Amongst Mathematicians

MATHEMATICS TEACHER EDUCATION

VOLUME 3

SCOPE

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Amongst Mathematicians

Teaching and Learning Mathematics at University Level

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'I was formed by nature to be a mathematically curious entity:
not one but half of two'
[the character of] Niels Bohr in *Copenhagen* by Michael Frayn

'La rêve de l'individu est d' être deux; la rêve de l'état est d' être seul.'
[The dream of the individual is to be two; the dream of the state is to be on its own.]
Jean-Luc Godard, *Notre Musique*

ACKNOWLEDGEMENTS

The main bulk of this book consists of fictional, yet data-grounded, dialogues between two characters: M, mathematician, and RME, researcher in mathematics education. The discussion in these dialogues is triggered by samples of the writing or speaking of mathematics undergraduates at the beginning of their university studies. The data on which these dialogues are grounded, from mathematicians and their students, was collected over a long period: from 1992, when I embarked on my doctoral studies at Oxford, to 2004, when the last study reported in this book was completed. To all of these mathematicians and students I extend my warmest thanks – particularly those mathematicians from the latest study who 'became' M more directly: thank you for giving your valuable time and energy and for trusting me with your words. I hope you will find that, however playful, the 'processing' of your words was never done with anything less than respect for the complexity of the ideas you intended to put across and a desire for understanding.

The 'playfulness' I cite above refers to the somewhat unconventional choice of format for mediating the ideas in the book, the dialogue. Given my origins as a mathematician, this choice may appear surprising. For a mathematician however who has always been fond of words and …wordiness of all sorts, I am lucky to have found myself in an environment where this penchant – for telling stories that aspire to reconcile vividness with subtlety and rigour – was treated as a methodological …oddity that was not only never discouraged; it was allowed to grow: University of East Anglia's School of Education has a long tradition of nurturing this kind of openness and I have been counting myself a fortunate beneficiary of this tradition since 1998.

Since 2000 I have been working closely with Paola Iannone whose commitment, talent and generosity have helped shape the most essential of the studies on which I draw in this book. I am profoundly grateful.

Finally a word for my friends and family*. For years now, with energy and method, we have managed a far from trivial feat: keeping the tie alive and crucial across several cities and countries, and two continents. For proving that distance can be reduced to nothing more than a construct of geography this book is for you. If there is any wisdom in the statement that a step into the unknown is taken by people either who have nothing to lose or who have solid safety nets to fall back into when the step proves too tough or foolhardy, then your faith, unflinching and energising, makes my case fit the latter. Thank you. Always.

* I felt that a listing of names at this stage would dull the intended effect: expressing my deeply felt gratitude. The story behind this work is told in several places across the book, mostly in Chapter 2 and in the Post-script. Most of the names that should be listed here appear, in footnotes, within the 'scenes' of the story there – namely at the *heart* of the story, where they rightfully belong.

TABLE OF CONTENTS

PROLOGUE

The main part of this book, **Chapters 3-8**, consists of dialogues between two characters M, a mathematician, and RME, a researcher in mathematics education. These dialogues focus on a range of issues regarding the learning and teaching of mathematics at university level. Each dialogue starts from a discussion of a sample of data (students' writing or speaking) that exemplify these issues. Both the samples that M and RME discuss as well as their discussions are grounded in data collected in the course of several studies that I have been involved in since 1992. In **Chapter 1** I outline the studies that formed the raw material for the book as well as the wider area of mathematics education research these studies are embedded in and the book aims to contribute to. In **Chapter 2** I outline the processing that the data collected in these previous studies has gone through in order to reach the dialogic form in which it is presented in Chapters 3-8. **Chapter 3** focuses on students' mathematical reasoning and in particular their conceptualization of the necessity for Proof and their enactment of various proving techniques. **Chapter 4** shifts the focus towards the students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams. **Chapters 5** and **6** offer accounts of the students' encounter with some fundamental concepts of advanced mathematics – Functions (across the domains of Analysis, Linear Algebra and Group Theory) and Limits. **Chapter 7** revisits many of the 'learning stories' told in Chapters 3-6 in order to highlight issues of university-level pedagogy. Finally, in **Chapter 8** M and RME, starting from the experience of working together in the context of the studies on which the book is based – and as showcased in Chapters 3-7 – discuss their often fragile relationship as well as the necessary and sufficient conditions for a collaboration between mathematicians and researchers in mathematics education. The book concludes with: a brief **Epilogue** in which I reflect on the experience of engaging with the research behind this book, and with its production, and I outline some steps that, at the final stages of writing, this research was beginning to take; and, with a **Post-script** in which I offer a chronological and reflexive account of the events that led to the production of the book.

Note to the reader regarding Chapters 1 and 2. Chapter 1 describes the theorerical background and previous studies on which the book is based and Chapter 2 the method through which the dialogues in Chapters 3-8 came to be. Readers less interested in these may wish to skip these chapters and go directly to Chapter 3. For minimal familiarisation there is a one-page summary at their beginning.

Note to the reader regarding Chapters 3-8. Each of these consists of Episodes which I recommend that you read as follows: engage briefly with the mathematics in the Episode (problem, solution and examples of student response); reflect on the learning/teaching issues these may generate; and, read and reflect on the dialogue between M and RME. There is a more elaborate version of this recommendation between Chapters 2 and 3 and a one-page summary of the issues covered at the beginning of each chapter.

CHAPTER 1
BACKGROUND AND CONTEXT

SUMMARY

The main part of this book, Chapters 3 – 8, consists of dialogues between two characters M, a mathematician, and RME, a researcher in mathematics education. M and RME discuss issues regarding the learning and teaching of mathematics at university level. The starting points of their discussions are data samples that exemplify these issues. Both data samples and discussion between M and RME originate in data collected in the course of studies that I have been involved in since 1992. In this chapter I introduce the international scene of mathematics education research that the book aims to contribute to and then outline these studies. The presentation is in three parts.

Part 1. *TALUM: the Teaching And Learning of Undergraduate Mathematics.*
TALUM, is a relatively new and rapidly developing field of mathematics education research. As, particularly in the 1990s, mathematics departments started to respond to the decline in the number of students who opt for mathematical studies at university level the realisation that, beyond syllabus change, there is also the need to reflect upon tertiary pedagogical practice began to grow. The research on which this book draws was conceived and carried out with the aim to address this need in a systematic and original way.

Part 2. *A rationale for a certain type of TALUM research*
The research on which this book is based draws on several traditions of educational research and has the following characteristics: it is collaborative, context-specific and data-grounded and, through being non-prescriptive and non-deficit, it aims to address the often difficult relationship between the communities of mathematics and mathematics education. A fundamental underlying belief of this work is that development in the practice of university-level mathematics teaching is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it.

Part 3. *The series of studies on which this book draws*
These aimed to explore students' learning in the first, and sometimes, second year of their undergraduate studies (mostly in Analysis, Linear Algebra and Group Theory and mostly through observing them in tutorials and analysing their written work); and, to engage their lecturers in reflection upon learning issues and pedagogical practice (mostly in individual or group interviews).

1. TALUM: A GENERAL INTRODUCTION

The area in which the book aims to contribute, the teaching and learning of mathematics at the undergraduate level, is a relatively new and rapidly developing field of mathematics education research. One rationale for this emergence of interest originates in the alarming decrease in the number of students who opt for exclusively mathematical studies beyond the compulsory level[1] (Hillel, 2001) – a decrease largely due to the alienation from the mathematics they experience at school (e.g. Smith, 2004; Boaler, 1998; Nardi & Steward, 2003). The number of mathematics graduates who choose to go into school teaching is also declining (French, 2004; LMS, 1995). This is partly due to the allure of careers in industry and IT (Hillel, 2001), but it is due also to the lacklustre and often demoralising learning experience that their university studies have been (Goulding et al, 2003; HEFCE, 1996; Burn & Wood, 1995; Kaput & Dubinsky, 1994; Harel & Trgalová, 1996)[2].

The decline in the numbers of mathematics graduates and well-qualified mathematicians willing to become teachers is likely to affect the quality of young people's experience of learning mathematics at school in ways that limit their appreciation and enjoyment of the subject (Johnston, 1994; MA, 2005; Perkins, 2005[3]). In turn this is likely to reduce the number of students who are willing and able to study mathematics at university level (Dick & Rallis, 1991) – and so the spiral of decline is perpetuated.

Mathematics departments have begun to respond to this decline in numbers by attempting to improve the students' learning experience from a cognitive as well as a socio-affective point of view (Petocz & Reid, 2005). Holton (2001) gives numerous examples of this trend. As a first step, mostly in the 1990s, and in order to adjust to the learning needs of new student intakes, many university mathematics departments undertook modifications of the syllabus (e.g. Kahn & Hoyles, 1997, in the UK; Ganter, 2000, in the USA). Beyond syllabus change, there has also been recognition of the need to reflect upon pedagogical practice (Anderson et al, 2000; McCallum, 2003) – and UK researchers have demonstrated significant initiative in this area (e.g.: Mason, 2002; Burton, 2004; Rodd, 2002; Alcock & Simpson, 2001; Povey & Angier, 2004; Brown & Rodd, 2004; Jaworski, 2002; Nardi et al, 2005; Iannone & Nardi, 2005a).

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[1] See, for example, reporting of this in the London Mathematical Society Newsletter, e.g. (Falconer, 2005a & b; LMS, 2005a&c, Qadir, 2005).

[2] It is almost paradoxical that this decline has been occurring in parallel with an unprecedented interest in mathematics and mathematicians in the media – as a browsing of recent titles in fiction, biography, film, theatre and television confirms. The mathematical community is starting to see that in this widespread, yet often stereotype-ridden (Davis, 2006), attention mathematics has been conferred, lies an opportunity for showcasing the relevance and beauty of the subject (e.g. Farley, 2006) and, thus, improve recruitment of young, committed and able mathematicians.

[3] 'there aren't enough teachers to put across the beauty, joy and excitement of mathematics' as well as its wide and increasing use in public debate (p31) bemoans Marcus du Sautoy (2006), a mathematician at the forefront of the community's attempts to communicate with the media (LMS 2006b; Mann 2005a and b), and others, in this brief report. Reporting of these attempts in other outlets such as the *Notices* of the American Mathematical Society, *The Mathematical Gazette* etc has also increased.

Nardi et al (2005) have described the changing climate within the mathematical community as follows (p284):

> 'Many academic mathematicians are aware of the changing perception of their pedagogical responsibility[4] and of experimentation with different teaching approaches, but they have limited opportunities to embrace change owing to faculty structures and organization. Often university teachers have joint responsibility for research and teaching. This is clearly beneficial, but it can cause more emphasis to be placed on mathematical research in places where that is the main criterion for promotion. Teachers of university mathematics courses, on the whole, have not been trained in pedagogy and do not often consider pedagogical issues beyond the determination of the syllabus; few have been provided with incentives or encouragement to seek out the findings of research in mathematics education. In days gone by, it was assumed that the faculty's responsibilities were primarily to present material clearly, and that "good" students would pass and "poor" ones fail. Of course, given the current climate of accountability, this is no longer the case (Alsina, 2001). Further, the relationships between mathematicians in mathematics departments and their colleagues in mathematics education are often strained, with less productive dialogue between them than could be beneficial (Artigue, 2001). The same can be said of relationships between mathematicians and engineers, economists, etc., although mathematics service teaching to students in other disciplines is an enormous enterprise (Hillel, 2001)'.

We then suggest (p285) that:

> 'These general factors tend to work against, or delay, improvements in the teaching and learning of mathematics at the undergraduate level. In this sense, research that builds the foundations of collaboration between university mathematics teachers and mathematics educators is crucial and, given the pressure currently exercised on universities regarding the need for a scrutiny of their teaching practices, timely. But reform of pedagogical practice can only follow from developing pedagogical awareness in the first place.'

And, in agreement with Artigue (2001) we then make a case (p285) for research which aims at

> '…exploring the *professional craft knowledge* of undergraduate mathematics teachers […] and uses a methodology that, we believe, can be used as a tool toward developing such awareness, [can] contribute in this much needed enterprise.'

The above suggest a certain type of TALUM research that is urgently needed. In Part 2 I describe certain characteristics of this type of research and in Part 3 I introduce the studies that the work presented in this book draws on – and in the process I highlight how they have some of these characteristics.

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[4] An acknowledgement of this responsibility has always been there to a certain, non-negligible degree. For example, in the UK, the LMS Newsletter briefly but regularly reports on activity relating to mathematics education whether it is on the work of ACME, the Advisory Committee on Mathematics Education (LMS, 2005b; LMS, 2006a) the Polya Prize (LMS, 2005d), via commenting on the teaching part of the lives of renowned mathematicians (Lickorish, 2006; Ostaszewski, 2005) etc.. The *Notices* of the American Mathematical Society also offer articles on the pedagogy of undergraduate mathematics regularly (I am referring to several of them across this book). The urgency of the issue of recruitment however dictates the need for intensifying these efforts.

2. A CERTAIN TYPE OF TALUM RESEARCH

Despite the focus of most mathematics education research being on the teaching and learning of mathematics at primary and secondary levels, the compulsory educational levels, interest in post-compulsory mathematics education has always been present to some extent. The *Handbook of International Research in Mathematics Education*, the *International Handbook of Mathematics Education*, the *Handbook of Research on Mathematics Teaching and Learning* all have chapters that in one way or another refer to issues on the teaching and learning of mathematics at the undergraduate level (Mamona-Downs & Downs, 2002; Harel & Trgalová, 1996; Tall, 1992, respectively). In most cases the focus is on reviewing the research in the field with a particular emphasis on the distinctive features of mathematics at this level, abstraction and formalism, and refer to pedagogical issues mostly with regard to pointing out the traditionalism of undergraduate mathematics pedagogy or citing the exceptions to this rule. Increasingly, and in the more recent reviews (e.g. Mamona-Downs & Downs, 2002, p169/70) references to the relationship between researchers and practitioners in the field is also starting to be addressed – and the *Second International Handbook of Mathematics Education* has at least three chapters (William, 2003; Breen, 2003; Begg et al, 2003) that address this issue, albeit on a broader canvas, not strictly in relation to undergraduate mathematics education.

The heftiest so far attempt at a re-acquaintance between the communities of mathematics education research and mathematics[5] in this type of overarching publication is in (Sierpinska & Kilpatrick, 1998), *Part VI (Mathematics Education and Mathematics)*. I return to this part of the book in due course in this and other chapters (mainly Chapter 8) but here I will highlight some of the ideas in it which are relevant to the work I present in this book and which, to me, constitute evidence that a mature, non-simplistic approach to this issue is beginning to emerge.

Boero & Szendrei (1998), for example, while outlining the contradictions and potential conflicts between the worlds of mathematics and mathematics education research, offer a useful distinction: energizers of practice / economizers of thought / demolishers of illusion is a construct that they use to describe qualitative, quantitative, practical and theoretical research results and that goes, helpfully, beyond the usual distinction between practical and theoretical. Carolyn Kieran's (1998) demonstration of the role of theoretical models in mathematics education research is also very helpful: she takes the Process-Object idea (I revisit this key theoretical construct in Chapter 5) and lays out its trajectory since its inception (with

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[5] Studies such as the one Leone Burton has conducted with seventy mathematicians in the UK (Burton, 1999; 2001; 2004) can help this re-acquaintance. Particularly an exploration of the deeply personal and creative ways in which mathematicians gain mathematical insight (their thinking styles, their working with others, the role of intuition – all matters that Burton's study offers evidence on –, the role of sociocultural and affective matters) can be instructive in terms of helping us shape pedagogies that are inclusive of mathematicians' perspectives as well as offer learners some of the riches mathematicians obviously gain through their particular ways of engaging with the subject. Other works of this ilk are by Sfard (1998a), Mura (1993), Alcock (2004) and Weber (2004).

a focus mostly on Sfard's version but inclusive of others). Her overall point seems to be that the distinction between empirical and theoretical work in mathematics education research is artificial and that there is a strong dialectic relationship between the two. The book offers thoughtful essays on Developmental Research and Action Research as potent paradigms within mathematics education research and Steve Lerman (1998, p346/7) in his contribution makes the observation that mathematics education research is not characterised by succession of paradigms in the Kuhnian / Popperian sense, but by a cohabitation / coexistence of many (and this coexistence depends on many other factors outside the stricter evolution we largely see in the natural sciences).

It is exactly this refreshing openness by Lerman that Mason’s (1998) chapter takes to its almost logical conclusion – and in a way which relates deeply to the principles that underlie the work presented in this book. Mason advocates the significance of *Research from the Inside*. Before I cite his proposition in a bit more detail, I would like to stress that the fresh perspective on what constitutes the theory and practice of mathematics education that comes through the ideas I cited above implies a rationale for collaborative, practitioner-engaging and context-specific research. Barbara Jaworski’s Co-Learning Partnerships (2002, 2003; 2004) are a good illustration of this rationale: when practitioners of mathematics teaching engage with research they, along with the researchers, become co-producers of knowledge about learning and teaching[6]. Jaworski makes a cautious case – she acknowledges the benefits as well as the pitfalls that may lie in the partnership – and she proposes a shift from a discourse on community of practice to a discourse on community of enquiry (Wenger, 1998). A fundamental belief in this type of work is that development in the practice of mathematics teaching – e.g. integration of innovation in the largely traditional contexts of undergraduate mathematics – is manageable, and sustainable, if driven and owned by the mathematicians who are expected to implement it. In Part 3 of this chapter and in Chapter 2 I link this idea to the studies that the work presented in this book draws on – in which mathematicians played a crucial role as participants with a gradually heavier involvement as educational co-researchers (Wagner, 1997). I also link this idea to my choice for an illustration of the outcomes of these studies in a format, the Dialogue, that I believe circumvents some of the obstacles that the dissemination of educational research results to the mathematical community has suffered. Finally in the Epilogue I make briefly a case for a type of Developmental Research (van den Akker, 1999) that I believe meets some of the needs of undergraduate mathematics education through adding a further shift to the one proposed by Jaworski: the shift from community-of-practice to community-of-enquiry can be followed by a return to an enriched community of practice for the purpose of implementing and evaluating innovative

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[6] Regarding this dialogue between theory and practice see for example (Bartolini-Bussi & Bazzini, 2003) for the many and varied ways in which mathematics education research relates to other disciplines including mathematics. See also (English, 2003) for an example of an ambitious multi-tiered teaching experiment which aimed to bring together researchers, teachers, student teachers and students.

practice that has been conceived and created through the collaborative processes described above.

Let me now return to Mason's advocating of Inner Research (1998). The two most important products of mathematics education research, Mason claims, are: the transformation in the being of the researcher; and, the provision of stimuli to others to test out conjectures for themselves in their own context. This way of researching can be as systematic as more traditional formats. The idea of researching from the inside is of course not totally new to mathematics education research but it is increasingly present (partly because researching from the outside has generally failed to generate effective and enduring reform of practice). Schön's (1987) Reflective Practice is very much in the above vein – as long as we stay cautiously clear of illusions of certainty (which breed prescription, dogma and stagnation), Mason continues. In terms of establishing validity for this type of research, unlike the natural sciences where Popperian ideas on falsification offer powerful tools for doing so, in education we need to seek ways of making assertions that are locally valid, make sense of past experience (fit), are in the form of conjectures (only), are testable to a wider population and situations and are as precisely stated as possible. Mason cites Action Research as a form of research akin to his account of Inner Research (but warns against its occasional tendency to turn into outsider research). The call for Inner research instead is a call for research as the development of sensitivity (noticing) and awareness and of course contravenes any approach that resembles the search for cause-and-effect in educational phenomena. Mason then re-tells the classic characteristics of research[7] in the terms of Inner Research and describes the outcomes of Inner Research. Whereas traditional research outputs go through a transformation similar to the transformation mathematical knowledge goes through in order to become learnable / teachable (he draws a parallel with Chevallard's *transposition didactique* (1985) there) – and in the process loses touch with the primary aim of research (to convey a sense of a particular experience, e.g. of a child learning mathematics) – Inner Research aims to stay close to this aim. This is why it is mostly defined as *research as working on being.*

In my eyes the dialogues presented in Chapters 3 – 8 demonstrate a case of Mason's transformation (of the characters of M and RME) from within and aspire to 'stay in touch' with conveying a sense of a particular experience (one such sense being to witness the emergence and demonstration of M's pedagogical awareness). Particularly with regard to M, the data propel M towards *noticing* and *'spection'* of all sorts, and heighten his awareness. RME's questioning creates space for / propels M towards a richer *mathematical being* (p373) – one with an overt, conscious pedagogical dimension, a *pedagogic being*. And RME's *ability to bear* is essential in this – and an essential lesson in the craft of establishing a balanced practitioner/researcher relationship.

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[7] Researchers: participate in a community; are systematic and disciplined; seek invariance within a specified domain of change; distance themselves from phenomena; observe and inspect; notice and mark; make distinctions, develop frameworks; collect and analyse data; have personal sensitivities; seek resonance with others; learn the most; and, research themselves.

The idea for the character of M of course is not new – neither is the idea of a conversation between a researcher in mathematics education and a mathematician (Sfard, 1998a). Sfard's Typical Mathematician (1998b, p495) and Davis & Hersh's Ideal Mathematician (1981) pre-date this book's M. In Chapter 2, Part 3 – where I elaborate the principles underlying (and the process of) the construction of the characters of M, RME and their dialogues – the fictional, yet strictly data-grounded, nature of this enterprise becomes, I hope, obvious… as does the aspiration for the emergence of a 'beautiful friendship' (Sfard, 1998b, p508) between M and RME.

In Part 3 I describe briefly the studies that the work presented in this book draws on.

3. THE TALUM STUDIES THE BOOK DRAWS ON

The book draws on a series of studies that I have been involved in since 1992 when I started my doctorate. Here I outline these studies with a particular focus on the aspects that relate to the work presented in the book.

In sum the database the material for the book draws on is the following:

- Observations of tutorials between university mathematics lecturers and first-year mathematics undergraduates (200 hours of audio recorded and transcribed material)
- Interviews with mathematics undergraduates (20 hours of audio recorded and transcribed material)
- Individual interviews with university mathematics lecturers (50 hours of audio recorded and transcribed material)
- Year 1 and Year 2 mathematics undergraduates' written responses to mathematical problems (approximately sixty students, first in their first and then in their second year responding to approximately 90 problems in a variety of pure mathematical topics)
- Focused group interviews between mathematicians and mathematics educators (twelve recordings of approximately 200 minutes each, with groups of 4-6 mathematicians, audio recorded and transcribed material)

Below I outline the theoretical origins of these studies and describe them briefly.

The work described in this section draws theoretically on the growing body of research into the learning of advanced mathematical topics (e.g. Tall, 1991a; Kaput & Dubinsky, 1994; Holton, 2001). It also draws on the extensive literature on teachers' thinking processes and practices at earlier educational levels (e.g. Jaworski, 1994): the basic idea from this literature can be largely described in the words of Brown & McIntyre (1993) as 'making sense of teaching from the perspective of teachers themselves':

> . . . how they construe and evaluate their own teaching, how they make judgements, and why in their own understanding, they choose to act in particular ways in specific circumstances to achieve their successes. (p. 1)

This theoretical perspective is relatively new. In (Nardi et al, 2005) there is a brief evolutionary account of how the field has come to this perspective. I would roughly outline these shifts of focus as follows:

- Until mid-20th century: methods of teaching of various mathematical topics
- 1950s and 60s: what teachers and their pupils actually do in classrooms
- 1970s: systematic study (modelling) of teachers' thinking

Since then there have been at least two developments of significance:

- First, most of these models turned out to be inadequate descriptors of teachers' decision making in the midst of an interaction in the classroom; hence the need for more descriptive research in the area. This led to:
- Second, largely case study research on knowledge of experienced teachers.

This latter type of studies however have often been interview studies, not studies of what actually occurs in the classroom. And therein lies a caveat: in interviews expert teachers tend to focus on atypical situations of their teaching, perhaps because they perceive most of their classroom actions as so ordinary and so obvious as not to merit any comment; hence researchers' attention too has tended to be directed mostly towards the *problems* that teachers experience: This tendency has become known in the field as 'deficit' and 'prescriptive' discourse on pedagogy (where the emphasis is on the identification of what it is thought teachers ought to be doing and are not doing, and on appropriate remedial action (Dawson, 1999)). This is clearly unsatisfactory as innovation needs to take account of what is already being done in classrooms. Moreover, an evaluation of teaching can certainly gain from extensive and systematic observation of actual teaching and of knowledge on how teachers conceptualise their actions.

Furthermore from a methodological point of view, qualitative methods seem more appropriate for illuminating the ways in which teachers construe what they are doing than surveys or experimental designs. The above shift of focus coincides historically also with the emergence of qualitative methodologies.

The work outlined here is located explicitly within this non-deficit discourse. It seeks to explore both the *professional craft* knowledge of undergraduate mathematics teachers (Shulman, 1986) and also ways in which pedagogical awareness can be raised.

Beyond the two theoretical strands mentioned above (work in the area of Advanced Mathematical Thinking, Non-Deficit Discourses on Pedagogy) the deeper theoretical origins of the work I will be discussing here lie in the three theoretical perspectives on the learning and teaching of mathematics I outline briefly next – these are slight variations of the outlines in (Nardi et al, 2005). I note that the studies have been drawing on these complementarily and with the intention to avoid the conflicts, polarisations or disputes that have characterized the discourse on at least the first two in recent times (Kieran et al, 2002; Even & Schwarz, 2003):

Sociocultural Theory: Enculturation

In this theory's enculturative dimension participants in a social community are drawn into the language and practices of the community and develop knowledge through communication and practice (e.g., Vygotsky, 1962; Sierpinska, 1994; Lerman, 1996; Wenger, 1998). In the context of the studies I introduce here participants are mathematics undergraduates, their lecturers and researchers in mathematics education. The enculturative dimension of this theory is employed in these studies in at least two senses: the students' mathematical learning is discussed by their lecturers and the researchers in terms of appropriation of the cultural practices of university mathematics; the mathematicians' pedagogical perspectives and knowledge are discussed by the researchers and the mathematicians' themselves in terms of the cultural practices of the two communities (of mathematics and of mathematics education research). In the first sense the lecturers, as experienced practitioners from within the mathematics community, encourage the students (newcomers, peripheral participants) into participation in a community (Lave & Wenger, 1991), i.e., into behaving as mathematicians through gaining the requisite knowledge through practice (Cobb, 1996). In the second sense the mathematicians are encouraged by the researchers to engage with the pedagogical discourses that characterise the community of mathematics education research. In the process the researchers also are acquainted (or in the particular case of these studies re-acquainted, as the researchers in these studies are also mathematicians) with the cultural practices of the mathematics community.

Constructivist Theory: Individual Sense-making

In this theory's account of individual sense-making of experience (e.g., Cobb, 1996; Confrey, 2000) in learning mathematics, undergraduates need to encounter ideas and to work personally on those ideas in order to make sense of them – for example, by means of reflective abstractions (von Glasersfeld, 1995, p. 374). In the interpretations of tutorial situations, student writing or the words of lecturers from the interviews, there are often references to the sense undergraduates are making of mathematical ideas, many of which depend on successive layers of abstraction. While certain aspects of mathematical activity develop through an enculturative process as part of doing and thinking mathematically in a mathematical community (such as pattern seeking, generalization, proof and proving) others (such as the concepts of function, limit, group etc) require the individual to encounter and grapple with ideas. Communication with others, particularly the sharing of imagery and language, are important to this grappling. Ultimately students might be seen to establish versions of the concepts through their own reflective abstraction and communicative reconciliation with the versions of their peers, lecturers and the wider mathematical community.

Enactivist Theory: Codetermination

In this theory's aspect of codetermination, living beings and their environment are seen to stand in relation to each other through mutual specification or codetermination (e.g., Dawson, 1999; Varela et al, 1991). This perspective offers an alternative to the two above in linking individuals fundamentally with their environment: the two are mutually constitutive. Dawson, following Varela et al.

(1991), speaks of a 'path laid while walking' (Dawson, 1999, p. 157). What this means for students' developing mathematical understandings in lectures, seminars, tutorials etc. alongside their lecturers, is that nothing is predetermined: knowledge and understanding grow relative to the interactions between participants in the domain of their activity. A consequence is that lecturers cannot plan exactly for the understanding of students. Rather the process of understanding is evolutionary and its directions are dependent on past actions and understandings. Both actors within the domain will have intentions, but these intentions are modified in the light of ongoing activity. So a lecturer might explain a mathematical concept, with the intention that a student will understand, but the student's response indicates some other direction of thinking than that envisaged by the lecturer. The above perspective on interaction between student and lecturer can be applied analogously to the interactions between mathematicians and researchers in the interviews the work presented in this book draws on. Exploring learning and teaching situations in terms of Episodes, such as the ones I present in Chapters 3 – 8 (whether these 'episodes' refer to instances that document the students' learning or instances that document their lecturers' articulation of pedagogical thinking) is a methodological decision that reflects the aspiration to capture the specificity and richness of the above processes of codetermination.

The material presented in Chapters 3 – 8 draws on the following studies:

Study D My doctoral study[8] (1992 - 1996) entitled *The Novice Mathematician's Encounter With Mathematical Abstraction: Tensions in Concept Image Construction and Formalisation* (Nardi, 1996; 2000a; 2000b) examined mathematics undergraduates' learning processes in the context of several Year 1 pure mathematics courses through observation of weekly tutorials and interviews. Specifically, twenty first-year mathematics undergraduates were observed in their weekly, one-to-one or two-to-one tutorials in four Oxford Colleges during the first two terms of Year 1 in 1993 (ten students from one College had also been informally observed over two terms in 1992 as part of a Pilot Study of the doctorate's Main Study). Tutorials were audio-recorded and field-notes kept during observation. The students were also interviewed twice in the middle and at the end of the period of observation. The recordings of the observed tutorials and the interviews were transcribed and submitted to an analytical process (in the spirit of Glaser & Strauss' Data Grounded Theory, 1967)[9] of filtering out episodes that aimed to illuminate the students' learning processes. An analytical framework consisting of cognitive and sociocultural theories on learning was applied on sets of episodes within the mathematical areas of Foundational Analysis, Calculus, Topology[10], Linear Algebra and Group Theory. This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the students' learning processes.

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[8] Available at http://www.uea.ac.uk/~m011. The study was supported by the Economic and Social Research Council in the UK and the British Federation of Women Graduates.
[9] a spirit followed in the analyses within all of the studies outlined here.
[10] Omitted from the thesis due to limitations of space.

Study PD1 In this small-scale follow up[11] (February – July 1998) three of the students' tutors from Study D reflected on and evaluated samples of the doctorate's data and analysis in three lengthy, one-to-one interviews (Nardi, 1999). The samples had been distributed to them in advance.

Study PD2 This 12-month study[12] (1998-1999), directed by Barbara Jaworski (with Stephen Hegedus as Research Associate), focused on the analysis of six undergraduate mathematics tutors' conceptualisations of their first-year students' difficulties; their descriptive accounts of strategies for facilitating the students' overcoming of these difficulties; and, their self-reflective accounts regarding these teaching practices. These were recorded in 45 semi-structured interviews conducted during one eight-week Oxford University term and followed minimally-participant observation of their tutorials. The study explored the professional craft knowledge of undergraduate mathematics teachers and its methodology offered an opportunity for pedagogical reflection, for raising pedagogical awareness and for demonstrating the potential of a closer collaboration between mathematicians and mathematics educators (Jaworski, 2002; Nardi et al, 2005).

Study N This is a series of three small-scale studies[13] (six, three and six months respectively from 2000 to 2002) conducted at the University of East Anglia in collaboration with Paola Iannone. These studies examined sixty Year 1 and 2 undergraduates' mathematical writing as evident in their written work submitted on a fortnightly basis. The studies were conducted in twelve cycles of Data Collection and Processing (six on Analysis and Linear Algebra, three on Probability and three on Abstract Algebra). Within each 2-week cycle students attended lectures and problem sheets were handed out; they participated in Question Clinics, a forum of questions from students to lecturers; they submitted written work on the problem sheet; they attended tutorials in groups of six and discussed the now marked work with their tutor. Analysis of the student scripts collated descriptions, patterns and interpretations of student responses within and across the above mentioned mathematical topics. The analyses of the students' mathematical writing with regard to conceptual, symbolic and reasoning difficulties (e.g. Iannone & Nardi, 2002)[14] expanded those that had emerged from the Oxford studies, in particular Study D.

Study L This 16-month study (October 2002 - January 2004)[15] was also conducted at the University of East Anglia in collaboration with Paola Iannone and engaged groups of twenty mathematicians from institutions across the UK as educational co-researchers (e.g. Iannone & Nardi, 2005a)[16]. For each one of six cycles of data collection, six Data Sets were produced on the themes *Formal*

[11] Funded by the Wingate Foundation in the UK.
[12] Funded by the Economic and Social Research Council in the UK.
[13] Funded by the Nuffield Foundation in the UK.
[14] There are references to several publications from Study N across Chapters 3 – 8 as most of the Episodes in these chapters use the data from this study as the starting point of the discussion between M and RME.
[15] Funded by the Learning and Teaching Support Network in the UK.
[16] There are references to several publications from Study L across Chapters 3 – 8 that report on preliminary analyses of the data collected during this study. This body of data formed the basis for the dialogues presented in Chapters 3 – 8.

Mathematical Reasoning I: Students' Perceptions of Proof and Its Necessity; Mathematical Objects I: the Concept of Limit Across Mathematical Contexts; Mediating Mathematical Meaning: Symbols and Graphs; Mathematical Objects II: the Concept of Function Across Mathematical Topics; Formal Mathematical Reasoning II: Students' Enactment of Proving Techniques and Construction of Mathematical Arguments; and, *A Meta-Cycle: Collaborative Generation of Research Findings in Mathematics Education*[17]. Each Dataset consisted of: a short literature review and bibliography; samples of student data (e.g.: written work, interview transcripts, observation protocols) collected in the course of Studies D and N; and, a short list of issues to consider. Participants were asked to study the Dataset in preparation for a half-day group interview. Analysis of the verbatim transcripts led to eighty *Episodes*, self-contained excerpts of the conversation with a particular focus, which were then transformed into *Stories*, narrative accounts in which content was summarised, occasionally quoting the interviewees verbatim, and conceptual significance was highlighted. The eighty *Stories* were grouped in terms of five *Categories*: *students' attempts to adopt the 'genre speech' of university mathematics*; *pedagogical insight: tutors as initiators in 'genre speech'*; *the impact of school mathematics on students' skills, perceptions and attitudes*; *mathematicians' own mathematical thinking and the culture of professional mathematics*; and, *the relationship, and its potential, between mathematicians and mathematics educators* (25, 25, 4, 20 and 6 *Stories* respectively).

In the Chapters that follow, starting from a discussion of samples of student writing (collected in the course of Study N) or passages of transcribed conversations between students and their tutors (collected in the course of Study D), mathematicians and researchers in mathematics education engage in a collective consideration of pedagogical issues (data collected in the course of Study L). Their discussions are presented in the slightly unconventional format of a dialogue between two fictional – yet firmly data-grounded characters – M and RME (mathematician and researcher in mathematics education respectively). Before we launch into the dialogues themselves I would like to do the following:

- Introduce the data samples that M and RME discuss in the dialogues
- Introduce how the dialogues between M and RME came to be. I am doing so as follows: first introduce the Narrative Approach as a method of analysing and presenting qualitative data; then outline the rationale (that started building up in the course of Studies PD1 and PD2 and culminated in the design for Study L) for using this method; and, finally, describe the technique I have used towards constructing these dialogues.
- Introduce the style, format and thematic breakdown of Chapters 3 – 8.

I will do the above in Chapter 2.

[17] I note that these themes emanated from the analyses in the previous studies and their use as focal points coincides chronologically with the initial planning for the structure of this book (see references to the December 2002 version of the book proposal to the publisher in the Post-script)

CHAPTER 2
METHOD, PROCESS AND PRESENTATION

SUMMARY

In this chapter I explain how the data samples that M and RME discuss came to be and who 'is' M; the dialogue composition process; and, the style and thematic breakdown of Chapters 3 – 8. The presentation is in three parts.

Part 1. *Data samples and M*
How the data samples that M and RME discuss came to be; the participating mathematicians that 'became' M.

The dialogues between M and RME in Chapters 3 – 8 originate in half-day focused group interviews with mathematicians of varying experience and backgrounds from across the UK. In the interviews discussion was triggered by data samples consisting of students' written work, interview transcripts and observation protocols collected during (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory.

Part 2. *The dialogic format*
The Narrative Approach adopted in this work; the composition process through which the dialogues between M and RME came to be.

The dialogues between M and RME are fictional, yet data-grounded: they were constructed entirely out of the raw transcripts of the interviews with the mathematicians and then thematically arranged in *Episodes*. For an example of the construction process see p27-28.

Part 3. *Style, format and thematic breakdown of Chapters 3 – 8*

Chapters were constructed as series of *Episodes* (sometimes also broken in *Scenes*). Each Episode starts with a mathematical problem and (usually) two student responses. A dialogue between M and RME on issues exemplified by the student responses follows. Other examples of relevant student work are interspersed in the dialogue and links with relevant mathematics education research literature are made in the footnotes. *Special Episodes* are episodes that supplement the discussion in the main *Episodes* and *Out-Takes* are slightly peculiar or too specific incidents that stand alone and outside the more 'paradigmatic' material of the main *Episodes* but somehow address the wider theme of a chapter.

Notes to the reader:

- The account is in, more or less, chronological order and it is intended to be as transparent as possible. By revealing – hopefully without too much pedantry – the details of the construction process (of data samples, dialogues and chapters), I am opening up this process to critique, a quintessential element to its validation.
- In what follows I use the same abbreviations for the studies I introduced in Chapter 1, Part 3.

1. DATA SAMPLES AND M

The bulk of the data used for the composition of the dialogues between M and RME in Chapters 3 – 8 originates in the Focused Group Interviews with mathematicians in the course of Study L. Here I return to the brief description of that Study in Chapter 1, Part 3 in order to zoom-in on some of that study's features and provide information that is a necessary prerequisite for understanding the context and background of the dialogues between M and RME in Chapters 3 – 8.

Chapters 3 – 8 consist of a series of Episodes in which M and RME set out from a discussion of a data sample which, in most cases, operates as a trigger for addressing an issue on the learning and teaching of mathematics at the undergraduate level. These data samples are parts of the Datasets used for the same purpose in the interviews during Study L. In Chapter 1 I outline these Datasets as consisting of a short literature review and bibliography; samples of student data (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of Studies D and N; and, a short list of issues to consider. Here I provide a bit more information.

As studies D and N had focused primarily on collecting data from Year 1 mathematics undergraduates, the focus of the Datasets (which built on the data and analyses of those studies) reflects the students' experience in Year 1[1]. The courses, in particular the parts referred to in the dialogues in Chapters 3 – 8, are mostly typical parts of introductory courses in Analysis / Calculus, Linear Algebra and Group Theory[2]. Paola Iannone, who conducted Studies N and L with me, and I were occasionally asked to provide further information about the stage of the students' studies that a particular data sample was exemplifying. However participants

[1] With the exception of Study N's foray into Year 2 for the Group Theory course that had then recently been moved from Year 1. As Group Theory is a splendid context for exploring the issues of abstraction and formalism that the previous studies had explored extensively, for continuity and coherence, we decided that this 'digression' into Year 2 would be most pertinent.

[2] For Study N the relevant courses were: *Analysis and Algebra*, *Probability* and *Groups and Rings* (content available at http://www.mth.uea.ac.uk/maths/syllabuses/0506/). For Study D the relevant courses were: *Continuity and Differentiability, Linear Algebra* and *Group Theory* (content available at http://www.maths.ox.ac.uk/current-students/undergraduates/lecture-material/). The parts of the courses that the discussion of M and RME focuses in Chapters 3 – 8 have not changed radically since the studies were conducted but I hold details of the courses as offered in the years in which the studies were conducted (2000-1, 2001-2 for Study N; 1993-4 for Study D)

generally recognised that the material discussed in the samples was typical of the students' early experiences of university mathematics. Of course the reader needs to bear in mind that all studies referred to here were conducted in the UK.

There were six Datasets, five 'mathematical' (as listed in the Study L outline in Chapter 1, Part 3) and one 'meta-mathematical' (on the theme of collaboration between mathematicians and mathematics educators and mathematicians' engagement with educational research).

The eleven half-day interviews that constitute the interview material of Study L were conducted as follows: six, one for each Dataset, at the University of East Anglia, where the data for Study N had been collected. The five 'mathematical' Datasets were also used in five analogous events in non-UEA institutions across the UK: two in England, one in Scotland, one in Wales and one[3] in the context of a group-work session at the Annual Conference of the Mathematical Association[4]. In total there were twenty participants, pure and applied mathematicians, all-bar-one male, white and European (but several with significant international experience), of age ranging from early thirties to late fifties and of teaching experience varying from a few years to a few decades.

Interviews were conducted during the academic year 2002 – 2003. They lasted approximately four hours and revolved around the above described Datasets. These had been distributed to participants at least a week prior to the interview. Most participants arrived at the interviews well prepared, with comments and questions scribbled in the margins and keen on a close examination of the Datasets. The interviews were conducted according to the principles of Focused Group Interviews (Wilson, 1997; Madriz, 2001) – see (Iannone & Nardi, 2005a) for a rationale and a description of our use of this tool.

As you can see in Chapters 3 – 8, each Episode starts with the discussion of a data sample that, in most cases, consists of:

- a mathematical problem (including its formulation as well as the suggested solution distributed to the students once they had submitted their written responses to their tutor[5])
- two typical student responses[6]

In the course of the discussion between M and RME, the latter presents M with more student responses[7]. All of the data samples originate in the Datasets and in the extra examples that were used in the course of the interviews. In the next Part of this chapter I explain how the raw material of the interviews was turned into the dialogues in Chapters 3 – 8.

[3] Replacing one institution which was happy to participate in principle but with which scheduling negotiations failed to result in an arrangement within the timeline of the project's data collection period.
[4] In this we used the Dataset we would have used in the missing institution mentioned above.
[5] See description of the data collection in Study N in Chapter 1, Part 3.
[6] Namely responses that typify the issue that the part of the Dataset aimed to address in the original interviews with the mathematicians.
[7] Replicating the events in the original interviews.

2. THE DIALOGIC FORMAT

In what follows (Part 2(i)) I place the choice of the dialogic format within the increasingly vigorous methodological tradition of Narrative Approaches in Qualitative Research (Denzin & Lincoln, 2001) and zoom in on my particular rationale for this choice. I then explain[8] (Part 2(ii)) how I applied some of the techniques of the Narrative Approach towards the composition of the dialogues presented in Chapters 3 – 8.

(i) The Narrative Approach

> '…all you can do, if you really want to be truthful, is to tell a story.'
> John Mason (1998, p367) consolidating Paul Feyerabend's epistemological position as elaborated in his *Three Dialogues on Knowledge* (1991)

Qualitative data analysis aims to produce generalisations embedded in the contextual richness of individual experience. Coding and categorising techniques, for example, a method I use and generally respect, often results in texts sorted out into units of like meaning. Despite evident benefits of this, such as facilitating access to interpretation of the situation in question etc., this sorting out can strip contextual richness away. A more holistic account, based on the rapidly developing Narrative Approach, can often be even more illuminating.

> 'People tell stories about their life experiences. Telling stories helps people to think about, and understand, their personal or another individual's, thinking, actions, and reactions (Bruner, 1986, 1990; Polkinghorne, 1988; Ricoeur, 1991). Thus, it is not surprising that collecting stories has emerged as a popular form of interpretive or qualitative research (Gudmundsdottir, 1997). It has rapidly gained legitimacy in education and has flourished at research conferences and in professional development activities in schools (Connelly & Clandinin, 2000).
> Over the past 20 years, the popularity of narrative research in the social sciences and education is evident from an increase in narrative publications having to do with narrative questions, phenomena, or methods (Lieblich et al, 1998). Narrative brings researchers and educators together collaboratively to construct school experiences (Connelly & Clandinin, 1990). It provides a voice for teachers and students (Errante, 2000), and it places emphasis on the value of stories in all aspects of life (McEwan & Egan, 1995).' (Ollerenshaw & Creswell 2002, p329-330)

My own general understanding of the potency of this approach has come largely from Clandinin & Connelly's *Narrative Inquiry* (2000), an extension of their influential 1990 article in the *Educational Researcher* and their other work at the time.

> 'Connelly and Clandinin's advocacy for this form of qualitative inquiry has deep roots in the social sciences and the humanities (Casey, 1995-1996; Cortazzi, 1993; Polanyi, 1989; Polkinghorne, 1988). Procedures for finding tellers and collecting their stories has

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[8] Sometimes I do so with reference to the process of conceptualising the book as a whole, a process I give an account of in the Post-script. However I have intended the account in this part to be self-contained.

> emerged from cultural studies, oral history, folklore, anthropology, literature, sociology, and psychotherapy. Interdisciplinary efforts at narrative research have been encouraged by Sage Publications through their *Narrative Study of Lives* annual series that began in 1993 (Josselson & Lieblich, 1993)'. (Ollerenshaw & Creswell 2002, p331)

Understandings on what constitutes narrative research, or 'narratology' (Connelly & Clandinin, 1990), are very diverse. Ollerenshaw & Creswell outline some common characteristics as follows:

> 'The inquirer emphasizes the importance of learning from participants in a setting. This learning occurs through individual stories told by individuals, such as teachers or students. For Clandinin and Connelly (2000), these stories report personal experiences in narrative inquiry (what the individual experiences) as well as social experiences (the individual interacting with others). This focus on experience draws on the philosophical thoughts of John Dewey, who saw that an individual's experience was a central lens for understanding a person. One aspect of Dewey's thinking was to view experience as continuous (Clandinin & Connelly, 2000), where one experience led to another experience. The stories constitute the data, and the researcher typically gathers it through interviews or informal conversations. These stories, called field texts (Clandinin & Connelly, 2000), provide the raw data for researchers to analyze as they retell or restory the story based on narrative elements such as the problem, characters, setting, actions, and resolution […]' (Ollerenshaw & Creswell 2002, p332)

Re-storying is the central feature of *holistic-content analysis*, one of Lieblich et al's (1998) four-type attempt to classify narrative analytic approaches, and the one that is most relevant to the work presented in this book.

> '[*holistic-content analysis* is] a narrative approach for understanding the meaning of an individual's stories'. […] The holistic-content analysis of field texts (e.g., transcripts, documents, and observational field notes) includes more than description and thematic development [..]. It involves a complex set of analysis steps based on the central feature of "re-storying" a story from the original raw data. The process of re-storying includes reading the transcript, analyzing this story to understand the lived experiences (Clandinin & Connelly, 2000) and then retelling the story.' (Ollerenshaw & Creswell 2002, p330).

Ollerenshaw & Creswell (2002, p332) describe the process of re-storying as follows:

> 'Re-storying is the process of gathering stories, analyzing them for key elements of the story (e.g., time, place, plot, and scene), and then rewriting the story to place it within a chronological sequence. Often when individuals tell a story, this sequence may be missing or not logically developed, and by re-storying, the researcher provides a causal link among ideas. In the re-storying of the participant's story and the telling of the themes, the narrative researcher includes rich detail about the setting or context of the participant's experiences. This setting in narrative research may be friends, family, workplace, home, social organization, or school—the place in which a story physically occurs.
>
> A story in narrative research is a first-person oral telling or retelling of events related to the personal or social experiences of an individual. Often these stories have a beginning, middle, and an end. Similar to basic elements found in good novels, these aspects involve a predicament, conflict, or struggle; a protagonist or character; and a sequence with implied causality (i.e., a plot) during which the predicament is resolved in some fashion (Carter, 1993). In a more general sense, the story might include the elements

> typically found in novels, such as time, place, plot, and scene (Connelly & Clandinin, 1990). In this process, researchers narrate the story and often identify themes or categories that emerge from the story. Thus, the qualitative data analysis may be both descriptions of the story and themes that emerge from it. In addition, the researcher often writes into the reconstituted story a chronology of events describing the individual's past, present, and future experiences lodged within specific settings or contexts. Cortazzi (1993) suggested that it is the chronology of narrative research with an emphasis on sequence that sets narrative apart from other genres of research. Throughout this process of collecting and analyzing data, the researcher collaborates with the participant by checking the story and negotiating the meaning of the database. Within the story may also be the story of the researcher interwoven as she or he gains insight into himself or herself'.

A particularly helpful way of seeing the brand of (re)story-ing I have used is Jerome Bruner's account of how the mind constructs a sense of reality through 'cultural products, like language and other symbolic systems' (p3) in his *Critical Inquiry* article 'The Narrative Construction of Reality' (1991)[9]. Bruner proposes narrative as one of these cultural products and defines it in terms of the following ten characteristics:

1. Diachronicity (narratives deal with events taking place over a period of time)
2. Particularity (narratives deal with particular events)
3. Intentional State Entailment (characters within a narrative have 'beliefs, desires, theories, values, and so on')
4. Hermeneutic Composability (narratives can be interpreted as playing a role in a series of events that constitute a 'story')
5. Canonicity and breach (stories are about something unusual happening that 'breaches' a normal or canonical state)
6. Referentiality (a story references reality although it does not offer verisimilitude in any direct way)
7. Genericness (as a flipside to particularity, this is a characteristic of narrative whereby the story can be classified as representing a genre, as being paradigmatic)
8. Normativeness (as a follow up to 'canonicity and breach' a narrative may also make claims to how one *ought to* act)
9. Context Sensitivity and Negotiability (relating to hermeneutic composability, an understanding of a narrative requires that between the author, the text and the reader there is a negotiation regarding the contextual boundaries within which the narrative works)
10. Narrative accrual (stories are cumulative; new stories follow from older ones).

In this work I use Narrative in at least two senses: in the literal sense of Narrative as the form of processing and presenting the data collected in the studies I am drawing on. In this sense the Episodes in Chapters 3 – 8 are the stories I am constructing as a researcher in order to make sense of how the participants in my

[9] I would like to thank Lulu Healy for bringing this part of Bruner's work to my attention.

studies experience learning, teaching etc. There is another sense parallel to this one though and one that is perhaps a bit closer to the one Bruner directly refers to in the above. Bruner talked about the stories children, for example, construct in order to make sense of the mathematics they are taught in school. Analogously the mathematicians participating in the studies I draw on here have their own 'stories', their own interpretive frames for making sense of things like their students' learning (in Chapters 3 – 6), their own pedagogical practices (in Chapter 7), the way they relate to mathematics educators and educational researchers (in Chapter 8) etc.. For example, 'landscapes' (an idea that recurs in their description of students' perception of mathematical concepts and that often resembles Vinner & Tall's (1981) 'concept image', a widely used descriptor in mathematics education research) seems to be one of the 'stories' employed in their accounts of their students' learning; 'mathematics as a language to master' seems to be their 'story' for interpreting students' writing; 'communication / interaction / gradual and negotiated induction into the practices of university mathematics' seems to be their 'story' for most of the teaching practices they express preference for; 'us and them' seems to be their 'story' for expressing a cautious attitude towards mathematics education research; etc.

Having placed this work within a certain strand of narrative inquiry I conclude this part with a presentation of my rationale for the use of the dialogic format as a way of presenting the 'stories' in Chapters 3 – 8.

My fascination with the dialogic format can be traced back in the 1980s and my first contact in school with the texts of Plato[10], first master of the format who consolidated earlier attempts at its use into what remain to this day some of the liveliest and most lucid philosophical texts. In his dialogues Plato (c400BC/1999), while tackling some of the hardest questions ever asked, offers perceptively drawn characters, minute contextual detail and a great sense of often deeply ironic humour. His dialectic, an exchange of theses and anti-theses between interlocutors resulting in a syn-thesis, particularly in its *Socratic* version[11], became a major literary form. A form often chosen by philosophers as a means to present their work all the way through to, for instance, Galileo (1638/1991) and Berkeley (1713) – and, in our days, Feyerabend (1991).

Feyerabend playfully dedicated his sometimes seen as incendiary *Against Method* (1975) to the man who brought the dialogic format closer to the preoccupations of mathematics educators than anyone else, Imre Lakatos. *Proofs and Refutations* (1976) is a fictional dialogue set in a mathematics classroom. It depicts students' attempts to prove the formula for Euler's characteristic. Through their successive attempts they re-live the trials and tribulations of the mathematicians who had previously engaged with it – most famously through the successive construction and employment of counterexamples that refute hitherto versions of the conjecture they are trying to prove.

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[10] I was educated in Greece where some of his texts are included in the Ancient Greek syllabus.

[11] The process of refuting or verifying an initial conjecture through searching for the contradictions its acceptance may yield.

The way I use the dialogic format in this work draws inspiration from the illustrative, evocative and suggestive powers of the medium that is evident in the above works. Lakatos, for example, aimed at demonstrating his view of the creative processes through which mathematics comes to be – and on the way, also inspired by Pólya (1945), he challenged our perception of how it is learnt and therefore how it ought to be taught. Plato employed the dialogic format, partly, as a platform to introduce us to the character of his teacher and mentor Socrates (and in his later works his own ideas as transformed by, and gradually independently of, Socrates' teaching). My aim here is to employ the dialogic format as a platform to introduce the character of M and, through his exchanges with the auxiliary, constantly prompting character of RME, showcase the rich canvas of perspectives on learning and teaching that have emerged from my collaborative work with mathematicians. In doing so I aim to demonstrate the potential that lies within this type of collaborative work (the main characteristics of which I describe in Chapter 1, Part 2).

Of course, unlike Plato / Socrates, there is no truth to be reached at the end of my lane and, unlike Lakatos, I am not proving any theorem. Alan Bishop (1998, p33) sums up a part of my rationale deftly in the summary of his involvement with an ICMI Study that aimed to offer a 'state of the nation' look at the field of mathematics education research at the time: '… I could detect certain emphases in the discourses together with some important silences' he observes and lists the silences as follows: syntheses, consensus-building, awareness of other audiences, researched arguing (to convince), over-arching structures, global theory, well-articulated similarities, agreements. This work aspires to address some of these silences – with the exception perhaps of 'global theory' on the teaching and learning of mathematics at the undergraduate level…!

Apart from the above, mostly philosophical, works another source of inspiration in my use of the dialogic format lies in theatre, particularly in a small number of plays where I feel that the subtle and the artful meet effectively. I am constantly fascinated by the capacity of great writers such as Tom Stoppard (particularly *Arcadia*) and Michael Frayn (particularly *Copenhagen* and *Democracy*) to touch on 'big issues' in ways that are accessible but are *'ni récréation ni vulgarisation'*, a difficult and in many ways problematic dual objective [12]. I have learnt a lot from the ways in which these authors manage to weave complex, multi-layered information into their characters' discourse: the superb marriage of the personal, political and scientific in *Copenhagen*; more the obvious, thus perhaps less elegant, way in which, say, Catherine's questions in David Pinner's *Newton's Hooke* prompt exposition on the part of Isaac Newton on his then developing ideas on Planetary Motion, Force, Fluxions / Calculus etc.. Without digressing into too lengthy a waxing lyrical about the virtues of these works let me exemplify with a short excerpt from Tom Stoppard's *Arcadia* (p48-50). In it Thomasina, age 13 and a prodigious

[12] as reviewer Arnaud Spire observed in *L'Humanité* (http://www.humanite.presse.fr/journal/2001-04-18/2001-04-18-243018) in the context of discussing Isabelle Stengers' 2001 '*scientifiction*' foray into the Newton - Leibniz priority rift about the invention of Calculus.

student of in-house tutor Septimus is amazed to find out he has given her work an A Minus. 'What is the minus for?' she asks.

Septimus:	For doing more than was asked.
Thomasina:	You did not like my discovery?
Septimus:	A fancy is not a discovery.
Thomasina:	A jibe is not a rebuttal. […] I think it is an excellent discovery. Each week I plot your equations dot for dot, *x*s against *y*s in all manner of algebraical relation, and every week they draw themselves as commonplace geometry, as if the world of forms were nothing but arcs and angles. God's truth, Septimus, if there is an equation for a curve like a bell, there must be an equation for one like a bluebell, why not a rose? Do we believe nature is written in numbers?
Septimus:	We do.
Thomasina:	Then why do your equations only describe the shapes of manufacture?
Septimus:	I do not know.
Thomasina:	Armed thus, God could only make a cabinet.
Septimus:	He has mastery of equations which lead into infinities where we cannot follow.
Thomasina:	What a faint-heart! We must work outward from the middle of the maze. We will start with something simple *[She picks us an apple leaf.]* I will plot this leaf and deduce its equation […]! *[Septimus firmly orders Thomasina to return to the piece of poetry they were supposed to discuss next in the lesson.]*

Thirst for knowledge 19th century-style? Awakening of a mathematical mind albeit – unthinkable! – in the body of a young girl? Pedagogy as uninspired reduction and drudgery? … whatever discourse one chooses to trace in the above exchange, one thing that comes across clearly is the power of the dialogic format to carry through complex, multi-layered conversation in engaging and thought-provoking ways.

'This is not science, this is story-telling' (p125), Septimus exclaims later at the perplexing sight of Thomasina's writings, whose prose combines well-ahead-of-her-time mathematical preoccupations with grander speculations about the universe. His narrow-minded treatment of his student's work is not pertinent here just for the obvious pedagogical reasons but also, as an acknowledgement by Tom Stoppard of the dangerous waters his own enterprise is attempting to tread. I share Stoppard's qualms but I cannot help but acknowledge that we have been living, often productively, in a literary world of such hybrids for quite some time now, at least since Truman Capote's non-fiction novel *In Cold Blood.* At the end of the day the dialogic format is something that I feel a 'natural affinity for'[13]. It also suits perfectly well that of which M and RME speak of. In Part 2(ii) I describe the genetic process through which their dialogues came to be[14].

[13] quoting *Copenhagen*'s Margrethe speculation about the reason why Heisenberg 'did Uncertainty', p78 – also see quotation following the title of this book.

[14] I would like to extend my warmest thanks to four friends, one of them also a colleague, with whom at this stage I shared the fledgling idea of using the dialogic format. At this very insecure stage had they cringed, had they insinuated anything like doubt about the idea, I may have never started. They are: Barbara Jaworski who supervised my work in Studies D and PD1, directed Study PD2, was consulted at various stages of Studies N and L, whose judgment I wholly trust and who insisted throughout that the

(ii) From interview transcripts to Dialogue: an application of the Narrative Approach

> *Vanbrugh:* […] The plot already exists… in real life. The play and all its scenes.
> *Cibber:* A drama documenting facts? […] Will you allow yourself the same liberties as Shakespeare? Taking liberties with facts converts facts into plays.
> *Vanbrugh:* No liberties… just facts in this play.
> Carl Djerassi, *Calculus* (Djerassi & Pinner, 2003, aka 'Newton's Whores'), *Scene I*

Study L was completed in January 2004. From February to July 2004 I spent the largest part of a Study Leave on searching for a method to be used towards the composition of the dialogues between M and RME and on composing first drafts of the dialogues. In this part I describe how I arrived at this method[15] and how I used it for producing these first drafts. Then in Part 3 I describe how I turned these first drafts into the episodic Chapters 3 – 8.

Study L's eleven half-day Focused Group Interviews produced material that amounted to about 30,000 - 40,000 words per interview. The order of discussion in these, in most cases, followed the structure of the Datasets. That structure was as follows – see (Sangwin et al 2004) for a complete Dataset:

- two introductory pages: cover, interview scheduling details, a few lines on the theme of the Dataset;
- four sections[16] each entitled Example I, Example II, Example III and Example IV. Each Example was about a couple of pages long and contained scanned images of
    - a mathematical problem and a lecturer's recommended response (both from course materials)
    - usually two student responses (more were used in the course of the interview) from the data collected in the course of Study N (and /or in several occasions Study D)

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practitioners' perspectives and priorities should stay right at the forefront of my priorities as the author; Panos Karnezis and James Ferron Anderson – both writers (of the literary kind!) whose trade, I felt, my choice of the dialogic format would be seen as intruding – for helping me to clarify the distinction between using the format for literary and for academic purposes; and, Margarita Angelou, mathematician, for her overall encouragement and for reminding me that, even though the focus of the dialogue is deliberately on M, 'whoever reads the book they will want to know what *you* make of M as well'. Within the dialogues this is kept to a minimum – in symmetry with how the original interviews were conducted. But in the choice of themes of the Episodes and Chapters, in highlighting of M's characteristic 'behaviour' and in the overall synthesis of the character, I believe the balance has been redressed towards my not fleeing the responsibility Margarita's comment assigns to me.

[15] In a February 2004 entry of my research diary I have scribbled the words 'Re-conceptualising Our Discipline As Conversation'. The scribbling looks like the title of something; it maybe a part of a sentence from something I read at the time. I have failed to locate the origin – so, with apologies, could the owner, if any, please stand up?! – but it seems to encapsulate my thinking and soul-searching at the time with regard to the direction this work was going to take. So I feel it is worth mentioning here.

[16] Except Datasets 3 and 4 for which there were five Examples.

- a mini literature review and bibliography on the theme of the Dataset
- concluding page with thanks to the participants (and, in the case of UEA interviews, arrangements for the next interview)

In the five 'mathematical' Datasets there were twenty two Examples and in the one 'meta-mathematical' Dataset there were three. For ease of access – and a much needed at this stage sense of preliminary structure! – I created twenty five folders, one for each Example, in which I filed the following materials:

- Transcripts
- Narrative Accounts, the descriptive summaries of the interviews produced in the course of Study L's data analysis,
- Scanned images of relevant materials and student writing or other student-related data

that had been used towards the creation and discussion of the Example in the two half-day interviews (one at UEA, one at an institution outside UEA). These folders were labeled Narratives X.Y, where X stood for the Dataset they came from and Y for the number of the Example they came from. So, for instance, the folder labeled 'Narrative 2.III' contains the materials revolving around Example III in Dataset 2. The materials within each folder formed the basis for a text, symmetrically labeled Narrative X.Y, in which the 'story' of this Example's discussion was told, in most cases[17], as follows. First I presented the mathematical problem and its recommended solution; then the student responses that had been used as triggers of the discussion in the original interviews; then a list of issues that the interviewees had been asked to consider (copied from the original Dataset); finally, and most significantly, a dialogue between two characters, M and RME, each consolidating the contributions in the interviews by the participating mathematicians (for M) and the researchers conducting the interviews (for RME).

In the previous section I explained the appeal that the dialogic format has exerted on me and my rationale for using it in this work. Here is how I presented this rationale in my very first attempt at composing a dialogue from Narrative 1.I. The outcome was intended as a chapter (that later became the opening scene of Chapter 4 – see Post-script) in a book aimed for mathematics undergraduates in Rio De Janeiro, Brazil where I spent the first weeks of my Study Leave in 2004 and where much of the conceptualisation described above took place[18]. The following is from the chapter submitted to the Brazilian colleagues responsible for the volume in April 2004:

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[17] This process differs slightly for the three Narrative Folders from the 'meta-mathematical' Dataset.

[18] I would like to thank my friend and colleague Victor Giraldo for his hospitality and support during this period, a perhaps not surprisingly muddled period during which I started the writing. Combining the purely academic parts of the visit with the unadulterated fun – that the city of Rio De Janeiro and its inhabitants seem to have a unique and magic recipe for! – succeeded entirely because of him. With the hindsight of the amount of work that followed that visit, the experience further supported a dearly-held conjecture of mine, that there are times when there is nothing more productive than a break away!

> ‘The presentation is in the somewhat unconventional format of a dialogue (My fascination with the dialogic format dates back to my school days in Greece and the experience of the works of Plato. However, within mathematics education literature, an early and defining influence was Imre Lakatos’ *Proofs and Refutations* (1976)) between a mathematician and a researcher in mathematics education – M and RME respectively. The dialogue is fictional but based on data collected in a series of studies that the author and her associates have been conducting in recent years. The set up of the conversation is as follows. M is presented with: a mathematical question given to Year 1 students in the early weeks of their course; the suggested answer expected by the lecturer of the course; two examples of students’ written responses to the question; a list of issues to consider in preparation for the discussion with RME. […] Relevant research literature is referred to within the dialogue as footnotes – as is information on the students’ mathematical background. The studies where the data that formed the basis for the dialogue originate from are introduced briefly in [an] Appendix.
> The exchange between M and RME sets out from the concrete context of a specific mathematical question and examples of student writing. However it soon becomes about an issue that is commonly known to cause some difficulty amongst mathematics undergraduates in the beginning of their studies: […]

The rationale for using the dialogic format is then explained as follows:

> ‘The dialogic format of the presentation is not a merely stylistic choice, even though the idea of improving its readability is appealing. After all it is not an easy task to represent the various layers of data and analysis (a mathematical question, students’ responses to the question, researchers’ analyses of these responses and distillation of cognitive and pedagogical issues from these analyses, university teachers’ reflections on the student data, on the analyses and on pedagogical practices relevant to these issues and researchers’ analyses of the teachers’ reflections…!) that form the basis of the work presented here. Beyond a stylistic choice, the dialogic format of the presentation is above all intended as a reminder of the need of the worlds of M and RME – both intrigued by and having a commitment to improving mathematical learning – to meet and confer more often. It is intended also as a response to stereotypical views that see researchers as irrelevant theorisers with a suspiciously loose commitment to the cause of mathematics (RME) and practitioners as non-reflective actors who insensitively rush through content-coverage and have no pedagogical ambition other than that related to success in exams and audits (M). In the realm of these stereotypes M and RME, deaf and blind to each other’s needs, skills but also idiosyncrasies of their respective epistemological worlds, have no choice other than also of being mute, remain silent, indifferent and even hostile to each other’s presence. Even though there is little pretense of constructing M and RME in a naturalistic way – after all their words are consolidations of those of the numerous mathematicians and researchers who participated in the studies that form the basis of the dialogue presented here – the effect is intended to be as close as possible to a realistic proposition: one of partnership.’

Between then and now I have had two more opportunities to present succinctly the rationale for (and way of) my using of the dialogic format. One was the CERME4 paper I mention in the Post-script; the other the Delphi Summer School sessions I also mention in the Post-script. The following is from the script I used for my introduction to those sessions (July 2005):

> ‘The dialogue consists […] of M and RME’s utterances, let’s call them ‘quotations’ – please note the inverted commas. The text within M’s ‘quotations’ is a consolidation of verbatim quotations from across the board of the twenty participants in the last study I

mentioned. […]. The text within RME's 'quotations' is a consolidation of the minimally leading interventions of the researchers in the group interviews on which the dialogues are based.'

The links of the material in the dialogue with relevant works in the field is then described as follows:

'The references to literature, attached to the dialogue in the form of footnotes, aim at highlighting places where I believe there is resonance between the views expressed in the text and other relevant works. To suggest that one unified perspective on M, RME and the literature is possible – or even desirable – would be facile and deprive the conversation this work wishes to contribute to of the richness that often emerges from difference. I invite you to see the aim of this exercise as two-fold and approach it having this multi-layered set of intentions in mind: to contribute to the substantive conversation regarding student learning and pedagogical practice at university level by bringing to the fore M's views on these issues; and, to represent the complexity and sensitivity of the pedagogical perspectives demonstrated by M […]'.

The rationale for using the dialogic format varies slightly but largely repeats the points made in its April 2004 version:

'The presentation in the form of a dialogue is not merely a stylistic choice: it serves as a reminder of the overall intention of the study to contribute to the highly needed rapprochement of the worlds of mathematics and research in mathematics education. These two worlds – whose members are both intrigued by and having a commitment to the improvement of mathematical learning – need to meet, confer and generate negotiated, mutually acceptable perspectives more often. Through a demonstration of the rich pedagogical perspectives that are evident in these 'quotations' this heavy-on-data presentation is intended also as a response to stereotypical views that see practitioners as non-reflective actors who rush through content-coverage in ways often insensitive to their students' needs and have no pedagogical ambition other than that related to success in examinations and audits (analogously these stereotypes also see researchers as irrelevant theorisers with a suspiciously loose commitment to the cause of mathematics and incapable of 'connecting' with practitioners). In the realm of these stereotypes M and RME, oblivious to each other's needs, skills but also idiosyncrasies of their respective epistemological worlds, have no choice other than to remain indifferent, and even hostile, to each other. Through the presentation in the form of a dialogue the effect is intended to be a hopefully not too unrealistic proposition: that of partnership.'

Let me now exemplify the process through which the dialogues in Chapters 3 – 8 came to be. I will do so by picking a dialogue excerpt and juxtaposing it to the pieces of transcript it originates from. The following excerpt is from E3.5, Scene I (first Scene of the fifth Episode in Chapter 3) in which M comments on a student's response to a question, Student L's (see details within Chapter 3):

M: …there are issues in the ways students engage with constructing mathematical arguments that need attention: often students will write down the thing that they are asked to prove and manipulate it. I see that they can't avoid doing this at some point in a contradiction proof but I would be much happier if the word *suppose* figured firmly in the beginning of their sentence. To allow this to go without comment would be doing the student a disservice. I would still like to stress though the originality of Student L's thought. That's something she thought up herself, not something she copied from a tutor

> or her lecture notes or a book. She believes in her claim totally! It is the type of informal, intuitive claim students may have been conditioned to find satisfying at school level, this idea that *well, we cannot keep on cancelling forever, can we?* The descending argument, by the way, is an approach I am perfectly happy with. Historically the habit of choosing a minimum counterexample, *m* and *n* having no common factors and reaching contradiction because of that, is a modern habit.

This originates in Narrative 5.I, namely in Example 1 of Dataset used in the fifth Cycle of Data Collection (*Students' Enactment of Proving Techniques*). The original piece of transcript used to produce this brief monologue is the following[19]:

M1: …[I am quite pleased with what is written here] apart from one or two things that need rescuing like the initial declaration. And what slightly depresses me about her is that often the students will write down the thing that they are asked to prove and manipulate it.
R1: Hum…
M2: Hum…
M3: But they can't avoid doing this at some point in a contradiction proof.
M4: Yes, but there isn't the word "suppose"…
M1: Yes, it is all in that first line, isn't it? She is guessing a lot more… So on the face of it this is very good but of course to allow this to go without comment it would be doing the student a disservice so… lots of comments about that first line, I didn't believe…
M3: Yes… I think that L's solution … she didn't get it from her advisor or a book, she actually thought it up…you see what I mean? I think that they believe it.
M2: But isn't this the sort of thing that would have appeared on the blackboard in an A-level class?
R2: Yes, this could be the only example of a proof by contradiction they see …
M4: Yes, but how do they do it? Do they do it by m and n being co-prime or do they do it by this method of saying, well, you cannot keep cancelling forever and raise their hands and say…?
R2: I don't know but I know that this is in a lot of cases the only example of formal proof that they would have of one sort of another…
M3: And this descending thing is fine. I mean it is a modern habit to say, ok, choose a minimum counterexample to get the contradiction. Whereas I would have just said: this is the descent argument, so we are done.

By the end of June 2004[20] twenty two Narratives, one for each Example had been produced[21]. In the time that followed these Narratives turned into the themed episodic entities that constitute Chapters 3 – 8. In Part 3 I tell this part of the story.

[19] For the sake of simplicity for this demonstration I have chosen a piece of dialogue that originates in one interview transcript only. I believe however that it is illustrative as several participants, denoted M1, M2, … express views in it. These views are consolidated into this monologue by M. The researchers facilitating the group interview are denoted R1 and R2. Their contributions did not seem to influence the course of discussion amongst M1,…, M4 therefore their voices have been removed – hence the monologue in the final product is, in this case, a mini-monologue.

[20] The twenty two Narratives, the preliminary and major groundwork for the text presented in Chapters 3 – 8, was done largely during eight weeks of my 2004 Study Leave in our family home in Thessaloniki, Greece. I deeply thank my mother Nicole and my father Christophoros for providing the tranquil and nurturing environment that was so important at that stage.

[21] The three Narratives from the sixth Dataset were produced later as at this stage I did not plan to use this material in the way I ended up using it in Chapter 8.

3. STYLE, FORMAT AND THEMATIC BREAKDOWN OF CHAPTERS 3 – 8

The twenty-two Narratives contained the first attempts at converting the material from each Example into a dialogue between M and RME. Having created the Narratives an increasingly precise understanding began to emerge of the themes and issues the dialogues were revolving around. Having selected the foci of the Datasets for the original interviews with the mathematicians on the evidence of relevant literature and findings from the previous studies, it was now a good moment to revisit, and update, this literature in order to start drawing tighter connections between it and the dialogues. Thus followed a period of searching, reading and summarising of relevant literature in July, August and September 2004[22]. I quote from my research diary in order to illustrate my thinking at the time on how the dialogues in the Narratives – which, in the natural course of conversation in the interviews, ebbed and flowed across many different issues – could be handled from now on so that the strength of the material (authenticity, richness and naturalistic flow) could be maintained while offering the reader a sense of focus, structure and direction(s) towards which the conversation is heading. The following is from the November 15th, 2004 diary entry and documents initial ideas on how to structure the Narratives internally as well as organise them under chapter headings:

> 'Narratives: sharpen the focus on each of the 22 Narratives until each is about 'something', a focal point. Write each Narrative with sharpened focus in mind as follows:
>
> - Introduce the focal point with reference to previous studies (mine and others')
> - Main perspective is M on student learning but M perspectives on teaching, on own thinking, on cultural / institutional issues and M/RME can be brief (abbreviated perhaps?) digressions. Signal these digressions or 'other' perspectives?
> - Insert as footnotes further references (mainly on focal point but on digressions too)
> - Conclude with a brief analysis of M-perspective (e.g. 'judging' the sensitivity of the discourse etc.)
>
> Chapters: within each chapter introduce the issue generally, justify the selection of the focal points. Title of each of the five[23] chapters: poetic plus 'On some issues regarding the learning of…'.
>
> - Five mathematical chapters: mathematical reasoning I (necessity of proof) and II (techniques of proving), mathematical language (notation and graphs), mathematical concepts I (limits) and II (functions).
> - One introductory chapter on aims and methodology
> - One chapter on M's discourse and on M/RME'

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[22] Having collected material for reading in July, a substantial part of the reading took place on the Aegean island of Kythnos in August 2004. I warmly thank my friend Alexis Spanos for sharing the slightly surreal experience of delving into piles of journal papers whilst inhabiting some of the most breathtaking Aegean landscapes and in a locale (a Greek island!) that, unsurprisingly, he and I, and most of our friends, had associated till then with total escape from the clutter of our lives in the city.

[23] Reminder: 'five' at this stage refers to the themes of the five datasets (see list in Part 1 and Chapter 1, Part 3).

After a period of minimal engagement with the material (largely the first semester of academic year 2004-5) further formation of the chapters took place from the spring of 2005 onwards. In what follows I illustrate these stages of the process. Landmarks of this process were the versions of the chapters in May, July and August 2005 and the work towards the submission of the manuscript for reviewing in February and March 2006.

A first crucial shift from the thinking that the above quotation reveals occurred in mid-March 2005. Following a re-reading of the material – at the state it was left in November 2004 when it had been last engaged with – a March 15th, 2005 diary entry lists the contents of the Narratives (apparently in the order that I read them at the time) as follows:

> 'In terms of what the Narratives so far are about:
> 1.I What is a mathematical argument to students?
> 1.II Students' mathematical reasoning: resort to familiarity of numbers, example construction and the tension between the specific and the general, [section on Question Setter's Intentions, QSI], [teaching and curriculum], implication / deduction, [against examples]
> 1.III Use of definition towards building an argument or not, concept images (det(A)), use of deduction (e.g. substitution), [mathematics], [QSI], writing
> 1.IV [QSI], role of definition in building an argument, writing random mathematics, the non-linear and sudden nature of mathematical understanding, premature compression in students argument?, the limiting process, the meta-theme
> 5.I Difficulties with Proof by Contradiction: spotting and logical leaps (the syndrome of the obvious); proof in school (UK): perceptions of proof in school mathematics, of $\sqrt{2}$, proof as algebraic manipulation, spotting contradiction at all cost; the contextual meaning of *prove*; on the significance of $\sqrt{2}\in$R-Q; a critique of the Proof by Contradiction that $\sqrt{2}\in$R-Q; on counter-examples
> 5.II Difficulties with Proof by Mathematical Induction; Syndrome of the Obvious: premature compression; difficulties with inequalities; school mathematics: lamentable state of proof, rationale for, attracting mathematicians in current cultural ambience, teacher recruitment (quality of degree, social status of the profession), value of articulating ideas (writing / speaking), (mathematics as one case in the school curriculum)
> 5.III Difficulty with applying the general to the particular (solve a Group Theory question on symmetries by resorting to the relevant axioms and theorems), [QSI]
> 5.IV In trouble with some properties in Group Theory, on counterexamples, the meta-theme
> 3.I to-ing and fro-ing between mathematics and language, [teaching]
> 3.II the use of graphs and graphic calculators in mathematical reasoning, [teaching: is absolute rigour pedagogically viable?]
> 3.III [QSI] an attempt to link matrices, vectors and linear equations, to connect the apparently distinct worlds of Algebra and Geometry.
> 3.IV Use of group tables to construct meaning about groups
> 3.V [teaching: pros and pitfalls in the use of pictures to convey meaning in Group Theory; teacher – student communication issues]
> 2.I Students' use of (and problems with) the definition of convergence, $\geq$ versus $>$ in the definition of convergence, convergence of a series, [teaching: definition of convergence, numerical experiments, the tension between formal and informal, use of pictures, foundational issues in the teaching of Analysis], students and pictures, [QSI]
> 2.II an episode in the learning about convergence: ignoring the head of a sequence, also on writing, symbols etc.

2.III students' logical misuse of LCT, [teaching: theorems as a toolbox, provoking student participation, the art of clever choice, lecturing], students' strategies on determining convergence
2.IV [QSI], students' perceptions of the definition of limit, symbolic writing, [teaching: interaction]
4.I [QSI], students' conceptions of the concept of function, [teaching: names for new concepts], using representations, students' misuse of logic, symbolic writing
4.II students' perceptions of function, [teaching: via rhymes and memory triggers, acquisition of mental representations, against indefinite integration, symbolism for Fourier series]
4.III students' perceptions of function, [teaching: fostering new definitions, context-bound usefulness of graphs, abstraction, critique of insensitivity of tutor, Oxford tutorials], [QSI]
4.IV [QSI], polynomials, [teaching new objects], [teaching: use of libraries and books]
4.V student difficulty with Group Theory concepts, [teaching: Group Theory, abstraction, attracting students, student participation, coping with content, the price of a vision].

And an idea for chapter headings as follows:

'proof (necessity and technique); function across topics; limiting process; mediating meaning (words, symbols, diagrams); methodology and introduction; meta-theme'

This chapter breakdown differs to the one up until then in that Proof is not dealt with in two (necessity, enactment) but in one chapter, Mediation of Mathematical Meaning is similarly not dealt with in two (words / symbols, diagrams) but in one and the idea for a 'meta-theme' chapters has also emerged. Obviously the 'data spoke' at this stage and dictated this reconsideration which is further consolidated in a March 17th, 2005 diary entry as follows:

'NEW CHAPTER STRUCTURE:
Reasoning (based on Narratives 1, 5)
Communicating (Narratives 3)
Limits (Narratives 2)
Functions (Narratives 4)
Teaching (across)*
M-RME relationship (across)
Intro
Methodology
Plus spin-off episodes? E.g. school, the equals sign
* e.g. sections on: [QSI] and mismatch between M and student perspectives, tutorials, writing, rigour as pedagogically viable?, pictures in Group Theory etc.'

In this I observe several developments:

- The new chapter breakdown specifies which of the 22 Narratives will form the basis for which chapter. So the four Narratives 1 and the four Narratives 5 will form the basis for the Reasoning Chapter etc.

- The idea for a separate chapter on teaching makes a first appearance – apparently because of the substantial number of [teaching] entries across the March 15th breakdown.
- The 'meta-theme' chapter is now called 'M-RME relationship' chapter and, similarly to the one on teaching will draw on material from across the Narratives[24].
- There is a first reference to 'spin-off episodes', an acknowledgement that there is substantial part of the material that doesn't fit neatly under the headings used at the time. This material later the Special Episodes and Out-Takes in Chapters 3 – 8.

Next day's entry, March 18th 2005, consolidates the above even further[25]:

> '- A new structure: one chapter on reasoning (both necessity for proof and reasoning techniques); one on mediating mathematical meaning through writing in words, diagrams and symbols); then two chapters where we see all of the previous stuff in the concrete context of two fundamental concepts: function and the limiting process; one chapter on the meta-theme and one chapter on teaching (before the meta-theme chapter). Finally the introduction should tell the story of the book (from PhD onwards) and there should be one methodology chapter where the construction of the dialogues is made transparent. In there, there needs to be a section on failed / unfocused (e.g. Narratives 4) dialogues where the interviewees get distracted or changed course. This would stress the value of the successful ones and strengthen our faith in them[26].
> - Regarding the chapter on teaching: put in there all the digressions from talking about students in the main narratives in the form of tactics etc.
> - On the [QSI] digressions: put them as vignettes in the main narratives' chapters or the teaching one.
> - In the introduction: say as much as possible about the background of the course, the students. There needs to be enough anchoring into the reality of the course but enough for the reader to take off into the more general.
> - When the narrative takes a turn to discuss thesis extracts, use [Study PD1] data to enrich the discussion[27].'

Subsequent work on 'chopping' the Narratives and forming the Chapters took place in May 2005 as follows.

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[24] But soon, in May 2005, Narratives 6, based on the material from Dataset 6, was added to this database.

[25] A month after this entry an updated proposal was submitted to the publisher (see Post-script).

[26] A worthy idea which I have not followed through in Part 2(ii) because its illustration is too lengthy. It requires use of extensive parts of the data so that we can have a macroscopic view of the trajectory the discussion between M and RME is taking, substantiate the 'failure' that the diary entry perhaps a bit too non-challantly highlights, and reflect on possible causes. It is an idea worth pursuing in another, more methodologically-geared piece of writing. The starting point for this could be diary entries such as this from May 9th, 2005: 'Narrative 4.I, where RME makes genuine attempts to steer the conversation towards the concept of function but the conversation keeps 'slipping' into the other themes (language, writing, logic, history, etc.!), is a good example of a dialogue that didn't work very well. For the methodology chapter: show and speculate why… Narrative 4.II fluctuates too… (and so do III, IV and V…).'

[27] Another worthy idea that never materialised. One of the reasons was that as time went by the material inflated to an extent that adding more data in it seemed less and like a good idea less and less.

> '…each Narrative should be broken to pieces[28], each labelled content-wise and with a reference where each piece belongs to (namely which chapter). The 'Mathematical Problem' page would of course only appear once and will be accompanied by a text where the choice of this problem / student examples / issue would be explained. This explanation will come from the analysis in the previous studies and the bibliography'. *Diary entry May 3rd, 2005*
> '…created the files called Chapters 1-8. I am putting the chunks from last week's etc chunks in each of the files for Chapters 3-7 [as numbered in the April 2005 outline for the publisher]'. *Diary entry May 17th, 2005*
> 'Now that I am planning a chapter on the M/RME relationship, Chapter 8, I need to go back to the data from Dataset 6 and produce Narratives 6.I, 6.II and 6.III.' *Diary entry May 18th, 2005*

In the weeks that followed work on the material now within each one of the data chapters, Chapters 3 – 8, is recorded in the diary as follows:

> '…breakdown each chapter in episodes according to the issues raised; relate each issue to relevant literature; write for each chapter (or for conclusions chapter? Epilogue?) my analysis of what the mathematicians say'. *Diary entry June 8th, 2005*

One outcome of this work was Content Maps for each of Chapters 3 – 8 followed by a construction of Chapter Summaries / Flow Texts (translations of the Maps into rough prose). The process of constructing the Summaries is recorded in the diary as follows:

> '… how to construct the summary – maps: I need an episode breakdown. Start from a smaller chapter, e.g. Limits, put all on the floor, read map, see clusters, reorganize material accordingly, title them…' *Diary entry June 27th, 2005*

At that stage I had also collected the material that I had earlier labelled 'Spin-Off Episodes' (see March 17th diary entry quoted above) under the heading Other Matters and was toying with ideas on how to handle this material from now on. One of these ideas was to insert them as separate Vignettes across the text. One example of this material, as recorded in a June 18th entry of the diary, was what has now become Out-Take 7.1. Analogously most of this material is now in Special Episodes and Out-Takes. Further rumination on the different status of the Episodes, the main carrier of the themes in each chapter, and this other material is in the same entry of the diary, June 27th, 2005:

> 'A note on the narrative approach of the book (and the studies it draws on): this is not a comprehensive coverage of all topics (e.g. why just limit, function etc?) and I often cover 'peculiar' cases (Head of a Sequence? Students and Bikes?[29]) BUT what these 'peculiarities' do have is a capacity to throw light on a paradigmatic issue [dealt with in the Episodes]. Is this then where the narrative becomes paradigmatic?'[30]

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[28] A preliminary breakdown into these 'pieces' is suggested in the March 15th, 2005 diary entry quoted earlier in this section.
[29] referring to two examples from the material that has now become an Episode (E4.1, Scene II) and a Special Episode (SE6.1) respectively.
[30] The distinction uses the terminology referred to in Part 2(i) of this chapter.

The arrangement of the material within Chapters proved to be an arduous experience as the 'taming' of naturalistic data typically tends to be. It is recorded in the diary entries between June 27th and July 5th, 2005 as 'slow', as being necessarily 'much rougher way than the June 27th, 2005 notes suggest' and 'very rough' at this stage. My feelings are recorded as needing to 'check whether this way works', as 'an even stronger urge to start seeing the final episode breakdown of the chapters' etc.. To this aim I keep 'rearranging the episodes until some flow/coherence started to emerge'. By July 5th I appear to have achieved some clarification on what is now called Special Episodes, namely different from the main Episodes but supplementing them in some way:

> '…and today I […] put these chapters in order. The main text, Special Episodes (they look special but in fact extend the discussion in the main text) and the Out-takes (special as well but somewhat outside the main flow – still interesting enough to keep). As the richness of the discussion makes it inevitable that there will be some digressions – and for helping the reader – I am thinking about numbering the text as follows: x.y (x for the chapter, y for the section) and insert [x.y] anywhere else in the text where the same issue is touched upon[31].' *Diary entry July 5th, 2005*

Crucially at this time the opportunity to trial some first outcomes of this work appeared in the shape of the Delphi Summer School sessions (see Post-script). For the three sessions I selected one Episode from each of the six Chapters. The preparation of the six Delphi Summer School Excerpts is described in the diary as follows:

> '…trim to the essentials, insert missing info (such as questions and answers etc), weave in relevant literature and write a brief chapter overview based on the Flow Texts. I will introduce each excerpt with such overviews to give a flavour of where each excerpt comes from' *Diary entry July 8th, 2005*
>
> ' Start with short [excerpts] for practice… of course I am doing this slightly differently to how I initially planned. I have trimmed the Functions [excerpt] one but then started writing little introductions to each of the six episodes: what went on before, which larger issue they maybe coming from, what we could be expecting from this episode. They are intended to be written in a relaxed way. I am also figuring out what the reader (and the students in the Summer School) are expected to do when faced with these dialogues. Given time / space constraints I can only insert literature indicatively, not exhaustively, in the footnotes of the text.' *Diary entry July 9th, 2005*
>
> 'It takes a lot longer to work on these excerpts and I am still figuring out the status of the footnotes: links to literature, analytical comments etc. It's hard but enjoyable.' *Diary entry for period July 14th – 18th, 2005*

The exercise was significant in several ways: it helped me clarify what 'episode' was starting to mean; and, it highlighted issues around the status and content of the footnotes (an essential component of the Episodes as the footnotes play the role of

[31] These evolved into Ex.y, SEx.y and OTx.y for Episodes, Special Episodes and Out-Takes from Chapter x and y standing for the number of Episode, Special Episode or Out-Take within the chapter.

associating the very context-specific discussion in the dialogue with its more general addressing in the literature[32]).

The experience of and feedback from the sessions would, I hoped, indicate ways in which these issues could be resolved.

And largely it did. Most of the decisions regarding the state of the text as it currently stands were finalised in the aftermath of these sessions[33] and in the work that took place in August 2005[34].

Alongside work on Chapters 3 – 8 ideas about the structuring of the *other* story-telling in the book (background, methodology etc.) are also beginning to emerge. I quote from the August 13th, 2005 diary entry:

> '… this diary […] tells, I believe, the Making of… story […]. [Tell the story…] also with scanned excerpts of the diary […]. The to-ing and fro-ing, the fluctuations, indecision, the 'worth noting' attempts all show […] in here'

This subsequent work on Chapters 3 – 8 consisted largely of the following: defining the boundaries of the Episodes and separating these from the Special Episodes and the Out-Takes; and, taking cue from the way I had worked on the Delphi Summer School Excerpts, starting to weave in preliminary indications for where Footnotes need to be inserted. In the light of the feedback on the Delphi material and my own further reading, observations, such as the ones recorded below in the diary, led to the next steps the refinement of the text needed to take.

On flow of the dialogues and the status of footnotes:

> 'I notice, now that a few weeks passed since I wrote [the Delphi Excerpts], where the text jars, where faithfulness to the transcript compromises flow or readability, where repetition may annoy or confuse the reader (is it repetition? Or does M or RME try to say something different?). Same applies to their footnotes: their status fluctuates…' *Diary entry August 16th, 2005*
> '…the type of footnote that simply says 'I say this and this other paper says that too' is kind of cheating, it's not enough. Add what that 'other paper' actually says that's relevant to the discussion here'. *Diary entry August 18th, 2005 (reminder repeated in the February 8th, 2006 entry)*

And on M's 'character':

> '…reading all the Delphi excerpts – perhaps experiencing these excerpts en masse similarly to how the students and faculty in Delphi did – I see now that M is a particularly 'nice' pedagogue. Part of the metaphor I think is that this kind of richness, while not plausibly attainable by each one of us, is however communally obtainable: it's the kind of richness we can achieve as a field if we learn from each other, practitioners and researchers alike'. *Diary entry August 18th, 2005*

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[32] Several of these references originate in the literature surveys carried out in the course of the studies introduced in Chapter 1, Part (iii) and the summaries distributed to the participants in Study L. Most were added in the course of constructing the dialogue.

[33] For this I would like to thank the students and faculty who participated in these sessions as well as Paola Iannone for their constructive feedback on the Delphi Excerpts.

[34] This work took place in two locations: on the Greek island of Alonissos in the company of my sister Danai Nardi and in the beach house of my friend Margarita (also mentioned in Part 2(i) of this chapter) in Chalkidiki, Greece. I warmly thank them both for their support.

By the end of August 2005 the structure of Chapters 3 – 8 had stabilised to one that is quite close to the current. The next level of refinement[35] took place in the eight weeks leading to the submission of the manuscript on March 31st, 2006. I now list observations and decisions made in this period (February and March 2006) through quotations from the diary[36].

On the Episode / Special Episode / Out-Take structure:

> '… in a naturalistic enquiry tidying up things in boxes doesn't work. There is always the spill over effect and what spills over is often equally interesting to what's inside. There are two spill over types in this book: the Special Episode and the Out-take. The former is just a slightly peculiar (too specific, at face value too distant from the mainstream of topics and themes frequenting our discussion in the field but still revolving around the Episodes' themes. The second is the type that doesn't fit the themes but it still says something about participants' point of view that I thought it deserves to be seen. Whereas what's in the E and SE represents or synthesises the point of view of the majority of participants, the OT is more of a lone-ranger category.' *Diary entry February 3rd, 2006*

On cross-referencing of Episodes and breaking particularly long or complex Episodes in Scenes:

> '…the same piece of data can appear in different Episodes and Chapters with a different function, in a different capacity, illustrating a different issue. However I cannot afford repeating the same piece of data: certain Episodes, e.g. E3.1 is already too long. So here are two suggestions: first, to avoid repetition, make cross-references, e.g. in Episode 3.x to Episode 6.y. Chapters 3 (Reasoning) and 4 (Writing) will then be introducing cross-topical issues and Chapters 5, 6 will be putting some of these within the topical context of Function and Limit. Also, across Chapters 3 and 4, where the dynamic between thought and language is often so intense that it becomes almost impossible to separate the discussion of one from the discussion of the other, I will need to exercise judgement about where every piece of data fits and apply cross-referencing, relentlessly if I have to. Second, as in Chapter 3, for example, Episodes are very long and move across mathematical problems and occasions, it may be a good idea to break them into Scenes [as I have already experimented in Chapter 6 back in August 2005]'. *Diary entry February 9th, 2006*

On strengthening the coherent flow of the text:

> 'I read and insert footnote signs where I feel there is a necessity for a comment or a reference. I also insert the connective sentences (including references) and the introduction so that the text flow is improved [and goes beyond] a collation of Episodes / Scenes with a synopsis at the front.' *Diary entry February 27th, 2006*
>
> 'Write one continuous text as the opening page of each data chapter. Start with a statement of the overall issue with a seminal reference in the field, preferably a quotation. In Chapter 3, for example, this would be Mathematical Reasoning as the defining activity of mathematics (in Chapter 4 then would start with a quotation on putting this thinking in communicable mode and this thinking being influenced by this

[35] The first semester of the academic year (2005-6) allowed minimal engagement with the material.

[36] In this intensive and isolated period of Writing Leave contact with the … outside world was mostly through the telephone. To the list of friends I have already thanked in other parts of the text I need to add here my sister Anthi Nardi, resident of the USA, who spent much of her valuable time on expensive long-distance phone-calls during this period. Her positive spirit and encouragement are priceless.

> mode of expression - *if we didn't have these symbols we may have never achieved the level of abstraction mathematics has achieved… etc*). Within each *Setting the Scene* section of Episodes / Scenes start similarly with highlighting the issue that takes centre-stage. […] I need to explain the non exhaustive, selective nature of the Episodes as they emerged from the data: M's preference guided by the datasets that were in turn guided by our and other studies. *Diary entry February 28th, 2006*

On the nature of footnotes in Chapters 3 – 8:

> 'Indicative, non exhaustive references to literature. The range of topics addressed in the book is so broad […] that the aim of exhaustiveness is unattainable'. *Diary entry March 7th, 2006*

This phase also included 'injecting references and cross-references in brackets or footnotes', 'inserting missing images, ensuring flow, cutting repetition' (all February 2006 entries)[37] from Chapters 3 – 8. Particularly, given the scope of issues raised in these Chapters, the task of choosing the location for the footnotes, identifying a salient piece or relevant literature to refer to and composing its brief but dense text was almost at times overwhelming:

> 'Sometimes I feel that my reading which I never really stop doing seems to be happening in a parallel universe to that of my writing. I am sure these two connect but I do need to find the connections, and fluently so…' *Diary entry March 8th, 2006*

Any 'proof' of whether the above targets have been hit can only be, as the saying goes, 'in the pudding' – Chapters 3 to 8.

[37] The diary records the trials and tribulations of this phase with extended examples that I omit here. Perhaps they are more pertinent to a more methodologically inclined piece of writing.

CHAPTER 3
THE ENCOUNTER WITH FORMAL MATHEMATICAL REASONING: CONCEPTUALISING ITS SIGNIFICANCE AND ENACTING ITS TECHNIQUES

Students' encounter with formal mathematical reasoning (in particular in terms of realising the necessity for proof and enacting proving techniques) is a cornerstone of their encounter with university mathematics[1]. In this and the next chapter M and RME explore how the students experience this encounter: here the focus is more explicitly on the students' *reasoning*, as evident in their writing / speaking etc; in Chapter 4 the focus is more explicitly on the ways in which students choose to express this reasoning and *mediate mathematical meaning* through words, symbols and various forms of graphical representation.

In the discussion that follows M and RME describe the students' encounter with formal mathematical reasoning as follows: at this stage students experience at least two tensions: that between the familiar (numerical, concrete) and the unfamiliar (rigorous, abstract) (E3.1); and, that between the general and the particular (E3.2). In the course of constructing mathematical arguments they often face the choice between seeking recourse to a concept definition (appropriate in the case discussed in E3.3, Scene I) and favoring manipulation of algebraic expressions (appropriate in the case discussed in E3.3, Scene II); they find the use of Logic difficult (E3.4); and, they are often unwilling to provide proof / justification for parts of their arguments that seem to them to be intuitively obvious (E3.5, Scene II).

When enacting classical proving techniques students often have difficulty with spotting contradiction in a Proof by Contradiction (E3.5, Scene I) and with the step from n to n+1 in Proof by Mathematical Induction (E3.6). They also appear ambivalent about whether all counterexamples that refute a statement in a Proof by Counterexample are of equal value (E3.7).

In the four Special Episodes that conclude the chapter M and RME explore several other influences on students' construction of a mathematical argument: that of the dominant discourses of school mathematics (SE3.1); that of technical difficulties (e.g. algebraic, SE3.2); that of conceptual novelty (e.g. Group Theory , SE3.3); and that of often elusive to the students links between mathematical topics (e.g. Algebra and Geometry, SE3.4).

[1] At least in the UK within which all of the studies this work draws on were conducted.

EPISODE 3.1
THE TENSION BETWEEN THE FAMILIAR (NUMERICAL, CONCRETE) AND THE UNFAMILIAR (RIGOROUS, ABSTRACT)[2]: RESORTING TO THE FAMILIARITY OF NUMBER[3]

Setting the scene: The following takes place in the context of discussing the question below as well as two examples of students' written responses to this question, Student W's and Student WD's:

Example from Exercise Sheet 5, Week 3, Autumn Semester 2000

Suppose A is an $n \times n$ matrix which satisfies $A^2 = 0$ (the $n \times n$ zero matrix).

1. Show that A is not invertible.
2. Show that $I_n + A$ has inverse $I_n - A$.
3. Give an example of a non-zero 2×2 matrix A with $A^2 = 0$.

Suggested solution

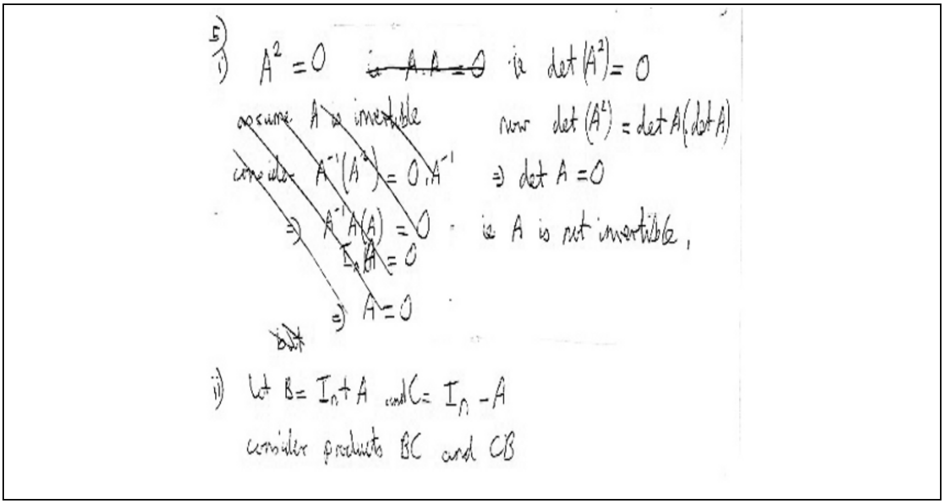
1. If $BA = AB = I_n$ then $0 = B^2A^2 = BBAA = I_n$. Contradiction.
2. $$(I_n + A)(I_n - A) = I_n - A + A - A^2 = I_n$$ $$(I_n - A)(I_n + A) = I_n + A - A - A^2 = I_n.$$ So $(I_n + A)^{-1} = (I_n - A)$.
3. $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

---

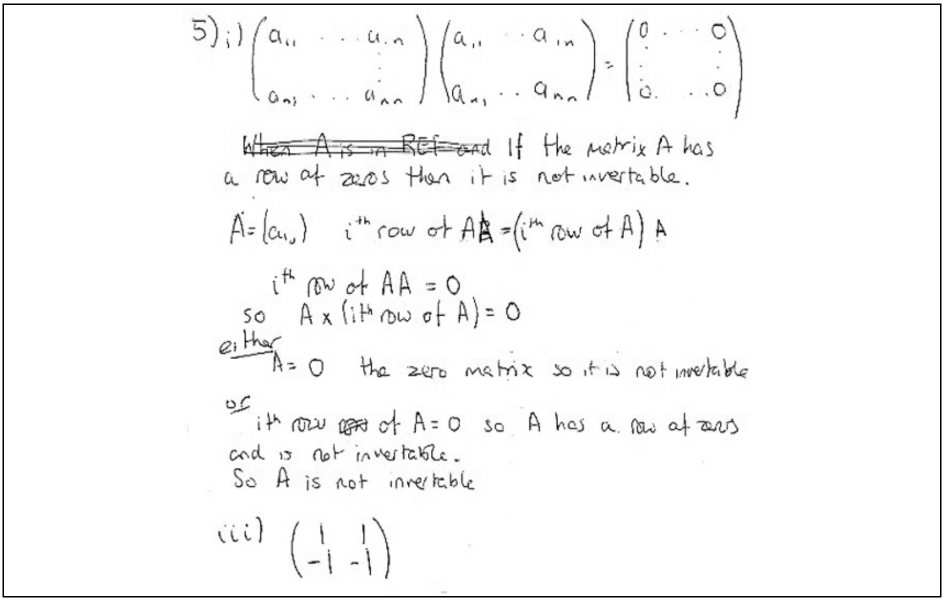
[2] Studies of the problematic aspects of the transition from the experimental and intuitive habits of school mathematical reasoning to the formal requirements of advanced mathematical thinking can be traced back in the 1970s (e.g. Bell, 1976 and 1979). Across the chapter I return to several of these studies. Some consolidations of this work can be found e.g. in (Moore, 1994).

[3] In (Nardi & Iannone, 2000) we discuss the issue of students' employment of previously 'established' mathematical results in terms of their emerging 'desire to be mathematical' (E4.0). Specifically to the question discussed in this Episode we observed the following with regard to the 'choice of method and context in the students' proofs':

> '…'to be mathematical' is an aspiration that the students materialise with hesitation when it comes to adopting a mathematical *modus operandi*. For example, in [this question] proof by contradiction was desirable – but not explicitly requested – as was an argument within the context of matrix operations. Few student responses though matched the question setter's intentions. Instead of this neatly contextualised Method-Context (1: Contradiction - Matrix operations), the majority of students made a 'reductive' choice of another Method-Context (2: 'straight' deduction - Arithmetic of Determinants). (p59)


5) i) $A^2 = 0$ ~~ie A.A = 0~~ ie $\det(A^2) = 0$
~~assume A is invertible~~ now $\det(A^2) = \det A(\det A)$
~~consider $A^{-1}(A^2) = 0.A^{-1}$~~ $\Rightarrow \det A = 0$
~~$\Rightarrow A^{-1}A(A) = 0$~~ ie A is not invertible,
~~$I_n A = 0$~~
~~$\Rightarrow A = 0$~~
~~but~~
ii) let $B = I_n + A$ and $C = I_n - A$
consider products BC and CB

Student W


5) i) $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$
~~When A is in REF and~~ If the matrix A has a row of zeros then it is not invertable.
$A = (a_{ij})$ i^{th} row of AA = (i^{th} row of A) A
i^{th} row of AA = 0
so A x (i^{th} row of A) = 0
either A = 0 the zero matrix so it is not invertable
or i^{th} row of A = 0 so A has a row of zeros and is not invertable.
So A is not invertable
iii) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$

Student WD

(To start with M offers a critique of the question and its proposed answer as follows:

M: I like the lecturer's part (iii): none of this sentences nonsense... Just zero, one, zero, zero!

RME: Most students too did just that in fact.

M: And most probably paid little attention to how this part connects with whatever you have proved in parts (i) and (ii), right? It would be interesting to ask this question with (iii) and (i) transposed. That would put the student in the A-is-not-the-zero-matrix frame of mind. In a sense there is a problem with the language of the question: the first line needs a lot of unpacking and it may even be interpreted as if the brackets referred to A. It could have been interpreted as *A* is in fact the *n*x*n* zero matrix. Maybe.

RME: It could be, even though most students worked with A as not necessarily being the zero matrix.)

M: I think the issue here is being exposed to things that do not behave like numbers[4]. Yes, students react with unease to this novelty. Cancellation is something they are so used to when it comes to products and the idea that it may not apply to these new creatures they are asked to work with is strange. So, in a somewhat reasonable manner, you get reactions like that of Student W who is using something he is familiar with, numbers, but in the form of determinants. Except that doing that is rather more than what is needed and is more

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[4] With regard to the distinction between Contexts 1 and 2 and Methods 1 and 2 in (Nardi & Iannone, 2000) we also observed:

> 'Context 2 is closer to their school mathematical knowledge and rings bells of familiarity. It is a safer choice. But does it provide the students with the benefits [from doing the question in the way] intended by the question setter? The question was intended as an exercise in matrix operations, where matrices are treated as objects, their multiplication is non-commutative and their inverse must be written/checked from both sides; where the identity matrix is decomposed as the product of a matrix and its inverse and associativity is a property that helps us reshape an expression with brackets. Reducing this argument from an argument about matrices to an argument about their determinants (themselves numbers where all the above properties have been used in a trivial manner by the students throughout their years of schooling; AND a method [formally proved in the lectures after the question sheet was given out]; from school they only 'know' the 3x3 case) is slightly missing the point of engaging with the question. As for choosing Method 2 at the expense of Method 1, this does not have the same grave repercussions as the choice of Context 2 at the expense of Context 1, even though it can be alarming that the efficiency of Method 1 eludes the students. Finally, if Method 2 had been formally introduced, the flavour of the above may have been slightly different but not substantially so: in fact it is quite natural for a learner to resort to the resources that are more familiar, that yield a sense of ease and confidence. It is perhaps more of a criticism of the question if it doesn't succeed in triggering the student's choices towards the more beneficial. The intentions of the question may have not been transparent enough for doing so and the students' use of Method 2 is just another case of *the confusion with what knowledge they are allowed to assume'* . (p59)

complicated. They do learn about determinants quite early but this was a question that is supposed to be dealt with without them (E7.4, Scene IIb).

RME: Quite a few students resorted to determinants though…

M: … which, in a sense, is fine. It must have to do with hearing about them recently and with this psychologically comforting feeling that they switch from matrices to numbers which students feel they know how they behave. If numbers are zero divisors then they are zero. If matrices are zero divisors, well… that is very mysterious. So maybe they do help themselves by referring to something that they can understand better. This is all right: look at the erased bit in Student W's answer too, where he does try to work within the realm of matrices. He almost manages to provide a proof by contradiction but then he doesn't know what to do with the A = 0 bit. He doesn't know what this actually tells him, why he has reached a contradiction (E3.5, Scene I) and he gives up. But, as I said, *he* is alright. I am a bit more concerned about Student WD: starting by writing out the tables in $\{a_{ij}\}$ format is likely to get him nowhere.

RME: Maybe it's his way of resorting to the familiarity of numbers?

M: Probably. Hopelessly trying to translate this into something familiar and, unlike Student W whose resorting to some sort of numbers is resourceful if somewhat heavy-handed, unfortunately, Student WD is still hooked down in doing calculations and doesn't get very far, does he?

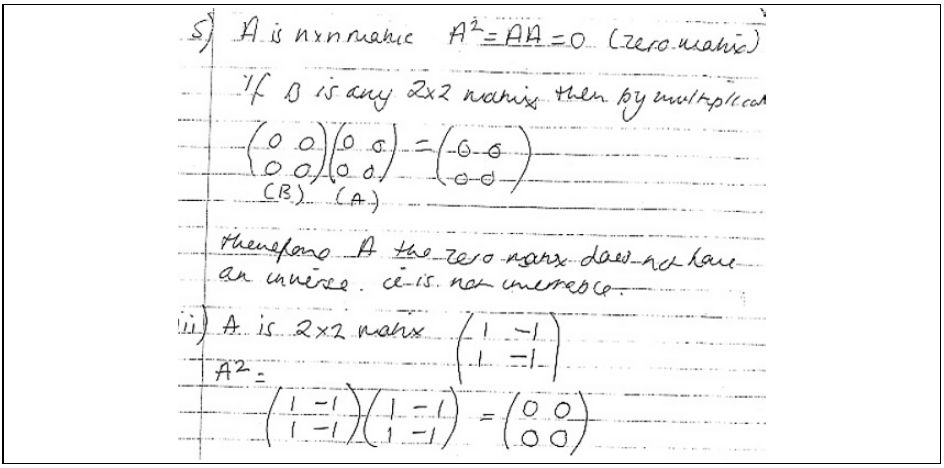
RME: It seems he is trying to connect invertibility of a matrix with having a row of zeros.

M: He starts from the more or less reasonable – but rather irrelevant here – idea that if a matrix has a row of zeros then it is not invertible. However it goes haywire from then on, as he is flipping the argument the wrong way around – his use of implication is a mess (E3.4) and I am seeing this a lot in students by the way – and assumes that, if the product is zero, then one of these two things, the row or matrix A, is zero. This is a wrong assumption but his argument would work if that were true! Oh, and by the way, I find mis-spelling the word *invertible* on the script annoying!

RME: Here is another response, from Student L.

M: Underlying Student L's response – which I admit I find rather disturbing in its lack of logical consequence and disregard for the world of matrices – is the complete impossibility to accept that the product of things can be zero even if none of them is zero (E4.2, Scene II). And everything else is revolving around accommodating this apparently preposterous idea, that this could be any different.

RME: Having worked with numbers for such a long time, to believe that what applies to numbers applies to all other objects one is asked to work with is an understandable propensity.



Student L

M: And I think you would find it surviving at a more sophisticated level too. You could ask mathematicians if it is possible that you multiply a matrix by its transpose and get zero. And everybody would say only if the matrix is zero. But that is actually false too. Just work with complex numbers. So, even for the more experienced of us, there is a certain resistance to the idea[5]. But there is another scenario that may be at play here. It may be the case Student L thinks that A is the zero matrix and that what she has to show is the zero matrix is not invertible. Students seem to have a very understandable inability to accept that something can be an expression for zero without the symbol of zero being used for it (E4.1). I am surprised that so few of them resorted to determinants by the way because they would have seen them recently or at school. Then, of course, you have all the issues arising from them using things that have not been properly introduced in the lectures (E7.4, Scene IIb). I probably wouldn't mark down Student W for resorting to determinants, for having that extra knowledge – well, my marking would probably have been influenced by the general standard of his script – but part of the point of them doing questions on a weekly basis is that they gradually introduce into their reasoning the elements introduced in the lectures. I often find

[5] Jones (2000) describes some naïve ideas that persist in the minds of students beyond graduation (and often follow them into their careers in teaching) by drawing on students' concept maps.

this problematic when I deal with questions in the sheets myself because I have to place myself in that frame of what I am sort of allowed to resort to. And I am not sure that we set the ground rules clearly enough to the students about this (E7.4, Scene IIb).

M and RME extend this discussion of the tension between the familiar (to which I would tend to resort as a problem solver because it is a terrain within which I feel comfortable) and the unfamiliar (which I know somehow I am expected to resort to but I do not know yet how. Also my level of comfort within this terrain is not sufficient and I am not even clear which of my hitherto practices / knowledge I am allowed to use) in several episodes to come: in Episode 6.1 the unfamiliarity revolves around the employment of definitions (What does the formal definition of convergence, for example, mean and how does it relate to how I understand convergence? What am I allowed to do in order to use this formal definition more efficiently? For example, is 'solving backwards' the inequality at the heart of the definition a legitimate way of identifying an N that fits the purpose of my proof?); in Episode 6.2 the discussion turns to the tension between an intuitive understanding of what the limit of a sequence may be (is it acceptable for me to 'guess' a limit?) and the need to formally prove it; and, in Episode 4.3 the discussion of the students' ambiguity extends to the realm of visualisation (Am I allowed to use pictures? Am I not constantly told that I am supposed to always prove any claim I make? How is this requirement compatible with using pictures? Etc.). M encapsulates below how this 'setting the ground rules' cannot be treated casually. His point is made in the context of Analysis – convergence – and is elaborated in Chapter 6:

M: One issue that I would like to raise […] is the interchangeable use of prove and show that in most of our writing. Yet prove is stronger and more mathematical for students. I think it is our obligation to define terms the same way terms are defined, say, in an insurance policy, namely in bold and clearly. If we expect students to produce proofs, we might as well have to explain exactly what we mean by proof. Whether we mean demonstrate that something works, that something is true or that something exists we should make all these meanings clear. And I am annoyed whenever I see words *prove* and *rigorously* side by side. What does this mean? How else can we prove other than rigorously? Of course rigour is context-dependent, often dramatically. What I can take for granted in one context, I need to prove in another. It is because of this dependence on context that we need to be as clear as possible. In Year 3 for example, *1/n goes to zero* can be assumed; in Year 1 it needs to be proved.

EPISODE 3.2
THE TENSION BETWEEN THE GENERAL AND THE PARTICULAR[6]: CONSTRUCTING EXAMPLES AND APPLYING THEORETICAL KNOWLEDGE IN CONCRETE CONTEXTS

A dominant focal point of research in the area is students' difficulty to abstract, to detect the general in the particular, to establish the generality of a statement not merely by resorting to a few examples[7]. However in the two scenes below the focus is on a somewhat opposite direction: applying theoretical knowledge (the general) towards the construction of examples[8] (the particular, Scene I) or towards drawing inferences about properties of a specific mathematical object (a certain group of permutations in Scene II).

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[6] Students find proof difficult, unnecessary and meaningless. They view empirical evidence as proof and use more empirical arguments than deductive arguments (Martin & Harel, 1989; Porteous, 1990; Williams, 1980; Yerushalmy et al, 1990; Almeida, 2000; Dreyfus, 1999; Selden & Selden, 2003; Raman, 2003). Various hierarchical models discuss the development of proof in terms of a dichotomy between the empirical and the deductive. Amongst these are: van Dormolen's (1977) (based on van Hiele's levels of geometric development) distinction between specific, common-properties and reason-about-reasoning proofs; and, Bell's (1976; 1979) triad of functions for a proof (verification, illumination, systematisation). In the latter Bell suggests that learners proving practices develop from regularity and rationality to explanatory quality and logical sophistication. In this context students' proving practices develop from the empirical to the deductive as follows: from failure to generate correct examples or to comply with given conditions, to extrapolation and non-systematic/partially systematic/systematic checking out of full finite set of cases. In his account students become progressively more able to make connections and present connected arguments with explanatory qualities. (Galbraith (1981) supplemented Bell's and van Dormolen's hierarchies with a reference to socio-emotional factors influencing students' proving behaviour. This is touched upon in the context of students' preferences for certain types of counterexamples in E3.7). As Fischbein (1982) notes, students are possibly not aware of the distinction between empirical and deductive arguments. Even when they are, says Schoenfeld (1987) who calls learners at this stage pure empiricists, they decline using deduction as a constructive tool for problem solving. Coe & Ruthven (1994) also described students' strategies as primarily empirical (e.g. validating conjectures through resorting to a few examples).

[7] A significant part of the discussion in the literature of students' construction of examples (a recent and quite comprehensive account is (Recio & Godino, 2001)) concerns the ways in which students relate to generic examples. Students do not always appreciate the 'generic' aspect of proof, namely, according to Balacheff (1990), the reliance of a proof on an object that is there not in its own right but as a representative of a class. Harel & Tall (1991), Tall (1989) and Mason & Pimm (1984) have also discussed the difficulties that learners have in abstracting from a generic example these elements that constitute its genericity and this implies, as Williams (1980) describes, that students cannot see the generalisation in a deductive proof. The tension between the general and the specific is often also present in students' employment of diagrammatic representations: for example, Martin & Harel (1989) found that students see proofs as figure-particular and described students' use of features of non-generic figures as having variable influence on their overall judgements of the proof.

[8] Constructing examples is a vital aspect of a learner's experience of mathematics as a constructive activity (Watson & Mason, 2002 and 2005) and students' ability to generate examples is linked to the overall effectiveness of their reasoning (Dahlberg & Housman, 1997). In E3.7 M and RME return to this issue in the context of counterexamples and in E7.4 they discuss practices that emphasise and facilitate the development of this skill (Mason, 2002).

Scene I: Constructing Examples

Setting the scene: The following takes place in the context of discussing part (iii) of the question in E3.1. RME has invited M's comments on the construction of the example that this part of the question asks for.

M: Most students came up with 2, 2, -2, -2 – it's a mystery to me why 2 and not 1! – and when asked how they got there, most said *randomly*! Well, I did try to explain that choosing one entry to start with and then determining the others is not random at all! And I am not sure the model solution provides any clue as to how the example suggested in part (iii) came to be – the example is coming out of the blue. So producing the example appears disconnected from the theory in part (ii).

RME: In some scripts the students' examples in part (iii) contradicted how they claimed these matrices should look like in parts (i) and (ii). In the example Student WD offers for part (iii), where is the row of zeros that, in part (i) he seemed to be convinced *A* ought to have?

M: Typical! Quite often their examples do not fit the theory at all[9]. This disregard for consistency, for a commonsensical match between the theory and the examples is to me evidence of how alienating this game in the abstract can be to them (E3.7). I find the ease with which they indulge in self-contradiction shocking sometimes. I even find myself having behaviourist thoughts from time to time: condition the students to behave in certain ways and make them behave like that until instructed otherwise! It is also interesting that most offered a specific example and almost nobody tried to produce a general example[10], starting from entries *a*, *b*, *c* and *d* and narrowing down the choices by multiplying and solving equations.

RME: Shall we look at what Student L did? Even though the question in parts (i) and (ii) refers to nxn matrices, she treats the whole thing in terms of 2x2 matrices. Maybe because part (iii) asks for an 2x2 example?

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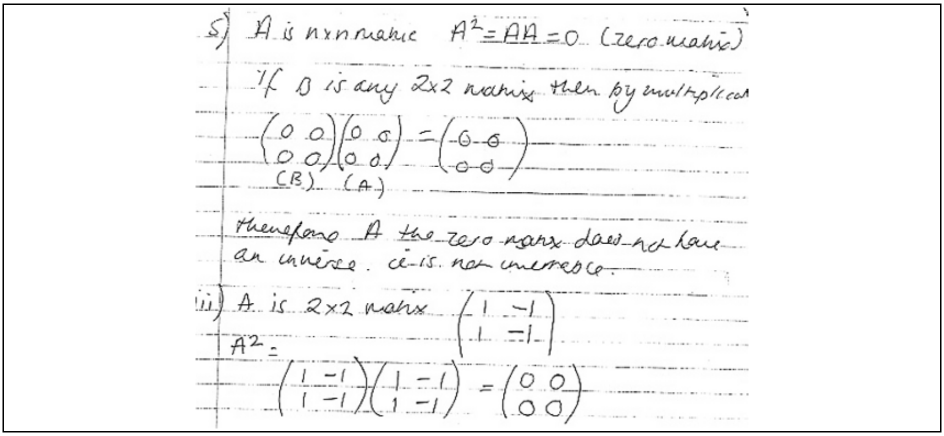
[9] This contradiction has been described by Vinner & Tall in their seminal 1980s work (e.g. 1981) as co-existing but not necessarily compatible parts of a student's Concept Image (where a student's Concept Image includes his or her understandings with regard to a particular concept – often ignoring, exceeding and rarely coinciding in a straightforward manner with the Concept Definition.

[10] Even though the question asked for one example only, M here makes a quintessentially mathematical point: the search for an example has the potential to raise a mathematician's curiosity about what *types* of matrices, in this case, satisfy a certain property. The method he proposes would have produced a description of this class of examples. In this way a search for a particularity would turn into a search for a generality.



5) A is nxn matrix $A^2 = AA = 0$ (zero matrix)

If B is any 2x2 matrix then by multiplication

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(B) (A)

therefore A the zero matrix does not have an inverse. ie is not invertible.

ii) A is 2x2 matrix $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

$A^2 =$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Student L

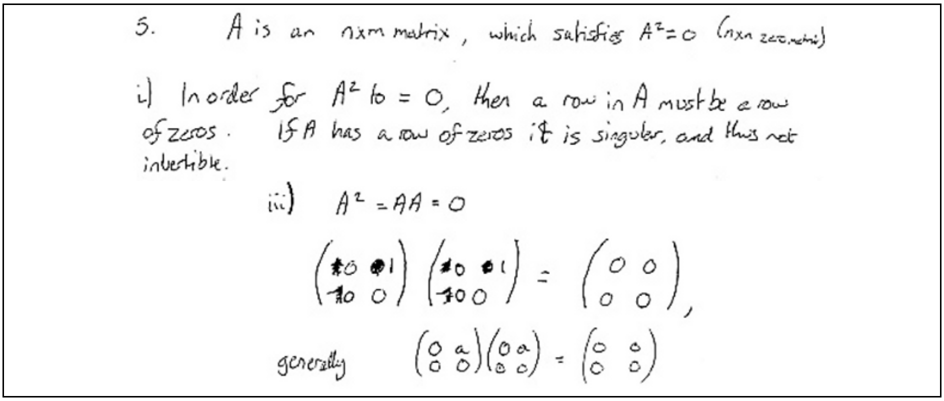
M: I am often asked by students who recognise that doing the 2x2 case only might not be enough in the end but then again they often behave as if it is… it was enough for Cayley and Hamilton by the way[11]. You know, they did the case 2x2 and then in the appendix they offered the 3x3 case. That's it. That's their proof! And students are often happy to live in Flatland!

But there is a slightly different issue here as well: in this question I wouldn't be surprised if they resisted the invitation to construct an example. They often resist hard our efforts to get them to give examples. They don't like doing examples[12]. They want somehow the characters in their solutions to be in the question already. They don't want to be required to construct any new characters! Let alone explain how these characters came to be.

RME: Here is one example where the student is attempting to present the thinking behind his claims.

[11] M draws a parallel here between students' employment of examples as sufficient support for their argument (see, for example, categories 'resort to examples', 'examples and statement of generality' in Recio & Godino 's (2001) analysis of this behaviour) and an example from the history of mathematics where mathematicians seemed at the time content with proving the 2x2 and 3x3 cases (see also OT5.1 for a brief discussion of such parallels). The (contemporary) mathematicians in Leone Burton's study (2004) have a similarly flexible view: they refer to generality as an aesthetic attribute that ranks very highly in their appreciation of a piece of mathematics (p70/71), a final product that is; yet, when they refer to the processes through which they gain mathematical insight, trial and error, resorting to pictures and several other mental actions on particular cases of what is examined make repeated appearances (p73 – 80).

[12] Students' resistance reflects their difficulty to generate examples. See E3.7 (Zaslavsky & Peled's work, e.g. 1996) for a discussion of difficulty with the construction of counterexamples.


5. A is an nxm matrix, which satisfies $A^2 = 0$ (nxn zero matrix)

i) In order for A^2 to = 0, then a row in A must be a row of zeros. If A has a row of zeros it is singular, and thus not invertible.

iii) $A^2 = AA = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

generally $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Student MR

M: I think he has worked it out and actually shows us the working out (E4.2). Lovely! And, look at part (iii): his example doesn't contradict the *A must have a row of zeros* bit that underlies his part (i). Do you think he may even have started from a row of zeros and then build up the rest of the example? That is what you would actually do. You sort of sit there and say: ok, let's try this and let's try this…

RME: You can probably see the construction process through the erasing and replacing in the *A*x*A* part.

Scene II: Applying the general to the particular[13]

Setting the scene: The following takes place in the context of discussing the question below as well as two examples of students' written responses to this question, Student E's and Student H's:

[13] Chazan (1993) proposes two types of problematic predispositions towards proof: students either see empirical evidence as proof or deductive proof simply as evidence. Elaborating on the above distinction, Chazan notes that students seem to recognise, especially when their attention is drawn to it, that empirical arguments rely on examples which are special, measurements are not exact and there may be counterexamples. Instead of becoming sceptical about empirical arguments they prefer to modify their empirical strategies in order to accommodate some of the limitations and counterexamples do not disturb their notion of mathematical truth which is not characterised by universality. Fischbein (1982) described the concept of formal proof as 'completely outside the main stream of behavior' (p17), distinguished between accepting a proof and accepting the universality of the statement proved by a proof and stresses the learner's need for accepting a proof also intuitively, in a complementary and inextricable way. Moreover in the cases where students object to deductive proof as mere evidence (Miyazaki, 1991) and ignore its universality, they are also not safe from counterexamples: as a result they may produce proofs based on one example.

Example from Exercise Sheet 2, Week 4-Group Theory, Autumn Semester 2001

(10) Show that $V = \{id, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of $Sym(4)$ and G/V is isomorphic to $Sym(3)$.

Notes on Solutions

Show that if $g \in Sym(4)$ is any permutation then $g^{-1}(12)(34)g$ is again a permutation composed of two 2-cycles, so belongs to V. The same is true for the other two non-identity elements. Now use the last part of the previous question to show that V is normal. For the reminder you might experiment in several ways. For instance if we use the same letters as in Question 2 for the permutations $e = id$, $a = (23)(1)(4)$, $b = (13)(2)(4)$ etc, then we can show by multiplying out that the cosets eV, aV, bV, cV, ..., fV are all distinct. (This is a fair amount of work and you should do it for practice.) As we expect 6 cosets, by Lagrange's Theorem, we know that we have all elements of G/V. So now the isomorphism is quite clear! However, this needs to be checked carefully. Alternatively wait until we have proved the relevant theorems, i.e. the First Isomorphism theorem.

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10. $V = \{id, (12)(34), (13)(24), (14)(23)\}$ $G = Sym(4)$

V is a subgroup of G

$V \neq \emptyset$ $|V| = 4$ $|G| = 12$ $12/4 = 3$

$\forall x, y \in V, xy^{-1} \in V$

V is normal $\forall g \in G, \forall h \in V$ $g^{-1}hg \in V$.

Student E

---

10. Show $V = \{id, (12)(34), (13)(24), (14)(23)\}$ is normal subgroup of $Sym(4)$

to be normal subgroup: $\forall K \in Sym(4)$, $kV = Vk$ must be true.

where $kV = \{kv : v \in V\}$

let $K^{-1}VK = \{K^{-1}vK : v \in V\} = V$, $\forall K \in Sym(4)$

so $K(\underbrace{K^{-1}VK}_{V}) = (KK^{-1})VK = VK$ by associativity of the $Sym(4)$ group.

by def. above $= kV$

so $kV = Vk$

Show G/V is isomorphic to $Sym(3)$

let $G = \langle g \rangle$ so $\frac{G}{V} = \langle gV \rangle = (gV)\cdot(kV) = (gk)V$

$\psi : \frac{G}{V} \to Sym(3)$

$\forall g \cdot g' \in \frac{G}{V}$, $\psi(g \cdot g') = \psi(g)\psi(g')$ respecting the binary operations so ψ is homomorphic

$\psi\ \frac{G}{V} \to Sym(3)$ with kernel K so that $\psi(a) = 1$

$\psi(aK) = \psi(a)$ as if $aK = a'K$ for $a^{-1}a' \in K$

$\psi(a^{-1}a') = 1$

so $\psi(a) = \psi(a')$

and $\psi(aK) = \psi(a'K)$

Student H

M: Students, like Student H here, often have difficulty with applying the general to the particular. We often talk about their difficulty with abstracting things from generic examples[14] and I think that this is an example of doing the opposite thing: having the abstract thing and then actually being able to produce an example from that or understand what that means in an example. I guess that they can follow through the steps in the abstract notion but then when it comes to…I don't think they understand this enough to be able to say what that actually means in an example. So, as in Student H's response, they quote a bit of abstract stuff which is mostly kind of ok, I guess, but then they haven't actually done anything with this specific group in this specific example. Like with the reproduction of the Mathematical Induction mechanism, where this type of proof is on one hand some formal machinery to be put in some sort of application to prove a property of numbers or whatever (E3.6), here the reasoning required involves working out an example and the particularity of this example. In some way the intention in a question like this is to bring these two things – the theorem/property and example of a specific group – together. Follow through the thinking and see how they connect. But, on the basis of these examples of student writing, no connection seems to have occurred. The students definitely did not see that you can work out the particular from the theoretical and know how these two things will relate. There is complete absence of any sense of connection.

RME: Student E seems to apply Lagrange's Theorem implicitly.

M: She also seems to claim that $Sym(4)$ has twelve elements… Or maybe she is generalising from $Sym\ (3) = 3$ (using the factorial argument, $Sym\ (n) = n!$ to say that if $Sym(3) = 3$ then $Sym(4) = 4x3 = 12$)? The simple intention of this exercise is to elicit the elementary observation that the subgroup is normal and previous discussions ought to have suggested to the students this normality. Maybe. But if she was thinking $Sym(3)$ has only three elements, shouldn't she stop and think this is nonsense? But probably she wasn't thinking that $Sym(3)$ has only three elements because the one formula every student seems to remember is that $Sym(n)$ has $n!$ elements.

RME: I promised to show you someone's work who may not have bothered with extracting observations and quoting theorems but just got down to work and produced seven pages of calculations! *[Student LD's response, not reproduced here]* Well, the lecturer did ask for it: *this is a fair amount of work and you should do it for practice*?

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[14] See examples of studies mentioned early in Scene I.

M: Blimey! Well, I would be tempted to say that this is worth more than the void thinking in Student E and H's responses – Student H did not engage at all with the content of the specific group – but still no property use is evident here at all. The student did not use any theorems to reduce the number of calculations but seems to know that *Sym*(4) = 24. Like the lecturer's notes seem to suggest. I think the amount of work is enormous and understanding of the context is somewhat impressive, even though the Group Theory done here is oblivious to theorems[15]. She's even done the conjugating for the identity – it's sort of pre-Jordan, pre-Cayley kind of Group Theory! Not much thinking in there, is there? But back to the Student E and H responses, there is hardly anything in them, is there?

RME: But is it possible that the students would interpret this question as a request to carry out all these calculations?

M: A conscientious student may have. And get full marks for completion.

RME: Which is probably what happened here with Student LD – *this is what I can do, I might as well do it*! I wonder however which marker would not be tempted to only check a random selection!

M: Wouldn't it be nice though if at the end the student had wondered whether this workload could have been reduced via some theorems?

RME: Indeed. It would be nice to see the students applying group theoretical knowledge in the particular context of the question.

M: I think there is a tension between the abstract and the specific in students' reactions to what they perceive is expected of them in Group Theory.

RME: But the trouble Student LD went through can be instructive: it could generate a discussion of the *look what happens when you ignore the power these theorems/tools can give you* type.

M: I agree but I am also confident that at the end of the day things fall into place and students begin to see the usefulness of these tools (OT7.1). At this moment though Student H seems to simply reproduce various ways of writing things probably from some notes, like something that looks like the definition of normality. Something like quoting Theorem X.Y but letting it hang in the air

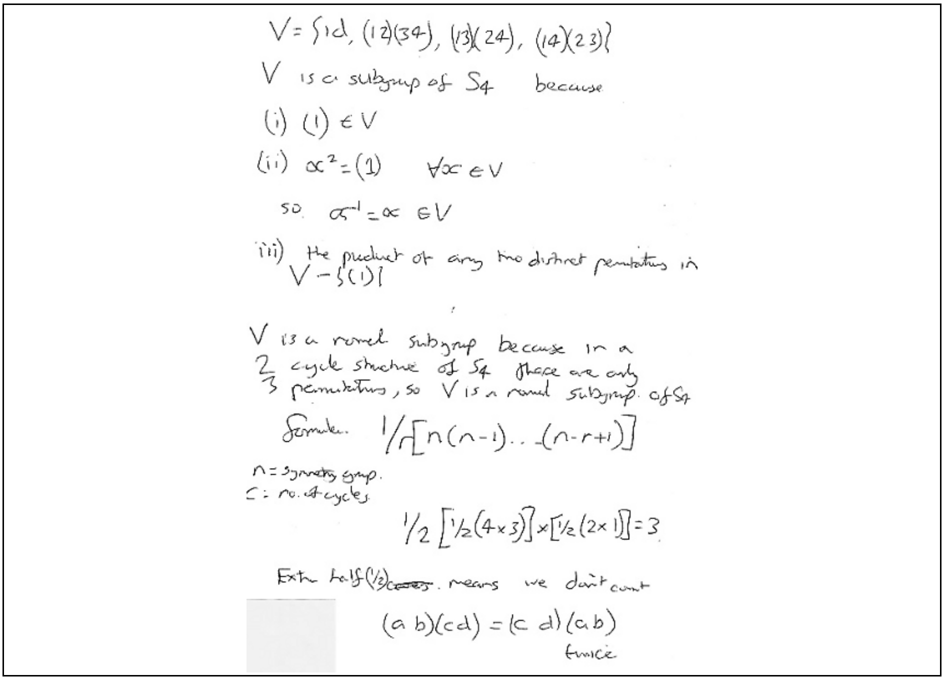
[15] Sheer brute force, Group Theory oblivious to theorems. The distinction between syntactic and semantic proofs made by Weber & Alcock (2004) indicates one useful way of evaluating the student's work here. See E3.3 for a more detailed employment of the distinction.

without any link whatsoever to the context of the question. I would probably call it nonsense written in apparently mathematical language.

RME: In contrast to the *random mathematics* you mention elsewhere (E4.1) where the language is all over the place but is attempting to convey some reasonable mathematical meaning?

M: Exactly. No references, not really any mathematics related to the question. I am a bit amazed these are Year 2 students to be frank with you.

RME: OK, here is a student response you might be less disappointed with, Student M's.


$V = \{id, (12)(34), (13)(24), (14)(23)\}$

V is a subgroup of S_4 because

(i) $(1) \in V$

(ii) $\alpha^2 = (1)$ $\forall \alpha \in V$

so $\alpha^{-1} = \alpha \in V$

iii) the product of any two distinct permutations in $V - \{(1)\}$

V is a normal subgroup because in a 2 cycle structure of S_4 there are only 3 permutations, so V is a normal subgroup of S_4

Formula. $1/r[n(n-1)\ldots(n-r+1)]$

n = symmetry group.

r: no. of cycles.

$1/2\,[1/2(4\times 3)] \times [1/2(2\times 1)] = 3$

Extra half (1/2) ~~cases~~. means we don't count

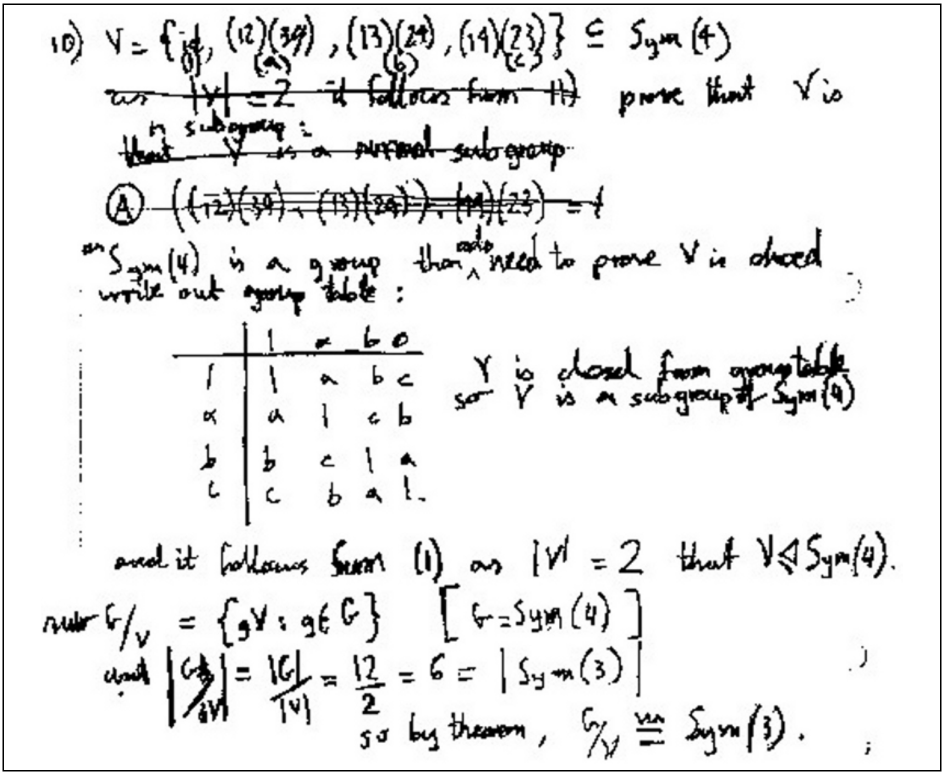
$(a\,b)(c\,d) = (c\,d)(a\,b)$

twice

Student M

M: This is essentially OK and an improvement on Student H's. There is an application of the conjugation argument which the lecturer aimed at having the students applying here. And of other previous knowledge.

RME: In the next example I would like to show you Student W resorts to the tabular representation of groups to show that V is a subgroup of G.


10) V = {id, (12)(34), (13)(24), (14)(23)} ⊆ Sym(4)
(a) (b) (c)
~~as |V| = 2 it follows from (1)~~ prove that V is a subgroup:
~~that V is a normal subgroup~~
Ⓐ ~~((12)(34) · (13)(24)) · (14)(23) = 1~~
as Sym(4) is a group then only need to prove V is closed
write out group table:

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1.

V is closed from group table so V is a subgroup of Sym(4)
and it follows from (1) as |V| = 2 that V ◁ Sym(4).
now G/V = {gV : g ∈ G} [G = Sym(4)]
and |G/V| = |G|/|V| = 12/2 = 6 = |Sym(3)|
so by theorem, G/V ≅ Sym(3).

Student W

M: To start with I would praise the theorem quoting here even though it is not all correct. And some of it is even cute, albeit mostly incorrect, such as the claim that the order of the group is the same as the order of its elements. The way he attempts to make the theory fit the hypothesis is almost endearing… Or chilling, depending on your point of view, or your mood!

RME: Quite a few students seemed to think that |*Sym*(4)| = 12. I wonder where this comes from.

M: Well, as I said, the *if Sym(3) = 3 then Sym(4) = 4x3 = 12* scenario is likely for me. Of course do not ask what happens to various group theoretical properties if these claims are taken to their logical conclusion. Probably *Sym*(1) has -1 elements! In any case I must recognize the merit in some of Student M's claims, even the incorrect ones. Actually I have a feeling the student may have suspected that something was wrong in the edifice of his claims but still wrote them down – why, you may ask.

RME: Well, he may suffer a bit from the *by theorem such and such* tactic: resorting to the theorems' authority? The sight of student work with evidence of the habit of grounding claims on some vague reference to some lemma or theorem, usually not fully referenced or reproduced is not a rare sight.

M: There is some sense in the way the student has written things out. Maybe he is working on the grounds of a meta-theorem: I will never be asked to prove something that is not true therefore I will somehow get to what apparently looks like an extraction of the truth.

RME: Or you might be in a generous mood?!

M: OK, maybe a bit. Strictly speaking this is not a very good response. at all. I always wonder when I see writing like this, what the student's reaction would be if asked about his writing two years after he wrote this (E7.1, OT7.1). Stand by it? Demolish it?

Towards the end of this Scene the conversation has turned towards the students' employment of theorems towards the construction of a mathematical argument[16]. In the following Episode M and RME examine several issues pertaining to students' recourse to the use of definitions as deductive steps towards such construction.

EPISODE 3.3
USING DEFINITIONS
TOWARDS THE CONSTRUCTION OF MATHEMATICAL ARGUMENTS

In what follows M and RME discuss two cases where recourse (or not) to a concept definition could facilitate students' construction of a mathematical argument[17].

[16] Keith Weber (2001) described the fluency to introduce theorems effectively into a mathematical argument (identifying the available range then selecting appropriately etc) as missing from beginning students. He terms this knowledge `strategic knowledge' – knowing how to choose the appropriate facts and theorems to apply – and observes that the doctoral students who participated in his study appeared to possess this knowledge to a higher degree than the undergraduates. The doctoral students had a stronger understanding of which theorems are most important, when particular facts and theorems are likely to be useful, and when one should or should not try and prove theorems using symbol manipulation. For dilemmas of this sort see E3.3.

[17] Weber & Alcock (2004) distinguished between syntactic proofs (drawing logically acceptable inferences by algebraic/symbolic manipulation) and semantic proofs (drawing logically acceptable inferences by seeking recourse to my understanding of concepts, e.g. concept definitions). Another relevant distinction is Balacheff's (1986): pragmatic proofs involve recourse to action or some kind of demonstration and conceptual proofs rely on manipulation of properties and relationships between them. Pragmatic proofs are either naive empiricisms (truth assertion through verification a several cases, see

Scene I: Weaving in the use of definitions towards the construction of a mathematical argument

Setting the scene: The following takes place in the context of discussing the question as in E3.1. M returns to commenting on Student WD's response.

M: Then again we have this peculiar claim that if $A^2 = 0$ then it must have a row of zeros, and therefore it is singular, thus non-invertible. Is this something like a fraction off the lecture notes that he is interpreting without much thinking? Where does the *if* $A^2 = 0$ *then it must have a row of zeros* bit come from? Maybe, because they have seen in the lectures that if A has a row of zeros, then its determinant is zero, therefore A is not invertible, they are trying to find vaguely related assertions that would imply the conclusion. They want to use a result available in their lecture notes and there is no such result. And they are unwilling to use the definition of the property you are talking about. So they will not look at the definition of invertibility. It is very difficult to get them to work from definitions and that again is something that they are not used to (E6.1, E6.2).

RME: We have already talked a bit about the students' variable willingness to provide explanation – here is one more example, Student LD's response, I would like to hear your views on.

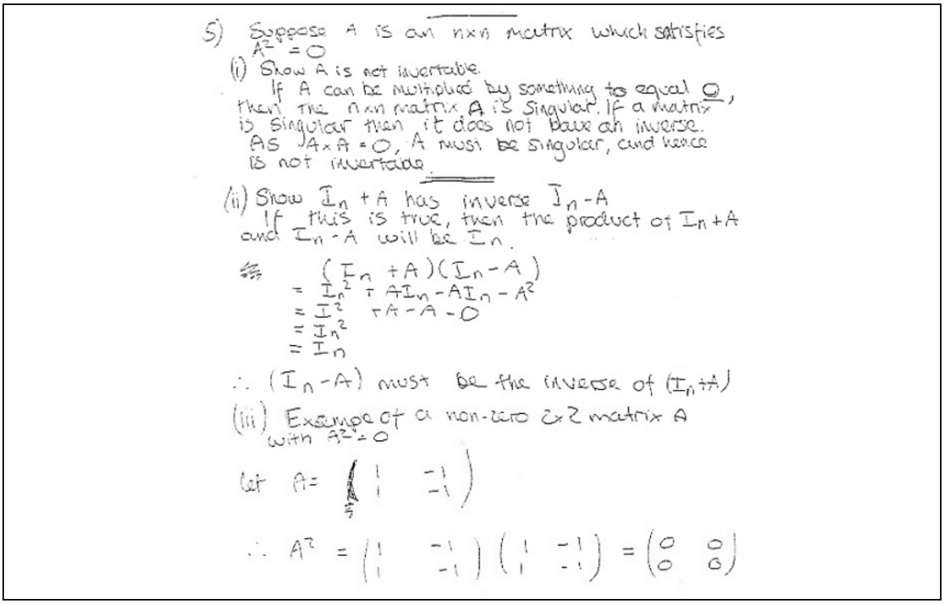
M: Here we go again with the spelling of *invertible*! What do these students have against spelling?! I can cite four different spellings of *symmetric* I have seen in their scripts. Anyway… This is a very verbal response. I like the confidence in it and the fact she demonstrates the validity of her example in part (iii) (E4.2). But what does she mean by *singular*? Probably determinant zero, right? As I see it now, I would give it good marks but let's look at it a bit more closely. She starts from *if A can be multiplied by something to equal 0, then A is singular*. Followed by: *if A is singular then it does not have an inverse*. As A can be multiplied by something to equal zero, itself, A does not have an inverse. That's sensible. However, where does the *if A can be multiplied by something to equal 0, then A is singular* bit come from? It's a bit like Student W where the determinants are coming into the argument (E3.1) – maybe.

RME: What do you think of her part (ii)?

M: It's fine – even though she is only dealing with it one way around. They always do that (E3.4). It must come from school where, in the 2x2 context, you find the inverse by using determinants and it is supposed to work both ways. So you

E3.2, Scene I) or crucial experiments (conducting an experiment in order to choose between two hypotheses).

never really check – and, in a sense, since it can be proven that the matrix you find this way does work both sides, why force the students through an unnecessary loop? But this is where a reminder to the students that they should always check both ways would also serve as a reminder of the more abstract, beyond 2x2, context they are now asked to work in. Finally, can I also say that I am not very keen on *if this is true, then the product of I-A and I+A will be I*, even though she is doing the absolutely right thing, starting from the product and ending up with I. You know why? Because it's getting so close to appearing as if she is assuming what she is supposed to be proving. What she wants to be saying is really *this is true because….*


5) Suppose A is an n×n matrix which satisfies $A^2 = O$
(i) Show A is not invertable.
If A can be multiplied by something to equal O, then the n×n matrix A is singular. If a matrix is singular then it does not have an inverse.
As $A \times A = O$, A must be singular, and hence is not invertable
(ii) Show $I_n + A$ has inverse $I_n - A$
If this is true, then the product of $I_n + A$ and $I_n - A$ will be I_n.
$(I_n + A)(I_n - A)$
$= I_n^2 + AI_n - AI_n - A^2$
$= I^2 + A - A - O$
$= I_n^2$
$= I_n$
$\therefore (I_n - A)$ must be the inverse of $(I_n + A)$
(iii) Example of a non-zero 2×2 matrix A with $A^2 = O$
Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
$\therefore A^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Student LD

RME: Yes, it is a slightly dubious way of putting her thought on paper, I agree.

M: There is a subtlety missing there regarding the converse statement and their Grammar is not up to scratch to help them see the difference (E4.4). She is alright here but I am talking more generally, people being unable to distinguish between a main and a subordinate clause. And the absence of teaching logic exacerbates the students' inability to see these subtle differences (SE3.1). But it is so bloody difficult to teach logic anyway! All this stuff about examples like *if you are in a running shower then you are wet, but if you are wet, it doesn't follow that you are in a shower* and so on…

Scene II: Making the fine choice between insightful algebraic manipulation and employment of a definition

Setting the scene: The following takes place in the context of discussing the question below as well as two examples of students' written responses to this question, Student L's and Student J's (the latter appears in E4.2):

Example from Exercise Sheet 5, Week 10, Autumn Semester 2000

Suppose $n \geq 2$ and A is an $n \times n$ matrix with $det(A) \neq 0$. In the following $adj(A)$ denotes the *adjoint* (or adjugate) matrix of A.

(i) Use the fact that $(adj(A))A = det(A)I_n$ and the product formula for determinants to show that $det(adj(A)) = (det(A))^{n-1}$.

(ii) Prove that $adj(adj(A)) = (det(A))^{n-2}A$.

Suggested solution

(i) By the product formula and the quoted equation we have $det(adj(A))det(A) = det(aI_n)$ where $a = det(A)$. As $det(aI_n) = a^n$, and $a \neq 0$ we obtain the required result.

(ii) By the equation in (i) applied to the matrix $adj(A)$ we have $adj(adj(A))adj(A) = (det(adj(A))I_n$. So $adj(adj(A))$ is equal to $det(adj(A))$ times the inverse of $adj(A)$, i.e. $adj(adj(A)) = (det(A))^{n-1}(adj(A))^{-1}$. But also $A = det(A)(adj(A))^{-1}$. We obtain the required result.

3i). $A = (a_{ij})_{n \times n}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $adj(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

multiply.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$A^{-1} = \frac{1}{detA} A^{adj}$

$det(A^{adj})$. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$det(A^{adj}) = det\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = ad-bc = (det(A))^{2-1}$

ii)?

2/7

Student L

(…regarding the question M observes:

M: I wonder whether we need n to be ≥ 2. I guess yes, because it is pretty pointless to look at determinants and all, for $n = 1$ or 0! And I guess you also need the invertibility because you need to cancel $det(A)$ so it needs to be $\neq 0$. I would need to check if the statement holds for $det(A) = 0$ too.)

M: What the teacher is trying to do is give the students some fairly sophisticated mathematical machinery and increase their confidence in dealing with this machinery. The adjoint of the adjoint is an impossible thing to understand by definition. However it turns out to be the matrix you had, multiplied by something. Right? If you see that, you can do such seemingly impossible things, walk away and say well, these things work. These are things you can do. Confidence is really important here, in particular with regard to being able to manipulate very formal expressions and then use some common sense at the same time, for example in seeing what the determinant of a diagonal matrix is[18]. This flexibility is crucial: insisting on working with the definition of the adjoint would be a guaranteed cul-de-sac – there are not enough sigmas in the world for this, believe me! Almost suspending the definition and resorting to insightful algebraic manipulation is a more promising approach. Of course you may turn around and say to me: isn't there an irony in there about this proposition – suspend the definition?! – especially in the light of what we usually say[19] about the importance of starting from the definition, of considering what these things you are working with are supposed to be. But, of course, the issue here is flexibility in knowing when to do so and when not to[20]. The more, so to speak, existential approach – what is the adjoint? – would rather detract you from the efficient approach in this case, the algebraic one. In a sense to me a more instrumental definition of the adjoint is much more helpful. What is the adjoint of a matrix A? It is another matrix, B, which, when you multiply by matrix A, gives you the determinant times I_n.

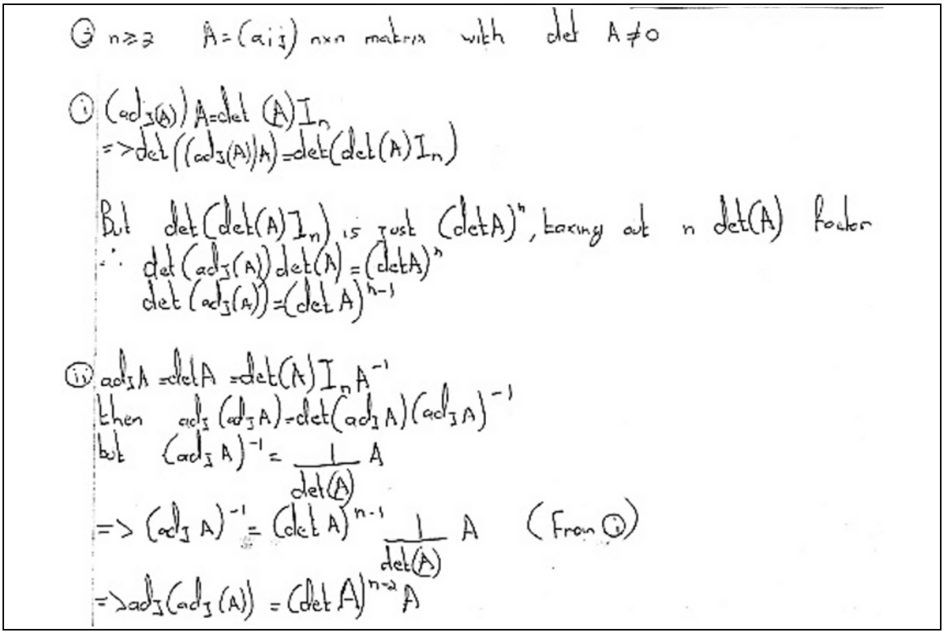
RME: This more helpful definition is on offer here in part (i) of the question and using part (i) almost renders part (ii) trivial. Almost…

M: Almost… Student L writes A equals that and adjoint A equals that … and she has no idea how to proceed so she reverts to naked calculation[21] and realises immediately that there is no hope unless she restricts these calculations to 2x2. She proves the expression in part (i) alright for $n = 2$ via these calculations but this is not an approach she can carry through for $n > 2$. I am guessing here but it would be very interesting to see the rough work for this or whatever she put down as part of her answer. For me, if you are successful at this question you have almost acquired the meta-mathematical ability to look out for the things

[18] But see E4.2, Scene I for a problematising of this 'common sense'.
[19] For example in the previous Scene.
[20] See earlier reference to Weber (2001).
[21] This is the same student who resorts to the 2x2 case in E3.1.

that would be helpful rather than trying to understand what is this thing[22]. So you are looking for the strategy by which you can enter and win this battle. There is next to nothing in terms of required previous knowledge here but the question is demanding exactly in terms of this meta-mathematical skill. It would be interesting to see if, at a later stage of their studies, the students have actually acquired this skill (OT7.1).

RME: I agree with you about the minimal requirements in terms of previous knowledge. I myself, when I had a go at the question, had no recollection of the definition but still completed it by simply using the property in part (i). Here is another response in which I think Student N connects the various bits in the question more or less fine.



Q $n \geq 2$ $A = (a_{ij})$ $n \times n$ matrix with $\det A \neq 0$

(i) $(adj(A))A = \det(A)I_n$

$\Rightarrow \det((adj(A))A) = \det(\det(A)I_n)$

But $\det(\det(A)I_n)$ is just $(\det A)^n$, taking out n $\det(A)$ factor

$\therefore \det(adj(A))\det(A) = (\det A)^n$

$\det(adj(A)) = (\det A)^{n-1}$

(ii) $adj\,A = \det A = \det(A)I_n A^{-1}$

then $adj(adj\,A) = \det(adj\,A)(adj\,A)^{-1}$

but $(adj\,A)^{-1} = \frac{1}{\det(A)}A$

$\Rightarrow (adj\,A)^{-1} = (\det A)^{n-1}\frac{1}{\det(A)}A$ (From (i))

$\Rightarrow adj(adj(A)) = (\det A)^{n-2}A$

Student N

M: Student N seems alright. In part (i) he writes this slightly mysterious *taking out n det(A) factor* – but I think he means *taking out of the brackets the factor det(A) n times*. So he is using a *det* property there. Let me stress that I am not terribly keen on his use of ⇒. I think he uses it as a postmark trying to say *this is what I am thinking about doing next*. In part (ii), in his probably mis-typed first line, I think he is trying to refer back to part (i) and express the adjoint of matrix A in

[22] See earlier reference to Weber (2001).

accordance with what part (i) allows us. Then he carries out the substitution of A by $adj(A)$ and uses the fact for $adj(A)$ from part (i). It makes sense even though it would benefit from some explaining. At this stage he is simply mimicking the use of logical notation and just about, only just, gets away with it (E4.1)!

RME: I guess one of the saving graces of his script is how deftly he carries out the substitution, seeing an appropriate particular case within the general case, not a trivial feat at this early stage (E3.2).

M: Oh yes! I see even postgraduate students tripping over things like this, for instance when the name of the variable changes. Try sometimes to get them to differentiate with t (time) and not x. It's just a name for Christ's sake, I could call them bananas if I like! Well, as long as this doesn't trick students into thinking that then the variable ranges in a finite set!

RME: Back to what we were saying earlier about the importance, or not, of knowing the formal definition of the adjoint. There are parts of mathematics I assume where not knowing what an adjoint is but using facts about it – like I admitted doing when approaching the problem – would not get you very far.

M: Indeed. In Code Theory, for example. The question here is really testing logical thought, arguably a rarity since most questions given to students at this stage are about ways of solving problems. Here there is actually only one thing that is possible to do and that thing will give you the answer almost immediately. And you would like the students to do something like that on auto pilot almost, via low maintenance common sense. That is why, because of the logical requirement and not the requirement of previous knowledge, final year students would probably do well, I think in this type of question. But then again by the time they reach their final year, students have all sorts of hang ups about the power of previous knowledge and are tied by these hang ups – too many stars in their skies, I guess[23]!

M's commentary on Student N's train of thought and representation of this thought in his writing make this Scene equally relevant here and in Chapter 4 where the students' employment of mathematical language and notation is examined in detail. There the discussion turns to students' often inscrutable attempts at verbalisation (E4.4) and at mathematical writing (E4.0) – intertwined with what M and RME, in the context of discussing examples from Group Theory, call 'desperate juggling of axioms', even 'random mathematics' (E4.1).

[23] See Weber (2001) and Jones (2000) quoted earlier, and (Schoenfeld & Hermann, 1982) for a discussion of habits that survive and habits that fade out as the students' experience increases.

Previously M and RME have discussed students' deductive steps and have noticed a tendency towards logical fallacy (e.g. Scene I in this Episode). In what follows the discussion turns to a particular logical fallacy in the context of employing Tests of Convergence to determine the convergence or divergence of series in Analysis.

EPISODE 3.4
LOGIC AS A BUILDING BLOCK OF MATHEMATICAL ARGUMENTS[24]: RECONCILING WITH INCONCLUSIVENESS[25]

Setting the scene: The following takes place in the context of discussing the following question as well as one examples of students' written responses to this question, Student H's:

For each of the following series decide whether or not it converges. State cerefully any test for convergence that you use. (1) $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$. (2) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$. (3) $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$. (4) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$. (5) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

Notes on Solutions

(2) Convergent by Comparison Test: (Compare with the series with terms $\frac{2}{n^{\frac{3}{2}}}$ which is convergent by the Integral Test).

(4) Divergent by Limit Comparison Test: ratio of nth term to $\frac{1}{n}$ clearly converges to 1, and the harmonic series is known to diverge.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2}$ by LCT, let $\sum b_n = \sum \frac{1}{n}$

$\frac{a_n}{b_n} = \frac{(\sqrt{n}+1)\,n^2}{n^2} = \sqrt{n}+1 \longrightarrow \infty$ does not converge.

$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ by LCT. $\sum b_n = \sum (1+n)$

$\frac{a_n}{b_n} = \frac{1}{n^2+1} \longrightarrow 0$ ∴ does not converge.

Student H

[24] The influence of students' grasp of Logic on their construction of mathematical arguments has been researched widely. Some works (e.g. Duval, 1991 and O'Brien, 1972; 1973) that attributes learners' difficulties with mathematical proof to their confusion of deductive thinking with ordinary argumentation. According to others (e.g. Selden & Selden, 1995) the difficulty of dealing with the logic behind formal proof lies also in the fact that learners are overwhelmed by the content of the mathematical statements in a proof and are not able to move beyond content and into the realm of logical manipulation of the statements. Anderson (1994), for example, refers to persistent inconsistencies in the students' proving behaviour when they fail to disentangle the logic from the analytic aspect of the concept involved.

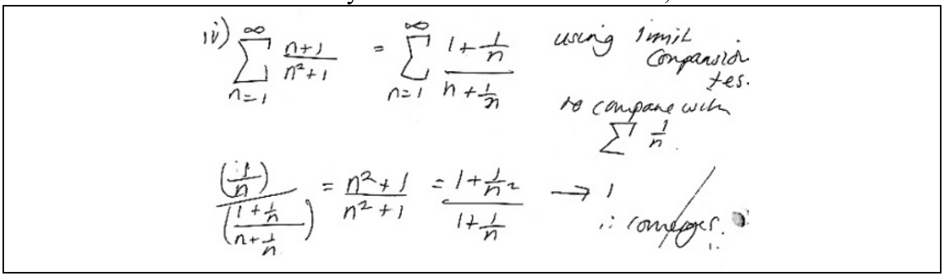
[25] See (Nardi & Iannone, 2001) for a preliminary discussion of some of the data here: there we describe the students' tendency to transform statements about convergence, for instance the Limit Comparison Test, into statements about divergence in ways that suggest a multiplicity of difficulties with Logic and a resistance to the idea that, in certain occasions, convergence tests are inconclusive.

M: Student H's response is pretty awful! It is sort of fascinating too, it is two lines. Students often concoct fantasy theorems, sometimes sensible, sometimes, like here, not and put then into application The Limit Comparison Test is clearly misinterpreted here.

RME: What do you think she is doing here? She seems to have chosen an inappropriate tool for the job, the tool doesn't work but she refuses to acknowledge that.

M: Her problem lies in the fact she does not see that you can never deduce non convergence from a convergence test.

RME: And she is not the only one. Have a look at this one, Student L's.



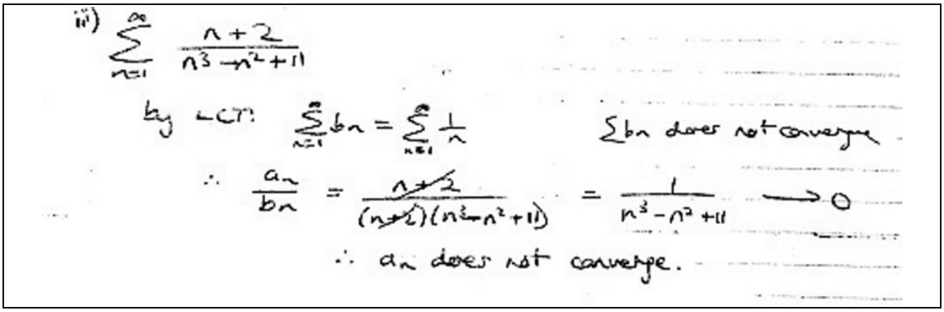
Student L

M: This is so deviant from sense – this reading of the LCT as *if the series* b_n *converges but the limit of* a_n *to* b_n *is infinity then the series* a_n *diverges* is a very odd distortion of the theorem's words. Students seem to think that something is either true or false. The idea that there is something in between is a fault in the theory. This is probably part of the same issue of them most of the times not understanding the status of the converse of a statement[26]. And I am sorry: this is

[26] Research has traced these difficulties in school experiences. A study in the UK by Hoyles & Kuchemann (2002) explored students' use and understanding of logical implication (if p then q) in the curricular context of secondary school mathematics where limited experience of proof was available. The study looked at how students' understanding of logical implication develops in school and changes over time. Amongst the findings are the following: 71% of students in Year 8 stated that 'if p then q' is the same as 'if q then p'. About half could start reasoning assuming the truth of a statement (regardless of its actual truth or falsity). A third supported the falsity of a statement with counterexamples (one only but some did attempt a general description of the class of examples for which this statement does not hold – an indication of an important shift towards the 'why' of the falsity). A quarter however supported the truth of a statement also with examples: with just one or with a general attempt (the latter often cryptic with regard to its deductive or inductive origins). Finally, some but modest progress was observed from Year 8 to Year 9. To determine the truth of a statement of logical implication students used empirical, focused-empirical and focused-deductive strategies with the last two likelier to generate valid arguments. A high proportion of students could work at purely hypothetical level. Using Toulmin's (1958) scheme (conclusion, data, warrant, backing) analysis showed four types of meaning that students had created

the type of thing I emphasise in my teaching all the time so allow to me to assume rather cynically Student H must have been missing many lectures!

RME: She is far from unique in this however. And the issue seems to persist a few weeks later.



Student H – a few weeks later

M: What can I say? It seems that some of our students graduate without a grasp of hypothetical reasoning, acquire important jobs, they probably end up running the economy…!

RME: May I invite your views on the student difficulty to deal with the inconclusiveness of a test?

M: There is the classic example from school mathematics: how the second derivative being zero at a point implying the point being an inflection point. They lack an awareness of the *third possibility*: true, false, inconclusive.

RME: But in the writing we have been looking at there has been no explanation why the particular test was chosen. Even though I recognize that the question does not ask them to justify their choices, the students still do not seem to see their implicit obligation to do so.

M: Maybe they should be asked to justify as well as write down the test they are using (E7.1, Scene I).

RME: Maybe they would then realise the distortion they have subjected the test to in order to fit their argument!

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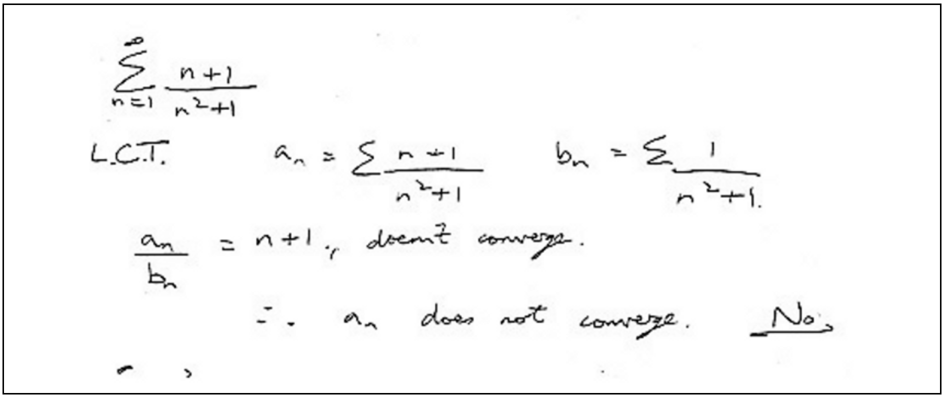
about the relationship between 'if p then q' and 'if q then p': (A) they are the same, (B) same by reference to data, (C) different by reference to data (D) different.

M: Or maybe not. Moreover the students need to have seen many examples of using the tests to begin to have a feel of what it means to use the tests (E7.4, Scene II). I know we are doing a lot of this in the lectures but as I said I am confident some of the students whose writing we are looking at here have not attended many. By the way I believe there is a correlation between lecture attendance and failure at exams![27]

RME: As I said she is not unique in her tendency to deform tests such as the LCT. Here is Student JR's response:



Student JR

M: I am honestly terrified at such muddled hypothetical reasoning. A lot of this is just horrendous. This is probably the first time students have to deal with statements that have so many hypotheses. And they are just completely unused to dealing with long series of hypothetical conditions.

RME: It looks as if students do not see a theorem as a statement that applies under certain conditions, that the conditions are often ignored.

M: I think I can point at two sources of this difficulty: first, the absence of a stock of illustrative examples in students' minds. I wonder whether the students' experience of things like functions has been so limited (E5.2) that they have never been challenged to look at examples where things go badly wrong, even for quite straight forward things like the availability of the derivative. And that lacking a collection of interesting examples is something that we need to help

[27] M often returns to the issue of lecture attendance as a factor influencing students' performance and overall quality of understanding. For further elaboration see Chapter 7.

them with (E7.4, Scenes III, IV and V). Second source of difficulty: their clinging to mathematics as a calculational activity (SE3.1).

RME: *if and only if* statements are highly problematic for the students.

M: I am afraid I can offer more frustrating evidence of this. Second years have more affinity with this. I set a few years ago an if and only if problem about two circles in the complex plane, are tangent, if and only if, and then I gave them an equation regulating the complex numbers for the centre of the stuff. And the students made a terrible mess about this. So much so that the next year I removed the *if and only if* and I only gave them the fact that these two circles were tangent and they had to show that the equation was true. And that suddenly became a little bit more straight forward. But there were still students who were working from the equation backward towards tangency … which is frustrating. But the year when I had the *if and only if* it was very illuminating. Students were virtually spiraling into the middle of the pages and pages of algebra not appreciating this *if and only if* at all.

RME: Let me close with a charitable attempt. Maybe we are underestimating Student H's cleverness: maybe she was attempting to construct a divergence argument from the convergence argument within LCT! And it just didn't come off this way!

M: Charitable indeed…

M and RME return to more incidents of students' tendency towards logical fallacy in the midst of discussions that focus on other issues – for example confusing must-not and need-not in Student LW's response to part (iii) of this question from Episode E5.1:

Example from Exercise Sheet 2, Week 4, Question 4

Suppose A, B and C are sets and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions. Let $h : A \longrightarrow C$ be the composition $g \circ f$.

(i) Prove that if f and g are onto, so is h.

(ii) Prove that if f and g are one-to-one so is h.

(iii) Give examples of functions where $A = B = C = R$ and h is one-to-one, but g is not one-to-one. Must f be one-to-one in this case?

Notes on solutions

(i) Let $c \in C$. As g is onto there exists $b \in B$ with $g(b) = c$. As f is onto there exists $a \in A$ with $f(a) = b$. Then $h(a) = g(f(a)) = g(b) = c$.

(ii) Suppose $h(a_1) = h(a_2)$. Thus $g(f(a_1)) = g(f(a_2))$. As g is one-to-one, it follows that $f(a_1) = f(a_2)$. As f is one-to-one, it then follows that $a_1 = a_2$.

(iii) The function f has to be one-to-one, but g need not to be. For example take $f, g : R \longrightarrow R$ given by $f(x) = e^x$ and $g(x) = x^2$. Then $h(x) = e^{2x}$ is one-to-one, but g is not.

Q7) v). $f_5(x) = \frac{x}{(1+x^2)}$

This is not onto as there are no solutions for $f_5(x) = 10$

This is not one-to-one as there are two solutions for $f_5(x) = 0.25$.

vi). $f_6(x) = x^3 - 5x$ is onto but not one to one.

Q4) i) If f is onto $\forall a \in A\ \exists b \in B$.
if g is onto $\forall b \in B\ \exists c \in C$.
if f and g are onto then
h is onto as $\forall a \in A\ \exists c \in C$.

ii) If f is onto one $a' \in A \Rightarrow b' \in B$
If g is one to one $b' \in B \Rightarrow c' \in C$.
so if f and g are one-to-one
h is one-to-one as
$a' \in A \Rightarrow c' \in C$.

iii) if h is one to one
and g is not one-to-one (i.e [illegible] x^2)
f must not be one-to-one (ie [illegible] $\sqrt{x}$)
so h must be $h(x) = x$. (which is one to on

Student LW

M: Oops! This *must* towards the end worries me no end! Classic: *must not* as opposed to *need not*, something not being true is the same as it always not being true.

RME: *must* is also in the question.

M: The student is confusing *not always* with *always not*. And generally students have problems with this interplay between proving a theorem, making a general statement, and identifying an example, a special case[28]. These are the substantially different tasks here in the three different parts. The negation is wrong here[29].

In the next three Episodes the discussion turns to the students' enacting of three common proving techniques: Contradiction, Mathematical Induction and Counter-Example.

EPISODE 3.5
PROOF BY CONTRADICTION:
SPOTTING CONTRADICTION AND SYNDROME OF THE OBVIOUS

Scene I: Spotting Contradiction

Setting the scene: The following takes place in the context of discussing the following question[30]

> Write down a careful proof that $\sqrt{2} \notin \mathbf{Q}$.
> (do this by contradiction: assume that $\sqrt{2} = m/n$ with n, m having no common factors and see where you get after squaring and clearing fractions).

and two examples of students' written responses to this question, Student L's and Student J's:

[28] See earlier reference to Hoyles & Kuchemann (2002)
[29] See Barnard (1995).
[30] See Tall (1979) and (Alibert & Thomas, 1991, p217) for a discussion of the same question in terms of comparing generic, standard and proof by contradiction of this statement. Generic proofs seemed to generate less confusion and be more welcome by the students at the early stage of their studies.

① $\sqrt{2} = \frac{m}{n}$ where m and n have no common factors.

$2 = \frac{m^2}{n^2}$ ⇒ $m^2 = 2n^2$

∴ m^2 is even

⇒ $m^2 = 2r$

∴ $2n^2 = 4r^2$

$n^2 = 2r^2$

⇒ n^2 is also even

thus m and n must have common factors. ※

∴ By contradiction $\sqrt{2} \neq \frac{m}{n}$ ∴ is irrational.

Student J

MTH - 1A11/13/15 Autumn 2000 : Exersise 1

1) $\sqrt{2} = \frac{m}{n}$

$2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

Any number multiplied by 2 becomes even.
∴ m^2 must be even.

If the square of a number is even then the number is even.
∴ m ~~must~~ be even and can be written as 2x, where x is some interger.

So $2n^2 = (2x)^2$
$2n^2 = 4x^2$
$n^2 = 2x^2$

∴ For the same reason n^2 must be even.
∴ n must be even and can be written as 2y, where y is some interger.

If $\sqrt{2} = \frac{m}{n}$

then $\sqrt{2} = \frac{2x}{2y} = \frac{x}{y}$

where $\frac{x}{y}$ is a ~~simplier~~ more simple fraction than $\frac{m}{n}$

But this process can be repeated on $\frac{x}{y}$ to obtain an even simpler fraction, which can be made into an even simpler fraction, and so on forever.

However, we know a fraction cannot be simplified forever, and so the above statement is absurd. This leads to the conclusion that $\sqrt{2}$ cannot be written as a fraction.

Student L

M's comments at first are quite general.

M: Students often find proof difficult and the difficulties we are discussing here are common.

RME: In terms of necessity or enactment?

M: Both but certainly the latter. Students can understand simple proofs when demonstrated to them (if done well of course!) but cannot reproduce or construct them themselves.

RME: Would you expect your students to have difficulties with a question such as this one?

M: It depends largely on the educational context: in Scotland[31], for example, the students are exposed to more Calculus first and then around the end of Year 1, proofs such as this begin to emerge as examples of Proof by Contradiction – even though the students hope these will never emerge again! As I said students can follow, not produce proofs.

RME: The set of examples I wish to share with you demonstrates the diversity of student responses to a production task and in particular to a task that required spotting a contradiction in an argument.

M: From the two examples I can see here the problems the students have are in the specific details, not in the actual concept of proof. There are proofs which they have seen and they are trying to reproduce such as the one here. That is why you often see things that, to the untrained eye, look like mathematics but the reasoning behind them is not solid at all. There are proofs where this citing of fragments of thinking is something you can get away with but there are ones where you can put down nothing. Once you show students the proof of the irrationality of $\sqrt{2}$ and then ask them to prove the irrationality of $\sqrt{3}$ you get all sorts of irrelevant thoughts on odd and even numbers. And, even worse they can even try to transfer this to proving the irrationality of $\sqrt{4}$!

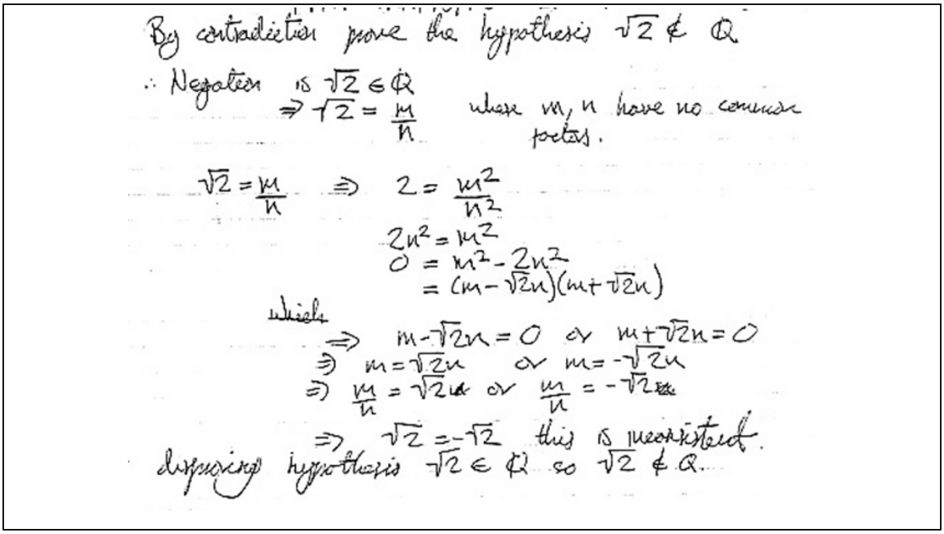
RME: …and then by Mathematical Induction prove the irrationality of all numbers!

M: There is a point here. They are missing where the proving point lies. If I attempted to explain where the principal difficulty with Proof by Contradiction lies I would have a go as follows: students are used to a linear proving argument where they start from one point and arrive at another in a sequence of logical

[31] As data collection was carried out in mathematics departments across the UK, participants sometimes contextualised their contributions with reference to the state of things in their own region. I have kept some of these if I thought they enriched the argument that M is putting forward.

steps. In Proof by Contradiction this linearity does not apply. Then again in my experience it is not always the logic behind this technique they have a problem with. And of course we always teach various groups of students with very diverse backgrounds and therefore the views I express here will vary in accordance with this diversity. In this sense the technicalities are more of a problem than the non linearity I mentioned earlier; the interim details. On top of which is writing the first line correctly and knowing where you are starting from, what is the assumption you are starting from. Then once you read the last line you must be able to see what this is actually saying and why it contradicts what you assumed. Proving linear independence in some exercises in Linear Algebra is an example where you need this kind of grasp.

RME: The majority of the students whose work I have examined had a problem with spotting where the contradiction lies in the sequence of claims they write out. Look at Student JWT's response: the difference of squares, the algebra, the logical error. I see the latter as possibly an outcome of a desperate desire, an urgency to complete the process of contradiction by spotting one.


By contradiction prove the hypothesis $\sqrt{2} \notin \mathbb{Q}$

$\therefore$ Negation is $\sqrt{2} \in \mathbb{Q}$

$\Rightarrow \sqrt{2} = \frac{m}{n}$ where m, n have no common factors.

$\sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

$0 = m^2 - 2n^2$

$= (m - \sqrt{2}n)(m + \sqrt{2}n)$

which $\Rightarrow m - \sqrt{2}n = 0$ or $m + \sqrt{2}n = 0$

$\Rightarrow m = \sqrt{2}n$ or $m = -\sqrt{2}n$

$\Rightarrow \frac{m}{n} = \sqrt{2}$ or $\frac{m}{n} = -\sqrt{2}$

$\Rightarrow \sqrt{2} = -\sqrt{2}$ this is inconsistent.

disproving hypothesis $\sqrt{2} \in \mathbb{Q}$ so $\sqrt{2} \notin \mathbb{Q}$.

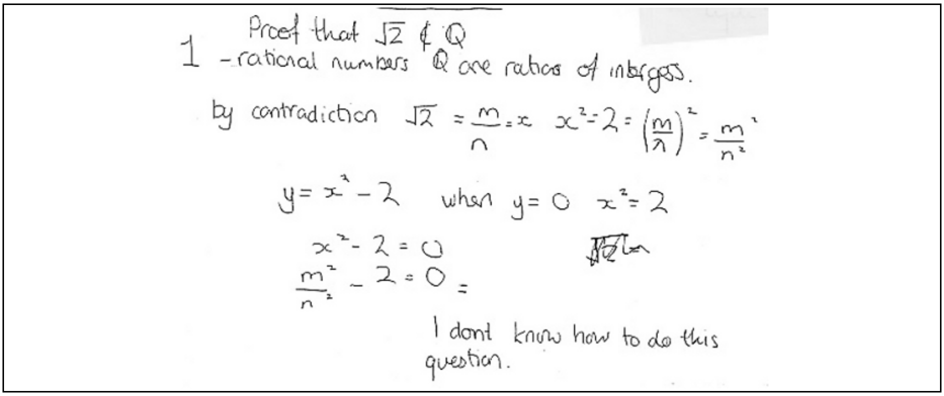
Student JWT

M: If you also look at Student L's response she is grinding rather knowingly and this grinding seems to be leading her into not knowing where to end up, how to wrap the argument up and close it. But we will come back to Student L soon. Have they seen other ways of proving the irrationality of √2 by the way and how many?

RME: This depends on their backgrounds and these vary considerably. Some may have seen this at A level.

M: As I said if you do not state at the top of your argument the non existence of common factors then you find yourself in the middle of a joggling act ending up with the final sentences which are a muddled and desperate attempt to complete the argument with something that more or less makes sense.

RME: The students have been given the tip, they know how to start off the proof. Off they go but still they get stuck at spotting the contradiction. Here is another example, Student WD who ended her script with *I don't know how to do this question*. It should all have been a simple execution of the hint, right?



1 Proof that $\sqrt{2} \notin \mathbb{Q}$

- rational numbers $\mathbb{Q}$ are ratios of integers.

by contradiction $\sqrt{2} = \frac{m}{n} = x$ $x^2 = 2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$

$y = x^2 - 2$ when $y = 0$ $x^2 = 2$

$x^2 - 2 = 0$

$\frac{m^2}{n^2} - 2 = 0 =$

I dont know how to do this question.

Student WD

M: In this sense I find Student L's *fractions cannot be simplified forever* really reasonable. I am reasonably happy assuming the student had not seen any proof like this before. Am I right to make this assumption?

RME: Probably. There is something sensible in Student L's *a fraction cannot be simplified forever*, I agree. But are we being too generous?

M: Probably a bit. There is some sense but not enough and clear demonstration of understanding. The problem with her response is that she hasn't put at the beginning of the sheet that *m* and *n* have no common factors. As a result she cannot get to the contradiction emerging from the fact that they end up appearing to have common factors. Instead she finds a different contradiction. Her argument has the scent of Analysis in it, this idea of simplifying for ever and ever! I appreciate that underlying aspect of infinity being introduced into the discussion. I am more concerned about her not proving that if *m* squared is even then *m* must be even (Scene II). But then again this is a simple idea that could be taken for granted? Because it is so simple to see that if *n* is odd then *n* square is odd? I am not sure.

RME: This is debatable isn't it? Is this a simple fact we are happy to let it be assumed or would we be happier if all claims were explicitly proved? Especially at this early stage where this notion of the obligation to justify every claim is a large part of the game?

M: I am happy to let it be assumed but at the same time I recognize there are issues in the ways students engage with constructing mathematical arguments that need attention: often students will write down the thing that they are asked to prove and manipulate it. I see that they can't avoid doing this at some point in a contradiction proof but I would be much happier if the word *suppose* figured firmly in the beginning of their sentence[32]. To allow this to go without comment would be doing the student a disservice. I would still like to stress though the originality of Student L's thought[33]. That's something she thought up herself, not something she copied from a tutor or her lecture notes or a book. She believes in her claim totally! It is the type of informal, intuitive claim students may have been conditioned to find satisfying at school level, this idea that *well, we cannot keep on canceling forever, can we?* The descending argument, by the way, is an approach I am perfectly happy with. Historically the habit of choosing a minimum counterexample, *m* and *n* having no common factors and reaching contradiction because of that, is a modern habit.

M then turns to commenting on the 'less convincing' Student J's response – in particular with regard to not proving that m^2 *and* n^2 *are even* implies that *m and n are even* (Scene II). In the context of those comments he also offers two more comments on this student's script, one on the quality of the writing:

M: I can elaborate on details of this piece of writing and there are typographical errors in there too. This may be an attempt to recall something from A level and the student maybe hopes for the contradiction to emerge like a little miracle.

And one more comment that compares Student J's and Student L's response:

M: Plus I have doubts about the ownership of the argument[34] because they have *a* and *b* written and then crossed out. That makes it less certain that this is their own argument. Unlike Student L's argument which is completely her own, as her

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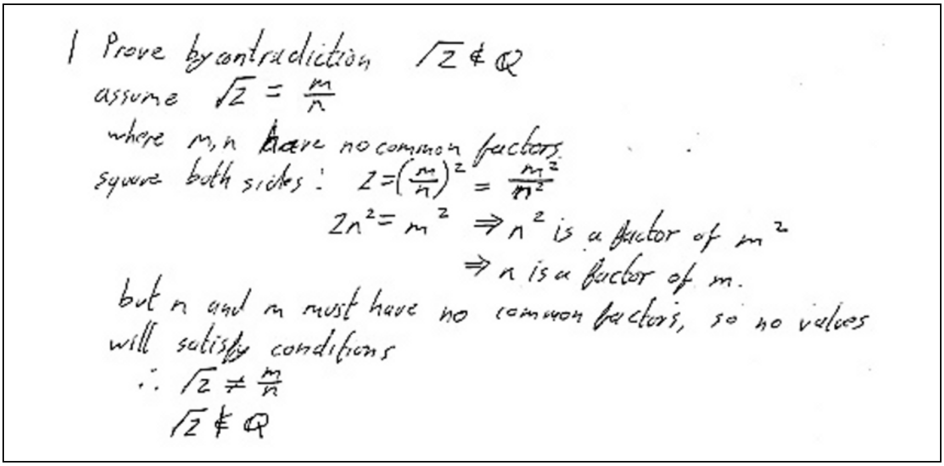
[32] See Chapter 4 where M and RME return repeatedly to issues of writing.
[33] M in several places associates originality of a student's argument (or presentation of) with its ownership (in the sense that a student who does not simply emulate a lecturer's writing on the board from their notes is likelier to have achieved a better understanding of the argument).
[34] See also M's earlier appreciation of the originality in Student L's argument.

use of letters *x* and *y* suggests. This is probably a rough reproduction from an advisor's writing, actually reproduced with not much understanding involved. As you can see I am still quite impressed by what Student L did! I guess I am a bit gullible towards a personal and original idea…

RME: Maybe it is a matter of personal taste but I find this, Student M's even less convincing.


1 Prove by contradiction $\sqrt{2} \notin \mathbb{Q}$

assume $\sqrt{2} = \frac{m}{n}$

where m, n have no common factors

square both sides: $2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$

$2n^2 = m^2 \Rightarrow n^2$ is a factor of m^2

$\Rightarrow n$ is a factor of m.

but n and m must have no common factors, so no values will satisfy conditions

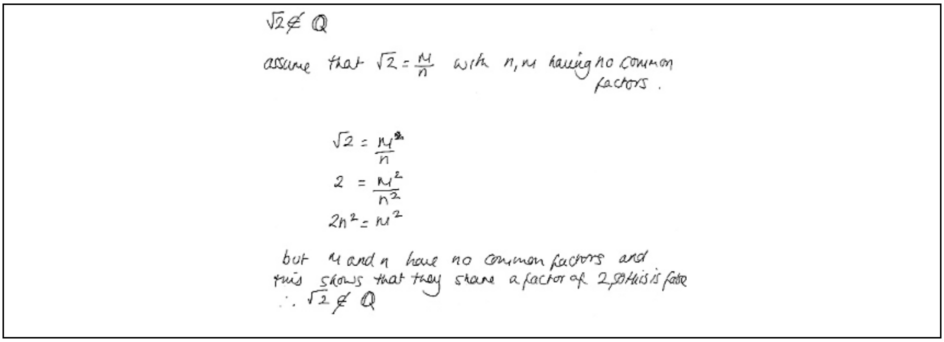
$\therefore \sqrt{2} \neq \frac{m}{n}$

$\sqrt{2} \notin \mathbb{Q}$

Student M

M: Yes, there is a real step missing.

RME: …as is in Student LSB's response where the student concluded contradiction because she knew she had to but had little idea of where the contradiction was lying in her writing.


$\sqrt{2} \notin \mathbb{Q}$

assume that $\sqrt{2} = \frac{m}{n}$ with n, m having no common factors.

$\sqrt{2} = \frac{m}{n}$

$2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

but m and n have no common factors and this shows that they share a factor of 2, so this is false

$\therefore \sqrt{2} \notin \mathbb{Q}$

Student LSB

M: I agree. This is the garbled version of something that they have heard, I think. Constructed or semi remembered, I wonder?[35] In a sense Student WD's response, where the student simply reached a similar stage after the manipulation and admitted on paper *I do not know how to do this question* is more honest. So, on the basis of what we see in these examples, your statement that students have a problem with spotting where the contradiction lies seems to be true. And the contradiction often lies in the leap to conclusion (Scene II) they perform.

RME: Most students reached a certain point in the proof and that point was up to where the instructions in the hint could take them but stopped where the creative part started.

M: So you can teach algebraic manipulation but not this creative part, right (E7.4)?

RME: Do you have any other examples of proof by contradiction which the students would encounter at this stage or would be good for them to encounter?

M: Unlimited number of primes maybe?

RME: Do you consciously ask them to engage with Proof by Contradiction so that they get used to it, become familiar with it? Or does it just happen?

M: We have seen another example in our discussions together, haven't we (E3.1)? *If A is an nxn matrix such that* $A^2=0$ *then the matrix is singular*. I am not sure how to answer your question[36]. At this level I don't know. Later on Proof by Contradiction is so commonplace that you don't think about it being as such. In this particular example I am concerned that the students, after following the hint, get to the point $2m^2=n^2$ and then they are not sure what they should be doing. The situation becomes a complete mental barrier. The proof that there is an unlimited number of primes is similar in the sense of containing the potential for a similar mental barrier. Or maybe it is even simpler? Here there is a real step between $2m^2=n^2$ and knowing that this implies that m and n are even therefore they must share a common factor (Scene II). And that therein lies the desired contradiction. There is a real step and it's a pity they didn't pick up on that.

RME: A fact that takes away from also consolidating the very important idea that $\sqrt{2}$ is irrational. The difficulties around this proof have raised the interest of mathematics educators who have written about it to some extent: there is a study with Year 1 and A level students. Barnard and Tall (1997) have de-composed the

[35] See earlier comment on M's frequent emphasis on ownership and originality.
[36] This would require a meta-mathematical course, or a part of a course on Proof, which is not a practice M generally reported as being a large part of his experience or practice.

proof to its elementary items according to their conversations with students and according to their own more advanced experience.

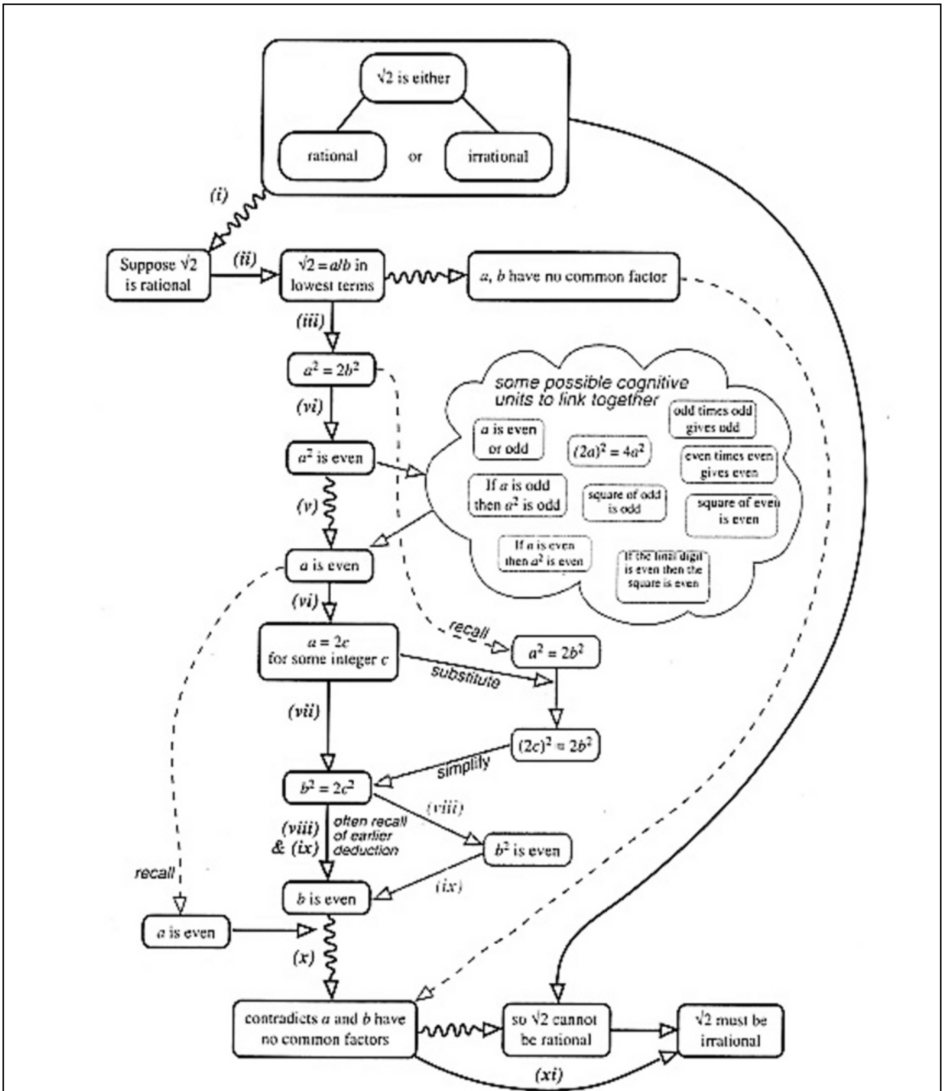


Figure 1: Observed cognitive units and connections in a typical initial proof of √2 irrational

Proving the irrationality of √2: a de-composition by Barnard & Tall (1997)

M: [browsing the diagram] Well, this diagram sums up how difficult mathematics can be. And it sums up our conversation so far nicely[37].

The discussion then turns to Interspersed in the above discussion are references to a small but crucial leap in the students' writing. Below is a synthesis of M and RME's views on the acceptability or not of this leap in their writing at this stage.

Scene II: Syndrome of the Obvious[38]

Setting the scene:. Why would anyone bother with arguing about – let alone proving – the truth of an obviously true statement? If this is a substantial element of the economy of the human brain, is asking students to unpack the thinking behind their endorsement of obviously true statements an act that veers against their natural tendencies? And if so, is this act then doomed to fail? M and RME reveal why this unpacking is such a fundamental element of mathematical thinking and hence internalising it is non-negotiable … and hugely instructive. The Scene was collated from several comments made by M and RME in the course of Scene I. For example Student L's sentence 'if the square of a number is even then the number is even' – stated, yet left unproven[39] – steers the discussion towards this, small yet significant, step of the proof. Is assuming the truth of this statement acceptable? Are the students at this stage expected to provide the proof for every step in a proof, even the apparently obvious? Other student responses to the same question – and towards the end a student's attempt at a Proof by Mathematical Induction that $2^n \geq n^2$, for $n \geq 4$ – enter the discussion of the same issue.

RME: Students seem to be able to see that both m^2 and n^2 are even but do not state explicitly that this implies that m and n are even, therefore both m and n have two as a factor, a fact which contradicts the initial assumption that m and n have no common factors.

M: I agree that this lack of explicitness is dubious. Here there is a real step between $2m^2 = n^2$ and knowing that this implies that m and n are even therefore they must share a common factor. And that therein lies the desired contradiction. There is a real step and it's a pity they didn't pick up on that. And I would like to observe the indecisiveness in the script of another student we have talked about, Student

[37] M is complimentary but uncharacteristically brief about the diagram. RME explores M's attitudes towards the theoretical constructs of mathematics education research in more detail in E8.2, Scene II where the discussion also returns to this diagram.

[38] Students are often undecided about the degree of detail they are expected to provide (see Nardi, 1996; 1999; and, Nardi & Iannone, 2001 for a discussion of this in the context of the concept of Didactical Contract for undergraduate mathematics).

[39] Studies that report this common student proclivity include (Almeida, 2000; Dreyfus, 1999; Healy & Hoyles, 2000; Recio & Godino, 2001 etc.).

J [whose response M found 'less convincing' than Student L's] , about the extent to which she needs to present that m^2 *is even* implying that *m is even*. Students have a problem in deciding what level of sophistication they need to present in their writing (E4.0). Writing out a proof for every single claim can result in a long script, something students are not used to. Since their school days they have been expected to do small things in response to small chunks of the syllabus (SE3.1).

RME: Would you expect to see a full proof of the claim *if* $2m^2 = n^2$ *then n is even*?

M: I am afraid I would, yes. And the question did ask for a *careful proof*!

M later returns to the particular incident of spotting the evenness of *m* and *n* and the commonality of factor 2 in them.

M: [In Student L's response] There maybe an underlying reference to prime decomposition of number there but I think the crucial point is the set up of the first line and the lack of caution in writing it out properly. But *writing out properly* is a whole other issue, isn't it, with students (E4.1)? They often do not repeat segments from the question but they mean the segments to be there. We should be teaching tidiness, clarity and readability (E7.1, Scene VI).

RME: Speaking of clarity, most students wrote down m^2 *is even implies m is even* without a proof.

M: This depends on the students' previous knowledge. I would be happy with such a leap if it followed other occasions where similar simple arguments had been proved. But there is a firmer approach: in a first course such as this I would expect a complete proof of this. In itself it is a small but substantial proof.

RME: It is unlikely that the students know about prime decomposition at this stage…

M: …which makes our judgements on this valid only if we have a clear picture of the students' previous experiences and of knowledge they are allowed to draw on. So at this stage this leap of faith is not completely satisfactory.

RME: So let me rehearse the necessary steps here: having followed the hint and once you prove both *m* and *n* are even, you deduce that *m* and *n* are multiples of 2, therefore they have a common factor that is 2, therefore the initial assumption of the rationality of $\sqrt{2}$ must be false.

M: Hearing you do this reminds me that distinguishing what knowledge can be assumed, compressed, omitted from a script is a difficulty in itself (E4.2). I often like brief student responses but I am not sure they are always very revealing.

RME: Student M's response which we looked at briefly earlier contains the claim that if m^2 divides n^2 then m divides n…

M: … a leap which demonstrates the students' looking with a vengeance for a contradiction in the argument. Again, I wonder, what do students know already? I feel my generosity with regard to these leaps is running out somewhere here.

RME: Yes, students often imitate the expert's style and in this case they do not seem to learn much about the necessity to prove even claims that are intuitively almost obvious.

M: Obviousness creates problems and potential misunderstandings about what level of detail is expected[40]. There are many simple proofs students can learn a lot from – for, example proving that the product of two even numbers is an even number. And perhaps in the writing you can let them get away without all of those within an argument but, if asked, the expectation would be that they can do them, right? In public, on the board …!

RME: So simplicity obstructs a more accurate perception of what is necessary? Do the students see proof as a tool for convincing themselves and others[41]?

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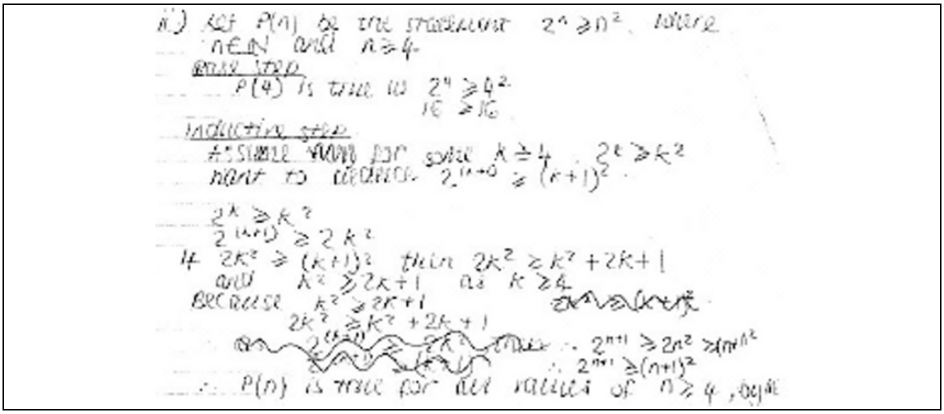
[40] The issue raised earlier becomes more specific here: not only students are unclear about what can be taken for granted, stated and be left unproven but the tendency to do may be exacerbated by the simplicity of the proposition in question: a syndrome of the obvious then emerges and avoidance of explanation or proof becomes the greatest casualty of this obviousness.

[41] The distinction between proof as a tool to convince oneself and others (whether friends or foes, Mason et al 1982) is longstanding in the field – see, for example, Recio & Godino's (2001) distinction between students' institutional and personal meanings for what constitutes proof. Another recent example is the study by Healy & Hoyles (2000) which examined students' conceptions of proof in the curricular context of high attaining 14/15 year old students in the UK. Their findings showed that there is a substantial difference between the students' personal preferences and the ones they would make in order to improve their chances for a better grade. (and there was evidence that teachers were not always aware of this). Students were rather poor at constructing proofs, often relied on empirical arguments and their narrative often had little algebra or formalism. They preferred empirical arguments but acknowledged more was expected of them. If they believed the conjecture in the first place, then they tended to rely more heavily on and be convinced by empirical argument. Algebraic arguments were highly valued by students but they also found them difficult to produce or understand. Narrative arguments were popular for their explanatory power and the students' own narrative constructions were often characterized by higher incidence of deductive reasoning. Their pattern was more or less: empirical evidence to convince, then words or pictures to complete the argument. Not algebra though. Dominant views of proof were: truth, explanation, (some, few) discovery. Use of examples was seen as important – so, often, a valid proof would be seen as not general and examples would be used to 'complete' it. Finally their views on what constitutes an acceptable argument influenced their ability to construct (and complete) a proof.

M: Students are accustomed to being told in school about the truth or falsity of statements, not to deciding themselves and this is a step they need to take at this stage: the desire to know why something is true. The focus in school is on algebraic manipulation, not proof. Which may foster a misleading image of mathematics as all being about manipulation only, missing thus what is at the heart of mathematics, namely proof. This unbalanced stock of experience explains why they find proof so difficult (SE3.1).

RME: Here is another instance where I think perceived obviousness interferes with clarity and transparency in the student's script in a Proof by Mathematical Induction (E3.6).


ii) Let P(n) be the statement $2^n \geq n^2$ where $n \in \mathbb{N}$ and $n \geq 4$.
Base step
P(4) is true as $2^4 \geq 4^2$
$16 \geq 16$
Induction step
Assume true for some $k \geq 4$, $2^k \geq k^2$
want to deduce $2^{(k+1)} \geq (k+1)^2$
$2^k \geq k^2$
$2^{(k+1)} \geq 2k^2$
If $2k^2 \geq (k+1)^2$ then $2k^2 \geq k^2+2k+1$
and $k^2 \geq 2k+1$ as $k \geq 4$
Because $k^2 \geq 2k+1$
$2k^2 \geq k^2+2k+1$
$\therefore 2^{n+1} \geq 2n^2 \geq (n+1)^2$
$\therefore 2^{n+1} \geq (n+1)^2$
$\therefore$ P(n) is true for all values of $n \geq 4$, $n \in \mathbb{N}$

Student JR

M: Well, there is quite a bit of backward reasoning and not much about why in our case $2k^2 > (k+1)^2$ is true. Perhaps they checked by substituting the numbers? How do they know this is true? Yes, this is symptomatic of the same issue. That their claims are suspiciously compact, not a compressed version of clear reasoning but a muddled packaging of some sort of reasoning. They are convinced themselves but this writing cannot convince many others (E4.2).

RME: In some cases the compression does not bother me but in many others, like here, I would like to see more explanation.

M: A sketch or something at least (E4.3). Sometimes what I find disappointing is when there is a hint that the students may have done so in their rough papers but have not felt compelled to reproduce the evidence that generated their own conviction in the first place it in their submitted script. This student seems to have all the right lines in the wrong order. She or he is somehow convinced that the inequality is true but has not bothered to clarify this conviction in the writing. And of course all that matters is what is on this piece of paper (E4.2).

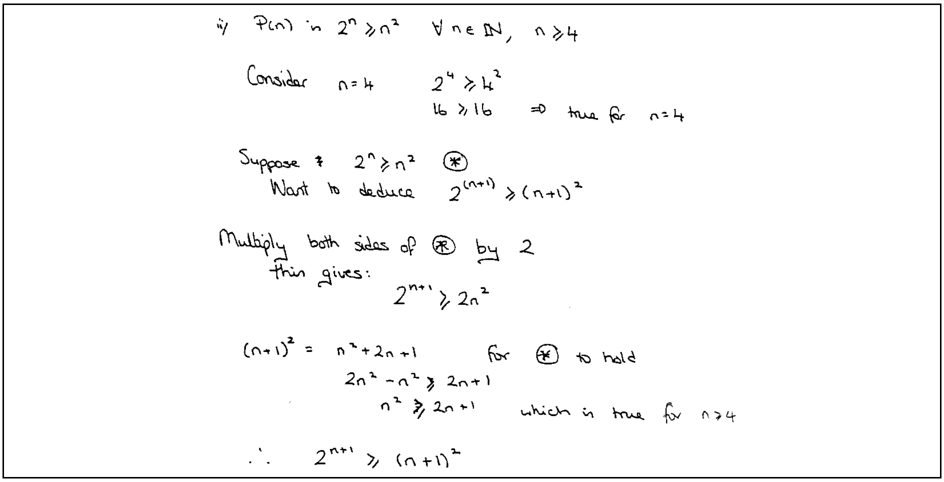
EPISODE 3.6
PROOF BY MATHEMATICAL INDUCTION[42]: FROM *N* TO *N*+1

Setting the scene: The following takes place in the context of discussing the following question below as well as two examples of students' written responses part (ii), Student JU's and Student E's. Later they also discuss part (iv).

(4) Give proofs by induction of the following statements:
(i) $1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$ for all natural numbers n.
(ii) $2^n \geq n^2$ for all natural numbers $n \geq 4$.
(iii) For all natural numbers n:

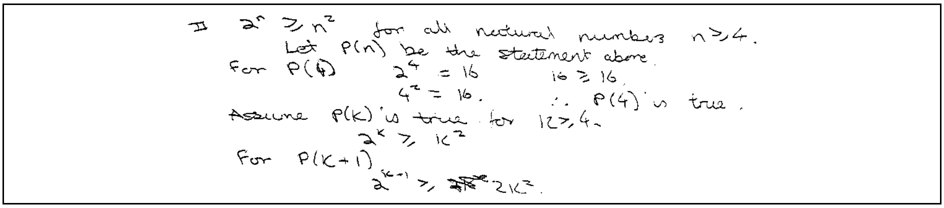
$$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

(iv) For all natural numbers n, the sum of the first n odd natural numbers is n^2.


ii) $P(n)$ is $2^n \geqslant n^2$ $\forall n \in \mathbb{N}$, $n \geqslant 4$
Consider $n = 4$ $2^4 \geqslant 4^2$
$16 \geqslant 16$ $\Rightarrow$ true for $n = 4$
Suppose $2^n \geqslant n^2$ (*)
Want to deduce $2^{(n+1)} \geqslant (n+1)^2$
Multiply both sides of (*) by 2
this gives: $2^{n+1} \geqslant 2n^2$
$(n+1)^2 = n^2 + 2n + 1$ for (*) to hold
$2n^2 - n^2 \geqslant 2n + 1$
$n^2 \geqslant 2n + 1$ which is true for $n > 4$
$\therefore$ $2^{n+1} \geqslant (n+1)^2$

Student JU

[42] For a preliminary discussion of the data in this episode see (Nardi & Iannone, 2003b). Our discussion there resonates with the literature on student difficulties with Mathematical Induction (e.g. Movshovitz-Hadar, 1993; Fischbein & Engel, 1989; Thompson, 1991). Segal (1998), for example, found that a constant factor of the difficulty the students have with Mathematical Induction is a conceptualisation of the central role of implication in the third step of the inductive mechanism. Moreover the examples we examine reinforce Baker's (1996) suggestion that in the conceptual and procedural difficulties evident in the students' proof writing, mathematical content knowledge (in our case this included manipulation of inequalities, use of Σ etc.) plays an inextricable and significant part.


II $2^n \geq n^2$ for all natural numbers $n \geq 4$.
Let P(n) be the statement above.
For P(4) $2^4 = 16$ $16 \geq 16$.
$4^2 = 16$. ∴ P(4) is true.
Assume P(k) is true for $k \geq 4$.
$2^k \geq k^2$
For P(k+1)
$2^{k+1} \geq 2k^2$.

Student E

RME: May I invite your comments on student difficulties with Mathematical Induction?

M: I would highlight several big difficulties with Mathematical Induction. These regard both those who have not seen it at all and those who have seen some examples of it. One difficulty is that somehow one proves a statement for n by assuming it in the second step. And this somehow contrasts with everything else that we have said to them often in the same week, maybe even in the same lecture, that there is something special about Proof by Induction that students have little understanding of. How can you possibly prove something in this way by saying it is true in the first place? It is hard work to get by this notion. And also I think another difficulty is beginning to appreciate that one needs to replace n with n+1, a difficulty also seen in the context of $f(x)$ and $f(x+3)$. This is always proving surprisingly hard work. I often think it would be easier to use n and k instead. Already that means that substituting k+1 to n is easier than substituting n+1 to n. Also seeing part (ii) reminds me of inequalities as another place of difficulty (SE3.2): you can twiddle it here and you can twiddle it there but the inequalities are hard I think. Inequalities are difficult because there is no right hand side in them which students would know how big or small they ought to make the expression on that side in order to continue producing the chain. Overall however I would point my finger at the main difficulty as being able to see the transition from the n to the n+1 case, this idea that you need to prove your right to the claim that the statement is true for k+1, assuming the truth for k.

RME: Students appear to reproduce accurately the 3-step mechanism in their writing but implementing it in the context of the question is a more difficult task (E3.2, Scene II). May I invite your comments on Student JU's response?

M: Let's see. I noticed *suppose that* $2^n \geq n^2$ and then I thought that the student was using the thing that they need to prove …that is $2^{n+1} \geq (n+1)^2$ and then multiply by two together to get this result further down $2n^2 - n^2 \geq 2n+1$. I think I would like to focus on the backwards reworking of the inequality in question (SE3.2). This is a very common occurrence. The rearrangement poses problems here. Especially because we have the difficulty of having to go back and forth twice in order to complete the proof. I think the student wrote all that she needed but not in the right order. And luck settles the trouble when the order doesn't matter a lot, namely when the claims are reversible. Like in equalities and $\Leftrightarrow$, and unlike in $\Rightarrow$. So reversibility saves the day more or less here. It is very difficult for students to understand the importance of order – by the way the conversational ones are the easier to convince of course (E7.1)! So, especially if the student got the rest of the question right, then we can reasonably assume that her problem is with dealing with inequalities, not the inductive mechanism. But this is what you mentioned in your text[43], the idea that to prove A=B, I start from A=C and prove that C=B. In this case they could have looked at the graphs of n^2 and $2n+1$ and consider when this is true…

RME: … which nobody did…

M: … which would help them find out more easily why the $n \geq 4$ is necessary here. Did the students wonder where this condition comes from, I wonder…

RME: As I was looking through the four parts of the exercise, one of my first questions was why the proof in part (ii) starts from $n = 4$…

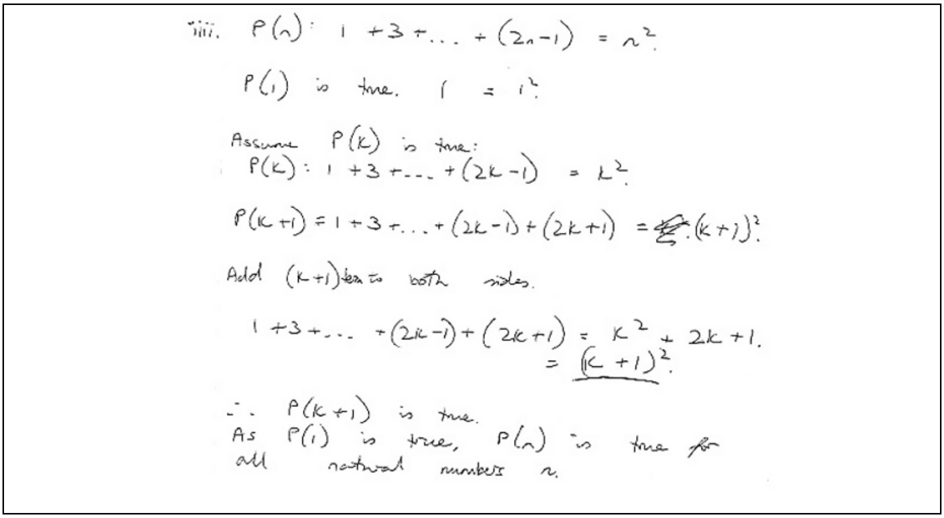
M: … and you soon found out that the statement holds for 1 and for 2 but not for 3!

RME: There were students who started from $n = 1$.

M: The ones who only checked $n = 1$ were tricked into believing the statement is true for all n! It is a bit monstrous that the statement is true for 1 and 2 but not for 3 and then true again for 4 onwards.

RME: Shall we move on to the example that involves the sum? There, quite a few students seemed fine with constructing the statement for n+1 out of the one for n by adding the (n+1)-th term. Here is Student JK's response to part (iv).

[43] In the Dataset we had included an observation we had made in (Nardi & Iannone, 2003b). That, when $P(n)$ was an inequality, a significant number of students had simply given up. The remaining students demonstrated intense uncertainty about their claims and a range of difficulties with completing the proof such as starting from the inequality one is supposed to arrive at. A number of students also presented answers where the manipulation of the inequalities involved was particularly convoluted. A conjecture that emerged in our analysis was that the students' proofs by Mathematical Induction often seem to follow the pattern: 'if A = B, to prove A = C, I start from C = B and end up with an identity'. We propose that this is an approach that the students appear to be at ease with when P(n) is in the form of an equality but not in the cases that it is in the form of an inequality.


iii. $P(n): 1 + 3 + \ldots + (2n-1) = n^2$.

$P(1)$ is true. $1 = 1^2$.

Assume $P(k)$ is true:

$P(k): 1 + 3 + \ldots + (2k-1) = k^2$.

$P(k+1) = 1 + 3 + \ldots + (2k-1) + (2k+1) = (k+1)^2$.

Add $(k+1)$ term to both sides.

$1 + 3 + \ldots + (2k-1) + (2k+1) = k^2 + 2k + 1$.

$= (k+1)^2$.

$\therefore P(k+1)$ is true.

As $P(1)$ is true, $P(n)$ is true for all natural numbers n.

Student JK

M: It's relatively easy. By the way I think the questions in the problem sheet are presented in reverse order of difficulty. I think part (ii) is the hardest. Part (iv) relies on writing down the first step properly.

RME: Some students chose the *...dot, dot dot* and others the Σ approach.

M: In Student JK's case the choice makes no difference: it is clear that the student understands how the final claim is derived, by adding the final term to the equality that you have assumed as true. She has written $P(k)$, then $P(k+1)$ so that she knows where she needs to get at, and then constructed $P(k+1)$ out of $P(k)$ by adding a term. So there is no need for the Σ notation then. Somehow the Σ notation would obscure the clarity of this student's answer. She clearly knows what is going on even though the writing could do with a bit more clarity. This is a good example but when teaching this material, I have often been deceived by students' nodding during the sessions and then their writing ending up being gibberish (E7.1)! Anyway students find the Mathematical Induction mechanism difficult overall.

RME: Yes, I have an example in mind where a student thought it would be sufficient to simply re-write the four statements in the exercise alongside the three steps of the Mathematical Induction for each and then stop, Student LW. Yes, the one who typed it all up (OT4.1)!

M: So there is a reproduction of the mechanism but not within the context of the specific propositions (E3.2, Scene II). The student only conducted the first step, *P*(1). By the way one way to convince students that this is not sufficient is to expose them to cases where the statement beyond $n = 1$ is ludicrous. The student has missed the inductive step! Well, I am sorry but the assumption in the middle step is at the heart of the mechanism.

RME: Do the students understand why the Mathematical Induction mechanism is a proof[44]?

M: It's a good question. Well, to start with, some of them come to university already familiar with it and it is an easy-to-follow recipe. I am thinking of places we use it commonly in Linear Algebra, for example.

RME: A kind of domino effect, I guess[45].

M: Yes! Which they accept easily. Especially in the context of proving equalities, like in part (iv), to which they are used to.

RME: There are plenty of examples in Geometry where the mechanism turns out to be useful.

M: Sure but they are much more accustomed to seeing this in the context of Algebra. And as a recipe that is easy to apply. But why is it a proof? I am not sure how far into thinking about this they go but it is easier for them than for us, in a sense. The less mathematics you know the more convincing Induction is. It is pretty obvious: $P(1)$ is true and, if you can prove that $P(n)$ implies $P(n+1)$, *P(2) is true* implies *P(3) is true*… and so on for all n. If you know lots of mathematics it is not… pretty obvious at all[46]!

RME: There is just one more example of student work on this exercise I would like to invite your comments on, Student H's response where there is somehow a use of the Σ notation.

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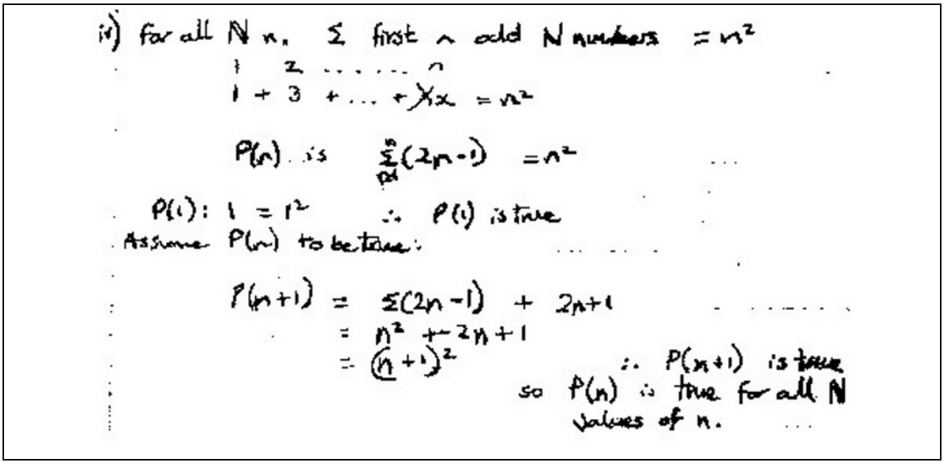
[44] Baker, in the study quoted earlier (1996), concluded that the participants in his study focused on the procedural aspects of mathematical induction far more often than on conceptual aspects. The data here do not allow more than speculation on this matter as the mathematical task the students engaged with was purely procedural and left little space for reflection on their part about conceptual aspects of Mathematical Induction: they were asked to use the technique to demonstrate the truth of certain propositions; they were not asked to consider the concept of Mathematical Induction. Hence their responses can be used to examine their application of the technique and little beyond that. In any case a study on Mathematical Induction as a meta-theorem (to use Lowenthal & Eisenberg's (1992) term) is a direction that certainly merits further study.

[45] A brief 'googling' of the term yielded numerous examples of notes to undergraduates from around the world that use the term to introduce the effect achieved by Proof by Mathematical Induction.

[46] M may allude here to philosophical issues of Incompleteness.



Student H

M: Well, as usual there is a problem with the idea that $P(n)$ is a statement, not a number[47]. Or an equation. I think there is an interesting interplay in this otherwise fantastic script – the student knows exactly what she is doing – between $P(n)$ being a statement and an algebra expression. $P(n)$ is is followed after a couple of lines by $P(n+1) = \ldots$. To me it is very crucial that they see $P(n)$ as a statement evaluated at n and not as a number or an expression. The whole logic of Mathematical Induction depends on that. I know $P(n)$ makes it look a bit like a function but in a sense there is a correspondence between statements and values of n so this symbolism is not off at all But of course, given the dominant understanding of function as an algebraic expression that arranges the assignment of values of a quantity for various values of x, this symbolism may invite misunderstandings. And I don't want to go into thinking that yes, there is a function but it is the one that assigns true or false to $P(n)$…

RME: Maybe not now… But I see what you mean. On a different note, as in other scripts we have seen that the student loses Σ soon enough on the way for the writing to suffer little from the ambivalence in the use of the dummy variable etc. (E4.1, Scene II). I recall another student who ignored the sum of… part of the question and proved the whole thing without it! But here she completes the proof unscathed.

M: It is interesting how Σ and the word *list* and the use of N are interspersed across the script. Still from the examples we have seen so far on students' enactment of proving techniques Student L's (E3.5) is the fascinating one. I can happily reproduce how Student L would have responded here as a tribute to her style!

[47] See Chapter 4 for a discussion of semantic issues such as the one raised here by M.

EPISODE 3.7
PROOF BY COUNTEREXAMPLE: THE VARIABLE EFFECT[48] OF DIFFERENT TYPES OF COUNTEREXAMPLE[49]

Setting the scene: The following takes place in the context of discussing the following excerpt (data and analysis) from (Nardi 1996, Chapter 6) entitled *The Unsettling Character of the Logical Conjunctions in the Definitions of $S\cup T$ and $S\cap T$ and the Complexity of the Notion of Supremum: the Varying Persuasion of Mathematical Arguments and the Importance of Semantic and Linguistic Clarity*. The discussion in the passage is between Student Alan and his tutor and it revolves around the following question[50]:

> (i) If S and T have suprema, prove $sup(S\cup T)$ = max $\{supS, supT\}$.
> (ii) Is it true that if S and T have suprema then $sup(S\cap T)$ = min $\{supS, supT\}$? Justify your answer.

Here is the excerpt (data part):

> …The tutor and Alan then agree that (ii) is not true. Alan cannot recall the counterexample he used. The tutor offers a counterexample ($S=\{1,2\}$, $T=\{1,3\}$) and asks Alan to think where exactly he realised that (ii) was not true.
>
> A1: It's true that $supS$ is an upper bound for the intersection but it's just saying that... it breaks down when I prove that it is the least upper bound... because if there is an element in between...

[48] Degrees of persuasion vary given that depending on where the proof comes from and how it is presented, learners may feel obliged to accept a proof which they do not necessarily believe. This reflects the reality in the mathematical community where acceptance of a proof is often a result of a variety of sociocultural factors other than its sheer formality. Sekiguchi (1992) points at the social dimension of proof presentation as a communication process and Hanna (1989a and b) enumerates a variety of dimensions in the social process of accepting a proof.

[49] For a preliminary discussion of this part of the data see (Iannone & Nardi, 2005b). In this we distinguish three, at times conflicting, roles that counterexamples play in learning and doing mathematics. The first role is an affective role. The counterexample has to be emotionally convincing for the students (strengthen their certainty). If it is based on what the student perceives as some minor technicality (as is the case in this Episode) the counterexample may not convince the student that the proposition is false. The cognitive role consists in conveying to the students that all counterexamples *are* the same, as far as mathematical logic is concerned; that a single counterexample can refute a proposition; that a proposition does not need to be *always* false in order to be false; and, that one occasion of falsity suffices. The epistemological-cum-pedagogical role has to do with what can be learned from a good counterexample – for example in mathematics we use counterexamples to identify which elements of a false statement would need to be amended in order to transform this statement into a theorem. Zaslavsky & Peled – who stress the pedagogical role that counterexamples have to play (1996), especially given the distinction they employ between counterexamples that (only) prove and counterexamples that (also) explain, (Peled & Zaslavsky, 1997) – locate students' difficulty with this type of proof in the wider arena of their difficulty to construct examples (often due to limited content space).

[50] In what follows in this Episode this question is referred as Question CD2.5.

T1: It would look awfully like this again! [refers to student's response to another part of this question]

A2: Oh, it's the same... probably... so *supS* is 2...

T2: Hmm... so alpha could go one and a half from there...[Alan nods] It's the problem about that you are considering two alternatives for all x as opposed to it holding for a particular x... I'm not certain...

A3: I'm saying... that x here... I think it's this... it's quite difficult to say...

The tutor says there is a 'shakier' counterexample which requires a bit of 'trickery': choosing S and T such that $S \cap T = \emptyset$ then $S \cap T$ has no supremum. He however suggested the one given above because he thought of it as 'slightly more convincing'.

Below is a part of the analysis of the above in (Nardi 1996):

The Non-Equivalent Psychological Power of Two Counterexamples. To prove that a set has a supremum, one has to prove that the set is non-empty and that it is bounded above. If any of these two conditions does not hold then the set has no supremum. In CD2.5ii to prove that $sup(S \cap T)$ is not necessarily equal to the $min\{supS, supT\}$, one can either show that $S \cap T$ does not necessarily have a supremum or that even if it does, this does not have to be $min\{supS, supT\}$. In the first case one can prove that $S \cap T$ has no supremum by illustrating a case in which for instance $S \cap T$ is the empty set; in the second case by pointing at two sets such as $\{1,2\}$ and $\{1,3\}$. Both are logically equivalent and satisfactory counterexamples. They nevertheless seem to differ in persuasiveness: in the tutor's words (end of Extract) the latter is 'slightly more convincing'. Similarly, students in other tutorials, that I observed but are not reported here, claimed that they did not stop looking for a counterexample for CD2.5ii until they finally thought of a pair of sets, such as $\{1,2\}$ and $\{1,3\}$, even though they kept coming across more examples of the first case, namely pairs of sets with an empty intersection.

A possible interpretation for this reluctance towards counterexamples of the first kind, that is pairs of sets of an empty intersection, is

First, that the counterexamples of the second kind refute $sup(S \cap T)=min\{supS, supT\}$ as opposed to the more formalistic refutation achieved by the counterexamples of the first kind. In other words, in answering a question such as 'is it true that $a=b$?', students and tutor here have expressed a psychological preference for

showing a case where $a \neq b$ (second kind)

instead of

showing a case where a does not exist (first kind).

Secondly, in the case of CD2.5ii the counterexamples of the first kind did not question the upper-boundedness of the intersection but its non-emptiness. When discussing however the existence of the supremum it seems that, even though the two conditions, upper-boundedness and non-emptiness, are logically equivalent, in these mathematicians' minds the former carries more weight than the latter.

So, in the above, I have tried to illustrate the difference between the epistemological and the psychological grounds of the tutor's and the students' preference. Epistemologically the counterexamples of the first kind are equivalent to the counterexamples of the second kind; psychologically however they do not seem to be equivalent. The significance of this distinction lies in the fact that the students, even after finding counterexamples of the first kind, continued to search for a psychologically satisfactory answer, that is counterexamples of the second kind. That is the priority of their own sense of conviction overshadowed the execution of the strictly mathematical task (to find a formally acceptable counterexample). In other words, the personal and subjective took over the priority from the impersonal and objective.

RME: May I invite your comments on this excerpt and in particular on the issue of more and less convincing counterexamples?

M: I think that the less convincing example is the one I would have thought of! But in any case choosing a couple of sets which are disjoint seems to be taking the toys away. I feel there is not even opportunity to start with toying with this idea. I think that from the students' point of view I would be dissatisfied. After all not all counterexamples are the same: they are if their use is to disclaim a statement but they are not as they highlight different things. For example the minimal counterexample, the least number that violates something is often a good indicator of a property or a phenomenon. A reasonable counterexample that shows up in how many cases, not simply one extreme one, the statement does not apply is far more acceptable. But is this a personal, psychological thing or something to do with the mathematics? I wonder. I am referring to the distinction you suggested in your text, *the non equivalent psychological power of two counterexamples*.

RME: Yes, I meant the qualitative difference between them in epistemological and psychological/affective terms.

M: But would this distinction apply for two students in the same way? What one finds convincing, the other one may not (E8.2, Scene II).

RME: Sure. This individuality is something we always have to consider in our analyses.

M: I think I can offer another example: convergence/divergence of a series when the absolute value of its sequence tends to zero. There are more and less instructive counterexamples you can use to illustrate the difference between a convergent and a divergent sequence or series. A counterexample may suggest to you how you can reformulate a statement to include fewer or more cases. So, if the condition about the intersection was that the two sets do not intersect, then the students' search for a counterexample would have been more productive as they would have been steered away from the *less convincing* one, I think.

RME: Do you get occasions where students say: well… ok, we have found one or two counterexamples but actually for most of the cases the statement applies, therefore the statement is true?

M: Isn't that what we all say?! [*laughs]*

RME: Because if that were the case then the students would not easily accept Proof by Counterexample.

M: Yes, students often think that to prove falsity of a statement they have to prove it is always false. Take *matrix multiplication is not commutative*, for example. They cannot always see that finding just one pair of matrices that do not commute is enough. Other interesting problems in this respect are: the probability of randomly picking two matrices that commute or a rational number off the real line. In fact it is a real problem to convey to the students how one goes from a state of doubt to a state of persuasion via examples or what not. I think there is potency in the idea of providing the opportunity to the students to explore the reasonable, meaningful cases (E7.4, Scene VI).

RME: You mention elsewhere (SE3.1) a type of proof that helps us modify an argument, a technique and so on… Can we revisit this in the context of the above excerpt?

M: Yes, but I need to know what definition of supremum the students had been given and whether this included an equal emphasis on upper boundedness and non emptiness of the set. Knowing this would tell us more about why the students respond to the counterexamples in these ways. The interest in the non emptiness is limited, it is just something that makes the definition more precise when you write things down. The essence lies in the bounded-aboveness, I would suggest. Therefore there is no surprise in the students' preference for counterexamples that refer to the latter. And I would agree with them. Moreover I think the empty intersection counterexample is pathological, an unproductive resolution of the question that does not illuminate us any further about the assertion. The other one is good. If the second part of the question put the non empty intersection as a requirement then attention would have been steered away from this unproductiveness. In this sense I would disagree with your comment in the text about the two counterexamples being of equal value.

RME: Well, epistemologically, if an example refutes any of the two conditions (non emptiness, bounded-aboveness), it is good enough, right? It *is* a counterexample?

M: Still, pedagogically the latter would be a better one. And beyond this psychological argument there is the issue of what is the role of a counterexample: its illuminatory powers. But as I said before to me it is important to know about the emphasis on the two conditions of the definition in appreciating reactions to these counterexamples.

SPECIAL EPISODE SE3.1: 'SCHOOL MATHEMATICS, UK'[51]

Setting the scene: Across their discussions M and RME often return to the students experience of school mathematics and the impact of this experience on the students' encounter with the requirements for rigour in university mathematics – in the UK[52]. Below is a synthesis of these discussions. In what follows the proof discussed in E3.5 is the starting point.

RME: Talking about what they have seen at school, this proof could be the only one they have seen.

M: …which makes me wonder what would be the point of doing that: what does exactly *having seen one proof* means and where does *having never seen a proof* leave the rest of the population. But I guess this is the natural outcome of giving up Geometry. Few doable proofs are left at this level if you leave Geometry aside. Students still experience some Co-ordinate Geometry but it is mostly Trigonometry. There are some opportunities for Proof by Mathematical Induction e.g. in proving some binomial formulae but this is rarely done explicitly – and how many students have been taught Mathematical Induction anyway? The obvious place for a proper encounter with Proof is Geometry.

RME: There is some implicit experience of proving in algebraic manipulation, starting from one side of an equality and ending up on the other. Which is, I think, what Student WD (E3.5, Scene I) is sort of up to.

M: I agree but in most cases students find it hard to see the mathematical sentence implicit in an algebraic expression. And to see that this expression needs to be embedded within words and with conditions under which it applies expressed verbally (E4.4). I also think that few students will have seen this proof. Perhaps in Further Maths, not A level Maths. Which leaves me thinking that, on the basis of what the students' background is, all you can expect from the word *Proof*, when bereft of all Geometry, would probably be some algebraic manipulation – even if you are using, in some vaguely defined sense, the words *theorem, statement, proof*. Moreover, given the culture of school mathematics, you cannot

[51] For a condensed presentation of most of the issues in this episode see (Nardi et al, 2003). For some of the tensions characterising the views of mathematicians on the school curriculum see, for example, (Vassiliev, 2004). A subtle and very worthy, in my view, amalgamation of propositions of how some of these tensions can be resolved is (Sierpinska, 1995).

[52] A significant recent report (2004) is *Making Mathematics Count*, the Report of Professor Adrian Smith's Inquiry into Post-14 Mathematics Education. 'Mathematics is of central importance to modern society. I have therefore found it deeply disturbing that so many stakeholders report a crisis in the teaching and learning of mathematics' is amongst the lines Smith uses to introduce the Report…

expect the students to wonder about the existence of irrational numbers in the same way this question would be fundamental in a culture like the Ancient Greek one – or would not be in a culture like the Ancient Chinese one. I am not even sure this issue of existence is what the question intends to raise here. Maybe this question simply aims to demonstrate one example of Proof by Contradiction. If this is the case, please allow me to think that this intention simply goes over the students' head: they are probably left with the fact and do not necessarily perceive it as an important idea of proof as a notion. Also, since they probably have already encountered the fact that √2 is irrational in the context of other problems, the whole exercise is probably devoid of meaning to them[53].

RME: Seeing the theorem that √2 is irrational is important also because on their calculators they will have 'seen' this as a decimal number[54]? I doubt whether all this subtext comes across though…

M: They are usually aware and comfortable with the idea that √2 is irrational but then again this may be just a vague feeling that there is something more complicated about √2 than meets the eye and that's it.

RME: Back to the issue you raised earlier, of algebraic manipulation. I think Student JWT's response (E3.5, Scene I) is an extreme example of adherence to that mode of thinking.

M: A proof for all seasons! It proves anything you want to say… cool…*and* its negation! This is a powerful thing we mustn't let out of the room! Its inconsistency is its one true thing! Sorry… On a less excited note, this is a good example of something else that has been said: us battling with the word *to prove* (E3.1). What is the meaning of this in English? *Something you test* seems to be this student's interpretation and she is having a go at this. And the basic flaw in this attempt lies in the leap from *or* to *and*, and the student's desperation to reach a contradiction after all this calculating effort (E3.5, Scene I).

RME: *Finding contradiction at whatever expense* would also be my consolidation of the situation here too. Even though I was intrigued by the originality of the first few lines to start with! I seriously thought it was getting somewhere! Or being open to new ideas often makes me gullible to students' writing…

[53] See earlier references, for example to Healy & Hoyles (2000), with regard to students' perceptions of necessity and nature of proof. In the case M discusses here the students' sense of personal conviction about the existence of √2 would interfere with their need to prove its existence.
[54] See E4.3 and E5.2 for some discussion of the role of calculators in the students' construction of mathematical meanings.

M: There is probably some way of getting through this that involves some manipulation of the quadratic ending up with a smaller terms solution or something. But back to the issue of students realising the significance of proof, I have mentioned elsewhere (E3.5, Scene II) about the step they need to take at this stage towards the desire to know why something is true and about how the focus on algebraic manipulation in school, not proof, may lead away from what is essential about mathematical reasoning, namely proof. And the lack of this particular sort of experience may explain why they find proof so difficult – and the limited popularity of proof oriented courses[55]! We need to justify proof to our students: proof helps us modify tools and arguments to our advantage. They don't ask again once this rationale for proof has been given to them[56].

RME: The importance of knowing why!

M: Proofs give us the mechanisms to getting to the why and I would hope that knowing why has some value per se[57]. This is a major difference between school and university mathematics.

RME: I would also say transferability of technique is another reason for knowing why something is true: the mechanism could be used in analogous arguments of other statements.

M: Convergence tests are a case in hand. But beyond pragmatics one would hope the beauty of knowing why is sufficient per se to convince of its value. And the pragmatic rationale can be a stepping stone to appreciation. I would also suggest that an explicit juxtaposition of scientific – inductive, experimental – and mathematical methods can enhance student appreciation.

RME: There is a lack of such an appreciation and I consider this as a loss. May I invite your comments on the fact that students in school do not have many opportunities for such an appreciation and therefore may be in for a surprise when they encounter university mathematics?

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[55] See Chapter 1, Part (i) for a brief discussion of students' disaffection from mathematical studies at post-compulsory level.

[56] Difficulties with mathematical reasoning imply that students may, in view of these difficulties, avoid formalisation, but they do not imply that students demonstrate no cognitive need for conviction and explanation (de Villiers, 1991). More specifically students seem to acknowledge the need to provide proof for their claims, especially in assessment-related contexts, such as examinations, relatively soon (Almeida, 2000).

[57] Amitsur in his interview with Sfard (1998a) – to which I return several times in Chapter 8 – also seems to propagate the idea that mathematics is for all.: because 'mathematics is the language in which modern society speaks', p453, and because use of logic is essential in everyday life.

M: I agree. There is a transition. Students arrive with much experience in computational mathematics. I can exemplify from Differential Calculus: the techniques the students know about do not include the Chain Rule, for example. So you have to teach what they do not know about and start from the mathematics they know about, be it computational. They arrive without a first inkling of proof. So their reasons for choosing mathematics are mostly to do with the view of mathematics they acquired in school. Therefore they find Year 2 (in the Scottish system for example), where proof begins to loom, difficult. If I stick to the example of the Scottish system for a moment, I can demonstrate to you how it accommodates the transition from school to university more smoothly than the English system: transition in teaching practices, content, environment, campus life, facing lecturing as a learning method (where even imposing lecturers like me may need to learn to tolerate questioning from the students!) etc.. with a Year 1 that focuses on this transition. So the culture shock is more gradual.

By the way, talking about this transition I find the regular exercise of browsing school textbooks in order to stay in touch with students' previous knowledge very informative – and often devastating. This knowledge mostly consists of instrumental elements: there is little effort in these books to try to create … concept images (E7.3). It is just: you do this and then you do this and that's it'. And if the student has any sort of personalized attitude to what they have just being doing I think this is of no concern whatsoever in these books. Also, and besides the algorithmic character of the syllabus, there is another substantial difference: in the old days a bad teacher or a bad student did all the syllabus badly, and had in fact seen all of it. Whereas nowadays the system pressures the good or bad teacher and the medium student to actually just omit sections of the syllabus, in their entirety. So students at A level may do chunks of the syllabus and at GCSE is even worse. If you are aiming for the B/C threshold or the D/C threshold you just omit chunks of the syllabus[58]. This is more significant than thinning the syllabus.

[58] See http://www.mathsinquiry.org.uk/report/chapter-3.html for a description, and reflection on some consequences of, the tiered GCSE system. (GCSE, the General Certificate of Secondary Education, is the qualification with which students leave school at 16. According to their projected performance in this final exam students will be allocated to sitting one of the three, for mathematics, papers of varying difficulty). Says Adrian Smith (2004): 'GCSE Mathematics has had overlapping tiered papers since its first examination in 1988. Pupils cannot be entered for more than one tier in any given examination period. From 1998, most major entry subjects, with the exception of mathematics, have been examined through a Higher Tier covering grades A*–D and a Foundation Tier covering grades C–G. Mathematics is the only subject to have retained more than two tiers. A small number of subjects, including art, music, PE, and history, have one tier. The intent of the three-tiered papers in mathematics was to cover a range of GCSE grades, so that candidates can attempt questions that are matched to their broad ability and enable them to demonstrate positive achievement. (the Foundation Tier awards grades D, E, F and G; the Intermediate Tier awards grades B, C, D, E; the Higher Tier awards grades A*, A, B, and C'.

RME: Is this in some ways an immediate consequence of teaching in sets in primary school?

M: Well, yes and no – generally the lower sets in primary school mathematics don't end up doing A-level mathematics. I am more concerned about people who will come out with an A-level mathematics at grade B with a chunk of the syllabus that they never saw. And it is a shame that students are offered a linear and often disconnected view of mathematics. I think that they are missing the connections between topics that they are being taught at the same time.

RME: So why would I choose to study mathematics at university level if I had been kept away in school from an understanding of what is at the heart of it?

M: It is a pragmatic decision based on the employability of mathematicians in other professions[59], not based on a desire to be a professional mathematician. And you probably enjoyed that slice of computational mathematics you experienced at school which is in itself a good indicator that you may enjoy the more advanced manipulation that is proof. And let us not exaggerate the gap between school and university: there are more similarities between school and university mathematics than this conversation may suggest – and which seems to be taking on the perspective of a pure mathematician than an applied one. Being able to do it, rule following, solving equations etc is not that different to doing proof by rule following. I can resort to my own schooling for examples. Plus the mathematical way of thinking runs through other disciplines such as Physics and Chemistry and I have always thought the ways of doing all of those are quite similar. But then again there we go – a classic case of contaminating the thoughts of a pure mathematician with the thoughts of an applied one! On the case in point, may I also point out that the simplicity of the idea behind the mechanism of Mathematical Induction should imply its presence in a school mathematics classroom? Yet it does not. I recall seeing this at school early and being fascinated.

RME: Oh yes! But Mathematical Induction has disappeared with a whole lot of other reasoning-related parts of the curriculum.

M: The fact that often it cannot be recorded in exams is actually the reason why it has disappeared, I think. It is the consistent failure of the students at these exam questions that led to their removal first from the exam and then gradually from being discussed in the classroom: why discuss something that will almost certainly not appear on an exam paper? In a modern school there is no marginal

[59] See Chapter 1, Part (i) for a reference to this.

effect on the students in the D/C border-line[60]. Proof is an ornament that is a waste of time. To interest the students in doing anything that doesn't immediately contribute to them getting an A is just a waste of breath. I think this is also the picture that practitioners would sketch for us[61]. In fact I am trying to think of anywhere else in any other subject, say in the sixth form, where one learns something as powerful as the notion of Proof as A level mathematics used to be done. With this gone pre-eighteen students have lost something major. I think it is quite bizarre that we have to re-introduce – or indeed introduce! – Proof in Year 1: isn't experiencing the way of thinking mathematically at A level what brought students to finding mathematics so beautiful in the first place and wanting to do it? Supposedly… In fact I suspect they have never seen what might be beautiful. And one of the things I am mostly worried about is that if, overall, school graduates know less mathematics, then mathematics teachers will know less mathematics and find things like the mechanism of Mathematical Induction very hard[62]. In sum, students nowadays find something like a proof of Pythagoras' Theorem, say the Indian Proof, too hard and irrelevant and as a result teachers – themselves not finding these things too appealing either – find them a waste of time and are unwilling to teach them. What a thorough catastrophe…

RME: I would sadly agree. I have been told by teachers too that the algebraic manipulations within the Indian proof of Pythagoras' Theorem are out of the reach of GCSE students. But may I ask you to consider a hypothetical scenario: if you were to make an argument for a reintroduction of this type of proving technique, Mathematical Induction for example, in school, what kind of argument would you use? Simplicity is one?

M: Simplicity is one: I recall how simple and beautiful and illustrative was my first encounter with the proof for the sum of the first n integers. I recall feeling moved. I recall thinking *hey, this is something new and it's called mathematics*!

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[60] See earlier note that explains the tiered GCSE system to which M's comment refers to here.

[61] A point extensively discussed in the aforementioned Smith Report.

[62] Knuth (2002) examined secondary school teachers' perceptions of proof following the five-fold theoretical framework that discusses proof as (1) verification, (2) explanation, (3) communication, (4) discovery / creation, (5) systematisation. With regard to (1) the teachers demonstrated belief in this but were less confident about their belief in the generality of proof. Regarding (2) they expressed an appreciation of procedural explanations more than relational or as insight into why a statement is true. Regarding (3) the social aspects of proving and accepting proof emerged. Regarding (4) and (5) there was less evidence of this in the teachers' views. What constitutes proof for teachers then? They were given various arguments and asked to rate them as proofs or not. Their confidence and success varied. Their criteria included: valid methods, mathematically sound, sufficient detail, knowledge dependent. Furthermore what teachers find convincing is concreteness, familiarity (with method from previous experience), generality and capacity to show why. Overall a certain lack of robustness and some reliance on empirical evidence was observed. In conclusion, Knuth proposed that, to reinforce these teachers' views of proof, better contact with proof, also more personalised, during their university education (and preparation for teaching) maybe required.

Gauss was just ten when he had his first encounter with this! But the answer to your question would depend largely on the audience you are addressing your argument towards: in the UK the utilitarian approach has been tested to destruction, and no-one is doing mathematics, or engineering or physics or chemistry[63]. So, if the initiating of this question comes from someone who is concerned about the fact that no one is doing mathematics, then there is a reasonable answer. If this comes from someone who doesn't care that no one is doing mathematics then there is no point talking to them. If they are concerned about the fact that no one does mathematics, I cannot stress any harder the fact that we have tested utilitarian education to destruction. So, I would ask the interested interlocutor, why not let someone with some scholarly view have an input in this catastrophic situation? Such as mathematicians themselves[64]? In which case it is obvious why you teach things that the Greeks found fascinating. But the fundamentally sad element of the situation is that there is a need for a cultural change. Then the question I just formulated can be asked again. In an ambience where market rules and the value of a university course is estimated in terms of salary increase for graduates[65], I think it is futile to even ask questions like these.

RME: I share your emotional and intellectual surprise when I recall my first encounter with Proof too. But then what about the rest of the 99% of the population who were not fascinated and carried on not only not doing mathematics but actually disliking it thereafter?

M: I guess making mathematics intellectually attractive is an issue here, isn't it? Making the profound impact it had on people like you and me at the age of seventeen or whatever available to kids today. But, hell, isn't that an almost impossible task? Here is an ironic situation I am experiencing here in the UK: funding would be easy to get for organizing mathematics days out to our university from local schools. If however I asked for a reconsideration of the mathematics pupils experience in school, the response would be totally different. Even though the cost could be virtually zero, right?

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[63] See Chapter 1, Part (i) for a brief account of, and relevant references to, this situation.

[64] See Chapter 8 for a discussion of this previous involvement by mathematicians. (Steen, 2004) discusses the up-surging interest in this involvement and makes some suggestions on what mathematicians can contribute towards the improvement of school mathematics education. These include: mathematical preparation of teachers; preserve momentum for high standards by advocating policies that judge students etc. using multiple criteria and not single high stake tests; advocate balance not only of content but also of strands of mathematical proficiency (conceptual understanding, adaptive reasoning, procedural fluency etc.); and, showcase the power of mathematical thinking by creating exemplary problems that convey this power. In the UK the recent establishment of the Further Mathematics Centres that aim to promote mathematics in schools through systematic collaboration with local mathematics departments is an initiative pretty much in this spirit.

[65] See http://www.prospects.ac.uk

RME: Well, not zero… for example, a change in the syllabus would have repercussions for teacher training in terms of adjusting the courses etc.. Especially if it involved an enrichment of the syllabus. Let us not forget that currently only about 40% of secondary mathematics teachers in the UK have a degree in mathematics[66]. The training of those people to a more mathematical curriculum is costly…

M: … and of these 40% not all mathematics degrees are the same, right? I shall name no names but there are some pretty lower quality institutions. Plus let us not hide the fact that the majority of mathematics graduates joining the teaching profession in the UK are mediocre[67].

RME: This is not the case in all other countries where good mathematics graduates are often encouraged to join the profession …

M: … and where also mathematics teachers are highly paid and respected in society. Do not even get me started on the issue of what is the social and media status associated with the teaching profession in this country[68]!

RME: You mentioned something about the attraction to mathematics and using the appeal of certain topics to lure students into the subject. Has this virtually disappeared from school mathematics?

M: It certainly has. And in a way mathematics appears less attractive because the audience is much tougher. We grew up in a time where expectation times were longer and delayed rewards were more acceptable. And, I think it is harder and harder to sell the idea that it may take ten minutes of thinking before you get that reward for something aesthetically pleasing…

RME: …an analysis that goes beyond mathematics….

M: …and to which I would also add, emphatically, a difference between mathematics and other subjects. Other subjects can effortlessly slide down that scale whereas mathematics cannot. We don't have a smooth slide. We have an extremely rocky sequence of bumps that we go down, conceptual bumps. So I think the way in which we adjust how you teach, say film, you just effortlessly accommodate this decline in attention span, decline in literacy, the fact that the intake doesn't read anything substantial at all. You just accommodate that with

[66] See details in the Smith Report (2004) at: http://www.mathsinquiry.org.uk/report/chapter-2.html
[67] See previous footnote particularly the section on the 'short of specialist mathematics teachers'.
[68] At the time of writing Cambridge University's *Teacher Status Project* was nearing completion. For preliminary results and discussion: http://www.educ.cam.ac.uk/status/#teaching

small changes here and here and here. Whereas for us it is a bit all or nothing, one disaster follows another and each of these disasters implies large conceptual gaps in the knowledge of our intake. I don't know a way around it but that is the problem. Mathematical knowledge just doesn't come in this soft chain[69].

RME: This is maybe part of the disappearance of an emphasis on reasoning from school work in general?

M: The case of Grammar and Syntax is another loss: Grammar and Syntax provide you with the ways in which arguments can be put together – the power behind the capacity to construct a sentence knowing where the main clause lies and how it connects with subordinate clauses! Probably this was just one of these things which happen to be a little bit more difficult so it has also been removed.

RME: I was told recently by a colleague that students come to university without any understanding of structure.

M: I agree and I wish that recent government initiatives such as the National Numeracy Strategy for both primary and secondary education[70] will start to reverse a situation that seems to have bottomed out… Anyway I couldn't stress more the benefit in speaking in what can be perceived as slightly unnatural but full sentences, with pauses for thought and with encouraging the students to exchange such sentences amongst them (E7.1, Scene I). There is ample space in the curriculum to practice writing in delightful, popular topics. But of course that implies grammatically literate teachers too… How more ardently can I stress that the personal well-being of our students will be improved, not simply in salary terms, by an enhanced power of persuasion in speaking and writing? The issue of developing ways of convincing is crucial, seeing that mathematics offers you ways to construct convincing arguments – for yourself and for the reader – and that there is power and pleasure in that[71]!

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[69] Davis & Hersh's (1981) account of this remains amongst the most lucid. See also (Flato, 1990).
[70] An initiative (http://www.standards.dfes.gov.uk/primary/) taken by the UK government with the aim to raise the reportedly falling standards of mathematical learning in primary and secondary schools.
[71] For further elaboration see, for example, (Flato, 1990).

SPECIAL EPISODE SE3.2: 'INEQUALITIES'

Setting the scene: The following takes place in the context of discussing the question in E3.6. . In their discussion M and RME touch upon how certain algebraic difficulties, in this case the manipulation of inequalities, may obstruct the students' construction and presentation of a mathematical argument.

RME: Can I come back to what you said earlier about Student JU: all the correct lines in the script but in the wrong order. Would it be different if this were an equality, not an inequality? The student would simply start from one side and end up on the other? What do you think?

M: A good one would. Most would start from what is to be proved and try to reason about it, reason backwards, hoping that these backward steps are allowed. Which is where the difficulty with dealing with inequalities lies[72]: a good sense of direction is crucial in order to avoid assuming what you are trying to prove or taking forbidden backward steps. In this sense this set of exercises is good because it allows all the usual wrinkles of reasoning to show up. Here is another one: I love this *let P(n) be the statement* $2^n \geq n^2$ in Student JR's response (cited within E3.5, Scene II). This is a virtually incorrect statement as it misses the $n \in N$ *where* $n \geq 4$ bit – which brings to my mind how incredibly difficult is for students to see that a statement is not necessarily true, that it has the potential to be true or false (or inconclusive as in E3.4). I think it comes from school where they only see true statements, they are never asked to prove or disprove anything (SE3.1). And they don't like that ambiguity, this potential for either truth or falsity to hold. But, returning to the issue of inequalities, I recall beginning to grasp how I ought to be teaching them by osmosis or by experience but often being troubled about what I need to say to students about what they are allowed to do with them. Students need to practice with inequalities a lot: simple ones, one that involve absolute values or quadratics etc. Jordan's inequality is an example where the graphing of the curves and seeing where the direction of an inequality changes can be helpful but overall the area is full of traps.

RME: … and I need not even mention the case of definitions such as convergence in Analysis where a good grasp of inequalities is absolutely crucial… (E6.1, E6.2)

[72] An issue usually considered in the wider context of students' difficulties with algebraic manipulation (Sfard & Linchevski, 1994).

SPECIAL EPISODE SE3.3: MATHEMATICAL REASONING IN THE CONTEXT OF GROUP THEORY[73]

Setting the scene: The following takes place in the context of discussing the question below as well as two examples of students' written responses to this question, Student H's and Student E's. In their discussion M and RME touch upon a variety of conceptual issues revolving around the students' encounter with the novel concepts of Group Theory and how this novelty may interfere with their construction of mathematical arguments.

Example from Exercise Sheet 3, Week 6, group Theory Course, Question 2

Let $\phi : G \rightarrow M$ be a map from the group G to the group M. When is ϕ said to be a homomorphism? How is the kernel of ϕ defined?

(i) Let ϕ be a homomorphism with kernel K. Show that the order of $\phi(g)$ divides the order of g, for all $g \in G$.

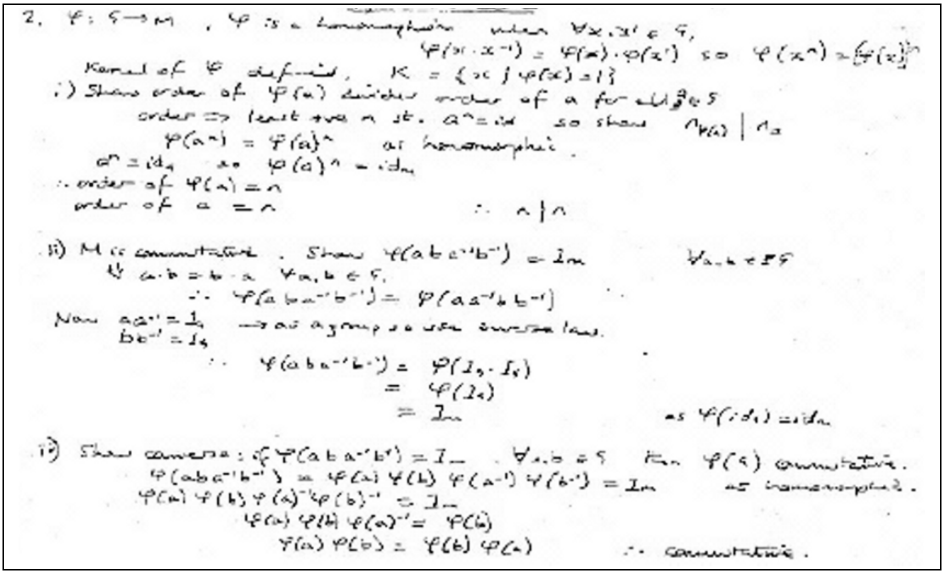
(ii) Suppose that M is commutative. Show that $\phi(aba^{-1}b^{-1}) = 1_M$ for all $a, b \in G$.

(iii) Show the converse: If $\phi(aba^{-1}b^{-1}) = 1_M$ for all $a, b \in G$ then $\phi(G)$ is commutative. (The group $< aba^{-1}b^{-1} : a, b \in G >$ is the *commutator subgroup*. One can show that it is a normal subgroup.)

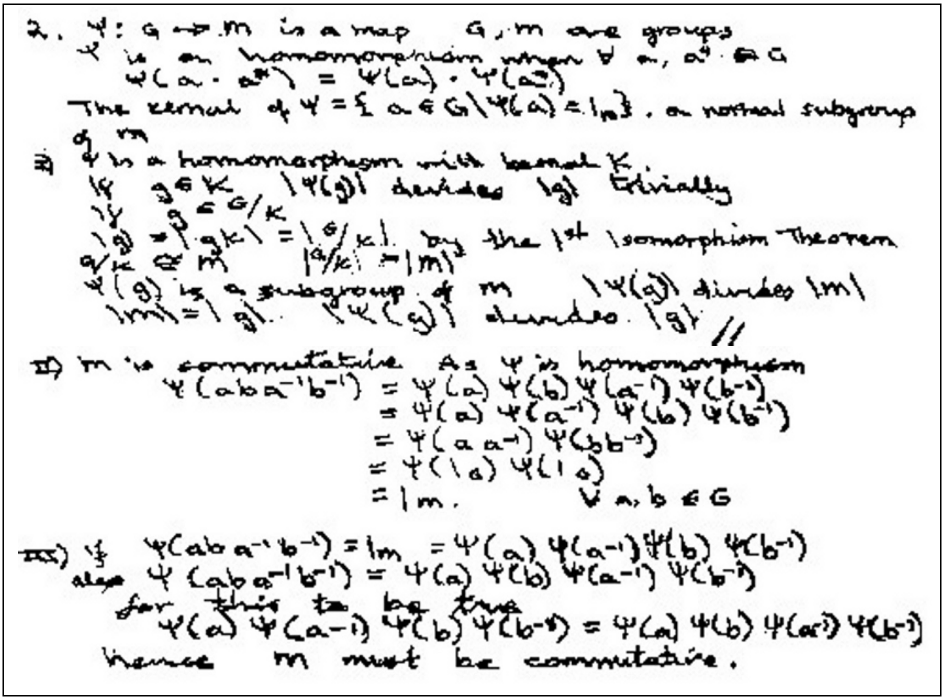
Notes on Solutions

See your notes for the first two items. (ii) If M is commutative we see that then $\phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a^{-1})\phi(b^{-1}) = \phi(a)\phi(b)(\phi(a^{-1})\phi(b^{-1})) = \phi(a)\phi(a^{-1})\phi(b)\phi(b^{-1}) = 1_M$, by commutativity of M. (iii) Note that $\phi(aba^{-1}b^{-1}) = 1_M$ iff $\phi(a)\phi(b) = \phi(b)\phi(a)$. If ϕ is surjective then any $x, y \in M$ is of the form $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in G$.

[73] The discussion here suggests, in the spirit of Dorier et al (2002) who argued this albeit in the context of Linear Algebra, that context / topic specific difficulties interfere with the student's completion of the argument. See Dubinsky et al (1994; 1997), (Hazzan, 1999) and (Nardi, 2000a) for an account of some of these difficulties.



Student H



Student E

The discussion touches on several student difficulties with Group Theory concepts that appear to interfere with the students' attempts at mathematical reasoning when engaging with Group Theory questions.

M: This is a reasonable question I think – much more so than the one we discussed earlier (E3.2, Scene II). Even though it contains substantial elements such as homomorphisms and proving the converse of a statement. The order of an element is a problem for Student H, missing the *least* condition in the definition. Isomorphism too. Showing the orders dividing is another thorny issue. But overall her response makes some sense, don't you think? Maybe not part (ii) but certainly part (i).

RME: I was struck by the student applying commutativity on group *M* whereas the property applies on group *G*. I often see problems with domain, range etc. in students' work (E5.1).

M: Well, life would be easier if *G* were commutative so the student goes for this wishful thinking route, thinking backwards. I often wish students had some artillery of examples to cope with the what-if implications of their claims and see that some of them simply do not stand scrutiny (E3.2, Scene I).

RME: I recall that, as a student, if a conclusion followed almost trivially, I would be suspicious and wonder whether it is possible for the question setter to ask for something so trivial. Of course this gets you into trouble if *are* asked something simple!

M: But this material is now perceived as too difficult for the students – so no chance of them seeing as trivial anything at all we ask of them! Therefore the material needs to be removed or transformed.

RME: I am hearing this a lot actually. And Group Theory seems to be at the heart of this discussion on syllabus adjustment. What do you make of Student E's response?

M: I am astonished to discover that the student has been confusing the quotient symbol with the difference-of-sets symbol! Still her part (ii), albeit more mechanical, is ok. And there is a certain potential in splitting cases for *g* being or not being in the kernel. This could be a reasonable approach. Or maybe not…

RME: There is also a subtlety in the script: she uses different symbols for the identity elements in groups *G* and *M*.

M: At least some drilling in the lectures works! I am bit skeptical about the *according to the First Isomorphism Theorem* line. I am not sure it makes sense. Order of group and order of element are often confused.

RME: The symbol does not carry well: the abbreviation order of g actually means the order of the cyclic group generated by element g[74].

M: Oh, well! We cannot change the language of the books and the students need to learn how to understand those, ok?

RME: Of course but we may need to clarify this more emphatically when we teach this topic.

The discussion concludes with looking at another example, Student LF (three pages, omitted here) where the Greek letter *phi* has been replaced by the Latin letter *p* or Greek letter *rau*. And where the Euclidean algorithm has been used (but with no assumption about whether *n* is greater than *m*). M finds this 'more promising – even though there are some errors in the manipulations' and concludes that this is not a bad question, unlike the one discussed in E3.2, Scene II towards which he had expressed reservations.

SPECIAL EPISODE SE3.4:
'ALGEBRA / GEOMETRY'

Setting the scene: In what follows M and RME discuss the question below as well as two examples of students' written responses to this question, Student J's and Student JU's. The question attempts to link matrices with vectors and systems of equations, one of the important links between Linear Algebra and Geometry which often remain elusive to students.

[74] See (Nardi, 2000a) for further substantiation of this issue.

Example from Exercise Sheet 3, Week 6, Autumn Semester 2000

For a natural number m let R^m denote the set of column vectors with m real entries (i.e. $m \times 1$ matrices). Show that if $v_1, v_2, v_3, v_4 \in R^3$ then there exist $r_1, r_2, r_3, r_4 \in R$ which are not all zero and such that $r_1v_1 + r_2v_2 + r_3v_3 + r_4v_4 = 0$, the zero column vector. (Hint: Write this as a system of linear equations in the variables r_i and use Theorem 2.2 from the notes.) Can you generalise this result to R^m?

Notes on Solutions

The general result is that if $v_1, \ldots, v_n \in R^m$ and $n > m$ then there exist $r_1, \ldots, r_n \in R$, not all zero, such that $r_1v_1 + \ldots + r_nv_n = 0$. Here is a proof (the first part of the question is the special case $m = 3$, $n = 4$): Write $v_j = \begin{pmatrix} a_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{pmatrix} \in R^m$, for $j = 1, \ldots, n$. Then the equation $r_1v_1 + \ldots + r_nv_n = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$ is a system of m linear equations in the n variables $r_1, \ldots, r_n$:

$$\left.\begin{array}{c} r_1a_{11} + \ldots + r_na_{1n} = 0 \\ \cdot \\ \cdot \\ \cdot \\ r_1a_{m1} + \ldots + r_na_{mn} = 0 \end{array}\right\}.$$

These are homogeneous and $m < n$, so by Theorem 2.2 there is a non-trivial solution i.e. one in which not all the $r_1, \ldots, r_n$ are zero.

Let $v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$, $v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$, $v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$, $v_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}$

If $r_1v_1 + r_2v_2 + r_3v_3 + r_4v_4 = 0$
and $r_1, r_2, r_3, r_4 \in R$ and are not all zero

This implies that

$$r_1\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + r_2\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + r_3\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} + r_4\begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} r_1a_{11} + r_2a_{12} + r_3a_{13} + r_4a_{14} \\ r_1a_{21} + r_2a_{22} + r_3a_{23} + r_4a_{24} \\ r_1a_{31} + r_2a_{32} + r_3a_{33} + r_4a_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

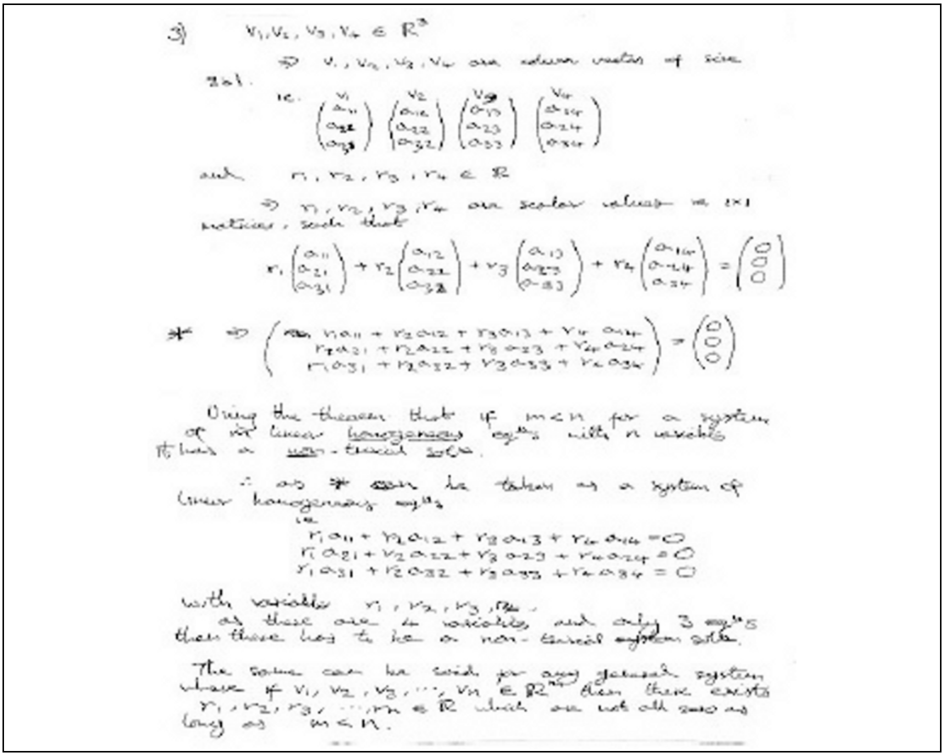
Which shows r_1, r_2, r_3, r_4 are the variables
and for a system of m linear homogeneous equations in
n variables if $m<n$ there is a non trivial solution.
The system of equations is

$$r_1a_{11} + r_2a_{12} + r_3a_{13} + r_4a_{14} = 0$$
$$r_1a_{21} + r_2a_{22} + r_3a_{23} + r_4a_{24} = 0$$
$$r_1a_{31} + r_2a_{32} + r_3a_{33} + r_4a_{34} = 0$$

$\therefore$ $m = 3$
as r_1, r_2, r_3, r_4 are the variables $n = 4$

$\therefore$ $m < n$ as $3 < 4$
So there is a non trivial solution.

Student JU



Student J

M and RME explore possible rationales for the question and RME wonders whether the question was intending to encourage some use of visualisation on the part of the students.

M: A possible rationale is that there may be a way of approaching the question by visualizing the vectors in the three dimensional space in order to see that if four vectors are in a three-dimensional space then one must be linearly dependent on the other three. But I do not see any diagrams in the student responses or the lecturer's so I guess this perspective was not taken by many?

RME: You are right, it was not.

M: I guess underlying the question are tacit notions of basis and dimension leading up to the central idea that having more variables than equations, there must be more than one solution. But isn't this a bit of an eccentric way of initiating a discussion of this? If I attempted a pictorial explanation of the same point, I would probably come up with something like row–reducing the matrix results in running out of rows. Therefore there must be more than one solution. Which is more or less what the quoted Theorem[75] 2.2 says, right?

RME: Yes. I think behind all this is the intention to link matrices and linear equations, a link that seems to be missed by the students in their writing[76].

M: I wouldn't be surprised if the abstraction in the question led the students in not seeing the intention and the relevance of all these things at all. And I am not sure there is much gain from writing down all these indices, they do not seem to serve much of a purpose here so it is likely the students have been resisting any essential engagement with the task because they do not see the point.

RME: The two systems, equations and matrices, have representation systems that may appear unrelated to the students.

M: I am ambivalent about how to determine whether the student responses are correct: this could be a one line proof if the students are allowed to use Theorem 2.2 from the course. Was the intention here that the students learn something about transferring their thinking between R^3 and matrices? I do not see much of this in the notes on solutions. Student JU has attempted something along those lines, hasn't he? He expresses the relationship among the four vectors in a list of equations. Beyond quoting Theorem 2.2 I cannot see what else one can do here. Or maybe this is not so bad: so often students find it difficult to write out correctly *mxn* matrices with all the indices involved[77]. So, given that there is not much else to do and that they have a very initial knowledge of row operations at this stage, perhaps this question intended to prompt the students to practise exactly that. And the application of a theorem that appears to have been proved elsewhere. On those grounds Student J seems to know what is going on and, if asked, probably a reasonable proof would emerge. I still wonder however whether students at this stage see these things in terms of linear (in)dependence. Probably they can make sense of linear (in)dependence in R^3 but not necessarily in R^n. And this leads me to ask what is it then, in the absence of a linear (independence) perspective, that the students learn here. Perhaps it is what we said earlier about the content and application of Theorem 2.2? But not much about why this theorem works? The fact of the theorem is a fairly deep fact and I am not sure this is the way to illustrate this depth or any thinking about Rank and Nullity. My guess is that faced with all these subscripts and this to-ing and fro-ing between matrices and vectors, students may grind to a halt or have no clue as to where the answer is complete: this is the sort of question with which they

[75] A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

[76] See relevant reference to (Burn, 2003) in E7.2.

[77] See Chapter 4 for further elaboration on these issues.

knock at your door, show you and say: *and am I done now?*! Both Students J and JU have more or less answered but I think Student J has moved more towards establishing the generalisation to (m, n). *The same can be said* is more or less fair because there is nothing substantially different between the proof for 3 and the proof for n. The natural thought for many of them is: *if this is the case, why should I then have to work all this out all over again for the n case?* And of course the question asks for a mere statement of the generalisation, not its proof. They are not asked in my view to do anything in particular! I think the intention here is to place the emphasis on this fantastic connection between geometry and solving systems of equations, on this beautiful interpretation about a potential perspective on vectors.

RME: Can we discuss a bit more this link then?

M: Sure. I think Student J makes this link: look at the *as there are four variables and only three equations then this has to be a non-trivial system solution* bit. And to a satisfactory degree Student JU too. The situation is not too uncomfortable for them anyway because it is about things having solutions and most of their experiences are where there are as many variables as equations. So this is an opportunity to consider a scenario where there maybe more than one solution which is what they are used to.

RME: Is there any other concern you would like to share?

M: Yes, just briefly I would like to stress my concern that we cannot assume that the students go to the mxn case unproblematically[78]. At this stage an exercise in doing so would be good for them, even in terms of simply practicing a precise use of the indices. Also a final concern: I am not sure that seeing the $m>n$ case only, without any reference to the other cases, to the total picture, is very instructive. They are getting a subset of the whole picture here without anyone saying *this is a subset of the whole picture*. So overall I would say I have quite a few objections to the question: there is an uncertainty of point of view here; linear independence is not in the picture as emphatically and openly as it should be; there is nothing more required of the students than to re-express the vectors in terms of equations and then quote Theorem 2.2 – therefore possibly without much understanding being necessary or resulting as, at this stage, the subtleties of rank, bases etc. probably fly over the students' heads and Theorem 2.2 is simply quoted, not discussed in terms of its significance. In a sense I would prefer the question to be anchored in the world of equations, linear independence etc. and with more emphasis on the link which – I repeat – is a point probably missed by the students; and, doing so by example, not by making these abstract statements.

[78] See earlier references regarding the tension between the general and the particular in E3.2.

CHAPTER 4
MEDIATING MATHEMATICAL MEANING THROUGH SYMBOLISATION, VERBALISATION AND VISUALISATION

'The limits of my language mean the limits of my world' (Wittgenstein, [1922] 2003 – Section 5.6) is perhaps an aphorism that the maverick philosopher re-thought in his later work (in which he attempted to be more inclusive of perceptions that he admitted as perhaps unutterable). However it is an apt introduction to the theme M and RME discuss in this chapter: students' attempts to mediate mathematical meaning through words, symbols and diagrams.

Mathematics is often described as the activity of constructing metaphors that reveal the answers to extraordinary questions relating to abstraction – itself a form of imagination (Bullock, 1994). It might as well follow then that if language is an expression of imagination then mathematics is the language of abstraction. Language is also often seen as a major determinant of communication: whether in its ordinary form, as the formal syntax and notation of mathematics or through visualisation, language is the architecture of thought as well as its carrier. Most of the discussion in this chapter focuses on these three forms of communication.

In the discussion that follows M and RME describe the students' attempts to mediate mathematical meaning: at this stage students experience the tension between the need to *appear* (use the norms of formal mathematical notation such as quantifiers etc.) and the need to *be* (use the norms of formal mathematical reasoning, such as providing proof etc.) *mathematical* in their writing (E4.0). Their attempts at producing what they see as acceptable mathematical writing often result in inconsistent use of the symbolic language of mathematics (what M calls 'gibberish' and 'random mathematics' in E4.1, Scene I and in the extreme case examined in OT4.1). They are often lost in the translation between the symbolic language of mathematics and ordinary language (E4.1, Scene II). Their writing is prematurely compressed, even gap-ridden, (both scenes in E4.2) and their understanding of the role visualisation plays in gaining mathematical insight ambivalent (E4.3 and SE4.1). Students seem at this stage to undervalue, even avoid entirely, expressing their mathematical thoughts verbally in their writing which does not yet achieve a balanced integration of words, symbols and diagrams (E4.4).

A note on the numbering of the Episodes in this chapter: numbering here starts from 0, not 1. One reason for this is that in E4.0, the Episode that opens the chapter many of the issues examined in Chapter 3 and Chapter 4 do not merely intersect; they are almost impossible to disentangle. I have therefore placed E4.0 within the chapter that it looks more, but not exclusively, comfortable in; yet numbered it in a way that suggests its close relation to the material examined in Chapter 3.

EPISODE 4.0
TO APPEAR AND TO BE[1]:
CONQUERING THE 'GENRE' SPEECH[2] OF UNIVERSITY MATHEMATICS

Setting the scene: The following takes place in the context of discussing the question below as well as two examples of student responses, Student J's and Student W's[3]:

Example from Exercise Sheet 1, Week 2, Autumn Semester 2000

Let $x \in R$ have the properties that $x \geq 0$ and $\forall n \in N$, $x < \frac{1}{n}$. What is x?

Suggested solution

Adequate answer for now: $x = 0$. Better answer: since $x \geq 0$, if $x \neq 0$ we must have $x > 0$. But if $x > 0$ then for n large enough (bigger than $\frac{1}{n}$), $\frac{1}{n} < x$, which contradicts the assumption about x. So x must be equal to 0.

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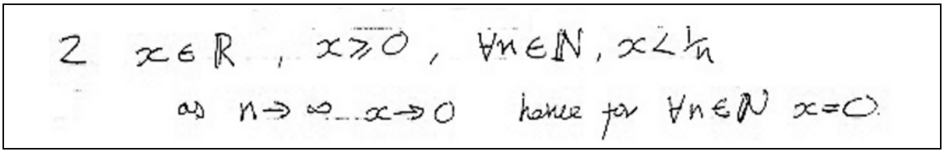
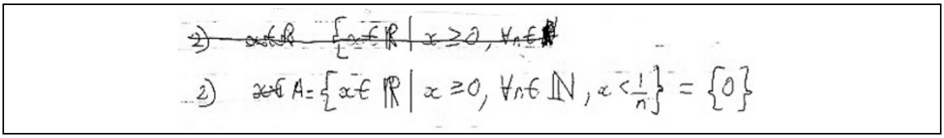
[1] Formal proof is the driving force and the aim of official mathematical communication. As any other communication it is characterised by a number of conventions the adoption of which is synonymous to a learner's advanced mathematical enculturation. The notion of enculturation employed here (see also Chapter 1, Part (iii)) originates in Michel Foucault's (1973) and E T Hall's (1981/1959) cultural triads. Hall, for example, recognises 'three types of consciousness, three types of emotional relations to things' (Sierpinska, 1994, p161): the 'formal', the 'informal' and the 'technical'. In the context of mathematical culture the 'technical' level is the level 'of mathematical theories, of knowledge that is verbalised and justified in a way that is widely accepted by the community of mathematicians' (p163). 'At the 'formal' level, our understanding is grounded in beliefs; at the 'informal' level — in schemes of action and thought; at the 'technical' level — in rationally, justified explicit knowledge' (p163**)**. Central to the purposes of this chapter are processes taking place within the informal level of Hall's triad. This is 'the level of tacit knowledge […], of unspoken ways of approaching and solving problems. This is also the level of canons of rigour and implicit conventions about how, for example, to justify and present a mathematical result' (p163**)**. A student's enculturation is treated here as taking place at the informal level: through the accumulation of mathematical experience shared with the expert and in the process of appropriation by an internalising/imitating the expert's cultural practices.

[2] In the essay 'The Problem of Speech Genres', Bakhtin (1986) proposes that 'genre', a construct typically used towards distinguishing between types of literature, can be also used towards distinguishing between ordinary language and specialist languages such as mathematics, science, law etc.. Bakhtin makes the distinction between primary genres and secondary genres: primary genres are those which legislate those words, phrases, and expressions that are acceptable in everyday life, and secondary genres are those which legislate which words, phrases and expressions are acceptable in these other extra-ordinary types of language, for example, mathematics. In this sense the students' contact with textbooks, their lecturers' writing etc is their contact with the secondary genres that legislate what are the acceptable ways of writing and speaking mathematically. The issues arising from this contact are the focus of this and most other Episodes in this chapter. For a preliminary discussion see also (Nardi & Iannone, 2000); for a review with a particular section on writing see (Ellerton & Clarkson, 1996; p1007-1021). Lithner (2003) and Pugalee (2004) also focus on how students relate to mathematical writing, e.g. in textbooks.

[3] Earlier versions of this episode appear in the Brazilian book mentioned in the Post-script; in (Nardi & Iannone, 2005); and, with a focus on the transition from school to university in (Iannone, 2004).



Student J



Student W

RME informs M that this exercise was given to the students in the second week of their first semester at university. The question does not ask explicitly for a proof that $x = 0$; it simply asks *what is x?*. Still, asks RME, can we infer anything regarding the students' beliefs about proof from their responses? In the student responses RME examined most students identified zero as the value of x; several students simply offered $x = 0$; others attempted a justification of the claim. RME then poses the following questions to M:

- What can we infer about student W's employment of set-theoretic language above? Why would he choose to write his answer in set-theoretic language?
- What can we infer about student J's employment of convergence symbolism above? Why would she choose to write her answer in such as way?

RME: To start off, as the question setter, what were you trying to achieve here[4]?

M: Well, I had several things in mind. Sooner or later they will be running into this kind of idea and I wanted them to start thinking about it. I wanted to flush out the people who imagined that x might just be an incredibly small positive number, smaller that anything that can be thought of. Which they do think still, some of them. Once a Year 1 lecturer said 'x might be something like one minus point nine recurring' and I thought this kind of comment was fantastic. It opens the door to the whole of Analysis! The other sort of motivation behind setting the question was that I was trying to do something that is less formal than presenting the Archimedean axiom at the beginning. I wanted them to convince themselves that they believe it without even hearing it stated. And also I wanted to flush out whether any of them somehow would see this as an opportunity to use an argument by contradiction. These are the things I was looking for.

[4] Amongst the participating mathematicians often sat the person who *had* set the question to the students. *In this Episode only* the use of 'I' refers to that person. The reason for this exception is that having the presence of the question setter provided a rare opportunity for discussion of the intentions themselves as well as how they were interpreted by the students at the time.

RME: So did they use an argument by contradiction?

M: Some did – but often only after consultation with their adviser. It didn't even occur to them that this was a method of proof (E3.5)! Or that this was a mathematical question at all!

RME: The question doesn't really seem to be asking for a proof though, does it?

M: A proof would need the Archimedean Axiom and, if the grounds are not clear, students may get confused about what is asked of them (SE3.4). I guess by suggesting in the sheet at least two ways of providing an answer that is satisfactory at this stage, I am more or less admitting that things are pretty ambiguous as to what is expected, at least at this stage (E7.4, Scene IIb)! One could come up with things like 'x could be infinitesimal' but then again, I am saying x is a real number and we have to assume that the game is fair even if not all the rules are there! Overall I would say the question is doing a slightly better job at developing intuition on real numbers, but doesn't necessarily help the understanding of the game of proof very much.... But you did say you wanted to discuss some students' writing, right?

RME: Yes, most answers were that $x = 0$ with no justification provided and I thought this could be a good place to look at their ideas at this stage of what is a mathematical argument.

M: I think that for this question most of them thought that just the answer was all right. I don't think they recognise the need for proof, but they recognise that we want them to give a proof[5]. Most of them do not just say $x = 0$, they try to say this in different ways, which is something more than just stating the answer. Still they do not necessarily know what to do. After all they are used to being asked to show their workings in school, not provide logical explanations (SE3.1). There is a switch from the one to the other that needs to take place at this stage of their studying. Something that I think is common to both the answers we are looking at here is that the students are having a go at giving something that looks like an argument but they are terribly greedy at using the symbols that are provided in the question and are reluctant to provide anything extra. Sometimes they seem to think *this stuff must be hard so we have to write longer* but most of the times they provide short answers when we, mathematicians, want more (E4.2).

RME: This seems to be a common problem with Year 1 students.

[5] See Chapter 3. Almeida (2000) observes that students seem to acknowledge the need to provide proof for their claims, especially in assessment-related contexts such as examinations, relatively soon. Here the focus is on how this realisation materialises in their writing or speaking.

M: It’s probably going on beyond Year 1. When I bring this to their attention they often say *well, maybe you have a point there*. What I mean is that a string of symbols is not a sentence in English: so, if you throw in a few words in English, why would you expect anyone to make sense of it as if it is in English (E4.1)? But we do speak and communicate mostly in sentences and when you confront the students with this fact they admit *well, we just drop the words which are sort of obvious or superfluous*. So I think there may be a misunderstanding here about what mathematical writing means.

RME: Where do you think this misunderstanding comes from?

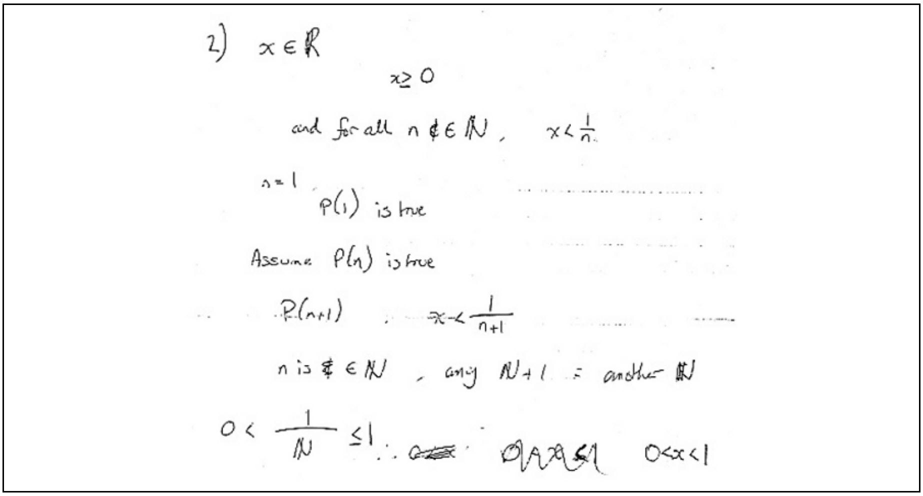
M: Maybe it comes from school where the English around the mathematical writing is irrelevant because it cannot be wrong or right (SE3.1). It is seen as unimportant, they have dropped the sentence in which the statement is embedded and therefore all they write down, stripped of syntax and punctuation, is in fact unreadable. *But* in school this kind of writing would probably get full marks. Once I confronted a student who produced a particularly terrible thing for an induction proof that had somewhere inside it the algebraic step that was going to be needed; and I said it was unreadable under all accounts. You know, this was not the proof of anything. And he said *oh, you know, this would be fine at school and I do it like that and I get full marks because the algebra is right*. Argh! And it is very difficult to get them to see the logical framework of a proof, the shape of an argument separately from the little calculations. Especially when the calculations are absolutely trivial and the triviality tricks you into circumventing the logical flow of the argument (E3.5, Scene II). I must admit that, as professional mathematicians, we do this sometimes when we present trivial results to students because we ourselves have compacted the argument into an apparently obvious concise statement. But they, the students, still have to understand the logic behind it all and therefore have to stick, for a while, with these often longer, unpacked statements (E4.2).

RME: Is that the case here with the more or less obvious fact that x must be zero?

M: Yes, I think so. Look at the extraordinary and very, very weird response of student W. Something is going on in his mind, just before that final equals sign. But he hasn’t made the point, he hasn’t written it down. Where is the sentence which makes the point underlying the equals sign? He was probably under the influence of some introduction of set theory notation in the first weeks of the course and casually drifted into using this notation. It is impressive also because it is syntactically correct but I suspect that they think that putting it somehow in mathematical language makes it more convincing.

RME: Another student, MR, went for presenting their point in terms of Mathematical Induction!

M: Jeepers! Now, this is essentially random mathematics. I am not even sure that by the end of this the student actually believed x is zero. This is probably taking things from recent lectures and applying them (3.5, Scene I). Or maybe it is a case of previous questions asking students to use Mathematical Induction and they arrive at this one assuming it will be another one of those questions and force their answer to fit the Mathematical Induction mould. There is a lesson in there probably about being careful with the order of questions in the problem sheets!



Student MR

RME: So why do you think they do that, trying to find a mathematical mould that they think fits?

M: Well, you see the students do not wish to take the risk of explanation in ordinary English because, the more they write, the more likely it is they will get marked down for the wrong explanation. So what do they do? They provide the answer, zero is after all the right number here, and are as economical as possible with justifying it. They thus minimise the risk of having their logic faulted by the marker!

RME: So they just provide the answer also dressing it up in, for example, set theoretical language?

M: Yes, to me writing down a set[6] whose description solves the problem was quite impressive and … perfectly inscrutable! I cannot follow the reasoning that makes him write this down. I think that what is being said here is that it is obvious that the answer is zero, but I cannot say that obviously the answer is zero so how do I write the answer in a posh fashion. That will give me credibility, and this is what I think he has done. The thought is very frightening but it is very clear.

RME: You said earlier it is hard to convince students to unpack the argument behind statements they make, especially ones they perceive as obvious (E3.5, Scene II; E4.2).

M: Yes. The more trivial the mathematics the tougher it is to convince them. Say you set a triangle, give the two angles and ask for the third. They know that there is a calculation that they have to do. The bright students just write down the answer because they have that calculation straight away in their heads. Persuading them to write down the *trivial* calculations is tough. And I think this is symptomatic of the same thing going on here. The brighter students are unable to write down the convincing part of the answer. I think they lose track of the fact that their task is to communicate this conviction. And that the argument, of course, in mathematics is at least as important as the number in the end[7].

RME: Answers like Student W's almost beg the question *what do you mean here, why do you say so*?

M: Absolutely. And this is what is frustrating about the monologue within a piece of homework like this. When I look at this, I envisage something more interactive (E7.1), a kind of dialogue where they read the question and say to me *x is obviously zero* and then I ask them why – because I want them to convince themselves that they really believe this. Faced with this piece of writing it is very hard to work out what they mean, what they are understanding.

RME: I guess you would feel the same way about Student J's use of the language of limits here?

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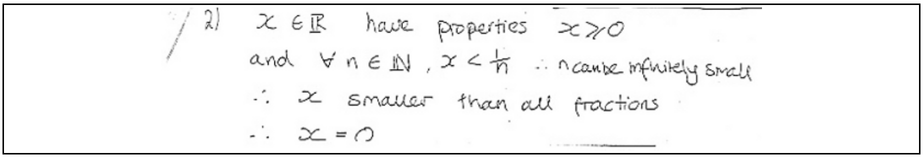
[6] Starting from defining the set may not be as straightforward for the students who often find the mathematical notion of 'set' far from easy. Fischbein & Baltsan (1999) analysed the various 'misconceptions' held by students with regard to the mathematical concept of set. They hypothesized that these misunderstandings may be explained by the initial 'collection' model. Even after learning the formal properties of a set in the mathematical sense, the students were still influenced in their reactions by the collection representation, which acts 'from behind the scenes' as a tacit model. They claimed that if the mathematical concept is not continually reinforced through systematic use, it is the initial figural interpretation which will replace, as an effect of time, the formal one. In this work they also juxtaposed their analysis to that of Lakoff & Núñez (2000), who use a 'pot' analogy for sets.

[7] An issue extensively discussed in Chapter 3.

M: It looks like something they may have seen at school, some primary contact with the language of convergence which makes them feel comfortable with using it. Again it is kind of impressive even though it doesn't mean much. … x goes to zero and well, what is x? Yes, I don't know, it is zero, it goes to zero. At least nowhere here is infinity attached to an equals sign, so this person has been well schooled in how to write this kind of thing. The *hence*, the quantifiers are there, you could possibly think of even rescuing this! Oh well,…

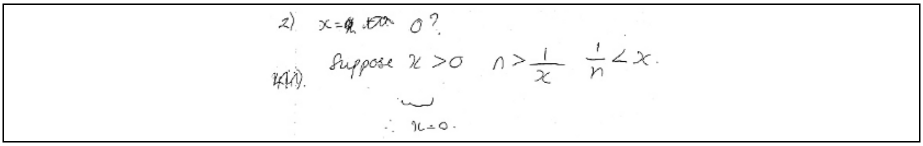
RME: For a change let's look at another, slightly more verbal answer, Student LF's.



Student LF

M: Nice! And you can get into non-standard analysis within two minutes from this argument… Or you can avoid non-standard analysis from this argument!

RME: I am inclined to think Student LF has thought through the answer more than, say, Student J who also seems to think that $1/n$ can get very, very small. Look at this quasi-articulate expression. n *can be infinitely small…smaller than all fractions* … there is some attempt there to grasp the idea of why x must be zero. I sort of like that. A bit like Student L's response where there is some trace of engaging with an argument by contradiction with the *?* and *suppose* in there.



Student L

M: I agree. In Students L and LF there is an effort to communicate which Students J and W have forgotten. Student LF does not seem to mind the risk of putting in a possibly wrong explanation. Of course there are problems here: I would feel more positive towards this if the *therefore n can be* infinitely small bit was not there. If because was there instead of therefore, and n was x. I am sure that is what they meant in their minds… also commas would make all the difference…

RME: It seems that with generous editing of the text, you could make it really good!

M: Come on, technically speaking there is little here to suggest any syntactic structure: no commas, no beginning of sentence, no full stops. Is this a transcription, I wonder, of the sort of ways of writing what you think in some unedited fashion? Which is clearly not in terms of sentences. Where does this way of writing come from? Think for a moment about when we are doing research, new research, and we are writing some odd notes, grappling with the problem, and we are not terribly fussed whether someone will ever see it; it is not the polished version. Then we experience something similar to some of these students grappling with what to them is the unknown. And I think that somehow students do forget that this is ever going to be read and marked by some human being. And there are these bits and pieces floating around in their minds and there is pressure and difficulty to write them down somehow.

RME: I hear a bit of a longing in your words for a mathematical writing consisting of fully-fledged, clear sentences. The type of writing that Social Sciences or Humanities students are more encouraged and expected to develop.

M: Oh yes! I think it is our obligation to convey to them as clearly as possible the benefits of communicating their mathematical thoughts in this way. I am not saying we do not but it takes a long time to get through to them. And perhaps we should be doing this more systematically. I must tell you the story of one of our students whose work is a joy to look at! He came in the first year having abandoned an English degree. He came about six weeks late or so, this is why I recall having to look after him a little bit. And this person is interested in language. He writes fully comprehensive, beautifully argued, fantastically correct English sentences through which his mathematics improves phenomenally. If you want to see the advantages of being able to express yourself clearly, study the work of this guy. It is phenomenal. Other students should take his homework and copy it out so at least they have seen it once how to write a complete homework …oh, and the handwriting is beautiful too! (E7.4, Scene IIa)

EPISODE 4.1
STRINGS OF SYMBOLS AND 'GIBBERISH'[8]
SYMBOLISATION AND EFFICIENCY[9]

Setting the scene: In E3.3, Scene II, particularly in the context of discussing Student N's response, M comments on the student's use of ⇒ ('a postmark trying to say *this is what I am thinking about doing next*); he speculates on N's 'mis-typing the first line' and his 'simply mimicking the use of logical notation'. He also mentions there that typically students 'trip over' things such as changing the name of a variable. On a different occasion, M comments on Student LW's writing (her response to the question at the end of E3.4) as follows:

M: I would also like to point out the use of quantifiers in LW's writing, a sort of truncated note-form solution. *if f is onto* $\forall a \in A\ \exists b \in B$? Before we know it, she would insert minuses across the sentence to denote negation. And, oh those dreaded arrows which sometimes mean convergence, other times mean maps, and often just all those words the student can't quite identify but should be in there! Here the arrow seems to be a lazy implication sign, I think.

RME: If I made a list with all your little obsessions that keep cropping up in your comments on the students' writing…

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[8] Appropriation of the symbolism of university mathematics is fundamental part of the students' enculturation and a focus of significant parts of research work in the area. Berger (2004), for example, examines the question of how a mathematics student at university-level makes sense of a new mathematical sign. Using an analogy with Vygotsky's theory of how a child learns a new word, Berger argues that learners use a new mathematical sign both as an object with which to communicate (like a word is used) and as an object on which to focus and to organise their mathematical ideas (again as a word is used) even before they fully comprehend the meaning of this sign. Through this sign usage, the claim is here, the mathematical concept evolves for that learner so that it eventually has personal meaning, like the meaning of a new word does for a child; furthermore, because the usage is socially regulated, the concept evolves for the learner so that its usage concurs with its usage in the mathematical community. In line with Vygotsky, this usage of the mathematical sign before mature understanding can be termed `functional use'. The paper demonstrates `functional use' of signs (manipulations, imitations, template-matching and associations) through an analysis of an interview in which a mathematics university student engages with a `new' mathematical sign, the improper integral, using pedagogically designed tasks and a standard Calculus textbook as resources.

[9] Students experience difficulty with translating from ordinary language into mathematics - see e.g. (Clements et al, 1981), (Ghosh & Giri, 1987) and (Burton, 1988) for evidence that students express differently on the blackboard, in writing and in exams. Clements et al (1981), as well as Janvier (1987), also refer to interference from everyday logic and language (e.g. in the well known case of the student-professor problem). In this vein Bjorkqvist (1993), who studied the students' personal conceptions of logical necessity and possibility, found influence from everyday conceptions and a partial dependence on key elements in the structure of the sentence (for example, double modalities, the categorical form of propositions that contain two negations). Students were found to be deeply confused with these linguistic structures. Add to these structures some syntactic and semantic *mathematical* content of mathematical expressions and it should be no surprise that, this confusion is further compounded. See also about the need to introduce these potential conflicts to the newcomer in (Byers & Erlwanger, 1984), (Abkemeier & Bell, 1976) and (Davis & McKnight, 1984).

M: …the *arrows* would definitely make the list! By the way the definition is reproduced wrongly too, this is rubbish. I am about to laugh with myself because the neatness and symbolism of this makes it plausible at first glance!

(For more on this issue see also E6.2 for M and RME's close examination of students' attempts to reproduce what they see as acceptable mathematical writing in the context of Analysis). Here M and RME address this issue in two more occasions, one in the context of Group Theory (Scene I) and another in the context of discussing students' responses that asked them to engage explicitly with a translation of a mathematical statement expressed verbally into one expressed symbolically (Scene II).

Scene I: Desperate juggling of axioms and random mathematics

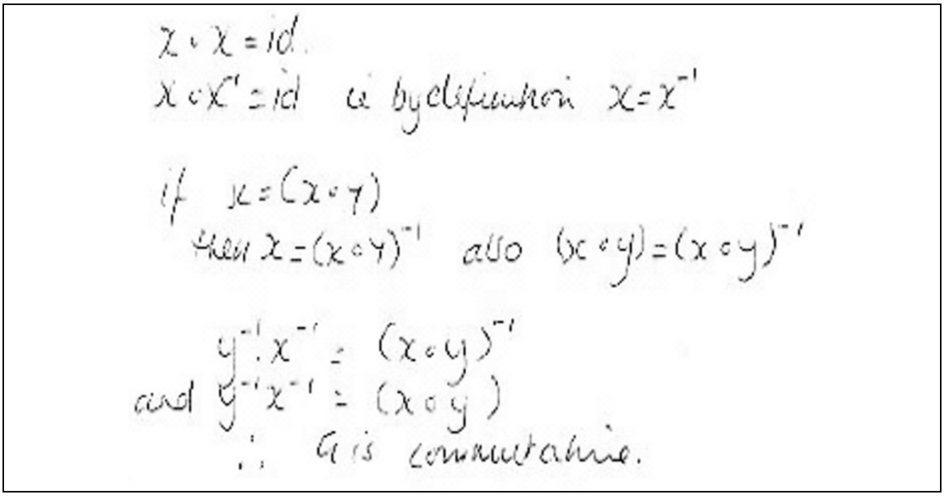
The following takes place in the context of discussing the question below as well as two examples of student response to this question, Student's WD and Student's L:

Example from Exercise Sheet 1, Week 2, Autumn Semester 2001 - Group Theory

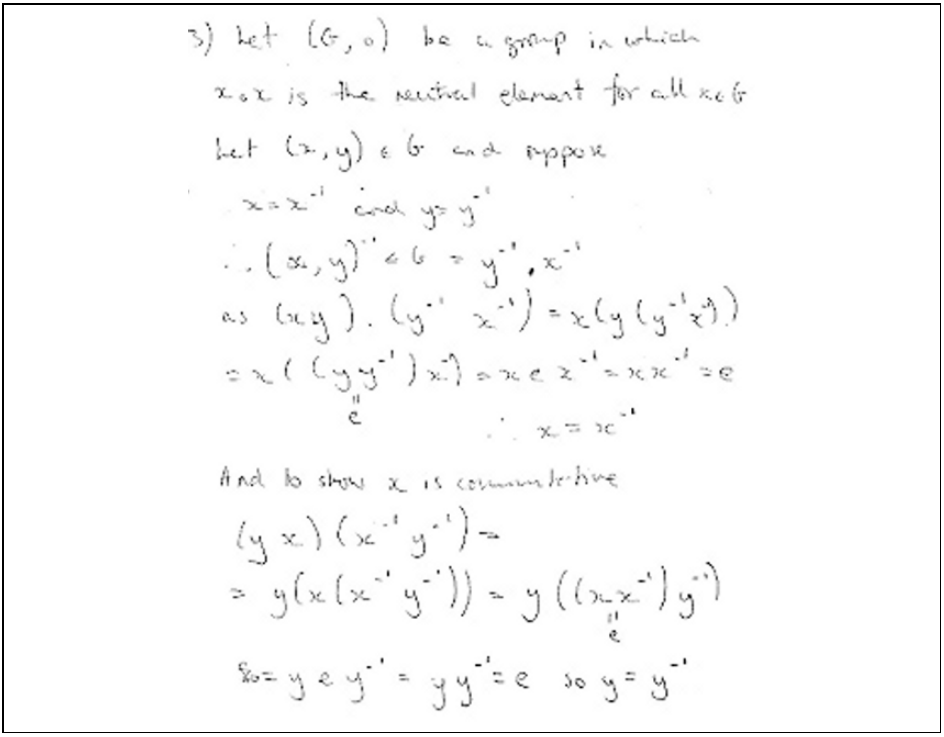
Let $(G, \circ)$ be a group in which $x \circ x$ is the neutral element for all $x \in G$. Prove that $x = x^{-1}$ for all $x \in G$, and that G is commutative. HINT: Work out $(x \circ y)^{-1}$ in two different ways.

Suggested solution

If $x \circ x = 1$ then x is its own inverse, by definition. In any group $(xy)^{-1} = y^{-1}x^{-1}$ but here also $(xy)^{-1} = (xy) = y^{-1}x^{-1} = yx$.


$x \circ x = id$.
$x \circ x^{-1} = id$ ∴ by definition $x = x^{-1}$
if $x = (x \circ y)$
then $x = (x \circ y)^{-1}$ also $(x \circ y) = (x \circ y)^{-1}$
$y^{-1}.x^{-1} = (x \circ y)^{-1}$
and $y^{-1}x^{-1} = (x \circ y)$
∴ G is commutative.

Student L



3) Let $(G, \circ)$ be a group in which
$x \circ x$ is the neutral element for all $x \in G$
Let $(x, y) \in G$ and suppose
$x = x^{-1}$ and $y = y^{-1}$
$\therefore (x, y)^{-1} \in G = y^{-1}, x^{-1}$
as $(xy).(y^{-1}\ x^{-1}) = x(y(y^{-1}x^{-1}))$
$= x((yy^{-1})x^{-1}) = x e x^{-1} = xx^{-1} = e$
$\therefore x = x^{-1}$
And to show x is commutative
$(yx)(x^{-1}y^{-1}) =$
$= y(x(x^{-1}y^{-1})) = y((xx^{-1})y^{-1})$
So $= y e y^{-1} = yy^{-1} = e$ so $y = y^{-1}$

Student WD

M: I guess that the three dot symbol is used here to denote implication but overall my impression is that there is a circular argument at play here. I wonder what Student WD means by $(x,y) \in G$: probably he means *consider x, y as elements of group G*, a bit like writing this in LaTex. But the brackets shouldn't be there, he is confused about how the bracketing is used. He then supposes that $x = x^{-1}$ and $y = y^{-1}$ and tries to say something about the inverse of (x, y)? Which means (x, y) means xy to him and that the brackets are not alluding to the binary operation in a group. Then inserting an equals sign prior to G, the symbol for the group? Manipulating the inverse –1 symbol in and out of brackets? Concluding that $x = x^{-1}$ and $y = y^{-1}$ having assumed it? Oh God…

RME: I think half way through the brackets become a way of noting in which order the operation in the group has been implemented. What do you think?

M: I agree. There are several steps where the group operation is applied on various elements, probably in ways reminiscent of their lecture notes, to end up verifying that $(xy)^{-1}$ is $y^{-1}x^{-1}$ … Hum. Then, in the second part, is *and to show that x is* *commutative* a slip or a genuine misunderstanding about *commutative* applying to elements or the group itself?

RME: I have been tempted to call this *desperate juggling of axioms…*

M: And I am tempted to call this *random mathematics*! What a hopeless chain of a circular argument this is. And what an overkill in using $\therefore$ for *therefore* or *because* instead of the words themselves. Plus there is little consequence in the script anyway so I do not see how the right to use $\therefore$ has been earned! This is so badly written, it is depressingly inscrutable. But overall maybe we should take into account that there is a minority of our students, who have no clue at all on what is going on. At some point we lose them and, by some miracle, they carry on into the second year[10] and normally there things go horribly wrong because attending and writing, writing in the gibberish we are witnessing here in Student WD without understanding creates these enormous gaps in their learning. You could joke they could still probably pull off a PhD in computing with these gaps but the issue remains we are losing them somehow in the early days and the repercussions are felt later and severely!

The discussion turns to whether these repercussions are irreversible or not (OT7.1) and then M returns to discussing Student WD's response:

M: Back to Student WD though, in a peculiar way, locally, always all that he writes is sensible. The whole thing, of course, makes little sense and the writing demonstrates disregard of the conventions of formal mathematical writing and of concentrating on carrying out these sort of calculations, if you can call them that way in Group Theory. And what about Student L's $x = (xoy)$? I see in her use of brackets the same disregard for notational convention as Student WD's.

RME: I guess she meant *substitute x for xy* in whatever statement we have made about x. So if $x = x^{-1}$ then $xy = (xy)^{-1}$. Or am I being generous here?

M: Yes, ok. So, in her writing, x is a sort of meta-variable. I think I agree there is some evidence here of lack of ability in writing mathematics, not necessarily lack of ability in doing mathematics. It is laconic and disrespectful of conventional formalism but not incorrect. But may I say something general about introducing students to mathematical writing?

RME: Please do.

M: We often touch in our discussion on the students' resistance to writing in full sentences where this resistance comes from – their perceptions of mathematics as

[10] In the case we are examining here Group Theory is taught in the beginning of Year 2.

a non-verbal activity, the almost wordless ways in which they have been writing mathematics in school, their lack of a strong background in Grammar/Syntax and Logic and Proof, this last one being a casualty of the disappearance of Euclidean Geometry from school mathematics (SE3.1) – and how you get them to do so beyond marking them down for not doing so e.g. by encouraging them to write essays etc., having introductory courses in mathematical reasoning, establishing early on what is mathematical proof etc. (E7.4, Scene II). But, of course, their attempts to introduce mathematical notation is problematic to start with. It is more often than not that I see a script in which symbols are used parsimoniously, sometimes repeating a letter that the author has used for symbolising something else too. I feel I need to bring this to their attention all the time.

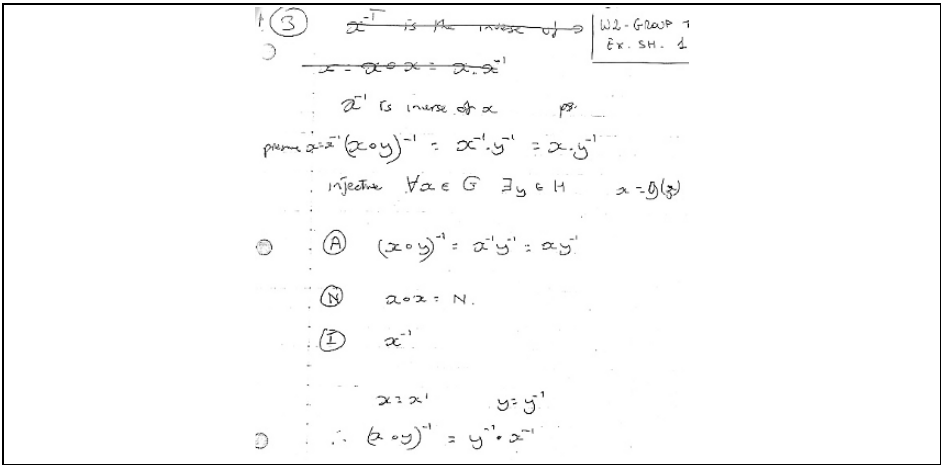
RME: Notational inconsistency seems to be something we can all be accused of. Look at the symbol for the group operation: it fluctuates between a circle and a gap in the question setter's writing. This may cause confusion.

M: Well, gradually in Group Theory the symbol in *x*o*y* does go away and the multiplication gap (*xy*) comes in.

RME: Yes, but maybe it is a bit too early and too confusing to start doing this in the middle of a question!

M: Yes, of course. Having said that, funnily enough it is easier to convince students that a group operation is not necessarily commutative when the symbol o is used (*x*o*y*) rather than when the one resembling multiplication (*xy*) is used.

RME: Here is another response Student M's. What do you think?



Student M

M: This doesn't look good, does it? It is also strange because, if this fragment of a statement around *injective* meant anything, it would have to be *surjective.* Gibberish I'm afraid! I know we have been critical of WD's writing but there was a trace of thought in there that merited some attention on our part. But this one…OK, if we assume there is something to be said about all of them, student responses, let's take a deep breath and look at this… $(x\text{o}y)^{-1}$ becomes $x^{-1}.y^{-1}$ so here we have first of all a problem with bracketing and what it means to take an inverse of the product of two elements and also the o symbol for the group operation becoming a dot. Then a bit later it becomes o again. A, N and I stand for Associative, Neutral, Inverse, I assume. Oh, now, this is probably rubbish because he is claiming he is using the associative and other laws in these lines and does nothing of the sort. Yes, this is completely devoid of meaning. And God knows where *injective* comes from! Probably from some reference to injectivity in the discussion of homomorphisms during the course. But overall this is such a desperate throwing of meaningless things on paper in the futile hope that some marks can be gained just by virtue of not leaving the piece of paper blank – an attitude very common in exams I must say!

Scene II: The dicey game of to-ing and fro-ing between mathematics and language[11]

The following takes place in the context of discussing student responses to this question:

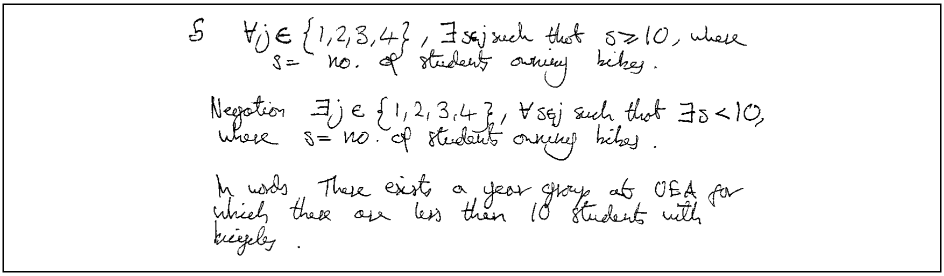
> Write down using quantifiers the statement:
> " In each year group in the university there are at least ten students with bicycles."
> Using your quantified statement, write the negation of the statement down, both in words and using quantifiers.

The lecturer's suggested response distributed to the students had been:

> The statement:
> "In each year group in the university there are at least ten students with bicycles" may be written using quantifiers as follows:
> $\forall j \in \{1,2,3\}$ $\exists$ ten students in year j that have bicycles.
> The negation is then
> $\exists j \in \{1,2,3\}$ such that no more than nine students in year j have bicycles.
> In words " there is a year group in the university with no more than nine students in that years group having bicycles".
> [There are many different ways to write these answers.]

[11] For a preliminary discussion of this data see also (Nardi & Iannone, 2002). Barnard (1995) discusses students' considerable difficulty with negating logical statements as follows: the perceived 'meaning' of the statement interferes with the action of negation and prevents students from acting on a logical statement regardless of its content. Not at ease with reading formal mathematical language as something more than a meaningless citation of symbols, students often find it difficult to engage with a logical manipulation of the statements (Barnard, 1995; Almeida, 2000; Jones, 2000).

The conversation between M and RME starts with looking at Student J's and Student L's responses:

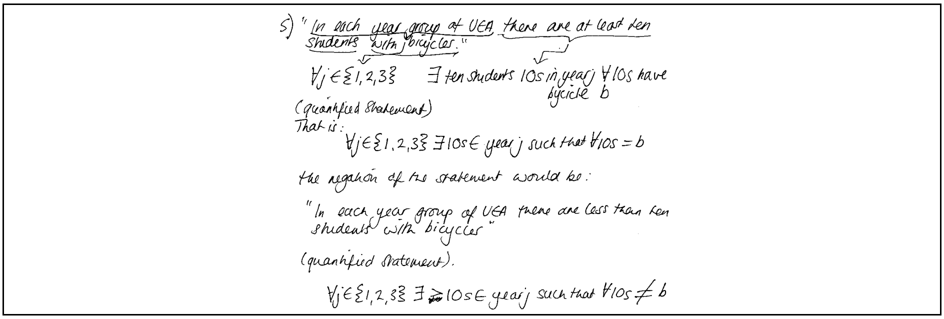


5 $\forall j \in \{1,2,3,4\}$, $\exists s \in j$ such that $s \geq 10$, where $s =$ no. of students owning bikes.

Negation $\exists j \in \{1,2,3,4\}$, $\forall s \in j$ such that $\exists s < 10$, where $s =$ no. of students owning bikes.

In words There exist a year group at UEA for which there are less than 10 students with bicycles.

Student J



5) "In each year group at UEA there are at least ten students with bicycles."

$\forall j \in \{1,2,3\}$ $\exists$ ten students 10s in year j $\forall$ 10s have bicycle b

(quantified statement)

That is:

$\forall j \in \{1,2,3\}$ $\exists 10s \in$ year j such that $\forall 10s = b$

the negation of the statement would be:

"In each year group of UEA there are less than ten students with bicycles"

(quantified statement).

$\forall j \in \{1,2,3\}$ $\exists \not\geq 10s \in$ year j such that $\forall 10s \neq b$

Student L

M: I am interested in the strange ways students represent the meanings they have in their heads which are perhaps definitely good but don't get translated in the appropriate symbols; particularly the use of the equals sign. This is a difficult issue for me as I don't get the impression that the students understand what are the ways and expectations of symbol use and they just scatter them over their script. I have always thought students are reluctant to use new symbols: they use the symbols that are already used in the question. That is one of the reasons why this is a wonderful example of a question as it is totally free of symbols. As a result the students need to rise up to the occasion and introduce symbols following their own initiative. The question is intended as an opportunity to practise writing logically, practise the algorithms of logical writing, for example the algorithm of negating a quantified statement. The students' success in this question depends on their capacity for and previous experience of this translating into logic, this to-ing and fro-ing between mathematics and language.

RME: It seems that once the logical writing is there, the negation algorithm is usually carried out fine. But it is the logical writing that suffers to start with most times.

M: These gymnastics with quantifiers are important: define the set whose cardinality you are concerned with, then construct an inequality that involves this cardinality etc. This is a difficult question for students as they may be defeated by the fact they do not know yet what is the acceptable and expected level of symbolism or, significantly, what is the gain from using symbolism.

RME: You hinted at the severe imprecision in the students' writing. Is Student L's writing an example of this?

M: A vivid one. Especially with regard to what the actual negation of the statement is which she gets wrong. And this is regardless of the use of symbolism – did you notice she wrote *quantified statement* twice in the script as if to remind herself of what is asked? Anyway, in a lecture or a seminar, converting the students' often vague views and verbal statements into precise statements, ones that have one and only meaning, is the main task (E7.1). And this is a task often complicated by the fact that this same meaning maybe disguised in different symbolic clothes according to which textbook, notes etc one is reading. Eventually I believe students achieve this precision but their first experiences are usually difficult because the conversion of vague verbal statements into precise verbal statements and then the jumping between verbal statements and logical statements are difficult. At some basic level naming and labeling things in an assertively unambiguous way is still difficult: *s* for year group *and* an individual student – with or without a bike! This reminds me of the apples, apricots and avocados set of equations I give students sometimes. They choose *a* and find out that all three fruits would thus get confused. They choose *ap* and soon confuse apples with apricots. Choosing something sensible (*a*, *b*, *c* or whatever) escapes them. There seems to be a lack of decisiveness about the meaning of the symbols they use: the meanings are changing as you go through the statement and they don't go back to re-read what they wrote down.

RME: One of the reasons I find Student L's response interesting is because – in her own idiosyncratic way, to say the least! – she is trying to engage with this translating exercise: she has broken the statement in bits and has attempted a correspondence between mathematical and ordinary language.

M: Do you honestly think that if she thought really hard about what the negation of this statement is, she would have come up with this? Do you think she actually thinks this is the true negation of this statement? The flaw here is not to do so much with what is written as with what is thought. She is not thinking hard enough that this means something because the intensity of meaning in mathematical sentences is not something that they are used to, it is a totally alien idea. They are so used to this vague connection between a sentence and its meaning in most of what they have seen before. They do not yet fully realise that you could change the order of two words and the sentence totally changes its meaning…Did you notice that the sentence Student L wrote down cannot be read out? Try! Nothing that has actually been written down can be reasonably read as a sentence. I see what you are saying about her attempts to engage with a translating exercise but she hasn't done so successfully, has she? *10s in each j* to denote the existence of ten students with bicycles in each year? Please! But that's a mere detail here: the attempt is falling apart the moment the negation does not start with *in each year*… That is what she needs to understand first. And the rest is somehow, well, detail. Even though I screamed when I realised the use of the equals sign to mean ownership of a bike! And, don't you find entertaining the use of ≠ to mean *not owning a bike*? Points for consistency there! Student J's response by the way, even though not as overtly problematic as Student L's response, is not light years away from trouble in the first line either: $\exists s \in j$?

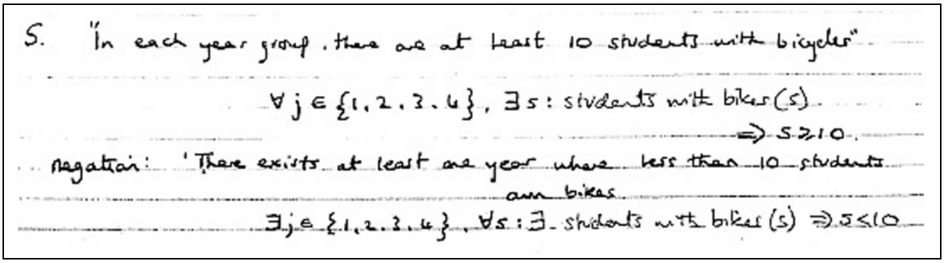
RME: There seems to be a dual meaning attached to *s*: a set *and* the number of its elements.

M: Cardinality is something they would have very little knowledge of at this stage – seen but not used and practiced the manipulation of very frequently. The underlying purpose here is to learn the language of mathematics. Still, just like when you are trying to teach a child writing stories you will not correct every single grammatical error but you would expect some minimally meaningful engagement with the rules of language, here this minimal expectation is something that begins with *there exist*…. But I do not want to sound gloomy: at least in these examples the students seem to understand that they have to engage with some kind of translation, they just do not do it adequately. What they need to see is the point of carrying out such translation: that it helps you see clearly what the negation of the statement is here.

RME: They appear compelled to offer highly symbolic answers.

M: Well, they were asked to do so overtly here, weren't they?

RME: What do you think of this one?



5. "In each year group, there are at least 10 students with bicycles".

$\forall j \in \{1, 2, 3, 4\}, \exists s$: students with bikes(s).

$\Rightarrow s \geq 10$.

negation: 'There exists at least one year where less than 10 students own bikes.

$\exists j \in \{1, 2, 3, 4\}, \forall s : \exists$ students with bikes (s) $\Rightarrow s \leq 10$

Student H

M: Fantastic. For me automatically changing the *for-all* to *there-exists* – even perhaps not thinking it through – is a good demonstration of understanding the underlying logic that must be used. Even though I am a bit concerned about the *for all s, colon, there exists…* bit and about the fact that *j* is defined and never used again and about that *bikes (s)* bit which seems to suggest some kind of function. But that's like writing a story in French and missing a few accents. With some constructive feedback this would be perfect. The battle is half over with this student. Whereas with Student L it hasn't even begun. That I found alarming – even though I liked the diagrammatic demonstration of the translating process. This is a good start. With seeing and practicing the detail of many examples – from scratch: define the sets and the years, label the sets, discuss the role of j etc. – the problems I listed earlier could be resolved.

RME: How do you shift a student away from using $\in$ as meaning ownership of a bike?!

M: It is important to get the underlying concept for each of these symbols. The purpose of this question was I think to stress the process of negating a statement. OK, first by constructing a precise verbal statement and then translating this into a precise mathematical statement. Accepting on the way that ordinary language will be interspersed into one's writing to clarify one's meaning. But the primary aim was to practice negation.

RME: Why then ask for it in a quantified statement?

M: Because they need to be initiated into the necessity of mathematical language: without its precision mathematics is very difficult to express. I am not saying it's a mindless task – quite the contrary and I have a grain of sympathy for the students who are first timers in handling this necessity. Finding a half way in between mathematical and ordinary language maybe the way at this sensitive moments of their mathematical career.

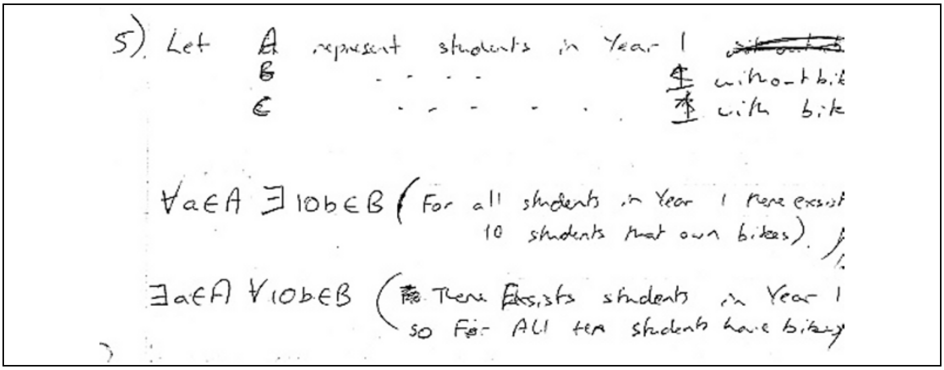
RME: It strikes me that Student H has written down the negation of the statement despite, not because of the quantified statement. What I am saying is that I am not sure that the necessity of mathematical language in this case comes across because the sentence of the negation is absolutely fine – concise and unambiguous.

M: You are probably right. And the question seems to try to achieve too many things at once.

RME: I am interested in what you said a little while ago about meeting the students half way, steering a course between verbal and mathematical expression.

M: I think in lectures we do that a lot, we are never going all the way one way or the other.

RME: Here is another example, Student LW's response.


5) Let A represent students in Year 1 ~~students~~
B . . . . without bike
C . . . . . with bike

$\forall a \in A \ \exists 10 b \in B$ (For all students in Year 1 there exist 10 students that own bikes).

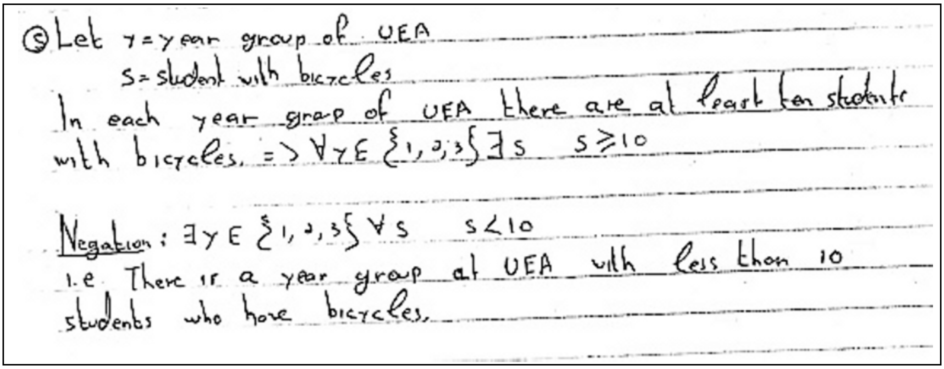
$\exists a \in A \ \forall 10 b \in B$ (There exists students in Year 1 so for all ten students have bikes)

Student LW

M: How weird! This looks like nothing you would see in a lecture. Creating the three sets and never coming back to them. No carrying through of the initial idea. I can try to be generous in my approaching this but there doesn't seem to be any real content here.

RME: Neither does this? Student N's response.

M: So here at least there is the comfort that this person's intuition at least intervened, even if for writing gibberish. Oh, the dreaded use of the *implies* $\Rightarrow$ sign…


⑤ Let y = year group of UEA
s = student with bicycles
In each year group of UEA there are at least ten students
with bicycles. => ∀y ∈ {1, 2, 3} ∃s s ≥ 10

Negation: ∃y ∈ {1, 2, 3} ∀s s < 10
i.e. There is a year group at UEA with less than 10
students who have bicycles.

Student N

RME: And the complete absence of anything before the *there exists*. Something like *such that* is missing. No relationship between *y* and *s* is established.

M: Yes. It is a computer scientist's writing, if I am allowed some speculation on the specialism this student may have chosen. Define your variables, e.g. year group from a list 1 through to 3. But, oh dear, I cannot have a list for students with bikes therefore I am stuck. Which reminds me of another classical difficulty: if the set is bounded, say between 10 and 20 then students are ok; if say it is greater than something, they are not. The unboundedness, the impossibility to list and name the contents is problematic for them. All students put something like $s \geq 10$ or $s<10$ in their writing. This is what they capture as required from the question. If they knew exactly how many students, say 100, they could list them and test, I believe.

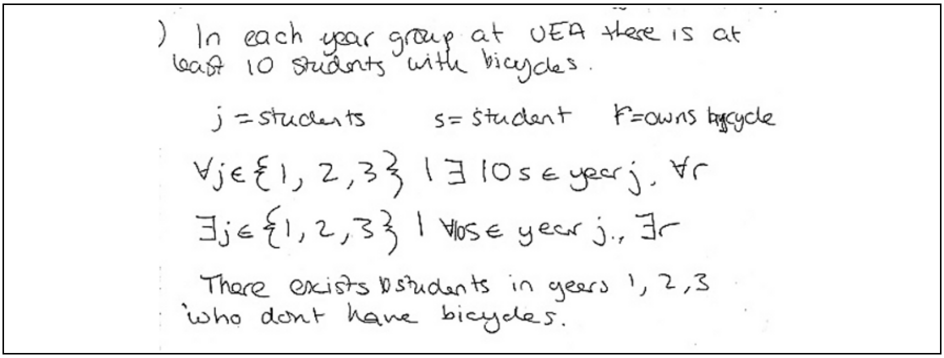
RME: I haven't seen much of this in their writing but it is a speculation worth checking.

M: Yes! Back to the use of the *implies* ⇒: this is almost an emotional thing, I think. So, thinks the student, I am now going to write down something which will compel the marker that the next bit follows naturally. And there is also an odd writing of the *belongs* symbol here too. And *y* is a year group *and* a number here, isn't it? If it wasn't so muddled this approach of assigning multiple meanings to the same symbol could take this student places! On a more serious note, I think that the students need to get used to the idea of switching swiftly between ordinary and symbolic language having pinned down the precise meaning they are trying to convey. There is precious little of that done overtly in the teaching, I admit (E7.1, Scene VI). I often wonder whether, once comfortable, the students would then gravitate towards one or the other. Of course the point is to fluctuate between the two: take the example of *arbitrarily large* (E6.2). This is something better expressed in language than symbols. To me language in mathematical writing is for inserting nuance into the symbolic writing.

RME: The definition of convergence is one such example of a compressed mathematical statement. But what you are saying is that one needs to learn how to choose an economical way of saying something every time?

M: Including diagrams (E4.3).

RME: Have a look at this: Student WD's response, a word by word translation, I reckon?


) In each year group at UEA there is at least 10 students with bicycles.

j = students   s = student   r = owns bycycle

$\forall j \in \{1, 2, 3\} \mid \exists\, 10 s \in$ year j . $\forall r$

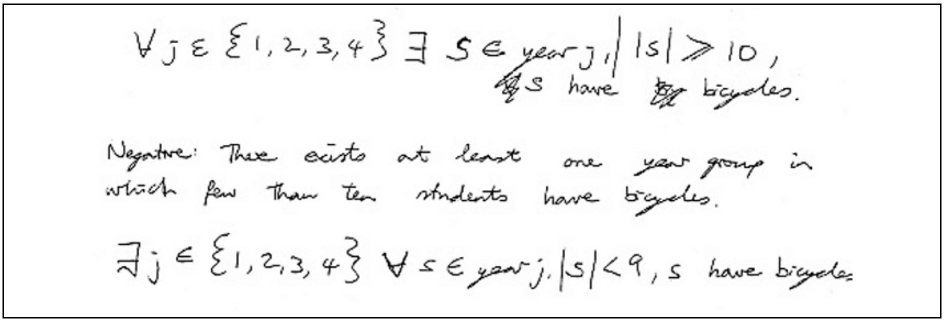
$\exists j \in \{1, 2, 3\} \mid \forall 10 s \in$ year j ., $\exists r$

There exists 10 students in years 1, 2, 3 who dont have bicycles.

Student WD

M: Oh God, where do I start? The use of quantifiers, the assignment of variables, above all the unsuccessful way in which the relevant sets are not defined. There is a remotely evident attempt here to consider the set of students in one year group and intersect it with the set of students who have bikes and then consider the population of this set as being at least 10 etc. but the encoding here is far from accomplished. And the vertical lines in the writing and the pair recall the probabilistic use *on condition that*, *such that* which I find odd. But above all like in most of the examples we have discussed there is no talk in terms of sets here.

RME: I have one more example for you to look at, Student JK's response, in which cardinality makes an appearance.


∀ j ∈ {1, 2, 3, 4} ∃ S ∈ year j, | |S| ≥ 10, S have bicycles.

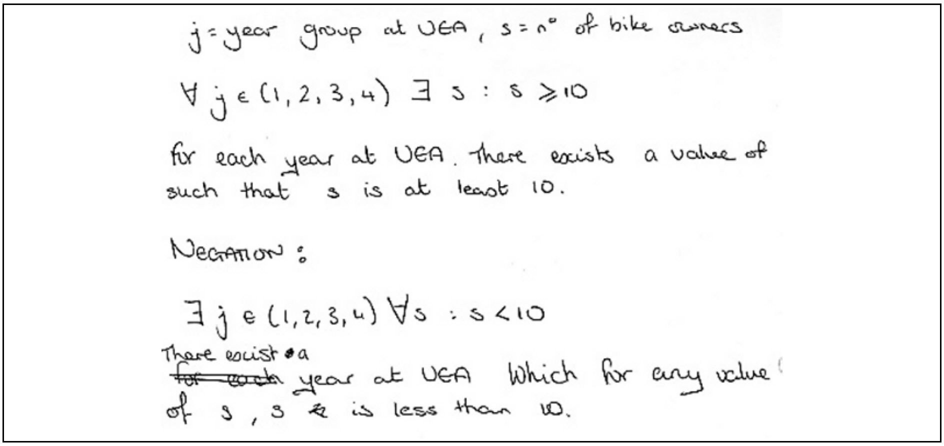
Negative: There exists at least one year group in which few than ten students have bicycles.

∃ j ∈ {1, 2, 3, 4} ∀ s ∈ year j, |S| < 9, s have bicycles

Student JK

M: … who I hope is one of those who come and knock on doors and ask questions. The writing still doesn't make sense, but never mind … In a sense it can be rescued because the student has introduced subsets in a way. Even though I would question the student's understanding of the difference between ⊂ and ∈.

RME: Here is a final example, Student JU's response, where the need to use cardinality was bypassed by defining *s* as the number of students owning bikes.


j = year group at UEA, s = n° of bike owners

∀ j ∈ (1, 2, 3, 4) ∃ s : s ≥ 10

for each year at UEA. there exists a value of such that s is at least 10.

NEGATION:

∃ j ∈ (1, 2, 3, 4) ∀ s : s < 10

There exist a ~~for each~~ year at UEA Which for any value of s, s is less than 10.

Student JU

M: This is so close to being so good. One subscript *j* away…!

EPISODE 4.2
PREMATURE COMPRESSION

Setting the scene: Gaps in student writing are often mind-boggling. But what is exactly missing from the students' writing? And do they know there is something missing? Do they consciously compress their writing to the size they think is expected of their responses? Or does the gap-ridden text reflect gap-ridden thinking? If indeed a process of compression is taking place, what are the casualties of this compression? And are they casualties really? Or are they just part of a rites-of-passage towards embracing a common practice in professional mathematics, that of implicit meanings conveyed in short-handed, heavily symbolic writing[12]?

Scene I: Why is $det(aI_n) = a^n$ true?

The following takes place in the context of discussing Student J's response to the question discussed in E3.3, Scene II:

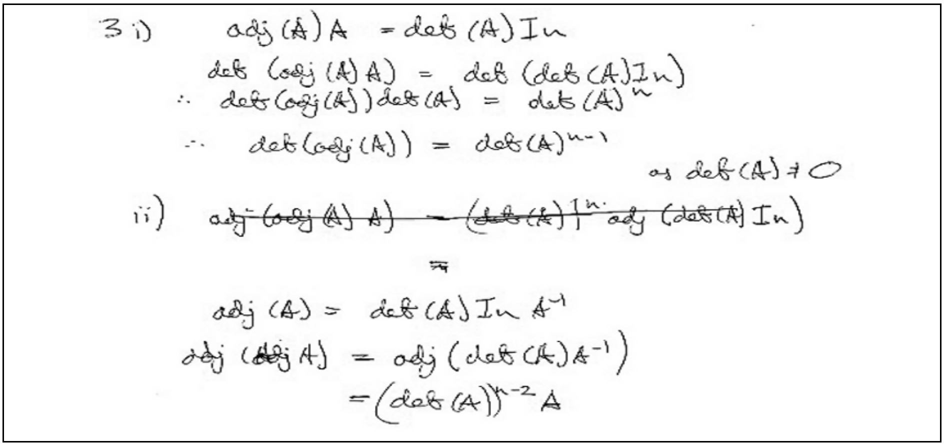
M: In a vein similar to our discussion of what is required of students in terms of reasoning[13] I wonder what their perception of *to show* is at this stage. Student J is alright, isn't she? This is a typically wordless answer[14] but it doesn't matter very much here. OK, maybe in part (ii), there is a jump between the last two lines – it took me a couple of lines to put down how she got there via part (i). Also I would be happier if she had said a bit more about why $det(aI_n) = a^n$ is true (E3.5, Scene II), perhaps even with a little picture or something. But then again she may think of this as an obvious fact and she clearly arrives at the completion of the argument.

RME: There are several ways of proving that – by Laplace expansion or by Mathematical Induction – which also give good opportunities for working with these novel-to-them creatures, determinants!

M: Yes, this is exactly the point here: that by asking them to prove even these trivial facts, you basically ask them to de-compress arguments that we ourselves may have inadvertently compressed because, of course, we need to allow ourselves some shorthand[15]. Maybe in their writing they just reproduce this kind of compression that they see in us. I often wonder to what extent I should take this issue with them because it could be time-consuming and pedantic and produce cumbersome, over-long responses. But, in a sense, this is instructive and once

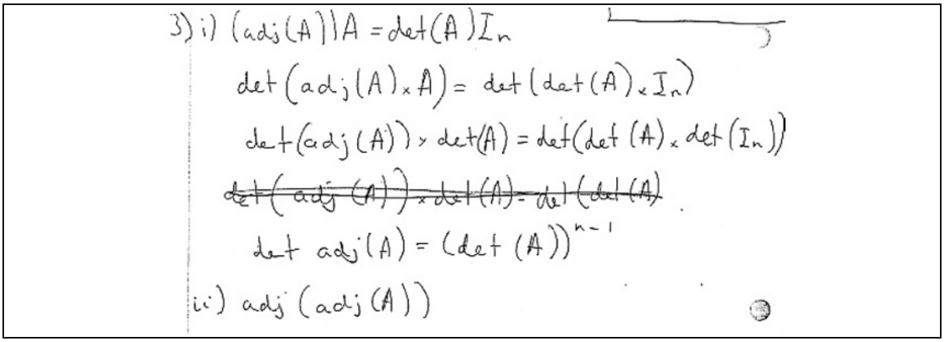
[12] For this common mathematical practice see (Burton & Morgan, 2000).
[13] See Chapter 3. It is difficult, even futile, to attempt a distinction between students' reasoning and its demonstrations through writing, speech etc.: 'the wall between language and thought has crumbled to the point that now we no longer know where one ends and the other begins' (Radford , 2003, p123) .
[14] See discussion of wordless student writing in E6.2 and (Pugalee, 2004)
[15] See (Burton & Morgan, 2000) and discussion within E3.5, Scene II and E6.2.

you have seen a fact properly proven a few times, then we can all begin to relax a bit. But at this stage it's a bit early – and confusing – to be too relaxed! Give me some signs, please, that you know why $det(aI_n) = a^n$ is true , a picture or something, to appease my mind. And if it is in your rough work, do include it, do not keep it a secret[16]!



Student J

RME: Speaking of responses that are very economical with words…here is Student WD's.



Student WD

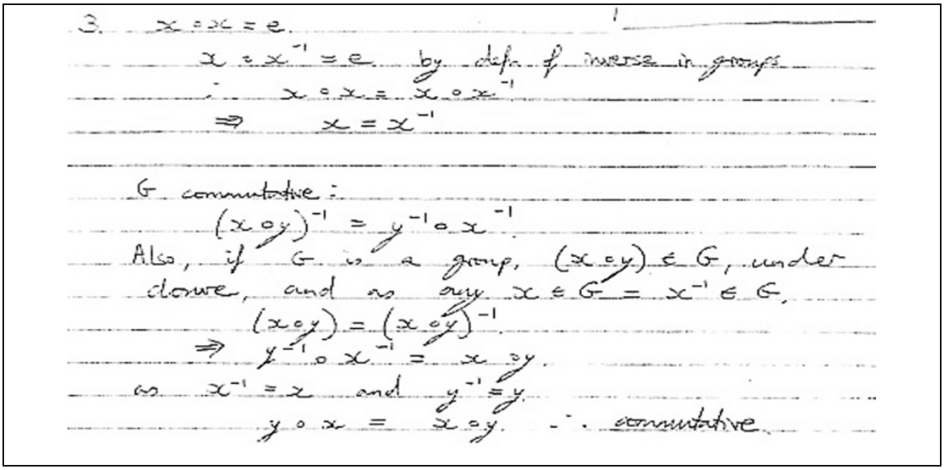
M: This almost makes sense if you delete a couple of *det* in the right hand side of his second line. Of course the fact that they know where they are expected to arrive implies to me, without wanting to sound cynical, that less weight can be attached to the fact that they reach the destination. Putting on paper a series of

[16] See similar plea at the end of E3.5, Scene II and (Bullock, 1994).

transformations of the available expressions is not very satisfactory here, is it? In a sense highlighting that they are using $det(aI_n) = a^n$ would make this a clearer demonstration that they know what they are doing.

Scene II[17] *Why is* $xox = xox^{-1} \Rightarrow x = x^{-1}$ *true?*

The following takes place in the context of discussing Student JK's response to the question in E4.1, Scene I


3. $x \circ x = e$.
$x \circ x^{-1} = e$ by def. of inverse in groups
$\therefore x \circ x = x \circ x^{-1}$
$\Rightarrow x = x^{-1}$

G commutative:
$(x \circ y)^{-1} = y^{-1} \circ x^{-1}$
Also, if G is a group, $(x \circ y) \in G$, under closure, and so any $x \in G = x^{-1} \in G$,
$(x \circ y) = (x \circ y)^{-1}$
$\Rightarrow y^{-1} \circ x^{-1} = x \circ y$.
as $x^{-1} = x$ and $y^{-1} = y$
$y \circ x = x \circ y$. $\therefore$ commutative

Student JK

M: He is using e for the neutral element, instead of 1 or 0. That's OK.

RME: You think he is simplifying *x* from both sides in the first part?

M: I think I am happy with the short cut.

RME: But isn't he supposed to demonstrate the use of group theoretical laws that are implicit in canceling at this stage? Multiply both sides with x^{-1}, apply associativity etc and arrive at $x = x^{-1}$?

M: Yes, he could. But I am happy with this because *x* can be nothing else other than x^{-1} from his first two lines. Sooner or later one needs to stop thinking in those terms and start doing manipulative algebra in groups. I am absolutely fine with canceling *x* without comments.

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[17] This scene is untypical in one way: RME usually takes a listener position rather than one of an argumentative interlocutor as the overall aim is to explore, not modify M's position (see Chapter 2, Part IIb and Chapter 1, Parts II & III).

RME: But, if the question was intended to be an exercise of demonstrating the use of group theoretical laws, isn't skipping them for a shortcut paradoxical? Isn't this compression a bit premature[18]?

M: Yes, it may be a bit too early for that but once they have seen it several times they should be allowed this shortcut. Generally I wish the students were more explicit about the steps they take, which element operates upon which and by which laws a statement follows from its predecessor. Yes, here it is a bit early for this compression. However I would like to point at the confident use of the axioms by Student JK a few lines below where the writing is absolutely explicit about the use of group theoretical laws. Absolutely perfect.

RME: Why then not act analogously in the first part? Perhaps because student JK simply cancelled without realising the series of implicit steps this action requires? He is quoting – perfectly as you said – every single group theoretical law he is using in the second part.

M: Because the first part is too easy to require such explicitness. After all this flexibility is almost the reason why you do Group Theory[19]. The suggested solution itself is very succinct regarding the first part – *by definition*!

RME: I can see why you think this is sufficient, even though it implies the use of a uniqueness theorem but I still struggle with seeing the point of this compression at this early stage. I can only see it muddling the waters.

M: For want of staff time we have decided to give suggested notes on solutions and not model solutions and here we may have a case of a casualty of this preference for succinctness. In Linear Algebra we struggle to convince students to think twice regarding the conditions in which canceling is allowed – think of $A^2 = 0$ as an example. We keep telling them that deducing $A = 0$ by canceling is wrong! So, yes. At this stage de-compression is crucial. And I know we are talking about groups here but the students may naively see some overlap here and get confused. And this is exactly where their distinguishing between rings and groups etc. should start, from seeing what makes groups…groups. Discussing canceling is a good way to start this process of appreciation[20].

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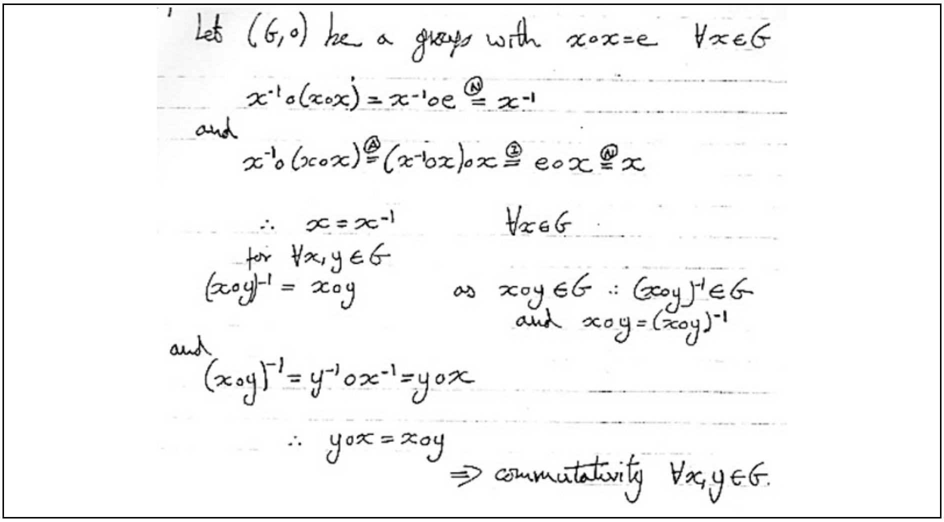
[18] See also preceding footnote: labelling of the issue discussed (here: 'premature compression') usually originates in M. Here it seems to be more on RME's agenda but it gradually becomes M's as well.

[19] Across this scene there seems to be a tension in M between the pedagogical and the epistemological aspects of the point he is putting forward. For a discussion of this tension, in the context of students' enactment of proving techniques (Proof by Counterexample) see (Iannone & Nardi, 2005b).

[20] As in E6.2 the tension noted in the preceding footnote seems to be shifting towards a better coordination of pedagogical and epistemological perspectives.

RME: There is one student response, J’s, where I found the degree of explicitness satisfying.


Let $(G,\circ)$ be a group with $x\circ x=e\ \forall x\in G$

$x^{-1}\circ(x\circ x)=x^{-1}\circ e \overset{N}{=} x^{-1}$

and

$x^{-1}\circ(x\circ x)\overset{A}{=}(x^{-1}\circ x)\circ x\overset{I}{=} e\circ x\overset{N}{=}x$

$\therefore\ x=x^{-1}\quad \forall x\in G$

for $\forall x,y\in G$

$(x\circ y)^{-1}=x\circ y$ as $x\circ y\in G\ \therefore (x\circ y)^{-1}\in G$ and $x\circ y=(x\circ y)^{-1}$

and

$(x\circ y)^{-1}=y^{-1}\circ x^{-1}=y\circ x$

$\therefore\ y\circ x=x\circ y$

$\Rightarrow$ commutativity $\forall x,y\in G$.

Student J

M: Cool! There is a bit of backward reading required here but that’s lovely.

RME: Is writing the same thing in two different ways and then deducing that the two outcomes must be the same something they see you doing often?

M: Yes, they have seen this many times. It is one of the tricks of the trade, so to speak! What we see here is of course the polished version, the back-of-an-envelope rough work is missing and there is hardly any commentary or verbal explanation. But this is nice work. This is what the lecturer should have written in the suggested solution! This student gets more marks from me than the lecturer!

RME: I see! What I like about J’s response is this confidence in dealing with the properties and that there is no apparent interference from numbers and arithmetical operations here at all[21]. He fully justifies every single step he is taking and that gives me a sense of comfort that he knows the terrain he is operating in.

[21] For an example of this concern in the context of Linear Algebra (and the influence on students’ treatment of algebraic objects of their habitual treatment of numbers) see E3.1 and (Nardi & Iannone, 2000). For a wider consideration of the role of particulars (e.g. examples) see (Mason & Pimm, 1984).

M: May I come back to this issue of compactness … I understand what you are saying about the need for clarity but this is getting them used to the necessary shortcuts, to the idea that it is not always best to be very explicit about things, that completely rigorous shortcuts are absolutely crucial: without them mathematics would be impossible. Of course they have to learn what exactly constitutes a rigorous shortcut and of course they have to be aware they are inserting a shortcut in their argument. For example we would never expect them to mention commutativity or associativity of … numbers, right?! We expect them to constantly manipulate numbers with the usual rules that are allowed and, in the context of Group Theory, we hope they will reach an analogous type of facility with the manipulating the elements of a group – or matrices, or elements of rings etc.. And this common understanding of what shortcuts are allowed is built on a tacit agreement that, if challenged, the author of what I read will be able to unpack the argument for me[22]!

EPISODE 4.3
VISUALISATION[23] AND THE ROLE OF DIAGRAMS

Setting the scene: At the end of E6.3, Scene I M initiates a discussion on the importance of building bridges between the formal and the informal in a dialogic negotiation with the students (also discussed in E7.1, Scene II). As that conversation is in the context of convergence and the definition of limit he uses the example of a 'pictorial' understanding of || to illustrate the value in 'all sorts of visual representation' (E7.4, Scene IVb). There, as the discussion unfolds, the potency of visualisation emerges in several examples ranging from juxtaposing a long calculational, but counterintuitive and esoteric, task with the instantaneous mediation of meaning in a picture ('a little picture tells me everything', example: the indelible 'first impression' of the obvious truth at the heart of the picture for the Intermediate Value Theorem); to the more obvious rationale for visualisation in Geometry (e.g. in the complex plane). The discussion in that scene concludes as follows:

RME: How do students respond to this use of visual representation in your experience?

M: Variably. Students often mistrust pictures as *not mathematics* – they see mathematics as being about writing down long sequences of symbols, not

[22] Ultimately the community's practice which the students need to adopt (Burton & Morgan, 2000).

[23] Students also often have a turbulent relationship with visual means of mathematical expression (e.g. Bishop, 1989; Davis, 1989; Janvier, 1987; and, Presmeg, 1986): while there is a strong visual element in mathematical cognition at all levels, when the students find difficulty in connecting different representations (for instance formal definitions and visual representations), they often abandon visual representations – which tend to be personal and idiosyncratic – for ones they perceive as mathematically acceptable. M and RME return to this point repeatedly in this episode.

drawing pictures – and they also seem to have developed limited geometric intuition perhaps since their school years (SE3.1). I assume that, because intuition is very difficult to examine in a written paper, in a way it is written out of the teaching experience, sadly. And, by implication, out of the students' experience. It is stupefying sometimes to see their numb response to requests such as imagining facts about lines in space or what certain equations in Complex Analysis mean as loci on the plane.

RME: Do you think this numbness may also be a repercussion of them being bombarded at the beginning of their studies with the necessity for proving everything etc (E3.1)?

M: Yes, they somehow end up believing that they need to belong exclusively to one of the two camps, the informal or the formal, and they do not understand that they need to learn how to move comfortably between them.

In SE6.1 M's interpretation for the difficulty Student N has with not leaving out of his argument a small but significant number of terms in the sequence he is working on contains analogous elements of appreciation for visualisation: 'had the student drawn a picture, he would have seen he had left them out' (a comment he repeats more or less later in the Episode with regard to the work of Student J). He carries on to appreciate another student's, Student H, emulation of the type of picture drawing seen in lectures (she however 'needed a more helpful picture') and to stress that he is impressed with how both students pinned down an understanding of || as a 'distance between things'. He is not similarly impressed with Student E who, despite leaving some of the terms of the sequence (which should have acted like a cue for her to not-leave them out of her formal argument), has not 'used this diagram as a source of inspiration for answering the question'. Instead 'she drew this, on cue from recommendations that are probably on frequent offer during the lectures, and then returned to the symbol mode unaffected'. So 'there is no real connection between the picture and the writing'.

The following takes place in the context of M and RME's discussion of the question below as well as the lecturer's suggested responses and two student responses, Student WD's and Student LW's:

(3) For each of the following functions $\mathbb{R} \longrightarrow \mathbb{R}$ decide whether it is one-to-one, onto (or both, or neither). Give brief explanations for your answers.

(i) $f_1(x) = \sin x + \cos x$

(ii) $f_2(x) = 7x + 3$

(iii) $f_3(x) = e^x$

(iv) $f_4(x) = x^3$

(v) $f_5(x) = x/(1+x^2)$.

Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is onto but not one-to-one.

3)(i) As $\sin x \leq 1$ and $\cos x \leq 1$ for $x \in \mathbb{R}$, we have $f_1(x) \leq 2$ $\forall x \in \mathbb{R}$. Thus f_1 is not onto. Also, $f_1(0) = f_1(2\pi)$, so f_1 is not one-to-one.

(ii) For every $y \in \mathbb{R}$ there is a unique $x \in \mathbb{R}$ with $f_2(x) = y$, namely $x = \frac{1}{7}(y-3)$. Thus f_2 is one-to-one and onto.

(iii) Not onto (as $e^x > 0$ for all $x \in \mathbb{R}$): one-to-one (if $y \in \mathbb{R}$ the only real solution to $e^x = y$ is $x = \ln y$).

(iv) One-to-one and onto (a bijection): any real number has a unique real cube root.

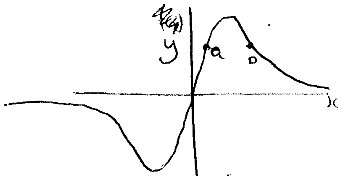
(v) Neither one-to-one nor onto: $f_5(1/2) = f_5(2) = 2/5$, so not one-to-one. Also, $f_5(x) \leq 1$ (as this is equivalent to $x \leq 1 + x^2$, and $1 - x + x^2 = (x - 1/2)^2 + 3/4 \geq 0$.

Remark: You might like to think about what bits of calculus can be used to justify more fully the fact that the functions in (iii) and (iv) are one-to-one.

Last part: $f(x) = x(x-1)(x+1)$ is onto but not one-to-one.

Lecturer's suggested response

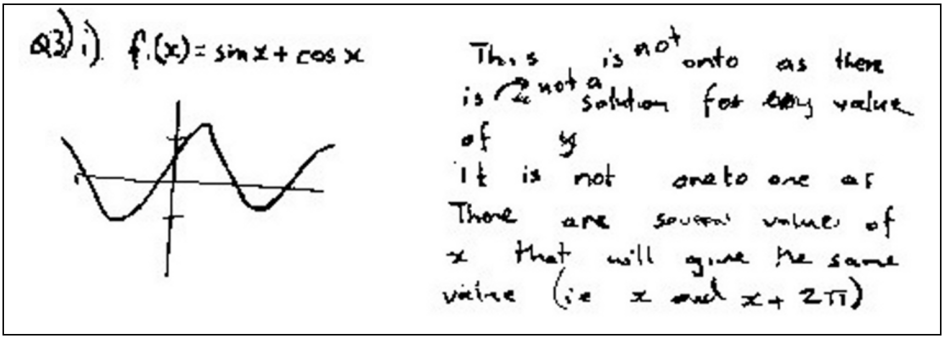
v) This is not onto and ~~but~~ not one to one. f(x) = y



~~b~~ not one to one because ~~[illegible]~~ ~~[illegible]~~ = a = b in the y-axis

not onto because f(x) has maximum and minimum

Student WD



Student LW

M: Of course if one uses the calculator then it makes no difference: all parts of the question are equally simple or complex. But that's a big issue in itself, the use of the graphic calculator!

RME: I sense you fervently want to raise it!

M: Well, a calculator could give you a good picture in part (i), for example. The students here seem to have access to a calculator and it seems to me they have missed a terrific amount of practice in the Analysis which is how these questions ought to be tackled. Generally students lack graphing skills and awareness of what to look for when asked to produce a graph[24].

RME: But surely the technology should help them see more examples and develop such expertise.

M: Yes, they have access to more pictures, and more easily, but have no feel for what to look out for.

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[24] Intertwined here are at least two issues: constructing graphs and extracting mathematical meanings from interpreting graphs – for a review on graph comprehension and definitions of what constitutes good graph sense see (Friel et al, 2001) and the reference to (Roth & Bowen, 2001) in E7.4, Scene III. The two are linked as, particularly in the absence of a graphic calculator, for example, resorting to an understanding of a function's properties is a necessary step towards the construction of its graph. Researchers have used graphing tasks to investigate graphing skills and students' developing comprehension. E.g. drawing on the theoretical construct of APOS (Action, Process, Object, Schema) as well as Piaget & Garcia's triad of levels for schema development (intra, inter, trans) Baker et al (2000) studied undergraduates' comprehension of a non-routine graphing problem in Calculus. The triad was applied for properties (condition-property schema) and for intervals (domain, interval schema). Several student difficulties were observed: with cusp point, vertical tangent, removal of the continuity condition and second derivative. Overall co-ordinating information about properties and intervals was a problem – as was the resilience of incorrect images and overly emphasis on first derivative. Generally the two-schema interplay (property, interval) was difficult for the students

RME: So if the students had no access to the calculator how would they reach an intuition about these properties? There are no pictures in the lecturer's response after all.

M: From previously known facts and practice: that's why I said this is a question in Analysis. You can build up this stock of facts starting from simple functions. And this is serious business because students arrive with little practice, limited capacity for manipulation and a small if any nap-sack of examples.

RME: So your overall take on the usefulness of the calculator is rather negative then?

M: Well, we will see details as we look at the examples of student work – for example, there is no evidence of Analysis, of construction in Student WD's response, just of reproducing something from a screen, and in dubious scaling! – but I am a bit old school in that sense. I do not allow them in the exam room, I admit they can be used to illustrate an answer but this illustration is not the answer and I demand a justification for an answer. I think an argument is necessary. Calculators are nothing more than a useful source of quick illustrations.

RME: Whatever ways the students have used to produce the graphs, I would like to hear your views on whether they make a good use of the information compressed in a graph.

M: I encourage them to draw graphs, see what the answer is and then prove it afterwards. The graph is fantastic to get the answer but there is actually not enough in their writing, once they have done that. It may be a bit of a surprise that I do even though in the suggested solutions in this particular question you don't see much resorting to graphs. It probably does say quite a lot that the lecturer thinks in terms of domain etc. and not overtly about graphs. I would be drawn towards a low-tech approach, roughly draw them and insert them on the side but I wouldn't find them necessary for answering this question. I would like the students though to carry the graphs of all these functions in their heads straightaway and have them immediately available. But then again there are less and more visual people and the more visual may think that a question like this can only be done by producing the graph, using it and then proving the claim formally. Graphs are good ways to communicate mathematical thought and I do not wish to underplay that at all.

RME: Are you worried when the students rely too much on the graph in order to demonstrate their claims?

M: Well, let's look at Student WD's response. In this case I am concerned about the answer being provided before the graph is produced but I also observe that the answer has been modified on the way – which may mean the graph did play some part after all in the student's decision making. If the student had drawn a line through points a and b, I would be a bit more convinced that the student is actually building the argument from what they see in the graph. I am also disappointed by the absence of a transition from the picture to some appropriate words and with the use of $a = b$ to denote that points a and b on the curve have same y. What a use of the equals sign! In this sense I am more sympathetic to Student LW who may need the Intermediate Value Theorem to complete the argument in part (i) – the IVT is true after all –, the picture is almost perfect, all the shifting etc. is there, but this is still an incomplete answer. Still there is no construction evidence. Unless it is in the rough work that they haven't submitted which is unlikely.

RME: There is a bit more writing: the definition in the first part and definition plus counterexample in the second part. Is this more satisfactory?

M: Yes, I see these merits in the writing too despite the use of the calculator suggested by the counterexample: another case reflecting the limits of the calculator screen.

RME: May I invite your views on the role of the diagram in deciding whether a function is one to one and onto (Chapter 5)?

M: Important indeed. For some of these questions there is no way to start other than draw the graph, look at a few values and decide. The students of course are often unsure about how to use a diagram and I am a bit ambivalent about them using a graphing calculator to produce these graphs, especially since they are rather weak in producing graphs by hand.

RME: I sense that Student WD may have produced the graph on a calculator and then reproduced it on paper by hand.

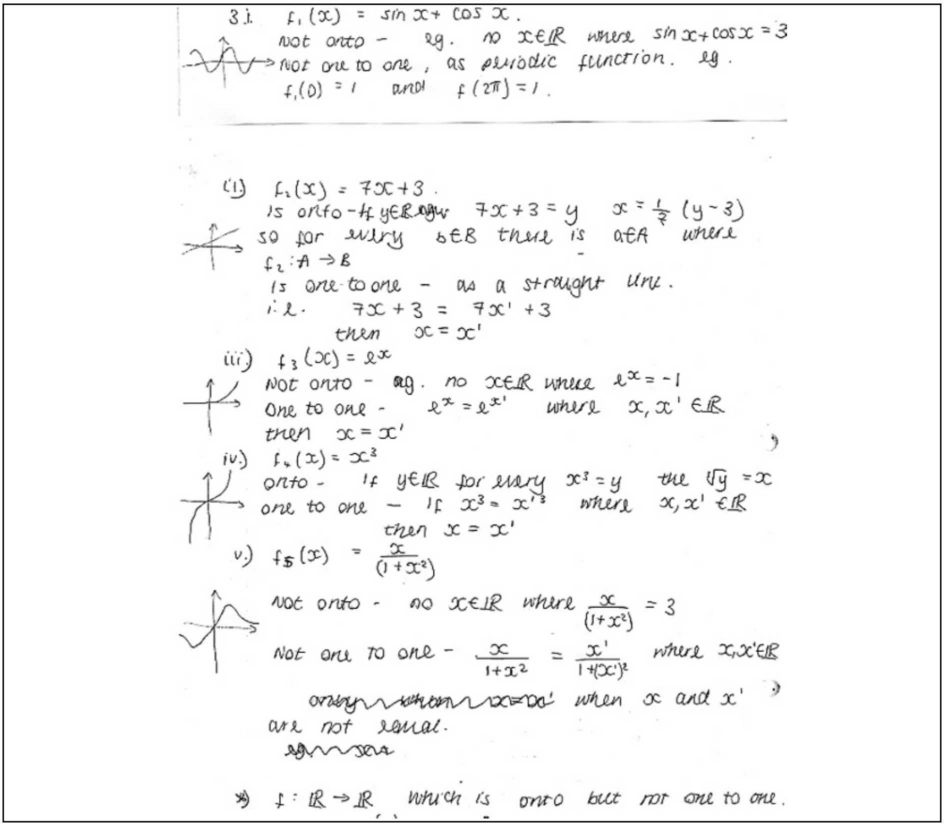
M: I think there is some irony in using the graph to produce evidence that a function is one to one or onto and I find this evidence compelling but still this is not a complete answer. This picture is potent and I see a certain danger in its sophistication: the fact, for example, that, if a function has a maximum, it cannot be onto is immediately graspable from the graph. However some unpacking is still necessary in order to provide a full justification of the claim. I am a proponent of starting with a diagram but I do not wish to see this placing value on starting with a diagram giving the students a false sense of obligation to do so, another hurdle to get over. I want them to think of doing so as a totally natural procedure to follow but also do it correctly – have you noticed that so often they either ignore labeling or do it incorrectly? Students should be allowed at this stage to use the graphs for something more than simply identifying the answer because after all they allowed to use all sorts of other facts – the uniqueness of cubic roots is one of those facts – that have not been formally established yet (E7.4, Scene IIb). So if the IVT is implicit in their finding the answer by looking at the graph, then let that be! Of course one needs to check: an actual value of a and b there would be very reassuring. At this stage I feel sympathy for them and want to let them say this function is onto because of the uniqueness of the cubic root. Because at this stage, well, I don't want to tell you what the cube root of two is … I want to tell you the cube root of eight is. I am not sure I even know how to exhibit the cube root of two without resorting to some quite sophisticated ideas. It goes without saying I would be far less frustrated if I could find evidence in the students' writing that the diagram is used almost as a third type of language, where the other two are words and symbols, as an extension of their power to understand: just drawing a diagram bigger, or, for example in the first picture, putting in a horizontal line that goes through the points a and b. I am afraid students do not use pictures to their full potential. Of course I see that relying on their power therein lies a danger but I would like to see students make a sophisticated use of this power and be alert to their potential to be misleading too.

RME: Do you think these graphs were produced on a graphing calculator to start with? This is Student JR's response.

M: Yes, a school classic, isn't it? Students not labeling the axes. OK, it may not be strictly necessary in this case. And from what is written they seem to understand what is going on. *No x in R where sinx+cosx=3*, well, this is true. And I think I can often say whether the students have used the graphic calculator because of the range of *x*. What I am frustrated about is that they do not engage with the required down to earth calculations (e.g. solving the quadratic for a particular value of *y* in part (v)) that will help them build an argument.

RME: This particular one was given to us in school almost as the one and only way to test whether a function is onto.

M: May I return to my earlier point about the dangers of jumping between rigour and non rigour: most of the functions appearing in the question have not been formally established; for the moment they are just mystical things, buttons on the calculator: sine, cosine, exponentials etc. Yet I am saying the students should be allowed to use them and their properties casually. Of course I cannot avoid the skepticism of how difficult it would be to introduce Analysis from an entirely foundational perspective. Strictly speaking parts of this question would be undoable if we did not allow for this use.


3.i. $f_1(x) = \sin x + \cos x$.
not onto – eg. no $x \in \mathbb{R}$ where $\sin x + \cos x = 3$
not one to one, as periodic function. eg.
$f_1(0) = 1$ and $f(2\pi) = 1$.

(ii) $f_2(x) = 7x + 3$.
is onto – if $y \in \mathbb{R}$ ~~sgw~~ $7x + 3 = y$ $x = \frac{1}{7}(y-3)$
so for every $b \in B$ there is $a \in A$ where
$f_2 : A \to B$
is one-to-one – as a straight line.
i.e. $7x + 3 = 7x' + 3$
then $x = x'$
iii) $f_3(x) = e^x$
Not onto – eg. no $x \in \mathbb{R}$ where $e^x = -1$
One to one – $e^x = e^{x'}$ where $x, x' \in \mathbb{R}$
then $x = x'$
iv) $f_4(x) = x^3$
onto – if $y \in \mathbb{R}$ for every $x^3 = y$ the $\sqrt[3]{y} = x$
one to one – if $x^3 = x'^3$ where $x, x' \in \mathbb{R}$
then $x = x'$
v.) $f_5(x) = \frac{x}{(1+x^2)}$
Not onto – no $x \in \mathbb{R}$ where $\frac{x}{(1+x^2)} = 3$
Not one to one – $\frac{x}{1+x^2} = \frac{x'}{1+(x')^2}$ where $x, x' \in \mathbb{R}$
~~only when x=x'~~ when x and x' are not equal.

*) $f : \mathbb{R} \to \mathbb{R}$ which is onto but not one to one.

Student JR

RME: In my mind I make a distinction between working on polynomials – perhaps familiar from school in the form of solving equations – and working on other functions, such as exponentials, not formally defined yet.

M: I guess I am happy with using the ingredients for proving a claim and then, at some later stage, spending some time on establishing those ingredients formally (E7.4, Scene II). So prove that e^x is injective via the IVT and then later on prove the IVT. This to me is fine as long as I know that all along I have been leaving some business-to-be-finished on the side. That kind of rigour is fine with me. On the other hand the moment I start thinking this way I am also starting to think, I am really keen on seeing some evidence of thinking, not just seeing on a graph. Some actual calculation of the maximum and the minimum, not just some pointing at a graph sketched on the basis of what is on a calculator's screen. I want them to be able to produce an accurate, elaborate graph and I want them to see the use of the calculator as a privilege that allows them easier access to this elaboration and as a privilege they ought to learn how to make the most of. That is much more convincing of their understanding.

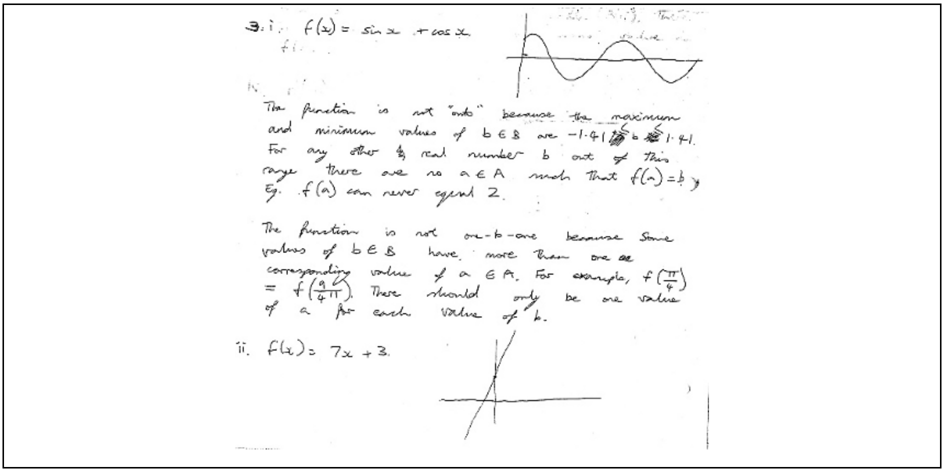
RME: Features such as tracing for identifying max and min are substantial help provided by the graphic calculator.

M: I hope we are united in praising the students for a beautiful diagram. Via the calculator or MAPLE or whatever.

RME: If they have used a calculator to graph the function I am surprised they haven't put the max and min down as this information is usually on the top of the screen on most GCs. I think this student, Student JK, is doing exactly that.

M: Before we move on to Student JK, let me make an observation on the use of language by Student JR in part (ii), in the argument for onto: I know what they mean but it is not accurately written down. And I am not happy with the claim that the function is one to one as a straight line. Across the response there is dubious use of previously un-established knowledge – the inverse, the exponential etc. – as well as too much relying on the graph. It's better than the other too but there is substantial detail missing. OK, now let's have a look at Student JK's response.

RME: Another case of the student looking at the screen of the calculator, being convinced, reproducing the appropriate definitions but not proceeding with producing justification for the reader. I noticed that Student JK produces examples to refute a claim but only produces the definition to accept a claim E3.3).


3. i. $f(x) = \sin x + \cos x$

The function is not "onto" because the maximum and minimum values of $b \in B$ are $-1.41 \leq b \leq 1.41$. For any other real number b out of this range there are no $a \in A$ such that $f(a) = b$, Eg. $f(a)$ can never equal 2.

The function is not one-to-one because some values of $b \in B$ have more than one corresponding value of $a \in A$. For example, $f(\frac{\pi}{4}) = f(\frac{9}{4}\pi)$. There should only be one value of a for each value of b.

ii. $f(x) = 7x + 3$.

Student JK Part (i)

It is one-to-one, as there is no $f(a)$ with the same value of b as $f(a')$.
ie. If $f(a) = f(a')$ then $a = a'$.

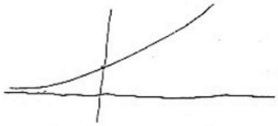
~~Thus~~ $f(x) = \tan x$ is onto but not one-to-o

Student JK Part (v)

Sets etc.

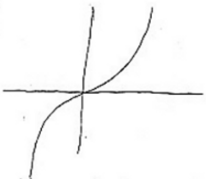
It is onto. – For every $b \in B$, where b is a real number, there is an $a \in A$ such that $f(a) = b$.
It is also one-to-one – Every value of a has only one corresponding value of b when $f(a) = b$.

iii. $f(x) = e^x$.



It is not onto, as there is no value of $b \leq 0$ such that $f(a) = b$. Eg $f(a) \neq 0$.
It is one-to-one because, as with (3(ii)), there is no $f(a)$ which has the same value as $f(a')$.

iv. $f(x) = x^3$.



It is onto, for every real number $b \in B$ there is a value such that $f(a) = b$.
It is one-to-one, for the same reasons as (iii).

1. $f(x) = \frac{x}{(1 + x^2)}$.

Not onto, as there is no value $b \in B$ where $-0{\cdot}5 > b > 0{\cdot}5$, where $f(a) = b$.
For example $f(a) \neq 1$.

Student JK Parts (ii), (iii) and (iv)

M: This is the classic problem with pictures. And with the murky ground of using mathematics that have not been proved yet: there is an assumption of continuity for example. And where is the physical solving of the equation to construct the inverse? But above all… Fascinating… 1.41… Clearly by inspection on a wretched graphing calculator. That is defying the whole point of learning Analysis. Boy, that is so infuriating, it brings up the worst in me. I would be so tempted to say that this is actually wrong! It is not true that the variable is bounded between -1.41 and 1.41. Look, if they write down exact things like that, like a specific value such as 1.41 they have to make sure that they are correct.

RME: It looks as if the calculator was set by default on two decimal points…

M: … and that went unquestioned by the student who could have just as well used -3 and 9 for the bounds and still be right if this line of argument was to be followed. Why did they bother finding out a more exact set of bounds, I wonder? Using the calculator in this fashion is stopping the students from thinking – they entered the function in part (i), graphed it on the calculator and did not stop to think about things such as the similarities of the graph with *sinx*. This approach is quite pathetically switching off their brains and it is a shame. At least the diagrams are bigger compared to the almost decorative small ones of Student JR – even though this looks like a rough sketch that could have been drawn by hand in the first place and doesn't seem to have gained much from the student's use of the calculator. I am sorry I am being sarcastic but I cannot help but notice the inappropriate scaling of the graphs presented on the script: I guess this may have to do with thoughtless use of the scaling on the calculator. Missing crucial parts of the graph because you have not bothered defining the range of values that the screen captures can be disastrous or misleading to say the least. And look at part (ii), look at part (v). I'm afraid I would give little mark to this kind of work: nothing here has been proved. The least I would expect is some explanation, or some indication on the graph, maybe some plotting of points that illustrate the claim and none of this is here. At least some assertion about the absence of values of the function above a certain value, not a mere repetition of the definitions. Maybe I would give them something little for doing that accurately but that's all.

RME: Is there anything else on Student JK you wanted to discuss[25]?

[25] Here much of the discussion of issues of visualisation revolves around the tension, paradox even (Kirshner & Whitson, 1997) between visual means as tools with mathematical legitimacy but without a ready capacity to assist student understanding and tools that allow student creativity but maybe somewhat more controversial in terms of mathematical legitimacy. Another issue underlying the discussion here is a questioning of the often rehearsed view of resorting to the visual means as 'panacea'. For a voice of doubt, if not dissent, see, for example (Aspinwall, 1997) which elaborates the case of Tim, Calculus student, whose uncontrollable mental images obstructed the complete resolution of a problem that involved

M then offers the comment discussed in SE5.1. Many of the issues discussed in this episode are revisited, but in a different context, that of Group Theory, in SE4.1 and E7.4, Scene IV.e. Within that scene M construes the juxtaposition of the tutor's use of a 'broad, brush representations' (while she 'had a very clear idea in her head about what idea she was trying to convey') and the student's 'entirely different meaning' for these representations as evidence that the tutor and the student are 'residing in different sets of thought and understanding'. 'It goes back to the same old idea of: if you are going to do a diagram, then do a good diagram' M exclaims and proceeds with a hefty critique of the tutor's diagrams. He then contrasts these diagrams to an 'absolutely wonderful' one suggested by the student later on, one that is 'suggestive of a fair number of properties' and is 'the compromised outcome of a fantastic fight between the student's own notions and the tutor's figures'. 'Any student that would come up with this […] deserves to get a degree immediately!' he raves. He then suggests an image of cosets that involves slices of a loaf of bread and balances his previously relentless critique of the tutor's approach with the 'somewhat justified' concern about the group-theoretical understandings the student's figure may generate (regarding the concept of coset). He is now worried though as he sees the figure as 'evidence of a very good step the student is taking towards understanding that the coset is not some external thing, it is a lump of stuff that comes out of some act between elements of the group. And that there are other ones in the group as a result of this act'. 'To her great credit' he concludes 'the student is transforming these not exactly right images [of the tutor's] into something more meaningful'. As the discussion unfolds M appears more willing to offer 'a more understanding interpretation of the tutor's reticence towards embracing the student's diagram' and to accept the tutor's diagrams 'as not so inappropriate'. In his view the student's diagram remains 'the more sophisticated one' and he marvels that 'to have a student drawing a picture that they haven't seen, at this level of abstraction is worth something'. '…coming up with four cosets, not two' is 'a pleasant surprise' as 'the only way to get four is to think of this whole lump being moved and moved and moved and moved…and that is so cool!' he concludes. The discussion continues with acknowledging the difficulty to represent visually some concepts in mathematics (e.g. compactness in Topology) and M expresses concern that he 'could nudge students in the wrong direction' as 'constructing helpful pictures is also a deeply personal issue' as the student's quest amply demonstrates in the Scene. The discussion closes with a discussion of images of the concept that the word *coset* may generate and a series of metaphors that can be used to represent concepts in Group Theory.

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derivatives. One of the limitations of resorting to the visual means comes in the paper in the shape of the notion of an *uncontrollable image*, an image which may persist, thereby hindering the opening up of more fruitful avenues of thought. The more vivid the image, the more acute this hindrance may turn out to be. In Tim's case, although his images generally supported high levels of mathematical functioning, occasionally his vivid images became uncontrollable, and the power of these images did more to obscure than to explain.

EPISODE 4.4: UNDERVALUED OR ABSENT VERBALISATION[26] AND THE INTEGRATION OF WORDS, SYMBOLS AND DIAGRAMS

Setting the scene: M and RME discuss their concern about students' inadequate appreciation of expressing their mathematical thoughts verbally on several occasions[27]: extensively so in E3.5, SE3.1, E6.2, SE6.1 and E7.1 Scene VI. For example M here is a reminder of the comment with which M concludes the discussion of what could possibly trigger students' noticing that they are omitting a small but significant number of terms in the sequence they are working on in SE6.1:

M: It seems that after all the presence of the quantifiers themselves in the text of the question is not emphatic enough to suggest universality or existence to the students. And words, sentences, those creatures ever-absent from students' writing exist exactly for this purpose: of emphasis, of clarification, of explanation, of unpacking the information within the symbols. Especially within complex and subtle statements such as the definition of convergence.

But the value of this realisation often escapes the students, M claims. The following takes place in the context of the discussion in E3.3, Scene I.

RME: Could you say a bit more about how a good grasp of Grammar helps the students' mathematical expression?

M: It should be made clear to the students that this type of command of the language is not irrelevant to good mathematical writing. And that applies all the way through to completion of their studies. I sometimes see final year students and I wonder whether they deserve marks for a response that I could only detect as correct amidst grammatically incorrect statements. When you put things on paper with such ambiguity and inconsistency, such as sentences without verb or subject etc., maybe you should expect a lesser reward too.

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[26] Furinghetti & Paola (1991) have discussed students' understanding of mathematical texts (e.g. problem sheets, lecture notes) in the context of difficulties with formal proof. Others have focused on particular aspects: e.g. Dee-Lucas & Larkin (1991) found that proofs written in a verbal, ordinary language produced better performance than equation-based proofs on problems related to both equation and non-equational proof content. Equations cause students to shift attention away from non-equational content and learners have more difficulty processing equations than verbal statements of the same content. Similarly (MacGregor, 1990) writing sentences helps students write correct equations (incidentally, contrary to expectations, the most successful students in that study were those who used common idiomatic forms of English that could not be directly translated into mathematical notation).

[27] The issue of absence of verbalisation in the students' writing is treated in the discussion here mostly as an indication of the problematic relationship students often have with the notion of making their thinking transparent to the reader of their work. However 'proofs without words' – see (Jaroma, 2005) and (Osler, 2005) for two recent examples in *The Mathematical Gazette* – exert a certain fascination on mathematicians, perhaps because of the inherent elegance in their capacity to convey meaning implicitly.

RME: I suspect students may not even be aware that putting things on paper in a grammatically and syntactically correct manner actually matters at all!

In E6.2 M and RME discuss the connection – often eluding students – between the verbal, formal / symbolic and visual representation of a concept; in this case convergence of a sequence. 'Students don't quite understand the relation between this expression and what convergence ought to mean exactly' M observes there. 'Making this link between this image [previously in the discussion he is proposing an image of convergence involving a box] and the formalisation behind this is utterly important. Otherwise the definition is nothing other than formalistic nonsense' he stresses. He then praises the use of words 'such as *eventually* and *arbitrarily*' to establish this link further – words that unfortunately are often 'ignored by students as irrelevant waffle'.

In that scene M recommends caution about the above integrated approach: 'But using words is risky: I have seen verbal explanations of the definition which are in fact wrong!' he warns. 'Verbalising, geometrisising it etc. is fine as long as we stay this side of correctness!' he concludes and stresses that at times getting the students to work through strings of quantifiers can be effective: 'steering clear of intuitions and pictures […] even though one may not be so sure of what is going on, can be seen as less messy, less risky'. This following of formalistic steps may not be to everyone's taste, he concludes, but since 'there is not always a good intuitive picture of everything' we may sometimes have little choice about avoiding it!

SPECIAL EPISODE 4.1: THE GROUP TABLE[28]

Setting the scene: Group Theory is a topic often perceived as not lending itself to visual representations easily (see also E7.4, Scene IV.e). The following takes place in the context of M and RME's discussion of the question below the lecturer's suggested responses and two student responses, Student W's and Student H's:

> **Q1.5:** Write down all group tables for a group of four elements. Hence show that there are two essentially different such groups, both commutative. (Consider group tables obtained by merely renaming elements as essentially the same). How are they best distinguished? For each make a list of all the subgroups.

[28] For a preliminary discussion of the data used here see (Iannone & Nardi, 2002). For discussion of conceptual difficulties in the area of Group Theory see (Dubinsky et al, 1994) and (Nardi, 2000a).

Example from Exercise Sheet 1, Week 1, Autumn Semester 2001 - Group Theory

A little more difficult: Write down all group tables for a group of 4 elements. Hence show that there are two essentially different such groups, both commutative. (Consider group tables obtained by merely renaming elements as essentially the same). How are they best distinguished? For each make a list of all their subgroups.

Notes on Solutions

In the first table below, the first element to decide about is x. If $x \neq 1$, say $x = b$ then you will find that there is just one way to complete and this is the second table. If you select $x = 1$ then you can complete the first two rows/columns uniquely. For the third diagonal entry y you have a choice: $y = a$ gets you back to the first table. This is easy but you need to check this carefully. (Have in mind that elements can be renamed.) For $y = 1$ you can complete uniquely, to get the third table. This doesn't say that these are actually groups, only that there are at most two. However, $\{\text{id}, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$ is a group that satisfies the first table. To prove this use the theorem that a non-empty subset H of a group (here the symmetric group on four letters) is a group if and only if $a, b \in H$ implies that $ab^{-1} \in H$. You need to check this for all pairs $a, b \in \{\text{id}, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$. Observe that you can identify $a := (1,2,3,4)$, $b := (1,3)(2,4)$, $c := (1,4,3,2)\}$. On the other hand, and by very similar arguments, $\{\text{id}, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$ is a group that satisfies the third table. The first group has only three subgroups: itself, $\{1\}$ and $\{1, b\}$ while the second has more: itself, $\{1\}$, $\{1, a\}$, $\{1, b\}$ and $\{1, c\}$. What distinguishes the groups is that $x^2 = 1$ holds for all elements in the second group, but only for two elements of the first group, have a look at Question 3. Both groups are commutative as their tables are symmetric about the diagonal.

$\circ$	1	a	b	c
1	1	a	b	c
a	a	x		
b	b		y	
c	c			

can be completed only to

$\circ$	1	a	b	c
1	1	a	b	c
a	a	b	c	1
b	b	c	1	a
c	c	1	a	b

or to

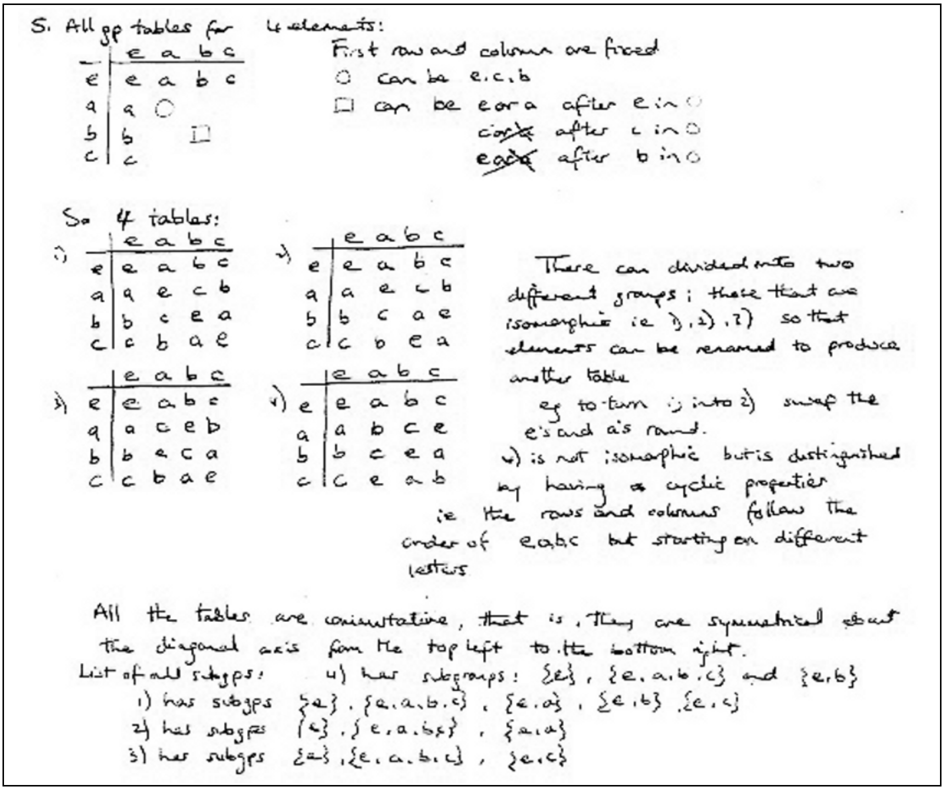
	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

.

The lecturer's suggested response

Can divide into 2 ~~groups~~ different groups
i) Group has 2 self-inverting elements (tables 1, 2, 3)
ii) Group has all self-inverting elements (tables 4)
Both commutative as the tables are symmetric di the left diagonal.
Subgroups for i)
(e), (a), (b), (c) and (e, a*) [and (e, a, b, c)] * a, b, or c depending on which element is self inverse.
Subgroups for ii)
(e)(a), (b), (c), (e, a), (e, b), (e, c) [(e, a, b, c)].

Student W



5. All gp tables for 4 elements:

	e	a	b	c
e	e	a	b	c
a	a	○		
b	b		□	
c	c			

First row and column are fixed
○ can be e, c, b
□ can be e or a after e in ○
c or b after c in ○
e or a after b in ○

So 4 tables:

1)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

2)

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

3)

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

4)

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

These can divided into two different groups; those that are isomorphic ie 1), 2), 3) so that elements can be renamed to produce another table
eg to turn 1) into 2) swap the e's and a's round.
4) is not isomorphic but is distinguished by having a cyclic properties ie the rows and columns follow the order of e,a,b,c but starting on different letters

All the tables are commutative, that is, they are symmetrical about the diagonal axis from the top left to the bottom right.
List of all subgps: 4) has subgroups: {e}, {e, a, b, c} and {e, b}
1) has subgps {e}, {e, a, b, c}, {e, a}, {e, b}, {e, c}
2) has subgps {e}, {e, a, b, c}, {e, a}
3) has subgps {e}, {e, a, b, c}, {e, c}

Student H

RME: In this example the use of picture is in the form of a table – for representing a group.

M: I can locate the origin of using a pictorial representation as far back as in Burnside's book[29]. I don't think I know of any other book that, that early on, attempted to introduce this image. But the pedagogical interest of this picture is in the fact that it makes the task of constructing a group appear doable. And that is something students find very appealing. So we feel compelled to include this type of task. There is a lot of valid information about the group to be learned just by constructing this table. Not that you need to understand what a group is at all to complete this task: you just need the instruction that in each row and in each column you must have every symbol. That's all. And this sense of do-ability is compelling. Even Burnside, who was writing for professional mathematicians,

[29] William Burnside's classic work *Theory of Groups of Finite Order* was published in 1897. The second edition (1911) was for many decades a standard work in the field, and is still useful today.

himself being a leading group theorist at the time, thought that it was necessary to do that. While Peano certainly would have never thought of introducing such a thing. Or any of the continental group theorists would have had such a thing. The appeal of this is something specific in the cultural environment of attraction to the concrete and the specific. At this early stage when students know not much more than the definition of a group – also a bit about the Isomorphism Theorem, permutations, not much about things like a group of order p squared must be Abelian or about groups whose order is a power of two – this tabular task can help them realise that there is just one group of order three. But, frankly, for groups of order four I think this method almost fails. Still this type of doable task is popular.

RME: Does this emphasis on do-ability risk overlooking other significant things they need to learn at this stage?

M: Students seem to enjoy working on this type of task but personally this is culturally a bit outside my own way of being interested in Group Theory. I feel a bit distant from this way of building up an understanding of groups in terms of their order: 1, 2, 3, 4, 5, 6… etc. when things begin to become complicated and various rules and theorems must be put to use. I know a course which builds Group Theory around this task up to really high orders is possible and I can see that having a sense of purpose is the benefit of this linear, gradual approach. It just doesn't fully appeal to me.

RME: This neatness removes the sense of uselessness from Group Theory students often complain about. May I invite your views on the kinds of understanding about the concept of group that the use of the table promotes?

M: As I said the concreteness of the representation is congenial. And I think it promotes the understanding that every group has a permutation representation.

RME: Do students miss the fact that a perfectly reasonable table may exist that does not represent a group – this is the case when you are looking for groups of order 6 – because the operation is not associative?

M: Yes! The question requires them to first construct all possible tables and then decide which of these represent groups. But the construction of the tables often takes over, students perceive the task to be about that. This misleading impression may be the outcome of perceiving the task as being about filling the table, a perception that overruns the intended one which is to construct the group(s) of four elements. Of course you may ask: how are the students to see that a particular table does not represent a group? Well, construct the table and check associativity – which is not fun most times to be honest! And which brings me to my first major concern about the use of the table: it does not help you at all to see associativity; it helps you see commutativity and it may help you identify subgroups. But associativity, no! So I am concerned that students may think the existence of the table implies the existence of the group. Then again engaging with the construction and then discussing this crucial distinction that there are more tables than groups once you have controlled the issue of the isomorphism, for example in the case of order six, could be educational, don't you think?

RME: Sure.

M: You could give this is as a problem in order to get students to appreciate the distinction… Well, maybe not students that you would see again! I am joking but only half way here: filling the table all the way for the case of order six can be a daunting task. But I do think this is a really nice problem and a neat way to think about some elementary properties of groups. Shall we look at the particular student examples then?

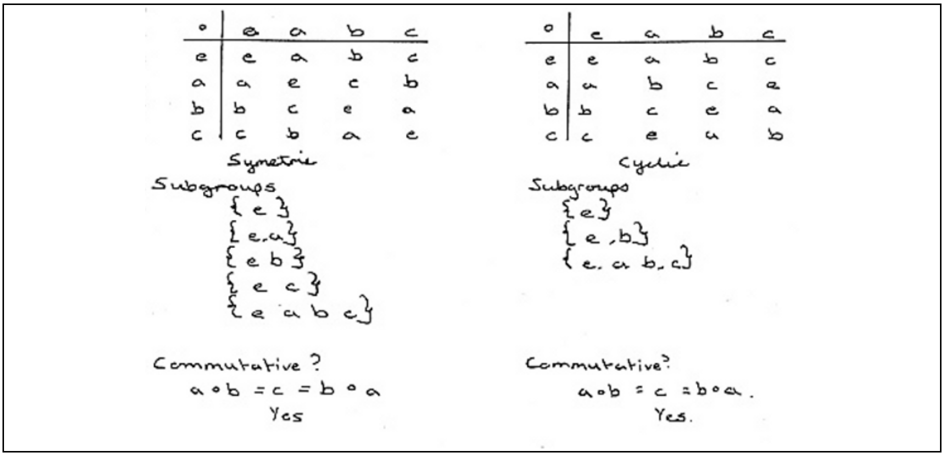
RME: I was concerned about Student H's *4 is not isomorphic*. To what, I would ask.

M: My guess is she means to some of the other groups she has cited. But I am being generous here: she should at least say something along New Testament lines, *to that which came before*! From this piece of writing I am not necessarily obtaining the impression that the student understands what an isomorphism is; for example, that it suggests identical subgroup lattices for the two groups. This lack of awareness that groups with different subgroup structure cannot be isomorphic is typical of most students anyway. I am not underestimating the sophistication of this but it is crucial to be aware of this fact.

RME: Yes, this is all about what *essentially the same* means. Student W seems to be using somewhat sloppy notation: () as opposed to {} for sets etc..

M: On top of this, the response is wrong anyway.

RME: Shall we talk a bit more about what aspects of the concept of a group are possibly concealed by the tabular representation. This is Student E's response.



Student E

M: I think the difficulty to check associativity is a major concern as I said earlier. I wonder about the status of the word *cyclic* in this. At least students have not come up with any suggestions for subgroups of order three! I find the fact that all that the students have to say about this question – for example, decide on commutativity just by checking for two elements – depressing. This is similar to Student W…

RME: Well, there is something more in there in terms of explaining. Here table-based actions have taken over and the student sees little in the task beyond the table.

M: Yes. And of course there is no evidence that they know why these groups are different which, of course, in the long run is a very deep part of the topic. What is even more disappointing is that the table is not used in order to contextualise the knowledge to come – for example, Lagrange's Theorem. There are quite a few opportunities for property noticing where the table could be used: notice we found no subgroups of order three and then, when Lagrange is introduced, link this discovery to the fact that the order of a subgroup must divide the order of a group. So, since three does not divide four, we now know we couldn't have found a subgroup of order three. The table is also a good tool for discussing observations on commutativity: that there is only one commutative group of order four, two for groups of order six etc.. The table in this case is indispensable.

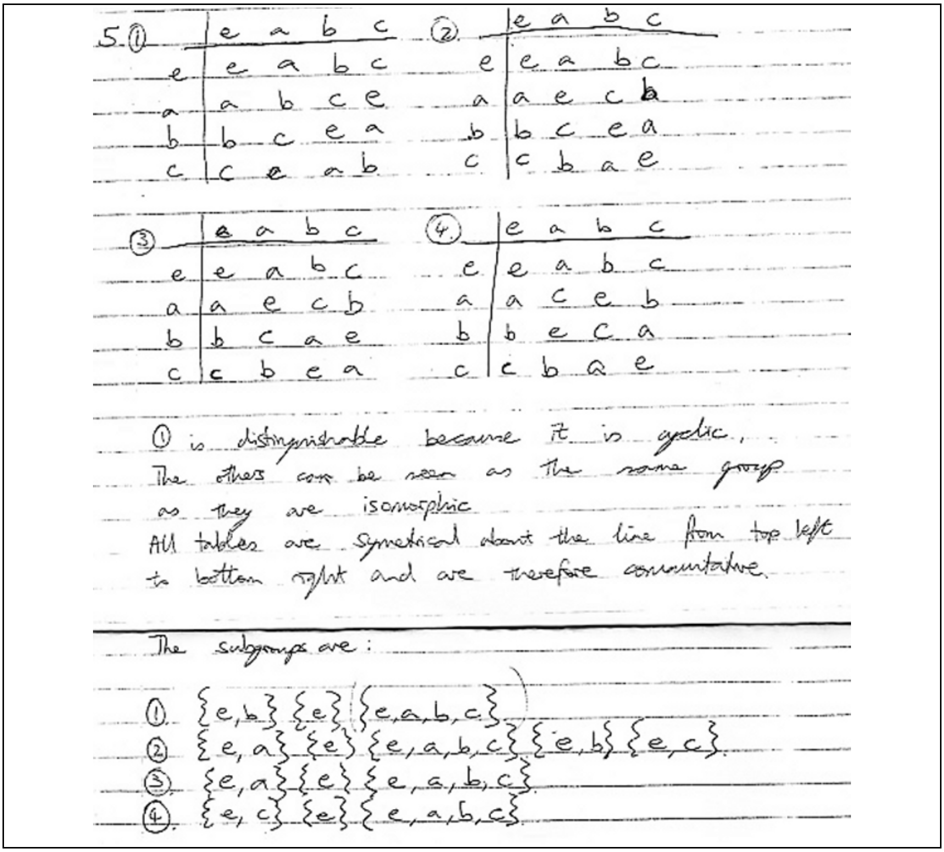
RME: We seem to agree that the table is to be taken as scaffolding for building up an understanding of groups and their properties. When would you then remove the scaffolding, if at all?

M: Fairly early on I think. This is useful at the early stages where the students need to practice and clarify notions of operating between the elements of the set, achieving a better grasp of the rule for the operation. But then, once this has been achieved, you can forget about the table more or less.

RME: To me the table is putting forward an image of a group as a set *with* an operation as opposed to simply a set.

M: I agree. And you can do the tables – for example, using colours – so that subgroup lattices and cosets can be identified.

RME: You did say before that students often do not see that isomorphic groups cannot have different subgroup lattices. This is Student JK's response.


5. ①

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

②

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

③

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

④

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

① is distinguishable because it is cyclic,
The others can be seen as the same group
as they are isomorphic
All tables are symetrical about the line from top left
to bottom right and are therefore commutative.

The subgroups are:

① {e,b} {e} {e,a,b,c}
② {e,a} {e} {e,a,b,c} {e,b} {e,c}
③ {e,a} {e} {e,a,b,c}
④ {e,c} {e} {e,a,b,c}

Student JK

M: I am not sure about the correctness of the tables and I would have liked to see more explanation of why these groups are isomorphic. Also there are similarities with the problems Student H had in writing out the subgroup lattices. I realise this is the very beginning of their studies in Abstract Algebra but I would like to see more overt checking of properties – perhaps exactly because they are at the beginning!. Let me also stress my concern that using the table is within the more general frame of being driven by what you can do. And I am concerned because somehow how far such an approach gets you in terms of understanding must be more prominent than what this line of argument – this priority of do-ability – usually allows you to consider. This approach takes for granted, for example, that a definition – proof – lemma leads to lesser learning and I am not necessarily happy with this idea. Without excluding the use of the table, I would like to say that I favour, at least equally if not more, an approach to teaching Group Theory that includes a good stock of examples, strong association with permutations and an emphasis on building this stock from the rules that define the concept of a group. Tables may come in but only in auxiliary roles.

OUT-TAKE 4.1:
TYPED UP

Setting the scene: The following takes place in the midst of the discussion in E3.6.

RME: Before we move on, there is another example of student work I would like to share with you, Student LW. It is almost wordless.

M: But typed in Microsoft *Word*!

RME: Indeed!

M: It completely lacks punctuation… oh, hold on there is full stop over there. But maybe this is just a typo…!

RME: … and its layout resembles that of a computer programme. May I have your comment on *since these have no common factors this cannot be a quotient*?

M: Maybe this student is hoping that we will read it so quickly without absorbing anything that the student is saying! Maybe this student has been planted by the auditors to check that we actually read the work that we mark! Maybe this student is not offering us his handwriting because he is downloading answers from the Internet! Or maybe he thinks we will think that because what he writes is typed, it must then be true! I am so annoyed by the spelling – oh, that poor *therefore* – by the cut and paste approach across part (i) through to (iii) of the script, by the shortcut spirit of it. All it contains is a description of the Mathematical Induction mechanism and let me at least praise – for the sake of not sounding like a complete bigot against this piece of writing – the fact that at least in 4(ii) the student is starting with n = 4 and not n = 1.

RME: Which, I assure you, is not to be taken for granted as other students have not (E3.6) .

M: This person's writing is like watching somebody else doing mathematics and simply noting the various stages of the proof. That's repulsive!

RME: Please don't upset yourself too much. This is the only Word-processed piece of student work I have ever seen! It is an extreme case and I am only presenting this to you as a crash test for your nerves!

MTH-1A11/13/15 Autumn 2000: Exercises 1

(1)
$\sqrt{2} = M/N$
$2 = M^2/N^2$
$2N^2 = M^2$
$N^2 = M^2/2$
$N = \sqrt{(M^2/2)}$

Since these have no common factors this cannot be a quotient.

(2)
X is a real number
It is bigger than or equal to 0
All values of n are natural numbers
X is smaller than 1/n
There for X must be zero as it is smaller than 1/∞

(4i)
$1^3 + 2^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4$
For n = 1 this is true as $n^2 (n + 1)^2 / 4$ also equals 1
$P(1) \Rightarrow P(2)$
P(2) is true
$P(2) \Rightarrow P(n)$
P(n) is true for all natural numbers

(4ii)
$2^n \geq n^2$ for all natural numbers ≥ 4
P(4) $2^n = 16$ and $n^2 = 16$
Which is true
If P(n) is true it follows that P(n+1) is also true
$P(n) \Rightarrow P(n+1)$
Therefore P(n) must be true for all values of n and so the equation holds true for all natural numbers greater than 4

(4iii)
1/1x2 + 1/2x3 + … + 1/n(n+1) = n/n+1
For n = 1
LHS = 1/1x2 = ½
RHS = 1/1+1 = ½
If P(1) is true it follows that P(2) is also true
$P(1) \Rightarrow P(2)$
And it also follows that from P(n), P(n+1) is also true
$P(n) \Rightarrow P(n+1)$
Therefore P(n) is true for all natural values of n

Student LW

CHAPTER 5
THE ENCOUNTER WITH
THE CONCEPT OF FUNCTION

The concept of function as commonly known through the Bourbaki definition (a correspondence between two sets which assigns every one of the elements in the first set to an element in the second set), plays a central and unifying role in mathematics. However most students do not generally associate function with this formal definition. Their images of the concept (e.g. Vinner, 1983) are varied and numerous. Most of the studies that examine these images do so in an Analysis / Calculus context (e.g. Dubinsky & Harel, 1992); in the Episodes that follow the focus is cross-topical and the students' encounter with the concept is seen in a variety of contexts from Analysis, Linear Algebra and Group Theory.

One element that runs through most of the discussion in this chapter is how lack of flexibility in working across different representations influences students' encounter with the concept of function. Anna Sfard (e.g. 1991) discussed the importance of this flexibility in terms of establishing connections between static (object) and dynamic (process) aspects of the concept. This duality, which is also part of this concept's epistemological power, is at the heart of the discussion in many of the examples M and RME discuss in this chapter (most prominently in E5.3 and SE5.2).

Furthermore M and RME describe the students' encounter with the concept as follows: one particular perception of function, function as a formula/rule, seems to dominate students' concept image often at the expense of engaging with essential constituent elements of the concept such as domain and range (E5.1 and SE5.1). In terms of how students relate to graphs, the immediate appeal of a graph that reveals several features of a function in an instant conflicts with students' uncertainty about the legitimacy of resorting to it for such information – as well as with their often limited ability to construct graphs in the first place (E5.2). In terms of how students relate to the symbolism and mathematical language used in the context of functions M and RME revisit some of the issues discussed in Chapter 4 as they comment briefly on the many and variably effective terms used for one-to-one and onto functions (OT5.2) and the potentially damaging effect of using R^R as a symbol for the set of real functions from R to R (OT5.3). Finally M offers a fleeting but illustrative exposition on the history of the concept of function in order to support a parallel between the phases students need to go through in their encounter with the concept and its epistemological evolution (OT5.1).

EPISODE E5.1
CONCEPT IMAGES AND CONCEPT DEFINITION

Scene I Domineering presences (function-as-formula)[1], *conspicuous absences (domain-range)*

Setting the scene: In what follows an incident revolving around whether using n_ε. is more appropriate than using $n(\varepsilon)$ in a proof of the Archimedean property becomes the starting point for discussing one particular perception of function, function as a formula, that seems to dominate the students' concept image often at the expense of engaging with essential constituent elements of the concept such as domain and range. The following takes place in the context of discussing the following excerpt from (Nardi, 1996) as presented in (Nardi, 2000b).

> (Week 2, Year 1 Analysis course) A first-year mathematics undergraduate is presenting a proof of the Archimedean Property:
>
> $$\forall x \in \Re, \exists n \in \aleph \text{ such that } x<n.$$
>
> In this he is reproducing, from his lecture notes, the definition of a supremum a for the set of natural numbers $\aleph$:
>
> $$n \leq a \ \forall n \in \aleph \text{ and if } \varepsilon>0 \text{ then } \exists n_\varepsilon \in \aleph: a-\varepsilon<n_\varepsilon.$$
>
> The student points at n_ε and says he does not 'understand this notation'. His tutor replies that the lecturer was trying to 'produce a bit of inflection':
>
> Tutor: You specify ε first and then you are choosing n, therefore your choice will have to depend upon ε. But that doesn't mean there is a function.
>
> He then calls this notation a bit 'perverse' and recommends its use for specific values of ε: for instance for ε=1, write n_1. The student is puzzled and wonders whether he should avoid using this notation altogether.

RME: The reason I am showing this to you is in order to invite your comments on the meaning of n_ε as seen by the student and discussed with the tutor, for example in *but that doesn't mean that there is a function*.

M: I think most people would agree that the relationship between epsilon and n here is not necessarily a function $n(\varepsilon)$ but I would say there is at least one function. Even though I see that seeing this as a function can be immensely irrelevant and confusing. All you need here is simply the idea that for every interval defined by

[1] Students often adhere to the 'function as formula' (Nardi, 1992), as rules with regularities and hey often identify function with just one representation – usually either the symbolic or the graphical. In Ferrini-Mundy & Graham's study first semester calculus students (1991) 'were not able to provide any type of general definition of function but readily gave examples of functions by writing formulas. There is little evidence that the students see functions as objects of study in mathematics; rather, when a function is given, in equation form, usually one is expected to do something to it, such as substitute in a value. This part of studying functions (plugging in values) seems to be firmly established and becomes their way of working with other calculus concepts such as limit' (p630). (Even (1993) reported similar tendencies of more advanced learners such as prospective mathematics teachers).

your choice of epsilon, there exists a natural number which lies beyond this interval. The important thing is to do things in the right order – first define your epsilon, then find your n so that its relation to your choice of epsilon is clear. You can have an approach where this relationship is defined as a function but it is not necessary to pull this kind of trick. A mere relationship is fine. In a sense it is better to stay clear of the word *function* as so often in English we use it to mean *something depending upon something* and I am not sure we want to go heedlessly down this avenue. The next thing students might expect then is a formula for the construction of n_ε! Which is their dominant image of a function anyway, regardless of domain etc., away from the definition of a function as a quantified statement: for every x there is a y such that $y = 2$, so y is a function of x here, even though it appears not to be. It is because of this wanting perspective of a function as a formula, not as a relationship between elements of specific sets that students often do not see the constant function as a function at all[2]. And I would not like creating a situation here where this flawed perspective is perpetuated in the apparently innocuous and different environment of a discussion of *supremum*!

RME: I noticed the tutor did not elaborate on finding any one of those precious n.

M: It makes a difference if you think you are looking for that *one n* or that it is one amongst many out there. I favour the bracketed $n(\varepsilon)$ because it highlights the dependence between the two without suggesting uniqueness or whatever. Of course you may retort this symbolism is not essentially necessary as, in fact, n does not need to depend upon ε.

RME: You touched briefly on students' images of the concept of function with which they arrive at university. Shall we talk a bit more about this?

M suggests that, in a discussion like this, revisiting briefly the history of the concept of function might be useful (OT5.1) and then offers the following comments on the images of function students arrive at university with.

M: It is difficult for them to perceive the importance of sets and axis labeling. It is the measurement of a quantity in terms of another that is the students' focus. Observe and measure is what they have associated functions with: if I measure the temperature of every point in this room – or the electric field in every point in this room – then I have a function of three variables (for the location of the point in space). Range/domain are almost irrelevant to them because they are defined by the general problem you look at. Say, if you are solving a differential equation, your range and domain are somehow given by the context.

[2] Students see a change in the independent variable as causing a change in the dependent variable, with the consequence that constant functions are often not considered as functions (Nardi, 1992).

Temperature will not be smaller than, say, 280 degrees Celsius. So range and domain are not so important in a way. And this kind or approach marks the beginning of a battle for me because they have to see the importance of range, domain and co-domain sooner or later. For example, they have to start seeing the mechanism of Mathematical Induction as a function running across *N*.

RME: You mentioned earlier the standard student expectation of a formula for a function.

M: In fact one formula for every function. A function like $1/xsinx$ defined as zero at zero is interesting because understanding its definition involves seeing this expression in terms of the various values of x in *R*, realising the need to define it via a different formula or value at a point where this formula doesn't work, $x = 0$. The problem is that they would see this as more than one function because it is defined in terms of more than one formulae or relationships.

RME: Here is an exercise (the one cited at the end of E3.4) given to the students in the first weeks of the course when the definitions of one-to-one and onto were being discussed. And here is Student N's response who is replicating a picture used in the lectures.

M: I would certainly commend his extensive use of language. This is far from common for most students (E4.4).

RME: He is slightly uncommon in the sense that he was not educated in the UK.

Before commenting on Student N's writing M offers a critique of the question.

M: In any case this is not an easy question. Especially part (iii). I guess an easier alternative for part (iii) is: prove h is bijective if either of f and g are? To be able to talk about the injectivity etc of composition of functions is a high demand. The order of things makes it easy to make mistakes and as a lecturer you would often look at your notes to make sure your order is correct and then pretend it is… obviously so[3]! You are pulling a stunt there that is almost a lie. I think there is a difference between parts (i) and (ii) and part (iii): the former are about practicing a definition and the latter is more sophisticated. Perhaps it is more suitable for the more able.

RME: What is then a satisfactory answer to part (iii)?

[3] In Chapter 7 M returns to the issue of transparency and de-compression of the mathematical arguments we present to students on several occasions.

M: A picture (E7.4, Scene IVc) for sure with bubbles and arrows[4]! Going the formula way may confuse students.

RME: Does the question, by requiring *f*, *g* and *h* in part (iii) to be from *R* to *R*, take away an opportunity to think in terms of the significance of domain, range etc.? I wonder.

M: I agree.

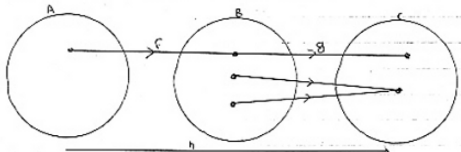
RME: And the bubbles in Student N's writing do not look like *R* at all!

M: It's not long and thin!

(4)(i) If f is onto, then every element in A has a corresponding element in B.
If g is onto, then every element in B has a corresponding element in C.
It follows then that every element in A is have a corresponding element in C => h is onto

(ii) If f is one-to-one then every element in A has one and only one corresponding element in B.
If g is one-to-one then every element in B has one and only one corresponding element in C.
It follows then that every element in A must have one and only one corresponding element in C => h is one-to-one.

(iii) A=B=C=R h is one-to-one, g is not one-to-one



For h to be one-to-one f must be one to one

It follows also that f cannot be onto as an element in g.A must correspond to one and only one element in B which is shown must correspond to one and only one element in C. But as we have said that g is not one-to-one then the one-to-one rule specified above must be seperate from the "many-to-one" part of g (as shown in Venn diagram)

$\Rightarrow f(x) = e^x$ and $g(x) = \begin{cases} 2x+1 & \text{if } x>0 \\ 0 & \text{if } x\le 0 \end{cases}$

Student N

[4] Student N seems to have successfully spotted this and has offered 'bubbles' in his part (iii).

Scene II The students' turbulent relationship with the concept definition[5]

Setting the scene: The following takes place in the context of discussing students' responses to the same question as the one at the end of Scene I. The starting point of the discussion is (Nardi, 2001), a short paper that M had read prior to this discussion with RME. The paper focuses on students' difficulty to conceive of the relationship, the domain and the co-domain as three equally important aspects of the concept of function.

RME: I would like to hear your views on how the students build up some understanding of the definitions. For example, towards the end of the paper, one student claims: *It is a bijection but there is one and only one solution to the equation* $y=7x+3\ \forall y \in R$.

M: They could have used *for each y* instead of *for all y*. I am concerned about the use of quantifiers here. I also think the appearance of y for the values of the function indicates that they are perhaps thinking graphically. And I detect a certain indecisiveness in the student's writing [p51 of the paper] where the initial claim *onto* is changed somewhere down the line into *not onto*. And [on p53] there is the statement *there is no f(a) which has the same value of f(a')* without specifying the relationship between *a* and *a*'. I wonder what the intrinsic difficulty of the concept of function is. I wonder whether a Bourbaki definition of this concept is too weak a building for the students at this stage: do the students have too few previously acquired building blocks in order to deal with the Bourbaki definition[6]? Of course I would like to stay clear from predicating on the assumption that our students fail to learn much. They do eventually get there but perhaps too slowly for such a basic concept, a concept that soon enough must become second nature, a word which the student can say without having to recall definitions or any other props. This is such a basic notion that it cannot be done early enough. Maybe students' ideas that they have fixed already need to be challenged a bit? If they write down a formula of a graph and that is part of the

[5] Vinner & Dreyfus (1989), in their application of the *concept image* - *concept definition* schema on the learning of function, suggest that concept images are not simply formed by definitions but by experiences. This explains the diversity of concept images associated with the concept of function: a correspondence between two variables, a rule of correspondence, a manipulation or operation (put one number, take another), formula/algebraic term/equation, graph.

[6] Function is a concept that has been evolving in the last 4000 years, from a complex network of conceptions as a graph, a formula, a relationship, an input-output machine to the unified and rigorous Bourbaki set-theoretic definition. In fact research interest in the concept was heightened on the aftermath of the New Math era when the definition was used widely as the main way of introducing the concept to students – with often unimpressive results. Malik (1980) identified the gap between the Bourbaki definition and the rule-based relationship between dependent and independent variables as a source of these difficulties and Sierpinska (1994, p50) contended that the latter, as opposed to the former, is fundamental for understanding functions within the meaningful contexts of formulas and graphs that students are likely to be using them in.

problem, we may then have to challenge them: if you want to find the inverse of this function, in what circumstances can you do that? And get these things hard at them and things start to break down. It's no good if they go through all this first year stuff and they come up thinking, well, that was pretty tough, wasn't it, but I will just carry on with what I was doing at A level. It would be very interesting to ask them *what is a function?* on the first day of the first year and then the second year because I am sure that on the first day of the first year they would know and on the first day of the second year they wouldn't know. They have seen this building collapse several times. All the previous notions of function would be declared invalid. And in the third year they would resort – there is the proud algebraist that lurks in all of us! – to morphism or something like that and be absolutely safe! OK, ok I am joking: a context-bound notion of function may be more useful. Trying to handle a tool at the pure, abstract, generic level would be meaningless to them.

Concluding the discussion of the material in the paper M also comments on the role of pictures (graphs etc.) can play in forming students' concept images thus introducing the discussion in E5.2.

M: You see a graph as a concept image of function is perfectly suitable in Analysis but rather useless in Algebra. Nobody would ever draw a graph of a morphism or a permutation. And the student is supposed to jump between these paradigms at great speed and use the essence in each paradigm to the best effect. Is that reasonable? I don't think it is reasonable. A picture, a graph in Analysis is a fundamental tool, mathematically it is absolutely essential that you can think in those terms. In Algebra it is almost meaningless. The word graph is a completely superfluous notion. I guess the expectation from the students is that they learn how to distinguish between what are the important images that are supposed to work in one field but not in another. From a teaching point of view I would of course question whether it is acceptable to call all these things functions in the different contexts but not discuss that in fact they are all the same thing. Can we reasonably expect that the commonality across all these uses of the concept will naturally emerge in the students' brains? You may as well say that prematurely indoctrinating students about that commonality is futile. And we have seen what happened to the Bourbaki approach, haven't we?

EPISODE E5.2
RELATIONSHIP WITH GRAPHS:
ATTRACTION, REPULSION, UNEASE AND UNCERTAINTY[7]

Setting the scene: In the following the discussion revolves around students' understanding of the concept of function across several mathematical topics such as Analysis, Linear Algebra and Group Theory. In the sample below (collated from evidence in (Nardi, 1996) and some analysis in (Nardi, 2000b)) the focus is on students' relationship with graphs – largely how the immediate appeal of a graph conflicts with students' uncertainty about the legitimacy of resorting to graphs and with their limited ability to construct graphs in the first place.

> Typically by the time students arrive at university the are aware of functions as **analytical expressions** and as **graphs**.
> **Issue to consider:** How do students perceive the relationship between these two?
> Examples of students' perceptions of how a graph illustrates the 'behaviour' of a function: the existence of *sine* in $f(x) = x\sin 1/x$ suggests a type of periodicity for the values and the graph of f; or, the existence of e^x in $f(x)= x^{-1}e^x$ suggests a type of 'bigness'. For these students the 'behaviour' of a function is illustrated in its graph[8].
> **Issue to consider:** But what happens when students see a function as *identified* with its graph? (For example: a continuous function identified with a smooth, uninterrupted curve)
> **Example 2.1** Consider $f(x)= x^{-1}e^x$, where the students have been asked to produce the inverse of f on a particular interval (this was intended by the question setter as an application of the Inverse Function Theorem). A student, Frances, bypasses using *IFT* and, when asked by the tutor about her finding f^{-1}, she exclaims:
> Frances: Because you can tell from the graph! *[the tutor asks why again and a pause follows]* If...you reflect it on *y*...

[7] (In addition to those discussed in E5.1) students' dominant images of the concept with relation to its graphical representation are reported as follows: students are reluctant to employ graphs, interpret graphical information poorly and can generally answer 'point-wise questions' fairly well but have trouble with 'across-time questions' (Monk, 1992). Because of the often constrained ways in which students learn about functions in school – e.g. Markovits et al (1986) claim that the emphasis put by school mathematics on functions as rules/relationships is often at the expense of the equally important notions of domain and range – students experience difficulties with transformations, e.g. stretches and shifts, as well as with change of variables or change of domain (Dreyfus & Eisenberg, 1983).

[8] Students' biased domination of the function concept image by straight lines is an effect of the extreme emphasis on linearity in the school mathematics curriculum, claim Markovits et al (1986). Students' persistence on linearity was evident in their experiment in which students were presented with two points on a Cartesian graph and asked to define a function whose graph passes through these two points. Most students drew the line connecting the two points, and, when given scattered points, some of them still tried to fit linear functions. In Barnes' (1988) experiment students claimed that $y=4$ is not a function because x does not depend on y. When shown the graph of $y=4$, a line parallel to the x-axis, most changed their minds. Students from the same group then claimed that $x^2+y^2=1$ is a function because it's a circle; also that a function with a split domain is not really one function but several. In general 'strange' or broken, that is non-smooth and non-continuous graphs, are not easily considered by students as functions (Vinner, 1983).

M: Have these functions been defined formally or do students have a naïve view of these functions?

RME: The latter. I plug in *x* and I get *e* to the *x* either by pressing a button on my calculator or by having a rough idea in my head on how these two points are found.

M: Which is a starting point. This is what they come with from school. Along with a tendency for point-wise understanding[9]. Graphs begin to suggest something about overall behaviour and I try to highlight this in the lectures (E7.4, Scene IVd) but the reflection of properties in a graph is something they need to hear stressed at this point. This is different to how it used to be before the advent of technologies such as graphic calculators because they do less graph drawing now: graphs are available at the press of a button and students seem to have less of a stock of graphs in their heads[10]. I had to spend masses of time drawing graph after graph and this task is now out of fashion. Look at the first example in your text, xsin1/x. Producing the graph of this is such a good opportunity to reflect on a number of properties of the functions that constitute this function, sine, 1/x, the product of functions, composition of functions. I always find that asking students to produce a graph provides telling information about their actual understanding of the subject when I interview them at the admission stage. As it is something they are not necessarily familiar with from school, you are forcing them to think rather than regurgitate information. Build up a picture from smaller, more basic pictures, such as consider the impact of mixing a sine or a cosine in the composition of a function. This is the type of task they find tremendously difficult: seeing, for example, that the sine graph is a mere shift of the cosine graph and that this information should sit comfortably next to whatever rhymes or memory triggers they have learnt in school in order to memorise what sine and cosine are!

RME: Rhymes?

M: You know, all these things about the cosine being the opposite over the hypotenuse in a right-angled triangle etc.. To their shock – they probably thought it was so unprofessional of me! – I had no idea what they were rhyming about in

[9] See (Monk, 1992).

[10] Williams (1998) also reported a prevalence of algebraic images and shortage of this stock. The debate about the optimal ways to use the opportunities that technology provides is a vast field of mathematics education research. To mention one example only, of a favourable report: Schwartz & Hershkowitz (1999) report the role of prototypical examples that students acquire in the process of learning about functions in an interactive environment that included the use of multi-representational software in rather positive light (even though they acknowledge that the use of *some* prototypes can be detrimental – for example, prototypes of linear functions when used as a basis for generalising the concept of function). They reported that students use prototypes as levers to handle a variety of other examples; articulate justifications often accounting for context; and, understand certain attributes of function.

a recent session. In which I was stunned by a whole group of them rhyming at the cue of my mentioning sines and cosines! It should then be no surprise to me that, given this type of understanding of trigonometric functions, they often treat the information on sine as a shift of cosine in graphing terms as some kind of major revelation. Or that adding sine and cosine becomes another sine function. They see a function like $x\sin 1/x$ as a complicated expression, period. No guts for deconstructing it down to and reflecting upon its elements. Which is what you want them to do. Beyond whatever verbal memory tricks, have a mental representation of these things[11].

RME: Is the connection between an image of e^x in terms of its graph going a certain way and *bigness* as one of its properties, one of these healthy mental representations you are talking about?

M: Yes, there is something reassuring in identifying *e* to the *x* as big in relation to a polynomial. However I am far less comfortable with a student who sees periodicity in any graph that involves a trigonometric function. This is way too crude.

RME: What is the level of crudeness you would tolerate in order to at least initiate the students' involvement with graphs?

M: Well, let's think about the so called *bigness* of exponentials. This is just about the time that students ought to start realising how faster exponentials grow in comparison to powers. Through a picture like this they can start making this realisation and then, of course, prove it by induction or whatever method is appropriate in every case. A more formal grasp of exponentials comes subsequently when Taylor expansions and Riemann integration are introduced. Then exponentials are seen as inverses of logarithms. These more formal images are not necessarily substantially helped by early pictures but for a primary understanding of exponentials this picture is good enough. And necessary. I don't want to wait until the Riemann integral is formally introduced for the students to have an image of what the graph of $x^{-1}e^x$ looks like. This is a totally unnecessary and unproductive delay. Of course exponentials are also buttons on the calculator and in that sense the point-wise view comes to the rescue – at least temporarily. And I am terribly worried about the unrigorous messages conveyed by acts such as raising a real number to a real power, when we have hardly introduced real numbers, or powers! I know I sound pedantic but we find ourselves in this impossible situation of somehow requiring both linearity and rigour. I know that overall the exercise of drawing the graph of $x^{-1}e^x$ is

[11] Often reported (Tall, 1996) as the Rule of Three (algebraic, numerical, graphical) representations.

probably wonderful and very good for them despite these big conceptual holes. And you can get students to generalise x^{-n} for any integer *n* which also has this nice property of getting bigger as x gets bigger. You can explore how fast this happens. And every now and then you get a surprising insightful comment from a student such as *the degree is not high enough* for our function to do this or that. And that is impressive and reassuring, that they can get to see the dominant term in a polynomial when you are trying to produce its graph. And in fact see these things beyond the range of your graph paper. Use their imagination on how this thing would go beyond what fits in your piece of paper[12].

RME: Students often rely heavily on the graph to infer properties of the function, say. Frances in our example.

M: Well, if the telling from the graph is followed by a translation into formal writing and argumentation, this is fine. It is not enough just to write you can tell from the graph. I suspect Frances has a way of seeing the reflection on the y axis in a formally satisfactory way. If only she were here to finish her sentence! The question seems to be aiming at enacting the students' use of the IFT. Which, I think, remains an ambiguously stated intention of the question setter. The emphasis overall is to get the students to engage with the importance of the interval in deciding the inversion of the function.

The focus of the discussion turns to the process-object duality at the heart of the concept (E5.3) but returns to focusing on dominant images of function, this time in the context of another data example (same source as the previous one).

Example 2.2 Students perceive piece-wise functions such as $f(x)= x/|x|$ as two functions. Or when confronted with families of functions f_n such as $f_n(x)=x^n cos(1/x^2)$ instances like the following occur: student Camille's eloquent answer on the meaning of the question 'For which n, f_n is differentiable at zero?' was:

Camille: The question is to find for which *n* to have the limit of this derivative to be zero (*she points at the derivative in her notes and the proof for the right answer n=4*).

However she then asked what would happen if $x>1$ — ignoring that the limits involved in the process are all taken when $x \to 0$.

[12] Studies discussing students' graphing skills (both in term of graph construction and interpretation of information provided by a graph) abound. With regard to the focus of M's comment here, the creativity with which students treat the graphs in order to elicit information about functional properties, one relevant example of these studies is (Zbiek, 1998): participants were given access to curve fitters and graphing utilities for the purpose of developing and validating function models. The creativity with which they produced and interpreted these models varied. One of the foci of the analysis was on the degree of dependence on the tool to make creative decisions: this creativity increased (and the user's dependence on the tool decreased), for example, when the user could resort to previously well known functions.

RME: You mention elsewhere (E5.1) that students may see functions such as $x/|x|$ as two functions, not one. Can I have your comment on the latter part of this example, students' views of families of functions at this stage: I have a vague sense they may somehow the n and the x being the variables or something like that….

M: Oh, that's a different ball game altogether. Families of functions, $f_n(x)$, are also treated like, because of n, as something that goes to infinity – they have a strong instinct towards seeing n as going to infinity, or asking for proof by induction to be used, as in your examples. But maybe these are just misunderstandings of a superficial nature that are resolved once their image of function is suitably enriched? (E5.3)

EPISODE E5.3

THE TROUBLING / POWERFUL DUALITY AT THE HEART OF A CONCEPT: FUNCTION AS A PROCESS, FUNCTION AS AN OBJECT[13]

Setting the scene: In what follows appearances of the concept of function across topics as diverse as Differential Equations, Linear Algebra, Group Theory and Topology become the starting point for discussing the ways in which students relate to the duality at the heart of the concept: function as a process (an image they are mostly familiar with) and function as an object (an image they are required to become familiar with, and fast). RME initiates a discussion of this duality following from a comment by M in E5.2 on the aim of the exercise in Example 2.1. That aim was to activate students' use of the Inverse Function Theorem and help them realise the importance of domain when constructing the inverse of a function.

RME: In the various contexts we present functions to the students – only in Calculus I can name Taylor expansions, Fourier Series, Differential Equations etc. – we often require students to fluctuate between notions of function as a process and as an object.

M: In the context of Differential Equations taking the necessary steps towards selecting which members from the larger family of available functions comply with the boundary conditions is a difficulty for students.

RME: So here I am, solving a differential equation and, until now, whenever I solve an equation, the answer is a number. And all of a sudden you are asking

[13] A major source of student difficulty with the concept is their lack of flexibility in switching representations or working on the relationships between them; often students tend to view algebraic data and graphical data as being independent. Sfard discussed this difficulty with shifting between different representations of function in terms of establishing connections between its static (object) and dynamic (process) aspects of the concept (1991). This duality, which is also part of its epistemological power, is at the heart of the discussion in this Episode (also in (Nardi, 2000b)).

me to solve an equation and the answer is a … function? Or a family of functions?! I wouldn't be sure what you mean! Functions are formulas I use to evaluate an *y* in terms of an *x* (E5.1). They are not solutions to equations! Can you actually take for granted that I see a function as an object that can be a solution to an equation? Somehow this shift of gear needs to be clarified to me?

M: But that depends on the context in which the problem is set and on the notion of function one may need to activate. Think, for example, of the pendulum and how it works. This suggests a function in terms of a process. When I solve a differential equation I need to think of functions as objects, not as rules, not as processes. I think I want the students to be close enough to what the problem is about. And I don't think it's ever too early for discussing these differentiations within the concept itself. The case of the pendulum is a good one to reinforce an understanding of functions as solutions to differential equations, to raise the issues related to solving differential equations in closed form – as is the solving of integrals in closed form too – to certainly go beyond an understanding of functions as buttons on calculators and graphs in the head! OK, maybe graphs in the head are not so totally bad but… under certain conditions only (E5.2)! By the way not seeing function as an object makes things like *dx*, an operator, something I do to a function, ghastly to teach. *dx* is almost unfathomable in any meaningful way without seeing what it operates upon! Writing *dx* with a capital D or as *d*-brackets-over *dx* and placing the function on which to operate within the brackets are possibly good symbolic ways to express the operational role of the differential but, God, that's a tough one[14]. Sometimes a good way to show the relationship between a function and its derivative is to draw the graph of the function and also draw on that picture the graph of the slope of the function and show how changes in *y* reflect changes on the slope but frankly this is far easier said than done. In the context of differential equations, or morphisms in Group Theory, or whatever, seeing a function as an object you put between brackets in, say, $\varphi(e^{-x})$, generates responses from students such as: OK, I can see e^{-x} as a function. What is φ then? Oh, Lord! Then you are in trouble.

RME: Is the type of complexity you are referring to the same one as the one involved in understanding the role of families of functions emerging from the constant in the solution of an integral?

M: How the integration constant acts on the shape of the things that you are plotting is an idea that is too complex to grasp, especially in some contexts such as when all functions pass through one point, or are not just conveniently stacked up on top of each other. This is the main reason I oppose – in fact I deliberately avoid –

[14] Thurston (2005) proposes a replacement of traditional introductions with a definition of *dy* and *dx* as functions in order to illustrate the operational elements that M refers to here. In this vein he then proposes to replace indefinite integrals with anti-derivatives (see footnote on this also in SE7.2).

teaching indefinite integrals: the idea that you have to talk about an equivalence class of functions to students who have no grasp of a function as an object is ludicrous! To understand indefinite integrals requires a rigorous understanding of =, a notion of = that goes beyond telling what the left- and right-hand side of an equality are and that they are the same. And if you stick with that limited notion it is very hard to do indefinite integrals! There is no need for indefinite integrals anyway, at this stage.

RME: But don't you want to use this as an opportunity to start thinking of functions as members of families, and ultimately as objects?

M: You may as well want to avoid indefinite integrals. I just don't think that there is any need for them.

The discussion of this continues in SE7.2. M then turns to discussing Example 2.2 (E5.2) but returns to the issue of duality in the context of discussing the students' reaction to the following problem in Linear Algebra (example taken from (Nardi, 1996; Nardi, 2000b)):

The Concept of Function in the Context of Linear Algebra
Applying the Subspace Test and Looking for the Zero Element of $\boldsymbol{R}^{\boldsymbol{R}}$

The four tutorials quoted here are on Linear Algebra. The tutor explains that a common way to prove that a subset S of a vector space V is a subspace of V ($S<V$) is to use the Subspace Test, namely prove that $0\in S$, and, $af+bg\in S$, f, $g\in S$ and scalars a and b, i.e. addition and scalar multiplication are closed in S.

Two applications of the Test for two subsets of $M_n(\boldsymbol{R})$, the set of nxn matrices, follow. A third example is: **Prove that $U=\{f: \boldsymbol{R}\to\boldsymbol{R}, f(0)=f(1)\}$ is a subspace of $\boldsymbol{R}^{\boldsymbol{R}}$, the set of all real functions from $\boldsymbol{R}$ to $\boldsymbol{R}$.**

The students look as if they are not familiar with $\boldsymbol{R}^{\boldsymbol{R}}$. The tutor realises their unease and asks what is the zero element of $\boldsymbol{R}^{\boldsymbol{R}}$. She reminds them that the zero element is an element of U and she asks them what is the property it has to satisfy. Silence follows. In Tutorial 2, Eleanor says about the zero element of $\boldsymbol{R}^{\boldsymbol{R}}$ that **'It will stay the same'.** The tutor disagrees, reminds her of the definition of the zero vector in a general vector space ($a+0=0+a=a$, $a\in V$) and asks for the zero element z in U and $\boldsymbol{R}^{\boldsymbol{R}}$ again. Abidul says **$z(x)=x$** and Eleanor says **'nought'.** The tutor agrees with Eleanor and adds that this is a function they have been dealing with 'for ages' in Analysis 'in a slightly more abstract context'. In Tutorial 4 Patricia says that in a general vector space **'if you add zero to any vector you end up with the same vector'** but is unable to apply this to $\boldsymbol{R}^{\boldsymbol{R}}$. In Tutorial 1 the students simply remain silent to all the tutor's prompts. Finally in all tutorials, in proving closure, the second condition of the Subspace Test, the students frequently confuse f with $f(x)$ in evaluating $af+bg$.

The first frictions with regard to the zero element of a vector space appear in the discussion of the first two applications of the Subspace Test: for instance when Abidul uses the term 'nought' for the zero vector of $M_n(\boldsymbol{R})$, the tutor corrects it to 'zero' (meaning: the zero matrix). Subsequently Abidul asks whether she can write **'just 0'**,

> not the full matrix with zeros everywhere. The tutor says it's fine as long as it is clear what she means. The application of the Subspace Test then continues almost trouble-free.
>
> In the third example however there is a problem: while the previous two applications of the Subspace Test involved thinking in terms of the elements of $M_n(\boldsymbol{R})$, the vector space in question here, $\boldsymbol{R}^{\boldsymbol{R}}$, appears less familiar. Even though some of the students can recall the formal definition of the zero element in a vector space, they stumble upon the students the identification of the zero element in this particular vector space. Moreover, even once the hurdle of identifying the zero element of $\boldsymbol{R}^{\boldsymbol{R}}$ is overcome, the students are still uncomfortable with its contents – as their frequent confusion of f with $f(x)$ illustrates in their considering separate values of f as elements of $\boldsymbol{R}^{\boldsymbol{R}}$.

RME: Ironically the ostensibly more familiar R^R is the one that caused more trouble in our example (OT5.3)!

M: I see. Zero function should be familiar, at this stage, but of course it is too difficult: it is constant so it doesn't have the variation expected in a function, it is also the zero element in some space and it has some existence in some other form – as this very concrete thing, a matrix with zero everywhere. There are some powerful connections that are not explicitly made in the example[15]!

RME: So why do the students have such a difficulty with it?

M: The problem is, I think, that the students have not seen this a lot and function is, at this level, not understood as an object. It is still a process or something else. All of a sudden a function has to be an element of a set and, to be an element of a set, it has to be a thing. And if you haven't made this transition towards function being a thing, then coping with this novel situation is impossible[16].

RME: The tutor in the example also sounds puzzled at the students' inability to recognise this very familiar object, the zero function, in a new context.

M: Later on, when the students are introduced to Functional Analysis, where they see functions as points, this problem is resolved. But, at this early stage, it is the points of a graph that the students are considering – at best. They are staring all

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[15] As many authors in (Dubinsky & Harel, 1992) point out Year 1 students' experience of and capacity to see through these connections is limited. Accepting a function as an element of a vector space can be an expectation too far at this stage.

[16] Here M encapsulates an issue that has been revisited frequently by researchers in the area at least since the late 1980s – early 1990s. Extending the Piagetian notion of *reflective abstraction*, namely the process of interiorising physical operations on objects, to include cognitive structures (not only physical objects as Piaget initially intended) numerous authors in the field – but primarily Sfard (1991) and Dubinsky & Harel (1992: their own chapter there as well as others') – have described *encapsulation*, namely the acting upon something that may hitherto have been perceived as a process as if it is an object. Encapsulation requires a reconceptualisation, a leap or a 'transgression', in M's words, that has been identified as a seminal phase in the student's growth of understanding.

wide-eyed and stunned at the idea of considering all functions from *R* to *R*. Abidul's comment however suggests that some of this maturing process may be in place already[17]. Also, given the use of the term 'identity' meaning different things in Analysis ($f(x) = x$) and in Linear Algebra ($a+0=a$) there is another conflict here, that of terms used for multiple purposes. As for this maturity process I believe there is a parallel with mathematical history here: functions were not really understood as objects until Banach's study of spaces of functions (OT5.1). Lie groups is a significant area in this respect too.

RME: If what you are saying is true then Topology is a crucial topic to study for facilitating this transition, right?

M: Absolutely. In Banach's book metric spaces are totally natural examples.

RME: And a great scaffolding for looking at spaces in Abstract Topology.

M: Yes! It's packed with meaty examples. Like the ones involving the Contraction Theorem.

SPECIAL EPISODE SE5.1
THE TREMENDOUS FUNCTION-LOOKALIKE THAT IS *TANX*

Setting the scene: In the following M is concerned about a repercussion of the students' disregard for essential constituent elements of the concept such as domain and range (E5.1), students' regarding *tanx* as a function (comment in the context of discussing Student JK's part (v), question on 1-1 and onto functions in E4.3).

M: By the way, by any standards of rigour, *tanx* is not a function and should not be used as an example for the concluding part of the question! No marks there, sorry! The reason I mention this is that this falls exactly into the trap of a graph being misleading: *tanx* looks tremendously onto and tremendously not 1-1 but it is not a function defined on R! The classic problem of not thinking of functions in terms of their domain, range etc. (E5.1). I have some sympathy for the victims of this trap but…

[17] Moschkovich (1999) describes some evidence of students' understandings as 'transitional conceptions': conceptions which are the result of sense-making, reflect the complexity of the domain, are productive in some contexts and have the potential for refinement. Here Abidul's suggestion to write 'just zero' may have some of the characteristics of a transitional conception. Moschkovich's account of such conceptions is particularly strong in the context of linear functions.

SPECIAL EPISODE SE5.2
POLYNOMIALS AND THE DECEPTIVE FAMILIARITY OF ESSENTIALLY UNKNOWN OBJECTS

Setting the scene: In the following the apparent familiarity of polynomials (that at this stage of the course have not been introduced formally) plays a central role in an incident from Linear Algebra that captures the students' unease with conceptualising polynomial functions as objects (as also in E5.3). The data example below from (Nardi, 1996, as adapted for Nardi, 2000b) is the starting point of the discussion.

Looking for the 'Usual' Basis of P_3(R): In the four tutorials quoted here the discussion is on finding a matrix A for T: $V \rightarrow W$, a linear mapping between two vector spaces V and W. As an application, the tutor suggests finding the matrix for *T:* $P_3(R) \rightarrow P_3(R)$ *where* $Tp(x)=p(x+1)$. All agree that $dimP_3(R)=4$ but some of the students have problems with identifying the 'usual' basis of $P_3(R)$, as the tutor calls $\{1, x, x^2, x^3\}$. In one occasion the basis is identified once the students are reminded of the general expression of $p(x)$: ax^3+bx^2+cx+d. Patricia offers 'x^3, x^2, x and the constant' and then explains that **'constant' is 'just one'.** In another occasion Beth gives **'1 0 0 0, 0 1 0 0... '.** The tutor reminds her that they are talking about polynomials, not matrices. Beth then recalls the general expression of $p(x)$ and the tutor asks her what is $p(x)$ a linear combination of: 'Er**,... to the *s* from 1 to 3**...' replies Beth. The tutor insists that they are 'looking for some simple looking polynomials, nice simple things', and repeats her question. 'Mmm... **it's just made by linear ones**...' she says and adds, when asked about the tutor what she means: 'It's **a product of three**...'. The tutor realises that Beth is talking about factorisation and she stresses that multiplication of polynomials is not linear. Asking Beth to 'take a step back' she repeats that they are trying to write $p(x)$ as a linear combination of four other polynomials $d_1,..., d_4$, which form a basis, and asks Beth to compare $ad_1+bd_2+cd_3+dd_4$ with ax^3+bx^2+cx+d. Beth then dictates the 'usual basis' $\{1, x, x^2, x^3\}$ for $P_3(R)$. Subsequently the tutor asks the students to calculate the values of $1, x, x^2, x^3$ via T and to start with $T(1)$. The discussion in the four tutorials is as follows:

Tutorial 1	Tutorial 2	Tutorial 3	Tutorial 4
Patricia: Two. *The tutor asks for the definition of T. Patricia says it is x+1 and the tutor wonders what happens if there is no x to map. To the students' silence she notes that* 'in fact there is nothing to do'. *Patricia exclaims* 'ah!' *and says that then T(1) is* 'just one'.	*Silence. Then Abidul suggests (*'1+*x*'*) for which she can however give no reason. The tutor repeats that we have to replace x with x+1 and asks what* 'problem they have with it'. Abidul: *x* doesn't appear... One?	Cleo: Two. *The tutor disagrees and asks what* 'problem they have with it'. 'Take your polynomial and wherever *x* appears, replace it with *x*+1' *she suggests.* Cleo: Just one.	B5 (Beth): Polynomial 2. *Asked what she means she replies:* B6: Because we are adding one. *The tutor repeats the definition of T and notes that to add 1 you first have to have an x.* B7: Is it one?

M: Is it possible that one problem they have is that the students don't know what a polynomial is? If they think of a polynomial as a polynomial function…. Which they might do… then it is easier to write down why x, x^2 and x^3 is a basis. I doubt that they have seen a definition of a polynomial like this, of such a formal, shorthand notion. A definition I could be happy with would be an expression with powers of x. Maybe? But I doubt whether the students have such a notion in mind. Rather I would expect they would have some ideas of the polynomial as something where you have just powers of x and they would think of this as a function. I think they would think of a polynomial as a thing with polynomial solutions. Because that is what they would have come across at this stage. It may be the case that an understanding through the formal expression shouldn't be too difficult to achieve but, if it hasn't, as is probably the case in these tutorials, then the operator in the Example is too difficult for the students.

RME: And there is the added difficulty here of the tension between point-wise and object-wise uses of the polynomial (E5.3).

M: It is really quite horrible. They have to think of these polynomials as objects to be shifted. Golly! In fact the only way you can cope with what the question asks is to write, instead of ax^3, $a(x+1)^3$ plus bx^2 … $b(x+1)$…. and so on. So you cannot actually solve the problem unless you think of it as a formal process.

RME: So the tutor's casually calling this *the usual basis* is slightly ironic…

M: There are too many complications here generated by the indecisiveness about whether we will dare to be formal or stick to some vague casual approach (E7.4, Scene IIb). I therefore express sympathy for the students who claimed that $T(1)$ is 2! And I admire the fact that by the end of the episode all of them realise $T(1)$ is 1. I am a bit less convinced by Patricia's success, but Abidul seems to have a notion of these things not as objects but as …doing stuff. This is very early in the course and the expectation of them to handle this seems to be too high.

RME: Back to your query about underlying notions of a polynomial here: as the tutor unproblematically invites the students to identify these simple, familiar objects as elements of a basis, she eventually is shocked when the students come up with factorisation or things like 1,0,0,0 etc..

M: In a sense she is thinking ahead: the basis she is after can be expressed in terms of vectors. A polynomial in this case can be expressed as the vector (a, b, c, d), where a, .., d are the coefficients. This could have been a good, concrete example of representation through linear combinations of vectors in a basis etc. Judging from the table of these various tutorials, the students are really struggling with what kind of object they are handling all in all.

OUT-TAKE OT5.1
HISTORY RELIVED[18]

Setting the scene: In the midst of the discussion in E7.4, Scene IVc M comments on students' dominant perception of a function as a rule as potentially having 'a historical basis': 'most functions in Physics were precisely that to start with!', he concludes. M elaborates the idea that revisiting the history of the concept of function can be instructive when we try to understand how certain images dominate students' perceptions in E5.1.

M: According to the Oxford English Dictionary the first use of the term *function* in a mathematical sense, not an ordinary one such as *purpose* etc. was as *a variable quantity regarded in its relation to one or more other variables in terms of which it might be expressed or on the value of which its own value depends*. This was in 1779. In the 1911 edition of the Encyclopedia Britannica there are 26 pages on *function* of extremely high level discussion and it starts off with that sentence – perhaps a copy of the 18th century version? So, by the beginning of the 20th century, the image of the concept represented by this sentence was in full use. But back in the 1804 edition of the Encyclopedia there is nothing at all, even though there are pages on fluxions etc. So, somewhere in between these editions, the term came into full use. I guess this is around the time when Euler and others were debating whether a graph with a break represents a function – in fact I see this gravitation towards continuous functions in students too: they are often reluctant to consider discontinuous functions as functions at all. Of course you can trace this fully-fledged use of the term back in Newton's locus as a function of time but the term wasn't put in its full use for another hundred years at least. It took so long to evolve that we can only see its complicated nature by tracing this journey. The notion of the Fourier Series of a function is another intriguing stop in this journey: in particular, the relationship between the function and the thing to which the Fourier Series converges and in which cases this thing is the same as the function. The students more or less have to go through the motions that 19th century mathematicians had to go through, around 1820 probably. They

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[18] For solid accounts of this history see the Sfard and Sierpinska chapters in (Dubinsky & Harel, 1992). The theoretical origins of this belief can be traced back to the debates within biogenetics on whether ontogenesis recapitulates phylogenesis. Freudenthal (1983) and others have transferred this debate within the realm of mathematics (its learning and its culture) by suggesting that this recapitulation does occur but in a modified way: the individual experiences the obstacles that have been phylogenetically experienced but the experience of the individual is modified by the added value of the developments bequeathed by the mathematical culture. In this sense it is not surprising that students may have difficulty with perceiving a function as an object or that their initial concept image may be dominated by more procedural elements. However as the mathematical culture around them (their teachers, the textbooks etc.) are aware of the necessity for the students to move towards a richer concept image, these students are in a more privileged position that their 18th century predecessors: their understanding can go through these necessary evolutionary steps but with the knowing support of the culture around them. For the role history of mathematics in teaching can play in teaching see also (Mason, 2002).

have to start seeing the subtleties, going beyond continuous graphs, for example, and consider functions that may not even have an immediately graspable graph. Or consider the concept of a polynomial in a finite field where two polynomials, one of degree 18 and one of degree 16, may have exactly the same values but they are two completely different entities. They originate from different worlds. It is this kind of phenomenal flexibility we aspire to foster at this point.

OUT-TAKE OT5.2
EVOCATIVE TERMS FOR 1-1 AND ONTO

Setting the scene: M offers the following comment in the context of the discussion of Student N's writing in E5.1.

M: I know there are more than one terms for the concepts involved here: *one to one*, *onto*, *into*, *injective*, *surjective*, *bijective* and I am as generally dissatisfied with one- to-one as I am generally fond of the idea that, when you expect the students to do sophisticated and hitherto unfamiliar things, then the introduction of new, unfamiliar words such as *injective* and *surjective* is helpful. *Into* is a perfect word for what it aims to depict! And *onto* is too! Then again I have seen *into* being used for bijections so maybe the slightly more formal *injective* and *surjective* are appropriate terms. But I am awfully concerned with the even more formal *(iso)(endo)(...)morphisms* where confusion is almost inevitable...

OUT-TAKE OT5.3
R^R: A GROTESQUE AND VULGAR SYMBOL?

Setting the scene: M's comment below is in the context of the discussion in E5.3.

M: I think symbolising the notion of a set of all functions from A to B with B^A or A^B is grotesque. I dislike R^R for similar reasons. And I am furious about the use of the Subspace Test here: it contains redundant proving work. I like the very useful $M_n(R)$ example. Unlike R^R – it is monstrous to use examples from this context as it probably will not be seen again until Year 3 – it is important to understand that $M_n(R)$ is a linear space, so it is important to have examples from that context. Its sort of unfamiliarity is also helpful in addressing issues like multiplying entities in the context of Algebra – whereas with numbers the familiarity gets in the way and muddles the conversation (E3.1). Not that matrix multiplication is easy but its difficulty makes us notice things that need to be addressed. But back to the R^R symbolism: it is so crude! It says nothing! What is this resemblance to the symbolism of power trying to hint at? I guess its origins lie somewhere within Combinatorics? There is some statement in there about how many elements you have in A^B? $|A||B|$ or something? $2|A|$? But of course all of this alleged association suggested by this symbol would fly completely over the students' heads, so what is the point of using it?

CHAPTER 6
THE ENCOUNTER WITH THE CONCEPT OF LIMIT

'the sequence $\{a_n\}$ converges to A as $n \to \infty$'
means that
$\forall \varepsilon > 0\ \exists N \in \mathbf{N}$ *such that* $n > N \Rightarrow |a_n - A| < \varepsilon$

Similarly to the concept of function examined in Chapter 5, the limiting process is another fundamental building block of advanced mathematical thinking. It is then of no surprise that a considerable portion of studies in the area focus on students' first encounters with it. In most studies (e.g. Ferrini-Mundy & Graham, 1991) students appear to be in difficulty with the *delta-epsilon* mathematical formalism as well as with assigning meaning to the formal definition. In particular, their difficulty seems to be with comprehending the mechanism of the proposition that is contained in the formal definition and with how this mechanism provides a tool for proving limits. For most students the concept of limit is veiled with a certain amount of mystique: there are no computational recipes for finding limits and its understanding impinges upon a very complex network of ideas and an equally complex novel notation.

In the discussion that follows M and RME describe the students' encounter with the concept of limit often by revisiting several of the themes explored in Chapters 3 and 4: at this early stage students are not always at ease with beginning to conceptualise the necessity for a formal definition of convergence (E6.1); and, with the ways in which they relate to the formal definition in terms of its symbolisation (for example, with regard to constituent elements of the formal definition such as quantifiers, modulo, inequalities and the nature of ε), its verbalisation (for example, with regard to the role of ordinary words such as *arbitrarily* and *eventually* in their attempts to verbalise the meaning of the definition) and visualisation (for example, with regard to the role of real-line based pictures in attempts to visualise the meaning of the definition) (E6.2, OT6.1). Students are also not yet at ease with the mechanics of proving the existence of the limit of a sequence (with or without necessarily identifying it) through use of the definition (E6.3, Scene I); and, through invoking pictures, conducting numerical experiments or employing relevant theorems (E6.3, Scene II). Particularly with regard to visualisation, while apt pictures or evocative sentences appear to be able to turn a student's attention to parts of the argument that s/he may have otherwise ignored, students, often responding to what they perceive as customary expectation, include pictures in their writing but do not then use them to support the generation of their argument (SE6.1).

Many of the above issues turn out to be pertinent also in the context of series (OT6.2) and continuity/differentiability (OT6.3).

EPISODE 6.1
BEGINNING TO UNDERSTAND THE NECESSITY FOR THE FORMAL DEFINITION OF CONVERGENCE

Setting the scene: M and RME discuss the following excerpt from (Nardi, 1996, Chapter 6) entitled: "*Preliminary Conceptions of Limit and Infinite Largeness. The Two-Step Battle Between Intuition and Formalisation: Conceptualising and Materialising the Necessity for Formal Proof*". Student Kelle and his tutor discuss the student's written response to the question:

If $0 < \theta < 1$, prove that, given any $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that $\theta^{n\varepsilon} < \varepsilon$.

According to the tutor, Kelle has not proved 'what they want'. What they want, says the tutor, is to prove [the question] for 'as small epsilon as you can get' via [a previous question] and the Approximation Lemma. This is equivalent to proving that $\theta^n \to 0$, she adds. She then asks him whether sequences have been mentioned in the lectures. Kelle says hesitantly yes and, when the tutor asks him how he will prove that $\theta^n \to 0$, he responds reluctantly, referring to sequences, that he doesn't 'think they are meant to read anything more'. The tutor then returns to the terminology of the question and asks what is it they have to prove.

K1: We have to prove that there exists an *n*,... that there exists an epsilon...
T1:... not that there exists an epsilon, no...
K2:... there exists an *n* in...
T2: The tutor shows then on paper that what they want to prove is that starting with any 'kind of slice', any epsilon, all θ^n will 'eventually lie in between' $-\varepsilon$ and $+\varepsilon$. So, how will he write that? she asks.
K3: Er,... you can find an element such that if *m* equals *N* this will be less than epsilon, eh for $m>N$ θ^n will be less than...
T3: Yes! We are getting there. Now what is this epsilon, is it some particular epsilon or... what?
K4: It can be any... You can choose…You can find an epsilon such that...
T4. The tutor explains that for every epsilon they are given they should find an n_ε such that $\forall n>n_\varepsilon$ $\theta^n<\varepsilon$. This is something he intuitively knows about but it is imperative he proves for all ε. She then asks Kelle why.
K5: Because it might be greater than that before that, beyond that n_ε tries to be smaller for all the rest of the numbers.
T5: Why?
K6: Because if it's converging they should have been higher...

The tutor says she is not pleased that Kelle is using the convergence of θ^n in his justification because this is what they are trying to prove. She repeats the statement and stresses that it is important to see that because $\theta^{n\varepsilon}<\varepsilon$ and the sequence is decreasing $\theta^n<\varepsilon$ will be true $\forall n>n_\varepsilon$. 'It's so obvious that it goes to nought' remarks Kelle. She says that she agrees but they 'still have to prove it'.

Below is a passage from the analysis that accompanied the above piece of data in (Nardi, 1996) and was included in the Dataset that M examined prior to the discussion:

> *Kelle's Problematic Handling of the ε-Definition of lims$_n$*. Kelle in the beginning of the Episode is rather uncomfortable with the tutor's rephrasing of the question in terms of limits, sequences and convergence. Despite his implicit request to stay within a familiar lexical territory, during the Episode the tutor occasionally returns to this terminology. However Kelle's major difficulty seems to be the manipulation of the ε-definition of limit.
>
> K1 and K2 are hesitant and confused attempts to reproduce the proposition in [the question] and Kelle sounds baffled with the quantifiers for n and ε. K1, K2 are followed by the tutor's objection (T1) and her pictorial exposition on what [the question] is actually about. K3 is then received by the tutor as progress (T3) and she proceeds with elaborating on the nature of ε. K4 is evidence of how undecided Kelle is about the nature of ε: ε can be any number, ε is a number we can choose, ε is a number we must find. T4 then is an attempt to establish a connection between Kelle's intuitive knowledge that $\theta^n \to 0$ and ε as a tool to formally express this knowledge. K5 and K6 indicate that this connection has not been established. Kelle is convinced that $\theta^n \to 0$ and he does not hesitate to use this to-him-established fact (K6) in his vague attempt to answer the tutor's formal question about proving it. When the tutor intervenes in order to interrupt the vicious circle of his thinking and completes the proof, he listens and concludes in a dramatically expressive frustrated tone 'It's so obvious that it goes to nought'.
>
> As an observer I was concerned that Kelle has not been persuaded of the necessity (and technique) to present his intuitive ideas formally, namely via the ε-definition of limit. The discussion of his problematic handling of the ε-definition of limit possibly reveals his reluctance and difficulty in formalising what he thinks as an obviously true proposition. In an approach that appears to be rather confused, the student seems to be engaged in a vicious circle of assuming in his proof what is to him intuitively obvious and what he is actually being asked to prove.

The above piece of data and analysis was accompanied in the Dataset by the following note:

> At least two intertwined issues emerge from the above example: sensitisation to the requirements for rigour (conceptualisation of the necessity for proof) and enactment of proving techniques (materialisation of proof). With regard to the former the student here seems to be increasingly, but not necessarily willingly, engaging with formalising his argument. In other Episodes in (Nardi, 1996) students often appear reluctant towards (or even avoid) formalisation and sometimes demonstrate a preference for concrete, intuitive arguments. Here Kelle's reluctance can be partly attributed to a belief that proof is not necessary when a proposition is perceived as obviously true.

M's first observation is on 'the trajectory the teaching seems to be taking here': 'gradually the tutor understands there is a problem but then does not offer appropriate help'. RME invites his comments on the particular issue of 'how did the student after all end up asking *isn't it obvious it is zero?*' (E3.5, Scene II; E5.2).

M: I find using n_ε instead of n unnecessarily heavy – and I know n_ε is a way of emphasising the dependency on our choice of ε. I also have serious doubts about whether the dialogue is getting the student to see the argument.

RME: I agree that probably the question setter's intentions are frustrated as the necessity of this formalism escapes the students after doing the question.

M: I am troubled by these references to logarithms but I can see she is attempting to introduce students to the limiting process. But I think there are better ways of doing that: seeing the limit by doing examples on the calculator, drawing a picture (E7.4, Scene IV). I think the dialogue suffers from a misunderstanding: using the convergence intuition in an argument that is about proving the convergence. There is of course an interesting element in this type of one-to-one interaction where the student is offered the experience of conversing with the more experienced mathematician in this language that is distinctly different, so much more intimate and not to be found anywhere else: the books, the lecture notes or even the lectures (E7.1, Scene IV). Still I find the tutor's approach totally inappropriate: this is an opportunity lost for genuine interaction and I admire the student for carrying on the conversation despite the flawed quality of the tutor's responses.

RME: You briefly mentioned the question setter's intentions earlier. What do you think these are?

M: It is hard to guess outside the context of the course but I presume this is a part of a more generally orchestrated effort to introduce the foundations of analysis formally?

RME: Yes, it was around the time students had been introduced to the Approximation Lemma.

EPISODE 6.2
BEYOND THE 'FORMALISTIC NONSENSE': UNDERSTANDING THE DEFINITION OF CONVERGENCE[1] THROUGH ITS VERBALISATION AND VISUALISATION – SYMBOLISATION AS A SAFER ROUTE?[2]

Setting the scene…what is it about the definition of limit? Below M and RME deconstruct *that* one line of the definition and explore the problems students usually encounter with its constituent elements (e.g. quantifiers, the choice of N, the nature of ε, $||$, inequalities, the notion of implication)[3]. In what follows M and RME discuss the question below as well as two examples of student responses, Student N's and Student MR's:

[1] Studies aiming to explore this understanding date as far back as the late 70s. Amongst the earliest observations is that of Tall & Schwarzenberger (1978) that a common informal interpretation of the formal definition of $\lim s_n = l$ is 'we can make s_n as close to l as we please by making n sufficiently large' where 'close' means near but not coincident. The words associated with the concept of limit seem to conjure up ideas that feed a rich and complex concept image – to use a term that has defined the discourse on the concept of limits since its inception in the late 70s / early 80s (Vinner & Tall, 1981). Robert (1982), for example, offered a taxonomy of the students' conceptions of the limit of a sequence: monotonic and dynamic monotonic (limit associated with the monotonicity of the sequence), dynamic (limit associated with approaching, closeness, tendency), static (terms around the limit or close to the limit), limit as bound. She also reports that students seem to have a diversity of images that co-exist in their concept image: some of these images are stronger; they dominate and hence they are evoked more easily. Remarkably the concept definition is not the strongest of these images, even when the students seem to have acquired it. Vinner & Tall (1981) found that the students who recollect a dynamic informal definition are more likely to recollect it correctly than the ones who attempt to recollect the formal definition. Davis & Vinner (1986) say that parts of the concept image of limit cannot be evoked instantaneously and in a complete and mature way. As a result adequate representation for some parts occurs earlier than for others; this is their explanation of what they perceive as the inevitability of some obstacles (for example, in her work on the epistemological obstacles related to limits, Sierpinska (1985; 1987) embedded more generally the students' difficulties in their attitudes towards mathematics and infinity). Davis & Vinner then suggest that part of learning is about making these obstacles visible and conscious: concept images of limit are dominated by examples, for instance; convergent sequences are mostly seen as monotonic ones; another strong concept image is the dynamic notion of limit as a value where the terms of a sequence approach. The latter has the implication of what Tall terms *generic limit property* (Tall, 1991b): all the properties of the terms of the sequence also hold for its limit. As a result students believe $\lim 0.999.... < 1$ even when they can prove that $\sum 9/10^n = 1$: 1 is not of the form $9/10^n$, so it cannot be the limit of $\sum 9/10^n$. Cornu (1991) reports that students in this case also claim that $\sum 9/10^n$ tends to 0.999..., but has the limit of 1.

[2] I use the terms verbalisation, visualisation and symbolisation is in symmetry with Sierpinska's 'model of natural language', 'model of geometrical representations' and 'formalisation' (Sierpinska, 1994, p87)

[3] Apart from often being seen by the students as meaningless and redundant the notation associated with limits seems to generate unanticipated ideas by the students: dx, for example, seems to evoke the idea of a number smaller than all positive real numbers but not equal to zero. There is evidence (Tall & Schwarzenberger, 1978; Orton, 1983a and b; Cornu, 1991) of the strong influence these symbols have on students' building up concept images from the definition.

(a) Write out carefully the meaning of the statement " the sequence (a_n) converges to A as $n \to \infty$."

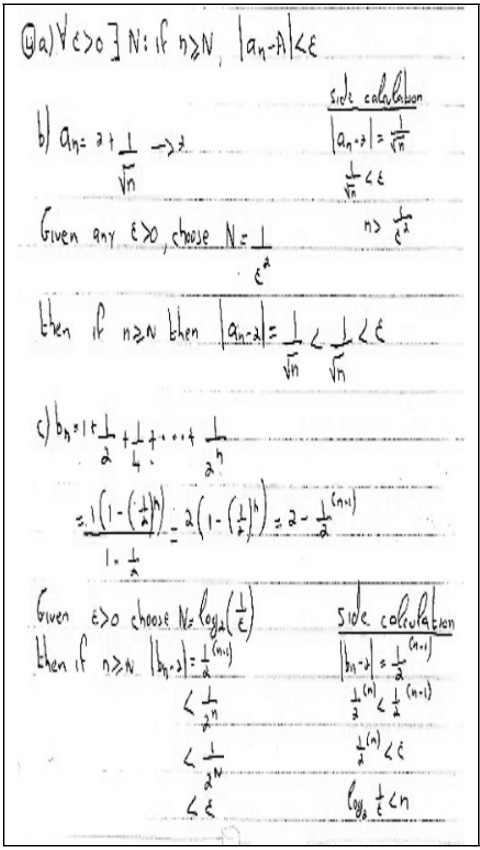
(b) Prove using (a) that the sequence $a_n = 2 + \frac{1}{\sqrt{n}}$ converges to 2.

(c) Prove using (a) that the sequence $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$ converges to 2.

Notes on solutions

(a) "The sequence (a_n) converges to A as $n \to \infty$" means: $\forall \epsilon > 0$, $\exists N$ such that $n \geq N \Rightarrow |a_n - A| < \epsilon$.

(b) Given any $\epsilon > 0$ choose $N = \frac{1}{\epsilon^2} + 1$. Then $n \geq N \Rightarrow n > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon$. On the other hand $|a_n - 2| = \frac{1}{\sqrt{n}}$ so we have shown that $n \geq N \Rightarrow |a_n - 2| < \epsilon$ which proves that a_n converges to 2 as $n \to \infty$.



4 a) $\forall \varepsilon > 0$ $\exists N$: if $n \geq N$, $|a_n - A| < \varepsilon$

b) $a_n = 2 + \frac{1}{\sqrt{n}} \to 2$

side calculation: $|a_n - 2| = \frac{1}{\sqrt{n}}$, $\frac{1}{\sqrt{n}} < \varepsilon$, $n > \frac{1}{\varepsilon^2}$

Given any $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon^2}$

then if $n \geq N$ then $|a_n - 2| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$

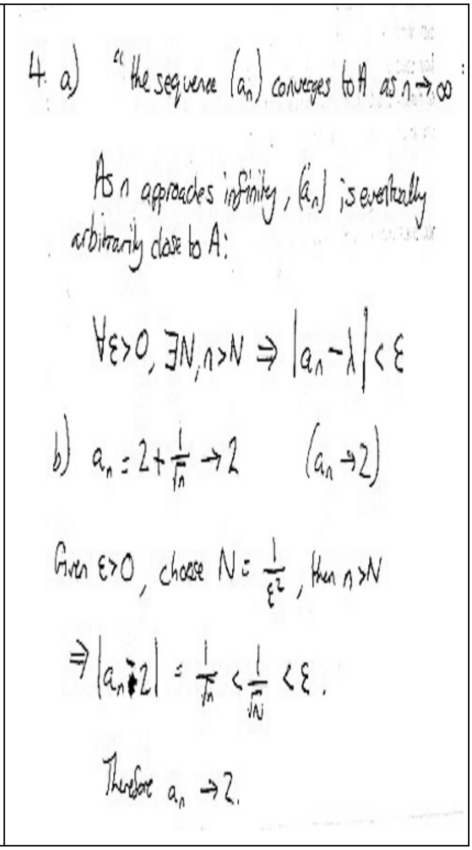
c) $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$

$= \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^n\right) = 2 - \frac{1}{2}^{(n-1)}$

Given $\varepsilon > 0$ choose $N = \log_2\left(\frac{1}{\varepsilon}\right)$

then if $n \geq N$ $|b_n - 2| = \frac{1}{2}^{(n-1)} < \frac{1}{2^n} < \frac{1}{2^N} < \varepsilon$

side calculation: $|b_n - 2| = \frac{1}{2}^{(n-1)}$, $\frac{1}{2}^{(n)} < \frac{1}{2}^{(n-1)}$, $\frac{1}{2}^{(n)} < \varepsilon$, $\log_2 \frac{1}{\varepsilon} < n$



4. a) "the sequence (a_n) converges to A as $n \to \infty$":

As n approaches infinity, (a_n) is eventually arbitrarily close to A:

$\forall \varepsilon > 0, \exists N, n > N \Rightarrow |a_n - A| < \varepsilon$

b) $a_n = 2 + \frac{1}{\sqrt{n}} \to 2$ $(a_n \to 2)$

Given $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon^2}$, then $n > N$

$\Rightarrow |a_n - 2| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \varepsilon$.

Therefore $a_n \to 2$.

Student N ***Student MR***

To start with M critiqued the question as follows:

M: *Write out carefully the exact meaning of* can be quite a treacherous way of asking this question and I wonder what the question setter means actually!

RME: At that point in the course the students have been introduced to the formal definition of convergence. My take on this is that this is a request for a verbal explanation of this formal definition using words like *eventually* and *arbitrarily*[4] maybe?

M: But then in the *Notes on Solutions* the question setter simply rehearses the formal definition! Did students react in different ways to this request?

RME: They did. I have seen a mixture of responses from blending words with symbols all the way down to reproducing the definition or attempting a word-only response.

In what follows M and RME focus on the utter necessity of symbolisation and how it needs to sit comfortably alongside attempts at verbalisation and visualisation of the definition of convergence[5]. Elsewhere M uses the words 'what students see as madness' (OT6.1) with reference to the students' first impressions when they realise that heavy symbolisation[6], as evident in the one line of the definition of convergence, is the norm of mathematical writing – and a norm they are expected to accustom to quite quickly.

RME: Talking about this madness, I am interested in your views on what is often seen by students as *this* ε *nonsense ...*

M: Your question brings to my mind a recent example of a student who wrote down a neat response to a convergence question – applying the definition impeccably – and then asked *why does this prove the convergence?*! What an excellent question! I tried to explain that this is the *definition* of convergence but students don't quite understand the relation between this expression and what convergence ought to mean exactly. I try to convey the idea that *by definition the last line of the proof is the definition of something*. That *prove that something is*

[4] A number of studies (e.g. Robert, 1982) suggest that due to the vernacular uses of the word *limit*, students find it hard to distinguish between limit and bound. Moreover *tends to*, *approaches*, *gets close to* are expressions underlain by the assumption that a limit cannot be attained. In this sense a linguistically informal expression may reinforce the ambiguity of the notion in a way that students actually evade a confrontation with their contradictory ideas.

[5] As explained in detail in (Tall, 1996).

[6] Students' shock at this realisation has been well documented, e.g. in much of the work in (Tall, 1991a).

something implies the use of a definition[7]. This is, I'm afraid, an idea that escapes most students (E3.3) and the Analysis course at these early stages should aspire to help them understand this.

RME: I think my concern here is a bit beyond this more general issue of how students deal with definitions: do they see that the expression of the definition says something about where the terms of the sequence gather under certain conditions? I have seen occasions[8] where the student is asking, after twenty minutes of discussing this expression of the definition, and working with it, *oh, this is what we have been doing, we have been proving that that is going there!* While at the same time using the words *converging, limit, definition* and everything.

M: In my view the difficulty lies with the successive appearance of quantifiers in the definition whereas the primary notion for the students ought to be that *no matter what I specify the ε region about the A, from a certain point onwards everything fits inside this box*. Making this link between this image and the formalisation[9] behind this is utterly important. Otherwise the definition is nothing other than formalistic nonsense.

RME: The difficulty of making this connection was exactly my concern.

M: But this is exactly the meaning into words such as *eventually* and *arbitrarily*, which I constantly put in my writing in my attempt to convey the idea underlying the formal expression of the definition. Most students however simply ignore them as irrelevant waffle (E4.4) and copy the definitions only in their notes. I even try grouping the various parts of the definition with different colours of chalk! Later on, in the final years, the students seem to retrospectively understand this though, and to an amazing extent. But using words is risky[10]: I have seen verbal explanations of the definition which are in fact wrong! Like

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[7] For a meta-mathematical exploration of students' perception of the role of definitions see for example (Zaslavsky & Shir, 2005). In this study students were asked to consider several possible definitions of some mathematical concepts. In contemplating alternative definitions students distinguished various roles for a definition but its procedural aspects – which would be useful in the case of our example – were not always given high priority. Correctness (necessary and sufficient characteristics) also did not seem to completely satisfy Zaslavsky & Shir's 12th graders who did not attribute the status of a definition to a correct and equivalent statement and were not very comfortable with acknowledging the possibility of more than one definition for a concept.

[8] See one such occasion in (Nardi, 1996) at : http://www.uea.ac.uk/~m011/thesis/chapter7/7i.htm)

[9] A point aptly developed in (Przenioslo, 2004) where classes of images of the limit concept are described according to the students' mainly focusing on notions of neighbourhood, on graphical notions of approaching, on notions of values approaching the limit value, the function being defined at x_0, the limit of the function at x_0 being equal to $f(x_0)$ and algorithmic procedures for calculating limits.

[10] Mathematicians' concern about the potential ambiguity within verbalisation has tended to imply a preference for the concision and compactness of symbolisation (Burton & Morgan, 2000).

many textbooks are! Which is not embarrassing given the long debates[11] about acceptable verbalisations of the definition. Verbalising, geometrisising it etc. is fine as long as we stay this side of correctness!

RME: Often the dominant approach though is to just get the students working through strings of quantifiers[12].

M: You see sometimes I think steering clear of intuitions and pictures[13] etc, yes, working through strings of quantifiers, even though one may not be so sure of what is going on, can be seen as less messy, less risky. I think some students may in fact see it this way and be happy to work this way and just do what they are told. You can view this as the recipe, you can do this, you do this and you do this… You just follow the steps. And in some ways they are safer this way because they will not make mistakes as long as they are technically doing the right steps. Of course this depends a lot on what you think doing mathematics is and we all may have our differences about that… I realise this and this is largely the reason why I encourage them to use pictures (E7.4, Scene IV)[14]. But, hell, there is not always a good intuitive picture of everything[15]: try a good intuitive geometrical or pictorial view of what the statement *the series does not converge* means, for example. Or of what the statement *this function is not uniformly continuous means*!

RME invites M's comments on Student MR's response:

M: Absolutely horrendous – in particular with regard to how he uses *arbitrarily* and *eventually*. I know at face value this is more or less an acceptable response and I would have to give it some good mark but a closer look would reveal that there is quite a bit of non-sense in these few lines. I appreciate that there is an attempt here to employ words to supplement the explanation of what the definition means and that Student MR does not simply reproduce something from the lecture notes. However even the order of things is wrong and I don't think he really pulls off a meaningful explanation. Sorry. The right picture seems to be more or less there but it is hardly conveyed. I use *arbitrarily* and *eventually* in specific ways in lectures, not far removed from their ordinary use and as bridges between the symbolic and verbal interpretations of the definition of convergence, as verbal expressions for quantifiers[16]. Because as we keep repeating (E4.1) there needs to be some mediation between what is to them at least initially an obscure

[11] For example, within the American Mathematical Society (e.g. Cajori, 1917).
[12] A tendency reinforced also by a belief in the efficiency of the symbolic mode (Burton, 2004)
[13] Illustrations such as the one of the limiting procedure as a staircase have been reported as prone to evoking ambiguous perceptions of the concept (Tall, 1992).
[14] Fervent arguments of this can be traced at least in the early 1990s e.g. (Zimmermann, 1991).
[15] For the difficulties of this type see, for example, the case study in (Aspinwall et al, 1997)
[16] A bridging consistently recommended in the literature (e.g. Dee-Lucas & Larkin 1991).

string of symbols and a sentence that explains what this string of symbols mean. Once I have done this and the mediation has taken place we can all happily dispense with this scaffolding of words[17] and carry on with the string of symbols that has its well known advantages. And I am not saying that we as mathematicians are not often culpable of expressing ourselves in imprecise ways that are then emulated by our students[18] (try the ways in which applied mathematicians serve up the notion of a Taylor series to unsuspecting Year 1 students…!). But I often despair of how complicated this task of verbalisation is[19] – how, for example, we often read a mathematical quotation backwards when we try to express it in words. I, as more experienced, am aware of doing that and of the jeopardy to which this puts the logic of my sentence. But students often don't[20]. A mere change in the order of quantifiers – and the order in which you use words such as *arbitrarily* and *eventually* in a sentence – and the whole meaning of the thing changes totally (and often disappears altogether…!). Student MR's words seem to be doing a bit of this.

RME: You don't sound very happy with this response…

M: What does *arbitrarily* mean here? This use of *arbitrarily* verges on waffle. It is possible in words to express the idea of convergence correctly, and it is possible in words to express it wrongly. I think it is more complicated to write this in words correctly than writing it in symbols, but, unsurprisingly, and given my experience, I am very familiar with the symbols. But, even in words, you can just state it. Say, you can say: given any box, from a certain point onwards everything falls into the box. Understanding the particular notion of closeness involved in the definition is what I am after here. In this sense Student MR's response is irredeemable – unless of course they come up with an appropriate use of the definition in the second part. Unless they handle epsilon and the quantifiers alright in the second part and the whole first part thing turns out to be one more of those *I say one thing but I mean another and my use of the definition proves that what I mean is alright*[21] instances.

RME: What do you think of this one, Student LW's response?

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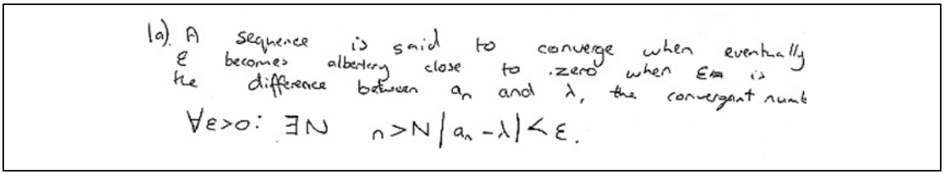
[17] See (Dee-Lucas & Larkin, 1991)
[18] See (Burton & Morgan, 2000)
[19] See (Burton & Morgan, 2000)
[20] Across the examples we are examining here M often juxtaposes his practices, those of an expert, to those of the students. In Chapters 3 and 4 I have placed these references within the discourse of introducing the students to the cultural practices of university mathematics (e.g. Sierpinska, 1994)
[21] See (Pugalee, 2004). Also with regard to verbalisations of the definition of limit (Williams, 2001).


1a). A sequence is said to converge when eventually ε becomes alberterry close to zero when ε is the difference between a_n and λ, the convergant numb

$\forall \varepsilon > 0 : \exists N \quad n > N \,|\, a_n - \lambda| < \varepsilon.$

Student LW

M: I would say a wordy but genuine attempt to translate what is in their minds in their own words – a tall order in any case – and not simply cobbling together words from the lectures and the notes. The ship is sinking here however because *when ε is the difference between a_n and λ* is nonsense. Because of ε being viewed as anything other than a number. Because I am unsure about the punctuation in there, especially the role of this colon between the quantified bits about ε and N. Is it just punctuation for its own sake, perhaps meaning *such as*? Maybe the student meant is as a comma? In fact there is a need for some punctuation in this line: not where the student put it though but between N and the modulo. So I think this colon is there because it has to go somewhere and the student has not read it again to make sure that placing it where she did actually makes sense. Do you want some more reasons why this ship is sinking? Because of the wrong order of quantifiers – and the positioning of the word *eventually*. The very first word is *eventually* and from that point they are doomed. Within *eventually* lies a second quantifier, that's how it is usually employed in lectures. And the order of quantifiers is so essential in mathematical writing[22]. Before you know it what you have written down is dire because you tinkered unwisely with this order.

RME: I also detect an absence of a sense of implication in Student LW's response.

M: That is the other problem. For me, even if Student LW is using the symbols, it is not clear that they confidently know what the definition is. The only remotely meaningful thing I think has come out of this is that a_n in some sense has to be near λ, but it has been expressed rather poorly. I am a proponent of minimal use of quantifiers in a sentence – a quantifier kills the sentence, I think – and of consistently pursuing full sentences in English when writing on the board to deter students from falling into the trap of elliptic, oblique writing like in these examples[23]. Of course the whole point of using symbols is to do things that are

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[22] M translates the students' use of *arbitrarily* and *eventually* into what they possibly mean as quantifiers and condemns the logic behind this use as flawed. See (Williams, 2001) for associations between these words and students' understanding of 'reaching', 'approaching' within the notion of convergence.

[23] M captures that elusive characteristic of quality mathematical writing, one that manages an equilibrium between succinct and precise symbolisation and adequately explanatory verbalisation (Bullock, 1994).

more easily done when in symbolic form, such as the negation of a statement. Overall the students need to appreciate the need for this kind of fluency[24].

RME: Symbolic writing ought to be seen as alleviating the burden of complexity, not adding to it. Students need to appreciate its utility.

M: This compression – compactness (E4.2) is hard to cope with in the beginning but utterly important to learn how to. This is pure mathematics! As it proceeds, pure mathematics invests more and more meaning in purer and purer symbols and so you end up with this capital K subgroup stuff, let's say. And, you know, you can literally write down a one-line statement that would take ten years to explain. And that is very unfamiliar to students. They think that a calculation that is twice as hard is much longer and the integral is twice as difficult if it has twice many steps in it. And a differential equation that is twice as difficult has twice as many steps in it. But this is just not the way in which pure mathematics is written. From their point of view of course in many occasions length in some way is comparable to difficulty[25]. One of the difficulties I think that the students have is that they have all these new symbols to deal with in the problem that seem to have nothing to do with the calculations involved in answering the question. So the students see these as obstacles rather than things that are cutting down on the number of words that they need to write. Also using a symbol doesn't always quite reveal much about what one is qualifying for: take *there exists N*. Well, OK, you wish to consider an N. Where does this come from? Which pool am I picking from? Etc.

RME: Would you agree that to produce a totally wordy, verbal explanation of the definition of convergence would take a good four or five lines of words?

M: Not necessarily. But you would have to allow space to expand on the idea of ε being something that is beyond the control of the person judging the process. That *for all ε* comes first means subsequently you can make no assumptions about this. Half of your writing could be about this. At this stage I think it is absolutely essential to demystify all parts of the definition to the students using plain language. And of course never forget that often our own writing can be ambiguous and the subject of controversy[26] – for example regarding quantifiers. Or regarding the order we write symbolically which is different to that of ordinary language. And writing in LaTex sometimes exacerbates these differences. There seems to be little misunderstanding amongst mathematicians regarding the writing when we write in journals but overall conveying what we mean to students is not a straightforward business (e.g.E7.4ii).

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[24] Or in Sierpinska's (1994) to adopt to the mathematical community's established discursive practices.
[25] For an exploration of students' judgements of this kind see, e.g., (Schoenfeld & Hermann, 1982)
[26] See (Burton & Morgan, 2000)

EPISODE 6.3
THE MECHANICS OF IDENTIFYING AND PROVING A LIMIT

In what follows M and RME discuss the difficulties students have with proving the existence of the limit of a sequence (with or without necessarily identifying it) through invoking pictures, conducting numerical experiments or employing relevant theorems; they also discuss the issue of proving an identified number as the limit of a sequence through the use of the definition in the context of discussing the mechanics of searching for *N*.

Scene I: In search of N

Setting the scene: The discussion here follows from M's exposition (E7.4, Scene III) on the strengths of using concrete tools such as graphs, numerical experiments and certain types of calculation towards establishing the existence (and proof of) a limit.

RME: Student N uses this type of side calculation (E6.2) you are talking about, doesn't he? To help him define *N*. He sort of solves the inequality backwards until he gets to an idea of what this *N* could look like.

M: Which is absolutely fine by me. I know what the limit I am trying to prove is and I am trying to get to this by solving backwards – great! I notice that even though *N* is supposed to be a positive integer, the *N* on offer here does not include the integer part [ ], but what I essentially appreciate mostly is that *N* is not coming out of the blue, it is constructed through a very sensible *solving backwards* procedure. I am very happy as long as this *solving backwards* does not become some sort of gambit that is followed blindly of course[27]!

RME: Oh yes! Anything they do should contribute and link to an understanding of what the formal expression of the definition means.

M: They have a tremendous difficulty with these first encounters with the formal machinery (E6.1, E6.2). And so often they simply do not wish to bother proving something that is pretty obvious to them (E6.1). What I say in moments like that is (E7.1, Scene II): ok, this is your informal understanding, you are expecting that this is generally true, but hey, here is an example where this is not the case, so you see? Your thinking doesn't cover all the ground. To do that you need to address this formally, you need to use the formal apparatus because this helps you sort out the general case, it helps you not miss things.

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[27] As in E7.4, Scene V, this is in tune with M's frequent advocating a balanced development of students' abilities in terms of algorithmic skills as well as conceptual understanding.

Scene II: Identifying the limit of a sequence

Setting the scene: In the context of M's comment on the variable usefulness of pictures in E6.2 (there is not always a good intuitive picture of everything) RME explores M's views on how best students can take the step from intuitively guessing a limit[28] towards proving it.

RME: How would a student, I wonder, take the step between the intuition, e.g. guessing a limit when this is not given, and the formalisation, proving it?

M: Not easily. That is why I always believed Cauchy sequences, where you can talk about these limits without knowing what these limits are, are really tremendous inventions! And I think a syllabus that stays clear of them for a long time can be seriously impoverished. They get you out of a situation which is almost impossible: if I don't know what the sequence converges to, all this is meaningless. How can I ascertain the existence of such a thing, a limit, unless I have a constructive procedure? The bounded-and-monotonic theorem is another one of those tools that serve this purpose.

RME: In most work we do in the area of convergence, often the point is to prove convergence without ever specifying the limit. Not in the question we are discussing here though where the limit, 2, is a given and the request is to prove it *is* the limit.

M: In some sense this is part of the nature of the subject. One thing that I try and get students to do – which again they are very resistant to, ironically – is numerical experiments (E7.4, Scene III).

[28] For example Ferrini-Mundy & Graham (1991) report that students are often unwilling to employ graphical information or geometrical understandings in this process.

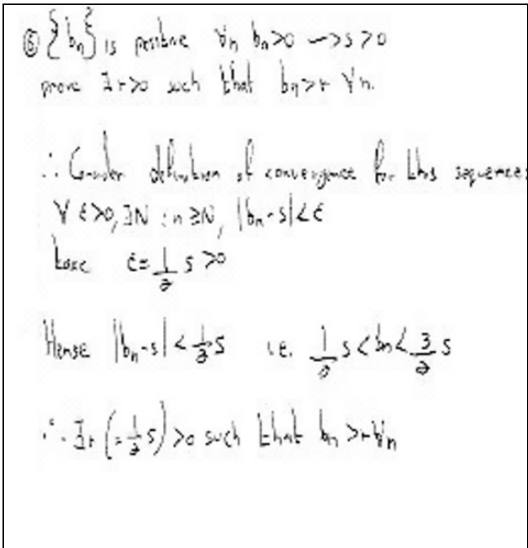
SPECIAL EPISODE SE6.1:
IGNORING THE 'HEAD' OF A SEQUENCE[29]

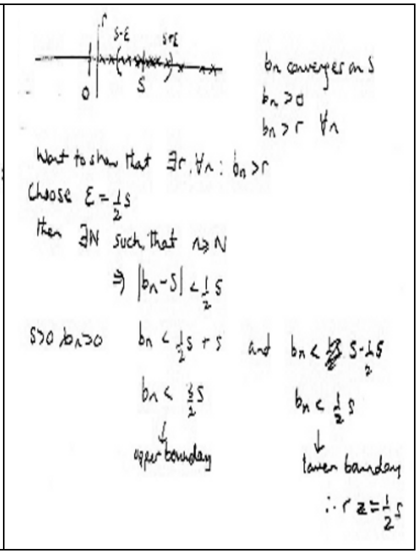
Setting the scene: The following takes place in the context of discussing the question below as well as two examples of students' responses, Student N's and Student H's:

Write down a careful proof of the following useful lemma sketched in the lectures. If $\{b_n\}$ is a positive sequence (for each n, $b_n > 0$) that converges to a number $s > 0$, then the sequence is bounded away from 0: there exists a number $r > 0$ such that $b_n > r$ for all n. (Hint on how to start: Since $s > 0$, you might take $\frac{1}{2}s = \epsilon > 0$ in the definition of convergence.)

Notes on Solutions

Let $\epsilon = \frac{s}{2}$ in the definition of convergence. Then there is an N such that $n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n \geq \frac{s}{2}$. Then, for any n, $b_n \geq r = \min\{b_1, \ldots, b_N, \frac{s}{2}\}$ which is the minimum of finitely many positive quantities, hence is positive.





Student N ***Student H***

First M offers a critical comment on the question.

M: This is a particularly difficult question to be given to students that early. The hint I guess is a bit of a comfort, a resort to the familiarity of dealing with an

[29] See (Iannone & Nardi, 2001) for a preliminary consideration of the issues M and RME discuss in this Episode. There we consider more widely students' use of the universal quantifier, ∀, and suggest that students may neglect the 'universality' in its meaning (we include more examples in which students' application of the definition of convergence did not cover all cases for ε).

algebraic expression, to the safety of following instructions on some algebraic manipulation they are confident they can follow through somewhat blindly (E6.2). And on the way the students, taken by these manipulations, ignore the words through which they would have to phrase what is happening with the *head* of the sequence here (E4.4).

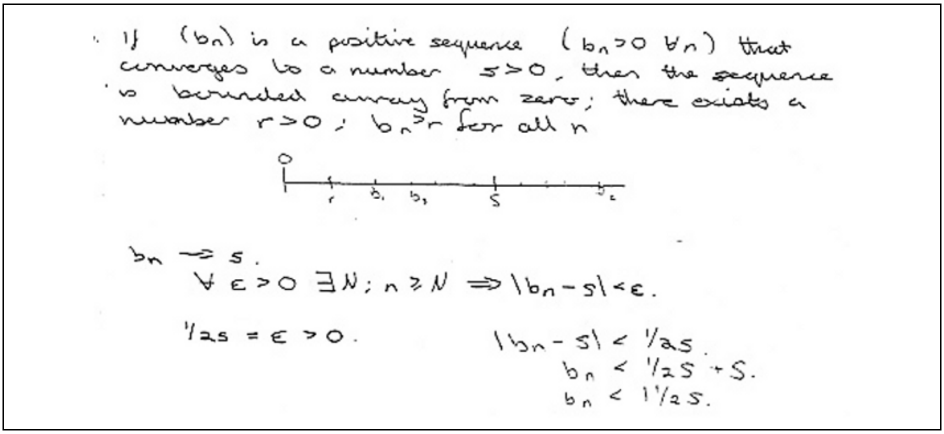
RME: Is this your take on why Student N is ignoring the *head* then?

M: He has followed the hint, established boundedness for most of the terms of the sequence but then ignored the ones below N: had the student drawn a picture (E4.3), he would have seen he had left them out.

RME: To draw a picture is what Student H did.

M: Yes, the kind you would see on the board during a lecture. By the way I am impressed how both Student N and Student H go from expressions that involve absolute value || to perfectly sensible statements about the distance between things. But I reckon she needed a more helpful picture, one that would have enhanced the possibility of seeing that this otherwise sensible argument was leaving terms of the sequence remaining unaccounted for. For example a graph, not simply putting the terms on a line, because the *head* is a much bigger and more beautiful thing if you draw a graph. Of course essentially the importance of the question lies in noticing the words *for all n* whether on a picture or in the words. That is probably why I find Student N's response more compelling.

RME: What do you think of what this student did then, Student E?


If (b_n) is a positive sequence $(b_n > 0\ \forall n)$ that converges to a number $s > 0$, then the sequence is bounded away from zero; there exists a number $r > 0$: $b_n \geq r$ for all n

$b_n \to s$.
$\forall \epsilon > 0\ \exists N;\ n \geq N \Rightarrow |b_n - s| < \epsilon$.

$1/2 s = \epsilon > 0$.
$|b_n - s| < 1/2 s$.
$b_n < 1/2 s + s$.
$b_n < 1\,1/2 s$.

Student E

M: She has indeed left some *crosses* outside the s-ε, s+ε area. I am happy with this to start with: Student E has at least summarised the bits and pieces of information in the question on this picture. However I doubt whether this picture has been of any help at all and I have concerns about this as evidence of understanding of the absolute value (E4.3) and the bound in the definition. She doesn't see that the inequality with the modulus implies the existence of an upper bound. In fact I do not think the student used this diagram as a source of inspiration for answering the question. I think she drew this, on cue from recommendations that are probably on frequent offer during the lectures, and then returned to the symbol mode unaffected. And without one word for explanation (E4.4). I am also not sure that the modulus has been accepted as a distance[30].

RME: The students seem to ignore the *head* when they concentrate on the inequalities and ignore the quantifiers part of the statement. Do you think there are any contradictions between the drawing and the writing in the students' responses? Say Students E and H?

M: Well, in Student E there is no real connection between the picture and the writing. In Student H it is a good sign that the s-ε, s+ε area is bracketed. I wonder how these students would respond to a similar question such as *here is a finite list of positive numbers. Prove there is a positive number that is smaller than all of them.* Because I think this is essentially a question about the understanding of boundedness, not convergence. I am looking at the question and I am thinking: is there anything in the question driving the students to forget about the *head*?

RME: You think maybe the smallness, the finiteness of the *head* made it insignificant to the students as they are being trained to look at the infinite majority of terms. If the picture used by Student H was originally hers and not simply reproduced on cue from the lectures then maybe we could consider this possibility. But… it is not. I feel that the density of crosses in the s-ε, s+ε area in Student H's picture, and the scarcity of the crosses outside the area may reflect the student's sense of how many lie inside and outside the bracketed area.

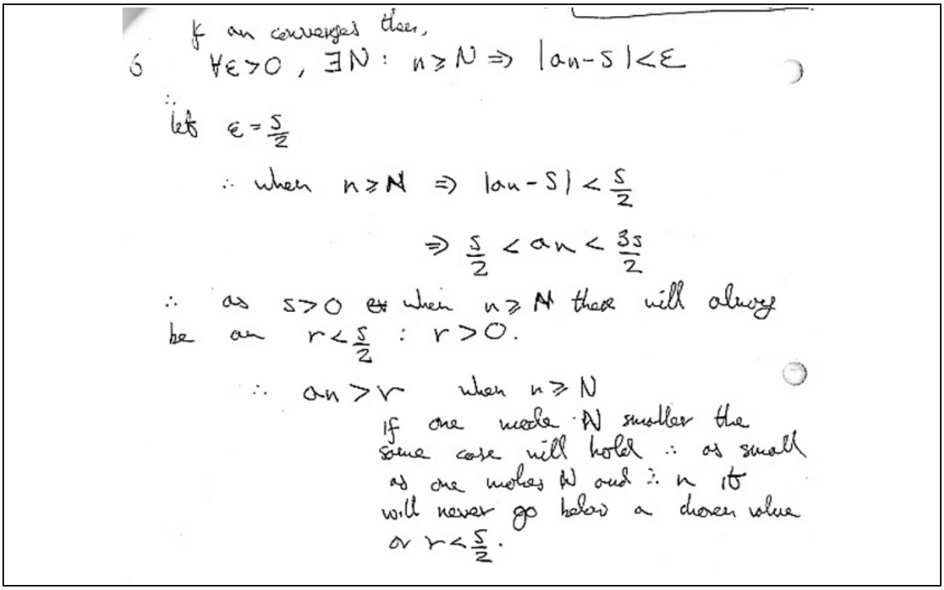
M: I see what you are saying about some evidence that the student may be concerned about most – hence the density – but not all terms of the sequence. However I would probably insist that more or less Student H did the diagram as often recommended but then abandoned it. So I wouldn't associate the ignoring of the *head* or any part of her thinking too much with the existence of the diagram in her script.

---

[30] As in Chapter 5 M and RME discuss the importance of a fluent movement between multiple representations – here in the context of analytic and geometric notions of ||.

RME: I see. There is one student who actually did not ignore the *head*, Student J.

M: Oh, there is a clear attempt – through the verbal explanations and the clarity regarding the status of N and what it does in terms of excluding some terms of the sequence from the boundedness argument – to include all the terms of the sequence so that more and more *head* can be captured. This is just a hair's distance from being complete and it is one of those occasions where a brief but focused exchange with the student would suffice to clarify what remains to be done (E7.1). And with a picture that somehow indicates what is going on in these sentences, it would be even easier. I am a bit worried about the *if one made N smaller* bit, because N is determined by the choice of epsilon in the definition but what I think he is trying to say is more or less within the *will never go below a chosen value* bit. There is recognition here that certain terms are not included and an attempt to include them. Pushing N to smaller values, leaves fewer and fewer terms out. This is a rather nice thought.



6 If a_n converges then,
$\forall \varepsilon > 0,\ \exists N : n \geq N \Rightarrow |a_n - S| < \varepsilon$

∴ let $\varepsilon = \frac{S}{2}$

∴ when $n \geq N \Rightarrow |a_n - S| < \frac{S}{2}$

$\Rightarrow \frac{S}{2} < a_n < \frac{3S}{2}$

∴ as $S > 0$ when $n \geq N$ there will always be an $r < \frac{S}{2} : r > 0$.

∴ $a_n > r$ when $n \geq N$
if one made N smaller the same case will hold ∴ as small as one makes N and ∴ n it will never go below a chosen value or $r < \frac{S}{2}$.

Student J

RME: Still this was the only example I could dig out where there was a hint of awareness by a student that the argument has left some terms of the sequence out.

M: Yes, it is clearly a difficult question for them. I am wondering how it could be reformulated to drive them towards the thought of full coverage more emphatically. Maybe ask them to conduct some concrete calculations (E7.4, Scene III) such as identify bounds for specific values of epsilon. Maybe ask the same question for a specific s. Maybe saying $n \geq 1$ instead of *for all n* may have driven the students towards thinking of covering all n before and after N, which is a very common mistake they make. Underlying all this is the issue of understanding quantified statements (E4.1, Scene II). It seems that after all the presence of the quantifiers themselves in the text of the question is not emphatic enough to suggest universality or existence to the students. And words, sentences, those creatures ever-absent from students' writing exist exactly for this purpose: of emphasis, of clarification, of explanation, of unpacking the information within the symbols. Especially within complex and subtle statements such as the definition of convergence (E4.4).

OUT-TAKE OT6.1
$\geq$ OR $> N$?

Setting the scene. The following takes place in the context of the discussion in E6.2 and comes after M's critique of the question early in that Episode. It is a statement on the part of M about the apparently inexplicable preponderance of $\geq$ over $> N$ in the definition of convergence and the unnecessary (for students already overwhelmed by the density of the definition) conceptual burden $\geq$ carries

M: And I am afraid this is not the only point of ambiguity here. I think there is a bit of a nightmarish situation also regarding whether n ought to be taken as $\geq$ or $> N$. There is quite a lot of inconsistency in many books, authors fluctuate between $\geq$ and $>$, and I feel this could generate confusion to a student who is experiencing their first encounter with the definition. Even though, for some reason that remains totally mysterious to me, $\geq$ seems to be so ingrained in us, I am a proponent of $>$ myself. And here is why: guaranteeing $\geq$ is a two-step process, one step for $>$ and one step for $=$. Also, if you go along with $\geq$, then once you have identified a potential N you need to add 1 to it to make sure $\geq$ holds. And in actual fact in terms of the meaning of the definition $\geq$ or $>$ makes no difference. Therefore, what is the point of having a two-step procedure, which risks confusing the novice about an incredibly minor thing, instead of the simpler and equally satisfying one-step procedure? Fluctuating between the two in the various texts also may make students, especially good ones who would be more alert to details such as this, think there is a meaningful difference. Frankly I don't know why we still bother with $\geq$! This is what I was taught, it lingers in textbooks but in fact it takes you down weird side alleys. And I just don't know where it comes from. It's weird.

OUT-TAKE OT6.2
SERIES[31]

Setting the scene. The following takes place in the context of the discussion in E6.2. M comments on the students' handling of a series at this early stage of their encounter with the concept.

M: Initially one may feel disappointed about the approach in part c, where the students may think they are required to use a mechanism for calculating the sum in b_n and then consider the convergence of b_n – which, by the way, is an approach that in the things to come in Analysis regarding series, is not an approach that will get them very far. They would need to get used to treating series as entities on their own – and not convert them into a number or a function or whatever they are familiar with – and have a bag of tricks for dealing with them (E7.4, Scene V), like knowing about Cauchy sequences. But in this particular case, to be honest, I think the underlying intention is for them to do exactly that: find a way of calculating the sum in b_n and then consider the convergence of b_n. One way to calculate would be by Mathematical Induction, or most likely here, the formula for the geometric progression, given their familiarity with it from school. OK, this latter one would involve a bit of cheating since we are not exactly clear about whether they are allowed to use this or not yet (E7.4, Scene IIb), but there we go! So the underlying intention is, I think, to exploit this very rare opportunity in Analysis for some basic, but hugely gratifying to the novice, algebraic manipulation, and one that they have seen before: convert the sum within b_n into the manageable expression that involves $(1/2)^n$. Then consider ways in which they can deduce whether b_n converges or not and, if yes, to what. Granted, we do not wish them to think that this is a typical approach when we are discussing the convergence of series but I feel they are entitled to this moment of gratification through familiarity before in Analysis it all goes to what they see as madness. Soon they will be parting company with this sort of approach so why not enjoy a sort of last shout?!

OUT-TAKE OT6.3
CONTINUITY AND DIFFERENTIABILITY[32]

Setting the scene: In their discussions of students' encounter with the concept of limit M and RME made occasional associations with students' understanding of continuity and differentiability. Below they discuss:

[31] González-Martín (2005; González-Martín & Camacho, 2004) links students' difficulties with integration with the understanding of this largely under-researched concept. Of particular interest is his view that some of these originate in the learning of series within the constraints of the 'algebraic register'.
[32] This is part of the (Nardi, 1996) Episode referred to in E6.2.

- the potentially unfortunate use of $\delta(a, \varepsilon, f)$ in the definition of continuity to stress the dependence of δ on a, ε and f, where a is the value of a function at a certain point;
- the potentially damaging triviality of choosing $f(x) = x$ as an example of a continuous function, since in this case the fact that δ can be chosen as equal to ε nulls the opportunity to explore the nature of δ; and,
- the conceptually necessary association between the concepts of derivative, convergence and continuity at an early stage of the students' encounter with these notions

In the following M and RME discuss an excerpt from (Nardi, 1996): Chapter 7, Section (i) (*Constructing a Meaning of the Concept of Limit: Concept Definition and the Formalism of Mathematical Notation, Concept Image and Visualisation*). In the excerpt the discussion between the tutor and the students on their conceptions of limit takes place soon after the students have been introduced to the concept in Week 4 of their first term. Therefore the conversation captures in freshness the genesis of their conceptions.

The tutor asks for a definition of limit. Cathy sounds hesitant and asks the tutor whether he wants it 'properly defined'. The tutor says yes. After a few moments of hesitation George writes on the b/b: $lim_{x \to a} f(x)=L$ given $\varepsilon>0$ $|f(x)-L| < \varepsilon$ (1). The tutor stresses 'it cannot always be $<\varepsilon$' and George then adds below (1): there exists $\delta(a,\varepsilon)$ whenever $0<|x-a|<\delta$ (2). The tutor then asks for 'the picture of this new caption'. George replies:

G1: When you...as your...x approaches a where the limit is then...approaches that point...then the difference between that $f(x)$ and the limit is...you could...it is smaller than any epsilon...well not any epsilon...epsilon you...

The tutor then turns to Cathy and asks for her 'intuitive concept'.

C1: Well, I think...I mean it doesn't really give me one. I mean I don't understand what delta is supposed to be. Is it a number or is it a function or what?
T1: Number, number.
C2: Number. In that interval?
T2: What interval?
C3: a comma epsilon.
T3: Is that the notation the lecturer has used?

The students nod. The tutor sounds quite surprised with Cathy's interpretation and 'coming back to what George said' he explains that all they need to say is that whenever x approaches a, $f(x)$ approaches L. $F(x)$, he explains, 'doesn't bounce off to all sorts of different directions'. To 'pin down' the notion of 'approaches', or 'close to', we use ε. Depending on how accurate we want to be, how close we want to get, we take a small ε. Then 'the smaller the epsilon is the harder is to get that close'. Cathy then asks 'What does delta depend on?'.
The tutor turns to the b/b and rewrites the definition: $L = lim_{x \to a} f(x)$ if for any $\varepsilon>0$ there exists $\delta>0$ such that whenever $0<|x-a|<\delta$, $|f(x)-L| < \varepsilon$ (3). He stresses that he wishes to delete $\delta(a,\varepsilon)$ which has 'given rise to concern'. Then Cathy asks:

> C4: Well, what would happen if it didn't exist, this delta? I mean the way it's said kind of...I mean if there wasn't the limit you wouldn't be able to do that.
> The tutor says that her question is about the case where L is not the limit and asks them to turn to question CD4.1. The question is about, he says, whether there exists a number L such that $lim_{x \to 0} f = L$. Namely, such that (3) on the b/b is true. Cathy asks him to repeat and he does. Discussion then is as follows:
>
> C5: So say I am given an epsilon and we want $f(x)$-L to be this epsilon and you...then there would be a delta for which $f(x)$-L is always less than epsilon.
> G2: It's that you have to find the delta to...
> T4: Hold on. This is the definition...
> C6: So it could never be true that if L existed, then you couldn't find a delta for any given epsilon...I mean...
> T5: Well, you are trying to deny a definition, aren't you?
> C7: Deny a definition?
>
> The tutor stresses that this is the definition of continuity: 'If we cannot prove that this condition holds then we have one of two possibilities: either the function doesn't have a limit at L or we're just being unable to prove it'. George then is concerned about the dependence of δ on ε and the tutor explains that dependence here is taken in the general sense (that is given an ε beforehand, one should be able to find a δ) and does not suggest the existence of a function between δ and ε. He suggests looking at an example. Cathy then adds about the definition of a limit:
>
> C8: Is it alright saying that when x is close enough to a, then $f(x)$ is close enough to L?
>
> The tutor enthusiastically agrees and asks them to 'prove the conjecture that if $f(x)=x$ then $lim_{x \to 1} f = 1$'. The students are silent and the tutor asks them to identify a δ for $\varepsilon=10^{-472}$. Cathy hesitantly suggests $\delta=\varepsilon$. The tutor substitutes $\delta=\varepsilon$. It works. George then points out that 'it normally works to choose your delta in terms of that epsilon' and Cathy that 'then you also choose your delta so that...you...depending on the function'. The tutor adds that if he had chosen a more complicated function, the choice of delta would be probably a more complex process. Then Cathy asks:
>
> C9: Why should we take limits? I mean if we put 1 in there then we see what value the function takes and...
>
> The discussion then focuses on students' 'prejudice for' continuous functions[33]. […]

Below is a passage from the analysis that accompanied the above piece of data in (Nardi, 1996) and was included in the Dataset that M examined prior to the discussion:

> Cathy's hesitation in the beginning signifies quite eloquently the paralysing effect that the request for formalism has on novices at this stage. George's reaction is to fully reproduce in (1) and (2) the definition given in the lectures giving however the impression that he does so in a mechanistic, uncritical way: he completely ignores the

[33] Students often calculate lim$f$ more easily when f is continuous (Ferrini-Mundy & Graham, 1991). More generally concept images of function have a strong influence on the ways in which students choose to find limits e.g. when 0 1 0 1/2 0 1/3 0... is taken as two functions, not one.

conditions under which $|f(x)-l|<\varepsilon$ holds in (1). Then, as a result of the tutor's objection, he restores the conditions impeccably in (2). That the formalistic integrity of G1 and G2 do not reflect the depth of his understanding, is illustrated in G1 which, on one hand, contains the main implication of the definition ($x \to a \Rightarrow f(x) \to l$) and, on the other hand, shows a very problematic perception of the nature of ε. Translating the $\Rightarrow$ of the above implication into the quantification of δ and ε is probably the most cognitively problematic aspect of the definition of limit observed in these students.

Moreover C1-C3 are indications of Cathy's semantic misinterpretation of the notation used by the lecturer. In order to denote the dependence of δ on a and ε, the lecturer used $\delta(a,\varepsilon)$ which Cathy mistook, first, for δ being a function of a and ε (C1) and, then, for an interval (a,ε) of which δ is an element. In sum, so far Cathy appears to be completely at sea with the concept. The subsequent shift of Cathy's perception of the form and content of the definition of limit seems remarkable and is presented in the following along with further elaboration on the concept. The conversation starts immediately after the tutor has explained the definition of limit (first verbally, in colloquial terms such as 'approaches' and 'close', then formally in (3), introducing δ and ε, and graphically – not included here). While listening to him, Cathy sounds preoccupied with the definition of δ (C4).

In C4 Cathy appears as if she is trying to understand the definition of limit from its negation without realising that her words constitute parts of the negation. C5 is a verbalisation of the definition that seems to reflect a growing image of the limiting process as a machine (input: ε, output: δ). George also accentuates this with 'find a δ' in G2 as opposed to 'there would be a δ' in C5. C6 is more illuminating about Cathy's intentions: she seems to be struggling with the implication *if the limit exists then* $\forall\varepsilon>0\ \exists\delta>0...$ and wants to find out if it is likely at all for the limit to exist and at the same time to be unable to find a δ. The tutor (T5) still thinks she is trying to construct the negation but she seems surprised at this interpretation (C7). Her surprise in C7 might also be attributed to the common belief that definitions are undeniable (by 'denying' the tutor possibly refers to negating the categorical proposition contained in the definition). Then in C8 it seems that, for the time, Cathy has resolved her perplexity at the definition of limit. Her 'close enough' verbalisation in C8 is an articulate one and is enthusiastically received by the tutor. However Cathy's enlightenment proves very short-lived (in the rest of the Episode not included here).
Led by the tutor through applying the definition of limit on $f(x)=x$, Cathy appears capable of finding a δ on her own (I note that whereas ε is given by the tutor as equal to 10^{-472} she replies 'take $\delta=\varepsilon$' which shows that she has generalised the process of finding δ and doesn't need the specificity of 10^{-472}). Moreover both students appear to be beginning to comprehend the mechanism of the definition as seen in their observations on δ and its dependence on ε and the function.

M: I am intrigued by *is it a number or a function or what?* in the dialogue between the tutor, Cathy and George. Also I am impressed by the use of a specific, very small ε and by *the smaller the ε the harder it is to get close to it.* A tutorial is an ideal place to explore these things: look at the part of the dialogue on the nature of δ which, without a tutorial, would have been left un-discussed (E7.1, Scene IV). I think the *$\delta(a,\varepsilon, f)$...* choice of notation is unfortunate as it invites a misunderstanding of delta as a function of a and epsilon which is not the case at all. δ is not a function, it is a number (E5.1). If you read the quantified statement precisely that is all that it is. It depends on anything that precedes it. That's what these quantifiers mean. I am always weary of the treatment of delta as a function. If you like there are many more than one possible functions if you think of this dependence on a and epsilon but it is genuinely confusing to treat the choice of delta as a function. Not for everybody of course. Some people may find it unproblematic to pick one of those functions and make this function their way of choosing delta but why complicate things this way? And how exactly would you negate the definition if you go along with this function idea? Negate a statement written in quantifiers is a much more straightforward task.

RME: I think I would go along with keeping the choice of delta away from any notion of *functional* dependence on a and epsilon but of course fully engaging with it in terms of this dependence.

M: And I don't think choosing $f(x) = x$ – the most boring choice of example (E7.4, Scene VI, SE7.1) I have ever seen and one that is almost destroying the point of doing Analysis! – helps with this clarification as in this case the sensible choice is to take delta as equal to epsilon. But how typical is this choice? How many times would that be a good idea? So, why start with an example which is so untypical?

RME: It's probably intended as a nice and simple first foray into the use of the definition of continuity kind of example.

M: In any case I think it is very difficult to conceptualise the limit of a function at a point outside the concept of derivative.

RME: If one assumes that students see the derivative as a limit. On the grounds of their school mathematical experience (SE3.1) their images of the concept is of something you calculate; a derivative is simply the opposite of the integral, maybe supported with a bit of imagery on areas under curves etc.

M: I am amazed that students are allowed to differentiate without understanding what a derivative is. I am appalled that someone would differentiate without having the faintest clue that they are dealing with the issue of local change.

RME: Anyway, to return to the excerpt, the dialogue here seems to suggest that some misunderstandings are resolved.

M: I agree. This is an effective example of the sort of Socratic dialogue I am in principle a proponent of (E7.1, Scene II). There are some opportunities lost in there but yes, I think overall the exchange, catching the student in the moment of grappling with the understanding of something, worked.

CHAPTER 7
UNDERGRADUATE MATHEMATICS PEDAGOGY

The debate over various approaches to the teaching of mathematics at university level has been gaining momentum in recent years (see Chapter 1, Part I). Before launching into a consideration of the pedagogical practices M states that he employs or that he wishes to employ in his discussion with RME, I would like to refer to two strong influences on this work: John Mason's (2002) *Mathematics Teaching Practice*, which has impressed me[1] with its lucidity, sophistication and succinctness (all crucial when addressing practitioners of mathematics teaching at this level, as M stresses repeatedly in Chapter 8) and the 2001 volume edited by Derek Holton and based on the 1998 ICMI Study *The Teaching and Learning of Mathematics at University Level*, that not only filled a glaring gap in the literature but is also very impressive with its international and thematic scope. Across this chapter I refer to many of the recommendations in both.

Particularly the former's views are very akin to the spirit in which undergraduate mathematics pedagogy is discussed here. For example, Mason kicks off with a chapter on classic student difficulties bringing home the point that a consideration of pedagogical practice is meaningless if it doesn't follow from a thorough and sensitive understanding of students' needs. In exactly this spirit M and RME's discussion of pedagogical practice in the Episodes that follow comes on the tail of the discussion of learning issues in Chapters 3 – 6.

In this discussion M's pedagogical role emerges as involving at least three major responsibilities: encouraging an interactive/participatory approach to student learning[2] (E7.1, SE7.5); introducing and contextualizing important new ideas/ways of thinking/concepts (E7.2), for example in ways that facilitate students' concept image construction (E7.3, SE7.3); and fostering a fluent interplay between the formalism/abstraction/rigour of university mathematics and the concrete/informal/intuitive/example-based ways of thinking the students seem to be better accustomed to when they arrive at university[3] (E7.4, SE7.1, SE7.2, SE7.4).

Finally, on a slightly paradoxical end-note to the otherwise very pro-active tone that runs throughout the discussion in this chapter, M and RME discuss whether, despite whatever pedagogical choices one makes, ... 'learning happens anyway' (OT7.1)...!

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[1] For a summary and review of the book see (Nardi, 2002).

[2] Approaches include: interactive interrogation of student thinking and expression; 'Socratic dialogue'; encouraging personal responsibility for learning; benefits of one-to-one contact; effective conditioning of interactive sessions; and, coping with student resistance to interactive approaches.

[3] Elements of mathematical thinking and expression covered include: abstraction, formalism, theorem use, numerical experiments, visualization, trial-and-error , appropriate / 'clever' choices.

EPISODE 7.1: INTERACTION / PARTICIPATION[4]

The type of pedagogical practice M and RME discuss in this Episode – all setting out from the principle that students need to actively engage with their learning of mathematics – resembles in some ways what Mason (2002) labels *tactics* (a word he is choosing for its evoking of *tacking* in sailing, which is decision-making when *specific* conditions are considered). He describes teacher-student interaction in terms of six modes: *expounding; explaining; exploring; examining; exercising; expressing*. In what follows M refers to several of these modes.

Scene I: Enhancing students' mathematical expression through interactive interrogation of their thinking [5] *(in the context of the discussion in E3.3, Scene I)*

M: Sometimes I ask them to say why they think their answer is true and the few who attempt an explanation struggle. What you really need is to do this exploration in an interaction with them and it is frustrating that we get to do so little of that talking over solutions with students. I do it occasionally to confirm that each one of them understands what they have put on paper, that they haven't simply been copying from each other, but this is slightly different purpose to the one I am talking about here and these chats I have with them are often an one-off.

Scene II: Building students' understanding through 'Socratic dialogue' (in the context of the discussion in E6.3, Scene I)

RME: Building these bridges between the formal and the informal is the hardest thing for you, right?

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[4] For a discussion of some of the data in this Episode see also (Iannone & Nardi, 2005a).

[5] Selden & Selden (2003) report how an activity in which students read and reflect on student-generated arguments purported to be proofs of a single theorem can provide an opportunity for discussing ways in which proving strategies can improve. In another report of the value of exposing students to weaknesses in their thinking Tsamir & Tirosh, (1999) – albeit in an upper secondary context – exposed students to different representations of the same mathematical idea (infinite sets) in order to raise their awareness about inconsistencies in their reasoning. Here the inconsistency was about two infinite sets having different number of elements, e.g. N and {4x, x in N}, mostly because the latter is a subset of the former. The exposure to cognitive conflict worked for a substantial number of students who felt they had to clarify the criteria for their judgement (one to one correspondence? Part / Whole?); who declared different criteria apply to infinite sets; who declared there is only one kind of infinity, therefore the sets must have the same cardinality. All these declarations could be used to resolve inconsistency.

M: I believe these bridges are achieved in a Socratic dialogue[6] kind of teaching situation. You address the issue with the students as follows: so, you tell them, you claim it is obvious that this converges: convince me. I bet you cannot make it within ten to the minus six. Show me! What about within ten to the minus twelve? And so on. And what effectively emerges from this exchange is the definition. OK, it would probably take you half an hour but it is efficient use of the time because ultimately they are becoming fluent with this type of argument by using the formal definition in the context of specific numbers, specific epsilons. By challenging them with different epsilons again and again, I am representing what the definition is trying to say, *for whatever epsilon, you need to find me an N that…* I think it is quite an effective way of assigning to epsilon the status that the universal quantifier implies. *Any* epsilon, none of this *very small positive number* nonsense. Yes, of course we end up bothering only with very small positive values of epsilon because this is where the point we are trying to prove is actually becoming controversial. But in fact epsilon is really *any positive number.*

Scene III: Facilitating students' realisation of their responsibility towards their own learning[7] (in the context of the discussion in E3.4 and E7.4, Scene V)

M explains a tactic he uses in order to trigger and maintain students' engagement during lectures: calling out students' names and asking them to provide a definition in the midst of a lecture.

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[6] An initiative relevant to what Mason (2002) describes as the benefits of a conjecturing atmosphere is *Scientific Debates* (Legrand, 2001a). An antidote to the passivity of a traditional, transmissive style of undergraduate mathematics teaching, Legrand describes the didactical contract of a scientific debate as follows: students assume epistemological responsibility for the claims they make; statements by the lecturer are not to be assumed as valid; the lecturer only at the end offers a consolidated version of the findings produced during the session; unlike polemic debates participants change their minds on the way but they have to justify this change. The method aims at emulating the mathematical community where, every time we make a claim, we engage with demonstrating the truth or falsity of our claim. In this sense scientific debate offers opportunities for learning to distinguish between two kinds of rationality: that of everyday life (where one maybe penalized for being wrong) and that of science (where progress, *a la* Lakatos, is often achieved through modification of false statements - socio-constructivist principle). Also it offers an opportunity to understand the discipline of mathematics at a deeper level (socio-ethicist principle). Overall it is a good vehicle for practising communication skills. Legrand stresses that it is imperative that the teacher is a non authoritarian, co-coordinating, equity-guaranteeing presence. He offers two types of debate: the unplanned debate (e.g. triggered by a question in a class); and, a planned situation intended to introduce a new concept or overcome an epistemological obstacle (he offers as examples the Riemann-Lebesgue Integral transition, the relationship between Linear Algebra and Analysis and the deepening of a concept or a theory).

[7] Yusof & Tall (1999) report the positive effect on students' attitudes (and perceptions of university mathematics) of a course encouraging co-operative problem-solving and reflection on a series of mathematical thinking activities.

M: My colleagues are a bit scandalised by such things – they probably think this is too intimidating and too radical – and I am doing this less and less these days. But I can definitely recall some positive results! Attendance was improved too as students were afraid of being called out and not being there. Giving a wrong answer but being there was seen as better than just not being there!

RME: You were not concerned that the tactic could be seen as somewhat threatening?

M: They don't seem to see it this way and I will still do that when I can. It is one way that brings you face to face with the fact that ultimately learning mathematics involves a little bit of this kind of learning, that it is your responsibility to learn some of these things and a completely casual approach is not appropriate. In a sense we need to be willing to stretch those who need to be stretched as well as those who want to. The substantially interesting parts of mathematics cannot be done while watching television – an investment of your own personal concentration is essential. Only through doing it yourself, through participating in the process you can appreciate how much imagination an act of clever choice in mathematics involves. I seriously believe that teaching the choice of a convergence test ought to be a highly participatory experience[8].

Scene IV: Benefiting from the rich environment of a one-to-one tutorial

M and RME discuss the following excerpt from (Nardi 1996)[9]. The discussion follows the one in E5.3.

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[8] See, for example, Millett's (2001) suggestions on how to improve lectures, this much-maligned but probably there-to-stay form of teaching university mathematics. Mason (2002, Chapter 2) distinguishes between: lectures that resemble or are drafts of textbooks; lectures based on students' queries on material distributed for study at the end of the previous lecture; lectures that present new mathematical topics as emergent from the need to resolve a problem. He then recommends that, whatever the format, introduction to any topic must vary between going from the specific to the general and vice versa in order to increase students' adaptability and flexibility and enact their participation in their own learning. Justified versatility is a theme running across his chapter. In employing screens (blackboards, overheard projectors, epidiascopes, fixed video cameras, computer screens and smart boards) and in introducing diagrams and symbols the emphasis needs to be on the ways in which these resources allow student thinking to focus and mathematical understanding to emerge. E.g. is a diagram transparent in terms of highlighting the various dependencies amongst its elements? Is it generic? Have you engaged with decoding the meaning assigned to a string of symbols on a board? Have you called upon your students' use of their *mental* screens before resorting to a physical one? Amongst the numerous tactics Mason proposes, most relating to enacting student participation, issues addressed include: *Punctuation; Being interested and stimulating; Keeping fresh; Handouts; 'Mixed ability'; Providing additional support.*

[9] What follows is a close examination of the one-to-one tutorial in the context of its almost unique use at Oxford. Even though the institutional origin of the data is inevitably obvious (the doctorate (Nardi, 1996) this data originates in acknowledges the university where the study was conducted) all colleges and individuals involved have been anonymised.

In the following extract the discussion starts with student Camille's question about R^R : 'Is it the same as $\Re^2$?' (C1). The tutor defines R^R, 2^R and R^2 (starting from the definition of A^B as the set of functions from the set B to the set A). Camille asks whether these are 'transposes'. These are the mappings from one set to the other, she replies, and are vector spaces over addition of functions and over scalar multiplication, both point-wise. She then asks what is the zero in R^R but Camille is still trying to understand what R^R consists of:

C2: It's a mapping from R to R... and each element of $\Re$ has a correspondent... with the mapping... the graph... it's a mapping from $\Re$ to $\Re$ so it's...
T1: No, no, no. Each mapping is a subset of R x R. It's not...
C3: And U is a subset...
T2: Yes. No. No, you are not looking at the individual f. Yes, it's true that f is a subset of R^2. That's true but I'm not looking at the individual f. I'm looking at the set of all fs. [*The tutor returns to the question about the zero element of R^R.*] What's your mapping from R to R with that property? It's not in there. U is not a subset of R^2.
C4: Do you have an example of a function f that isn't in R^2?
T3: Yeah, I mean any of them... if you like you could have... *cos2πx*... this is a function that... I mean to say that something belongs to R^2... what you are saying is that f belongs to R^2. I mean that f belongs to R^2. But it doesn't because that... to say that f belongs to R^2 is to say that f is an ordered pair. It's a set of ordered pairs.
C5: $(x, f(x))$... but isn't f an ordered pair?
T4: No, f doesn't belong to R^2. f is a subset of R^2. And it belongs to the set R^R. It belongs to a set of functions.
C6: It's very hard to imagine that... a set is usually a set of elements or matrices...
T5: Ah,... yes, that makes it harder. But I mean you could do it with equations.

The tutor stresses that all the things they are talking about here are casual things in Analysis and that their trouble is with the vector space context.

C7: So R is a subset of R^R...
T6: The elements are...
C8: How about $R^2 \rightarrow R$... is it a subset of this?

Subsequently the tutor repeats the definition of R^R and draws the parallel with A^B, the set of functions between sets A and B, where A has k elements and B has m. She stresses that R^R contains the functions they have been dealing with usually in Analysis and that it is the vector space context used in the example that makes things look more complicated. She then repeats the question about the zero vector in R^R: it has to be a function in R^R that satisfies some kind of property. Then Camille asks:

C9: Is the zero vector a function?
T7: Zero vector is a function because all of them are functions here... all the elements...
C10: They are not vectors? *[The tutor repeats the definition of the zero element of a vector space and explains that in this case this element is a function that has the properties of the zero element.]*
C11: So we are not looking for the zero vector anymore but for the zero function.

The tutor accepts that $z(x)=0$ is the zero element of R^R, Camille explains that $z \in U$ because $z(0)=0=z(1)$ and checks out closure in U. The proof that $U < R^R$ is completed.

RME: Shall we have a look at this dialogue between the very vocal Camille and the tutor? I think it is rather lovely that some excerpts of conversation with students encapsulate some of the things we are saying in such a vivid way!

M: Indeed! This is an amazing piece! I love the *so R is a subset of* R^R*? How about* R^2 *?* and the *should be an element of R cross R* bit. The tutor seems to be doing exactly what one should not do!

M queries about Camille's background (educated outside the UK) and work after her graduation (the City). 'That much comes across' comments M referring to the confidence which she exudes in the excerpt[10].

RME: As a student she had a quite confident approach: in moments where everybody else would remain silent she would just stand up for everybody else and say: look, give us some more here, explain to me!

M: T1 is a real moment of madness. And it gets worse later. The tutor gets really mixed up. *Yes, no…* sums it up pretty much, really. *Yes, no, no…*! She seems to be getting Camille off the notion of vectors following *it is true that f is in a subset of* R^2. This is fast and beautiful! I think *f is a subset of* R^2 suggests a graph. But what does she mean with *is*? Can I step back into a query about the Oxford tutorial system: if such a use of symbolism has created such an ubiquitous difficulty, would the lecturer be aware of such a fact? I guess s/he would be one of the tutors, right? So, with that much experience of the students' reaction, isn't it insane to stick to this kind of symbolism?

RME: There were about 30-40 tutors across the colleges at the time of the study and she is one of them but, yes: she has a significant amount of contact with students to allow her to see this kind of difficulty.

M: I am amazed she would think this symbolism is good for the students then. But then again maybe she is not thinking at all about this.

RME: Still towards the end Camille seems to have come up with a distinction between a zero vector and the zero function. *So we are not looking for the zero vector anymore but for the zero function*, she says. Yet we have little evidence of

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[10] Camille's query about R^R and her exchanges with the tutor triggers a multi-faceted discussion between M and RME. One facet of the discussion is the influence of Camille's personality on the course her discussion with the tutor takes. At first sight it may appear odd to present the reader with personal details on the student. However the discussion of her personality brings about comments by M on the type of student who is more willing to participate and benefit from the opportunity for interaction with an experienced mathematician that a tutorial offers. These details thus acquire some significance and this is why I have kept them in the dialogue.

whether she has grasped the notion of the latter being the former in a particular context.

M: She is fantastic. But I think the tutor has not been keeping up with Camille's thought process. Camille got away on her own trail and on her own afford. And the whole thing wouldn't have happened in the first place if they had called it … D instead of R^R . Or the tutor had said no to that question! Anyway, are all Oxford tutorials one to one?

RME: Most were at the time of the study but this one was two to one. However the other student in this case remained silent through most observed tutorials, almost totally non-participant, perhaps due to the vocal and outwardly personality of Camille!

M: Camille is the type that is easier to teach because of this outward-ness. I would love to see Year 2 students' reactions to such a piece of dialogue! I guess they would keep their cool, face the symbolism, go confidently down this blind alley and dissolve the confusion with little trouble. Or so I wish! Tell me a bit about the questions in these tutorials. Did the tutor come up with them or did the students have time to work on them in advance?

RME: These are questions the students had time to think about and they become the focus of the tutorial because they students stated they had problems with them.

M: Students do ask at Oxford[11], don't they? And they do use libraries. A big difference is also that the work is not assessed[12]. The sooner you start assessing the work, the sooner what you get are absolutely perfect answers to a few questions and then they are not bothered about the rest. Even though using books to sort out your queries could be problematic: just the sheer intimidation by the different notation used in different books is a big issue. If you go to the lecturer to ask, then you get an answer also in the language and notation you are expected to produce it yourself. But then there is a skill in selecting a book that is appropriate to your level: make sure it doesn't simply say *Algebra* on the title – there are some books you couldn't go beyond the first page – but also *Linear*! And maybe *Introduction* too! And *Not from Bourbaki*! Or *Anyone French*[13]! I wish my students felt more inclined to use books. Every college having a library helps at Oxford of course. Polynomials by the way is a good example to look at

[11] M here alludes to the fact that Oxford and Cambridge attract the highest-performing students.
[12] Unlike the students' written work M and RME have been discussing in Chapters 3 – 6 where the mark counted towards the students' overall result for the course.
[13] At face value this may appear is a frivolous comment by M. However it does reflect some of the intra-European tensions (Anglo-saxon and Continental) on issues of syllabus and presentation of choice (for example in Analysis, see E7.4, Scene IIb).

in various books because it is explored to an extent impossible to achieve in a tutorial. Of course the fact that the students get there by the end is evidence of the efficiency of tutorials [14].

Scene V: Students' resistance to participatory teaching[15] (in the context of the discussion in E7.4, Scene I and SE7.4)

M: And of course participation of the students cannot be taken for granted either: I have had a student walking out when asked to consider presenting on a blackboard. Neither is the university's support for any addressing of these issues. The students are conditioned to non-participation and to heavy examining. Oral examination, student presentations, the potential of one-to-one interaction – I need to have ten minutes to try to understand what they haven't understood! – student-led projects are things we need to think about more thoroughly[16]. Just some good-looking presentation, some colourful overheads don't do the trick. Students need to own the material and, however spectacular a presentation they sit through passively, won't help them acquire this sense of ownership. It is very easy to see fantastically presented things, what is hard is to ask some really simple question like write on the board for me a non-Abelian group and its conjugacy classes. And some could do it, some not and those could learn something from those who could. And that somehow would increase the confidence of the group.

Scene VI: Conditioning interaction effectively (in the context of the discussion in E4.1, Scene II)

M: The weakness of the lecturer's response makes me think that it is very difficult to interpret the students' work in isolation from the lectures the students have attended and that we ourselves may be encouraging bad habits. I guess when students interact with a lecturer in a one-to-one tutorial on a regular basis, there is the very important difference that they somehow have to be able to say or

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[14] Tutoring is the teacher's opportunity to enter the student's world – as opposed to lecturing during which the student most often enters the lecturer's world (Mason, 2002). Amongst the tactics he proposes for enhancing the student's learning experience in tutorials are: *Conjecturing Atmosphere; Scientific Debate; Asking Students Questions; Getting Students to Ask Questions; Worked Examples; Assent – Assert; Collaboration Between Students;* and, *Advising Students How to Study.* These aim to help students with transgressing a common perception of tutorials as the place where the tutor demonstrates correct solutions and towards a perception of tutorials where they are actively engaged with learning. *Tactic: Anti-funnelling*, aimed to tackle the overall rather unsuccessful tendency of tutors to be 'drawn into a sequence of ever-increasingly specific and explicit questions, searching for something that the student can actually answer' (p75) is a good example of this. The valuable role by a sensitive and challenging tutor has been highlighted (Carlson, 1999) by successful graduates as a contributing factor to their success.

[15] See (Alcock & Simpson, 2001) for some, initial, student resistance to their innovative Warwick Analysis Project.

[16] Several authors in (Holton, 2001) report on the use of those: e.g. Haines & Houston, Niss and Smith.

write something clearly on a regular basis. Submitting some homework of the type we are looking at here is another example of this regular obligation. You can catch this while it is happening and amend it accordingly by providing an appropriate comment. Of course catching them expressing themselves this way implies you have allowed them to say something and not reduced the session into another one of your lectures. Catching them in the act implies allowing an act to unfold! And when the issue is still hot, not weeks, not months later, and not in the exams! There is often a communication gap regarding where the students are and where we as lecturers expect them to be. I realise the problem sheets are there to play this part and I am disappointed when I see the treatment of the students' submissions as mark aggregation exercises. The fewer marks one gets the more we, the teachers, have identified problems we need to deal with, and paradoxically, we reward the person who has given us more of this evidence with the fewer marks! As I always say a Socratic dialogue is the only way one can show to students that there are more than one ways of dealing with a response to a question and what are the pros and cons of each way.

RME: When you see something as shocking – forgive the excessive language – as Student L's response, does this inform at all what you are going to be doing next? Not necessarily in terms of content but in terms of presentation, in terms of thinking *I have actually a very clear vision of where I want the students to be by a certain time, they seem to be behind and I need to be doing something about that*.

M: It depends on how frequent are the appearances of these shocking responses such as Student L's. But to me Student L is fantastic. Give me ten minutes alone with Student L and she would have really learned something. She would walk away from this in far better shape than Student H because student H would probably not get picked up. But Student L will end up saying: of course! You know, the opposite of *all people in Norwich are tall* is that *there is at least one short person in Norwich*. That is assuming that you can have this student talk to you. Of course it's difficult because with someone like Student LW whose writing hardly makes any sense, I wouldn't be sure where to start. A certain order could be that we write down the statement, we clarify what it means and what the opposite of it means and then try to write it down symbolically. And the student should get rewarded for engaging with this dialogue, no matter what the results. So that they overcome the constraints that hurdle this process. Oh, I can already here auditors asking for a mark out of a hundred of the student's engagement[17]. Argh…

[17] M here refers to the escalating climate of an accountability-through-monitoring audit culture in the UK (Charlton, 2002).

RME: So you prefer someone admitting openly to not knowing how to use the symbols precisely as opposed to someone just using the symbols and desperately trying to appear as if they know what they are doing[18]?

M: I admire and look forward to work being handed to me, at any level, with questions on it. If you can show some sort of attitude and distil your thinking down to a question for me, we are half way there. I can then teach efficiently.

RME: So the point is to negotiate common understandings of the language then?

M: Yes, and engaging with convincing others about your views. Presenting a convincing account of why you think this series converges to this number or something[19]. Furthermore I have found that responding to collective errors[20] is a tactic that is highly appreciated by the students: instead of writing out model solutions to a question, I collect their responses, collate their errors, problems etc. and tailor my suggested solution on the issues that emerge from their responses. So, instead of presenting them with a solution that for some hardly defined reason is seen as a model one, I present them with one that seems to accommodate the problems they had to tackle on their initial attempts at the question. Plus let us not forget that a collective error is less painful at a personal level. And because you are not addressing the students personally anymore you can say some pretty harsh things about this error without hurting anyone's feelings.

The discussion returns to the type of student who is willing to engage in what M calls 'a little to and fro on a piece of paper or the blackboard' until the student 'comes up with something better' than what they submitted in their written responses. Often students are not so keen on this to-ing and fro-ing, are they? asks RME.

M: But then there are also always the ones who would use the time in an interactive session to ask questions. They might not get involved in the dialogue but at least they would recognise that they are stuck and come up with some questions and maybe even a reasonable subset of those you could then have a dialogue with. I think that that is still better than an awkward tutorial situation in which one is addressing the entire group and doing things on the board. You may occasionally achieve little or no dialogue but in my experience almost invariably some interaction always happens. Also, when I see students having large problems, I am just wondering: my feeling is that they are like a toddler who manages

[18] So, since students may not 'get picked up', they have to put themselves forward for help. See also Chapter 4 for the discussion of their attempts to 'appear mathematical'.
[19] See extensive discussion of this issue in Chapters 3 and 4.
[20] See Tactic *Using Common Errors* (p10) in (Mason, 2002).

somehow to get from this side of the room to that side and he falls over five times and the parent stands there and observes the toddler making progress. And a year later he can walk. And this is a little bit how I think about these problems: they are falling over all the time but somehow it sorts itself out as time goes on. And I just wonder: is there research evidence for this process, of falling over to maybe make the next five steps correctly and eventually walking confidently, essentially without proper instruction. It seems to me that if you take the same student now, who is now saying the things we discussed in Year and in Year 2 you give him this bicycle problem, he may not be able to do this problem yet but by the time he is graduating he will do so competently! So, certain things just get known! Their work at this early stage when they have had minimal experience is, yes, full of problems but they do get over those problems (OT7.1). The dialogue we have been talking about and the responding to their errors and own needs are ways to accelerate this process. So instead of focusing the coursework on the marking, you can set the scene as a place where one comes to practise and model solutions are discussed. You thus take off the pressure completely and then ask the students to work on similar problem sheets where you do the inevitable marking etc. There is a caveat here though: that once marking is part of the game, they are almost always keener on engaging with both content and presentation of their scripts. They are still trying to impress a marker after all. So you have to cultivate a certain climate to prevent them from relaxing a bit too much[21].

EPISODE 7.2: INTRODUCING, CONTEXTUALISING THE IMPORTANCE OF NEW IDEAS[22]

Setting the scene: The following takes place is in the context of the discussion in E3.5, Scene I

M: May I return to a previous comment about the importance of the fact that √2 is irrational?

[21] Mason (2002) describes some of this conditioning in terms of a Vygotskian ‘scaffold and fade’ model. Moschkovich (2004), for example, used a sociocultural perspective (the concept of appropriation) to describe how a student learned to work with linear functions in close and highly influential interaction with a tutor during which the learner, through active participation, appropriated goals, actions, perspectives, and meanings that are part of mathematical practices, and how the learner was active in transforming several of the goals that she appropriated. The particular context of the case study was *mathematical practices that are crucial for working with functions* (learning to treat lines as objects and to connect a line to its corresponding equation in the form $y = mx + b$). The tasks used by the tutor (estimating y-intercepts, evaluating slopes, and exploring parameters) reflected these mathematical practices.

[22] Mason (2002) offers a perspective on motivation of this kind with a particular focus on: sensitive responses to ‘why are we doing this?’; locating and exploiting the surprises and intrigue that are inherent in mathematics; and, embroidering exercises that aim to develop technical facility without reducing learning to a mechanistic procedure).

RME: Please do.

M: You said it has raised the interest of mathematics educators. But is it also raising the interest of the students? I wonder how students feel once they have seen this proof: just the feeling that *I have just seen a proof working*? Or something more, along the lines of *I have just seen a really important bit of the universe*? I wonder…

RME: I recall my amazement about discovering the existence of irrational numbers, of so many more numbers I thought thus far were around when I first saw the proof.

M: I wonder how typical you were though: many of our students may walk away simply having seen this fact proved and extrapolating nothing from this. Or having a gut feeling that it is essential enough to appear as an exam question. One sort of motivation can be that you can't prove the Intermediate Value Theorem without an axiom that separates the rationals from the reals. And the simplest way to get that across is to know that $\sqrt{2}$ is irrational or something like it. But surely students do not think about this deeper reason when they have seen this proof! And this is a truly deep reason: in Analysis it is absolutely crucial to see the distinction between real and rational numbers and the students have the opportunity to see this in the IVT, in the convergence of increasing, bounded sequences and elsewhere. Seeing that $\sqrt{2}$ is irrational is a good and relatively easy start – unless you implicate Set Theory and Cantor's proof! Which, in the context we are talking about, can have no bigger ambition other than being a joke…

RME: But students could just simply see this as an isolated fact: an irrational number exists. This does not imply that others exist too. I know it is a sort pessimistic view.

M: Sure. That's why I insist we have to stress the meaning of this fact. And one way of doing so is in the context, for example, of *the probability of picking a random real number that is rational is equal to zero*. This is a fact that the students learn at some stage but it usually takes a long time to actually be understood. Out of context, this fact is almost meaningless. But in the context of exploring the various kinds of numbers that are around, even at A level, this type of question could start them off wondering about these things[23].

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[23] In the context of Linear Algebra one example of this is Uhlig's (2002) method of introducing new ideas not through the rigorous but alienating definition - lemma - proof - theorem - proof – corollary (DLPTPC) but from a gentler, more intuitive starting point. Uhlig's proposal has triggered several responses: Burn (2003) stressed that studying R^2 and R^3 is a sound basis for generalising properties of vector spaces and for highlighting the links between Algebra and Geometry; in their response however Dorier et al (2002) expressed their preference for a more stand-back, abstraction-friendly approach (see SE7.1).

EPISODE 7.3: CONCEPT IMAGE CONSTRUCTION[24]

Setting the scene: In several episodes M and RME discuss students' concept images and how these relate to the students' handling of definitions, for example in the context of functions (E5.1). The following takes place in the context of the discussion in E3.3, Scene II.

M: Think of the concept of determinant here, for example. How does one relate to it[25]? I wonder. Is it just a bit of garbage coming your way when you have to apply certain rules or do you have a mental image of what it might mean? Do you think of the determinant as a volume or as something to be worked out, or something else? For example, to me, the adjoint is a cute way of getting the inverse of a matrix. So that could be my way of relating the concept of the adjoint to what I already know and use: when I see adjoint I think of inverse and I work from there. Because I happen to be one of those people who cannot handle very complicated things so I need to have simple justifications at hand for new things such as the adjoint. Probably these simplifications take away some of the true power of the concepts but, to start with, they are good enough for me. So I am looking at the student's work and thinking: what were the pictures in her mind when she put this or that down? In a sense I am trying to understand and appreciate the student's landscape. Everyone has their own personal landscape. For example, regarding determinants, I wouldn't be surprised that, like an integral, a determinant is seen by most students as a number to be worked out. And then I am contemplating whether this is what I would like to instill in them about determinants or whether I can give them a more structural *raison-d'-être* for the concept.

RME: Do you think that there is some sharing of landscapes between you and the students? You have been working with these things for a long time and your landscape is probably richer than theirs. How do you think this imparting of landscapes gets on?

M: For example, when I am introducing the Intermediate Value Theorem, or when I am showing that a 3x3 matrix as a transformation of a three space has an invariant axis. This is somehow a tremendously important fact. And it is very clear in talking to them in the lecture when I do this, that they are surprised, they

[24] In tune with Vinner & Tall's (1981) *Concept Image – Concept Definition* framework. For a preliminary discussion of the data here see also (Nardi & Iannone, 2003a). This is the paper that M scrutinises in the meta-discussions in Chapter 8. See also the Meta-Scene appended to this Episode.

[25] As Leone Burton amply demonstrated (2004) reflection upon M's own perspectives on fundamental concepts of mathematics (even as 'simple' to and casually used by him as $det(A)$; actually, *particularly* these concepts) can provide illuminating insights into the ideas that form his 'mathematical', and by implication his 'pedagogic' (Mason, 1998), being.

find it difficult to imagine that a pure mathematics lecturer can think of a 3x3 matrix as a transformation of a three space. For them that is incongruous!

RME: Maybe it is less surprising if you think they see so few links amongst the various topics in the courses.

M: I agree that this is probably behind their reaction. But I am always surprised by their funny characterisations of things as sophisticated or not: they seem to think of IVT and determinants as sophisticated. But transformations – without the word *linear* attached – not! They think of transformations, they may not call them such, as quite elementary things. I would love to know whether or when the notion of a determinant as a volume kicks in. And they see determinants in relation to cross products later – that's another connection to make. The question we are looking at is in fact a good example of an opportunity to make that link between determinants as something which just saves you writing down a large number of elements. For instance, you can write down the adjoint of A in terms of $det(A)$. I see this business of sharing landscapes as my main business as a lecturer. We are not just communicating facts, we are saying that this is one way you can view it and that is another way you can view it, let's put these together somehow. And it's not an easy job, believe me! Say you are thinking of determinants as volumes, what the heck is a 4x4 determinant then? And, of course, mathematically you want them to think of an *nxn* determinant. Plus the fact that making the connection is a personal issue because you need to establish links between your pictures, not somebody else's pictures. And I am very, very weary of plots about this forced, so to speak, networking of all of mathematics where everything relates to everything.

RME: That assumes there's just one network!

M: Exactly. You need to have your own tailor-made brain version of what the thing is, ordered according to your view, in your own time, without anyone's landscape imposed on you… but of course with some guidance. I love these moments in later years – and it completely eludes me how they come to happen (OT7.1)! – when a student makes a connection between two seemingly unrelated things from two different courses. To me it is a sign they are achieving mathematical maturity.

RME: What concept images of a determinant would you expect students to have at the end of their degree.

M: Volume, images from physics, integration etc: any image would be welcome as long as the students don't say this is a thing that I can compute. If that happens then I would say that I failed. If they say it is volume, or it helps to solve systems of linear equations… I would sort of be surprised but yes, this is also good. I recognize that the emphasis on calculational tasks is sometimes a necessary evil of devising exam questions and making the exam passable for students but I also happen to believe it is a mistaken attitude. Rather than algorithmic ones, maybe we should be setting questions that foster understanding where students need to … showcase their concept image, rather than algorithmic understanding. I can recall incidents where students stared in bafflement about going from one idea to another and were very uncomfortable suddenly going from numbers to vectors. This crucial navigating between these things is often very difficult – and it shouldn't – but maybe achieved by Year 3.

A 'meta-scene' in the middle of a scene. M and RME discuss concept image construction in the context of the discussion in E8.2, Scene IV[26]*.*

M: I am fascinated by this excerpt in the paper, by how mathematical it is and how un-pedagogical it is. Which is not as surprising… we think like mathematicians because I suppose… we are mathematicians! After all the excerpt is an example of us talking about shared or less shared notions on a mathematical concept – there seems to be surprisingly little common ground amongst us about the notion of a matrix I would expect an algebraist first situate a matrix in a group. I wonder actually whether enough of this is being done between teachers and students. Sharing an exactly common ground is not in fact the issue at all, indeed it is interesting that we differ but at least we are talking about something that is coming from a common experience that we have. Hunting for some common ground with the students would be great but we don't. To elaborate a bit upon the algebraist's arguably preferred perception of a matrix – and I am aware that our conversation is shifting from methodological to contextual aspects of the paper – the idea that a pure mathematician thinks of a matrix as a thing that relates to physics I would find surprising. But anyway while most of us would agree that the teaching needs to address the multiple contexts within which a concept can be encountered while maintaining the notion that it is actually one and the same thing, we would find it hard to agree about whether the teaching actually achieves that. There could be better linking between matrices and physics in the first year for example. Vectors can be placed in the context of physics, economics etc. this would be an enriching experience. For example in Analysis one can encourage students to think of a 3x3 matrix as representing a transformation of a three dimensional space. In Algebra we do this through physical examples to illustrate rotations etc. knowing of course that these examples do not necessarily apply to 2 or 4 dimensions. But then of course maybe the connection is lost.

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[26] I label this part of the text as 'meta-scene' for the following reason: M offers these comments on concept images in the context of the discussion in *E8.2, Scene IV*. That discussion is initiated by his reading of a conference paper written by RME and focusing on M's comments on concept image construction discussed in the scene here in E7.3. I honestly hope this makes sense…

Shall we maybe talk a bit about which stages of their studies the students see various applications of matrices?

RME: Sure, go ahead.

M: Within the context of multiple integrals and Jacobeans, a determinant is seen as a distortion of volume. Which by the way that is what they are. I don't know if this is the way they are always talked about but that is what they are. Stress vectors come much later. Within computer science a student might first of all be interested in a determinant just as an array of numbers. And then maybe go through the whole degree without dealing with them as anything else. And we should never forget the difficulty of learning to deal with all this multiplicity of contexts or that the things we want to connect are actually not known to the students. You see, it would be nice at one stage to say, ok, this is what matrices do in geometry and this is what they do in economics and so on and now we can potentially do it in a unified way, but we don't know each one of the topics we want to connect. And I don't mean to underestimate the knowledge with which the students arrive from GCSE but some have never seen these things before at all; or seen them at GCSE and then not at A level so by the time they are here they haven't seen them for a long time. So do not be surprised if you ask them about the rotation of R cubed, what is that in terms of a matrix, and what you receive is a very blank look. (M turns the discussion towards the influence of school mathematics on the students. This part of the discussion has been absorbed in SE3.1)

EPISODE 7.4
ABSTRACTION AND RIGOUR
VERSUS CONCRETISATION, INTUITION AND EXEMPLIFICATION[27]

Scene I: Abstraction

Setting the scene: Struggling for a Meaningful Interpretation of the Definitions of Centraliser and Conjugacy Class (excerpt from (Nardi, 1996)).

> 'I don't really understand what they are' (C1) says Connie of the centraliser of an element *x* in a group. The tutor then defines it as *the set C(x) of elements that commute with x:* $C(x) = \{y \in G / xy=yx\}$. Connie's reaction to this definition is:

[27] For a discussion of this theme based on some of these data see also (Iannone & Nardi, 2005a). With regard to this theme Mason (2002) describes the complexity of steering a balanced course in terms of three tensions:

Agenda and expectations: negotiating expectations; varying pace of coverage; alternating between serialist and holist views of a topic; varying the degree of challenge*;*

Doing, Construing and Wanting: steering a course between knowing and understanding while nurturing all three key perspectives on learning: affective, cognitive and behavioural)*;* and,

Being Subtle and Being Explicit: considering the paradoxical tension between the two.

C2: You can also swap back to y over there, can't you?
T1: But this is a different thing though: you can say that the centraliser of y, if y is an element of G is the set of x in G such that $xy=yx$, but is that what you mean? *[Connie nods]* It's what I'm saying here I just changed the dummies so to speak.
C3: Can you give me an example so that I can understand?

The tutor offers an example from permutations. Subsequently the discussion turns to C_g, the set of conjugates of g:

C4: We were introduced to... We've defined conjugate as x times g times x^{-1}... I mean how do you choose g? g is an element in the group...?
T2. [The tutor defines the set C_g of conjugates of g. In an Abelian group an element g is only conjugate to itself.]
C5: So it's for every g belonging to G?
T3: For all g belonging to G you run through them all and you get what you get as a set here.
C6: x has several conjugates then?

The tutor nods and says that if G is Abelian then $C_x=\{x\}$. Connie is confused:

C7: What does the conjugate actually mean? Because the inverse is actually going to send it back to itself... what are you using it for... I mean... because I am always getting confused with inverses and conjugates.

M: I am terrified by the idea that a few years ago an appreciation of conjugacy classes was possible in Year 1 and now they need to be moved to Year 3 – or removed altogether. Because it is too difficult. Actually I cannot understand why it is seen as too difficult because some years ago nobody seemed to have any difficulties with conjugacy classes: the group acts on itself, there you are! And it was a good illustration of how you can use this idea in order to find out how the group looks like. Class Equation comes out of these things beautifully. Until two years ago I did the Class Equation and at the time the better students certainly thought that this was a high-class item in their education. And I would say that thirty percent of people were able to handle this thing. And now for strange reasons it is too difficult. I would say between five years. And I cannot pin down why on earth it is this way. Frightening. Because this is such an absolutely elementary idea, an idea you can apply so easily on finite groups – at least.

RME: Maybe a very instrumental introduction could help?

M: It is anything you want it to be, you know. It is a concrete idea, you can work out relevant examples, you can prove good theorems such as *groups of order p^2 are abelian* and so on. It is beautiful and now impossible to teach. It has to be the abstraction in it, not the difficulty as such. Or one level below: to the problems our students have with fundamental concepts such as the notion of set.

RME: You are probably right. We usually rush towards accusing the tutor in these examples of not being in touch of what the students' difficulties are but here the tutor's *for all g belonging to G you run through them all and you get what you get as a set here* is a clear, instrumental instruction to which the students have very confused responses.

M: Exactly. Add to this the suspicion of the students towards the possibility of having more than one conjugate to an element. I am concerned about the implications of using $C(x)$ to suggest the set of conjugates of x, because it recalls functions. Another notation that drives them away from the notion of a function that has one value, would be better, I think.

RME: $C(x)$ actually denotes the centraliser of an element.

M: Ooops! You are right. C_x is the one for conjugacy classes. Anyway it may just be the inability of understanding sets.

RME: In what sense an ordinary understanding of set as a collection of things would be inadequate here?

M: The commonality of a certain property amongst the elements of a set is crucial here. So a set usually appears as a subset of something larger. So would the centralizer or the conjugacy classes be here. And it is something more than knowledge we are talking about: we are talking about naturalness, a natural, hard wired understanding of a set as something that is absolutely fundamental.

RME: Tell me more about this *more than knowledge* idea.

M: Look at Connie's confusion with the nature of things involved here and the name of the thing running through this set. Names mean a lot to the students. And they are also confused when variables disappear somehow, e.g. y in the centraliser of x. You define the centraliser to have y index and when you compute the centraliser there is no y in it. That depends on x.

I am still impressed however by the profundity of some observations made by the students: if x is in the centraliser of y then y is in the centraliser of x. This is a good, non-trivial observation. You see that the student is making an effort at breaking down the instrumental rule which defines the thing.

RME: There may be a dangerous interplay there between x and y as respectively fixed and variable, but I can see why you are impressed.

M: And integration is another example where the role of variables is problematically perceived by the students. When you change the dummy variable. I pushed it to the limit and said *let's call it Fred*! To get away from these hated creatures, x and *y*! In our context here I am concerned about discussing conjugate classes and centralisers so close to each other as, in a sense. They are rather opposite notions. The notation also suggests proximity rather than opposition. And the tutor's bringing into the conversation of the exceptional case of Abelian groups is rather unfortunate: it seems to complicate matters.

RME: As a mathematician the tutor may have felt compelled to cover all cases and to counteract a possibly emerging perception by the student that it is always the case that conjugates are more than one.

M: Mentioning the identity element as an element that has only one conjugate would suffice. *The inverse of x is actually going to send it back to itself* is definite proof to me that it was a bad idea to introduce Abelian groups in the discussion. Jeepers! Isn't that awful…

RME: And almost invariably all these convoluted conversations end up with the student asking – what does this [concept] actually mean?

M: Yes, look at C7. *What does it actually mean?* Students can write down definitions without really understanding them. But they do wonder what the whole thing is about. They are in search for raisons d'être, examples, characteristic properties; they are no longer happy with introducing a concept for its own sake. I know we are teaching them an abstract subject – which is exactly what I like about it! – but we cannot afford introducing concepts in this way any longer. Appealing to the immediately useful aspects of a subject is what seems to be imposed by the ambient culture. I am not sure we can carry on saying *this is mathematics, if you don't like it then do something else.* There is also something else which may complicate our position even further: the trend towards undergraduate learning through reading research papers! This is possible in most subjects but virtually impossible in mathematics. However I should still be able to show various cool arithmetic relationships, for example in S_3, and then generalise from there. The real job in our hands: to actually communicate what our subject is about, something not required in a subject like advertising or management. We need to be prepared to insist that our subject must play by different rules. I can see there are aspects of the culture that make this task increasingly difficult: supporting engineers, the manufacturing industry, the cold war used to be good enough justifications. Nowadays the discourse is customer perception of the necessity and appeal of a product. Anyway, let me return to possible ways in which mathematical concepts are introduced to the students: my personal preference is through example, justification for its introduction and then formal definition.

RME: Perhaps more than one example is necessary so that the students begin to see what these things have in common which we can then name accordingly?

M: Yes, and I propose we take it even further: an unashamedly cool theorems approach in which the usefulness of concepts arises from their use in putting together arguments and consolidated in results published in accessible places like *Scientific American*. This would have a substantial impact on motivation. Of course you could have some doubt about whether a specific question like the one posed by the student in the example is catered for in this scheme. I am not saying that we can get over issues like time constraints and introduce the entire scientific story but in some sense it is nice to show it at some level.

RME: I wonder whether all courses could be rewritten with a clear aiming towards the proof of a fundamental theorem.

M: I don't think so – maybe Galois Theory is one of the few, good examples of where this is possible. I admit that we mathematicians keep retreating into application in order to justify the subject as fewer and fewer students find a discussion at the abstract level gratifying. And some of these applications are in fact so boring – decide how many coloured necklaces we have of three colours!

RME: Sorry – that wouldn't motivate me terribly… I think that part of the beauty of the whole thing is that we are looking at something beyond necklaces, you know.

M: It has become increasingly more difficult to convince students to see things such as the capacity to engage with abstract ideas as an important part of our history. Employers views of what constitutes usefulness is central here too, I think: not the content but the thinking that goes with it. But let me stress again the idea of rewriting courses that use the theorem-as-objective as a smokescreen for doing pretty good mathematics along the way. You know devise a course that aims at the solvability of the quintic. Or at the Fundamental Theorem of Calculus. Which is an example that has been done, right? Maybe as an optional course or something. But you know what I mean.

Scene II: Formalism

Sub Scene II.a Fostering the significance of mathematical literacy[28]

The following takes place in the context of the discussion in E4.0. Towards the end of that discussion RME asks 'what can we do to help the students?'

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[28] This scene first appeared in the chapter prepared for the book aimed at Brazilian mathematics undergraduates mentioned in the Prologue. For an interpretation of the data presented here with a focus on the difficult transition from school to university mathematics see (Iannone, 2004). A condensed version of this scene is also contained in (Nardi & Iannone, 2005).

M: To start with we always put a positive spin on what we read… see through what they do, the things which are sort of right and smooth out the junk pieces of reasoning. It alarms me that a student at that level is not yet able to put longer thought experiments together in a coherent fashion, but, yes, we should stop at the positive things, look for that little nugget of understanding and then work backwards…

RME: So, look for the essence, so to speak, of what they are trying to say and work from there.

M: Yes, throughout… quite often, while doing the marking, I kind of insert the key linking words that are missing, in red, and to suggest to them that maybe next time you could write *if*, *then*, *it follows that* and so on. And eventually they do! I mean, occasionally you get third year students and they write extremely coherently argued mathematics that you could actually read out and it would make sense. That's another thing I do actually – and that's one reason small group tutorials are good: I get them to read out what they have written, and then they think it is nonsense! They immediately start to question what they have written. But this is not something you can do easily with large tutorial groups.

RME: Do you have some detail from your experience of how this works?

M: Yes, I can offer some examples from some mathematics-as-a-service-course experiences I have had, where there were many opportunities for seminar work. Students work on exercises which are not going to be assessed, to build up their confidence and get to know how to approach what will be assessed work. It works. It does involve of course some preparation time for me and a teaching assistant, a doctoral student for example, both of us going around the class and giving students a lot of verbal encouragement and indications on what they should be writing down in that type of exercise. And it is very intimate, even if it is a group of twenty students because, by being over their shoulder, you immediately see what students are doing and get them to write more or question their writing. And I have got the feeling that, in these two examples we looked at earlier, if the students had made their first attempt in such a seminar, where someone had helped them, they could have learned a lot from that half stage that they reached on paper. So I have got the feeling that, in learning mathematics, they do sometimes need that little bit of contact at the right time, not too early and not too late…

RME: And you think students are comfortable with this type of contact?

M: They can get flustered a bit to start with but they soon get used to it and begin to appreciate the opportunity. Let me add a cautionary note however: there is always the danger that, in talking to them, basically you tell them what to write down. And they write it down but they still don't know what is going on. You see, I can imagine one of my students this year having produced work like some of the examples we looked at here, rather word-less. Because you know, this is essentially what I would have said to them. I would have said *more words in between*, but this is maybe all I would have ended up writing down on a piece of rough paper which they ended up just copying down. And I am not sure what I want to do is to write down the whole solution which they will then just copy down. And given their school background[29], I wonder whether they feel that they should always be calculating and that it is the calculations that count. I do believe that students can be gently pushed on the blackboard until they write down a coherent argument[30].

RME: You seem to be thinking it is the teacher's responsibility to foster this literacy.

M: Yes, I have always tried to present model solutions, proper and typed. But, especially with the first year students, I may have been a bit lazy about putting words in because it just seems a bit trivial. It just feels a bit naff to say *the derivative of the function is*. Coming to think of it of course maybe I should, because, yes, this is trivial in a sense, but I am asking them to do something trivial in order to stress that I am asking them to do something they have always been doing – in their graphic calculators, on MAPLE or wherever – differently. To look at this same thing they have been doing for a long time from a philosophically different perspective. The tricky bit here is that I am asking them to engage with something that looks deceptively familiar but to do so in an almost totally unfamiliar way.

RME: That's exactly the reason I wanted to discuss this question with you. Because it asks the students to engage with something only now they are beginning to experience: the need to have to explain and justify even the most

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[29] See http://www.nc.uk.net/index.html for information on UK's National Curriculum. Also (Nardi et al, 2003) and SE3.1 for more views from UK mathematicians on this.

[30] See (Bullock, 1994) and, for example, Houston's (2001) advocating of journal writing. Despite general support for this idea there are, of course, sceptics. Porter & Masingila (2000), for example, set out from the idea that Writing to Learn Mathematics (WTLM) is often seen as a panacea for improving students' understanding but sometimes scarce evidence is available for the actual improvement on students' performance. Their study aimed to do that: they classified errors in Calculus and tested whether writing activities had an effect on students' conceptual understanding by comparing the performances of two groups (WTLM and non-WTML). They found no significant difference. I would however tend to support a subtler discourse on this matter and suggest that a slower effect of tactics like this may have on student learning. Take, for example, *Tactic: Muddiest and Most Important* (Mason, 2002 p51) where the students are asked on a regular basis to write down their views on the least clear concept, definition, example etc. or, at the end of a lecture, to write down their views on the salient points made in the lecture. A longer term systematic investigation of the impact of this type of tactic would be more apt than a controlled, and short-term, experiment.

apparently obvious statement. Starting to get used to the idea of explaining things and not just finding the answers.

M: I don't want however to overestimate the value of presenting what I called earlier *model solutions*. In fact very often if I look at my model solutions I feel – golly – I would get myself very bad marks for writing down such things … so much so that I never call them model solutions ever again, I call them *notes on solutions*. Because I actually think that the element of difficulty in solutions is not something that you can pin down on a piece of paper. An argument, between the student and the teacher, ideally, why is this so and by the time I distil it in the model solutions the argument is dead, it becomes part of script. We should have oral examination, and we should have lots and lots of oral interaction with the students to see that mathematics is about arguments. Yes, arguments can be written down but it is far more important they see the argument coming together.

RME: You think students are comfortable with this type of interaction?

M: From my experience typically the more able would be more likely to. And there is some proportion of those who would initiate discussion – for example through seeking help by coming to me with queries because they are enthusiastic about the subject – but then, having listened to what I have to say, would say *I have not got the foggiest idea what are you talking about*. Still this is preferable to someone who sits at the back and just nods. With them I have little clue what is happening and the potential of interaction is minimal. I need some signal to work from.

RME: You think this is something students can do for each other?

M: Absolutely! This is what is going on in these small tables sometimes in the seminar room, where the students are doing exactly what I did, but they are listening to it somehow, because it's coming from one of them maybe? It appears to me that they learn more there than in many of our lectures – particularly when these discussions are supplemented by one of them going and asking the lecturer a question. Of course in this type of environment we need to make sure that the presence of a member of faculty will not disrupt the interaction amongst the students[31].

[31] Following experimentation with peer tutoring Evans et al (2001) elaborate the benefits (for peer-teachers and peer-learners alike) as follows: easier to focus on specific difficulties; pace the work tailored for a specific audience; better understanding than in a traditional lecture; learners are 'more comfortable asking their peer-teachers questions that they might have been embarrassed to ask in large-group lectures' (p170). Finally they stress the importance of 'access to staff to resolve tricky questions' (p170).

RME: You seem to be very clear about how productive this way is for the students' learning. What's stopping us then from incorporating this more extensively into their experience?

M: I don't think we can do more than provide them the space, say how wonderful this is, hope they see how wonderful this is and let them get on with it. You can't make interaction obligatory! You can however have this happening in connection with homework. There needs to be some work that they have to do in order to trigger them to do this.

RME: Do you have administrative freedom to do so?

M: Huh! We customarily have some slight rumblings from university administration about student collaboration and the repercussions it has on our ability to assess each individual's work. Still this does not change my mind because they learn a lot this way. What is cheating, copying and plagiarism, in the eyes of the university authorities, can be high quality tutorials in mine. I understand there are differences in how different departments perceive assessment differently. I guess judging whether two literary essays by two students are substantially different, and not a product of copying from each other, is a substantially different task to judging whether the mathematical answer submitted by a group of students is understood by all students involved. Our students work together and copy, in some sense, a lot, but it is not a problem. They are learning stuff.

RME: Mostly through articulating mathematical ideas to each other as you said earlier…?

M: Yes, I think that there are several levels of language at work here and students talking to each other is one such level. At another level there is the lecturer giving a class, maybe one to one to the student who has come to the lecturer saying *I don't understand this question*. And there is a third level of language, the one in a question like this one here and in textbooks as well. Now I sympathise with students that struggle with this very concise language especially when rather formal lecturers sometimes speak to them in a language that is appropriate to a lecture hall and not to one-to-one advice. It is not being informal enough. And something is happening among students when they are talking to each other and there is a kind … I don't know what it is, some vibe that they are able to tune in to each other to get to appreciate what is going on. See this example here when you mentioned the word *induction*: the very word makes the students fall into a cold sweat and they need a lot of work done on them and by them to unravel what this single word means. And I don't know how they get there but it is a different language to what we would normally use. It is the potency of this language, their language, I am keen on having them benefit from.

Sub Scene II.b The fuzzy didactical contract of university mathematics[32]

The following takes place is in the context of the discussion in E4.1, Scene I.

M: I feel like clarifying to them that I feel very awkward about marking work that doesn't appear to be written in any given language! By the way we have tried to teach the language of propositional calculus and we failed. In the current cultural ambience it is probably even more difficult but let me stress here that children, especially very young ones, often do find logical, minimalist games very appealing. But this way of introducing students to formal mathematical reasoning is not fashionable anymore. We are under pressure to redesign the whole Analysis course[33] – that used to be built around some clean, beautiful arguments – towards more relevant, accessible topics. Not that this fashion would stop me from believing that there is some learning value in exercises with truth tables and the like! But given the circumstances, I guess I would go back to what I said earlier about the potency of interacting with the students and negotiating the detail of their work[34]: if you point to the student that they are using what they are supposed to be proving then the student will almost certainly alter their course of action.

RME: There still maybe intelligent ways for adjusting to this change of fashion as you called it!

M: I am not so sure about this so-called modernising movement towards fewer and fewer definitions, theorems and proofs in the first year. Maybe we cannot modernise in that fashion because we might end up with little left in fact…We will be following the disastrous ways school mathematics has gone… is this part of post-modernist destructive deconstructivist culture to remove all…

RME: M, I am getting a bit scared now… Shall we leave it there then?

M: But there is something in Student N's response that brings my mind back to this tension I mentioned earlier: he is in fact using logarithms at this stage when logarithms have not been formally established, isn't he? I am concerned about students' resorting to tools not yet established but then again being so fussy

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[32] Borrowing Brousseau's (1997) term - see also (Nardi, 1999). Herbst (2002), for example, uses the example of a traditional practice, two-column proofs (albeit in an upper secondary, not tertiary, context) to raise issues around the didactical contract with the students and uncover a paradoxical situation in which this practice places the teacher (described here as a 'double bind', a condition imposed on a person made of contradictory requests and from which the person cannot escape).

[33] See a well regarded attempt at such redesign in (Alcock & Simpson, 2001), the *Warwick Analysis Project* (despite some initial scepticism from colleagues). Keynes & Olson (2001) also report similarly on the success of the modernising approach of their *Calculus Initiative.* Of course the *Calculus Reform* in the USA is a well documented initiative - see (Robert & Speer, 2001) for a brief account.

[34] See E8.1, Scene II for more on the value of detailed feedback to students.

about the foundations of Analysis would probably make parts b and c of the question we are discussing here impossible to manage. One of the great questions I always struggle with is: should the real numbers be constructed from scratch in this first Analysis course? I guess this should not be out of the question. However at the same time in other courses students are solving differential equations without anybody wondering about the foundations and this is a not a reality we can ignore. We cannot just simply take away these tools and say *you are not allowed to use them until we have established formally what real numbers, functions, logarithms etc. are*.

RME: When are references to foundational issues, such as the Archimedean property, coming in these courses usually?

M: I am weary of introducing them too early and as anything more than mere hints. Things like a definition of real numbers as those things that you can write as infinite decimal fractions; or the notion that we build what we do on sets of axioms and the Completeness is one of them. But I feel it is a rather half-hearted approach that may confuse rather than clarify. Internationally there is a great variability in what different courses do. I am aware that in France traditionally there has been a stronger emphasis on foundational issues and that this has generated some student difficulties.

RME: In my own experience, I found the experience of Foundational Analysis an initially intimidating but ultimately rewarding experience to those who survive it.

M: For sure I can see that in discussions with the students such as the difference between 0.9999… and 1. In fact I often think how challenging and intriguing discussing the exact definition of function can be. And what a definitely different discourse that would be to a technique course.

RME: I guess there is a tension between the logical with the psychological elements here: for example the well-recorded student difficulties with the notion of infinity.

M: Axiomatics was largely removed from Analysis primarily in the States and as a result of a pump-not-filter view of the subject. This is a huge project in America and its slogan was that undergraduate calculus is a pump not a filter. Of course this involved a great stripping out of axiomatics completely and a great deal of graphing, calculating and so on. I find this infuriating. Jeepers! One shouldn't be shy about what your subject is and ours is largely about reasoning. However I am skeptical about both extremes as the logical progression and gradual build up can be a very slow and intellectually hard process few would survive. Whereas in the current, more formally fluid state of things a few weeks into the course students can solve impressive-looking differential equations in some sense, benefit from this gratification and then in the second year see the theorems and proofs that allowed them to do so earlier. Maybe. Then again you could retort that there is a potentially damaging for the student contradiction in the insistence on proof in one context but then reliance on unproved facts and tools in another. Of course I can always highlight the contradiction to the students and promise them a later gratification via proof – even though it sounds a bit like propagating a certain logic of complacency! I am pretty sure that a fully blown foundational course on Analysis is not impossible (other countries do it) but I have serious doubts about whether our students can take it. And how many hours of teaching such an approach would take. I guess undertaking such a reform requires rethinking totally the way in which we allocate time in the Analysis course. Issues like the war between the formal and the informal or questions such as *what is the actual purpose of rigor if I am doing something completely un-rigorous at the same time?* acquire a new significance and centrality in this reconceptualisation.

RME: In my experience of talking with undergraduates there is an intriguing impact of the contradiction you are referring to: the students sometimes develop a mistrust towards things they have been doing for ages in school or are suspicious towards the algebra of limits and call it things such as *imprecise* because they are confused about has and what has not been formally established.

M: I can see this point but still I don't see quite how where one goes with it. To me the huge difference is quite a pragmatic one. If one is non-rigorous with limits and convergence and Taylor's series and so on one makes mistakes. What you write down is wrong and easily counter-exampled. If you rely on an as-yet-not-proved but true property, you do not make any mistakes. It is not logically constructed but you will never say anything that is not true. And that is actually the fundamental difference as far as I am happy with students using logarithms or root *n*. It is not like using logarithms is ripping the heart out of a rigorous approach to mathematics. In a sense it maybe does that. At some logical level of course it does, for example working in this context with functions without any prior experience of Set Theory. Some able students who perceive these subtle differences may be disadvantaged in this sense but, hey, it looks pretty impractical to me from a teaching point of you to follow through and implement a pure axiomatic approach.

M: May I return briefly to a contradiction I identified earlier?

RME: Please do.

M: We are asking the students to engage with a question that requires proof of certain facts by using sometimes facts not yet proved. Part (iii) is an example of this because logarithms and exponentials have not been defined formally yet. Part (v), where the point is to demonstrate capacity for manipulating a complex algebraic expression, is not. I am concerned about this paradoxical situation but I think uniform rigour is just impossible, pedagogically impossible.

RME: Perhaps you can be more explicit to the students about the issue? And clarify to them that some of the tools to be used are not yet fully formally established. This way they get the benefit of using them, yet they are not misled about the formal status of these tools.

M: I cannot say I have any solutions on this and this is frustrating. Instinctively my expectation in marking part (v) would actually be for a proper argument. So, for example, show that $f_5(x)=2$ has no solutions. But somehow, in marking part (iii), of course I would be happy for them just to present a name for the inverse of this function … that's fine. So why not? Why not let them invent names for the inverse of any function and use this when they need to have an inverse. I don't know, I have no solutions, it is something in Analysis you grapple with all the time: if you want to use interesting examples[35] and you want to use available tools and you want to be rigorous and you cannot be rigorous and simultaneously meet the whole target somehow of teaching all these things. I don't know! In fact the question is interesting in its vagueness. It is asking the student to merely determine the answer and then justify it, not fully prove it. And justifying is murky ground. Maybe this is why we are getting solutions which look much more verbal: the student is checking the answer out, touching base, making a decision, stating this decision. I think that the question is begging this sort of answer.

RME: I noticed that at the bottom of notes on solutions the students are invited to consider 'bits of calculus that can be used to justify more fully…'. This is in some way towards the explicitness I mentioned earlier.

M: And it is also about a distinction between Calculus – where indisputable algebra is acceptable – and Analysis – where a discussion on foundations is required. But in most courses the Analysis bit comes later. Therefore using facts from Calculus is not strictly allowed here, bar certain algebraic manipulations that are expected at this stage. Or an understanding that actually surjectivity has to do with solving an equation and injectivity has to do with unique solvability of an equation. This to me would be an educationally sound step. Part (iv) requires some knowledge of cubics, of solvability of equations, of uniqueness of solutions etc.. Of course this type of rigour ties you down so much. The rigour with which some of these properties and facts have been introduced at school level, for example, is so questionable that I see the task of solving this question rigorously as very vague. What is, for example, the prerequisite knowledge for this question: the IVT, the

[35] For coursework that develops students' skills in constructing examples (polynomials in this case) see, for example, Sangwin's (2005) Computer Algebra System.

definitions of exponentials, logarithms and trigonometric functions? Is the definition of the square root or the exponential as just a symbol of an operation on a number adequate at this stage? I could go on for a while. Analysis *is* possible and a foundational approach would resolve this lack of clarity.

RME: You mean where Cantor, Dedekind cuts, the Archimedean property etc. are coming in?

M: Yes! I mean a programme of Analysis where the reals are constructed and Calculus is built on first principles. Where the deal is we teach the students to be absolutely correct, where they don't talk about or use things we will meet later. But I am not entirely sure about the pedagogical viability or popularity of this process – amongst teachers and students alike. Also in actual mathematical practice, many times the definition does not always precede the first discussion of things. So maybe I am asking for way too much if I want to make this instructive yet tough austerity happen in an introductory course. But then again there is a certain fascination in this kind of austere schooling, a first principles approach done alongside what we would call techniques and also done across a few years. I see little problem in pulling this off but one thing needs to be emphasized: these distinctions need to be made clear to the students. At the moment we do not do that and students don't know what is what! This is where the complexity of the task lies.

RME: If we take a fairly restricted view of the question, could we see this as simply aiming at checking the students' knowledge of the definitions of onto and one to one? Would then their previous knowledge be sufficient for dealing with this?

M: Yes, we could see it this way – these functions and their graphs are up the students' territory and that allows students to check out a couple of definitions. But you cannot call this *knowledge*. You can call it working information or some picture of the situation. At a low level the students can still come to the right decision about which of these functions are one-to-one and onto, even if I find their reasoning not particularly persuasive. But they need to have their intuitions, the trust bestowed upon a graph challenged, for example. It is time that their attention is alerted to the risk of relying on these intuitions too much (try the picture of the rational circle and the proof that there are lines starting from zero and going off that will never meet the circle – marvel at their surprise!) and of using certain as-yet-not-established facts. Perhaps the question was trying to achieve too many things at the same time while not being very clear in which world the whole thing is happening: where is the clarification here that in one world you need to have rigorous arguments and we can do them and in other worlds you cannot yet do them and maybe this is the situation where we will learn more precisely what 1-1 and onto mean. The discourse on the definitions of onto and one to one is quite distinct from the discourse on certain functions and their origins, properties etc. You see, when you teach Algebra you are privileged in that sense. You never actually have to do things which aren't true. You define the reals as a field and you are done, you can start doing things with them without bothering too much about their nature.

Scene III: Numerical experiments[36]

Setting the scene: The following takes place in the context of the discussion in E6.3, Scene II.

M: One thing that I try and get students to do – which again they are very resistant to, ironically – is numerical experiments. What is a reasonable way to guess the limit of a mysterious sequence? Evaluate when n is very large. This is not a proof but it is a very reasonable thing to do. They will not do it and I usually have to enforce it on them: use the few, scattered tools school mathematics has equipped you with – and doing numerical experiments is one of those! Of course we have to be cautious here: seeing n as a specific number may be prone to misunderstanding. I have occasionally heard suggestions floating around about responding to the students' difficulty with the formal definition of convergence: simplify the procedure by removing it. Approach the whole notion of convergence via examples of sequences whose limit we know, zero for example, and examine different ways to show that something is converging towards zero. So not bother with the notion of the existence of the limit for the moment. But, you know what? This way, I think, you lose the one truly magical thing you can do at this stage, use γ, the Euler-Mascheroni constant – monotonic and bounded converges – to make your case that the limit exists or not. And this is so cool about mathematics, the deeply exotic, fascinating idea that we are able to deal with this thing without having to specify exactly what this is, but knowing that it is there. Yes. Somehow this business of unknown limits is tedious but rather rare too!

RME: Do you encourage the students to conduct numerical experiments by using technology? For example, to conduct some calculation for very large n.

M: I do – MAPLE for example. Sure, why not? Some of these calculations take no longer than a few seconds. It is part of my overall idea of using finite methods to deal with infinite problems. Like the use of the geometric progression formula in the question we are discussing here.

[36] The discussion here is intertwined with the issue of using new technologies to support the construction of mathematical meaning. Therefore the references to the use of these technologies in this chapter and in Chapter 4 would also be relevant here.

RME: Thank you. But you may wish to return to discussing numerical experiments.

M: Yes, please. Because I would like to address what some colleagues may call their *personal prejudice*, about whether numerical evidence is important in Analysis: people are known to say that numerical things are very separate from the algorithm by which certain formulae can be worked out. My response to this is that an analyst always does examples when trying to gain new insight into something and I do not see why, if I am doing so when I am exploring ideas as a professional mathematician, this strategy shouldn't treacle down to the way I teach. Of course an algebraist may say otherwise, they may say that you do examples if they create structure insight and in Abstract Algebra this may not always be the case. But to me again this is only a variation of my argument, is not an essentially different argument. In Abstract Algebra the process of computation is such that, at each stage, I formulate a conjecture and I see what this computation tells me structurally: computations are extremely useful if I can verify parts of conjectures. Fine, I agree that they are far less useful if I am just trying this number or that number without aiming to gain some structural insight.

RME: Can you exemplify this difference for me?

M: Graphs such as $n!$ divided by n to the n or e to the minus n are spectacularly converging graphs. You can easily construct them on MAPLE and is quite a stunning view. I don't know how else you could get to that kind of insight about their convergence so easily. Of course the whole point is to gain that insight which will make you crave for a rigorous proof of why this is so. And I know that this may not be the case with students to whom seeing this graph would suffice as conviction.

RME: I was just about to say that!

M: *Dom spiro sperum…* And let me stress that my support for the technology or the numerical experiment is on the proviso that they offer insights I do not already have in my head[37], that they move my thinking forward. You may not need a computer to tell you one over square root of n is getting smaller when n gets large but for a number of other cases you do. The machinery is not always

[37] For one example of facilitating students' acquaintance with such insights (in this case structural conceptions of function) see (Hollar & Norwood, 1999). They report on the use of CIA (Computer Intensive Algebra) as follows: the IT used was the TI – 82 graphic calculator; the O'Callaghan test (modelling, interpreting, translating, reifying) was used to compare performance between two groups, one that used IT and one that did not. The IT group performed better and their attitudes improved, if only slightly. In the final exam no significant difference was observed. Two interesting observations were: IT students performed better at reification (traditionally the most difficult of all four parts of the test) and their computational abilities (an oft reported casualty of using IT) did not suffer.

indispensable but it is there not to let students do Analysis after a lobotomy, but to allow exploration of expressions which you cannot deal with by hand. To offer intuitions about the size of things, for example, that are not achievable simply by imagining them. And, also one hopes, it increases their mental store of examples. Even further: one should not underestimate the historical importance of numerical calculations in the field. Look at Stirling's work as well as the numerous doctorates in mathematics that are nothing other than lengthy, sophisticated calculations. Numerical calculations in Analysis can be informative in situations where nothing else is informative.

RME: This is fine but there may still some cautionary mileage in what you said before about the seeing-it-on-the-screen-is-believing-it-full-stop issue. School mathematics is often fostering this attitude.

M: Well, there is certainly a need for some … un-learning here and for teaching them the difference between a proof and a numerical investigation. But I would be sad not to use numerical investigations because I fear the resilience of these school-fostered attitudes. I will endeavour to change these attitudes and make it absolutely clear that my ultimate aim is to prove why the idea I have gained access to via calculation or graphing or whatever is true. Producing the evidence is different to producing the theorem.

RME: In any case accumulation of evidence does not always secure you from missing what happens in sort of crazy cases.

M: Indeed but quite a few of those by the way exist almost exclusively in the bizarre realm of Analysis. I usually try to get away from these luridly pathological examples that only exist in the subject of Analysis and to offer students examples that are common, are relevant to other situations, examples that resonate in mathematics[38] like the Euler constant or the formula in this example. Or, you know, the sum of one over *nlogn* diverges. That really matters in Number Theory, so it is something that they should see. This is the kind of things I feel I am after most of the times, not the exceptional, exotic creatures.

RME: Are you concerned that students, accustomed to trusting graphs, for example, for a long time, may feel a bit frustrated about how restricted this relying on graphs needs to be from now on?

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[38] Amongst the criteria Mason (2002) cites for what makes a good example is what is exemplary about an example, an area his work has extensively focused across a number of years. He often uses *Michener's distinction* of mathematical understanding in terms of example spaces (start up, reference, model and counter-examples), result spaces (basic, key, culminating, transitional results) and concept spaces (definitions, heuristics, mega principles, counter-principles).

M: Yes, and it is exactly this potential frustration I want to avoid by stressing that we cannot underestimate that a good graph does tend to give an idea of what is going on. And students do use graphs and calculations on the side, literally on the side, to form a claim, all the time and I am happy with them doing that, saying nonetheless *look, I have been looking at the graph, I believe this is true and I will try to prove it to you*[39].

Scene IV: Pictures[40]

Sub Scene IV.a: The pedagogical potential, and the strongly personal nature, of pictures (in the context of the discussion in E6.2)

M: Yes, they somehow end up believing that they need to belong exclusively to one of the two camps, the informal or the formal, and they do not understand that they need to learn how to move comfortably between them (E7.4, Scene II). Because in fact this is how mathematicians work! I still remember acutely my own teachers' explanations of some Group Theory concepts via their very own, very personal pictures. I am a total believer in the Aristotelian *no soul thinks without mental images.* In our teaching we ought to communicate this aspect of our thinking and inculcate it in the students. Bring these pictures, these informal toolboxes to the overt conscious, make students aware of them and help them build their own. And I cannot stress the last point strongly enough: we need to maintain that these pictures are of a strictly personal nature and that students should develop their own. All I can do is describe vividly and precisely my own pictures and, in turn, you pick and mix and accommodate them according to your own needs.

RME: Still, you must surely think that some pictures are more potent than others and that those deserve more attention, right?

M: … and that these should somehow be imposed! Yes. I do that all the time, I submit students to intricate demonstrations for some of mine!

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[39] Here is an example of work that can help in this matter: Roth & Bowen (2001) use semiotics and hermeneutic phenomenology to link perceptions of real world (experience) with mathematical worlds of scientists (ecologists) in the context of graph reading. Previous research has identified problems with slope and with iconic interpretation. The premise is that graph reading is heavily contextual. This study uses a semiotic model based on the processes of grounding (a sign into the real world) and structuring (constructing an understanding from the plethora of signs by selecting etc.). Two case studies (one of a scientist reading graphs not from his own work -; and, one of a scientist whose reading of familiar, personally constructed and frequently used graphs is transparent) are used to illustrate the contextual nature of graph reading. A relationship within the quadruplet (S, R, r, c) (sign, referent, context, restrictions / connections of mathematics) is sought to explain the differences between the two cases. The implications are embedded into a discussion of how learning activities regarding graphing should relate to the types of transparency achieved by the reading of the more successful readers of the study.

[40] Many of the references in E4.3 are relevant to this Scene and vice versa.

Sub Scene IV.b: Building students' understanding of convergence through the use of visual representations (in the context of the discussion in E6.3, Scene I and E7.1, Scene II)

RME: I am delighted with this …deconstruction of the elements, symbolic or other, in the definition! What's your take on the presence of || in it? This is another quite packed symbol.

M: What the students really need to be thinking about is what || means on the number line and as a distance. But they so often get stuck to the algorithmic habit of *solving* this without knowing what it means. And that stubbornness can be a nightmare. What I mean by *what it means* is, for example, seeing, what an equality or inequality involving $|x\text{-}1|$ means pictorially on the real line. Once you have seen it on the line, the answer to your question is obvious. That is why I am a huge fan of them doing using all sorts of visual representation (E7.4, Scene IV): because the ones who do, almost invariably are the ones who end up writing down proper proofs.

RME: This is a remarkable conjecture.

M: And allow me to think it is a strong one too. Also I think, if you have to present the material to students, the only way to get things across really is to draw lots and lots pictures. Earlier I expressed reservations about the usefulness of some computational aspects – a long chain of complicated numerical results usually tells me nothing, even though I exclude from those illustrative computational exercises that demonstrate to the students something they have difficulty understanding through its slightly counterintuitive and esoteric proof (*multiplying small errors results in unboundedly large errors* is one of those) – but a little picture tells me everything. As long as I can create in the students a feeling that this is maybe so for them, then I am happy. And I think that there would be strong evidence that this is so.

RME: Are you concerned that some pictures may be a wee too convincing? That they may offer blinding intuitions that prevent students from thinking that the statement still needs a proof. I am thinking of the classic picture for the Intermediate Value Theorem.

M: Lest we forget some very clever people regarded this not needing a proof either! People like Newton. But of course I agree there is a risk there: you need to constantly interrogate your picture and its goodness. By the way there is an irony in the fact that validating the truth of the statement in IVT means that all the pictures that students have been drawing are retrospectively true – like drawing the solutions of an equation. This irony in fact is nothing other than another piece of evidence of a constant tension within pure mathematics: that you want to use these methods and occasionally you need a theory to come along and make them valid. And you need these means, diagrams etc., so badly. Yes, they are not proofs but they do help students acquire first impressions, start inventing some suitable notation. And of course there are things that are very concrete like geometric problems in the complex plane that I frankly cannot think of a better way of presenting other than through a diagram, for instance to see that $|z-a|=1$ is a disk.

Sub Scene IV.c: Strengthening students' understanding of injective and surjective functions using Venn diagrams (in the context of the discussion in E5.1)

M: And I guess Venn diagrams come from school, right? That's fine I can see how helpful these diagrams can be for showing relationships between three sets – could I extend their use to five or even seven I wonder? Oh well, probably true but not useful in this context. Anyway Venn diagrams are a good idea: I often wonder how we can make helpful representations of notions like 1-1 and onto for the students. I noticed there are no pictures in the lecturer's response. This is a bit poor I think considering how telling pictures can be in this context and how the negotiation of their meaning could bring about a good understanding of these properties of functions. These definitions are hard to read but you can see what they are trying to say in a diagram *pronto*!

RME: Do you think that a discussion of the concepts of 1-1 and onto in the context of an abstract definition of function as a correspondence between two sets could be equally useful here?

M: Probably as it is difficult to discuss these concepts in clear terms when using actual functions from the physical reality or in statistics and probability where injectivity and surjectivity are less clear notions.

RME: I would also add that producing drawings that represent these concepts for functions with a smaller domain, for example, is also easier when you teach. I can illustrate the concepts by showing where all the arrows go, how having arrows that start from different places in one set and end up on the same blob means the function is not one to one. How having blobs in the second set where no arrow ends up means the function is not onto, how having arrows starting from all blobs in the first set is a prerequisite for having a function in the first place.

M: I agree. I want to go briefly back to something we mentioned earlier, the students' dominant perception of a function as a rule (E4.1). This perception may in fact have a historical basis: most functions in Physics were precisely that to start with (OT4.1).

Sub Scene IV.d: Strengthening students' understanding of functional properties through construction and examination of function graphs[41] *(in the context of the discussion in E5.2)*

M: They see a function like $x\sin 1/x$ as a complicated expression, period. No guts for deconstructing it down to and reflecting upon its elements. Which is what you want them to do. Beyond whatever verbal memory tricks, have a mental representation of these things.

RME: Is the connection between an image of e^x in terms of its graph going a certain way and *bigness* as one of its properties, one of these healthy mental representations you are talking about?

M: Yes, there is something reassuring in identifying *e* to the *x* as big in relation to polynomial. However I am far less comfortable with a student who sees periodicity in any graph that involves a trigonometric function. This is way too crude.

RME: What is the level of crudeness you would tolerate in order to at least initiate the students' involvement with graphs?

M: Well, let's think about the so called *bigness* of exponentials. This is just about the time that students ought to start realizing how faster exponentials grow in comparison to powers. Through a picture like this they can start seeing this and then, of course, prove it by induction or whatever. A more formal grasp of exponentials comes subsequently when Taylor expansions and Riemann integration are introduced. Then exponentials are seen as inverses of logarithms. These more formal images are not necessarily substantially helped by early pictures but for a primary understanding of exponentials this picture is good enough. And necessary. I don't want to wait until the Riemann integral is formally introduced for the students to have an image of what the graph of $x^{-1}e^x$ looks like. This is a totally unnecessary and unproductive delay. Of course exponentials are also buttons on the calculator and in that sense the point-wise view comes to the rescue – at least temporarily. I am terribly worried about the unrigorous messages conveyed by acts such as raising a real number to a real

[41] See, for example, Slavit's (1997) alternate perspective for utilizing the action/process/object framework towards discussing student development of conceptions of function. Slavit presents a property-oriented view of function based on visual aspects of functional growth and differing from theories on covariance and correspondence - the two main differences being: less emphasis is placed on the manner in which the variables are changing and more emphasis is placed on the properties that result from these changes; functional properties such as invertibility and domain give rise to a different kind of thinking about functions than do properties such as symmetry, linearity, continuity, etc. Another example (in an upper secondary context) of activities that facilitate the growth of functional knowledge is (Yerushalmy, 1997).

power, when we have hardly introduced real numbers, or powers (E7.4, Scene II)! I know I sound pedantic but we find ourselves in this impossible situation of somehow requiring both linearity and rigour. I know that overall the exercise of drawing the graph of $x^{-1}e^{x}$ is probably wonderful and very good for them despite these big conceptual holes. And you can get students to generalise x^{-n} for any integer n which also has this nice property of getting bigger as x gets bigger. You can explore how fast this happens. And every now and then you get a surprising insightful comment from a student such as *the degree is not high enough* for our function to do this or that. And that is impressive and reassuring. That they can get to see the dominant term in a polynomial when you are trying to produce its graph. And in fact see these things beyond the range of your graph paper. Use their imagination on how this thing would go beyond what fits in your piece of paper.

Sub Scene IV.e: Negotiating meanings and appropriateness of pictures as a means of strengthening students' concept images in Group Theory.

The following takes place in the context of the discussion between M and RME of the following excerpt:

Bestowing Meaning On the Concept of Coset
Through Ambivalent Uses of Geometric Images
from (Nardi, 2000a) and based in material from (Nardi, 1996)

As addressed in Harel and Kaput's work on Object-Valued Operators (1991), students often construct concept images of cosets in Group Theory via familiar geometric figures. The students' intensive need to resort to this familiarity originates largely, not to the rather 'simple' relationships between certain mathematical objects (Leron and Dubinsky, 1995) that characterise theorems such as the Homomorphism Theorems or Lagrange's Theorem (their consistently problematic conceptualisation by students is reported in (Hazzan & Leron, 1996)) but to the abstract nature of the mathematical objects involved. For example, Lagrange's Theorem is about 'easy' things such as one number dividing another, two sets having the same cardinality. Leron & Dubinsky attribute the students' difficulty to confusion about the nature of cosets: students do not see cosets as objects to be measured, counted and compared. These observations on the students' conceptualising processes resonate with the evidence presented in this Episode.

Background to the Episode. In the beginning of the tutorial student Camille declares her confusion with the notion of equivalent classes defined by "$a \sim b$ when $f(a)=f(b)$ where G is a group and $f\colon G \to G$" (Note: in more accurate terms Camille ought to have said 'if and only if' instead of 'when' and also specify what f is: a homomorphism from G to G or an injection as suggested in the Episode). The tutor draws fig.1 to illustrate that 'if b and b' are in different equivalence classes then $f(b)$and $f(b')$ are different'. In the subsequent discussion (C1-C5 and T1-T3 lines of transcript, omitted here) Camille appears puzzled with fig.1 ('Why are they all straight lines?', she enquires).

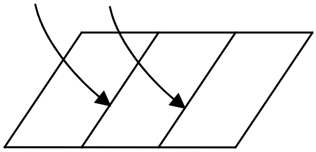

fig. 1

Because these lines 'do not mean anything', the tutor replies and replaces fig.1 with fig.2, one with 'squigges' as the equivalence classes:

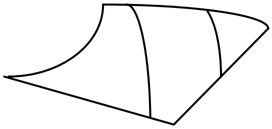

fig. 2

Fig.1 is the tutor's image of equivalence classes where an element *a* of the domain is a dot and its equivalence class (defined as the set of elements in the domain that are mapped on the same value as *a*) is a line segment. The metaphorical elements of this image however seem to escape Camille who interprets fig.1 literally. In the *Episode* below she offers analogous interpretations with regard to the notion of *coset*.

The Episode. Subsequently the discussion of the correspondence between the elements of a group and their equivalence classes evokes in Camille a query on another correspondence: 'the 1-1 correspondence between the conjugates of *x* and *x*'. Remarkably Camille demonstrates precise knowledge of the relevant definitions (centraliser, conjugate) as well as a relation between the two concepts (unlike most of the students who were incapable of reproducing definitions of even simpler group-theoretical constructs mentioned in the lectures). The tutor initiates a discussion of the 1-1 relation between cosets of the centraliser and conjugates of *x* but Camille is quiet and looks skeptical. Then she asks:

C6: What are cosets materially?
T4: What do you mean by that?
C7: If we have a group *G* and a subgroup *H* why do we bother to find the cosets?
T5: Because of results like this. They turn up naturally.
C8: Cosets are a group multiplied by an element in the big group.
T6: *[hesitantly]* Yes...it's a set...
C9: Cosets are just a moving...
T7: That's right. That's one way... you can look at it as translates of a subgroup... sort of multiplying *g* with everything in *H* and it shifts it...
C10: *[after a pause]* So if we have a square of size one and then the group *G* is like this...

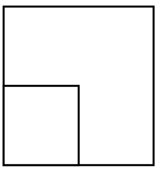

fig. 3

T8: You have to be slightly careful... It is slightly... don't think about in... you're not thinking of applying it on squares, are you?
C11: So then it would have four cosets?
T9: Mmm... if the subgroup was a quarter of the size of the whole thing, yes... it would have four cosets... that's right.
C12: And the cosets are always the same size as the original.
T10: That's right. As we know they partition the group.

Camille has not used the term *coset* at all. The term occurs for the first time in the tutor's words and captures Camille's attention. Subsequently, and in the rest of the Episode, it seems that the notion of *coset* constitutes a large part of her preoccupation : C6-C12 seem to be persistent, multiple attempts to imbue it with some meaning. C6 comes through as a surprisingly philosophical and abstract question which raises a very fundamental existential issue with regard to the notion of coset: what is surprising about C6 is that it comes in the middle of the tutor's describing a quite sophisticated construction (establishing a correspondence between the cosets of the centraliser and the conjugates of an element x in a group) and shifts the conversation from the strictly and specifically mathematical (represented by the tutor) to the metamathematical. Camille has been attentively listening to the tutor's demonstration of the construction and has given the very strong impression that, throughout, she has been processing the dense information provided by the tutor. C6 however illustrates that this processing must have been motivated mostly by the desire to construct an image of coset — visual, 'material' — than consume the tutor's argument. From then on, as said earlier, C6-C12 is a series of successive attempts at interpreting the concept of coset.

C6 is a nearly platonistic enquiry on the nature of cosets as objects, as entities. Camille's entities in C6 do not necessarily act or interact. In C7 the questioning of the nature of these objects takes the form of an exploration of their *raison-d'-être*. C8 is a dissection of a coset that equates a coset with how it comes into existence. I note that so far T4-T6 do not seem to have a direct impact on Camille's generating of ideas of what a *coset* is. C9 is a geometric interpretation of C8 derived from the notion (and notation for) transformations, and in particular translations. The tutor carefully tunes in, using 'translates', a classical description (T7), but Camille accelerates her tentative condensation of her conception of coset in a geometric image in an unusual way. C10 (in parallel with her 'straight lines' in the *Background*) illustrate the thin line between a metaphorical and a literal interpretation of a picture. It is perhaps reasonable to assume here that Camille operates under the strong visual impact of the four-sided figures used by the tutor earlier in the tutorial (fig.1 and fig.2). The tutor is surprised and alarmed (T8) by Camille's intention to 'apply [this idea] on squares'. C11 is evidence that Camille is too preoccupied with her image construction to be influenced by T8 and she furthers the interpretation of her fig.3 in a less controversial but highly ambivalent way. T9 is one more effort on the tutor's side to tune in and transform the student's images from within. Surprisingly then Camille turns in a shift to a more abstract property of cosets in which however the geometric/numerical jargon ('size' in C12) is maintained. The tutor (T10) has completely adopted Camille's metaphor and contributes another observation on cosets.

Finally Camille ceases the effort to interpret further the notion of *coset* once she acquires an image of *cosets* that is satisfying and clear to her. That Camille is content with what she has acquired can be assumed on the basis of the evidence, given during the study, that this student does not bring a conversation to an end until she acquires a satisfactory (to her) understanding. The issue that C6-C12 raise is whether the quality of the acquired perception of a coset – via these visual images – justifies Camille's eventual sense of content. Given that the tutor cautiously surrenders in adopting Camille's image but does not cross-check whether the intended (by the tutor) and the acquired (by Camille) image of a *coset* coincide, the questions raised by this issue ought to remain open.

In sum, in the above, a student, who exhibits a remarkable memory of the definitions of the concepts involved in the discussion, is engaging in a meaning bestowing process with regard to the notion of coset. The student asks the tutor about the raison-d'-être of the concept and her efforts are characterised by a tendency to use images of familiar regular geometric shapes in order to construct a mental image of new concepts (equivalence classes as parallel straight lines, cosets as squares). Evidence was given that these geometric images are interpreted literally by the student. This raises the issue of a potential cognitive danger built in their use — despite their undeniable, and widely acknowledged in the literature, pedagogical value .

From a teaching point of view, the tutor has demonstrated a certain degree of flexibility in thinking in the terms of the student's images (actually it is the tutor who sparks off the use of geometric images in this tutorial) but, in the end, there doesn't seem to exist any guarantee that the particular use of these images has resulted in the tutor's intended concept image of the notion of coset. The student's plea may have been for concrete illustrations ('materially'?) of the utility of the new concept of coset but this plea here seems to have remained un-responded to.

RME: The representation of concepts from Group Theory in this example are idiosyncratic to say the least. What do you think?

M: I am fully with the student on this one. The student is confronted with a tutor who sketched this broad, brush representation and who had a very clear idea in her head about what idea she was trying to convey. But the student had an entirely different meaning for Fig. 1 and 2. And what I find interesting is that during the whole exchange the student and the tutor were just residing in different sets of thought and understanding. It goes back to the same old idea of: if you are going to do a diagram, then do a good diagram and make sure it is properly labelled and you can see where the parts work. And, in this respect, for me Fig. 1 and 2 are hopeless in communicating any new ideas to the student. OK, maybe I am being a bit too harsh, maybe I can see the link between the diagram and the notion of vector space and of cosets as parallel lines as not too bad. Then Fig.3 is absolutely wonderful: it is suggestive of a fair number of properties of subgroups and cosets – such as the equal size of cosets – and is the compromised outcome of a fantastic fight between the student's own notions and the tutor's figures. It is a greatly diplomatic attempt at building common understanding. This doesn't change my view of course of Fig.1 and Fig.2 as horrendous, as pretty ghastly images in their imprecision and vagueness: if you have to bring in an image of cosets – and why not? – I vastly prefer seeing them as slices of a loaf of bread. And a subgroup is a slice and the cosets are the other slices and so you must check that each slice has the same number of crumbs in it and the slices are disjoint and there is a finite number of them, and you check those axioms and properties. Actually fantastic! I am hugely appreciative to the lecturer who introduced this to me when I was a student. Fig. 2 fares better in the sense that it is an attempt, albeit not very successful, to get the student away from the damaging effect of Fig. 1. Still I simply cannot see the point of using these two figures: some illustration in different colours would have had much more and better impact. Or of cosets as balls within a bigger ball. In any case I am always a bit flabbergasted by how difficult this concept of coset is for students: a simple, in my view, concept that involves the multiplication of all elements in a group by one of its elements, a simple act of chunking the elements of a group on the basis of a selective criterion.

RME: I think I can see that Camille seems to understand that.

M: Yes, any student that would come up with this Fig. 3 and thinks that there are four cosets deserves to get a degree immediately! But it is odd that the teacher is so reticent in following this idea through, She is probably terrified that Camille is thinking of this as the group table or something. In which case the worry would be somewhat justified.

RME: Shall we have a closer look at the process that student and tutor seem to be going through? Especially with regard to Camille's questioning of what is *actually*, *materially* a coset, as this questioning initiates the exchange. Also I would like to hear your views on how Fig.3 may be originating in Fig.1 and Fig.2.

M: Fig.3 is evidence of a very good step the student is taking towards understanding that the coset is not some external thing, it is a lump of stuff that comes out of some act between elements of the group. And that there are other ones in the group as a result of this act. But the tutor is not paying attention. I mean, T10 is supposed to be a response to C12 and I don't know what she is on about. *And the cosets are always the same size as the original* is right; *We know that they partition the group* is a completely different assertion. They could partition the group and be all of different sizes. Of course the ending is only a repercussion of the discomfort the tutor feels with regard to her communication with the student already evident in T4, *what do you mean by that?* This is where we know Fig.1 and Fig.2 have completely failed to communicate to the student what the tutor aimed at communicating. To her great credit, the student is transforming these not exactly right images into something more meaningful.

RME: You seem to have an account of how the mis-communication developed.

M: I was under the impression that the student had already started a train of thought along the lines of C6, C7 and C8 and the tutor is on another planet. They are just not connecting. And, if anything, the tutor is trying to follow the exciting feeling that the student may be transmitting. That collection of thoughts is going around C10 and it is certainly clear from that last exchange right down T9, T10 that the tutor is learning more than the student has done! But there is such a thing as answering this question *what are cosets materially?* and you can do a diagram that communicates something. Fig.3 is to me closer to being such an image and it sort of displaces in the tutor's mind the far less successful Fig.1 and Fig.2. This is not necessarily a pleasant realisation for the tutor.

RME: She is also a bit panicky about how far Camille will take this image: *you are not thinking of applying it on squares, are you?* Is, I think, the culmination of the tutor's sort of panic.

M: I would like to praise the sensibility and rationality that seems to come across from C9 onwards but, as I said, I have doubts about the final tutor utterance *and we know cosets partition the group*. But may I offer an observation on Fig.2?

RME: Please do.

M: I think the curved lines are there to emphasise to the student that the lines are in fact just a mere representation that is not to be taken literally. Following from that I think I may be able to offer a more understanding interpretation of the tutor's reticence towards embracing the student's Fig.3: yes, in this case she should have because things turned out well in terms of the attained concept image of a coset by the student, but in many cases this could have spiraled into a nightmarishly confusing situation. Given the unknown outcome, a lot of experienced tutors would have also chosen to backtrack from it at the early stage or would have been suspicious of embracing it – exactly like the tutor here. And, after all, one may see Fig.1 and Fig. 2 as representations of an attempt to construct the factor group and in that sense these images are not so inappropriate. Of course Fig.3 is still the more sophisticated one: to have a student drawing a picture that they haven't seen, at this level of abstraction is worth something. And then she is coming up with four cosets, not two. This is a pleasant surprise. The only way to get four is to think of this whole lump being moved and moved and moved and moved…and that is so cool!

RME: Trying to represent visually some concepts in mathematics is occasionally futile: I recall an image I have seen of a compact set in Topology where, of course, on paper all a cover for the set can be is finite!

M: Add to this the ordinary notions of *compact* in English and you are in trouble! With regard to Group Theory there is little space for visualization to be honest, especially above a certain level. Group Theory concepts are not always amenable to plane representation. I am always worried I could nudge students in the wrong direction. Constructing helpful pictures is also a deeply personal issue. To me *what are cosets materially?* signals a desire on the part of the student for an image of the concept of cosets that goes beyond what was thus far offered to her through the lecture and through Fig.1 and Fig.2. And beyond that she seems to be seeking an understanding of what is the function of this new concept. By the way I think the English word *coset* is a bit weird: it is not very suggestive of the nature of the concept. The German word is *side groups* or *beside classes* which is much more evocative – especially the *classes* bit which suggests a bit more than *sets*.

RME: In other languages such as Greek *side groups* is used to.

M: Anyway, let me return to the student's *why* do we bother to find the cosets?. To me this is a plea for contextualising cosets in the big picture of Group Theory and a cry out for a worked example. A cry that's not being responded to. A cry in fact for many examples: from permutations, from the small, manageable groups of symmetries and, a bit more controversially – because numbers are deceptively familiar and like Fig.1 may evoke misleadingly familiar images – from modulo arithmetic. I now recall the way isomorphisms were introduced to me through a charismatic lecturer who started with two … isomorphic jokes. This is similar to the appeal I find in the loaf of bread image: this is an image you can never mix up with something else! It is distant from anything else and this distance helps as there is nothing else that is closely related to it – unlike Fig.1 and Fig.2 where the parallels with other mathematical topics can be distracting. Another one I like is that of cosets as buses with the same number of people, where the people in the same bus have something in common: they go to the same place. Sheets of paper is another one too! These are examples of how a diagram comes to the rescue and saves all our souls from being thrown into a long twisty journey!

Scene V: The Toolbox Perspective[42]

Setting the scene: The following takes place in the context of the discussion in E3.4.

M: I am a firm believer in this case in the potency of concrete, calculation-based reasoning: it helps the student discover the ludicrouness of their claim. In the course of their studies they probably get to see more of the asymptotic behaviour of some sequences and series but I am concerned about Year 1 though: the students' unwillingness to throw away parts that look big when they are trying to set up a comparison, for example. Or have a grasp of speed of convergence.

RME: We tend to use expressions like *this goes to zero very fast* and I often wonder what this means to students and how one conveys an idea such as speed of convergence.

[42] There is analogous discussion of this perspective in (Nardi et al, 2005)

M: I recall an experiment some time ago where the students were asked to put together, *distil what matters* on a couple of pages and bring this summary along to the exams. Making them write down the theorems may invoke in them the reflection on what these theorems really mean. Even Student H would reconsider some of her ideas, I reckon! Maybe if she was exposed clearly to a theorem as something being valid under certain conditions, with the emphasis on the conditions?

RME: I wonder whether students have these theorems as tools in their minds. Something they can retrieve and put into use? Associate certain theorems with a certain purpose, or a name of a famous mathematician or something? I often catch myself having a preference, a bias if you like towards tools I can retrieve more easily because I associate them with something.

M: They should. And an Analysis course ought to definitely aim at achieving this toolbox perspective. But this ultimately involves an effort on the part of the students to learn the associated definitions and theorems and have them handy. I sometimes experiment in my lectures: a student name is called out and the student is asked to provide a definition!

Scene VI: The skill and art in trial-and-error – making appropriate / clever choices when deciding the steps of a proof

Setting the scene: The following takes place in the context of the discussion in E7.4, Scene V. Demystification of mathematical thinking is part of M's mission. Here he elaborates why this is not a mission-impossible. Earlier in the dialogue, and in the context of a discussion in which M critiqued students' choice of a convergence test as inappropriate – or in some cases as plainly wrong – M used the words 'art of clever choice'. Here RME pursues an elaboration of this statement.

RME: Can you expand on *the art of clever choice*? Whether regarding a convergence test or a theorem that will lead to an insight in a proof generally.

M: This is about this very complex business of learning about ways of learning[43]. I usually propose that through trying many things that do not work you develop a sense of what works[44].

RME: I recall tricks that were on offer in school: when you see an equation of the 4th degree, with no cubes but with squares in it, think of an $y = x^2$ substitution. Is this spoon feeding or is it suggesting examples to the students of the kinds of good practices they ought to be building themselves?

[43] Also in accordance with literature on study skills and meta-cognition (e.g. Mason, 2002; 1999).
[44] For the benefits of this trial and error approach see (Mason et al, 1982)

M: I am not worried about spoon feeding. I always try to get the students to state whether they believe a series or a sequence converges and then discuss with them the choice of an appropriate test. And then making an appropriate choice of, for example, a sequence to compare with, in case they are using one of the comparison tests. Because they would often go as far as making a wise choice of test but then an unwise choice of series or sequence to compare with. This guess-then-prove approach can be powerful[45]. Often combined with collecting tabular evidence or working on MAPLE for sort of crude technicalities such as considering what the value of the sequence could be for large n – not helping with selecting a test of convergence of course.

RME: Would you expect the students to have a gut feeling on whether the sequence converges to something or not?

M: There is a big number of them who would just find access to these gut feelings difficult – for example making an educated guess about the behaviour of a series or a sequence for large n. Students cling to the idea of looking for something to be worked out and to the objective of all this as somehow being to get a number[46]. To decide whether or not the series or sequence converges is entirely secondary to them. Teaching the art of choice – Integral Calculus is another similar case in point – is a slow and painful process. Getting the students to see some unsuccessful substitutions in Integral Calculus, for example, can be instructive. Try to integrate one over *x logx* by the obvious side calculation and it is horrendous! So you have to consider other more viable scenarios. It is not rocket science but it is important of course to do so with them in the context of what they have seen, of what they are capable of and of what is the kind of mathematics they are good at[47]. Your own familiarity with the subject, your perception of these processes as almost trivial because you yourself have been involved in them for years, should not be part of the equation at all. It is pretty hard actually to unpack and then convey how some of these gut feelings come to be, what information your mind draws on in order to form these intuitions[48]. And I am afraid school mathematics does not emphasise these acts of choice enough. Not only that, but students are normally protected, even turned away, from unsuccessful choices. Thus missing this absolutely crucial development of a feeling for how these things work. Training, achieving that ever-elusive mathematical maturity, through looking at things that do not work is valuable.

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[45] See (Mason et al, 1982). In other episodes M elaborates this idea by expanding on potential uses of IT, numerical experiments and pictures.
[46] For the impact of school mathematics see SE3.1 and (Nardi et al, 2003)
[47] M's statement has a distinct constructivist flavour (von Glasersfeld, 1995). For extensive examples of mathematicians' similar tendencies see also (Nardi et al, 2005).
[48] Leone Burton's interviewed mathematicians (2004, Chapter 5) capture the elusive nature of mathematical intuition and the difficulty of its mediation to students.

RME: Is the choice of N in the definition of convergence one of those crucial acts of choice?

M: At a much more algorithmic level. This almost entirely depends on solving the inequality. There lies more creativity in choosing a test for convergence.

SPECIAL EPISODE SE7.1: TEACHING WITHOUT EXAMPLES[49]

Setting the scene: The following takes place is in the context of the discussion in E7.1, Scene I and starts when M says he would like to raise something in connection with the role of examples in the development of the students' thinking.

M: I have a proposition to make regarding ways in which we can discuss with the students what the implications of $A^2 = 0$ are: not through examples but driving them towards a sense of internal coherence.

RME: And how exactly do you prove that $A^2 = 0$ does not imply $A = 0$, other than showing a non-zero matrix whose square is zero?

M: Exactly by not doing so, I propose. And here is why I think that. This reliance on concrete, practical work has led us where we are today: look at the school situation. I propose pushing the brain structure, wired from childhood to resorting to examples, beyond its hitherto capacity. So, I propose you build up the students' stock of mathematical results only by proving true theorems. At research level, yes, counterexamples are necessary for shielding you from trying to prove a theorem that is in fact false. But in the well researched domains of school mathematics, we know what the facts are, what the true theorems are. So resorting to examples for the purpose of conviction must be developmentally a regressive, not progressive way. Other cultures do so: there are a lot of students out there who have never seen that most bizarre of all constructions, a group table. Which of course you cannot use for anything other than the very small groups, which are not terribly interesting, so what exactly is the use of this concrete representation?

[49] In Dorier et al's (2002) response to Uhlig mentioned in E7.2 they highlight that a major difficulty the students need to cope with at this stage is that most concepts in Linear Algebra are unifying - generalizing concepts. For the students to understand, a step back and reflection is necessary. The authors list their experience of teaching when this student need is considered: they make their case for a teaching method that utilises links of the subject with History and converges towards proofs that highlight conceptual significance (but can be, unlike Uhlig's, more computationally difficult). (Dorier & Sierpinska, 2001) and (Dorier, 2000) elaborate these views further.

RME: I could see many immediate benefits in exploring concrete representations, especially when one is learning about a new concept!

M: Fine. I can see this too as a starting point for good mathematical thoughts. But I can maybe see this as part of the empiricist educational tradition of this country which contrasts with the experiences of visiting students and fellows from, for example, continental countries where tradition is different.

RME: I am always impressed however with the conspicuous consensus regarding student difficulties I experience in international conferences, almost regardless of the traditions you are describing.

M: What I am saying is that for the yet untrained mind of the student to say you cannot prove with examples but you can (dis)prove with counterexamples may contain paradoxical confusion. But then again you may say that this is exactly what we want them to clarify: the logical subtlety of this difference. And that that is paramount to their learning! I guess part (iii) in the question sort of aims at doing that in its own little way, right?

SPECIAL EPISODE SE7.2: DO NOT TEACH INDEFINITE INTEGRATION[50]

Setting the scene: The following takes place in the context of the discussion in E5.3.

RME: But don't you want to use this as an opportunity to start thinking of functions as members of families, as objects?

M: You may as well want to avoid that. I just don't think that there is any need for them. The integral from a to x of x^2 is $a^3 - 3 + x^3/3$ or… whatever. That is just a true statement, valid for all x bigger or smaller than a, if you like a is nought, put a as nought, x could be negative, it is just a true statement. Whereas $\int x^2 dx = x^3/3 + c$ I don't really know what it means. The equal sign there for sure it is not something that I recognise.

RME: But how far can you go avoiding that? What about integration by parts? Or seeing integration in the complex field?

M: There is no problem. Integration by parts is the integral from a to x of … it is a certain formula. If you were taken into a darkened room and asked to explain what the equal sign means in an indefinite integral expression you write down …

[50] See, for example, Thurston's (2005) views according to which Calculus is taught afresh with a new definition of dy and dx as functions and not via f, where y is defined as $f(x)$; replace indefinite integrals with anti-derivatives etc.

it is very hard to relate it to the equal sign in the sense of 3+4=7. By the way let me mention another example of a debatable, by the students, use of =: when writing out a series for $cosz^2$ via replacing z with z^2 in a series for *cosz*.

RME: What about Fourier Series and the notion of = in there? In the twiddle ≈.

M: Allow some fanaticism there. I am rather dismissive of the use of = or twiddle in the definition of the Fourier series. $S_n(f)$ or something along those lines would be sufficient and far less confusing even if under certain conditions the series and the function may coincide. Let me also emphasise the importance of the context you are working in: for physicists the equation applies because *almost everywhere* is good enough for them. The analyst however needs to make the distinction, to highlight the level of subtlety in the expansion denoted by n. Something that denotes that the Taylor expansion of a function, for example, and the function are generally different things coinciding under certain conditions.

RME: No dodgy *dot, dot, dot* for you, heh?

M: No!

RME: I always know which button to press with you!

M: Anyway, all I mean – and I totally understand the need that applied mathematics or an introductory course has for sometimes skipping the consideration of these finer details – is that I am quite keen on an exact association between a function and its Fourier expansion, on saying exactly when these things, if ever, coincide. There are certain dangers in approximately equating these two things. I sometimes like ≡, it's like a wake up call that we are not talking about equality in a conventional sense, but then again the approximation sign may be used in other contexts, therefore confusing the reader (e.g. isomorphism). And, no arrows please, not another blurry use of something that reminds us of the implication sign!

SPECIAL EPISODE SE7.3: TEACHING OF FUNCTIONS, PROCESS – OBJECT, POLYNOMIALS

Setting the scene: The following takes place in the context of the discussion in SE5.2.

M: Judging from the table of these various tutorials, the students are really struggling with what kind of object they are handling at all.

RME: I confess I would have been tempted to ask the students: what do you think $P_3(R)$ is?

M: Actually there is another level to this problem. As this is an Algebra class at this stage to bring in functions is ill-advised. *T* should be the only function here and bringing in functions that need to be operated upon as objects complicates matters. And what is worse is what the connotation little *x* comes with: students are too familiar with it. Capital *X* is better because it has hardly any luggage.

RME: Maybe then the tutor's turn towards looking for polynomials d_i is like such a distancing machine.

M: Possibly. There are I guess alternative powerful approaches that highlight the duality in the concept without obscuring the algebraic objective of the exercise. That put the emphasis on the possibility to express polynomials in terms of different bases; on polynomials as points, as elements of a set; on the connection with R^4; on the fact that, yes, after all polynomials are functions as you have always known them but this is not necessarily the most useful perspective on their nature in this context.

RME: I think some of this ambition runs through the question. You have seen only one part of it but across its several parts the interplay amongst the various contexts, R^n, matrices, polynomials is evident but maybe a bit too ambitious for the question's own good.

M: I guess one way of formulating the question could be: show that the set of all polynomials of degree less than *n* is a vector space of dimension n+1. Thus the students are instructed to think of polynomials as elements of vector spaces. And then you can ask them something along the lines of linear dependence in terms of a basis etc.. Thankfully the students in this example get there.

RME: And I think the tutor's responses across Tutorials 1-4 become more helpful. As the story unfolds, she begins to understand that she has to go back and explore what a polynomial is. And what is exactly what she asks the students to do here.

SPECIAL EPISODE SE7.4: RULES OF ATTRACTION

Setting the Scene: The following takes place in the context of the discussion in SE7.2 and starts with RME's question about how many students in the current cohort are attracted to mathematics for its slight out-of-worldliness.

M: Very few. These days I am often confronted by the very sad fact of students who actually dislike mathematics and admit that they chose it because it will lead them more easily to a good job. But if they are doing that in order to get a job that they may not even like at the end just because it pays a lot, this choice of values is their problem, not ours!

RME: Well, maybe they see studying mathematics as the un-enjoyable way to an enjoyable job. What about graduate students?

M: We have few who in fact have been under the pressure of the you-are-too-good-not-to-do-this argument. Students who were frustrated in Year 1 however often come round later on: the frustration of the first years is significantly decreased by Year 3. It has something to do, I think, with the hard work involved in the beginning and the gratification that results from this hard work later. And the capacity to see the bigger picture. Converted on some road to Damascus! Or maybe our enthusiastic lecturing to the third years on things that are closer to our research interests has an infectious effect. And the overall improvement in their confidence and participation: they are more confident, they will put their hands up and interrupt you. I think they are quite different people. Also the material for grinding is less at this stage than Year 1: like the difference between learning irregular verbs and reading a French play. The joy and beauty of mathematics may not be immediately discernible to beginners. Of course I would love to see more and more students willing to engage with mathematics and its slightly otherworldly ways but it is now almost impossible given the instrumentalisation of school mathematics (SE3.1). I cannot actually make you enjoy something unless you are prepared to. During a lecture where we discussed the argument whether 0.999 recurring is 1, some of them enjoyed this. And usually at the end someone says to me that they are not used to thinking like that. All that happens in school at the moment is: prepare / conduct / recover from exam.

RME: And the beauty of mathematics cannot be conveyed in that environment.

M: It would be nice to hear somehow more about the students' perceptions of the qualitative difference in the mathematics they experience at university, at the end of the first year. Perhaps through an interview. But I haven't searched what the absolute limits are of what the university will let us do.

SPECIAL EPISODE SE7.5: CONTENT COVERAGE[51]

Setting the scene: The following takes place in the context of the discussion in E7.1, Scene V.

RME: Is the frustration you talked about earlier compounded by the fact we are compacting a lot of material in the first year?

M: Absolutely. I always think: how on earth can our students cope? But apart from this we need to prioritise thinking over content knowledge – do you understand what a vector space is? Do you know what continuous means? – and thinking skills like flexibility. Even make this type of questions part of the exam. This is not a trivial task at all. And we need to deal with and the unwillingness of agencies like the LTQ[52] or employers to embrace such initiatives, to understand that it is also to their interest that the students achieve some kind of meaningful understanding and this kind of flexibility.

RME: It seems to me you are outlining a beautiful educational project.

M: To watch LTQ close a department…Or we are so successful that we close down LTQ!

RME: Prove them wrong by having every single student in Year 1 end up with a PhD in pure or applied mathematics! And then all of these PhD holders decide to take up mathematics teaching and change the face of mathematics teaching as we know it!

M: Oh yeah…

OUT-TAKE OT7.1 DOES LEARNING HAPPEN ANYWAY?

Throughout M almost invariably maintains a tough but fair, often optimistic and largely sympathetic attitude towards the students' trials and tribulations. On one occasion however he digresses from this overall attitude (E3.5, Scene I).

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[51] The issue of content coverage is at the forefront of discussion in undergraduate mathematics pedagogy. I was constantly impressed though throughout Studies PD1, PD2 and L by the economy and restraint statements about content coverage were made. I believe that the specificity and focus of the data samples – coupled with the commitment of the participants to work hard towards thoughtful commentary on the samples – diverted them from general statements that can sometimes sound evasive (Nardi et al, 2005).
[52] The Learning and Teaching Quality agency (alluding to the audit culture mentioned in E7.4, Scene VI).

M: I think that there are people who, given an infinite amount of time, will never understand certain mathematical arguments.

RME: It depends on what level of mathematical thinking you are talking about. I surely believe that up to quite a good level of mathematical thinking everything is in principle accessible to everyone.

M: OK, maybe the borderline is somewhere between the proof for the irrationality of $\sqrt{2}$ and the proof for the irrationality of π. I think there you will hit a level at which there are people who will never follow these arguments. But you cannot simply identify and then eliminate obstacles for learners and then, if they do not understand, deduce that you have not eliminated all obstacles.

Moreover there are very few – but too intriguing to ignore – occasions where, despite (and often in the midst of!) extensive discussion of teaching practices that have the potential to help students overcome difficulty, M almost shrugs off the capacity of teaching to have any influence whatsoever on students as they ...learn anyway (either simply on their own resources or as a natural consequence of their day-in, day-out contact with the subject). '... at the end of the day things fall into place and students begin to see the usefulness of these tools' says M (E7.4, Scene V). The primary example of this attitude is M's monologue that concludes E7.1, Scene VI. In that appears the metaphor of 'students like toddlers falling over' (also in E4.1, Scene I cited below). M appears to be in some awe of the complexity of the learning process but, ultimately, acknowledges the facilitating role that his actions can play.

M: I never give up completely on a student. They are like toddlers falling over. They fall over and they fall over and all of a sudden they demonstrate some sort of strength and understanding and they come up with something very good. I don't know what it is that suddenly helps them but I insist I have seen it happening way too many times to ignore this. Some sort of digestion? Making connections between things they initially perceive as disjoint. I often think of this non-linear process of assembling understood items in the same way as the process by which you begin to put together the pieces of a puzzle. Definitely this does not work in a linear way: you understand this, and then you understand this, then you do this, and so on. It doesn't work like that.[53]

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[53] Nardi et al (2005), in their four-level *Spectrum of Pedagogical* Awareness, characterise incidents of a similar attitude as Level I and observe that in the course of lecturers' engagement with pedagogical reflection these incidents tend to become less frequent, even fade away. In any case 'students' learning *should not* be accidental…' [my emphasis], write Marton et al (2004, p331, as quoted in (Jaworski, 2004, p27)). 'Teachers' opportunities to learn are a key factor affecting […] practice' (p331), they add. Jaworski counts 'developing teaching through inquiry' (p27) as one of these opportunities. The research M is participating is aimed to be as one such type of inquiry.

CHAPTER 8
FRAGILE, YET CRUCIAL: THE RELATIONSHIP BETWEEN MATHEMATICIANS AND RESEARCHERS IN MATHEMATICS EDUCATION[1]

As I mentioned in Chapter 1, Part (ii), the idea of a conversation between a mathematician and a researcher in mathematics education – usually on general matters of teaching, learning and research rather than a scrutiny of the type of data M and RME have been discussing in the preceding chapters – is not new. Anna Sfard (1998a), for example, discussed such matters with mathematician Shimshon A. Amitsur. I return to some of the views Amitsur expressed in that interview in several occasions in this chapter in order to encourage the reader to compare and contrast Amitsur's views with those expressed by M[2]. In several of the works referred to in this chapter – of which strong influences on this work have been those by Michèle Artigue (1998), Anthony Ralston (2004), Gerry Goldin (2003) and Anna Sfard (1998b) – it is not hard to discern the fragility of the relationship between the communities of mathematics education research and mathematics.

In what follows[3] M and RME discuss this fragile relationship while repeatedly acknowledging it as also crucial. M cites the benefits for pedagogical practice ensuing from using the findings and recommendations of educational research (E8.1, Scene I) and from engagement with educational research (E8.1, Scene II). M also reflects on, and often critiques, the practices of RME (E8.2): how RME is done (Scene Ia) and how it could be done (Scene Ib); how RME theory is being built (Scene II); how RME is written up (Scene III); and, how it is being disseminated (Scene IV). Finally M and RME acknowledge the stereotypical perceptions of mathematics, mathematicians and educational research that tantalise their relationship (E8.2, Scene IV; SE8.1).

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[1] For a preliminary presentation of M's views in this chapter see (Nardi & Iannone, 2004 and 2007).

[2] who of course expressed his views in the course – and on the tail – of participating in a piece of educational research for a quite lengthy period of time.

[3] Even though the dialogues have been collated from material from across the data, their main bulk comes from the final meta-cycle of data collection (Study L). This aimed to elicit participant perspectives on the above relationship through discussing the experience of participating in the study and through examining examples of RME literature. One of these examples was the Moore (1994) diagram on proof; another was (Nardi & Iannone, 2003a), a PME27 Research Report (based on the interview data that are included in E7.3). The latter was accompanied by the anonymised three reviews that PME Research Reports customarily receive.

EPISODE 8.1
BENEFITS

Scene I : Benefits from using mathematics education research

Setting the scene: The following takes place in the midst of the discussion in E8.2, Scene Ia. There the focus is on M's critique of RME's choices of methodology and how these choices, mostly for qualitative inquiries – because of their unfamiliarity – may alienate M from engaging with research that he generally feels methodologically distant from. RME suggests a distinction between *engaging with* such research (which is discussed in Scene II of this Episode) and *using* its outcomes. She then invites M's comments on the latter:

RME: As a potential user of this research, as a practitioner, would you find any use in such research[4]?

M: I would be interested in the recommendations at the end, I think. When the collation of all of our data is over, the interest in them would be only in terms of whether they offer any good ideas on how to overcome the problems. The solutions!

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[4] Amitsur, in his interview with Sfard (1998a), describes how his interest in the field started when he realised teachers had so many different ways of thinking about mathematics. He wants mathematics education research to try and help teachers cope with the diversity of their students' ways of thinking mathematically instead of engaging in labyrinthine theorising. Teachers are important as students are mirrors of their teachers. In the 60s he was involved with a curriculum change initiative (negotiated but ultimately rather imposed on teachers; its activities not systematically recorded) because he believes in trying to change teacher practice from within and with a consideration of educational contexts. In the 70s he found most of the published work in mathematics education too *ad hoc* and uninteresting. He is interested in whole-system work, not details. Regarding more current work he is disappointed with studies that report certain learning phenomena and then, instead of probing further, they move on to hasty recommendations for repair-and-teach-better, thus disregarding that perhaps these phenomena are natural and must be part of the teaching. Regarding what aims mathematics education research must have: theoretical aims (explorations of the different ways people think mathematically) and practical aims (supporting teachers' practice). Aims RME should not have: for example, deficit studies on student ability! Content decisions must be grounded on what type of mathematical thinking and what type of skills we wish learners to acquire. Because of certain mathematics education research findings in the 1970s that suggested it was unfeasible to teach mathematical thinking via Geometry the subject was subsided or removed. The validity of this claim is today very debatable, he concludes, and states that the aims of research in mathematics education research should be to improve the teaching of mathematics by developing learners' skills in calculation, logical thinking and problem solving; and, to cultivate the learner's development of intuition which is imperative in moving forward in mathematics. He also seems to propagate that reform in mathematics education should be carried by mathematicians because mathematics education researchers and teachers are often consumed by fear. The latter typically resist change and the former are not providing helpful answers. When Sfard reminds him of the quasi-fiasco that was New Math, an approach mostly instigated by mathematicians, he retorts that in any case a mathematician is capable more than anyone else to say what constitutes mathematics (for example proof) and what is not (in some cases visual arguments) and therefore what needs to be included in learners' mathematical experiences...!

RME: Well, there must be some value in articulating the problem. This surely must be a good start.

M: Sure but beyond generic teacher training there is a domain specific need to understand these problems.

RME: The research we have been engaged in here is in a sense basic: identify the problem, possibly explain it and then proceed with making potentially useful recommendations. So far we have not conducted any experiments such as the ones you are proposing.

M: You must understand that my background drives my preference towards the experimental method but my emphasis is clearly on identifying sources of difficulty, helping students overcome difficulty, boosting confidence, achieving learning. I have attended events where these issues are being touched upon but for me there is a need to translate those into specific recommendations for mathematics. I can give you an example of a forum where these things would certainly make a difference: [names a UK university's training course for new lecturers] if it were not bogged down to epitomizing the worst aspects of professional education by being content-less[5]. If I had gone to one of those

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[5] Falconer (2006, p2), in the London Mathematical Society Newsletter, reported 'strong dissatisfaction with the widespread reliance on generic methods and learning theory that is often inappropriate to mathematics' in the training for new lecturers. Friedberg (2005) also raises a similar issue and reports some positive outcomes from an innovative approach through a discussion of Case Studies on teaching and learning (as written up in (Friedberg et al, 2001)). For how this preparation takes place in different countries see various chapters in (Holton, 2001): for example, Marc Legrand (2001b) reports on the *Centres d' Initiation à l' Enseignement Supérieur* founded in 1989 in France. In these, prospective lecturers are called *moniteurs*. Despite a risk that such a training plays only a symbolic role the sessions aim at fostering reflective practice and at highlighting the relevance of educational theory. CIES training aspires to go beyond superficial prescription and to allow the exploration of complexity but it faces several issues: moniteurs often appear willing to embrace reform, initially. Soon however, in their desire to be accepted and promoted within the system, they regress to attitudes they initially resented. The programme addresses above resistance and contradictions through sessions on: the nature of a lecturer's job; pedagogical theory (e.g. *epistemological obstacles*, *didactical contract* etc.); didactics (e.g. analysing teaching situations from three perspectives: student, teacher, teacher educator), etc.. Training, concludes Legrand, must maintain the balance between empowering the new professionals, encouraging them to identify and use relevant theory, illuminating practice and fostering self-reflection that has a chance to survive friction with a conservative system. John Mason (2001) also reports on the foundation of the ILT (Institute for Learning and Teaching) in 1999, following recommendations in the Dearing Report (1997), in the UK. Membership to the ILT is dependent upon response, via nominated referees, to a number of questions regarding: teaching and the support of learning; contribution to the design and planning of learning activities; assessment and feedback to students; developing effective learning environments and student learning support systems; reflective practice and professional development. What is available to new lecturers is rather generic. Issues that need urgent consideration include: that mathematicians realise the need to engage more in pedagogical debate; that promotion in universities considers excellence in teaching more; that mathematicians see engagement with teaching as having a far from adverse effect on their research capacity; that mathematicians accept that the epistemologies of the two fields are different; that the relevant educational literature, currently growing with a pace that makes it unattractively big for mathematicians, is concisely 'rewritten' for this audience.

things and someone had said: about the ε-δ definition of continuity or the definition of a group or a specific thing in Chemistry or had talked in these terms about it I would have been fully attentive and I would have got something from it. So… it is not that the audience is not there, and somehow those are the kind of places in which this kind of research should be disseminated.

RME: I agree. The audience is there because the problem (of teaching, of recruitment etc.) is there.

M: Absolutely! But you know what I mean? There is exactly that difference between the specific, the meaningful and the… I don't know … and the vague!

RME: I agree that courses like [the one mentioned before] can be seen as useless because of this contentless-ness. And it is often related to tenure or the end of a new lecturer's probationary period so you have practically no choice but to attend them…

M: And I am not saying that talking across contents of the various disciplines cannot be illuminating. Even if these disciplines employ different educational or psychological terms small glossaries could facilitate the exchange amongst interlocutors from different disciplines. And some of these things, I am sure, translate across disciplines even though not all brick walls to progress would be the same for all disciplines. In fact I am sure mathematics has some of its very own brick walls!

Scene II: Benefits from engagement with mathematics education research

Setting the scene: The following comment was offered by M in the midst of the discussion in E8.2, Scene III where the focus is on the communicational gaps between the two communities, largely, according to M's critique there, due to the 'indecipherability' of the texts in which RME outcomes are disseminated. In the midst of what is often in that Scene a scathing critique M contrasts the futility and irrelevance of those texts with the potent experience of participating in the discussions for the purposes of this study:

M: May I say that it is in these discussions exactly that these sessions have proved enormously valuable already. There are things I will teach differently. There are things that I feel like I understand better of mathematics students than I did before. And I appreciate the questioning aspects of the discussion and I realise how one should be liaising with the other lecturers simultaneously lecturing the students and discussing what things we are doing that confuse them.

The following[6] takes place at the end of E6.1. Chronologically the discussion of the data in that Scene took place in the very first cycle of the study's data collection. This is worth noting as it is evidence that M was beginning to have these meta-comments on the experience of participating in the study from so early on.

M: Well, I would like to say that these discussions are already beginning to influence the way I think about my teaching.

RME: In what ways?

M: I think discussing the examples is a very good starting point, and a well-structured one. By seeing these often terrifying pieces of writing I am faced with

[6] In this scene the focus is mostly on the benefits of M's engagement with educational research in terms of the impact this engagement has on raising his pedagogical awareness and sensitivity. It is worth reading this Scene in the light of a historical account of M's previous, and often turbulent, involvement in mathematics education. I find the account on this matter by Michèle Artigue (1998) particularly perceptive. Artigue (1998) – an RME who is inside the world of M and is active in mathematics – is taking a dialectic perspective on the relationship between M and RME as something continually evolving and is placing M's relationship with RME in the wider spectrum of M's relationship with mathematics education. She outlines M's influence on mathematics education in terms of what and how is taught. She uses the case of secondary mathematics in France to map out several landmarks in M's involvement:
The 1902 Reform, a strong and rewarding reform: an approach to Calculus that aimed to balance intuition with rigour and include applications. It seems to have brought about a more comprehensive understanding by the students of the time. And: *the New Math Reform*, a strong but not rewarding reform. Structuralism and formalism took over and results were disappointing. By the early 1970s the realisation that a more comprehensive, systemic approach to the enterprise of mathematics teaching, one that involves an understanding of the complexity of the situation and does not take cue only from epistemological concerns and objectives, was necessary (e.g. Chevallard's (1985) *transposition didactique*).
Artigue highlights several consequences of the New Math disappointment as follows: the distancing between ME (e.g. teachers) and M, thankfully somehow compensated (in France) by the launch of IREMs (Institut de Recherche en l'Enseignement Mathématique); the consolidation of the RME community that realised that the enterprise of mathematics teaching is something to be taken on by specialists and not as a marginal activity of M; and, a restricting, as a result, of the role of M in mathematics education matters.

Ralston (2004) also considers the relationship between M and RME, particularly in the context of a tension between teachers and researchers of mathematics (in terms of 'traditional' vs 'reform' or 'progressive' mathematics teaching) that became known in the USA as the Math Wars and has been intensified since the early 1990s and an upsurge of involvement of mathematicians in mathematics education. Looking at the strengths and weaknesses of this involvement he acknowledges M's contribution in terms of: advice on topics to include in the curriculum; errors in the publications; methods for teaching some topics; support for the in/pre-service teacher education. In terms of weaknesses he is less impressed by: mathematicians' often arrogant and aggressive attitudes (with regard to unfamiliar to them epistemologies of RME and with regard to the mathematical credentials of mathematics education researchers); mathematicians' assigning unjustifiably high credit to test scores as evidence of worsening / improving standards of mathematics education in schools; the treatment of IT in school mathematics with unnecessary and ill-informed suspicion; mathematicians' suffering from the One Right Answer Syndrome (resulting in accusations that reform mathematics curricula promote images of mathematics as a fuzzy subject). He concludes with a recommendation for humility and constructive engagement from now on... I believe M's tenor in this Scene reflects this recommendation.

the harsh reality of the extent of the students' difficulties. Too often I see colleagues who are in denial and opportunities like this are poignant reality checks! In most European countries we carry on about the value of conceptual understanding, especially in juxtaposition to the instrumentalism of the American educational system but we do not necessarily do our best to foster it. I have seen evidence of students' mathematical thinking from a variety of institutions, from average all the way to right down excellent, and, even in more advanced years, it is often shocking. I am therefore grateful for this opportunity to face the music, so to speak.

In that instant M concludes with volunteering an idea of how these discussions can be taken further and engage him even more (E8.2, Scene Ib). He returns to the significance of participating in this type of research at the end of E8.2, Scene III – the following takes place soon after M has expounded the excitement of engaging in the particular type of data this study has given him access to. It is the type of writing he claims there that can render the communication between the two communities easier.

M: Look, this is a significant enterprise. There is substance in this; it is important. Suppose you have a schoolteacher. So, here is someone who has to run classes and for some reason or another their view of mathematics is no other than an instrumental one: you apply this rule, you put this in and you get this out. Suppose that such a person one day meets Concept Image and all that. All of a sudden he learns that these things are all out there and that changes that person's professional view entirely. It can change the whole classroom, it can change the whole mathematical process. That is precisely what we want. A lot of the problems you have to deal with when you meet our students is that they have a very singular view of mathematics, a rather poor view of mathematics. So, I mean, that sort of debate that is happening here is on some of the building blocks around which, it seems to me, if made available at the school level for practitioners, would be hugely interesting. To get away from this sort of mathematics which is, you know, quite poor in a way.

While praising the subtlety of the data discussed in the study (E8.2, Scene Ia) M also discusses how he has been changing in any way due to this experience.

M: I think now I don't have any more answers than when I started but certainly I don't take things for granted anymore, from colleagues or from students. I think I am much more open-minded on what might be going on inside other people brains. The material that you have got here has given the evidence that sure, it is fascinating glancing in other people's heads. And I have become much more conscious about the spoken word. What I say can have an impact, saying the right thing at the right time when you get one opportunity to introduce the students for the first time to how mathematics works and not fluff the line. That I think has made a big influence on the way I lecture. Some of us may choose to introduce silences into a lecture while writing on the board, to help build up to a sentence. Taking a long conscious silence and coming up with a gem of a sentence! There are at least two extremes: cautious prose and full sentences is one of them. Others would choose the other extreme: not allow any silence at all; gibber while writing things on the board. Also examining these pieces of data was something of a reminder, if not a revelation, of the devastating importance of detailed responses to written work[7]. In some sense every not totally perfect piece of written work has an interesting important story to tell that needs to be engaged with and responded to.

Finally on another part of the same Scene, around where he describes the institutional resistance of mathematics departments to creating space for listening to what RME has to say he offers the following:

M: Also, to make for you more concrete the value I see in participating in a study like this[8], I would say that in mathematics we got people who, without question, have a lot of knowledge but the problem, the intellectual challenge for me is, when I am teaching, converting what I know into communicating that to students. To me it is a real challenge to try to get to the fourth different way to explain to the student some idea. That is never routine to me, the imagination and effort it takes. Maybe this is another thing that is never written down or I have never been conscious of until coming to these meetings… And I think that what I find interesting is trying to get a better discipline in understanding that process of communication with the students. I think we should take an interest in this because obviously we are spending a lot of effort on things that aren't working. The routine milling that we have been doing since the middle ages probably doesn't produce students who have a broad view of mathematics as an exciting intellectual challenge. I think we can do this much better and I think we can start by mathematicians talking to each other about their own prejudices, and I know I have plenty, and scaling things up a bit. Getting to see how each one of us understands the subject is useful: and communicating our own misconceptions amongst each other as mathematicians before having a go at innocent students!

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[7] Mason (2002, Chapter 5) stresses: *never* allocate uncommented marks - it is not useful at all! Consulting with the other markers about the clarity of the marking scheme, establishing an atmosphere of mutual support and exchanging scripts are ways to cope with the potential isolation of the marking role. He also recommends engaging students in the marking process too (e.g. *Tactics: Using group conveners* and *Collaborative work*) for the sake of their own (*and* the teacher's) learning. With regard to providing feedback to students in ways that reflect as accurately as possible the student's needs Mason elaborates the following Tactics: *Focusing on what is mathematical; Developing a language; Finding something positive to say; Selecting what to mark; Summarising your observations;* and, *Providing a list of common errors or a 'corrected' sample of student argument.*

[8] As Artigue (1998) notes '...we, mathematicians as well as didacticians […] have to act energetically in order to create the positive synergy between our respective competences which is necessary for a real improvement of mathematical education, both at secondary and at tertiary levels. Obviously such a positive synergy is not easy to create and is strongly dependent on the quality of the relationship between mathematicians and didacticians'. (p482/3)

EPISODE 8.2
REFLECTION AND CRITIQUE OF THE PRACTICES OF RME

Scene Ia: There is something about the way you ... Do Research (an evaluation of Qualitative Inquiry and conditions under which it could work for mathematicians[9])

Setting the scene: The main bone of contention in the suspicion, even hostility often characterising the relationship between mathematicians and researchers in mathematics education is the substantially different epistemologies of the two communities. Unsurprisingly M initially hints at a preference for generalisable (statistically valid experiment-based) and readily applicable (stripped off the detachment from immediate applicability that comes with theorising) pedagogical recommendations. In the course of participating in the study a new and …methodologically reconstructed M seems to emerge[10].

[9] Artigue (1998) writes about these conditions: despite achieving acceptability as an academic discipline RME's 'privileged links with the mathematics community remain fragile' (p483). She then makes a case for issues that need immediate tackling:

- why RME needs to stay as close to M as possible (e.g RMEs need to be Ms);
- the suspicion and isolation of M generated by the explosion of theories in RME that methodologically and epistemologically M cannot understand or follow;
- the didactician = sub-mathematician syndrome M suffers from;
- the increasing alienation of M and student needs in contemporary culture leading M to seek help from RME, help that RME has been unable to offer (she lists Brousseau's criteria for RME output such as relevance, immediacy, freedom from jargon etc on p484)
- RME often exposes us to the weaknesses of our teaching and our complicity in the malfunctioning and ineffectiveness of the educational system; it can therefore be unpleasant, even disturbing and thus ignored or disregard by M. In the process M may also disregard the complexity of learning issues and RME's often convoluted recommendations may annoy M.
- RME needs to engage more systematically with disseminating its outputs to less specialist, but interested like M, audiences. RME is essentially a field that is both basic and applied.

[10] In his interview with Sfard (1998a) Amitsur expressed concerns on educational research methods that are often similar to those expressed here by M: they often verge on the unsound (statistics), the unethical (do the researchers make sure that the students get something learning-wise out of participating in the study?) or the inconsiderate towards bias. Mathematics education research will only become an academic discipline, he claims, if it manages to provide 'proof' (explanatory propositions) of what constitutes mathematical thinking. The individual stories that mathematics education research usually provides are too far away from the general picture about the whole population. Studies of misconceptions and errors are mostly culpable of this partial and inconclusive reporting, for example: we do not need to know what mathematics education research considers unlearnable; it doesn't help the cause of mathematics education. Regarding research methods mathematics education research needs to develop the equivalent that the sciences have for controllability. This is difficult as there are way too many parameters to consider and mathematics education experiments are not easily (or at all) replicable. M and RME can work together towards the above described goals but never-ever should their methods be confused: M is concerned with abstract ideas, RME with human beings. In this Scene M speaks in ways that may have the potential to alleviate some of these concerns.

Here RME witnesses this emergence[11].

M: As a mathematician I want mathematics education papers to follow a scientific manner: citing verifiable results, differences in performance that various teaching methods can generate. For example I am interested in clarifying intentions and achieved aims and observing the differences between them. I believe that to achieve anything in terms of learning, your aims must be absolutely clear.

RME: I believe there is potential for exactly that in the types of data we have been discussing: identify the lecturer's / question setter's intentions, juxtapose these intentions with the ways in which the students responded to a particular question

[11] Like Ralston (2004) Goldin's (2003) positioning on the conditions for a collaboration between M and RME draws on the events of the Math Wars in the USA. His thesis on the fundamental divide between mathematics and mathematics education research is that the gap is primarily epistemological (2003) – crucially this thesis comes from someone who has been steering a course between the two communities for thirty-plus years. His thesis is that the divide can be mostly attributed to a victim-of-fashion, ultra-relativist paradigms that dominate mathematics education research. RME's tendency to 'deny or dismiss the integrity of fundamental aspects of knowledge in mathematics and science' (p174) may imply that, unless this attitude is severely curtailed, the divide may never be bridged. Goldin then assigns different kinds of integrity to knowledge the two fields generate: the power and generality of mathematical and scientific knowledge cannot be easily dismissed by RME's various –isms, he claims; and, the achievements of educational knowledge that have moved educational discourse beyond simplistic views on content, performance assessment and teaching methods often held by mathematicians cannot be dismissed by their often unnecessary suspicion. He concludes with a call for a more truth-intensive type of mathematics education research.

Sfard (1998b) also acknowledges epistemological differences: these differences can be traced in the ways in which the two fields have drifted apart in the last 30 to 40 years (during which mathematics education research started coming to its own as a field). RME's perspective is closer to social science rather than science, she proposes. Both have reasons to explore and clarify these differences. Consider, for example, the question 'what is mathematics?' and consider the term 'visualisation'. This as an illustrative example of an area where the differences between M's and RME's epistemological perspectives on what mathematics is can become more than evident. These differences reflect also differences on 'basic epistemological beliefs and general conception of human knowledge' (p494). This emanating from 'different paradigms is likely to make their views on mathematics incommensurable rather than merely differing in some points'(p494).

As the few studies in the area (see some references in Chapter 1, Part II) suggest M's perspective remains more or less (crypto)-Platonist. RME's perspective however has gradually drifted away from this perspective (a rough sketch of this journey away from Platonism would have to include the influence of Constructivism, Social Constructivism, Post-Structuralism and Situated Cognition). But, given this difference / incommensurability, can M and RME learn to live with these differences and work together towards solving the vital problems of mathematics education? (A question also asked – followed by genuinely noble attempts at an answer! - by authors of several articles in the *Notices of the American Mathematical Society* (e.g. Sultan & Arzt, 2005; Ball et al, 2005). Sfard's call for a step forward is slightly different to Goldin's (who seems to assign more responsibility to RME than M for bridging the gap): M's views must become significantly more inclusive of the constantly evolving, complex and contemporary understandings of how mathematical learning takes place. And mathematics education research needs to study more extensively and more systematically M's ways of thinking as they can be substantial and illuminating for RME's cause.

and identify whether these intentions materialised as well as explore what the students' perceptions were of the lecturer's intentions.

M: This is certainly important at least in terms of helping to modify one's practice as a teacher.

RME: I would be cautious in saying that this can be implied readily as different participants may suggest different points in every question as meriting special attention so, by implication, any hypothetical reform of practice could take many different directions.

M: As I said it is important to have clear aims, to evaluate the learning outcomes and of course to link these aims and outcomes to assessment.

RME: Let me ask a methodological question: our data is of a more qualitative nature. Would you find these data illuminating with regard to the same research purposes you outlined earlier?

M: Yes, but as a mathematician I wouldn't be able to emulate this style of sociological / psychological research on such interactions. However as a *user of this type of research…*

M lists several of these uses (E8.1, Scene I) and the conversation then turns towards a deep-seated epistemological concern of M. M raises this concern on the occasion of recounting the experience of attending a PME conference:

M: I have little experience of mathematics education conferences, but the one occasion [names PME session] I seriously attempted attendance was a thoroughly discouraging experience. There was this curious argument between [names prominent member of the field] and someone in the audience and the tenor and the content of the argument left me thinking I was wrong to attend that talk and right to miss all the others. Which is unfair… but you know what I mean? This is just nonsense. This is egotistical nonsense. And I thought of somehow connecting what I was hearing with what I was doing in a lecture theatre or in a seminar group… but, to do that, first I would want them to kind of not be arguing about what they are meaning. If they don't know, I am not going to pay attention to what they are saying because they should know. And they seemed to be arguing about what they meant which was unfortunate…

RME: How come you picked this particular session? Because of the tertiary level topic?

M: It was also a timetabling thing. So it was a semi-random choice.

RME: And a rather unfortunate one! Even more unfortunate because the members of that group are mostly mathematicians who have shifted towards mathematics education and are more likely to be doing things that are directly relevant to the learning and teaching of mathematics at the undergraduate level. You know I sometimes feel there is a sort of class distinction within mathematics education between the people who are mathematicians and the people who are not. The people who are not mathematicians have a kind of disregard for the other ones because they think they don't know enough about educational psychology, students' needs, pedagogy. And the former accuse the latter of not knowing the mathematics in the first place. And often mathematics educators are guilty of being judgemental towards mathematicians and accusing them of not doing the job properly. This is a discourse that in this study we have deliberately aimed at staying clear of. The question of course of how I treat all this naturalistic material I am collecting, which often resembles a stream of consciousness, and turn it into something useful, remains as pertinent as ever. I have a note from Barbara Jaworski in which she comments on the PME27 paper I gave you to read [discussed in E8.2, Scene IV] on the issue of formulating a 'mutual agenda': 'I wondered to what extent the mathematicians contribute to the analysis or consider it when it is written'. We, as mathematics educators, can plan and follow a path of action that is efficient in terms of our needs, careers etc. But how about the mathematicians' side of things?

M: Merely participating in these discussions has proved enormously valuable already.

M lists benefits to pedagogical practice emanating from engaging with educational research (E8.1, Scene II). He then turns to an obstacle in the relationship between M and RME: the institutional resistance to allowing educational research to influence reform of practice at university level:

M: But there is another difficulty in the relationship you outlined: this is a department with a commitment to the idea that research driven tertiary education is the only and best way to teach mathematics, to the terribly arrogant idea that we are the holders of the knowledge of how tertiary mathematics should be taught and to the idea that universities should not be giving degrees in mathematics to people if they don't have active research faculty. Why? Why do we think that is a bad thing? And we do think it is a bad thing…This makes attempt at reform extremely difficult.

He then turns to a more philosophical evaluation of his participation in educational research.

M: For me participating in an educational study is an occasion to explore the beliefs I already hold as well as talk about the beliefs I already hold. It is not necessarily about questioning the beliefs I have but maybe just making me even more efficient at having them. In fact I appreciate the richness and diversity of views exchanged in the duration of the study and I do not necessarily think we need to come up with one right one.

M then returns to elaborating the above mentioned institutional resistance:

M: And, while I appreciate this opportunity, it is still the case that the image of a mathematics department that pays a lot of attention and contributes to research in mathematics education would be poor from other mathematicians' point of view: the mathematics community does not in its bulk look to this type of research as a source of knowledge or ideas about mathematics teaching. It just doesn't… whether it should or not is a different matter of course.

– as well as how exploring the naturalistic material made to him available during this study refined his capacity to reflect on students' learning needs and on his own practice (E8.1, Scene II).

M: This is the right starting point [before engaging with reform of practice]; it is starting at the right end: what mathematicians can say to each other when all the guards are down[12]. Also this is a good starting point because it is very mathematically specific, which is not always the case in your field. But still mathematicians would never read anything like that, I am afraid I have to say. Given our priorities somehow mathematicians are not engaged in the discourse that this is. Or putting in another way you have this volume of data, who is the intended audience?

RME: Yes, that is a crucial question. I have at least two audiences in mind: researchers in mathematics education and mathematicians. And I know I need to find ways to communicate to both audiences. Generally I think that both of us have a lot to learn from talking to each other. I certainly know that as a mathematics educator I have masses to learn from what mathematicians do. Not the least because of the educational chain reaction that is taking place here: some mathematicians will become teachers etc., etc. As for a mutually appealing and beneficial focus of the research I believe that a tight focus on the psychology of mathematical learning is crucial. I understand so much better what the students are doing when I know how you, who is teaching mathematics, is producing mathematics, and so on are doing all of those things. To me it is very clear why I find this an incredibly useful and productive exercise. Mind you this implies the need for substantial numbers in the field who can play this dual role.

[12] For the benefits (e.g. relaxed, reflective and unthreatening environment) of this method of collecting data (focused group interviews) see (Madriz, 2001) and an account of how it was used in this particular study see (Iannone & Nardi, 2005a).

M: The material you describe would certainly be useful. Another one would be a longitudinal study[13] of a single student, one that would focus on what in fact are the conceptual hurdles students must overcome, what happens to them. As tutors, we all have an intuitive feeling on what goes or what doesn't go. You look at these students, you look at their faces, you know that they are lost. This project is like is a platform by which we start to rationalize. And this platform will inevitably include some terminology, some new concepts that will open up the discussion.

RME: I am pleased to hear this openness towards the language of mathematics education – resistance to this is more often than not the case – and your willingness to engage with all this. It can be very frustrating you know to engage with this type of work and the people you are mostly hoping to address remain unaware, indifferent or even hostile to it.

M offers new lecturers training courses as at least one place where this work can make a difference (E8.1, Scene I), comments on the necessity of some educational jargon (Scene III) and turns to commenting on the qualitative data the study offered him access to:

M: May I add a compliment to the type of data we have been looking at?

RME: Indulge me!

M: I remember distinctly the excitement of my first encounter with some of the transcripts and the samples of student writing – the excitement and the interest that kept me reading that came from the fact that these are real people struggling with a real difficulty, in real time. And I think that anyone coming to this new, not having seen a piece of data like this before, would not only find that interesting personally, but also because of the way in which these data would click with their own experiences. So we must never lose sight of the value in this type of data – unusual for me but vastly revealing[14].

RME: I must admit that initially I had to nurse fears that the debate in our meetings would never go anywhere near the subtlety you have suggested and, instead, stagnate in discussing marks and right/wrong answers. And… we haven't!

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[13] an interesting suggestion as many AMT studies in the field take a snapshot approach (due to convenience, finance etc). Margaret Brown and others (e.g. Brown 1998) have expressed concerns about the repercussions this has on the quality of RME's outputs.

[14] The shift from the preference expressed in the beginning of the Episode is remarkable. Engagement, indeed regular and un-diverted immersion into naturalistic data is a potent way to achieve this shift of perspective (Greenwood & Levin, 2001).

M: Exploring why things are in a certain way is what is incredibly valuable about what is coming out of this. It is really a confirmation that somehow the quantitative story about learning is devoid of meaning. A tick box of the average understanding of the concept of group among the students is a figure that has absolutely no meaning at all in it, in comparison with exploring the individual detailed discourse of each student – exploring what is happening in this student's brain. It is different from what is happening in every other student's brain, and there are similar hurdles and I think that defining the hurdles is important. But the whole quantitative side of research of this sort, you know, on reflection of all what we have gone through, it makes me think that it is all nonsense, really…

RME: I recall examples from the data where we would have sixteen students responding in sixteen different ways…

M: …and this diversity is actually a wonderful thing, from the lecturer's point of view. You really know that they are doing it themselves, they are thinking about it. Not necessarily the case if all responses are the same! But may I return to something I mentioned earlier, regarding the mathematical specificity of the samples we have been discussing?

RME: Please do.

M: To be honest I think I should confess some prejudice. I admit that your being a mathematician alleviated some initial suspicion. Because right from the start we can talk having this common ground, the actual understanding of the mathematical concepts we are talking about. And I think that if I had serious anxiety that there was no common ground that it would have rapidly started to feel this could be a waste of time; or that you would end up having nothing interesting to say – or interesting for me. To me the fact that we both understood what we were talking about made a huge difference.

RME: I have heard this a lot actually.

M: And to be honest if people want to find open doors in a mathematics department they need to be able to talk to mathematicians about mathematics[15], and if they can't maybe they are in the wrong business. Maybe they should be in education, not mathematics education. Having said all this, let me stress that I absolutely mean to be friendly towards the community of mathematics educators and have a more exploratory rather than absolutist spirit when discussing issues of teaching and learning. But, as I said, practically not being able to understand the

[15] Ronald Brown (1998) suggests that, apart from gaining credibility with mathematicians, mathematics education researchers who stay in touch with the subject are likely to maintain a more vivid sense of what the encounter with mathematics feels like and are thus in a good position to develop empathy with the learners their work is intended to support. See also Artigue's (1998) chapter quoted earlier in this Episode for a similar position on this matter.

mathematics around which the conversation on the students' learning etc revolves can be a huge barrier.

RME: I agree that such understanding and ability to converse in mathematical terms in a sense is a minimum requirement for such conversation to have some meaning. In fact the high degree of mathematical content in our papers has meant that warnings have to be issued for the less mathematically confident in the mathematics education community, e.g. when we are presenting at conferences. So the problem goes both ways.

[The discussion appears to take a turn towards a different issue, that of gender representation in the study and in the mathematics community generally, but soon returns to M's evaluative comments on the qualitative research methods used in the study]

RME: Can I raise another issue now? The issue of gender representation amongst participants in the study. A colleague asked me how many female mathematicians are participating … ooops! When I said *none* she said *you seem to be quite sort of happy working with them. Don't you think they have a lot to answer for?*

M: Yes. Some more than others!

RME: As this colleague noted, one might say you cannot not-address the gender issue at this level of mathematics education given the terrible numbers of women in the profession, especially at its higher echelons. This colleague said to me rather dramatically that she seriously doubted whether there is any point in examining other aspects of undergraduate mathematics education (I suspect she meant the thinking / learning / understanding oriented studies like the one we have been involved with here) if such basic issues have not yet been properly addressed.

M: Of course, but it is important to bring questions, not answers to the table. The latter makes the conversation far less interesting. I agree there may be interesting aspects to the gender issue that merit exploration: for example, is there any qualitative research evidence of differences in the ways male and female mathematics students experience undergraduate mathematics that are not socially driven? Performance-wise female mathematics undergraduates are doing extremely well.

RME: Yes, there are social issues beyond the strictly cognitive, such as *do women and men understand things in the same way?* Or, *do they respond differently to how they have been taught?* And these involve issues that have to do with things happening before a woman even reaches the stage of becoming a lecturer in mathematics, things she has gone through that are influenced by social barriers, life choices etc. The issue of gender is much wider than simply an identification of cognitive differences.

M: … in which differences I would include a noticeable difference in the approach female and male undergraduates have with regard to asking for help. Female students tend to be more verbal, more willing to talk to you and to explain better and be more up-front about what they don't understand. Unlike their male counterparts who are less willing to show any such vulnerability. In any case, yes, all these issues are here. But the idea that we can simultaneously look at all these issues is dubious. What would worry me about this person that spoke to you is the danger of smudging all these differences and different things together. And smudging together the effect of all the many, conceivably very many, different things under the same banner would be worrying. I prefer a more focused, clinical approach.

RME: Perhaps I haven't found shut doors so far in my research because I am looking at the apparently less controversial cognitive issues and not so much at the 'hot' social ones like the gender representation in mathematics faculties. I think that there is something relatively safe in the kind of questions that I have been asking you. And I haven't yet asked you to consider acting upon the problems we discussed either. At this stage I prefer to take the safer route because it is leading me towards an understanding of how people involved in an undergraduate mathematics course are thinking. For me this is the starting point.

M: Exploring how people involved in the teaching of mathematics at university level are thinking is a starting point that is an absolutely necessary predecessor to any discussion about reform of practice. We have been talking about mathematical content and learning / teaching issues covered across a period of time with as little as possible change on how we view them. If we were sort of on the fly during the process changing how, for example, we presented certain concepts, what we would be saying would change very rapidly and we would be somehow observing something, while participating in it … Possibly this could be to the more immediate benefit of our current undergraduates but not to the benefit of understanding what is going on. I think it would be like trying to hit a moving target.

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RME: That is an extremely interesting comment which I believe we will have the opportunity to elaborate while looking at the three reviews that the paper I gave you to read has received (SE8.1)..

M: Oh, I see. That would be fantastic!

In a vein similar to the above discussion M highlights another methodological characteristic he finds appealing in this type of educational research. He offered the following comment in the midst of the discussion of issues of sampling participants for this type of study in Scene 1b – in response to RME's following question:

RME: … a question I have often been asked about this project 'don't you think that your sampling is biased? Because people have accepted participating they are therefore people who, in a sense, are already interested in pedagogical issues'. And I understand that and I understand that maybe there is another group of mathematicians that we really need to talk to and with whom we haven't established the connection yet.

M: I think there is a clear distinction here between how you are trying to effect change or how you are trying to study something[16]. And I think to me there is still something that is very refreshing about the whole thing you have been trying to do which actually has elements of control where control is appropriate, and un-control where control is inappropriate. For example, the mathematicians you have been collaborating with are not a random sample but nonetheless we have been quite disciplined about the specificity, mathematical and other, of what questions we are asking and what answers we are seeking. I think we have stayed clear of the unhelpful smudging together of parameters that your naturalistic data in sometimes in danger of pushing us towards. I would hate this if it was all vague and unfocused.

RME: I am pleased to hear you didn't think it was!

M expands on ideas about types of educational studies he would be interested in (Scene Ib) and concludes with a comment on the originality within the experience of participating in this type of study:

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M: Overall I must say that I find your idea of looking at these pieces of students' writing very original. And we have made it possible because we have grown towards trusting each other and trust has a lot to do with the success of this[17].

RME: Thank you. I find your comment very encouraging.

Scene Ib: …and other ways you could be doing it!

Setting the scene: At the discussion cited within E8.1, Scene II (and following from the discussion in E6.1), M is beginning to show an interest in extensions, elaborations etc of the ways the project could go:

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[16] M's comment here reflects the classical methodological distinction between exploration, interpretation and intervention often discussed in the context of Action Research (Greenwood & Levin, 2001)

[17] …an almost painful truth also stressed by McCallum (2003) and most of the authors quoted in this chapter.

M. …maybe at some point I could bring along examples that I think merit discussion myself. Or even the students themselves! Maybe we can ask them to participate and comment on other students' work. Do you think that's possible?

On another occasion, in the midst of reflecting on potential uses of the paper discussed in E8.2, Scene IV, he suggests further engagement of students in the study:

M: I actually believe that students should have access to the things we do in this project (e.g. by putting this paper on the website).

RME: Perhaps through seminars to students once analysis is over as the paper would appear a little bit out of the blue … or even this book…!

M: What a perfect role reversal, of exposing this to the students, having lecturers thinking hard about what they are understanding when we write determinants on the board! They may be surprised to discover the depth this type of conversation can have!

RME: I am happy with involving them but let's be cautious about making them part of the process at this late stage. I am sure there are ways of engaging them early enough in the process.

Finally M devotes considerable thought to types of educational research he would like to see more of. The following takes place soon after the discussion in E8.2, Scene IV is over.

M: There is so much more to be looked at in terms of students' thinking than simply looking at their writing in these pieces of homework. And so qualitatively meaningful as well. I hesitate to raise this question but there are vast sources of data within the department in the form of exam papers. I wonder whether it is possible to use these papers as data. Possibly administrators would not be happy with the prospect of doing so and would come up with many technical reasons for this! Still course work, as we have done here, might be more negotiable with individual students in terms of using it as data. And an agreement with individual students that their coursework and their exam papers would be copied through the whole year to be used as data is not out of the question either with timely organisation of this. I even wonder whether we could find students who have kept all of their course work. Or at least some percentage of their course work. Looking at the work of individuals as it develops across the year could be, I believe, illuminating.

M then offers the comment that concludes Scene Ia. RME thanks him and returns to his suggestions for the future.

RME: In the longitudinal study you propose I would add other sources of data such as interviews with the lecturers, observation of the lectures and certainly interviews with the students.

M: But even at a smaller scale, just looking at the coursework would be quite something. Say, the students who are now graduating and would perhaps be prepared to share their work and their thoughts with us. Of course the sampling for this kind of exercise would be by necessity iffy, right? Only talk to the ones who are willing to talk to us!

RME: There is no need to worry about this. We even have a term for this in qualitative methodology, Opportunistic Sampling, and let me remind you that it happened in this project too!

M: Granted!

The discussion switches briefly to the particular sample of participants in this study (towards the end of Scene Ia[18]) but M soon returns to discussing the engagement of students.

M: Back to recruiting students in future projects may I suggest emailing students to explain the aims of the project and invite participation? I guess written consent, especially when it comes to consenting to photocopying their work would be essential at the recruitment stage but as a first stage of approaching them perhaps this could work.

RME: I cannot recall ever having problems with students about this type of consent but, yes, seeking their written consent is an ethical requirement.

M: Good. And in a sense we could simply seek their agreement and file away these copies for whenever we need them for research purposes, even for a small number of students and for across the three years of the course. I guess also some type of monetary compensation, in line with what you are offering me as a Participant's Reward[19] can be used to involve them, especially if they agree to spend some time helping us. Would rewarding them be ethically controversial? I would certainly offer my reward towards experimenting with this idea!

RME: Your commitment is touching! Of course it is totally up to you to spend your reward in any way you like and that is why I have budgeted this in the

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[18] … where the drawback of bias (participants being volunteers) is compensated for by benefits such as the enthusiasm of those committed to the study.

[19] Participants were offered some financial reward for their participation. This was intended more as a gesture of acknowledgement and gratitude for their commitment rather than a fiscally accurate remuneration for their time investment.

project in this non-committal way. Just let me know how you wish to spend it and I will divert the fund in the right direction!

M: Thank you. One way to use the reward would be mathematics education books to be placed in the departmental library for future reference! I think this would also be smart tax-wise!

Scene II: There is something about the way you… Theorise (Or: On the Moore diagram)

Setting the scene: In the context of the discussion of students' enactment of proving techniques (discussed in E3.5, E3.6 and E3.7) M had been offered a copy of the Robert Moore (1994) diagram on students' difficulties with mathematical proof. RME's intention was to use this diagram as a concrete context in which M would express his views on the theorising practices (and outcomes of this theorising) of mathematics education research. The following takes place immediately after the completion of the discussion in E3.7.

RME: May I invite your comments on this attempt to illustrate student difficulty with mathematical proof, a diagram by Robert Moore?

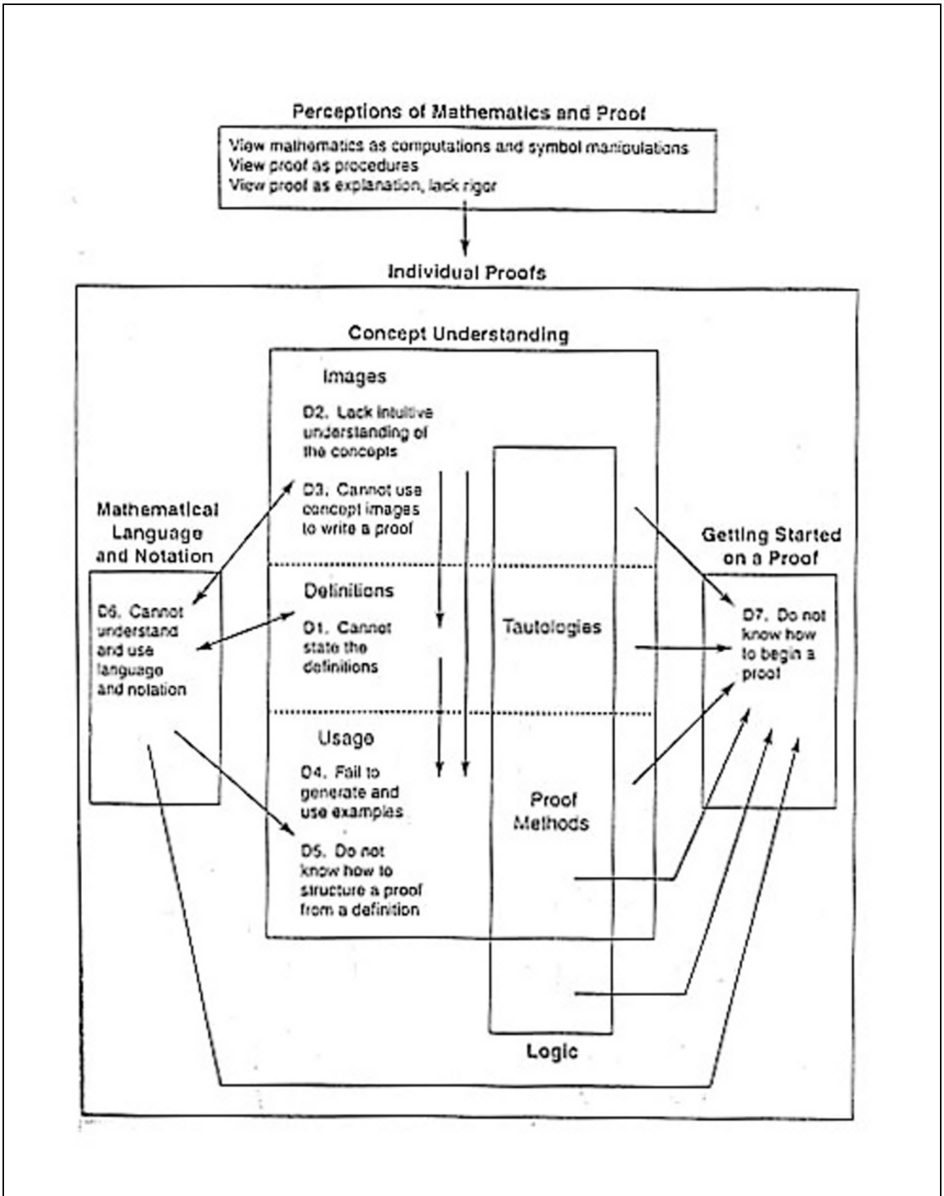
M: I have quite a few queries here: is there some natural pathway through this? Is this a flowchart? I am struggling with this diagram, I am not exactly a fan of flowcharts.

RME: I am not sure it is a flowchart.

M: There seem to be some sets identified. I am living in the hope that they are disjoint sets of misunderstanding. I think they are. So I guess that the thing on the right hand side is a place where a lot of students end up not producing course work and there might be lots of origins for that position. Do the boxes suggest disjoint areas of difficulty?

RME: There seems to be like a big box around *concept understanding* and that is where the images, definitions and usage come in.

M: So it is like the students would be hopping around between these different difficulties and not getting anywhere. But they do know how to do some things like in the top one *perceptions of mathematical proof*. They could do some stuff if they could only get started. I am not so sure that this is accurate, that our students normally have these problems. They might get stuck in the middle of this picture, say when they have not only one definition but two.



Model of the major sources of students' difficulties in doing proofs (Moore, 1994)

RME: Like the Mathematical Induction examples we have discussed (E3.6) where they get stuck with inequalities, right? This was the case where they probably have a general idea of what the tool offered them but they got stuck implementing the particular thing from algebra that they needed.

M: I think the guy who produced this diagram is very optimistic – he is stunned by the bewildering array of misunderstanding. Can we interpret the examples we have just discussed in terms of this diagram?

RME: That's a good question. A colleague produces a model. We collect data and ask how or whether this model actually fits any of the things that we see in the classroom or the tutorials and so on. Do you see any way of linking the mathematical thinking of the students as we saw it unfolding in these examples and the model presented in this diagram?

M: Does this diagram cover all the possible reasons students may have problems with proof? Why does someone run fast or slow, or not at all? There could be thousands of reasons for this and I am not sure any such model can point at all of these. And where would all the exceptional cases of all sorts go. I would struggle to get much use from this. And what do these arrows, say between *tautologies, proof methods and logic*, mean? That *tautologies* are somehow linked to *definitions*? Or that definitions are tautologies? This diagram is unclear in many ways. Is it legitimate, for example, to distinguish proof and formal proof as two different animals? What is then informal proof? Would there have to be a box about the student not understanding what is to be proved? Or maybe this is covered in the *language, notation* box. Am I being too insulting towards this by the way?

RME: No, carry on.

M: Well, may I ask then: does this diagram completely miss the complexity of mathematical thinking? To me this smacks of something that I find incredibly frustrating. That there are people who do very little to get across how difficult some arguments in mathematics just are. So somewhere else, actually, you need an idea that in fact, not so much in first year of mathematics but in school at some level, the idea of this proof and the steps needed might be just incredibly sophisticated. Just beyond, you know – I don't see the word *hard* here anywhere!

RME: … unlike the Barnard and Tall (1997) one we examined earlier (E3.5, Scene I)? Where the complexity of the situation was more satisfactorily portrayed maybe?

M: Yes, slightly, but yes! That diagram was far more meaningful, I think. It reflected complexity and in years to come we may even have a huge one on the irrationality of π and π+e too! The proofs for these do not fit in this diagram at all and to me this means that this diagram is very limited in its scope. I know I sound a bit harsh and I fully acknowledge the difficulty of trying to map out student difficulty: in fact I could find a resemblance between this effort and the effort made by an undergraduate trying to cope with the syllabus – an act itself I often think of as learning by imitation of expert practice. This is another element missing from the diagram even though the author is probably, but a bit surprisingly, a mathematician. I am sorry but this diagram is not presenting me with a thesis about learning and for a scientist, say an economist, not having a thesis after having done a lot of work is inconceivable.

RME: Well, this is a diagram that reflects how students see proof in Year 1, not how it is generally…

M: I still see little use or interest in it.

RME: Well, I would expect experienced people like you would probably come up with a more comprehensive list of student difficulties with proof than any diagram?

M: Allow me to doubt this. As Tolstoy said: all happy families are alike but unhappy families are unique[20].

RME: You are not saying that research in mathematics education is impossible?

M: Very difficult for sure, especially if your underlying belief is that anybody can understand anything in mathematics. Returning to the Moore diagram, I would conclude by saying this is a reasonable, if not comprehensive, attempt at portraying the issues.

RME: Well, I am not necessarily saying it has been the key that opened all the doors in my analyses but there are certainly parts of the diagram that ring very true, if not exhaustively so, when I examine my data. Like *not able to start a proof* or not *stating definitions properly*. These are things I encounter in student work all the time.

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[20] Opening line of *Anna Karenina* (Tolstoy, 1877/2003). As Mason (2002) wonders towards the end of his book on undergraduate mathematics teaching practice, *are there any theorems in mathematics education*? Given the diversity of contingencies in a teaching-learning situation and the idiosyncrasies of human nature (a comment that brings back to mind the *tacking* metaphor from the beginning pages of the book, see E7.1) they do not – at least not in the strictly contemporary sense of the word (expressing a generality etc.). But in its original sense, the Greek *therein* (*θεωρείν*, a looking, a way of seeing) theorems in mathematics education do exist.

M: I think I would happily stick with my earlier analogy between thinking mathematically and thinking about the learning of mathematics. Through refinement of your arguments, your proofs become better and better. Same for models like these. You need to constantly revisit them and refine them until they reach a satisfactory level or representing the situation you want to represent.

RME: I think I would be keen to discuss with you at some point some of the theoretical tools we use in order to interpret our data in mathematics education. We have seen numerous examples of student work but I think it is now time we discuss some of the theoretical artillery of mathematics education!

M: Sure.

Scene III: There is something about the way you… Write up

Setting the scene: The following takes place immediately after Scene II. Each dataset M was given during data collection focused on a theme on student learning selected by RME on the evidence of relevant bibliography and her previous work. Some of this bibliography was succinctly outlined in each dataset (see Chapter 2, Part (i)). The discussion starts with RME inviting M's comments on these outlines.

M: I have hardly paid any attention to be honest. I think I would have wished for more specific examples to explain the dense, jargon laden text.

RME: Would you say that the writing up of mathematics education results in this way enlarges the gap between mathematics educators and mathematicians?

M: Well, I can see an analogy there between mathematicians and their undergraduates and the way the latter relate to mathematical texts!

RME: In what format would you find the presentation of mathematics education research results readable?

M: In total frankness, the noble cause of involving both communities cannot be served by a presentation that is almost indecipherable. Because, you know, mathematics education uses another language and it takes a vast amount of time to, I don't know, split the sentences and understand the meaning of this or that. Because I have not read that much from this discipline. And I do not mean this as a personal comment about either your writing or my reading ability!

RME: If this is the case, if there is no common language, then the aim of communication and collaboration will never be achieved. And I have noticed that throughout the project the one section of the dataset you have devoted next to no discussion is the literature reviews at the end. While, significantly, at the same time you have keenly engaged in lengthy discussions of particular student responses.

M: Back to your earlier comment on resistance to jargon can I add that I understand that at this level of sophistication some jargon is inevitable because there are subtle concepts involved? Engaging with thoughts expressed in different languages is part of our academic life, for God's sake! And we do it automatically for other subjects in science, in engineering etc.

RME: So it's only fair to ask mathematicians to do it when engaged in mathematics education.

M: The problem is getting mathematicians interested in reading mathematics education. I actually think that trying to understand mathematics education is actually a fun exercise, in the same way in which you can read a political essay and so on and you can take a certain pleasure just from that!

Scene IV: There is something about the way you…Disseminate

Setting the scene: Having discussed five datasets on issues of student learning over several months, in the final meeting RME invites M to comment on a paper submitted and accepted for presentation at PME27.

RME: And here we are, our final meeting … Can I first thank you for sticking with these meetings? Your commitment throughout never ceased to amaze me and I must say the study has already been welcomed by colleagues elsewhere too.

M: The great secret is to use mathematicians!

RME: Aah! OK, this final meeting is different as there is no student work to discuss directly but a first attempt at a paper we wrote only a month into the project and which, happily, has been accepted for presentation at PME27. I would love to hear your views about it!

M: OK, I will have a go. What I found surprising from a personal point of view is the central extract that you have in the paper, Figure One, reading what I recall are my own words… I think I have identified who I am…despite your anonymising!

RME: I am not surprised. The anonymity in any case is for protecting you from others, not yourself!

M: The fact that we have got transcripts in front of us of what we actually said is an entirely new medium to me. I guess I am typical in the sense that we view mathematics as something that is definitely written down and anything that is spoken is kind of transitory or motivating material or… Things you might say in lectures for example, that stuff that gets everybody going while the raw meat of it is actually what you write on the board or what you have on the overhead or as a handout. And reading this paper made me realize that I have a very biased upbringing and I am probably continuing like this… What is written on paper and the fact that is sacred… almost. So it is interesting to see all of us who participated in the study thinking about mathematics and trying to build in the gaps between what we are saying and … what is going on around this table… Of course I think it is impossible for a transcript to mirror exactly each individual's reactions to everything that was being said. Perhaps we need some notation between the chunks of spoken vibes and the transcript here indicating what kind of tension there is in the room … There is in other words great novelty in seeing one's own words on a transcript. I realized the substantial differences between speaking about and writing mathematics, especially in a teaching context. And I wonder whether a transcript accurately and richly enough reflects the tone, intensity etc of a conversation.

RME: I agree that the transcript offers no clues as to the inflection of the speaker's voice, tone etc. There are more exact ways to code a transcript in order to evoke inflection etc but I find them tiresome and hard to read.

M: Of course and then students' thought is even more inaccessible than mathematicians.

RME: Perhaps. May I now invite your critical comments on the paper?

M: In what sense do you use the term Focus Group? It rings some eerie bells involving politicians and spin and all that!

RME: Oh no – let me clarify: it is used in its research-methods, not its media sense!

M: Thanks! I think the paper teaches you more about us, I think, than … how our students learn about determinants. Is this because it's from a very early stage of the project[21]?

RME: Well, we chose this excerpt for quite exceptional reasons, with the limitations of space in mind more than anything else: in the hundreds of thousands of transcribed words there is ample material where the students are discussed. But in a PME paper there is no way a mathematical question, its

[21] The data for the paper did indeed come from the first cycle of data collection.

answer, student responses, transcripts of teachers commenting on those, analysis and relevant literature to fit in. One thing I guess you don't have in mathematics conferences' papers is a page limit, right?

M: Depends… but we don't publish much in conferences anyway….

RME: So in sum we had to find something which was short enough and provided a flavour of the discussions taking place in the course of the study to readers who know nothing about the project. In that case talking about one isolated mathematical concept is more convenient than a whole mathematical question.

M: May I say I think the project is quite amazing. I imagine there are very few research projects that end up with this kind of data?

RME: Originality of the data is the one thing not missing here, isn't it? …of course given that the data has given us anything worth remembering afterwards! Of which I can already think of dozens!

M: Really? Good!

RME: So the excerpt is indeed on mathematicians talking about a concept and less about their students' learning but for very good reasons. Still pedagogical discussion is underlying the whole excerpt and one can always place the emphasis strongly enough on these during a conference presentation.

M: Yes. These do come across. You see we certainly do not wish to perpetuate the stereotype of self-indulgent, navel-gazing, white, Anglo-Saxon mathematicians with no concern whatsoever for the predicament of their students! Students? What students? All they can talk is *det(A)* this and *det(A)* that…Otherwise I am confident that the paper would make an interesting read for colleagues in other places who have their own experiences of students' work and would project them on the discussion presented in the paper.

RME: I would hope so. Thank you.

M: Can I discuss briefly the use of the word 'landscape' in the title of the paper[22]: I wouldn't like views of a concept to be seen as something static: I see these views being in constant flux. I am not sure 'landscape' does that well.

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[22] The paper was entitled *Mathematicians on concept-image construction: single 'landscape' vs 'your own tailor-made brain version'*.

RME: I agree that we need to maintain that dynamic and fluid character in the words we use to describe the concept. I can personally see a landscape as having that fluidity so I am not worried about the use of this metaphor!

M: OK, good. As long as the dichotomy in the paper is not misleading the reader…I think I know who I am in the paper by the way. We all know who uses which words like 'tricky' and 'what the heck'!

RME: You do? Well, the paper aimed to contrast not the views of the individuals who used the words in the title but to place the notion of uniqueness versus that notion of multiplicity and diversity.

M: I am indeed impressed by the bewildering variety of different reasoning even amongst practicing mathematicians coming across the paper. And I think that the readers will also be able to bring some of their own experience to reading this.

RME: There is another layer of interest in the title: mathematicians talking about concept images?! Alluding to the originality of the project that managed to gather mathematicians in such a systematic and subtle inquiry into students' mathematical thinking? This collective commitment is an achievement to celebrate, I insist. So many mathematics educators have found closed doors…
By the way would you like to tell me a bit more about how you relate to mathematics educators?

M: Well, I routinely converse with mathematicians but mathematics educators? I believe you are the first I have ever talked to! And as far as I know the ratio of mathematicians within PME is not that impressive either.

RME: There is a small but substantial number. In any case I cannot see how a non-mathematician would even begin to get to grips with the data we have been discussing here.

M: Sure. Actually there is a regular column in the *Notices* of the AMS on teaching so I guess there is another publication outlet you could consider. I guess the people who write in this column think of mathematicians as interested in mathematics education as well!

RME: Would you ever read a paper like our PME paper?

M: I would never come across it in one of our journals – perhaps the *Mathematical Gazette* where there are occasionally contributions to teaching both at school and sometimes undergraduate level. Or the *Notices* I mentioned above.

RME: So the mechanics of the problem seems to be that both mathematicians and mathematics educators need to publish in and read journals in their own areas (e.g. for RAE purposes[23]) and there is precious little time for reading each other's journals. The two worlds don't meet a lot… unfortunately…. I think this is at the heart of the problem.

M: Also, we mathematicians are more likely to read pedagogically thoughtful books than journals say Pólya, Stewart & Tall[24] and the like.

M then turns to commenting on mathematics education conferences as another source of learning about the outcomes of mathematics education research. His comment on PME conferences in E8.2, Scene Ia becomes the starting point of a discussion of deeper epistemological differences between the two communities.

SPECIAL EPISODE 8.1:
THE REVIEWS

Setting the scene: The following takes place towards the end of E8.2, Scene Ia. RME presents M with the three reviews that the PME27 paper received. In the process the discussion escalates towards an exchange of views on how both communities perceive mathematicians, mathematics and educational research.

RME starts with outlining the review process.

RME: The drill is more or less as follows… Every year papers are submitted to the conference programme committee by mid-January. The committee then allocates three reviewers according to a list of categories that reflect the areas and research methods of the field. If the paper attains three *accepted* verdicts then it is accepted; if it attains three *rejected* verdicts is rejected; if it attains 2:1 or 1:2 verdicts the committee considers reviewing it once more. This is a delicate and sometimes painfully difficult process. The paper you read attained three *Accept* verdicts. This is a verdict received by a minority of the submitted papers as generally consensus of this type is rare amongst researchers in mathematics education. I am pleased of course. I think there is, to say the least, something about the data that captured the reviewers' imagination and please accept my thanks for this! Shall we look at the reviews one at a time?

M: Yes, fine. By the way is it difficult for the committee to get enough referees? You must need a vast number of them *and* with a very wide range of expertise.

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[23] As Court (2005) observes, in the UK, with the 2008 Research Assessment Exercise (a national exercise during which grades to universities are assigned to university departments according to the quality of their research output) looming (at the time of writing), academics are discouraged from publishing practice / policy related articles and are pressurised to publish in highly regarded academic journals. This pressure is likely to widen the gap between M and RME as it may steer their energies away from the difficult task of disseminating to multiple communities. Multiple-purpose research outputs (like this book!) tread dangerous waters...

[24] M here refers to (Pólya, 1945) and (Stewart & Tall, 1977)

RME: Indeed. Furthermore in recent years the number of submissions has escalated[25] and the organization is struggling to keep the number of papers sent to each reviewer to a maximum of four or five. And as we said we need three reviewers for each paper!

M: Very hard work.

RME: Plus typically reviewers need themselves to have had at least two papers accepted at the conference before they are invited to become reviewers – even though the committee may accept some exceptions to this rule. Reviews are, of course, anonymous. I understand the intentions behind this choice even though sometimes the urge to respond to certain annoying comments makes me think otherwise about it! By the way reviewers are asked to comment on the following:

1. The theoretical framework and related literature
2. The methodology (if appropriate)
3. The statements and discussion of results
4. Clarity
5. Relevance to PME audience
6. Detailed reasons for recommendation and general comments (e.g. suggestions for presentation if recommended for acceptance or clear reasons for rejection if recommended for rejection).

The six criteria for PME reviewers

RME: So, there we go, here is the first review[26]. Please ignore the fact that there is an unfinished sentence in there!

1. While the title and the analysis refer to concept images, this does not seem to be part of the [sic]
2. Very clearly described
3. The text of the data was hard to read, and the problem was not clearly enough described for the reader to understand the interpretation. It is, however, all there.
4. Well written
5. Relevant
6. The paper seem to have potential interest and I am happy for it to be accepted. It is a very interesting idea and the data and the interpretation are innovative. In reality the framework does not match the data. The problem and its discussion are difficult to follow and the conclusion something of a leap. The author has bent the rules of font size, I actually recommend the paper not to be included in the proceedings as it is. The above issues should be addressed in the presentation.

Review 1

M: They seem to have some problem with the interpretation or the data in the figure.

[25] from about 270 in 2002 to about 400 in 2006!

[26] What follows is the typed version of the three reviews. The text has not been edited for grammar etc..

RME: So this reviewer seems to be interested in what we are doing in terms of data and methodology…but…

M: … has some trouble with the framework. What do they mean by *framework* by the way? Perhaps that the title does not fully reflect the issues developed in the paper? Somehow the title suggests a debate between "tailor-made brain versions" and "single landscapes" while in fact the paper is about much more (E8.2, Scene IV). That is just one thing that emerges, but there is a lot else in there. I think that the Abstract describes much better the paper than the title does. And while "The problem and its discussion are difficult to follow" is a bit harsh the overall stance of the reviewer is positive: 'interesting', 'innovative'?

RME: Well, they did accept the paper and the comments, however harsh, should be used to enhance the presentation. That 'difficult to follow' issue can be a serious one though. We need to be careful at the presentation stage and in later write ups.

M: I am also less happy about the 'leap of a conclusion' remark in the review when in fact the authors have clearly gone for an explicit absence of a conclusion.

RME: I am assuming they mean the shift from the very particular to the general that happens in the concluding paragraphs.

M: But it is exactly this open-endedness that is quite impressive, may I say? Leaving aside the fact that you obviously ran out of space…! [laughs]

RME: At the conference we have 20 minutes for presentation and then 20 for discussion. So an open ending to the paper in fact works well as an opener for the discussion part of the presentation at the conference itself.

M: Interesting. In which case you could change your concluding words to questions in order to initiate discussion.

RME: Exactly. Rather than present fixed claims. So here is the second review.

1. The theoretical framework and the related literature are appropriate. In particular there is a good balance between the literature necessary to state the problem of research and the literature referring to the means of investigation.
2. The methodology is appropriate, both in the way of gathering the data and of analyzing them.
3. I note that the references are written in such a small caractere [sic] that in the printed version will not be possible to read them. The available space was not used in an efficient way.
4. The paper is really clearly written.
5. The theme is interesting for the PME audience and it offers hints for further developments.
6. N/A

Review 2

M: That's quite positive I think. But not mind-blowingly insightful… By the way may I say that the limited use of LaTex by mathematics educators could be another obstacle in the communication between the two communities?

RME: I accept that mathematically heavy text looks massively better on LaTex but a substantial number of texts in our field is not mathematically heavy and they look much better in Word. In any case with PDF an acceptable format of submission for the camera ready copies this is much less of a problem from now on.

M: The ugliness, instability, cumbersomeness and heaviness of equation editors in some software such as Word is beyond description. Let alone the problems of compatibility between different versions of Word. Argh… I have an endless list of anecdotes on these matters. LaTex is clearly superior for mathematical writing but cannot be used as it is for Powerpoint presentations unless you PDF the file and use it as slides – but not animated. Returning to the review, I believe this referee doesn't understand the process that is involved in this.

RME: I agrees it is a disappointing review. It is insubstantial.

M: At least they acknowledged the paper is very clearly written…

RME: Here is the third review.

1. The work reported here is part of a larger project devoted to understanding how mathematicians think about mathematics—the underlying assumption being that they think differently from the hoi polloi. (A hidden agenda seems to be to get them involved in math education research.) So this work engaged a small group of mathematicians (from seven universities in the UK) and they discussed their perceptions, attitudes, beliefs, emotions, etc on pre-chosen topics. In this case, on the notion of the determinate of a matrix. The idea was to determine their concept image of a determinate.
2. Mathematicians discussed their impressions of what a determinate is, and various notions defined on it. For example the adjoint of a matrix is the transpose of the matrix of co-factors. A major theorem in linear algebra is that $A^{-1} = (1/\det A)(\mathrm{adj}A)$, and provides a nice way to get the inverse of a matrix. Or is it more than that? Listening to the mathematicians discuss their thoughts on this gave the researchers insights into how they were thinking about this notion. Another notion discussed was whether or not they them selves thought of a matrix in terms of its applications. (E.g., for example the volume of certain three dimensional bodies can be thought of in terms of 3x3 matrices. The advantages and disadvantages of thinking of them in this way were discussed and then the steered by the researchers into such perceptions can mean to the students.)
3. The researchers obtained the concept images of the mathematicians. What happens next is not discussed – and this I find disturbing.
4. I found this paper to be understandable, but not so clearly written. The Themes are not clear as to what the authors are trying to accomplish. A few sentences about each theme might have helped. The cycles of data collection were particularly confusing to me.
5. There will be many at PME who will be interested in this paper. I believe that there is an error in grammer in 'An example from cycle 1'. I believe it should read: 'In this part of the recording, Part III, the group has been discussing a Linear Algebra question…' And not '…the group have been discussing a Linear Algebra question…'
6. I recommend that this paper **BE ACCEPTED** for presentation at PME.

Review 3

M: I am impressed by the size of this. The reviewer has actually read and engaged with the paper. But I find the 'hidden agenda' comment odd – I wish to return to this at some point – and I don't like their use of 'hoi polloi' and 'determinate'.

RME: Please bear in mind that reviewers are likely to be non-native speakers of English…

M: Still is 'I find disturbing' a fair comment?

RME: It is a bit strong but it expresses concern.

M: 'Concerning' would have been better.

RME: Well, most reviewers seem to want the whole world in a paper of eight pages! Yet these are the rules of the game we have agreed to play by so… And I do think there are things that can be said in eight pages and it's up to us the writers of this to find those things!

M: This referee wants some nice, clear conclusions and it is very, very clear in the paper that that is not the purpose.

RME: Of course a reviewer can ask for anything! And this one has asked for more on many aspects of methodology etc. Some reviewers simply do not keep a realistic hold on what they ask for. Still it is good to know what they are missing in the paper so that we can address these on further write ups of this or at the conference. Even though we cannot fix this now – as I said the paper has gone to the printers in its initially submitted form, as PME wishes to prepare proceedings for distribution to the delegates on arrival. They can read, select and prepare for attending sessions that for most of them are in a different language – there are 60-odd countries participating in PME after all. I think it is a kind of give and take – give up some of the presentational standards for the proceedings so you can have them ready on time for the conference to facilitate the participation of non native speakers – that I find fair.

M: Sure but then the role of the referee is peculiar to say the least. You might as well just have accept or reject, without other comments.

RME: Well, apart from the obvious role of sifting through all the variable quality submissions for the conference and doing so with some justification – imagine having your work accepted or, particularly, rejected without explanation! – the reviews do feed into the presentation and subsequent forms of life of the paper. In qualitative research there is also a strong need for peer validation: a lot of times I have taken papers to different directions because of what referees or people at conferences were saying and so on.

M: I see. It carries on the dialogue, it encourages openness.

RME: And of course referees have a lot of power in terms of deciding who goes to the conference and who doesn't given the rules in many education departments about covering expenses only when you present. Which, for me, makes the second review harder to respect given that the reviewer does not seem to engage with the paper very much. Still I personally overall take them very seriously. What do you think I should be careful about given these reviews?

M: There are several things in this that are very, very peculiar. In particular the comment: "the underlying assumption being that they think differently from the hoi polloi". Of course! These are mathematicians we are talking about, they are the people who actually teach mathematics at this level, so are they not automatically relevant or think differently from the hoi polloi? It think the comment is nonsense.

RME: But it does say something about some of the attitudes towards mathematicians, no?

M: Oh yes! It is just amazing, this idea: what are you doing talking to mathematicians?! And the reviewer seems to have a strangely static view of what mathematics is. To them mathematics seems to exist and then everybody can talk about it, which is a completely different way to how most of us would think of mathematics. In fact I think that in section 2 of the review somehow the whole plot is lost. "Mathematicians discuss their impressions of what a determinant is."'... No! They discuss their view of somehow what happens when you try to get across in a useful way what a determinant is. And if that is the same as 'their impressions of what a determinant is', then the discourse that I believe is mathematics education is gone out of the window. These two things are not identical! Then look at section 3: it is just horrendous. "The researchers obtained the concept images of the mathematicians."'! This is so odd.

RME: I think it is an odd notion of research … surgery-like: open and examine the brain, take some pieces away and write about them!

M: That sort of … surgical image implied in the sentence is hilarious…

RME: You said you wanted to return to the 'hidden agenda' comment in the review?

M: … which by the way is not hidden at all! The paper says clearly it aims to engage mathematicians in educational research!

RME: I am afraid, even though I appreciate this reviewer's engagement with the paper in detail, the underlying issue may be their suspicion towards collaboration with mathematicians.

M: Overall I get the impression that PME reviews are usually very reasonable especially when one needs to make a case for rejection. Acceptance reviews are always bound to be less interesting, even bland.

RME: Yes, you feel ethically inclined to make a sharper case for rejection than acceptance.

M: Even though not possible within the PME framework, a rejection review can function as a trigger to improve an interesting but under-materialised piece of work.

RME: And in any case PME papers are only glimpses into the world of mathematics education but good starting points. One should always read the extended journal papers of the same authors even though referring to PME papers is often convenient.

M: The reviewer does find the paper highly relevant in section 5 of the review but still seems baffled as to why you would want to talk to mathematicians! In any case congratulations!

RME: Thank you. Thank you also for the comments and the typographical work. Would you consider co-authoring papers for conferences with me and even joining me for attendance and presentation?

M: I would seriously consider this. Let's check diaries. Any ideas of what we would do?

RME: There is so much to choose from. And, by the way , let's make sure we cobble together some funding for this![27]

[27] Since the completion of Study L two conference papers based on data collected in its duration have been co-authored with one of the participating mathematicians who 'became' M. For the work with M following Study L see Epilogue.

EPILOGUE

What isn't and what can / will be…

There are so many things that the dialogues between M and RME are not about; so many angles and issues on the learning and teaching of undergraduate mathematics that have been left out – some might say glaringly so.

Take *Affect*: there are *some* references to building student confidence but M and RME hardly pursue affective matters to the extent these matters certainly merit. For example the advocating of interactive / participatory pedagogical practices they discuss in Chapter 7 is far more grounded on the beneficial influence these practices may have on improving student learning rather than overtly addressing their potential as emotional motivators.

Or take *Gender*, a thorny issue in so many respects in the context of university mathematics: despite *some* comments in Chapter 8 – for example, the distinction between issues that concern the different ways in which male and female students respond to the challenges of university mathematics and more socio-cultural issues that concern the worrying levels of recruitment and retention of women in mathematical careers[1] – M and RME soon return to discussing methodological issues concerning mathematics education research.

I could offer more examples (*Equity* – the influence of socioeconomic, racial etc background on access and success in mathematics – would be another obvious one). However I will agree with what M described as this particular collaboration steering clear of the 'smudging' effect on our discourse that a simultaneous consideration of all of these issues could have (E8.2, Scene Ia). Focussing on what I would describe as the nitty-gritty of the business of learning and teaching mathematics at university, throwing a magnifying lens on it and inviting M to reflect on it relentlessly, I aspired to meet at least two objectives:

- obviously, listen to what M, a very experienced learner, doer and teacher of mathematics, may have to say about this 'business';
- less obviously, through my scrutiny of the data, allow a certain image of M to emerge, one that contains evidence of pedagogical awareness, perceptiveness and sensitivity.

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[1] In Chapters 3 – 8 I have followed Sfard's (1998b, p508) convention, RME as a She, M as a He, also for a reason of (sobering) verisimilitude: that there was one female participant in Study L (the study on which the dialogues have largely been based). There were two in Study D, one in Studies PD1 and PD2. Study N did not involve lecturers; also across the studies students were deliberately chosen in almost equal halves. Furthermore the mathematics education researchers involved in the studies were almost all female, for example in Study L Iannone and I (and, a bit more obliquely and a bit less confidently, the choice may also resonate with Sultan & Arzt's (2005) provocatively titled *Mathematicians Are From Mars, Math Educators Are From Venus!...*)

I am fully aware that this image strikes a contrast with the widespread stereotype of M as indifferent, even cynical, towards demonstrating or developing such pedagogical sensibilities. And – one may ask – didn't M manifest any trace of these stereotypes? Of course he did: see *Does learning happen anyway?* (OT7.1) for a small collection of examples that one might say have a whiff of the elitism mathematicians are typically accused of. Didn't M attempt to absolve himself of any responsibility for the students' difficulties? Of course he did. *I always tell them so! Were they not in the lectures?* was uttered more than once (and I have left it in the text occasionally as a reminder). Didn't M obsess with the minutiae of student writing and digress from what appeared to be the focal point of the discussion in certain occasions? Didn't he express sometimes hollow-sounding sympathy for the predicament of the students who have to cope with the complex and novel ideas he is teaching them about just because, perhaps, he wanted to endear himself to RME? Didn't he insist, sometimes just a tiny bit more than reasonably on knowing about the course details of the institution the student examples originated from in a way that could be construed as diverting attention from the learning issues in question?

Yes, yes and yes!…But, overwhelmingly, the data I and colleagues have collected in the course of the studies on which the book draws were of high quality (in terms of precision, substance and focus) and the dialogues in Chapters 3 – 8, I believe, amply convey that. In this sense the data we have collected do beg to differ from the above mentioned stereotypes. What is also different, significantly, is the milieu in which we collected this data: context-specific, example-heavy, mathematically-centred samples of data discussed in the relaxed yet focused, unthreatening and mutually respectful ambience of the interviews.

Another thing the book is not about is direct recommendations for practice. Even in Chapter 7 where numerous practices are presented in a positive light, I feel the need to remind the reader that much of the content of the dialogues there consist of M and RME's speculation of what *may* work or what M wishes he could do in his own teaching. In this sense the speculative, exploratory nature of his comments begins to suggest directions for the future of the longer 'project' documented in this book (I use the inverted commas to suggest a sense of continuity of purpose that runs across Studies D, PD1, PD2, N and L) – and directions that at the time of writing I and colleagues are starting to follow: a collaborative consideration, implementation and evaluation of innovative practice. Returning to Barbara Jaworski's (2003; Chapter 1, Part (ii)) proposition for a shift from community of practice to community of enquiry I am looking forward to more opportunities that bring M and RME back to a community of – enriched – *practice*[2].

[2] In the months leading to the completion of the book Paola Iannone and I, with the support of a small grant from UK's Higher Education Academy, produced *How To Prove It,* (Nardi & Iannone, 2006) which collated recommendations for the teaching of Proof to Year 1 mathematics undergraduates. We have also been designing a 24-month developmental research study in collaboration with six UK mathematics departments that focuses on implementing and evaluating practices that support mathematics students' improvement of writing and reasoning skills. Furthermore, through the work of several of our doctoral students, we are hoping to expand the project in other university-level subject areas with strong mathematical elements (e.g. Physics, Engineering and Business Studies).

Still beneath the shadow of doubt…

Regardless of the shapes and directions the project takes in the near as well as the less foreseeable future, for me the major joy in producing this book has been immersing myself into the wor(l)ds[3] of M in the playful way of de-constructing and then re-constructing his words in a format that has fascinated me for so long. Full of the joy this playfulness has given me as I am, I still cannot say that my qualms, about whether my right to 'play' so has been earned, have in any way diminished. A diary entry from February 2004, when I started the search for the way in which to compose these dialogues, documents these qualms as follows:

> 'Why 'dialogues'? Because the one thing I have learnt all these years [of doing research] is that all I can do is tell a story? And tell it from as many sides as I can?'

And from the diary entries of the same period the tantalising question about how legitimate is my acting upon the words, that the colleagues participating in the studies trusted me with, persists. On a very doubtful day of the same month I imagined their reaction – and their regret for this trust! – as follows:

> 'What if they read this and feel like Ripley seeing incomplete versions of her cloned self in *Alien Resurrection*, or like an evictee from the *Big Brother* house seeing what the programmers have made of them (the 'grumpy one', the 'boring one', the 'smart one', the 'sexy one', the 'dumb one'…). Are they supposed to feel happy and proud of helping in this? About allowing themselves to be made to measure?'
> '…the characters are all un-natural constructs (even though the dialogues are as natural as they can be, even though they are rooted in things said and done) that embody the ideas I wish to convey…'

The characters, I continue in the diary, are 'methodological constructs but, as the book progresses, they are bound to start taking a 'human' shape'… and to even start talking back (!):

> '…throughout this exercise you've been calling me names: I am a construct, I am an artifact that embodies characteristics of all those you've worked with (and the one that you've been) but now, please, let me take the face of one of them, let me be, let me have a life, one life, not twenty, thirty or forty, hundred odd lives of all those people! Even if I end up being just one, untypical and methodologically unsound, can't you let me be just C, just L, just W, I am *so* tired of being all of them, all the time – please! Can't you see how exhausted I am? What an overblown relic you have made of me! Who is going to believe me anyway?'

These 'voices' never went away. They did fade however into the buzz and the busi-ness of a writing process that aimed to be as meticulous as possible and turned out to be also equally engrossing.

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[3] Inspired by Sfard's 'changing worlds with words' (2002, p49).

POST-SCRIPT
AMONGST MATHEMATICIANS:
MAKING OF, COMING TO BE

This is a chronological and reflexive account of the events that led to the production of the book. As a reader myself I am always curious about – and look for clues that add to the transparency of – the process through which a piece of writing has come to be. This chapter is aimed at those who share this kind of curiosity.

BEGINNINGS…

The book was contracted in December 2002. The idea for it had been incubating in some way or another since 1999. At the time it would have been based on my doctorate which I completed in 1996 and for which I had observed, and interviewed twice, twenty mathematics undergraduates at Oxford in their first encounters with the formalism and abstraction of university mathematics. The observations had taken place in the one to one or two to one tutorials that this university offered the students on a weekly basis and in which the students had the opportunity to discuss problem sheets that they had worked on during that week as well as queries that had emerged from their attendance at the lectures, their reading etc. The rich student – tutor dialogues turned out to be a goldmine of evidence that showcased the range and depth of issues that characterise the students' first encounters with university mathematics. The themes that emerged from the analysis of that data more or less influenced or determined the focal points of the studies that followed. Furthermore some of the data from the doctorate, excerpts of tutor – student dialogue, and some of the analysis presented in the thesis were used as triggers for the focus group interviews with the mathematicians in the later studies that became the ground material for the contents of this book.

A short follow-up of the doctorate, in the form of a respondent validation exercise, followed in 1998. In that samples of data and analysis from the doctorate were distributed to the tutors – whose students had been observed during the doctoral study – prior to an approximately ninety minute interview in which they were invited to comment on the samples. The samples turned out to be particularly effective triggers for reflection on the part of the tutors both in terms of reflection on their students' difficulties and in terms of reflection on their own pedagogical practice. Thematically and structurally, they are the first draft of the Datasets that were to be used for similar purposes in the focus group interviews with the mathematicians in the studies that followed a few years later and became the ground material for the contents of this book.

Almost immediately after this brief respondent validation exercise a one-year study that led back into the tutorials – this time focusing on the tutors, selecting critical incidents from the tutorial to be discussed in a post-tutorial thirty minute interview – followed in 1999. The six participating tutors were observed over eight weekly tutorials and interviewed after each tutorial. Again focusing the interviews on concrete and critical incidents from their teaching turned out to be an effective springboard for their raising and extensively reflecting on issues of teaching and learning. Their reflection added at least two valuable perspectives to the analyses in the doctorate: that of the practitioner who has been teaching and coming to know, in considerable depth, the needs of the students that the doctorate had focused on; and that of the professional mathematician, steeped into the epistemological and curricular dimensions of the mathematical content the students were experiencing in their first year. The potency of amalgamating the perspectives of mathematicians into the perspectives emerging from mathematics education research to produce complex and subtle accounts of the teaching and learning of mathematics at university level was beginning to emerge from the data very vividly.

As invaluable as this close scrutiny of student – tutor interaction was, the sense that this interaction should probably be studied in contexts other than the rich but rarefied context of the Oxford tutorial was also beginning to emerge. As it happens, while the last Oxford-based study was still in progress, I was offered a lectureship at the University of East Anglia (Norwich, UK) a newer and quite different institution to that of Oxford but with a well regarded mathematics department that I hoped was going to welcome my aspirations for collaborative research. Fortunately it did. Following the pattern that had worked successfully at Oxford (a study focusing on the students, followed closely by a study focusing on the mathematicians) two studies took place at UEA from 2000 onwards.

The first of these studies focused on the written work submitted on a fortnightly basis by the sixty Year 1 mathematics undergraduates throughout their first year. The analysis of this written work followed more or less the thematic structure that had emerged from the analyses in the doctorate and the studies that had followed. This data added a further layer on these analyses: the doctorate had focused, through the tutorial observations and the interviews, on a scrutiny mostly of the students' spoken mathematical word; these data consisted of ample evidence of the written one.

The second of these studies realised the plan for an overt pursuit and integration of the co-ordinated perspectives of mathematicians and researchers in mathematics education that had begun its formation during the Oxford-based studies. Mathematicians from six institutions across the UK were invited to participate in half-a-day group interviews in order to discuss datasets distributed to them in advance of the interview. Each dataset focused on a theme that had emerged in the previous studies as pertinent to the students' learning particularly in the beginning of their university-level mathematical studies. Each consisted of: examples of student work (written or other); a short list of issues and questions the participants may wish to consider as starting points for their discussion; and, a brief review of mathematics education research literature on the theme. The examples of student work were, primarily, from the then recently completed study of the UEA students' written work and, secondarily, from the doctorate. The interviews around these datasets were completed by the end of 2003 and provided the raw material for the dialogues presented in Chapters 3 – 8 of this book.

The idea for a presentation of issues regarding the teaching and learning of mathematics at university level in the dialogic format had been incubating alongside the idea of converting the student-tutor interactions as recorded in the doctorate into what at the time I envisaged as short stories, telling 'tales on the learning of advanced mathematics'. Quoting from a research diary of the time (1999) the plan was to select a small number of the thirty-two 'stories' told in the doctorate (each of the four data analysis chapters – on Analysis, Calculus, Linear Algebra and Group Theory – consisted of eight Episodes of Learning each capturing an issue as encapsulated in paradigmatic samples from the data) and

> '…tell the most impressive stories of stumbling blocks the students have when they are learning mathematics at university; introduce the book with a map of the difficulties to be talked about and recommended sequence of reading the stories; weave in historical aspects and explain the learning/Advanced Mathematical Thinking theories associated with the analysis of each story in plain but intelligent terms; introduce the mathematics in each story, recommend to the reader to engage with the exercise first and then ask them to compare their approach to the ones presented in the Story; conclude each Story with advice on how to avoid the stumbling blocks (as a learner but also as a teacher), and, with further reading on the learning / AMT theories and on the mathematics; conclude the book with a portrait of a developing mathematician and the rewards of becoming a mathematician; aim for a readership of mathematicians, mathematics undergraduates and mathematics educators'.

This initial idea for a book (1999) was growing alongside my increasing fascination with the mathematicians' perspectives and my realisation of the potency in co-ordinating these perspectives with those of researchers in mathematics education. The two – this attraction towards the story-telling approach to issues of teaching and learning and the intrigue by the idea of a mathematician / researcher in mathematics education co-ordinated perspective – came together in the planning for this book…

INITIAL PROPOSAL

… a plan the traces of which began to appear in the course of the last study, the one with the extended group interviews with mathematicians from across the UK. In the summer of 2002 UEA hosted PME26, the 26th Annual Conference of the International Group for Psychology in Mathematics Education. At the book exhibition of that conference where Kluwer Academic Publishers had a prominent stand I made my first contact with the publisher. By the end of 2002 my mumblings at that meeting had transformed into a book proposal entitled *Amongst mathematicians: conversations of a teacher, a student and a researcher* with the subtitle *Bridging the gap between the theory and practice of mathematics education* and intended for an audience of mathematicians and mathematics students (undergraduates and postgraduates), researchers in mathematics education, upper secondary education mathematics teachers and mathematics teacher educators. It was to originate in my previous work and named David Tall's *Advanced Mathematical Thinking* (1991a), Derek Holton's *The Teaching and Learning of*

Mathematics at University Level: An ICMI Study (2001) and John Mason's then just published *Mathematics Teaching Practice: A Guide for University and College Lecturers* (2002) as the seminal works in the area in which the book aspired to contribute. In the following I quote from the December 19th, 2002 text of the proposal submitted to the publisher:

> 'The book aims at extending the pedagogical discourses developed in these and other relevant works in a direction that demonstrates the substantial benefits of a dialogue amongst the interested parties, namely the learners of mathematics at pre-, under- and post- graduate level, their teachers and researchers in mathematics education. The conversations, based on extensive bodies of data collected and analysed in the last 10 years, aim at steering a course between theory and practice and at highlighting how potent self-reflection and active participation of all of the above can be with regard to developing a mathematical pedagogy. Transcending the often prescriptive nature of the literature in the area teachers and learners of mathematics are invited here to engage-with and form such development.'

As evidence of its genuine intention to address audiences both in mathematics and in mathematics education research the proposal quoted major associations in both fields as the forums which the book would aim to reach (for example: the London Mathematical Society and Mathematical Association of America, the International Group for Psychology in Mathematics Education, the British Society for Research into the Learning of Mathematics etc.). As further evidence of this intention the proposal put forward the idea that the reviewers of the manuscript include members of both communities, particularly ones who have demonstrated in their own work a penchant for a collaboration between the two communities. Furthermore, given the relative methodological novelty – narrative approach, dialogic format – of the proposed book, a wider, extra-mathematical audience of researchers with an interest in this type of novelty was also mentioned. The estimated length of the monograph was deliberately kept at a minimal 60,000 words as it hoped to address mathematicians accustomed to dense yet brief texts.

The rationale for the book was presented as follows:

> 'From Plato's and Galileo's dialogues through to contemporary works like *Proofs and Refutations* and *Copenhagen* […], philosophers, mathematicians and scientists – contrary to a common stereotype of creativity being at its best when in isolation – have been engaged in fictional, yet reality-grounded conversations with themselves and the public about the nature and value of their activity. The proposed book aims at extending this *dialogic* tradition into a new, hitherto under-reported (in this format) area of discussion:
>
> Ways in which we understand mathematics
> Ways in which we teach mathematics
> Ways in which we evaluate this teaching
>
> The book will do so by bringing three fictional characters into a conversation on the above issues:
>
> a learner of mathematics in the beginning of her undergraduate studies,
> a university mathematics lecturer, and,
> a researcher in mathematics education.

> The mathematical content of the conversations and the three characters are based on extensive bodies of data collected and analysed by the author in the last 10 years.'

A description of the data follows and concludes with the promise that the proposed book

> '…will take the analyses of the above material a step further and towards the demonstration of an argument for collaborative, multi-laterally engaging research and action in an area that desperately needs so.'

The topicality of the book is then described as springing from the following rationale:

> 'Teaching mathematics at university and college level is rapidly changing. Fewer and fewer students opt for exclusively mathematical studies. At least in the UK, recruitment of good mathematics graduates to mathematics teaching is at an all-time low. Given the substantial gap between secondary and tertiary mathematics teaching approaches, students feel increasingly alienated from the traditionalism of university-level teaching. Moreover universities are more than ever accountable to society regarding the quality of their teaching. By the late 90s most responses to these changes were in terms of modifying the tertiary syllabus. It is now becoming evident that reform should be: focusing on teaching both in terms of underlying principles and practices; and, moving away from the deficit and prescriptive discourses that tantalise previous works in this area and disaffect practitioners of mathematics teaching at this level. In content and form the proposed book aims to contribute in exactly this discourse for reform.'

In that proposal the structure of the book was envisaged as follows: six chapters, approximately 10,000 words each and on the following themes – these were the themes of the six datasets for the then-in-progress group interviews with the mathematicians from across the UK:

Theme 1	Formal Mathematical Reasoning I: Students' Perceptions of Proof and Its Necessity
Theme 2	Formal Mathematical Reasoning II: Students' Enactment of Proving Techniques and Construction of Mathematical Arguments
Theme 3	Mathematical Objects I: the Concept of Function Across Mathematical Topics
Theme 4	Mathematical Objects II: the Concept of Limit Across Mathematical Contexts
Theme 5	Mediating Mathematical Meaning I: Language and Notation
Theme 6	Mediating Mathematical Meaning II: Diagrams as Metaphors

The contents of each chapter were outlined as follows:

> 'In each chapter, starting from a set of mathematical problems (for the ease of the readers these will be presented with suggested solutions at an appendix but the readers will be encouraged to try the problems themselves so that they can begin to engage actively with some of the issues brought by the three characters into the conversation) the three characters raise a variety of issues that bring together findings from the data mentioned above on learning processes and teaching practices; and, theories from previous relevant research in mathematics education.

> As the dialogues themselves are intended to be as jargon-free as possible, but are essentially research and theory based, the relevant references and brief theoretical outlines will be provided for the benefit of the reader who wishes to examine the sources in more detail in footnote or endnote form. It is hoped that this format will be reader friendly and rigorous in equal measures.
>
> To put forward the central argument of the book – that of how vital and productive such a dialogue can be and of how essential a consideration of the needs of the learner is – and to reflect the history of suspicion and unease in the relationship between mathematicians and mathematics educators, from Chapter 1 to Chapter 6 the characters are intended to move from wariness to mutual trust and genuine enjoyment of the interaction. In particular:
>
> **The student:** she is concerned about the reasons behind the invitation to participate in the dialogue as she initially perceives the questioning as one more form of assessment of her learning. She gradually moves away from a belief that the other two characters are merely interested in assessment and have no regard for the actual processes of her mathematical thinking, beliefs and emotions. By the end she is a willing and confident contributor with an invigorated interest in mathematics teaching as well as in mathematics education research!
>
> **The university mathematics lecturer:** he is initially mistrustful of educational and psychological discourse as he does not rate it as 'hard core science'. Gradually his interest is intrigued as is the articulacy and vividness with which he offers examples from his experience, his views, interpretations and pedagogical strategies. By the end he is even seen using some of the idioms of mathematics education research!
>
> **The researcher in mathematics education:** her tone is personal and revealing of her background, philosophical, epistemological beliefs etc. She pays attention to methodological details such as the generality of claims made by the participants of the dialogue. The character is changing from rather austere, overly theoretically inclined and slightly negligent of routine details of the realities of mathematics teaching to more aware of the subtleties of this reality and more willing to 'translate' educational and psychological theory into widely communicable terms!
>
> Given the upper secondary/tertiary level mathematics that constitutes the basis for the conversations in the book, an analogous mathematical background of the reader is desirable. However it is hoped that the delivery, through a humorous look that incorporates popular references (for example public images of mathematicians in films like *Pi* and *A Beautiful Mind*) and classical polarities such as logical vs psycho-logical ways of thinking; quantitative vs qualitative methods of educational and psychological research; masculine vs feminine perspectives; and, Anglo-Saxon vs continental perspectives will help carry forward the essential message of the book across a wide range of audiences.'

I am rather stunned and slightly embarrassed by the ambition of this initial proposal. So... let me flash-forward to the present and inform the reader of two things.

FLASH FORWARD…

First, the character of the student is still very much present in the dialogue; albeit only through the examples of written work or utterances in the numerous data samples that are the starting points in the discussion in each episode and every chapter. Every word of the dialogues in the chapters that follow is grounded on words uttered by participants in the studies the book draws on; apart from the interviews in the doctorate (the content of which has influenced but does not always coincide with the focus of the episodes in this book) the studies on which the book draws have not given me enough of the students' direct perspective on these issues. Weaving in a version of this voice that stands equally vividly amongst those of the mathematician and the researcher in mathematics education (for whom the raw material was ample) would mean altering the nature of the dialogues from totally data-based to more mixed creatures. I felt this would muddle the methodological waters which the book is attempting to tread and I decided against it. To give you a sense of how I gradually reached this decision I quote from my research diary, entry February 2004. In it I think a sense of the muddling of the methodological waters that the creation of three characters may entail is beginning to creep in. Initially, instead of abandoning any of the three however, I considered creating separate dialogues:

> 'Within each chapter: Dialogue Mathematics Educator – Student, then Mathematics Educator – Tutor? But there can be no Mathematics Educator – Student – Tutor Dialogue. It never happened. I need to explain this. The reader would prefer a three-way dialogue. Which I (the data) cannot offer.'

Later in the month a further degree of separation is recorded in the diary and the idea for the three-way dialogue seems to have been abandoned:

> 'Alternatively perhaps the book could be in three parts as follows
> I: Dialogue Student – Mathematics Educator
> II: Dialogue Mathematics Educator – Tutor
> III: Dialogue Student – Tutor
> …across the six themes promised in the proposal. But then again I liked how the thematic thing looked in the proposal…'

Secondly, the promise for character development has been severely compromised: at the time of that initial proposal to the publisher the intention was to keep the calls on each one of the six themes in the chronological order in which they had occurred in the course of the study the book would mostly draw on. By doing so the trust and goodwill that had gradually emerged between the interlocutors in the course of that study – receding suspicion, developing pedagogical awareness and articulacy etc. – would be documented. Alongside the addressing of learning and teaching issues, a drama would also evolve, a drama showcasing the evolution of these characters. I quote from my research diary, entry February 2004 (at the beginning of Study Leave and before I had engaged at all with the composition of the dialogues):

> 'Within each chapter [insert] meta comments: the characters showing awareness of the process that takes place, of the story that's unfolding, of their becoming something…'

I believe this compromise was inevitable: as the analysis of the data and the composition of the dialogues progressed, especially as the dialogues started to be grouped in clusters around the learning issues that Chapters 3 – 6 focus on (or the teaching issues that Chapter 7 focuses on, or the collaboration issues that Chapter 8 focuses on), weaving in the dramatic elements (character evolution) required a skill that I have personally seen possessed only by the most expert playwrights – and whom I would certainly not even dream of emulating. Let me mention here two of them that have been a tremendous influence: Michael Frayn, particularly his work in *Copenhagen* and *Democracy*, and Tom Stoppard, particularly his stunning *Arcadia*. Delving into these works, as well as works of similar ilk – albeit, perhaps, of less accomplishment or ambition, such as *Newton's Darkness (*the two plays on the Newton-Hooke and Newton-Leibniz 17th century priority rifts by David Pinner and Carl Djerassi), David Auburn's *Proof* and, perhaps surprisingly, Terry Johnson's *Hitchcock Blonde* – I clarified and drew the line between the facts (the data) the dialogues would be grounded in and the fiction (the dramatic promise) I would hope they convey. In drawing this line, the detailed process of which I describe in Chapter 2, the above compromise was sealed.

BACK TO INITIAL PLANNING

Acceptance of the December 2002 proposal and completion of the study on which the book would mostly draw followed and by the end of 2003 I had also secured a six month Study Leave during which the manuscript would begin to materialise. In that time I worked on searching for and shaping the method for composing the dialogues and on the transformation of the raw data into the dialogic material that forms the bulk of this book. (I elaborate this method in Chapter 2.)

A first trial of the newly formed material – especially with the intention to see whether the addressing of a mathematical but non-RME audience works – took the form of a chapter on students' emerging understandings of the necessity for formal mathematical reasoning and writing prepared for a book intended for mathematics undergraduates by colleagues in Rio De Janeiro with whom I worked during one of the academic visits of the Study Leave. At that stage the challenge of composing the dialogues was starting to be compounded by the challenge of embedding the dialogue into relevant work from mathematics education research, of lifting it above the specificity of the student / course / university context which it referred to while keeping it in touch with the definitive detail and richness of that contextual fabric. Between the summer of 2004 and the spring of 2005 meeting these challenges as well as the evolving thematic clustering of the dialogues (all processes reported in detail in Chapter 2) resulted in an updated proposal submitted at the request of the new publisher (now Springer as the Kluwer / Springer merge had taken place in the meantime). I quote from that proposal, of April 18th 2005, in order to map out how the conceptualisation of the book was evolving at the time.

A MODIFIED PROPOSAL

The title in this new proposal was amended to *Amongst mathematicians: conversations between a mathematician and a researcher in mathematics education*. The subtitle remained *Bridging the gap between the theory and practice of* *undergraduate mathematics education*. The realities of composing the dialogues and embedding into relevant words have started to dawn and the estimated length of the monograph has risen to a more realistic 80,000 words. Of course the completion date (from December 2004 in the initial proposal and already passed…) is being pushed back (December 2005). While the references to Plato, Lakatos etc remain as the background influences on the proposed book, the areas of discussion are subtly, but significantly rephrased: the December 2002 version

> Ways in which we understand mathematics
> Ways in which we teach mathematics
> Ways in which we evaluate this teaching

is now

> Ways in which mathematicians perceive their students' learning
> Ways in which they teach
> Ways in which they reflect on their teaching practice

and the book promises to offer this discussion

> 'by bringing two fictional characters into a conversation on the above issues:
> a university mathematics lecturer, and,
> a researcher in mathematics education.
> Their conversation is triggered by the work of a third character, a learner of mathematics in the beginning of their undergraduate studies, documented in written or oral evidence.'

A description of the data follows as in the initial proposal but with the emphasis on the statement that

> 'the dialogues presented in the book are based on the material listed above, mostly the last [study] in this list.'

The intention of the book to demonstrate the pedagogical potential of a research collaboration between the communities of mathematics and mathematics education research remains – as does the rationale for the topicality of the book (an extended version of this has now become part of Chapter 1). The new proposal elaborates the intended discourse for reform (described in the original one merely as non-deficit and non-prescriptive) as follows:

> '…the book aims to demonstrate the feasibility and potential of a collaboration between mathematicians and researchers in mathematics education by engaging mathematicians as educational co-researchers;
> it aims to develop
> **a non-deficit discourse** on the pedagogy of undergraduate mathematics. This book is not intended as an exposé of incorrect pedagogical practice at university level but a demonstration that engaging mathematicians in a pedagogically sensitive conversation is possible and full of potential. And,
> **a non-prescriptive discourse** on the pedagogy of undergraduate mathematics. This book is not intended as a listing of the do-s and don't-s of mathematics teaching at university level as prescribed by pedagogues but a collaborative exploration of

> potentially effective pedagogical practice based on the vast experience and reflection of both practitioners and researchers.'

While the intended audiences remain the same as in the initial proposal, the new one elaborates its reaching out to mathematicians, particularly new ones, as follows:

> 'I believe the book can be used as a textbook for
> **undergraduate and graduate courses in mathematics education**. As it is intended as mathematician-friendly and jargon-free (or minimal), it can serve the purpose of introducing complex issues in the learning and teaching of mathematics to readerships not yet well-versed in the field's jargon and theories. […] the book could be used as a textbook in particular with the aim to trigger discussion as well as to be an entry point into the literature of the field.
> **the now increasing new-lecturer training courses.** There is a significant paucity of subject-specific texts for these courses. Given the idiosyncratic nature of mathematics, new and in-service university lecturers of mathematics often complain that their training is content-free (hence of limited use). The book can fill this gap: not only because its style and content is particularly mathematician-friendly but also because the learning situations and evidence discussed in the book are drawn, as authentically and realistically as possible, from situations the trainees will be facing in their daily practice as university lecturers of mathematics.'

As evident in Chapter 8 these elaborations became possible because in the meantime the mathematicians participating in the latest study, where such issues had been explicitly discussed (see Chapters 1 and 2), had suggested that the materials from the study (data and analyses) could be used so.

Another subtle but significant change can be observed in the new proposal with regard to listing the titles of other books in the area that address similar issues: the brief mention of Tall, Holton and Mason in the original proposal is replaced by the following:

> 'A seminal work in the area in the area of learning and teaching mathematics at the upper secondary and tertiary level has been (Tall 1991). The book remains a major entry point for those wishing to familiarise themselves with the field. This has recently been complemented by the substantial contribution of (Holton 2001). In my view these texts are classical both in terms of the potential longevity of their influence in the field and in terms of the academic style in which they are written. This style, the mainstream style of reporting findings and theorising in the social sciences, is met with often-notorious weariness by mathematicians (lecturers and students, a significant portion of the targeted audience of this book) who prefer (as repeatedly reported by the mathematicians participating in the studies the book will draw on) to read less jargon-laden, more context / situation – specific, naturalistic accounts of teaching and learning. In this sense the following book has made a significant contribution with regard to, at least, one issue: it transforms theoretical findings in the field into a substantial list of tactics that a university lecturer of mathematics can employ in their teaching in order to improve their practice: (Mason 2002). Largely, however, all above works engage mathematicians (their perspectives, needs etc.) in a tangential, peripheral way: at heart the views expressed in these works are those of researchers in mathematics education. The following book, based on interviews with 70 mathematicians in the UK has been a good attempt to introduce in more detail the voice of mathematicians: (Burton 2004). However, as Burton acknowledges herself in the book, the focus was on what these mathematicians' career trajectories as academics have been; what their research practices are and how they come to know mathematics. Especially within the last of these three lie implications about the ways they teach and these are sometimes touched

upon in the book. However Burton's aim was not to convey these mathematicians' perspectives on teaching or, crucially, their perspectives on their students' learning and how these perspectives influence their practice. In fact the mathematicians' accounts here are presented at a macro-level as broad brushes on a landscape of their practices and experiences. The two issues that I raise in the above,

paucity of mathematicians' perspectives in the academic literature on mathematics education; and,

paucity of mathematicians' perspectives in particular with regard to their views on learning and how these views influence their teaching practice,

are major in the following sense:

reform of practice, urgently needed in the field of undergraduate mathematics education, proves futile when not originating from and owned by the practitioners themselves. A prerequisite for engaging with reform of practice (designing and implementing it) is raising awareness about the need to do so. In this sense research that engages mathematicians in reflection on pedagogical issues is likely to raise their pedagogical awareness and alert then to the need for reform. Even likelier to achieve this aim is research, such as the one reported in the book outlined here, where the mathematicians are invited to a discussion of mathematically-specific, concrete, example-based learning and teaching situations. As my studies with them over the last few years have shown, they are keen, sharp and sensitive participants in this type of research. The amazing wealth of data they have supplied in these studies is the ultimate evidence base that this book draws on. My intention is to do justice to the commitment and sensitivity evident in these data.'

In summarising the ways in which the book is intended to be different from other current offerings the proposal mentions

'**scarcity** of texts in the area of undergraduate mathematics education

paucity of texts that explicitly introduce the pedagogical perspectives of mathematicians

paucity of texts that explore the pedagogical perspectives of practitioners in undergraduate mathematics education, not in the abstract, but in the context of specific and realistic learning and teaching situations

urgent need to develop a realistic, relevant, mutually engaging (practitioners and researchers) discourse on undergraduate mathematics pedagogy in order to facilitate and accelerate reform of practice'

and stresses 'the avid expression of interest' in this type of work by the mathematical community (as evident in the invitations for contributions to seminars, conferences, publications etc that had accompanied the then recently completed collaborative study).

Finally the structure of the book is envisaged in the new proposal as follows:

Chapter 1	Introduction
Chapter 2	On methodology: the fact and fiction of narrative research
Chapter 3	Formal Mathematical Reasoning: Students' Perceptions of Proof and Its Necessity; Students' Enactment of Proving Techniques and Construction of Mathematical Arguments
Chapter 4	Mediating Mathematical Meaning: Language and Notation; Diagrams as Metaphors
Chapter 5	Mathematical Objects I: the Concept of Function Across Mathematical Contexts
Chapter 6	Mathematical Objects II: the Concept of Limit Across Mathematical Contexts

The format (dialogues with footnotes) remains more or less as in the initial proposal. And of course there is nothing on the dramatic elements (character development) promised in the original…

FIRST TRIALS AND REVIEWS…

In the first half of 2005 two opportunities emerged for trialing parts of the evolving manuscript (the progress of the manuscript in the intermediate months is elaborated in Chapter 2). The first was preparing a paper for CERME4, the 4th Conference of European Researchers in Mathematics Education in February 2005. The paper was a condensed version of the chapter produced for the Brazilian book I mentioned above (and eventually became the opening Episode in Chapter 4 and Episode 7.4, Scene IIa in Chapter 7). Its extensive discussion made possible in the (hugely beneficial for work in progress) interactive / group-work format of this conference was a much needed at the time reminder of my methodological obligation to offer a transparent account of how the dialogue, and specifically the character of M, was being composed; it also supplied evidence that the discursive, jargon-light dialogic format may have considerable appeal to the mathematical community. (It also provided evidence that merging the characters of M and RME into one voice – as an early version of the paper foolishly and prematurely attempted to do – was doomed to be met with justified reticence by members of both communities who stated clearly to me that, however close the collaboration between the two becomes, there can never be one such voice. I drew consolation from the fact that I had never intended to do so in the book, examined the reviews carefully and rewrote the paper in a format that is much closer to the book.)

The second of the opportunities mentioned above offered a platform for an even more direct and extensive trial of material from the book. Towards preparing materials for teaching three sessions at the Annual Graduate Summer School of Mathematics Education (Delphi, Greece) in July 2004, I selected six Episodes, one from the then-rough versions of each of Chapters 3 – 8, and used it as the starting point for the discussion with the mathematics graduates and their lecturers in the sessions. The sessions were introduced as follows – I cite from the session handouts:

> **Session Summary**
> In these three sessions we will explore seminal issues in the learning and teaching of mathematics at university level by drawing on a series of studies conducted by Elena Nardi and Paola Iannone at the University of East Anglia in Norwich, UK as well as on Elena's previous work at Oxford University.
>
> In these studies the experience of learning and teaching mathematics at university level was examined from the perspective of mathematics undergraduates – in terms of what they find difficult or not, rewarding etc. – and from the perspective of the mathematicians who teach them – in terms of how they believe their students learn, what their difficulties are, what practices they employ to help their students overcome

> these difficulties, etc.. In the course of these studies, this invitation towards mathematicians and their students to contribute to a discussion on issues of learning and teaching has acquired further significance: raising pedagogical awareness and offering what many mathematicians have described to us as a first entry into the world of research in mathematics education.
>
> In the sessions we will examine excerpts from a book Elena is currently working on: in this, starting from a discussion of samples of student writing or passages of transcribed conversations between students and their tutors, mathematicians and researchers in mathematics education engage in a collective consideration of pedagogical issues. Their discussions are presented in the slightly unconventional format of a dialogue between two fictional – yet firmly data-grounded characters – M and RME (mathematician and researcher in mathematics education respectively).
>
> Six excerpts of this dialogue will be available for these sessions. Each represents a theme that the book aims to examine: four themes on student learning – the learning of specific topics or concepts such as function and convergence; mathematical reasoning; and, mediating mathematical meaning – one theme on pedagogical practice at university level; and, finally, a meta-theme, the relationship between M and RME. *Excerpt: Function* and *Excerpt: Limit*, along with a brief introduction to the background studies and the narrative method used for the construction of the dialogue will be discussed in **Session 1**; *Excerpt: Reasoning* and *Excerpt: Writing* in **Session 2**; and, *Excerpt: Teaching* and *Excerpt: M / RME*, along with a discussion of how collective efforts, such as the ones that these studies represent, may evolve into a collaborative consideration of reforming pedagogical practice at university level, in **Session 3**.
>
> Elena will introduce and co-ordinate each session and the style of the sessions is intended to be highly participatory. Copies of relevant papers will also be available for you to take away and read at your own time.

The level and fervor of discussion in the sessions was thoroughly encouraging. The main aims of the exercise of using the Excepts in the sessions were: to evaluate the use of the dialogue / footnote format; and, to evaluate how clear (and relevant) were the Excerpts to an audience of mathematicians who had little knowledge of the context (UK undergraduate mathematics courses) in which the data was collected and were just beginning to get to grips with the mathematics education research literature (the mathematics graduates in the Delphi Summer School were in the beginning of their studies towards a Masters in Mathematics Education). For both aims the feedback was constructive and encouraging. I describe how the feedback propelled further progress of the manuscript in Chapter 2.

A first version of the manuscript (March 31st, 2006), a slightly modified version of the April 2005 proposal, was finalised during an eight-week Writing Leave that UEA granted me for February and March 2006. It was structured as follows:

Chapter 1	Background and Context
Chapter 2	On Methods, Process and Presentation
Chapter 3	The Encounter With Formal Mathematical Reasoning: Conceptualising Its Significance and Enacting Its Techniques
Chapter 4	Mediating Mathematical Meaning Through Symbolisation, Verbalisation and Visualisation
Chapter 5	The Encounter With the Concept of Function
Chapter 6	The Encounter With the Concept of Limit
Chapter 7	On the Pedagogy for Undergraduate Mathematics
Chapter 8	The Fragile, Yet Crucial, Relationship Between Mathematicians and Researchers in Mathematics Education

Reviews arrived on June 7th, 2006. They were enthusiastic and appreciative of the manuscript (its content, its style, its ambition, its topicality) as well as relentless in pinpointing aspects of it that worked less well (one of the reviews is a meticulously and enthusiastically composed twelve pages of line by line commentary on every chapter!). With immense gratitude these reviews, as well as discussions with the publisher[1], formed the basis of the work towards the final version of the manuscript[2] (for example: the book now starts with a brief Prologue; all chapters start with one-page summaries – instead of, in the case of Chapters 3 – 8, lengthy-ish literature reviews; parts of the dialogue have been condensed or deleted if dragging, etc.). In the meantime several invitations to present parts of the manuscript have provided more opportunity for mulling over the text. Between the March 2006 and June 2007 versions of the manuscript these invitations, for which I am also grateful, were:

2006	Session entitled *'You look at these students, you look at their faces, you know they are lost…': Towards a new pedagogy for undergraduate mathematics'*, The Mathematical Association Annual Conference, Loughborough. 11 April.
2006	Seminar entitled *'Adventurous Partnerships and Turbulent Transitions: Engaging Mathematicians as Educational Co-Researchers'*, Mathematics Education Centre, Loughborough University. 12 April.
2006	Plenary session entitled *The pedagogical role of the mathematician as de-mystifier and en-culturator* at the 3rd International Conference on the Teaching of Mathematics at the Undergraduate Level (Istanbul, Turkey) July 1.
2006	Invited lecture at the Third Irish Symposium for Undergraduate Mathematics Education, entitled *'Proof by Mathematical Induction: conveying a sense of the domino effect* (or: don't worry, you are not assuming what you are supposed to be proving…)', National University of Ireland, Galway. 15-16 December
2007	Invited presentation at a symposium entitled 'Mathematicians and Mathematics Educationalists: Can We Collaborate?', University of Warwick, Department of Mathematics, 25 January. Followed by a contribution to the *First Workshop on Advanced Mathematical Thinking*, 26 January

I hope the discussion will carry on as vibrantly as it seems to have started.

Elena Nardi
June 30th, 2007.

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For details on the films *Notre Musique*, *Pi, Alien Resurrection* and *A Beautiful Mind* and the television programme *Big Brother:* Internet Movie DataBase at http://us.imdb.com/ .

THEMATIC INDEX: MATHEMATICS

Below I am listing the mathematical content each Episode, Special Episode and Out-take in Chapters 3 – 8 focuses on in terms of:

- general mathematical domain (e.g. Analysis, Group Theory, Linear Algebra)
- specific mathematical topics (e.g. Functions, Proof by Counterexample)

Episodes are denoted *x.y*. Special Episodes *SEx.y* and Out-takes *OTx.y* where *x* is the number of the Chapter and *y* the number of the Episode, Special Episode or Out-take. So SE3.4 denotes the fourth Special Episode in Chapter 3.

General mathematical domains

Specific mathematical topics

THEMATIC INDEX: LEARNING AND TEACHING

Below I am listing the learning and teaching themes that each Episode, Special Episode (SE) and Out-take (OT) in **Chapters 3 – 8** focuses on. Entries in italics denote location where the theme is mainly tackled. In order to locate references to the main mathematical topics discussed see *Thematic Index: Mathematics*.

AUTHOR INDEX

Numbers refer to pages where author names appear.

There are also references to the films *Notre Musique (*first page after the title), *Pi* (p302)*, Alien Resurrection* (p295) and *A Beautiful Mind* (p302) and the television programme *Big Brother* (p295).

MATHEMATICS TEACHER EDUCATION

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