoint subsets of $[n]$ with union $[n]$ and with $|S_U|=r_1$, $|S_D|=r_2$, $|S_{\mathscr{N}}|=r_3$, $|S_{\mathscr{Z}}|=r_4$, and $|S_{\mathscr{R}}|=r_5$.

Lemma 1 *Let* $(i,j,t,s)\in\mathscr{I}_p(n)$ *and* $r=(i-t,j-t,s,n+t-i-j,t-s)$. *Then*

$$\mathscr{M}^{t,s}_{i,j}=\sum_{\mathscr{X}\in\Pi(r)}F(\mathscr{X}). \tag{31}$$

Proof Let $b=(b_1,\ldots,b_n)\in B_p(n)$ and $\mathscr{X}=(S_U,S_D,S_{\mathscr{N}},S_{\mathscr{Z}},S_{\mathscr{R}})\in\Pi(r)$. We consider two cases:

(i) $|S(b)|\neq j$: In this case we have $\mathscr{M}^{t,s}_{i,j}(\bar{b})=0$. Now $|S_D|+|S_{\mathscr{N}}|+|S_{\mathscr{R}}|=j-t+s+t-s=j$. Thus, from (30) we also have $F(\mathscr{X})(\bar{b})=0$.

(ii) $|S(b)|=j$: Assume that $F(\mathscr{X})(\bar{b})\neq 0$. Then, from (30) we have that $F(\mathscr{X})(\bar{b})=\sum_a\bar{a}$, where the sum is over all $a=(a_1,\ldots,a_n)\in B_p(n)_i$ with $S(a)=S_U\cup S_{\mathscr{N}}\cup S_{\mathscr{R}}$, $a_k\neq b_k, k\in S_{\mathscr{R}}$, and $a_k=b_k, k\in S_{\mathscr{N}}$.

Going over all elements of $\Pi(r)$ and summing, we see that both sides of (31) evaluate to the same element on $\bar{b}$. □

Theorem 8 *Let* $(A,f,B)\in\mathscr{K}_p(n)$ *with* $|A|=l$, *and* $(i,j,t,s)\in\mathscr{I}_p(n)$.

(i) $\mathscr{M}^{t,s}_{i,j}(v(A,f,B))=0$ *if* $|B|\neq j-l$.
(ii) *If* $|B|=j-l$, *then*

$$\mathscr{M}^{t,s}_{i,j}\big(v(A,f,B)\big)=(p-1)^{\frac{i+j}{2}-t}\left\{\sum_{g=0}^{l}(-1)^{l-g}\binom{l}{g}\binom{t-l}{s-g}(p-2)^{t-l-s+g}\right\}\times\Big(\sum_{B'}v\big(A,f,B'\big)\Big),$$

where the sum is over all $B'\subseteq([n]-A)$ *with* $|B'|=i-l$ *and* $|B\cap B'|=t-l$.

Proof Let $r=(i-t,j-t,s,n+t-i-j,t-s)$, and let $\mathscr{X}=(S_U,S_D,S_{\mathscr{N}},S_{\mathscr{Z}},S_{\mathscr{R}})\in\Pi(r)$. Assume that $F(\mathscr{X})(v(A,f,B))\neq 0$. Then we must have (using (15) and the definitions of $\mathscr{N}$, $\mathscr{Z}$, and $\mathscr{R}$)

$$S_U\cup S_{\mathscr{Z}}=[n]-A-B,\qquad A\subseteq S_{\mathscr{N}}\cup S_{\mathscr{R}},\qquad S_D\subseteq B,$$
$$S_{\mathscr{N}}\cup S_D\cup S_{\mathscr{R}}=A\cup B.$$

Thus, $|B|=n-l-|S_U\cup S_{\mathscr{Z}}|=n-l-(i-t+n+t-i-j)=j-l$ (so part (i) follows).

Put $|A\cap S_{\mathscr{R}}|=l-g$. Then $|A\cap S_{\mathscr{N}}|=g$, and thus $|B\cap S_{\mathscr{N}}|=s-g$. We have $|B\cap S_{\mathscr{R}}|=|B-S_D-(B\cap S_{\mathscr{N}})|=j-l-j+t-s+g=t-l-s+g$.

We now have (using (17), (28), and (29))

$$F(\mathscr{X})\big(v(A,f,B)\big)=(-1)^{l-g}(p-2)^{t-s-l+g}(p-1)^{\frac{i+j}{2}-t}v\big(A,f,B'\big), \tag{32}$$

where $B' = S_U \cup (B - S_D)$ and $|B'| = i - t + j - l - j + t = i - l$, $|B \cap B'| = |B - S_D| = j - l - j + t = t - l$.

Formula (32) depends only on S_U, S_D, and $|A \cap S_{\mathscr{R}}|$. Once S_U, S_D are fixed, the number of choices for $S_{\mathscr{R}}$ with $|A \cap S_{\mathscr{R}}| = l - g$ is clearly $\binom{l}{g}\binom{t-l}{s-g}$.

Going over all elements of $\Pi(r)$ and summing, we get the result. □

For $i, j, k, t, s, l \in \{0, \dots, n\}$, define

$$\alpha_{i,j,k,l}^{n,t,s} = (p-1)^{\frac{1}{2}(i+j)-t} \left\{ \sum_{g=0}^{l} (-1)^{l-g} \binom{l}{g} \binom{t-l}{s-g} (p-2)^{t-l-s+g} \right\} \beta_{i-l,j-l,k-l}^{n-l,t-l}.$$

For $0 \le k \le n$ and $k^- \le l \le k$, define $E_{i,j,k,l}$ to be the $(n+l-2k+1) \times (n+l-2k+1)$ matrix, with rows and columns indexed by $\{k, k+1, \dots, n+l-k\}$ and with entry in row i and column j equal to 1 and all other entries 0.

The following result is due to Gijswijt, Schrijver and Tanaka [9].

Theorem 9 *Let $p \ge 3$, and let $J_p(n)$ be an orthogonal SSJB of $V(B_p(n))$ satisfying the conditions of Theorem 7. Define a $B_p(n) \times J'_p(n)$ unitary matrix $M(n)$ as follows: for $v \in J'_p(n)$, the column of $M(n)$ indexed by v is the coordinate vector of v in the standard basis $B_p(n)$. Then*

(i) *$M(n)^* \mathscr{A}_p(n) M(n)$ consists of all $J'_p(n) \times J'_p(n)$ block diagonal matrices with a block corresponding to each (normalized) SSJC in $J_p(n)$ and any two SSJCs starting and ending at the same rank give rise to identical blocks. Thus, for each $0 \le k \le n$, $k^- \le l \le k$, there are $\mu(n,k,l)$ identical blocks of size $(n+l-2k+1) \times (n+l-2k+1)$.*

(ii) *Conjugating by $M(n)$ and dropping duplicate blocks thus gives a positive semidefiniteness-preserving C^*-algebra isomorphism*

$$\Phi : \mathscr{A}_p(n) \cong \bigoplus_{k=0}^{n} \bigoplus_{l=k^-}^{k} \mathrm{Mat}\big((n+l-2k+1) \times (n+l-2k+1)\big).$$

It will be convenient to reindex the rows and columns of a block corresponding to an SSJC starting at rank k and having offset l by the set $\{k, k+1, \dots, n+l-k\}$. Let $i, j, t, s \in \{0, \dots, n\}$. Write

$$\Phi\big(M_{i,j}^{t,s}\big) = (N_{k,l}), \quad 0 \le k \le n, k^- \le l \le k.$$

Then

$$N_{k,l} = \begin{cases} \binom{n+l-2k}{i-k}^{-\frac{1}{2}} \binom{n+l-2k}{j-k}^{-\frac{1}{2}} \alpha_{i,j,k,l}^{n,t,s} E_{i,j,k,l} & \text{if } k \le i, j \le n+l-k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Follows from Theorem 5 using Theorems 6 and 8 and the dimension counts (26), (27). □

Remark 1 In Srinivasan [18] an explicit orthogonal SJB $J(n)$ of $V(B(n))$ was constructed and given an S_n-representation theoretic interpretation as the canonically defined Gelfand–Tsetlin basis of $V(B(n))$. This explicit basis from the binary case, together with a choice of the $(p-1)\times(p-1)$ unitary matrix P, leads to an explicit orthogonal SSJB $J_p(n)$ in the nonbinary case. Different choices of P lead to different SSJBs $J_p(n)$. One natural choice, used in Gijswijt, Schrijver and Tanaka [9], is the Fourier matrix. Another natural choice is the following. Consider the action of the symmetric group S_{p-1} on $V = V(\{L_1,\dots,L_{p-1}\})$. Under this action, V splits into two irreducibles, the one-dimensional trivial representation and the $(p-2)$-dimensional standard representation consisting of all linear combinations of $L_1,\dots,L_{p-1}$ with coefficients summing to 0. The first row of P is a basis of the trivial representation, and rows $2,\dots,p-1$ of P are a basis of the standard representation. Choose rows $2,\dots,p-1$ to be the canonical Gelfand–Tsetlin basis of this representation. (Up to order) we can write them down explicitly as follows (see [18]): for $i=2,\dots,p-1$, define $v_i = (i-1)L_i - (L_1+\cdots+L_{i-1})$ and $u_i = \frac{v_i}{\|v_i\|}$. So rows $2,\dots,p-1$ of P are $u_2,\dots,u_{p-1}$. The resulting matrix P is called the Helmert matrix (see Sect. 7.6 in Rohatgi and Saleh [16]). It is interesting to study the resulting orthogonal SSJB $J_p(n)$ from the point of view of representation theory of the wreath product $S_p(n)$ (for which, see Appendix B of Chap. 1 in Macdonald [11]).

We now explicitly diagonalize $\mathscr{B}_p(n,i)$.

Lemma 2 *Let* $0\le i\le n$. *Set*

$$L(i) = \big\{(k,l) : i^- \le k \le i, 0 \le l \le k\big\},$$
$$R(i) = \big\{(k,l) : 0 \le k \le n, k^- \le l \le k, k \le i \le n+l-k\big\}.$$

Then $|L(i)| = |R(i)|$.

Proof The identity is clearly true when $i \le n/2$. Now assume that $i > n/2$. Then the set $L(i)$ has cardinality $\sum_{k=2i-n}^{i}(k+1)$. The defining conditions on pairs (k,l) for membership in $R(i)$ can be rewritten as $0\le l\le k\le i, 0\le k-l\le n-i$. For $0\le j\le n-i$, the pairs (k,l) with $0\le l\le k\le i$ and $k-l=j$ are $(j,0),(j+1,1),\dots,(i,i-j)$, and their number is $i-j+1$. Thus, for $i>n/2$, $|R(i)| = \sum_{j=0}^{n-i}(i-j+1) = \sum_{t=2i-n}^{i}(t+1)$. The result follows. ☐

Let $0\le i\le n$. It follows from (25) that $\mathscr{B}_p(n,i)$ has a basis consisting of $M_{i,i}^{t,s}$, for $(t,s)\in L(i)$ (here we think of $M_{i,i}^{t,s}$ as $B_p(n)_i \times B_p(n)_i$ matrices). The cardinality of this basis, by Lemma 2, is $\tau(i)$ (where $\tau(i) = |R(i)|$). Since $\mathscr{B}_p(n,i)$ is commutative, it follows that $V(B_p(n)_i)$ is a canonical orthogonal direct sum of $\tau(i)$ common eigenspaces of the $M_{i,i}^{t,s}$, $(t,s)\in L(i)$ (these eigenspaces are the irreducible $S_p(n)$-submodules of $V(B_p(n)_i)$).

Let $0\le i\le n$. For $(k,l)\in R(i)$, define

$$J_p(n,i,k,l) = \big\{ v \in J_p(n) : r(v) = i, \text{ and the Jordan chain containing } v \text{ starts at rank } k \text{ and has offset } l \big\}. \tag{33}$$

Let $W_p(n,i,k,l)$ be the subspace spanned by $J_p(n,i,k,l)$ (note that this subspace is nonzero). We have an orthogonal direct sum decomposition

$$V\big(B_p(n)_i\big) = \bigoplus_{(k,l)\in R(i)} W_p(n,i,k,l).$$

It now follows from Theorem 9 that the $W_p(n,i,k,l)$ are the common eigenspaces of the $M_{i,i}^{t,s}$. The following result is due to Tarnanen, Aaltonen and Goethals [21].

Theorem 10 *Let $0 \le i \le n$. For $(t,s) \in L(i)$ and $(k,l) \in R(i)$, the eigenvalue of $M_{i,i}^{t,s}$ on $W_p(n,i,k,l)$ is*

$$(p-1)^{i-t}\left\{\sum_{g=0}^{l}(-1)^{l-g}\binom{l}{g}\binom{t-l}{s-g}(p-2)^{t-l-s+g}\right\}$$
$$\times\left\{\sum_{u=0}^{n-l}(-1)^{u-t+l}\binom{u}{t-l}\binom{n-k-u}{i-l-u}\binom{i-k}{i-l-u}\right\}.$$

Proof Follows from substituting $j=i$ in Theorem 9 and noting that

$$\binom{n+l-2k}{i-k}^{-1}\binom{n+l-2k}{u+l-k}\binom{n-k-u}{i-l-u} = \binom{i-k}{i-l-u}. \qquad \square$$

References

1. Aigner, M.: A Course in Enumeration. Springer, Berlin (2007)
2. Bachoc, C.: Semidefinite programming, harmonic analysis, and coding theory. In: Lecture Notes of a Course Given at the CIMPA Summer School—Semidefinite Programming in Algebraic Combinatorics (2009). arXiv:0909.4767
3. Ceccherini-Silberstein, T., Scarabotti, F., Tolli, F.: Harmonic Analysis on Finite Groups. Cambridge University Press, Cambridge (2008)
4. Delsarte, P.: Association schemes and t-designs in regular semilattices. J. Comb. Theory, Ser. A **20**, 230–243 (1976)
5. Delsarte, P.: Hahn polynomials, discrete harmonics, and t-designs. SIAM J. Appl. Math. **34**, 157–166 (1978)
6. Dunkl, C.F.: A Krawtchouk polynomial addition theorem and wreath product of symmetric groups. Indiana Univ. Math. J. **26**, 335–358 (1976)
7. Dunkl, C.F.: An addition theorem for some q-Hahn polynomials. Monatshefte Math. **85**, 5–37 (1978)
8. Eisfeld, J.: The eigenspaces of the Bose–Mesner algebras of the association schemes corresponding to projective spaces and polar spaces. Des. Codes Cryptogr. **17**, 129–150 (1999)
9. Gijswijt, D., Schrijver, A., Tanaka, H.: New upper bounds for nonbinary codes based on the Terwilliger algebra and semidefinite programming. J. Comb. Theory, Ser. A **113**, 1719–1731 (2006)
10. Go, J.T.: The Terwilliger algebra of the hypercube. Eur. J. Comb. **23**, 399–429 (2002)

11. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
12. Marco, J.M., Parcet, J.: On the natural representation of $S(\Omega)$ into $L^2(\mathscr{P}(\Omega))$: Discrete harmonics and Fourier transform. J. Comb. Theory, Ser. A **100**, 153–175 (2002)
13. Marco, J.M., Parcet, J.: A new approach to the theory of classical hypergeometric polynomials. Trans. Am. Math. Soc. **358**, 183–214 (2006)
14. Marco, J.M., Parcet, J.: Laplacian operators and Radon transforms on Grassmann graphs. Monatshefte Math. **150**, 97–132 (2007)
15. Proctor, R.A.: Representations of $\mathfrak{sl}(2, \mathscr{C})$ on posets and the Sperner property. SIAM J. Algebr. Discrete Methods **3**, 275–280 (1982)
16. Rohatgi, V.K., Ehsanes Saleh, A.K.Md.: An Introduction to Probability and Statistics, 2nd edn. Wiley, New York (2001)
17. Schrijver, A.: New code upper bounds from the Terwilliger algebra and semidefinite programming. IEEE Trans. Inf. Theory **51**, 2859–2866 (2005)
18. Srinivasan, M.K.: Symmetric chains, Gelfand–Tsetlin chains, and the Terwilliger algebra of the binary Hamming scheme. J. Algebr. Comb. **34**, 301–322 (2011)
19. Srinivasan, M.K.: A positive combinatorial formula for the complexity of the q-analog of the n-cube. Electron. J. Comb. **19**(2), #P34 (2012)
20. Stanley, R.P.: Enumerative Combinatorics, vol. 1, 2nd edn. Cambridge University Press, Cambridge (2012)
21. Tarnanen, H., Aaltonen, M., Goethals, J.M.: On the nonbinary Johnson scheme. Eur. J. Comb. **6**, 279–285 (1985)
22. Terwilliger, P.: The incidence algebra of a uniform poset. In: Coding Theory and Design Theory, Part I. IMA Vol. Math. Appl., vol. 20, pp. 193–212. Springer, New York (1990)
23. Vallentin, F.: Symmetry in semidefinite programs. Linear Algebra Appl. **430**, 360–369 (2009)

The Third Immanant of q-Laplacian Matrices of Trees and Laplacians of Regular Graphs

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Abstract Let $A = (a_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix where $n \ge 3$. Let $\det2(A)$ and $\det3(A)$ be its second and third immanants corresponding to the partitions $\lambda_2 = 2, 1^{n-2}$ and $\lambda_3 = 3, 1^{n-3}$, respectively. We give explicit formulae for $\det2(A)$ and $\det3(A)$ when A is the q-analogue of the Laplacian of a tree T on n vertices and when A is the Laplacian of a connected r-regular graph G.

Keywords Regular graphs · Laplacian matrices · Immanants of matrices · q-Analogue

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1 Introduction

Let T be a tree with vertex set $[n] = \{1, 2, \ldots, n\}$. Define $\mathscr{L}_q$, the q-analogue of T's Laplacian, as

$$\mathscr{L}_q = I - qA + q^2(\Delta - I), \tag{1}$$

where q is a variable, A is the adjacency matrix of T, and $\Delta = (d_{i,j})_{1 \le i,j \le n}$ is a diagonal matrix with $d_{i,i} = \deg(i)$, where $\deg(i)$ is the degree of vertex i in T (see Bapat, Lal and Pati [2, Proposition 3.3]). Note that on setting $q = 1$, we get $\mathscr{L}_q = L$, where L is the usual combinatorial Laplacian matrix of T. $\mathscr{L}_q$ is known to have the following tree-independent property (see [2, Proposition 3.4]).

Lemma 1 *Let T be a tree on n vertices. Then* $\det(\mathscr{L}_q) = 1 - q^2$.

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$\mathscr{L}_q$ is thus an interesting matrix, and in this work, we determine both its second and third immanants (see Theorems 5 and 6). We also determine the third immanant of the combinatorial Laplacian matrix of r-regular graphs on n vertices. We mention two contexts where $\mathscr{L}_q$ occurs and then give a brief introduction to immanants to make this work self-contained.

The definition of $\mathscr{L}_q$ can be extended to graphs that are not trees in a straightforward manner using Eq. (1). When the graph G is connected, but not necessarily a tree, $\mathscr{L}_q$ has connections to the number of spanning trees of G. Northshield [9] showed the following about the derivative of the determinant of $\mathscr{L}_q$.

Theorem 1 *Let G be a connected graph with m edges, n vertices, and κ spanning trees. Let $\mathscr{L}_q$ be the q-analogue of its Laplacian matrix, and let $f(q) = \det(\mathscr{L}_q)$. Then $f'(1) = 2(m-n)\kappa$.*

The polynomial $\det(\mathscr{L}_q)$ has also occurred in connection with the Ihara–Selberg zeta function of G, see Bass [4]. Foata and Zeilberger [6] have given combinatorial proofs of the results of Bass. We refer the reader to [6] for a pleasant introduction to this topic.

1.1 Immanants of Matrices

In this subsection, we briefly state the needed background from the representation theory of the symmetric group $\mathfrak{S}_n$ on the set $[n] = \{1, 2, \dots, n\}$. Let $A = (a_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix with entries from a commutative ring R, and let $f : \mathfrak{S}_n \to \mathbb{Z}$ be a function. Define the matrix function $\det_f(A) = \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$. We only consider functions f that arise as characters of irreducible representations of $\mathfrak{S}_n$. $\det_f(A)$ is called an immanant if f arises in this manner.

The number of distinct irreducible representations of $\mathfrak{S}_n$ is $p(n)$, the number of partitions of the positive integer n (see Sagan's book [10, Proposition 1.10.1]). Thus, if λ is a partition of n, we have functions $\chi_\lambda : \mathfrak{S}_n \to \mathbb{Z}$. Let $n \ge 3$ and λ_2 and λ_3 be the partitions $(2, 1^{n-2})$ and $(3, 1^{n-3})$ of n. Denote by $\chi_2 : \mathfrak{S}_n \to \mathbb{Z}$ and $\chi_3 : \mathfrak{S}_n \to \mathbb{Z}$ the irreducible characters of $\mathfrak{S}_n$ corresponding to the partition λ_2 and λ_3, respectively. Define the second immanant of A to be

$$\mathrm{det2}(A) = \det_{\chi_2}(A) = \sum_{\pi \in \mathfrak{S}_n} \chi_2(\pi) \prod_{i=1}^n a_{i,\pi(i)}.$$

Likewise, define the third immanant of A to be

$$\mathrm{det3}(A) = \det_{\chi_3}(A) = \sum_{\pi \in \mathfrak{S}_n} \chi_3(\pi) \prod_{i=1}^n a_{i,\pi(i)}.$$

Such characters χ have the important property that for any $\pi, \sigma \in \mathfrak{S}_n$, $\chi(\pi) = \chi(\sigma^{-1}\pi\sigma)$. In linear algebraic graph theory, we are usually interested in properties of matrices that are independent of the order in which its rows and columns are listed. All immanants enjoy this property, and to illustrate this, for completeness, we show this known fact for $\mathrm{det2}(A)$ (see Littlewood's book [7]).

Lemma 2 *If A is an $n \times n$ matrix and P is an $n \times n$ permutation matrix, then* $\mathrm{det2}(A) = \mathrm{det2}(P^{-1}AP)$.

Proof By definition, $\mathrm{det2}(A) = \sum_{\pi \in \mathfrak{S}_n} \chi_2(\pi) T_\pi$ where $T_\pi = \prod_{i=1}^n a_{i,\pi(i)}$. We call $\chi_2(\pi) T_\pi$ as the term corresponding to π in the expansion of $\mathrm{det2}(A)$. Let σ be the permutation corresponding to the permutation matrix P. In $P^{-1}AP$, the term corresponding to π would be $\chi_2(\sigma^{-1}\pi\sigma) T_{\sigma^{-1}\pi\sigma}$. This is again a sum over all permutations in $\mathfrak{S}_n$, and since $\chi_2(\pi) = \chi_2(\sigma^{-1}\pi\sigma)$, the second immanant is unchanged. □

For a matrix A, both $\mathrm{det2}(A)$ and $\mathrm{det3}(A)$ can be computed in polynomial time. The existence of such an algorithm follows from a connection between identities on symmetric functions of degree n and immanants of $n \times n$ matrices. Littlewood's book [7, Chap. 6.5] contains an exposition of this connection. See, Merris and Watkins [8] as well.

By applying this connection to the Jacobi–Trudi identity, the following results for the second immanant and third immanants are obtained. We just state the identities that are obtained in this manner (see Merris and Watkins [8, p. 239]). For an $n \times n$ matrix A and for $1 \le i \le n$, let $A(i)$ be the $(n-1) \times (n-1)$ principal matrix obtained from A by deleting its ith row and its ith column.

Theorem 2 *Let $A = (a_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Then,*

$$\mathrm{det2}(A) = \sum_{i=1}^{n} a_{i,i} \det\big(A(i)\big) - \det(A).$$

A similar identity for $\mathrm{det3}(A)$ is stated next. For $1 \le i < j \le n$, let $A(i, j)$ be the $(n-2) \times (n-2)$ principal matrix obtained from A by deleting both its ith and jth rows and its ith and jth columns. Similarly, let $A[i, j]$ be the principal 2×2 matrix obtained by restricting A to its ith and jth rows and columns. For a square matrix $A = (a_{i,j})_{1 \le i,j \le n}$, let $\mathrm{perm}(A)$ denote its permanent, $\mathrm{perm}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n a_{i,\sigma_i}$.

Theorem 3 *Let $A = (a_{i,j})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Then,*

$$\mathrm{det3}(A) = \sum_{1 \le i < j \le n} \Big(\mathrm{perm}\big(A[i,j]\big) \times \det\big(A(i,j)\big)\Big) - \mathrm{det2}(A).$$

2 The q-Analogue of T's Laplacian

Let $\mathscr{L}_q(T)$ be the q-analogue of the Laplacian of a tree T. We need the following result of Bapat and Sivasubramanian [3, Theorem 15] for obtaining the value of

$\det 2(\mathscr{L}_q)$ of a tree T. We recall that for the tree T on the vertex set $[n]$, we have its distance matrix $D = (d_{i,j})_{1 \le i,j \le n}$, where $d_{i,j}$ is the length of the unique path from i to j. Let D_q be the q-analogue of D which is obtained from D as follows. For a variable q and a positive integer i, change all entries i in D to $[i]_q = 1 + q + q^2 + \cdots + q^{i-1}$. Assuming $[0]_q = 0$, we get D_q from D by changing each entry i of D to $[i]_q$. Let $S, K \subseteq [n]$ with $|S| = |K|$. Let $\alpha(S) = \sum_{i \in S} i$ and recall that $D_q(S, K)$ is the submatrix of D_q obtained by omitting the rows in S and the columns in K. For a square matrix A, define $\mathrm{cofsum}(A)$ as the sum of its cofactors. If A is a 1×1 matrix, then $\mathrm{cofsum}(A) = 1$. Define $\mathrm{qcofsum}(D_q(S, K)) = \mathrm{cofsum}(D_q(S, K) - (1 - q)\det(D_q(S, K)))$.

Theorem 4 *Let T be a tree with vertex set $[n]$. Let $S, K \subseteq [n]$ with $|S| = |K|$. Then,*

$$(-1)^{\alpha(S)+\alpha(K)}(-1-q)^{n-|S|-1}\det \mathscr{L}_q[S, K] = \mathrm{qcofsum} D_q(S, K). \tag{2}$$

With these preliminaries, we can show the following.

Theorem 5 *Let T be a tree on n vertices with $\mathscr{L}_q(T)$ being the q-analogue of its Laplacian. Then,* $\det 2(\mathscr{L}_q(T)) = (n-1)(1+q^2)$.

Proof Using Theorem 2, we see that $\det 2(\mathscr{L}_q) = \sum_{i=1}^{n} \big(1 + \big(\deg(i) - 1\big)q^2\big) \times \det\big(\mathscr{L}_q(i)\big) - \det(\mathscr{L}_q)$. Using Theorem 4, we see that for all $i \in [n]$, we have $\det(\mathscr{L}_q(i)) = \mathrm{qcofsum}(D_q[i, i]) = 1$. Combining with Lemma 1, we see that

$$\det 2(\mathscr{L}_q) = \left(\sum_{i=1}^{n}\big(1 + \big(\deg(i) - 1\big)q^2\big)\right) - \big(1 - q^2\big) = (n-1)\big(1 + q^2\big). \qquad \square$$

For any nonnegative integer i, we know its s-analogue $[i]_s = 1 + i + \cdots + i^{s-1}$, where $[0]_s = 0$. We will at the end, substitute $s = q^2$ and derive q^2-analogues of known results.

Lemma 3 *Let T be a tree with vertex set $[n]$, and let $i \in [n]$. Then,*

$$\sum_{j \in [n]} [d_{i,j}]_s - (n-1) = s\left(\sum_{k}\big(\deg(k) - 1\big)[d_{i,k}]_s\right). \tag{3}$$

Proof Fix $i \in [n]$. We consider the sum $[d_{i,j}]_s$. For each vertex j, let k be the vertex closest to j on the unique i, j path. Clearly, we have $[d_{i,j}]_s = 1 + s\,[d_{i,k}]_s$. We root the tree at vertex i. Since T is rooted, we can use terms like child, parent, etc. Let $C(k)$ be the children of k. Clearly,

$$\sum_{j \in C(k)} (d_{i,j})_s = \begin{cases} (\deg(k) - 1)(1 + s\,[d_{i,k}]_s) & \text{if } k \neq i, \\ \deg(i) & \text{if } k = i. \end{cases} \tag{4}$$

Thus,

$$\begin{aligned}\sum_{j\in[n]}[d_{i,j}]_s &= \sum_k \sum_{j\in C(k)}[d_{i,j}]_s\\ &= \sum_{k\neq i}\big(\deg(k)-1\big)s[d_{i,k}]_s + \sum_{k\neq i}\big(\deg(k)-1\big)+\deg(i),\end{aligned}$$

i.e.,

$$\sum_{k\in[n]} s\big(\deg(k)-1\big)[d_{i,k}]_s = \sum_j [d_{i,j}]_s - (n-1).$$

In the last line, we have $\sum_{k\neq i}(\deg(k)-1)+\deg(i) = \sum_{k\in V}\deg(k)-(n-1) = n-1$. □

We next show the following result about the third immanant of the q-analogue of the Laplacian $\mathscr{L}_q(T)$ of a tree T with vertex set $[n]$.

Theorem 6 *Let $\mathscr{L}_q(T)$ be the q-analogue of the Laplacian of T. Then,*

$$\begin{aligned}\operatorname{det3}\big(\mathscr{L}_q(T)\big) = 3\Bigg(\sum_{1\le i<j\le n}\big([d_{i,j}]_{q^2}-1\big)\Bigg) + \Bigg(\sum_{1\le i<j\le n}\big([d_{i,j}]_{q^2}-1\big)-q^2\Bigg)\\ -(n-1)\big(1-q^2\big).\end{aligned}$$

Proof For this proof, we denote $\mathscr{L}_q(T)$ as R. Applying Theorem 3 to $R = (\ell_{i,j})_{1\le i,j\le n}$, we see that

$$\begin{aligned}\operatorname{det3}(R) &= \sum_{i=1}^n\sum_{j>i}[\ell_{i,i}\ell_{j,j}+\ell_{i,j}\ell_{j,i}]\det\big(R(i,j;i,j)\big) - (n-1)\big(1+q^2\big)\\ &= \frac{1}{2}\Bigg[\sum_{i=1}^n\sum_{j=1}^n\big\{1+q^2\big(\deg(i)-1\big)\big\}\big\{1+q^2\big(\deg(j)-1\big)\big\}[d_{i,j}]_{q^2}\Bigg]\\ &\quad + q^2\Bigg(\sum_{\{i,j\}\in E(T)} 1\Bigg) - (n-1)\big(1+q^2\big)\\ &= \frac{1}{2}\Bigg[\sum_{i=1}^n\big\{1+q^2\big(\deg(i)-1\big)\big\}\sum_{j=1}^n[d_{i,j}]_{q^2}\big\{1+q^2\big(\deg(j)-1\big)\big\}\Bigg]\\ &\quad -(n-1)\\ &= \frac{1}{2}\Bigg[\sum_{i=1}^n\big\{1+q^2\big(\deg(i)-1\big)\big\}\Bigg[2\sum_{j=1}^n[d_{i,j}]_{q^2}-(n-1)\Bigg]\Bigg]-(n-1)\\ &= \sum_{i=1}^n\big\{1+q^2\big(\deg(i)-1\big)\big\}\sum_{j=1}^n[d_{i,j}]_{q^2}\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{n-1}{2}\left[\sum_{i=1}^{n}\left\{1+q^2\big(\deg(i)-1\big)\right\}\right] - (n-1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}[d_{i,j}]_{q^2} + \sum_{i=1}^{n} q^2\big(\deg(i)-1\big)\sum_{j=1}^{n}[d_{i,j}]_{q^2} \\
&\quad - \frac{n-1}{2}\big(n+q^2(n-2)\big) - (n-1) \\
&= 2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[d_{i,j}]_{q^2}\right) - n(n-1) - \frac{n-1}{2}\big(n+q^2(n-2)\big) - (n-1) \\
&= 4\left(\sum_{i=1}^{n}\sum_{j>i}[d_{i,j}]_{q^2}\right) - \binom{n}{2}(3+q^2) - (n-1)\big(1-q^2\big).
\end{aligned}
$$

□

Note that we get the following result of Chan and Lam [5, Theorem 4.3] as a corollary. A simple proof follows by setting $q=1$ in Theorem 6.

Corollary 1 *Let T be a tree on n vertices with Laplacian $L(T)$. For vertices i, j, let $d_{i,j}$ be the distance between them in T. Then,*

$$\det3\big(L(T)\big) = 4\left(\sum_{1\le i<j\le n}(d_{i,j}-1)\right).$$

When we plug in $q=-1$, we get the matrix $L' = D + A$, which is called the signless Laplacian as it resembles the Laplacian $L = D - A$ but has no negative sign. L' is also positive semidefinite. Another corollary of Lemma 1, Theorem 5, and Theorem 6 is the following.

Corollary 2 *For a tree T, let $L = D - A$ be its Laplacian, and $L' = D + A$. Then,* $\det(L) = \det(L')$, $\det2(L) = \det2(L')$ *and* $\det3(L) = \det3(L')$.

Proof The results of Lemma 1, Theorem 5, and Theorem 6 all have q^2 appearing in them. Thus, setting $q = a$ and $q = -a$ gives the same immanant values in $\mathcal{L}_q$. Setting $q=-1$ in $\mathcal{L}_q$ gives L', while setting $q=1$ in $\mathcal{L}_q$ gives L. Finally, $q=\pm1$, so we have $q^2=1$. □

3 Laplacian of Connected r-Regular Graphs

Let L be the Laplacian of a connected r-regular graph. We do a similar computation of the third immanant of L. We need Theorem 2 and the Matrix-Tree theorem, which we state below for easy reference. We refer the reader to West's book [11] for its proof.

Theorem 7 (Matrix-Tree Theorem) *Let G be a connected graph with κ spanning trees. Let L_v be the $(n-1)\times(n-1)$ matrix obtained from its Laplacian L by omitting the row and column corresponding to vertex v. Then, for all $v \in V$,* $\det(L_v) = \kappa$.

Lemma 4 *Let G be an r-regular graph on n vertices with κ spanning trees. Let L be its Laplacian. Then,* $\det2(L) = \kappa r n$.

Proof It is well known that $\det(L) = 0$ for all graphs. By Theorem 2 we have

$$\det2(L) = \sum_{i=1}^{n} L[i] \det\bigl(L(i)\bigr) = \sum_{i=1}^{n} \deg(i)\kappa = nr\kappa. \qquad \square$$

For a square matrix A, let $\mathrm{Trace}(A)$ denote its trace. A more interesting result is the evaluation of the third immanant of L. We first state a few preliminaries involving the notion of resistance that we need for the proof of Theorem 8.

For a connected graph G on n vertices, there exists a notion of "resistance distance," which is related to the $(n-2)$-sized minors of its Laplacian L as follows. We will use the notation $L(i)$ and $L(i, j)$ described in Sect. 1.1. Define the resistance $r_{i,j}$ between vertices i, j of G as

$$r_{i,j} = \frac{\det(L(i, j))}{\det(L(i))}.$$

By the Matrix-Tree theorem, see West [11], we see that $\det(L(i)) = \kappa$, where κ is the number of spanning trees of G. Thus, for any pair of vertices i, j of G, we have $\det(L(i, j)) = \kappa r_{i,j}$. Let L^+ be the Moore–Penrose inverse of L. We further need the following two results.

Lemma 5 (Bapat [1], Lemma 9.9) *Let $R = (r_{i,j})_{1\le i,j\le n}$ be the resistance matrix of a connected graph $G = (V, E)$ with $|V| = n$. Then,* $\sum_{\{i,j\}\in E} r_{i,j} = n - 1$.

Lemma 6 (Bapat [1], Eq. (9.20), p. 122) *Let $R = (r_{i,j})_{1\le i,j\le n}$ be the resistance matrix of a connected graph G on n vertices. Then,* $\sum_{1\le i<j\le n} r_{i,j} = n\,\mathrm{Trace}(L^+)$.

Theorem 8 *Let G be a connected r-regular graph with Laplacian L. Let L^+ be the Moore–Penrose inverse of L. Then,* $\det3(L) = rn\kappa + 2r^2 n\kappa \mathrm{Trace}(L^+) + (n-1)\kappa$.

Proof By Theorem 3 we see that

$$\begin{aligned}
\det2(L) + \det3(L) &= \sum_{1\le i<j\le n} \mathrm{perm}\bigl(L[i, j]\bigr)\det\bigl(L(i, j)\bigr) \\
&= \sum_{\{i,j\}\in E} \bigl(r^2+1\bigr)\det\bigl(L(i, j)\bigr) + \sum_{\{i,j\}\notin E} r^2 \det\bigl(L(i, j)\bigr) \\
&= r^2\Biggl(\sum_{1\le i<j\le n} \det\bigl(L(i, j)\bigr)\Biggr) + \sum_{\{i,j\}\in E} \det\bigl(L(i, j)\bigr)
\end{aligned}$$

$$= r^2 n\kappa \operatorname{Trace}\bigl(L^+\bigr) + (n-1)\kappa,$$
$$\det3(L) = rn\kappa + r^2 n\kappa \operatorname{Trace}\bigl(L^+\bigr) + (n-1)\kappa. \qquad \square$$

We end this work with a question.

Question 1 Can the third immanant of the q-analogue of the Laplacian of an r-regular graph be found out?

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References

1. Bapat, R.B.: Graphs and Matrices. Hindusthan Book Agency (2010)
2. Bapat, R.B., Lal, A.K., Pati, S.: A q-analogue of the distance matrix of a tree. Linear Algebra Appl. **416**, 799–814 (2006)
3. Bapat, R.B., Sivasubramanian, S.: Identities for minors of Laplacian, resistance and distance matrices. Linear Algebra Appl. **435**, 1479–1489 (2011)
4. Bass, H.: The Ihara–Selberg zeta function of a tree lattice. Int. J. Math. **3**, 717–797 (1992)
5. Chan, O., Lam, T.K.: Hook immanant inequalities for Laplacians of trees. Linear Algebra Appl. **261**, 23–47 (1997)
6. Foata, D., Zeilberger, D.: Combinatorial proofs of Bass's evaluations of the Ihara–Selberg zeta function of a graph. Trans. Am. Math. Soc. **351**, 2257–2274 (1999)
7. Littlewood, D.E.: The Theory of Group Characters and Matrix Representation of Groups, 2nd edn. Chelsea, New York (2002)
8. Merris, R., Watkins, W.: Inequalities and identities for generalized matrix functions. Linear Algebra Appl. **64**, 223–242 (1985)
9. Northshield, S.: A note on zeta function of a graph. J. Comb. Theory, Ser. B **74**(2), 408–410 (1998)
10. Sagan, B.E.: The Symmetric Group: Representation, Combinatorial Algorithms and Symmetric Functions, 2nd edn. Springer, Berlin (2001)
11. West, D.B.: Graph Theory, 2nd edn. Prentice-Hall of India, New Delhi (1997)

Matrix Product of Graphs

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Abstract In this paper, we characterize the graphs G and H for which the product of the adjacency matrices $A(G)A(H)$ is graphical. We continue to define matrix product of two graphs and study a few properties of the same product. Further, we consider the case of regular graphs to study the graphical property of the product of adjacency matrices.

Keywords Adjacency matrix · Matrix product · Realizability

Mathematics Subject Classification (2010) 05C50

1 Introduction

Graphs considered in this paper are simple, undirected, and without self loops. Let G be a graph on the set of vertices $\{v_1, v_2, \ldots, v_n\}$. Two vertices v_i and v_j, $i \neq j$, are said to be adjacent to each other if there is an edge between them. An adjacency between the vertices v_i and v_j is denoted by $v_i \sim_G v_j$, and we denote $v_i \nsim_G v_j$ to represent that v_i is not adjacent to v_j in the graph G. For a graph G on the set of vertices $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix of G is the matrix $A(G) = (a_{ij}) \in M_n(\mathbb{R})$ in which $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Given

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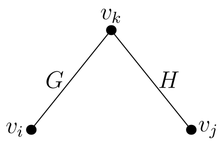


Fig. 1 GH path

two graphs G and H on the same set of vertices $\{v_1, v_2, \dots, v_n\}$, $G \cup H$ represents the union of graphs G and H on the same set of vertices, where two vertices are adjacent in $G \cup H$ if they are adjacent in at least one of G and H. Graphs G and H on the same set of vertices are said to be (mutually) edge disjoint if $u \sim_G v$ implies that $u \nsim_H v$. Equivalently, H is a subgraph of $\overline{G}$ and vice versa.

For the terminologies and notation that are not defined but used in this paper, we refer to the books by West [3] and Buckley and Harary [2].

Definition 1 (Graphical matrix; Akbari [1]) A symmetric (0, 1)-matrix is said to be graphical if all its diagonal entries equal zero.

In fact, a graphical matrix is an adjacency matrix of some graph. If B is a graphical matrix and there exists a graph G such that $B = A(G)$, then we often say that G is the realization of graphical matrix B. In this paper, our focus is on studying the case where the product of adjacency matrices is graphical.

If G is any graph and H is a totally disconnected graph on the same set of vertices as that of G, then the product $A(G)A(H)$ is obviously a graphical as $A(H)$ is a null matrix. In this case, the resulting graph is a null graph. So, the case where either G or H is a totally disconnected graph is a trivial case, and we consider only the nontrivial cases, where both the graphs G and H are non-null, for the further discussion.

Similarly, an isolated vertex in G or H remains as an isolated vertex in Γ whenever there exists a graph Γ such that $A(G)A(H) = A(\Gamma)$. This can be verified easily through matrix multiplication, as the column and row in the adjacency matrix corresponding to an isolated vertex contains only zeros. Therefore, the problem of realization of the product $A(G)A(H)$ reduces to the case of realization of the product of the block matrices of $A(G)$ and $A(H)$ corresponding to the set of vertices excluding the isolated vertices. Therefore, for all further discussion, we consider the graphs G and H that have no isolated vertices.

In the following, we introduce a concept of GH path between two vertices, which is very useful in the future discussion.

Definition 2 (GH path) Given graphs G and H on the same set of vertices $\{v_1, \dots, v_n\}$, two vertices v_i and v_j $(i \neq j)$ are said to have a GH path if there exists a vertex v_k, different from v_i and v_j, such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$ (see Fig. 1).

The capital letters G, H, or Γ occur beside the lines/edges joining vertices, in the following and the later figures, does not represent the label of the edge but represents the graph in which the end vertices are adjacent.

Note that a GH path between two vertices v_i and v_j, whenever it exists, is a path $v_i v_k v_j$ of length two in $G \cup H$. The same GH path also represents an HG path $v_j v_k v_i$ between v_j and v_i through v_k.

2 Characterization

In this section, we answer to the question:

> What are the characteristics of the graphs G and H such that the matrix product $A(G)A(H)$ is graphical?

In order to answer the above question, we need the following lemmas.

Lemma 1 *Given two graphs G and H on the same set of vertices $\{v_1, \ldots, v_n\}$, the principal diagonal entries of $A(G)A(H)$ are all zeros if and only if H is a subgraph of $\overline{G}$ (the complement of G).*

Proof Consider $C = A(G)A(H) = (c_{ij}),\ 1 \leq i, j \leq n$. Since $A(G)$ and $A(H)$ are symmetric and nonnegative, $C_{ii} = 0 \Leftrightarrow$ the ith row of $A(G)$ is orthogonal to the ith row of $A(H)$ for every i, $1 \leq i \leq n$. In other words, b_{ki} the (k, i)th entry of $A(H)$ is nonzero and equals to one only when a_{ik} the (i, k)th entry of $A(G)$ is zero for all k, $1 \leq k \leq n$. This proves the lemma. □

Lemma 2 *Let G and H be graphs on the same set $\{v_1, \ldots, v_n\}$ of vertices. Then the matrix $C = A(G)A(H)$, the product of adjacency matrices, is a symmetric $(0, 1)$-matrix if and only if the following statements are true for every pair of vertices v_i and v_j:*

(i) *There exists at most one GH path from v_i to v_j.*
(ii) *If there exists a GH path from v_i to v_j, then so does one from v_j to v_i.*

Proof Suppose that $C = A(G)A(H)$ is a symmetric (0,1)-matrix where $A(G) = (a_{ij}),\ A(H) = (b_{ij})$, and $C = (c_{ij}),\ 1 \leq i, j \leq n$.

If there exist more than one GH path from v_i to v_j, for some i and j, then there exist more than one k ($k \neq i, j$) such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$. That is, for these values of k, $a_{ik} = b_{kj} = 1$. Hence, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \geq 2$, a contradiction. This proves part (i).

Part (ii) follows from the symmetry of $C = A(G)A(H)$.

The converse is verified easily through matrix multiplication. □

Remark 1 In the above Lemma 2, statements (i) and (ii) can be replaced by the following statements (i)′ and (ii)′. The proof is straightforward from the fact that a GH path from the vertex v_i to the vertex v_j is an HG path from v_j to v_i.

(i)′ There exists at most one HG path between v_i and v_j.

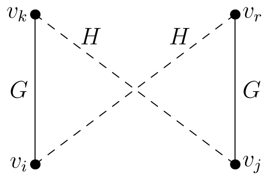


Fig. 2 Adjacency of v_i and v_j in Γ

(ii)$'$ For every HG path from the vertex v_i to the vertex v_j, there exists an HG path from v_j to v_i.

Now we prove the main theorem of this section.

Theorem 1 *Let G and H be graphs on the same set of vertices $\{v_1, \ldots, v_n\}$ with the adjacency matrices $A(G)$ and $A(H)$, respectively. Then $C = A(G)A(H)$ is graphical if and only if the following three statements hold.*

(i) *G and H are edge disjoint.*
(ii) *For every two vertices v_i and v_j, there exists at most one GH path from the vertex v_i to the vertex v_j.*
(iii) *For every two vertices v_i and v_j, if there exists a GH path from the vertex v_i to the vertex v_j, then there exists an HG path from v_i to v_j.*

In fact, the (i, j)th entry of the matrix C correspond to the unique GH path (as well HG path) between v_i and v_j.

Proof Noting that a GH path from v_i to v_j is an HG path from v_j to v_i, the proof of the theorem follows from Lemma 1 and Lemma 2. □

The above results lead us to the following definition of *matrix product of two graphs*.

Definition 3 (Matrix product of graphs) Given graphs G, H and Γ on the same set of vertices, the graph Γ is said to be the *matrix product of graphs* G and H if $A(G)A(H)$ is graphical and $A(\Gamma) = A(G)A(H)$. Further, we say that G and H are the graph factors of a matrix product Γ.

Remark 2 If $A(G)A(H)$ is graphical, then from statements (i) and (ii) of Theorem 1, it is clear that the GH path and HG path from the vertex v_i to the vertex v_j, whenever they exist, are not through the same vertex.

Remark 3 Let G and H be two graphs on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$. Let Γ be the matrix product of G and H. From (iii) of Theorem 1 it is clear that the vertices v_i and v_j are adjacent in Γ if and only if $G \cup H$ has the structure described in Fig. 2.

In fact, the degree of any vertex v_i in Γ is the number of distinct GH paths between v_i and other vertices in Γ.

In the following theorem, we prove that if graphs G and H are regular and Γ is the matrix product of G and H, then Γ is also regular.

Theorem 2 *Let the graphs G and H be defined on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$, and graph Γ be the matrix product of G and H. If G and H are regular graphs with regularities r and s, respectively, then Γ is regular with regularity rs.*

Proof As noted in Remark 3, the degree of any vertex v_i in Γ is given by the number of GH paths of length two from v_i to other vertices in $G \cup H$. Therefore,

$$\begin{aligned}\deg_\Gamma v_i &= \sum_k [A_{ik} \deg_H v_k] \\ &= \sum_k [A_{ik} s] \\ &= s\left(\sum_k A_{ik}\right) \\ &= s(\deg_G v_i) \\ &= sr.\end{aligned}$$

Hence, Γ is regular with regularity rs. □

Corollary 1 *If G and H are regular graphs on the same set of n vertices with regularity r and s, respectively, and $rs > (n-1)$, then the product $A(G)A(H)$ is not graphical.*

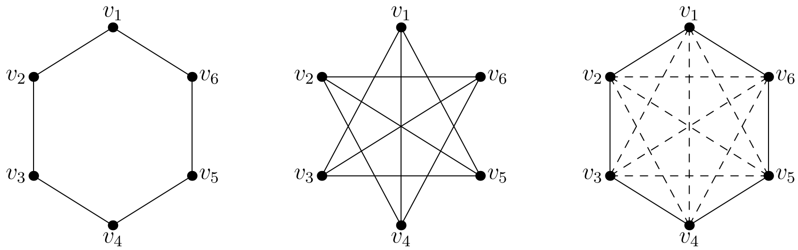
Proof Proof follows from Theorem 2. □

The following example illustrates Corollary 1.

Example 1 Consider $G = C_6$ and $H = \overline{C_6}$ on six vertices, i.e., $n = 6$. Here G is 2-regular (i.e., $r = 2$), and H is 3-regular (i.e., $s = 3$). Hence, $rs = 6 > 6 - 1 = 5$, satisfying the condition discussed in Corollary 1.

Now, consider the corresponding adjacency matrices and their product:

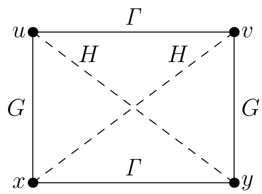
$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad A(H) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \text{and}$$



(a) G (b) H (c) $G \cup H$

Fig. 3 $A(G)A(H)$ is not graphical

Fig. 4 Parallel edges in Γ



$$
A(G)A(H) = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{bmatrix},
$$

where the matrix $A(G)A(H)$ is not graphical.

Now, we shall also observe that the graphs considered in this example are not satisfying the conditions stated in Theorem 1. Consider the graphs G, H, and $G \cup H$ given in Fig. 3 and the vertices v_1 and v_4. Clearly, the number of GH paths between v_1 and v_4 is more than one. In fact, there are two such GH paths through v_2 and v_6.

Definition 4 Given two disjoint edges $e = (x, y)$ and $f = (u, v)$ in the matrix product Γ of the graphs G and H, f is said to be parallel to e if xuy is a GH path (HG path) and xvy is an HG path (GH path) between the vertices x and y, as shown in Fig. 4.

Theorem 3 *Given an edge $e = (x, y)$ in the matrix product Γ of the graphs G and H, the following statements are true*:

(i) *There exists an edge $f = (u, v)$ in Γ, disjoint from e, such that f is parallel to e.*
(ii) *If f is parallel to e then e is parallel to f.*
(iii) *The edge f parallel to e is unique.*

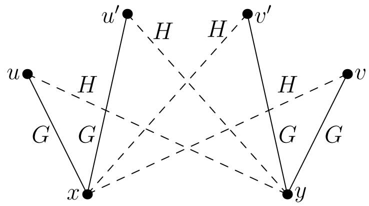


Fig. 5 Two GH and HG paths between x and y

Proof (i): Let $e = (x, y)$ be an edge in Γ. Since Γ is the matrix product of G and H, there exist vertices u and v in Γ such that

$$x \sim_G u \sim_H y \quad \text{and} \quad x \sim_H v \sim_G y. \tag{1}$$

However, this implies

$$u \sim_G x \sim_H v \quad \text{and} \quad u \sim_H y \sim_G v. \tag{2}$$

In other words, $(u, v) = f$ is an edge in Γ, and, by the definition of the parallel edge and Eq. (1), f is parallel to e.

(ii): From the equivalence of Eqs. (1) and (2) we get that f is parallel to e if and only if e is parallel to f.

(iii): Suppose that $f = (u, v)$ and $f' = (u', v')$ are two edges parallel to e in Γ. Then $x \sim_G u \sim_H y$, $x \sim_H v \sim_G y$, and $x \sim_G u' \sim_H y$, $x \sim_H v' \sim_G y$ (see Fig. 5).

So, we get two GH paths xuv and $xu'y$ between x and y, and similarly, we get two HG paths xvy and $xv'y$ between x and y. This contradicts (ii) of Theorem 1. Hence the uniqueness. □

Theorem 4 *If graph Γ is the matrix product of graphs G and H, then Γ has an even number of edges.*

Proof From Theorem 3 it is very clear that every edge (x, y) in Γ has unique parallel edge and the relation "is parallel to" is symmetric. Though the relation is not an equivalence relation, it partitions the set of edges of Γ, and each partition consists of exactly two edges that are parallel to each other. So, the number of edges in Γ is even. □

3 Further Results

In this section, we find an expression for the degree of a vertex in the matrix product. We observe that if a graph factor of a matrix product is connected, then the other graph factor is a regular graph. Also, we find that the graph such as a path graph is not a graph factor for any matrix product. Finally, we characterize all regular graphs G for which $A(G)A(\overline{G})$ is graphical.

3.1 Degree of a Vertex in Matrix Product

Theorem 5 *Given graphs G, H, and Γ on the same set of vertices such that Γ is the matrix product of G and H, and a vertex v_i, we have that*

$$\deg_\Gamma v_i = \sum_{v_j \sim_H v_i} \deg_G v_j. \tag{3}$$

Proof Let $A(\Gamma) = C = (c_{ij})$ be the adjacency matrix of the matrix product Γ of G and H. Define $e_i = [0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T$, a column vector of length n with 1 in the ith position and all other entries equal to zero. For a vertex v_i, its degree in Γ is given by

$$\begin{aligned}
\deg_\Gamma v_i &= \sum_{j=1}^{n} c_{ji} \\
&= [1 \quad 1 \quad \ldots \quad 1]Ce_i \\
&= [1 \quad 1 \quad \ldots \quad 1]A(G)A(H)e_i \\
&= \big([1 \quad 1 \quad \ldots \quad 1]A(G)\big)\big(A(H)e_i\big) \\
&= [\deg_G v_1 \quad \deg_G v_2 \quad \ldots \quad \deg_G v_n] \begin{bmatrix} A(H)_{1i} \\ A(H)_{2i} \\ \vdots \\ A(H)_{ni} \end{bmatrix} \\
&= \sum_{j=1} (\deg_G v_j) A(H)_{ji}.
\end{aligned}$$

Since $A(H)_{ji} = 1$ only when $v_i \sim_H v_j$, we get $\deg_\Gamma v_i = \sum_{v_j \sim_H v_i} \deg_G v_j$. $\square$

Remark 4 Since the adjacency matrix C of Γ is symmetric, $A(G)$ and $A(H)$ commute, and therefore $\deg_\Gamma v_i$ is also given by

$$\deg_\Gamma v_i = \sum_{v_j \sim_G v_i} \deg_H v_j. \tag{4}$$

Theorem 6 *Let Γ be the matrix product of the graphs G and H. If one among the graphs G and H is connected, then the other graph is regular.*

Proof First, we shall prove that $\deg_H v_1 = \deg_H v_2$ for every pair of vertices v_1 and v_2 adjacent in G, i.e., $v_1 \sim_G v_2$. For the purpose, we shall prove that there is a 1–1 correspondence between H adjacencies of v_1 and H adjacencies of v_2. The case $\deg_H v_1 = \deg_H v_2 = 1$ is trivial. To consider the nontrivial case, let $\deg_H v_2 \geq 2$.

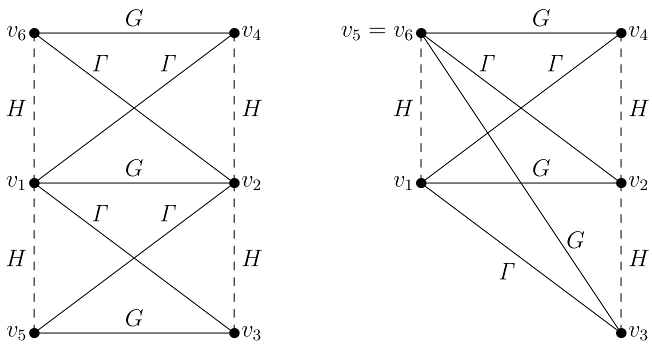


Fig. 6 GH paths between v_5 and v_2

Consider any two vertices v_3 and v_4 that are adjacent with v_2 in H. Since $v_1 \sim_G v_2 \sim_H v_3$ and $v_1 \sim_G v_2 \sim_H v_4$ are GH paths and Γ is the matrix product of G and H, we get that the vertices v_3 and v_4 are adjacent with v_1 in Γ.

Now $v_1 \sim_\Gamma v_3$ implies, as in Remark 3, that there exists an HG path $v_1 \sim_H v_5 \sim_G v_3$ through a vertex v_5 $(\neq v_2)$. For the same reason, $v_1 \sim_\Gamma v_4$ implies that there exists an HG path $v_1 \sim_H v_6 \sim_G v_4$ through a vertex v_6 $(\neq v_2)$. So, we have vertices v_5 and v_6 adjacent with v_1 in H, respectively, correspond to the vertices v_3 and v_4 adjacent with v_2 in H. Now, we shall prove that $v_5 \neq v_6$ in order to prove the 1–1 correspondence between adjacencies of v_1 and v_2 in H.

Suppose that $v_5 = v_6$. Then, v_5 is adjacent with v_3 and v_4 in G, and both v_3 and v_4 are adjacent with v_2 in H. Then there are two GH paths $v_5 \sim_G v_3 \sim_H v_2$ and $v_5 \sim_G v_4 \sim_H v_2$ from v_5 to v_2 (refer Fig. 6), which contradicts (ii) of Theorem 1.

This proves that $\deg_H v_1 = \deg_H v_2$ whenever v_1 and v_2 are adjacent in G. So, if G is connected, there exists a path between every pair of vertices, and the degrees of all vertices of H are identical. In other words, H is regular. $\square$

From the lines of the proof of Theorem 6, the following corollary is immediate.

Corollary 2 *Let Γ be the matrix product of G and H. If v_i and v_j are any two vertices connected by a path in G, then* $\deg_H v_i = \deg_H v_j$.

Corollary 3 *A graph G with unique full degree vertex cannot be a graph factor for a nontrivial matrix product.*

Proof Let G be a connected graph with unique full degree vertex, say v. In other words, the vertex v is adjacent with all the remaining vertices in G. Suppose that there exists a graph H and a graph Γ such that Γ is a matrix product of G and H. By Theorem 1, H is a subgraph of $\overline{G}$, and therefore v is an isolated vertex in H. Since the vertex v is adjacent with all the remaining vertices in G, from Corollary 2, degree of all the vertices in H equals to $\deg_H v (= 0)$. Hence, the graph H can only be a null graph. $\square$

The definition of degree partition of a graph, given below, is useful in describing a particular characteristic of graphs that explains the relation between the set of vertices S of a component of a graph to which given vertex belongs to and the degree of the vertices from S in the other graph, whenever the corresponding matrix product exists.

Definition 5 (Degree partition of a graph) For a graph G on the set of vertices $\{v_1, v_2, \ldots, v_n\}$, the degree partition of the graph G is a partition $\{V_0, V_1, \ldots, V_k\}$ of the vertex set $V(G)$ with the property that $V_j = \{v_i \mid \deg_G v_i = j\}$, where $0 \le j \le k$ and $k = \Delta(G) = \max_{1 \le i \le n}\{\deg_G v_i\}$.

Corollary 4 *Let Γ be the matrix product of G and H. If $\{V_0, V_1, \ldots, V_k\}$, where $k = \Delta(G)$, is the degree partition of the vertex set of G, then the vertex set corresponding to each component of H is entirely in one of the partite sets of the degree partition.*

Corollary 5 *Let graphs G, H, and Γ be the graphs such that Γ is the matrix product of G and H. Let G be connected, and let $\{V_0, V_1, \ldots, V_k\}$ where $k = \Delta(G)$ be the degree partition of the vertex set of G. Further, let $|V_i| = n_i$, $G_i = \langle V_i \rangle$, the graph induced by the vertices in the partite set V_i, and $\Delta_i = \max_{w \in V_i}\{\deg_{G_i} w\}$, $0 \le i \le k$. Then the regularity of H is less than $\min_{0 \le i \le k}\{n_i - \Delta_i\}$.*

Proof Follows from Theorem 6, Corollary 4, and the fact that H is a subgraph of $\overline{G}$. □

Theorem 7 *Let G, H, and Γ be the graphs such that Γ is the matrix product of G and H. Then*

$$\deg_\Gamma v_i = \deg_G v_i \cdot \deg_H v_i \tag{5}$$

for all vertices v_i.

Proof From Theorem 5 we have $\deg_\Gamma v_i = \sum_{v_j \sim_H v_i} \deg v_j$. In the summation on the right-hand side, we consider all j such that $v_i \sim_H v_j$. Therefore, from Corollary 2 we get

$$\deg_G v_i = \deg_G v_j$$

for all j with $v_i \sim_H v_j$. Therefore,

$$\begin{aligned}\deg_\Gamma v_i &= \sum_{v_i \sim_H v_j} \deg_G v_j \\ &= (\deg_G v_i)\left(\sum_{v_i \sim_H v_j} 1\right) \\ &= \deg_G v_i \cdot \deg_H v_i.\end{aligned}$$

Hence the theorem. □

The following corollary is immediate from Theorem 7.

Corollary 6 *Let G, H, and Γ be graphs such that Γ is the matrix product of G and H, Then the following statements are true.*

(i) *If any two of G, H and Γ are regular, then so is the third.*
(ii) *Γ is regular if and only if $\deg_G v_i \cdot \deg_H v_i$ is a constant for all i, $1 \le i \le n$.*
(iii) *A connected graph Γ is Eulerian if and only if the degree of each vertex is even in at least one among the graphs G and H.*

Corollary 7 *If Γ is a matrix product of G and H, where one among G and H is a regular graph, then Γ cannot be a tree.*

Proof Suppose that Γ is a tree. Then Γ has a pendant vertex. Let v be a vertex such that $\deg_\Gamma v = 1$. By Theorem 7, $\deg_\Gamma v = \deg_G v \cdot \deg_H v$. This implies that $\deg_G v = \deg_H v = 1$. Without loss of generality, let H be regular. So, H is 1-regular and contains an even number of vertices. If Γ is a tree, then it contains an odd number of edges (which is less by one than the number of vertices). This contradicts the fact that Γ has an even number of edges (see Theorem 4). □

Remark 5 From Corollary 7 and Theorem 6 it is clear that if either of G and H is connected, then their matrix product Γ cannot be a tree.

In the following theorem, we shall show that a path graph is not a factor graph for any matrix product.

Theorem 8 *Let G be a path graph on n vertices $\{v_1, v_2, \ldots, v_n\}$. Then, there exists no nontrivial graph H such that $A(G)A(H)$ is graphical.*

Proof Suppose that there exists no nontrivial graph H such that $A(G)A(H)$ is graphical. From Theorem 1 it is clear that G and H are edge disjoint.

Let the vertices $v_1, v_2, \ldots, v_n$ of G be such that $v_1 \sim_G v_2 \sim_G \cdots \sim_G v_{n-1} \sim_G v_n$. Then, v_1 and v_n are the only vertices of degree 1 in G. Since G is connected, from Theorem 6 we get that H is regular. Clearly, the vertices v_1 and v_n are not adjacent with any of v_j, $2 \le j \le n-1$, in H. Otherwise, by Corollary 2 the degree of vertices among v_1 and v_n in G will be same as the degree of adjacent vertices among v_j, which is not true. So, if v_1 and v_n are not adjacent in H, then those vertices are isolated, and by regularity of H we get that H is a null graph. Since H is considered to be non-null, v_1 and v_n are adjacent in H and are of degree one in H. Now, by regularity of H, the degree of each vertex is one, and therefore H is a 1-factor (union of disjoint K_2s) of the complement of G. Hence, n is even.

Since $v_2 \sim_G v_1$ and $v_1 \sim_H v_n$, we obtain a GH path between v_2 and v_n, and therefore $v_2 \sim_\Gamma v_n$. Similarly, $v_1 \sim_H v_n$ and $v_{n-1} \sim_G v_n$ implies $v_{n-1} \sim_\Gamma v_1$.

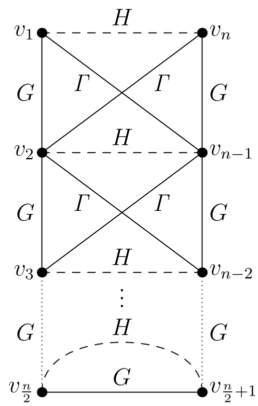


Fig. 7 Adjacency between $v_{n/2}$ and $v_{n/2+1}$ in both G and H

Noting that $v_2 \sim_\Gamma v_n$ and $v_2 \sim_G v_1 \sim_H v_n$ is a GH path, as in Remark 2, there exists a vertex v_j ($\neq v_1$) such that $v_2 \sim_H v_j$ and $v_j \sim_G v_n$. But v_{n-1} is the only vertex such that $v_{n-1} \sim_G v_n$, and thus $v_j = v_{n-1}$. This implies $v_2 \sim_H v_{n-1}$.

Continuing similarly, we get $v_3 \sim_H v_{n-2},\ v_4 \sim_H v_{n-3},\ \ldots,\ v_{\frac{n}{2}} \sim_H v_{\frac{n}{2}+1}$ as in Fig. 7. But G being a path, we have $v_{\frac{n}{2}} \sim_G v_{\frac{n}{2}+1}$ as in Fig. 7, which contradicts the fact that G and H are edge disjoint.

Thus, there is no nontrivial graph H such that $A(G)A(H)$ is graphical. $\square$

In the following remarks, we observe a few more types of graphs G that do not admit a graph H such that $A(G)A(H)$ is graphical.

Remark 6 Let G be a connected graph with degree partition $\{V_0, V_1, \ldots, V_k\}$, where $k = \Delta(G)$. If the degree partition satisfy any of the following properties, then there is no nontrivial graph H such that $A(G)A(H)$ is graphical.

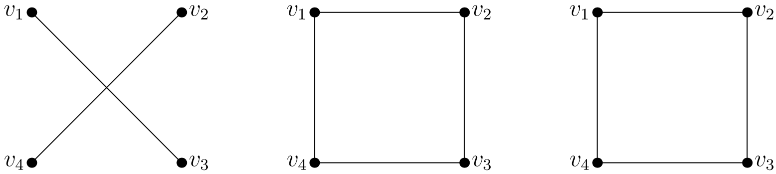
(i) A partite set among $V_i,\ 0 \le i \le k$, has exactly one vertex.
(ii) A partite set among $V_i,\ 0 \le i \le k$, has exactly two vertices, and $\overline{G}$ has no 1-factor.

Proof is immediate from Corollary 2 and Theorem 6.

3.2 *Regular Graphs G for Which $A(G)A(\overline{G})$ Is Graphical*

In this subsection, we shall characterize all regular graphs G such that $A(G)A(\overline{G})$ is graphical. In the following theorem, we shall consider the graphs with 1 or $n-2$ regularity and prove that the product $A(G)A(\overline{G})$ in such a case is always graphical.

Theorem 9 *If G is a regular graph on the set of vertices $\{v_1, v_2, \ldots, v_n\}$, with regularity 1 or $n-2$, then the product $A(G)A(\overline{G})$ is graphical. In fact, if G is*



(a) G (b) $\overline{G}$ (c) Γ

Fig. 8 $G = \overline{C_4}$ a 1-regular graph

1-*regular, then*

$$A(G)A(\overline{G}) = A(\overline{G})A(G) = A(\overline{G}),$$

and if G is $(n-2)$-*regular, then*

$$A(G)A(\overline{G}) = A(\overline{G})A(G) = A(G).$$

Proof Note that the case $n = 2$ is trivial. Further, $n = 3$ case does not arise, in which case neither G nor $\overline{G}$ can be 1-regular.

We shall prove the theorem when $n > 3$ and G is $(n-2)$-regular. The case of 1-regular G follows by symmetry as $\overline{G}$ is $(n-2)$-regular.

Consider any two vertices v_1 and v_2 in G and two distinct cases where v_1 and v_2 are adjacent and not adjacent in G.

Case I: v_1 and v_2 are adjacent in G. Since G is $(n-2)$-regular, there exists a unique vertex u such that v_2 and u are adjacent in $\overline{G}$, but u is adjacent with v_1 in G. Hence, we get a $G\overline{G}$ path $v_1 \sim_G u \sim_{\overline{G}} v_2$ between v_1 and v_2. Again by $(n-2)$-regularity of G, there is no vertex w different from u such that v_2 and w are adjacent in $\overline{G}$. Hence the uniqueness of $G\overline{G}$ path from v_1 to v_2. Similarly, we get a unique $\overline{G}G$ path from v_1 to v_2.

Case II: v_1 and v_2 are not adjacent in G. Since v_1 and v_2 are not adjacent in G, they are adjacent in $\overline{G}$ and not adjacent with any other vertex in $\overline{G}$. Therefore, there is no $G\overline{G}$ or $\overline{G}G$ path from v_1 to v_2.

From Case I and Case II above it is clear that for every pair of vertices, there exists at most a pair of $G\overline{G}$ and $\overline{G}G$ paths, and from Theorem 1 we get that $A(G)A(\overline{G})$ is graphical. Again referring to the cases, we observe that a pair of vertices are adjacent (not adjacent) in G implies that the vertices are adjacent (not adjacent) in the matrix product Γ of G and $\overline{G}$. Therefore, we get that $\Gamma = G$ or $A(G)A(\overline{G}) = A(G) = A(\overline{G})A(G)$. □

In Figs. 8 and 9, we demonstrate the above theorem for the cases of 1-regular and $(n-2)$-regular G.

Remark 7 In Theorem 9, we have observed that $A(G)A(\overline{G}) = A(G)$ whenever G is $(n-2)$-regular. In other words, $A(\overline{G})$, where $\overline{G}$ is a 1-factor, acts like a projector on the column and row spaces of $A(G)$. But it is not true that $A(\overline{G})$ is a projector.

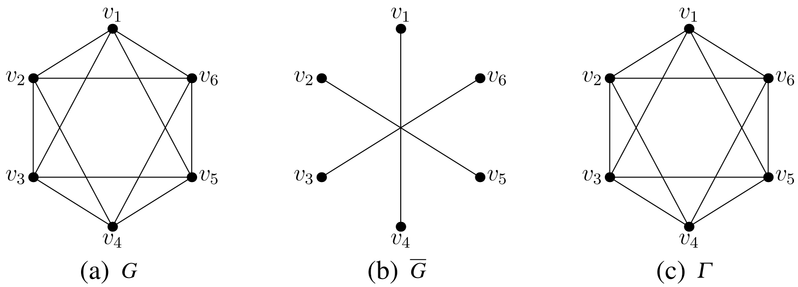


Fig. 9 G a cocktail party graph on six vertices, a $(6-2)$-regular graph

The following theorem characterizes the regular graphs G for which $A(G)A(\overline{G})$ is graphical.

Theorem 10 *Given a regular graph G, if the nontrivial product $A(G)A(\overline{G})$ is graphical, then G is exactly one of the following*:

(i) *G is* 1*-regular*
(ii) *G is* $(n-2)$*-regular* (*called a cocktail party graph on n vertices*)
(iii) $G = C_5$.

Proof Consider a regular graph G with regularity k such that $A(G)A(\overline{G})$ is graphical. Clearly, $\overline{G}$ is also regular with regularity $n-k-1$. By Theorem 2, the graph Γ realizing $A(G)A(\overline{G})$ is also regular with regularity $k(n-k-1)$. Therefore, $k(n-k-1) \leq n-1$, and we get

$$n \leq \frac{k}{k-1} + (k+1) \tag{6}$$

for $k \neq 1$.

In the case of $k = 1$, i.e., G is 1-regular in the class of graphs described in (i) of the theorem. In fact, from Theorem 9 we have that $A(G)A(\overline{G})$ is graphical for every 1-regular graph G.

If $k \neq 1$, note that $1 \leq \dfrac{k}{k-1} \leq 2$, and by substituting the upper limit in (6), we get that $n \leq k+3$ or $k \geq n-3$. In other words,

$$k = n-1,\ n-2,\ \text{or}\ n-3.$$

If $k = n-1$, G is the complete graph and $\overline{G}$ is the null graph, which is a trivial case.

If $k = n-2$, then the graph G is $(n-2)$-regular in the class of graphs described in (ii). Again referring to Theorem 9, we know that for every $(n-2)$-regular graph G, $A(G)A(\overline{G})$ is graphical.

If $k = n-3$ and $k \neq 1$, then the regularity of $\overline{G}$ is 2, and therefore,

$$(n-3)2 \leq n-1 \quad \Longrightarrow \quad n = 5.$$

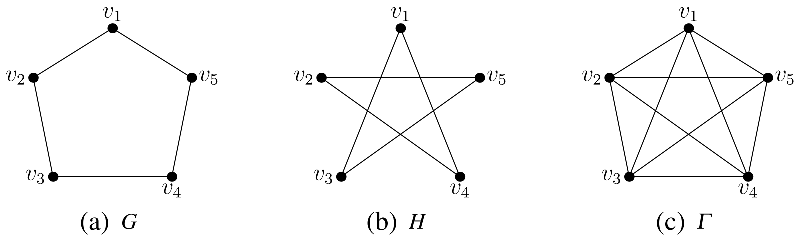


Fig. 10 $\Gamma = K_5$, which is the product of $G = C_5$ and $H = \overline{C_5}$

In other words, $G = C_5$.

Note that for a graph $G = C_5$, also $\overline{G} = \overline{C_5} \simeq C_5$. In $G \cup \overline{G}$, it is easy to observe that between every two vertices in $\{v_1, v_2, v_3, v_4, v_5\}$, there exists a unique $G\overline{G}$ path and a unique $\overline{G}G$ path. Thus, $A(G)A(\overline{G})$ is graphical, and the matrix product Γ is a complete graph K_5 (see Fig. 10).

Hence the theorem. □

Remark 8 Whenever H is a 1-factor graph and Γ is the matrix product of G and H, then it is also true that G is the matrix product of Γ and H, that is, $A(\Gamma)A(H) = A(G)$. This follows from Theorem 9 and the fact that $A(H)$ is a permutation matrix and is its own inverse.

References

1. Akbari, S., Moazami, F., Mohammadian, A.: Commutativity of the adjacency matrices of graphs. Discrete Math. **309**(3), 595–600 (2009)
2. Buckley, A., Harary, F.: Distance in Graphs. Addison-Wesley, Reading (1990)
3. West, D.B.: Introduction to Graph Theory. Pearson Education, Singapore (2002)

Determinant of the Laplacian Matrix of a Weighted Directed Graph

Debajit Kalita

Abstract The notion of weighted directed graph is a generalization of mixed graphs. In this article a formula for the determinant of the Laplacian matrix of a weighted directed graph is obtained. It is a generalization of the formula for the determinant of the Laplacian matrix of a mixed graph obtained by Bapat et al. (Linear Multilinear Algebra 46:299–312, 1999).

Keywords Laplacian matrix · Mixed graph · Weighted directed graph · 3-Colored digraph · Essential spanning subgraph

Mathematics Subject Classification (2010) 05C50 · 05C05 · 15A18

1 Introduction

Throughout this article all our graphs are simple. All our directed graphs have simple *underlying* undirected graphs. At times we use $V(G)$ (resp. $E(G)$) to denote the set of vertices (resp. edges) of a graph G (directed or undirected). In the absence of any specification, $V(G)$ is assumed to be the set $\{1, 2, \dots, n\}$. We write $(i, j) \in E(G)$ to mean the existence of the directed edge from a vertex i to a vertex j. Sometimes it is convenient to denote $(i, j) \in E(G)$ by $i \rightsquigarrow j$. Throughout this article, $\imath = \sqrt{-1}$.

Definition 1 (Bapat et al. [1]) Let G be a directed graph. With each edge (i, j) in $E(G)$, we associate a complex number w_{ij} of unit absolute value and nonnegative imaginary part. We call it the weight of that edge. A directed graph G with such a weight function w is called a *weighted directed graph*. The *adjacency matrix* $A(G)$ of G is the matrix with ijth entry

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$$a_{ij} = \begin{cases} \mathsf{w}_{ij} & \text{if } i \rightsquigarrow j, \\ \overline{\mathsf{w}}_{ji} & \text{if } j \rightsquigarrow i, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a weighted directed graph. In defining subgraph, walk, path, component, connectedness, and degree of a vertex in G, we focus only on the underlying unweighted undirected graph of G. Thus the *degree* d_i of a vertex i in a weighted directed graph may be viewed as the sum of absolute values of the weights of the edges incident with vertex i.

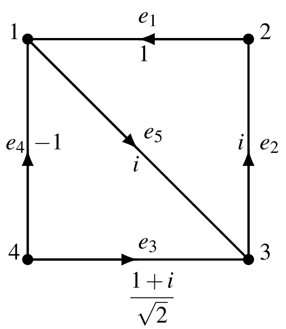
Definition 2 (Bapat et al. [1]) Let G be a weighted directed graph. The *Laplacian matrix* $L(G)$ of G is defined as the matrix $D(G) - A(G)$, where $D(G)$ is the diagonal matrix with d_i as the ith diagonal entry.

Definition 3 (Bapat et al. [1]) The *vertex edge incidence* matrix $M = M(G) = [m_{i,e}]$ of a weighted directed graph G is defined as the matrix with rows labeled by the vertices and columns labeled by the edges in G satisfying

$$m_{i,e} = \begin{cases} 1 & \text{if } e = (i, j) \text{ for some vertex } j, \\ -\overline{\mathsf{w}}_{ij} & \text{if } e = (j, i) \text{ for some vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1 Consider the weighted directed graph G as shown below. Weights of the edges are written beside them. The vertex edge incidence matrix $M(G)$ of G is supplied.

G



$$M(G) = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & i & 0 & 0 & 0 \\ 0 & 1 & \frac{-1+i}{\sqrt{2}} & 0 & i \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A *mixed graph* is a graph with some directed and some undirected edges. Let G be a weighted directed graph. If the weights of the edges in G are ± 1, then *viewing the edges of weight* 1 *as directed and the edges of weight* -1 *as undirected*, we see that $M(G)$ coincides with the vertex edge incidence matrix of a mixed graph introduced by Bapat et al. [2]. Notice that $(MM^*)_{ii} = d_i$ for $i \neq j$, $(MM^*)_{ij} = -\mathsf{w}_{ij}$ if $i \rightsquigarrow j$, $(MM^*)_{ij} = -\overline{\mathsf{w}}_{ji}$ if $j \rightsquigarrow i$, and $(MM^*)_{ij} = 0$ otherwise. Thus we see that $L(G) = MM^*$, which implies that the Laplacian matrix of a weighted directed graph is positive semi-definite.

It was observed in Bapat et al. [1] that unlike the Laplacian matrix of an unweighted undirected graph, the Laplacian matrix of a weighted directed graph is sometimes non-singular. A weighted directed graph is said to be *singular* (*resp. non-singular*) if its Laplacian matrix is singular (resp. non-singular). Several characterizations of singularity of the weighted directed graphs were provided in [1].

Definition 4 Let G be a weighted directed graph. An i_1–i_k-*walk* W in G is a finite sequence $i_1, \ldots, i_k$ of vertices such that, for $1 \le p \le k-1$, either $i_p \rightsquigarrow i_{p+1}$ or $i_{p+1} \rightsquigarrow i_p$. If $e = (i_p, i_{p+1}) \in E(G)$, then we say that e is *directed along* the walk. Otherwise, we say that e is *directed opposite* to the walk. We call

$$\mathsf{w}_W = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$$

the weight of the walk W, where a_{ij} are the entries of $A(G)$.

2 Determinant of the Laplacian Matrix of a Weighted Directed Graph

In this section we describe the determinant of the Laplacian matrix of a weighted directed graph. The following lemma gives the determinant of the Laplacian matrix of a cycle in a weighted directed graph.

Lemma 1 *Let C be a weighted directed graph whose underlying unweighted undirected graph is a cycle. Then* $\det(L(C)) = 2(1 - \operatorname{Re} \mathsf{w}_C)$.

Proof Consider the vertex–edge incident matrix $M(C)$ corresponding to C. We may assume that $C = [1, 2, \ldots, n, 1]$ so that the edges e_i in C have end vertices i and $i+1$ for $i \in \mathbb{Z}_n$ and $m_{1,e_1} = 1$ after relabeling the vertices if necessary. Note that the non-zero entries of $M(C)$ occur precisely at the positions m_{i,e_i} and m_{i+1,e_i} for each i. In that case, expanding along the first row of $M(C)$, we see that

$$\det\bigl(M(C)\bigr) = \prod_{\substack{(i+1,i) \in E(C) \\ i \in \mathbb{Z}_n}} (-\overline{\mathsf{w}}_{i+1,i}) - (-1)^n \prod_{\substack{(i,i+1) \in E(C) \\ i \in \mathbb{Z}_k}} (-\overline{\mathsf{w}}_{i,i+1}).$$

Since $L(C) = M(C)M(C)^*$, we see that $\det(L(C)) = 2(1 - \operatorname{Re} \mathsf{w}_C)$. □

In view of Lemma 1, we call a cycle in a weighted directed graph *singular* (resp. non-singular) if its weight is different from 1.

The next lemma gives the determinant of the Laplacian matrix of a unicyclic weighted directed graph.

Lemma 2 *Let G be a connected unicyclic weighted directed graph with a cycle C. Then* $\det(L(G)) = 2(1 - \operatorname{Re} \mathsf{w}_C)$.

Proof If G is the cycle C itself, then the result follows immediately from Lemma 1. Otherwise, G has a pendent vertex, say i. Let j be the vertex adjacent to i in G with an edge e of weight w. We may assume, after a permutation similarity, that the first row and the first column of $M(G)$ correspond to the vertex i and the edge e, respectively. Then expanding along the first row, we see that if $e = (i, j)$, then $\det(M(G)) = \det(M(G'))$, otherwise $\det(M(G)) = (-\overline{\mathsf{w}}) \det(M(G'))$, where G' is the weighted directed graph obtained from G by deleting the vertex i. Hence, in any case, $\det(L(G)) = \det(L(G'))$. Continuing similarly, after finitely many steps we see that $\det(L(G)) = \det(L(C))$. Hence the result holds by Lemma 1. □

Remark 1 By Lemma 2, a connected unicyclic weighted directed graph G is non-singular if and only if the cycle in G is non-singular.

Definition 5 Let G be a connected weighted directed graph. We call a subgraph H an *essential spanning subgraph* of G if $V(G) = V(H)$ and every component of H is a non-singular unicyclic weighted directed graph. By $\mathscr{E}(G)$ we denote the class of all essential spanning subgraphs of G.

The following lemma contained in Bapat et al. [1] provides the class of non-singular weighted directed graphs.

Lemma 3 (Bapat et al. [1], Corollary 9) *Let G be a connected weighted directed graph. Then G is non-singular if and only if it contains a non-singular cycle. In particular, a weighted directed tree is always singular.*

The next lemma says that a non-singular weighted directed graph contains at least one essential spanning subgraph, and vice versa.

Lemma 4 *Let G be a connected weighted directed graph. Then G is non-singular if and only if $\mathscr{E}(G)$ is nonempty.*

Proof Assume that G is non-singular. By Lemma 3, G contains a non-singular cycle. Thus, G has a connected non-singular unicyclic spanning subgraph. Hence, $\mathscr{E}(G)$ is nonempty.

Conversely, suppose that $\mathscr{E}(G)$ is nonempty. Thus, G contains a non-singular cycle. Hence, G is non-singular by Lemma 3. □

The next result describes the determinant of the Laplacian matrix of a weighted directed graph in terms of the determinants of its essential spanning subgraphs.

Lemma 5 *Let G be a non-singular connected weighted directed graph. Then*

$$\det\big(L(G)\big) = \sum_{H\in\mathscr{E}(G)} \det L(H).$$

Proof Since $L(G) = MM^*$, by the Cauchy–Binet theorem (see [3]) we know that

$$\det L(G) = \sum_{\substack{E'\subseteq E(G)\\ |E'|=|V(G)|}} \det M\big[V(G), E'\big] \det M\big[V(G), E'\big]^*,$$

where $M[V(G), E']$ is a square submatrix of M. Note that $M[V(G), E']$ is the vertex–edge incident matrix of a spanning subgraph of G, say $H_{E'}$, with the edge set E' such that $|E'| = |V(G)|$. Thus, $L(H_{E'}) = M[V(G), E']M[V(G), E']^*$. Note that $\det L(H_{E'}) \neq 0$ if and only if each component of $H_{E'}$ is non-singular. Since $|E'| = |V(G)|$, we see that each component of $H_{E'}$ is a unicyclic weighted directed graph. Thus, $\det L(H'_E) \neq 0$ if and only if $H_{E'} \in \mathscr{E}(G)$. Hence the result holds. □

Let G be a weighted directed graph, and let H be an essential spanning subgraph of G. By $\omega(H)$ we denote the number of components of H. By $C_i(H)$, $1 \leq i \leq \omega(H)$, we denote the cycles contained in H.

The following theorem is our main result of this section, which provides a formula for the determinant of the Laplacian matrix of a weighted directed graph.

Theorem 1 *Let G be a non-singular connected weighted directed graph. Then*

$$\det\big(L(G)\big) = \sum_{H\in\mathscr{E}(G)} 2^{\omega(H)} \prod_{i=1}^{\omega(H)} (1 - \operatorname{Re} \mathrm{w}_{C_i(H)}).$$

Proof Proof follows from Lemma 5 and Lemma 2. □

Definition 6 (Bapat et al. [1]) Let G be a directed graph with edges having colors red, blue, or green. Assign a weight 1 to each red edge, a weight -1 to each blue edge, and a weight $\imath$ to each green edge in G. We call the resulting graph a 3-*colored digraph.*

Observe that the weight of a cycle in a 3-colored digraph is either 1, -1, or $\pm\imath$. Thus, the weight of a non-singular cycle in a 3-colored digraph is either -1 or $\pm\imath$. Hence, the following result is an immediate consequence of Theorem 1.

Corollary 1 *Let G be a non-singular connected* 3*-colored digraph. Then*

$$\det L(G) = \sum_{H\in\mathscr{E}(G)} 2^{2\omega_1(H)+\omega_2(H)},$$

where $\omega_1(H)$ *and* $\omega_2(H)$ *denotes the number of cycles of weight* -1 *and* ± 1 *in* H*, respectively.*

Note that a mixed graph may be viewed as a 3-colored digraph without green edges. Thus, the following result obtained by Bapat et al. [2] is an immediate consequence of Corollary 1.

Corollary 2 (Bapat et al. [2], Corollary 2) *Let G be a non-singular connected mixed graph. Then*

$$\det L(G) = \sum_{H \in \mathscr{E}(G)} 4^{\omega_1(H)},$$

where $\omega_1(H)$ *denotes the number of cycles of weight* -1 *in* H.

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References

1. Bapat, R.B., Kalita, D., Pati, S.: On weighted directed graphs. Linear Algebra Appl. **436**, 99–111 (2012)
2. Bapat, R.B., Grossman, J.W., Kulkarni, D.M.: Generalized matrix tree theorem for mixed graphs. Linear Multilinear Algebra **46**, 299–312 (1999)
3. Marcus, M., Minc, H.: Survey of Matrix Theory and Matrix Inequalities. Allyan and Bacon, Boston (1964)

From Multivariate Skewed Distributions to Copulas

Tõnu Kollo, Anne Selart, and Helle Visk

Abstract In this paper, a methodology is presented for constructing skewed multivariate copulas to model data with possibly different marginal distributions. Multivariate skew elliptical distributions are transformed into corresponding copulas in the similar way as the Gaussian copula and the multivariate t-copula are constructed. Three-parameter skew elliptical distributions are under consideration. For parameter estimation of the skewed distributions, the method of moments is used. To transform mixed third-order moments into a parameter vector, the star product of matrices is used; for star product and its applications, see, for example, Kollo (J. Multivar. Anal. 99:2328–2338, 2008) or Visk (Commun. Stat. 38:461–470, 2009). Results of the first applications are shortly described and referred to.

Keywords Method of moments · Multivariate skewness · Skew normal copula · Skew normal distribution · Skew t-copula · Skew t-distribution

Mathematics Subject Classification (2010) 62E17 · 62F10 · 62H12

1 Introduction

In this paper, we consider a construction of copulas from multivariate skew symmetric elliptical distributions. Copula models have become extremely popular in applications, especially in financial and environmental studies. In applications, marginal distributions are often skewed, heavy tailed, and belong to different parametric families. In such situations, the copula approach gives us, in fact, the only way to model

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the data with certain dependence structure. In Sect. 2, we consider three families of skew elliptical distributions and apply the method of moments to find estimates of the shifted asymmetric Laplace distribution, skew normal distribution, and skew t-distribution. The method of moments gives us biased estimates, and as these are easy to find, they can be used as initial values of parameters for further study when applying, for instance, maximum likelihood method. In Sect. 3, we define the skew normal copula and skew t-copula. Estimation of the copula parameters is considered, and results of some applications are discussed.

2 Skew-Symmetric Distributions: Notions, Notation, Estimation

Since the multivariate skew normal distribution was introduced in Azzalini and Dalla Valle [2], the model has become popular, and their construction has been generalized in different ways. Several generalizations up to 2004 can be found in the collective monograph Genton [3]. The idea to transform a symmetric multivariate elliptical distribution by a distribution function or a function with similar mathematical properties has become extremely fruitful. There are at least two important properties to be pointed out. Firstly, the distributions are easy to simulate; secondly, the moment generating function $M(\mathbf{t})$ often has a relatively simple analytic form, and therefore the first moments can be easily found by differentiation of $M(\mathbf{t})$. This gives us a possibility to estimate parameters by the method of moments. In the two-parameter case, a shape parameter is a vector, and a scale parameter is a $p \times p$ positive definite matrix. Often a parameter-vector of shift is added, and then a three-parameter distribution family is under consideration. In some generalizations, the number of parameters is much bigger (see, for instance, Gonzáles-Farías et al. [4] and Gupta et al. [5]). To apply the method of moments in the case of three-parameter family, the first three moments are needed. Typically, these three parameters are two vectors and one positive definite matrix. The first three moments form matrices with growing dimensionalities. Let a p-vector $\mathbf{x}$ have a skew-symmetric distribution with shift parameter $\boldsymbol{\mu}$: $p \times 1$, the shape parameter $\boldsymbol{\alpha}$: $p \times 1$, the scale parameter $\boldsymbol{\Sigma} > 0$: $p \times p$, and the moment generating function $M(\mathbf{t})$. The moments $m_k(\mathbf{x})$ can be found as the matrix derivatives of $M(\mathbf{t})$:

$$m_k(\mathbf{x}) = \left.\frac{d^k M(\mathbf{t})}{d\mathbf{t}^k}\right|_{\mathbf{t}=\mathbf{0}}, \quad k = 1, 2, \ldots .$$

The matrix derivative of an $r \times s$-matrix $\mathbf{Y}$ by a $p \times q$-matrix $\mathbf{X}$ is defined as the $rs \times pq$-matrix (for properties and higher-order derivatives, see Magnus and Neudecker [14] or Kollo and von Rosen [9, Sect. 1.4])

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \frac{1}{d'\mathbf{X}} \otimes \mathbf{Y}.$$

The central moments $\overline{m}_k(\mathbf{x})$ are defined via the moments:

$$\overline{m}_k(\mathbf{x}) = m_k(\mathbf{x} - E\mathbf{x}).$$

It follows from the definitions of moments and the matrix derivative that $m_1(\mathbf{x})$ is a transposed p-vector, $m_2(\mathbf{x})$ is a $p \times p$ matrix, and $m_3(\mathbf{x})$ is given by a $p^2 \times p$-matrix of the third-order mixed moments. To apply the method of moments, we have to transform the $p^2 \times p$-matrix of the third-order moments into a p-vector in a meaningful way. This idea can be realized by the star-product of matrices. This operation appeared to be useful when defining multivariate skewness and kurtosis characteristics (Kollo [7]).

Classical measures of skewness and kurtosis were introduced by Mardia [15]. Unfortunately, these scalar characteristics may have the same numerical values for multivariate distributions with different shape. There have been suggestions to solve this problem by defining multivariate characteristics of skewness and kurtosis (see, for example, Móri, Rohatgi and Székely [16] and Koziol [11]). But these suggested characteristics do not take into account all mixed moments/cumulants of the third and fourth orders. Kollo [7] suggests a multivariate skewness measure as a p-vector and a kurtosis characteristic in the form of a $p \times p$-matrix. These characteristics take into account all mixed moments of the third and fourth orders and are defined with help of the star-product of matrices. This notion has been introduced by MacRae [13] as a generalization of the trace function.

Definition 1 Let $\mathbf{A}$ be an $m \times n$ matrix, and $\mathbf{B}$ an $mr \times ns$ partitioned matrix consisting of $r \times s$-blocks $\mathbf{B}_{ij}$, $i = 1, \dots, m,\ \ j = 1, \dots n$. The star-product $\mathbf{A} \star \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is the $r \times s$-matrix

$$\mathbf{A} \star \mathbf{B} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \mathbf{B}_{ij}.$$

The big matrix of the third-order moments will be compressed into a p-vector with help of the star-product in the same way as the skewness vector is defined (Kollo [7]).

Definition 2 Let $\mathbf{x}$ be a random p-vector. Then a p-vector $\mathbf{b}(\mathbf{x})$ is called the skewness vector of $\mathbf{x}$ if

$$\mathbf{b}(\mathbf{x}) = \mathbf{1}_{p \times p} \star m_3(\mathbf{y}),$$

where

$$\mathbf{y} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{x} - \boldsymbol{\mu}),$$

$\boldsymbol{\Sigma}^{1/2}$ stands for any symmetric square root of $\boldsymbol{\Sigma}$, and $\mathbf{1}_{p \times p}$ is the $p \times p$-matrix of ones.

Let us apply the method of moments to three-parameter distributions. First, we consider estimation of the parameters of the shifted Laplace distribution (Visk [19]) with the characteristic function

$$\varphi_{\mathbf{x}+\mathbf{a}}(\mathbf{t}) = \frac{e^{i\mathbf{t}'\mathbf{a}}}{1 - i\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}.$$

The first three moments are of the form

$$E\mathbf{x} = \mathbf{a} + \boldsymbol{\mu}, \qquad D\mathbf{x} = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}',$$
$$\overline{m}_3(\mathbf{x}) = 2\boldsymbol{\mu} \otimes \boldsymbol{\mu}\boldsymbol{\mu}' + \operatorname{vec}\boldsymbol{\Sigma} \cdot \boldsymbol{\mu}' + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}.$$

We have three parameters, and we need to transform the third central moment into a p-vector. We shall use the star-product:

$$\mathbf{1}_{p\times p} \star \overline{m}_3(\mathbf{x}) = \mathbf{1}_{p\times p} \star \left(2\boldsymbol{\mu} \otimes \boldsymbol{\mu}\boldsymbol{\mu}' + \operatorname{vec}\boldsymbol{\Sigma} \cdot \boldsymbol{\mu}' + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}\right).$$

Denote

$$M = \sum_{i=1}^{p} \mu_i; \qquad S = \sum_{i=1}^{p}\sum_{j=1}^{p} (D\mathbf{x})_{ij}; \qquad H = \sum_{i=1}^{p^2}\sum_{j=1}^{p} \left(\overline{m}_3(\mathbf{x})\right)_{ij}.$$

Then the star products of interest are

$$\mathbf{1}_{p\times p} \star \left(\boldsymbol{\mu} \otimes \boldsymbol{\mu}\boldsymbol{\mu}'\right) = M^2\,\boldsymbol{\mu};$$
$$\mathbf{1}_{p\times p} \star (\operatorname{vec}\boldsymbol{\Sigma} \cdot \boldsymbol{\mu}) = M \cdot D\mathbf{x} \cdot \mathbf{1}_p - M^2\,\boldsymbol{\mu};$$
$$\mathbf{1}_{p\times p} \star (\boldsymbol{\mu} \otimes \boldsymbol{\Sigma}) = M \cdot D\mathbf{x} \cdot \mathbf{1}_p - M^2\boldsymbol{\mu};$$
$$\mathbf{1}_{p\times p} \star (\boldsymbol{\Sigma} \otimes \boldsymbol{\mu}) = \left(S - M^2\right)\boldsymbol{\mu},$$

where $\mathbf{1}_p$ is the p-vector of ones. The parameter $\boldsymbol{\mu}$ can be expressed as

$$\boldsymbol{\mu} = \frac{1}{S - M^2}(\mathbf{s} - 2M \cdot D\mathbf{x} \cdot \mathbf{1}_p),$$

where $\mathbf{s} = \mathbf{1}_{p\times p} \star \overline{m}_3(\mathbf{x})$. To estimate $\boldsymbol{\mu}$, we need to know M.

Summing the coordinates of $\boldsymbol{\mu}$ brings us to the cubic equation

$$M^3 - 3MS + H = 0.$$

The only real solution of the equation is

$$M_* = \sqrt[3]{-4H + \sqrt{H^2 - 4S^3}},$$

and the parameters can be expressed as follows:

$$\boldsymbol{\mu} = \frac{1}{S - M_*^2}(\mathbf{s} - 2M_* \cdot D\mathbf{x} \cdot \mathbf{1}_p); \qquad \boldsymbol{\Sigma} = D\mathbf{x} - \boldsymbol{\mu}\boldsymbol{\mu}'; \qquad \mathbf{a} = E\mathbf{x} - \boldsymbol{\mu}.$$

Let us examine now the skew normal distribution $SN_p(\boldsymbol{\mu}, \mathbf{R}, \boldsymbol{\alpha})$ with the density

$$f_{p,SN}(\mathbf{x}, \boldsymbol{\mu}, \mathbf{R}, \boldsymbol{\alpha}) = 2 f_{N_p(\boldsymbol{\mu},\mathbf{R})}(\mathbf{x}) \Phi\left(\boldsymbol{\alpha}'(\mathbf{x} - \boldsymbol{\mu})\right), \tag{1}$$

where $\mathbf{R}$ is the correlation matrix, and $\Phi(\cdot)$ the distribution function of the univariate standard normal distribution. Expectation and dispersion matrix are of the form (Gupta and Kollo [6])

$$E\mathbf{x} = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}, \qquad D\mathbf{x} = \mathbf{R} - \frac{2}{\pi}\boldsymbol{\delta}\boldsymbol{\delta}',$$

where

$$\boldsymbol{\delta} = \frac{\mathbf{R}\boldsymbol{\alpha}}{\sqrt{1+\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha}}}.$$

The third central moment equals

$$\overline{m}_3(\mathbf{x}) = \sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\boldsymbol{\delta}\otimes\boldsymbol{\delta}'\otimes\boldsymbol{\delta}.$$

Then

$$\mathbf{1}_{p\times p}\star\overline{m}_3(\mathbf{x}) = \mathbf{1}_{p\times p}\star\sqrt{\frac{2}{\pi}}\left(\frac{4}{\pi}-1\right)\boldsymbol{\delta}\otimes\boldsymbol{\delta}'\otimes\boldsymbol{\delta}.$$

Fortunately,

$$\mathbf{1}_{p\times p}\star\boldsymbol{\delta}\otimes\boldsymbol{\delta}'\otimes\boldsymbol{\delta} = M^2\,\boldsymbol{\delta};\quad M=\sum_{i=1}^{p}\delta_i.$$

From here we have

$$\boldsymbol{\delta} = \mathbf{1}_{p\times p}\star\overline{m}_3(\mathbf{X})\frac{\pi^{3/2}}{M^2\sqrt{2}(4-\pi)}.$$

Summing the coordinates on both sides of the last equality gives us

$$M=\frac{H\pi^{3/2}}{M^2\sqrt{2}(4-\pi)};\quad H=\sum_{i=1}^{p^2}\sum_{j=1}^{p}\big(\overline{m}_3(\mathbf{x})\big)_{ij}.$$

Then

$$M = H^{1/3}\frac{\sqrt{\pi}}{(\sqrt{2}(4-\pi))^{1/3}}$$

and

$$\boldsymbol{\delta} = \mathbf{1}_{p\times p}\star\overline{m}_3(\mathbf{x})\frac{\pi H^{2/3}}{(2(4-\pi)^2)^{1/3}}.$$

The estimates are obtained from the sample estimates of moments:

$$\hat{\boldsymbol{\delta}} = \mathbf{1}_{p\times p}\star\widehat{\overline{m}_3}(\mathbf{x})\frac{\pi\hat{H}^{2/3}}{(2(4-\pi)^2)^{1/3}}, \tag{2}$$

$$\hat{\boldsymbol{\mu}} = \overline{\mathbf{x}} - \sqrt{\frac{2}{\pi}}\hat{\boldsymbol{\delta}}, \tag{3}$$

$$\hat{\boldsymbol{\Sigma}} = \mathbf{S} + \frac{2}{\pi}\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\delta}}', \tag{4}$$

$$\hat{\boldsymbol{\alpha}} = \frac{(\mathbf{S}+\frac{2}{\pi}\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\delta}}')^{-1}\hat{\boldsymbol{\delta}}}{\sqrt{1-\hat{\boldsymbol{\delta}}'(\mathbf{S}+\frac{2}{\pi}\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\delta}}')^{-1}\hat{\boldsymbol{\delta}}}}. \tag{5}$$

Here $\overline{\mathbf{x}}$ and $\mathbf{S}$ denote the sample mean and the sample covariance matrix, respectively. Next, we consider the p-dimensional skew $t_{p,\nu}$-distribution where ν stands for the number of degrees of freedom. There are several modifications and extensions of the standard multivariate $t_{p,\nu}$-distribution; an overview can be found in Kotz and Nadarajah [10, Chap. 5] We shall follow the definition of Azzalini and Capitanio [1]. The density of the multivariate $t_{p,\nu}$-distribution is

$$t_{p,\nu}(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{\Gamma(\frac{\nu+p}{2})}{(\pi\nu)^{\frac{p}{2}}\Gamma(\frac{\nu}{2})|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\left[1+\frac{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right]^{-\frac{\nu+p}{2}},$$

where the scale parameter $\boldsymbol{\Sigma}$ is a positive definite $p \times p$-matrix, and the location parameter $\boldsymbol{\mu}$ is a p-vector.

Definition 3 A random p-vector $\mathbf{x} = (X_1, \ldots, X_p)'$ has p-variate skew $t_{p,\nu}$-distribution with parameters $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\Sigma}$ if its density function is of the form

$$g_{p,\nu}(\mathbf{x};\boldsymbol{\mu},\ \boldsymbol{\Sigma},\boldsymbol{\alpha}) = 2\cdot t_{p,\nu}(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})\cdot T_{1,\nu+p}\left[\boldsymbol{\alpha}^T\ \boldsymbol{\Sigma}_d^{-1/2}(\mathbf{x}-\ \boldsymbol{\mu})\left(\frac{\nu+p}{Q+\nu}\right)^{\frac{1}{2}}\right], \tag{6}$$

where Q denotes the quadratic form

$$Q = (\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}),$$

$T_{1,\nu+p}(\cdot)$ is the distribution function of the central univariate t-distribution with $\nu+p$ degrees of freedom, and $\boldsymbol{\Sigma}_d$ is the diagonalized matrix $\boldsymbol{\Sigma}$.

Later we shall consider the case $\boldsymbol{\mu} = \mathbf{0}$. For estimation, we need the first two moments:

$$E\mathbf{x} = \boldsymbol{\Sigma}_d^{1/2}\xi,$$
$$D\mathbf{x} = \frac{\nu}{\nu-2}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_d^{1/2}\xi\xi'\boldsymbol{\Sigma}_d^{1/2},$$

where

$$\xi = \left[\frac{\nu}{\pi(1+\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha})}\right]^{1/2}\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\mathbf{R}\boldsymbol{\alpha},$$

and $\mathbf{R}$ is the correlation matrix. The estimates from these moment expressions are of the form (Kollo and Pettere [8]):

$$\widehat{\boldsymbol{\Sigma}} = \frac{\nu-2}{\nu}\left(\mathbf{S}+\overline{\mathbf{x}}\,\overline{\mathbf{x}}'\right); \tag{7}$$

$$\widehat{\boldsymbol{\alpha}} = \frac{b(\nu)\boldsymbol{\beta}}{\sqrt{b^2(\nu)-\overline{\mathbf{x}}'\widehat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{x}}}}, \tag{8}$$

where $\overline{\mathbf{x}}$ and $\mathbf{S}$ are the sample mean and the sample covariance matrix,

$$\boldsymbol{\beta} = \frac{1}{b(\nu)}\widehat{\boldsymbol{\Sigma}}_d^{1/2}\widehat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{x}} \tag{9}$$

and $\widehat{\boldsymbol{\Sigma}}_d^{1/2} = (\delta_{ij}\sqrt{\widehat{\sigma}_{ij}})$, $i, j = 1, \dots p$, where δ_{ij} is the Kronecker delta, and

$$b(\nu) = \left[\frac{\nu}{\pi}\right]^{\frac{1}{2}} \cdot \frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}. \tag{10}$$

3 Copulas from Skewed Distributions

There are only few skewed copulas in use. Archimedean copulas are axial symmetric by construction, the copulas built via elliptical multivariate distributions are also symmetric (for instance, Gaussian and t-copula). It seems natural to join skewed marginals into a multivariate distribution by a skewed copula density. Skew elliptical families give a good possibility for that. In the following, we shall apply this approach to multivariate skew normal and skew t-distributions.

Let $X_1, \dots, X_p$ be continuous random variables with strictly monotone distribution functions $F_i(x_i)$ and the density functions $f_i(x_i)$, respectively. Let their joint distribution function and the density function be $F_{\mathbf{x}}(x_1, \dots, x_p)$ and $f_{\mathbf{x}}(x_1, .., x_p)$. By Sklar's theorem (see, for example, Nelsen [17]), the distribution function $F_{\mathbf{x}}(x_1, \dots, x_p)$ can be presented through a copula $C(u_1, \dots, u_p) : [0, 1]^p \to [0, 1]$, $\mathbf{u} = (u_1, \dots, u_p)' \in [0, 1]^p$:

$$F_{\mathbf{x}}(x_1, \dots, x_p) = C\big(F_1(x_1), \dots, F_p(x_p)\big). \tag{11}$$

The copula density $c(u_1, \dots, u_p)$ is obtained from the copula $C(u_1, \dots, u_p)$ by differentiation:

$$c(u_1, \dots, u_p) = \frac{\partial^p C(u_1, \dots, u_p)}{\partial u_1 \cdots \partial u_p}.$$

Taking into account (11), we can present the density $f_{\mathbf{x}}(x_1, \dots, x_p)$ through the copula density

$$f_{\mathbf{x}}(x_1, \dots, x_p) = c\big(F_1(x_1), \dots, F_p(x_p)\big) f_1(x_1) \times \cdots \times f_p(x_p)\ .$$

From here the copula density $c(\mathbf{u}) : I^p \to R$ can be expressed through the densities of $\mathbf{x}$ and X_i, $i = 1, \dots, p$:

$$c(\mathbf{u}) = \frac{f_{\mathbf{x}}(F_1^{-1}(u_1), \dots, F_p^{-1}(u_p))}{f_1(F_1^{-1}(u_1)) \times \cdots \times f_p(F_p^{-1}(u_p))},$$

where $F_1(\cdot), \dots, F_p(\cdot)$ are the univariate marginal distribution functions, and $f_1(\cdot), \dots, f_p(\cdot)$ are the corresponding marginal densities. We are going to construct skewed copulas that are based on the multivariate skew elliptical distributions. Let us first define a skew normal copula.

Definition 4 A p-dimensional copula $C_{p,SN}$ is called a skew normal copula with parameters $\boldsymbol{\mu}$, $\mathbf{R}$, and $\boldsymbol{\alpha}$ if

$$C_{p,SN}(\mathbf{u}; \boldsymbol{\mu}, \mathbf{R}, \boldsymbol{\alpha}) = F_{p,SN}\big(F_1^{-1}(u_1; \mu_1, 1, \alpha_1), \dots, F_1^{-1}(u_p; \mu_p, 1, \alpha_p); \boldsymbol{\mu}, \mathbf{R}, \boldsymbol{\alpha}\big),$$

where $F_1^{-1}(u_i;\mu_i,1,\alpha_i)$ denotes the inverse of the distribution function of the univariate skew normal distribution $SN(\mu_i,1,\alpha_i)$, and $F_{p,SN}(\cdot)$ is the distribution function of the p-variate skew normal distribution with the density given in (1).

The corresponding copula density is

$$c_{p,SN}(\mathbf{u};\boldsymbol{\mu},\mathbf{R},\boldsymbol{\alpha})=\frac{f_{p,SN}(F_1^{-1}(u_1;\mu_1,1,\alpha_1),\dots,F_1^{-1}(u_p;\mu_p,1,\alpha_p);\boldsymbol{\mu},\mathbf{R},\boldsymbol{\alpha})}{\prod_{i=1}^p f_{1,SN}(F_1^{-1}(u_i;\mu_i,1,\alpha_i))},$$

where the density $f_{p,SN}(\cdot)$ is given in (1), and the functions $F_1^{-1}(u_i;\mu_i,1,\alpha_i)$ are as in Definition 4.

The skew $t_{p,\nu}$-copula is defined in the same way (Kollo and Pettere [8]).

Definition 5 A copula $C_{p,\nu}$ is called a skew $t_{p,\nu}$-copula with parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\boldsymbol{\alpha}$ if

$$\begin{aligned}&C_{p,\nu}(u_1,\dots,u_p:\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha})\\&\quad=G_{p,\nu}\big(G_{1,\nu}^{-1}(u_1;\mu_1,\sigma_{11},\alpha_1),\dots,G_{1,\nu}^{-1}(u_p;\mu_p,\sigma_{pp},\alpha_p);\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}\big),\end{aligned}$$

where $G_{1,\nu}^{-1}(u_i;\mu_i,\sigma_{ii},\alpha_i)$, $i\in\{1,\dots,p\}$ denotes the inverse of the univariate skew $t_{1,\nu}$-distribution function, and $G_{p,\nu}$ is the distribution function of the p-variate skew $t_{p,\nu}$-distribution with the density as given in (6).

The corresponding copula density is

$$\begin{aligned}&c_{p,\nu}(\mathbf{u};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha})\\&\quad=\frac{g_{p,\nu}[\{G_{1,\nu}^{-1}(u_1;\mu_1,\sigma_{11},\alpha_1),\dots,G_{1,\nu}^{-1}(u_p;\mu_p,\sigma_{pp},\alpha_p)\};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}]}{\prod_{i=1}^p g_{1,\nu}[G_{1,\nu}^{-1}(u_i;\mu_i,\sigma_{ii},\alpha_i);\mu_i,\sigma_{ii},\alpha_i]},\end{aligned}\tag{12}$$

where the density function $g_{p,\nu}(\mathbf{u};\boldsymbol{\mu},\boldsymbol{\Sigma},\boldsymbol{\alpha}):R^p\to R$ is defined by (6), and the functions $G_{1,\nu}^{-1}(u_i;\mu_i,\sigma_{ii},\alpha_i)$ are as in Definition 5. As in the numerator of the density expression in (12) stands the density of the multivariate skew t-distribution, the resulting copula density also represents a skewed multivariate distribution. Several properties of the skew t-copula are still under investigation. For example, what is the maximum value of the Mardia's multivariate skewness characteristic of the copula; how tail dependence can be expressed between bivariate marginals, etc.

How to fit a multivariate skew elliptical copula to data? At the first step, we find models for marginals and calculate the dependence measures between them. The matrix of Pearson correlation coefficients would be the first idea, especially when using the method of moments, as then the parameters of skew-elliptical distributions can be expressed through the Pearson correlation matrix. Unfortunately, the Pearson correlations are not invariant under the nonlinear transformations that we use when constructing copulas via inverse distribution functions. The rank correlations would be invariant, but then we need to know the transformation between the linear and the rank correlations. In the case of normal distribution, the transformation is known for a long time; for the multivariate t-distribution, the corresponding result is more recent (Lindskog, McNeil and Schmock [12]). The same time simulation studies have shown that for small and "average" values (up to 0.5) of the Spearman and Pearson correlations the estimates do not differ much when using inverse functions of distribution functions. Estimates (2)–(5) for skew normal copulas and (7)–(10) for skew t-copulas can be used as initial values for further study by the maximum likelihood method. Some first applications of these skewed copulas have shown that the estimates by the method of moments can be used in practice for construction of data models. In Kollo and Pettere [8], a three-variate skew t-copula was applied to anthropometric data, where two marginals followed gamma distribution, and one was modeled by lognormal distribution. The Pearson correlations between marginals were 0.47, 0.62, and 0.86. The skew t-copula with estimated parameters (7)–(10) gave better fit with the data compared with the Gaussian copula. In the second application, Pettere and Kollo [18] predicted the annual cash flow of an insurance company on the basis of three main business lines of the company. The best models for the marginals were obtained by gamma, Pareto, and lognormal distributions. The marginals were correlated with Pearson correlations 0.14, 0.60, 0.11. The joint distribution was modeled by several Archimedean copulas, Gaussian copula, and skew t-copula. The best fit was obtained by the skew t-copula with three degrees of freedom, and the parameters were estimated by (7)–(10). Summarizing, we can say that the first examples have shown applicability of skew elliptical copulas in data modeling.

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References

1. Azzalini, A., Capitanio, A.: Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. J. R. Stat. Soc., Ser. B, Stat. Methodol. **65**, 367–389 (2003)
2. Azzalini, A., Dalla Valle, A.: The multivariate skew normal distribution. Biometrika **83**, 715–726 (1996)
3. Genton, M.G. (ed.): Skew-Elliptical Distributions and Their Applications—A Journey Beyond Normality. Chapman & Hall/CRC, Boca Raton (2004)
4. González-Farías, G., Domínguez-Molina, J.A., Gupta, A.K.: The closed skew-normal distribution. In: Genton, M.G. (ed.) Skew-Elliptical Distributions and Their Applications—A Journey Beyond Normality, pp. 25–42. Chapman & Hall/CRC, Boca Raton (2004)
5. Gupta, A.K., González-Farías, G., Domínguez-Molina, J.A.: A multivariate skew normal distribution. J. Multivar. Anal. **89**, 181–190 (2004)
6. Gupta, A.K., Kollo, T.: Density expansions based on the multivariate skew normal distribution. Sankhya **65**, 821–835 (2003)
7. Kollo, T.: Multivariate skewness and kurtosis measures with an application in ICA. J. Multivar. Anal. **99**, 2328–2338 (2008)
8. Kollo, T., Pettere, G.: Parameter estimation and application of the multivariate skew t-copula. In: Jaworski, P., et al. (eds.) Copula Theory and Its Applications, pp. 289–298. Springer, Berlin (2010)
9. Kollo, T., von Rosen, D.: Advanced Multivariate Statistics with Matrices. Springer, Dordrecht (2005)

10. Kotz, S., Nadarajah, S.: Multivariate t Distributions and Their Applications. Cambridge University Press, Cambridge (2004)
11. Koziol, J.A.: A note on measures of multivariate kurtosis. Biom. J. **31**, 619–624 (1989)
12. Lindskog, F., McNeil, A.J., Schmock, U.: Kendall's tau for elliptical distributions. In: Bol, G., Nakhaeizadeh, G., et al. (eds.) Credit Risk: Measurement, Evaluation and Management—Contributions to Economics, pp. 149–156. Physica-Verlag, Heidelberg (2003)
13. MacRae, E.C.: Matrix derivatives with an application to an adaptive linear decision problem. Ann. Stat. **2**, 337–346 (1974)
14. Magnus, J.R., Neudecker, H.: Matrix Differential Calculus with Applications in Statistics and Econometrics, 2nd edn. Wiley, Chichester (1999)
15. Mardia, K.V.: Measures of multivariate skewness and kurtosis with applications. Biometrika **57**, 519–530 (1970)
16. Móri, T.F., Rohatgi, V.K., Székely, G.J.: On multivariate skewness and kurtosis. Theory Probab. Appl. **38**, 547–551 (1993)
17. Nelsen, R.B.: An Introduction to Copulas. Springer, New York (1999)
18. Pettere, G., Kollo, T.: Risk modeling for future cash flow using skew t-copula. Commun. Stat., Theory Methods **40**(16), 2919–2925 (2011)
19. Visk, H.: Shifted multivariate Laplace distribution. Commun. Stat. **38**, 461–470 (2009)

Revisiting the BLUE in a Linear Model via Proper Eigenvectors

Jan Hauke, Augustyn Markiewicz, and Simo Puntanen

Abstract We consider two linear models, $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ and $\mathscr{M}_2 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$, having different covariance matrices. Our main interest lies in question whether a particular given BLUE under $\mathscr{M}_1$ continues to be a BLUE under $\mathscr{M}_2$. We give a thorough proof of a result originally due to Mitra and Moore (Sankhyā, Ser. A 35:139–152, 1973). While doing this, we will review some useful properties of the proper eigenvalues in the spirit of Rao and Mitra (Generalized Inverse of Matrices and Its Applications, 1971).

Keywords Best linear unbiased estimator · Gauss–Markov model · Linear model · Löwner ordering · Orthogonal projector · Proper eigenvalues

Mathematics Subject Classification (2010) 15A42 · 62J05 · 62H12 · 62H20

1 Introduction

In this article, we consider the general linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, denoted as a triplet $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, where $\mathbf{X}$ is a known $n \times p$ model matrix, the vector $\mathbf{y}$ is an observable n-dimensional random vector, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, and $\boldsymbol{\varepsilon}$ is an unobservable vector of random errors with expectation $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$ and covariance matrix $\mathrm{cov}(\boldsymbol{\varepsilon}) = \mathbf{V}$. The nonnegative definite matrix $\mathbf{V}$ is known.

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The symbol $\mathbb{R}^{n\times m}$ stands for the set of real $n \times m$ matrices; all matrices in this paper have real elements. We will use the standard notation; for example, $\mathbf{A}'$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})^\perp$, and $\mathscr{N}(\mathbf{A})$ denote, respectively, the transpose, a generalized inverse, the Moore–Penrose inverse, the column space, the orthogonal complement of the column space, and the null space of a matrix $\mathbf{A}$. By $(\mathbf{A} : \mathbf{B})$ we denote the partitioned matrix with $\mathbf{A}$ and $\mathbf{B}$ as submatrices. By $\mathbf{A}^\perp$ we denote any matrix satisfying $\mathscr{C}(\mathbf{A}^\perp) = \mathscr{N}(\mathbf{A}') = \mathscr{C}(\mathbf{A})^\perp$. Furthermore, $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ denotes the orthogonal projector (with respect to the standard inner product) onto $\mathscr{C}(\mathbf{A})$. In particular, we denote $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X}$. We also denote

$$\mathbf{P}_{\mathbf{A};\mathbf{N}} = \mathbf{A}\big(\mathbf{A}'\mathbf{N}\mathbf{A}\big)^-\mathbf{A}'\mathbf{N}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $\mathbf{N} \in \mathrm{NND}_n$, the set of $n \times n$ nonnegative definite symmetric matrices. The matrix $\mathbf{P}_{\mathbf{A};\mathbf{N}}$ is invariant with respect to the choice of $(\mathbf{A}'\mathbf{N}\mathbf{A})^-$ if and only if $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}'\mathbf{N})$; see Rao and Mitra [23, Lemma 2.2.4].

An unbiased linear estimator $\mathbf{G}\mathbf{y}$ for $\mathbf{X}\boldsymbol{\beta}$ is defined to be the *best* linear unbiased estimator, BLUE, for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ if

$$\operatorname{cov}(\mathbf{G}\mathbf{y}) \le_\mathrm{L} \operatorname{cov}(\mathbf{L}\mathbf{y}) \quad \text{for all } \mathbf{L}\colon \mathbf{L}\mathbf{X} = \mathbf{X}, \tag{2}$$

where "$\le_\mathrm{L}$" refers to the Löwner partial ordering.

In Sect. 2, we introduce some preliminary results that will be needed later on. In Lemma 2, we give an equation that $\mathbf{G}$ has to satisfy so that $\mathbf{G}\mathbf{y}$ would be a BLUE for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$. The symbol $\mathscr{W}$ will refer to a specific set of nonnegative definite matrices:

$$\mathscr{W} = \big\{\mathbf{W} : \mathbf{W} = \mathbf{V} + \mathbf{X}\mathbf{L}\mathbf{L}'\mathbf{X}',\ \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})\big\}. \tag{3}$$

In (3), $\mathbf{L}$ can be any (conformable) matrix as long as $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$ is satisfied. Writing

$$\mathbf{P}_{\mathbf{X};\mathbf{W}^+} = \mathbf{X}\big(\mathbf{X}'\mathbf{W}^+\mathbf{X}\big)^-\mathbf{X}'\mathbf{W}^+, \quad \text{where } \mathbf{W} \in \mathscr{W}, \tag{4}$$

we get the well-known expression for the BLUE of $\mathbf{X}\boldsymbol{\beta}$:

$$\mathbf{P}_{\mathbf{X};\mathbf{W}^+}\mathbf{y} = \mathbf{X}\big(\mathbf{X}'\mathbf{W}^+\mathbf{X}\big)^-\mathbf{X}'\mathbf{W}^+\mathbf{y}. \tag{5}$$

In (3), we have all members of $\mathscr{W}$ nonnegative definite, but this is not necessary in all considerations to follow. Similarly, expression (5) is invariant with respect to the choice of all generalized inverses involved (supposing that $\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$ with probability 1). For convenience, we use the Moore–Penrose inverse of $\mathbf{W}$ in (5). In Lemma 1, we have collected, to simplify the referencing, some properties of $\mathscr{W}$ (not assuming nonnegative definiteness though). For references to Lemma 1, we may mention Baksalary and Puntanen [4], Baksalary, Puntanen and Styan [5, Theorem 2], Baksalary and Mathew [3, Theorem 2], and Harville [10, p. 468].

Lemma 1 *Let* $\mathbf{V}$ *be an* $n \times n$ *nonnegative definite matrix*, *let* $\mathbf{X}$ *be an* $n \times p$ *matrix*, *and define* $\mathbf{W}$ *as* $\mathbf{W} = \mathbf{V} + \mathbf{X}\mathbf{U}\mathbf{X}'$, *where* $\mathbf{U}$ *is a* $p \times p$ *matrix. Then the following statements are equivalent*:

(a) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W})$,
(b) $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$,
(c) $\mathbf{X}'\mathbf{W}^-\mathbf{X}$ *is invariant for any choice of* $\mathbf{W}^-$,
(d) $\mathscr{C}(\mathbf{X}'\mathbf{W}^-\mathbf{X}) = \mathscr{C}(\mathbf{X}')$ *for any choice of* $\mathbf{W}^-$,
(e) $\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{X} = \mathbf{X}$ *for any choices of the generalized inverses involved.*

Moreover, each of these statements is equivalent to (a$'$) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}')$ *and hence to statements* (b$'$)–(e$'$) *obtained from* (b)–(e) *by setting* $\mathbf{W}'$ *in place of* $\mathbf{W}$.

2 Preliminaries

The following lemma gives the "Fundamental BLUE equation"; see, e.g., Rao [19], Zyskind [29], Baksalary [1], and Baksalary and Trenkler [7, 8].

Lemma 2 *Consider the general linear model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$. *Then the estimator* $\mathbf{Gy}$ *is the* BLUE *for* $\mathbf{X}\boldsymbol{\beta}$ *if and only if* $\mathbf{G}$ *satisfies the equation*

$$\mathbf{G}\big(\mathbf{X} : \mathbf{V}\mathbf{X}^{\perp}\big) = (\mathbf{X} : \mathbf{0}). \tag{6}$$

The set of matrices $\mathbf{G}$ satisfying (6) may be denoted as $\{\mathbf{P}_{\mathbf{X}|\mathbf{V}\mathbf{X}^{\perp}}\}$, indicating that $\mathbf{G}$ is a projector onto $\mathscr{C}(\mathbf{X})$ along $\mathscr{C}(\mathbf{V}\mathbf{X}^{\perp})$. Obviously, $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}}$ is one choice for $\mathbf{X}^{\perp}$. Notice also that even though $\mathbf{G}$ may not be unique, the numerical observed value of $\mathbf{Gy}$ is unique (with probability 1) once the random vector $\mathbf{y}$ has realized its value in the permissible space $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM})$. The consistency of the model $\mathscr{M}$ means that the observed $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{X} : \mathbf{V})$, which is assumed to hold whatever model we have; for the consistency concept, see, for example, Baksalary, Rao and Markiewicz [6].

Characterizing the equality of the ordinary least squares estimator $\mathbf{P}_{\mathbf{X}}\mathbf{y}$ and the BLUE has received a lot of attention in the statistical literature, the major breakthroughs being made by Rao [19] and Zyskind [29]; for a detailed review, see Puntanen and Styan [18].

Consider now two models $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ and $\mathscr{M}_2 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$ that differ in their covariance matrices. As in (3), define a nonnegative definite matrix $\mathbf{W}_1$ so that

$$\mathbf{W}_1 = \mathbf{V}_1 + \mathbf{XLL}'\mathbf{X}', \quad \text{where } \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X} : \mathbf{V}_1), \tag{7}$$

and denote the set of all such matrices as $\mathscr{W}_1$. Then one representation for the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_1$ is

$$\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y} = \mathbf{X}\big(\mathbf{X}'\mathbf{W}_1^+\mathbf{X}\big)^-\mathbf{X}'\mathbf{W}_1^+\mathbf{y}, \tag{8}$$

while the general representation for the BLUE is $\mathbf{By}$, where $\mathbf{B} = \mathbf{P}_{\mathbf{X};\mathbf{W}_1^+} + \mathbf{F}(\mathbf{I}_n - \mathbf{P}_{\mathbf{W}_1})$, with $\mathbf{F}$ being free to vary.

We can now ask whether $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y}$ continues to be BLUE under $\mathscr{M}_2$. If $\mathbf{V}_1$ and $\mathbf{V}_2$ are positive definite, then the BLUE of $\mathbf{X}\boldsymbol{\beta}$ has unique representations under $\mathscr{M}_1$ and $\mathscr{M}_2$, and the equality between the BLUEs becomes the equality between the multipliers $\mathbf{P}_{\mathbf{X};\mathbf{V}_1^{-1}}$ and $\mathbf{P}_{\mathbf{X};\mathbf{V}_2^{-1}}$. The first solution to this problem appears to be by Thomas [25], who in an abstract considered the equality of the BLUEs with both $\mathbf{V}_1$ and $\mathbf{V}_2$ being positive definite. For further references in positive definite case, we may mention Harville [10, p. 265], Luati and Proietti [13, Theorem 1, Corollary 1], and Hauke, Markiewicz and Puntanen [11]. In particular, Hauke, Markiewicz and Puntanen [11] studied the conditions for $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y}$ to be the BLUE under $\mathscr{M}_2$ using a particular transformation technique and the concept of linear sufficiency.

Subsequently, we will utilize the *proper eigenvalues* and *proper eigenvectors* following Rao and Mitra [23, §6.3]; see also Mitra and Rao [17], as well as de Leeuw [9], Mitra and Moore [16, Appendix], Scott and Styan [24], and Isotalo, Puntanen and Styan [12, §2]. For this purpose, let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric $n \times n$ matrices of which $\mathbf{B}$ is nonnegative definite. Let $\lambda \in \mathbb{R}$ be a scalar, and $\mathbf{w}$ a vector such that

$$\mathbf{A}\mathbf{w} = \lambda\mathbf{B}\mathbf{w}, \quad \mathbf{B}\mathbf{w} \neq \mathbf{0}. \tag{9}$$

Following Rao and Mitra [23, §6.3], we call λ a proper eigenvalue and $\mathbf{w}$ a proper eigenvector of $\mathbf{A}$ with respect to $\mathbf{B}$, or shortly, $(\lambda, \mathbf{w})$ is a proper eigenpair for $(\mathbf{A}, \mathbf{B})$. The set of all proper eigenvalues of pair $(\mathbf{A}, \mathbf{B})$ is denoted as $\operatorname{ch}(\mathbf{A}, \mathbf{B})$. If $\mathbf{B}$ is singular, there may exist a vector $\mathbf{w} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{w} = \mathbf{B}\mathbf{w} = \mathbf{0}$, in which case

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{w} = \mathbf{0} \tag{10}$$

for arbitrary λ. We call such a vector $\mathbf{w} \in \mathbb{R}^n$ an improper eigenvector of $\mathbf{A}$ with respect to $\mathbf{B}$. The space of improper eigenvectors is precisely $\mathscr{N}(\mathbf{A}) \cap \mathscr{N}(\mathbf{B}) = \mathscr{C}(\mathbf{A} : \mathbf{B})^{\perp}$.

Notice that if (10) holds for some nonzero $\mathbf{w}$, then necessarily

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0 \tag{11}$$

for any $\lambda \in \mathbb{R}$. Then the pencil $\mathbf{A} - \lambda\mathbf{B}$ (with indeterminate λ) is said to be singular; otherwise the pencil is regular. For a positive definite $\mathbf{B}$, we obviously have

$$\operatorname{ch}(\mathbf{A}, \mathbf{B}) = \operatorname{ch}\big(\mathbf{B}^{-1}\mathbf{A}, \mathbf{I}_n\big) = \operatorname{ch}\big(\mathbf{B}^{-1}\mathbf{A}\big) = \operatorname{ch}\big(\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\big), \tag{12}$$

where $\operatorname{ch}(\cdot)$ refers to the set of the eigenvalues (including multiplicities) of the matrix argument. If $\mathbf{B}$ is singular, we might wonder whether, for example, the following might be true:

$$\operatorname{ch}(\mathbf{A}, \mathbf{B}) = \operatorname{ch}\big(\mathbf{B}^{+}\mathbf{A}\big)? \tag{13}$$

Statement (13) does not always hold, but if $\mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{B})$, then indeed (13) holds for nonzero proper eigenvalues; see Lemma 5.

3 When Is $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y}$ BLUE Under $\mathscr{M}_2$?

The following lemma is due to Mitra and Moore [16, p. 140].

Lemma 3 *Consider the linear model* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *and denote* $\mathbf{W} = \mathbf{V} + \mathbf{XLL'X'}$, *where* $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$, *and let* $\mathbf{W}^-$ *be an arbitrary generalized inverse of* $\mathbf{W}$. *Then*

$$\mathscr{C}(\mathbf{W}^-\mathbf{X}) \oplus \mathscr{C}(\mathbf{X})^\perp = \mathbb{R}^n, \qquad \mathscr{C}(\mathbf{W}^-\mathbf{X})^\perp \oplus \mathscr{C}(\mathbf{X}) = \mathbb{R}^n,$$
$$\mathscr{C}\big[(\mathbf{W}^-)'\mathbf{X}\big] \oplus \mathscr{C}(\mathbf{X})^\perp = \mathbb{R}^n, \qquad \mathscr{C}\big[(\mathbf{W}^-)'\mathbf{X}\big]^\perp \oplus \mathscr{C}(\mathbf{X}) = \mathbb{R}^n,$$

where $\oplus$ *refers to the direct sum of subspaces.*

The above lemma is needed in the proof of Theorem 1. For completeness, we also state the following two lemmas appearing in Rao and Mitra [23, §6.3]; see also Mitra and Moore [17].

Lemma 4 *Let* $\mathbf{A}_{n\times n}$ *and* $\mathbf{B}_{n\times n}$ *both be symmetric matrices of which* $\mathbf{B}$ *is nonnegative definite*, $\operatorname{rank}(\mathbf{B}) = b$, *and suppose that*

$$\operatorname{rank}\big(\mathbf{N'AN}\big) = \operatorname{rank}\big(\mathbf{N'A}\big), \quad \text{where } \mathbf{N} = \mathbf{B}^\perp. \tag{14}$$

Then there are precisely b proper eigenvalues of $\mathbf{A}$ *with respect to* $\mathbf{B}$,

$$\operatorname{ch}(\mathbf{A}, \mathbf{B}) = \{\lambda_1, \ldots, \lambda_b\}, \tag{15}$$

some of which may be repeated or null. Also the corresponding eigenvectors $\mathbf{w}_1, \ldots, \mathbf{w}_b$ *can be chosen so that if* $\mathbf{W}$ *is an* $n \times b$ *matrix with* $\mathbf{w}_i$ *being its ith column, then*

$$\mathbf{W'BW} = \mathbf{I}_b, \qquad \mathbf{W'AW} = \boldsymbol{\Lambda}_1, \qquad \mathscr{C}(\mathbf{AW}) \subset \mathscr{C}(\mathbf{B}), \tag{16}$$

where $\boldsymbol{\Lambda}_1$ *is the diagonal matrix of* $\lambda_1, \ldots, \lambda_b$.

Condition (14) holds, for example, if $\mathbf{A}$ is nonnegative definite or if $\mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{B})$. Using the notation $\operatorname{nzch}(\cdot)$ for the set of the nonzero eigenvalues and $\operatorname{nzch}(\mathbf{A}, \mathbf{B})$ for the set of the nonzero proper eigenvalues of $\mathbf{A}$ with respect to $\mathbf{B}$, we have the following lemma.

Lemma 5 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be as in Lemma 4 and assume that (14) holds. Then the nonzero proper eigenvalues of* $\mathbf{A}$ *with respect to* $\mathbf{B}$ *are the same as the nonzero eigenvalues of* $[\mathbf{A} - \mathbf{AN}(\mathbf{N'AN})^-\mathbf{N'A}]\mathbf{B}^-$ *and vice versa for any generalized inverses involved, i.e.*,

$$\operatorname{nzch}(\mathbf{A}, \mathbf{B}) = \operatorname{nzch}\big(\big[\mathbf{A} - \mathbf{AN}\big(\mathbf{N'AN}\big)^-\mathbf{N'A}\big]\mathbf{B}^-\big). \tag{17}$$

In particular,

$$\mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{B}) \quad \Longrightarrow \quad \operatorname{nzch}(\mathbf{A}, \mathbf{B}) = \operatorname{nzch}\big(\mathbf{A}\mathbf{B}^-\big). \tag{18}$$

Now we are ready to introduce the main target of our paper, Theorem 1. This theorem appears in Mitra and Moore [16, Theorems 2.1 and 2.2, Note 1]. Some parts of it are not proved in detail, and then only hints for the proofs are given. Our purpose is to provide a complete proof, which we believe to be of interest.

Theorem 1 *Consider the linear models* $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ *and* $\mathscr{M}_2 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$ *where* $\operatorname{rank}(\mathbf{X}) = r$. *Denote* $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+} = \mathbf{X}(\mathbf{X}'\mathbf{W}_1^+\mathbf{X})^-\mathbf{X}'\mathbf{W}_1^+$, *where* $\mathbf{W}_1$ *is defined as in* (7) *so that* $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y}$ *is the* BLUE *for* $\mathbf{X}\boldsymbol{\beta}$ *under* $\mathscr{M}_1$. *Denote* $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X}$ *and let* $\mathbf{Z}$ *be a matrix whose column space is* $\mathscr{N}(\mathbf{X}'\mathbf{W}_1^+)$, *i.e.,* $\mathbf{Z} \in \{(\mathbf{W}_1^+\mathbf{X})^\perp\}$. *Then* $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{y}$ *is the* BLUE *for* $\mathbf{X}\boldsymbol{\beta}$ *also under* $\mathscr{M}_2$ *if and only if any of the following equivalent conditions holds*:

(a) $\mathbf{X}'\mathbf{W}_1^+\mathbf{V}_2\mathbf{M} = \mathbf{0}$,
(b) $\mathscr{C}(\mathbf{V}_2\mathbf{W}_1^+\mathbf{X}) \subset \mathscr{C}(\mathbf{X})$,
(c) $\mathscr{C}(\mathbf{V}_2\mathbf{M}) \subset \mathscr{N}(\mathbf{X}'\mathbf{W}_1^+) = \mathscr{C}(\mathbf{W}_1^+\mathbf{X})^\perp = \mathscr{C}(\mathbf{Z})$,
(d) $\mathbf{V}_2 \in \{\mathbf{V}_2 \in \mathrm{NND}_n : \mathbf{V}_2 = \mathbf{X}\mathbf{K}_{11}\mathbf{X}' + \mathbf{Z}\mathbf{K}_{22}\mathbf{Z}'$ *for some* $\mathbf{K}_{11}$ *and* $\mathbf{K}_{22}\}$,
(e) $\mathbf{P}_{\mathbf{X};\mathbf{W}_1^+}\mathbf{V}_2$ *is symmetric*,
(f) $\mathscr{C}(\mathbf{W}_1^+\mathbf{X})$ *is spanned by a set of* r *proper eigenvectors of* $\mathbf{V}_2$ *with respect to* $\mathbf{W}_1$,
(g) $\mathscr{C}(\mathbf{X})$ *is spanned by a set of* r *eigenvectors of* $\mathbf{V}_2\mathbf{W}_1^+$.

Proof The estimator $\mathbf{X}(\mathbf{X}'\mathbf{W}_1^+\mathbf{X})^-\mathbf{X}'\mathbf{W}_1^+\mathbf{y}$ is also a BLUE under $\mathscr{M}_2$ if and only if

$$\mathbf{X}\big(\mathbf{X}'\mathbf{W}_1^+\mathbf{X}\big)^-\mathbf{X}'\mathbf{W}_1^+(\mathbf{X} : \mathbf{V}_2\mathbf{M}) = (\mathbf{X} : \mathbf{0}), \tag{19}$$

which in view of Lemma 1(e) is equivalent to

$$\mathbf{X}\big(\mathbf{X}'\mathbf{W}_1^+\mathbf{X}\big)^-\mathbf{X}'\mathbf{W}_1^+\mathbf{V}_2\mathbf{M} = \mathbf{0}. \tag{20}$$

Premultiplying (20) by $\mathbf{X}'\mathbf{W}_1^+$ and using again Lemma 1(e) show that (20) is equivalent to

$$\mathbf{X}'\mathbf{W}_1^+\mathbf{V}_2\mathbf{M} = \mathbf{0}, \tag{21}$$

and so we obtain (a). Statements (b) and (c) are obvious alternative ways to express condition (a).

According to Lemma 3, $\mathscr{C}(\mathbf{Z}) \oplus \mathscr{C}(\mathbf{X}) = \mathbb{R}^n$, where $\mathbf{Z} \in \{(\mathbf{W}_1^+\mathbf{X})^\perp\}$ and hence every nonnegative definite $\mathbf{V}_2$ can be written as $\mathbf{V}_2 = \mathbf{F}\mathbf{F}'$ for some $\mathbf{K}_1$ and $\mathbf{K}_2$ such that $\mathbf{F} = \mathbf{X}\mathbf{K}_1 + \mathbf{Z}\mathbf{K}_2$, and so

$$\begin{aligned}\mathbf{V}_2 &= \mathbf{X}\mathbf{K}_1\mathbf{K}_1'\mathbf{X}' + \mathbf{Z}\mathbf{K}_2\mathbf{K}_2'\mathbf{Z}' + \mathbf{X}\mathbf{K}_1\mathbf{K}_2'\mathbf{Z}' + \mathbf{Z}\mathbf{K}_2\mathbf{K}_1'\mathbf{X}' \\ &:= \mathbf{X}\mathbf{K}_{11}\mathbf{X}' + \mathbf{Z}\mathbf{K}_{22}\mathbf{Z}' + \mathbf{X}\mathbf{K}_{12}\mathbf{Z}' + \mathbf{Z}\mathbf{K}_{12}'\mathbf{X}'.\end{aligned} \tag{22}$$

Suppose that (a) holds. Then substituting (22) into (21) yields

$$\mathbf{X}'\mathbf{W}_1^{+}\big(\mathbf{Z}\mathbf{K}_2\mathbf{K}_2'\mathbf{Z}' + \mathbf{X}\mathbf{K}_1\mathbf{K}_2'\mathbf{Z}'\big)\mathbf{M} = \mathbf{0}, \tag{23}$$

which, in view of $\mathbf{X}'\mathbf{W}_1^{+}\mathbf{Z} = \mathbf{0}$, further simplifies into

$$\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{K}_1\mathbf{K}_2'\mathbf{Z}'\mathbf{M} = \mathbf{0}. \tag{24}$$

By Lemma 1(d) we have $\operatorname{rank}(\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}) = \operatorname{rank}(\mathbf{X})$, and by Marsaglia and Styan [15, Corollary 6.2],

$$\operatorname{rank}\big(\mathbf{Z}'\mathbf{M}\big) = \operatorname{rank}(\mathbf{Z}) - \dim \mathscr{C}(\mathbf{Z}) \cap \mathscr{C}(\mathbf{X}) = \operatorname{rank}(\mathbf{Z}), \tag{25}$$

and hence we can use the rank cancellation rule of Marsaglia and Styan [15, Theorem 2] and cancel $\mathbf{X}'\mathbf{W}_1^{+}$ and $\mathbf{M}$ from (24) and thus obtain

$$\mathbf{X}\mathbf{K}_1\mathbf{K}_2'\mathbf{Z}' = \mathbf{0}. \tag{26}$$

Substituting (26) into (22) proves that (a) implies (d). The reverse relation is obvious.

Consider condition (e), i.e.,

$$\mathbf{P}_{\mathbf{X};\mathbf{W}_1^{+}}\mathbf{V}_2 = \mathbf{V}_2\mathbf{P}'_{\mathbf{X};\mathbf{W}_1^{+}}. \tag{27}$$

Substituting (22) into (27) yields $\mathbf{X}\mathbf{K}_{12}\mathbf{Z}' = \mathbf{Z}\mathbf{K}_{12}'\mathbf{X}'$ and thereby

$$\mathbf{V}_2 = \mathbf{X}\mathbf{K}_{11}\mathbf{X}' + \mathbf{Z}\mathbf{K}_{22}\mathbf{Z}' + 2\mathbf{Z}'\mathbf{K}_{12}'\mathbf{X}'. \tag{28}$$

Pre- and postmultiplying (28) by $\mathbf{X}'\mathbf{W}_1^{+}$ and $\mathbf{M}$, respectively, yields (a), i.e., we have shown that (e) implies (a). The reverse relation is again obvious.

Let us then take a look at (f) and (g). In view of (22), we observe that

$$\mathbf{X}'\mathbf{W}^{+}\mathbf{V}_2\mathbf{W}^{+}\mathbf{X} = \mathbf{X}'\mathbf{W}^{+}\mathbf{X}\mathbf{K}_{11}\mathbf{X}'\mathbf{W}^{+}\mathbf{X}. \tag{29}$$

Consider the proper eigenvalues and eigenvectors of

$$\mathbf{A} = \mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{K}_{11}\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X} \tag{30}$$

with respect to

$$\mathbf{B} = \mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}. \tag{31}$$

Obviously, $\mathscr{C}(\mathbf{A}) \subset \mathscr{C}(\mathbf{B})$ and $\operatorname{rank}(\mathbf{B}) = r$. Moreover, in view of Lemma 4, there exists a matrix $\mathbf{T}_{p\times r}$ such that

$$\mathbf{T}'\mathbf{B}\mathbf{T} = \mathbf{T}'\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{T} = \mathbf{I}_r, \qquad \mathbf{A}\mathbf{T} = \mathbf{B}\mathbf{T}\mathbf{D}, \tag{32a}$$

$$\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{K}_{11}\mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{T} = \mathbf{X}'\mathbf{W}_1^{+}\mathbf{X}\mathbf{T}\mathbf{D}, \tag{32b}$$

where $\mathbf{D}_{r\times r} = \mathrm{diag}(d_1,\dots,d_r)$ is a diagonal matrix. The columns of $\mathbf{T}$ are the proper eigenvectors of $\mathbf{A}$ with respect to $\mathbf{B}$, and the d_is are the corresponding proper eigenvalues.

We can cancel the left-most $\mathbf{X}'\mathbf{W}^+$ from each side of (32b), and so we have

$$\mathbf{X}\mathbf{K}_{11}\mathbf{X}'\mathbf{W}_1^+\mathbf{X}\mathbf{T} = \mathbf{X}\mathbf{T}\mathbf{D}. \tag{33}$$

Suppose that (d) holds. Then

$$\mathbf{V}_2\mathbf{W}_1^+\mathbf{X} = \big(\mathbf{X}\mathbf{K}_{11}\mathbf{X}' + \mathbf{Z}\mathbf{K}_{22}\mathbf{Z}'\big)\mathbf{W}_1^+\mathbf{X} = \mathbf{X}\mathbf{K}_{11}\mathbf{X}'\mathbf{W}_1^+\mathbf{X}, \tag{34}$$

and so (33) becomes

$$\mathbf{V}_2\mathbf{W}_1^+\mathbf{X}\mathbf{T} = \mathbf{X}\mathbf{T}\mathbf{D}. \tag{35}$$

From $\mathbf{T}'\mathbf{X}'\mathbf{W}_1^+\mathbf{X}\mathbf{T} = \mathbf{I}_r$ it follows that $\mathrm{rank}(\mathbf{X}\mathbf{T}) = \mathrm{rank}(\mathbf{X}) = r$, and so the columns of $\mathbf{X}\mathbf{T} \in \mathbb{R}^{n\times r}$ are nonzero, and thereby they comprise a set of r eigenvectors of $\mathbf{V}_2\mathbf{W}_1^+$ corresponding to the eigenvalues $d_1,\dots,d_r$. In view of $\mathscr{C}(\mathbf{X}\mathbf{T}) = \mathscr{C}(\mathbf{X})$, we can conclude that $\mathscr{C}(\mathbf{X})$ has a basis comprising a set of r eigenvectors of $\mathbf{V}_2\mathbf{W}_1^+$, i.e., we have shown that (d) implies (g).

On the other hand, because $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W}_1)$, (35) can be equivalently written as

$$\mathbf{V}_2\big(\mathbf{W}_1^+\mathbf{X}\mathbf{T}\big) = \mathbf{W}_1\big(\mathbf{W}_1^+\mathbf{X}\mathbf{T}\big)\mathbf{D}, \tag{36}$$

and so the columns of $\mathbf{W}_1^+\mathbf{X}\mathbf{T} \in \mathbb{R}^{n\times r}$ are r proper eigenvectors of $\mathbf{V}_2$ with respect to $\mathbf{W}_1$. Because $\mathrm{rank}(\mathbf{W}_1^+\mathbf{V}_2\mathbf{T}) = r = \mathrm{rank}(\mathbf{W}_1^+\mathbf{V}_2)$, we necessarily have $\mathscr{C}(\mathbf{W}_1^+\mathbf{X}\mathbf{T}) = \mathscr{C}(\mathbf{W}_1^+\mathbf{X})$. Hence, $\mathscr{C}(\mathbf{W}_1^+\mathbf{X})$ has a basis comprising a set of r proper eigenvectors of $\mathbf{V}_2$ with respect to $\mathbf{W}_1$, i.e., (g) and (f) are equivalent.

Assume then that (35) holds. Because $\mathscr{C}(\mathbf{W}_1^+\mathbf{X}\mathbf{T}) = \mathscr{C}(\mathbf{W}_1^+\mathbf{X})$, (35) implies that

$$\mathscr{C}\big(\mathbf{V}_2\mathbf{W}_1^+\mathbf{X}\mathbf{T}\big) = \mathscr{C}\big(\mathbf{V}_2\mathbf{W}_1^+\mathbf{X}\big) \subset \mathscr{C}(\mathbf{X}), \tag{37}$$

i.e., (f) implies (a). Thus, the proof is completed. □

Let us summarize the above findings concerning the proper eigenvalues. Denoting $\mathbf{A} = \mathbf{X}'\mathbf{R}\mathbf{X}$ and $\mathbf{B} = \mathbf{X}'\mathbf{S}\mathbf{X}$, where

$$\mathbf{R} = \mathbf{W}_1^+\mathbf{V}_2\mathbf{W}_1^+, \qquad \mathbf{S} = \mathbf{W}_1^+, \qquad r = \mathrm{rank}(\mathbf{B}) = \mathrm{rank}(\mathbf{X}), \tag{38}$$

and $a = \mathrm{rank}(\mathbf{A})$, we observe that if any of the conditions in Theorem 1 holds, then

$$\begin{aligned} \mathrm{ch}(\mathbf{A},\mathbf{B}) &= \{d_1,\dots,d_a, d_{a+1} = 0,\dots,d_r = 0\} \\ &= \{\text{the set of all } r \text{ proper eigenvalues of } \mathbf{A} \text{ w.r.t. } \mathbf{B}\} \\ &= \{\text{a set of } r \text{ proper eigenvalues of } \mathbf{V}_2 \text{ w.r.t. } \mathbf{W}_1\} \\ &= \big\{\text{a set of } r \text{ eigenvalues of } \mathbf{V}_2\mathbf{W}_1^+\big\}, \end{aligned} \tag{39a}$$

$$\operatorname{nzch}\big(\mathbf{AB}^{+}\big) = \{d_1, \ldots, d_a\}, \tag{39b}$$

$$\begin{aligned} \operatorname{ch}\big(\mathbf{AB}^{+}\big) &= \{d_1, \ldots, d_a, d_{a+1} = 0, \ldots, d_p = 0\} \\ &= \operatorname{ch}\big[\mathbf{X}'\mathbf{RX}\big(\mathbf{X}'\mathbf{SX}\big)^{+}\big]. \end{aligned} \tag{39c}$$

Moreover, denoting $s = \operatorname{rank}(\mathbf{W}_1)$ and $m = \operatorname{rank}(\mathbf{V}_2\mathbf{W}_1)$, we obviously always have

$$\begin{aligned} \operatorname{ch}(\mathbf{R}, \mathbf{S}) &= \{e_1, \ldots, e_m, e_{m+1} = 0, \ldots, e_s = 0\} \\ &= \{\text{the set of all } s \text{ proper eigenvalues of } \mathbf{R} \text{ w.r.t. } \mathbf{S}\} \\ &= \big\{\text{the set of } s \text{ largest eigenvalues of } \mathbf{RS}^{+}\big\} \\ &= \big\{\text{the set of } s \text{ largest eigenvalues of } \mathbf{V}_2\mathbf{W}_1^{+}\big\}, \end{aligned} \tag{40a}$$

$$\operatorname{nzch}\big(\mathbf{V}_2\mathbf{W}_1^{+}\big) = \{e_1, \ldots, e_m\} = \operatorname{nzch}\big(\mathbf{RS}^{+}\big). \tag{40b}$$

4 When Does Every Representation of the BLUE Under $\mathscr{M}_1$ Continue to Be BLUE Under $\mathscr{M}_2$?

In Theorem 1 we consider whether a *particular* BLUE under $\mathscr{M}_1$ continues to be BLUE under $\mathscr{M}_2$. What about if we require that *every* representation of the BLUE under $\mathscr{M}_1$ continues to be BLUE under $\mathscr{M}_2$? The answer is given in Theorem 2. For the proof and related discussion, see, e.g., Baksalary and Mathew [2, Theorem 3], Mitra and Moore [16, Theorems 4.1 and 4.2], Rao [20, Lemma 5], Rao [21, Theorems 5.2 and 5.5], Rao [22, p. 289], Tian [26], Tian and Takane [27, 28], and Markiewicz, Puntanen and Styan [14].

Theorem 2 *Consider the linear models $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_1\}$ and $\mathscr{M}_2 = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_2\}$. Then every representation of the* BLUE *for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_1$ remains the* BLUE *for $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_2$ if and only if any of the following equivalent conditions hold*:

(a) $\mathscr{C}(\mathbf{V}_2\mathbf{X}^{\perp}) \subset \mathscr{C}(\mathbf{V}_1\mathbf{X}^{\perp})$,
(b) $\mathbf{V}_2 = \mathbf{V}_1 + \mathbf{XN}_1\mathbf{X}' + \mathbf{V}_1\mathbf{MN}_2\mathbf{MV}_1$ *for some* $\mathbf{N}_1$ *and* $\mathbf{N}_2$,
(c) $\mathbf{V}_2 = \mathbf{XN}_3\mathbf{X}' + \mathbf{V}_1\mathbf{MN}_4\mathbf{MV}_1$ *for some* $\mathbf{N}_3$ *and* $\mathbf{N}_4$,

where $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}}$.

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References

1. Baksalary, J.K.: An elementary development of the equation characterizing best linear unbiased estimators. Linear Algebra Appl. **388**, 3–6 (2004)
2. Baksalary, J.K., Mathew, T.: Linear sufficiency and completeness in an incorrectly specified general Gauss–Markov model. Sankhyā, Ser. A **48**, 169–180 (1986)

3. Baksalary, J.K., Mathew, T.: Rank invariance criterion and its application to the unified theory of least squares. Linear Algebra Appl. **127**, 393–401 (1990)
4. Baksalary, J.K., Puntanen, S.: Weighted-least-squares estimation in the general Gauss–Markov model. In: Dodge, Y. (ed.) Statistical Data Analysis and Inference, pp. 355–368. North-Holland, Amsterdam (1989)
5. Baksalary, J.K., Puntanen, S., Styan, G.P.H.: A property of the dispersion matrix of the best linear unbiased estimator in the general Gauss–Markov model. Sankhyā, Ser. A **52**, 279–296 (1990)
6. Baksalary, J.K., Rao, C.R., Markiewicz, A.: A study of the influence of the "natural restrictions" on estimation problems in the singular Gauss–Markov model. J. Stat. Plan. Inference **31**, 335–351 (1992)
7. Baksalary, O.M., Trenkler, G.: A projector oriented approach to the best linear unbiased estimator. Stat. Pap. **50**, 721–733 (2009)
8. Baksalary, O.M., Trenkler, G.: Between OLSE and BLUE. Aust. N. Z. J. Stat. **53**, 289–303 (2011)
9. de Leeuw, J.: Generalized eigenvalue problems with positive semidefinite matrices. Psychometrika **1**, 87–93 (1982)
10. Harville, D.A.: Matrix Algebra from a Statistician's Perspective. Springer, New York (1997)
11. Hauke, J., Markiewicz, A., Puntanen, S.: Comparing the BLUEs under two linear models. Commun. Stat. Theory Methods. **41**, 2405–2418 (2012)
12. Isotalo, J., Puntanen, S., Styan, G.P.H.: The BLUE's covariance matrix revisited: A review. J. Stat. Plan. Inference **138**, 2722–2737 (2008)
13. Luati, A., Proietti, T.: On the equivalence of the weighted least squares and the generalised least squares estimators, with applications to kernel smoothing. Ann. Inst. Stat. Math. **62**, 851–871 (2011)
14. Markiewicz, A., Puntanen, S., Styan, G.P.H.: A note on the interpretation of the equality of OLSE and BLUE. Pak. J. Stat., Ser. A **26**, 127–134 (2010)
15. Marsaglia, G., Styan, G.P.H.: Equalities and inequalities for ranks of matrices. Linear Multilinear Algebra **2**, 269–292 (1974)
16. Mitra, S.K., Moore, B.J.: Gauss–Markov estimation with an incorrect dispersion matrix. Sankhyā, Ser. A **35**, 139–152 (1973)
17. Mitra, S.K., Rao, C.R.: Simultaneous reduction of a pair of quadratic forms. Sankhyā, Ser. A **30**, 313–322 (1968)
18. Puntanen, S., Styan, G.P.H.: The equality of the ordinary least squares estimator and the best linear unbiased estimator (with discussion). Am. Stat. **43**, 151–161 (1989) [Commented by O. Kempthorne on pp. 161–162 and by S.R. Searle on pp. 162–163, Reply by the authors on p. 164]
19. Rao, C.R.: Least squares theory using an estimated dispersion matrix and its application to measurement of signals. In: Le Cam, L.M., Neyman, J. (eds.) Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, California, 1965/1966, vol. 1, pp. 355–372. University of California Press, Berkeley (1967)
20. Rao, C.R.: A note on a previous lemma in the theory of least squares and some further results. Sankhyā, Ser. A **30**, 259–266 (1968)
21. Rao, C.R.: Unified theory of linear estimation. Sankhyā, Ser. A **33**, 371–394 (1971) [Corrigendum (1972), **34**, p. 194 and p. 477]
22. Rao, C.R.: Representations of best linear unbiased estimators in the Gauss–Markoff model with a singular dispersion matrix. J. Multivar. Anal. **3**, 276–292 (1973)
23. Rao, C.R., Mitra, S.K.: Generalized Inverse of Matrices and Its Applications. Wiley, New York (1971)
24. Scott, A.J., Styan, G.P.H.: On a separation theorem for generalized eigenvalues and a problem in the analysis of sample surveys. Linear Algebra Appl. **70**, 209–224 (1985)
25. Thomas, D.H.: When do minimum variance estimators coincide? [Abstract] Ann. Math. Stat. **39**, 1365 (1968)

26. Tian, Y.: On equalities for BLUEs under misspecified Gauss–Markov models. Acta Math. Sin. Engl. Ser. **25**, 1907–1920 (2009)
27. Tian, Y., Takane, Y.: Some properties of projectors associated with the WLSE under a general linear model. J. Multivar. Anal. **99**, 1070–1082 (2008)
28. Tian, Y., Takane, Y.: On V-orthogonal projectors associated with a semi-norm. Ann. Inst. Stat. Math. **61**, 517–530 (2009)
29. Zyskind, G.: On canonical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models. Ann. Math. Stat. **38**, 1092–1109 (1967)

Inference in Error Orthogonal Models

Francisco Carvalho and João Tiago Mexia

Abstract Error Orthogonal Models constitute a very interesting class of models very useful in the design of experiments. The use of commutative Jordan algebras of symmetric matrices is used in order to perform statistical inference. The concept of segregation is introduced thus allowing the estimation of variance components.

Keywords Error Orthogonal · Jordan algebras · Segregation · Variance components

Mathematics Subject Classification (2010) 62K99 · 62J10 · 62H12

1 Introduction

Error Orthogonal Models (EO), as defined in VanLeuwen et al. [21], are mixed models such that the Least Square Estimators (LSE) for their estimable vector are Best Linear Unbiased Estimators (BLUE) whatever the variance components. We say that the LSE is a Uniformly Best Linear Unbiased Estimator (UBLUE).

An alternative definition is given by requiring that the model has the variance–covariance matrix

$$\mathbf{V} = \sum_{j=1}^{m} \gamma_j \mathbf{K}_j,$$

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where the $\mathbf{K}_1, \ldots, \mathbf{K}_m$ are pairwise orthogonal projection matrices (POOPM) that add up to $\mathbf{I}_n$ and commute with the orthogonal projection matrix $\mathbf{P}$ on the space Ω spanned by the mean vector $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$. Actually, this alternative definition is already mentioned in Houtman and Speed [10]. The equivalence between both definitions is established by VanLeeuwen et al. [21]. It may be of interest to point out that, according to the second definition, $\mathbf{P}$ and $\mathbf{V}$ commute, and so, see Zmyślony [23], the LSE for estimable vectors are UBLUE. This observation points to the interest of this definition in deriving interesting results.

This definition will lead to the key feature in our treatment, which will be defining homoscedastic submodels. In the study of these submodels, we will use the notion of quadratic sufficiency, see Mueller [15] to obtain Best Quadratic Unbiased Estimators (BQUE) for the $\gamma_1, \ldots, \gamma_m$. This study will be continued for mixed models

$$\mathbf{Y} = \sum_{i=0}^{w} \mathbf{X}_i \boldsymbol{\beta}_i$$

with $\boldsymbol{\beta}_0$ fixed and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ independent with null mean vectors and variance–covariance matrices $\sigma_1^2 \mathbf{I}_{i_1}, \ldots, \sigma_w^2 \mathbf{I}_{i_w}$. When the matrices $\mathbf{M}_i = \mathbf{X}_i \mathbf{X}_i^\top$, $i = 1, \ldots, w$, the mixed model is EO, see, e.g., Fonseca et al. [9].

The study of mixed models that are EO has rested on the use of Commutative Jordan Algebras (CJA). These are linear spaces constituted by symmetric matrices that commute and containing the squares of their matrices. These structures were introduced by Jordan et al. [11], who used them in a reformulation of Quantum Mechanics. They were rediscovered by Seely, who used them to obtain many interesting results on linear statistical inference, see Seely [17–19]. These were the initial papers on a very fruitful research line, see, e.g., Seely and Zyskind [20], Drygas and Zmyślony [3], VanLeeuwen et al. [21, 22], Fonseca et al. [7–9], Ferreira et al. [4], and Carvalho et al. [1, 2].

We matured the use of CJA since our approach without explicitly using them is nevertheless related. Thus, see Seely [19], for every CJA, there is one and only one basis, the principal basis, constituted by POOPM, $\mathbf{K}_1, \ldots, \mathbf{K}_m$ such as those that appear also in the expression of $\mathbf{V}$. Let the $\mathbf{K}_1, \ldots, \mathbf{K}_m$ have range spaces $\nabla_j = R(\mathbf{K}_j)$, $j = 1, \ldots, m$. If they add up to $\mathbf{I}_n$, we have the orthogonal partition

$$\mathbb{R}^n = \boxplus_{j=1}^{m} \nabla_j,$$

where $\boxplus$ indicates the orthogonal direct sum of subspaces. So the principal basis will be associated to this orthogonal partition. Let now the row vectors of $\mathbf{A}_j$ constitute an orthonormal basis for $\mathbf{V}_j$, $j = 1, \ldots, m$. For the model with mean vector $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ and variance–covariance matrix $\mathbf{V}$, we have the homoscedastic submodels with observation vectors

$$\mathbf{y}_j = \mathbf{A}_j \mathbf{y}, \quad j = 1, \ldots, m,$$

and variance–covariance matrices $\gamma_j \mathbf{I}_{g_j}$ with $g_j = \text{rank}(\mathbf{K}_j)$, $j = 1, \ldots, m$. These submodels will play a central role in our treatment. To lighten the discussion of these submodels, we will start with a section on homocedastic submodels. In this section, we will consider the normal one. Next, we go over to EO models, first in general, and next when they are written as normal models

$$\mathbf{y} = \sum_{i=0}^{w} \mathbf{X}_i \boldsymbol{\beta}_i,$$

where $\boldsymbol{\beta}_0$ is fixed, and the $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ are independent with null mean vectors and variance–covariance matrices $\sigma_i^2 \mathbf{I}_{c_i}$, $i = 1, \ldots, w$. When assuming normality, we will discuss the existence of complete and sufficient statistics.

2 Homoscedastic Models

We start by introducing quadratic sufficiency. If $\mathbf{y}$ is quasi-normal, has mean vector $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ and variance–covariance matrix $\mathbf{V} = \sigma^2 \mathbf{V}_\circ$, the pair $(\mathbf{L}\mathbf{y}, \mathbf{y}^\top \mathbf{T}\mathbf{y})$ is quadratic sufficient (QS) if there exists a symmetric matrix and a real ε such that $\mathbf{y}^\top \mathbf{L}^\top \mathbf{C}\mathbf{L}\mathbf{y} + \varepsilon \mathbf{y}^\top \mathbf{T}\mathbf{y}$ is BQUE for $f\sigma^2$ with $f = \dim(\Omega^\perp)$, with $\Omega^\perp$ being the orthogonal complement of the space Ω spanned by $\boldsymbol{\mu}$.

A necessary and sufficient condition for the pair $(\mathbf{L}\mathbf{y}, \mathbf{y}^\top \mathbf{T}\mathbf{y})$ to be QS is that

$$\mathscr{N}(\mathbf{L}) \cap R(\mathbf{W}) \subseteq \mathbf{V}\mathscr{N}\left(\mathbf{X}^\top\right) \cap \mathscr{N}\left(\mathbf{X}^\top \mathbf{T}\right) \cap \mathscr{N}(\mathbf{I} - \varepsilon \mathbf{V}_\circ \mathbf{T}),$$

where $\mathscr{N}(\cdot)$ indicates the nullity space

$$\mathbf{V}\mathscr{N}\left(\mathbf{X}^\top\right) = \left\{\mathbf{V}\mathbf{u} : \mathbf{u} \in \mathscr{N}\left(\mathbf{X}^\top\right)\right\},$$

and

$$\mathbf{W} = \mathbf{V}_\circ + \mathbf{X}\mathbf{X}^\top.$$

When $\mathbf{V}_\circ = \mathbf{I}_n$, we have $R(\mathbf{W}) = \mathbb{R}^n$, and the previous condition reduces to

$$\mathscr{N}(\mathbf{L}) \subseteq \mathscr{N}\left(\mathbf{X}^\top\right) \cap \mathscr{N}\left(\mathbf{X}^\top \mathbf{T}\right) \cap \mathscr{N}(\mathbf{I} - \varepsilon \mathbf{T}).$$

Moreover, if we take $\mathbf{T} = \mathbf{I}_n$, since we can also take $\varepsilon = 1$, the condition is now given by

$$\mathscr{N}(\mathbf{L}) \subseteq \mathscr{N}\left(\mathbf{X}^\top\right).$$

So, assuming that the model is quasi-normal and homoscedastic, the pair $(\mathbf{L}\mathbf{y}, \mathbf{y}^\top \mathbf{y})$ is QS whenever $\mathscr{N}(\mathbf{L}) \subseteq \mathscr{N}(\mathbf{X}^\top)$. The orthogonal projection matrix $\mathbf{P}$ on Ω can be written as

$$\mathbf{P} = \mathbf{W}^\top \mathbf{W}$$

whenever the row vectors of $\mathbf{W}$ constitute an orthonormal basis for Ω. We now have the canonical estimable vector

$$\boldsymbol{\eta} = \mathbf{W}\boldsymbol{\mu} = \mathbf{W}\mathbf{X}\boldsymbol{\beta},$$

for which we have the LSE $\widetilde{\boldsymbol{\eta}} = \mathbf{W}\mathbf{y}$.

Clearly, the Zmyślony (1978) version of the Gauss–Markov theorem holds for homoscedastic models, so the LSE of $\boldsymbol{\eta}$ will be BLUE. Moreover, we may establish the following proposition.

Proposition 1 *We have* $\mathscr{N}(\mathbf{L}) = \mathscr{N}(\mathbf{X}^\top)$.

Proof $R(\mathbf{P}) = R(\mathbf{X})$, so $\mathscr{N}(\mathbf{P}) = R(\mathbf{P})^\perp = R(\mathbf{X}^\perp) = \mathscr{N}(\mathbf{X}^\top)$. Moreover, $\mathbf{P} = \mathbf{W}^\top\mathbf{W}$, so $\mathscr{N}(\mathbf{W}) \subseteq \mathscr{N}(\mathbf{P}) = \mathscr{N}(\mathbf{X}^\top)$ and $\mathbf{W} = \mathbf{W}\mathbf{W}^\top\mathbf{W} = \mathbf{W}\mathbf{P}$, so $\mathscr{N}(\mathbf{X}^\top) = \mathscr{N}(\mathbf{P}) \subseteq \mathscr{N}(\mathbf{W})$, thus $\mathscr{N}(\mathbf{P}) = \mathscr{N}(\mathbf{X}^\top)$, and the proof is complete. □

Corollary 1 *If the model is quasi-normal homoscedastic, then the pair* $(\mathbf{W}\mathbf{y}, \mathbf{y}^\top\mathbf{y})$ *is QS.*

Thus, for $\boldsymbol{\sigma}^2$, we have the BQUE

$$\widetilde{\boldsymbol{\sigma}}^2 = \frac{1}{f}\left(\mathbf{y}^\top\mathbf{y} - \mathbf{y}^\top\mathbf{W}^\top\mathbf{W}\mathbf{y}\right) = \frac{1}{f}\left(\mathbf{y}^\top\mathbf{y} - \mathbf{y}^\top\mathbf{P}\mathbf{y}\right) = \frac{1}{f}\mathbf{y}^\top\mathbf{Q}\mathbf{y},$$

where $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ is the orthogonal projection matrix on $\Omega^\perp$.

Let us now assume normality. Since $\boldsymbol{\mu}^\top\mathbf{y} = \boldsymbol{\mu}^\top\mathbf{P}\mathbf{y} = \widetilde{\boldsymbol{\eta}}^\top\boldsymbol{\eta} = \boldsymbol{\eta}^\top\widetilde{\boldsymbol{\eta}}$, we have

$$\|\mathbf{y} - \boldsymbol{\mu}\|^2 = \mathbf{y}^\top\mathbf{y} - 2\boldsymbol{\mu}^\top\mathbf{y} + \boldsymbol{\mu}^\top\boldsymbol{\mu} = S - 2\boldsymbol{\eta}^\top\widetilde{\boldsymbol{\eta}} + \|\boldsymbol{\mu}\|^2$$

with $S = \mathbf{y}^\top\mathbf{y}$ and thus the density

$$n(\mathbf{y}) = \frac{e^{-\frac{1}{2\sigma^2}(S - 2\boldsymbol{\eta}^\top\widetilde{\boldsymbol{\eta}} + \|\boldsymbol{\mu}\|^2)}}{(2\pi\sigma^2)^{n/2}};$$

so, we will have the sufficient complete statistics S and $\widetilde{\boldsymbol{\eta}}$, the natural statistics $\theta = \frac{1}{\sigma^2}$, and

$$\boldsymbol{\xi} = \frac{1}{\sigma^2}\boldsymbol{\eta},$$

see Lehmann–Casella [14].

We now assume that the matrix $\mathbf{X}$ is not null. If $\mathbf{X} = \mathbf{0}$, the sole natural parameter would be θ with the corresponding complete sufficient statistic S.

Thus, we have the Uniformly Minimum Variance Unbiased Estimator (UMVUE), $\widetilde{\boldsymbol{\eta}}$, and

$$\widetilde{\boldsymbol{\sigma}}^2 = \frac{1}{f}\left(S - \widetilde{\boldsymbol{\eta}}^\top\widetilde{\boldsymbol{\eta}}\right) = \frac{1}{f}\mathbf{y}^\top\mathbf{Q}\mathbf{y}.$$

3 EO Models

We now assume that $\mathbf{y}$ has the mean vector $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ and variance–covariance matrix

$$\mathbf{V} = \sum_{j=1}^{m} \gamma_j \mathbf{K}_j = \sum_{j=1}^{m} \gamma_j \mathbf{A}_j^\top \mathbf{A}_j,$$

where the $\mathbf{K}_1, \ldots, \mathbf{K}_m$, are POOPM that add up to $\mathbf{I}_n$, and the row vectors of $\mathbf{A}_j$ constitute a orthonormal basis for $\nabla_j = R(\mathbf{K}_j)$, $j = 1, \ldots, m$.

We now have the homoscedastic submodels observation vectors

$$\mathbf{y}_j = \mathbf{A}_j \mathbf{y}, \quad j = 1, \ldots, m,$$

which, with $\mathbf{X}_j = \mathbf{A}_j \mathbf{X}$, have the mean vectors $\boldsymbol{\mu}_j = \mathbf{X}_j \boldsymbol{\beta}$ and variance–covariance matrices $\mathbf{V}_j = \gamma_j \mathbf{I}_{g_j}$ with

$$g_j = \mathrm{rank}(\mathbf{K}_j) = \mathrm{rank}(\mathbf{A}_j), \quad j = 1, \ldots, m.$$

For the submodels, we have the spaces $\Omega_j = R(\mathbf{X}_j)$, $j = 1, \ldots, m$, spanned by their mean vectors, and the orthogonal projection matrices on these subspaces can be written as

$$\mathbf{P}_j = \mathbf{W}^\dagger \mathbf{W}, \quad j = 1, \ldots, m,$$

once we assume that the row vectors of $\mathbf{W}_j$ constitute an orthonormal basis for Ω_j, $j = 1, \ldots, m$. We assume that $\mathbf{X}_j$ are not null, so that we will have the canonic estimable vectors

$$\boldsymbol{\eta} = \mathbf{W}_j \boldsymbol{\mu}_j, \quad j = 1, \ldots, m,$$

with LSE derived from the submodels

$$\widetilde{\boldsymbol{\eta}} = \mathbf{W}_j \mathbf{y}_j, \quad j = 1, \ldots, m.$$

If $\mathbf{X}_j$ is null, we take $\boldsymbol{\eta}_j$ and $\widetilde{\boldsymbol{\eta}}_j$ to be null. Now $\boldsymbol{\psi} = \mathbf{G}\boldsymbol{\beta}$ is an estimable vector for the full model if $\mathbf{G} = \mathbf{U}\mathbf{X}$, so that we will have

$$\boldsymbol{\psi} = \mathbf{U}\mathbf{A} = \mathbf{U} \sum_{j=1}^{m} \mathbf{K}_j \boldsymbol{\mu}_j = \mathbf{U} \sum_{j=1}^{m} \mathbf{A}_j^\top \boldsymbol{\mu}_j = \sum_{j=1}^{m} \mathbf{U}_j \boldsymbol{\eta}_j$$

with

$$\mathbf{U}_j = \mathbf{U}\mathbf{A}_j^\top \mathbf{W}_j^\top, \quad j = 1, \ldots, m,$$

since $\boldsymbol{\mu}_j = \mathbf{W}_j^\top \boldsymbol{\eta}_j$, $j = 1, \ldots, m$. Thus, the estimable vectors are generalized linear combinations of the canonical estimable vectors.

We recall that the LSE will be BLUE, and thus, for $\boldsymbol{\psi}$, we will have the BLUE

$$\widetilde{\boldsymbol{\psi}} = \mathbf{G}\widetilde{\boldsymbol{\beta}},$$

where

$$\widetilde{\boldsymbol{\beta}} = \left(\mathbf{X}^\top \mathbf{X}\right)^\dagger \mathbf{X}^\top \mathbf{y}$$

with $\mathbf{L}^\dagger$ the Moore–Penrose inverse of matrix $\mathbf{L}$. Now, with

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1^\top & \dots & \mathbf{A}_m^\top \end{bmatrix}^\top,$$

we have

$$\begin{bmatrix} \mathbf{y}_1^\top & \dots & \mathbf{y}_m^\top \end{bmatrix}^\top = \mathbf{A}\mathbf{y},$$

and so to have

$$\mathbf{L}\mathbf{y} = \begin{bmatrix} (\mathbf{L}_1\mathbf{y}_1)^\top & \dots & (\mathbf{L}_m\mathbf{y}_m)^\top \end{bmatrix}^\top = \mathbf{D}\begin{pmatrix} \mathbf{L}_1 & \dots & \mathbf{L}_m \end{pmatrix} \mathbf{A}\mathbf{y}$$

with $\mathbf{D}(\dots)$ indicating a blockwise diagonal matrix, we must have

$$\mathbf{L} = \mathbf{D}(\mathbf{L}_1 \quad \dots \quad \mathbf{L}_m)\mathbf{A}.$$

Likewise, to have

$$\mathbf{y}^\top \mathbf{T}\mathbf{y} = \begin{bmatrix} \mathbf{y}_1^\top \mathbf{T}_1 \mathbf{y}_1 \\ \dots \\ \mathbf{y}_m^\top \mathbf{T}_m \mathbf{y}_m \end{bmatrix} = \mathbf{y}^\top \mathbf{A}^\top \mathbf{D}(\mathbf{T}_1 \quad \dots \quad \mathbf{T}_m)\mathbf{A}\mathbf{y},$$

we must have

$$\mathbf{T} = \mathbf{A}^\top \mathbf{D}(\mathbf{T}_1 \quad \dots \quad \mathbf{T}_m)\mathbf{A}.$$

We now define quadratic sufficiency for error orthogonal models. We point out that this concept, see, e.g., Mueller [15] or Kornacki [13], has only been considered for models with one variance component, so we are now presenting its extension. Thus, $(\mathbf{L}\mathbf{y}, \mathbf{y}^\top \mathbf{T}\mathbf{y})$ is QS for EO models if

$$\begin{cases} \mathbf{L} = \mathbf{D}(\mathbf{L}_1 \quad \dots \quad \mathbf{L}_m)\mathbf{A}, \\ \mathbf{T} = \mathbf{A}^\top \mathbf{D}(\mathbf{T}_1 \quad \dots \quad \mathbf{T}_m)\mathbf{A}, \end{cases}$$

with pairs $(\mathbf{L}_j\mathbf{y}_j, \mathbf{y}^\top \mathbf{T}_j \mathbf{y}_j)$, $j = 1, \dots, m$, QS for the homoscedastic submodels. For instance, if we take $\mathbf{T} = \mathbf{I}_n$, we can also take $\mathbf{T}_j = \mathbf{I}_{g_j}$, $j = 1, \dots, m$, and whenever $\mathscr{N}(\mathbf{L}_j) \subseteq \mathscr{N}(\mathbf{X}_j)$, $j = 1, \dots, m$, taking

$$\mathbf{L} = \mathbf{D}(\mathbf{L}_1 \quad \dots \quad \mathbf{L}_m)\mathbf{A},$$

the pair $(\mathbf{L}\mathbf{y}, \mathbf{y}^\top \mathbf{T}\mathbf{y})$ will be QS. Namely, we can take

$$\mathbf{L}_j = \mathbf{W}_j, \quad j = 1, \dots, m,$$

the row vectors of $\mathbf{W}_j$ constituting an orthonormal basis for Ω_j, $j = 1, \ldots, m$, to have a QS pair $(\mathbf{Wy}, \mathbf{y}^\top\mathbf{y})$, where

$$\mathbf{W} = \mathbf{D}\begin{pmatrix}\mathbf{W}_1 & \ldots & \mathbf{W}_m\end{pmatrix}\mathbf{A}.$$

When we use this pair, from the submodels we get the estimator

$$\widetilde{\gamma}_j = \frac{1}{f_j}\mathbf{y}_j^\top\mathbf{Q}_j\mathbf{y}_j = \frac{1}{f_j}\mathbf{y}^\top\overline{\mathbf{Q}}_j\mathbf{y}, \quad j = z+1, \ldots, m',$$

once we assume that, with $z \geq 0$, the first z submodels are the ones with null matrices $\mathbf{X}_j$. In the last expression, we have

$$\overline{\mathbf{Q}}_j = \mathbf{A}_j\mathbf{Q}_j\mathbf{A}_j, \quad j = z+1, \ldots, m.$$

Moreover, see Schott [16], if we assume $\mathbf{y}$ to be quasi-normal, we have

$$\mathrm{cov}(\widetilde{\gamma}_j; \widetilde{\gamma}_\ell) = \frac{2}{f_j f_\ell}\,\mathrm{tr}(\overline{\mathbf{Q}}_j\mathbf{V}\overline{\mathbf{Q}}_\ell\mathbf{V}) + \frac{4}{f_j f_\ell}\boldsymbol{\mu}^\top\overline{\mathbf{Q}}_j\mathbf{V}\overline{\mathbf{Q}}_\ell\boldsymbol{\mu} \geq 0, \quad \begin{cases} j \neq \ell \\ j > z \\ \ell > z \end{cases}$$

since the matrices

$$\overline{\mathbf{Q}}_j\mathbf{V}\overline{\mathbf{Q}}_\ell = \mathbf{A}_j^\top\mathbf{Q}_j\mathbf{A}_j\mathbf{V}\mathbf{A}_\ell^\top\mathbf{Q}_\ell\mathbf{A}_\ell = \gamma_j\mathbf{A}_j^\top\mathbf{Q}_j\mathbf{A}_j\mathbf{A}_j\mathbf{A}_\ell^\top\mathbf{Q}_\ell\mathbf{A}_\ell = 0, \quad j \neq \ell.$$

We now establish the following proposition.

Proposition 2 *If, with* $\overline{\mathbf{H}}_j = \mathbf{A}_j^\top\mathbf{H}_j\mathbf{A}_j$, $j = z+1, \ldots, m$, *the*

$$\gamma_j^* = \mathbf{y}_j^\top S_j\mathbf{y}_j = \mathbf{y}^\top\overline{S}_j\mathbf{y}, \quad j = z+1, \ldots, m,$$

are unbiased estimators for the γ_j, $j = z+1, \ldots, m$, *and we have*

$$\mathrm{var}\left(\sum_{j=z+1}^{m} c_j\widetilde{\gamma}_j\right) \leq \mathrm{var}\left(\sum_{j=z+1}^{m} c_j\gamma_j^*\right).$$

Moreover, with $\mathbf{C} = [c_{i,j}]$, *and* $\widetilde{\boldsymbol{\gamma}}$ *and* $\boldsymbol{\gamma}^*$ *the vectors with components* $\widetilde{\gamma}_{z+1}, \ldots, \widetilde{\gamma}_m$ *and* $\gamma_{z+1}^*, \ldots, \gamma_m^*$, *respectively, we have*

$$\boldsymbol{\Sigma}(\mathbf{C}\widetilde{\boldsymbol{\gamma}}) \leq \boldsymbol{\Sigma}\big(\mathbf{C}\boldsymbol{\gamma}^*\big),$$

and $\boldsymbol{\Sigma}(\mathbf{C}\boldsymbol{\gamma}^*) - \boldsymbol{\Sigma}(\mathbf{C}\widetilde{\boldsymbol{\gamma}})$ *is positive semi-definitive.*

Proof Reasoning as above, we show that

$$\mathrm{cov}\big(\gamma_j^*; \gamma_\ell^*\big) = \mathrm{cov}(\widetilde{\gamma}_j; \widetilde{\gamma}_\ell), \quad j \neq \ell;\ j > z;\ \ell > z,$$

so

$$\operatorname{var}\left(\sum_{j=z+1}^{m} c_j \widetilde{\gamma}_j\right) = \sum_{j=z+1}^{m} c_j^2 \operatorname{var}(\widetilde{\gamma}_j) \le \sum_{j=z+1}^{m} c_j^2 \operatorname{var}\big(\gamma_j^*\big) = \operatorname{var}\big(c_j \gamma_j^*\big).$$

In the same way, we may show that for any $\mathbf{v}$, we have

$$\operatorname{var}\big(\mathbf{v}^\top \mathbf{C}\widetilde{\boldsymbol{\gamma}}\big) \le \operatorname{var}\big(\mathbf{v}^\top \mathbf{C}\boldsymbol{\gamma}^*\big),$$

and $\operatorname{var}(\mathbf{v}^\top \mathbf{C}\boldsymbol{\gamma}^*) - \operatorname{var}(\mathbf{v}^\top \mathbf{C}\widetilde{\boldsymbol{\gamma}})$ is positive semi-definitive. □

If we assume the normality, since the cross-covariance matrices of the submodels are

$$\begin{aligned}\boldsymbol{\Sigma}(\mathbf{y}_j; \boldsymbol{\Sigma}_\ell) = \boldsymbol{\Sigma}(\mathbf{A}_j\mathbf{y}; \mathbf{A}_\ell\mathbf{y}) &= \mathbf{A}_j\left(\sum_{h=1}^{m} \gamma_h \mathbf{K}_h\right)\mathbf{A}_\ell^\top \\ &= \gamma_j \mathbf{A}_j \mathbf{A}_\ell^\top = \mathbf{0}_{g_j \times g_\ell}, \quad j \neq \ell,\end{aligned}$$

the $\mathbf{y}_1, \ldots, \mathbf{y}_m$ will be normal independent with, when $z > 0$, densities

$$\begin{cases} n_j(\mathbf{y}) = \dfrac{e^{-\frac{S_j}{2\gamma_j}}}{(2\pi\gamma_j)^{g_j/2}}, & j = 1, \ldots, z, \\[2ex] n_j(\mathbf{y}) = \dfrac{e^{-\frac{1}{2\gamma_j}(S_j - 2\boldsymbol{\eta}_j^\top \widetilde{\boldsymbol{\eta}} + \|\boldsymbol{\eta}_j\|^2)}}{(2\pi\gamma_j)^{g_j/2}}, & j = z+1, \ldots, m, \end{cases}$$

where $\boldsymbol{\eta}_j = \mathbf{A}_j\boldsymbol{\mu}$ and $\widetilde{\boldsymbol{\eta}}_j = \mathbf{A}_j\mathbf{y}$, $j = z+1, \ldots, m$.

Due to the independence of the submodels, the joint density is $\prod_{j=1}^{m} n_j(\mathbf{y}_j)$. Thus, see Lehmann and Casella [14], we have the sufficient statistics $S_1, \ldots, S_m$ and $\widetilde{\boldsymbol{\eta}}_{z+1}, \ldots, \widetilde{\boldsymbol{\eta}}_m$ and the natural parameter $\boldsymbol{\theta}$ with components $\theta_j = \frac{1}{\gamma_j}$, $j = 1, \ldots, m$, and

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}_{z+1}^\top & \cdots & \boldsymbol{\xi}_m^\top \end{bmatrix}^\top$$

with $\boldsymbol{\xi}_j = \frac{1}{\gamma_j}\boldsymbol{\eta}_j$, $j = z+1, \ldots, m$. We now establish the following lemma.

Lemma 1 *There are no linear restrictions on the components of* $\boldsymbol{\xi}$ *or of* $\boldsymbol{\eta} = [\boldsymbol{\eta}_{z+1}^\top \cdots \boldsymbol{\eta}_m^\top]^\top$.

Proof It suffices to establish the thesis for $\boldsymbol{\eta}$ pointing out that $\boldsymbol{\mu}$ spans $R(\mathbf{X})$ of dimension $p = \sum_{j=z+1}^{m} p_j$ with $p_j = \operatorname{rank}(\mathbf{X}_j) = \operatorname{rank}(\mathbf{P}_j) = \operatorname{rank}(\mathbf{W}_j)$, $j = z+1, \ldots, m$.

Now since, when $z > 0$, $\mathbf{K}_j\boldsymbol{\mu} = \mathbf{0}$, $j = 1, \ldots, z$, we have

$$\boldsymbol{\mu} = \sum_{j=z+1}^{m} \mathbf{K}_j\boldsymbol{\mu} = \sum_{j=z+1}^{m} \mathbf{A}_j^\top \boldsymbol{\mu}_j = \sum_{j=z+1}^{m} \mathbf{A}_j^\top \mathbf{W}_j^\top \boldsymbol{\eta}_j$$

and

$$\begin{aligned}\operatorname{rank}\big(\mathbf{A}_j^\top \mathbf{W}_j^\top\big) &= \operatorname{rank}\big(\mathbf{A}_j^\top \mathbf{W}_j^\top \mathbf{W}_j \mathbf{A}_j\big) = \operatorname{rank}\big(\mathbf{A}_j^\top \mathbf{A}_j\big)\\ &= \operatorname{rank}(\mathbf{I}_{p_j}) = p_j, \quad j = 1, \ldots, m.\end{aligned}$$

We have only to point out that

$$\operatorname{rank}(\mathbf{X}) = \operatorname{rank}(\mathbf{A}\mathbf{X}) = \operatorname{rank}\Big((\mathbf{A}_{z+1}\mathbf{X}_{z+1})^\top \quad \ldots \quad (\mathbf{A}_m\mathbf{X}_m)^\top \Big)^\top = \sum_{i=1}^{m} p_j$$

to see that if $\boldsymbol{\mu}$ is to span $\Omega = R(\mathbf{X})$, there can be no restriction on the components of $\boldsymbol{\eta}$. □

If, moreover, there are no restrictions on the components of $\boldsymbol{\gamma}$, the sufficient statistics presented above will be, see again Lehmann and Casella [14], complete, and we can establish the following proposition.

Proposition 3 *If the normality is assumed and there are no restrictions on the components of $\boldsymbol{\gamma}$, the unbiased estimators derived from the statistics $S_1, \ldots, S_m$ and $\widetilde{\boldsymbol{\eta}}_{z+1}, \ldots, \widetilde{\boldsymbol{\eta}}_m$ will be UMVUE.*

Proof The proposition follows directly from the Blackwell, Lehmann and Scheffé theorem. □

4 Mixed Models

We now consider mixed models written in their usual form

$$\mathbf{y} = \sum_{i=0}^{w} \mathbf{X}_i \boldsymbol{\beta}_i,$$

where $\boldsymbol{\beta}_0$ is fixed, and the $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ are independent with null mean vectors

$$\boldsymbol{\mu} = \mathbf{X}_0 \boldsymbol{\beta}_0$$

and variance–covariance matrix

$$\mathbf{V} = \sum_{i=1}^{w} \sigma_i^2 \mathbf{M}_i$$

with $\mathbf{M}_i = \mathbf{X}_i \mathbf{X}_i^\top$, $i = 1, \ldots, w$. When these matrices commute, they are diagonalized by the same orthogonal matrix, see Schott [16]. The row vectors

$\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ of the orthogonal matrix will be eigenvectors for the $\mathbf{M}_1, \ldots, \mathbf{M}_w$. Writing $\boldsymbol{\alpha}_h \tau \boldsymbol{\alpha}_\ell$ when $\boldsymbol{\alpha}_h$ and $\boldsymbol{\alpha}_\ell$ are associated to identical eigenvectors for all matrices $\mathbf{M}_1, \ldots, \mathbf{M}_w$, we define an equivalence relation between eigenvectors. Let $\mathscr{C}_1, \ldots, \mathscr{C}_m$ be the sets of indexes of the eigenvectors belonging to the different equivalence classes. Then

$$\mathbf{K}_j = \sum_{\ell \in \mathscr{C}_j} \boldsymbol{\alpha}_j \boldsymbol{\alpha}_j^\top$$

will be POOPM. Moreover, we will have

$$\mathbf{M}_i = \sum_{j=1}^{m} b_{i,j} \mathbf{K}_j, \quad i = 1, \ldots, w,$$

with $b_{i,j}$ the eigenvalue for $\mathbf{M}_i$, $i = 1, \ldots, w$, and the eigenvectors with indexes in $\mathscr{C}_j$, $j = 1, \ldots, m$.

Then

$$\mathbf{V} = \sum_{i=1}^{w} \sigma_i^2 \sum_{j=1}^{m} b_{i,j} \mathbf{K}_j = \sum_{j=1}^{m} \gamma_j \mathbf{K}_j$$

with

$$\gamma_j = \sum_{i=1}^{w} b_{i,j} \sigma_i^2, \quad j = 1, \ldots, m.$$

Now with, when $z > 0$,

$$\boldsymbol{\gamma}(1) = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_z \end{bmatrix}, \qquad \boldsymbol{\gamma}(2) = \begin{bmatrix} \gamma_{z+1} \\ \vdots \\ \gamma_m \end{bmatrix}, \qquad \boldsymbol{\sigma}^2 = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_w^2 \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}(1) & \mathbf{B}(2) \end{bmatrix},$$

where $\mathbf{B}(1)$ has z columns, and $\mathbf{B}(2)$ has $m - z$ columns, we have

$$\boldsymbol{\gamma}(\ell) = \mathbf{B}(\ell)^\top \boldsymbol{\sigma}^2, \quad \ell = 1, 2.$$

Of course, if $z = 0$, we have $\boldsymbol{\gamma} = \boldsymbol{\gamma}(2)$, $\mathbf{B} = \mathbf{B}(2)$, and the last expression is replaced by $\boldsymbol{\gamma} = \mathbf{B}^\top \boldsymbol{\sigma}^2$.

We now assume $\mathbf{y}$ to be quasi-normal, so, according to Proposition 2, we have

$$\boldsymbol{\Sigma}\big(\widetilde{\boldsymbol{\gamma}}(2)\big) \leq \boldsymbol{\Sigma}\big(\boldsymbol{\gamma}^*(2)\big)$$

with $\gamma_{z+1}^*, \ldots, \gamma_m^*$, the components of $\boldsymbol{\gamma}^*(2)$, derived from the submodels. If $z = 0$, the last expression would be replaced by $\boldsymbol{\Sigma}(\widetilde{\boldsymbol{\gamma}}) \leq \boldsymbol{\Sigma}(\boldsymbol{\gamma}^*)$.

If the row vectors of $\mathbf{B}(2)$ are linearly independent, we have

$$\boldsymbol{\sigma}^2 = \mathbf{B}(2)^{\top^-}\boldsymbol{\gamma}(2)$$

with $\mathbf{B}(2)^{\top^-}$ a generalized inverse of $\mathbf{B}(2)^\top$. Also, according to Proposition 2, we have

$$\boldsymbol{\Sigma}\big(\mathbf{B}(2)^{\top^-}\widetilde{\boldsymbol{\gamma}}(2)\big) \le \boldsymbol{\Sigma}\big(\mathbf{B}(2)^{\top^-}\boldsymbol{\gamma}^*(2)\big),$$

so, we must use $\widetilde{\boldsymbol{\gamma}}(2)$ to estimate $\boldsymbol{\sigma}^2$. When $z = 0$, we replace $\widetilde{\boldsymbol{\gamma}}(2)$ $[\boldsymbol{\gamma}^*(2), \mathbf{B}(2)]$ by $\widetilde{\boldsymbol{\gamma}}$ $[\boldsymbol{\gamma}^*, \mathbf{B}]$ to obtain the corresponding results.

Since generalized inverses are not unique, we must also consider their choice. To lighten the writing, we assume that $z = 0$ since we have only to do straightforward adjustments for the case where $z > 0$.

We assume the row vectors of $\mathbf{B}$ to be linearly independent, so $w \le m$. Now

$$\mathbf{B}^\top = \mathbf{P}^\top \begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix} \mathbf{P}^\circ$$

with $\mathbf{P}$ and $\mathbf{P}^\circ$ orthogonal and $\mathbf{D}$ the diagonal matrix whose principal elements are the singular values $\mathbf{v}_1, \ldots, \mathbf{v}_w$ of $\mathbf{B}$. Thus,

$$\mathbf{B}^{\top^-} = \mathbf{P}^{\circ^\top} \begin{bmatrix} \mathbf{D}^\dagger & \mathbf{U} \end{bmatrix} \mathbf{P},$$

where $\mathbf{U}$ is arbitrary, and $\mathbf{D}^\dagger$ is the diagonal matrix with principal elements $\mathbf{v}_i^\dagger = \mathbf{v}_i^{-1}$ $[= 0]$ when $\mathbf{v}_i \ne 0$ $[= 0]$, $i = 1, \ldots, w$.

It is also easy to see that the Moore–Penrose inverse of $\mathbf{B}^\top$ is

$$\mathbf{B}^{\top^\dagger} = \mathbf{P}^{\circ^\top} \begin{bmatrix} \mathbf{D}^\dagger & \mathbf{0} \end{bmatrix} \mathbf{P}^\top,$$

so,

$$\left\|\mathbf{B}^{\top^\dagger}\right\| \le \left\|\mathbf{B}^{\top^-}\right\|,$$

and thus, to minimize the upper bound of $\|\boldsymbol{\Sigma}(\mathbf{B}(2))^{\top^-}\widetilde{\boldsymbol{\gamma}}(2)\|$, we take $\mathbf{B}^{\top^-} = \mathbf{B}^{\top^\dagger}$.

5 An Application

We now apply our results to a model with balanced cross-nesting. This model will have a first factor with u levels that nests a factor with v levels, and the first factor crosses with the third one with ℓ levels. These will have r observations for each of the $u \times v \times \ell$ treatments.

We assume the first and third factors to have fixed effects and the second one to have random effects. This model may be written, see Khury et al. [12] as

$$\mathbf{y} = \sum_{i=1}^{7} \mathbf{X}_i^\circ \boldsymbol{\beta}_i^\circ + \boldsymbol{\varepsilon}^\circ,$$

Table 1 Matrices, factors, and interactions

Matrices	Factors and interactions
$\mathbf{X}_1^\circ = \mathbf{1}_u \otimes \mathbf{1}_v \otimes \mathbf{1}_\ell \otimes \mathbf{1}_r$	General mean value
$\mathbf{X}_2^\circ = \mathbf{I}_u \otimes \mathbf{1}_v \otimes \mathbf{1}_\ell \otimes \mathbf{1}_r$	First factor
$\mathbf{X}_3^\circ = \mathbf{I}_u \otimes \mathbf{I}_v \otimes \mathbf{1}_\ell \otimes \mathbf{1}_r$	Second factor
$\mathbf{X}_4^\circ = \mathbf{1}_u \otimes \mathbf{1}_v \otimes \mathbf{I}_\ell \otimes \mathbf{1}_r$	Third factor
$\mathbf{X}_5^\circ = \mathbf{I}_u \otimes \mathbf{1}_v \otimes \mathbf{I}_\ell \otimes \mathbf{1}_r$	Interaction between the first and second factors
$\mathbf{X}_6^\circ = \mathbf{I}_u \otimes \mathbf{I}_v \otimes \mathbf{I}_\ell \otimes \mathbf{1}_r$	Interaction between the second and third factors
$\mathbf{X}_7^\circ = \mathbf{I}_u \otimes \mathbf{I}_v \otimes \mathbf{I}_\ell \otimes \mathbf{I}_r$	Technical error

where, with $\otimes$ indicating the Kronecker matrix product, we have that the mean vector of this model is

$$\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$$

with

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1^\circ & \mathbf{X}_2^\circ & \mathbf{X}_3^\circ & \mathbf{X}_4^\circ & \mathbf{X}_5^\circ \end{bmatrix},$$

and, taking $\mathbf{X}_1 = \mathbf{X}_4^\circ$, $\mathbf{X}_2 = \mathbf{X}_6^\circ$ and $\mathbf{X}_3 = \mathbf{X}_7^\circ$, we may write the model as

$$\mathbf{y} = \sum_{i=0}^{3} \mathbf{X}_i \boldsymbol{\beta}_i$$

with $\boldsymbol{\beta}_0 = \beta$ fixed and, as before, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$ independent with null mean vector and variance–covariance matrices $\sigma_1^2\mathbf{I}_\ell$, $\sigma_2^2\mathbf{I}_{\ell\times u\times v}$, and $\sigma_3^2\mathbf{I}_{\ell\times u\times v\times r}$. In Table 1 we define how the matrices $\mathbf{X}_i^\circ$, $i = 1, \dots, 7$, are constructed, and what they represent.

Now, see Fonseca et al. [5, 7], we have

$$\begin{cases} \mathbf{M}_1 = \mathbf{X}_1\mathbf{X}_1^\top = \ell \times r(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3), \\ \mathbf{M}_2 = \mathbf{X}_2\mathbf{X}_2^\top = r(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + \mathbf{Q}_6), \\ \mathbf{M}_3 = \mathbf{X}_3\mathbf{X}_3^\top = \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5\mathbf{Q}_6 + \mathbf{Q}_7, \end{cases}$$

with $\mathbf{K}_d = \mathbf{I}_d - \frac{1}{d}\mathbf{I}_d$. In Table 2 we define how the matrices $\mathbf{Q}_i^\circ$, $i = 1, \dots, 7$, are constructed, and what they represent.

Now, the orthogonal projection matrix on the range space spanned by the mean vector is

$$\mathbf{T} = \sum_{j=1}^{4} \mathbf{Q}_j,$$

so the γ_1, γ_2, γ_3, and γ_4 cannot be estimated from

$$\mathbf{y}_j = \mathbf{A}_j\mathbf{y}, \quad j = 1, \dots, 4.$$

Moreover, since the $\mathbf{Q}_1, \dots, \mathbf{Q}_7$, are pairwise orthogonal, from the expression of $\mathbf{T}$ we see that the $\mathbf{y}_5$, $\mathbf{y}_6$, and $\mathbf{y}_7$ have null mean vectors and variance–covariance

Table 2 $\mathbf{Q}_j$ matrices, factors and interactions

Matrices	Factors and interactions
$\mathbf{Q}_1 = \frac{1}{u}\mathbf{J}_u \otimes \frac{1}{v}\mathbf{J}_v \otimes \frac{1}{\ell}\mathbf{J}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	General mean value
$\mathbf{Q}_2 = \mathbf{K}_u \otimes \frac{1}{v}\mathbf{J}_v \otimes \frac{1}{\ell}\mathbf{J}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	First factor
$\mathbf{Q}_3 = \frac{1}{u}\mathbf{J}_u \otimes \frac{1}{v}\mathbf{J}_v \otimes \mathbf{K}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	Second factor
$\mathbf{Q}_4 = \mathbf{K}_u \otimes \frac{1}{v}\mathbf{J}_v \otimes \mathbf{K}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	Third factor
$\mathbf{Q}_5 = \mathbf{I}_u \otimes \mathbf{K}_v \otimes \frac{1}{\ell}\mathbf{J}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	Interaction between the first and second factor
$\mathbf{Q}_6 = \mathbf{I}_u \otimes \mathbf{K}_v \otimes \mathbf{I}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	Interaction between second and third factor
$\mathbf{Q}_7 = \mathbf{I}_u \otimes \mathbf{I}_v \otimes \mathbf{I}_\ell \otimes \frac{1}{r}\mathbf{J}_r$	Technical error

matrices $\gamma_5\mathbf{I}_{u(v-1)}$, $\gamma_6\mathbf{I}_{u(v-1)(\ell-1)}$, and $\gamma_7\mathbf{I}_{uv\ell(r-1)}$, so we get the estimators

$$\begin{cases} \widetilde{\gamma}_5 = \frac{\|\mathbf{y}_5\|^2}{u(v-1)}, \\ \widetilde{\gamma}_6 = \frac{\|\mathbf{y}_6\|^2}{u(v-1)(\ell-1)}, \\ \widetilde{\gamma}_7 = \frac{\|\mathbf{y}_7\|^2}{uv\ell(r-1)}. \end{cases}$$

To obtain the $\mathbf{y}_5$, $\mathbf{y}_6$, and $\mathbf{y}_7$, we use, see Fonseca et al. [5], the matrices

$$\begin{cases} \mathbf{A}_5 = \mathbf{I}_u \otimes \mathbf{T}_v \otimes \frac{1}{\sqrt{\ell}}\mathbf{1}_\ell^\top \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r^\top, \\ \mathbf{A}_6 = \mathbf{I}_u \otimes \mathbf{T}_v \otimes \mathbf{T}_\ell^\top \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r^\top, \\ \mathbf{A}_7 = \mathbf{I}_u \otimes \mathbf{I}_v \otimes \mathbf{I}_\ell \otimes \mathbf{T}_r, \end{cases}$$

where $\mathbf{T}_h$ is obtained deleting the first row equal to $\frac{1}{\sqrt{h}}\mathbf{1}_h^\top$ from an $h \times h$ orthogonal matrix.

From the expressions of matrices $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{T}$ we clearly get the matrix

$$\mathbf{B}(2) = \begin{bmatrix} \ell r & 0 & 0 \\ r & r & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

so,

$$\begin{cases} \gamma_5 = \ell r\sigma_1^2 + r\sigma_2^2 + \sigma_3^2, \\ \gamma_6 = \sigma_2^2 + \sigma_3^2, \\ \gamma_7 = \sigma_3^2, \end{cases}$$

and thus,

$$\begin{cases} \widetilde{\sigma}_3^2 = \widetilde{\gamma}_7, \\ \widetilde{\sigma}_2^2 = \frac{(\widetilde{\gamma}_6 - \widetilde{\gamma}_7)}{r}, \\ \widetilde{\sigma}_1^2 = \frac{(\widetilde{\gamma}_5 - \widetilde{\gamma}_6)}{\ell r}. \end{cases}$$

Table 3 Yields in kg

Location	Origin 1			Origin 2		
	Clone 1	Clone 2	Clone 3	Clone 1	Clone 2	Clone 3
1	3.00	1.00	1.10	1.75	1.10	1.05
	1.85	1.10	1.50	3.50	1.05	1.25
	0.75	1.00	1.80	2.50	0.50	2.00
	1.35	1.60	1.45	2.00	1.05	1.50
	1.45	1.50	1.25	0.65	1.25	2.10
2	1.80	1.60	0.85	2.00	1.20	1.00
	0.70	1.75	0.65	3.00	1.35	2.70
	2.50	0.50	0.55	2.55	1.20	2.15
	1.70	1.35	0.90	3.00	0.30	2.10
	0.40	1.10	0.90	2.65	2.50	2.70
3	1.05	0.75	0.90	1.60	1.05	1.60
	1.50	0.65	0.90	3.05	1.95	1.10
	1.15	0.90	0.55	0.25	2.00	2.05
	0.85	0.85	0.70	1.66	2.20	1.50
	1.15	1.05	0.35	2.65	2.35	3.00

In order to illustrate the results presented in the previous sections, we will use an example, see Fonseca et al. [6].

In this example, we do not intend to evaluate the grapevine castes homogeneity. These castes are constituted by clones, and these clones have a common origin, i.e., probably a common ancestor. In this way, one would think that there is a genetic homogeneity. This claim is not shared by the wine produces since, for them, the genetic homogeneity does not always prevail.

In this example, the Touriga Nacional (Portuguese caste from the family of *Vitis Viniferas*) caste was used. In the design of the experiment, two groups from different origins of three clones were grown side by side.

The grapevines were displayed in a rectangular grid following a fertility gradient where each column is occupied by different clones. Three locations were chosen to cover the experimental field, and, in each location, the grapevines in five rows were considered. The data can be seen in Table 3.

Using the previous results, taking $u = 2$, $v = 3$, $\ell = 3$, and $r = 5$, we obtained

$$\begin{cases} \widetilde{\gamma}_5 = 1.5648, \\ \widetilde{\gamma}_6 = 0.4270, \\ \widetilde{\gamma}_7 = 0.3671, \end{cases}$$

and

$$\begin{cases} \widetilde{\sigma}_1 = 0.0759, \\ \widetilde{\sigma}_2 = 0.0120, \\ \widetilde{\sigma}_3 = 0.3671. \end{cases}$$

We may conclude that neither the third factor nor its interaction with the second factor is relevant.

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References

1. Carvalho, F., Mexia, J.T., Covas, R.: Structured families of models with commutative orthogonal block structures. AIP Conf. Proc. **1281**, 1256–1259 (2010)
2. Carvalho, F., Mexia, J.T., Covas, R., Fernandes, C.: A fundamental partition in models with commutative orthogonal block structure. AIP Conf. Proc. **1389**, 1615–1618 (2011)
3. Drygas, H., Zmyślony, R.: Jordan algebras and Bayesian quadratic estimation of variance components. Linear Algebra Appl. **168**, 259–275 (1992)
4. Ferreira, S., Ferreira, D., Nunes, C., Mexia, J.T.: Nesting segregated mixed models. J. Stat. Theory Pract. **4**(2), 233–242 (2010)
5. Fonseca, M., Mexia, J.T., Zmyślony, R.: Estimators and tests for variance components in cross nested orthogonal models. Discuss. Math., Probab. Stat. **23**, 175–201 (2003)
6. Fonseca, M., Mexia, J.T., Zmyślony, R.: Estimating and testing of variance components: an application to grapevine experiments. Biom. Lett. **40**(1), 1–7 (2003)
7. Fonseca, M., Mexia, J.T., Zmyślony, R.: Binary operation on Jordan algebras and orthogonal normal models. Linear Algebra Appl. **417**, 75–86 (2006)
8. Fonseca, M., Mexia, J.T., Zmyślony, R.: Jordan algebras. Generating pivot variables and orthogonal normal models. J. Interdiscip. Math. **10**(2), 305–326 (2007)
9. Fonseca, M., Mexia, J.T., Zmyślony, R.: Inference in normal models with commutative orthogonal block structure. Acta Comment. Univ. Tartu Math. **12**, 3–16 (2008)
10. Houtman, A.M., Speed, T.P.: Balance in designed experiments with orthogonal block structure. Ann. Stat. **11**(4), 1069–1085 (1983)
11. Jordan, P., von Neumann, J., Wigner, E.: On the algebraic generalization of the quantum mechanical formalism. Ann. Math. **36**, 26–64 (1934)
12. Khuri, A.I., Mathew, T., Sinha, B.K.: Statistical Tests for Mixed Linear Models. Wiley Series in Probability and Statistics. Wiley, New York (1998)
13. Kornacki, A.: Different kinds of sufficiency in the general Gauss–Markov model. Math. Slovaca **57**(4), 389–392 (2007)
14. Lehmann, E.L., Casella, G.: Theory of Point Estimation, 2nd edn. Springer Texts in Statistics. Springer, Berlin (2003)
15. Mueller, J.: Sufficiency and completeness in the linear model. J. Multivar. Anal. **21**(2), 312–323 (1987)
16. Schott, J.R.: Matrix Analysis for Statistics. Wiley Series in Probability and Statistics. Wiley, New York (1997)
17. Seely, J.: Linear spaces and unbiased estimation. Ann. Math. Stat. **41**, 1725–1734 (1970)
18. Seely, J.: Linear spaces and unbiased estimation—application to a mixed linear model. Ann. Math. Stat. **41**, 1735–1748 (1970)
19. Seely, J.: Quadratic subspaces and completeness. Ann. Math. Stat. **42**, 710–721 (1971)

20. Seely, J., Zyskind, G.: Linear spaces and minimum variance unbiased estimation. Ann. Math. Stat. **42**, 691–703 (1971)
21. VanLeeuwen, D.M., Seely, J.F., Birkes, D.S.: Sufficient conditions for orthogonal designs in mixed linear models. J. Stat. Plan. Inference **73**, 373–389 (1998)
22. Vanleeuwen, D.M., Birkes, D.S., Seely, J.F.: Balance and orthogonality in designs for mixed classification models. Ann. Stat. **27**, 1927–1947 (1999)
23. Zmyślony, R.: A characterization of best linear unbiased estimators in the general linear model. Lect. Notes Stat. **2**, 365–373 (1978)

On the Entries of Orthogonal Projection Matrices

Oskar Maria Baksalary and Götz Trenkler

Abstract The present paper is concerned with characterizing entries of orthogonal projectors (i.e., a Hermitian idempotent matrices). On the one hand, several bounds for the values of the entries are identified. On the other hand, particular attention is paid to the question of how an orthogonal projector changes when its entries are modified. The modifications considered are those of a single entry and of an entire row or column. Some applications of the results in the linear regression model are pointed out as well.

Keywords Orthogonal projector · Idempotent matrices · Oblique projector · Moore–Penrose inverse · Linear model

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1 Preliminaries

Let $\mathbb{C}_{m,n}$ ($\mathbb{R}_{m,n}$) denote the set of $m \times n$ complex (real) matrices. The symbols $\mathbf{M}^*$, $\mathscr{R}(\mathbf{M})$, $\mathscr{N}(\mathbf{M})$, and $\mathrm{rk}(\mathbf{M})$ will stand for the conjugate transpose, column space (range), null space, and rank of $\mathbf{M} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n, and for a given $\mathbf{M} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{M}} = \mathbf{I}_n - \mathbf{M}$. Additionally, $\mathrm{tr}(\mathbf{M})$ will stand for the trace of $\mathbf{M} \in \mathbb{C}_{n,n}$. For two matrices $\mathbf{M}$ and $\mathbf{N}$ having the same number of rows, the columnwise partitioned matrix obtained by juxtaposing the two matrices will be denoted by $(\mathbf{M} : \mathbf{N})$. In Sect. 4, which provides some results dealing with real matrices, we will use the symbol $\mathbf{M}'$ to denote the transpose of $\mathbf{M} \in \mathbb{R}_{m,n}$.

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We define two particular vectors in $\mathbb{C}_{n,1}$, namely $\mathbf{1}_n$, which denotes $n \times 1$ vector of ones, and $\mathbf{e}_i$, which stands for a vector having 1 in the ith row and all other entries equal to zero, i.e.,

$$\mathbf{1}_n = (1, 1, \ldots, 1)^* \quad \text{and} \quad \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^*, \quad i = 1, 2, \ldots, n.$$

Recall that any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}_{n,1}$ satisfy the Cauchy–Schwarz inequality, which reads $|\mathbf{x}^*\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$.

A matrix $\mathbf{M} \in \mathbb{C}_{n,n}$ satisfying $\mathbf{M}^2 = \mathbf{M}$ is called idempotent. It is known that every idempotent matrix represents an oblique projector onto its column space $\mathscr{R}(\mathbf{M})$ along its null space $\mathscr{N}(\mathbf{M})$. A key role in the subsequent considerations will be played by a subset of idempotent matrices consisting of matrices which are additionally Hermitian. Such matrices are called orthogonal projectors, and their set will be denoted by $\mathbb{C}_n^{\mathsf{OP}}$, i.e.,

$$\mathbb{C}_n^{\mathsf{OP}} = \left\{\mathbf{M} \in \mathbb{C}_{n,n} : \ \mathbf{M}^2 = \mathbf{M} = \mathbf{M}^*\right\}.$$

A projector $\mathbf{M} \in \mathbb{C}_n^{\mathsf{OP}}$ projects onto $\mathscr{R}(\mathbf{M})$ along $\mathscr{R}(\mathbf{M})^\perp$, where $\mathscr{R}(\mathbf{M})^\perp$ denotes the orthogonal complement of $\mathscr{R}(\mathbf{M})$.

An important role in considerations dealing with projectors is played by the notion of the Moore–Penrose inverse, for $\mathbf{M} \in \mathbb{C}_{m,n}$ defined to be the unique matrix $\mathbf{M}^\dagger \in \mathbb{C}_{n,m}$ satisfying the equations:

$$\mathbf{M}\mathbf{M}^\dagger\mathbf{M} = \mathbf{M}, \quad \mathbf{M}^\dagger\mathbf{M}\mathbf{M}^\dagger = \mathbf{M}^\dagger, \quad \mathbf{M}\mathbf{M}^\dagger = \left(\mathbf{M}\mathbf{M}^\dagger\right)^*, \quad \mathbf{M}^\dagger\mathbf{M} = \left(\mathbf{M}^\dagger\mathbf{M}\right)^*.$$

The Moore–Penrose inverse can be used to represent orthogonal projectors onto subspaces determined by $\mathbf{M} \in \mathbb{C}_{m,n}$. To be precise:

(i) $\mathbf{P}_\mathbf{M} = \mathbf{M}\mathbf{M}^\dagger$ is the orthogonal projector onto $\mathscr{R}(\mathbf{M})$,
(ii) $\mathbf{Q}_\mathbf{M} = \mathbf{I}_m - \mathbf{M}\mathbf{M}^\dagger$ is the orthogonal projector onto $\mathscr{R}(\mathbf{M})^\perp = \mathscr{N}(\mathbf{M}^*)$,
(iii) $\mathbf{P}_{\mathbf{M}^*} = \mathbf{M}^\dagger\mathbf{M}$ is the orthogonal projector onto $\mathscr{R}(\mathbf{M}^*)$,
(iv) $\mathbf{Q}_{\mathbf{M}^*} = \mathbf{I}_n - \mathbf{M}^\dagger\mathbf{M}$ is the orthogonal projector onto $\mathscr{R}(\mathbf{M}^*)^\perp = \mathscr{N}(\mathbf{M})$.

It is known that the trace of an idempotent matrix is equal to its rank; see Baksalary, Bernstein, and Trenkler [3]. The trace of a product of $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ satisfies

$$\mathrm{tr}(\mathbf{P}\mathbf{Q}) \leq \min\left\{\mathrm{rk}(\mathbf{P}), \mathrm{rk}(\mathbf{Q})\right\} = \min\left\{\mathrm{tr}(\mathbf{P}), \mathrm{tr}(\mathbf{Q})\right\}; \tag{1}$$

see Baksalary and Trenker [4, inequality (3.29)].

The lemma below is recalled following Baksalary, Baksalary, and Trenkler [2, Lemma 2.1].

Lemma 1 *For any nonzero* $\mathbf{u}, \mathbf{v} \in \mathbb{C}_{n,1}$, *the product* $\mathbf{u}\mathbf{v}^*$ *is an oblique projector if and only if* $\mathbf{v}^*\mathbf{u} = 1$, *in which case* $\mathbf{u}\mathbf{v}^*$ *is an oblique projector onto* $\mathscr{R}(\mathbf{u})$ *along* $\mathscr{R}(\mathbf{v})^\perp$. *Moreover, if* $\mathbf{v}^*\mathbf{u} = 1$ *and* $\mathbf{P}$ *is an oblique projector, then*:

(i) $\mathbf{P} + \mathbf{u}\mathbf{v}^*$ *is an oblique projector if and only if* $\mathbf{u} \in \mathscr{R}(\mathbf{P}^*)^\perp$ *and* $\mathbf{v} \in \mathscr{R}(\mathbf{P})^\perp$, *in which case* $\mathbf{P} + \mathbf{u}\mathbf{v}^*$ *is an oblique projector onto* $\mathscr{R}(\mathbf{P}) \oplus \mathscr{R}(\mathbf{u})$ *along* $\mathscr{R}(\mathbf{P}^*)^\perp \cap \mathscr{R}(\mathbf{v})^\perp$,

(ii) $\mathbf{P}-\mathbf{uv}^*$ *is an oblique projector if and only if* $\mathbf{u}\in\mathscr{R}(\mathbf{P})$ *and* $\mathbf{v}\in\mathscr{R}(\mathbf{P}^*)$, *in which case* $\mathbf{P}-\mathbf{uv}^*$ *is an oblique projector onto* $\mathscr{R}(\mathbf{P})\cap\mathscr{R}(\mathbf{v})^{\perp}$ *along* $\mathscr{R}(\mathbf{P}^*)^{\perp}\oplus\mathscr{R}(\mathbf{u})$.

Another result of interest given in Baksalary, Baksalary, and Trenkler [2, Theorem 1.1] characterizes possible changes of a rank of a matrix when it is modified by a matrix of rank one. The result is recalled in what follows.

Lemma 2 *For given* $\mathbf{A}\in\mathbb{C}_{m,n}$ *and nonzero* $\mathbf{b}\in\mathbb{C}_{m,1}$, $\mathbf{c}\in\mathbb{C}_{n,1}$, *let* $\mathbf{M}$ *be the modification of* $\mathbf{A}$ *to the form*

$$\mathbf{M}=\mathbf{A}+\mathbf{bc}^*, \tag{2}$$

and let $\lambda=1+\mathbf{c}^*\mathbf{A}^{\dagger}\mathbf{b}$. *Then*:

$$\mathrm{rk}(\mathbf{M})=\mathrm{rk}(\mathbf{A})-1 \iff \mathbf{b}\in\mathscr{R}(\mathbf{A}),\ \mathbf{c}\in\mathscr{R}\big(\mathbf{A}^*\big),\ \lambda=0, \tag{$\downarrow$}$$

$$\mathrm{rk}(\mathbf{M})=\mathrm{rk}(\mathbf{A}) \iff \begin{cases}\mathbf{b}\in\mathscr{R}(\mathbf{A}),\ \mathbf{c}\in\mathscr{R}(\mathbf{A}^*),\ \lambda\neq 0, & (\leftrightarrow_1)\\ \mathbf{b}\in\mathscr{R}(\mathbf{A}),\ \mathbf{c}\notin\mathscr{R}(\mathbf{A}^*), & (\leftrightarrow_2)\\ \mathbf{b}\notin\mathscr{R}(\mathbf{A}),\ \mathbf{c}\in\mathscr{R}(\mathbf{A}^*), & (\leftrightarrow_3)\end{cases}$$

$$\mathrm{rk}(\mathbf{M})=\mathrm{rk}(\mathbf{A})+1 \iff \mathbf{b}\notin\mathscr{R}(\mathbf{A}),\ \mathbf{c}\notin\mathscr{R}\big(\mathbf{A}^*\big). \tag{$\uparrow$}$$

The symbols $(\downarrow)$, $(\leftrightarrow_1)$, $(\leftrightarrow_2)$, $(\leftrightarrow_3)$, and $(\uparrow)$ occurring in Lemma 2 were originally used in [2] to distinguish the five possible situations that can happen when a matrix is rank-one modified.

In the next section we focus on the entries of an orthogonal projector, providing various bounds for its entries. In Sect. 3, we consider modifications of an orthogonal projector by a matrix of rank one. The last section of the paper deals with the linear regression model. The considerations in this part are concerned with situations in which some observations are omitted or new observations are added to the model.

2 Entries of Orthogonal Projectors

All nine theorems given in the present section are satisfied trivially when the projector $\mathbf{P}\in\mathbb{C}_n^{\mathsf{OP}}$ involved in them is the zero matrix. For this reason, we exclude the case $\mathbf{P}=\mathbf{0}$ from the considerations.

The first theorem of the paper provides a characteristic of diagonal entries of an orthogonal projector, which was also given in Chatterjee and Hadi [6, Property 2.5].

Theorem 1 *Let* $\mathbf{P}=(p_{ij})\in\mathbb{C}_n^{\mathsf{OP}}$ *be nonzero. Then the diagonal entries* p_{ii}, $i=1,2,\ldots,n$, *are real and such that* $0\leq p_{ii}\leq 1$.

Proof Consider the $n\times 1$ vector $\mathbf{e}_i=(0,\ldots,0,1,0,\ldots,0)^*$ with the only nonzero entry in the ith row. Then, for every i, we have $\mathbf{e}_i^*\mathbf{P}\mathbf{e}_i=p_{ii}$, $i=1,2,\ldots,n$. Hence, it is seen that the diagonal entries of $\mathbf{P}$ are necessarily real. Furthermore, from the fact that $\mathbf{P}$ is nonnegative definite it follows that $p_{ii} \geq 0$. To show that the diagonal entries cannot be larger than 1, observe that $p_{ii} = \mathrm{tr}(p_{ii}) = \mathrm{tr}(\mathbf{e}_i^* \mathbf{P} \mathbf{e}_i) = \mathrm{tr}(\mathbf{P} \mathbf{e}_i \mathbf{e}_i^*)$. The product $\mathbf{e}_i \mathbf{e}_i^*$ is clearly an orthogonal projector. In consequence, on account of (1), we get $\mathrm{tr}(\mathbf{P} \mathbf{e}_i \mathbf{e}_i^*) \leq \min\{\mathrm{rk}(\mathbf{P}), \mathrm{rk}(\mathbf{e}_i \mathbf{e}_i^*)\}$. Since $\mathrm{rk}(\mathbf{P}) \geq 1$ and $\mathrm{rk}(\mathbf{e}_i \mathbf{e}_i^*) = 1$, we arrive at $p_{ii} \leq 1$ for all i. □

Theorem 1 leads to the following characteristics.

Corollary 1 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero, and let* $\mathbf{a} = (a_i) \in \mathbb{C}_{n,1}$ *be such that* $\|\mathbf{a}\| = 1$.

(i) *If* $\mathbf{a} \in \mathscr{R}(\mathbf{P})$, *then* $0 \leq p_{ii} - |a_i|^2 \leq 1$;
(ii) *If* $\mathbf{a} \in \mathscr{N}(\mathbf{P})$, *then* $0 \leq p_{ii} + |a_i|^2 \leq 1$;

where $i = 1, 2, \ldots, n$.

Proof In the light of Lemma 1, $\mathbf{a} \in \mathscr{R}(\mathbf{P})$ ensures that $\mathbf{P} - \mathbf{a}\mathbf{a}^*$ is an orthogonal projector. Hence, point (i) of the corollary is established by Theorem 1 on account of an observation that the elements on the main diagonal of $\mathbf{a}\mathbf{a}^*$ are $a_i \overline{a}_i = |a_i|^2$.

Similarly, the assumption $\mathbf{a} \in \mathscr{N}(\mathbf{P})$ implies that $\mathbf{P} + \mathbf{a}\mathbf{a}^*$ is an orthogonal projector. In consequence, point (ii) of the corollary is established by Theorem 1. □

An alternative formulation of the two implications established in Corollary 1 reads:

(i) if $\mathbf{a} \in \mathscr{R}(\mathbf{P})$, then $|a_i|^2 \leq p_{ii} \leq 1$,
(ii) if $\mathbf{a} \in \mathscr{N}(\mathbf{P})$, then $0 \leq p_{ii} \leq 1 - |a_i|^2$,

where $i = 1, 2, \ldots, n$.

The next theorem identifies a lower bound for the product of two diagonal entries of an orthogonal projector.

Theorem 2 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero. Then* $|p_{ij}|^2 \leq p_{ii} p_{jj}, i, j = 1, 2, \ldots, n$.

Proof Consider the $n \times 1$ vector $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^*$ with the only nonzero entry in the ith row. Then $\mathbf{e}_i^* \mathbf{P} \mathbf{e}_j = p_{ij}$, $i, j = 1, 2, \ldots, n$. On account of $\mathbf{P} = \mathbf{P}^* \mathbf{P}$, we get

$$|p_{ij}|^2 = \left|\mathbf{e}_i^* \mathbf{P} \mathbf{e}_j\right|^2 = \left|\mathbf{e}_i^* \mathbf{P}^* \mathbf{P} \mathbf{e}_j\right|^2 \leq \|\mathbf{P} \mathbf{e}_i\|^2 \, \|\mathbf{P} \mathbf{e}_j\|^2,$$

with the last inequality being a consequence of the Cauchy–Schwarz inequality. Hence, observing that $\|\mathbf{P} \mathbf{e}_i\|^2 = \mathbf{e}_i^* \mathbf{P} \mathbf{e}_i = p_{ii}$ completes the proof. □

Corollary 2 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero, and let* $\mathbf{a} = (a_i) \in \mathbb{C}_{n,1}$ *be such that* $\|\mathbf{a}\| = 1$.

1. *If* $\mathbf{a} \in \mathscr{R}(\mathbf{P})$, *then*:
   (i) $|p_{ij} - a_i\overline{a}_j|^2 \leq (p_{ii} - |a_i|^2)(p_{jj} - |a_j|^2)$,
   (ii) $|p_{ij}|^2 - 2\,\mathrm{Re}(p_{ij}\overline{a}_i a_j) \leq p_{ii}p_{jj} - p_{ii}|a_j|^2 - p_{jj}|a_i|^2$;
2. *If* $\mathbf{a} \in \mathscr{N}(\mathbf{P})$, *then*:
   (i) $|p_{ij} + a_i\overline{a}_j|^2 \leq (p_{ii} + |a_i|^2)(p_{jj} + |a_j|^2)$,
   (ii) $|p_{ij}|^2 + 2\,\mathrm{Re}(p_{ij}\overline{a}_i a_j) \leq p_{ii}p_{jj} + p_{ii}|a_j|^2 + p_{jj}|a_i|^2$;

where $i, j = 1, 2, \ldots, n$.

Proof To establish part 1 of the corollary, consider the orthogonal projector $\mathbf{P} - \mathbf{aa}^*$. Point 1(i) follows directly from Theorem 2. To establish point 1(ii), observe that

$$|p_{ij} - a_i\overline{a}_j|^2 = (p_{ij} - a_i\overline{a}_j)(\overline{p}_{ij} - \overline{a}_i a_j) = |p_{ij}|^2 - 2\,\mathrm{Re}(p_{ij}\overline{a}_i a_j) + |a_i|^2|a_j|^2,$$

which leads to the assertion by point 1(i).

Part 2 of the corollary is derived in a similar fashion. Note that the entries of the orthogonal projector $\mathbf{P} + \mathbf{aa}^*$ are $p_{ij} + a_i\overline{a}_j$. Hence, inequality in point 2(i) follows by Theorem 2. The proof of point 2(ii) is based on the equalities

$$|p_{ij} + a_i\overline{a}_j|^2 = (p_{ij} + a_i\overline{a}_j)(\overline{p}_{ij} + \overline{a}_i a_j) = |p_{ij}|^2 + 2\,\mathrm{Re}(p_{ij}\overline{a}_i a_j) + |a_i|^2|a_j|^2,$$

whence point 2(ii) is obtained on account of point 2(i). □

Yet another upper bound for a square of a single entry of an orthogonal projector is provided in the theorem below.

Theorem 3 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero. Then* $|p_{ij}|^2 \leq (1 - p_{ii})(1 - p_{jj})$, $i, j = 1, 2, \ldots, n$.

Proof The entries of the orthogonal projector $\overline{\mathbf{P}} = \mathbf{I}_n - \mathbf{P}$ are $1 - p_{ii}$ if $i = j$ and $-p_{ij}$ if $i \neq j$. This observation combined with Theorem 2 leads to the assertion. □

From Theorems 2 and 3 we obtain

$$|p_{ij}|^2 \leq \min\{p_{ii}p_{jj}, (1 - p_{ii})(1 - p_{jj})\}, \quad i, j = 1, 2, \ldots, n.$$

The next theorem identifies an upper bound for the sum of all entries of an orthogonal projector.

Theorem 4 *Let* $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero. Then* $\mathbf{1}_n^*\mathbf{P}\mathbf{1}_n \leq n$.

Proof Let $\mathbf{Q} \in \mathbb{C}_n^{\mathsf{OP}}$ be given by $\mathbf{Q} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^*$. In the light of (1), it is seen that $\mathrm{tr}(\mathbf{PQ}) \leq 1$. On the other hand, $\mathrm{tr}(\mathbf{PQ}) = \mathrm{tr}(\mathbf{P}\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^*) = \frac{1}{n}\mathbf{1}_n^*\mathbf{P}\mathbf{1}_n$, which proves the assertion. □

An alternative proof of Theorem 4 can be based on the fact that for all nonzero vectors $\mathbf{x} \in \mathbb{C}_{n,1}$, we have

$$\frac{\mathbf{x}^*\mathbf{P}\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \leq \lambda_{\max}(\mathbf{P}),$$

where $\lambda_{\max}(\mathbf{P})$ is the maximal eigenvalue of $\mathbf{P}$. Taking $\mathbf{x} = \mathbf{1}_n$ and referring to the known fact that $\lambda_{\max}(\mathbf{P}) = 1$ for nonzero $\mathbf{P}$ establishes the theorem. A related observation is that for all $\mathbf{x} \in \mathbb{C}_{n,1}$, we have $\mathbf{x}^*\mathbf{P}\mathbf{x} \leq \mathbf{x}^*\mathbf{x}$, which can be alternatively expressed as $\mathbf{x}^*\overline{\mathbf{P}}\mathbf{x} \geq 0$.

Theorem 5 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero, and let* $\mathbf{p}_{(i)}^*$ *be its* i*th row. Then* $|\mathbf{p}_{(i)}^*\mathbf{1}_n|^2 \leq p_{ii}(\mathbf{1}_n^*\mathbf{P}\mathbf{1}_n)$, $i = 1, 2, \ldots, n$.

Proof Clearly, $\mathbf{p}_{(i)}^* = \mathbf{e}_i^*\mathbf{P}$, from which we obtain $|\mathbf{p}_{(i)}^*\mathbf{1}_n|^2 = |\mathbf{e}_i^*\mathbf{P}\mathbf{1}_n|^2 = |\mathbf{e}_i^*\mathbf{P}^*\mathbf{P}\mathbf{1}_n|^2$. Further, by the Cauchy–Schwarz inequality, $|\mathbf{e}_i^*\mathbf{P}^*\mathbf{P}\mathbf{1}_n|^2 \leq \|\mathbf{P}\mathbf{e}_i\|^2\|\mathbf{P}\mathbf{1}_n\|^2 = p_{ii}(\mathbf{1}_n^*\mathbf{P}\mathbf{1}_n)$, which completes the proof. □

In the light of Theorem 4, an alternative expression of Theorem 5 is $|\mathbf{p}_{(i)}^*\mathbf{1}_n|^2 \leq p_{ii}n \leq n$. Two particular situations, namely when $\mathbf{1}_n \in \mathcal{R}(\mathbf{P})$ and $\mathbf{1}_n \in \mathcal{N}(\mathbf{P})$, lead to interesting conclusions. In the former of them, $\mathbf{p}_{(i)}^*\mathbf{1}_n = \mathbf{e}_i^*\mathbf{P}\mathbf{1}_n = \mathbf{e}_i^*\mathbf{1}_n = 1$, which means that the entries in each row of $\mathbf{P}$ add up to 1. On the other hand, when $\mathbf{1}_n \in \mathcal{N}(\mathbf{P})$, then $\mathbf{p}_{(i)}^*\mathbf{1}_n = \mathbf{e}_i^*\mathbf{P}\mathbf{1}_n = 0$, i.e., the entries in each row of $\mathbf{P}$ add up to 0.

An upper bound for a sum of moduli of entries in a row of an orthogonal projector is identified in what follows.

Theorem 6 *Let nonzero* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be of rank* r. *Then*

$$\sum_{j=1}^{n} |p_{ij}| \leq \sqrt{p_{ii}nr}, \quad i = 1, 2, \ldots, n. \tag{3}$$

Proof Let $\mathbf{p}_{(i)}^* = (|p_{i1}|, |p_{i2}|, \ldots, |p_{in}|)$, $i = 1, 2, \ldots, n$. Hence, by the Cauchy–Schwarz inequality,

$$\left|\mathbf{p}_{(i)}^*\mathbf{1}_n\right|^2 \leq \mathbf{p}_{(i)}^*\mathbf{p}_{(i)}\mathbf{1}_n^*\mathbf{1}_n = \sum_{j=1}^{n} |p_{ij}|^2 n.$$

From Theorem 2 we conclude that

$$\sum_{j=1}^{n} |p_{ij}|^2 n \leq \left(\sum_{j=1}^{n} p_{ii}p_{jj}\right) n \leq np_{ii}\sum_{j=1}^{n} p_{jj},$$

from where the assertion follows on account of the fact that the sum of diagonal elements is rank of $\mathbf{P}$. □

The next theorem in some sense extends Theorem 6 and provides an upper bound for a sum of moduli of all entries of an orthogonal projector. Even though the sum of all entries of an orthogonal projector is real, it is not necessary equal to the sum of moduli of all of its entries.

Theorem 7 *Let nonzero* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be of rank* r. *Then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}| \leq n\sqrt{r}. \tag{4}$$

Proof It is clear that

$$\mathrm{rk}(\mathbf{P}) = \mathrm{tr}(\mathbf{P}) = \mathrm{tr}\big(\mathbf{P}^*\mathbf{P}\big) = \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}|^2.$$

Consider now the real matrix $|\mathbf{P}| = (|p_{ij}|)$ and the vector $\mathbf{p} = \mathrm{Vec}|\mathbf{P}|$. On account of the Cauchy–Schwarz inequality, we obtain

$$\big(\mathbf{1}_{n^2}^*\mathbf{p}\big)^2 \leq \mathbf{1}_{n^2}^*\mathbf{1}_{n^2}\mathbf{p}^*\mathbf{p}.$$

Now

$$\mathbf{1}_{n^2}^*\mathbf{1}_{n^2} = n^2, \quad \mathbf{p}^*\mathbf{p} = \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}|^2 = r, \quad \text{and} \quad \mathbf{1}_{n^2}^*\mathbf{p} = \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}|,$$

whence the assertion follows. □

From (3) we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}| \leq \sum_{i=1}^{n} \sqrt{p_{ii}nr} = \sqrt{nr} \sum_{i=1}^{n} \sqrt{p_{ii}}. \tag{5}$$

A natural question arises which bound is better for the sum of moduli of all entries of an orthogonal projector, (4) or (5). The example below shows that neither of them.

Example 1 Consider the orthogonal projectors

$$\mathbf{P} = \frac{1}{14}\begin{pmatrix} 1 & 2 & -3i \\ 2 & 4 & -6i \\ 3i & 6i & 9 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

It is easy to verify that $\mathrm{rk}(\mathbf{P}) = 1$ and $\mathrm{rk}(\mathbf{Q}) = 2$, whence:

$$n\sqrt{\mathrm{rk}(\mathbf{P})} = 3 \quad \text{and} \quad \sqrt{n\mathrm{rk}(\mathbf{P})} \sum_{i=1}^{3} \sqrt{p_{ii}} = 2.78,$$

$$n\sqrt{\mathrm{rk}(\mathbf{Q})} = 3\sqrt{2} \quad \text{and} \quad \sqrt{n\mathrm{rk}(\mathbf{Q})}\sum_{i=1}^{3}\sqrt{p_{ii}} = 6.$$

It is seen that with respect to the projector $\mathbf{P}$, the bound defined in (5) is better than the one specified in (4), whereas with respect to the projector $\mathbf{Q}$, it is the other way round.

Recall that Theorem 4 asserts that the sum of all entries of $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ cannot be greater than n, whereas Theorem 7 claims that the sum of moduli of the entries cannot exceed $n\sqrt{\mathrm{rk}(\mathbf{P})}$. The two results are clearly in accordance with each other. This follows from the fact that for a Hermitian $\mathbf{P}$ we have $p_{ji} = \overline{p}_{ij}$, whence $p_{ij} + p_{ji} = p_{ij} + \overline{p}_{ij} = 2\,\mathrm{Re}(p_{ij})$ can be negative.

Subsequently, we provide an upper bound for the sum of squares of moduli of nondiagonal entries of an orthogonal projector.

Theorem 8 *Let nonzero* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be of rank* r. *Then*

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} |p_{ij}|^2 \leq 0.25, \quad j = 1, 2, \ldots, n.$$

Proof Let $\mathbf{p}_{(j)}^{*} = (\overline{p}_{j1}, \overline{p}_{j2}, \ldots, \overline{p}_{jn})$ denote the jth row of $\mathbf{P}$. Since $\mathbf{P}$ is Hermitian,

$$\mathbf{p}_{(j)} = \begin{pmatrix} p_{j1} \\ p_{j2} \\ \vdots \\ p_{jn} \end{pmatrix} = \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix} = \left(\mathbf{p}_{(j)}^{*}\right)^{*}$$

is the jth column of $\mathbf{P}$. In consequence, $p_{ij} = \mathbf{p}_{(i)}^{*}\mathbf{p}_{(j)}$, i.e., $\mathbf{P} = (p_{ij}) = (\mathbf{p}_{(i)}^{*}\mathbf{p}_{(j)})$, whence

$$p_{ii} = \mathbf{p}_{(i)}^{*}\mathbf{p}_{(i)} = \sum_{j=1}^{n}|p_{ij}|^2 = \sum_{\substack{j=1 \\ j\neq i}}^{n}|p_{ij}|^2 + p_{ii}^2.$$

It is thus clear that

$$\sum_{\substack{j=1 \\ j\neq i}}^{n}|p_{ij}|^2 = p_{ii}(1 - p_{ii}),$$

and taking into account that $p_{ii} \in [0, 1]$ yields $p_{ii}(1 - p_{ii}) \in [0, 0.25]$ the assertion is derived. $\square$

In a comment to Theorem 8 it is worth pointing out that since for each $k \neq i$

$$|p_{ik}|^2 \leq \sum_{\substack{i=1\\i\neq j}}^{n} |p_{ij}|^2 \leq 0.25,$$

it follows that $|p_{ik}| \leq 0.5$ for all $i \neq k$. Furthermore, if $\mathbf{P}$ is of real entries, then $-0.5 \leq p_{ik} \leq 0.5$ for all $i \neq k$, which was also observed in Chatterjee and Hadi [6, Property 2.5].

From Theorem 8 we obtain the following result.

Corollary 3 *Let* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be nonzero, and let* $\mathbf{a} = (a_i) \in \mathbb{C}_{n,1}$ *be such that* $\|\mathbf{a}\| = 1$.

(i) *If* $\mathbf{a} \in \mathscr{R}(\mathbf{P})$, *then*

$$\sum_{\substack{i=1\\i\neq j}}^{n} |p_{ij} - a_i\overline{a}_j|^2 \leq 0.25;$$

(ii) *If* $\mathbf{a} \in \mathscr{N}(\mathbf{P})$, *then*

$$\sum_{\substack{i=1\\i\neq j}}^{n} |p_{ij} + a_i\overline{a}_j|^2 \leq 0.25,$$

where $j = 1, 2, \ldots, n$.

Proof To establish part (ii) note that $\mathbf{a} \in \mathscr{R}(\mathbf{P})$ ensures that $\mathbf{P} - \mathbf{aa}^*$ is an orthogonal projector whose (i, j)th entry is $p_{ij} - a_i\overline{a}_j$. The assertion is now obtained by Theorem 8. The remaining part of the corollary is established in a similar fashion. □

From Corollary 3 it follows that

$$|p_{ij} + a_i\overline{a}_j| \leq 0.5 \quad \text{and} \quad |p_{ij} - a_i\overline{a}_j| \leq 0.5 \quad \text{for all } i \neq j.$$

The last result of the present section concerns an upper bound for the sum of any number of diagonal elements of an orthogonal projector.

Theorem 9 *Let nonzero* $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ *be of rank* r. *Then*

$$\sum_{i=1}^{k} p_{ii} \leq \min\{k, r\}, \quad 1 \leq k \leq n.$$

Proof First, observe that $\sum_{i=1}^{k} p_{ii} = \sum_{i=1}^{k} \mathbf{e}_i^* \mathbf{P} \mathbf{e}_i$, from where we obtain $\sum_{i=1}^{k} p_{ii} = \mathrm{tr}[\mathbf{P}(\sum_{i=1}^{k} \mathbf{e}_i \mathbf{e}_i^*)]$. Since $\sum_{i=1}^{k} \mathbf{e}_i \mathbf{e}_i^*$ is an orthogonal projector, the assertion follows on account of (1). □

Note that the bound given in Theorem 9 is sharper than $\sum_{i=1}^{k} p_{ii} \leq k$, which could be expected from Theorem 1.

3 Changes of Entries of Orthogonal Projectors

Let the (i, j)th element of $\mathbf{P} = (p_{ij}) \in \mathbb{C}_n^{\mathsf{OP}}$ be denoted by α, i.e., $p_{ij} = \alpha$. It is of interest to inquire how does $\mathbf{P}$ change when we replace α with $\beta \in \mathbb{C}$, $\beta \neq \alpha$. Is it possible that the resultant matrix remains an orthogonal projector? To answer this question, let

$$\mathbf{P}_{\mathrm{mod}} = \mathbf{P} - \alpha \mathbf{e}_i \mathbf{e}_j^* + \beta \mathbf{e}_i \mathbf{e}_j^* = \mathbf{P} - \gamma \mathbf{e}_i \mathbf{e}_j^*, \tag{6}$$

where $\gamma = \alpha - \beta$. Direct calculations show that when $i \neq j$, in which case $\mathbf{e}_i^* \mathbf{e}_j = 0$, then $\mathbf{P}_{\mathrm{mod}}$ cannot be an orthogonal projector. On the other hand, when $i = j$, then $\mathbf{P}_{\mathrm{mod}}$ is an orthogonal projector provided that $\mathbf{e}_i, \mathbf{e}_j \in \mathscr{R}(\mathbf{P})$ and $\gamma = 1$. This fact follows from Lemma 1 combined with Baksalary and Baksalary [1, Corollary 2].

In an analysis of the situations corresponding to $\gamma = 1$, it is useful to refer to Baksalary, Baksalary, and Trenkler [2, Theorem 2.1]. List 2 therein provides five representations of the projector $\mathbf{M}\mathbf{M}^\dagger$ when $\mathbf{M}$ is of the form (2), each corresponding to one of the five situations listed in Lemma 2. On account of these formulae we conclude that the orthogonal projector $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})}$ onto the column space of matrix $\mathbf{P}_{\mathrm{mod}}$ defined in (6) has one of the following representations:

(i) if $\mathbf{e}_i \in \mathscr{R}(\mathbf{P})$, $\mathbf{e}_j \in \mathscr{R}(\mathbf{P})$, $1 + \mathbf{e}_j^* \mathbf{P} \mathbf{e}_i = 0$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})} = \mathbf{P} - \eta^{-1} \mathbf{e}\mathbf{e}^*$, where $\mathbf{e} = \mathbf{P}\mathbf{e}_j$, $\eta = \mathbf{e}^*\mathbf{e}$,

(ii) if $\mathbf{e}_i \in \mathscr{R}(\mathbf{P})$, $\mathbf{e}_j \in \mathscr{R}(\mathbf{P})$, $1 + \mathbf{e}_j^* \mathbf{P} \mathbf{e}_i \neq 0$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})} = \mathbf{P}$,

(iii) if $\mathbf{e}_i \notin \mathscr{R}(\mathbf{P})$, $\mathbf{e}_j \in \mathscr{R}(\mathbf{P})$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})} = \mathbf{P}$,

(iv) if $\mathbf{e}_i \in \mathscr{R}(\mathbf{P})$, $\mathbf{e}_j \notin \mathscr{R}(\mathbf{P})$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})} = \mathbf{P} - \eta^{-1}\mathbf{e}\mathbf{e}^* + \nu^{-1}\eta^{-1}\mathbf{q}\mathbf{q}^*$, where $\mathbf{e} = \mathbf{P}\mathbf{e}_j$, $\mathbf{f} = \overline{\mathbf{P}}\mathbf{e}_i$, $\mathbf{q} = \lambda\mathbf{e} + \eta\mathbf{f}$, $\eta = \mathbf{e}^*\mathbf{e}$, $\phi = \mathbf{f}^*\mathbf{f}$, $\lambda = 1 + \mathbf{e}_j^*\mathbf{P}\mathbf{e}_i$, $\nu = |\lambda|^2 + \eta\phi$,

(v) if $\mathbf{e}_i \notin \mathscr{R}(\mathbf{P})$, $\mathbf{e}_j \notin \mathscr{R}(\mathbf{P})$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}_{\mathrm{mod}})} = \mathbf{P} + \phi^{-1}\mathbf{f}\mathbf{f}^*$, where $\mathbf{f} = \overline{\mathbf{P}}\mathbf{e}_i$, $\phi = \mathbf{f}^*\mathbf{f}$.

Let us now consider the problem of how $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$ changes when we add or subtract from it the orthogonal projector $\mathbf{a}\mathbf{a}^*$, without requesting that the resultant matrix remains an orthogonal projector. The theorem below provides a formula for an orthogonal projector onto $\mathscr{R}(\mathbf{P} + \mathbf{a}\mathbf{a}^*)$ when $\mathbf{P} + \mathbf{a}\mathbf{a}^*$ is not an orthogonal projector.

Theorem 10 *Let* $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$, *and let* $\mathbf{a} \in \mathbb{C}_{n,1}$ *be such that* $\|\mathbf{a}\| = 1$ *and* $\mathbf{a} \notin \mathscr{R}(\mathbf{P})$. *Then*

$$\mathbf{P}_{\mathscr{R}(\mathbf{P}+\mathbf{a}\mathbf{a}^*)} = \mathbf{P} + \frac{\overline{\mathbf{P}}\mathbf{a}\mathbf{a}^*\overline{\mathbf{P}}}{\mathbf{a}^*\overline{\mathbf{P}}\mathbf{a}}.$$

Proof The result is based on two known representations of orthogonal projectors. Firstly, we refer to the fact that the orthogonal projector onto $\mathscr{R}(\mathbf{P} : \mathbf{a})$ is given by

$$\mathbf{P}_{\mathscr{R}(\mathbf{P}:\mathbf{a})} = \mathbf{P} + \mathbf{P}_{\overline{\mathbf{P}}\mathbf{a}}; \tag{7}$$

see Puntanen, Styan, and Isotalo [8, Chap. 8]. Secondly, we exploit the fact that the latter projector on the right-hand side of (7) can be expressed as

$$\mathbf{P}_{\overline{\mathbf{P}}\mathbf{a}} = \frac{\overline{\mathbf{P}}\mathbf{a}\mathbf{a}^*\overline{\mathbf{P}}}{\mathbf{a}^*\overline{\mathbf{P}}\mathbf{a}};$$

see Meyer [7, Sect. 5.13]. Hence, the equalities

$$\mathscr{R}(\mathbf{P} : \mathbf{a}) = \mathscr{R}\left((\mathbf{P} : \mathbf{a})\begin{pmatrix}\mathbf{P}^*\\ \mathbf{a}^*\end{pmatrix}\right) = \mathscr{R}\big(\mathbf{P}\mathbf{P}^* + \mathbf{a}\mathbf{a}^*\big) = \mathscr{R}\big(\mathbf{P} + \mathbf{a}\mathbf{a}^*\big)$$

lead to the assertion. □

Equation (7) illustrates the sensitivity of an orthogonal projector $\mathbf{P}$ when it is extended by one column, i.e., a vector $\mathbf{a}$. Needless to say, it does not matter whether the vector $\mathbf{a}$ stands beside the projector $\mathbf{P}$ or is inserted between the columns of $\mathbf{P}$.

Theorem 11 *Let* $\mathbf{P} \in \mathbb{C}_n^{\mathsf{OP}}$, *and let* $\mathbf{a} \in \mathbb{C}_{n,1}$ *be such that* $\|\mathbf{a}\| = 1$ *and* $\mathbf{a} \notin \mathscr{R}(\mathbf{P})$. *Then*

$$\mathscr{R}\big(\mathbf{P} + \mathbf{a}\mathbf{a}^*\big) = \mathscr{R}\big(\mathbf{P} - \mathbf{a}\mathbf{a}^*\big).$$

Proof Note that

$$\mathbf{P} - \mathbf{a}\mathbf{a}^* = (\mathbf{P} : \mathbf{a})\begin{pmatrix}\mathbf{P}^*\\ -\mathbf{a}^*\end{pmatrix}, \tag{8}$$

from where we obtain $\mathscr{R}(\mathbf{P} - \mathbf{a}\mathbf{a}^*) \subseteq \mathscr{R}(\mathbf{P} : \mathbf{a}) = \mathscr{R}(\mathbf{P} + \mathbf{a}\mathbf{a}^*)$. Since, on the one hand, Eq. (8) yields $\mathrm{rk}(\mathbf{P} - \mathbf{a}\mathbf{a}^*) = \mathrm{rk}(\mathbf{P}) + 1$, and, on the other hand, case ($\uparrow$) of Lemma 2 entails $\mathrm{rk}(\mathbf{P} + \mathbf{a}\mathbf{a}^*) = \mathrm{rk}(\mathbf{P}) + 1$, the assertion follows. □

An alternative proof of Theorem 11 can be based on Baksalary, Baksalary, and Trenkler [2, Theorem 2.1].

Combining Theorems 10 and 11 leads to the conclusion that when $\mathbf{a} \notin \mathscr{R}(\mathbf{P})$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}+\mathbf{a}\mathbf{a}^*)} = \mathbf{P}_{\mathscr{R}(\mathbf{P}-\mathbf{a}\mathbf{a}^*)}$. On the other hand, when $\mathbf{a} \in \mathscr{R}(\mathbf{P})$, then $\mathbf{P}_{\mathscr{R}(\mathbf{P}+\mathbf{a}\mathbf{a}^*)} = \mathbf{P}$, and $\mathbf{P}_{\mathscr{R}(\mathbf{P}-\mathbf{a}\mathbf{a}^*)}$ is the orthogonal projector onto $\mathscr{R}(\mathbf{P}) \cap \mathscr{N}(\mathbf{a})$; see Lemma 1.

4 Linear Regression Model

Consider the linear regression model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \tag{9}$$

where $\mathbf{y}$ is an $n \times 1$ observable random vector, $\mathbf{X}$ is an $n \times p$ matrix of regressors (predictors), $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters to be estimated, and $\mathbf{u}$ is an $n \times 1$ unobservable random error vector. In order to find a unique estimate of $\boldsymbol{\beta}$, it is necessary that $\mathbf{X}'\mathbf{X}$ is nonsingular or, in other words, that $\mathbf{X}$ is of full column rank, i.e., $\text{rk}(\mathbf{X}) = p$. If this is the case, then $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is the least squares estimator of $\boldsymbol{\beta}$. It is known that $\hat{\boldsymbol{\beta}}$ is unbiased for $\boldsymbol{\beta}$ and has the smallest variance among all linear unbiased estimators of $\boldsymbol{\beta}$.

The matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}\mathbf{X}^\dagger$, often called *hat matrix*, plays an important role in linear regression analysis and other multivariate analysis techniques. It is clear that $\mathbf{H}$ is the orthogonal projector onto $\mathscr{R}(\mathbf{H}) = \mathscr{R}(\mathbf{X})$. A number of relevant properties of the hat matrix were described in Chatterjee and Hadi [6, Chaps. 1 and 2]. For instance, several applications of the entries of $\mathbf{H} = (h_{ij})$ given by

$$h_{ij} = \mathbf{x}'_{(i)}\big(\mathbf{X}'\mathbf{X}\big)^{-1}\mathbf{x}_{(j)}, \quad i, j = 1, 2, \ldots, n,$$

where $\mathbf{x}'_{(i)}$ is the ith row of $\mathbf{X}$, were pointed out in [6, Chap. 2].

The subsequent two theorems and corollary provide characteristics of entries of the hat matrix without the assumption that $\mathbf{X}$ is of full column rank. In other words, these results are concerned with the matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^\dagger\mathbf{X}' = \mathbf{X}\mathbf{X}^\dagger$.

Theorem 12 *Let $\mathbf{X} \in \mathbb{R}_{n,p}$ be the matrix of regressors involved in model (9), and let $\mathbf{H} = (h_{ij}) \in \mathbb{R}_{n,n}$ be given by $\mathbf{H} = \mathbf{X}\mathbf{X}^\dagger$.*

(i) *If $\mathbf{1}_n \in \mathscr{R}(\mathbf{X})$, then*

$$h_{ij}^2 - \frac{2}{n}h_{ij} \le h_{ii}h_{jj} - \frac{1}{n}(h_{ii} + h_{jj});$$

(ii) *If $\mathbf{X}'\mathbf{1}_n = \mathbf{0}$, then*

$$h_{ij}^2 + \frac{2}{n}h_{ij} \le h_{ii}h_{jj} + \frac{1}{n}(h_{ii} + h_{jj}),$$

where $i, j = 1, 2, \ldots, n$.

Proof The characteristics follow directly from Corollary 2 by setting $\mathbf{a} = n^{-1/2}\mathbf{1}_n$. □

It should be pointed out that both situations considered in Theorem 12 are realistic. The first one, for in many regression models it is assumed that the vector $\mathbf{1}_n$ occurs explicitly as the first column of $\mathbf{X}$. If the first column of $\mathbf{X}$ is $\mathbf{1}_n$, then centering transforms it into the zero column which is removed from the regression model. The second one, i.e., $\mathbf{X}'\mathbf{1}_n = \mathbf{0}$, happens if the regression matrix is centered, i.e., the columns of $\mathbf{X}$ are modified by subtracting the column means. This leads to a new matrix of regressors $\mathbf{Z} = \mathbf{C}\mathbf{X}$, where $\mathbf{C} = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}'_n$. In such a situation, $\mathbf{Z}' = \mathbf{X}'\mathbf{C}$ and $\mathbf{Z}'\mathbf{1}_n = \mathbf{0}$.

Note that under the assumption of point (i) of Theorem 12 we have $0 \leq h_{ii} - \frac{1}{n} \leq 1$ (cf. Chatterjee and Hadi [6, Property 2.5]), whereas in point (ii) of the theorem, $0 \leq h_{ii} + \frac{1}{n} \leq 1$.

Theorem 13 *Let* $\mathbf{X} \in \mathbb{R}_{n,p}$ *be the matrix of regressors involved in model* (9). *Moreover, let* $\mathbf{x}_k = \mathbf{X}\mathbf{e}_k$ *denote the kth column of* $\mathbf{X}$, *and let* $\mathbf{H} = (h_{ij}) \in \mathbb{R}_{n,n}$ *be given by* $\mathbf{H} = \mathbf{X}\mathbf{X}^{\dagger}$. *Then*:

(i)

$$h_{ii} \geq \frac{x_{ik}^2}{\mathbf{x}_k'\mathbf{x}_k}, \quad i, k = 1, 2, \ldots, n, \ \mathbf{x}_k \neq \mathbf{0},$$

(ii)

$$h_{ij} - \frac{x_{ik}x_{jk}}{\mathbf{x}_k'\mathbf{x}_k} \in [-0.5, 0.5], \quad i, j, k = 1, 2, \ldots, n, i \neq j, \ \mathbf{x}_k \neq \mathbf{0}.$$

Proof Consider

$$\mathbf{H} - \frac{\mathbf{x}_k\mathbf{x}_k'}{\mathbf{x}_k'\mathbf{x}_k} = \left(h_{ij} - \frac{x_{ik}x_{jk}}{\mathbf{x}_k'\mathbf{x}_k}\right), \quad k = 1, 2, \ldots, n,$$

which is an orthogonal projector since $\mathbf{x}_k \in \mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{H})$. Now point (i) of the theorem follows from Corollary 1, whereas point (ii) is a consequence of Corollary 3. □

Corollary 4 *Let* $\mathbf{X} \in \mathbb{R}_{n,p}$ *be the matrix of regressors involved in model* (9). *Moreover, let* $\mathbf{x}_k = \mathbf{X}\mathbf{e}_k$ *denote the kth column of* $\mathbf{X}$, *and let* $\mathbf{H} = (h_{ij}) \in \mathbb{R}_{n,n}$ *be given by* $\mathbf{H} = \mathbf{X}\mathbf{X}^{\dagger}$. *Then*:

(i)

$$h_{ii} \geq \max_{1 \leq k \leq n} \frac{x_{ik}^2}{\mathbf{x}_k'\mathbf{x}_k}, \quad i = 1, 2, \ldots, n, \ \mathbf{x}_k \neq \mathbf{0},$$

(ii)

$$h_{ij} \geq -0.5 + \max_{1 \leq k \leq n} \frac{x_{ik}x_{jk}}{\mathbf{x}_k'\mathbf{x}_k}, \quad i, j = 1, 2, \ldots, n, i \neq j, \mathbf{x}_k \neq \mathbf{0}.$$

Proof The result follows straightforwardly from Theorem 13. □

Subsequently, we consider a general situation in which $\mathbf{X}$ is not only not requested to be of full column rank but can also have complex entries. In such a case $\mathbf{H}$ is of the form

$$\mathbf{H} = \mathbf{X}\left(\mathbf{X}^*\mathbf{X}\right)^{\dagger}\mathbf{X}^* = \mathbf{X}\mathbf{X}^{\dagger}. \tag{10}$$

It is of interest to investigate how the orthogonal projector $\mathbf{X}\mathbf{X}^\dagger$ changes when one row is omitted or a new observation row is added to the model. Let $\mathbf{X} \in \mathbb{C}_{n,p}$ have the representation

$$\mathbf{X}^* = (\mathbf{x}_{(1)} : \mathbf{x}_{(2)} : \ldots : \mathbf{x}_{(n)}),$$

where $\mathbf{x}_{(i)}^* \in \mathbb{C}_{1,p}$, $i = 1, 2, \ldots, n$, are the rows of $\mathbf{X}$. When one row of $\mathbf{X}$, say the kth, is deleted, then we obtain the matrix $\mathbf{X}_k \in \mathbb{C}_{n-1,p}$ such that

$$\mathbf{X}_k^* = (\mathbf{x}_{(1)} : \mathbf{x}_{(2)} : \ldots : \mathbf{x}_{(k-1)} : \mathbf{x}_{(k+1)} : \ldots : \mathbf{x}_{(n)}). \tag{11}$$

Let $\mathbf{J}_k$ be the $(n-1) \times n$ matrix obtained from $\mathbf{I}_n$ by deleting the kth row. Then we readily have $\mathbf{X}_k = \mathbf{J}_k\mathbf{X}$. Observe that $\mathbf{J}_k\mathbf{J}_k^* = \mathbf{I}_{n-1}$ and $\mathbf{J}_k^*\mathbf{J}_k$ results from $\mathbf{I}_n$ by replacing 1 in the (k,k)th position with 0, i.e., $\mathbf{J}_k^*\mathbf{J}_k = \mathbf{I}_n - \mathbf{e}_k\mathbf{e}_k^*$. Moreover,

$$\mathbf{X}^*\mathbf{X} = \sum_{j=1}^{n} \mathbf{x}_{(j)}\mathbf{x}_{(j)}^*, \qquad \mathbf{X}_k^*\mathbf{X}_k = \sum_{\substack{j=1\\ j\neq k}}^{n} \mathbf{x}_{(j)}\mathbf{x}_{(j)}^* = \mathbf{X}^*\mathbf{J}_k^*\mathbf{J}_k\mathbf{X}.$$

Hence,

$$\mathbf{X}_k^*\mathbf{X}_k = \mathbf{X}^*\mathbf{X} - \mathbf{x}_{(k)}\mathbf{x}_{(k)}^*, \tag{12}$$

i.e., $\mathbf{X}_k^*\mathbf{X}_k$ is a rank-one modification of $\mathbf{X}^*\mathbf{X}$. It is seen that (12) is obtained from (2) by taking $\mathbf{M} = \mathbf{X}_k^*\mathbf{X}_k$, $\mathbf{A} = \mathbf{X}^*\mathbf{X}$, and $\mathbf{b} = -\mathbf{x}_{(k)} = -\mathbf{c}$. Clearly, $\mathbf{x}_{(k)} \in \mathscr{R}(\mathbf{X}^*)$, so that the only possible cases described in Lemma 2 are $(\downarrow)$ and $(\leftrightarrow_1)$, of which the former one corresponds to a decrease of rank of $\mathbf{X}^*\mathbf{X}$ by 1, and the latter leaves rank of $\mathbf{X}^*\mathbf{X}$ unaffected upon subtraction of $\mathbf{x}_{(k)}\mathbf{x}_{(k)}^*$. In both situations λ defined in Lemma 2 is given by

$$\lambda = 1 + \mathbf{c}^*\mathbf{A}^\dagger\mathbf{b} = 1 - \mathbf{x}_{(k)}^*\big(\mathbf{X}^*\mathbf{X}\big)^\dagger\mathbf{x}_{(k)} = 1 - \mathbf{e}_k^*\mathbf{X}\big(\mathbf{X}^*\mathbf{X}\big)^\dagger\mathbf{X}^*\mathbf{e}_k,$$

with the last equality obtained on account of $\mathbf{x}_{(k)}^* = \mathbf{e}_k^*\mathbf{X}$. Referring now to the hat matrix specified in (10) and setting $\mathbf{H} = (h_{ij})$ lead to

$$\lambda = 1 - \mathbf{e}_k^*\mathbf{H}\mathbf{e}_k = 1 - h_{kk}. \tag{13}$$

The matrix $\mathbf{X}_k$ defined in (11) generates the corresponding hat matrix, which is of the form

$$\mathbf{H}_k = \mathbf{X}_k\big(\mathbf{X}_k^*\mathbf{X}_k\big)^\dagger\mathbf{X}_k^* = \mathbf{J}_k\mathbf{X}\big(\mathbf{X}_k^*\mathbf{X}_k\big)^\dagger\mathbf{X}^*\mathbf{J}_k^*. \tag{14}$$

Our aim is to derive expressions for $\mathbf{H}_k$ separately in each of the two cases $(\downarrow)$ and $(\leftrightarrow_1)$ of Lemma 2. In the former of them $\lambda = 0$ which is equivalent to $h_{kk} = 1$. Then, on account of Baksalary, Baksalary, and Trenkler [2, Theorem 2.1],

$$\big(\mathbf{X}_k^*\mathbf{X}_k\big)^\dagger = \mathbf{A}^\dagger - \delta^{-1}\mathbf{d}\mathbf{d}^*\mathbf{A}^\dagger - \eta^{-1}\mathbf{A}^\dagger\mathbf{e}\mathbf{e}^* + \delta^{-1}\eta^{-1}\rho\mathbf{d}\mathbf{e}^*,$$

where

$$\mathbf{d} = \mathbf{A}^{\dagger}\mathbf{b} = -\big(\mathbf{X}^*\mathbf{X}\big)^{\dagger}\mathbf{x}_{(k)} = -\big(\mathbf{X}^*\mathbf{X}\big)^{\dagger}\mathbf{X}^*\mathbf{e}_k = -\mathbf{X}^{\dagger}\mathbf{e}_k \tag{15}$$

and

$$\mathbf{e} = \big(\mathbf{A}^{\dagger}\big)^*\mathbf{c} = -\mathbf{d} = \mathbf{X}^{\dagger}\mathbf{e}_k. \tag{16}$$

Furthermore,

$$\delta = \mathbf{d}^*\mathbf{d} = \mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*\mathbf{X}^{\dagger}\mathbf{e}_k = \mathbf{e}_k^*\big(\mathbf{X}\mathbf{X}^*\big)^{\dagger}\mathbf{e}_k,$$
$$\rho = \mathbf{d}^*\mathbf{A}^{\dagger}\mathbf{e} = -\mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*\big(\mathbf{X}^*\mathbf{X}\big)^{\dagger}\mathbf{X}^{\dagger}\mathbf{e}_k, \quad \text{and} \quad \eta = \delta.$$

Hence, it can be verified that

$$\mathbf{d}\mathbf{d}^*\mathbf{A}^{\dagger} = \mathbf{X}^{\dagger}\mathbf{e}_k\mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*\big(\mathbf{X}^*\mathbf{X}\big)^{\dagger}, \qquad \mathbf{A}^{\dagger}\mathbf{e}\mathbf{e}^* = \big(\mathbf{X}^*\mathbf{X}\big)^{\dagger}\mathbf{X}^{\dagger}\mathbf{e}_k\mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*,$$

and

$$\mathbf{d}\mathbf{e}^* = -\mathbf{X}^{\dagger}\mathbf{e}_k\mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*.$$

In consequence, (14) entails

$$\mathbf{H}_k = \mathbf{J}_k\mathbf{H}\mathbf{J}_k^* - \delta^{-1}\mathbf{J}_k\big[\mathbf{H}\mathbf{e}_k\mathbf{e}_k^*\big(\mathbf{X}\mathbf{X}^*\big)^{\dagger} + \big(\mathbf{X}\mathbf{X}^*\big)^{\dagger}\mathbf{e}_k\mathbf{e}_k^*\mathbf{H} + \delta^{-1}\rho\mathbf{H}\mathbf{e}_k\mathbf{e}_k^*\mathbf{H}\big]\mathbf{J}_k^*.$$

Note that $\mathbf{J}_k\mathbf{H}\mathbf{J}_k^*$ is the hat matrix $\mathbf{H}$ with the kth row and kth column removed, whereas $\mathbf{J}_k\mathbf{H}\mathbf{e}_k$ is the kth column of $\mathbf{H}$ with the kth entry removed.

Let us now consider the case $(\leftrightarrow)_1$. Then $\lambda \neq 0$, which is equivalent to $h_{kk} < 1$. In the light of Baksalary, Baksalary, and Trenkler [2, Theorem 2.1]

$$\big(\mathbf{X}_k^*\mathbf{X}_k\big)^{\dagger} = \mathbf{A}^{\dagger} - \lambda^{-1}\mathbf{d}\mathbf{e}^*,$$

where λ, $\mathbf{d}$, and $\mathbf{e}$ are as specified in (13), (15), and (16). Hence,

$$\big(\mathbf{X}_k^*\mathbf{X}_k\big)^{\dagger} = \big(\mathbf{X}^*\mathbf{X}\big)^{\dagger} + \lambda^{-1}\mathbf{X}^{\dagger}\mathbf{e}_k\mathbf{e}_k^*\big(\mathbf{X}^{\dagger}\big)^*,$$

which inserted in (14) gives

$$\mathbf{H}_k = \mathbf{J}_k\mathbf{H}\mathbf{J}_k^* + \lambda^{-1}\mathbf{J}_k\mathbf{H}\mathbf{e}_k\mathbf{e}_k^*\mathbf{H}\mathbf{J}_k^*.$$

It should be emphasized that according to our best knowledge, the case corresponding to $h_{kk} = 1$ was so far never considered in the literature, whereas the case corresponding to $h_{kk} < 1$ was dealt with exclusively under the assumption that $\mathbf{X}$ is of real entries and such that $\mathbf{X}'\mathbf{X}$ is nonsingular; see Chatterjee and Hadi [6, Chap. 2].

Let us now consider another situation, namely where the model matrix $\mathbf{X}$ is extended by an additional row. For the clarity of notation, we denote the model matrix by $\mathbf{X}_n \in \mathbb{C}_{n,p}$ and assume that $\mathbf{X}_n$ is obtained from $\mathbf{X}_{n-1} \in \mathbb{C}_{n-1,p}$ by addition of a row vector $\mathbf{r}^* \in \mathbb{C}_{1,p}$, i.e.,

$$\mathbf{X}_n^* = \left(\mathbf{X}_{n-1}^* : \mathbf{r}\right). \tag{17}$$

The question arises, how can we express $\mathbf{H}_n = \mathbf{X}_n\mathbf{X}_n^\dagger$ in terms of $\mathbf{H}_{n-1} = \mathbf{X}_{n-1}\mathbf{X}_{n-1}^\dagger$? To answer it, we introduce the vector $\mathbf{t} = (\mathbf{I}_p - \mathbf{X}_{n-1}^*(\mathbf{X}_{n-1}^*)^\dagger)\mathbf{r} = (\mathbf{I}_p - \mathbf{X}_{n-1}^\dagger\mathbf{X}_{n-1})\mathbf{r}$. Then, in view of Campbell and Meyer [5, Theorem 3.3.1], two disjoint situations are to be considered, namely where $\mathbf{t} \neq \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$, which can be alternatively expressed as $\mathbf{r} \notin \mathscr{R}(\mathbf{X}_{n-1}^*)$ and $\mathbf{r} \in \mathscr{R}(\mathbf{X}_{n-1}^*)$, respectively. According to Theorem 3.3.1 in [5], in the former of these situations

$$\left(\mathbf{X}_n^*\right)^\dagger = \begin{pmatrix} (\mathbf{X}_{n-1}^*)^\dagger - (\mathbf{X}_{n-1}^*)^\dagger\mathbf{r}\mathbf{t}^\dagger \\ \mathbf{t}^\dagger \end{pmatrix}. \tag{18}$$

Hence, representations (17) and (18) entail

$$\left(\mathbf{X}_n^*\right)^\dagger\mathbf{X}_n^* = \begin{pmatrix} (\mathbf{X}_{n-1}^*)^\dagger(\mathbf{I}_p - \mathbf{r}\mathbf{t}^\dagger)\mathbf{X}_{n-1}^* & (\mathbf{X}_{n-1}^*)^\dagger\mathbf{r}(1 - \mathbf{t}^\dagger\mathbf{r}) \\ \mathbf{t}^\dagger\mathbf{X}_{n-1}^* & \mathbf{t}^\dagger\mathbf{r} \end{pmatrix}. \tag{19}$$

Since $\mathbf{t} \neq \mathbf{0}$, we have

$$\mathbf{t}^\dagger\mathbf{X}_{n-1}^* = \frac{1}{\mathbf{t}^*\mathbf{t}}\mathbf{t}^*\mathbf{X}_{n-1}^* = \frac{1}{\mathbf{t}^*\mathbf{t}}\left(\mathbf{I}_p - \mathbf{X}_{n-1}^\dagger\mathbf{X}_{n-1}\right)\mathbf{X}_{n-1}^* = \mathbf{0}$$

and

$$\mathbf{t}^\dagger\mathbf{r} = \frac{\mathbf{t}^*\mathbf{r}}{\mathbf{t}^*\mathbf{t}} = \frac{\mathbf{r}^*(\mathbf{I}_p - \mathbf{X}_{n-1}^\dagger\mathbf{X}_{n-1})\mathbf{r}}{\mathbf{r}^*(\mathbf{I}_p - \mathbf{X}_{n-1}^\dagger\mathbf{X}_{n-1})\mathbf{r}} = 1.$$

Consequently, (19) can be rewritten as

$$\mathbf{H}_n = \mathbf{X}_n\mathbf{X}_n^\dagger = \begin{pmatrix} \mathbf{X}_{n-1}\mathbf{X}_{n-1}^\dagger & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{H}_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},$$

from where we arrive at $\mathbf{J}_n\mathbf{H}_n\mathbf{J}_n^* = \mathbf{H}_{n-1}$.

Let us now consider the situation where $\mathbf{t} = \mathbf{0}$. Then by Campbell and Meyer [5, Theorem 3.3.1] we get

$$\left(\mathbf{X}_n^*\right)^\dagger = \begin{pmatrix} (\mathbf{X}_{n-1}^*)^\dagger - \mathbf{s}\mathbf{Z}^* \\ \mathbf{Z}^* \end{pmatrix}, \tag{20}$$

where $\mathbf{s} = (\mathbf{X}_{n-1}^*)^\dagger\mathbf{r}$ and $\mathbf{Z}^* = \gamma\mathbf{s}^*(\mathbf{X}_{n-1}^*)^\dagger$ with $\gamma^{-1} = 1 + \mathbf{r}^*\mathbf{X}_{n-1}^\dagger(\mathbf{X}_{n-1}^*)^\dagger\mathbf{r} = 1 + \mathbf{r}^*(\mathbf{X}_{n-1}^*\mathbf{X}_{n-1})^\dagger\mathbf{r}$. From (17) and (20) we straightforwardly obtain

$$\mathbf{X}_n\mathbf{X}_n^\dagger = \begin{pmatrix} (\mathbf{X}_{n-1}^*)^\dagger\mathbf{X}_{n-1}^* - \mathbf{s}\mathbf{Z}^*\mathbf{X}_{n-1}^* & \mathbf{s} - \mathbf{s}\mathbf{Z}^*\mathbf{r} \\ \mathbf{Z}^*\mathbf{X}_{n-1}^* & \mathbf{Z}^*\mathbf{r} \end{pmatrix}.$$

The south–west entry of the matrix above satisfies

$$\begin{aligned}\mathbf{Z}^*\mathbf{r} &= \gamma\mathbf{s}^*\left(\mathbf{X}_{n-1}^*\right)^\dagger\mathbf{r} = \gamma\mathbf{r}^*\mathbf{X}_{n-1}^\dagger\left(\mathbf{X}_{n-1}^*\right)^\dagger\mathbf{r} = \gamma\mathbf{r}^*\left(\mathbf{X}_{n-1}^*\mathbf{X}_{n-1}\right)^\dagger\mathbf{r} = \gamma\left(\gamma^{-1}-1\right)\\ &= 1-\gamma.\end{aligned}$$

The other entries of $\mathbf{H}_n = \mathbf{X}_n\mathbf{X}_n^\dagger$ can be simplified due to the following relationships:

$$\mathbf{s}\mathbf{Z}^*\mathbf{X}_{n-1}^* = \gamma\mathbf{s}\mathbf{s}^*\left(\mathbf{X}_{n-1}^*\right)^\dagger\mathbf{X}_{n-1}^* = \gamma\mathbf{s}\mathbf{s}^*\mathbf{X}_{n-1}\mathbf{X}_{n-1}^\dagger = \gamma\mathbf{s}\mathbf{r}^*\mathbf{X}_{n-1}^\dagger\mathbf{X}_{n-1}\mathbf{X}_{n-1}^\dagger = \gamma\mathbf{s}\mathbf{s}^*,$$

$$\mathbf{Z}^*\mathbf{X}_{n-1}^* = \gamma\mathbf{s}^*\left(\mathbf{X}_{n-1}^*\right)^\dagger\mathbf{X}_{n-1}^* = \gamma\mathbf{s}^*,$$

and

$$\mathbf{s}\mathbf{Z}^*\mathbf{r} = \gamma\mathbf{s}\mathbf{r}^*\mathbf{X}_{n-1}^\dagger\left(\mathbf{X}_{n-1}^*\right)^\dagger\mathbf{r} = (1-\gamma)\mathbf{s}.$$

In consequence,

$$\mathbf{H}_n = \begin{pmatrix}\mathbf{H}_{n-1}-\gamma\mathbf{s}\mathbf{s}^* & \gamma\mathbf{s}\\ \gamma\mathbf{s}^* & 1-\gamma\end{pmatrix},$$

whence we arrive at $\mathbf{J}_n\mathbf{H}_n\mathbf{J}_n^* = \mathbf{H}_{n-1}-\gamma\mathbf{s}\mathbf{s}^*$.

The paper is concluded with some comments dealing with the situation in which the matrix $\mathbf{X}\in\mathbb{C}_{n,p}$ is extended by a vector $\mathbf{a}\in\mathbb{C}_{n,1}$. On account of the result already recalled in the proof of Theorem 10 (see Puntanen, Styan, and Isotalo [8, Chap. 8]), the orthogonal projector onto the range of $(\mathbf{X}:\mathbf{a})$ is

$$\mathbf{P}_{\mathscr{R}(\mathbf{X}:\mathbf{a})} = \mathbf{P}_{\mathbf{X}} + \mathbf{P}_{\mathbf{Ma}}, \tag{21}$$

where $\mathbf{M} = \mathbf{I}_n - \mathbf{H}$ and $\mathbf{H} = \mathbf{X}\mathbf{X}^\dagger$. Since $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{H})$, formula (21) can be rewritten in the form

$$\mathbf{P}_{\mathscr{R}(\mathbf{X}:\mathbf{a})} = \mathbf{H} + \mathbf{Ma}(\mathbf{Ma})^\dagger.$$

Observe that $\mathbf{P}_{\mathscr{R}(\mathbf{X}:\mathbf{a})} = \mathbf{H}$ if and only if $\mathbf{Ma} = \mathbf{0}$, which is equivalent to $\mathbf{a}\in\mathscr{R}(\mathbf{X})$. On the other hand, when $\mathbf{a}\notin\mathscr{R}(\mathbf{X})$, then

$$\mathbf{P}_{\mathscr{R}(\mathbf{X}:\mathbf{a})} = \mathbf{H} + \frac{\mathbf{M}\mathbf{a}\mathbf{a}^*\mathbf{M}}{\mathbf{a}^*\mathbf{M}\mathbf{a}}.$$

In consequence, the diagonal and nondiagonal elements of $\mathbf{P}_{\mathscr{R}(\mathbf{X}:\mathbf{a})}$ are given by

$$h_{ii} + \frac{|\mathbf{e}_i^*\mathbf{M}\mathbf{a}|^2}{\mathbf{a}^*\mathbf{M}\mathbf{a}} \quad \text{and} \quad h_{ij} + \frac{\mathbf{e}_i^*\mathbf{M}\mathbf{a}\mathbf{a}^*\mathbf{M}\mathbf{e}_j}{\mathbf{a}^*\mathbf{M}\mathbf{a}}, \quad i\neq j,$$

respectively. Parenthetically, note that

$$\begin{aligned}\mathscr{R}(\mathbf{X}:\mathbf{a}) &= \mathscr{R}\left(\mathbf{X}\mathbf{X}^* + \mathbf{a}\mathbf{a}^*\right) = \mathscr{R}\left(\mathbf{X}\mathbf{X}^*\right) + \mathscr{R}\left(\mathbf{a}\mathbf{a}^*\right)\\ &= \mathscr{R}(\mathbf{X}) + \mathscr{R}\left(\mathbf{a}\mathbf{a}^*\right) = \mathscr{R}(\mathbf{H}) + \mathscr{R}\left(\mathbf{a}\mathbf{a}^*\right) = \mathscr{R}\left(\mathbf{H}+\mathbf{a}\mathbf{a}^*\right),\end{aligned}$$

which is in accordance with the formulae given in the proof of Theorem 10.

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References

1. Baksalary, J.K., Baksalary, O.M.: Idempotency of linear combinations of two idempotent matrices. Linear Algebra Appl. **321**, 3–7 (2000)
2. Baksalary, J.K., Baksalary, O.M., Trenkler, G.: A revisitation of formulae for the Moore–Penrose inverse of modified matrices. Linear Algebra Appl. **372**, 207–224 (2003)
3. Baksalary, O.M., Bernstein, D.S., Trenkler, G.: On the equality between rank and trace of an idempotent matrix. Appl. Math. Comput. **217**, 4076–4080 (2010)
4. Baksalary, O.M., Trenkler, G.: Rank formulae from the perspective of orthogonal projectors. Linear Multilinear Algebra **59**, 607–625 (2011)
5. Campbell, S.L., Meyer, C.D.: Generalized Inverses of Linear Transformations. SIAM, Philadelphia (2009)
6. Chatterjee, S., Hadi, A.S.: Sensitivity Analysis in Linear Regression. Wiley, New York (1988)
7. Meyer, C.D.: Matrix Analysis and Applied Linear Algebra. SIAM, Philadelphia (2000)
8. Puntanen, S., Styan, G.P.H., Isotalo, J.: Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty. Springer, Berlin (2011)

Moore–Penrose Inverse of Perturbed Operators on Hilbert Spaces

Shani Jose and K.C. Sivakumar

Abstract Rank-one perturbations of closed range bounded linear operators on Hilbert space are considered. The Moore–Penrose inverses of these operators are obtained. The results are generalized to obtain the Moore–Penrose inverse of operators of the form $A + V_1 G V_2^*$. Applications to nonnegativity of the Moore–Penrose inverse and operator partial orders are considered.

Keywords Moore–Penrose inverse · Bounded linear operator · Rank-one perturbation · Partial order

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1 Introduction and Preliminaries

There are many physical models where successive computation of matrix inverses must be performed. Naturally, in all these methods, the amount of computation increases rapidly as the order of the matrix increases. In most cases, the matrix that we deal with may not be the exact matrix representing the data. Truncation and round-off errors give rise to perturbations. If we have a nonsingular matrix and its inverse and suppose that some of the entries of the matrix are altered, then it is desirable to have an efficient computational method for finding the inverse of the new matrix from the known inverse without computing the inverse of the new matrix, afresh. In this regard, there is a well-known and perhaps the most widely used formula called the Sherman, Morrison and Woodbury (SMW) formula. It gives an explicit formula for the inverse of matrices of the form $A + V_1 G V_2{}^*$, where A and G are nonsingular matrices. More precisely, let A and G be $n \times n$ and $r \times r$ nonsingular matrices with $r \leq n$. Also, let V_1 and V_2 be $n \times r$ matrices such that $(G^{-1} + V_2{}^* A^{-1} V_1)^{-1}$ exists.

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The formula for the inverse of $A + V_1 G V_2{}^*$ is given by

$$\left(A + V_1 G V_2^*\right)^{-1} = A^{-1} - A^{-1} V_1 \left(G^{-1} + V_2{}^* A^{-1} V_1\right)^{-1} V_2{}^* A^{-1}. \tag{1}$$

Note that the matrix $G^{-1} + V_2{}^* A^{-1} V_1$ is of order $r \times r$. This formula is useful in situations where r is much smaller than n and in situations where A has certain structural properties that could be exploited so that the effort involved in evaluating $A^{-1} V_1 (G^{-1} + V_2{}^* A^{-1} V_1)^{-1} V_2{}^* A^{-1}$ is small relative to the effort involved in inverting a general $n \times n$ matrix. This formula has a wide range of applications in various fields (see, for instance, Hager [8] and Kennedy and Haynes [9]).

The formula was first presented by Sherman and Morrison [15] and extended by Woodbury [17]. In [15], Sherman and Morrison showed how the inverse of a matrix changes when one of the elements of the original matrix is changed. Bartlett [3] also developed a formula to find the inverse of matrices of the form $B = A + v_1 v_2'$, where v_1 and v_2 are column vectors.

The SMW formula (1) is valid only if the matrices A and $A + V_1 G V_2^*$ are invertible. Generalizations have been considered in the case of singular/rectangular matrices using the concept of Moore–Penrose generalized inverses (see Baksalary, Baksalary and Trenkler [2], Meyer [10], Mitra and Bhimasankaram [11], and Riedel [14]). Certain results on extending the formula to operators on Hilbert spaces are proved in Deng [5] and Ogawa [13].

The article is concerned with certain extensions of Sherman, Morrison and Woodbury (SMW) formula to operators between Hilbert spaces. The framework for the results of this article is as follows. Let H_1, H_2 be Hilbert spaces over the same field, and $\mathscr{B}(H_1, H_2)$ be the space of all bounded linear operators from H_1 to H_2. For $A \in \mathscr{B}(H_1, H_2)$, let $R(A)$, $N(A)$, $R(A)^{\perp}$, and A^* denote the range space, the null space, the orthogonal complement of the range space, and the adjoint of the operator A, respectively.

The Moore–Penrose inverse of $A \in \mathscr{B}(H_1, H_2)$ is the unique $X \in \mathscr{B}(H_2, H_1)$ (if it exists) satisfying the equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. The unique Moore–Penrose inverse of A is denoted by $A^{\dagger}$, and it coincides with A^{-1} when A is invertible. For $A \in \mathscr{B}(H_1, H_2)$, the Moore–Penrose inverse exists if and only if $R(A)$ is closed. Any X satisfying the equation $AXA = A$ is called an inner inverse of A. In general, an inner inverse of an operator A is not unique. Moreover, A has a bounded inner inverse if and only if $R(A)$ is closed. Throughout this paper, we will consider operators with closed range space.

The following are some of the well-known properties of the Moore–Penrose inverse: $R(A^*) = R(A^{\dagger})$, $N(A^*) = N(A^{\dagger})$, $A^{\dagger}A = P_{R(A^*)}$, $AA^{\dagger} = P_{R(A)}$. Here, for complementary closed subspaces L and M of a Hilbert space H, $P_{L,M}$ denotes the projection of A onto L along M. P_L denotes $P_{L,M}$ if $M = L^{\perp}$.

The following theorem, which is frequently used in the sequel, gives an equivalent definition for the Moore–Penrose inverse.

Theorem 1 (Groetsch [7], Theorem 2.2.2) *If $A \in \mathscr{B}(H_1, H_2)$ has closed range, then $A^{\dagger}$ is the unique operator $X \in \mathscr{B}(H_2, H_1)$ satisfying*:

(i) $XAx = x \ \ \forall x \in R(A^*)$;
(ii) $Xy = 0 \ \ \forall y \in N(A^*)$.

Let $A \in \mathscr{B}(H_1, H_2)$ with $R(A)$ closed. For fixed nonzero vectors $b \in H_2$ and $c \in H_1$, we define the rank-one bounded linear operator $B : H_1 \to H_2$ as $B(x) = \langle x, c\rangle b$. Set $M = A + B$, a rank-one perturbation of the operator A. Then $M \in \mathscr{B}(H_1, H_2)$. Though the range space of A is closed, the range space of the perturbed operator M need not be closed. Certain sufficient conditions for the perturbed operator to have a closed range space can be found in Christensen [4] and Wei and Ding [16]. The purpose of this article is to find the Moore–Penrose inverse of the operator M when $R(M)$ is closed. These results are also extended to more general perturbations of the type $A + V_1 G V_2^*$. In the finite-dimensional case, rank conditions are used as main tools. These cannot be used in the general infinite-dimensional setting. We provide linear algebraic proofs of our results.

The finite-dimensional version of the above problem was studied by Meyer [10], who found the formulae for the Moore–Penrose inverses of rank-one perturbed matrices, where he categorized the problem into six different cases. Recently, Baksalary et al. [1] improved this result and proved that these six cases can be reduced to five mutually exclusive and collectively exhaustive cases. Further, they found the Moore–Penrose inverse corresponding to each of the five different cases (see Baksalary, Baksalary and Trenkler [2]). We extend the results of Baksalary et al. [2] to operators on infinite-dimensional Hilbert space. The five different cases mentioned earlier are valid in the general setting too.

The paper is organized as follows. In Sect. 2, we give formulae for the Moore–Penrose inverse of rank-one updated operators. Section 3 includes generalizations of the rank-one perturbations to Hilbert space operators of the form $A + V_1 G V_2^*$. The generalization corresponding to all the cases in the rank-one perturbation case have been considered. Theorem 1 in Riedel [14] is obtained as a special case of our result. Section 4 concludes the paper by giving some applications of our results. Namely, we find bounds for a single element perturbation of two nonnegative operators so that the Moore–Penrose inverse of the perturbed operator is also nonnegative. Also, we use these formulae to verify whether the perturbed operator preserves the star partial order for operators on $\mathscr{B}(H)$. Certain necessary and sufficient conditions under which rank-one perturbed operator preserves star partial order are also obtained.

2 Moore–Penrose Inverse of Rank-One Perturbed Operators

In this section, we provide formulae for the Moore–Penrose inverse of the perturbed operator M. The following are the five mutually exclusive and collectively exhaustive cases considered for discussion, where $\lambda := 1 + \langle A^\dagger b, c\rangle$:

(i) $b \in R(A),\ c \in R(A^*)$, and $\lambda = 0$;
(ii) $b \in R(A),\ c \in R(A^*)$, and $\lambda \neq 0$;

(iii) $b \notin R(A),\ c \notin R(A^*)$;
(iv) $b \in R(A),\ c \notin R(A^*)$;
(v) $b \notin R(A),\ c \in R(A^*)$.

We consider the first four of the above cases and determine the Moore–Penrose inverse of the operator M. The fifth case is obtained by interchanging A, b, and c by A^*, c, and b, respectively in case (iv). The following notation will be used in the sequel:

$$d = A^\dagger b, \quad e = \left(A^\dagger\right)^* c, \quad f = P_{R(A)^\perp} b, \quad g = P_{R(A^*)^\perp} c,$$
$$\delta = \langle d, d\rangle, \quad \eta = \langle e, e\rangle, \quad \phi = \langle f, f\rangle, \quad \psi = \langle g, g\rangle,$$
$$\mu = |\lambda|^2 + \delta\psi, \quad \nu = |\lambda|^2 + \eta\phi, \quad p = \bar{\lambda} d + \delta g \in H_1, \quad q = \lambda e + \eta f \in H_2,$$

where $\lambda = 1 + \langle d, c\rangle = 1 + \langle b, e\rangle$.

From the above notation we infer the following, which we frequently used in the proofs:

$$A^* f = 0, \qquad Ag = 0, \qquad \langle b, f\rangle = \phi, \qquad \langle c, g\rangle = \psi. \tag{2}$$

Theorem 2 *Let $M = A + B$, $b \in R(A), c \in R(A^*)$, and $\lambda = 0$. Then $R(M)$ is closed. Set $D : H_1 \to H_1$ and $E : H_2 \to H_2$ by $D(x) = \langle x, d\rangle d$ and $E(x) = \langle x, e\rangle e$. Then*

$$M^\dagger = A^\dagger - \delta^{-1} D A^\dagger - \eta^{-1} A^\dagger E + \delta^{-1}\eta^{-1} D A^\dagger E. \tag{3}$$

Proof The conditions $b \in R(A)$ and $c \in R(A^*)$ imply

$$b = AA^\dagger b = Ad, \qquad A^\dagger A d = d, \qquad A^\dagger A c = c, \qquad AA^\dagger e = e. \tag{4}$$

Also, $BA^\dagger A = AA^\dagger B = B$ and $BA^\dagger B = -B$. Therefore, we have $MA^\dagger M = M$. This implies M has a bounded inner inverse, and hence $R(M)$ is closed.

Set $X = A^\dagger - \delta^{-1} D A^\dagger - \eta^{-1} A^\dagger E + \delta^{-1}\eta^{-1} D A^\dagger E$. The condition $\lambda = 0$ implies that $\langle d, c\rangle = \langle b, e\rangle = -1$. Using these, we get $DA^\dagger A = D$, $A^\dagger B = \delta^{-1} D A^\dagger B$, $A^\dagger E A = -A^\dagger E B$, and $DA^\dagger E A = -DA^\dagger E B$. Thus, $XM = A^\dagger A - \delta^{-1} D$.

Now, $x \in R(M^*)$ implies $x = M^* y = A^* y + B^* y$ for some $y \in H_2$. Since $Bd = \langle d, c\rangle b = -b$ and $Ad = b$, we have $\langle x, d\rangle = \langle A^* y, d\rangle + \langle B^* y, d\rangle = 0$. As $R(M^*) \subseteq R(A^*)$, $A^\dagger A x = x$. Hence, $XM(x) = A^\dagger A x - \delta^{-1}\langle x, d\rangle d = A^\dagger A x = x$.

For $z \in N(M^*)$, we have $A^\dagger z = -\langle z, b\rangle A^\dagger e$. Substituting this into the expression for Xz and simplifying yields $Xz = 0$. Thus, $X = M^\dagger$ by Theorem 1. □

Remark 1 Note that the operators $\delta^{-1} D$ and $\eta^{-1} E$ are orthogonal projections onto the one-dimensional subspaces, spanned by the vectors $A^\dagger b$ and $A^{\dagger *} c$, respectively. Formula (3) can also be written as $M^\dagger = (I - \delta^{-1} D) A^\dagger (I - \eta^{-1} E)$, where $I - \delta^{-1} D = P_{R(d)^\perp}$ and $I - \eta^{-1} E = P_{R(e)^\perp}$.

Theorem 3 *Let $M = A + B$ with $R(M)$ is closed. Let $b \in R(A),\ c \in R(A^*)$, and $\lambda \neq 0$. Then*

$$M^\dagger = A^\dagger - \lambda^{-1} A^\dagger B A^\dagger. \tag{5}$$

Proof Set $X = A^\dagger - \lambda^{-1} A^\dagger B A^\dagger$. Since $b \in R(A)$ and $c \in R(A^*)$, conditions in (4) hold in this case, too.

We have $XM = A^\dagger A - \lambda^{-1} A^\dagger B A^\dagger A + A^\dagger B - \lambda^{-1} A^\dagger B A^\dagger B$. But $A^\dagger B A^\dagger A = A^\dagger B$ and $A^\dagger B A^\dagger B = \langle d, c\rangle A^\dagger B = (\lambda - 1) A^\dagger B$. Therefore, $XM = A^\dagger A$. For $x \in R(M^*)$, $XM(x) = A^\dagger A(x) = x$, which follows as $R(M^*) \subseteq R(A^*)$.

Now, for $y \in N(M^*)$, $0 = \langle M^* y, A^\dagger b\rangle = \bar{\lambda}\langle y, b\rangle$. This implies that $\langle y, b\rangle = 0$ as $\bar{\lambda} \neq 0$. Then, $A^* y = -\langle y, b\rangle c = 0$, and hence $A^\dagger y = 0$. Therefore, $Xy = 0$. Thus, $X = M^\dagger$. □

Theorem 4 *Let $M = A + B$, $b \notin R(A)$, and $c \notin R(A^*)$ with $R(M)$ closed. Set $B_1 : H_2 \to H_2$ by $B_1(x) = \langle x, f\rangle b$, $G_1 : H_1 \to H_1$ by $G_1(x) = \langle x, c\rangle g$, and $G_2 : H_2 \to H_1$ by $G_2(x) = \langle x, f\rangle g$. Then*

$$M^\dagger = A^\dagger - \phi^{-1} A^\dagger B_1 - \psi^{-1} G_1 A^\dagger + \lambda \phi^{-1} \psi^{-1} G_2. \tag{6}$$

Proof Set $X = A^\dagger - \phi^{-1} A^\dagger B_1 - \psi^{-1} G_1 A^\dagger + \lambda\phi^{-1}\psi^{-1} G_2$. Simplifying using (2), we get $B_1 A = 0 = G_2 A$, $\phi^{-1} A^\dagger B_1 B = A^\dagger B$, $G_1 A^\dagger B = (\lambda - 1)\langle x, c\rangle g$, and $\phi^{-1} G_2 B = \langle x, c\rangle g$. Then, $XM(x) = A^\dagger A(x) + \psi^{-1}\langle x, g\rangle g$.

Now, for $x \in R(M^*), x = A^* y + \langle y, b\rangle c$ for some $y \in H_2$ and $XM(x) = A^\dagger A A^* y + \langle y, b\rangle A^\dagger A c + \psi^{-1}\langle A^* y, g\rangle g + \psi^{-1}\langle y, b\rangle\langle c, g\rangle g = A^* y + \langle y, b\rangle(A^\dagger A c + g) = x$.

For $y \in N(M^*)$, $A^* y = -\langle y, b\rangle c$. But $A^* y = A^\dagger A A^* y = -\langle y, b\rangle A^\dagger A c$. Equating the right-hand sides of both the equations for $A^* y$, we get $\langle y, b\rangle c = \langle y, b\rangle A^\dagger A c$. This implies that $\langle y, b\rangle g = 0$ and hence $\langle y, b\rangle = 0$ as $g \neq 0$. Hence, $A^* y = 0$. Therefore, $y \in N(A^*) = N(A^\dagger)$. Now, $\langle y, f\rangle = \langle y, P_{N(A^*)}\rangle b = \langle P_{N(A^*)} y, b\rangle = \langle y, b\rangle = 0$. Therefore, $Xy = A^\dagger y - \phi^{-1}\langle y, f\rangle A^\dagger b - \psi^{-1}\langle A^\dagger y, c\rangle g + \lambda\phi^{-1}\psi^{-1}\langle y, f\rangle g = 0$. Thus, $X = M^\dagger$. □

Theorem 5 *Let $M = A + B$, $b \in R(A)$, and $c \notin R(A^*)$ with $R(M)$ closed. Set $G_3 : H_1 \to H_1$ by $G_3(x) = \langle x, d\rangle g$. Let D and G_1 be defined as in Theorem 2 and Theorem 4, respectively. Then*

$$M^\dagger = A^\dagger - \mu^{-1}\big(\psi D A^\dagger + \delta G_1 A^\dagger - \lambda G_3 A^\dagger + \bar{\lambda} A^\dagger B A^\dagger\big). \tag{7}$$

Proof Set $X = A^\dagger - \mu^{-1}(\psi D A^\dagger + \delta G_1 A^\dagger - \lambda G_3 A^\dagger + \bar{\lambda} A^\dagger B A^\dagger)$. Clearly, $D A^\dagger A = D$, $D A^\dagger B = \delta A^\dagger B$, $G_3 A^\dagger A = G_3$, $G_3 A^\dagger B = \delta G_1$, $G_1 A^\dagger B = (\lambda - 1) G_1$, and $A^\dagger B A^\dagger B = (\lambda - 1) A^\dagger B$. Now,

$$\begin{aligned} XM(x) = A^\dagger A(x) - \mu^{-1}\big[&\psi\langle x, d\rangle d + \delta\big\langle A^\dagger A x, c\big\rangle g - \lambda\langle x, d\rangle g \\ &+ \bar{\lambda}\big\langle A^\dagger A x, c\big\rangle d + \psi\delta\langle x, c\rangle d + \delta(\lambda - 1)\langle x, c\rangle g - \lambda\delta\langle x, c\rangle g \end{aligned}$$

$$+ \bar{\lambda}(\lambda - 1)\langle x, c\rangle d\big] + \langle x, c\rangle d$$
$$= A^\dagger Ax - \mu^{-1}\big[\delta^{-1}\big(\mu - |\lambda|^2\big)\langle x, d\rangle d - \delta\langle x, g\rangle g - \lambda\langle x, d\rangle g - \bar{\lambda}\langle x, g\rangle d\big]$$
$$= A^\dagger Ax - \delta^{-1}\langle x, d\rangle d + \mu^{-1}\delta^{-1}\langle x, p\rangle p,$$

where p is the vector $\bar{\lambda}d + \delta g$.

For $x \in R(M^*), x = A^*y + \langle y, b\rangle c,\ A^\dagger A(x) = A^*y + \langle y, b\rangle A^\dagger Ac,\ \langle x, d\rangle = \langle A^*y, d\rangle + \langle y, b\rangle\langle c, d\rangle = \bar{\lambda}\langle y, b\rangle$, and $\langle x, p\rangle = \langle A^*y, \bar{\lambda}d + \delta g\rangle + \langle y, b\rangle\langle c, \bar{\lambda}d + \delta g\rangle = \mu\langle y, b\rangle$. Substituting into $XM(x)$ and simplifying, we get $XM(x) = x$.

Proving the second condition, $Xy = 0$ for $y \in N(M^*)$ is the same as that in the proof of Theorem 4. Thus, $X = M^\dagger$. □

For case (v), the result follows from Theorem 5 by replacing A, b, and c by A^*, c, and b respectively. We include the statement for completeness.

Theorem 6 *Let $M = A + B$, $b \notin R(A)$, and $c \in R(A^*)$ with $R(M)$ closed. Set $\tilde{B}(x) = \langle x, f\rangle b$ and $\tilde{E}(x) = \langle x, f\rangle e$. Let E be as in Theorem 2. Then*

$$M^\dagger = A^\dagger - \nu^{-1}\big(\phi A^\dagger E + \eta A^\dagger\tilde{B} - \lambda A^\dagger\tilde{E} + \bar{\lambda}A^\dagger BA^\dagger\big). \tag{8}$$

Remark 2 The range spaces of M and M^* in each of the above cases can be explicitly written using the expression for $MM^\dagger$ and $M^\dagger M$, respectively. In each of these cases, we get $MM^\dagger$ and $M^\dagger M$ as either a sum or a difference of certain orthogonal projections.

3 Formulae for $(A + V_1GV_2^*)^\dagger$

In this section, we present certain extensions of the representations of Moore–Penrose inverse obtained for the rank-one perturbation operator $M = A + B$. In particular, we study an operator of the form $A + V_1GV_2^*$. That is, we replace the vectors b and c in Sect. 2 by bounded linear operators V_1 and V_2. The following theorem gives a generalization of case (i) of Sect. 2. This is a particular case of Theorem 3.2 in Du and Xue [6]. Hence, we omit the proof.

Theorem 7 *Let $A \in B(H_1, H_2)$ with closed range space. Let $V_1 \in B(H_2)$ and $V_2 \in B(H_2, H_1)$ be such that $R(V_1) \subseteq R(A)$, $R(V_2) \subseteq R(A^*)$, and $V_2^*A^\dagger V_1 = -I_{H_2}$. Let $\hat{\Omega} = A + V_1V_2^*$. Then $R(\hat{\Omega})$ is closed, and*

$$\hat{\Omega}^\dagger = P_{R(A^\dagger V_1)^\perp}\, A^\dagger\, P_{R(A^{\dagger *}V_2)^\perp}, \tag{9}$$

where $P_{R(A)^\perp} = I - AA^\dagger$, the orthogonal projection onto the orthogonal complement of $R(A)$.

In order to generalize the result corresponding to case (ii) for the perturbed operator $A + V_1GV_2^*$, we observe that the analogous conditions are $R(V_1) \subseteq R(A)$ and $R(V_2) \subseteq R(A^*)$ with $G^{-1} + V_2^*A^\dagger V_1$ being nonsingular. The following theorem is obtained.

Theorem 8 (Deng [5], Theorem 2.1) *Let H_1, H_2 be Hilbert spaces, and $A \in B(H_1, H_2)$ with closed range space. Let $V_1 \in B(H_1, H_2)$ and $V_2 \in B(H_2, H_1)$ be such that*

$$R(V_1) \subseteq R(A) \quad \textit{and} \quad R(V_2) \subseteq R\big(A^*\big). \tag{10}$$

Let $G \in B(H_2, H_1)$ be such that G and $G^\dagger + V_2^*A^\dagger V_1$ have closed range spaces. Let $\Omega = A + V_1GV_2^*$ have a closed range space. If

$$R\big(V_1^*\big) \subseteq R\big(\big(G^\dagger + V_2^*A^\dagger V_1\big)^*\big), \qquad R\big(V_2^*\big) \subseteq R\big(G^\dagger + V_2^*A^\dagger V_1\big), \tag{11}$$

$$R\big(V_1^*\big) \subseteq R(G), \quad \textit{and} \quad R\big(V_2^*\big) \subseteq R\big(G^*\big), \tag{12}$$

then,

$$\Omega^\dagger = A^\dagger - A^\dagger V_1\big(G^\dagger + V_2^*A^\dagger V_1\big)^\dagger V_2^*A^\dagger. \tag{13}$$

The next result is a generalization corresponding to Theorem 4 in Sect. 2. The range conditions are $R(V_1) \nsubseteq R(A)$ and $R(V_2) \nsubseteq R(A^*)$. So, by the projection theorem for Hilbert spaces, we have $V_1 = U_1 + W_1$ and $V_2 = U_2 + W_2$, where $R(U_1) \subseteq R(A)$, $R(W_1) \subseteq R(A)^\perp$, $R(U_2) \subseteq R(A^*)$, and $R(W_2) \subseteq R(A^*)^\perp$.

Theorem 9 *Let $A \in B(H_1, H_2)$ with $R(A)$ closed. Let $U_1, W_1 \in B(H_1, H_2)$ be such that $R(U_1) \subseteq R(A)$ and $R(W_1) \subseteq R(A)^\perp$. Let $U_2, W_2 \in B(H_2, H_1)$ be such that $R(U_2) \in R(A^*)$ and $R(W_2) \in R(A^*)^\perp$. Let $G \in B(H_2, H_1)$ be such that $R(G)$ is closed. Let $\Omega = A + (U_1 + W_1)G(U_2 + W_2)^*$ with $R(\Omega)$ closed. If*

$$R\big((U_1 + W_1)^*\big) \subseteq R(G), \qquad R\big((U_2 + W_2)^*\big) \subseteq R\big(G^*\big), \tag{14}$$

$$R\big(G^*\big) \subseteq R\big(W_2^*\big), \quad \textit{and} \quad R(G) \subseteq R\big(W_1^*\big), \tag{15}$$

then

$$\Omega^\dagger = A^\dagger - {W_2}^{\dagger *}{U_2}^*A^\dagger - A^\dagger U_1{W_1}^\dagger + {W_2}^{\dagger *}\big(G^\dagger + U_2^*A^\dagger U_1\big){W_1}^\dagger. \tag{16}$$

Proof From the hypothesis we have $A^\dagger AU_2 = U_2$, $A^*W_1 = 0 = W_1^*U_1$, ${W_1}^\dagger W_1G = G$, and $G^\dagger G(U_2 + W_2)^* = (U_2 + W_2)^*$ to simplify $X\Omega$ in a direct computation. This yields $X\Omega = A^\dagger A + W_2W_2^\dagger$. Now for $x = A^*y + (U_2 + W_2)G^*(U_1 + W_1)^*y$,

$$\begin{aligned} X\Omega(x) = {} & AA^\dagger A^*y + A^\dagger A(U_2 + W_2)G^*(U_1 + W_1)^*y \\ & + W_2W_2^\dagger A^*y + W_2W_2^\dagger(U_2 + W_2)G^*(U_1 + W_1)^*y. \end{aligned}$$

Since $AW_2 = 0$, $X\Omega x$ simplifies to x.

For $y \in N(\Omega^*)$, $A^*y = -(U_2 + W_2)G^*(U_1 + W_1)^*y$. Since $W_2^*A^* = 0 = W_2^*U_2$, we get $G^*(U_1 + W_1)^*y \in N(W_2) \subseteq R(G^*)^{\perp}$. This implies $G^*(U_1 + W_1)^*y = 0$ by hypothesis, and hence $A^*y = 0$. Also, we get $(U_1 + W_1)^*y \in N(G^*)$, which implies $(U_1 + W_1)^*y = 0$, since by assumption $R((U_1 + W_1)^*) \subseteq R(G)$. Now, since $N(A^*) \subseteq N(U_1{}^*)$, $A^*y = 0$ implies $U_1^*y = 0$, which in turn implies $W_1^*y = 0$. Hence, $Xy = 0$ for $y \in N(\Omega^*)$. Thus, $X = \Omega^{\dagger}$. □

We obtain Riedel's result [14] as a special case of Theorem 9.

Corollary 1 (Theorem 1, [14]) *Let A be an $n \times n$ matrix of rank $n_1 \le n$, and U_1, W_1, U_2, W_2 be $n \times m$ matrices such that $W_1{}^*W_1$ and $W_2{}^*W_2$ are nonsingular. Let G be an $m \times m$ nonsingular matrix. Let $R(U_1) \subseteq R(A)$, $R(W_1) \subseteq R(A)^{\perp}$, $R(U_2) \subseteq R(A^*)$, and $R(W_2) \subseteq R(A^*)^{\perp}$. Let $\Omega = A + (U_1 + W_1)G(U_2 + W_2)^*$. Then*

$$\Omega^{\dagger} = A^{\dagger} - C_2U_2{}^*A^{\dagger} - A^{\dagger}U_1C_1{}^* + C_2\big(G^{\dagger} + U_2^*A^{\dagger}U_1\big)C_1{}^*, \tag{17}$$

where $C_i = W_i(W_i{}^*W_i)^{-1}$ for $i = 1, 2$.

Proof The conditions in (14) and (15) are satisfied as G, $W_1{}^*W_1$, and $W_2{}^*W_2$ are nonsingular. Thus, Eq. (17) follows from (16). □

Our next result is a generalization corresponding to the fourth case in Sect. 2. The corresponding conditions in the general case are $R(V_1) \subseteq R(A)$ and $R(V_2) \nsubseteq R(A^*)$. Let $V_2 = U_2 + W_2$, where $R(U_2) \subseteq R(A^*)$ and $R(W_2) \subseteq R(A^*)^{\perp}$. For $L = I + U_2{}^*A^{\dagger}V_1$, we assume that $LL^* = L^*L$. Also, we assume that $W_2{}^*W_2 = I$, $(A^{\dagger}V_1)^*(A^{\dagger}V_1) = I$, and $K = I + L^*L = I + LL^*$ is nonsingular. In the following theorem, we obtain a formula for $(A + V_1(U_2 + W_2)^*)^{\dagger}$.

Theorem 10 *Let $A \in \mathscr{B}(H_1, H_2)$ with closed range space. Let $V_1 \in \mathscr{B}(H_1, H_2)$ and $U_2, W_2 \in \mathscr{B}(H_2, H_1)$ be such that $R(V_1) \subseteq R(A)$, $R(U_2) \subseteq R(A^*)$, and $R(W_2) \subseteq R(A^*)^{\perp}$. Let $W_2{}^*W_2 = I$, $(A^{\dagger}V_1)^*(A^{\dagger}V_1) = I$, and $L = I + U_2^*A^{\dagger}V_1$. Assume that $L^*L = LL^*$ and let $K = I + L^*L$. Let $\Omega = A + V_1(U_2 + W_2)^*$ with $R(\Omega)$ closed. Then*

$$\begin{aligned}\Omega^{\dagger} = A^{\dagger} &- \big(A^{\dagger}V_1\big)K^{-1}\big(A^{\dagger}V_1\big)^*A^{\dagger} - W_2K^{-1}U_2{}^*A^{\dagger} \\ &+ W_2K^{-1}L\big(A^{\dagger}V_1\big)^*A^{\dagger} - \big(A^{\dagger}V_1\big)K^{-1}L^*U_2{}^*A^{\dagger}.\end{aligned} \tag{18}$$

Proof We have $AA^{\dagger}V_1 = V_1$, $A^{\dagger}AU_2 = U_2$, $AW_2 = 0$, $(A^{\dagger}V_1)^*A^{\dagger}A = (A^{\dagger}V_1)^*$, and $U_2{}^*A^{\dagger}A = U_2{}^*$. Now, set

$$\begin{aligned}X = A^{\dagger} &- \big(A^{\dagger}V_1\big)K^{-1}\big(A^{\dagger}V_1\big)^*A^{\dagger} - W_2K^{-1}U_2{}^*A^{\dagger} + W_2K^{-1}L\big(A^{\dagger}V_1\big)^*A^{\dagger} \\ &- (A^{\dagger}V_1)K^{-1}L^*U_2{}^*A^{\dagger}.\end{aligned}$$

We observe the following:

$$\begin{aligned}
&A^\dagger \Omega = A^\dagger A + A^\dagger V_1(U_2+W_2)^*,\\
&A^\dagger V_1 K^{-1}\big(A^\dagger\big)^*\Omega = A^\dagger V_1 K^{-1}\big(A^\dagger V_1\big)^* + A^\dagger V_1 K^{-1}(U_2+W_2)^*,\\
&W_2K^{-1}U_2{}^*A^\dagger\Omega = W_2K^{-1}L(U_2+W_2)^* - W_2K^{-1}W_2{}^*,\\
&W_2K^{-1}L\big(A^\dagger V_1\big)^*A^\dagger\Omega = W_2K^{-1}L\big(A^\dagger V_1\big)^* + W_2K^{-1}L(U_2+W_2)^*,\\
&A^\dagger V_1K^{-1}L^*U_2{}^*A^\dagger\Omega = A^\dagger V_1K^{-1}L^*L(U_2+W_2)^* - A^\dagger V_1K^{-1}L^*W_2{}^*.
\end{aligned}$$

Thus, $X\Omega$ simplifies to

$$\begin{aligned}
X\Omega = {}& A^\dagger A - A^\dagger V_1K^{-1}\big(A^\dagger V_1\big)^* + W_2K^{-1}W_2{}^*\\
&+ W_2K^{-1}L\big(A^\dagger V_1\big)^* + A^\dagger V_1K^{-1}L^*W_2{}^*.
\end{aligned}$$

Now, for $x \in R(\Omega^*), x = A^*y + (U_2+W_2)V_1{}^*y$ for some $y \in H_2$. Then, $A^\dagger Ax = A^*y + U_2V_1{}^*y$, $A^\dagger V_1K^{-1}(A^\dagger V_1)^*x = A^\dagger V_1K^{-1}L^*V_1{}^*y$, $W_2K^{-1}W_2{}^*x = W_2K^{-1}V_1{}^*y$, $W_2K^{-1}L(A^\dagger V_1)^*x = W_2K^{-1}LL^*V_1{}^*y$, and $A^\dagger V_1K^{-1}L^*W_2{}^*x = A^\dagger V_1K^{-1}L^*V_1{}^*y$. Simplifying using these expressions, we get $X\Omega x = x$ for every $x \in R(\Omega^*)$.

Now, for $y \in N(\Omega^*)$, $A^*y = -(U_2+W_2)V_1{}^*y$. Multiplying both sides by $A^\dagger A$, we get $A^*y = -U_2V_1{}^*y$. Hence, we have $W_2V_1{}^*y = 0$, which implies that $V_1{}^*y \in N(W_2) = \{0\}$. Thus, $V_1{}^*y = 0$, and so $A^*y = 0$. Therefore, $y \in N(A^\dagger)$, so $A^\dagger y = 0$, and hence $Xy = 0$. This proves that $X = \Omega^\dagger$. □

For generalizing case (v), we take $V_1 = U_1 + W_1$ such that $R(U_1) \subseteq R(A)$ and $R(W_1) \subseteq R(A)^\perp$ and V_2 such that $R(V_2) \subseteq R(A^*)$. Here, $\Omega = A + (U_1 + W_1)V_2^*$. We provide the corresponding result in the following theorem. We omit the proof as it is obtained by replacing A, V_1, U_2, and W_2 by A^*, V_2, U_1, and W_1, respectively.

Theorem 11 *Let $A \in \mathscr{B}(H_1, H_2)$ with $R(A)$ closed. Let $U_1, W_1 \in \mathscr{B}(H_1, H_2)$ and $V_2 \in \mathscr{B}(H_2, H_1)$ be such that $R(U_1) \subseteq R(A)$, $R(W_2) \subseteq R(A)^\perp$, and $R(V_2) \subseteq R(A^*)$. Let $W_1{}^*W_1 = I$, $(A^{\dagger *}V_2)^*(A^{\dagger *}V_2) = I$, and $\tilde{L} = I + V_2{}^*A^\dagger U_1$. Assume that $\tilde{L}^*\tilde{L} = \tilde{L}\tilde{L}^*$ and let $N = I + \tilde{L}^*\tilde{L}$. Let $\Omega = A + (U_1 + W_1)V_2{}^*$ with $R(\Omega)$ closed. Then*

$$\begin{aligned}
\Omega^\dagger = {}& A^\dagger - A^\dagger\big(A^{\dagger *}V_2\big)N^{-1}\big(A^{\dagger *}V_2\big)^* - A^\dagger U_1N^{-1}W_1{}^*\\
&+ A^\dagger A^{\dagger *}V_2\tilde{L}N^{-1}W_1{}^* - A^\dagger U_1\tilde{L}^*N^{-1}\big(A^{\dagger *}V_2\big)^*. \qquad (19)
\end{aligned}$$

4 Applications

4.1 Perturbation Bounds for Nonnegativity of Moore–Penrose Inverses

For the first application, we consider $A \in \mathscr{B}(l^2)$. Suppose that $A^\dagger \geq 0$ (meaning that $A^\dagger(x) \geq 0$ whenever $x \geq 0$, where for $x \in l^2,\ x \geq 0$ means that $x_i \geq 0$ for all i). Let $M = A + B$, where B is a rank-one operator in l^2. We study the case where $M^\dagger \geq 0$. The second application studies the same question for the perturbation of a Toeplitz matrix.

(I) Let $H_1 = H_2 = l^2$, and let A be the right shift operator defined on l^2, i.e., $Ax = (0, x_1, x_2, \ldots)$ for every $x = (x_1, x_2, x_3, \ldots) \in l^2$. Then $R(A)$ is closed, and hence it has a bounded M–P inverse. In fact, $A^\dagger x = A^* x = (x_2, x_3, x_4, \ldots)$ for every $x = (x_1, x_2, x_3, \ldots) \in l^2$, the left shift operator on l^2. Note that $A \geq 0$ and $A^\dagger \geq 0$ in the following sense: $Ax \geq 0$ whenever $x \geq 0$, where $x \geq 0$ denotes that $x_i \geq 0$ for all i. For fixed nonzero vectors $b, c \in l^2$, let $B(x) = \langle x, c\rangle b$ and $M = A + B$.

By Theorem 2.3 in Christensen [4], it can be shown that $\|b\|\|c\| < 1$ is a sufficient condition for $R(M)$ to be closed. Here we consider two different cases.

(1) Let $b = he_{i+1}$ and $c = -e_i$ for $i = 1, 2, 3, \ldots$ with $h \in (-1, 1)$, so that $R(M)$ is closed. Also, we have $b \in R(A)$, $c \in R(A^*)$, and $\lambda = 1 - h > 0$ since $h \in (-1, 1)$. By Theorem 3, $M^\dagger(x) = A^\dagger(x) + \frac{h}{1-h} x_{i-1} e_i$. Therefore, $M^\dagger \geq 0$ if and only if $h \geq 0$.

(2) Let $b = he_1$ and $c = e_3$ with $h \in (-1, 1)$. Here $b \notin R(A)$ and $c \in R(A^*)$, which comes under case (v). Then $M^\dagger \geq 0$ if and only if $h \in [0, 1)$.

(II) Let C be the circulant matrix generated by the row vector $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1})$. Consider

$$A = \begin{pmatrix} I \\ {e_1}^T \end{pmatrix} \left(I + e_1 {e_1}^T\right)^{-1} C^{-1} \left(I + e_1 {e_1}^T\right)^{-1} (I \quad e_1),$$

where I is the identity matrix of order $n-1$, e_1 is $(n-1) \times 1$ vector with 1 as the first entry and 0 elsewhere. A is an $n \times n$ matrix of real entries. Then $A^\dagger$ is the $n \times n$ Toeplitz matrix

$$A^\dagger = \begin{pmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{n-1} & \cdots & \frac{1}{3} & \frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n-1} & \frac{1}{n-2} & \frac{1}{n-3} & \cdots & 1 & \frac{1}{n-1} \\ 1 & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & \frac{1}{2} & 1 \end{pmatrix}.$$

Clearly $A^\dagger \geq 0$. Fix $b = he_i$ and $c = e_i$, where $i \in \{2, 3, \dots, n-1\}$. Set $M = A + bc^*$. Then $M^\dagger \geq 0$ if and only if

$$h \geq \begin{cases} \frac{(n+1)^2}{4-(n+1)^2} & \text{when } n \text{ is odd;} \\ \frac{n(n+2)}{4-n(n+2)} & \text{when } n \text{ is even.} \end{cases}$$

Proof We have $b \in R(A)$, $c \in R(A^*)$, and $\lambda = 1 + h \neq 0$ if $h \neq -1$. Therefore, $M^\dagger = A^\dagger - \lambda^{-1} A^\dagger B A^\dagger = A^\dagger - \frac{h}{1+h} A^\dagger e_i e_i{}^* A^\dagger$, where we have used Theorem 3. Now,

$$M^\dagger \geq 0 \quad \text{if and only if} \quad \frac{h}{1+h} A^\dagger e_i e_i{}^* A^\dagger \leq A^\dagger. \tag{20}$$

The second matrix inequality in (20) yields n^2 scalar inequalities. Using the unique Toeplitz structure of the matrix $A^\dagger$, we can deduce that such n^2 inequalities yield a bound of the form $\frac{h}{1+h} \leq k$. An inequality corresponding to the maximum value of k yields the minimum value of h.

Now, for a fixed n, we have the inequalities as

$$\frac{h}{1+h} \leq \begin{cases} (\frac{n+1}{2})^2 & \text{when } n \text{ is odd;} \\ \frac{n}{2}\frac{n+2}{2} & \text{when } n \text{ is even.} \end{cases}$$

Hence we get the lower bound for h as

$$h \geq \begin{cases} \frac{(n+1)^2}{4-(n+1)^2} & \text{when } n \text{ is odd;} \\ \frac{n(n+2)}{4-n(n+2)} & \text{when } n \text{ is even.} \end{cases}$$ □

4.2 Rank-One Modified Operator and Star Partial Order

Let $A_1, A_2 \in \mathscr{B}(H)$ with $R(A_1)$ and $R(A_2)$ closed. The star partial order relating A_1 and A_2 is defined as in Mitra, Bhimasankaram and Malik [12]:

$$A_1 \overset{*}{\leq} A_2 \quad \text{if and only if} \quad A_1 A_1^* = A_2 A_1^* \quad \text{and} \quad A_1^* A_1 = A_1^* A_2. \tag{21}$$

Equivalently,

$$A_1 \overset{*}{\leq} A_2 \quad \text{if and only if} \quad A_1 A_1{}^\dagger = A_2 A_1{}^\dagger \quad \text{and} \quad A_1{}^\dagger A_1 = A_1{}^\dagger A_2. \tag{22}$$

Let $A_1 \overset{*}{\leq} A_2$. Consider two nonzero vectors b and c in H. We let B be the rank-one operator defined as $B(x) = \langle x, c\rangle b$. Set $M_1 = A_1 + B$ and $M_2 = A_2 + B$. In this subsection, we study when $M_1 \overset{*}{\leq} M_2$ holds. We have the following result.

Theorem 12 *Any two of the following imply the other*:

(a) $A_1 \overset{*}{\leq} A_2$;
(b) $M_1 \overset{*}{\leq} M_2$;
(c) $R(B) \subseteq N(A_2 - A_1)^*$ *and* $R(B^*) \subseteq N(A_2 - A_1)$.

Proof Assume that (a) and (b) hold. Then we have $A_1 {A_1}^* = A_2 {A_1}^*$ and ${A_1}^* A_1 = {A_1}^* A_2$. Also, $M_1 {M_1}^* = M_2 {M_1}^*$ implies

$$A_1 {A_1}^* + A_1 B^* + B {A_1}^* + B B^* = A_2 {A_1}^* + A_2 B^* + B {A_1}^* + B B^*.$$

Therefore, $(A_1 - A_2)B^* = 0$. Hence, $R(B^*) \subseteq N(A_1 - A_2)$. In a similar way, simplifying the equation ${M_1}^* M_1 = {M_1}^* M_2$ gives the other condition in statement (c). The rest of the proof can also be done in a similar fashion. □

A similar result in the finite-dimensional case has been considered in [12].

Remark 3 We can also use the formulae for the Moore–Penrose inverse of the rank-one perturbed operators, given in Sect. 2, to check whether $M_1 \overset{*}{\leq} M_2$. Assuming that $A_1 \overset{*}{\leq} A_2$, from Eq. (22) we have

$$(A_2 - A_1){A_1}^\dagger = 0 \quad \text{and} \quad {A_1}^\dagger (A_2 - A_1) = 0.$$

For case (i) considered in Sect. 1, we can obtain $M_1 \overset{*}{\leq} M_2$ in the following way:

$$(A_2 - A_1)(A_1 + B)^\dagger = -\delta^{-1}(A_2 - A_1)D{A_1}^\dagger + \delta^{-1}\eta^{-1}(A_2 - A_1)D{A_1}^\dagger E.$$

Now, $(A_2 - A_1){A_1}^\dagger = 0$ implies $(A_2 - A_1){A_1}^\dagger b = (A_2 - A_1)d = 0$. Therefore,

$$(A_2 - A_1)D{A_1}^\dagger x = \langle {A_1}^\dagger x, d\rangle (A_2 - A_1)d = 0,$$

$$(A_2 - A_1)D{A_1}^\dagger E x = \langle x, e\rangle \langle A_1^\dagger e, d\rangle (A_2 - A_1)d = 0.$$

Thus, $(A_2 - A_1)(A_1 + B)^\dagger = 0$. Similarly, we can show that $(A_1 + B)^\dagger (A_2 - A_1) = 0$. Hence, $A_1 + B \overset{*}{\leq} A_2 + B$. The result can be obtained in a similar way for the other cases as well.

Remark 4 Note that the equivalent condition given in Eq. (22) can be written in terms of the adjoint when $A_1 \overset{*}{\leq} A_2$:

$$(A_2 - A_1){A_1}^* = 0 \quad \text{and} \quad {A_1}^*(A_2 - A_1) = 0.$$

From these equations $M_1 \overset{*}{\leq} M_2$ follows.

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References

1. Baksalary, J.K., Baksalary, O.M.: Relationship between generalized inverses of a matrix and generalized inverses of its rank-one modifications. Linear Algebra Appl. **388**, 31–44 (2004)
2. Baksalary, J.K., Baksalary, O.M., Trenkler, G.: A revisitation formulae for the Moore–Penrose inverse of modified matrices. Linear Algebra Appl. **372**, 207–224 (2003)
3. Bartelett, M.S.: An inverse matrix adjustment arising in discriminant analysis. Ann. Math. Stat. **22**, 107–111 (1951)
4. Christensen, O.: Operators with closed range, pseudo-inverses, and perturbation of frames for a subspace. Can. Math. Bull. **42**, 37–45 (1999)
5. Deng, C.Y.: A generalization of the Sherman–Morrison–Woodbury formula. Appl. Math. Lett. **24**, 1561–1564 (2011)
6. Du, F., Xue, Y.: Expression of the Moore–Penrose inverse of $A - XY^*$. J. East China Norm. Univ. Natur. Sci. Ed. **5**, 33–37 (2010)
7. Groetsch, C.W.: Generalized Inverses of Linear Operators: Representation and Approximation. Dekker, New York (1977)
8. Hager, W.W.: Updating the inverse of a matrix. SIAM Rev. **31**, 221–239 (1989)
9. Kennedy, S.C., Haynes, R.D.: Inverse positivity of perturbed tridiagonal M-matrices. Linear Algebra Appl. **430**, 2312–2323 (2009)
10. Meyer, C.D.: Generalized inversion of modified matrices. SIAM J. Appl. Math. **24**, 315–323 (1973)
11. Mitra, S.K., Bhimasankaram, P.: Generalized inverses of partitioned matrices and recalculation of least squares estimates for data model changes. Sankhya, Ser. A **33**, 395–410 (1971)
12. Mitra, S.K., Bhimasankaram, P., Malik, S.B.: Matrix Partial Orders, Shorted Operators and Applications. World Scientific, Singapore (2010)
13. Ogawa, H.: An operator pseudo-inversion lemma. SIAM J. Appl. Math. **48**, 1527–1531 (1988)
14. Riedel, K.S.: A Sherman–Morrison–Woodbury identity for rank augmenting matrices with application to centering. SIAM J. Matrix Anal. Appl. **13**, 659–662 (1992)
15. Sherman, J., Morrison, W.J.: Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. Ann. Math. Stat. **21**, 124–127 (1950)
16. Wei, Y., Ding, J.: Representations for Moore–Penrose inverses in Hilbert spaces. Appl. Math. Lett. **14**, 599–604 (2001)
17. Woodbury, M.A.: Inverting modified matrices. Memorandum Rept. 42, Statistical Research Group, Princeton University, Princeton, NJ (1950)

The Reverse Order Law in Indefinite Inner Product Spaces

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Abstract The aim of this short note is to present a few reverse order laws for the Moore–Penrose inverse and the group inverse (when it exists) in indefinite inner product spaces, with respect to the indefinite matrix product. We also point out its relationship with the star and sharp orders, respectively.

Keywords Indefinite matrix product · Indefinite inner product spaces · Star order · Sharp order · Reverse order laws

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1 Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0$ for all $y \in \mathbb{C}^n$ only when $x = 0$. Associated with any indefinite inner product is a unique invertible Hermitian matrix J (called a weight) with complex entries such that $[x, y] = \langle x, Jy \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$ and vice versa. Motivated by the notion of Minkowski space (as studied by physicists), we also make an additional assumption on J, namely, $J^2 = I$, which can be shown to be less restrictive as the results presented in this manuscript can also be deduced without this assumption on J, with appropriate modifications. It should be remarked that this assumption also allows us to compare our results with the Euclidean case, apart from allowing us to present the results with much algebraic ease.

As there are two different values for dot product of vectors in indefinite inner product spaces, Kamaraj, Ramanathan, and Sivakumar introduced a new matrix

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product called indefinite matrix multiplication and investigated some of its properties in Ramanathan, Kamaraj and Sivakumar [11]. More precisely, the indefinite matrix product of two complex matrices A and B of sizes $m \times n$ and $n \times l$, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$, is defined to be the matrix $J_nA^*J_m$, where J_m and J_n are weights in the appropriate spaces. Note that there is only one value for the indefinite matrix product of two vectors. Many properties of this product are similar to that of the usual matrix product and also enables one to recover some interesting results in indefinite inner product spaces in a manner analogous to that in the Euclidean case (refer to Lemmas 2.3 and 2.7 of [11]). Kamaraj, Ramanathan, and Sivakumar also pointed out that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate than that of the usual matrix product (refer to Theorem 3.6 of [11]). Recall that the Moore–Penrose inverse exists if and only if $\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A)$. If we take $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $AA^{[*]}$ and $A^{[*]}A$ are both zero, and so $\text{rank}(AA^{[*]}) < \text{rank}(A)$, thereby proving that the Moore–Penrose inverse does not exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $\text{rank}(A \circ A^{[*]}) = \text{rank}(A^{[*]} \circ A) = \text{rank}(A)$. Thus, the Moore–Penrose of a matrix with real or complex entries exists over an indefinite inner product space with respect to the indefinite matrix product, whereas a similar result is false with respect to the usual matrix multiplication. The indefinite matrix product has also been used in connection with nonnegativity of various generalized inverses (refer to Ramanathan and Sivakumar [10] and Jayaraman [5]) and also in studying theorems of the alternative, see in Ramanathan and Sivakumar [9].

One of the basic problems in linear algebra is to characterize reverse order laws for matrix products with regard to various generalized inverses. For instance, if A and B are matrices such that the product AB is defined, then $(AB)^\dagger = B^\dagger A^\dagger$ holds if and only if A^*ABB^* is EP. Werner studied the following problem: when a matrix of the form YX is a generalized inverse of AB, where X and Y are generalized inverses of A and B, respectively (see Theorems 2.3 and 2.6 in Werner [13]). Another interesting result is that given two matrices A and B, all matrices of the form YX, where Y is a generalized inverse of B, and X is a generalized inverse of A, are generalized inverses of the product AB if and only if $ABYXAB$ is invariant with respect to the choice of Y and X. This in turn is related to the invariance of certain matrix products (see Baksalary and Baksalary [1] and the references therein). Various partial orders were introduced on the set of all complex matrices, possibly with a view to study matrix decompositions and those of the associated generalized inverses. These were later used to study reverse order laws for generalized inverses and also in problems in Statistics (for instance, Fisher–Cochran-type theorems) and Electrical Engineering (shorted operators), see Mitra, Bhimasankaram and Malik [6] (also refer to Blackwood, Jain, Prasad and Srivastava [3] and the references therein). Three of the most prominently used partial orders are the minus, star, and sharp orders, which are used to study reverse order laws for a generalized inverse, the Moore–Penrose inverse, and the group inverse (when it exists), respectively. A wealth of information is available on this topic, and research on this topic was pioneered by Drazin, Mitra, Hartwig, J.K. Baksalary, O.M. Baksalary, Trenkler, Gross, to name a few. For some recent work, one can refer to Benitez, Liu and Zhong [2] and the references therein. A comprehensive source of reference is the recent monograph by Mitra, Bhimasankaram and Malik [6]. Recently, reverse order laws have also been studied over more general algebraic structures (refer Mosic and Djordjevic [7, 8] and the references therein).

The aim of this manuscript is to investigate reverse order laws in indefinite inner product spaces with respect to the indefinite matrix product. Also investigated are possible extensions of the star and sharp order in the above setting. It turns out that the definition of the star order carries over as such to indefinite inner product spaces, whereas the sharp order becomes a partial order in a restricted class of matrices that are a natural generalization of EP matrices to indefinite inner product spaces (called J-EP matrices). The manuscript is organized as follows. We recall the definitions and preliminary results in Sect. 2. Section 3 deals with the main results. We begin with the definition of J-EP matrices and list a few of its properties (Definition 5 and the remarks following it). Reverse order laws for the indefinite matrix product, where one or both factors are J-EP, are presented (without proofs); two reverse order laws for the triple product are also presented (Theorems 1, 2, 3, and 4). The star order with respect to the indefinite matrix product is taken up next, and two reverse order laws are presented (Theorems 7 and 8). We then take up the case of the sharp order. It turns out that this is a partial order in a restricted class of matrices, namely, J-EP matrices. A few reverse order laws are presented in this setting too (see Theorems 10, 11, and 12 and Corollary 1). Wherever possible, we give examples to illustrate our results.

2 Notation, Definitions, and Preliminaries

We first recall the notion of an indefinite multiplication of matrices. We refer the reader to Ramanathan, Kamaraj and Sivakumar [11], wherein various properties of this product have been discussed in detail. One of the main advantages of this product has been pointed out in the introduction, namely, the existence of the Moore–Penrose inverse.

Definition 1 Let A and B be $m \times n$ and $n \times l$ complex matrices, respectively. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_nB$.

Note that there is only one value for the indefinite product of vectors/matrices. When $J_n = I_n$, the above product becomes the usual product of matrices. Let A be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of A (relative to J_n, J_m) is defined by $A^{[*]} = J_nA^*J_m$, and whenever A is a square matrix, we drop the subscripts n, m and write $A^{[*]} = JA^*J$. $A^{[*]}$ satisfies the following: $[Ax, y] = [x, A^{[*]}y]$ and $[A \circ x, y] = [x, (I \circ A \circ I)^{[*]} \circ y]$.

Let A be an $m \times n$ complex matrix. Then the range space $\mathrm{Ra}(A)$ is defined by $\mathrm{Ra}(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$, and the null space $\mathrm{Nu}(A)$ of A is defined by $\mathrm{Nu}(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$ [11]. It is clear that $\mathrm{Ra}(A) = R(A)$ and that $\mathrm{Nu}(A^{[*]}) = N(A^*)$.

Definition 2 Let $A \in \mathbb{C}^{n\times n}$. A is said to be J-invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that $A \circ X = X \circ A = J$.

It is obvious that A is J-invertible if and only if A is invertible and in this case the J-inverse is given by $A^{[-1]} = JA^{-1}J$. We now pass on to the notion of the Moore–Penrose inverse in indefinite inner product spaces.

Definition 3 For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is called the Moore–Penrose inverse if it satisfies the following equations: $A \circ X \circ A = A, X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X, (X \circ A)^{[*]} = X \circ A$.

Such an X will be denoted by $A^{[\dagger]}$. As pointed out earlier, it can be shown that $A^{[\dagger]}$ exists if and only if $\mathrm{rank}(A) = \mathrm{rank}(A \circ A^{[*]}) = \mathrm{rank}(A^{[*]} \circ A)$ (see Ramanathan, Kamaraj and Sivakumar [11]). The Moore–Penrose has the representation $A^{[\dagger]} = J_n A^\dagger J_m$. We also have $\mathrm{Ra}(A \circ A^{[\dagger]}) = \mathrm{Ra}(A)$ and $\mathrm{Ra}(A^{[\dagger]} \circ A) = \mathrm{Ra}(A^{[*]})$. One can similarly define the notion of the group inverse in indefinite inner product spaces.

Definition 4 For $A \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ is called the group inverse of A if it satisfies the equations $A \circ X \circ A = A, X \circ A \circ X = X, A \circ X = X \circ A$.

As in the Euclidean setting, it can be proved that the group inverse exists in the indefinite setting if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^{[2]})$ and is denoted by $A^{[\#]}$. In particular, if $A = A^{[*]}$, then $A^{[\#]}$ exists. If $A = B \circ C$ is a rank factorization, then the group inverse of A exists if and only if $C \circ B$ is invertible and in this case, the group inverse is given by $A^{[\#]} = B \circ (C \circ B)^{[-2]} \circ C$. We shall denote by $\mathscr{I}_{1,n}$ the set of all square matrices of size n having index at most 1.

3 Main Results

We present our main results in this section. Our concentration is mainly on EP matrices and its generalization, J-EP matrices, to indefinite inner product spaces. The notion of EP matrices was extended recently to indefinite inner product spaces by the author [4].

Definition 5 A square matrix A is said to be J-EP if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.

The following results were proved in this connection. The proofs can be found in [4].

- A is J-EP if and only if AJ is EP (Remarks 3.2 of [4]).
- If $AJ = JA$, then A is EP if and only if A is J-EP (Theorem 3.7(a) of [4]).
- For square matrices A and B such that $AJ = JA$, AB is EP if and only if $A \circ B$ is J-EP (Theorem 3.7(b) of [4]).
- A is J-EP if and only if $\mathrm{Ra}(A^{[2]}) = \mathrm{Ra}(A^{[*]})$ (Theorem 3.7(c) of [4]).
- If A is J-EP, then $A^{[\dagger]}$ is a polynomial in A.

A few results concerning the reverse order law were also obtained. We state these below. The proofs can be found in [4].

Theorem 1 ([4], Theorem 3.14) *If A and B are J-EP with* $\mathrm{Ra}(A) = \mathrm{Ra}(B)$, *then* $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

Theorem 2 ([4], Theorem 3.17) *Let A be such that $AJ = JA$. Then*, $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ *if and only if $A^*A \circ BB^*$ is J-EP.*

Theorem 3 ([4], Theorem 3.20) *Let A, B, and C be square matrices, and J be a weight such that $AJ = JA$, $(A \circ B)J = J(A \circ B)$. Further, assume that A^*ABB^* and $(AB)^*(AB)CC^*$ are EP. Then*, $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$.

Theorem 4 ([4], Theorem 3.21) $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$ *if and only if* $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$, *assuming that $BJ = JB$.*

As was pointed out in the introduction, various partial orders in the space of all complex matrices have been used to study reverse order laws for various types of generalized inverses. Let us recall the definition of the star order and point out a few results. For matrices A and B of the same order, A is said to be below B in the star order, denoted by $A <^* B$, if $AA^* = BA^*$ and $A^*A = A^*B$. The star order, which is a partial order, was introduced by Drazin. The star order was studied in connection with the reverse order law for the Moore–Penrose inverse. It was shown by Drazin that $A <^* B$ if and only if $AA^\dagger = BA^\dagger$ and $A^\dagger A = A^\dagger B$ (see Mitra, Bhimasankaram and Malik [6]). The following theorems are well known.

Theorem 5 ([6], Theorem 5.4.3) *Let A and B be matrices of the same order such that $A <^* B$. Then, the following are equivalent*:

(1) *A is EP,*
(2) $A^\dagger B = BA^\dagger$, *and*
(3) $AB^\dagger = B^\dagger A$.

Theorem 6 ([6], Theorem 5.4.15) *Let A and B be EP matrices of the same order. Then, $A <^* B$ if and only if* $(AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger = (A^\dagger)^2$.

One can now define the star order with respect to the indefinite matrix product. A natural way to generalize the star order with respect to the indefinite matrix product is to define $A <^{[*]} B$ if and only if $A \circ A^{[*]} = B \circ A^{[*]}$ and $A^{[*]} \circ A = A^{[*]} \circ B$.

Simplifying these two equations, we see that $A <^{[*]} B$ if and only if $AA^*J_m = BA^*J_m$ and $J_nA^*A = J_nA^*B$. Thus, $A <^{[*]} B$ if and only if $A <^* B$. It also follows from the above that the relation $<^{[*]}$ is a partial order on $\mathbb{C}^{m\times n}$. One can now attempt to study reverse order laws with respect to the indefinite matrix product. A generalization of Theorem 5 to indefinite inner product spaces is the following. We skip the proof.

Theorem 7 ([4], Theorem 4.3) *Let A and B be square matrices of the same order such that $A <^* B$. Then, A is J-EP if and only if $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$.*

As a consequence of the above result, the following reverse order law can be proved.

Theorem 8 ([4], Theorem 4.4) *Let A and B be square matrices of the same size. Then, we have the following: If $A <^* B$, $AJ = JA$, $BJ = JB$, and $AB^* = B^*A$, then $A^{[\dagger]} \circ B = B \circ A^{[\dagger]} \Longrightarrow B^{[\dagger]} \circ A = A \circ B^{[\dagger]}$.*

The following result is on sums of J-EP matrices. For a remark on range additivity, refer Remarks 3.13 of [4].

Theorem 9 ([4], Theorem 3.12) *Let $A_1, \dots, A_m$ be J-EP, and let $A := A_1 + \cdots + A_m$. Suppose that $\mathrm{Nu}(A) \subseteq \mathrm{Nu}(A_i)$ for each i and that $A_i \circ A_j = 0$ for $i \neq j$. Then A is J-EP.*

Let us recall that in the Euclidean setting, for matrices $A, B \in \mathscr{I}_{1,n}$, $A <^{\#} B$ if there exist commuting generalized inverses G_1 and G_2 of A such that $AG_1 = BG_1$ and $G_2A = G_2B$. It can then be proved that $A <^{\#} B$ if and only if $A^{\#}A = A^{\#}B$ and $AA^{\#} = BA^{\#}$ (refer Theorem 4.2.5 in Mitra, Bhimasankaram and Malik [6]). This is in turn equivalent to $A^2 = AB = BA$ (refer to Theorem 4.2.8 of [6]); moreover, $(AB)^{\#} = B^{\#}A^{\#}$ (refer to Theorem 4.2.14 of [6]).

Our attempt is to generalize the above notion to indefinite inner product spaces. It is obvious that we have to look for a class of matrices, where each member possesses a group inverse (in the indefinite setting, with respect to the indefinite matrix product). With this in mind, we restrict our attention to the collection of all J-EP matrices. It is possible that the results presented below are true for a bigger class of matrices, too. For J-EP matrices A and B, define $A <^{[\#]} B$ if and only if $A^{[\#]} \circ A = A^{[\#]} \circ B$ and $A \circ A^{[\#]} = B \circ A^{[\#]}$. Note that as in the Euclidean case, we need two equations to define this relation. Then, $A <^{[\#]} B$ implies that $A^{[2]} = A \circ B = B \circ A$ and conversely (compare with the equivalence of statements (i) and (ii) in Theorem 4.2.8 of [6]). The proof of the above statement is along the same lines as that of statements (i) and (ii) of Theorem 4.2.8 in [6] and is therefore skipped. Since $\mathrm{rank}(A) = \mathrm{rank}(A^{[2]}) = \mathrm{rank}(A^{[4]})$ (as both $A^{[\#]}$ and $(A^{[2]})^{[\#]}$ exist), we see that $\mathrm{rank}(A \circ B) = \mathrm{rank}((A \circ B)^{[2]})$, thereby proving that $(A \circ B)^{[\#]}$ exists when $A <^{[\#]} B$. It is easy to prove that when A and B are J-EP, $A <^{[\#]} B$ if and only if $AJ <^{\#} BJ$. Therefore, $<^{[\#]}$ is a partial order on the collection of all J-EP matrices. It should be noted that the existence of $A^{[\#]}$ does not imply the existence of $A^{\#}$ and conversely. The following examples illustrate this.

Example 1 Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $A^{[\dagger]} = (1/4)A$, and so $A^{[\#]}$ exists; however, $A^{\#}$ does not exist as $A^2 = 0$. On the other hand, if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $A^{\#}$ exists, whereas $A^{[\#]}$ does not.

We notice, however, that if A is such that $A^{\#}$ exists and is such that $AJ = JA$, then $A^{[\#]}$ also exists and equals $A^{\#}$. Let us also observe that both $A^{\#}$ and $A^{[\#]}$ may exist without being equal to each other. We now have the following theorem.

Theorem 10 *If A and B are J-EP, $AJ = JA$, and $A <^{[\#]} B$, then the group inverse of AB is given by $J(BJ)^{\#}A^{\#}$.*

Proof The assumptions guarantee that A is EP, and hence $A^{\#}$ exists. Now, $A <^{[\#]} B$ is equivalent to $AJ <^{\#} BJ$, which in turn is equivalent to the two equations $A^2 = J(AB)J = BA$. From this it easily follows that the group inverse of AB exists. We also know that $((A \circ B)J)^{\#}$ exists as $AJ <^{\#} BJ$. We therefore get $((A \circ B)J)^{\#} = (AJBJ)^{\#} = (JABJ)^{\#} = J(AB)^{\#}J$. However, $((A \circ B)J)^{\#} = (BJ)^{\#}(AJ)^{\#}$. Combining everything, we get $(AB)^{\#} = J(BJ)^{\#}A^{\#}$. □

We now prove two more reverse order laws for the group inverse. Given two matrices A and B of the same order, we say that A is below B in the space preorder, denoted $A <^{s} B$, if $R(A) \subseteq R(B)$ and $R(A^t) \subseteq R(B^t)$ (refer to Definition 3.2.1 in Mitra, Bhimasankaram and Malik [6]). It is known that if $A <^{s} B$, then $AB^{-}A$ is invariant under all choices of B^{-}, where B^{-} is a generalized inverse of B (refer to Theorem 3.2.9 in [6]). It was proved by Malik that if $A, B \in \mathscr{I}_{1,n}$ are such that $A <^{s} B$ and if B commutes with $A^{\#}A$, then $(AB)^{\#} = B^{\#}A^{\#}$ (refer to Theorem 2.2 in Malik [12]). In particular, if $A <^{\#} B$, then the above reverse order law holds (refer to Theorem 2.3 in [12]). We now prove a similar result in the indefinite setting.

Theorem 11 *Let A and B be J-EP. Assume that $A \circ B = B \circ A$, that $AJ <^{s} BJ$, and that $J(A \circ B) = (A \circ B)J$. Then, $(A \circ B)^{[\#]} = J(BJ)^{\#} \circ (AJ)^{\#}$.*

Proof Since A and B are J-EP, AJ and BJ are EP and hence belong to $\mathscr{I}_{1,n}$. We also know that $A^{[\dagger]}$ is a polynomial in A as A is J-EP. Therefore, the commutativity assumption $A \circ B = B \circ A$ ensures that $B \circ A^{[\dagger]} \circ A = A^{[\dagger]} \circ A \circ B$. After simplification we get $BA^{\dagger}A = JA^{\dagger}AJB$. From this it follows that $BJ(AJ)^{\#}(AJ) =$

$(AJ)^{\#}(AJ)BJ$ as $(AJ)^{\#} = (AJ)^{\dagger}$ (since AJ is EP). Therefore, by the above remark, $(AJBJ)^{\#} = (BJ)^{\#}(AJ)^{\#}$. Since $(A \circ B)J$ commutes with J (as J commutes with $A \circ B$), we see that $((A \circ B)J)^{[\#]}$ exists and equals $((A \circ B)J)^{\#}$. Again, since J is invertible, $(J(A \circ B)J)^{\#}$ exists, and hence $(A \circ B)^{\#}$ exists. But $(J(A \circ B)J)^{\#} = ((A \circ B)J)^{\#}J = ((A \circ B)J)^{[\#]}J$. Since $(A \circ B)^{\#}$ exists and $A \circ B$ commutes with J, $(A \circ B)^{[\#]} = (A \circ B)^{\#}$. Combining everything, we get $(A \circ B)^{[\#]} = J(BJ)^{\#}(AJ)^{\#}$. □

We also observe that an additional assumption $BJ = JB$ in Theorem 11 implies $(A \circ B)^{\#} = B^{\#} \circ A^{\dagger}$. We now give another reverse order law (a formula) for the group inverse of the product of two J-EP matrices A and B when $A <^{[\#]} B$.

Theorem 12 *Suppose that A and B are J-EP and are such that $A <^{[\#]} B$. If $J(A \circ B) = (A \circ B)J$, then, $(A \circ B)^{[\#]} = J(BJ)^{\#}(AJ)^{\#}$.*

Proof Recall that $A <^{[\#]} B$ is equivalent to $AJ <^{\#} BJ$. Since AJ and BJ are EP, we get $(AJBJ)^{\#} = (BJ)^{\#}(AJ)^{\#}$. As before, $(AJBJ)^{\#}$ is $((A \circ B)J)^{\#}$. If the commutativity assumption $J(A \circ B) = (A \circ B)J$ holds, then it is easy to verify that the group inverse of $(A \circ B)J$ equals $J(A \circ B)^{[\#]}$. Therefore, $(A \circ B)^{[\#]} = J(BJ)^{\#}(AJ)^{\#}$. □

Remark 1 Since AJ is EP, $(AJ)^{\#} = (AJ)^{\dagger}$ (similarly for BJ). Therefore, under the assumptions of Theorem 12, we get that $(A \circ B)^{[\#]} = B^{\dagger} \circ A^{\dagger}$.

The following example shows that the commutativity assumption $J(A \circ B) = (A \circ B)J$ cannot be dispensed with in Theorems 11 and 12.

Example 2 Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, A and B are J-EP, $A^{[\dagger]} = A^{[\#]} = (1/4)A$, and $A <^{*} B$. From this it also follows that $A <^{[\#]} B$ (see also Corollary 1 below). Note that $AJ \neq JA$ and $J(A \circ B) \neq (A \circ B)J$. Note also that $B \circ (AJ)^{\#}A = (AJ)^{\#}A \circ B = A$. One can also easily compute $J(BJ)^{\#}(AJ)^{\#} = B^{-1}(AJ)^{\#}$ to be equal to $-(1/8)\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. However, $(A \circ B)^{[\#]} = -(1/8)A$.

An interesting result relating the star order and the sharp order is the following: If A and B are square matrices such that A is EP, then, $A <^{*} B$ if and only if $A <^{\#} B$ (refer to Theorem 4.2.8 in Mitra, Bhimasankaram and Malik [6]). A generalization of this does not hold in indefinite inner product spaces. However, it is easy to see that when A and B are J-EP and $A^{[\dagger]} = A^{[\#]}$, then $A <^{[\#]} B$ if and only if $A <^{*} B$. Consequently, the following mixed-type reverse order law can be obtained as a corollary.

Corollary 1 *Suppose that A and B are J-EP with $A^{[\dagger]} = A^{[\#]}$. Let $A <^* B$ and suppose that $J(A \circ B) = (A \circ B)J$. Then, $(A \circ B)^{[\#]} = B^{[\dagger]} \circ A^{[\dagger]}$.*

Proof From the above paragraph, the assumption $A <^* B$ is equivalent to $A <^{[\#]} B$, which in turn is equivalent to $AJ <^{\#} BJ$. Therefore, $(A \circ B)^{[\#]} = B^{\dagger} J A^{\dagger}$ (see Theorem 12 and Remarks 1). Since $J(A \circ B) = (A \circ B)J$, we see that $J(A \circ B)^{[\#]} = (A \circ B)^{[\#]}J$. Thus, $J(A \circ B)^{[\#]}J = (A \circ B)^{[\#]} = JB^{\dagger}JA^{\dagger}J = B^{[\dagger]} \circ A^{[\dagger]}$. □

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References

1. Baksalary, J.K., Baksalary, O.M.: An invariance property related to the reverse order law. Linear Algebra Appl. **410**, 64–69 (2005)
2. Benitez, J., Liu, X., Zhong, J.: Some results on matrix partial orderings and reverse order law. Electron. J. Linear Algebra **20**, 254–273 (2010)
3. Blackwood, B., Jain, S.K., Prasad, K.M., Srivastava, A.K.: Shorted operators relative to a partial order in a regular ring. Commun. Algebra **37**(11), 4141–4152 (2009)
4. Jayaraman, S.: EP matrices in indefinite inner product spaces. Funct. Anal. Approx. Comput. **4**(2), 23–31 (2012)
5. Jayaraman, S.: Nonnegative generalized inverses in indefinite inner product spaces. Filomat, accepted for publication
6. Mitra, S.K., Bhimasankaram, P., Malik, S.B.: Matrix Partial Orders, Shorted Operators and Applications. Series in Algebra, vol. 10. World Scientific, Singapore (2010)
7. Mosic, D., Djordjevic, D.S.: Reverse order law for the Moore–Penrose inverse in C* algebras. Electron. J. Linear Algebra **22**, 92–111 (2011)
8. Mosic, D., Djordjevic, D.S.: Further results on the reverse order law for the Moore–Penrose inverse in rings with involution. Appl. Math. Comput. **218**(4), 1476–1483 (2011)
9. Ramanathan, K., Sivakumar, K.C.: Theorems of the alternative in indefinite inner product spaces. J. Optim. Theory Appl. **137**(1), 99–104 (2008)
10. Ramanathan, K., Sivakumar, K.C.: Nonnegative Moore–Penrose inverse of a Gram matrix in an indefinite inner product space. J. Optim. Theory Appl. **140**(1), 189–196 (2009)
11. Ramanathan, K., Kamaraj, K., Sivakumar, K.C.: Indefinite product of matrices and applications to indefinite inner product spaces. J. Anal. **12**, 135–142 (2004)
12. Malik, S.B.: Matrix partial orders and reversal law. Indian J. Pure Appl. Math. **38**(6), 569–577 (2007)
13. Werner, H.J.: When is B^-A^- a generalized inverse of AB? Linear Algebra Appl. **210**, 255–263 (1994)

Generalized Inverses and Approximation Numbers

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Abstract We derive estimates for approximation numbers of bounded linear operators between normed linear spaces. As special cases of our general results, approximation numbers of some weighted shift operators on ℓ^p and those of isometries and projections of norm 1 are found. In the case of finite-rank operators, we obtain estimates for the smallest nonzero approximation number in terms of their generalized inverses. Also, we prove some results regarding the relation between approximation numbers and the closedness of the range of an operator. We recall that the closedness of the range is a necessary condition for the boundedness of a generalized inverse. We give examples illustrating the results and also show that certain inequalities need not hold.

Keywords Generalized inverse · Generalized inverse of operator · Normed linear space · Approximation numbers

Mathematics Subject Classification (2010) 15A09 · 47B06

1 Introduction

Let X and Y be normed linear spaces, and $BL(X,Y)$ be the class of all bounded linear operators from X to Y. We use the notations $BL(X)$ for $BL(X,X)$ and X' for $BL(X,\mathbb{C})$. We shall denote the set of all finite-rank operators $F \in BL(X,Y)$ with $\operatorname{rank}(F) < k$ by $\mathscr{F}_k(X,Y)$ and use the notation $\mathscr{F}_k(X)$ for $\mathscr{F}_k(X,X)$. Also,

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we denote by $\ell^p(n)$ the space $\mathbb{C}^n$ with norm $\|\cdot\|_p$, $1 \le p \le \infty$. We also use the notation

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases} \quad \text{for } i, j \in \mathbb{N},$$

and for $T \in BL(X, Y)$, we denote by $R(T)$ the range of T.

The concept of approximation numbers of operators from $BL(X, Y)$ is a generalization of the concept of singular values of compact operators between Hilbert spaces. For $T \in BL(X, Y)$ and $k \in \mathbb{N}$, the kth *approximation number* $s_k(T)$ of T is defined as

$$s_k(T) := \inf\bigl\{\|T - F\| : F \in \mathscr{F}_k(X, Y)\bigr\}.$$

It is clear that $\|T\| = s_1(T) \ge s_2(T) \ge \cdots \ge 0$ and if T is of finite rank, then $s_k(T) = 0$ for all $k > \operatorname{rank}(T)$.

Some studies on approximation numbers and their properties can be found in Pietsch [13–15]. Approximation numbers play an important role in the geometry of Banach spaces as they are used in defining certain subclasses (*ideals*) of operator spaces (see Pietsch [15]). The convergence properties of approximation numbers are found useful in estimating the error while solving operator equations (see Schock [17]).

Computation of approximation numbers is a very difficult task, even in the case of operators between finite-dimensional spaces. There have been very few attempts in literature to estimate the approximation numbers of bounded linear operators between normed linear spaces. For example, Hutton, Morrell and Retherford [7] and Pietsch [14] contain methods of computing approximation numbers of diagonal operators in $BL(\ell^q, \ell^p)$, $1 \le p \le q \le \infty$, with nonincreasing positive diagonal entries, diagonal operators between some finite-dimensional spaces, and embedding maps in $BL(\ell^p, \ell^q)$, $1 \le p \le q \le \infty$. In Lomakina [9, 10], some estimates were given for approximation numbers of certain classes of integral operators.

The purpose of this article is to give some estimates for approximation numbers of bounded linear operators between normed linear spaces. Since, for given $T \in BL(X, Y)$ and $k \in \mathbb{N}$, $s_k(T) \le \|T - F\|$ for each operator $F \in \mathscr{F}_k(X, Y)$, finding lower estimates of approximation numbers is of importance. We give some results in this regard in Sect. 2. Approximation numbers of isometries, projections of norm 1, and those of some weighted shift operators in $BL(\ell^p)$, $1 \le p \le \infty$, are specified in this section. For finite-rank operators in $BL(X, Y)$, we give an estimate for the least nonzero approximation number in terms of generalized inverses of the operator, and as a special case, we show that it coincides with the reciprocal of the norm of the Moore–Penrose inverse of the operator when X and Y are Hilbert spaces. This special case is a known result.

Let X, Y be Hilbert spaces, $T \in BL(X, Y)$, and let $T^* \in BL(Y, X)$ be the adjoint operator of T. In Kulkarni and Nair [8], it was shown that the closedness of $R(T)$, the range of T, can be characterized using the spectrum of T^*T. A question of interest is whether it is possible to study the closedness of $R(T)$ using $\{s_k(T)\}$ when X and Y are general normed linear spaces. It is relevant to note here that when X, Y

are Banach spaces and T has a bounded generalized inverse, then $R(T)$ is closed and when X, Y are Hilbert spaces, then $T^\dagger$ is bounded if and only if $R(T)$ is closed (see Ben-Israel and Greville [1]). In Sect. 3, we prove some results regarding the relation between $\{s_k(T)\}$ and closedness of $R(T)$ for $T \in BL(X,Y)$. We also give counter examples to show the inadequacy of approximation numbers in characterizing the closedness of $R(T)$.

2 Some Estimates for Approximation Numbers

The following elementary proposition is useful to identify approximation numbers of some operators.

Proposition 1 *Let X, X_1, Y, Y_1 be normed linear spaces, and $T \in BL(X,Y)$. Let $U \in BL(Y, Y_1)$ and $V \in BL(X_1, X)$ be surjective isometries. Then*

$$s_k(UTV) = s_k(T) \quad \textit{for all } k \in \mathbb{N}.$$

Proof Let $k \in \mathbb{N}$. Then

$$s_k(UTV) \leq \|U\| s_k(T) \|V\| = s_k(T).$$

Also,

$$s_k(T) = s_k\big(U^{-1}UTVV^{-1}\big) \leq \big\|U^{-1}\big\| s_k(UTV) \big\|V^{-1}\big\| = s_k(UTV).$$

Hence, $s_k(UTV) = s_k(T)$ for all $k \in \mathbb{N}$. □

Example 1 Let, for $1 \leq p \leq \infty$, $D \in BL(\ell^p(n))$ be the diagonal operator defined by

$$D(x_1, x_2, \ldots, x_n) = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n), \quad (x_1, x_2, \ldots, x_n) \in \ell^p(n),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers satisfying $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Then it is known that $s_k(D) = \alpha_k$ for all $k \in \{1, 2, \ldots, n\}$ (see Pietsch [15]).

Now, suppose that $(a_1, a_2, \ldots, a_n)$ is a rearrangement of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $A \in BL(\ell^p(n))$ is defined by

$$A(x_1, x_2, \ldots, x_n) = (a_1 x_1, a_2 x_2, \ldots, a_n x_n), \quad (x_1, x_2, \ldots, x_n) \in \ell^p(n).$$

Since $A = UDV$ for appropriate isometries U and V, by Proposition 1 we have $s_k(A) = \alpha_k$ for $k = 1, \ldots, n$. Also, if $B \in BL(\ell^p(n))$ is defined by

$$B(x_1, x_2, \ldots, x_n) = (\alpha_1 x_2, \alpha_2 x_3, \ldots, \alpha_{n-1} x_n, \alpha_n x_1), \quad (x_1, x_2, \ldots, x_n) \in \ell^p(n),$$

then $s_k(B) = \alpha_k$ for $k = 1, \ldots, n$. To see this, we first observe that $B = UC$, where C is the diagonal operator defined by

$$C(x_1, x_2, \ldots, x_n) = (\alpha_n x_1, \alpha_1 x_2, \ldots, \alpha_{n-1} x_n), \quad (x_1, x_2, \ldots, x_n) \in \ell^p(n),$$

and U is the surjective isometry on $\mathbb{C}^n$ defined by

$$U(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1), \quad (x_1, x_2, \ldots, x_n) \in \ell^p(n).$$

Taking $V = I$ in Proposition 1, $s_k(B) = s_k(UC) = s_k(C)$ and, since $(\alpha_n, \alpha_1, \dots, \alpha_{n-1})$ is a rearrangement of $(\alpha_1, \alpha_2, \dots, \alpha_n)$, $s_k(C) = s_k(D) = \alpha_k$. Since B is a compact operator, we also have $s_k(B') = \alpha_k$ (see Hutton, Morrell and Retherford [7]), where $B' \in BL(\ell^p)$ is the operator defined by

$$B'(x_1, x_2, \dots, x_n) = (\alpha_n x_n, \alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}), \quad (x_1, x_2, \dots, x_n) \in \ell^p.$$

In the literature, very little is known about the approximation numbers of general bounded linear operators, though the approximation numbers of compact operators on Hilbert spaces (known as singular values) and of diagonal operators in $BL(\ell^p, \ell^q)$ for $1 \le q \le p \le \infty$ are known (see Hutton, Morrell and Retherford [7] and Pietsch [14]). Also, some estimates are given for the approximation numbers of certain classes of integral operators in Lomakina [9] and [10].

Applicability of Proposition 1 is very limited since there may not be many isometries with the help of which one can transform a given operator into a simpler form whose approximation numbers are known. For example, even for an operator $T \in BL(\ell^p(n))$ with $p \neq 2$, the class of operators T for which there exist a diagonal operator B and surjective isometries U, V such that $T = UBV$ is not very large. In fact, for $n = 2$ and $p \neq 2$, the operators for which this is possible are operators having matrix representations $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right]$ with respect to the standard basis of $\mathbb{C}^2$ (see Böttcher [2]). Note that both these cases are already covered in Example 1.

Let $T \in BL(X, Y)$, and for each $n \in \mathbb{N}$, let $P_n \in BL(X)$ and $Q_n \in BL(Y)$ be projections with $\operatorname{rank}(P_n) = \operatorname{rank}(Q_n) = n$ and $\|P_n\|\|Q_n\| = 1$. Let $T_n := Q_n T P_n$, $n \in \mathbb{N}$. We have proved in Deepesh, Kulkarni and Nair [5] (see Theorem 3.3 in [5]) that if X is separable, Y is the dual space of a separable normed linear space and if $T_n x \to Tx$ as $n \to \infty$ for each $x \in X$ in the weak* sense of convergence, then $\lim_{n\to\infty} s_k(T_n) = s_k(T)$. In applications, the projections P_n and Q_n may be of finite rank, and one may know the approximation numbers of the operators

$$\tilde{T}_n := T_n|_{R(P_n)} : R(P_n) \to R(Q_n).$$

So, a natural question is whether $s_k(T_n) = s_k(\tilde{T}_n)$ for $n, k \in \mathbb{N}$. We answer this question affirmatively, as a consequence of the following two propositions.

Proposition 2 *Let $T \in BL(X, Y)$, X_0 be a nonzero subspace of X, and Y_0 be a nonzero subspace of Y such that $R(T) \subseteq Y_0$, $T_0 := T|_{X_0} : X_0 \to Y$, and $T_1 := T : X \to Y_0$. Then*

$$s_k(T_0) \le s_k(T) \le s_k(T_1).$$

Proof Let $I_0 : X_0 \to X$ and $I_1 : Y_0 \to Y$ be the inclusion operators. Then $T_0 = TI_0$ and $T = I_1 T_1$, and hence,

$$s_k(T_0) = s_k(TI_0) \le s_k(T)\|I_0\| = s_k(T),$$
$$s_k(T) = s_k(I_1 T_1) \le \|I_1\| s_k(T_1) = s_k(T_1).$$

□

In the next proposition, we use the notation $\widehat{T}$ to represent an operator $T \in BL(X, Y)$, considered as from X to $R(T)$, that is, $\widehat{T} := T : X \to R(T)$ defined by $\widehat{T}x = Tx$, $x \in X$.

Proposition 3 *Let $T \in BL(X, Y)$, and let $P \in BL(X)$ and $Q \in BL(Y)$ be nonzero projections. Then we have the following*:

(a) $\frac{1}{\|P\|} s_k(TP) \le s_k(TP|_{R(P)}) \le s_k(TP)$;
(b) $s_k(QT) \le s_k(\widehat{QT}) \le \|Q\| s_k(QT)$;
(c) $\frac{1}{\|P\|} s_k(QTP) \le s_k(\widehat{QTP}|_{R(P)}) \le \|Q\| s_k(QTP)$.

In particular, if $\|P\| = 1 = \|Q\|$, then

$$s_k(QTP) = s_k(\widehat{QTP}|_{R(P)}).$$

Proof (a) We have

$$s_k(TP) = s_k(TP|_{R(P)}P) \le \|P\| s_k(TP|_{R(P)})$$

and, by Proposition 2, $s_k(TP|_{R(P)}) \le s_k(TP)$.

(b) The inequality $s_k(QT) \le s_k(\widehat{QT})$ follows from Proposition 2. For $F \in \mathscr{F}_k(X, Y)$, we have

$$\left\| \widehat{QT} - \widehat{QF} \right\| = \|QT - QF\| \le \|Q\| \|QT - F\|.$$

Hence, $s_k(\widehat{QT}) \le \|Q\| \|QT - F\|$. Taking the infimum over $F \in \mathscr{F}_k(X, Y)$, we get $s_k(\widehat{QT}) \le \|Q\| s_k(QT)$.

(c) Taking $TP|_{R(P)}$ in place of T in (b) and using (a), we get

$$\begin{aligned} s_k(QTP|_{R(P)}) &\le s_k(\widehat{QTP}|_{R(P)}) \\ &\le \|Q\| s_k(QTP|_{R(P)}) \\ &\le \|Q\| s_k(QTP). \end{aligned}$$

Also, by taking QT in place of T in (a) we have $\frac{1}{\|P\|} s_k(QTP) \le s_k(QTP|_{R(P)})$. Hence,

$$\frac{1}{\|P\|} s_k(QTP) \le s_k(\widehat{QTP}|_{R(P)}) \le \|Q\| s_k(QTP).$$

The particular case is obvious from (c). □

The particular case in Proposition 3, together with Theorem 3.3 in Deepesh, Kulkarni and Nair [5], leads to the following.

Corollary 1 *Let X be separable, Y be the dual space of a separable space, and $T \in BL(X, Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be sequences of projection operators in $BL(X)$ and $BL(Y)$, respectively, such that $\|P_n\| = 1 = \|Q_n\|$, $n \in \mathbb{N}$. Let*

$$T_n := Q_n T P_n \quad \textit{and} \quad \tilde{T}_n := T_n|_{R(P_n)} : R(P_n) \to R(Q_n), \quad n \in \mathbb{N}.$$

If $T_n x \to Tx$ as $n \to \infty$ for each $x \in X$ in the weak sense of convergence, then for each $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(\tilde{T}_n) = s_k(T).$$

The above corollary helps us in identifying the approximation numbers of certain weighted shift operators on ℓ^p, $1 \le p \le \infty$, as illustrated in the following proposition.

Proposition 4 *Let $1 \le p \le \infty$, and let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_i \ge \alpha_{i+1} \ge 0$ for every $i \in \mathbb{N}$. Let $A \in BL(\ell^p)$ be the operator defined by*

$$Ax = (\alpha_1 x_2, \alpha_2 x_3, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Then $s_k(A) = \alpha_k$ for every $k \in \mathbb{N}$.

Proof For $n \in \mathbb{N}$, let $P_n \in BL(\ell^p)$ be the projection operator defined by

$$P_n x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

and let $A_n := P_n A P_n$ and $\tilde{A}_n := A_n|_{R(P_n)} : R(P_n) \to R(P_n)$. Since $A_n \to A$ as $n \to \infty$ in the weak* operator topology (considering ℓ^p as a dual space), by Corollary 1, we have $s_k(\tilde{A}_n) \to s_k(A)$ as $n \to \infty$. Now the operator $\tilde{A}_n : \ell^p(n) \to \ell^p(n)$ defined by

$$\tilde{A}_n x = (\alpha_1 x_2, \alpha_2 x_3, \ldots, \alpha_{n-1} x_n, 0), \quad x = (x_1, x_2, \ldots, x_n) \in \ell^p(n)$$

can be obtained from the diagonal operator $B \in BL(\ell^p(n))$ defined by

$$Bx = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_{n-1} x_{n-1}, 0), \quad x = (x_1, x_2, \ldots, x_n) \in \ell^p(n),$$

by composing with suitable isometries. Since the approximation numbers of an operator do not change if the operator is composed with surjective isometries, we have

$$s_k(\tilde{A}_n) = s_k(B) = \alpha_k \quad \text{for all } k = 1, 2, \ldots, n-1.$$

Hence,

$$s_k(A) = \lim_{n\to\infty} s_k(\tilde{A}_n) = \alpha_k \quad \forall k \in \mathbb{N}.$$ □

Remark 1 Let $1 \le p \le \infty$ and $\{\alpha_n\}$ be as in Proposition 4. Let $T, S \in BL(\ell^p)$ be defined by

$$Tx = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \qquad x = (x_1, x_2, \ldots) \in \ell^p,$$
$$Sx = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

respectively. Using similar arguments as in Proposition 4, it can be shown that

$$s_k(T) = \alpha_k = s_k(S) \quad \forall k \in \mathbb{N}.$$

Remark 2 It can be seen that $s_k(A)$ in Proposition 4 can also be found from the known values of $s_k(T)$ (see Pietsch [14]) by using the inequalities

$$s_k(T) = s_k(TV_-V_+) \le s_k(TV_-) \le s_k(T),$$

where T is as in Remark 1, V_- and V_+ are the left and right shift operators on ℓ^p, and $A = TV_-$. But we give Proposition 4 as an application of Corollary 1.

The following theorem helps in finding lower bounds for approximation numbers in certain cases.

Theorem 1 *Let $T \in BL(X, Y)$, and let M be a subspace of X. Suppose that there exists $\alpha > 0$ such that*

$$\|Tx\| \ge \alpha\|x\| \quad \forall x \in M.$$

Then

$$s_k(T) \ge \alpha \quad \forall k \le \dim(M).$$

In particular, if M is infinite-dimensional, then

$$s_k(T) \ge \alpha \quad \forall k \in \mathbb{N}.$$

Proof Suppose that $k \in \mathbb{N}$ with $k \le \dim(M)$ and $F \in \mathscr{F}_k(X, Y)$. Then, it follows that $N(F|_M) \ne \{0\}$, so that there exists $x \in M$ with $\|x\| = 1$ and $F(x) = 0$. Therefore,

$$\alpha \le \|Tx\| = \|Tx - Fx\| \le \|T - F\|.$$

This is true for all $F \in \mathscr{F}_k(X, Y)$. Hence, $\alpha \le s_k(T)$. □

Remark 3 Suppose that there exist $\alpha > 0$ and $k \in \mathbb{N}$ such that $s_k(T) \ge \alpha$. One may ask whether there exists a subspace M of X such that $\dim(M) \ge k$ and $\|Tx\| \ge \alpha\|x\|$ for all $x \in M$. The answer is negative. To see this, consider the inclusion operator $I : \ell^2 \to \ell^\infty$. Then $s_k(I) = 1$ for all $k \in \mathbb{N}$ (see Hutton, Morrell and Retherford [7]). Now assume that there exists a subspace M of ℓ^2 such that $\dim(M) \ge 2$ and $\|Ix\|_\infty \ge 1\,\|x\|_2$ for all $x \in M$. Since $\|x\|_\infty \le \|x\|_2$ for all $x \in \ell^2$, it follows that $\|x\|_\infty = \|x\|_2$ for all $x \in M$. This implies that for each $x \in M$, there exists $\beta \in \mathbb{C}$ and $j \in \mathbb{N}$ such that $x = \beta e_j$, where $e_j = (\delta_{jn})$, $j \in \mathbb{N}$. This j is independent of x. To see this, suppose that $x, y \in M$ are given by $x = \beta_1 e_{j_1}$ and $y = \beta_2 e_{j_2}$ for some nonzero $\beta_1, \beta_2 \in \mathbb{C}$ and $j_1, j_2 \in \mathbb{N}$. Then, since $x + y \in M$, we get $e_{j_1} = e_{j_2}$, so that $\dim(M) = 1$, which is a contradiction to our assumption that $\dim(M) \ge 2$. Thus, it is impossible to have the relation $\|x\|_\infty \ge \|x\|_2$ for all $x \in M$, if $\dim(M) \ge 2$.

Corollary 2 *Let $T \in BL(X, Y)$ be bounded below. Then $T^{-1} : R(T) \to X$ is continuous, and*

$$s_k(T) \ge \frac{1}{\|T^{-1}\|} \quad \forall k \le \operatorname{rank}(T).$$

In particular, if T is an isometry, then $s_k(T) = 1$ for all $k \le \operatorname{rank}(T)$.

Proof Since T is bounded below, T is injective, $\operatorname{rank}(T) = \dim(X)$, and $T^{-1} : R(T) \to X$ is continuous. In particular,

$$\|x\| = \|T^{-1}(Tx)\| \leq \|T^{-1}\| \, \|Tx\| \quad \forall x \in X.$$

Hence, by Theorem 1 we have $s_k(T) \geq \frac{1}{\|T^{-1}\|}$ for all $k \leq \dim(X)$. The particular case is obvious. □

Remark 4 By Corollary 2, we can infer that if X is infinite-dimensional and $T \in BL(X, Y)$ is bounded below, then $s_k(T) \geq \frac{1}{\|T^{-1}\|}$ for all $k \in \mathbb{N}$, and if $T \in BL(X, Y)$ is an isometry, then $s_k(T) = 1$ for all $k \in \mathbb{N}$.

Now we use Theorem 1 for showing that approximation numbers of projections of norm 1 are either 1 or 0.

Corollary 3 *If $P \in BL(X)$ is a nonzero projection, then*

$$1 \leq s_k(P) \leq \|P\| \quad \forall k \leq \operatorname{rank}(P),$$

and $s_k(P) = 0$ for every $k > \operatorname{rank}(P)$. In particular, if $\|P\| = 1$, then

$$s_k(P) = \begin{cases} 1, & k \leq \operatorname{rank}(P), \\ 0, & k > \operatorname{rank}(P). \end{cases}$$

Proof Since $Px = x$ for all $x \in R(P)$, Theorem 1 with $M = R(P)$ implies that $s_k(P) \geq 1$ for $k \leq \operatorname{rank}(P)$. Thus,

$$1 \leq s_k(P) \leq \|P\| \quad \forall k \leq \operatorname{rank}(P).$$

The particular case is obvious. □

Let $\alpha_i \in \mathbb{R}$ be such that $\alpha_i \geq \alpha_{i+1} \geq 0$ for all $i \in \mathbb{N}$. Let $D \in BL(\ell^p)$, $1 \leq p \leq \infty$, be the diagonal operator defined by

$$D(x_1, x_2, \ldots) = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad (x_1, x_2, \ldots) \in \ell^p.$$

We know from Pietsch [14] that $s_k(D) = \alpha_k$ for all $k \in \mathbb{N}$. We prove a generalization of this result as a corollary to Theorem 1, where $\{\alpha_n\}$ is assumed to be a sequence of complex numbers.

Corollary 4 *Let $D \in BL(\ell^p)$, $1 \leq p \leq \infty$, be the diagonal operator defined by*

$$Dx = (\alpha_1 x_1, \alpha_2 x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

where $\alpha_n \in \mathbb{C}$ satisfy $|\alpha_n| \geq |\alpha_{n+1}|$, $n \in \mathbb{N}$. Then

$$s_k(D) = |\alpha_k| \quad \forall k \in \mathbb{N}.$$

Proof Let $k \in \mathbb{N}$ and $M = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$, where $e_i = (\delta_{in})$, $i \in \mathbb{N}$. Then $\|Dx\| \geq |\alpha_k| \|x\|$ for all $x \in M$. Hence, by Theorem 1, $s_k(D) \geq |\alpha_k|$. Taking $P_{k-1} \in BL(\ell^p)$ as

$$P_{k-1}x = (x_1, x_2, \ldots, x_{k-1}, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p,$$

we have

$$s_k(D) \le \|D - P_{k-1}D\| = |\alpha_k|.$$

Thus, $s_k(D) = |\alpha_k|$ for all $k \in \mathbb{N}$. □

The next corollary gives a relation between certain approximation numbers and norms of certain generalized inverses. We refer to Ben-Israel and Greville [1] for theory and applications of generalized inverses. In particular, recall that for $T \in BL(X, Y)$, $S \in BL(Y, X)$ is called a $\{1\}$-*inverse of* T if $TST = T$.

Corollary 5 *Let $T \in BL(X, Y)$ be such that there exists $S \in BL(Y, X)$ satisfying $TST = T$. Then*

$$s_k(T) \ge \frac{1}{\|S\|} \quad \forall k \le \mathrm{rank}(T).$$

In particular, if X and Y are Hilbert spaces and T has closed range, then

$$s_k(T) \ge \frac{1}{\|T^\dagger\|} \quad \forall k \le \mathrm{rank}(T),$$

where $T^\dagger$ is the Moore–Penrose generalized inverse of T.

Proof Since $TST = T$, ST is a projection, and $\mathrm{rank}(ST) = \mathrm{rank}(T)$. Therefore, by Corollary 3,

$$1 \le s_k(ST) \le s_k(T)\|S\| \quad \forall k \le \mathrm{rank}(T).$$

The particular case follows by noticing that $TT^\dagger T = T$ whenever $R(T)$ is closed. □

Remark 5 We may observe that Corollary 2 is a particular case of Corollary 5.

Corollary 6 *Let $T \in BL(X)$, and let λ be an eigenvalue of T. Then*

$$s_k(T) \ge |\lambda| \quad \forall k \le \dim\big(N(\lambda I - T)\big).$$

In particular, if $\dim(N(\lambda I - T)) = \infty$, *then*

$$s_k(T) \ge |\lambda| \quad \forall k \in \mathbb{N}.$$

Proof Let $M_\lambda := N(\lambda I - T)$ and $k \in \mathbb{N}$ be such that $\dim(M_\lambda) \ge k$. Then $\|Tx\| = |\lambda|\|x\|$ for all $x \in M_\lambda$, and hence, by Theorem 1, $s_k(T) \ge |\lambda|$. □

Example 2 Let (α_n) be a bounded sequence in $\mathbb{C}$, and let $\beta \in \mathbb{C}$. Let $D \in BL(\ell^p)$, $1 \le p \le \infty$, be defined by

$$Dx = (\beta x_1, \alpha_1 x_2, \beta x_3, \alpha_2 x_4, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Then β is an eigenvalue of infinite geometric multiplicity, so that by Corollary 6, $s_k(D) \ge |\beta|$. If, in addition, $|\alpha_n| \le |\beta|$ for all $n \in \mathbb{N}$, then we have $s_k(D) \le \|D\| = |\beta|$, so that, in this case, $s_k(D) = |\beta|$ for all $k \in \mathbb{N}$.

If $T \in BL(X, Y)$ and M is a subspace of X, then we define

$$\nu_M(T) := \inf\big\{\|Tx\| : x \in M, \|x\| = 1\big\}.$$

Also, we denote by $\mathscr{M}_k(X)$ the set of all subspaces M of X such that $\dim(M) \geq k$. Then the quantity

$$\sup_{M \in \mathscr{M}_k(X)} \nu_M(T)$$

coincides with $u_k(T)$, the kth Bernstein Number of T (see Pietsch [15]). It is clear from Theorem 1 that

$$s_k(T) \geq \sup_{M \in \mathscr{M}_k(X)} \nu_M(T) = u_k(T). \tag{1}$$

Remark 6 Let X and Y be Hilbert spaces, and $T \in BL(X, Y)$ be compact. Then we have $s_k(T) = \nu_M(T)$ for some subspace M of X of dimension k (see Sunder [18], p. 165). Hence, in this case, if $s_k(T) \geq \alpha$ for some $\alpha > 0$, then there exists a subspace M of X such that $\dim(M) = k$ and $\|Tx\| \geq \alpha\|x\|$ for all $x \in M$. Hence, in this special case, we have a positive answer to the question asked in Remark 3.

Now we show that the inequality in (1) can be strict for some operator. We first prove a lemma in this regard by modifying the arguments in the proof of Lemma 11.11.4 in Pietsch [14]. In the following, card$K$ denotes the cardinality of a set K.

Lemma 1 *Let M be a subspace of ℓ^p, $1 \leq p < \infty$, such that $\dim(M) \geq n$. Then there exists $e \in M$ such that $\|e\|_\infty = 1$ and card$\{k : |e(k)| = 1\} \geq n$.*

Proof Let $1 \leq p < \infty$. Since $M \subset \ell^p \subset \ell^\infty$ is a finite-dimensional subspace, the closed unit ball U_M of M (with respect to $\|\cdot\|_\infty$) is compact and convex in M. Hence, by the Krein–Milman theorem [18], there exists an extreme point e. Now since $e \in \ell^p$, also, $\|e\|_\infty = 1 = |e(k_0)|$ for some $k_0 \in \mathbb{N}$.

Let $K := \{k : |e(k)| = 1\}$. Assume that card$\{K\} < n$. Then, since $e \in \ell^p$, the number $\alpha := \sup\{|e(k)| : k \notin K\} < 1$. Now let

$$N = \big\{x \in M : x(k) = 0 \text{ for all } k \in K\big\}.$$

Then N is a nontrivial subspace of M. Let $u \neq 0$ be an element of U_N and $\delta := 1 - \alpha$. We claim that $e \pm \delta u \in U_M$. To see this, note that $|e(k) \pm \delta u(k)| = 1$ for $k \in K$ and $|e(k) \pm \delta u(k)| \leq \alpha + \delta = 1$ for $k \notin K$. Hence,

$$\|e \pm \delta u\|_\infty = \max\big\{\big|e(k) \pm \delta u(k)\big| : k \in \mathbb{N}\big\} \leq 1.$$

Thus, $e \pm \delta u \in U_M$. But then e cannot be an extreme point of U_M, and the assumption card$\{K\} < n$ cannot be true. Thus, card$\{k : |e(k)| = 1\} \geq n$. □

The following example shows that the inequality in (1) is strict for the inclusion operator $I : \ell^2 \to \ell^\infty$.

Example 3 Let $I : \ell^2 \to \ell^\infty$ be the natural injection. Then $\nu_M(I) \le \frac{1}{\sqrt{2}}$ for all $M \in \mathscr{M}_2(\ell^2)$, and hence, $s_2(I) \neq \sup_{M\in\mathscr{M}_2(\ell^2)} \nu_M(I)$. To see this, note that, by Lemma 1, any subspace of ℓ^2 of dimension 2 contains an element e with $\|e\|_\infty = 1$ and $\operatorname{card}\{k : |e(k)| = 1\} \ge 2$. Hence, $\|e\|_2 \ge \sqrt{2}$. Then the element $u := \frac{e}{\|e\|_2}$ satisfies $\|u\|_2 = 1$ and $\|Iu\|_\infty \le \frac{1}{\sqrt{2}}$. Therefore, $\nu_M(I) \le \frac{1}{\sqrt{2}}$ for all $M \in \mathscr{M}_2(\ell^2)$, whereas $s_2(I) = 1$ (see Rogozhin and Silbermann [7]).

It has been proved in Böttcher [2] (see Proposition 9.2 in [2]) that if X is a finite-dimensional space, say with $\dim(X) = n$, then for $T \in BL(X)$,

$$s_n(T) = \begin{cases} \frac{1}{\|T^{-1}\|} & \text{if } T \text{is invertible,} \\ 0 & \text{if } T \text{ is not invertible.} \end{cases}$$

This is a particular case of the following general result for any finite-rank operator.

Theorem 2 *Let $T \in BL(X, Y)$ be a finite rank operator, say* $\operatorname{rank}(T) = n$, *and let $S \in BL(R(T), X)$ satisfy $TST = T$. Then*

$$\frac{1}{\|S\|} \le s_n(T) \le \frac{\|ST\|}{\|S\|}.$$

In particular, if X and Y are Hilbert spaces, then $s_n(T) = \frac{1}{\|T^\dagger\|}$, where $T^\dagger$ is the Moore–Penrose inverse of T.

Proof By Corollary 5, we have $s_n(T) \ge \frac{1}{\|S\|}$. Now, since $R(T)$ is finite-dimensional, there exists $y \in R(T)$ such that $\|y\| = 1$ and $\|Sy\| = \|S\|$. By the Hahn–Banach theorem, there exists $f \in X'$ such that $\|f\| = 1$ and $f(Sy) = \|Sy\| = \|S\|$. Define $P : X \to Y$ by

$$Pu = \frac{1}{\|S\|} f(STu)y, \quad u \in X.$$

Then $\|P\| \le \frac{\|ST\|}{\|S\|}$. Let $F = T - P$. Note that $Fu = 0$ for all $u \in N(T)$. Hence, $N(T) \subseteq N(F)$. Since $TSy = y$, we have

$$F(Sy) = TSy - \frac{1}{\|S\|} f(STSy)y = 0.$$

Thus, $Sy \in N(F)$, but $TSy = y \neq 0$. Hence, $N(F) \supsetneq N(T)$. Thus, $\operatorname{rank}(F) \le n-1$, so that

$$s_n(T) \le \|T - F\| = \|P\| \le \frac{\|ST\|}{\|S\|}.$$

If X and Y are Hilbert spaces, then we can take $S = T^\dagger$, and in that case $T^\dagger T$ is an orthogonal projection onto $N(T)^\perp$. Thus, we obtain $\|ST\| = 1$ and, consequently, $s_n(T) = 1/\|T^\dagger\|$. □

For $T \in BL(X, Y)$ and $k \in \mathbb{N}$, we have $s_k(T) = \operatorname{dist}(T, \mathscr{F}_k(X, Y))$. A question of interest is whether one can replace the set $\mathscr{F}_k(X, Y)$ by a smaller collection. In this regard we have the following proposition proved by Carl and Stephani [4] (see also Pietsch [14]).

Theorem 3 (Carl and Stephani [4], pp. 67, 71) *Let X and Y be Banach spaces, $T \in BL(X,Y)$, and $k \in \mathbb{N}$. Then we have the following.*

(i) *If X is a Hilbert space, then*

$$s_k(T) = \inf\big\{\|T - TP\| : \, P \in \mathscr{F}_k(X) \text{ is an orthogonal projection}\big\}.$$

(ii) *If Y is a Hilbert space, then*

$$s_k(T) = \inf\big\{\|T - PT\| : P \in \mathscr{F}_k(Y) \text{ is an orthogonal projection}\big\}.$$

In view of Theorem 3, to obtain further estimates for $s_k(T)$, we introduce a few more quantities.

Definition 1 For $T \in BL(X,Y)$ and $k \in \mathbb{N}$, we define

$$\gamma_k(T) := \inf\big\{\|T - PT\| : P \in \mathscr{F}_k(Y) \text{ is a projection}\big\}.$$

Also, for a finite dimensional subspace M of X, we define

$$\eta(M,X) := \inf\big\{\|I - P\| : \, P \in BL(X) \text{ is a projection with } R(P) = M\big\},$$

and for $n \in \mathbb{N}$, we define

$$\widehat{\eta}_n(X) := \sup\big\{\eta(M,X) : \, M \text{ subspace of } X \text{ with } \dim(M) = n\big\}.$$

Note that $\{\gamma_k(T)\}$ is a nonincreasing sequence and $s_k(T) \leq \gamma_k(T)$ for all $k \in \mathbb{N}$.

Clearly, if X is a Hilbert space, then $\widehat{\eta}_k(X) = 1$ for all $k \in \mathbb{N}$. For a general normed linear space X, it is known (Pietsch [14], p. 386) that if $\check{\eta}_k(X)$ is the quantity defined by

$$\check{\eta}_k(X) := \sup_{M \in \widehat{\mathscr{N}_k}(X)} \inf\big\{\|P\| : \, P \text{ is a projection on } X \text{ with } R(P) = M\big\},$$

then $\check{\eta}_k(X) \leq \sqrt{k}$, so that

$$\widehat{\eta}_k(X) \leq 1 + \check{\eta}_k(X) \leq 1 + \sqrt{k}.$$

For a real Banach space X, it was shown in Makai and Martini [11] that

$$\check{\eta}_k(X) \leq \frac{2 + (k-1)\sqrt{k+2}}{k+1} \leq \sqrt{k}.$$

This leads to an improved estimate for $\widehat{\eta}_k(X)$, for a real Banach space X, namely,

$$\widehat{\eta}_k(X) \leq 1 + \frac{2 + (k-1)\sqrt{k+2}}{k+1} \leq 1 + \sqrt{k}.$$

In terms of $\widehat{\eta}_k$, we give a general relation between $s_k(T)$ and $\gamma_k(T)$ in the following.

Proposition 5 *Let $T \in BL(X, Y)$ and $k \in \mathbb{N}$. Then*

$$s_k(T) \leq \gamma_k(T) \leq \widehat{\eta}_{k-1}(Y) s_k(T) \quad \forall k \in \mathbb{N}.$$

Proof Clearly, $s_k(T) \leq \gamma_k(T)$ for all $k \in \mathbb{N}$. Now let $\varepsilon > 0$ be given. Let $F \in \mathscr{F}_k(X, Y)$ be such that $\|T - F\| \leq s_k(T) + \varepsilon$. Then there exists a projection $P \in BL(Y)$ with $R(P) = R(F)$ and $\|I - P\| \leq \widehat{\eta}_{k-1}(Y)$. Hence,

$$\begin{aligned} \|T - PT\| = \big\|(I - P)T\big\| &\leq \big\|(I - P)(T - F)\big\| \\ &\leq \|I - P\| \, \|T - F\| \\ &\leq \widehat{\eta}_{k-1}(Y)\big(s_k(T) + \varepsilon\big). \end{aligned}$$

Thus, $s_k(T) \leq \gamma_k(T) \leq \widehat{\eta}_{k-1}(Y) s_k(T)$. □

Remark 7 Since for a Hilbert space X, $\widehat{\eta}_k(X) = 1$ for all $k \in \mathbb{N}$, we get Theorem 3(ii) as a corollary of Proposition 5.

Remark 8 Let X and Y be Banach spaces, and $T \in BL(X, Y)$. We may recall that the essential norm of T, denoted by $\|T\|_{\text{ess}}$, is defined by

$$\|T\|_{\text{ess}} := \inf\big\{\|T - K\| : K \in BL(X, Y) \text{ is compact}\big\}.$$

Since $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq \|T\|_{\text{ess}}$, it follows that for those operators T for which $\|T\| = \|T\|_{\text{ess}}$ we have

$$s_k(T) = \|T\| \quad \forall k \in \mathbb{N}.$$

In particular, for Toeplitz operators in $BL(\ell^p)$ with $1 < p < \infty$, $s_k(T) = \|T\|$ for all $k \in \mathbb{N}$, as for such operators, we have $\|T\| = \|T\|_{\text{ess}}$ (see Böttcher [2] as well as Böttcher and Silbermann [3]). Of course, there are operators other than Toeplitz operators that satisfy $\|T\| = \|T\|_{\text{ess}}$. For example, consider $X = C[a, b]$ with $\|\cdot\|_\infty$ and $A : X \to X$ defined by

$$(Ax)(t) = tx(t), \qquad t \in [a, b].$$

Then it can be seen that $\|A\| = \|A\|_{\text{ess}}$.

We also infer that if $T \in BL(X, Y)$ satisfies $\|T\| = \|T\|_{\text{ess}}$ and if X is separable, Y is a dual space of some separable normed linear space and $\{T_n\}$ and $\{\tilde{T}_n\}$ are as in Corollary 1, then $\lim_{n\to\infty} s_k(\tilde{T}_n) = \lim_{n\to\infty} s_k(T_n) = s_k(T) = \|T\|$.

Now, let X be a Banach space, and for $k \in \mathbb{N}$, let

$$\mathscr{A}_k := \big\{A \in BL(X) : A + F \text{ is not invertible for any } F \in \mathscr{F}_k(X)\big\}.$$

In the following, we give estimates for the approximation numbers of operators of the form $\lambda I - A$ for $A \in \mathscr{A}_k$.

Proposition 6 *Let X be a Banach space, $\lambda \in \mathbb{C}$, and $A \in \mathscr{A}_k$ for some $k \in \mathbb{N}$. Then*

$$s_k(\lambda I - A) \geq |\lambda|.$$

In particular, if $A \in \bigcap_{k=1}^{\infty} \mathscr{A}_k$ and $\lambda \in \mathbb{C}$, then

$$s_k(\lambda I - A) \geq |\lambda| \quad \forall k \in \mathbb{N}.$$

Proof Let $k \in \mathbb{N}$, $A \in \mathscr{A}_k$, and $F \in \mathscr{F}_k(X)$. Since $0 \in \sigma(A+F)$, for any $\lambda \in \mathbb{C}$, we have (see Nair [12], Theorem 10.10(i))

$$\|(\lambda I - A) - F\| \geq r_\sigma(\lambda I - A - F) \geq |\lambda|.$$

Thus, $s_k(\lambda I - A) \geq |\lambda|$. The remaining part of the theorem is obvious. □

Let X and Y be Banach spaces, and $T \in BL(X, Y)$. We recall that T is said to be a *Fredholm operator* if $R(T)$ is closed and $\dim(N(T))$ and $\operatorname{codim}(R(T))$ are finite, and in that case, the *index* of T is defined as the number

$$\operatorname{ind}(T) := \dim\big(N(T)\big) - \operatorname{codim}\big(R(T)\big).$$

We may recall that an operator $T \in BL(X, Y)$ is a Fredholm operator of index zero if and only if there exists a finite-rank operator F such that $T + F$ is invertible (Gohberg, Goldberg and Kaashoek [6], p. 191). Hence, it follows that

$T \in BL(X)$ is a Fredholm operator of index zero if and only if there exists $k \in \mathbb{N}$ such that $T \notin \mathscr{A}_k$.

Thus, as a consequence of Proposition 6, if $T \in BL(X)$ is not a Fredholm operator of index zero, then for every $\lambda \in \mathbb{C}$,

$$s_k(\lambda I - T) \geq |\lambda| \quad \forall\, k \in \mathbb{N}.$$

3 Closed Range Operators and Approximation Numbers

Let X and Y be Hilbert spaces, and $T \in BL(X, Y)$. If the operator T is compact, then the set of all nonzero singular values of T coincides with the set of square roots of nonzero elements of $\sigma(T^*T)$. For a general $T \in BL(X, Y)$, it was shown in Kulkarni and Nair [8] that

$R(T)$ is closed if and only if 0 is not an accumulation point of $\sigma(T^*T)$.

A question of interest is whether it is possible to study the closedness of $R(T)$ using the approximation numbers of T when X and Y are not necessarily Hilbert spaces. Again it is worth recalling here that the closedness of range is connected with the boundedness of a generalized inverse. It is known that if X, Y are Banach spaces and $T \in BL(X, Y)$ has a bounded $\{1\}$-inverse, then $R(T)$ is closed. Also, if X, Y are Hilbert spaces, then the Moore–Penrose inverse $T^\dagger$ is bounded if and only if $R(T)$ is closed.

Suppose that X and Y are Banach spaces and $T \in BL(X, Y)$ is such that $\lim_{k\to\infty} s_k(T) = 0$. Then we know that T is a compact operator, and in that case, $R(T)$ is not closed whenever T is of infinite rank.

But the converse need not be true. That is, if T has nonclosed range, then it is not necessary that $\lim_{k\to\infty} s_k(T) = 0$. To see this, consider the diagonal operator $D \in BL(\ell^2)$ defined by

$$Dx = \left(2x_1, \frac{1}{2}x_2, 2x_3, \frac{1}{3}x_4, 2x_5, \ldots\right), \quad x = (x_1, x_2, x_3, \ldots) \in \ell^2.$$

In this case, $R(D)$ is not closed since 0 is an accumulation point of the spectrum of D^*D (cf. Kulkarni and Nair [8]). We observe that $s_k(D) = 2$ for all $k \in \mathbb{N}$ (taking $\beta = 2$ and $\alpha_n = \frac{1}{n}$, $n \in \mathbb{N}$, in Example 2). However, we have $s_n(P_n D P_n) = \frac{1}{n}$ for all $n \geq 2$, so that $s_n(P_n D P_n) \to 0$ as $n \to \infty$, where $\{P_n\}$ is the sequence of projection operators in $BL(\ell^2)$ defined by

$$P_n x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^2.$$

In fact, the above result is true in a more general setting.

Proposition 7 *Let X be a Banach space, $T \in BL(X)$, and for each $n \in \mathbb{N}$, let $P_n \in \mathscr{F}_n(X)$ be projections such that $P_n \to I$ pointwise. If $R(T)$ is not closed, then* $\lim_{n\to\infty} s_n(P_n T P_n) = 0$.

Proof For $n \in \mathbb{N}$, let $T_n := P_n T P_n$ and $\tilde{T}_n := T_n|_{R(P_n)} : R(P_n) \to R(P_n)$. Assume that $\{s_n(T_n)\}$ does not converge to 0. Then there exist $d > 0$ and a subsequence $\{s_{n_k}(T_{n_k})\}$ of $\{s_n(T_n)\}$ such that $s_{n_k}(T_{n_k}) \geq d$ for all $k \in \mathbb{N}$. Hence,

$$s_{n_k}(\tilde{T}_{n_k}) \geq \frac{1}{M} s_{n_k}(T_{n_k}) \geq \frac{d}{M} > 0,$$

where $M \geq 1$ is such that $\|P_n\| \leq M$ for all $n \in \mathbb{N}$ (see Proposition 3). Then $\tilde{T}_{n_k}$ are invertible, and $s_{n_k}(\tilde{T}_{n_k}) = 1/\|\tilde{T}_{n_k}^{-1}\|$ for each $k \in \mathbb{N}$. In particular, $\|\tilde{T}_{n_k}^{-1}\| \leq \frac{M}{d}$. Hence, for $x \in X$,

$$\|P_{n_k} x\| = \left\|\tilde{T}_{n_k}^{-1} \tilde{T}_{n_k} P_{n_k} x\right\| \leq \left\|\tilde{T}_{n_k}^{-1}\right\| \|\tilde{T}_{n_k} P_{n_k} x\| \leq \frac{M}{d} \|\tilde{T}_{n_k} P_{n_k} x\|.$$

Letting $k \to \infty$, we have $\|x\| \leq \frac{M}{d} \|T x\|$ for all $x \in X$. Thus, T is bounded below, and hence, $R(T)$ closed. Thus, $s_n(T_n) \to 0$ as $n \to \infty$ if $R(T)$ is not closed. □

The above result generalizes the last part of the following result proved by Rogozhin and Silbermann [16].

Theorem 4 (Cf. [16]) *Let $T \in BL(\ell^p)$, $1 < p < \infty$, and for $n \in \mathbb{N}$, let $P_n \in BL(\ell^p)$ be defined by*

$$P_n x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \quad x = (x_1, x_2, \ldots) \in \ell^p.$$

Let $\{T_n\}$ be a sequence of operators in $BL(R(P_n))$ such that $T_n P_n \to T$ pointwise as $n \to \infty$. If $R(T)$ is not closed, then for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} s_{n-k+1}(T_n) = 0.$$

In particular, if $R(T)$ is not closed, then $\lim_{n\to\infty} s_n(P_n T P_n) = 0$.

We would like to mention that the converse of Theorem 4 need not be true in general, i.e., $s_{n-k+1}(T_n) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$ does not imply that $R(T)$ is not closed. To see this, consider the following example.

Example 4 For $n \in \mathbb{N}$, let $T_n \in BL(\ell^2)$ be defined by

$$T_n x = \left(x_1, x_2, \ldots, \frac{x_m}{n}, \frac{x_{m+1}}{n}, \ldots, \frac{x_n}{n}, 0, 0, \ldots\right), \quad x = (x_1, x_2, \ldots) \in \ell^2,$$

where $m = [\frac{n}{2}]$, the greatest integer less than or equal to $\frac{n}{2}$. For $n \in \mathbb{N}$, let $\tilde{T}_n := T_n|_{R(P_n)} : R(P_n) \to R(P_n)$, where P_n is as in Theorem 4. Then for each $k \in \mathbb{N}$, we have $s_{n-k+1}(\tilde{T}_n) = \frac{1}{n}$ for all sufficiently large n, and hence $s_{n-k+1}(\tilde{T}_n) \to 0$ as $n \to \infty$, whereas the pointwise limit of $\tilde{T}_n P_n$ is the identity operator, which has closed range.

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References

1. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses, 3rd edn. Springer, New York (2003)
2. Böttcher, A.: On the approximation numbers of large Toeplitz matrices. Doc. Math. **2**, 1–29 (1997)
3. Böttcher, A., Silbermann, B.: Analysis of Toeplitz Operators, 2nd edn. Springer, Berlin (2006)
4. Carl, B., Stephani, I.: Entropy, Compactness and the Approximation of Operators. Cambridge Tracts in Mathematics, vol. 98. Cambridge University Press, Cambridge (1990)
5. Deepesh, K.P., Kulkarni, S.H., Nair, M.T.: Approximation numbers of operators on normed linear spaces. Integral Equ. Oper. Theory **65**, 529–542 (2009)
6. Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators, Vol. I. Operator Theory: Advances and Applications, vol. 49. Birkhäuser, Basel (1990)
7. Hutton, C.V., Morrell, J.S., Retherford, J.R.: Diagonal operators, approximation numbers, and Kolmogoroff diameters. J. Approx. Theory **16**(1), 48–80 (1976)
8. Kulkarni, S.H., Nair, M.T.: A characterization of closed range operators. Indian J. Pure Appl. Math. **31**(4), 353–361 (2000)
9. Lomakina, E.N.: Estimates for the approximation numbers of a class of integral operators. I. Sib. Mat. Zh. **44**(1), 178–192 (2003) (in Russian). Translation in Sib. Math. J. **44**(1), 147–159 (2003)
10. Lomakina, E.N.: Estimates for the approximation numbers of a class of integral operators. II. Sib. Mat. Zh. **44**(2), 372–388 (2003) (in Russian). Translation in Sib. Math. J. **44**(2), 298–310 (2003)
11. Makai, E. Jr., Martini, H.: Projections of normed linear spaces with closed subspaces of finite codimension as kernels. Period. Math. Hung. **52**(1), 41–46 (2006)
12. Nair, M.T.: Functional Analysis: A First Course. Prentice-Hall of India, New Delhi (2002)
13. Pietsch, A.: Eigenvalues and s-Numbers. Cambridge Studies in Advanced Mathematics, vol. 13 (1985)
14. Pietsch, A.: Operator Ideals. North-Holland, Amsterdam (1980). Translated from German by the author
15. Pietsch, A.: s-numbers of operators in Banach spaces. Stud. Math. **51**, 201–223 (1974)
16. Rogozhin, A., Silbermann, B.: On the approximation numbers for the finite sections of block Toeplitz matrices. Bull. Lond. Math. Soc. **38**(2), 301–313 (2006)
17. Schock, E.: On projection methods for linear equations of the second kind. J. Math. Anal. Appl. **45**, 293–299 (1974)
18. Sunder, V.S.: Functional Analysis: Spectral Theory. Hindustan Book Agency, New Delhi (1997)

On the Level-2 Condition Number for Moore–Penrose Inverse in Hilbert Space

Huaian Diao and Yimin Wei

Abstract We prove that $\mathrm{cond}_\dagger(T) - 1 \le \mathrm{cond}_\dagger^{[2]}(T) \le \mathrm{cond}_\dagger(T) + 1$, where T is a linear operator in a Hilbert space, $\mathrm{cond}_\dagger(T)$ is the condition number of computing its Moore–Penrose inverse, and $\mathrm{cond}_\dagger^{[2]}(T)$ is the level-2 condition number of this problem.

Keywords Condition number · Linear operator · Moore–Penrose inverse · Perturbation

Mathematics Subject Classification (2010) 15A09 · 15A60 · 15A35

1 Introduction

Condition numbers are a classical theme in numerical analysis (Cline, Moler and Stewart [7], Higham [16], Kahan [17], Rump [21]). They occur as a parameter in both complexity and round-off analysis and hence, there is an obvious interest in their computation. In this way, a problem Π induces a new problem, namely, the computation of its condition number cond_Π. Two natural questions arise: how difficult is to compute cond_Π? and how sensitive is this computation for a given input d?

For the first question, Renegar conjectured that computing $\mathrm{cond}_\Pi(d)$ is as difficult as solving Π with input d. He elaborated on this conjecture in Renegar [20].

The second question can be restated as what is the condition number of d for the problem of computing $\mathrm{cond}_\Pi(d)$? This "condition of the condition," called *level-2*

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condition number and denoted by $\text{cond}_{\Pi}^{[2]}(d)$, was introduced by Demmel [11], who proved, for some specific problems, that their level-2 condition numbers coincided with their original condition numbers up to a multiplicative constant. Subsequently, Higham [15] improved this result by sharpening the bounds for the problems of matrix inversion and linear systems solving. More recently, it was shown that for a large class of problems—those whose condition number equals the relativised inverse of the distance to ill-posedness—the level-2 condition number coincides with the original condition number modulo adding or subtracting 1 (see Cheung and Cucker [6]).

The goal of this paper is to further extend this result to the computation of the Moore–Penrose inverse (Ben-Israel and Greville [1]) (for maps on infinite-dimensional space). We note that the result in [6] does not directly apply to this problem, since its condition number does not belong to the class dealt with in [6]. Yet, the method of proof used therein can be adapted to our case.

1.1 Condition Numbers

Condition numbers measure the sensitivity of the output of a problem with respect to small perturbations of the input data. For instance, the condition number for the problem of computing the inverse of a square matrix A is

$$\kappa(A) = \lim_{\varepsilon \to 0} \sup_{\|\Delta A\|_2 \le \varepsilon} \frac{\|(A+\Delta A)^{-1} - A^{-1}\|_2 \|A\|_2}{\|A^{-1}\|_2 \|\Delta A\|_2}. \tag{1}$$

Here $\|A\|_2$ denotes the operator norm of A with respect to the Euclidean norm in both domain and target spaces.

Classical results in linear algebra characterize $\kappa(A)$ in two different ways, namely,

$$\kappa(A) = \|A\|_2 \left\|A^{-1}\right\|_2$$

and

$$\kappa(A) = \frac{\|A\|_2}{\inf_{E \in \Sigma} \|A - E\|_2},$$

where Σ denotes the set of singular matrices. Note that the elements E in Σ are precisely those for which the inverses do not exist. They are *ill-posed* for the problem of matrix inversion, and, by convention, they have $\kappa(E) = \infty$.

1.2 Moore–Penrose Inverse and Its Condition Number

Let A be an $m \times n$ real matrix. Then (see Chen and Xue [5], Campbell and Meyer [3], and Wang, Wei and Qiao [23]) there exists a unique $n \times m$ real matrix $A^{\dagger}$

satisfying the following four matrix equations:

$$AXA = A, \qquad XAX = X, \qquad (AX)^{\mathrm{T}} = AX, \qquad (XA)^{\mathrm{T}} = XA.$$

Here, for a real matrix M, M^{T} denotes its transpose. We call $A^\dagger$ the *Moore–Penrose inverse* of A.

It follows from the singular value decomposition (see in Golub and Van Loan [14]) that we may write A as

$$A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^{\mathrm{T}}, \tag{2}$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and $D = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, where $r = \mathrm{rank}(A)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values of A. It is easy then to check that $A^\dagger$ can be expressed by

$$A^\dagger = V \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}}. \tag{3}$$

To define a condition number for the Moore–Penrose inverse that could be of some use, one faces the following difficulty (see Campbell and Meyer [3], p. 247).

Unpleasant fact. Suppose that $A \in \mathbb{R}^{m\times n}$ is of neither full column nor full row rank. Then, for any number K and any $\varepsilon > 0$, there exists a matrix E, $\|E\| < \varepsilon$, such that $\|(A+E)^\dagger - A^\dagger\| \geq K$.

Therefore, definition (1) would make ill-posed matrices that have a perfectly well-defined Moore–Penrose inverse. A way out is to restrict definition (1) to structured perturbations, i.e., perturbations that do not alter the rank of A.

For $\ell \in \mathbb{N}$, denote by $\Sigma(\ell)$ the set of $m \times n$ matrices with rank ℓ. Let $\mathrm{rank}(A) = r$. Then we define

$$\kappa_\dagger(A) = \lim_{\varepsilon\to 0} \sup_{\substack{\|\Delta A\|_2 \leq \varepsilon \\ A+\Delta A \in \Sigma(r)}} \frac{\|(A+\Delta A)^\dagger - A^\dagger\|_2 \|A\|_2}{\|A^\dagger\|_2 \|\Delta A\|_2}.$$

Also, we define

$$\mathrm{cond}_\dagger(A) = \|A\|_2 \left\|A^\dagger\right\|_2.$$

In contrast with the case of matrix inversion, we do not necessarily have $\kappa_\dagger(A) = \mathrm{cond}_\dagger(A)$. It can be proven, however, that these numbers are closely related (see Stewart and Sun [22], Section III.3, and Wang, Wei and Qiao [23], Chap. 7),

$$\kappa_\dagger(A) \leq \mathrm{cond}_\dagger(A) \leq \mu\kappa_\dagger(A),$$

where $\mu \leq 3$ is a constant depending on the considered norm. The condition number $\mathrm{cond}_\dagger(A)$ can also be characterized as a relative inverse of the distance to rank deficiency. If $A \neq 0$ and $r = \mathrm{rank}(A)$, we let

$$\varrho(A) = \inf_{E\in\Sigma(r-1)} \|A - E\|_2.$$

Proposition 1 *For all $m \times n$ real matrix $A \neq 0$, we have*

$$\operatorname{cond}_{\dagger}(A) = \frac{\|A\|_2}{\varrho(A)}.$$

Proof The statement follows from the chain

$$\varrho(A) = \inf_{E \in \Sigma(r-1)} \|A - E\|_2 = \sigma_r(A) = \|A^{\dagger}\|_2^{-1},$$

where the middle equality is due to Golub and Van Loan [14, Theorem 2.5.3]. □

Remark 1 Note that there is no set of "ill-posed" inputs. Here each $\Sigma(\ell)$ plays a similar role for the elements in $\Sigma(\ell + 1)$, but there is no a single set Σ whose elements are ill-posed and which is used for all input data.

These notions can be extended to the infinite-dimensional case. Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces over $\mathbb{R}$. We denote by $\mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ the Banach space of all bounded linear operators $T : \mathbb{H}_1 \to \mathbb{H}_2$ with the operator norm

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

Let $T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$, and let $\operatorname{Im}(T)$ and $\operatorname{Ker}(T)$ be the range and null space of T, respectively. According to Nashed [19, Theorem 5.7], any $T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ with $\operatorname{Im}(T)$ closed has a Moore–Penrose inverse $T^{\dagger}$ satisfying the equalities

$$TT^{\dagger}T = T, \qquad T^{\dagger}TT^{\dagger} = T^{\dagger}, \qquad \left(TT^{\dagger}\right)^* = TT^{\dagger}, \qquad \left(T^{\dagger}T\right)^* = T^{\dagger}T,$$

where T^* is the adjoint operator of T.

For $T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$, the *reduced minimum module* of T, denoted by $\mathrm{r}(T)$, is defined by

$$\mathrm{r}(T) = \inf\bigl\{\|Tx\| \mid \operatorname{dist}\bigl(x, \operatorname{Ker}(T)\bigr) = 1\bigr\}, \tag{4}$$

where $\operatorname{dist}(x, \operatorname{Ker}(T)) = \inf_{y \in \operatorname{Ker}(T)} \|x - y\|$.

It is shown in Chen and Xue [5] that $\operatorname{Im}(T)$ is closed if and only if $\mathrm{r}(T) > 0$. If this holds, according to Ding and Huang [13],

$$\mathrm{r}(T) = \bigl\|T^{\dagger}\bigr\|^{-1} = \inf_{\substack{x \in \operatorname{Ker}(T)^{\perp} \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}. \tag{5}$$

We extend the definition of $\operatorname{cond}_{\dagger}$ to bounded operators in Hilbert space by taking, when $\operatorname{Im}(T)$ is closed,

$$\operatorname{cond}_{\dagger}(T) = \frac{\|T\|}{\mathrm{r}(T)} = \|T\|\bigl\|T^{\dagger}\bigr\|. \tag{6}$$

When $\operatorname{Im}(T)$ is not closed, we define $\operatorname{cond}_{\dagger}(T) = \infty$. This is consistent with the fact (see Nashed [19, p. 63]) that $T^{\dagger}$ is bounded if and only if $\operatorname{Im}(T)$ is closed.

1.3 Level-2 Condition Numbers

To a problem Π one can associate a new problem, namely, the computation of its condition number cond_Π. The condition number of this new problem, called *level-2 condition number* for Π, is defined, for an input d of Π, by

$$\mathrm{cond}_\Pi^{[2]}(d) = \lim_{\varepsilon\to 0} \sup_{\|\Delta d\|\le\varepsilon} \frac{|\mathrm{cond}_\Pi(d+\Delta d) - \mathrm{cond}_\Pi(d)|\|d\|}{\mathrm{cond}_\Pi(d)\|\Delta d\|}.$$

In the definition above, there are no restrictions for Δd. But in the level-2 condition numbers for the Moore–Penrose inverse, we want to restrict Δd much in the same manner as we do it for its condition number $\kappa_\dagger$ or for the distance ϱ.

The level-2 condition number of the Moore–Penrose inverse of a linear operator in Hilbert space. For $d \in \mathbb{N}$, let $\Sigma(d) = \{T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2) \mid \dim \mathrm{Ker}(T) = d\}$. We define the level-2 condition number for an operator $T \in \Sigma(d)$ by

$$\mathrm{cond}_\dagger^{[2]}(T) = \lim_{\varepsilon\to 0} \sup_{\substack{\|\Delta T\|\le\varepsilon \\ T+\Delta T\in\Sigma(d)}} \frac{|\mathrm{cond}_\dagger(T+\Delta T) - \mathrm{cond}_\dagger(T)|\|T\|}{\mathrm{cond}_\dagger(T)\|\Delta T\|}$$

if $\mathrm{cond}_\dagger(T) < \infty$ and $\mathrm{cond}_\dagger^{[2]}(T) = \infty$ otherwise.

2 Main Results

The main results of this paper are the following.

Theorem 1 *Let* $T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ *be such that* $\dim \mathrm{Ker}(T) < \infty$. *Then*

$$\mathrm{cond}_\dagger(T) - 1 \le \mathrm{cond}_\dagger^{[2]}(T) \le \mathrm{cond}_\dagger(T) + 1.$$

To bound the level-2 condition number of operator T, we use the following results.

Lemma 1 (Chen and Xue [5], Chen, Wei and Xue [4], Xue [28]) *Let* $T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ *with* $\mathrm{Im}(T)$ *closed and Moore–Penrose inverse* $T^\dagger$, *and let* $\Delta T \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_2)$ *be such that* $\|T^\dagger\|\|\Delta T\| < 1$ *and* $\dim \mathrm{Ker}(T+\Delta T) = \dim \mathrm{Ker}(T) < \infty$. *Then*

$$\left|\mathrm{r}(T+\Delta T) - \mathrm{r}(T)\right| \le \|\Delta T\|.$$

Proof We first prove that $(T+\Delta T)^\dagger$ exists and

$$\frac{\|T^\dagger\|}{1+\|T^\dagger\|\|\Delta T\|} \le \left\|(T+\Delta T)^\dagger\right\| \le \frac{\|T^\dagger\|}{1-\|T^\dagger\|\|\Delta T\|}. \tag{7}$$

The right-hand side of this inequality is Chen and Xue [5, Theorem 1]. It follows from it that $\|(T+\Delta T)^{\dagger}\|$ is bounded, which implies that $\mathrm{Im}(T+\Delta T)$ is closed, see Nashed [19, p. 63]. Using this upper bound with T replaced by $T+\Delta T$ and ΔT replaced by $-\Delta T$, we obtain the left-hand side.

Since $\mathrm{r}(T)=\frac{1}{\|T^{\dagger}\|}$ and $\mathrm{r}(T+\Delta T)=\frac{1}{\|(T+\Delta T)^{\dagger}\|}$, we may rewrite (7) as

$$\frac{1}{\mathrm{r}(T)+\|\Delta T\|}\le\frac{1}{\mathrm{r}(T+\Delta T)}\le\frac{1}{\mathrm{r}(T)-\|\Delta T\|}$$

or, equivalently, $|\mathrm{r}(T+\Delta T)-\mathrm{r}(T)|\le\|\Delta T\|$. □

The following proposition is of Wei and Ding [25, Theorem 1].

Proposition 2 *Let* $T,\Delta T\in\mathcal{L}(\mathbb{H}_1,\mathbb{H}_2)$ *such that* $\mathrm{Im}(\Delta T)\subseteq\mathrm{Im}(T)$ *and* $\mathrm{Ker}(T)\subseteq\mathrm{Ker}(\Delta T)$. *If either* $\|T^{\dagger}\Delta T\|<1$ *or* $\|\Delta T T^{\dagger}\|<1$, *then* $(T+\Delta T)^{\dagger}$ *is well defined, and*

$$(T+\Delta T)^{\dagger}=\big(\mathrm{Id}+T^{\dagger}\Delta T\big)^{-1}T^{\dagger}=T^{\dagger}\big(\mathrm{Id}+\Delta T T^{\dagger}\big)^{-1}.$$

Here Id *denotes the identity map*.

Proof of Theorem 1 If $\mathrm{Im}(T)$ is not closed, then $\mathrm{cond}_{\dagger}(T)=\mathrm{cond}_{\dagger}^{[2]}(T)=\infty$, and the statement trivially holds. We next assume that $\mathrm{Im}(T)$ is closed. Note that, for $\|\Delta T\|$ sufficiently small, the inequality $\|T^{\dagger}\|\|\Delta T\|<1$ holds and that we are considering ΔT such that $\dim\mathrm{Ker}(T+\Delta T)=\dim\mathrm{Ker}(T)<\infty$. Therefore, the hypothesis of Lemma 1 holds.

Using the definition of $\mathrm{cond}_{\dagger}(T)$, we have

$$\begin{aligned}\mathrm{cond}_{\dagger}^{[2]}(T)&=\lim_{\varepsilon\to0}\sup_{\substack{(T+\Delta T)\in\Sigma(d)\\ \|\Delta T\|_2\le\varepsilon}}\frac{|\mathrm{cond}_{\dagger}(T+\Delta T)-\mathrm{cond}_{\dagger}(T)|\|T\|_2}{\mathrm{cond}_{\dagger}(T)\|\Delta T\|_2}\\&=\lim_{\varepsilon\to0}\sup_{\substack{(T+\Delta T)\in\Sigma(d)\\ \|\Delta T\|_2\le\varepsilon}}\frac{\left|\frac{\|T+\Delta T\|_2}{\mathrm{r}(T+\Delta T)}-\frac{\|T\|_2}{\mathrm{r}(T)}\right|\|T\|_2}{\frac{\|T\|_2}{\mathrm{r}(T)}\|\Delta T\|_2}\\&=\lim_{\varepsilon\to0}\sup_{\substack{(T+\Delta T)\in\Sigma(d)\\ \|\Delta T\|_2\le\varepsilon}}\left|\frac{\|T+\Delta T\|_2\mathrm{r}(T)-\|T\|_2\mathrm{r}(T+\Delta T)}{\mathrm{r}(T+\Delta T)\|\Delta T\|_2}\right|.\end{aligned}$$

To prove the upper bound, note that, for all ΔT,

$$\big|\|T+\Delta T\|_2-\|T\|_2\big|\le\|\Delta T\|_2,$$

and, if $T+\Delta T\in\Sigma(d)$, from Lemma 1 we have

$$\big|\mathrm{r}(T+\Delta T)-\mathrm{r}(T)\big|\le\|\Delta T\|_2.$$

Therefore, for all ΔT such that $T + \Delta T \in \Sigma(d)$,

$$\big| \|T + \Delta T\|_2 \mathrm{r}(T) - \|T\|_2 \mathrm{r}(T) \big| \le \|\Delta T\|_2 \mathrm{r}(T)$$

and

$$\big| \|T\|_2 \mathrm{r}(T + \Delta T) - \|T\|_2 \mathrm{r}(T) \big| \le \|T\|_2 \|\Delta T\|_2.$$

It follows that

$$\big| \|T + \Delta T\|_2 \mathrm{r}(T) - \|T\|_2 \mathrm{r}(T + \Delta T) \big| \le \|\Delta T\|_2 \mathrm{r}(T) + \|T\|_2 \|\Delta T\|_2$$

and consequently that, for sufficiently small ΔT,

$$\begin{aligned} \left| \frac{\|T + \Delta T\|_2 \mathrm{r}(T) - \|T\|_2 \mathrm{r}(T + \Delta T)}{\mathrm{r}(T + \Delta T)\|\Delta T\|_2} \right| &\le \frac{\|\Delta T\|_2 \mathrm{r}(T) + \|T\|_2 \Delta T\|_2}{(\mathrm{r}(T) - \|\Delta T\|_2)\|\Delta T\|_2} \\ &= \frac{\mathrm{r}(T) + \|T\|_2}{\mathrm{r}(T) - \|\Delta T\|_2}. \end{aligned}$$

Now use this inequality together with the definition of $\mathrm{cond}_\dagger^{[2]}(T)$ to obtain

$$\begin{aligned} \mathrm{cond}_\dagger^{[2]}(T) &= \lim_{\varepsilon \to 0} \sup_{\substack{(T+\Delta T) \in \Sigma(d) \\ \|\Delta T\|_2 \le \varepsilon}} \left| \frac{\|T + \Delta T\|_2 \mathrm{r}(T) - \|T\|_2 \mathrm{r}(T + \Delta T)}{\mathrm{r}(T + \Delta T)\|\Delta T\|_2} \right| \\ &\le \lim_{\varepsilon \to 0} \sup_{\substack{(T+\Delta T) \in \Sigma(d) \\ \|\Delta T\|_2 \le \varepsilon}} \frac{\mathrm{r}(T) + \|T\|_2}{\mathrm{r}(T) - \|\Delta T\|_2} \\ &= \frac{\mathrm{r}(T) + \|T\|_2}{\mathrm{r}(T)} \\ &= 1 + \frac{\|T\|_2}{\mathrm{r}(T)} \\ &= 1 + \mathrm{cond}_\dagger(T). \end{aligned}$$

This proves the upper bound. We now proceed with the lower bound.

Since

$$\left\| T^\dagger \right\| = \sup_{\substack{x \in \mathbb{H}_2 \\ \|x\|=1}} \left\| T^\dagger x \right\| = \sup_{\substack{x \in \mathrm{Im}(T) \\ \|x\|=1}} \left\| T^\dagger x \right\|,$$

we can find $u \in \mathrm{Im}(T)$ and $v \in \mathrm{Im}(T^\dagger) = \mathrm{Ker}(T)^\perp$ with $\|u\| = \|v\| = 1$ and such that

$$T^\dagger u = \left\| T^\dagger \right\| v.$$

Now define ΔT by

$$\Delta T(x) = -\varepsilon \langle x, v \rangle u \quad \forall x \in \mathbb{H}_1,$$

where $\langle\ ,\ \rangle$ denotes the inner product of $\mathbb{H}_1$, and $0 < \varepsilon < \min\{\|T\|, \|T^\dagger\|^{-1}\}$.

Let $x \in \mathrm{Ker}(T)$. Since $v \in \mathrm{Im}(T^\dagger) = \mathrm{Ker}(T)^\perp$, we have $\langle x, v\rangle = 0$ and $\Delta T(x) = 0$. Therefore,

$$\mathrm{Ker}(T) \subseteq \mathrm{Ker}(\Delta T),$$

from which it follows that

$$\mathrm{Ker}(T) \subseteq \mathrm{Ker}(T + \Delta T). \tag{8}$$

In addition, $\|\Delta T\| \le \varepsilon\|v\|\|u\| = \varepsilon$, and thus,

$$\left\|T^\dagger \Delta T\right\| \le \left\|T^\dagger\right\| \|\Delta T\| < \frac{1}{\varepsilon}\varepsilon = 1.$$

We can therefore apply Proposition 2 to deduce

$$(T + \Delta T)^\dagger = T^\dagger\big(\mathrm{Id} + \Delta T T^\dagger\big)^{-1},$$

from which it follows that

$$T^\dagger = (T + \Delta T)^\dagger\big(\mathrm{Id} + \Delta T T^\dagger\big),$$

which in turn implies that $\mathrm{Im}(T^\dagger) \subseteq \mathrm{Im}((T + \Delta T)^\dagger)$. Since $\mathrm{Im}(T^\dagger) = \mathrm{Ker}(T)^\perp$, we deduce that $\mathrm{Ker}(T + \Delta T) \subseteq \mathrm{Ker}(T)$. Together with (8), this shows that $\mathrm{Ker}(T + \Delta T) = \mathrm{Ker}(T)$ and, in particular, that $\dim \mathrm{Ker}(T + \Delta T) = \dim \mathrm{Ker}(T)$.

We now focus on the reduced minimum module of $T + \Delta T$. We have

$$\begin{aligned}
\mathrm{r}(T + \Delta T) &= \inf_{\substack{x \in \mathrm{Ker}(T+\Delta T)^\perp \\ x \neq 0}} \frac{\|(T + \Delta T)x\|}{\|x\|} \\
&= \inf_{\substack{x \in \mathrm{Ker}(T)^\perp \\ x \neq 0}} \frac{\|Tx - \varepsilon\langle x, v\rangle u\|}{\|x\|} \\
&\ge \inf_{\substack{x \in \mathrm{Ker}(T)^\perp \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} - \varepsilon \frac{|\langle x, v\rangle|}{\|x\|} \\
&\ge \inf_{\substack{x \in \mathrm{Ker}(T)^\perp \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} - \varepsilon \\
&= \mathrm{r}(T) - \varepsilon.
\end{aligned}$$

To prove the reverse inequality, consider $w \in \mathbb{H}_1$ such that $u = Tw$. Then

$$Tv = T\big(\left\|T^\dagger\right\|^{-1} T^\dagger u\big) = \left\|T^\dagger\right\|^{-1}\big(TT^\dagger Tw\big) = \left\|T^\dagger\right\|^{-1}(Tw) = \mathrm{r}(T)u.$$

Choosing $x = v \in \mathrm{Ker}(T)^\perp$, we have

$$(T + \Delta T)x = Tv - \varepsilon\|v\|u = \big(\mathrm{r}(T) - \varepsilon\big)u,$$

from which it follows that $\mathrm{r}(T+\Delta T)=\mathrm{r}(T)-\varepsilon$.

The lower bound now easily follows. Using that $\mathrm{cond}_\dagger(T)=\frac{\|T\|}{\mathrm{r}(T)}$, we get

$$\begin{aligned}
\mathrm{cond}_\dagger^{[2]}(T) &= \lim_{\varepsilon\to 0} \sup_{\substack{\|\Delta T\|\le\varepsilon\\ T+\Delta T\in\Sigma(d)}} \frac{|\mathrm{cond}_\dagger(T+\Delta T)-\mathrm{cond}_\dagger(T)|\|T\|}{\mathrm{cond}_\dagger(T)\|\Delta T\|}\\
&= \lim_{\varepsilon\to 0} \sup_{\substack{\|\Delta T\|\le\varepsilon\\ T+\Delta T\in\Sigma(d)}} \frac{|\|T+\Delta T\|\mathrm{r}(T)-\|T\|\mathrm{r}(T+\Delta T)|}{\|\Delta T\|\mathrm{r}(T+\Delta T)}\\
&\ge \lim_{\varepsilon\to 0} \frac{|\|T+\Delta T\|\mathrm{r}(T)-\|T\|(\mathrm{r}(T)-\varepsilon)|}{\|\Delta T\|(\mathrm{r}(T)-\varepsilon)}\\
&\ge \lim_{\varepsilon\to 0} \frac{\|T+\Delta T\|\mathrm{r}(T)-\|T\|(\mathrm{r}(T)-\varepsilon)}{\|\Delta T\|(\mathrm{r}(T)-\varepsilon)}\\
&\ge \frac{(\|T\|-\|\Delta T\|)\mathrm{r}(T)-\|T\|(\mathrm{r}(T)-\varepsilon)}{\|\Delta T\|(\mathrm{r}(T)-\varepsilon)}\\
&= \lim_{\varepsilon\to 0} \frac{(\|T\|-\varepsilon)\mathrm{r}(T)-\|T\|(\mathrm{r}(T)-\varepsilon)}{\|\Delta T\|(\mathrm{r}(T)-\varepsilon)}\\
&= \lim_{\varepsilon\to 0} \frac{-\varepsilon\mathrm{r}(T)+\|T\|\varepsilon}{\varepsilon(\mathrm{r}(T)-\varepsilon)}\\
&= \mathrm{cond}_\dagger(T)-1.
\end{aligned}$$

□

Our result can be applied to the Moore–Penrose inverse of a matrix in finite-dimensional space. The level-2 condition number of the Moore–Penrose inverse, $\mathrm{cond}_\dagger^{[2]}(A)$ is defined, for a matrix A with $\mathrm{rank}(A)=r$, by

$$\mathrm{cond}_\dagger^{[2]}(A) = \lim_{\varepsilon\to 0} \sup_{\substack{\|\Delta A\|_2\le\varepsilon\\ A+\Delta A\in\Sigma(r)}} \frac{|\mathrm{cond}_\dagger(A+\Delta A)-\mathrm{cond}_\dagger(A)|\|A\|_2}{\mathrm{cond}_\dagger(A)\|\Delta A\|_2}.$$

By similar techniques as in the proof of Theorem 1, we have the following theorem for the matrix Moore–Penrose inverse's level-2 condition number.

Theorem 2 *For all* $A\in\mathbb{R}^{m\times n}$,

$$\mathrm{cond}_\dagger(A)-1\le \mathrm{cond}_\dagger^{[2]}(A)\le \mathrm{cond}_\dagger(A)+1.$$

3 Concluding Remark

In this paper, we investigate the level-2 condition number for Moore–Penrose inverse in Hilbert space. We will explore the smoothed analysis for the Moore–Penrose inverse (see Bürgisser and Cucker [2] and Cucker, Diao and Wei [9]) and structured/componentwise perturbation for the Moore–Penrose inverse of the rank-deficient case (see Cucker and Diao [8], Cucker, Diao and Wei [10], Diao and Wei [12], Li, Xu and Wei [18], Wei [24], Wei and Wang [26], and Xu, Wei and Gu [27]). Since the main result is a pair of inequalities, it is natural to ask under what condition any of these inequalities becomes an equality. This will be our future research topic.

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References

1. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer, New York (2003)
2. Bürgisser, P., Cucker, F.: Smoothed analysis of Moore–Penrose inversion. SIAM J. Matrix Anal. Appl. **31**, 2769–2783 (2010)
3. Campbell, S.L., Meyer, C.D.: Generalized Inverse of Linear Transformations. SIAM, Philadelphia (2008)
4. Chen, G., Wei, M., Xue, Y.: Perturbation analysis of the least squares solution in Hilbert spaces. Linear Algebra Appl. **244**, 69–80 (1996)
5. Chen, G., Xue, Y.: The expression of the generalized inverse of the perturbed operator under Type I perturbation in Hilbert space. Linear Algebra Appl. **285**, 1–6 (1998)
6. Cheung, D., Cucker, F.: A note on level-2 condition numbers. J. Complex. **21**, 314–319 (2005)
7. Cline, A.C., Moler, C.B., Stewart, G.W., Wilkinson, J.H.: An estimate for the condition number of a matrix. SIAM J. Numer. Anal. **16**, 368–375 (1979)
8. Cucker, F., Diao, H.: Mixed and componentwise condition numbers for rectangular structured matrices. Calcolo **44**, 89–115 (2007)
9. Cucker, F., Diao, H., Wei, Y.: Smoothed analysis of some condition numbers. Numer. Linear Algebra Appl. **13**, 71–84 (2006)
10. Cucker, F., Diao, H., Wei, Y.: On mixed and componentwise condition numbers for Moore–Penrose inverse and linear least squares problems. Math. Comp. **76**, 947–963 (2007)
11. Demmel, J.W.: On condition numbers and the distance to the nearest ill-posed problem. Numer. Math. **51**, 251–289 (1987)
12. Diao, H., Wei, Y.: On Frobenius normwise condition numbers for Moore–Penrose inverse and linear least-squares problems. Numer. Linear Algebra Appl. **14**, 603–610 (2007)
13. Ding, J., Huang, L.J.: On perturbation of linear squares solution in Hilbert space. Linear Algebra Appl. **212/213**, 487–500 (1994)
14. Golub, G.H., Van Loan, C.F.: Matrix Computations, 3rd edn. Johns Hopkins University Press, Baltimore (1996)

15. Higham, D.J.: Condition numbers and their condition numbers. Linear Algebra Appl. **214**, 193–213 (1995)
16. Higham, N.J.: Accuracy and Stability of Numerical Algorithms, 2nd edn. SIAM, Philadelphia (2002)
17. Kahan, W.: Conserving confluence curbs ill-condition. Computer Science Department Report, University of California. Berkeley (1972). Available via DIALOG. www.dtic.mil/dtic/tr/fulltext/u2/766916.pdf
18. Li, Z., Xu, Q., Wei, Y.: A note on stable perturbations of Moore–Penrose inverses. Numer. Linear Algebra Appl. **20**, 18–26 (2013)
19. Nashed, M.Z. (ed.): Generalized Inverses and Applications. Academic Press, New York (1976)
20. Renegar, J.: Is it possible to know a problem instance is ill-posed? J. Complex. **10**, 1–56 (1994)
21. Rump, S.: Ill-conditioned matrices are componentwise near to singularity. SIAM Rev. **41**, 102–112 (1999)
22. Stewart, G.W., Sun, J.G.: Matrix Perturbation Theory. Academic Press, San Diego (1990)
23. Wang, G., Wei, Y., Qiao, S.: Generalized Inverses: Theory and Computations. Science Press, Beijing (2004)
24. Wei, Y.: Generalized inverses of matrices. In: Hogben, L. (ed.) Handbook of Linear Algebra, 2nd edn. Chapman & Hall/CRC, London (2013)
25. Wei, Y., Ding, J.: Representation for Moore–Penrose inverses in Hilbert spaces. Appl. Math. Lett. **14**, 599–604 (2001)
26. Wei, Y., Wang, D.: Condition numbers and perturbation of weighted Moore–Penrose inverses and weighted linear least squares problem. Appl. Math. Comput. **145**, 45–58 (2003)
27. Xu, Q., Wei, Y., Gu, Y.: Sharp norm-estimations for Moore–Penrose inverses of stable perturbations of Hilbert C^*-module operators. SIAM J. Numer. Anal. **47**, 4735–4758 (2010)
28. Xue, Y.: Stable Perturbations of Operators and Related Topics. World Scientific, Singapore (2012)

Products and Sums of Idempotent Matrices over Principal Ideal Domains

K.P.S. Bhaskara Rao

Abstract Writing a square matrix as a product of idempotent matrices attracted the attention of several linear algebraists. Equally interesting is the problem of writing a square matrix as a sum of idempotent matrices. Much work was done for real matrices and for matrices over other algebraic structures. We shall consider some of this work and present some new results for matrices over projective free rings.

Keywords Idempotent matrices property · Field · Euclidean domain · Principal ideal domain · Projective free ring

Mathematics Subject Classification (2010) 15A33 · 15A23 · 13F07 · 13F10 · 11A55

1 Introduction

In 1966, J.M. Howie [6] showed that every noninvertible mapping from a finite set X to itself is a finite composition of idempotent mappings. Taking the cue from this result, J.A. Erdos [4] in 1967 showed that every singular real matrix is a finite product of idempotent matrices.

Now, several natural questions arise.

- How far can this result be extended to other algebraic structures? What about finding the maximum number of idempotent matrices that are required for writing a matrix as a product of idempotent matrices?
- Are there corresponding results for sums of idempotent matrices? What about finding the maximum number of idempotent matrices that are required for writing a matrix as a sum of idempotent matrices? Are there other ramifications for writing a matrix as a sum of idempotent matrices?

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In this paper we shall look at some of these problems.

An integral domain R with 1 is called a projective free ring (see Bhaskara Rao [2], p. 48) if every finitely generated projective module over R is free. Every principal ideal domain is a projective free ring. Over a projective free ring, every $m \times m$ idempotent matrix can be written as PQ^T, where P and Q are $m \times r$ matrices such that $Q^T P = I$ (see [2], Theorem 4.21(ii)). We shall say that a ring has idempotent matrices property if every singular square matrix is a product of idempotent matrices.

2 Products of Idempotent Matrices

Ballantine [1] showed that every square matrix over a field is a finite product of idempotent matrices. This was extended to Euclidean domains by Laffey [7]. Laffey showed that every square singular matrix over a Euclidean domain is a finite product of idempotents.

In Bhaskara Rao [3], Theorem 2, the present author showed the following result for principal ideal domains.

Theorem 1 *For a principal ideal domain R, the following are equivalent.*

(i) *Every square singular matrix over R is a product of idempotent matrices (also called the idempotent matrices property of R in [3]).*
(ii) *Every 2×2 matrix $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ is a product of idempotent matrices.*
(iii) *Every 2×2 matrix $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ with $\gcd(a, b) = 1$ is a product of idempotent matrices.*

The main part of the above theorem is the proof that (iii) implies (i).

Professors André Leroy and S.K. Jain brought to my attention that the proof of Theorem 2 of Bhaskara Rao [3] given in my paper is lacking in details. Here I shall complete the missing part of the proof of Theorem 2 of [3]. The following proposition completes the missing part of the proof.

Proposition 1 *Let a, b be elements of a principal ideal domain R with $\gcd(a, b) = 1$. If $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ is a product of idempotent matrices, there exist elementary matrices $U_1, U_2, \ldots, U_k$ such that $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = U_1 U_2 \cdots U_k \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.*

Proof Let $A = \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ with $\gcd(a, b) = 1$. Since this is a product of idempotent matrices and since every principal ideal domain is a projective free ring, there exist 2×1 matrices $P_0, Q_0, P_1, Q_1, \ldots, P_n, Q_n$ such that $A = P_0 Q_0^T P_1 Q_1^T \cdots P_n Q_n^T$ and $Q_i^T P_i = 1$ for $0 \le i \le n$.

Then, necessarily, $P_0 = \left[\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right]$ and $Q_n^T = [e \;\; f]$ for some c, e, and f. If we write $Q_0^T P_1 Q_1^T \cdots P_n = d$, we get that $a = cde$ and $b = cdf$. Since $\gcd(a, b) = 1$, it follows that $a = 1, e = a, f = b$, and $d = 1$. By induction on i it also follows that $Q_i^T P_{i+1} = 1$ for all i.

Let us write $P_i = \begin{bmatrix} a_{2i} \\ b_{2i} \end{bmatrix}$ and $Q_i = \begin{bmatrix} a_{2i+1} \\ b_{2i+1} \end{bmatrix}$. Then a_i's and b_i's satisfy the conditions $a_{2n+1} = a$, $b_{2n+1} = b$, $a_0 = 1$, $b_0 = 0$, and $a_i a_{i-1} + b_i b_{i-1} = 1$ for $1 \leq i \leq (2n+1)$.

Define $\{p_i, q_i\}, 0 \leq i \leq 2n+1$, by $p_i = a_i$ and $q_i = b_i$ if i is even, $p_i = -b_i$ and $q_i = a_i$ if i is odd. We also set $p_{-1} = 1$ and $q_{-1} = 0$.

Then $\{c_i, 1 \leq i \leq 2n+1\}$ defined by $c_i = (-1)^i (p_{i-2} q_i - q_{i-2} p_i)$ will satisfy the equations $p_i = c_i p_{i-1} + p_{i-2}$ and $q_i = c_i q_{i-1} + q_{i-2}$ for $i \geq 1$. As in the proof of Theorem 7 of Bhaskara Rao [3], we also get

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} c_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} c_{2n+1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

But the left-side matrices are all products of elementary matrices. This proves the proposition. □

In Landers and Xue [8], it was shown that for $n > 2$, every $n \times n$ singular integer matrix is a product of at most $3n + 1$ idempotent matrices. We raise the problem of determining the maximum number of idempotent matrices that are required for writing a square $n \times n$ matrix as a product of idempotent matrices for principal ideal domains with the idempotent matrices property. For $n = 2$, no such bound exists for integer matrices (see Laffey [7]).

3 Sums of Idempotent Matrices

If A is an $m \times m$ idempotent matrix of rank r over a projective free ring R, then $A = PQ^T$ for some P and Q, where P and Q are $m \times r$ matrices such that $Q^T P = eI$. Hence, we have the following result.

Theorem 2 *Over a projective free ring, a matrix A is a sum of idempotent matrices if and only if A has a decomposition $A = RS^T$, where $S^T R$ has a diagonal of e's.*

It follows that the trace of A is equal to ke, where e is the unit of the ring R, and k is an integer. Hence, over a projective free ring R, if a matrix A is a sum of idempotent matrices, the trace of A is some ke where e is the unit of R, k is an integer, and $k \geq \text{rank}(A)$.

In [5], Hartwig and Putcha showed that a real matrix is a sum of idempotent matrices if and only if trace(A) is an integer and trace$(A) \geq \text{rank}(A)$. They also showed that for matrices over fields of characteristic zero, an $n \times n$ matrix is a sum of idempotent matrices if and only the trace of A is ke where k is an integer, e is the unit of R, and $k \geq \text{rank}(A)$.

For matrices over the ring of integers, Hartwig and Putcha showed that for a matrix A, if $\text{rank}(A) < \text{trace}(A)$, A is a sum of idempotent matrices and that if $\text{rank}(A) = \text{trace}(A)$, A may or may not be a sum of idempotent matrices.

For matrices over a field K of characteristic $p \neq 0$, Pazzis [12] showed that a matrix is a sum of idempotent matrices if and only if $\text{trace}(A)$ is an element of the prime subfield of K.

A variation of the problem of writing a matrix as a sum of idempotent matrices is the problem of writing a matrix as a linear combination of idempotent matrices. Indeed, every matrix over any ring with 1 is a linear combination of idempotent matrices.

For example, $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$.

An interesting problem is to find the maximum number of matrices that are required to write every matrix as a linear combination of idempotent matrices.

In [13], Rabanovich showed that every matrix over a field of characteristic 0 is a linear combination of three idempotent matrices. This was extended by Pazzis [11] to matrices over any field.

We refer the reader to Laurie, Mathes and Radjavi [9], Rabanovich [13], Pazzis [10–12], Wang [14], and Wu [15] for further results on writing matrices as sums of idempotent matrices.

Problem Study the maximum number problem of writing a matrix as a linear combination of idempotent matrices for matrices over projective free rings and in particular for matrices over principal ideal domains.

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References

1. Ballantine, C.S.: Products of idempotent matrices. Linear Algebra Appl. **19**(1), 81–86 (1978)
2. Bhaskara Rao, K.P.S.: The Theory of Generalized Inverses over Commutative Rings. Algebra, Logic and Applications Series. Taylor and Francis, London (2002)
3. Bhaskara Rao, K.P.S.: Products of idempotent matrices over integral domains. Linear Algebra Appl. **430**, 2690–2695 (2009)
4. Erdos, J.A.: On products of idempotent matrices. Glasg. Math. J. **8**, 118–122 (1967)
5. Hartwig, R.E., Putcha, M.S.: When is a matrix a sum of idempotents? Linear Multilinear Algebra **26**, 279–286 (1990)
6. Howie, J.M.: The subsemigroup generated by the idempotents of a full transformation semigroup. J. Lond. Math. Soc. **41**(1), 707–716 (1966)
7. Laffey, T.J.: Products of idempotent matrices. Linear Multilinear Algebra **14**(4), 309–314 (1983)
8. Landers, P., Xue, J.: Factorization of singular integer matrices. Linear Algebra Appl. **428**, 1046–1055 (2008)

9. Laurie, C., Mathes, B., Radjavi, H.: Sums of idempotents. Linear Algebra Appl. **208/209**, 175–197 (1994)
10. Pazzis, A.S.: On linear combinations of two idempotent matrices over an arbitrary field. Linear Algebra Appl. **433**, 625–636 (2010)
11. Pazzis, A.S.: On decomposing any matrix as a linear combination of three idempotents. Linear Algebra Appl. **433**, 843–855 (2010)
12. Pazzis, A.S.: On sums of idempotent matrices over a field of positive characteristic. Linear Algebra Appl. **433**, 856–866 (2010)
13. Rabanovich, V.: Every matrix is a linear combination of three idempotents. Linear Algebra Appl. **390**, 137–143 (2004)
14. Wang, J.-H.: The length problem for a sum of idempotents. Linear Algebra Appl. **215**, 135–159 (1995)
15. Wu, P.Y.: Sums of idempotent matrices. Linear Algebra Appl. **142**, 43–54 (1990)

Perfect Semiring of Nonnegative Matrices

Adel Alahmedi, Yousef Alkhamees, and S.K. Jain

Abstract In this paper, it is shown that the semiring of nonnegative matrices satisfies descending chain condition on right and left ideals, i.e., it is left or right perfect if and only if it is closed under Drazin inverse of all elements. Furthermore, each nonnil right and left ideal contains a nonzero idempotent. This generalizes the known result on the characterization of finite semigroups of nonnegative matrices.

Keywords Perfect semiring · Artinian semiring · Nonnegative matrices · Drazin inverse

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1 Introduction

We consider semirings of nonnegative matrices that satisfy a descending chain condition on principal right or left ideals. Rings with this property are known as left or right perfect, respectively (see Bass [1]). It turns out that the condition of being right or left perfect on a semiring of nonnegative matrices is equivalent to having Drazin inverse of each element as [4] nonnegative. Thus, right or left perfect semiring becomes two-sided perfect. Using known results on the monotonicity of Drazin inverse (see Goel and Jain [5]), we give a complete description of the structure of the elements of this semiring.

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2 Definitions and Notation

Throughout, **R** will denote a semiring of $n \times n$ matrices over nonnegative real numbers. **R** is called right Artinian or right perfect respectively if every descending chain of right ideals or every descending chain of principal left ideals terminates after finite number of terms. One defines left Artinian semiring and left perfect semiring, similarly. For an element $a \in \mathbf{R}$, the index of a is defined as the smallest positive integer k such that $\operatorname{rank}(a^k) = \operatorname{rank}(a^{k+1})$. For an element $a \in \mathbf{R}$, there exists a unique solution x, called Drazin inverse of a [4], satisfying the following system of equations numbered in the usual notation: (1^n) $a^n x a = a^n$, (2) $xax = x$, (5) $xa = ax$, where $n \geq \operatorname{index}(a)$ (see p. 169, Ben-Israel and Greville [2]). We shall denote the Drazin inverse of a by $a^{(\mathrm{d})}$. It is known that the Drazin inverse of any matrix a over a field $\mathbb{F}$ is a polynomial in a over $\mathbb{F}$ (Theorem 5, p. 172, [2]).

The matrix $\beta x y^T$, where $\beta > 0$ and x, y are positive column vectors, is called a matrix of type (I). The $d \times d$ block matrix

$$\begin{pmatrix} 0 & \beta_{12}x_1y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\beta_{ij} > 0$, and x_i and y_j are positive column vectors, is called a matrix of type (II) (see, for example, Goel and Jain [5]).

For all definitions and results on generalized inverses, one may consult Ben-Israel and Greville [2].

3 Main Results

Theorem 1 *Let* **R** *be a semiring of all $n \times n$ nonnegative matrices. Then the following statements are equivalent*:

(i) **R** *is left perfect*.
(ii) *Each nonzero element in* **R** *has a nonnegative Drazin inverse*.
(iii) *Each element a of* **R** *is a sum $b_1 + b_2 + \cdots + b_r + t$, where $b_i, t \in \mathbf{R}$, $b_ib_j = 0$ for all $i \neq j$, $b_it = 0 = tb_i$, t is nilpotent, and there exist permutation matrices p_i such that*

$$p_ib_ip_i^T = \begin{pmatrix} G_i & G_iD_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_iG_i & C_iG_iD_i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where G_i is a matrix of type (*I*) *or* (*II*) *and C_i, $D_i \geq 0$.*

Remark 1 Under any of the equivalent statements of Theorem 1, the right perfect semiring in (i) will be two-sided perfect because the Drazin inverse is right–left symmetric.

Before we give the proof of Theorem 1, we prove two lemmas.

Lemma 1 *Let* $\mathbf{R}$ *be a left perfect semiring of nonnegative square matrices. Then for every* $a \in \mathbf{R}$, $a^{(\mathrm{d})}$ *exists in* $\mathbf{R}$.

Proof Consider the descending chain of right ideals

$$a\mathbf{R} \supset a^2\mathbf{R} \supset a^3\mathbf{R} \supset \cdots.$$

By hypothesis, there exists a positive integer n such that

$$a^n\mathbf{R} = a^{n+1}\mathbf{R} = a^{n+2}\mathbf{R} = \cdots.$$

This implies that $a^n \in a^{n+1}\mathbf{R}$. Without loss of generality we may assume that $n \geq \operatorname{index}(a)$. Now, $a^n = a^{n+1}x$ for some $x \in \mathbf{R}$. Premultiplying by $a^{(\mathrm{d})}$ on both sides, we obtain

$$a^{(\mathrm{d})}a^n = a^{(\mathrm{d})}a^{n+1}x = a^n x.$$

Again, premultiplying both sides by $a^{(\mathrm{d})}$, we get $(a^{(\mathrm{d})})^2 a^n = a^{(\mathrm{d})}a^n x = a^n x^2$. By repeatedly premultiplying with $a^{(\mathrm{d})}$, we get

$$\left(a^{(\mathrm{d})}\right)^n a^n = a^n x^n.$$

This implies $(a^{(\mathrm{d})}a)^n = a^n x^n$, and so

$$a^{(\mathrm{d})}a = a^n x^n.$$

By premultiplying the last equation by $a^{(\mathrm{d})}$ and using the fact that a and $a^{(\mathrm{d})}$ commute, we obtain $a^{(\mathrm{d})}aa^{(\mathrm{d})} = a^{(\mathrm{d})}a^n x^n$, yielding

$$a^{(\mathrm{d})} = a^{(\mathrm{d})}aa^{(\mathrm{d})} = a^n x x^n = a^n x^{n+1},$$

and thus $a^{(\mathrm{d})}$ $(= a^n x^{n+1})$ is nonnegative. This completes the proof. □

Remark 2 Under the notation in the proof of Lemma 1, we may note that $a^{(\mathrm{d})}a = a^n x^n = a^{n+1}x^{n+1} = \cdots$ is an idempotent.

Lemma 2 *Let* $\mathbf{R}$ *be a left perfect, and* $a \in \mathbf{R}$. *Then* $a^{(\mathrm{d})}a\mathbf{R} = a^{(\mathrm{d})}\mathbf{R} = a^k\mathbf{R}$, *where* k *is the index of* a.

In particular, every nonnil right ideal contains nonzero idempotent.

Proof From the definition of $a^{(\mathrm{d})}$ it is clear that $a^2a^{(\mathrm{d})}\mathbf{R} = aa^{(\mathrm{d})}\mathbf{R}$. Then $a\mathbf{R} \supset aa^{(\mathrm{d})}\mathbf{R} \supset a^{(\mathrm{d})}aa^{(\mathrm{d})}\mathbf{R} = a^{(\mathrm{d})}\mathbf{R}$, which implies

$$a^2\mathbf{R} \supset aa^{(\mathrm{d})}\mathbf{R} = a^{(\mathrm{d})}\mathbf{R}.$$

By repeatedly multiplying both sides by a, we obtain $a^k\mathbf{R} \supset a^{(\mathrm{d})}\mathbf{R}$. But $a^{(\mathrm{d})}\mathbf{R} \supset a^{(\mathrm{d})}a^{k+1}\mathbf{R} = a^k\mathbf{R}$. Therefore,

$$a^{(\mathrm{d})}a\mathbf{R} = a^{(\mathrm{d})}\mathbf{R} = a^k\mathbf{R}.$$

Since $aa^{(\mathrm{d})}$ is an idempotent, it follows that every right ideal contains an idempotent. Note that if a is nilpotent, then $a^{(\mathrm{d})} = 0$. This implies every nonnil right ideal contains a nonzero idempotent. This completes the proof. $\square$

We now proceed to prove the main theorem.

Proof (i) $\Rightarrow$ (ii): This follows from Lemma 1.

(ii) $\Rightarrow$ (i): Suppose that we have a descending chain of principal right ideals, $a\mathbf{R} \supset a^2\mathbf{R} \supset a^3\mathbf{R} \supset \cdots$. By axioms ($1^n$) for the Drazin inverse, $a^{n+1}a^{(\mathrm{d})} = a^n$. Since $a^{(\mathrm{d})} \in \mathbf{R}$, we get that $a^n \in a^{n+1}\mathbf{R}$. This yields $a^nR = a^{n+1}\mathbf{R}$, and thus the above descending chain terminates, proving (i).

(ii) $\Leftrightarrow$ (iii). This follows from Goel and Jain [5]. $\square$

4 Illustration

We illustrate the main theorem by choosing a simple example consisting of 2×2 nonnegative matrices. Firstly, note that any nonnegative nil element is of the form $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right)$, where a is any nonnegative real number. For structure of nonnegative nilpotent matrix of any order, one may refer to the paper by Jain and Goel [5]. Also, it is well known that nonnegative nonsingular matrices having nonnegative inverse are monomial matrices (see p. 69 in Berman and Plemmons [3]).

Following the structure of matrices as described in part (iii) of the theorem, the possible semirings of 2×2 matrices satisfying the hypotheses of the main theorem are the following semirings and semirings generated by appropriate elements belonging to these semirings:

$$\mathbf{R}_1 = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R}^+ \right\},$$

$$\mathbf{R}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \middle| a \in \mathbb{R}^+ \right\},$$

$$\mathbf{R}_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R}^+ \right\},$$

$$\mathbf{R}_4 = \left\{ \begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix} \middle| a \in \mathbb{R}^+ \right\},$$

and for a fixed nonnegative real number α,

$$\mathbf{R}_\alpha = \left\{ \begin{pmatrix} a & \alpha a \\ b & \alpha b \end{pmatrix} \middle| a, b \in \mathbb{R}^+ \right\}.$$

We remark that the semiring satisfying the hypothesis of the theorem cannot contain arbitrary elements in $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. For if it does, then we can produce an invertible matrix whose inverse is not nonnegative. We close with the following natural question:

Is the perfect semiring of $n \times n$ nonnegative matrices Artinian?

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References

1. Bass, H.: Finitistic dimension and homological generalization of semiprimary rings. Trans. Am. Math. Soc. **95**, 466–488 (1960)
2. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer, Heidelberg (2003)
3. Berman, A., Plemmons, R.L.: Nonnegative Matrices in Mathematical Sciences. SIAM, Philadelphia (1994)
4. Drazin, M.P.: Pseudo-inverses in associative rings and semigroups. Am. Math. Mon. **65**, 506–514 (1958)
5. Jain, S.K., Goel, V.K.: Nonnegative matrices having nonnegative Drazin pseudo-inverse. Linear Algebra Appl. **29**, 173–183 (1980)

Regular Matrices over an Incline

AR. Meenakshi

Abstract We discuss the invertibility of incline matrices over *DL*, the set of all idempotent elements in an incline, and for matrices over an integral incline. We discuss the regularity of matrices over *DL*. We obtain equivalent conditions for the existence of various generalized inverses of an incline matrix. We provide an algorithm for the regularity of matrices over *DL* and illustrate with suitable examples.

Keywords Incline · Regular incline · Integral incline

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1 Introduction

The concept of an incline, introduced by Cao, is an algebraic structure, a special type of a semiring. The notion of inclines and their applications are described comprehensively in Cao, Kim and Roush [1]. Kim and Roush [5] have surveyed and outlined algebraic properties of inclines and incline matrices. In Han and Li [2, 3], some invertible conditions of a matrix over a commutative incline R with additive identity "O_R" and multiplicative identity "1_R" are obtained. Invertible matrices over an incline and Cramer's rule are investigated in [3].

An element a in a semiring is said to be regular if a solution exists for the equation $aya = a$, and such a solution is called a generalized inverse (or g-inverse) of a. A semiring R is regular if and only if every element of R is regular. Recently, Meenakshi and Anbalagan [7], by using the incline axioms, obtained some characterizations of regular elements in an incline and proved that every commutative regular incline is a distributive lattice. In Meenakshi and Shakila Banu [8], we have obtained equivalent conditions for regularity of a matrix over an incline whose idempotent elements are linearly ordered. Recall that an incline matrix, that is, a matrix $A \in R_{mn}$, the set of $m \times n$ matrices over an incline R, is regular if and only if there

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exists $Y \in R_{nm}$ such that $AYA = A$, and such a Y is called a g-inverse of A (see Kim and Roush [5]). Regularity of matrices over a regular incline are discussed in Meenakshi and Shakila Banu [9]. For regularity of fuzzy matrices, that is, matrices over the max min fuzzy algebra F with the operations defined as $a + b = \max\{a, b\}$ and $a.b = \min\{a, b\}$ for all a, b in F and their g-inverses, one may refer to Kim and Roush [4] and Meenakshi [6].

In this paper, we discuss the invertibility of an incline matrix over *DL*, the set of idempotent elements of an incline R. We obtain equivalent conditions for an incline matrix over *DL* to be regular as a generalization of the results found in Meenakshi and Shakila Banu [8] for matrices over a regular incline and those of fuzzy matrices established by Kim and Roush [4]. We discuss the existence of various g-inverses of a matrix over *DL*. We provide an algorithm for the regularity of a matrix over *DL* as a generalization of algorithms available for matrices over a regular incline [9] and for matrices over a regular incline whose elements are linearly ordered (Kim and Roush [4, 5]).

2 Preliminaries

In this section, we present basic definitions and required results on incline matrices.

Definition 1 An incline is a nonempty set R with binary operations addition and multiplication denoted as $(+, \bullet)$ and satisfying the following (we usually suppress the dot $\bullet$ for multiplication) for $a, b, c \in R$:

(i) $a + b = b + a$;
(ii) $a + (b + c) = (a + b) + c$;
(iii) $a(bc) = (ab)c$;
(iv) $a(b + c) = ab + ac$;
(v) $(b + c)a = ba + ca$;
(vi) $a + a = a$;
(vii) $a + ac = a$;
(viii) $c + ac = c$.

In an incline $(R, +, \bullet)$ with order relation $\leq$ defined on R as $x \leq y$ if and only if $x + y = y$ for $x, y \in R$, by the incline axioms $x + xy = x$ and $y + xy = y$ we get $xy \leq x$ and $xy \leq y$. Thus, inclines are additively idempotent semirings in which products are less than or equal to either factor. This incline order relation has the following properties:

(P.1) $x + y \geq x$ and $x + y \geq y$ for $x, y \in R$;
(P.2) $xy \leq x$ and $xy \leq y$ for $x, y \in R$.

Definition 2 For $x, y \in R$, if $x \leq y$ for all $y \in R$, then x is called the least element of R and is denoted as 0. If $x \geq y$ for all $y \in R$, then x is called the greatest element of R and denoted as 1. However, 1 need not be the multiplicative identity I_R (refer the example in Remark 7).

Definition 3 (Cao et al. [1]) $A \in R_{mn}$ is regular $\Leftrightarrow$ $AYA = A$ for some $Y \in R_{nm}$.

In the sequel, we shall repeatedly use the following:

Lemma 1 (Meenakshi and Anbalagan [7]) *Let R be an incline. Then the following are equivalent*:

(i) *R is a regular incline*;
(ii) *Every element of R is regular*;
(iii) *Every element of R is idempotent.*

Let R_{mn} denote the set of all $m \times n$ matrices over an incline R. Let R_m denote the set of all $m \times m$ matrices over an incline R. For $A \in R_{mn}$, let A^T, A^-, A_{i*}, A_{*j}, $\mathscr{R}(A)$, and $\mathscr{C}(A)$ denote the transpose, g-inverse, ith row, jth column, row space, and column space of A, respectively. We shall follow the basic operations on matrices over R induced by the incline operations of R as in [1]. Let DL be the set of all idempotent elements of R. DL_{mn} denotes the set of $m \times n$ matrices over DL.

Lemma 2 *Let R be any semiring. For $A, B \in R_{mn}$, we have the following*:

(i) $\mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow B = XA$ *for some* $X \in R_m$;
(ii) $\mathscr{C}(B) \subseteq \mathscr{C}(A) \Leftrightarrow B = AY$ *for some* $Y \in R_n$.

Lemma 3 (Meenakshi et al. [9]) *Let R be an incline. If for $A, B \in R_{mn}$, $\mathscr{R}(A) = \mathscr{R}(B)$ (or) $\mathscr{C}(A) = \mathscr{C}(B)$, then A is a regular matrix $\Leftrightarrow$ B is a regular matrix.*

Theorem 1 (Meenakshi et al. [9]) *Let $A \in R_{mn}$.*

(i) *If $\mathscr{R}(A) = \mathscr{R}(A^T A)$, then A is a regular matrix $\Leftrightarrow$ $A^T A$ is a regular matrix.*
(ii) *If $\mathscr{C}(A) = \mathscr{C}(AA^T)$, then A is a regular matrix $\Leftrightarrow$ AA^T is a regular matrix.*

3 Invertible Matrices over an Incline

In this section, let us consider an incline R with least element 0, greatest element 1, and having no multiplicative identity. For, if R has the multiplicative identity I_R, then by Definition 2, $I_R \leq 1$, and by incline property (P.2) $1 = 1 \cdot I_R \leq 1_R$. Hence, I_R coincides with the greatest element. In general, the greatest element 1 is not idempotent (refer Remark 7). Here, we shall use the following results proved in our earlier work.

Lemma 4 (Meenakshi et al. [8]) *Let R be an incline with* 0 *and* 1. *Then* 1 *is the multiplicative identity for elements of DL.*

Remark 1 In particular, for a regular incline, by Lemma 1, $DL = R$, and by Lemma 4, 1 is the multiplicative identity of R.

Lemma 5 *Let R be a regular incline whose elements are all linearly ordered. Then, for $x, y \in R$, $x \leq y \Leftrightarrow x + y = y \Leftrightarrow xy = x$.*

Remark 2 From the above Lemma 5 it follows that, for a regular incline R whose elements are all linearly ordered, the incline operations reduce to the max min operations, and hence, $R = \{[0, 1], \max\{x, y\}, \min\{x, y\}\}$ is the max min fuzzy algebra.

Definition 4 Let R be an incline with 0 and 1. $A \in DL_n$ is invertible over DL if and only if there exists $X \in DL_n$ such that $AX = XA = I_n$, where I_n is the $n \times n$ matrix whose diagonal entries are the greatest element 1 and the remaining entries are all 0.

Remark 3 In particular, for a regular incline, by Lemma 1, Definition 4 reduces to the invertible matrices over an incline (Han and Li [2, 3]).

Theorem 2 (Meenakshi et al. [8], Theorem 3.20) *Let R be an incline with 0 and 1. Then $A \in DL_n$ is invertible over $DL \Leftrightarrow AA^T = A^T A = I_n$.*

Corollary 1 *Let R be an incline with 0 and 1. If the elements of DL are linearly ordered, then DL is an integral incline, and $A \in DL_n$ is invertible over $DL \Leftrightarrow AA^T = A^T A = I_n$.*

Proof Since elements of DL are linearly ordered, for any pair of nonzero elements $x, y \in DL$, either $x \leq y$ (or) $y \leq x$. Hence, either $x + y = y$ (or) $x + y = x$. We claim that (x, y) is not an integral pair. For, if it is an integral pair, then $x + y = 1$ and $xy = 0$, which implies that either $y = 1$ (or) $x = 1$. Since $x, y \in DL$, by Lemma 4, $x \cdot 1 = x$ and $y \cdot 1 = y$. Substituting $y = 1$ (or) $x = 1$ into $xy = 0$, we get $x = 0$ (or) $y = 0$, which is a contraction, and hence DL has no integral pair. Therefore, DL is an integral incline. The rest follows from Theorem 2. □

Corollary 2 (Meenakshi et al. [8], Corollary 3.38) *Let R be a regular incline whose elements are linearly ordered. Then $A \in R_n$ is invertible $\Leftrightarrow AA^T = A^T A = I_n \Leftrightarrow A$ is a permutation matrix.*

Remark 4 From Remark 2 we observe that Corollary 2 reduces to the result that "a fuzzy matrix is invertible if and only if it is a permutation matrix" (see Kim and Roush [4]).

Definition 5 Let R be an incline whose idempotent elements are linearly ordered. For $A \in DL_{mn}$ and $\alpha \in DL$, the zero pattern A_α of A is defined as

$$[A_\alpha]_{ij} = \begin{cases} 1 & \text{if } a_{ij} \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let ϕ_A denote the set of all nonzero entries of A. Then for each $\alpha \in \phi_A$, the zero pattern A_α of A is the matrix whose entries are 0 and 1.

Remark 5 If $1 \in DL$, then $A_\alpha \in DL_{mn}$. If not, then by Lemma 4, $\alpha A_\alpha \in DL_{mn}$ for each $\alpha \in \phi_A$. If R is a regular incline, then by Lemma 5, Definition 5 reduces to the zero patterns of a fuzzy matrix defined in [4].

Theorem 3 *Let R be an incline whose idempotent elements are linearly ordered. Let $A \in DL_{mn}$, and ϕ_A be the set of all nonzero entries of A. Then $A = \sum_{\alpha \in \phi_A} \alpha A_\alpha$.*

Proof Let $A = (a_{ij}) \in DL_{mn}$ and $B = \sum_{\alpha \in \phi_A} \alpha A_\alpha$. By Remark 5, it follows that $B = (b_{ij}) \in DL_{mn}$. It is enough to show that $a_{ij} = b_{ij}$ for all i, j. Since $a_{ij} \in \phi_A$, a_{ij} is the ijth entry of αA_α for some α, one of the summands of B,

$$\begin{aligned}
b_{ij} &= \left(\sum_{\alpha \in \phi_A} \alpha \cdot A_\alpha \right) \\
&= \sum_{\alpha \in \phi_A} ij\text{th entry of } (\alpha \cdot A_\alpha) \\
&= \sum_{\alpha \in \phi_A} \alpha \cdot ij\text{th entry of } A_\alpha \quad \text{(by Definition 5)} \\
&= \sum_{\alpha \in \phi_A} \alpha \cdot (0 \text{ or } 1) \quad \text{(by Lemma 4, } \alpha.1 = \alpha \in DL) \\
&= a_{ij}.
\end{aligned}$$

Thus $A = \sum_{\alpha \in \phi_A} \alpha \cdot A_\alpha$. Hence the theorem. □

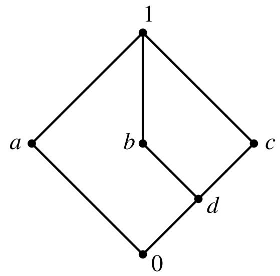
Theorem 4 *Let R be an incline whose idempotent elements are linearly ordered. Let $A, B \in DL_n$ and $\alpha \in DL$, then $(AB)_\alpha = A_\alpha \cdot B_\alpha$.*

Proof Let $A = (a_{ij}) \in DL_n$ and $B = (b_{ij}) \in DL_n$. Then

$$\begin{aligned}
\left[(AB)_\alpha\right]_{ij} = 1 \quad &\Longleftrightarrow \quad (AB)_{ij} \geq \alpha \quad \text{(by Definition 5)} \\
&\Longleftrightarrow \quad \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \geq \alpha \\
&\Longleftrightarrow \quad a_{ik} \cdot b_{kj} \geq \alpha \quad \text{for at least one } k \quad \text{(by P.1)} \\
&\Longleftrightarrow \quad a_{ir} \cdot b_{rj} \geq \alpha \quad \text{(for } k = r\text{, say)} \\
&\Longleftrightarrow \quad a_{ir} \geq \alpha \text{ and } b_{rj} \geq \alpha \quad \text{(by P.2)} \\
&\Longleftrightarrow \quad [A_\alpha]_{ir} = 1 \text{ and } [B_\alpha]_{rj} = 1 \quad \text{(by Definition 5)} \\
&\Longleftrightarrow \quad \sum_{k=1}^{n} (A_\alpha)_{ik} (B_\alpha)_{kj} = 1 \\
&\Longleftrightarrow \quad (A_\alpha B_\alpha)_{ij} = 1.
\end{aligned}$$

Thus, $(AB)_\alpha = A_\alpha B_\alpha$. Hence the theorem. □

Fig. 1 Hasse graph



Corollary 3 *Let R be an incline whose idempotent elements are linearly ordered. If $1 \in DL$, then $A \in DL_n$ is invertible over $DL \Leftrightarrow A_\alpha \in DL_n$ is invertible over DL for each $\alpha \in \phi_A$.*

Proof Note that

$$\begin{aligned} A \text{ is invertible over } DL \quad &\Longleftrightarrow \quad AA^T = A^T A = I_n \quad \text{(by Theorem 2)} \\ &\Longleftrightarrow \quad \left(AA^T\right)_\alpha = \left(A^T A\right)_\alpha = (I_n)_\alpha \quad \text{for each } \alpha \in \phi_A \\ &\Longleftrightarrow \quad A_\alpha A_\alpha^T = A_\alpha^T A_\alpha = I_n \\ &\qquad\quad \text{(by Theorem 4 and by Remark 5)} \\ &\Longleftrightarrow \quad A_\alpha \text{ is invertible over } DL \text{ for each } \alpha \in \phi_A \\ &\qquad\quad \text{(by Definition 4)} \end{aligned}$$

Hence the corollary. □

Remark 6 In particular, for a regular incline, by Remark 2, Theorems 3 and 4 and Corollary 1 reduce to the results of Kim and Roush [4] and proved in P. 41 in Meenakshi [6].

Theorem 5 *Let R be an incline whose idempotent elements are linearly ordered and $1 \in DL$. If $A \in DL_{mn}$ is regular over DL, then $A_\alpha \in DL_{mn}$ is regular over DL for each $\alpha \in \phi_A$.*

Proof Since $A \in DL_{mn}$ is regular over DL, there exists $X \in DL_{mn}$ such that $AXA = A$. For each $\alpha \in \phi_A$, $(AXA)_\alpha = A_\alpha$. By Theorem 4, $A_\alpha X_\alpha A_\alpha = A_\alpha$. By using $1 \in DL$, we get $A_\alpha \in DL_{mn}$ and $X_\alpha \in DL_{mn}$ for each $\alpha \in \phi_A$. Thus, A_α is regular over DL. □

Remark 7 We observe that both the conditions on DL are essential. Let us consider the incline $R = \{0, a, b, c, d, 1\}$ lattice-ordered by Hasse graph as given in Fig. 1.

Define $R \cdot R \to R$ as

$$xy = \begin{cases} d & \text{if } x, y \in \{b, c, d, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this incline, for $x, y \in R$, $xy = 0$ (or) d, $DL = \{0, d\}$, the elements of DL are linearly ordered. R is not regular by Lemma 1 and $1 \notin DL$. For $A = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \in DL_2$, $AA^T = A^T A \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

A is not invertible over DL. Here, $A^3 = A$. Hence, A is regular over DL.

For $0 \neq d \in \phi_A$, $A_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin DL_2$. By Lemma 4, $1 \cdot d = d \cdot 1 = d$, and hence, there exists no $Y \in R_2$ such that $A_d Y A_d = A_d$. Hence, A_d is not regular. Thus, the conclusions of Theorem 5 do not hold.

Remark 8 The condition $1 \in DL$ need not imply that R is a regular incline. Let us consider $R = \{[0, 1], \sup\{x, y\}, \bullet\}$ under usual multiplication of real numbers. Here $1 \in R$ is the multiplicative identity and the greatest element of R, and $DL = \{0, 1\} \neq R$. Hence, by Lemma 1, R is not regular.

4 Generalized Inverses of Matrices over an Incline

In this section, we discuss the existence and construction of various inverses associated with an incline matrix.

Definition 6 For $A \in R_{mn}$, consider the following four equations:

1. $AXA = A$;
2. $XAX = X$;
3. $(AX)^T = AX$;
4. $(XA)^T = XA$.

$X \in R_{nm}$ is said to be a λ-inverse of A and $X \in A\{\lambda\}$ if X-satisfies λ-equation where λ is a subset of $\{1, 2, 3, 4\}$. In particular, if $\lambda = \{1, 2, 3, 4\}$, then X is called the Moore–Penrose inverse of A and denoted $A^\dagger$.

Remark 9 For $A \in R_{mn}$, $X \in A\{1, 3\} \Leftrightarrow X^T \in A^T\{1, 4\}$.

Lemma 6 *For $A \in DL_{mn}$, $AA^T A \geq A$.*

Proof Let $A = (a_{ij}) \in DL_{mn}$. Then the ipth element of $AA^T A = (\sum_j (\sum_k a_{ik} a_{jk}) \cdot a_{jp})$. By incline property (P.1) this expression is greater than or equal to each term in the summation. Hence, for $k = p$ and $j = i$, we have that the ipth element of $AA^T A \geq a_{ik} \cdot a_{ik} \cdot a_{ip} = a_{ip}^3 = a_{ip}$, the ipth element of A, by using $a_{ij} \in DL$. Hence, $AA^T A \geq A$. □

Lemma 7 *For $A \in DL_{mn}$, $AA^T A = A \Leftrightarrow A^\dagger$ exists and equals A^T.*

Proof The existence of $A^\dagger$ directly follows from the fact that A^T satisfies the defining equations in Definition 6. The converse is trivial.

Just for sake of completeness, we shall state a part of Theorem 7 of Meenakshi and Shakila Banu [8], and applying it, we discuss the existence of various g-inverses of $A \in DL_{mn}$. □

Theorem 6 *Let R be an incline whose idempotent elements are linearly ordered and form a vector space over R. For $A \in DL_{mn}$, the following statements are equivalent*:

(i) *A is regular.*
(ii) *There exists an idempotent matrix $E \in R_n$ such that $\mathscr{R}(E) = \mathscr{R}(A) = (DL^n)E$.*
(iii) *There exists an idempotent matrix $F \in R_n$ such that $\mathscr{C}(A) = \mathscr{C}(F) = (DL^m)F$.*
In either case, row rank(A) = column rank(A).

Theorem 7 *Let R be an incline whose idempotent elements are linearly ordered. For $A \in DL_{mn}$, A has a $\{1,3\}$ inverse $\Leftrightarrow$ A^TA is a regular matrix and $\mathscr{R}(A^TA) = \mathscr{R}(A)$.*

Proof Let X be a $\{1,3\}$ inverse of A. Then by Definition 6, $AXA = A$ and $(AX)^T = AX$. Hence, $A^TA = A^TAXA$ and $A = AXA = (AX)^TA = X^TA^TA$. Then by Lemma 2, $\mathscr{R}(A) \subseteq \mathscr{R}(A^TA)$ and $\mathscr{R}(A^TA) \subseteq \mathscr{R}(A)$. Hence, $\mathscr{R}(A) = \mathscr{R}(A^TA)$. Since A has a $\{1,3\}$ inverse, this automatically implies that A has a $\{1\}$ inverse. Then if A is regular and $\mathscr{R}(A) = \mathscr{R}(A^TA)$, then A^TA is regular by Theorem 1(i). Conversely, if A^TA is a regular matrix and $\mathscr{R}(A) = \mathscr{R}(A^TA)$, then A is a regular matrix by Theorem 1(i). Since A^TA is regular, let us define $Y = (A^TA)^-A^T$ for some g-inverse of A^TA. Further, by Lemma 2,

$$\mathscr{R}(A) \subseteq \mathscr{R}\big(A^TA\big) \implies A = XA^TA \text{ for some } X \in R_m, \tag{4.1}$$

$$\begin{aligned} AY &= XA^TA\big(A^TA\big)^-A^T \\ &= XA^TA\big(A^TA\big)^-A^TAX^T \\ &= X\big(A^TA\big)\big(A^TA\big)^-\big(A^TA\big)X^T \\ &= XA^TAX^T \\ &= XA^T. \end{aligned} \tag{4.2}$$

Now post multiplying by A and using (4.1), we get $AYA = XA^TA = A$. Hence, $Y \in A\{1\}$, and $(AY)^T = (XA^T)^T = AX^T = XA^TAX^T = XA^T = AY$ by (4.2). Therefore, $Y \in A\{3\}$. Thus, A has a $\{1,3\}$ inverse. □

Theorem 8 *Let R be an incline whose idempotent elements are linearly ordered. For $A \in DL_{mn}$, A has a $\{1,4\}$ inverse $\Leftrightarrow$ AA^T is regular and $\mathscr{C}(AA^T) = \mathscr{C}(A)$.*

Proof Note that

$$\begin{aligned} A \text{ has a } \{1,4\} \text{ inverse} &\iff A^T \text{ has a } \{1,3\} \text{ inverse} \quad \text{(by Remark 9)} \\ &\iff AA^T \text{ is regular, and } \mathscr{R}\big(A^T\big) = \mathscr{R}\big(AA^T\big) \end{aligned}$$

$$\begin{aligned}&\text{(by Theorem 7)}\\ \iff\quad & AA^T \text{ is regular, and } \mathscr{C}(A)=\mathscr{C}\big(AA^T\big).\end{aligned}$$

Hence the theorem. □

Since for an incline R whose idempotent elements are linearly ordered and form a vector space over R, every finite subspace of DL^n has a unique standard basis, see Cao et al. [1]. Further, by Lemma 4, for $\alpha, \beta \in DL$, $\alpha \leq \beta \Leftrightarrow \alpha+\beta=\beta \Leftrightarrow \alpha\beta=\alpha$, which is precisely the max–min compositions in DL. Hence, equivalent conditions for the existence of $\{1,3\}$ ($\{1,4\}$) inverses involving row (column) basis vectors proved in Theorem 3.16 of Kim and Roush [4] (quoted in P. 99 of Meenakshi [6]) for fuzzy matrices remain valid for matrices over DL. Hence, we state the following theorems without proofs.

Theorem 9 *Let R be an incline whose idempotent elements are linearly ordered and form a vector space over R. For $A \in DL_{mn}$, the following statements are equivalent*:

(i) *A has a $\{1,3\}$ inverse.*
(ii) *For any two row basis vectors A_{i*} and A_{j*}, $xA_{i*}=xA_{j*}$ for some $x \in R$.*

Theorem 10 *Let R be an incline, whose idempotent elements are linearly ordered and form a vector space over R, then the following statements are equivalent*:

(i) *A has a $\{1,4\}$ inverse.*
(ii) *For any two column basis vectors A_{*i} and A_{*j}, $xA_{*i}=xA_{*j}$ for some $x \in R$.*

These theorems lead to the formula for the computation of the Moore–Penrose inverse of a matrix over DL.

Corollary 4 *Let R be an incline whose idempotent elements are linearly ordered and form a vector space over R. Then, $A^{(1,4)}AA^{(1,3)}=A^{\dagger}$ for some $A^{(1,4)} \in A\{1,4\}$ and $A^{(1,3)} \in A\{1,3\}$.*

Proof This follows directly by verifying the defining equations of the Moore–Penrose inverse in Definition 6. □

Theorem 11 *Let R be an incline whose idempotent elements are linearly ordered and form a vector space over R. For $A \in DL_{mn}$, the following statements are equivalent*:

(i) *AA^T and A^TA are regular, $\mathscr{R}(A)=\mathscr{R}(A^TA)$, and $\mathscr{C}(A)=\mathscr{C}(AA^T)$.*
(ii) *A has a $\{1,3\}$ inverse and $\{1,4\}$ inverse.*
(iii) *For any two row basis vectors A_{i*} and A_{j*} and any two column basis vectors A_{*i} and A_{*j}, $A_{i*}A_{j*}=xA_{i*}=xA_{j*}$ and $A_{*i}A_{*j}=yA_{*i}=yA_{*j}$ for $x, y \in R$.*
(iv) *A^T is a g-inverse of A.*

(v) *Each zero pattern of* A *has a Moore–Penrose inverse.*
(vi) $A^{\dagger}$ *exists and equals* A^T.

Proof (i) $\Leftrightarrow$ (ii): This follows directly from Theorems 7 and 8.

(ii) $\Leftrightarrow$ (iii): This follows from Theorems 9 and 10.

(ii) $\Leftrightarrow$ (vi): (ii) $\Rightarrow$ (vi) follows by Corollary 4, and the converse is trivial.

(iv) $\Leftrightarrow$ (vi): This equivalence is precisely Lemma 7.

(iv) $\Leftrightarrow$ (v): If each zero pattern A_α of A has a Moore–Penrose inverse, then by Lemma 7, $A_\alpha^{\dagger} = A_\alpha^T$. Now by Theorem 3, $A^T = (\sum_{\alpha \in \phi_A} \alpha A_\alpha)^T = \sum_{\alpha \in \phi_A} \alpha A_\alpha^T = \sum_{\alpha \in \phi_A} \alpha A_\alpha^{\dagger} = A^{\dagger}$. Thus, $A^T = A^{\dagger}$ is the Moore–Penrose inverse of A, and hence A^T is a g-inverse of A. □

Example 1 Let us consider the incline $R = [0, 1]$ under addition as supremum and usual multiplication of real numbers. Here, $DL = \{0, 1\}$, and the elements of DL are linearly ordered and form a vector space over R.

Example 2 For the incline R in Remark 7, $DL = \{0, d\}$. The elements of DL are linearly ordered. Since the product of any two elements of R is 0 (or) d, DL does not form a vector space over R.

5 Algorithm

In Meenaksi and Shakila Banu [9], we have given an algorithm for regularity of a matrix over a regular incline. Here, we observe that the same algorithm remains valid for the regularity of a matrix over DL whose elements are linearly ordered and form a vector space over R.

6 Conclusion

The main results in the present paper are generalization of the results on regular fuzzy matrices, construction of generalized inverse of fuzzy matrices found in Kim and Roush [4], and generalization of regular matrices over a regular incline in Meenaksi and Shakila Banu [8].

References

1. Cao, Z.Q., Kim, K.H., Roush, F.W.: Incline Algebra and Applications. Wiley, New York (1984)
2. Han, S.C., Li, H.X.: Some conditions for matrices over an incline to be invertible and general linear group on an incline. Acta Math. Sin. **21**, 1093–1098 (2005)
3. Han, S.C., Li, H.X.: Invertible incline matrices and Cramer's rule over inclines. Linear Algebra Appl. **389**, 121–138 (2004)

4. Kim, K.H., Roush, F.W.: Generalized fuzzy matrices. Fuzzy Sets Syst. **4**, 293–315 (1980)
5. Kim, K.H., Roush, F.W.: Inclines and incline matrices: a survey. Linear Algebra Appl. **379**, 457–473 (2004)
6. Meenakshi, AR.: Fuzzy Matrices: Theory and Applications. MJP Publishers, Chennai (2008)
7. Meenakshi, AR., Anbalagan, A.: On regular elements in an incline. Int. J. Math. Math. Sci. (2010). doi:101155/2010/903063
8. Meenakshi, AR., Shakila Banu, P.: On regularity of incline matrices. Int J. Algebra. **5**, 909–924 (2011)
9. Meenakshi, AR., Shakila Banu, P.: Matrices over a regular incline. Int. J. Comp. Cogn. **9**(4), 1–5 (2011)

Matrix Partial Orders Associated with Space Preorder

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Abstract In this expository article, we discuss some fundamentals of well-known matrix partial orders that are closely associated with space preorder on rectangular matrices. Particularly, we consider partial order defined by space decomposition, star ordering, and minus partial order for our discussion. These relations are closely associated with comparison of column spaces and row spaces of matrices. Results associated with selected matrix relation that are known in the literature along with some interesting observations are put together. At many places, though the proofs of several results are known in the past literature, by part or completely, for better reading purpose, independent proofs are provided.

Keywords Generalized inverse · Partial order · Preorder · Star partial order · Space decomposition · Minus partial order

Mathematics Subject Classification (2010) 15A09

1 Preliminaries

In this section, we provide preliminaries such as notation, definitions, and basic results required on matrices, generalized inverse, and partial order on a set.

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1.1 Matrices and Generalized Inverses

Let $\mathbb{C}, \mathbb{R}, \mathbb{F}$ denote the complex, real, and arbitrary fields, respectively. We use the notation $\mathbb{K}$ for a scalar field when the concerned scalar field can be either $\mathbb{C}$ or $\mathbb{R}$ by choice. Let $\mathbb{F}^n$ denote the vector space of all n-tuples (usually written columnwise) with entries from $\mathbb{F}$. $\mathbb{F}^{m\times n}$ denotes the set of all $m \times n$ matrices, which is also a vector space over $\mathbb{F}$. The dimension of a vector space $\mathscr{V}$ is denoted by $\mathscr{D}(\mathscr{V})$. The class of all matrices defined over $\mathbb{F}$ is denoted by Mat($\mathbb{F}$).

Given two vector spaces $\mathscr{V}$ and $\mathscr{W}$, a mapping

$$T : \mathscr{V} \to \mathscr{W}$$

is called a *linear mapping* if T is additive ($T(v + w) = T(v) + T(w)\ \forall\, v, w \in \mathscr{V}$) and preserves scalar multiplication ($T(\lambda v) = \lambda T(v)$ for all scalars λ and $v \in \mathscr{V}$). An additive mapping T on a complex vector space is called *conjugate linear* if $T(\lambda v) = \bar{\lambda} T(v)$.

$A \in \mathbb{F}^{m\times n}$ can also be treated as a linear transformation

$$A : \mathbb{F}^n \to \mathbb{F}^m.$$

The matrix denoted by A' (A^* in case $\mathbb{F} = \mathbb{C}$) is an $n \times m$ matrix such that the (i, j)th entry of A' equals the (j, i)th entry of A (conjugate of the (j, i)th entry in the complex case). The (i, j)th entry of a matrix A is generally denoted by A_{ij}. A' is called the *transpose* of A, and A^* in the case of complex numbers is called *conjugate transpose* of A. A complex (real) matrix is said to be Hermitian (symmetric) if $A = A^*$ ($A = A'$). The class of all $n \times n$ Hermitian matrices is denoted by $\mathbb{H}_n$.

The vector space spanned by the columns of A is called the *column space* (*column span*) of A and denoted by $\mathscr{C}(A)$. In general, the *row space* (*row span*) of a matrix A considered to be the subspace generated by rows of A, which is same as $\mathscr{C}(A')$. But in the case of complex numbers, the row space of A is $\mathscr{C}(A^*)$. The row space of A is denoted by $\mathscr{R}(A)$. The dimension of column space (which is incidentally the same as the dimension of row space) is called the *rank* of A and denoted by $\rho(A)$. The null space (kernel) of a matrix A, denoted by $\mathscr{K}(A)$, is the subspace $\{x \in \mathbb{F}^n : Ax = 0\}$. The dimension of $\mathscr{K}(A)$ is often referred to as *nullity* of A.

A square matrix A of size n is said to be invertible if there exists a matrix B such that $AB = I$, the identity matrix of size n. Such inverse always exists uniquely for a matrix of full rank and is denoted by A^{-1}. Given a matrix $A \in \mathbb{F}^{m\times n}$, we consider the following matrix equations:

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

The set of matrix equations (1)–(4) above is called the Moore–Penrose equations, and a matrix X satisfying all the four equations is called the Moore–Penrose inverse of A (see Rao and Mitra [27]). A matrix X satisfying condition (1) is called a generalized inverse, 1-inverse, g-inverse, or sometimes an inner inverse of A. An arbitrary g-inverse is denoted by A^-. Similarly, a matrix X satisfying (2) is called a 2-inverse or outer inverse of A. An arbitrary outer inverse is denoted by $A^=$. A matrix X satisfying both (1) and (2) is called a reflexive generalized inverse or $(1, 2)$-inverse of A. The Moore–Penrose inverse, which is unique when it exists, is denoted by A^+. Whenever $m = n$, i.e., for square matrices, we shall consider the following additional conditions:

$$AX = XA, \tag{5}$$

$$XA^{k+1} = A^k. \tag{1^k}$$

A square matrix X satisfying (1), (2), and (5) is called a group inverse (denoted by $A^\#$) of A, and X satisfying (2), (5), and (1^k) for some integer k is called a Drazin inverse of A (denoted by $A^\dagger$).

For subspaces $\mathscr{U}, \mathscr{V}$, the sum $\mathscr{W} = \mathscr{U} + \mathscr{V}$ is said to be a direct sum, and we write $\mathscr{W} = \mathscr{U} \oplus \mathscr{V}$ if $\mathscr{U} \cap \mathscr{V} = (0)$. In such a case, $\mathscr{U} \oplus \mathscr{V}$ is said to be a space decomposition of $\mathscr{W}$.

The concept of disjoint matrices given in the following definition is very useful.

Definition 1 (Disjoint matrices, Mitra [11]) Given $B, C \in \mathbb{C}^{m\times n}$ are said to be *disjoint* matrices if $\mathscr{C}(B) \cap \mathscr{C}(C) = (0)$ and $\mathscr{R}(B) \cap \mathscr{R}(C) = (0)$.

Let $\mathscr{L}_1$ be a subspace of a linear space $\mathscr{L}$. Let $\mathscr{L}_2$ be another subspace such that $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$. In fact, there are infinitely many choices for such a subspace $\mathscr{L}_2$ whenever $(0) \neq \mathscr{L}_1 \neq \mathscr{L}$. Every $x \in \mathscr{L}$ can be uniquely expressed as $x = x_1 + x_2$, where $x_1 \in \mathscr{L}_1$ and $x_2 \in \mathscr{L}_2$. Clearly, the mapping $P : x \to x_1$ is a linear mapping onto $\mathscr{L}_1$. Since $\mathscr{L}_1 \oplus \mathscr{L}_2 = \mathscr{L}$, we see that $P^2 = P$, i.e., P is an idempotent linear transformation with range $\mathscr{L}_1$ and kernel $\mathscr{L}_2$. Such an idempotent linear transformation is called *a projector* onto $\mathscr{L}_1$ along $\mathscr{L}_2$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are orthogonal to each other, then the projector P is called an orthogonal projector onto $\mathscr{L}_1$.

We refer to Ben-Israel and Greville [5] and Rao and Mitra [27] for all the preliminaries regarding matrices and generalized inverses. For the further reading on the initial developments on generalized inverses, one may refer to Moore [22], Penrose [24] and Rao [25, 26].

1.2 Partial Order on a Set

A relation $\sim$ on a set S is said to be:

(i) *reflexive* if $x \sim x \ \ \forall x \in S$,
(ii) *symmetric* if $x \sim y \Rightarrow y \sim x, \ \ x, y \in S$,

(iii) *antisymmetric* if $x \sim y$ and $y \sim x \Rightarrow x = y$,
(iv) *transitive* if $x \sim y$, $y \sim z \Rightarrow x \sim z$ for $x, y, z \in S$.

A relation $\sim$ on S is said to be an *equivalence relation* if it is reflexive, symmetric, and transitive.

A relation which is reflexive and transitive is called a *preorder* on S. A preorder that is also an antisymmetric is called a *partial order.*

Let $\leq$ be a partial order on S. We say that $x, y \in S$ are comparable if $x \leq y$ or $y \leq x$. A partial order $\leq$ is said to be a *total order on* S if all elements $x, y \in S$ are comparable.

A set S together with partial order (preorder; total order) $\leq$ is called a *partially ordered* (preordered; ordered) set.

Let $(S, \leq)$ be a partially ordered set or, shortly, a poset. If S_1 is a subset of S, $\leq$ induces a trivial partial ordering on S_1. A subset S_1 of S is called a *chain* if S_1 is a totally ordered set with respect to the induced partial order. If $a \leq b$ in S, an interval between these elements, generally denoted by $[a, b]$, is the subset of all elements z such that $a \leq z \leq b$.

Given any two partially ordered sets $(S, \leq_s)$ and $(T, \leq_t)$, a mapping

$$f : S \to T$$

is said to be:

(i) *order preserving* if for $x, y \in S$,

$$x \leq_s y \quad \Longrightarrow \quad f(x) \leq_t f(y);$$

(ii) *order reversing* if for $x, y \in S$,

$$x \leq_s y \quad \Longrightarrow \quad f(y) \leq_t f(x).$$

Theorem 1 *Let $(S, \leq_s)$ be a partially ordered set, and U be any set. Given a mapping $f : U \to S$, consider the induced relation $\leq_f$ defined on U such that for $u_1, u_2 \in U$,*

$$u_1 \leq_f u_2 \quad \text{if } f(u_1) \leq_s f(u_2). \tag{6}$$

Then:

(i) *$\leq_f$ is a preorder on U.*
(ii) *$\leq_f$ is a partial order on U if and only if f is a one-to-one mapping.*

A partial order $\leq_1$ on a set S_1 is said to be an *extended partial order* of a partial order $\leq_2$ on S_2 if $S_2 \subseteq S_1$ and for $a, b \in S_2$,

$$b \leq_1 a \quad \Longleftrightarrow \quad b \leq_2 a.$$

A partial order $\leq_2$ on a set S_2 is said to be *dominated* by a partial order $\leq_1$ defined on S_1 if $S_2 \subseteq S_1$ and for $a, b \in S_2$,

$$b \leq_2 a \quad \Longrightarrow \quad b \leq_1 a.$$

In such a case, often we say that $\leq_1$ is a dominant partial order with reference to other partial order $\leq_2$.

Let $(S, \leq)$ be a partially ordered set, and $S_1 \subseteq S$. An element $x \in S_1$ is said to be a *maximal* (*minimal*) element in S_1 if from $x \leq y$ $(y \leq x)$ for any $y \in S_1$ it follows that $x = y$. An element $z \in S$ is said to be a *lower bound* (an *upper bound*) of S_1 if $z \leq x$ $(x \leq z)$ for every $x \in S_1$. The *greatest lower bound* (also known as *infimum*) of S_1, when it exists, is the unique maximal element in the set of lower bounds of S_1. The g.l.b., i.e., the greatest lower bound of S_1, when it exists, is denoted by $\bigwedge S_1$ or $\inf S_1$. Similarly, the *least upper bound* (also known as *supremum*) of S_1, when it exists, is the unique minimal element in the set of all upper bounds of S_1. The l.u.b., i.e., least upper bound of S_1, when it exists, is denoted by $\bigvee S_1$ or $\sup S_1$. Whenever $S_1 = \{x_1, x_2\}$ is a two-element set, $\bigwedge S_1$ and $\bigvee S_1$ are also denoted by $x_1 \wedge x_2$ and $x_1 \vee x_2$, respectively.

A poset $(S, \leq)$ is said to be a *lattice* if $x_1 \wedge x_2$ and $x_1 \vee x_2$ are well defined for every pair of x_1, x_2 in S. A poset S is said to be a *lower semilattice* (*upper semilattice*) if $x_1 \wedge x_2$ $(x_1 \vee x_2)$ is well defined for every pair of x_1, x_2 in S. A subset S_1 of a lattice S is said to be a sublattice of S if S_1 itself is a lattice under the induced $\wedge$ and $\vee$ operations.

The set of all subsets of a set X is called the power set of X and denoted by $\mathscr{P}(X)$. $\mathscr{P}(X)$ is a poset with set inclusion relation $\subseteq$. $\mathscr{P}(X)$ is also a popular example for a lattice.

Theorem 2 *Given a poset* $(S, \leq)$, *there exists a set* X *and a one-to-one map*

$$f : S \to \mathscr{P}(X)$$

that is order preserving.

2 Partial Order on Matrices

The most well-known poset to every mathematician is the set of all real numbers $\mathbb{R}$, which is in fact an ordered field. This ordering on $\mathbb{R}$ can be easily utilized to have the simplest conceivable partial ordering on $\mathbb{R}^n$ by defining a relation $\leq_{\mathbb{R}}$ such that for $x, y \in \mathbb{R}^n$,

$$x \leq_{\mathbb{R}} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad 1 \leq i \leq n.$$

The class of all $m \times n$ real matrices, which has a one-to-one correspondence with $\mathbb{R}^{mn}$, the class of real mn -tuples, is partially ordered in a natural way. Thus, the relation $\leq_{\mathbb{R}}$ on $\mathbb{R}^{mn}$, when extended to a pair of matrices $A, B \in \mathbb{R}^{m\times n}$, would read as follows:

$$A \leq_{\mathbb{R}} B \quad \Longleftrightarrow \quad (A)_{ij} \leq (B)_{ij} \quad \forall\, i, j. \tag{7}$$

This is a partial order that plays an important role in combinatorial matrix theory.

Now consider the class $\mathbb{H}_n$ of all $n \times n$ hermitian complex matrices. For any $x \in \mathbb{C}^n$ and $H \in \mathbb{H}_n$, we have $x^*Hx \in \mathbb{R}$. So, resembling nonnegative real numbers, we have here a well-established concept of nonnegative elements in $\mathbb{H}_n$ called hermitian nonnegative definite matrices. Recall that a matrix $N \in \mathbb{H}_n$ is called hermitian nonnegative definite if $x^*Nx \geq 0$ for all $x \in \mathbb{C}^n$ and the cone of all $n \times n$ nonnegative definite matrices is denoted by $\mathbb{N}_n$:

$$\mathbb{N}_n = \left\{N : N \in \mathbb{H}_n \text{ and } x^*Nx \geq 0\ \forall x \in \mathbb{C}^n\right\}. \tag{8}$$

Now for $A, B \in \mathbb{N}_n$, define the relation $\leq_{\mathscr{L}}$ such that

$$A \leq_{\mathscr{L}} B \quad \text{if } B - A \in \mathbb{N}_n. \tag{9}$$

This relation $\leq_{\mathscr{L}}$ defines a partial order on $\mathbb{N}_n$ and is called the *Löewner partial order* on $\mathbb{N}_n$. $\leq_{\mathscr{L}}$ can be extended for $\mathbb{H}_n$ to define a relation which would remain a partial order. This partial order, which was initially introduced by Löwner (see Löwner [10]), has wide application in applied linear algebra and is well studied in the literature.

In sequel of the Löewner order, several partial orders have been studied on the subclasses of matrices. Star partial order, Minus partial order, Sharp partial order, and Core partial order are a few amongst prominent partial orders. Readers are referred to Mitra, Bhimasankaram and Malik [15] for the further reading.

2.1 *Column and Row Space-Dependent Relations*

Given a finite-dimensional vector space $\mathscr{V}$, let S be the set of all subspaces of $\mathscr{V}$. Interestingly, S is a poset with space inclusion relation providing the partial order. In fact, S is a lattice with $\mathscr{S}_1 \wedge \mathscr{S}_2 = \mathscr{S}_1 \cap \mathscr{S}_2$ and $\mathscr{S}_1 \vee \mathscr{S}_2 = \mathscr{S}_1 + \mathscr{S}_2$ for all $\mathscr{S}_1, \mathscr{S}_2 \in S$. So, this observation, together with Theorem 1, would lead to the following preorders:

Given $A, B \in \text{Mat}(\mathbb{C})$, consider the following relations:

(i) $\leq_{\mathscr{C}}$ defined by

$$B \leq_{\mathscr{C}} A \quad \text{if } \mathscr{C}(B) \subset \mathscr{C}(A). \tag{10}$$

(ii) $\leq_{\mathscr{R}}$ defined by

$$B \leq_{\mathscr{R}} A \quad \text{if } \mathscr{R}(B) \subset \mathscr{R}(A). \tag{11}$$

(iii) $\leq_{\text{sp}}$ defined by

$$B \leq_{\text{sp}} A \quad \text{if } B \leq_{\mathscr{C}} A \text{ and } B \leq_{\mathscr{R}} A. \tag{12}$$

All these three relations defined above are easily found to be reflexive and transitive but not antisymmetric. This can be seen easily by considering a pair of invertible matrices of the same size. These relations define just preorders on $\text{Mat}(\mathbb{C})$. $\leq_{\mathscr{C}}$ is called a *column space preorder*; $\le_{\mathscr{R}}$ is called a *row space preorder*, and $\le_{\text{sp}}$ is called simply a *space preorder*. Two matrices A, B are said to be *space equivalent* matrices if $A \le_{\text{sp}} B$ and $B \le_{\text{sp}} A$.

Proposition 1 *Let C be a matrix such that $C \le_{\mathscr{C}} A$ and $C \le_{\mathscr{R}} B$ for some nonnull A, B. Then there exists a matrix X such that $AXB = C$ and A, B are respectively left and right cancelable on X.*

Proof From the definition of C it is clear that $AA^-CB^-B = C \; \forall A^-, B^-$. Choose $G_A \in \{A_r^-\}$ and $G_B \in \{B_r^-\}$. Clearly, $X_0 = G_ACG_B$ is a solution for $AXB = C$ such that A, B are respectively left and right cancelable on X_0. □

Though $\le_{\text{sp}}$ fails to be a partial order on $\text{Mat}(\mathbb{C})$, interestingly, it defines a partial order on the class of idempotent matrices.

The following theorem characterizes the idempotent matrices with specified column and row spaces.

Theorem 3 *Let M and N be any two matrices such that $\mathscr{C}(M)$ and $\mathscr{R}(N)$ are subspaces of $\mathbb{C}^n$. Then*:

(i) *There exists an idempotent matrix $E \ne 0$ such that $E \le_{\mathscr{C}} M$ and $E \le_{\mathscr{R}} N$ if and only if $NM \ne 0$, i.e., $\mathscr{C}(M)$ and $\mathscr{R}(N)$ are not orthogonal subspaces.*
(ii) $\{E : E^2 = E, E \le_{\mathscr{C}} M \text{ and } E \le_{\mathscr{R}} N\} = \{M(NM)^{=}N\}$.
(iii) *There exists an idempotent matrix E such that $\mathscr{C}(E) = \mathscr{C}(M)$ and $\mathscr{R}(E) = \mathscr{R}(N)$ if and only if $\rho(NM) = \rho(N) = \rho(M)$. Such an idempotent matrix is unique, when it exists.*

Proof (i): If $NM = 0$, then so is any matrix E with $E \le_{\mathscr{C}} M$ and $E \le_{\mathscr{R}} N$.

Part (ii): Now let $NM \ne 0$, and $(NM)^{=}$ be any nonzero outer inverse of NM. Clearly, $(M(NM)^{=}N)(M(NM)^{=}M) = M(NM)^{=}N$ since $(NM)^{=}NM(NM)^{=} = (NM)^{=}$, and therefore

$$\left\{M(NM)^{=}N\right\} \subseteq \left\{E : E^2 = E, E \le_{\mathscr{C}} M, \text{ and } E \le_{\mathscr{R}} N\right\}.$$

To prove the reverse inclusion, consider any idempotent matrix E from the set on the RHS above. From Proposition 1 we get that $E = MXN$ is solvable such that M and N are respectively left and right cancelable on X. Now, $E^2 = E \Rightarrow X \in \{(NM)^{=}\}$, and this proves (ii).

Part (iii): Since the given idempotent matrix E is of the form $E = M(NM)^{=}N$, we obtain that

$$\begin{aligned} \rho(E) = \rho(M) = \rho(N) \quad \Longleftrightarrow \quad & \rho(NM) = \rho(M) = \rho(N) \\ & \text{and } (NM)^{=} \in \left\{(NM)_r^-\right\}. \end{aligned} \tag{13}$$

The uniqueness of E in such a case is immediate from the invariance of $M(NM)_r^- N$. □

The result given in the following corollary is well known and follows immediately from Theorem 3.

Corollary 1 *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be any two subspaces such that $\mathscr{S}_1 \oplus \mathscr{S}_2^{\perp} = \mathbb{C}^n$. Then*:

(i) *There exists a unique idempotent matrix $E \in \mathbb{C}^{n\times n}$ such that $\mathscr{C}(E) = \mathscr{S}_1$ and $\mathscr{R}(E) = \mathscr{S}_2$. In fact, $\mathscr{K}(E) = \mathscr{S}_2^{\perp}$.*
(ii) *An orthogonal projection on $\mathscr{S}_1$ exists and is unique.*

Note that the orthogonal projection is based on the decomposition $\mathscr{S}_1 \oplus \mathscr{S}_1^{\perp} = \mathbb{C}^n$.

Theorem 4 *Let E, E_1 be any two $n \times n$ idempotent matrices, and $E_2 = E - E_1$. Then the following statements are equivalent*:

(i) $E_1 \leq_{\text{sp}} E$,
(ii) $E_2 \leq_{\text{sp}} E$,
(iii) $E_1 = EE_1 = E_1E$,
(iv) $E_1E_2 = E_2E_1 = 0$,
(v) *E_2 is an idempotent matrix*,
(vi) $\rho(E_1) + \rho(E_2) = \rho(E)$ (*rank additivity*),
(vii) $\mathscr{C}(E) = \mathscr{C}(E_1) \oplus \mathscr{C}(E_2)$ (*column space decompositions*),
(viii) $\mathscr{R}(E) = \mathscr{R}(E_1) \oplus \mathscr{R}(E_2)$ (*row space decomposition*).

Proof (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) is easily verified. (v) ⇒ (vi) follows from the fact that the rank of an idempotent matrix is given by its trace.

(vi) ⇒ (vii): Since $E = E_1 + E_2$, we have that $\mathscr{C}(E) \subset \mathscr{C}(E_1) + \mathscr{C}(E_2)$ and

$$\rho(E) = \mathscr{D}\big(\mathscr{C}(E)\big) \leq \mathscr{D}\big(\mathscr{C}(E_1) + \mathscr{C}(E_2)\big) \tag{14}$$
$$= \mathscr{D}\big(\mathscr{C}(E_1)\big) + \mathscr{D}\big(\mathscr{C}(E_2)\big) + \mathscr{D}\big(\mathscr{C}(E_1) \cap \mathscr{C}(E_2)\big) \tag{15}$$
$$\leq \rho(E_1) + \rho(E_2). \tag{16}$$

Now from (vi) and the chain of inequalities just established we obtain that $\mathscr{C}(E_1) \cap \mathscr{C}(E_2) = (0)$ and in fact $\mathscr{C}(E) = \mathscr{C}(E_1) \oplus \mathscr{C}(E_2)$.

(vi) ⇒ (viii) is similarly established.

(vii) ⇒ (i): From (vii) it is clear that $E_1 \leq_{\mathscr{C}} E$. To prove $E_1 \leq_{\mathscr{R}} E$, observe that

$$E_1 + E_2 = (E_1 + E_2)^2 \tag{17}$$
$$= \big(E_1^2 + E_1E_2\big) + \big(E_2^2 + E_2E_1\big). \tag{18}$$

Since $\mathscr{C}(E_1) \cap \mathscr{C}(E_2) = (0)$, we obtain that $E_1 = E_1^2 + E_1E_2$, which in turn gives that $E_1E_2 = 0$ and $E_1E = E_1$. Therefore, $E_1 \leq_{\mathscr{R}} E$.

(viii) ⇒ (i) is similar. □

Remark 1 From the observation (i) $\Leftrightarrow$ (iii) of Theorem 4, the antisymmetry of the relation $\leq_{\text{sp}}$ can be easily seen on the set of idempotent matrices. So $\leq_{\text{sp}}$ defines a partial order on the class of idempotent matrices.

If we drop the condition that E_1 be an idempotent matrix, condition (i) does not remain equivalent to any of conditions (iv)–(viii). But conditions (vii) and (viii) of the theorem are stronger than space preorder condition, and (vii) or (viii) $\Rightarrow$ E_1 is an idempotent matrix whenever given E is an idempotent matrix.

Remark 2 Theorem 3(iii) plays a crucial role in forcing $\leq_{\text{sp}}$ to be a partial order on idempotents. Note that $\leq_{\text{sp}}$ is already a space preorder on Mat($\mathbb{C}$) and thus automatically a partial order on the space equivalent classes in $\mathbb{C}^{n\times n}$. Theorem 3(iii) shows that each such class contains a unique idempotent matrix. The null matrix in $\mathbb{C}^{n\times n}$ has the exclusive membership of the space equivalent class to which it belongs and the null matrix is already idempotent. Any other space equivalent class has only non-null members. Choose and fix any such member E to represent the class. In the same spirit, E could replace both N and M in the formulation of Theorem 3, whence the condition $\rho(N) = \rho(M) = \rho(NM)$ is seen to be equivalent to demanding that the matrix E be a core matrix, that is, $\rho(E^2) = \rho(E)$. This property is clearly invariant under the choice of the representative E. Hence, a space equivalent class containing a noncore matrix cannot at the same time include an idempotent matrix that is core. All other space equivalent classes include a unique idempotent matrix each, and the relation $\leq_{\text{sp}}$ is seen to be a partial order on the entire collection of space equivalent classes. $\leq_{\text{sp}}$ is thus also a partial order on the subcollection of space equivalent core matrices and concurrently on idempotent matrices.

Now we shall prove a theorem.

Theorem 5 *Let E be an $n \times n$ idempotent matrix, and $E = E_1 + E_2$. Then the following statements are equivalent*:

(i) *E_1 is an idempotent matrix such that $E_1 \leq_{\text{sp}} E$.*
(ii) *$\mathscr{C}(E) = \mathscr{C}(E_1) \oplus \mathscr{C}(E_2)$.*

Proof (i) $\Rightarrow$ (ii) is proved in Theorem 4.

(ii) $\Rightarrow$ (i): From column space decomposition we see that for every $x \in \mathbb{C}^n$, there exists $y = Ey$ such that

$$E_1 x = Ey = E_1 y.$$

So now we get $0 = E_2 y = E_2 E y = E_2 E_1 x$ for every x. Therefore, $E_2 E_1 = 0$, which in turn gives $E_1 = EE_1 = E_1^2$. Now the proof follows from Theorem 4(vii) $\Rightarrow$ (i) part. □

The class of all orthogonal projectors being a subclass of the class of idempotent matrices, $\leq_{\text{sp}}$ defines a partial order on this subset. In fact, $\leq_{\text{sp}}$ defines a lattice structure on the set of all orthogonal projectors of order n, denoted by $\mathscr{P}_n$.

A well-known result given in the following theorem for projectors is an analogue to Theorem 4 proved for the class of idempotents.

Theorem 6 *Let $P \in \mathbb{C}^{n\times n}$ be an orthogonal projector, and $P = P_1 + P_2$. Then the following statements are equivalent.*

(i) $P_1 \in \mathscr{P}_n$ *and* $P_1 \leq_{\text{sp}} P$.
(ii) $P_1 = P_1 P = P P_1 = P_1^*$.
(iii) $P_1 \in \mathbb{H}_n$ *and* $P_1 P_2 = P_2 P_1 = 0$.
(iv) $P_1 \in \mathbb{H}_n$ *and* $\mathscr{C}(P_1) \oplus \mathscr{C}(P_2)$.
(v) $\mathscr{C}(P_1) \perp \mathscr{C}(P_2)$ *and* $\mathscr{R}(P_1) \perp \mathscr{R}(P_2)$.

Proof (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) follows from easy verifications.

(iv) $\Rightarrow$ (v): Since P is a projector and $\mathscr{C}(\mathscr{P}) = \mathscr{C}(P_1) \oplus \mathscr{C}(P_2)$, (ii) $\Rightarrow$ (i) of Theorem 5 gives that P_1 is an idempotent matrix such that $P_1 \leq_{\text{sp}} P$. From (i) $\Rightarrow$ (iv) of Theorem 4 we obtain $P_1 P_2 = P P_1 = 0$. Since $P, P_1 \in \mathbb{H}_n$, eventually we get $\mathscr{C}(P_1) \perp \mathscr{C}(P_2)$ and $\mathscr{R}(P_1) \perp \mathscr{R}(P_2)$.

(v) $\Rightarrow$ (i): From (v) we see that $P_2 P_1^* = P_1^* P_2 = 0$ and therefore $P P_1^* = P_1 P_1^*$ and $P_1^* P = P_1^* P_1$. This in turn gives $P_1 \leq_{\text{sp}} P \Leftrightarrow P_1^* \leq_{\text{sp}} P$. Since $P \in \mathscr{P}_n$, we now obtain that $P_1 = P_1 P_1^* = P_1 P_1^*$, and hence $P_1 \in \mathbb{H}_n$ and $P_1 \in \mathscr{P}_n$. □

Theorem 7 *$(\mathscr{P}_n, \leq_{\text{sp}})$ is a lattice.*

Proof Let S denote the set of all subspaces of $\mathscr{C}^n$. S is a poset under the space inclusion relation. S is in fact a lattice with

$$\mathscr{S}_1 \wedge \mathscr{S}_2 = \mathscr{S}_1 \cap \mathscr{S}_2, \quad \mathscr{S}_1 \vee \mathscr{S}_2 = \mathscr{S}_1 + \mathscr{S}_2 \quad \forall \mathscr{S}_1, \mathscr{S}_2 \in S.$$

Consider the mapping

$$f : \mathscr{P}_n \to S$$

defined by $f(P) = \mathscr{C}(P)$. By Corollary 1(ii), clearly, f is bijective. Note that f also induces $\leq_{\text{sp}}$ on $\mathscr{P}_n$, and therefore $(\mathscr{P}_n, \leq_{\text{sp}})$ is a lattice.

Alternatively, one can define

$$P_1 \wedge P_2 = 2P_1(P_1 + P_2)^+ P_2 = 2P_2(P_1 + P_2)^+ P_1, \tag{19}$$

$$P_1 \vee P_2 = (P_1 + P_2)(P_1 + P_2)^+ \tag{20}$$

and verify that $(\mathscr{P}_n, \leq_{\text{sp}})$ is a lattice. This verification is left as an exercise to the reader. □

Remark 3 It is interesting to notice that Theorem 6 does not remain true if we drop the condition $P_1 \in \mathbb{H}_n$ in the statements (iii) and (iv). Observe that for any idempotent matrix $E \notin \mathbb{H}_n$, $I = E_1 + E_2$ such that $E_1 E_2 = E_2 E_1 = 0$.

Theorem 8 *Let* $P_1, P_2 \in \mathscr{P}_n$. *Then*

(i) $I - (P_1 \vee P_2) = (I - P_1) \wedge (I - P_2)$,
(ii) $I - (P_1 \wedge P_2) = (I - P_1) \vee (I - P_2)$.

Proof Observe that for any $P_1, P_2 \in \mathscr{P}_n$, $(I - P_1)$, $P_1 \wedge P_2$, and $P_1 \vee P_2$ are projections on $(\mathscr{C}(P_1))^{\perp}$, $\mathscr{C}(P_1) \cap \mathscr{C}(P_2)$, and $\mathscr{C}(P_1) + \mathscr{C}(P_2)$, respectively. Therefore, from

$$\mathscr{C}(I - P_1 \vee P_2) = \big(\mathscr{C}(P_1 \vee P_2)\big)^{\perp} \tag{21}$$

$$= \big(\mathscr{C}(P_1 + P_2)\big)^{\perp} \tag{22}$$

$$= \big(\mathscr{C}(P_1) + \mathscr{C}(P_2)\big)^{\perp} \tag{23}$$

$$= \big(\mathscr{C}(P_1)\big)^{\perp} \cap \big(\mathscr{C}(P_2)\big)^{\perp} \tag{24}$$

we get that

$$I - P_1 \vee P_2 = (I - P_1) \wedge (I - P_2).$$

Similarly, from

$$\mathscr{C}(I - P_1 \wedge P_2) = \big(\mathscr{C}(P_1) \cap \mathscr{C}(P_2)\big)^{\perp} \tag{25}$$

$$= \mathscr{C}(P_1)^{\perp} + \mathscr{C}(P_2)^{\perp} \tag{26}$$

we conclude that

$$I - P_1 \wedge P_2 = (I - P_1) \vee (I - P_2).$$ □

2.2 Partial Ordering via Space Decomposition

While each of $\leq_{\mathscr{C}}$, $\leq_{\mathscr{R}}$, and $\leq_{\text{sp}}$ fails to be a partial order on Mat($\mathbb{C}$), it would be interesting to see what further conditions in addition to this preorder make Mat($\mathbb{C}$) a partially ordered set. In fact, the equivalence of (i), (vi), (vii), and (viii) of Theorem 4 lead us in the right direction. In the following theorem we shall see that conditions (vi), (vii), and (viii) in Theorem 4 are still equivalent when we consider arbitrary matrices from Mat($\mathbb{C}$).

Theorem 9 (Rao and Mitra [27]) *The following statements are equivalent for any* $A, B \in \mathbb{C}^{m \times n}$:

(i) $\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(A - B)$ (*column space decomposition*),
(ii) $\rho(A) = \rho(B) + \rho(A - B)$ (*rank additivity*),
(iii) $\mathscr{R}(A) = \mathscr{R}(B) \oplus \mathscr{R}(A - B)$ (*row space decomposition*).

Proof (i) $\Rightarrow$ (ii) is trivial from the fact that $\rho(A) = \mathscr{D}(\mathscr{C}(A)) = \mathscr{D}(\mathscr{C}(B) \oplus \mathscr{C}(A-B)) = \mathscr{D}(\mathscr{C}(B)) + \mathscr{D}(\mathscr{C}(A-B)) = \rho(B) + \rho(A-B)$.

(ii) $\Rightarrow$ (i): Let us write $C = A - B$. Since $Ax = Bx + Cx \; \forall x \in \mathbb{C}^n$, we have $\mathscr{C}(A) \subset \mathscr{C}(B) + \mathscr{C}(C)$. Noting that

$$\mathscr{D}\big(\mathscr{C}(B) + \mathscr{C}(C)\big) = \mathscr{D}\big(\mathscr{C}(B)\big) + \mathscr{D}\big(\mathscr{C}(C)\big) - \mathscr{D}\big(\mathscr{C}(B) \cap \mathscr{C}(C)\big) \tag{27}$$

$$\leq \rho(B) + \rho(C), \tag{28}$$

the rank additivity is seen to imply that $\mathscr{C}(B) \cap \mathscr{C}(C) = 0$ and therefore $\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(A-B)$.

From the equivalence (i) $\Leftrightarrow$ (ii) just established it follows that $\mathscr{R}(A) = \mathscr{R}(B) \oplus \mathscr{R}(A-B) \Leftrightarrow \mathscr{C}(A^*) = \mathscr{C}(B^*) \oplus \mathscr{C}(A^* - B^*) \Leftrightarrow \rho(A^*) = \rho(B^*) + \rho(A^* - B^*) \Leftrightarrow \rho(A) = \rho(B) + \rho(A-B)$. Thus, (i) $\Leftrightarrow$ (ii). $\square$

Definition 2 The *space decomposition relation* $\leq_\oplus$ on $\mathrm{Mat}(\mathbb{C})$ is a relation defined by $B \leq_\oplus A$ if any of the equivalent conditions given in Theorem 9 holds.

Theorem 10 *The relation* $\leq_\oplus$ *on* $\mathrm{Mat}(\mathbb{C})$ *is a partial order.*

Proof By the definition, it is trivial that $\leq_\oplus$ is reflexive and transitive. To prove the antisymmetry, let $A \leq_\oplus B$ and $B \leq_\oplus A$. That is,

$$\mathscr{C}(B) = \mathscr{C}(A) \oplus \mathscr{C}(B - A), \tag{29}$$

$$\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(A - B). \tag{30}$$

Since $\mathscr{C}(B - A) = \mathscr{C}(A - B)$, from (29) it is seen that $\mathscr{C}(A - B)$ is a part of $\mathscr{C}(B)$, while (30) claims that $\mathscr{C}(B)$ and $\mathscr{C}(A - B)$ are virtually disjoint. These two contradictory statements can be reconciled only if $\mathscr{C}(A - B)$ is zero-dimensional, that is, if $A - B = 0$. $\square$

Theorem 11 *Given* $A, B \in \mathbb{C}^{m\times n}$, *the following statements are equivalent*:

(i) $B \leq_\oplus A$.
(ii) *B and $A - B$ are disjoint i.e.*,

$$\mathscr{C}(B) \cap \mathscr{C}(A-B) = (0), \qquad \mathscr{R}(B) \cap \mathscr{R}(A-B) = (0). \tag{31}$$

(iii)

$$\rho(A) = \rho(B) + \rho(A - B). \tag{32}$$

Proof (i) $\Rightarrow$ (ii) is a simple consequence of the definition of $\leq_\oplus$.

(ii) $\Rightarrow$ (iii): Put $C = A - B$ and let B and C be disjoint. To prove that this implies (iii), we shall prove that

$$\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(C). \tag{33}$$

Since $\mathscr{C}(B) \cap \mathscr{C}(A - B) = (0)$, it is enough to prove that $\mathscr{C}(B) \subset \mathscr{C}(A)$. For this, note that $\mathscr{R}(B) \cap \mathscr{R}(C) = (0)$ would mean

$$\mathscr{N}(B) + \mathscr{N}(C) = \mathbb{C}^n.$$

That is, every $x \in \mathbb{C}^n$ can be written as $x = y + z$ for some $y \in \mathscr{N}(B)$ and $z \in \mathscr{N}(C)$. Therefore, $Az = Bz = B(y + z) = Bx$ for every x, and hence $\mathscr{C}(B) \subset \mathscr{C}(A)$. By Theorem 9, (33) $\Rightarrow$(iii).

(iii) $\Rightarrow$ (i) is a simple consequence of the definition of $\leq_{\oplus}$. This proves the theorem. □

Corollary 2 *Let the representations $B = L_1 R_1$ and $C = L_2 R_2$ be rank factorizations of B and C, respectively. Then the following statements are equivalent*:

(i) $A = B \oplus C$.
(ii) *The representation* $A = (L_1\ L_2)\binom{R_1}{R_2}$ *is a rank factorization.*
(iii) $A = B + C$ *and* $B \leq_{\oplus} A$.

Proof The corollary follows from the fact that B and C are disjoint $\Leftrightarrow \mathscr{C}(L_1) \cap \mathscr{C}(L_2) = (0)$ and $\mathscr{R}(R_1) \cap \mathscr{R}(R_2) = (0) \Leftrightarrow \rho(L_1\ L_2) = \rho(L_1) + \rho(L_2) = \rho(R_1) + \rho(R_2) = \rho\binom{R_1}{R_2}$. □

Corollary 3 *Given $A = B + C \in \mathbb{C}^{m\times n}$ and $G \in \{A^-\}$, the following statements are equivalent*:

(i) $B \leq_{\oplus} A$.
(ii) $BG \leq_{\oplus} AG$ *(and/or* $GB \leq_{\oplus} GA$*) and* $G \in \{B^-\} \cap \{C^-\}$.
(iii) BG *and* CG *are idempotent matrices, and* $G \in \{B^-\} \cap \{C^-\}$.
(iv) GB *and* GC *are idempotent matrices, and* $G \in \{B^-\} \cap \{C^-\}$.

Proof Observe that each statement given in the corollary implies that $\rho(A) = \rho(B) + \rho(C)$. To complete the proof, we shall prove that $B \leq_{\oplus} A$ and $G \in \{A^-\}$ implies that $G \in \{B^-\} \cap \{C^-\}$. Since $\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(C)$, we obtain that $B = AGB = BGB + CGB$, which in turn gives that $BGB = B$. Similarly, $C = CGC$. □

3 Drazin's Star Partial Order

In Theorem 11, we have just seen that $B \leq_{\oplus} A$ can hold if and only if $\mathscr{C}(B) \cap \mathscr{C}(C) = (0)$ and $\mathscr{R}(B) \cap \mathscr{R}(C) = (0)$, where $C = A - B$. Now let us see what would happen if $\mathscr{C}(B) \perp \mathscr{C}(C)$ and $\mathscr{R}(B) \perp \mathscr{R}(C)$. That is, we expect $\mathscr{C}(B)$ and $\mathscr{C}(C)$ to decompose $\mathscr{C}(A)$ orthogonally and similarly $\mathscr{R}(B)$ and $\mathscr{R}(C)$ to decompose $\mathscr{R}(A)$ orthogonally.

Lemma 1 *Let* $A = B + C \in \mathbb{C}^{m\times n}$. *Then the following statements are equivalent*:

(i) $\mathscr{C}(B) \perp \mathscr{C}(C)$ *and* $\mathscr{R}(B) \perp \mathscr{R}(C)$.
(ii) $B^*C = CB^* = 0$.
(iii) $B^*A = B^*B$ *and* $AB^* = BB^*$.
(iv) $B^+A = B^+B$ *and* $AB^+ = BB^+$.

Proof The (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) of the lemma is trivial. (iii) $\Leftrightarrow$ (iv) follows from the fact that for any matrix X, X^* and X^+ are space equivalent matrices. □

Now define the *star relation* $\leq_*$ on Mat($\mathbb{C}$) such that for $A, B \in$ Mat($\mathbb{C}$),

$$B \leq_* A \quad \Longleftrightarrow \quad B^*B = B^*A \quad \text{and} \quad BB^* = AB^*. \tag{34}$$

In fact, $\leq_*$ defines a partial order on Mat($\mathbb{C}$) and is called a *star partial order*.

Drazin [6] was the first to notice that $\leq_*$ defines a partial order on a semigroup with proper involution, and Hartwig and Drazin [8] were the first to call this as star order. By Lemma 1, $\leq_*$ is clearly a partial order on Mat($\mathbb{C}$).

The star partial order is closely related with the Moore–Penrose inverse and singular value decomposition (S.V.D.).

Lemma 2 *Let* $A = B + C \in \mathbb{C}^{m\times n}$ *such that* $B \leq_* A$. *Then*:

(i) $C \leq_* A$.
(ii) $B^+ + C^+$ *is the Moore–Penrose inverse of* A.

Proof The first part of the lemma follows from the definition of star order. To prove the second part, observe that B^+C, CB^+, C^+B, and BC^+ are null matrices. Using direct computation, one can now verify that $B^+ + C^+$ satisfies the required conditions to be the Moore–Penrose inverse of A. □

Lemma 3 *The mapping* $A \to A^+$ *is an order-preserving map on* Mat($\mathbb{C}$) *under star partial order.*

Proof Since A^+ is a g-inverse of A with $\mathscr{C}(A^+) = \mathscr{R}(A)$ and $\mathscr{R}(A^+) = \mathscr{C}(A)$, the lemma follows immediately from the definition of star partial order. □

Lemma 4 *The following statements are equivalent*:

(i) $B \leq_* A$;
(ii) $B \leq_\oplus A$, *and* BA^* *and* A^*B *are hermitian*;
(iii) $B \leq_\oplus A$, *and* BA^+ *and* A^+B *are hermitian*;
(iv) $B \leq_\oplus A$, *and* B^+A *and* AB^+ *are hermitian.*

Proof From the definition of $\leq_*$ it is clear that B and $A - B$ are disjoint matrices. So from Theorem 11 we have that $B \leq_\oplus AA$. Therefore, (i) $\Rightarrow$ (ii) is immediate from the definition of star partial order. Let $AA = B + C$. We know that $B^+ = B^+ + C^+$

whenever $B \leq_* A$. Again from the definition of $\leq_*$ we get that BC^+, C^+A, B^+C, and CB^+ are null matrices, and therefore (i) $\Rightarrow$ (iii) and (iv) is immediate.

(ii) $\Rightarrow$ (i): Write $BA^* = BB^* + BC^*$, where $C = A_B$. Observe that if BA^* is a hermitian, then so is BC^*. Therefore, $BC^* = CB^*$ and

$$BA^* = BB^* + CB^*. \tag{35}$$

Since $B \leq_{\oplus} A$, we have that $\mathscr{C}(B) \cap \mathscr{C}(C) = 0$, and now (35) implies that $CB^* = 0$. Similarly, $B \leq_{\oplus} A$ and A^*B hermitian implies that $B^*C = 0$. So, $B \leq_* AA$.

(iii) $\Rightarrow$ (i): Write $AA^+ = BA^+ + CA^+$. Since $B \leq_{\oplus} A$, from Corollary 3 we get that BA^+ and CA^+ are idempotent matrices, and

$$AA^+ = BA^+ \oplus CA^+.$$

This in turn gives that $BA^+CA^+ = 0$. Since BA^+ is hermitian, we get that $(A^+)^*B^*CA^+ = 0$, and therefore $B^*C = 0$. Similarly, since $B \leq_{\oplus} A$ and A^+B is hermitian, it follows that $BC^* = 0$. So, $B \leq_* A$.

(iv) $\Rightarrow$ (i): Since B^+A hermitian, we get that

$$B^+A = B^+B + B^+C \tag{36}$$

$$= B^+B + C^*\big(B^*\big)^+. \tag{37}$$

Since $\mathscr{R}(A) = \mathscr{R}(B) \oplus \mathscr{R}(C)$, we get that $C^*(B^*)^+$ is a null matrix. Therefore, $C^*B = 0$. Similarly, since $B \leq_{\oplus} A$ and AB^+ is hermitian, it follows that $CB^* = 0$. Hence, $B \leq_* A$. □

Lemma 5 *Let $P_1, P_2 \in \mathbb{C}^{n\times n}$. Then we have the following*:

(i) *If $P_2 \in \mathscr{P}_n$ and $P_1 \leq_* P_2$, then $P_1 \in \mathscr{P}_n$.*
(ii) *If $P_1 \in \mathscr{P}_n$ and $P_1 \leq_* P_2$, then*

$$P_2 = P_1 + (I - P_1)X(I - P_1)$$

for some X.

Proof (i): Since $P_1 \leq_* P_2$, it is clear that $P_1 \leq_{\text{sp}} P_2$. Now $P_2 \in \mathscr{P}_n$ implies that $P_1 = P_1P_2 = P_1P_2^* = P_1P_1^*$, and therefore $P_1 \in \mathscr{P}_n$.

Part (ii) is immediate from the fact that $P_1 \leq_* P_2$ and $P_1 \in \mathscr{P}_n$ implies that $\mathscr{C}(P_2 - P_1) \subseteq \mathscr{C}(I - P_1)$ and $\mathscr{R}(P_2 - P_1) \subseteq \mathscr{R}(I - P_1)$. □

Theorem 12 *For $A, B, C \in \mathbb{C}^{m\times n}$, let $A = B + C$. Then the following statements are equivalent*:

(i) $B \leq_* A$.
(ii) *There exists a singular value decomposition of B that can be extended for A. That is, there exist unitary matrices U and V of order m and n, respectively,*

such that

$$B = U\begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} V, \qquad A = U\begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{38}$$

where $\Delta_1 = \mathrm{diag}(d_1, \ldots, d_k)$, $\Delta_2 = \mathrm{diag}(d_1, \ldots, d_k, d_{k+1}, \ldots, d_l)$, *and* d_i *are positive real numbers.*

(iii) $A^+ = B^+ + C^+$ *and* $\rho(A) = \rho(B) + \rho(C)$.

(iv) $A^+ = B^+ \oplus C^+$.

Proof (i) $\Rightarrow$ (ii): From S.V.D. of matrices B and C we obtain the respective representations

$$B = U_1\Delta_1 V_1^* \quad \text{and} \quad C = U_2 D_2 V_2^*,$$

where Δ_1 and D_2 are diagonal matrices with singular values of B and C as the respective diagonal entries; U_1, U_2, V_1, and V_2 are orthonormal matrices. Since $B^*C = 0$ and $BC^* = 0$, we get $U_1^*U_2 = 0$ $V_1^*V_2 = 0$. By completing $(\, U_1 \; U_2 \,)$ and $(\, V_1 \; V_2 \,)$ into square unitary matrices U and V, respectively, and $\Delta_2 = \mathrm{diag}(\Delta_1 D_2)$, we obtain the expression (38).

(ii) $\Rightarrow$ (iii) follows immediately from S.V.D. we obtained in (38).

(iii) $\Rightarrow$ (iv) follows from the fact that $\rho(X) = \rho(X^+)$ for every X and Theorem 9.

(iv) $\Rightarrow$ (i): Again from S.V.D. of B and C, write

$$B = U_1\Delta_1 V_1^*, \qquad C = U_2\Delta_2 V_2^*,$$

where Δ_1 and Δ_2 are diagonal matrices with singular values of B and C as the respective diagonal entries. Clearly,

$$B^+ = V_1\Delta_1^{-1}U_1^*, \qquad C^+ = V_2\Delta_2^{-1}U_2^*. \tag{39}$$

Since $A^+ = B^+ \oplus C^+$, from Theorems 11 and 9 we obtain that B and C are also disjoint. Therefore, we get that $\mathscr{C}(U_1) \cap \mathscr{C}(U_2) = 0$ and $\mathscr{C}(V_1) \cap \mathscr{C}(V_2) = (0)$. Now by Gram–Schmidt's orthogonalization process we obtain orthogonal matrices U_3 and V_3 such that $(\, U_1 \; U_3 \,)$ and $(\, V_1 \; V_3 \,)$ are orthogonal matrices and

$$\begin{pmatrix} U_1 & U_2 \end{pmatrix} = \begin{pmatrix} U_1 & U_3 \end{pmatrix}\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}, \qquad \begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} V_1 & V_3 \end{pmatrix}\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \tag{40}$$

for some suitable matrices P and Q. Since $A = B + C$, we can write

$$A = \begin{pmatrix} U_1 & U_3 \end{pmatrix}\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}\Delta\begin{pmatrix} I & 0 \\ Q^* & I \end{pmatrix}\begin{pmatrix} V_1^* \\ V_3^* \end{pmatrix}, \tag{41}$$

where $\Delta = \mathrm{diag}(\Delta_1, \Delta_2)$. Now we obtain

$$A^+ = \begin{pmatrix} V_1 & V_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ -Q^* & I \end{pmatrix}\Delta^{-1}\begin{pmatrix} I & -P \\ 0 & I \end{pmatrix}\begin{pmatrix} U_1^* \\ U_3^* \end{pmatrix}. \tag{42}$$

Note that the condition $A^+ = B^+ \oplus C^+$ and (39) give that

$$\begin{aligned} A^+ &= \begin{pmatrix} V_1 & V_2 \end{pmatrix} \Delta^{-1} \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} \\ &= \begin{pmatrix} V_1 & V_3 \end{pmatrix} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \Delta^{-1} \begin{pmatrix} I & 0 \\ P^* & I \end{pmatrix} \begin{pmatrix} U_1^* \\ U_3^* \end{pmatrix}. \end{aligned} \tag{43}$$

From (43) and (42) we obtain that P and Q are null matrices and eventually $U_2 = U_3$ and $V_2 = V_3$. From the definition of U_3 and V_3 we have $U_1^* U_3 = V_1^* V_3 = 0$. $U_1^* U_2 = U_1^* U_3 = 0$ implies $B^* C = 0$, and similarly $V_1^* V_2 = V_1^* V_3 = 0$ implies $BC^* = 0$. Thus, we proved $B^* A = B^* B$ and $AB^* = BB^*$. □

Theorem 13 *Given $A = B + C \in \mathbb{C}^{m \times n}$, the following statements are equivalent:*

(i) $B \leq_* A$.
(ii) $A^+ \in \{B^{\{1,3,4\}}\}$.
(iii) $BA^+ \leq_* AA^+$, $A^+ B \leq_* A^+ A$, *and* $B \leq_{\text{sp}} A$.
(iv) *There exist orthogonal projections P and Q such that*

$$B = PA = AQ.$$

Proof (i) ⇒ (ii): From Theorem 12 we have that $A^+ = B^+ + C^+$ whenever $B \leq_* A$. Since $B^* C$ and CB^* are null matrices, we observe that BC^+ and $C^+ B$ are also null matrices. Therefore,

$$A^+ B = B^+ B \quad \text{and} \quad BA^+ = BB^+,$$

and this in turn implies that $A^+ \in \{B^{\{1,3,4\}}\}$.

(ii) ⇒ (iii) follows easily from the identities $BA^+ = BB^+$ and $A^+ B = B^+ B$.

(iii) ⇒ (iv): Since $B \leq_{\text{sp}} A$ and $BA^+ \leq_* AA^+$, we get that BA^+ is an orthogonal projection, in fact on $\mathscr{C}(B)$. Now defining $P = BA^+$, again from $B \leq_{\text{sp}} A$ we get that $B = PA$. Similarly, defining $Q = A^+ B$, we obtain that Q^* is an orthogonal projection onto $\mathscr{R}(B)$ such that $B = AQ$.

(iv) ⇒ (i) is trivial since P and Q are orthogonal projectors. □

Corollary 4 *Given $A, B \in \mathbb{C}^{m \times n}$, $B \leq_* A$ if and only if there exist orthogonal projections P and Q such that*

$$B = PAQ \quad \textit{and} \quad PA(I - Q) = (I - P)AQ = 0.$$

3.1 Lattice Properties of Star Order

It would be quite interesting to see whether $(\mathbb{C}^{m \times n}, \leq_*)$ satisfy any of the lattice properties. In fact, Hartwig and Drazin [8] proved that g.l.b.$\{B, C\} = B \wedge C$ exists in the poset $(\mathbb{C}^{m\times n}, \leq_*)$ and hence $(\mathbb{C}^{m\times n}, \leq_*)$ is a lower semilattice. However, one can easily see that l.u.b.$\{B, C\}$ does not exist when B and C both are of full rank. In fact, in this case, there does not exist any A such that $B \leq_* A$ and $C \leq_* A$.

Lemma 6 *Let $A, B, C \in \mathbb{C}^{m\times n}$ be such that $B \leq_* A$ and $C \leq_{\mathrm{sp}} B$. Then $C \leq_* A$ if and only if $C \leq_* B$.*

Proof If $C \leq_* B$, by the transitivity of $\leq_*$, it is trivially true that $C \leq_* A$, thus proving the "if" part of the lemma. To prove the "only if" part of the lemma, let $C \leq_* A$. Therefore, we have

$$CC^+A = AC^+C = C. \tag{44}$$

Since $C \leq_{\mathrm{sp}} B$, clearly, $C^+ \leq_{\mathrm{sp}} B^+$, and further,

$$CC^+BB^+ = CC^+ \quad \text{and} \quad B^+BC^+C = C^+C. \tag{45}$$

From (44) and (45) we get that

$$CC^+B = CC^+BB^+A \ \text{ (because } B \leq_* A) \tag{46}$$

$$= CC^+A = C. \tag{47}$$

Similarly, we get that $BC^+C = C$, and therefore $C \leq_* B$. □

Lemma 7 *Let $P_1, P_2 \in \mathscr{P}_n$. Then* g.l.b.$\{P_1, P_2\}$ *exists in the poset* $(\mathbb{C}^{n\times n}, \leq_*)$ *and is the same as* g.l.b.$\{P_1, P_2\}$ *in the lattice* $(\mathscr{P}_n, \leq_{\mathrm{sp}})$.

Proof For $P \in \mathscr{P}_n$ and $Q \in \mathbb{C}^{n\times n}$, observe that $Q \leq_* P \Rightarrow Q \leq_{\mathrm{sp}} P$ and $Q^* = Q^*P = Q^*Q = PQ^* = QQ^*$. Therefore, we obtain

$$Q \leq_* P \quad \Longleftrightarrow \quad Q \leq_{\mathrm{sp}} P \quad \text{and} \quad Q \in \mathscr{P}_n, \tag{48}$$

and this eventually proves the lemma. □

Theorem 14 *An interval* $[0, A] \subseteq (\mathbb{C}^{m\times n} \leq_*)$ *is a lattice under star partial order.*

Proof To prove the theorem, we shall prove that $B_1 \wedge B_2$ and $B_1 \vee B_2$ are well defined in $[0, A]$ for every $B_1, B_2 \in [0, A]$.

Existence of $B_1 \wedge B_2$: For $i = 1, 2$, there exist an orthogonal projection $P_i \in \mathscr{P}_m$ on $\mathscr{C}(B_i)$ and orthogonal projection $Q_i \in \mathscr{P}_n$ on $\mathscr{R}(B_i)$ (in fact, we can take $P_i = B_iB_i^+$ and $Q_i = B_i^+B_i$) such that

$$B_i = P_iAQ_i \quad \text{and} \quad (I - P_i)AQ_i = P_iA(I - Q_i) = 0. \tag{49}$$

Now define $P_3 = P_1 \wedge P_2$ and $Q_3 = Q_1 \wedge Q_2$. From (49) it is clear that

$$P_3A(I - Q_3) = P_3A\big((I - Q_1) + (I - Q_2)\big)\big((I - Q_1) + (I - Q_2)\big)^+ = 0$$

and

$$(I - P_3)AQ_3 = \big((I - P_1) + (I - P_2)\big)^+\big((I - P_1) + (I - P_2)\big)AQ_3 = 0.$$

In other words,

$$P_3AQ_3 = P_3A = AQ_3.$$

Now from Corollary 4 we get that $C = P_3AQ_3 \leq_* A$, and from the definition of P_3 and Q_3 we get that $C \leq_{\text{sp}} B$, $C \leq_{\text{sp}} B_2$ Further, by appealing to Lemma 3, we get that $C \in L^*_{(B_1,B_2)}$, where

$$L^*_{(B_1,B_2)} = \{D : D \leq_* B_1, D \leq_* B_2\}.$$

Now, for any $D \in L^*_{(B_1,B_2)}$, observe that

$$D = DD^+A = AD^+D, DD^+ \leq_{\text{sp}} B_1B_1^+$$

and

$$DD^+ \leq_{\text{sp}} B_2B_2^+.$$

Therefore,

$$DD^+ \leq_{\text{sp}} \left(B_1B_1^+\right) \wedge \left(B_2B_2^+\right) = P_3.$$

Similarly, we get that $D^+D \leq_{\text{sp}} Q_3$. Therefore, D is a matrix such that $D \leq_* A$ and $D \leq_{\text{sp}} P_3A(= AQ_3 = C)$. Now, again appealing to Lemma 6, we get that $D \leq_* C$. This proves that $C = \text{g.l.b.}\{B_1, B_2\} = B_1 \wedge B_2$ in $[0, A]$.

Existence of $B_1 \vee B_2$: Now let

$$U^*_{(B_1,B_2)} = \{D : B_1 \leq_* D, B_2 \leq_* D \text{ and } D \leq_* A\}.$$

Define $E = P_4AQ_4$, where $P_4 = P_1 \vee P_2$ in $(\mathscr{P}_m, \leq_{\text{sp}})$ and $Q_4 = Q_1 \vee Q_2$ in $(\mathscr{P}_n, \leq_{\text{sp}})$. Since Q_4 is the projection on $\mathscr{C}(Q_1) + \mathscr{C}(Q_2)$, we get that $I - Q_4 \leq_{\text{sp}} (I - Q_1), (I - Q_2)$. So, $P_iA(I - Q_4) = 0$ for $i = 1, 2$, and therefore $P_4A(I - Q_4) = 0$. Similarly, we obtain that $(I - P_4)AQ_4 = 0$. This proves that $E \leq_* A$. From the definition of P_4 and Q_4 we have that $B_i = P_iE = EQ_i$, and therefore $B_i \leq_* E$. Hence, $E \in U^*_{(B_1,B_2)}$.

For any $D \in U^*_{(B_1,B_2)}$, $\mathscr{C}(B_i) \subseteq \mathscr{C}(D)$, and therefore,

$$\mathscr{C}(B_1) + \mathscr{C}(B_2) = \mathscr{C}(P_4) \subseteq \mathscr{C}(D) \tag{50}$$

and, similarly,

$$\mathscr{R}(B_1) + \mathscr{R}(B_2) = \mathscr{R}(Q_4) \subseteq \mathscr{R}(D). \tag{51}$$

From (50) and (51) we get that $E \leq_{\text{sp}} D$. Now appealing to Lemma 6, we get that $E \leq_* D$, and therefore $E = \text{l.u.b.}\{B_1, B_2\}$ in $[0, A]$. $\square$

Remark 4 Observe that $\mathbb{C}^{m\times n}$ with star partial order satisfies the chain condition. That is, for any ascending (descending) sequence of matrices $\{A_i\}$ such that $A_i \leq_* A_j$ for $i \leq j (j \leq i)$, we have that $A_k = A_{k+l}$ for some k and $l = 1, 2, \ldots$. This is simply due to the fact that if $A_i \neq A_j$ and $A_i \leq_* A_j$, then necessarily $\rho(A_j) \geq \rho(A_i) + 1$.

Theorem 15 *Given* $A, B \in \mathbb{C}^{m\times n}$, *let*

$$L^*_{(A,B)} = \{C : C \leq_* A, C \leq_* B\}.$$

Then $L^*_{(A,B)}$ *is sublattice of both* $[0, A]$ *and* $[0, B]$ *with unique maximal element.*

Proof For $M = A, B$, let $\wedge_M$ and $\vee_M$ respectively denote the $\wedge$ and $\vee$ operations in $[0, M]$ under star partial order. Observe that for $i = 1, 2,$ $\mathbb{C}_i \leq_* A$ and $C_i \leq_* B$ if and only if there exist $P_i\ (= C_i C_i^+)$ and $Q_i\ (= C_i^+ C_i)$ such that $C_i = P_i A = P_i B = B Q_i = A Q_i$. Now

$$\begin{aligned} C_1 \wedge_A C_2 &= (P_1 \wedge P_2) A (Q_1 \wedge Q_2) && (52)\\ &= 4P_1(P_1 + P_2)^+ P_2 A Q_1 (Q_1 + Q_2)^+ Q_2 && (53)\\ &= 4P_1(P_1 + P_2)^+ P_2 B Q_1 (Q_1 + Q_2)^+ Q_2 && (54)\\ &= (P_1 \wedge P_2) B (Q_1 \wedge Q_2) && (55)\\ &= C_1 \wedge_B C_2. && (56) \end{aligned}$$

Therefore, $C_1 \wedge C_2 = C_1 \wedge_A C_2 = C_1 \wedge_B C_2$ is well defined in $L^*_{(A,B)}$. Similarly,

$$\begin{aligned} C_1 \vee_A C_2 &= (P_1 \vee P_2) A (Q_1 \vee Q_2) && (57)\\ &= (P_1 + P_2)^+ (P_1 + P_2) A (Q_1 + Q_2)(Q_1 + Q_2)^+ && (58)\\ &= (P_1 + P_2)^+ (P_1 + P_2) B (Q_1 + Q_2)(Q_1 + Q_2)^+ && (59)\\ &= (P_1 \vee P_2) B (Q_1 \vee Q_2) && (60)\\ &= C_1 \vee_B C_2, && (61) \end{aligned}$$

and $C_1 \vee C_2 = C_1 \vee_A C_2 = C_1 \vee_B C_2$ is well defined in $L^*_{(A,B)}$. Therefore, $L^*_{(A,B)}$ is a lattice.

Further, if E and F are any two distinct maximal elements in $L^*_{(A,B)}$, then $E \vee F \in L^*_{(A,B)}$ is the element that dominates both E and F, which contradicts the maximality of E and F. □

Lemma 8 *Given* $P_1, P_2 \in \mathscr{P}_n$, l.u.b.$\{P_1, P_2\}$ *in* $(\mathscr{P}_n, \leq_{\text{sp}})$ *is the same as* l.u.b.$\{P_1, P_2\}$ *in* $(\mathbb{C}^{n\times n}, \leq_*)$.

Proof From the proof of Theorem 7 it is clear that the l.u.b.$\{P_1, P_2\}$ in $\mathscr{P}_n$ is the projection P_3 on $(\mathscr{C}(P_1) + \mathscr{C}(P_2))$. So, $P_3 \in U^*_{(P_1,P_2)}$ is the set of upper bounds of $\{P_1, P_2\}$ in $(\mathbb{C}^{n\times n}, \leq_*)$.

Now suppose that P_3 is not l.u.b.$\{P_1, P_2\}$ in $(\mathbb{C}^{n\times n}, \leq_*)$ and there exists a D in $U^*_{(P_1,P_2)}$ that is not comparable with P_3. By Theorem 15, $P_3 \wedge D$ exists in $(\mathbb{C}^{n\times n}, \leq_*)$. Since P_3 and D are not comparable, $P_3 \wedge D$ is strictly dominated by P_3 under star partial order. Therefore, we observe that $P_3 \wedge D$ is a projection that dominates P_1 and P_2 both in $(\mathscr{P}_n, \leq_{\text{sp}})$. This contradicts the fact that P_3 is the l.u.b.$\{P_1, P_2\}$ in $(\mathscr{P}_n, \leq_{\text{sp}})$. □

Lemma 9 *Let $A, B \in \mathbb{C}^{m\times n}$ such that $U^*_{(A,B)}$, the set of all upper bounds of $\{A, B\}$, is nonempty. Then $U^*_{(A,B)}$ has unique minimal element, and hence $A \vee B$ is well defined in $(\mathbb{C}^{m\times n}, \leq_*)$.*

Proof Let $C_1, C_2 \in U^*_{(A,B)}$. Now, by Lemma 8, $C_1 \wedge C_2$ exists in $(\mathbb{C}^{n\times n}, \leq_*)$, and observe that, by the definition of g.l.b., both $A, B \leq_* C_1 \wedge C_2$. Therefore, $C_1 \wedge C_2 \in U^*_{(A,B)}$. If E and F are any two distinct minimal elements in $U^*_{(A,B)}$, then $E \wedge F \in U^*_{(A,B)}$, which contradicts the minimality of E and F. This proves the lemma. □

Remark 5 In view of Lemma 9, let us consider an element ∞ called infinity from outer space to $\mathbb{C}^{m\times n}$, so that we define

$$A \vee B = \infty \tag{62}$$

if $U^*_{(A,B)}$ is empty. Denote the set $\mathbb{C}^{m\times n} \cup \{\infty\}$ by $\mathbb{C}^{m\times n}_\infty$. With the assumption that ∞ is the largest element in $\mathbb{C}^{m\times n}_\infty$, we get that $(\mathbb{C}^{m\times n}_\infty, \leq_*)$ is a poset and in fact a lattice.

From Theorem 15, Lemmas 8 and 9, and Remark 5 we derive the following theorem.

Theorem 16 *In* Mat($\mathbb{C}$) *with star partial order, the following statements are true*:

(i) *$(\mathbb{C}^{m\times n}, \leq_*)$ is a lower semilattice.*
(ii) *$(\mathbb{C}^{m\times n}_\infty, \leq_*)$ is a lattice.*
(iii) *$(\mathscr{P}_n, \leq_{\text{sp}})$ and $(\mathscr{P}_n, \leq_*)$ are identical posets and sublattices of lattice $(\mathbb{C}^{m\times n}_\infty, \leq_*)$.*

Theorem 17 *For $A \in (\text{Mat}(C), \leq_*)$, we have the following*:

(i) $[0, A^+] = \{A^{\{2,3,4\}}\}$.
(ii) *For $F \in [0, A^+]$, $AFA \in [0, A]$.*
(iii) $[A^+, \infty) = \{A^{\{1,3,4\}}\}$.
(iv) $A^+ = \min\{A^{\{1,3,4\}}\} = \max\{A^{\{2,3,4\}}\}$ *in* $(\text{Mat}(C), \leq_*)$.

Proof From Theorem 13 we can see that

$$F \leq_* A \iff A = \left(A^+\right)^+ \in \left\{F^{\{1,3,4\}}\right\} \tag{63}$$

$$\iff F \in \left\{A^{\{2,3,4\}}\right\}. \tag{64}$$

This proves part (i).

Part (ii): If $F \in [0, A^+]$, from part (i) it clear that $FAF = F$ and that AF and FA are orthogonal projectors onto subspaces $\mathscr{C}(A)$ and $\mathscr{R}(A)$, respectively. Now write $A_1 = AFA$. Observe that

$$A_1 F A_1 = AFAA^+AFA = AFAFA = AFA = A_1,$$

and further, $A_1 A^+ = AFAA^+ = AF$ and $A^+A = A^+AFA = FA$ are hermitian. Therefore, $A^+ \in \{A_1^{\{1,3,4\}}\}$, which proves that $A_1 \in [0, A]$.

Part (iii) follows immediately from the fact that

$$F \in [A^+, \infty) \iff F = A^+ + \big(I - A^+A\big)X\big(I - AA^+\big) \quad \text{for some } X \tag{65}$$

$$\iff F \in \big\{A^{\{1,3,4\}}\big\}. \tag{66}$$

Part (iv) follows from parts (i) and (iii). □

3.2 Left- and Right-Star Orders

When we drop out one among the conditions $B^*A = B^*B$ and $AB^* = BB^*$ in defining a relation between A and B, the new relation does not satisfy antisymmetry. Still, these relations with space preorder condition define a partial order on Mat($\mathbb{C}$).

Define two relations $\leq_{l(*)}$ and $\leq_{r(*)}$ on Mat($\mathbb{C}$) such that for any two matrices $A, B \in$ Mat($\mathbb{C}$),

$$B \leq_{l(*)} A \quad \text{if } B^*A = B^*B \text{ and } B \leq_{\mathscr{C}} A, \tag{67}$$

$$B \leq_{r(*)} A \quad \text{if } A^*B = B^*B \text{ and } B \leq_{\mathscr{R}} A. \tag{68}$$

Lemma 10 *Let $A, B \in \mathbb{C}^{m\times n}$. Then*:

(i) $B \leq_{l(*)} A \Leftrightarrow B^* \leq_{r(*)} A^*$.
(ii) $B \leq_{l(*)} A$ *and* $B \leq_{r(*)} A \Leftrightarrow B \leq_* A$.

Lemma 11 *Given $A, B \in \mathbb{C}^{m\times n}$, let $B \leq_{l(*)} A$ and $A \leq_{\mathscr{C}} B$. Then $A = B$.*

Proof Since $A \leq_{\mathscr{C}} B$, there exists a matrix M such that $A = BM$. Now from $B \leq_{l(*)} A$ we get that

$$B^*B = B^*A = B^*BM,$$

which in turn gives that $B = BM = A$. □

From Lemma 11 we observe that the relation $\leq_{l(*)}$ is antisymmetric.

Therefore, $\leq_{l(*)}$ defines a partial order on Mat($\mathbb{C}$), called a *left-star partial order*. Similarly, we observe that $\leq_{r(*)}$ defines a partial order on Mat($\mathbb{C}$), called a *right-star partial order*. Readers are referred to Baksalary and Mitra [3] and Baksalary et al. [4] for more results on left-star and right-star orders.

4 Partial Orders Based on g-inverses

4.1 Minus Partial Order

Inspired by Drazin's work (see Drazin 1978 [6]) on star partial order on semigroup with proper involution, Hartwig, 1980 [7] introduced a plus partial order (later renamed as a minus partial order) on the set of regular elements in a semigroup. In the context of matrices, it was seen that $B \leq_* A$ (B related to A under star relation) is equivalent to the condition $B^+A = B^+B$ and $AB^+ = BB^+$. The question was whether the Moore–Penrose inverse was really necessary in the condition to define a partial order on the set of regular elements in a semigroup. Hartwig 1980 [7] replaced B^+ in the condition with a reflexive g-inverse of B and later found that just a generalized inverse of B would suffice to obtain a partial order.

Definition 3 (Minus order, Hartwig [7] and Nambooripad [23]) Let $A, B \in \mathbb{F}^{m\times n}$. We write $B \leq^- A$ if for some choice of A^-,

$$B^-A = B^-B, \tag{69}$$

$$AB^- = BB^- \tag{70}$$

and say that the matrix B is less than the matrix A under minus order.

The following lemma provides a few alternative characterizations of minus partial order. For the proof, we refer to Mitra [13] and Jain–Manjunatha Prasad [9].

Lemma 12 *For $A, B \in \mathbb{F}^{m\times n}$, the following conditions are equivalent*:

(i) *B and $A-B$ are disjoint matrices*, *i.e.*, $\mathscr{C}(B)\cap\mathscr{C}(A-B) = (0)$ *and* $\mathscr{R}(B)\cap\mathscr{R}(A-B) = (0)$.
(ii) *$rank(B) + rank(A-B) = rank(A)$.*
(iii) *$B \leq^- A$ with reference to minus order.*
(iv) $\{A^-\} \subseteq \{B^-\}$.
(v) $\mathscr{C}(B)\oplus\mathscr{C}(A-B) = \mathscr{C}(A)$.
(vi) $\mathscr{R}(B)\oplus\mathscr{R}(A-B) = \mathscr{R}(A)$.

Let $\mathbf{S} = \mathscr{C}(E)$ and $\mathbf{T} = \mathscr{R}(F)$. Define the set

$$\begin{aligned}\mathfrak{C} &= \left\{C \in \mathbb{F}^{m\times n} : \mathscr{C}(C) \subseteq \mathbf{S}, \mathscr{R}(C) \subseteq \mathbf{T}\right\}\\ &= \left\{C = EXF : X \in \mathbb{F}^{p\times q}\right\}.\end{aligned} \tag{71}$$

Definition 4 (Shorted matrix, Mitra [12], Mitra and Puri [21]) A matrix $B \in \mathfrak{C}$ is called a shorted matrix of A relative to $\mathbf{S}$ and $\mathbf{T}$ and denoted by $\mathbb{S}[A|\mathbf{S},\mathbf{T}]$ or $\mathbb{S}[A|E, F]$ if

$$\text{rank}(A-B) = \min_{C\in\mathfrak{C}} \text{rank}(A-C). \tag{72}$$

An equivalent maximality condition (see Mitra and Manjunatha Prasad [17]) for the minimality condition given in (72) is

$$B = \max_{C \in \mathfrak{C}} C \leq^{-} A. \tag{73}$$

The shorted matrix is known to be unique under certain regularity conditions. The shorted matrix could also be unique in some pathological situations where the regularity conditions fail (see Mitra [14]). When regularity conditions hold, the unique shorted matrix has many attractive properties (see Mitra [12], Mitra [13], and Mitra and Puri [20, 21]). Some of these properties are lost when a shorted matrix is not unique.

The definition of shorted matrix given in (73) extends the notion of shorted operator introduced by Anderson and Trapp. Initially, Anderson [1] and Anderson and Trapp [2] introduced the concept of a shorted operator having significance in the context of electrical network. If A is the impedance matrix of a resistive n-port network, then A_s is the impedance matrix of the network obtained by shorting the last $n - s$ ports; thus, we call A_s a shorted operator. This shorted operator satisfies certain maximal properties under the Löewner partial order. The shorted operator defined under Löewner partial order and the shorted matrix under minus partial order coincides when we confine to the class of positive semidefinite matrices. The notion of a shorted operator has very interesting interpretation in linear model (see Mitra and Puntanen [18] and Mitra, Puntanen and Styan [19]).

In the context of matrices possibly rectangular, we define $B \leq^{-} A$ if there exists B^{-} such that

$$B^{-}A = B^{-}B \quad \text{and} \quad AB^{-} = BB^{-}. \tag{74}$$

In such a case, we introduce the class of matrices

$$\left\{B^{-}\right\}_A := \left\{B^{-} : B^{-}A = B^{-}B, AB^{-} = BB^{-}\right\}. \tag{75}$$

Condition (74) is equivalent to

$$B_r^{-}A = B_r^{-}B \quad \text{and} \quad AB_r^{-} = BB_r^{-}. \tag{76}$$

Now we have the following theorem.

Theorem 18 *Given $A, B \in \mathbb{C}^{m\times n}$, the following statements are equivalent*:

(i) *There exist $G_1, G_2 \in \{B^{-}\}$ such that*

$$G_1B = G_1A \quad \textit{and} \quad BG_2 = AG_2. \tag{77}$$

(ii) $B \leq^{-} A$.

(iii) *There exists a $G \in \{B^{-}\}$ such that*

$$B = BGA = AGB. \tag{78}$$

(iv) *There exist idempotent matrices E and F such that*

$$B = EA = AF. \tag{79}$$

(v) $B \leq_{\oplus} A$.
(vi) $(A - B) \leq^{-} A$.

Proof (i) $\Rightarrow$ (ii) follows from the observation that

$$G_2 B G_1 B = G_2 B G_1 A, \qquad B G_2 B G_1 = A G_2 B G_1,$$

and $G_2 B G_1 \in \{B^-\}$. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivially established.

(iv) $\Rightarrow$ (v): From (79) we have that $A - B = (I - E)A$, and therefore $\mathscr{C}(B) \cap \mathscr{C}(A - B) = 0$. Hence, $\mathscr{C}(B) \oplus \mathscr{C}(A - B) = \mathscr{C}(A)$.

(v) $\Rightarrow$ (i): Observe that $\mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(A - B)$ implies that $B = AA^- B = BA^- B + CA^- B$, and therefore $CA^- B = 0,\ B = BA^- B$. Similarly, from $\mathscr{R}(A) = \mathscr{R}(B) \oplus \mathscr{R}(C)$ we obtain that $BA^- C = 0$. Now, by writing $G_1 = G_2 = A^- BA^-$ for some A^- we get that

$$G_1 B = G_1 A, \qquad AG_2 = BG_2.$$

(vi) $\Leftrightarrow$ (i) follows trivially from the (v) $\Leftrightarrow$(i) part of the theorem and the definition of $\leq_{\oplus}$. $\square$

Now it is clear that $\leq^-$ defines a partial order and in fact coincides with $\leq_{\oplus}$ on Mat($\mathbb{C}$). $\leq^-$ is popularly known as a *minus partial order*.

Theorem 19 *Given $A, B \in \mathbb{C}^{m \times n}$, the following statements are equivalent*:

(i) $B \leq^- A$.
(ii) $\{A^-\} \subseteq \{B^-\}$.
(iii) *For $C = A - B$, $BA^- C = CA^- B = 0$ for all A^-.*

Proof (i) $\Rightarrow$ (ii): Let G be a g-inverse of B such that

$$GA = GB, \qquad AG_B = BG. \tag{80}$$

For any arbitrary g-inverse A^- of A, observe that

$$\begin{aligned} BA^- B &= (BGB)A^-(BGB) && (81)\\ &= BGAA^- AGB && (82)\\ &= BGAGB && (83)\\ &= BGBGB = B. && (84)\end{aligned}$$

Therefore, $\{A^-\} \subseteq \{B^-\}$.

(ii) $\Rightarrow$ (iii): Since $BA^-B = B$ for all A^-, from the matrix invariance of BA^-B we get that $B \leq_{\text{sp}} A$, which implies $BA^-A = AA^-B = B$ and in turn $BA^-C = BA^-A = 0$, $CA^-B = AA^-B - BA^-B = 0$.

(iii) $\Rightarrow$ (i): Since $BA^-C = CA^-B = 0$ for all A^-, from the invariance of BA^-C and CA^-B we get that $B \leq_{\text{sp}} A$, and in turn $BA^-A = B$, which implies $BA^-B = BA^-(A - C) = B + 0 = B$. Thus, $A^- \in \{B^-\}$. Now, clearly A^-BA^- is a choice of g-inverse of B satisfying (74). Hence, $B \leq^- A$. $\square$

Corollary 5 *Given a poset $\mathbb{C}^{m\times n}$ with $\leq^-$, the mapping*

$$\mathscr{G} : \mathbb{C}^{m\times n} \to \mathscr{P}\big(\mathbb{C}^{n\times m}\big),$$

where $\mathscr{G}(A) = \{A^-\}$, is injective and order reversing.

Corollary 6 *Let A, B, and C from $\mathbb{C}^{m\times n}$ be such that $A = B + C$. Then the following statements are equivalent*:

(i) *A, B, and C have a common g-inverse.*
(ii) *$B \leq^- A$.*

Proof If G is a common g-inverse of A, B, and C, then AG, BG, and CG are idempotent matrices such that

$$AG = BG + CG.$$

Therefore, $\operatorname{Tr}(AG) = \operatorname{Tr}(BG) + \operatorname{Tr}(CG) \Rightarrow \rho(A) = \rho(B) + \rho(C) \Rightarrow B \leq_{\oplus} A$. This proves (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Noting that

$$\begin{aligned} B \leq^- A \quad &\Longleftrightarrow \quad B \leq_{\oplus} A && (85)\\ &\Longleftrightarrow \quad \mathscr{C}(A) = \mathscr{C}(B) \oplus \mathscr{C}(C) && (86)\\ &\Longleftrightarrow \quad C \leq_{\oplus} A && (87)\\ &\Longleftrightarrow \quad C \leq^- A, && (88) \end{aligned}$$

(i) $\Rightarrow$ (ii) part of Theorem 19 proves the part (ii) $\Rightarrow$ (i) of the corollary. $\square$

Remark 6 In proving the part (i) $\Rightarrow$ (ii) of Corollary 6, the existence of a g-inverse of A, which is also a common g-inverse of B C, is very crucial. Just having a common g-inverse for B and C does not imply that $B \leq^- A$. Observe that the matrices $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ have the identity matrix as a common g-inverse but B is not dominated by $B + C$ under minus partial order.

In [16], it was seen that the minus partial order can also be defined via outer inverses.

Theorem 20 *Given $A, B \in \mathbb{C}^{m\times n}$, $B \leq^- A$ if and only if there exists an outer inverse $A^=$ of A such that*

$$B = AA^=A. \tag{89}$$

Proof If $B = AA^=A$ for some outer inverse of A, observe that for all A^-,

$$BA^-B = AA^=AA^-AA^=A = AA^=A = B. \tag{90}$$

So from Theorem 19 we get that $B \leq^- A$ since (90) could be restated as $\{A^-\} \subseteq \{B^-\}$.

If $B \leq^- A$, observe that A^-BA^- is a reflexive g-inverse of B for any choice of A^- and $B \leq_{\text{sp}} A$, the later being otherwise evident from Theorem 18, equivalence of (ii) and (iii). Therefore,

$$\left(A^-BA^-\right)A\left(A^-BA^-\right) = A^-BA^-,$$
$$A\left(A^-BA^-\right)A = AA^-BA^-A = B.$$

This proves the theorem. □

We have seen in Theorem 19 that $B \leq^- A$ if and only if $\{A^-\} \subseteq \{B^-\}$. It seems natural to ask whether a partial order could be defined through the set inclusion $\{A^=\} \subseteq \{B^=\}$. Unfortunately, this relation does not define any interesting partial order since 0 is the only matrix different from A that can be comparable with A under the present relation. This could be seen as follows. Let B be a nonzero matrix such that $B \neq A$. If $A = 0$, A^* is the only outer inverse of A that is also an outer inverse of B. Otherwise, there exists a vector x such that $Bx \neq Ax$ and $Ax \neq 0$. Now choose y such that $y^*Ax \neq y^*Bx$ but $y^*Ax = 1$. So we have $xy^* \in \{A^=\}$ but $xy^* \notin \{B^=\}$. Therefore, $\{B^=\} \nsubseteq \{A^=\}$. Similarly, we obtain that $\{A^=\} \nsubseteq \{B^=\}$. So, for any two nonnull A and B such that $A \neq B$, the class of outer inverses are not comparable.

4.2 Minus Partial Order on the Class of Inner and Outer Inverses

Let $S_A = \{A^-\} \cup \{A^=\}$ be the set of all inner and outer inverses of A. Given below are some interesting relations between elements in S_A, based on minus partial order.

Lemma 13 *Given a $G \in \{A^-\}$, there exists a $G_1 \in \{A_r^-\}$ such that $G_1 \leq^- G$.*

Proof Define $G_1 = GAG$. Clearly, G_1 is a reflexive g-inverse of A, and in fact A is a g-inverse of G_1. Observe that

$$AG_1 = AG \quad \text{and} \quad G_1A = GA.$$

Hence, $G_1 \leq^- G$. □

It may be noted that a relation $\sim$ on the set of generalized inverses of A, defined by $G_1 \sim G_2$ if $G_1AG_1 = G_2AG_2$, is an equivalence relation. Further, $[G]_g$ the equivalence class that contain $G \in \{A^-\}$ has unique reflexive generalized inverse of A. In the following Corollary 7, which follows immediately from the Lemma 13, we obtain that the reflexive g-inverse in the equivalence class is the unique minimal element.

Corollary 7 *An equivalence class $[G]_g$ determined by G in $\{A^-\}$ has unique minimal element which is equal to GAG.*

Lemma 14 *Given $O_1 \in \{A^=\}$, let O_2 be any matrix such that $O_2 \leq^- O_1$. Then $O_2 \in \{A^=\}$.*

Proof From Theorem 19 we see that $O_2 \leq^- O_1$ implies $\{O_1^-\} \subseteq \{O_2^-\}$. Since $A \in \{O_1^-\}$, we obtain that $A \in \{O_2^-\}$; in other words, $O_2 \in \{A^=\}$. □

Lemma 15 *Given any $O_1 \in \{A^=\}$, there exists a $G \in \{A_r^-\}$ such that $O_1 \leq^- G$.*

Proof From Theorem 20 we observe that $B = AO_1A \leq^- A$. Now choose any g-inverse A^- and define

$$G = O_1 + A^-(A - B)A^-. \tag{91}$$

Since $B = BA^-B = BA^-AAA^-B$, a straightforward computation gives that $AGA = A$ and $GAG = G$. Again, $B \leq^- A$ implies that $\rho(A^-(B - A)A^-) = \rho(A - B)$. Therefore,

$$\rho(G) = \rho(A) = \rho(B) + \rho(A - B) \tag{92}$$

$$= \rho(O) + \rho\big(A^-(A - B)A^-\big), \tag{93}$$

which in turn gives that $O_1 \leq^- G$. □

From Lemmas 13, 14, and 15 we have the following theorem.

Theorem 21 *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$\{A_r^-\} = \max\{A^=\} = \min\{A^-\} \tag{94}$$

under the minus partial order.

Lemma 16 *The space preorder $\leq_{sp}$ defines a partial order on the set $\{A^=\}$. Further, both $\leq_{sp}$ and $\leq^-$ are identical on $\{A^=\}$.*

Proof The reflexivity and transitivity of $\leq_{sp}$ are trivial, the antisymmetry of $\leq_{sp}$ on $\{A^=\}$ can be proved as follows. Let O_1 and O_2 are any two outer inverses of A such that $O_1 \leq_{sp} O_2$ and $O_2 \leq_{sp} O_1$. Since $A \in \{O_1^-\} \cap \{O_2^-\}$, we observe that

$$O_1 = O_2AO_1 = O_2,$$

proving antisymmetry.

To prove the second part of the lemma, consider any two $O_1, O_2 \in \{A^=\}$. Now, from the definition of $\leq^-$ it is obvious that

$$O_1 \leq^- O_2 \quad \Longrightarrow \quad O_1 \leq_{\mathrm{sp}} O_1.$$

Conversely, if $O_1 \leq_{\mathrm{sp}} O_2$, we get that $O_1 O_2^- O_1$ is invariant under the choice of O_2^-. Since $A \in \{O_1^-\} \cap \{O_2^-\}$,

$$O_1 O_2^- O_1 = O_1 A O_1 = O_1.$$

This proves that $O_1 \leq^- O_2$ and hence the lemma. □

Theorem 22 *Let $A \in \mathbb{C}^{m\times n}$, and G be an arbitrary but fixed g-inverse of A. Then*

$$\left\{A_1 : A \leq^- A_1,\ G \in \left\{A_1^-\right\}\right\} = \left\{A + (I - AG)U(I - GA)\right\},$$

where $U \in \{(G - GAG)^=\}$.

Proof Consider any A_1 such that $A \leq^- A_1$ and $G \in \{A_1^-\} \subseteq \{A^-\}$. Clearly, $A = A_1GA = AGA_1$. This in turn gives that

$$AG(A_1 - A) = (A_1 - A)GA = 0$$

and

$$\mathscr{C}(A_1 - A) \subseteq \mathscr{C}(I - AG), \qquad \mathscr{R}(A_1 - A) \subseteq \mathscr{R}(I - GA).$$

Therefore,

$$A_1 - A = (I - AG)(A_1 - A)(I - GA).$$

Now by computing $(A_1 - A)(G - GAG)(A_1 - A)$ we obtain that $A_1 - A \in \{(G - GAG)^=\}$, and hence,

$$\left\{A_1 : A \leq^- A_1,\ G \in \left\{A_1^-\right\}\right\} \subseteq \left\{A + (I - AG)U(I - GA)\right\},$$

where $U \in \{(G - GAG)^=\}$.

To prove the reverse inclusion, consider $A_2 = A + (I - AG)U(I - GA)$ for some $U \in \{(G - GAG)^=\}$. Since $G \in \{A^-\}$, $G_1 = GAG \in \{A^-\}$, and

$$G_1A_2 = G_1A, \qquad A_2G_2 = AG_1.$$

Therefore, $A \leq^- A_2$.

Now,

$$\begin{aligned} A_2GA_2 &= AGA + (I - AG)U(I - GA)G(I - AG)U(I - GA) && (95)\\ &= A + (I - AG)U(G - GAG)U(I - GA) && (96)\\ &= A + (I - AG)U(I - GA). && (97)\end{aligned}$$

Therefore, $G \in \{A_2^-\}$. □

Corollary 8 *Let $O \in \{A^=\}$. Then*

$$\left\{A^= : O \leq^=_A\right\} = \left\{O + (I - OA)U(I - AO)\right\}, \tag{98}$$

where $U \in \{(A - AOA)^=\}$. Further,

$$\left\{A_r^- : O_1 \leq_{\text{sp}} A_r^-\right\} = \left\{O_1 + (I - OA)(A - AO_1A)_r^-(I - AO)\right\}. \tag{99}$$

Corollary 9 (Rao and Mitra [27], Theorem 2.7.1) *Let $A \in \mathbb{C}^{m\times n}$ of rank r, and s be an integer such that $r \leq s \leq \min(m,n)$. Then $G \in \{A^-\}$ with $\rho(G) = s$ if and only if it is of the form*

$$(A + M)_r^-,$$

where M is a matrix disjoint with A.

Proof If A and M are disjoint matrices, from Theorem 11 we get that $A \leq^- (A + M)$. Therefore, $G \in \{(A + M)_r^-\} \Rightarrow G \in \{A^-\}$.

If $G \in \{A^-\}$, chose U to be any reflexive g-inverse of $G - GAG$ and define $M = (I - AG)U(I - GA)$. Since A and M are disjoint matrices, from Theorem 11 we get that $A \leq^- A + M$ and $G \in \{(A + M)_r^-\}$. So, clearly $AGM = MGA = 0$. Now by computing $G(A + M)G$ we find that $G \in \{(A + M)_r^-\}$. $\square$

Analogous to the equivalence relation $\sim_g$ on the set $\{A^-\}$, we introduce $\sim_o$ on the set $\{A^=\}$ in the following definition.

Definition 5 (Equivalence relation on $\{A^=\}$) For $O_1, O_2 \in \{A^=\}$, we say $O_1 \sim_o O_2$ if $AO_1A = AO_2A$. The equivalence class determined by $O \in \{A^=\}$ is denoted by $[O]_o$.

From Theorem 20 we have observe that $O \in \{A^=\}$ implies $AOA \leq^- A$. In fact, we have the following.

Theorem 23 *Given $O \in \{A^=\}$ and $B = AOA$, the following statements are true.*

(i) $[O]_o = \{B_r^-\} \cap \{B^-\}_A$.
(ii) $[O]_o = \{A^- BA^-\}$.

Proof Consider any $O_1 \in [O]_o$, i.e., $AO_A = AOA = B$. Since $O_1 \in \{A^=\}$, we obtain that $O_1BO_1 = O_1AO_1AO_1 = O_1$ and $BOB = AO_1AOAO_1A = AO_1A = B$. This proves $O_1 \in \{B_r^-\}$. Further,

$$BO = AO_1AO_1 = AO_1, \tag{100}$$

$$OB = O_1AO_1A = O_1A, \tag{101}$$

which in turn, gives that $O_1 \in \{B^-\}_A$, and therefore $[O]_o \subseteq \{B_r^-\} \cap \{B^-\}_A$. To prove the reverse inclusion, consider any reflexive g-inverse G of B in $\{B^-\}_A$ satisfying $GB = GA$ and $BG = AG$. This implies that $GAG = GBG = G$ and $AGA =$

$BGB = B$. This proves that $G \in [O]_o$, and therefore $[O]_o = \{B_r^-\} \cap \{B^-\}_A$, proving part (i).

Since $B = AOA \leq^- A$, by direct computation we obtain that $\{A^- BA^-\} \subseteq [O]_o$. To prove the reverse inclusion, for any $O - 1 \in [O]_o$, consider any $G = O_1 + (I - O_1 A)(A - B)_r^-(I - AO_1)$ (as in (99)). The direct computation gives that $O_1 = GBG$, proving the reverse inclusion. □

The following corollary is immediate from the above theorem.

Corollary 10 *Let A be an* $m \times n$ *matrix. Then*:

(i) *For any* $O \in \{A^=\}$ *and* $B = AOA$,

$$[O]_o = \{B_r^-\} \cap \{B^-\}_A = \{A^=\} \cap \{B^-\}. \tag{102}$$

(ii) *The mapping* $[O]_o \mapsto B$ *where* $B = AOA$ *is a bijective mapping between the class of equivalence classes in* $\{A^=\}$ *and the set* $\{B : B \leq^- A\}$.

References

1. Anderson, W.N.: Shorted operators. SIAM J. Appl. Math. **20**, 520–525 (1971)
2. Anderson, W.N., Trapp, G.E.: Shorted operators II. SIAM J. Appl. Math. **28**, 60–71 (1975)
3. Baksalary, J.K., Mitra, S.K.: Left-star and right-star partial ordering. Linear Algebra Appl. **149**, 73–89 (1991)
4. Baksalary, J.K., Baksalary, O.M., Liu, X.: Further properties of the star, left-star, right-star and minus partial ordering. Linear Algebra Appl. **375**, 83–94 (2003)
5. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications. Wiley-Interscience, New York (1974)
6. Drazin, M.P.: Natural Structures on semigroups with involution. Bull. Am. Math. Soc. **84**, 139–141 (1978)
7. Hartwig, R.E.: How to order regular elements? SIAM J. Appl. Math. **25**, 1–13 (1980)
8. Hartwig, R.E., Drazin, M.P.: Lattice properties of the $*$-order for complex matrices. J. Math. Anal. Appl. **86**, 359–378 (1982)
9. Jain, S.K., Manjunatha Prasad, K.: Right–left symmetry of $aR \oplus bR = (a + b)R$ in regular rings. J. Pure Appl. Algebra **133**, 141–142 (1998)
10. Löewner, K.: Über monotone matrixfunktionen. Math. Z. **38**, 177–216 (1934)
11. Mitra, S.K.: Fixed rank solutions of linear matrix equations. Sankhya, Ser. A **34**, 387–392 (1972)
12. Mitra, S.K.: Properties of the fundamental bordered matrix used in linear estimation. In: Kallianpur, G., et al. (eds.) Statistics and Probability: Essays in Honor of C.R. Rao, North-Holland, New York (1982)
13. Mitra, S.K.: The minus partial order and shorted matrix. Linear Algebra Appl. **81**, 207–236 (1986)
14. Mitra, S.K.: Separation theorems. Linear Algebra Appl. **208/209**, 238–256 (1994)
15. Mitra, S.K., Bhimasankarm, P., Malik, S.B.: Matrix Partial Orders. Shorted Operators and Applications. World Scientific, Singapore (2010)
16. Mitra, S.K., Hartwig, R.E.: Partial orders based on outer inverses. Linear Algebra Appl. **176**, 3–20 (1992)
17. Mitra, S.K., Manjunatha Prasad, K.: The nonunique shorted matrix. Linear Algebra Appl. **237/238**, 41–70 (1996)

18. Mitra, S.K., Puntanen, S.: The shorted operator statistically interpreted. Calcutta Stat. Assoc. Bull. **40**, 157–160 (1990–91). Prof. H.K. Nandi memorial volume
19. Mitra, S.K., Puntanen, S., Styan, G.P.H.: Shorted matrices and their applications in linear statistical models, a review. In: Tiit, M., et al. (eds.) New Trends in Probability and Statistics. Proceedings of the 5th Tartu Conference on Multivariate Statistics, pp. 289–311. TEV, Vilnius (1995)
20. Mitra, S.K., Puri, M.L.: Shorted operators and generalized inverses of matrices. Linear Algebra Appl. **25**, 45–56 (1979)
21. Mitra, S.K., Puri, M.L.: Shorted matrix: An extended concept and some applications. Linear Algebra Appl. **42**, 57–79 (1982)
22. Moore, E.H.: On the reciprocal of the general algebraic matrix. Bull. Am. Math. Soc. **26**, 394–395 (1920)
23. Nambooripad, K.S.S.: The natural partial order on a regular semigroup. Proc. Edinb. Math. Soc. **23**, 249–260 (1980)
24. Penrose, R.: Generalized inverse for matrices. Proc. Camb. Philos. Soc. **51**, 406–413 (1955)
25. Rao, C.R.: A note on a generalized inverse of a matrix with applications to problems in mathematical statistics. J. R. Stat. Soc. B **24**, 152–158 (1962)
26. Rao, C.R.: Unified theory of linear estimation. Sankhya, Ser. A **33**, 371–394 (1971)
27. Rao, C.R., Mitra, S.K.: Generalized Inverse of Matrices and Its Applications. Wiley, New York (1971)

An Illustrated Introduction to Some Old Magic Squares from India

George P.H. Styan and Ka Lok Chu

Abstract In this article we consider old magic squares from India associated with

1. Daivajna Varāhamihira (505–587 AD) and his *Bṛhat Saṁhitā* [39]: magic perfume;
2. Khajuraho 945 AD: Sir Alexander Cunningham (1814–1893) [14];
3. Dudhai (Jhansi district) early 11th century: Harold Hargreaves (b. 1876) [27];
4. Ṭhakkura Pherū (fl. 1291–1323): *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics* [1];
5. Simon de la Loubère (1642–1729): *Monsieur Vincent,* Surat [3,15];
6. Major-General Robert Shortrede (1800–1868) [16], Gwalior 1483 [11, (1842)]; Andrew Hollingworth Frost (1819–1907) [23], Nasik [17, (1877)];
7. Nārāyaṇa Paṇḍita (fl. 1340–1400): *Gaṇita Kaumudī* [2, (1356)];
8. Srinivasa Aiyangar Ramanujan (1887–1920) [34,35,40,43]; Prasantha Chandra Mahalanobis (1893–1972).

Magic squares were once part of occult philosophy, but more recently, however, they form part of recreational mathematics. For the past 50 years or so, they have been studied in a matrix-theoretic setting. Our main interest is in the history and philosophy of magic squares and the related magic matrices and in the related bibliography and biographies. We try to illustrate our findings as much as possible and, whenever feasible, with images of postage stamps and other philatelic items.

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Keywords Sir Alexander Cunningham (1814–1893) · Andrew Hollingworth Frost (1819–1907) · Harold Hargreaves (b. 1876) · Simon de la Loubère (1642–1729) · Prasantha Chandra Mahalanobis (1893–1972) · Nārāyaṇa Paṇḍita (fl. 1340–1400) · Ṭhakkura Pherū (fl. 1291–1323) · Srinivasa Aiyangar Ramanujan (1887–1920) · Major-General Robert Shortrede (1800–1868) · Daivajna Varāhamihira (505–587 AD) · Bibliography · Biography · Chota Surang shrine · Dudhai (Jhansi district) · Franklin's most magically magical of magic squares · Gwalior magic squares · Khajuraho magic square · Magic matrices · Magic perfume · Nasik (pandiagonal) magic squares · Recreational mathematics · Rhomboidal magic squares

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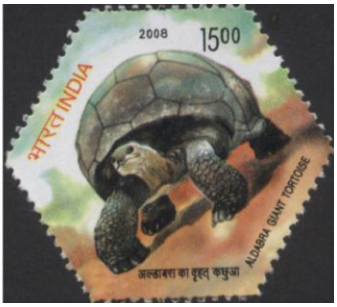
1 Introduction

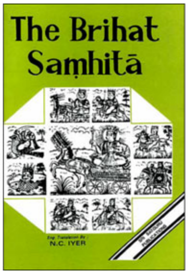
I want to tell you about some old magic squares from India and to show you images of some related postage stamps. But first a little history. Legend has it that the very first magic square, the Luoshu, was discovered in China near the Luoshu River about 4000 years ago on the back of a turtle or tortoise! The magic square known as Luoshu = Luo River Writing is based on the classic 3×3 *fully magic matrix*

$$\mathbf{L} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}.$$



In a classic fully magic matrix, the numbers in all the rows and columns and in the two main diagonals all add up to same magic sum m, here $m = 15$. When only the numbers in all the rows and columns but not the two main diagonals add up to the same *magic sum,* then we have a semi-magic matrix. We will say that an $n \times n$ magic square is classic when the n^2 entries are $1, 2, \ldots, n^2$ and adjusted-classic when the n^2 entries are one less: $0, 1, 2, \ldots, n^2 - 1$.

2 Daivajna Varāhamihira (505–587 AD) and His *Bṛhat Saṁhitā* [39]: Magic Perfume

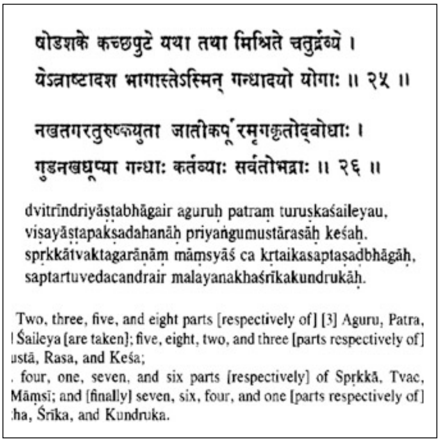


According to Hayashi [45, 47] the oldest datable magic square in India occurs in the encyclopedia *Bṛhat Saṁhitā* [19, 39] on divination by the 6th-century Indian astronomer, mathematician, and astrologer Daivajna Varāhamihira (505–587 AD) and is first published in Sanskrit. Varāhamihira showed how to make several varieties of perfume with precisely 4 substances selected from 16, including Agar wood (which yields Oud perfume oil), and used the magic square defined by the 4×4 magic matrix

$$\mathbf{V}_1 = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix} \tag{1}$$

with magic sum 18. In the matrix $\mathbf{V}_1$, the integers $1, 2, \ldots, 8$ each appear twice and correspond to:

aguru **2** *Agar wood* $\Rightarrow$ *oud oil*	**patra** **3** *bay* *leaves*	**turuṣka** **5** *frankincense* *resin*	**śaileya** **8** *lichen* *flakes*
priyaṅgu **5** *panic grass* *oil*	**mustā** **8** *cypriol* *root oil*	**bola** **2** *myrrh resin*	**keśa** **3** *lemongrass* *leaves*
spṛkkā **4** *fenugreek* *oil*	**tvac** **1** *cinnamon* *sticks*	**tagara** **7** *trangipani* *flower oil*	**maṁsī** **6** *spikenard* *leaves*
malaya **7** *sandal-* *wood*	**nakha** **6** *conch shell* *powder*	**śrīka** **4** *turpentine* *resin*	**kundruka** **1** *Deodar* *cedar resin*


षोडशके कच्छपुटे यथा तथा मिश्रिते चतुर्द्रव्ये ।
येऽत्राष्टादश भागास्तेऽस्मिन् गन्धादयो योगाः ॥ २५ ॥

नखतगरतुरुष्कयुता जातीकर्पूरमृगकृतोद्बोधाः ।
गुडनखधूप्या गन्धाः कर्तव्याः सर्वतोभद्राः ॥ २६ ॥

dvitrīndriyāṣṭabhāgair aguruḥ patraṃ turuṣkaśaileyau,
viṣayāṣṭapakṣadahanāḥ priyaṅgumustārasāḥ keśaḥ.
spṛkkātvaktagarāṇāṃ māṃsyāś ca kṛtaikasaptaṣaḍbhāgāḥ,
saptartuvedacandrair malayanakhaśrīkakundrukāḥ.

Two, three, five, and eight parts [respectively of] [3] Aguru, Patra,
l Śaileya [are taken]; five, eight, two, and three [parts respectively of]
ustā, Rasa, and Keśa;
. four, one, seven, and six parts [respectively] of Spṛkkā, Tvac,
Māṃsī; and [finally] seven, six, four, and one [parts respectively of]
:ha, Śrīka, and Kundruka.

The Sanskrit text above comes from the English translation [39] by Bhat (1993) of Varāhamihira's *Bṛhat Saṁhitā*. See also our "philatelic magic square" in Fig. 1 (and Table 1).

Fig. 1 A “philatelic magic square” for the 16 substances used by Varāhamihira [19]

To obtain a perfume mixture, Varāhamihira mixed together precisely 4 of the 16 substances, provided that the corresponding numbers add up to 18. For example, consider the first row of the Varāhamihira-perfume magic matrix $\mathbf{V}_1$

$$\begin{pmatrix} 2 & 3 & 5 & 8 \\ \text{Agar wood} & \text{bay leaves} & \text{frankincense} & \text{lichen} \end{pmatrix}. \tag{2}$$

Table 1 The stamps in the "philatelic magic square" (Fig. 1)

position	ratio 18	Hayashi 1987	My choice: English	stamp plant: English	stamp: country	stamp: year	stamp: *Scott*
11	2	aguru	Agar wood => oud oil	Agar wood	Qatar	2008	1035c
12	3	patra	bay leaves	Bay laurel, Grecian laurel	Greece: Mount Athos	2010	``2nd issue''
13	5	turuska	frankincense resin	frankincense	Oman	1985	285--286
14	8	saileya	lichen flakes	lichen	Liechtenstein	1981	713
21	5	priyangu	panic grass oil	Guinea grass	Bophuthatswana	1984	116
22	8	musta	cypriol root oil	"Sedge in Ninh Binh is mainly Cyperus tojet Jormis specie (white flower)."	Vietnam	1974	740
23	2	rasa	myrrh resin	myrrh	Liechtenstein	1985	822
24	3	kesa	lemongrass leaves	common bulrush, broadleaf cattail	Botswana	1987	427
31	4	sprkka	fenugreek oil	lanceleaf Thermopsis	USSR	1985	5380
32	1	tvac	cinnamon sticks	Indian spices: cinnamon	India	2009	*TBC*
33	7	tagara	frangipani flower oil	frangipani	Niue	1984	421
34	6	mamsi	spikenard leaves	Valerian	Poland	1980	2412
41	7	malaya	sandalwood	sandalwood	India	2007	TBC
42	6	nakha	conch shell powder	conch shell	Travancore & Cochin	1950	16
43	4	srika	turpentine resin	Mount Atlas mastic tree	Syria	2006	TBC
44	1	kundruka	Deodar cedar resin	Deodar cedar	Pakistan	2009	TBC

Then dividing (2) by 18 yields

$$\frac{2}{18}(\text{Agar wood}) + \frac{3}{18}(\text{bay leaves}) + \frac{5}{18}(\text{frankincense}) + \frac{8}{18}(\text{lichen}). \qquad (3)$$

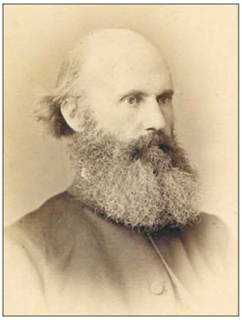
From Table 2 it seems that Varāhamihira could have made as many as 172 different perfume mixtures in all in c. 500 AD.

Varāhamihira used the magic square defined by the 4×4 fully magic matrix $\mathbf{V}_1$ (1) above and (4) below. The matrix $\mathbf{V}_1$ is not classic since each of the integers

Table 2 172 different perfume mixtures

1st substance	2nd substance	3rd substance	4th substance	Combinations
1	1	8	8	1
1	2	7	8	16
1	3	6	8	16
1	3	7	7	4
1	4	5	8	16
1	4	6	7	16
1	5	5	7	4
1	5	6	6	4
2	2	6	8	4
2	2	7	7	1
2	3	5	8	16
2	3	6	7	16
2	4	4	8	4
2	4	5	7	16
2	4	6	6	4
2	5	5	6	4
3	3	4	8	4
3	3	5	7	4
3	3	6	6	1
3	4	4	7	4
3	4	5	6	16
4	4	5	5	1
			GRAND TOTAL	172

$1, 2, \ldots, 8$ appears **twice**, but it is pandiagonal, the four numbers in each diagonal (with wrap-around) parallel to the main diagonals add up to the magic sum, here 18. Pandiagonal magic matrices are also called Nasik, following the seminal work [17] by Andrew Hollingworth Frost (1819–1907), while he was a British missionary in Nasik (Maharashtra), 1853–1869 [23].



Varāhamihira probably created the magic matrix $\mathbf{V}_1$ from a classic Nasik (pandiagonal) magic matrix with entries $1, 2, \ldots, 16$. A likely candidate [42] is $\mathbf{V}_2$ (4). Subtracting 8 from the eight largest entries in $\mathbf{V}_2$ yields $\mathbf{V}_1$. Rotating $\mathbf{V}_2$ by 90° counter-clockwise yields $\mathbf{V}_3$ (4). Cammann [36] calls $\mathbf{V}_3$ the "Plato magic matrix".

$$\mathbf{V}_1 = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix}, \qquad \mathbf{V}_2 = \begin{pmatrix} 10 & 3 & 13 & 8 \\ 5 & 16 & 2 & 11 \\ 4 & 9 & 7 & 14 \\ 15 & 6 & 12 & 1 \end{pmatrix}, \qquad \mathbf{V}_3 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix}. \tag{4}$$

3 Khajuraho and Dudhai (Jhansi district)

Our next oldest magic square was found at Khajuraho in the 9th century Jinanatha temple, "the largest and finest of the group of Jain temples at Khajuraho" ([14, No. 25, Plate XCV (p. 412/413), pp. 432–434 (1871)] and [21]). The Khajuraho magic square was found on the right door jamb, with the dedication: Bhata's son, Sri Deva Sarmma, may he be victorious. See also Fig. 2.

This magic square was also found "inscribed on a lintel, brought to light by a fall of masonry, in the 11th century Chota Surang shrine in Dudhai (Jhansi district)" [27, p. 3 (1915/1916)]; see also Fig. 3 and [27, Photograph 2035, Appendix B (1915/1916)]. Laxmi Bai, the Rani of Jhansi (c. 1835–1858), was the queen of the Maratha-ruled princely state of Jhansi. She was one of the leading figures of the Indian Rebellion of 1857 and a symbol of resistance to the rule of the British East India Company in the subcontinent. Jhalkari Bai (1830–1890) was a soldier in the women's army of Laxmi Bai and played an important role in the 1857 Rebellion during the battle of Jhansi. At the height of the battle, she disguised herself as the queen and fought on the front to let the queen escape safely out of the fort. (Easdale is one of the Slate Islands, in the Firth of Lorn, Scotland and was once the center of the British slate industry.)

No. 25 is the largest and finest of this group of Jain temples, being 60 feet in length by 30 feet in breadth. At the time of my first visit, in January 1852, it was fortunately deserted, and I was thus able to examine the inside with leisure. It was repaired five years ago by a Jain banker, and at my last visit in February 1865, I was not permitted to enter it. From the door-way, however, I could see that the whole of the statues, both large and small, which cover the entrance to the sanctum, had been most elaborately painted in blue, green, red, and yellow, and were still glistening as if recently varnished. The whole design of this temple is peculiar and novel. The interior consists of three chambers,—the *mandapa*, *antarâla*, and *garbhâgriha*,—or the open pillared hall, the vestibule and the sanctum, with a passage running all round the three. Externally the temple may be described as a simple oblong with a recess on each of the long sides, and a projection on each of the short sides, that to the east forming the entrance portico. The walls are decorated with numerous bands of mouldings, and with three rows of statues, as on No. 22, amongst which I recognised several of the Brahmanical gods. The ceiling of the portico is formed in the usual manner of cusped recesses, but the design is unusually beautiful. The square is reduced by overlapping stones, forming eight semi-circular recesses which are covered by three successive circles. In the middle circle and in the four corners there are five pendent drops, which terminate in flying figures of very graceful and original design. Over the entrance to the sanctum, there is a single naked figure seated, and on the sides there are two naked figures standing, while the front of the door-step is occupied with an elaborate design of the churning of the ocean. There is no inscription now attached to the temple recording its erection, but, on the left jamb of the entrance, there is a short inscription of eleven lines, announcing the gift of various gardens to the temple of *Jinânatha* by Bhavya Pâhila in the Samvat year 1011, or A. D. 954, during the reign of Raja Dhangga. The date of its erection may, therefore, be fixed as not later than A. D. 900, and, perhaps, even as early as A. D. 800. The pilgrims' records on this temple are both longer and more numerous than usual; and as two of them mention a Raja's son, it is probable that they may hereafter be of use in fixing the chronology of the kings as well as the date of the temple itself. I have numbered these records from I. to VIII. for easier reference.

On door-step.

I. *Mahârâja-putra Sri Jaya Singha khitah*
"The King's son, Sri Jaya Singha * * * "

On left jamb of door.

II. *Bhâta-putra Sri Goluna*
"Bhata's son, Sri Goluna.

III. *Sri Bhâta-putra Sri Mahula*
"Sri Bhata's son, Sri Mâhula."

IV. *Raja-putra Sri Jaya Singha*
Bhâta-putra Sri Pithan.
"The King's son, Sri Jaya Singha,"
"Bhata's son, Sri Pithan."

V. *Achârya Sri Deva Chandra, Sikhya Kumuda Chandra.*"
"The Teacher, Sri Deva Chandra; the Disciple, Kumuda Chandra."

On right jamb of door.

VII. *Bhata-putra Sri Deva Sarmma*
Viram (?) jayatu
"Bhata's son, Sri Deva Sarmma,
* * * * may he be victorious."

VIII. A magic square of 16 figures, which form 34 by addition every way, whether horizontally, perpendicularly, or diagonally. The figure 8 is remarkable for an additional stroke on the left side, which I take to be a mark of antiquity as it is a near approach to the figure in my Suhaniya numeral inscription. The figures are disposed thus—

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 2 The magic square in the 9th century Jinanatha temple at Khajuraho [14, No. 25, Plate XCV (p. 412/413), pp. 432–434 (1871)]

On the occasion of my inspection of the monuments of Dudhai, Jhansi District, I found at one end of, and on the underside of a fallen lintel in the shrine known locally as the Chota Surang, a so-called magic square. When the lintel was in position the square was hidden by the supporting column. The temple may be assigned with some certainty to the first half of the 11th Century A. D. The square is in the following form :—

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Mathematically it is interesting as possessing the following properties (*i*) the sum of each row, each column, and each diagonal is 34, (*ii*) the sum of all the numbers in each sub-square is also 34. The only other specimen known to me is the one as existing at Khujaraho where similar Chandel monuments are found. The lintel in question is seen lying in the foreground in Photograph 2035, Appendix B. Estampages of this epigraph have been supplied for publication to G. R. Kaye, Esqr., Curator, Bureau of Education, Government of India

Fig. 3 The magic square in the 11th century Chota Surang shrine in Dudhai (Jhansi district) [27, p. 3 (1915/1916)]

We define the magic square found in Khajuraho and in Dudhai by the Khajuraho–Dudhai magic matrix

$$\mathbf{K} = \begin{pmatrix} 7 & 12 & 1 & 14 \\ 2 & 13 & 8 & 11 \\ 16 & 3 & 10 & 5 \\ 9 & 6 & 15 & 4 \end{pmatrix}, \tag{5}$$

which, as we see, is Nasik (pandiagonal).

GSK 4:43–44. Magic squares of the odd order ($n = 2k + 1$): a construction method.

Here, Pherū's rule is very clear. It is summarized as follows.

(1) Construct the central column with the 'first sequence' (*paḍham-oli*, S. *prathama-āvali*) that consists of the arithmetical progression whose first term is one, common difference $(n + 1)$, and number of terms n.

n^2
$\vdots$
$2n + 3$
$n + 2$
1

(2) Fill the next left (or right) column by putting the number (q) that equals the sum, $(p+n)$, in the cell reached by the horse move (in chess, 'knight's move' in modern parlance) (*asu-kama*, S. *aśva-krama*) from the cell of p in the centre column. Note that the horse move to the second next column would not make a magic square.

q	
	p

or

	q
p	

where $q = p + n$.

(3) If the sum, $(p + n)$, is greater than n^2, then subtract n^2 from it.

(4) Repeat the same for the next columns. If the left (or the right) side is reached, go to the last (or the first) column, although this is not mentioned by Pherū.

Fig. 4 Algorithm for constructing odd-order magic squares [1, pp. 171–172]

4 Ṭhakkura Pherū (fl. 1291–1323)

In his *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics* [1, pp. 171–172], the Jain polymath Ṭhakkura Pherū (fl. 1291–1323) gave a method (Fig. 4) for constructing odd-order magic squares, e.g., the 5×5 Pherū magic matrix

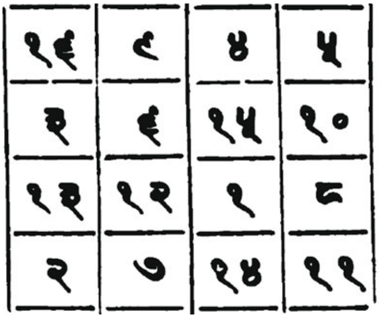
$$\mathbf{P} = \begin{pmatrix} 11 & 18 & 25 & 2 & 9 \\ 10 & 12 & 19 & 21 & 3 \\ 4 & 6 & 13 & 20 & 22 \\ 23 & 5 & 7 & 14 & 16 \\ 17 & 24 & 1 & 8 & 15 \end{pmatrix}, \tag{6}$$

where the numbers in the middle column are in arithmetic progression.

This method for the construction of odd-order magic squares was also given [3] in the late 17th century by the French diplomat and mathematician Simon de la Loubère (1642–1729), who learned the method from a *Monsieur Vincent, médecin provençal,* who (in turn) had learnt it in Surat, just north of Nasik; see also Frost [15, p. 93 (1865)].

5 Robert Shortrede (1800–1868), Andrew Hollingworth Frost (1819–1907), and Two Gwalior Magic Matrices

"As every thing tending to throw any certain light on the antiquities of India has an interest, I send you the following inscription of a Magic Square, which I copied last year [1841] from an old temple in the hill fort of Gwalior. It bears the date 1483 AD. The temple is on the northern side of the hill, and at one time it has been a very magnificent edifice, though now it be sorely dilapidated."



The 1842 article [11], which announced the discovery of this magic square (7), is signed by "Captain Shortreede", who we believe was (later Major-General) Robert Shortrede (1800–1868) [16], with the double "e" in "Shortreede" in the 1842 article, a typo. Robert Shortrede went to India in 1822 and was appointed to the Great Trigonometric Survey (GTS) with which he remained until 1845.

We define Shortrede's magic square by the Nasik (pandiagonal) Gwalior–Shortrede magic matrix

$$\mathbf{G}_1 = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix}. \tag{7}$$

Shortrede [11, (1842)] observed that the magic square defined by $\mathbf{G}_1$ has a "rhomboid" property: the places of the numbers

$$1,\ 2,\ 3,\ 4 \tag{8}$$

in $\mathbf{G}_1$ form a "rhomboid". So do the places of the numbers

$$5,\ 6,\ 7,\ 8; \quad 9, 10, 11, 12; \quad 13, 14, 15, 16. \tag{9}$$

We will say that $\mathbf{G}_1$ is "rhomboidal". Hayashi [47, (2008)] reports several 4×4 magic squares from the *Gaṇita Kaumudī* [2, (1356)] by Nārāyaṇa Paṇḍita (fl. 1340–

1400). Cammann [37, p. 274] gives Nārāyaṇa's favorite magic square defined by the Nasik (pandiagonal) magic matrix

$$\mathbf{N} = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix}, \tag{10}$$

which is also rhomboidal. Moreover, $\mathbf{N}$ is the "complement" of the Gwalior–Shortrede matrix $\mathbf{G}_1$ since

$$\mathbf{N} + \mathbf{G}_1 = \begin{pmatrix} 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \end{pmatrix}. \tag{11}$$

Andrew Hollingworth Frost (1819–1907) in his classic articles in *The Encyclopædia Britannica* [18, 9th ed. (1883); 24, 11th ed. (1911)] observed that a magic square like

$$\mathbf{G}_2 = \begin{pmatrix} 15 & 10 & 3 & 6 \\ 4 & 5 & 16 & 9 \\ 14 & 11 & 2 & 7 \\ 1 & 8 & 13 & 12 \end{pmatrix} \tag{12}$$

"is engraved in the Sanskrit character on the *gate* of the fort of Gwalior"; see also Strindberg [25, p. 307 (c. 1912)], Bragdon [26, pp. 47–48 (1915)], Kohtz and Brunner [30, p. 5 (1918)], Anderson [31, (1918); 32, 33, (1918/1919)], Bidev [38, Fig. 15, p. 51 (1981); 41, Fig. 15, p. 51 (1986)]. Frost [17, (1877)] gives the matrix $\mathbf{G}_2$ and observes that it is Nasik, but he does not connect $\mathbf{G}_2$ there with Gwalior. We will call $\mathbf{G}_2$ the Gwalior–Frost magic matrix.

We note that the Gwalior–Frost matrix $\mathbf{G}_2$ is also rhomboidal and that the difference matrix

$$\mathbf{G}_1 - \mathbf{G}_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \tag{13}$$

is fully magic with magic sum 0 and is Nasik (pandiagonal).

Frost [15, (1865)] wrote "That the idea of constructing [Nasik or pandiagonal] squares occurred centuries ago to Indian Mathematicians, is shewn [*sic*] by the existence of such a square engraven in the Sanscrit [*sic*] character upon the gate of the Fort at Gualior [*sic*], a record of which appears in the *Transactions of the Royal Asiatic Society*." Frost [15, (1865)], however, presents neither $\mathbf{G}_1$ nor $\mathbf{G}_2$, and we

I add a copy of the inscription in our common numerals, in case it may be wanted, as also a sample of the way in which it may be extended, which probably is similar to that in Dr. Franklin's Magical Square of Squares, but on this point I cannot speak positively, as I do not distinctly remember the particulars of Dr. Franklin's Square of Squares, and have at present no means of reference.

16	9	4	5	16			
3	6	15	10	3			
13	12	1	8	13	12		
2	7	14	11	2	7	14	11
	9	4	5	16	9	4	5
		15	10	3	6	15	10
				13	12	1	8
				2			

Fig. 5 Remarks by Robert Shortrede at the end of his article [11, p. 293 (1842)]

have not found any article on magic squares of any kind in the *Transactions of the Royal Asiatic Society of Great Britain and Ireland,* which was published between 1827 and 1834; the *Journal of the Royal Asiatic Society* began publication in 1834.

The Gwalior–Shortrede magic square $\mathbf{G}_1$ was copied by Shortrede [11, (1842)] from a "temple in the hill fort of Gwalior". Cammann [37, pp. 274–275 (1969)], commenting on the magic square $\mathbf{G}_1$ (7), wrote: "The more recent literature contains many references to 'the magic square on the gate of Gwalior Fort', apparently referring incorrectly to $\mathbf{G}_1$. I have personally twice climbed the citadel at Gwalior to examine the gates of the Old Fort, but found no magic squares on them". So were $\mathbf{G}_1$ and $\mathbf{G}_2$ both found in a "temple in the hill fort of Gwalior"?

Shortrede [11, (1842)] ends his article with some remarks, see our Fig. 5 above. The top left 4×4 corner in the "sample" above is the Gwalior–Shortreede matrix $\mathbf{G}_1$, while the submatrix boxed in red, $\mathbf{G}_3$ say, is the Gwalior–Frost matrix $\mathbf{G}_2$ with its rows flipped, i.e., $\mathbf{G}_3 = \mathbf{F}\mathbf{G}_2$, where

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is the 4×4 flip matrix. By "Dr. Franklin's "Magical Square of Squares", we believe that Shortrede [11, (1842)] was thinking of "A Magic Square of Squares" (our Fig. 5 above) by Benjamin Franklin is his (our Fig. 6 above) *Experiments and Observations on Electricity* [5, Plate IV, p. 353 (1769); 8, Plate IV, p. 360 (1774)], which Franklin called "the most magically magical of magic squares" [49, pp. 134–135].

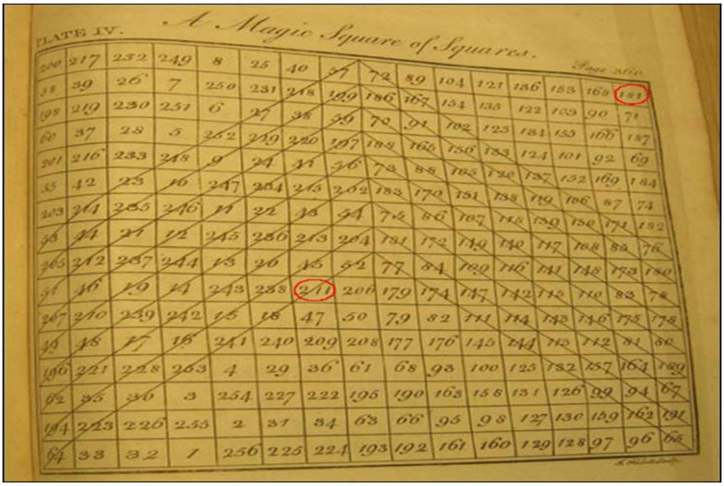
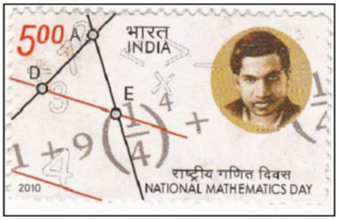


Fig. 6 "A Magic Square of Squares": Franklin [8, Plate IV, p. 360 (1774)], photograph from BrewBooks [44]. Entry (10, 7) corrected to 211 (from 241), but entry (1, 16) incorrect at 181 (should be 185). See also [6, 29]

6 Srinivasa Aiyangar Ramanujan (1887–1920)



Srinivasa Aiyangar Ramanujan (1887–1920) was born in Erode and lived in Kumbakonam (both then in Madras Presidency, both now in Tamil Nadu), and died in Chetput (Madras, now Chennai). Ramanujan lived most of his life in Kumbakonam, an ancient capital of the Chola Empire. Raja Raja Chola I, popularly known as Raja Raja the Great, ruled the Chola Empire between 985 and 1014 CE. The dozen or so major temples dating from this period made Kumbakonam a magnet to pilgrims from throughout South India.

Ramanujan was born on 22 December 1887, and in celebration of his birthday, on 22 December 1962 and on 22 December 2011, India Post issued a postage stamp in his honor. On 22 December 2011, Prime Minister Dr. Manmohan Singh in Chennai declared 22 December as National Mathematics Day, and declared 2012 as National Mathematics Year [51].

From 1914–1919 Ramanujan worked with G.H. Hardy at Trinity College, Cambridge, and lived in Whewell's Court [60].

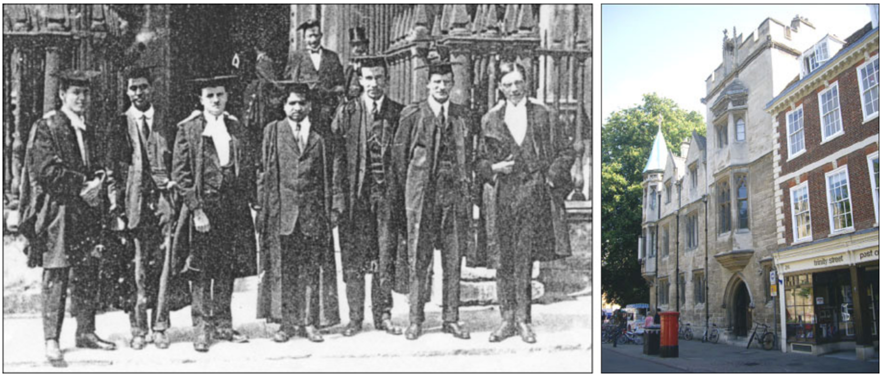
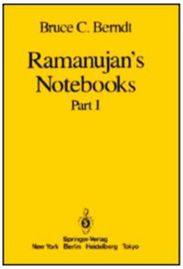
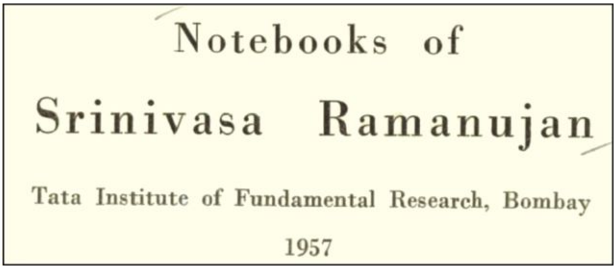


Fig. 7 (*Left panel*) Trinity College, Cambridge, with Ramanujan, center; (*right panel*) Whewell's Court [60]

Ramanujan's talent was said by the English mathematician Godfrey Harold Hardy (1877–1947) to be in the same league as that of Gauss, Euler, Cauchy, Newton, and Archimedes.

Ramanujan's work on magic squares is presented, in some detail, in Chap. 1 (pp. 16–24) of *Ramanujan's Notebooks, Part I*, by Bruce C. Berndt [40]. "The origin of Chap. 1 probably is found in Ramanujan's early school days and is therefore much earlier than the remainder of the notebooks." Ramanujan's work on magic squares was also presented, photographed from its original form, in *Notebooks of Srinivasa Ramanujan,* Volume I, Notebook 1, and Volume II, Notebook 2, pub. Tata Institute of Fundamental Research [34].





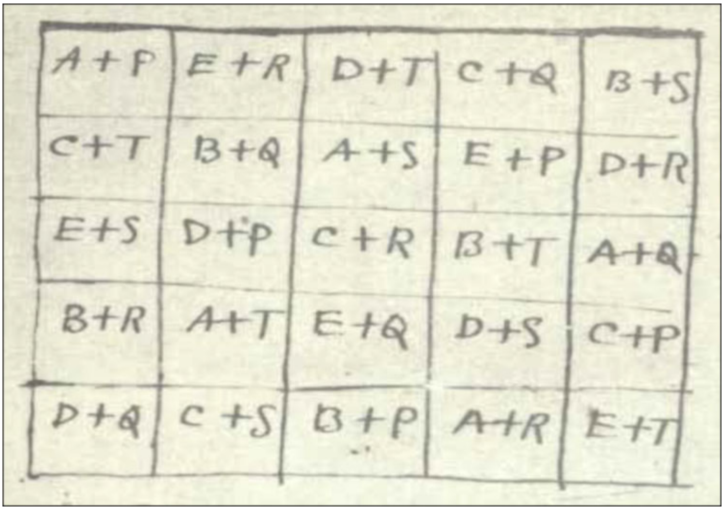
In Berndt [40, Corollary 1, p. 17], we find: In a 3×3 magic square, the elements in the middle row, middle column, and each [main] diagonal are in arithmetic progression. And so we have the general form for a 3×3 magic matrix

$$\begin{aligned}\mathbf{R}_3 &= \begin{pmatrix} h+u & h-u+v & h-v \\ h-u-v & h & h+u+v \\ h+v & h+u-v & h-u \end{pmatrix} \\ &= h\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + u\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} + v\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}. \qquad (14)\end{aligned}$$

Berndt [40, p. 21] presents

$$\begin{aligned}\mathbf{R}_4 &= \begin{pmatrix} a+p & d+s & c+q & b+r \\ c+r & b+q & a+s & d+p \\ b+s & c+p & d+r & a+q \\ d+q & a+r & b+p & c+s \end{pmatrix} \\ &= \begin{pmatrix} a & d & c & b \\ c & b & a & d \\ b & c & d & a \\ d & a & b & c \end{pmatrix} + \begin{pmatrix} p & s & q & r \\ r & q & s & p \\ s & p & r & q \\ q & r & p & s \end{pmatrix}, \qquad (15)\end{aligned}$$

the sum of two orthogonal Latin squares (Graeco–Latin square), while Ramanujan [34, Volume II, Notebook 2, p. 12 = original p. 8] gives the following 5×5 magic square, which is also the sum of two orthogonal Latin squares (Graeco–Latin square):



A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

Berndt [40] reports two 7×7 (p. 24) and two 8×8 (p. 22) magic squares (but apparently no 6×6) by Ramanujan, including

$$
\begin{aligned}
\mathbf{R}_7 &= \begin{pmatrix} 1 & 49 & 41 & 33 & 25 & 17 & 9 \\ 18 & 10 & 2 & 43 & 42 & 34 & 26 \\ 35 & 27 & 19 & 11 & 3 & 44 & 36 \\ 45 & 37 & 29 & 28 & 20 & 12 & 4 \\ 13 & 5 & 46 & 38 & 30 & 22 & 21 \\ 23 & 15 & 14 & 6 & 47 & 39 & 31 \\ 40 & 32 & 24 & 16 & 8 & 7 & 48 \end{pmatrix}, \\
\mathbf{R}_8 &= \begin{pmatrix} 1 & 62 & 59 & 8 & 9 & 54 & 51 & 16 \\ 60 & 7 & 2 & 61 & 52 & 15 & 10 & 53 \\ 6 & 57 & 64 & 3 & 14 & 49 & 56 & 11 \\ 63 & 4 & 5 & 58 & 55 & 12 & 13 & 50 \\ 17 & 46 & 43 & 24 & 25 & 38 & 35 & 32 \\ 44 & 23 & 18 & 45 & 36 & 31 & 26 & 37 \\ 22 & 41 & 48 & 19 & 30 & 33 & 40 & 27 \\ 47 & 20 & 21 & 42 & 39 & 28 & 29 & 34 \end{pmatrix},
\end{aligned}
\tag{16}
$$

and says that $\mathbf{R}_8$ is "constructed from four 4×4 magic squares". We find that $\mathbf{R}_8$ may be constructed from <u>two</u> 4×4 magic squares. We write

$$
\mathbf{R}_8 = \begin{pmatrix} \mathbf{R}_8^{(11)} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{R}_8^{(12)} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{0}_4 \\ \mathbf{R}_8^{(21)} & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{R}_8^{(22)} \end{pmatrix}, \tag{17}
$$

where $\mathbf{R}_8^{(11)}, \mathbf{R}_8^{(12)}, \mathbf{R}_8^{(21)}, \mathbf{R}_8^{(22)}$ are 4×4 fully-magic Nasik (pandiagonal) matrices, each with magic sum 130 equal to half the magic sum of $\mathbf{R}_8$. Moreover, the four 4×4 magic matrices $\mathbf{R}_8^{(11)}, \mathbf{R}_8^{(12)}, \mathbf{R}_8^{(21)}, \mathbf{R}_8^{(22)}$ are interchangeable, and so there are $4! = 24$ fully magic Nasik (pandiagonal) 8×8 matrices like $\mathbf{R}_8$. Barnard [20, p. 227] says that a magic matrix with this property is "tessellated". We write

$$\begin{aligned}
\mathbf{R}_8^{(12)} &= \mathbf{R}_8^{(11)} + 8\mathbf{X}, \\
\mathbf{R}_8^{(21)} &= \mathbf{R}_8^{(11)} + 16\mathbf{X}, \\
\mathbf{R}_8^{(22)} &= \mathbf{R}_8^{(11)} + 24\mathbf{X},
\end{aligned} \tag{18}$$

where the fully-magic Nasik (pandiagonal) 4×4 matrices

$$\begin{aligned}
\mathbf{R}_8^{(11)} &= \begin{pmatrix} 1 & 62 & 59 & 8 \\ 60 & 7 & 2 & 61 \\ 6 & 57 & 64 & 3 \\ 63 & 4 & 5 & 58 \end{pmatrix}, \\
\mathbf{X} &= \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix},
\end{aligned} \tag{19}$$

with magic sums $m(\mathbf{R}_8^{(11)}) = \frac{1}{2}m(\mathbf{R}_8) = 130$ and $m(\mathbf{X}) = 0$. So we see that Ramanujan's 8×8 magic matrix $\mathbf{R}_8$ may be constructed from two 4×4 magic matrices $\mathbf{R}_8^{(11)}$ and $\mathbf{X}$ (19), (20):

$$\begin{aligned}
\mathbf{R}_8 &= \begin{pmatrix} 1 & 62 & 59 & 8 & 9 & 54 & 51 & 16 \\ 60 & 7 & 2 & 61 & 52 & 15 & 10 & 53 \\ 6 & 57 & 64 & 3 & 14 & 49 & 56 & 11 \\ 63 & 4 & 5 & 58 & 55 & 12 & 13 & 50 \\ 17 & 46 & 43 & 24 & 25 & 38 & 35 & 32 \\ 44 & 23 & 18 & 45 & 36 & 31 & 26 & 37 \\ 22 & 41 & 48 & 19 & 30 & 33 & 40 & 27 \\ 47 & 20 & 21 & 42 & 39 & 28 & 29 & 34 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \mathbf{R}_8^{(11)} + 8\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \otimes \mathbf{X}.
\end{aligned} \tag{20}$$

A 16×16 tessellated magic square was constructed by Isaac Dalby (1744–1824) and published by Hutton [10, vol. 2, pp. 5–9, Plate XIX, Fig. 1 (1815)]:

$$\mathbf{R}_{16} = \begin{pmatrix} 226 & 255 & 18 & 15 & 194 & 223 & 50 & 47 & 162 & 191 & 82 & 79 & 130 & 159 & 114 & 111 \\ 32 & 1 & 240 & 241 & 64 & 33 & 208 & 209 & 96 & 65 & 176 & 177 & 128 & 97 & 144 & 145 \\ 239 & 242 & 31 & 2 & 207 & 210 & 63 & 34 & 175 & 178 & 95 & 66 & 143 & 146 & 127 & 98 \\ 17 & 16 & 225 & 256 & 49 & 48 & 193 & 224 & 81 & 80 & 161 & 192 & 113 & 112 & 129 & 160 \\ 228 & 253 & 20 & 13 & 196 & 221 & 52 & 45 & 164 & 189 & 84 & 77 & 132 & 157 & 116 & 109 \\ 30 & 3 & 238 & 243 & 62 & 35 & 206 & 211 & 94 & 67 & 174 & 179 & 126 & 99 & 142 & 147 \\ 237 & 244 & 29 & 4 & 205 & 212 & 61 & 36 & 173 & 180 & 93 & 68 & 141 & 148 & 125 & 100 \\ 19 & 14 & 227 & 254 & 51 & 46 & 195 & 222 & 83 & 78 & 163 & 190 & 115 & 110 & 131 & 158 \\ 230 & 251 & 22 & 11 & 198 & 219 & 54 & 43 & 166 & 187 & 86 & 75 & 134 & 155 & 118 & 107 \\ 28 & 5 & 236 & 245 & 60 & 37 & 204 & 213 & 92 & 69 & 172 & 181 & 124 & 101 & 140 & 149 \\ 235 & 246 & 27 & 6 & 203 & 214 & 59 & 38 & 171 & 182 & 91 & 70 & 139 & 150 & 123 & 102 \\ 21 & 12 & 229 & 252 & 53 & 44 & 197 & 220 & 85 & 76 & 165 & 188 & 117 & 108 & 133 & 156 \\ 232 & 249 & 24 & 9 & 200 & 217 & 56 & 41 & 168 & 185 & 88 & 73 & 136 & 153 & 120 & 105 \\ 26 & 7 & 234 & 247 & 58 & 39 & 202 & 215 & 90 & 71 & 170 & 183 & 122 & 103 & 138 & 151 \\ 233 & 248 & 25 & 8 & 201 & 216 & 57 & 40 & 169 & 184 & 89 & 72 & 137 & 152 & 121 & 104 \\ 23 & 10 & 231 & 250 & 55 & 42 & 199 & 218 & 87 & 74 & 167 & 186 & 119 & 106 & 135 & 154 \end{pmatrix}. \tag{21}$$

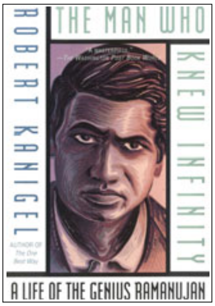
In $\mathbf{R}_{16}$, the sixteen 4×4 magic submatrices $\mathbf{R}_{16}^{(11)}, \mathbf{R}_{16}^{(12)}, \ldots, \mathbf{R}_{16}^{(44)}$ (starting with the top left 4×4 submatrix and then to the right and down) are interchangeable, and so there are $16! = 20,922,789,888,000$ fully magic Nasik (pandiagonal) 16×16 matrices like $\mathbf{R}_{16}$. See also Newton [12, (1844)]. We find that

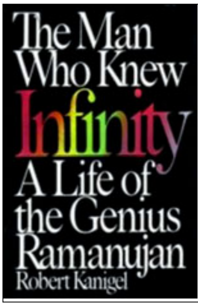
$$\mathbf{R}_{16} = \mathbf{E}_4 \otimes \mathbf{R}_{16}^{(11)} + 32\mathbf{G} \otimes \mathbf{X}_1 + 2\mathbf{G}' \otimes \mathbf{X}_2, \tag{22}$$

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

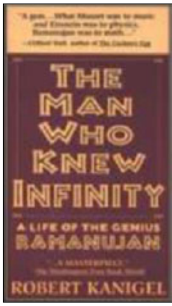
$$\mathbf{X}_1 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \tag{23}$$

$$\mathbf{X}_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$





Original Scribners hardcover edition, 1991



Original Washington Square Press paperback edition, 1992

German edition, Verlag Vieweg, 1993

Italian edition, *L'uomo che vide l'infinito*, Rizzoli, 2003

For a "model of the biographer's art", we recommend *The Man who Knew Infinity: A Life of the Genius Ramanujan* by Robert Kanigel [43]; see also [35]. There we learn that [43, Prologue, pp. 1–2]:

One day in the summer of 1913, a twenty-year-old Bengali from an old and prosperous Calcutta family stood in the chapel of King's College, Cambridge, England. Prasantha Chandra Mahalanobis (1893–1972) was smitten. Mahalanobis was an Indian scientist who founded the Indian Statistical Institute and contributed to the design of large-scale sample surveys.

Scarcely off the boat from India and planning to study in London, Mahalanobis had come up to Cambridge on the train for the day to sightsee. The next day he met with the provost, and soon, to his astonishment and delight, he was a student at King's College, Cambridge. Mahalanobis had been at Cambridge for about six months when his mathematics tutor asked him:

"Have you met your wonderful countryman Ramanujan?"

He had not yet met him, but he had heard of him. Soon Mahalanobis did meet Ramanujan, and the two became friends; on Sunday mornings, after breakfast, they would go for long walks, talk about life, philosophy, mathematics. Later, looking back, Mahalanobis would date the flowering of their friendship to one day in the fall following Ramanujan's arrival. He had gone to see him at his place in Whewell's Court. Cambridge (Fig. 7, right panel) was deserted. And cold.

> "Are you warm at night?",

asked Mahalanobis, seeing Ramanujan beside the fire.

> "No",

replied the mathematician from always warm Madras, where he slept with his overcoat on, wrapped in a shawl. Until early 1914, Ramanujan lived in a traditional home on Sarangapani Street in Kumbakonam. The family home is now a museum. From 1914–1919 Ramanujan lived in Whewell's Court, a 5-minute walk from Hardy's rooms. Whewell's Court was a 3-story stone warren of rooms laced with arched Gothic windows and pierced at intervals by staircases leading to rooms.

Figuring his friend had not enough blankets, Mahalanobis stepped back into the little sleeping alcove on the other side of the fireplace. The bedspread was loose, as if Ramanujan had just gotten up. Yet the blankets lay perfectly undisturbed, tucked neatly under the mattress. Yes, Ramanujan had enough blankets; he just did not know what to do with them. Gently, patiently, Mahalanobis showed him how you peeled them back, made a little hollow for yourself, slipped inside . . . .

For five years, walled off from India by World War One (1914–1918), Ramanujan would remain in strange, cold, distant England, fashioning, through 21 major papers, an enduring mathematical legacy. Then, he would go home to India to a hero's welcome.

> "Srinivasa Ramanujan", an Englishman would later say of him, "was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years."

Acknowledgements This paper is based, in part, on the invited talks [52, 53, 56] presented at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–7 & 10–11 January 2012; I am very grateful to Professor K. Manjunatha Prasad and Professor T.E.S. Raghavan who reminded me of Ramanujan's work on magic squares and to Dr. B. Chaluvaraju for drawing my attention to "Bangalore University's old collections in the library which deal with Yantras and magic squares." In addition, many thanks go to Miguel Angel Amela, Nicolas C. Ammerlaan, Thomas W. Ammerlaan, Pavel Chebotarev, Daniel J.H. Rosenthal, Evelyn Matheson Styan, and Götz Trenkler for their help. This article is based also, in part, on talks [50, 56] given at the Annual Meeting of the Canadian Society for the History and Philosophy of Mathematics (CSHPM), University of Waterloo, 27–29 May 2012, and in the Frederick V. Pohle Colloquium in the History of Mathematics, hosted by the Department of Mathematics & Computer Science at Adelphi University, Garden City, NY, 13 October 2010. This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

References

1. [c. 1315/2009-Pherū] *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics by* Ṭhakkura Pherū [Ṭhakkura Pherū (fl. 1291–1323)]. In: English and edited with Introduction, Translation & Mathematical Commentary, by SaKHYa [Sreeramula Rajeswara Sarma, Takanori Kusuba, Takao Hayashi and Michio Yano]. pub. Manohar, New Delhi (2009). First published in Sanskrit (early 14th century)
2. [1356/1942-Nārāyaṇa] *Gaṇita Kaumudī* by Nārāyaṇa Paṇḍita (fl. 1340–1400), edited by Padmakara Dvivedi, Princess of Wales Saraswvati Bhavana Texts, no. 57, part II, Banares, 1942: reprinted as *The Ganitha Koumudi Part-2,* by Narayana Pandita, Sanskrit Edition (with introduction in English by Padmakara Dvivedi), Nabu Press (2011). First published (1356)
3. [1693/1986-La Loubère] *The Kingdom of Siam* by Simon de La Loubère, with an Introduction by David K. Wyatt, pub. Oxford University Press (1986): online at Southeast Asia Visions. Reprint of *A New Historical Relation of the Kingdom of Siam, Tome II: Illustrated with Sculptures*, by Simon de La Loubère, pub. Tho. Horne at the Royal Exchange, Francis Saunders at the New Exchange & Tho. Bennet at the Half-Moon in St. Pauls Church-yard, London (1693)
4. [1769-Franklin/4th-Letter27] "Letter XXVII from Benj. Franklin, Esq; of Philadelphia [Benjamin Franklin (1706–1790)]. To Peter Collinson, Esq. at London [Peter Collinson (1694–1768)]: "According to your request I now send you the Arithmetical Curiosity [A Magic Square of Squares], of which this is the history ... ," In *Experiments and Observations on Electricity* [5, 4th ed., pp. 350–354 (1769)]. See also [7]
5. [1769/2010-Franklin/4th] *Experiments and Observations on Electricity, made at Philadelphia in America:* To which are added, letters and papers on philosophical subjects. The whole corrected, methodized, improved, and now first collected into one volume, and illustrated with copper plates, by Benjamin Franklin, [Fourth Edition] printed for David Henry and sold by Francis Newbery, at the Corner of St. Paul's Church-Yard, London, 496 pp (1769). Reprinted as Gale ECCO Print Edition (27 May 2010). For "A Magic Square of Squares", see also [4], and for the 5th Edition, see [8]
6. [1773/2012-Dubourg/Franklin] *Œuvres de M. Franklin* [Benjamin Franklin (1706–1790)], by Jacques Barbeu-Dubourg [Jacques Barbeu-Dubourg (1709–1779)], two volumes, Collection Léo Pariseau, pub. L'Auteur, Paris (1773). Reprinted by Nabu Press (2012)
7. [1774-Franklin/5th-Letter27] "Letter XXVII from Benj. Franklin, Esq; of Philadelphia [Benjamin Franklin (1706–1790)]. To Peter Collinson, Esq. at London [Peter Collinson (1694–1768)]: According to your request I now send you the Arithmetical Curiosity [A Magic Square of Squares], of which this is the history ... ," In *Experiments and Observations on Electricity* [8, 5th ed., pp. 358–362 (1774)]. See also [4]
8. [1774-Franklin/5th] *Experiments and Observations on Electricity, made at Philadelphia in America:* To which are added, letters and papers on philosophical subjects. The whole corrected, methodized, improved, and now collected into one volume, and illustrated with copper plates, by Benjamin Franklin, The Fifth Edition. Printed for F. Newbery, at the Corner of St. Paul's Church-Yard, London, 514 pp. (1774). Reprinted as Gale ECCO Print Edition (16 June 2010). For "A Magic Square of Squares", see also [7], and for the 4th Edition, see [5]
9. [1815-Dalby/Hutton-Magic] "Magic square" by Charles Hutton [Charles Hutton (1737–1823)], In *A Philosophical and Mathematical Dictionary* [10, vol. 2, pp. 5–9, Plate XIX, Fig. 1]
10. [1815-Hutton] *A Philosophical and Mathematical Dictionary, containing an explanation of the terms, and an account of the several subjects,* comprised under the heads mathematics [*sic*], astronomy, and philosophy both natural and experimental, with an historical account of the rise, progress and present state of these sciences; also memoirs of the lives and writings of the most eminent authors, both ancient and modern, who by their discoveries or improvements have contributed to the advancement of them, by Charles Hutton, New Edition, Volume 2, printed for the author, London (1815): online Google eBook. See also [9]

11. [1841/1842-Shortreede/Shortrede] “On an ancient magic square, cut in a temple at Gwalior” by “Captain Shortreede” [Robert Shortrede (1800–1868)], Journal of the Asiatic Society of Bengal, New Series, vol. 11, no. 124 (vol. 71, no. 37), pp. 292–293 (1842)
12. [1844/1845-Newton/magic] “Magic squares” by I. Newton [Israel Newton (1763–1856)], “Sept. 28, 1844, in the 82d year of his age”, In *A New Method of Ascertaining Interest and Discount* [13, pp. 15–16 (1845)]
13. [1845-Newton/Interest] *A New Method of Ascertaining Interest and Discount, at various rates, both simple and compound, and of interest on indorsements* [sic]: *also a few magic squares of a singular quality* by Israel Newton of Norwich, Vt. [Israel Newton (1763–1856) of Norwich, Vermont], E.P. Walton and Sons, printers, Montpelier [Vermont]. For “Magic squares”, see [12]
14. [1864/1871-Cunningham] *Four Reports made during the years 1862–63–64–65,* Volume II, by Alexander Cunningham [Sir Alexander Cunningham (1814–1893)], pub. Archæological Survey of India, Simla (1871): online Google eBook. “Khajurâho/Khajurâya”, pp. 412–439, with magic square on p. 434
15. [1865-Frost] “Invention of magic cubes, and construction of magic squares possessing additional properties” by Rev. A.H. Frost [Andrew Hollingworth Frost (1819–1907)] (communicated by Rev. Percival Frost (1817–1898)], The Quarterly Journal of Pure and Applied Mathematics, vol. 7, no. 25, pp. 92–96 (February 1865) & vol. 7, no. 26, pp. 97–102 (June 1865): online Google eBook
16. [1868-Shortrede/obit] Unsigned untitled obituary of Major-General Robert Shortrede (1800–1866), Monthly Notices of the Royal Astronomical Society, vol. 29, no. 1, pp. 120–121 (13 November 1868)
17. [1877-Frost] “On the general properties of Nasik squares” by A.H. Frost [Andrew Hollingworth Frost (1819–1907)], The Quarterly Journal of Pure and Applied Mathematics, vol. 15, no. 57, pp. 34–49 (June 1877): online Google eBook
18. [1883-Frost/EB-9th] “Magic square”, unsigned [by Andrew Hollingworth Frost (1819–1907)], In *The Encyclopædia Britannica: A Dictionary of Arts, Sciences, and General Literature,* Ninth Edition, pub. Charles Scribner’s Sons, New York, vol. 15, pp. 213–216 (1883)
19. [1885/1987-Iyer/Varāhamihira] *The Brihat Samhita of Varaha Mihira, Part II,* translated into English by N. Chidambaram Iyer, Aryan Miscellany, Samhita Series, printed at the Foster Press, Madras (1885). Reprinted as *The Bṛhat Saṁhitā of Varāh Mihira,* translated into English with notes by N.C. Iyer, Sri Garib Das Oriental Series No. 43, pub. Sri Satguru, Delhi (1987). See also Bhat [39]
20. [1888-Barnard] “Theory of magic squares and of magic cubes” by F.A.P. Barnard [Frederick Augustus Porter Barnard (1809–1889)], Memoirs of the National Academy of Sciences USA, vol. 4, Sixth Memoir, pp. 207–270 (1888)
21. [1904-Schilling/Kielhorn] “Mathematische Gesellschaft in Göttingen: Mitteilungen und Nachrichten”, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 13, pp. 383–384 (1904): online at DigiZeitschriften: Das Deutsche Digitale Zeitschriftenarchiv
22. [1905/1906-Cashmore] “XI. Chess magic squares” by M. Cashmore, M.P.S., *Addresses and Papers read at the Joint Meeting of the British and South African Associations for the Advancement of Science, held in South Africa, 1905,* vol. 1, pp. 83–90 (1906). See also [38]
23. [1907/2009-Frost/Boyer] “Andrew Hollingworth Frost [Hull 1819–Cambridge 1907], Biographical sketch” by Christian Boyer: online at `multimagie.com`. Includes 1870 portrait from the Church Mission Society archives
24. [1911/2010-Frost/EB-11th] “Magic square” by A.H. Frost [Andrew Hollingworth Frost (1819–1907)], In *The Encyclopædia Britannica: A Dictionary of Arts, Sciences, Literature and General Information,* 11th edition, vol. 17: Lord Chamberlain to Mecklenburg, pub. Cambridge University Press, pp. 310–313 (1911). Article reset as “1911 Encyclopædia Britannica/Magic Square” online at Wikisource
25. [1912/1921-Strindberg] “Schackspelets Hemligheter” [Chess Game Secrets], by August Strindberg [August Strindberg (1849–1912)], *En Blå Bok,* by August Strindberg, Del 1–4,

pp. 307–308 (c. 1912). Reprinted in *Samlade Skrifter av August Strindberg,* 46–48, pub. A. Bonnier, Stockholm (1918–1921): online at Project Runeberg

26. [1915-*Projective*] *Projective Ornament,* pub. The Manas Press, Rochester NY (1915): online at Hathi Trust
27. [1915/1916-*Progress*] *Annual Progress Report of the Superintendent, Hindu and Buddhist monuments, Northern Circle, for the year ending 31st March 1916,* by H. Hargreaves [Harold Hargreaves (b. 1876)], Superintendent, Hindu and Buddhist monuments, Northern Circle, Simla, pub. Archæological Survey of India, Lahore (18 September 1916): selected pages online at McGill
28. [1917-Andrews] *Magic Squares and Cubes* by W.S. Andrews [William Symes Andrews (1847–1929)], with chapters by other writers [Charles Albert Browne, Jr., Paul Carus, Lorraine Screven Frierson, H.M. Kingery, Charles Planck, D.F. Savage, Harry A. Sayles, Frederic A. Woodruff & John Worthington], 2nd edition, revised and enlarged, pub. Open Court, Chicago (1917): online at David Moews's home page
29. [1917-Carus/Franklin] "An analysis of the Franklin squares" by Paul Carus, In *Magic Squares and Cubes* [28, pp. 96–112 (1917)]
30. [1917/2012-KohtzBrunner] *Das Magische Quadrat und die Erfindung des Schachspiels* by Johannes Kohtz [Johannes Kohtz (1843–1918)] & Erich Brunner [Erich Anselm Brunner (1885–1938)], handwritten manuscript, ca. 100 pp., section by Kohtz (dated April 1917), Sächsische Landesbibliothek – Staats- und Universitätsbibliothek Dresden (SLUB), pdf file Mscr.Dresd.App.157 (2012): online in Dresden
31. [1918-Anderson/supermagic1] "The 34 supermagic square" by Brigadier-General F.J. Anderson, C.B. [Brigadier-General Sir Francis James Anderson, CB KBE (1860–1920)], Science Progress in the Twentieth Century: A Quarterly Review of Scientific Work, Thought & Affairs, vol. 13, no. 49, pp. 86–96 ("Popular Science" section, July 1918): online at Hathi Trust. Article reset and republished as [32, 33]
32. [1918-Anderson/supermagic2] "Survival of the mystical mathematician: riddle of a series of numbers found in ancient India", Current Opinion, vol. 65, no. 6, pp. 376–378 ("Science and Discovery" section, December 1918): online Google eBook. Article originally published as [31] , but this version has some introductory words though omits the last paragraphs on graphs with pretty patterns and Dürer's *Melencolia I*. Article reset and republished as [33]
33. [1919-Anderson/supermagic3] "The 34 supermagic square: mathematical recreations of ancient origin", Scientific American Supplement, vol. 87, no. 2246, pp. 44–45 (18 January 1919): online Google eBook. Article originally published as [31, 32]
34. [1957-Tata] *Notebooks of Srinivasa Ramamujan,* Volume I, Notebook 1, and Volume II, Notebook 2, pub. Tata Institute of Fundamental Research, Bombay (1957)
35. [1967-Ranganathan] *Ramanujan: The Man and the Mathematician,* by S.R. Ranganathan, pub. Asia Publishing House, Bombay (1967)
36. [1969-Cammann1] "Islamic and Indian magic squares, Part I" by Schuyler Cammann [Schuyler Van Rensselaer Cammann (1912–1991)], History of Religions, vol. 8, no. 3, pp. 181–209 (February 1969): online at JSTOR. For Part II, see [37]
37. [1969-Cammann2] "Islamic and Indian magic squares, Part II" by Schuyler Cammann, History of Religions, vol. 8, no. 4, pp. 271–299 (May 1969): online at JSTOR. For Part I, see [36]
38. [1981-Bidev/Kritik] "Bedenken gegen Anzahlbestimmung von Schach-Nasiks 8 × 8: Kritik der Arbeit von M. Cashmore, Capetown 1907: Chess magic squares [22]" by Pavle Bidev [Pavle Bidev (1912–1988)], typewritten manuscript [16 pp.], pub. Selbstverlag, YU-Igalo (c. 1981)
39. [1982/1987-Bhat] *Varāhamihira's Bṛhat Saṁhitā, Part Two* by M. Ramakrishna Bhat, with English translation, exhaustive notes and literary comments, pub. Motilal Banarsidass, Delhi, 1982; 2nd revised edition (1987), reprinted (1993). See also Iyer [19]
40. [1985-Berndt] *Ramanujan's Notebooks, Part 1* by Bruce C. Berndt, pub. Springer, New York (1985)

41. [1986-Bidev/*Stammt*] *Stammt Schach aus Alt Indien oder China? Schach-seine Urgeschichte, Schachmagische Mathematik, Schachmystische Philosophie,* Part 1: German, Part 2: English "Did chess originate in China or India? Chess and magic squares" by Pavle Bidev, pub. Selbstverlag, YU-Igalo (1986)
42. [1987-Hayashi/HistMath] "Varāhamihira's pandiagonal magic square of order four" by Takao Hayashi, Historia Mathematica, vol. 14, no. 2, pp. 159–166 (May 1987): online at ScienceDirect
43. [1991-Kanigel] *The Man Who Knew Infinity: A Life of the Genius Ramanujan* by Robert Kanigel, pub. Charles Scribner's Sons (1991)
44. [2006-BrewBooks/Franklin] Franklin's "most magically magical" 16×16 "magic square of squares" displayed [7, p. 360 (1774)] as exhibit i061706 346 at the British Library, London, photo by Fabrizio Pivari (1 June 2006): online at BrewBooks
45. [2007-Hayashi] "Magic squares in Indian mathematics" by Takao Hayashi, In *Encyclopaedia of Classical Indian Sciences* [46, pp. 222–233 (2007)]
46. [2007-SelinNarasimha] *Encyclopaedia of Classical Indian Sciences: Natural Science, Technology and Medicine,* edited by Helaine Selin & Roddam Narasimha, pub. Universities Press, Hyderabad (2007)
47. [2008-Hayashi/Selin] "Magic squares in Indian mathematics" by Takao Hayashi, In *Encyclopedia of the History of Science, Technology, and Medicine in Non-Western Cultures* [48, Part 13, pp. 1252–1259 (2008)]
48. [2008-Selin] *Encyclopedia of the History of Science, Technology, and Medicine in Non-Western Cultures,* 2nd edition, edited by Helaine Selin, pub. Springer, Berlin (2008): online at SpringerLink. For "Magic squares in Indian mathematics", see [47]
49. [2008-Pasles] *Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey* by Paul C. Pasles, pub. Princeton University Press (2008)
50. [2010-Styan/Pohle] "Some comments on old magic squares illustrated with postage stamps" by George P.H. Styan, invited talk presented in The Frederick V. Pohle Colloquium in the History of Mathematics, hosted by the Department of Mathematics & Computer Science at Adelphi University, Garden City NY (13 October 2010): beamer file online at McGill
51. [2011-The Hindu] "Year-long celebration: A statement by the Ramanujan Mathematical Society on the series of activities planned in 2012 to mark the anniversary", The Hindu (26 December 2011): online at The Hindu
52. [2012-Manipal1] "An illustrated introduction to some magic squares from India" by George P.H. Styan, Invited talk presented at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India (2–7 & 10–11 January 2012). Video (updated 17 February 2012): online at Vimeo. This talk is based, in part, on the talk [50] and then revised as the talk [56]
53. [2012-Manipal2] "An introduction to Ramanujan's magic squares" by George P.H. Styan, invited talk prepared for the International Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India (2–7 & 10–11 January 2012): beamer file online at McGill. This follows up on the talk [52] and then combined with [52] as the talk [56]
54. [2012-Manipal3] "An illustrated introduction to Yantra magic squares and Agrippa-type magic matrices" by George P.H. Styan, Götz Trenkler & Ka Lok Chu, In *Lectures on Matrix and Graph Methods* [55, pp. 159–220 (2012)]
55. [2012-Manipal/*Lectures*] *Lectures on Matrix and Graph Methods,* edited by Ravindra B. Bapat, Steve Kirkland, K. Manjunatha Prasad, and & Simo Puntanen, pub. Manipal University Press (2012). Lecture notes from the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India (2–7 & 10–11 January 2012). See also [54]
56. [2012-Waterloo] "An illustrated introduction to some magic squares from India" by George P.H. Styan, talk presented at the Annual Meeting of the Canadian Society for the History and Philosophy of Mathematics (CSHPM-2012), University of Waterloo, Waterloo, Ontario (27–

29 May 2012): beamer pdf file (updated 19 June 2012): 44 pp. This talk is based, in part, on the talks [50, 52, 53]

57. [2012-G75] "An illustrated introduction to Euler and Fitting factorizations and Anderson graphs for classic magic matrices" by George P.H. Styan, Special invited talk presented (on 19 July 2012) in the Special Session to celebrate George P.H. Styan's 75th Birthday at the International Conference on Trends and Perspectives in Linear Statistical Inference (LINSTAT-2012) & 21st International Workshop on Matrices and Statistics (IWMS-2012), Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences, Bedlewo (near Poznań), Poland (16–20 July 2012): handout [58]; see also [59]
58. [2012-G75bib] "An illustrated and annotated bibliography for Anderson graphs, Bragdon's "magic lines" diagrams and Moran's "sequence designs", Report 2012-06, Dept. of Mathematics and Statistics, McGill University, 10 pp. (17 July 2012). Handout (75 references) for the talk [57], expanded as [59]
59. [2012-MagicBib] "An illustrated, annotated and indexed bibliography for magic squares, with special emphasis on Anderson graphs, Knight's Tour diagrams, tessellated magic squares, and magic squares from India" by George P.H. Styan, Report 2012-07, Dept. of Mathematics and Statistics, McGill University, updated December 3, 2012. Expansion of Report 2012-06 [58]
60. [2012-Ramanujan/Wkp] "Srinivasa Ramanujan": online at *Wikipedia, the free encyclopedia,* last modified on 3 September 2012 at 00:11

A Report on CMTGIM 2012, Manipal

International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices 02–07 & 10–11 January 2012 Department of Statistics, Manipal University, Manipal, India

R.B. Bapat and K. Manjunatha Prasad

1 About Workshop and Conference

1.1 Objective of the Workshop

The objective of the workshop was to provide a platform for the young generation to have an exposure to lectures from leading mathematicians in the area of Combinatorial Matrix theory and Generalized inverses and its Applications. Apart from tutorial lecture on these topics, organizers' intention was to provide an exposure to all participants with allied linear algebra topics through some special lectures. To meet these objectives, several tutorial lectures, special lectures, formal and informal discussion hours were arranged. An environment for young scholars to interact with leading mathematicians was also created on and off the venue.

Photograph 1 Pro Vice Chancellor Dr. Vinod Bhat inaugurating the workshop by lighting the lamp. Accompanying are Professors S. Kirkland, K.M. Prasad, N.S.K. Nair and R. Balakrishnan

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1.2 Objective of the Conference

Generalized inverses of matrices and combinatorial matrix theory are among areas of matrix theory with a strong theoretical component and with applications in diverse areas that have seen rapid advances in recent years. The interaction between graph theory and matrix theory is known to be very fruitful, and example abound where generalized inverses of matrices arising from graphs play an important role. Just to give one example, an expression for the resistance distance between two vertices in a graph in terms of a generalized inverse of the Laplacian matrix is a key tool. The objective of two-day Conference was to provide an ideal opportunity for participants from India and abroad to interact and exchange ideas in pleasant and academic surroundings.

1.3 Participation

Having an initial response of 150 individuals (32 speakers, 56 students and 63 other participants), the actual attendance for the workshop and conferences were 114 (97 for workshop and 102 for the conference). We have received response from

Photograph 2 Ashma, Pavan, Suma, Nitin, Anitha, Shruti, Shavina, Pallavi, Serah and Banuchitra at Registration Counter

1. Bangladesh,
2. Canada,
3. Estonia,
4. Finland,
5. Germany,
6. Ireland,
7. Nepal,
8. New Zealand,
9. Poland,
10. Portugal,
11. United States of America,
12. PR China and
13. Denmark

apart from India. Most of the speakers who have given their initial consent made their presence possible, except in two cases, where in one case the speaker could not make it because of VISA, and in other case due to financial constraint. The organizing committee was very happy with the response from the speakers for the workshop and conferences.

2 Endorsement and Support

2.1 Societies and National Agencies

The organizers are thankful to:

- International Linear Algebra Society (http://www.ilasic.math.uregina.ca/iic/) for endorsing these events.

- Society for Industrial and Applied Mathematics (http://www.siam.org/) for being cooperating society.

The organizers place in record their thanks to Professor Steve Kirkland, the President of ILAS, for obtaining the endorsement and agreeing to be the representative of SIAM for these events.

While conducting an international event of this level apart from providing good academic environment and materials, local hospitality of high standard and the local travel facilities are equally important, which demand the involvement of very high amount of finance.

Many speakers from abroad and some from organizations like IITs and ISI have utilized their personal research grant to reach Manipal, and only their local expenses were taken care of by the organizers. Organizers take this opportunity to thank their respective organizations for the support that enabled those scientists of high repute to meet here and the scholars of the region to benefit from the formal and informal interaction with them.

In the interest of providing benefit for students and scientists in the underdeveloped countries of ASIA and help them in developing strong network with the rest of the world, ICTP (International Centre for Theoretical Physics) granted a sum of 3000 Euro. They have also permitted the organizers to utilize a part of the amount for a speaker from another country.

The Abdus Salam International Centre for Theoretical Physics (http://www.ictp.it/) supported us by awarding travel grant for selected speakers.

The speakers nominated for partial support of ICTP travel grant were

1. Prof. Yimin Wie, Fudan University, Shanghai, China.
2. Prof. Md. Maruf Ahmed, BRAC University, Dhaka, Bangladesh.
3. Prof. Oskar Maria Baksalary, Adam Mickiewicz University, Poland.
4. Prof. Om Prakash Niraula, Tribhuvan University, Kathmandu, Nepal.

Unfortunately, Prof. Yimin Wie could not turn up due to a problem with him getting VISA, but the other three have utilized the ICTP grant for their travel and local hospitality.

The following national agencies have supported through the grants which enabled the organizers to have the success in organizing the events:

- CENTRAL STATISTICAL OFFICE, MINISTRY OF STATISTICS AND PROGRAMME IMPLEMENTATION (http://www.mospi.nic.in)

- COUNCIL OF SCIENTIFIC AND INDUSTRIAL RESEARCH (http://www.csir.res.in) for supporting us by awarding separate grant-in-aid for workshop and conference.

- NATIONAL BOARD FOR HIGHER MATHEMATICS (http://www.nbhm.dae.gov.in) for supporting us by awarding grant-in-aid.

2.2 *Support from Manipal University*

MANIPAL UNIVERSITY is always keen on supporting research related activities. So, the university has provided the maximum possible support to the organizers, apart from regular administrative, secretarial and logistic support. It has provided the maximum possible local transportation to all the delegates and provided accommodation for about 17 speakers at FIVV (normally an accommodation at FIVV costs about INR 6,000 per day for a person) during the workshop and conference. Also, the conference dinner on the day of inauguration of conference was subsidized to the extent of 50 percent. The University has provided the dedicated lecture halls with audio and visual facilities. The support provided by the Manipal University is informally estimated for the cost about INR 10,00,000 (Ten Lakh only).

3 Feedback and the Response

In general, all the participants have expressed their happiness towards

- Hospitality
- Academic Environment
- Facilities available around and
- the opportunity they got for the interaction and learning new ideas from leading Mathematicians.

A comment from the participants is that

- with an assurance of travel support, many scholars not receiving support from any funding agencies would have attended and benefited from the program.

With the last comment, organizers find the reason for several dropout but felt helplessness as the response, for our request of fund support, from several agencies till November was not encouraging and could not commit firmly on support for the scholars. Of course, from the organizers end the registration fee was already subsidized to large extent.

4 International Workshop, 02–07 January 2012

The workshop was for the duration of 6 days consists of 10 tutorial lectures (21 Hrs.), 6 special lectures (7:30 Hrs.) and formal and informal discussion hours everyday. The afternoon session of Saturday (last day) was spent on taking feed back and valedictory session.

Based on the lectures delivered in the workshop, lecture notes titled "Lectures on Matrix and Graph Methods" were published. Editors for this volume are Professors R.B. Bapat, S. Kirkland, K. Manjunatha Prasad and Simo Puntanen, and the book was nicely printed and published by Manipal University Press. Organizers have arranged distribution of copies to all the participants of workshop at free of cost.

Photograph 3 Release of souvenir

Workshop was inaugurated by Pro Vice Chancellor Dr. H. Vinod Bhat. Prof. Steve Kirkland and Prof. R. Balakrishnan were the guests of honor for the function. Souvenir of the workshop was released by Prof. Kirkland.

The actual conduct of the workshop is as given in the following schedule:

Schedule for Workshop
02-07 January 2012
International Workshop & Conference
on
Combinatorial Matrix Theory & Generalized Inverses of Matrices

Venue: NLH 103, New Lecture Hall Complex, MIT

		Time Duration								
		9.00-10.00	10.00-11.15	11.15-11.45	11.45-12.45	12.45-14.00	14.00-15.00	15.00-15:30	15:30-16.30	16.30-17.30
Date	02-01-2012	Breakfast & Registration	SPL 1 (SSK)	Tea	TL 1 (RBK)	Lunch Break	TL 2 (SP)	Tea	TL 3(RBK)	Discussion

- 02-01-2012; 18.00 Inaugural Function Dinner at 20.00 ; Venue: AC Seminar Hall

		Time Duration								
		9:00-10:00		10:30-11:30	11:30-12:45		14:00-15:00		15:30-16:30	16:30-17:30
Date	03-01-2012	TL 4(OMB)	10:00-10:30 Tea Break	TL 5 (OMB)	SPL 2 (TESR)	12:45-14:00 Lunch Break	TL 6 (SRK)	15:00-15:30 Tea Break	TL 7 (SRK)	Problems & Discussion
	04-01-2012	TL 8 (SP)		TL 9 (SP)	SPL 3(SG)		TL 10 (GT)		TL 11 (GT)	
	05-01-2012	TL 12 (HJW)		TL 13 (HJW)	TL 14 (TESR)		TL 15 (TESR)		Break	
	06-01-2012	TL 16 (JJH)		TL 17 (JJH)	SPL 5 (KMP)		TL 18 (SSK)		TL 19 (SSK)	
	07-01-2012	TL 20 (SIMO)		TL 21 (SIMO)	SPL 6 (RB)		FB		Valedictory	

03-01-2012 Evening : Anatomy Museum & End Point visit

04-01-2012 Evening : Malpe Beach & Krishna Temple visit

05-01-2012 Evening : Photo Session; Basrur Palace visit

7:00 pm – GPHS lecture

- **GPHS - Styan G.P.H. - An Introduction to Yantra Magic Squares and Agrippa-type Magic Matrices**
  - **Lecture 1 : Thursday, January 5, 2012 Time: 7:00 – 8:00 pm**

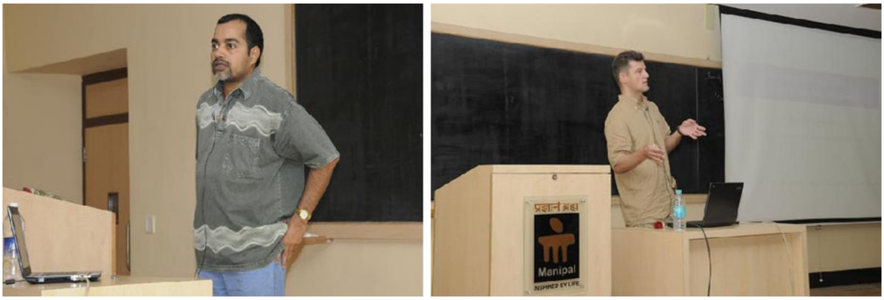
6 days: 21 Hrs of Tutorial Lecture (10 Speakers) ; 7:30 Hrs of Special Lecture (6 Speakers); 5 Hrs of Problems and Discussions

The titles of lectures delivered in the workshop are as given below:

- SSK – Steve Kirkland – Graph structure revealed by spectral graph theoretic methods
- SSK – Steve Kirkland – A structured condition number for an entry in the stationary distribution of a Markov chain
- SRK – S. Sivaramakrishnan – Applications of the Laplacian matrix of graphs
- SP – Sukanta Pati – Matrices and graphs
- SP – Sukanta Pati – Algebraic connectivity of graphs
- OMB – Baksalary O.M. – Hyderabad 2000–Manipal 2012—over 11 years with projectors
- TESR – T.E.S. Raghavan – Graph theoretic applications to computing the nucleolus of an assignment game
- RBK – R. Balakrishnan – Spectral properties of graphs

- GT – Trenkler Götz – Projectors in the linear regression model
- HJW – Hans Joachim Werner – G-inverses, projectors, and the general Gauss–Markov model
- JJH – Jeffrey Joseph Hunter – The derivation of Markov chain properties using generalized matrix inverses
- SIMO – Simo Puntanen – Matrix trix for statistical model
- KMP – K. Manjunatha Prasad – What are "generalized inverses" any way?
- RB – Rajendra Bhatia – Matrix inequalities
- SG – S. Ganesan – Covariance matrices in nuclear data
- GPHS – G.P.H. Styan – An introduction to Yantra magic squares and Agrippa-type magic matrices

  Scheduled on Thursday, 5 January 2012; Time: 7:00–8:00 pm; Video Lecture and discussion through Skype.



Photograph 4 Prof. Sukanta Pati (*on the left*) of IIT Guwahati delivering his lecture on "Algebraic Connectivity" and Prof. Oskar M Baksalary (*on the right*) of Adam Mickiewicz University, on his journey with "Projectors" for past 11 years

Apart from above academic activities, an informal arrangement of visiting the places around and cultural activities have been arranged for the delegates. The activities were as follows:

Photograph 5 Prof. Simo Puntanen with Prof. Jeffrey Hunter

02-01-2012 Evening: Bharatha Natya (Indian Classical Dance) programme by Shridhara Rao Bannanje's troupe, following the inaugural function

03-01-2012 Evening: Around University, Anatomy Museum & End Point visit

04-01-2012 Evening: Malpe Beach & Krishna Temple visit

05-01-2012 Evening: Photo Session; Barkur Palace visit

08-01-2012 St. Mary Island Picnic (during the break between workshop and conference).

The workshop part was concluded with feedback session and valedictory function. Registrar of Manipal University, Dr. G.K. Prabhu was the chairman of the valedictory function, and Prof. Rajendra Bhatia and Prof. Götz Trenkler were the chief guests of the function. Participation certificates for participants and the thanking letters for the resource persons were distributed on this occasion.

Photograph 6 Dr. H. Vinod Bhat—Pro Vice Chancellor lighting the lamp (*on the left*). Others accompanying are Prof. Krikland, Prof. Prasad, Prof. Nair and Prof. Balakrishnan. *On the right*—Prof. Raghavan of University of Illinois at Chicago being greeted by Prof. Sudhakara

Photograph 7 (*On the left*) Dr. H Vinod Bhat—Pro Vice Chancellor addressing the gathering at Inaugural function; (*on the right*) Shridhara Rao Bannanje and his team performing "Bharatha Natyam"

5 International Conference

Two days conference starting on 10 January was with four plenary sessions, 18 invited talks and 12 contributed talks. Invited talks and contributed talks were arranged in three parallel sessions.

Halls for plenary sessions and parallel sessions are named after three pioneers PROF. S.S. SHRIKHANDE, PROF. R.A. BRUALDI, PROF. M.P. DRAZIN and PROF.

Adi-Ben Israel in the areas of “Combinatorial Matrix Theory”, “Generalized Inverse of Matrices” and “Matrix Methods in Statistics”.

Organizers have arranged for an agreement with Springer in publishing the proceedings of the conference with an editorial board comprising Professors Bapat, Kirkland, Prasad and Puntanen. As a result, this volume with title “Combinatorial Matrix Theory and Generalized Inverses of Matrices” is before you.

Photograph 8 Halls for plenary sessions and parallel sessions are named after pioneers in the area of “Combinatorial Matrix Theory”, “Generalized Inverse of Matrices” and “Matrix Methods”. Apology for not able to obtain the photograph of M.P. Drazin in time

List of invited speakers presented papers in the conference:

1. Rafikul Alam, *Indian Institute of Technology, Guwahati*
2. Oskar Maria Baksalary, *Adam Mickiewicz University, Poland*
3. R. Balakrishnan, *Bharathidasan University, Tiruchirapalli*
4. Rajendra Bhatia, *Indian Statistical Institute, New Delhi*
5. Francisco Carvalho, *Polytechnic Institute of Tomar, Portugal*
6. Jeffrey Joseph Hunter, *Auckland University of Technology, New Zealand*
7. Surender Kumar Jain, *Ohio University, USA*
8. Steve J. Kirkland, *National University of Ireland Maynooth, Ireland*
9. Tõnu Kollo, *University of Tartu, Estonia*
10. Bhaskara Rao Kopparty, *Indiana State University, USA*
11. S.H. Kulkarni, *Indian Institute of Technology, Madras*
12. Augustyn Markiewicz, *Poznań University of Life Sciences, Poland*
13. AR. Meenakshi, *Annamalai University, Chennai*
14. Simo Puntanen, *University of Tampere, Finland*
15. T.E.S. Raghavan, *University of Illinois at Chicago, USA*
16. Sharad S. Sane, *Indian Institute of Technology, Bombay*
17. K.C. Sivakumar, *Indian Institute of Technology, Madras*
18. Sivaramakrishnan Sivasubramanian, *Indian Institute of Technology, Bombay*
19. Murali Srinivasan, *Indian Institute of Technology, Bombay*
20. George P.H. Styan, *McGill University, Canada*

21. Götz Trenkler, *Technische Universität Dortmund, Germany*
22. Hans Joachim Werner, *University of Bonn, Germany*

List of speakers presented contributed paper in the conference:

1. Mohammad Maruf Ahmed, *BRAC University, Dhaka*
2. A. Anuradha, *Bharathidasan University, Tiruchirapalli*
3. Ravi Shankar Bhat, *Manipal Institute of Technology, Manipal*
4. Sachindranath Jayaraman, *Indian Institute of Science Education and Research, Kolkata*
5. Debajit Kalita, *Indian Institute of Technology, Guwahati*
6. K. Kamaraj, *University College of Engineering, Arni*
7. Kuncham Syam Prasad, *Manipal Institute of Technology, Manipal*
8. K. Shreedhar, *K.V.G. College of Engineering, Sullia*
9. Kedukodi Babushri Srinivas, *Manipal Institute of Technology, Manipal*
10. G. Sudhakara, *Manipal Institute of Technology, Manipal*
11. H.S. Sujatha, *Manipal Institute of Technology, Manipal*

6 Conference Schedule and Activities

Venue: V Floor, Innovation Center

10th January 2012
9:00–9:30 **Registration**

9:30–10:20 **Plenary Session** **Prof. S.S. Srikhande Hall**

Steve J. Kirkland, National University of Ireland Maynooth, Ireland—SIAM Representative
The Group Inverse and Conditioning of Stationary Vectors for Stochastic Matrices

Chairman: Prof. R.B. Bapat

10:20–11:00 Tea Break

Invited Talks: Prof. R.A. Brualdi Hall

11:00–11:35
Sivaramakrishnan Sivasubramanian, Indian Institute of Technology, Bombay
The Second Immanant of Two Combinatorial Matrices

Photograph 9 Prof. Sivaramakrishnan Sivasubramanian of IIT, Bombay

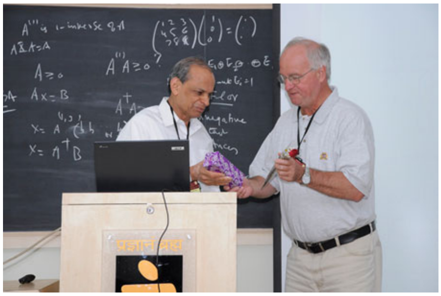
11:40–12:15
T.E.S. Raghavan, University of Illinois at Chicago, USA
Optimization Methods in Matrices

12:20–12:55
AR. Meenakshi, Professor Emeritus, Annamalai University, Annamalai Nagar
Regular Matrices over an Incline

Chairman: Prof. R. Balakrishnan

Invited Talks: Prof. M.P. Drazin Hall

11:00–11:35
Surender Kumar Jain, Ohio University, USA
Some Results on Semiring (Semigroup) of Nonnegative Matrices



Photograph 10 Chair of the session—Prof. Trenkler honoring the speaker Prof. S.K. Jain

11:40–12:15
Oskar Maria Baksalary, Adam Mickiewicz University, Poland
On Subspaces Attributed to Functions of Oblique Projectors

12:20–12:55
K.C. Sivakumar, Indian Institute of Technology, Madras
Weak Monotonicity of Interval Matrices

Chairman: Prof. Götz Trenkler

Invited Talks: Prof. Adi Ben-Israel Hall

Photograph 11 Prof. Carvalho being honored by Prof. Rao

11:00–11:35
Jeffrey Joseph Hunter, Auckland University of Technology, New Zealand
The Role of Kemeny's Constant in Properties of Markov Chains

11:40–12:15
Hans Joachim Werner, University of Bonn, Germany
Weak Complementarity, Non-Testability and Restricted Moore–Penrose Inverses

12:20–12:55
Francisco Carvalho, Polytechnic Institute of Tomar, Portugal
Models with Commutative Orthogonal Block Structure: Inference and Structured Families

Chairman: Prof. K.P.S. Bhaskara Rao

13:00–14:30 Lunch

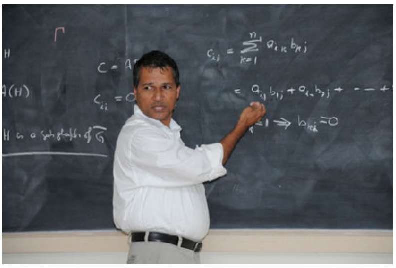
14:30–15:20 **Plenary Session Prof. S.S. Srikhande Hall**

Rajendra Bhatia, Indian Statistical Institute, New Delhi
The Sylvester Equation

Chairman: Prof. Simo Puntanen

15:20–16:00 Tea Break

Contributed Talks: Prof. R.A. Brualdi Hall



Photograph 12 Prof. Sudhakara G.

16:00 –16:20
G. Sudhakara, Manipal Institute of Technology, Manipal
Realization of Product of Adjacency Matrices of Graphs

16:20–16:40
A. Anuradha, Bharathidasan University, Tiruchirapalli
Nonisomorphic Cospectral Oriented Hypercubes

Chairman: Prof. Sivaramakrishnan

Contributed talks: Prof. M.P. Drazin Hall
16:00–16:20
Sachindranath Jayaraman, IISER, Thiruvanthapuram
Matrix Partial Orders in Indefinite Inner Product Spaces

16:20–16:40
Mohammad Maruf Ahmed, BRAC University, Dhaka
Assorted Representations of Generalized Inverse with Numerical Solutions

Chairman: Prof. Augustyn Markiewicz

Contributed Talks: Prof. Adi Ben-Israel Hall

16:00–16:20
Kuncham Syam Prasad, Manipal Institute of Technology, Manipal
Insertion of Factors Property in Matrix Nearrings

Chairperson: Prof. AR. Meenakshi

Photograph 13 Prof. AR. Meenakshi chairing the session

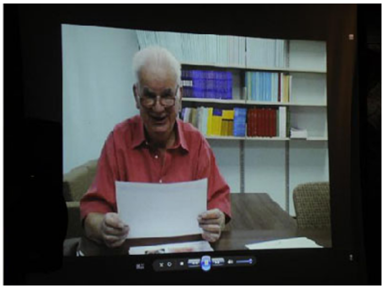
16:45–17:00 Photo Session

17:00–20:00 Inaugural Function Chaitya Hall, FIVV

Inaugural Function of Conference

Inaugural function of the conference was started with a video recorded talk by *Prof. G.P.H. Styan* from McGill University, Canada, and discussion with him over the Skype. Dr. H.S. Ballal—Pro Chancellor inaugurated the conference. Prof. K.P.S. Bhaskara Rao of Indiana University, USA, Prof. Simo Puntanen of University of Tampere, Finland, and Prof. T.E.S. Raghavan of University of Illinois at Chicago were the chief guests. Prof. Ravindra Bapat of ISI Delhi, who is the Chairman of Scientific committee, also presided over the function.

Inaugural function was followed by semi-classical dance by M.Sc. Biostatistics student, folk dance by kids Sushumna and Akshatha, and classical music by Ravikiran and his troupe.

Photograph 14 Dr. H.S. Ballal (*on the left*) lighting the lamp at Inaugural function of the conference. Others in the picture are Prof. K.M. Prasad, Prof. Simo Puntanen, Prof. K.P.S. Bhaskara Rao, Prof. T.E.S. Raghavan, Prof. R.B. Bapat and Prof. N.S.K. Nair. Prof. G.P.H. Styan (*on the right*) talking over the Skype

Conference Schedule contd..

Venue: V Floor, Innovation Center

11th January 2012

9:30–10:20 **Plenary Session** **Prof. S.S. Srikhande Hall**

Bhaskara Rao Kopparty, Indiana State University, USA
Sums of Idempotent Matrices

Chairman: Prof. S.K. Jain

10:20–11:00 Tea Break

Invited Talks: **Prof. R.A. Brualdi Hall**

Photograph 15 Prof. Sharad S. Sane with Prof. Steve Kirkland

11:00–11:35
R. Balakrishnan, Bharathidasan University, Tiruchirapalli
Polynomial Time Computation of the Hosoya Index of Some Families of Graphs

11:40–12:15
Sharad S. Sane, Indian Institute of Technology, Bombay
Linear Algebra of Strongly Regular Graphs and Related Objects

12:20–12:55
Murali Srinivasan, Indian Institute of Technology, Bombay
The Complexity of the q-Analog of the n-Cube

Chairman: Prof. Steve J. Kirkland

Invited Talks: Prof. M.P. Drazin Hall

11:00–11:35
Rafikul Alam, Indian Institute of Technology, Guwahati
On Structured Mapping Problems for Matrices

11:40–12:15
S.H. Kulkarni, Indian Institute of Technology, Madras
Generalized Inverses and Approximation Numbers

Photograph 16 Prof. Rafikul Alam

Chairman: Sivakumar

Invited Talks: Prof. Adi Ben-Israel Hall

Photograph 17 Prof. Tonu Kollo and Prof. Baksalary

11:00–11:35
Simo Puntanen, University of Tampere, Finland
Equalities of the BLUEs and/or BLUPs in Two Linear Models

11:40–12:15
Augustyn Markiewicz, Poznań University of Life Sciences, Poland
An Optimality of Neighbour Designs Under Interference Models

12:20–12:55
Tonu Kollo, University of Tartu, Estonia
Matrix Problems of Skewed Multivariate Distributions

Chairman: Oskar M. Baksalary

13:00–14:30 Lunch

14:30–15:20 **Plenary Session Prof. S.S. Srikhande Hall**

Götz Trenkler, Technische Universität Dortmund, Germany
Functions of Orthogonal Projectors Involving the Moore–Penrose Inverse

Photograph 18 Prof. Trenkler with Prof. Hunter.

Chairman: J.J. Hunter

Contributed Talks: Prof. R.A. Brualdi Hall

15:30–15:50
Debajit Kalita, Indian Institute of Technology, Guwahati
The Singularity of Weighted Directed Graphs

15:50–16:10
K. Shreedhar, K.V.G. College of Engineering, Sullia
Metric Dimension of Graphs by Using Distance Matrices

Photograph 19 Prof. Murali Srinivasan

Chairman: Prof. Murali Srinivasan

Contributed Talks: Prof. M.P. Drazin Hall

15:30–15:50
K. Kamaraj, University College of Engineering, Arni
Characterization of Normal Matrices in an Indefinite Inner Product Spaces

15:50–16:10
Kedukodi Babushri Srinivas, Manipal Institute of Technology, Manipal
Matrices Associated with GI(N)

Photograph 20 Prof. Hans J. Werner

Chairman: Prof. H.J. Werner

Contributed Talks: Prof. Adi Ben-Israel Hall

Photograph 21 Prof. Sreemathi S. Mayya

15:30–15:50
Ravi Shankar Bhat, Manipal Institute of Technology, Manipal
Strong/Weak Edge Vertex Mixed Domination Number of a Graph

15:50–16:10
H.S. Sujatha, Manipal Institute of Technology, Manipal
Realization of Modulo 2 Product of Adjacency Matrices of Graphs

Chairperson: Prof. Sreemathi S. Mayya

16:10–16:30 Tea Break

16:30 Valedictory Function: AC Seminar Hall

The valedictory function for the conference was chaired by Vice Chancellor, Dr. K. Ramnarayan, and the chief guests on the occasion were Prof. S.K. Jain, Prof. Jeffrey Hunter, Prof. H.J. Werner and Prof. Kumkum Garg. Prof. R.B. Bapat, the

Photograph 22 Dr. K. Ramnarayan, the Vice Chancellor of Manipal University, addressing the gathering. On the stage (*from the left*) are Prof. Jain, Prof. Hunter, Prof. Werner, Prof. Bapat, Prof. Kumkum Garg and Prof. N.S.K. Nair

Chairman of Scientific Committee, and Prof. N.S.K. Nair were also presided over the function. Prof. Sane from IITB, Prof. Om Prakash from Thribhuvan University Nepal and Prof. Kirkland, President of ILAS, are prominent among individuals given the feedback on conference. There was overwhelming appreciation on the conduct of conference, which gave an opportunity for Mathematicians from several countries to meet in person and discuss in a wonderful academic and beautiful environment. Everyone appreciated the academic reward benefited through the conference. Prof. Bapat announced on publication of lecture notes on selected workshop lectures, to be published by Manipal University Press, and proceedings of the conference published by the Springer.

7 Selected Photographs from the Events

Photograph 23 Workshop group photo. *Sitting from left*: Steve Kirkland, Francisco Carvalho, J.J. Hunter, T.E.S. Raghavan, H.J. Werner, O.M. Baksalary, N.S.K. Nair, K.M. Prasad, Sudhakara, Shreemathi Mayya. *Standing 1st row*: Nitin, Pavan, Amitha, Serah, Shavina, Vinitha, Divya, Reema, Pallavi, Sri Raksha, Melissa, Apporva Hegde, Apoorva, Vikas, Manjunath, Santhi, Banuchitra. *Standing 2nd row*: Biswajit Deb, Shahistha, Shruti, Ashma, G. Indulal, Milan Nath, Debajit Kalita, Sam Johnson, Bhargavarama Sarma, Maruf Ahmed, Kamaraj, Syed Asifulla, Manjunath, Madgala Werner. *Standing 3rd row*: Chaitra, Hamsa Nayak, Vasumathi, Anitha, Divya, Subramaniam, Somasundaram, Kalyan Sinha, Chaluvaraju, Vibekananda Dutta, Richard Werner, Narendra Shrimali, Mrs. Baksalary, Mrs. Carvalho. *Standing 4th row*: Ashwini, Suma, Shruthi Rao, Nivedita Baliga, Sowmya, Sujatha, Indira, Anuradha, Anuvarghese, Jais Kurian, Sanjay Kumar Patel, G. Vasudeva. Standing last row: Vinay, Devadas Nayak, Mohan, Kiran Hande, V.S. Binu, G. Ramu, Balaji, D. Sukumar, Ventakaramana, Vasanth, Mohandas

Photograph 24 Conference group photo. *1st row*: R.B. Bapat, Sachindranath Jayaram, Simo Puntanen, J.J. Hunter, H.J. Werner, Steve Kirkland, S.K. Jain, K. Manjunatha Prasad, K.P.S. Bhaskara Rao, Om Prakash Niraula, G. Trenkler, R. Balakrishnan, Augustyn Markiewicz, N. Sreekumaran Nair; *2nd row*: G. Ramu, Balaji, O.M. Baksalary, Francisco Carvalho, Tonu Kollo, Ashma, Vasumathi, Adelaide Rose Meryl, Anuvarghese, Anuradha, AR. Meenakshi; *3rd row*: D. Sukumar, Sam Johnson, S.H. Kulkarni, Rafikul Alam, Shreedar Kunikullayya, Sudhakara, A. Raghavendra, Jais Kurian; *4th row*: Binu Kumar, Debajit Kalita, Mohan, V.S. Binu, Maruf Ahmed, Sivaramakrishnan, Anjan Ray Chaudhury, Kamaraj; *last row*: Biswajit Deb, K.C. Sivakumar, Sharad S. Sane, Rajendra Bhatia, Murali K. Srinivasan

Photograph 25 From NW corner to the right: **a** Prof. Kirkland inspiring through his dedicated involvement throughout the workshop and conference and interactions with participants; **b** Duo Prof. Trenkler and Prof. Baksalary absorbing every moments of the workshop and conference; **c** Prof. Balakrishnan with a rose before his lecture; **d** Enjoying(!) food at Food Court; **e** Artists performing "Bharatha Natyam"; **f** Prof. Werner teaching g-inverses in Gauss–Markov linear models

Photograph 26 **a** Dr. G.K. Prabhu—Registrar, MU, greeting Prof. Tonu Kollo. **b** Dr. Prabhu with Prof. Bhatia. **c** Hunter couple with their soup. **d** Prof. Bhaskara Rao being greeted by Pro chancellor. **e** Markiewicz, Puntanen, Werner, Kollo, Steve and Hunter on chat and Bapat, Rafikul and Om Prakash are seen from the behind. **f** Organizers receiving an applause from the participants and delegates

Photograph 27 **a** Bharatha Natyam performance; **b** and **c** M.Sc. Biostatistics students; **d** Ravi Kiran and his team at performance of classical music; **e** Prof. Puntanen addressing the gathering in conference inauguration

Photograph 28 **a** Prof. Sivakumar with Prof. Trenkler, **b** Prof. Puntanen, **c** Prof. Jain received by Prof. Prasad; standing behind is Prof. Ashma, **d** Prof. Bhatia being greeted by Prof. Puntanen

Photograph 29 **a** Student volunteers, **b** Student participants, **c** Dr. G.K. Prabhu addressing the gathering at workshop valedictory function

8 Message from Pro Vice-Chancellor

Department of Statistics of Manipal University successfully conducted an International Workshop and Conference on Combinatorial Matrix Theory & Generalized Inverses of Matrices in January 2012. This well-attended academic event had presentation of invited papers from the likes of Steve Kirkland, Sharad Sane, Sivaramakrishnan, Jeffrey Hunter, Simo Puntanen, Balakrishnan, Rajendra Bhatia, T.E.S. Raghavan, George Styan, Oskar Baksalary, Götz Trenkler, etc.

Manipal Centre for Natural Sciences (MCNS) will soon be establishing a division of Mathematics & Computational Sciences. This will be one of many divisions that MCNS will initiate and incubate.

Dr. Manjunatha Prasad, Dr. Sreekumaran Nair and the faculty of Department of Statistics have strived hard to make a success of the conference and the publication of these proceedings is a testimony to that effort.

Thanks to all the authors for their excellent contributions and to the editorial team lead by Dr. R.B. Bapat for their effort in bringing out this volume.

Manipal, September 2012 *Dr. H. Vinod Bhat*
Pro Vice-Chancellor, Manipal University

9 Message from Joint Secretary, CSIR

Council of Scientific and Industrial Research (CSIR), as a premier scientific organization, takes pride in supporting and promoting scientific activities in the country.

We are delighted to note that the International Workshop and Conference on CMT-GIM received global representation and was conducted in a befitting manner.

We congratulate the organizers for bringing out proceedings on lectures delivered in the workshop and research articles, which would help in dissemination of the knowledge across a wide spectrum of academic fraternity, researchers and students.

Manipal, September 2012

Dr. K. Jayakumar, IAS
Joint Secretary, CSIR, New Delhi

10 Message from Director General, CSO

The Central Statistics Office (CSO) in the Ministry of Statistics and Programme Implementation has the responsibility to coordinate statistical activities in the country. The CSO is keen on encouraging organization of events such as workshop and conferences on pure and applied statistics, which strengthen the knowledge base of students and researcher in the field.

Generalized inverses of matrices and combinatorial matrix theory are among the areas of matrix theory with strong theoretical component. They have applications in diverse areas such as linear models, multivariate analysis, graphical statistical modeling and optimization.

The MOSPI congratulates the organizers for the success of the international Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices. CSO takes pride in being a partner of these events that attracted speakers from about twelve nations. Furthermore, the CSO expresses its pleasure in supporting the cause of publishing the present volume on *Combinatorial Matrix Theory and Generalized Inverses of Matrices*, which disseminates the research progress made by national and international scientists working in the area.

The volume will prove valuable to all the participants and researchers. The CSO appreciates the efforts of the editors in bringing out this volume.

Manipal, September 2012

Dr. S.K. Das
Director General, CSO

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