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Carl S. Helrich

# The Classical Theory of Fields

## Electromagnetism

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# The Classical Theory of Fields

## Electromagnetism

With 132 Figures

Springer

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# Preface

The study of classical electromagnetic fields is an adventure. The theory is complete mathematically and we are able to present it as an example of classical Newtonian experimental and mathematical philosophy. There is a set of foundational experiments on which most of the theory is constructed. And then there is the bold theoretical proposal of a field–field interaction from James Clerk Maxwell, the validity of which was established in Heinrich Hertz' laboratory.

It is my intention here to present the theory of classical fields as a mathematical structure based solidly on laboratory experiments. I try to introduce the reader – the student – to the beauty of classical field theory as a gem of theoretical physics.

To keep the discussion fluid I placed the history in a beginning chapter and some of the mathematical proofs in the Appendices. Helmholtz' Theorem determines the form that will be taken by the field equations and the way in which we must understand each experiment. To obtain Maxwell's field equations is the goal. If the reader also learns to work through exercises that is good. But that is not the goal. The problems the reader will encounter as a practioner will require thinking that must be based on a deep understanding of classical field theory.

And so I have tried to obtain Maxwell's Equations as soon as possible. I have not been completely successful because of my concerns about the reader's mathematical development. I felt compelled to include a rather extensive chapter on mathematical background for readers unfamiliar with some of the language. I have also included chapters on Green's Functions and Laplace's Equation between the static form of Maxwell's Equations and a discussion of Faraday's Experiment. These may be avoided by the reader already fluent in the mathematics.

The chapter on Einstein's relativity is an integral necessity to the text. This chapter is historically accurate and fairly complete for the level of the text. My treatment is based on original papers by Einstein, Hendrik A. Lorentz, and Hermann Minkowski, on the excellent historical analysis of Abraham Pais, and on some more modern treatments such as Wolfgang Pauli's and Wolfgang Rindler's. My goal is to demonstrate the covariance of Maxwell's Equations and to present the transformation theory, while not losing sight of the "step" that had been introduced. I do not suggest ignoring this chapter. It is good for the physicist's or engineer's soul to know about this step. But it is not absolutely required for much of the use to which a practitioner will put field theory.

I have tried to be honest with the reader about our microscopic picture of matter. I avoid quantum mechanical descriptions, but not the fact that these lie behind our treatment of matter.

Our models of plasmas provide a good testing ground for electrodynamic theory that does not require quantum mechanics. I have used this at points in the text. This has been my guide in the chapter on particle motion and in my final chapter on waves in a dispersive medium.

My discussions of particle motion are based on Hamiltonian mechanics, which I outline. This results in a symmetry, as well as simplicity in the equations of motion. My treatment of magnetic mirrors relies on numerical solutions, which are simplified by the canonical equations. And I have based my discussion of coherent particle motion verbally on what is known of the dynamics of plasmas.

I have not intended this treatment to be exhaustive. The topics I have chosen reflect my interests as well as what I felt my own education lacked. I will, probably, readily agree with any criticism claiming that I have missed an indispensable topic. I do, however, believe that after finishing this text the reader should be able to encounter that topic with confidence.

I am grateful to generations of students who have helped in the development of my course in classical field theory. Their patience and enthusiasm has been an inspiration.

I am also grateful to my teachers and the directors of programs in which I have been involved. Among these I particularly want to acknowledge Leslie Foldy, David Mintzer, Marvin Lewis, and Günter Ecker. From each of these people I have learned to be thorough, unrelenting, and even confident. The first three of these people were inspiring teachers, Lewis was my doctoral mentor, and Ecker was my director in Jülich.

I have discussed modern plasma theory, of which I am no longer a part, extensively with my friend Wei-li Lee of the Princeton Plasma Physics Laboratory. Lee contributed directly to my discussions of gyrokinetic theory and its application to magnetically confined fusion plasmas.

I am grateful for the patience and understanding of my wife, Betty Jane, who has endured more than I could have expected as I wrote this, and who remained a constant source of encouragement.

I am thankful for the encouragement and positive discussions from Dr. Thorsten Schneider and Ms. Birgit Münch of Springer-Verlag and the very careful work of Ms. Deepthi Mohan of SPi Technologies India.

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Goshen, Indiana *Carl Helrich*

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# Chapter 1
# Origins and Concepts

*Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this agent be material or immaterial, I have left to the Consideration of my Readers.*

*Isaac Newton*

*The field concept is the product of a highly original mind, a mind which never got stuck on formulas.*

*Albert Einstein*

## 1.1 Introduction

Classical field theory possesses a striking beauty in part because it comes to us as a complete theory in which we can tie almost each law and concept directly to a single experiment. Except for the qualifying "almost" we can present the subject as based on hard laboratory data and a very limited number of guiding ideas. But there is too much human thought that lies behind the word "almost" for us to drop it in our pursuit of understanding.

The history of science is a history of human thought. Because there is no simple understanding of the origin of ideas there can be no simple understanding of the development of any branch of science. And our understanding of a branch of science is linked to how well we comprehend the origin of the ideas on which it rests.

Classical field theory began as a logical extension of what we already knew from Isaac Newton's experimental and mathematical philosophy [79, pp. 214–215] and the mechanics that resulted. Newton, however, knew that his law of universal gravitation begged an explanation that lay beyond the experimental data.

Michael Faraday (1791–1867), whom we acknowledge as the greatest among experimentalists, imagined that lines of force permeated space and were responsible for electric and magnetic phenomena. This was heresy. But it was believed by William Thomson (1824–1907) and James Clerk Maxwell (1831–1879). Maxwell's

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mathematical development of Faraday's ideas, and the conviction that light waves must emerge from the theory, will bring us to a point where theory presses experiment.

Heinrich Hertz (1857–1894), in the laboratory in Karlsruhe, carried out the experiments that identified propagating electromagnetic waves. But this alone did not reveal the full truth, as Hertz knew.

We must also encounter the crisis in scientific thought that marked the beginning of the twentieth century. The answers presented by Albert Einstein in 1905 required a revision of the bedrock of Newtonian thought: the concept of time and consequently of space.

Classical field theory will then bring us new ideas that we could not have anticipated. In this chapter we will trace the twisted historical path with the intention of seeing the origin of ideas and the consequences. To fully understand classical field theory as a product of human thought we must encounter these origins and consequences.

We will have many occasions to reference this chapter.

## 1.2 Magnetism

In 1849 Thomson first used the term *field of force*, or simply *field*, in reference to magnetic effects [18, p. 146]. He was providing new words for Faraday's idea of lines of force. He was also, at least in part, giving words to the same phenomenon that had such an impression on Einstein when he was four years of age. Einstein had marveled at the fact that a compass needle responded to a magnet although there was nothing between the needle and the magnet [65, p. 3].

Faraday was a mature scientist in 1849. His lines of force gave expression to a conviction that we should not simply accept a description in the language of action at a distance. In Faraday's mind the space surrounding a magnet or an electrical charge was not empty. It was penetrated by magnetic or electric lines of force. These lines of force are responsible for what we experience when we bring the like poles of two permanent magnets close to one another. But Faraday also believed that such lines of force were present around any mass and were the source of the gravitational force [18, p. 147].

In a lecture he gave at the Royal Institution in 1834, three years after his discovery that electric currents result from variations in magnetism, Faraday claimed that, "We cannot say that any one is the cause of the others, but only that they are connected and due to a common cause" [18, p. 145].

In 1845 Faraday found the effect of magnetism on the polarization of light. In the paper reporting this effect he stated his conviction that the various forms of the forces of matter have one common origin. This was a great scientist and a very careful experimentalist speaking out of his experience. Faraday saw a universe permeated by fields where the nineteenth century theoreticians saw Newtonian particles and action at a distance.

Albert Einstein, the great twentieth century field theorist, wrote that the field concept was the product of "a highly original mind, a mind which never got stuck on formulas [18, p. 147]."

## 1.3 Gravitation

Isaac Newton had realized that there was a difficulty in his description of the universal gravitational force between two masses. The mathematical form of Newton's law of universal gravitation provides only the dependence of the force on the masses and the distance between them. There is no information about any agent that may cause the attractive force. Richard Bentley (1662–1742), who delivered the first of the Boyle Lectures in 1692, communicated with Newton about gravitation. In a letter to Bentley dated 25th February, 1692, regarding the origin of gravitation Newton expressed his own thoughts rather clearly.

> That gravity should be innate, inherent and essential to Matter … is to me so great an Absurdity, that I believe no Man, who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this agent be material or immaterial, I have left to the Consideration of my Readers [77, p. 302].

Newton refused to speculate. Faraday, however, did not refuse to speculate. Our physical understanding of the universe was more developed when Faraday was trying to give expression to his observations than when Newton expressed his ideas.

## 1.4 Faraday, Thomson, and Maxwell

Faraday's background was not remarkably unusual. Almost all great physicists have come from the broad economic region called the middle class. Faraday was only slightly below the bottom of that spectrum. He was born in a London slum and his schooling was, in his own words, "of the most ordinary description, consisting of little more than the rudiments of reading, writing, and arithmetic at a common day-school"[18, p. 137]. The result of this background was that Faraday did not express himself in the mathematical language of nineteenth century theoretical physics. His grasp and creation of theoretical ideas were those of a superb experimental scientist.

Faraday's ideas of fields were not immediately grasped, and certainly not immediately appreciated. He was considered a heretic by some of his contemporaries. This may have been a result of the fact that Faraday expressed ideas that were those of theoretical physics without the use of mathematics. It may also have been based in part on Faraday's firm belief that electric, magnetic and gravitational fields were united. But there were two younger scientists that believed Faraday's ideas to be correct. They were Thomson (Lord Kelvin) and Maxwell.

In more modern terminology Faraday's fields are vector fields. If a vector field permeates a certain region of space, then at each point in that region of space the field will have a magnitude and a direction. The value of the magnitude and the direction will normally change from point to point.

## 1.5 Gravitation a Vector Field

The gravitational field surrounding the sun is an example of a vector field. The earth is attracted to the sun, so the lines of force associated with the gravitational field point toward the sun. These force lines actually point toward the center of the sun and for our purposes we can replace the sun with a point mass. Newton's Fourth Law specifies that the magnitude of the gravitational force is proportional to the inverse square of the distance from this point.[1]

If we specify the (vector) direction $\hat{e}_{\mathrm{r}}$ as radially outward from the point mass representing the sun, then the force from the *gravitational field*, which we designate as $\mathbf{F}_{\mathrm{grav}}$, has the form

$$\mathbf{F}_{\mathrm{grav}}(\mathbf{r}) = -G\frac{M_{\mathrm{S}}}{r^2}\hat{e}_{\mathrm{r}}, \tag{1.1}$$

where $\mathbf{r}$ is the general (vector) point in space, $G$ is the universal gravitational constant, and $M_{\mathrm{S}}$ is the mass of the sun. The corresponding force on the earth is

$$\mathbf{F}_{\text{sun on earth}}(\mathbf{r}) = -G\frac{M_{\mathrm{S}}M_{\mathrm{E}}}{r^2}\hat{e}_{\mathrm{r}}, \tag{1.2}$$

where $M_{\mathrm{E}}$ is the mass of the earth.

Because the earth also has a gravitational field, which is of the same form as (1.1), the net field at the location of the earth will be slightly different from (1.1). But, since the earth is much less massive than the sun, we may neglect the effect of the gravitational field of the earth on the form of the gravitational field of the sun. The earth is then the small "test mass"[2] we use to measure the gravitational field. That is we can only observe the presence of a gravitational field by observing the effect of the field on a test mass.

The difference between this picture and that of action at a distance is fundamental, as was pointed out by John A. Wheeler (1911–2008) (quoted in [18], p. 147). The action at a distance picture identifies the two ponderable masses $M_{\mathrm{S}}$ and $M_{\mathrm{E}}$ and specifies their locations from which the magnitude and direction of the force is determined. In the field picture the sun creates a spherically symmetric field, which spreads out through space, decreasing in magnitude with distance. The test mass

[1] We shall treat here only Newtonian gravitation.

[2] A test mass is a small mass used to measure a gravitational field. The field is found as the force on the test mass divided by the mass of the test mass in the limit as the mass of the test mass goes to zero.

(earth) senses this field experiencing a force. The test mass does not, however, need to “know” that it is being attracted toward the sun. The origin of the field is of no importance.

But the field is real and contains an energy. The space is no longer a pure vacuum. In our ordinary discourse we may then speak of a vacuum as a region free of matter, but not free of fields.

## 1.6 Charges and Electric Fields

Analogously to the dependence of the gravitational field on mass, the time independent or static electric field is the result of the presence of electric charge. And a small “test charge”[3] will respond to the presence of an electric field just as our test mass responded to the gravitational field. Electric charge may, however, be either positive or negative. The force on the charge in an electric field will be in the direction of the field (positive charge) or opposite to the direction of the field (negative charge).

These electrostatic forces were known, although not understood in the eighteenth century by scientists like Charles du Fay (1698–1739) and Benjamin Franklin (1706–1790). The lack of understanding is indicated by the fact that du Fay and Franklin proposed different fundamental bases for what they were observing. Du Fay considered that the phenomena he was observing indicated the presence of two fluids, while Franklin decided that a single fluid was sufficient. Franklin thought that what du Fay had considered a second fluid was only the absence of the single fluid. Because the positive charges present in matter (protons) are not transferred at the energies associated with electrical experiments, Franklin was correct in claiming that only one fluid was present. And it would seem logical to identify that fluid as positive, as Franklin did. Electric current then became the flow of positive charge. We still use this convention. But we realize that in almost all circumstances the charges flowing are actually electrons, which are negative in sign. This is a peculiarity we now simply live with. It is not a problem that requires a solution [97, pp. 39–51].

Franklin also proposed that charge is conserved . The positive fluid may permeate matter, but it was neither created nor destroyed in the process [97, p. 51].

A real understanding of the nature of the electric force resulted from the experiments of Charles Augustin Coulomb (1736–1806). Coulomb was not a supporter of Franklin’s single fluid picture. He considered that there were two fluids that flowed and that the presence of a negative charge on a conductor was just that. The conductor had an excess of negatively charged electricity. It was not simply lacking in the positive charge of Franklin. And we shall later (tentatively) take this point of view in our analysis of Coulomb’s experiment.

[3]The field is found from the force on the test charge divided by the charge of the test charge in the limit as the charge goes to zero.

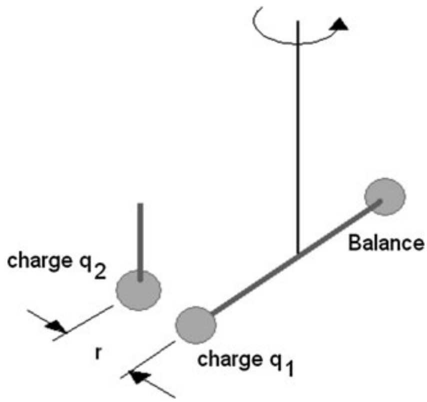


**Fig. 1.1** Basic apparatus designed and used by Charles Augustin Coulomb to discover the force law between electric charges

In 1785 Coulomb published three papers reporting his work on electric and magnetic interactions. Only the first of these three papers interests us here. This first paper carried a complete description of the apparatus he had designed and the experimental results for the force between two electrically charged spherical bodies [14].

In Fig. 1.1 we have drawn a picture of the basic torsion apparatus Coulomb designed and used to determine the form of the force law between electric charges.

He required only one charge $q_1$ mounted on the torsion arm and a single stationary charge $q_2$ because the electrostatic force is large. The second mass on the suspended bar balanced the charged body.

Coulomb could measure the distance $r$ separating the centers of the two spherical charges. The force between the charges he could then find from the angle of twist in the suspension cord. The relationship between torque in the cord and the angle of twist had to be measured separately.

In fact, however, the suspended bar did not come to rest in a reasonable time and measurements were made on the slowly swinging bar.

In Fig. 1.2 we present a facsimile of the apparatus as it appeared in Coulomb's paper.

Coulomb's law is essentially identical to Newton's law of universal gravitation except for the fact that the force may be either attractive or repulsive. In the field picture this fact provides the directionality of the force experienced by a test charge in the electric field.

Henry Cavendish (1731–1810) used a much larger version of the torsion apparatus to conduct similar measurements to determine the average density of the earth. His results were reported in 1798 [10].

## 1.7 Priestly's Speculation

Joseph Priestly (1733–1804) actually anticipated Coulomb's result. He published this in his book *The History and the Present State of Electricity with Original Experiments*, which was printed in London in 1767. Priestly's experiment had been

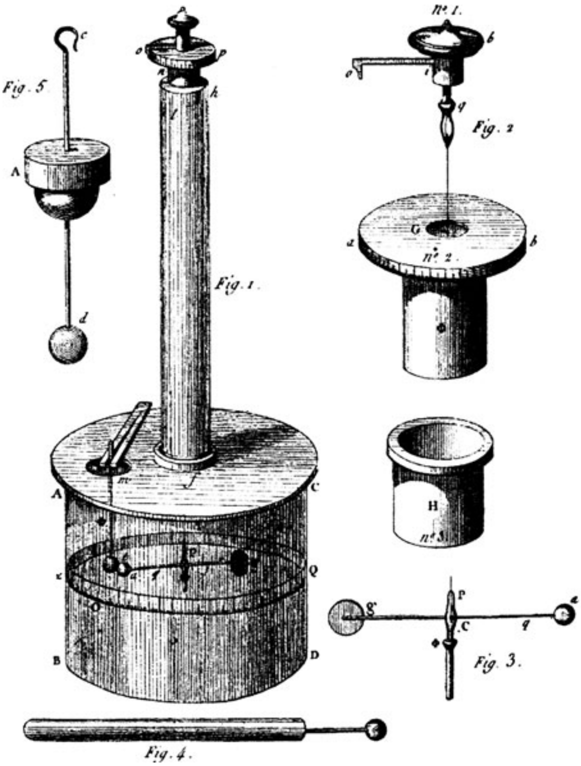


**Fig. 1.2** Coulomb's original apparatus from his 1785 paper. Source: Cnum – Conservatoire Numérique des Arts et Métiers, reprinted with the kind permission of The Bibliothèque centrale du Conservatoire National des Arts et Métiers

with a metal cup. He found no *electrification* inside the cup after it had been charged and compared this to the fact that "…were the earth in the form of a shell a body in the inside of it would not be attracted to one side of it more than another [98]." From that comparison he inferred that the electrical force must be of the same form as the gravitational, that is it must vary as $1/\left|\mathbf{r}_2 - \mathbf{r}_1\right|^2$. This represents masterful physical and theoretical insight on the part of Priestly. Nevertheless, in this text we shall continue to accept Coulomb's Law as the result of original experimental work carried out by Coulomb in 1785.

## 1.8 Voltaic Cell

The voltaic cell or voltaic pile was invented in 1800 by the Italian physicist Alessandro (Giuseppe Antonio Anastasio) Volta (1745–1827). Based on a scientific disagreement with his friend the physician and physicist Luigi Galvani (1737–1798), who had studied the electrically induced twitching of a frog's leg in


**Fig. 1.3** Three cells of a voltaic pile. In modern terminology the voltaic pile is a series connection of batteries

1786, Volta found that zinc and copper plates separated by cardboard soaked in brine produced a current. We have drawn three cells of a voltaic pile in Fig. 1.3.

The voltaic pile transferred energy to charges using a chemical reaction.

The importance of Volta's invention can hardly be over stated. With a voltaic pile a scientist could produce a current that could be regulated at the closing of a switch.

Modern batteries still carry the terminology electromotive force, which was applied originally to the voltaic pile. And we will apply this term to the induced electromagnetic field discovered by Faraday. These so-called electromotive forces are not conservative. The action of these electromotive forces through a closed circuit or contour is not zero.

The electrostatic force discovered by Coulomb is conservative. Electrical charges passing through wires lose energy in the heating of the wires. We can store energy in the separation of charges, which can be accomplished using an electrostatic generator, as Coulomb did. And we can dissipate that energy if we allow the charge to flow in a rapid pulse through a wire. But we cannot produce a continuous current of electrical charge through a circuit using only a conservative force. We must have a nonconservative energy input in the circuit to balance the energy losses. This we can accomplish with an electromotive force.

## 1.9 Currents and Magnetic Fields

### *1.9.1 Oersted*

The origins of our acquaintance with magnetism are ancient. Lodestone was known to the Greeks in 800 BCE. Thales and Anaxagoras (ca. 500 BCE–ca. 428 BCE) spoke of Lodestone as having a soul. More sophisticated theories considered effluvia, invisible emanations, or of a sort of dynamical field [65, p. 1]. There is also evidence, or weight of opinion, that the Chinese used a compass as early as 2637 BCE [65, p. 3]. What we may consider modern experimental study of magnetism, however, began with the discovery of the force caused by electric current on a magnetic compass needle by Hans Christian Oersted (Ørsted) (1777–1851) in Denmark in April of 1820 [50, p. 11].

Oersted had first begun to investigate the magnetic effects of electricity in 1807. But he apparently had no success until he was delivering a set of lectures on "Electricity, Galvanism, and Magnetism" in the winter of 1819–1820.

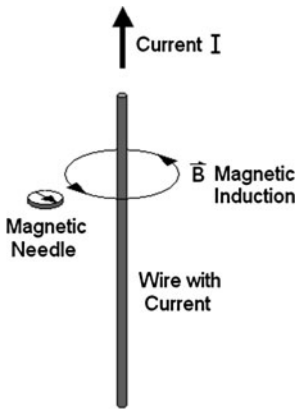


**Fig. 1.4** Schematic drawing of Oersted's experiment. The magnetic field **B** is circular around the current with orientation determined by the right hand rule

Oersted seems to have been inspired by the effect that electrical storms have on magnetic compass needles. He was able to demonstrate, and apparently did in one of the lectures, that a magnetic needle oriented perpendicularly to a wire experienced no force if an electric current was passed through the wire. Then after the lecture it occurred to him to place the needle parallel to the wire and then close the circuit to produce a current in the wire. The effect was dramatic. The needle quickly rotated to an orientation perpendicular to the wire. He had established the action of an electrical current on a (permanent) magnetic needle.

Oersted withheld publication of the results until July of 1820, after he had conducted confirming experiments with larger apparatus.[4] At that point he made no attempt at quantitative measurements of the forces and contented himself with a qualitative description of the observations.

We illustrate the situation schematically in Fig. 1.4.

What we would now call the magnetic field Oersted called the *conflict of electricity*. He showed that this magnetic field formed circles around the wire through which the current passed. These magnetic fields, as Oersted reported, passed through materials without affecting the electrical particles. They affected only magnetic particles.

Oersted explored as well the question of whether conductors carrying currents would experience forces from magnets and, subsequently, whether two conductors, each carrying a current, would experience forces from one another. This he confirmed by (qualitative) experimental test.[5]

Oersted published his results in Latin. Translations into common languages followed immediately and the reaction could even be called feverish [65, p. 12].

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[4]Schweigger's *Journal für Chemie und Physik*, xxix (1820), p. 364; Thomson's *Annals of Philosophy*, xvi (1820), p. 375; Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 63. (cited by [97]).

[5]Schweigger's *Journal für Chemie und Physik*, xxix (1820), p. 364; Thomson's *Annals of Philosophy*, xvi (1820), p. 375 (cited by [97]).

### *1.9.2 Ampère*

In September of 1820 Dominique F.J. Arago (1786–1853) reported the news of Oersted's experiments to the French Academy of Science. Within a week of hearing the report André-Marie Ampère (1775–1836) presented a paper on magnetism to the Academy which established the force between two wires through which electric currents were passed. This identification of a force between two wires is now known as *Ampère's law* and, together with Oersted's result, forms the basis of our understanding of static magnetic fields [65, p. 13].

Ampère continued to pursue the questions of what he called *electro-dynamics* and published the collected results in a memoir in 1825,[6] which is considered to be one of the greatest in the history of science. Maxwell said that in this memoir "The whole, theory and experiment, seems as if it had leaped, full-grown and full-formed, from the brain of the 'Newton of electricity [97, p. 92].'"

Ampère began the memoir by claiming that he was of the school for which all physical phenomena can be understood in terms of forces between particles. But then he hedged. He admitted that the forces between circuits carrying currents may be due to the reaction of the elastic fluid extending throughout all of space, the vibrations of which are responsible for light. This is the aether. Then he admitted the possibility of an intermolecular fluid in a metallic conductor, which consisted of non-equal amounts of the particles of electricity. This was an attempt to explain electric current ([97, pp. 87–88].

But the memoir did not dwell on speculation. The objective was to describe the experimental results. And there it succeeded.

In Fig. 1.5 we have drawn a basic representation of the principle Ampère was investigating.

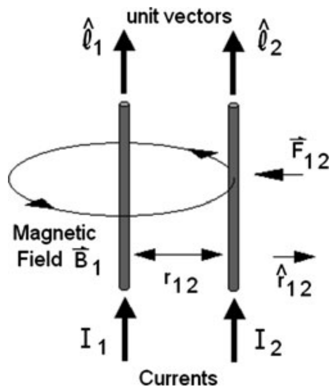
In Fig. 1.5 we have only drawn the magnetic field produced by $I_1$ (cf. Fig. 1.4) intersecting the current $I_2$. This causes, as Ampère found, a force $\mathbf{F}_{12}$ of $I_1$ on $I_2$. In field terminology, the magnetic induction $\mathbf{B}_1$ causes a force on current $I_2$. Because of symmetry and Newton's third law, this is identical to the situation viewed from the perspective of wire 1, with the magnetic field caused by wire 2. In either case we interpret the magnetic field as the intermediary agent producing the force, rather than simply claiming that the experiment determined a force between two currents.

Slightly over a month after Arago's report to the Academy on Orsted's experiment Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841) reported the results of their analysis of the magnetic force on a magnetic needle from a straight wire carrying an electrical current [97, p. 86].

Biot and Savart expressed their law in terms of an element of the current in an infinitesimal length of the straight wire. The force on a magnetic needle in the plane perpendicular to this infinitesimal length was at right angles to the length of wire and

[6]Mém. de l'Acad., vi, p.175 (cited by [97]).

**Fig. 1.5** Ampère's Experiment. The wires 1 and 2 carry currents $I_{1,2}$ in the directions specified by the unit vectors $\hat{\ell}_{1,2}$ respectively. The wires are separated by a distance $r_{12}$, and each are of a length $\ell$. We have represented the results in terms of the force on wire 2 from wire 1




to the line from the length to the point at which the magnetic needle was located. And it was inversely proportional to the distance from the wire to the location of the needle.

### *1.9.3 Electrical current*

There was no way to measure electrical current in 1820. Oersted published his results in the Journal für Chemie und Physik of which Johann Schweigger (1779–1857) of the University of Halle was editor. Schweigger immediately recognized the possibilities of using the phenomenon for the measurement of electrical current [52]. His device measured the deflection of a magnetic needle in the magnetic field produced by a coil through which the current to be measured passed.

The first mirror galvanometer was constructed in 1826 by Johann Christian Poggendorff (1796–1877). The name *galvanometer*, selected by Poggendorff, honors Luigi Galvani.

In 1879 Edwin Herbert Hall (1855–1938), then a doctoral student at the Johns Hopkins University in Baltimore, conducted a set of experiments that had been suggested to him by his thesis adviser, Henry A. Rowland (1848–1901).

In the experiments Hall placed a leaf of conductor (gold in the final experiments) in a magnetic field. As a current passed through the leaf the charge carriers experienced a force from the magnetic field. The electrical charge on the charge carriers could be deduced from the potential difference perpendicular to the current and the magnetic field.

Hall's experiments showed that the charge carriers were negative [38]. These would later be identified as electrons.

## 1.10 Induced Electric Field

Oersted's discovery that electric currents produced magnetism caused Faraday to wonder if there may be a corresponding relation between magnetism and electricity. Does magnetism produce electric currents? Faraday finally discovered the effect in 1831.

The path from any thoughts Faraday may have had in 1820 to the final set of experiments Faraday conducted over a period of ten days in 1831 was, however, not direct. In 1821 Faraday was working with gases, not electricity. He was also a consummate experimentalist and not a theorist or one given to hypotheses beyond the experimental facts [33, pp. 83–100].

In the experiments of 1831 Faraday first wound helices of wire around a wooden cylinder. In each helix the wire of one spire was isolated from the next by an interposed wrapping of twine. And each helix was covered with calico to isolate it from the next helix, which Faraday wound around it. The result was 12 helices superposed on one another.

The length of the wire in each of the helices was 27 ft. The first, third, fifth, seventh, ninth, and eleventh helices were connected as were the second fourth, sixth, eighth, tenth, and twelfth. The result was equivalent to two very long helices with one inside of the other. One set of helices Faraday connected to a galvanometer and the other he connected to a voltaic battery with ten pairs of plates each four inches square.

There was no observable deflection of the galvanometer [27, p. 2].

In another experiment Faraday wound 200 ft of copper wire around a wooden block with 200 ft of copper wire wound around that. The windings were isolated from one another as in the first experiment. One helix was again connected to a galvanometer and the other to a voltaic battery with 100 pairs of plates.

In this experiment Faraday observed a small deflection of the galvanometer when the battery was connected and an equally small deflection in the opposite direction when the battery was disconnected. He also observed that the battery heated the primary coil indicating that a current was flowing even when there was no longer any deflection in the secondary coil connected to the galvanometer [27, p. 3].

In a third experiment Faraday formed a circle of a soft iron bar and welded the ends together. He then wrapped two lengths of wire to form coils $A$ and $B$ around the iron ring.[7] In Fig. 1.6 we have a drawing of the basic apparatus used in this experiment.

The ends of coil $A$ he connected through a switch to a battery. The ends of coil $B$ he passed over a magnetic needle a distance of three feet from the iron ring. This distance ensured that the needle was not affected by any magnetic fields produced by coil $A$.

[7] These lengths of wire actually consisted of multiple lengths which could be connected together to change the effective length of the wire.

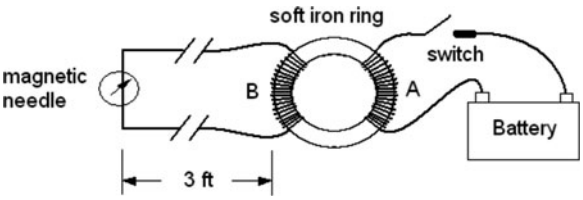



**Fig. 1.6** The apparatus used by Faraday in the discovery of electromagnetic induction

Faraday observed that the magnetic needle responded (oscillated) for a short time as the switch was closed or opened. But then the needle "settled at last in [its] original position." That is there was a current in coil $B$ only during the brief time required for the magnetic field in the soft iron ring to increase to its steady state value upon closing the switch, or to decrease from its steady state value upon opening the switch. A magnetic field alone does not cause a current [27, pp. 7–8].

Faraday also showed that he could produce the same transient current in a coil wrapped around a cardboard tube if he thrust a permanent magnet into the cardboard tube. The conclusion from this series of experiments was that a change in the number of magnetic lines of force, in Faraday's terminology, penetrating the central area of a coil of wire induced a current in the wire.

This induced current rapidly died out as the current in the primary coil reached a steady state or as the motion of the magnet ceased. The electric current was produced by a change in the magnetic field and not by the mere presence of the field. Faraday had discovered the importance of time in electromagnetism.

Only electric fields can cause charges to move. Magnetic fields can only deflect charges already in motion. So the presence of the current Faraday observed indicated that an electric field had been produced by the change in the magnetic field. But the wire provided a resistance to the flow of current in the wire, as had long been known.

Measurements of conductivity had been carried out by Humphry Davy (1778–1829) in 1821 and a theory had been developed by Georg Simon Ohm (1787–1854) in 1826 [97, pp. 94–98]. So the electrical field Faraday had discovered was of a different character from that studied by Coulomb.

## 1.11 The Mathematical Theory

### *1.11.1 The field equations*

Our discussion in Sects. 1.6–1.10 outlined the discovery of four fundamental facts about the electric and magnetic fields. These are almost the four basic field equations. To arrive at the field equations we must cast these statements in a particular mathematical form that we will develop in this text.

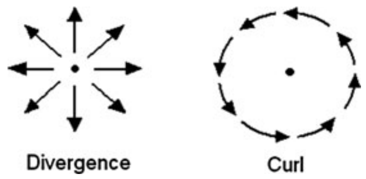



**Fig. 1.7** Illustration of the *divergence* and the *curl*. If the divergence of a field is nonzero in the neighborhood of a point the field lines diverge out from the point. If the curl of the field is nonzero in the neighborhood of a point the field lines form a countour around the point

The form that we require is not the form in which they were first obtained for historical reasons. The vector notation was not developed until later by Oliver Heaviside and Josiah Willard Gibbs [18, p. 162]. But the notation of the vector calculus, to which we refer here, is more transparent than the original form. And the discussion here will preserve the spirit of Maxwell's ideas.

The field equations involve the mathematical operations *divergence* (div) and *curl* (curl). Both of these operations represent the behavior of the field lines.

If the field lines diverge from sources, as discovered by Coulomb for electric fields produced by charged particles, then the field has zero curl and nonzero divergence. If the field lines form closed contours around the sources, as discovered by Oersted and Ampère for magnetic fields produced by electric currents, then the fields have zero divergence and nonzero curl.

This is, perhaps, best illustrated by a drawing first produced by Maxwell [68, p. 265]. We have a modification of this drawing in Fig. 1.7.

Maxwell's original notation is somewhat different from ours. And he spoke of a *convergence*, rather than a divergence, for which the field lines converged on the point rather than diverging from the point. But Maxwell's choice of the method to illustrate the meaning of these operations on vector fields is timeless.

In the language of fields, Coulomb discovered experimentally that electric charges cause electric fields. The result is that Faraday lines of electrical force **E** originate on positive charges and terminate on negative charges. The electric field lines then diverge from positive charges as

$$\operatorname{div}\mathbf{E} = \frac{1}{\varepsilon_0}\rho, \tag{1.3}$$

where $\rho$ is the density of charge. This is the result of Coulomb's experiment in the notation of the *vector calculus*.

In the language of fields, Oersted discovered that magnetic field lines form closed loops. They do not originate at one point nor do they terminate at another and so have zero divergence. That is

$$\operatorname{div}\mathbf{B} = 0, \tag{1.4}$$

where **B** is the magnetic field induction. This is *Oersted's Result*[8] in the notation of the *vector calculus*.

From the experimental studies of *Ampère* and the mathematical work of *Biot* and *Savart* we have the relationship between the *magnetic field induction* **B** and the current density **J** causing the induction as

$$\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J}. \tag{1.5}$$

This is *Ampère's Law* in the notation of the *vector calculus*.

*Faraday* discovered that a variation in the number of magnetic field lines passing through an area defined by a loop of wire produced an *electromotive force*[9] and a current in the wire. The induced field, which produced the current, is

$$\operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}. \tag{1.6}$$

This is *Faraday's Law* in the notation of the *vector calculus*.

Equations (1.3)–(1.6) are *almost* the complete field equations. There is one statement missing from this set of equations that has roots in Maxwell's thinking. This will emerge from our considerations here of Maxwell's contributions.

### *1.11.2 Maxwell*

#### 1.11.2.1 The Aether

Maxwell was deeply impressed by Faraday's idea of lines of force. Indeed Maxwell could be considered a faithful follower of Faraday in the development of a field theory. But as a trained mathematician Maxwell was able to do what Faraday could not. Maxwell placed Faraday's ideas on a firm mathematical basis.

We must, however, recognize that Maxwell's goals were not to form the field theory we presently have. For Maxwell the fields and the medium in which the fields existed, the aether, or luminiferous aether, were the primary quantities. The particles and the currents were secondary concepts that were manifestations of the aether [97, p. 279].

The aether has a nuanced history, as do many scientific concepts, which eventually become replaced by others. The aether has roots in an Aristotelian concept

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[8]This is a fundamental statement regarding the geometry of the magnetic field lines, but does not define a physical relationship among measurable physical quantities.

[9]The modern term *electromotive force* (emf) is the integral around a contour of this **E**. That the integral does not vanish indicates that **E** resulting from the rate of change of the magnetic field intensity **H** is not conservative.

to avoid the vacuum. We are tempted to claim that the aether was reintroduced into scientific discussion with the discovery that light was a transverse wave phenomenon [79, pp. 48, 188, 269] and then dropped with the failure of Albert A. Michelson (1852–1931) and Edward W. Morley (1838–1923) to measure the aether drift in 1887 [71], [79, p. 277]. But this story is inaccurate. Any attempt to gloss over the role of the aether in the development of Maxwell's theory does an injustice to his ideas and, as a consequence, to our understanding of the revolution in thought that accompanied the removal of the aether of the nineteenth century from modern physics.

If we attempt to make light of the aether we must also ignore the statement of Einstein's regarding the aether in a lecture at the University of Leyden in 1920. The aether made no appearance and was unnecessary in Einstein's 1905 paper on special relativity. But that paper considered only electromagnetic fields and not gravitational fields. The general relativity theory of 1915 resulted in an effect of mass on space. This effect Einstein noted was the equivalent of an aether, although not of an aether in the form developed in the nineteenth century.

> ...according to the general theory of relativity space without aether is unthinkable; for in such a space there would be not only no propagation of light, but also no possibility of existence for standards of space and time (measuring-rods and clocks), nor therefore any space-time intervals in the physical sense. But this aether may not be thought of as endowed with the physical quality characteristics of ponderable media ... The idea of motion may not be applied to it [79, p. 303], [78, p. 313].

For Faraday the particles were manifest only as terminus points of the lines of force. And currents were states of the aether. Then in Maxwell's theory the pressure of the particles on one another corresponded to the electrical potential, while the displacement of the particles from equilibrium resulted in a tangential action on the cells in the aether. Variations in the *displacement* of these cells was regarded as current. Displacement of the particles alone was not current [97, p. 278].

For Faraday and Maxwell the concepts of particles and currents emerged from the field dynamics, which involved the mechanics of the aether. The particles were not the sources of the fields.

Maxwell himself was never completely successful at keeping the fields primary and the particles and currents secondary. Charges and currents do appear as independent quantities in his later work. This, as D.M. Siegel points out, makes the totality of Maxwell's work difficult to interpret [86, p. 115]. We shall not labor over the details of Maxwell's development. But we shall not pass lightly to what we now refer to as Maxwell's Equations.

Hendrik A. Lorentz[10] combined the Maxwellian tradition with the Continental tradition, which regarded the particles and currents as primary. It is this mixed tradition that comes to us now [86, p. 115].

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[10]Hendrik Antoon Lorentz (1853–1928) was a Dutch theoretical physicist.

#### 1.11.2.2 The Field Theory

Maxwell's theory, as we consider it here, appeared as a paper entitled *On Physical Lines of Force*, which was published in a series of installments over an 11-month period between 1861 and 1862. This is a theory of *molecular vortices* and was a detailed and elaborate mechanical theory of the aether which was the medium sustaining the fields. Maxwell was influenced by Thomson's[11] arguments that magnetism was a *rotational phenomenon*. Tubes of force or vortices consisted of lines of magnetic force and each tube contained a fluid in motion about the tube axis [97, p. 276].

Thomson's ideas led him to an energy per unit volume in the magnetic field as $(1/(8\pi))\,\mu H^2$ by identifying $\mu$ (the permeability) as the medium density and $\mathbf{H}$ (the *magnetic intensity*[12]) as the tangential velocity at the surface of the tube. To allow the rotating tubes to be close to one another Maxwell introduced idler wheels between the tubes.

In addition, as we pointed out above, Maxwell regarded electrical current to be a translational phenomenon (displacement of cells in the aether) with a current density $\mathbf{J}$. He was then able to obtain a kinematical relationship between $\mathbf{H}$ and $\mathbf{J}$ as

$$4\pi\mathbf{J} = \operatorname{curl}\mathbf{H}. \tag{1.7}$$

This is Ampère's Law in the form it attained in the middle of the nineteenth century.

In the third installment of "Physical Lines," Maxwell modified (1.7) to

$$\mathbf{J} = \frac{1}{4\pi}\operatorname{curl}\mathbf{H} - \frac{1}{4\pi c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{1.8}$$

where $c$ is a constant, which is equal to the speed of light [86, p. 112]. This is a notable, and perhaps curious, equation. If we take the divergence of (1.8) and use Coulomb's Law we get the correct form of charge conservation for the time-dependent case. But there is no direct evidence that obtaining a consistency of this sort was foremost in Maxwell's mind. Had that been the case he would surely have mentioned this in the paper. But he did not.

Maxwell's primary concern was to develop a mechanically based theory of molecular vortices in the aether. In Maxwell's theory the variation of $\mathbf{E}$ produced variations in the displacement of cells in the aether, which he considered to be the current. The description of this mechanism is in Faraday's Law

$$\mu\frac{\partial}{\partial t}\mathbf{H} = -\operatorname{curl}\mathbf{E}. \tag{1.9}$$

[11]William Thomson, later Lord Kelvin, was a personal friend of Maxwell's, as well as a scientific collaborator.

[12]The magnetic field is designated by magnetic induction $\mathbf{B}$ or as magnetic intensity $\mathbf{H}$. These are related by the permeability $\mu$ as $\mathbf{H} = (1/\mu)\,\mathbf{B}$.

A time dependence of $\mathbf{H}$ will, through (1.9), produce a time dependence of $\mathbf{E}$, which will result in a current $\mathbf{J}$ through (1.8) [86, p. 139].

Faraday had only applied this idea to the polarization of dielectrics. In a constant electric field the dielectric medium becomes polarized with a positive charge concentration at one end and a negative charge concentration on the other. For Faraday, as we recall, charge was associated with the terminus of electric field lines.

In a modern picture polarization arises from shifts in electron charge densities on molecules and shifts in sublattices of ionic crystals, which result from variations in the external electric field. The variation in polarization charge density is a current.

Maxwell's picture is of an aether-based phenomenon and goes beyond this. For Maxwell the change in electric field is the source of the current.

If we write the polarization vector as $\mathbf{P}$ then Maxwell is saying that in the presence of the dielectric there is a (polarization) current

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}, \tag{1.10}$$

and Ampère's Law (1.7) must be modified to acknowledge this. That is (1.8) must become

$$\mathbf{J} = \frac{1}{4\pi}\,\mathrm{curl}\,\mathbf{H} + \frac{\partial \mathbf{P}}{\partial t}. \tag{1.11}$$

Comparing this with (1.8) we can identify the displacement as the vector

$$\mathbf{D} = \mathbf{P} = -\frac{1}{4\pi c^2}\mathbf{E} \tag{1.12}$$

and the displacement current as

$$\frac{\partial}{\partial t}\mathbf{D} = -\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\mathbf{E}. \tag{1.13}$$

Here $\mathbf{E}$ is a (vector) electromotive force. It is a nonconservative electric field. We have kept the signs consistent with Maxwell's picture in which $-\partial \mathbf{E}/\partial t$ is the source of a current. It is not itself a current.

With

$$\mathrm{div}\,\mathbf{H} = \mathbf{0}, \tag{1.14}$$

Maxwell had the complete set of equations of motion for the system of charges and vortices.

Considering a medium in which $\mathbf{J} = \mathbf{0}$, Maxwell was able to show that (1.8), (1.9) and (1.14) combined to give

$$\frac{\partial^2}{\partial t^2}\mathbf{H} = c^2\nabla^2\mathbf{H}, \tag{1.15}$$

with the same basic equation for **E** except that $\mathbf{E} \perp \mathbf{H}$. Equation (1.15) is the wave equation. The mathematical solutions to (1.15) are oscillatory (sinusoidal) disturbances propagating through the aether at a speed $c$. From the electromagnetic experiments of Wilhelm Weber (1804–1891) and Friedrich Kohlrausch (1840–1910) [95] Maxwell was able to calculate this speed of the disturbance in the aether ($3.107 \times 10^8\,\mathrm{m\,s^{-1}}$). This was very nearly the value Hippolyte Fizeau (1819–1896) [29] had found for the speed of light in air ($3.14858 \times 10^8\,\mathrm{m\,s^{-1}}$) and the more accurate value found by Léon Foucault (1819–1869) [30] ($3.08 \times 10^8\,\mathrm{m\,s^{-1}}$). Maxwell did not hesitate to point to the identity of this disturbance and light [97, p. 283], [66, p. 499].

In his memoir of 1865, "A Dynamical Theory of the Electromagnetic Field," Maxwell removed the aether-based architecture beneath the fields. The field equations alone remained [66]. And in 1868 he proposed to base the electromagnetic theory of light solely on the equations [67], [68, p. 125]

$$\operatorname{curl}\mathbf{H} = 4\pi\mathbf{S}, \tag{1.16}$$

where

$$\mathbf{S} = \mathbf{J} + \frac{1}{4\pi c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{1.17}$$

is the total current, and

$$-\operatorname{curl}\mathbf{E} = \mu\frac{\partial}{\partial t}\mathbf{H}. \tag{1.18}$$

This is an essentially modern picture of the fields. At the end of the 1868 paper Maxwell provided a detailed demonstration that the velocity of light is calculable from the equations of the electromagnetic field.

Maxwell's 1868 paper also provides us with the letters that we now use to designate the field vectors. Maxwell used an alphabetical order as the fields appeared in the text [44, p. 232]. These are

- Vector potential **A**
- Magnetic induction **B**
- Displacement **D**
- Electric field **E**
- Magnetic intensity **H**

Maxwell used **C** to designate the current density and **F** to designate electromagnetic force.

Here we see the source of the difficulty in trying to fit Maxwell's theory completely into the modern picture of classical field theory. As a theoretical physicist Maxwell realized that (1.16) and (1.17) must apply in empty space. If they did not there would be no consistent *electromagnetic theory* for light.

Maxwell based his argument for (1.17) on the reality of the aether and the belief that currents were manifestations of the motion of the aether. If we discard the aether we are left with no solid *mechanistic argument* that results in the pair of equations (1.16) and (1.17), except for the fact that they are together necessary for agreement with Hertz' experiments that we discuss in the succeeding section.

Physicists agree that the displacement current must appear in Ampère's Law even in free space. We cannot, however, point to a classical mechanistic basis for this current. The same difficulty occurs with Faraday's Law in empty space.

## 1.12 Experimental Evidence

### *1.12.1 Waves in the laboratory*

Heinrich Hertz was able to produce electromagnetic oscillations at one point in the laboratory at the Karlsruhe Technische Hochschule in Baden, Germany, where he had just arrived as professor in 1887. He was 30 years old, but rising rapidly in German academia [9, pp. 217–218].

Hertz was interested in the experimental problems set forth by the *Berlin Academy* (*Preussische Akademie der Wissenschaften zu Berlin*) in 1879 [19, p. 234]. The Berlin Academy had singled out the central problem in the Maxwell theory as that of treating the displacement current in (1.13) as an actual current.

A prize was offered for decisive experimental proof that (1) changes of dielectric polarization in nonconductors produce the same electromagnetic forces as do the currents which are equivalent to them, (2) electromagnetic forces as well as electrostatic are able to produce dielectric polarizations and (3) in all these respects air and empty space behave like all other dielectrics [46, p. 6].

The Academy had contented itself requiring confirmation of either one of the first two.

Hermann von Helmholtz, Hertz' doctoral advisor and post doctoral mentor, brought the prize to Hertz' attention and promised the assistance of the Physical Institute in Berlin should he take up the work. But, upon study of the experimental possibilities, Hertz reluctantly concluded that the effects would be just within the limits of observation and, therefore, decided against the undertaking [46, p. 1]. But Hertz wrote he was still "ambitious to discover it by some other method," and was, therefore, becoming interested in electrical oscillations.

Hertz' experiments at Karlsruhe began as lecture demonstrations with Riess or Knochenhauer spirals, which demonstrated induction between two spiral conductors in parallel planes. A group of Leyden jars (a Leyden battery) was charged to a high voltage, which was discharged across the primary coil. The voltage appearing in the secondary coil resulted in a spark between the terminals of the secondary [19, p. 239].

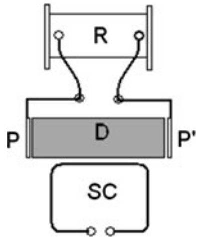


**Fig. 1.8** Hertz' apparatus for investigating the polarization of a dielectric. $R$ is the Ruhmkorff coil. The primary circuit is the capacitor with plates $P$ and $P'$. $D$ is the dielectric. And $SC$ is the secondary, which is a rectangle of 2 mm copper wire with a gap for sparking. [Permission by Dover Publications, Inc.]

Hertz eventually found that small Leyden jars sufficed. And he used a Ruhmkorff coil[13] to drive the primary Riess coil [46, p. 2], [19, p. 239].

Hertz became intrigued by the sparks appearing across the terminals in secondary Riess coil, which he termed *side sparks*. At first he thought the sparking was too irregular to be of any use beyond demonstration until he identified a neutral point (node) in one of the conductors. That meant that the oscillations were regular and the wavelength could be measured. He then believed that the first question of the Berlin Academy could be answered [46, p. 2].

We have drawn Hertz' apparatus for investigating the polarization of a dielectric in Fig. 1.8 (cf. [46], p. 5).

The Ruhmkorff coil $R$ produced high frequency (10 kHz) oscillating voltage across the dielectric $D$ located between the plates $P$ and $P'$. Hertz termed the secondary circuit $SC$ a *Nebenkreis* (side circuit).

If the oscillating voltage across $D$ produced a polarization current in $D$ the result would be an oscillating magnetic field in $SC$ and an electromotive force in $SC$ by Faraday's Law. The electromotive force would produce sparks across the gap in $SC$.

Hertz observed strong sparks across the gap in $SC$ [46, p. 5]. This was the experimental result he reported to the Berlin Academy on 10th November, 1887. The first experimental proof asked for by the Berlin Academy had been provided. The polarization current in the dielectric had produced the same effect as a standard current.

Hertz then prepared to provide the second experimental proof when it occurred to him that the central issue was not in the first two hypotheses. The third hypothesis contained the essence of Faraday's and, therefore, of Maxwell's position. What Hertz needed to do was demonstrate the propagation of electromagnetic waves at a finite velocity in air.

[13] A Ruhmkorff coil is an induction coil with a few turns in the primary, many turns in the secondary, and a core of iron threads.

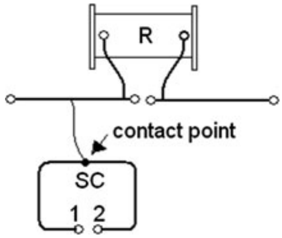


**Fig. 1.9** Hertz' circuit with Primary circuit connected to secondary [Permission by Dover Publications, Inc.]

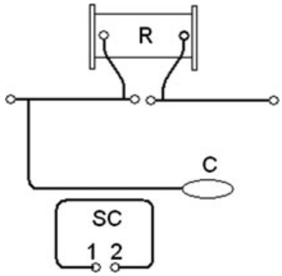


**Fig. 1.10** Hertz' apparatus with Nebenkreis $SC$ separated from discharge (primary) circuit by a small distance [Permission by Dover Publications, Inc.]

In this Hertz elected to be guided only by experiment. In part this was because of the difficulty in interpreting Maxwell. Hertz remarked, just as Siegel would a century later, that the difficulty in grasping the totality of Maxwell's ideas has caused many "to abandon the hope of forming … an altogether consistent conception of Maxwell's ideas."

Hertz said that he fared no better. He also adamantly claimed that the difficulty is not mathematical. Maxwell's theory, he wrote finally, is Maxwell's set of equations. It is not Maxwell's particular conceptions or methods [46, pp. 20–21].

In his book *Electric Waves* Hertz recounted in detail his reasoning and the series of experiments he conducted as he moved toward establishing the existence of Maxwellian electromagnetic waves in air. He began by linking the Nebenkreis $SC$ directly to a pole of the Ruhmkorff coil. We have drawn the apparatus in this configuration in Fig. 1.9 (cf. [46, p. 34]).

In this experiment Hertz moved the contact point for the wire with the Nebenkreis $SC$ and noted the sparks at the gap 1–2. If the contact point was equally distant from the points 1 and 2 the sparks disappeared.

This experiment convinced Hertz that a wave was excited in the parts of the Nebenkreis $SC$. This wave had a finite velocity in the wire of the Nebenkreis, or else it would have been impossible to regulate its appearance at the points 1 and 2 by shifting the contact point.

Hertz reasoned that the current in the discharge of the Ruhmkorff coil could then induce an electromotive force in the Nebenkreis $SC$ by mutual induction without the wire connecting the two. We have drawn the apparatus for this experiment in Fig. 1.10 (cf. [46, p. 37]).

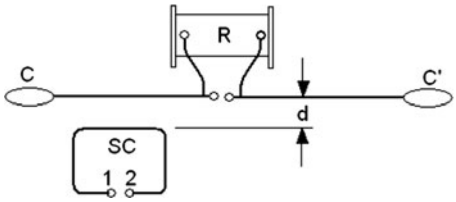


**Fig. 1.11** Hertz' symmetrized primary circuit. The distance $d$ was increased from 50 to 150 cm and sparks, although diminished, were still observed. [Permission by Dover Publications, Inc.]

Hertz formed a lengthened section of heavy (2 mm diameter) copper wire of the primary circuit and added a large insulated conductor to the end to increase the current in this extension. He observed sparks in the gap 1–2.

It also became clear to Hertz that the regularity of the oscillations was critical. So he symmetrized the experiment as we have shown in Fig. 1.11 (cf. [46, p. 40]).

In this experiment Hertz observed sparks at the gap 1–2 in the Nebenkreis when the separation $d$ was as much as 150 cm. He even walked between the primary and the Nebenkreis and noted no change in the sparks.

To us, from our perspective, it seems clear that Hertz had established the existence of electromagnetic waves that propagated through space. These had been produced by the antenna with its two insulated conductors $C$ and $C'$ and had been picked up by the receiving antenna $SC$. But Hertz was a careful experimentalist and jumped to no conclusions without experimental justification.

He followed these experiments with demonstrations that the waves had finite velocity, he showed that the waves could be reflected, could be focused, and could be diffracted. These were electromagnetic waves that behaved in the same manner as do light waves.

The concept of action at a distance, we may say, died in Hertz' laboratory at Karlsruhe. The electromagnetic fields could no longer be considered a mathematical crutch for calculations.

### *1.12.2 Wave energy and momentum*

Whether we consider them to move in the aether or in empty space we should expect that electromagnetic waves will have energy and momentum. The energy transported by electromagnetic fields had been discovered mathematically by John H. Poynting in 1884 [19, p. 182]. Hertz was aware of this and calculated the energy coming from his primary circuit [9, p. 320].

By implication there will be a momentum associated with this energy, which is transported at the speed of light. This momentum may be observable if we can direct a high energy light beam on a very light suspended object using a balance similar to Coulomb's.

Maxwell had also identified a momentum in the waves. He treated matter as an extension of the aether to be distinguished from the aether only by altered values of the constants. Particularly Maxwell made no distinction between stress in a material body and stress in the aether [97, p. 302].

From his theory of stresses in the aether Maxwell was able to deduce that an electromagnetic wave would exert a pressure on a conductor. Energy would then be transported to matter when an electromagnetic (Maxwellian) wave was incident on the matter.

At one time the adherents to the Newtonian corpuscular theory of light believed that demonstrating the pressure of light on a surface would be the final vindication of the corpuscular theory of light. And measurements to demonstrate the pressure of light had been conducted in the eighteenth century. These used intense beams of light and delicately suspended bodies. Light pressure was not observed in these experiments.

Final experimental confirmation of the pressure of light was in 1899 [97, p. 307]. By then the validation was not of Newton's, but of Maxwell's theory.

## 1.13 Michelson and Morley Experiment

A blow, but not the final blow, to the aether picture came with the failed attempt by Albert A. Michelson (1852–1931) and Edward W. Morley (1838–1923) to measure the aether drift in 1887 [71], [79, p. 277].

Michelson was Professor of Physics at Case School of Applied Science (presently the engineering and scientific school of Case Western Reserve University) and Morley was Professor of Chemistry at Western Reserve College (presently the liberal arts school of Case Western Reserve University).

Michelson was young and unknown and needed the prestige of Morley to fund the experiment he wanted to do. He had tried the experiment in Germany on a modest form of his interferometer in 1881. The idea now was to expand the size of the interferometer arms by introducing multiple reflections.

The issue was fundamental. The aether had become a central part of the Maxwell theory. However, as Hertz had noted in his attempts to understand exactly what the theory entailed, Maxwell's theory was Maxwell's Equations. That is, when you finally had the equations there was no need to mention the complex scaffolding of the aether that presumably was so critical. We may then ask whether or not we can actually measure effects of the aether. And the first thing that comes to mind is the effect the ether drift should have on the propagation of light.

The experimental problem had been to measure the speed of light in two directions and to compare the results. We have illustrated the experimental problem in Fig. 1.12.

If the aether is at rest in the universe (in Newtonian terms) the earth must be moving with respect to this stationary aether. Then at some time during the year the aether drift will be along the axis of our apparatus, as we have shown. If we

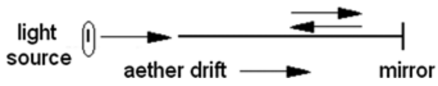


**Fig. 1.12** The speed of light in two directions

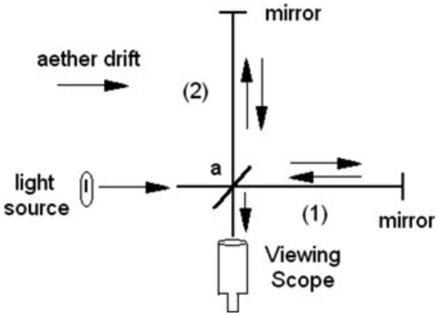


**Fig. 1.13** Schematic of Michelson and Morely apparatus

can measure the times taken for the light to pass first with the aether drift and then against it we will be able to see the effect of the aether drift on the speed of light.

We accept that light has a speed $c$ in the aether and that the earth has a speed $v$ in the aether. Then the light moves down the arm of our apparatus (with the aether drift) at a speed $c + v$ and it returns (against the aether drift) at a speed $c - v$ and our clocks located at the light source and at the mirror will record two times of passage for the light. These are the time for passage with the aether $t_1 = L/(c + v)$ and the time of passage against the aether $t_2 = L/(c - v)$. The difference in these two measured times is $t_2 - t_1 = \left(2Lv/c^2\right)/\left(1 - v^2/c^2\right)$.

We knew the speed of light from Fizeau's measurements at the middle of the nineteenth century. And we knew the velocity of the earth in orbit. So we knew the value of $v/c$. No pair of clocks could measure the time differences required for this experiment to be carried out.

But Michelson had a different approach that did not require clocks. Michelson decided to measure phase shifts. In Fig. 1.13 we have drawn the basic apparatus.

Again we assume that when the measurement is made, at some time during the year, the aether drift is to the right. Then the motion of light along the axis (1) is affected by the aether drift, but not along axis (2).

We split the single light beam coming from the source into two beams by the half-silvered mirror at $a$. One of these beams travels along axis (1) and the other along axis (2) and we view the recombination of the beams in the telescope.

What we see is a set of concentric rings. We can align a viewing micrometer with one of the rings, rotate the apparatus, and note any shift in the position of the ring.

There were practical experimental issues to be overcome.

- The arms (1) and (2) need to be long. This was accomplished by using sets of four mirrors each.
- The apparatus needed to be very stable so that the fringes remained stationary, but must be easily rotated. This was accomplished by mounting the apparatus on a stone block and floating the block in mercury.

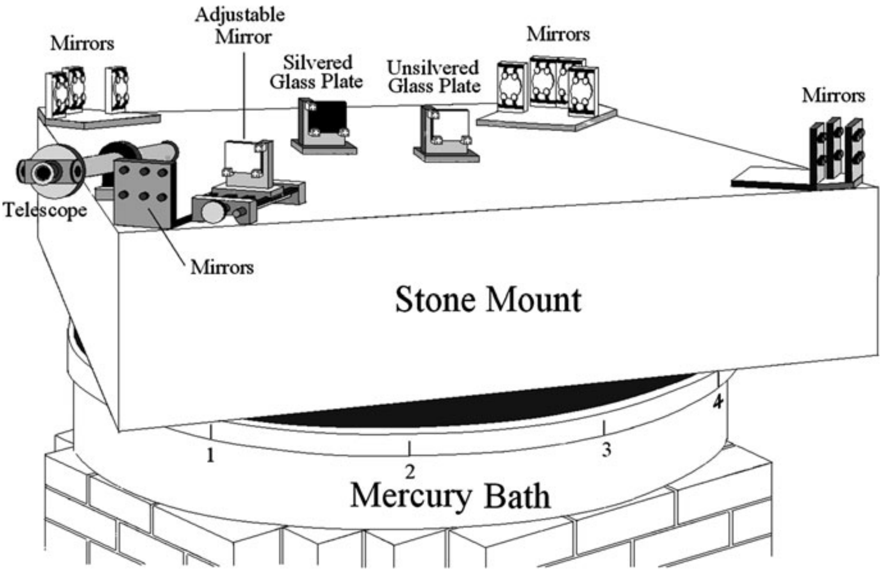


**Fig. 1.14** Michelson and Morely apparatus

- The light must be intense. For this an *Argand burner* was used, which has a central air flow to enhance the burning. The light was then very bright.

We show the resulting apparatus in Fig. 1.14, which is redrawn from Fig. 3 from the paper in which the results were reported.

This is a premier example of nineteenth century research in the physical sciences. The idea and the design of the apparatus represent genius. The conduct of the experiment was very carefully carried out. Michelson and Morley were also able to demonstrate that they would be able to detect the motion of the earth in the aether by measuring the motion of the fringes.

But they detected no motion of the earth in the aether. This is interpreted as the statement that there is no aether.

Again, however, the result is nuanced. In 1895 Hendrik A. Lorentz (1853–1928) analyzed the Michelson–Morley experiment showing that their null result could have been obtained even if the aether were present [24, pp. 3–7]. The molecular interactions, which are electromagnetic, would also be transmitted through the aether. This would result in a contraction of the length of the interferometer arms in the direction of motion. The Michelson–Morley experiment would then be unable to detect any aether drift because the apparatus dimensions simply changed to cancel out any possible effect.

Therefore, in spite of the fact that the Michelson–Morley experiment is a premier example of research in the physical sciences, it may have actually demonstrated nothing of great importance. At the end of the nineteenth century we at least had no definitive proof or refutation of the aether.

## 1.14 Relativity

The inability of Michelson and Morley to measure an aether drift was distressing to the physics community at the end of the nineteenth century. And some of the great minds of theoretical physics were occupied with the problem. In addition to Lorentz, Henri Poincaré (1854–1912) also contributed to attempts at resolution, as did George Francis FitzGerald (1851–1901), who in 1889 published a remarkable single paragraph paper[14] clarifying the Michelson–Morley result using the same mechanism for contraction that Lorentz would later (independently) use.

But there was also another problem that seemed to escape many familiar with electromagnetic fields. There was something deeply troubling about Faraday's Law. An electric field could be induced in a loop of wire by moving a magnet toward it. And the same result could be obtained by moving the wire loop while the magnet was stationary. The difficulty was that the interpretation of the experiment was different in the two cases.

If the magnet is in motion an electric field is produced in the vicinity of the magnet, with a definite energy, which produces a current in the wire. But if the magnet is stationary and the loop is moving there is no electric field. The charges experience a force and move in a direction perpendicular to the magnetic field and the motion of the wire resulting in a current. In this motion there is no energy imparted directly from the magnetic field to the charges.

It was this asymmetry in Faraday's Law that disturbed Einstein.

Einstein was also aware of the question of the aether drift. At least he also mentioned the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," which was the aether. But Einstein drew a different conclusion from the inability to measure an aether drift than did FitzGerald, Lorentz, or Poincaré. Einstein suggested that electrodynamics as well as Newtonian mechanics possess no properties corresponding to the idea of absolute rest.

Einstein then wrote that "the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good." This he elevated to the status of a postulate and called it the *Principle of Relativity*.

To this he added a second postulate, which he admitted is only apparently irreconcilable with the first. This second postulate is "that light is always propagated in empty space with a definite velocity $c$ that is independent of the state of motion of the emitting body [24, p. 38]."

Besides an acknowledgement at the beginning of the paper, there is no mention of the aether at any other point. The aether, on which Maxwell had labored, was an unnecessary construct for Einstein. He was certainly not basing his ideas on the failure of Michelson and Morley to discover an aether drift [71]. Einstein also later

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[14]The paper was *The Ether and the Earth's Atmosphere*, published in the American journal Science [Science, 1889, 13: 390].

denied any direct influence of the Michelson and Morley result on his thought [78, pp. 114–117].

Einstein had discovered that the problem lay in our lack of understanding of time. He began the first part of the paper, the kinematical part, with a discussion of time and the meaning of simultaneity. In a few short sentences he removed the Newtonian concept of absolute time [79, p. 231] and replaced it with the realization that all statements about time deal with *simultaneous events*. One of those events is something physical occurring in our immediate vicinity and the other is the number indicated on our timepiece.[15]

Here Einstein referred to the arrival of a train and the location of the hands on a clock (appearance of a number on a timepiece) [24, p. 39]. This defines what we mean by time in our immediate vicinity. But for time to have a meaning to us beyond that we must have a way of synchronizing our timepiece with timepieces located at points remote from us.

To do this Einstein defined the *synchronization* of timepieces. Two timepieces remote from one another are synchronized if the time taken for a light ray to travel from one to the other is the same as the time taken for the return. This requires the second postulate regarding the constancy of the velocity of light.

The frames of reference for which the laws of mechanics hold good, which are those traveling at constant velocity relatively to one another [24, p. 45], are called *inertial frames*. We may consider one such inertial frame to be fixed and the other(s) to be moving with respect to that frame at constant velocitie(s). Each inertial frame is equivalent as far as the laws of physics are concerned. But an event observed in one inertial frame will have a different appearance when viewed from another inertial frame, if they are moving relatively to one another.

Einstein considered particularly a timepiece synchronization experiment observed in two inertial frames and experiments in which light beams were sent down various axes. These were thought experiments. But thought experiments are not fantasy. It must be possible to perform each of the measurements required.

The result of these thought experiments was a space and time transformation between inertial reference frames that was *mathematically* the same as the Lorentz Transformation [24, pp. 11–34]. In 1905 Einstein was not aware of the Lorentz Transformation [78, p. 133]. The basis, however, was completely different. Einstein had recognized that time was the basis for the new kinematics.

The curious results of this transformation were the fact that timepieces in a moving inertial frame, when observed from a stationary inertial frame, appeared to be running slowly. The time intervals in different inertial frames were then not the same. Similarly the length of a rod in a moving inertial frame appeared shortened in the direction of motion, when observed from a stationary inertial frame.

[15]The terminology Einstein used was *clock*. We have used *timepiece* because the clock with hands is becoming less common.

The most important result was, however, that the Maxwell–Hertz[16] Equations for the electromagnetic field remained the same in all inertial frames [24, pp. 51–53]. That is Einstein's concept of time and the constancy of the speed of light provided the basis for what Einstein had called the Principle of Relativity. This also resolved the asymmetry of Faraday's Law.

The apparent asymmetry in Faraday's Law was not the result of a difficulty in the Maxwell Equations. It was a result of the fact that we did not understand time, and as a consequence, space.

Einstein's recognition that time was suspect and that an understanding of time resolved the issue came suddenly to him during an afternoon's discussion with his friend Michele Besso. Einstein said he returned the next day and, without even first greeting his friend, said to him, "Thank you. I've completely solved the problem."

The paper, which was completed five weeks later, contained a note of thanks to Besso, but no other acknowledgements or references.

The Principle of Relativity is of more importance than is apparent. The actual issue is, as we will later discuss, the covariance of the laws of physics. The laws of mechanics and of electrodynamics are covariant under Lorentz Transformation.

Nowhere in the paper is the aether introduced as a part of the argument. That is Einstein had effectively removed the aether from consideration. There was no need for it as long as we accept that space and time no longer exist as separate entities and that only a union of the two is preserved as an independent reality, as was pointed out by Hermann Minkowski [24, p. 75].

This is also the issue raised by Einstein in the Leyden lecture of 1920 referred to above. The aether is no longer a ponderable medium. Its existence is in the geometry of space-time.

## 1.15 Summary

In this chapter we have introduced the concept of a vector field and provided a brief overview of the history of the development of classical field theory. In this we have included a discussion of the basis of Maxwell's mathematical theory using the vector equations which will be completely developed in the text.

We culminated our discussion with Hertz' laboratory experiments, that demonstrated the reality of the fields ending serious discussion of action at a distance, and with Einstein's realization that classical field theory requires a new understanding of time and of space. A new door had opened at the back of the laboratory.

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[16]Einstein used the form of the Maxwell equations employed by von Helmholtz and Hertz [46, p. 201].

Our intention in the remainder of the text will be to develop the theory mathematically. In this we will show that Maxwell's field equations result logically from the experiments that we have outlined here. We will use a modern picture with no reference to the aether. Our presentation of Einstein's theory will also be based on a modern approach.

For the reader who wishes to understand classical field theory philosophically as well as mathematically, however, this chapter is an important step.

## Questions

**1.1.** Define the concept of the vector field and carefully note how it differs from the simpler concept of a vector.

**1.2.** How does the idea of vector field differ from that of action at a distance, or are they both really the same?

**1.3.** What, briefly, were Newton's thoughts on action at a distance? Why did Newton not propose the field?

**1.4.** Is there any part of Coulomb's experiment that requires the field concept? Oersted's experiments? Ampère's Experiment?

**1.5.** Faraday first used the term field in reference to magnetism after his discovery of induction. Why do you suppose Faraday would begin to think that a field was a viable concept? He clearly had not needed this concept for his work on induction.

**1.6.** The aether is not a trivial concept. Why did it seem necessary? Do you think Newton would have liked it?

**1.7.** Maxwell labored with the aether as a primary concept. Maxwell's ideas were ahead of his time, but he was also a product of nineteenth century science. As his friend William Thomson (Lord Kelvin) Maxwell believed in a mechanical universe. Why was the aether so important in the mechanical picture of the universe?

**1.8.** Faraday's Law is based on hard experimental data. We simply must accept it as fact. If we remove the aether from consideration what is the interpretation of Faraday's Law? Can you reconcile this with a mechanical universe?

**1.9.** The existence of the displacement current is critical in Maxwell's theory. The Prussian Academy recognized this and called for experimental proof. Why is the displacement current central to Maxwell's theory? Why not just drop it and avoid controversy?

**1.10.** Critique Hertz' comment that Maxwell's theory is Maxwell's Equations. Is something deep being revealed here? Or is this a throwaway comment? To what are we committing ourselves if we agree with Hertz?

**1.11.** Why were the results of the Michelson–Morley experiment considered important?

**1.12.** What was the principal concern of Einstein in his 1905 paper on special relativity? What was the role of the Michelson–Morley result?

**1.13.** Einstein himself spoke of special relativity as being the "step." He had realized something that set his ideas apart from those of FitzGerald, Lorentz, and Poincaré. What was this? Einstein's ideas resulted in the same transformation as Lorentz'. What was different?

**1.14.** The Prussian Academy realized that the displacement current was critical in field theory. Einstein realized that there was a problem with Faraday's Law. Faraday's experimental discovery of induction brought time into field theory as a variable. The displacement current is a time dependent *something*. Special relativity emerged from a new realization of the meaning of time. Write an essay tying these *neatly* together *scientifically*.

# Chapter 2
# Mathematical Background

*The place of mathematics in the physical sciences is not something that can be defined once and for all. The interrelations of mathematics with science are as rich and various as the texture of science itself.*

*Freeman Dyson*

## 2.1 Introduction

In its present form, after the intense intellectual effort that we outlined in the preceding chapter, classical field theory comes to us as an essentially complete science. We may now develop it as a masterpiece of absolute, mathematically based science, studying only the final form of the theory as a set of mathematical equations linked completely to a set of identifiable experiments.

Our study, of necessity, must then begin with an outline of the mathematics we will use. We have devoted this chapter to that task. Here we include the topics

- Vector analysis
- Multivariate functions and coordinate systems
- Vector calculus
- Laplace and Poisson partial differential equations
- Helmholtz' Theorem for the vector field
- The Dirac Delta Function

We have placed proofs and vector identities in the appendices in order to keep the discussion fluid. The reader who is already familiar with these topics may choose to only look over this chapter. Because coordinate notations and the representation of some vector operators are not universal, however, we do not recommend ignoring this chapter.

We will take up further mathematical developments as we need them. These include, for example, Green's Functions for the solution of nonhomogeneous

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_2, 

differential equations,[1] Fourier transforms for the study of waves, and tensors for the study of relativity.

## 2.2 Vectors

### *2.2.1 The Vector Space*

We begin with the concept of an *abstract vector space*, which is defined through a set of postulates. This concept contains the familiar displacement, momentum, and force vectors as special cases.

In modern physics vectors are the Dirac ket vectors, or simply kets introduced by Paul A.M. Dirac (1902–1984) in the first edition of his monograph on quantum theory [21]. The vector analysis and the vector calculus we shall use in classical field theory, however, retains the *dyadic* form of the vectors, such as **a**, **b**, .... Both ket vectors and dyadic vectors are examples of representations of the vectors in a space.

The dyadic representation and the vector calculus of three dimensions were the invention of Josiah Willard Gibbs (1839–1903), professor of mathematical physics at Yale College in the last decades of the nineteenth century. In Gibbs' own words,

> One of the principal objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity.

The dyadic notation did this for vector quantities in three dimensions and was a great step forward in the application of mathematics to physics.

The set of quantities **a**, **b**, ... are elements of a *vector space* **V** if they satisfy the postulates of a vector space, which are [20]

1. *Closure under addition:* for each **a** and **b** which are elements of $\boldsymbol{V}$ there is a unique sum **a**+ **b** that is a vector in the space. That is[2]

$$\text{For } \mathbf{a} \in \boldsymbol{V} \quad \text{and} \quad \mathbf{b} \in \boldsymbol{V}\ \exists\ \mathbf{a} + \mathbf{b} = \mathbf{c} \in \boldsymbol{V}. \tag{2.1}$$

2. *Addition is associative:*

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}). \tag{2.2}$$

3. *Addition is commutative:*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}. \tag{2.3}$$

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[1] The terms nonhomogeneous or inhomogeneous are equivalent for describing differential equations with sources.

[2] The symbol ∈ means "is an element of" and ∃ means "there exists." These symbols are commonly used as shorthand in physics just as in mathematics.

4. There exists a *zero vector* **0** defined by the requirement that the addition of this zero vector to any vector in the space results in the original vector. That is[3]

$$\exists\, \mathbf{0} \ni \mathbf{a} + \mathbf{0} = \mathbf{a}\ \forall\ \mathbf{a} \in V. \tag{2.4}$$

5. There exists a *negative* of a vector, −**a**, defined by the requirement that the sum of any vector in the space and the negative of that vector produces the zero vector. That is

$$\forall\ \mathbf{a} \in V\ \exists\ -\mathbf{a} \ni \mathbf{a} + (-\mathbf{a}) = \mathbf{0}. \tag{2.5}$$

6. *Closure under multiplication by a scalar:* For every number from the field of complex numbers, $C$ (a real number is a complex number with an imaginary part equal to zero), and every vector from the space, **a**, there is a unique vector $C\mathbf{a}$ that is also contained in the space. Multiplication by a scalar satisfies

   (a)

$$C\,(\mathbf{a} + \mathbf{b}) = C\mathbf{a} + C\mathbf{b}. \tag{2.6}$$

   (b)

$$(C + D)\,\mathbf{a} = C\mathbf{a} + D\mathbf{a}. \tag{2.7}$$

   (c)

$$(CD)\,\mathbf{a} = C\,(D\mathbf{a})\,. \tag{2.8}$$

   (d)

$$1\mathbf{a} = \mathbf{a}. \tag{2.9}$$

In the last expression 1 is the number one, known as unity.

We will also require that our vector space has a *scalar product*, which, between two vectors **u** and **v**, we write as $\mathbf{u} \cdot \mathbf{v}$. The scalar product of *real vectors* has the properties that

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}, \tag{2.10}$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}, \tag{2.11}$$

$$(a\mathbf{u}) \cdot \mathbf{v} = a\,(\mathbf{u} \cdot \mathbf{v})\,, \tag{2.12}$$

[3] The symbol ∀ means "for all" and the symbol ∋ means "such that."

$$\mathbf{u} \cdot \mathbf{u} \geq 0, \tag{2.13}$$

$$\mathbf{u} \cdot \mathbf{u} = 0 \quad \text{if and only if} \quad \mathbf{u} = \mathbf{0}. \tag{2.14}$$

In (2.12) $a$ is a scalar.

The reader will recognize familiar manipulations in these postulates. Position and displacement vectors, with magnitude and direction, constitute a possible vector space defined by these postulates. Thinking of a vector as a quantity possessing magnitude and direction is often helpful in our mental picture of vectors in two or three spatial dimensions. But we will be prepared to extend our concept of vector space beyond the limits of two and three dimensions when we consider the representation of continuous functions, which also satisfy these postulates.

## *2.2.2 Representation*

### 2.2.2.1 Basis

We can make general statements in terms of abstract vectors. This includes the formulation of physical laws. The transition from abstract vectors to the forms of these vectors which we can apply to concrete situations requires that we represent the abstract vectors in a *basis*.

In rectangular Cartesian representation, with the familiar $(x, y, z)$ axes, the basis vectors are $\{\hat{e}_{\mathrm{x}}, \hat{e}_{\mathrm{y}}, \hat{e}_{\mathrm{z}}\}$. These are unit vectors, which means they are normalized to one, i.e. have magnitude one. This is the requirement that

$$\hat{e}_{\mathrm{x}} \cdot \hat{e}_{\mathrm{x}} = \hat{e}_{\mathrm{y}} \cdot \hat{e}_{\mathrm{y}} = \hat{e}_{\mathrm{z}} \cdot \hat{e}_{\mathrm{z}} = 1. \tag{2.15}$$

They are also orthogonal which means

$$\hat{e}_{\mathrm{x}} \cdot \hat{e}_{\mathrm{y}} = \hat{e}_{\mathrm{x}} \cdot \hat{e}_{\mathrm{z}} = \hat{e}_{\mathrm{y}} \cdot \hat{e}_{\mathrm{z}} = 0. \tag{2.16}$$

Basis vectors satisfying (2.15) and (2.16) are termed *orthonormal*. They are orthogonal and normalized.

We may combine the statements (2.15) and (2.16) into a single statement if we introduce the *Kronecker*[4] *delta* $\delta_{\mu\nu}$. The Kronecker delta is simply a counting index defined by

$$\boxed{\delta_{\mu\nu} = \begin{cases} 1 \text{ if } \mu = \nu, \\ 0 \text{ if } \mu \neq \nu. \end{cases}} \tag{2.17}$$

[4]Leopold Kronecker (1823–1891) was a German mathematician and logician who argued mathematics (arithmetic and analysis) must be founded on whole numbers. He claimed that "God made the integers," and that "all else is the work of man"[6].

With (2.17) we can write our conditions of orthonormality (2.15) and (2.16) as

$$\hat{e}_\mu \cdot \hat{e}_\nu = \delta_{\mu\nu}. \tag{2.18}$$

We may represent a three dimensional vector **a** (such as a distance, velocity, or a force) in the basis $\{\hat{e}_\mathrm{x}, \hat{e}_\mathrm{y}, \hat{e}_\mathrm{z}\}$ as

$$\mathbf{a} = a_\mathrm{x}\hat{e}_\mathrm{x} + a_\mathrm{y}\hat{e}_\mathrm{y} + a_\mathrm{z}\hat{e}_\mathrm{z}. \tag{2.19}$$

For three dimensional vectors we may also choose cylindrical or spherical representations. The basis vectors in cylindrical and spherical systems are $\{\hat{e}_\mathrm{r}, \hat{e}_\vartheta, \hat{e}_\mathrm{z}\}$ and $\{\hat{e}_\mathrm{r}, \hat{e}_\vartheta, \hat{e}_\phi\}$, respectively.

We represent the vector **a** in the basis $\{\hat{e}_\mathrm{x}, \hat{e}_\mathrm{y}, \hat{e}_\mathrm{z}\}$ by projecting the vector **a** onto the basis $\{\hat{e}_\mathrm{x}, \hat{e}_\mathrm{y}, \hat{e}_\mathrm{z}\}$. If we write the projector **P** as

$$\mathbf{P} = \hat{e}_\mathrm{x}\hat{e}_\mathrm{x} + \hat{e}_\mathrm{y}\hat{e}_\mathrm{y} + \hat{e}_\mathrm{z}\hat{e}_\mathrm{z}, \tag{2.20}$$

the projection of $a$ onto the basis $\{\hat{e}_\mathrm{x}, \hat{e}_\mathrm{y}, \hat{e}_\mathrm{z}\}$ is

$$\begin{aligned} \mathbf{P}\cdot\mathbf{a} &= \left(\hat{e}_\mathrm{x}\hat{e}_\mathrm{x} + \hat{e}_\mathrm{y}\hat{e}_\mathrm{y} + \hat{e}_\mathrm{z}\hat{e}_\mathrm{z}\right)\cdot\mathbf{a} \\ &= a_\mathrm{x}\hat{e}_\mathrm{x} + a_\mathrm{y}\hat{e}_\mathrm{y} + a_\mathrm{z}\hat{e}_\mathrm{z}, \end{aligned} \tag{2.21}$$

where $a_\nu = \mathbf{a}\cdot\hat{e}_\nu$ is the $\nu$th component of the vector **a**.

#### 2.2.2.2 Complete Basis

Vectors, the vector calculus (see [15, vol. II, pp. 88–93])[5], and vector analysis are all independent of the basis in which we represent the vectors, provided that basis is *complete*.

The mathematical question of completeness is not trivial if we are dealing with a vector space of arbitrary dimension (cf. [20, p. 37], [8, p. 302]). The requirement for completeness is finally, however, quite simply that the original (abstract) vector and the representation of that vector in the basis do not differ from one another.

If we have a complete representation of a vector in a basis we may then conduct all the required mathematical operations in that basis. For example, if we represent a three dimensional position vector **a** in the rectangular Cartesian basis $\{\hat{e}_\mathrm{x}, \hat{e}_\mathrm{y}, \hat{e}_\mathrm{z}\}$, we may conduct our mathematical operations on the algebraic quantities $\{a_\mathrm{x}, a_\mathrm{y}, a_\mathrm{z}\}$, which are the components of the vector.

For vectors that we may already realize remain in a plane (two dimensions) or in a three dimensional space, the requirement of completeness is that we have either two

[5] Courant's treatment of the independence of vector calculus on basis is particularly clear.

(for the planar vectors) or three basis vectors (for the spatial vectors). It is also very convenient, for our purposes, to require the basis vectors to be orthonormal. That is not, however, convenient in crystallography, where the basis vectors are imposed by the crystal symmetry.

Until we discuss relativity our spaces will be limited to two or three dimensions and we may avoid questions of completeness. In the four dimensional space of relativity time will become a dimension rather than a parameter. But that will not raise any questions of completeness. So we shall simply assume completeness for all of our representations of vectors.

When we come to the representation of continuous functions in the continuous basis of the Fourier transform we will have developed a simple proof of completeness specific for the Fourier transform.

In terms of the projector (2.20) our assumption of completeness means that the projector is the identity, i.e. $\mathbf{P} = \mathbf{1}$. If the projector is the identity then $\mathbf{P} \cdot \mathbf{a} = \mathbf{a}$, which is what is implied in (2.19).

Our development will be simpler if we are less explicit about the identity of our basis vectors. Therefore, we shall specify our *basis vectors* as $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ or simply as $\{\hat{e}_\mu\}_{\mu=1}^3$.

The vector $\mathbf{a}$ is then represented as

$$\mathbf{a} = \sum_{\mu=1}^{3} a_\mu \hat{e}_\mu. \tag{2.22}$$

A general representation in $N$ dimensions is

$$\mathbf{a} = \sum_{\mu=1}^{\mathrm{N}} a_\mu \hat{e}_\mu. \tag{2.23}$$

The *dimension* of a vector space is equal to the number of basis vectors required to represent a vector on the space.

#### 2.2.2.3 Sum Convention

In his publication *The Foundation of the General Theory of Relativity* in 1916, Einstein noted that in the equations he was obtaining repeated indices were always summed over three spatial coordinates and the one temporal coordinate, i.e. from 1 to 4 [24, p. 122]. These are the basis vectors in relativity.

It may appear premature to make the observation that Einstein made based only on what we have done so far. Nevertheless it is true that in our vector spaces repeated indices will always be summed from 1 to the number of basis vectors. In the interest of simplicity we shall, therefore, introduce this *Einstein sum convention*.

We will make clear instances when the sum convention is not used by either introducing Latin subscripts or by indicating that the sum convention is not used. Otherwise we will always sum repeated Greek indices from 1 to the number of basis vectors.

We may then write both (2.22) and (2.23) as

$$\mathbf{a} = a_\mu \hat{e}_\mu \tag{2.24}$$

as long as we remember the number of basis vectors that we have.

Introducing the Kronecker delta we may write (2.24) as

$$\mathbf{a} = \delta_{\mu\nu} a_\mu \hat{e}_\nu . \tag{2.25}$$

#### 2.2.2.4 Linear Independence

The vectors $\hat{e}_\mu$ in the set of basis vectors $\left\{\hat{e}_\mu\right\}_{\mu=1}^{\mathrm{N}}$ are *linearly independent* if

$$\mathbf{a} = a_\mu \hat{e}_\mu = \mathbf{0} \tag{2.26}$$

results only when

$$a_\mu = 0 \;\; \forall \mu .$$

That is only when none of the vectors in the set $\left\{\hat{e}_\mu\right\}_{\mu=1}^{\mathrm{N}}$ can be represented in terms of the others. This is a requirement for any complete set of basis vectors.

We can always construct a set of orthonormal basis vectors from a set of linearly independent basis vectors. The procedure by which we accomplish this is the Gram–Schmidt procedure.[6] We will not discuss this procedure here, since we have no direct use for it.

### *2.2.3 Scalar Product*

Using (2.24) and an equivalent representation for **b** we write the scalar product of **a** and **b** as

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= \left(a_\mu \hat{e}_\mu\right) \cdot \left(b_\nu \hat{e}_\nu\right) \\ &= \delta_{\mu\nu} a_\mu b_\nu \\ &= a_\mu b_\mu ,\end{aligned} \tag{2.27}$$

[6] The Gram–Schmidt procedure is named for Jørgen Pedersen Gram and Erhard Schmidt, although but it appeared earlier in the work of Laplace and Cauchy.

using (2.18). We can show that (2.27) satisfies the requirements for the scalar product (see exercises).

In elementary texts the scalar (or dot) product is defined in terms of the magnitudes of the vectors involved and the cosine of the angle between them. This definition of the scalar product and (2.27) are equivalent (see exercises).

### 2.2.4 Vector Product

The *vector* or *cross product* is defined only for three dimensional vectors. Two symbols are commonly used to designate the cross product. These are $\times$ and $\wedge$. We will use the symbol $\times$. And we will designate the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{a} \times \mathbf{b}$.

For two vectors $\mathbf{a}$ and $\mathbf{b}$, represented in the basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ with components $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$, the cross product is defined by the determinant

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \det \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\ &= \hat{e}_1 \left( a_2 b_3 - b_2 a_3 \right) + \hat{e}_2 \left( b_1 a_3 - a_1 b_3 \right) + \hat{e}_3 \left( a_1 b_2 - b_1 a_2 \right). \end{aligned} \tag{2.28}$$

The determinant of a square matrix may be written in terms of the *Levi-Civita density*[7] $\varepsilon_{\alpha\beta\gamma}$ [20], which we may consider here to be a counting index with slightly more complex properties than the Kronecker delta. The Levi-Civita density $\varepsilon_{\alpha\beta\gamma}$ is defined by

$$\boxed{\varepsilon_{\alpha\beta\gamma} = \begin{cases} +1 & \text{if } \alpha\beta\gamma \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } \alpha\beta\gamma \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if two or more of the indices } \alpha\beta\gamma \text{ are identical.} \end{cases}} \tag{2.29}$$

Even permutations of $(1,2,3)$ are $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$. Odd permutations are $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$.

In terms of the Levi-Civita density the determinant of a $3 \times 3$ matrix is

$$\begin{aligned} \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} &\equiv \varepsilon_{\alpha\beta\gamma} a_\alpha b_\beta c_\gamma \\ &= a_1 \left( b_2 c_3 - b_3 c_2 \right) + a_2 \left( b_3 c_1 - b_1 c_3 \right) + a_3 \left( b_1 c_2 - b_2 c_1 \right). \end{aligned} \tag{2.30}$$

[7] Tullio Levi-Civita (1873–1941) was an Italian mathematician most noted for his work on absolute differential calculus (tensor calculus).

Therefore, the cross product (2.28) is

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{\alpha\beta\gamma} \hat{e}_\alpha a_\beta b_\gamma . \tag{2.31}$$

In elementary texts the cross product is defined in terms of the magnitudes of the vectors involved and the sine of the angle between them. We can show that (2.31) reduces to the elementary definition (see exercises). Equation (2.31) is the more general definition.

We notice that in (2.31) the presence of the symbol for the basis $\hat{e}_\alpha$ only indicates that $\varepsilon_{\alpha\beta\gamma} a_\beta b_\gamma$ is the $\alpha$th component of the cross product. That is

$$[\mathbf{a} \times \mathbf{b}]_\alpha = \varepsilon_{\alpha\beta\gamma} a_\beta b_\gamma = \varepsilon_{\beta\gamma\alpha} a_\beta b_\gamma , \tag{2.32}$$

by a double (even) permutation of the $\alpha$ among the subscripts of the Levi-Civita density. Because it is usually only necessary to deal with individual components, (2.32) is often sufficient for a definition of the cross product.

We will find the subscript notation for the scalar and vector products indispensable when dealing with complex relationships or when establishing the validity of a vector identity. But we will often find that the physics is more clearly expressed in the dyadic notation. We will then elect to work generally in the dyadic notation. And we will use the subscript notation when convenient.

Among three vectors **a**, **b**, and **c** we can form two types of triple products. The *scalar triple product* is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and the *vector triple product* is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

The scalar triple product has a geometrical interpretation. It is the volume of the parallelepiped defined by the vectors **a**, **b**, and **c**. Because it is a volume

$$\boxed{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},} \tag{2.33}$$

which is referred to as "exchanging the dot and the cross."

The vector triple product has no simple geometrical interpretation. But it can be expanded into a form that is easily remembered and very useful. This is the so-called $bac - cab$ rule

$$\boxed{\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}).} \tag{2.34}$$

We leave the proof of both of these to the exercises.

## 2.3 Multivariate Functions

We will deal with mathematical functions of spatial coordinates and the time. We will generally indicate the three spatial coordinates using a position vector **r** with components $(x, y, z)$. In our treatment of relativity this will be replaced by a four vector with the additional component $ct$, which is the fourth (time) dimension in spatial form. In Fourier, or wave vector space the position vector $\mathbf{r}$ will be replaced by the wave vector $\mathbf{k}$ with the three components $(k_x, k_y, k_z)$ and the time $t$ will be replaced by the angular frequency $\omega$. We must then have an understanding of the treatment of these multivariate functions.

### *2.3.1 Differentials*

The differential of a function $\Phi$, dependent on three spatial coordinates $x$, $y$ and $z$ and the time $t$, is Pfaff's[8] differential form (see e.g. [40], p. 43), or the Pfaffian[9]

$$\boxed{\mathrm{d}\Phi = (\partial\Phi/\partial x)\,\mathrm{dx} + (\partial\Phi/\partial y)\,\mathrm{dy} + (\partial\Phi/\partial z)\,\mathrm{dz} + (\partial\Phi/\partial \mathrm{t})\,\mathrm{dt},} \tag{2.35}$$

in rectangular Cartesian coordinates $(x, y, z)$ and with the time $t$.

The Pfaffian is a linear differential form. That is the dependence of $\mathrm{d}\Phi$ on the differential of each coordinate is linear. We may think of the Pfaffian as the differential change in the function $\Phi$ resulting from a translation in coordinates $(x_1 \to x_1 + \mathrm{d}x_1, x_2 \to x_2 + \mathrm{d}x_2, x_3 \to x_3 + \mathrm{d}x_3)$ in the time interval $t \to t + \mathrm{d}t$. In this translation the differential $\mathrm{d}\Phi$ has a value, which is linearly dependent on the translation in each of the coordinates and the time. This is not different in principle from the expression of the differential of a function of a single variable $y = y(x)$ expressed in the form $\mathrm{d}y = y'\mathrm{d}x$, where $y'$ is the derivative of $y(x)$. We must now only think in more general terms.

In the other common spatial coordinate systems, cylindrical $(r, \vartheta, z)$ and spherical $(r, \vartheta, \phi)$, the Pfaffians are

$$\mathrm{d}\Phi = \frac{\partial\Phi}{\partial r}\mathrm{d}r + \frac{\partial\Phi}{\partial\vartheta}\mathrm{d}\vartheta + \frac{\partial\Phi}{\partial z}\mathrm{d}z + \frac{\partial\Phi}{\partial t}\mathrm{d}t \tag{2.36}$$

and

$$\mathrm{d}\Phi = \frac{\partial\Phi}{\partial r}\mathrm{d}r + \frac{\partial\Phi}{\partial\vartheta}\mathrm{d}\vartheta + \frac{\partial\Phi}{\partial\phi}\mathrm{d}\phi + \frac{\partial\Phi}{\partial t}\mathrm{d}t, \tag{2.37}$$

with the time indicated by $t$.

---

[8] Johann Friedrich Pfaff (1765–1825) was one of Germany's most eminent mathematicians during the nineteenth century. He is noted for his work on partial differential equations of the first order, which became part of the theory of differential forms. He was also Carl Friedrich Gauss's formal research supervisor.

[9] Pfaff's differential form for the function $\Psi(\xi_1, \ldots, \xi_n)$ is defined as

$$\mathrm{d}\Psi = \sum_j^n \frac{\partial\Psi}{\partial\xi_j}\mathrm{d}\xi_j.$$

Each of the partial derivatives is the differential change in $\Phi$ with respect to a single coordinate while all other coordinates are held constant. Each partial derivative is, in general, a function of all the independent variables. The geometrical meaning of each partial derivative as a slope is valid if we think in terms of a plot of $\Phi$ holding all but one variable constant, which is a planar representation of $\Phi$.

Relative extrema of a function $\Phi$, considered for the moment to be a function only of spatial coordinates, are points for which $\mathrm{d}\Phi = 0$ for all infinitesimal (differential) changes in the coordinates.

For example

$$\mathrm{d}\Phi = 0 = \frac{\partial \Phi}{\partial r}\mathrm{d}r + \frac{\partial \Phi}{\partial \vartheta}\mathrm{d}\vartheta + \frac{\partial \Phi}{\partial \phi}\mathrm{d}\phi \tag{2.38}$$

for all possible variations $(\mathrm{d}r, \mathrm{d}\vartheta, \mathrm{d}\phi)$. If the coordinates $(r, \vartheta, \phi)$ are linearly independent of one another, i.e. the system is not constrained by a particular relationship among these coordinates, then (2.38) can only be satisfied if each of the partial derivatives is independently equal to zero.

The spatial portions of the Pfaffians in (2.35)–(2.37) are the directional derivatives of the quantity $\Phi$ along directions specified by the differentials $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in the rectangular coordinate system, $(\mathrm{d}r, \mathrm{d}\vartheta, \mathrm{d}z)$ in the cylindrical system, and $(\mathrm{d}r, \mathrm{d}\vartheta, \mathrm{d}\phi)$ in the spherical system. We shall assume that the reader is familiar with the rectangular system and consider here details for only the cylindrical and spherical systems.

## 2.3.2 *Cylindrical Coordinates*

There is no ambiguity in the choice of coordinate definitions for the cylindrical coordinate system, which we have drawn in Fig. 2.1.

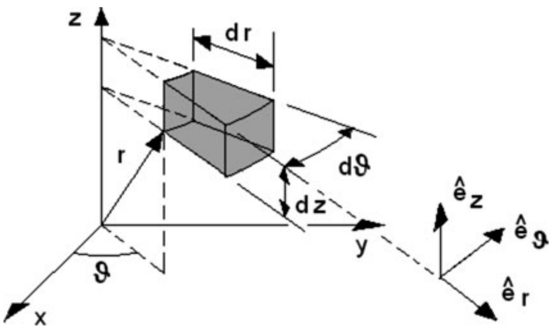



**Fig. 2.1** Cylindrical coordinates. The differential volume (shaded) and the vector triad $(\hat{e}_{\mathrm{r}}, \hat{e}_{\vartheta}, \hat{e}_{\mathrm{z}})$ are shown. The triad is separated from the vector point to limit clutter

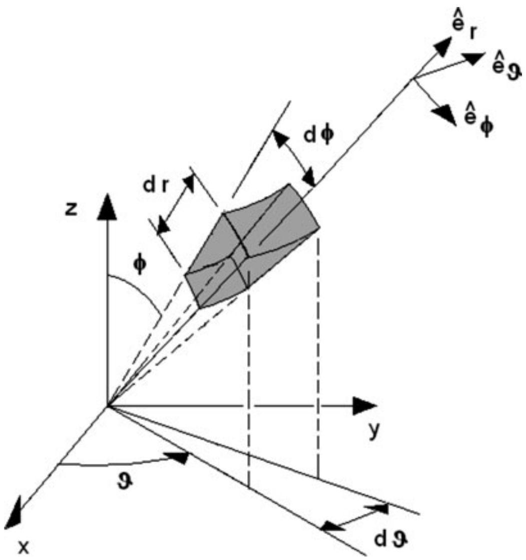



**Fig. 2.2** Spherical coordinates. The differential volume (shaded) and the vector triad $(\hat{e}_r, \hat{e}_\vartheta, \hat{e}_\phi)$ are shown. The triad is separated from the vector point to limit clutter

The angle $\vartheta$ is the *azimuthal* angle[10] in the $(x, y)$ plane and $z$ is the vertical. We have drawn the differential volume and the vector triad $(\hat{e}_r, \hat{e}_\vartheta, \hat{e}_z)$ in Fig. 2.1. The differential volume, which is shaded in Fig. 2.1, is

$$dV = r dr\, d\vartheta\, dz. \tag{2.39}$$

## *2.3.3 Spherical Coordinates*

There is an ambiguity in the definitions of the angles $\vartheta$ and $\phi$ in spherical coordinates. Authors differ in their choices of the symbols for the azimuthal and the polar angles. The reader must be aware of this and carefully note the form used in any particular text or table.

Here we shall use $\vartheta$ as the definition of the azimuthal angle in both cylindrical and spherical coordinates. The angle $\phi$ is then the *polar angle*, or angle measured down from the $z$-axis of the rectangular system. This is sometimes called the *zenith angle* or the *colatitude*. We have drawn the spherical coordinate system we will use in Fig. 2.2. The differential volume, which is shaded in Fig. 2.2, is

$$dV = r^2 \sin\phi\, d\vartheta\, d\phi\, dr. \tag{2.40}$$

[10] Azimuth comes from the Arabic word as-simt, which means direction, referring to the direction a person faces. The equatorial angle is the azimuthal angle defined such that a person facing East has an azimuthal angle of 90°, and a person facing South has an azimuthal angle of 180°.

## 2.4 Analytic Functions

When we are dealing with a function representing a physical quantity, for example a field component, we normally assume that the function will be continuous and have continuous derivatives, at least in a certain region of space, because we expect that of the field. Even when we consider fields produced by what we shall call point particles, we realize that the particle is still a physical particle and not a mathematical point.

We are using mathematics to represent what we observe in the universe. A one-to-one correspondence between our mathematical theory and the observable universe is not guaranteed by our intuition. At the very minimum, however, we must make certain that our mathematical theory is correct. In the next sections we develop the theorems of the vector calculus and for solutions to the most important differential equations we will encounter in our study.

Here we will assume that all functions are analytic (see [8, p. 59]). In the simplest language, a function is analytic in a certain domain if it can be expanded in a Taylor Series at every point within that domain (see [39, p. 23]).

### 2.4.1 Taylor Series

A Taylor series is a representation of the function in a power series. We cannot assume that this representation is valid everywhere. So let us assume that we want an approximation to the function $f(x)$ in a domain surrounding a point $x = a$. The Taylor Series near $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} c_{\mathrm{n}} (x - a)^{\mathrm{n}},$$

where the coefficients $c_{\mathrm{n}}$ are

$$c_0 = f(a),\; c_1 = \frac{1}{1!} \frac{d}{dx} f\bigg]_{\mathrm{x=a}},\; c_2 = \frac{1}{2!} \frac{d^2}{dx^2} f\bigg]_{\mathrm{x=a}}, \cdots\, c_{\mathrm{n}} = \frac{1}{n!} \frac{d^{\mathrm{n}}}{dx^{\mathrm{n}}} f\bigg]_{\mathrm{x=a}}, \cdots.$$

The subscript $x = a$ indicates that the derivatives are all to be evaluated at the point $x = a$. These coefficients are normally written with the shorthand notation

$$\begin{aligned}
&f^{(0)}(a) = f(a),\\
&f'(a) = \frac{d}{dx} f\bigg]_{\mathrm{x=a}},\; f''(a) = \frac{d^2}{dx^2} f\bigg]_{\mathrm{x=a}},\; \cdots f^{(\mathrm{n})}(a) = \frac{d^{\mathrm{n}}}{dx^{\mathrm{n}}} f\bigg]_{\mathrm{x=a}} \cdots.
\end{aligned}$$

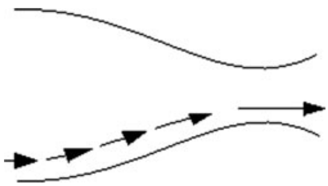


**Fig. 2.3** Fluid flowing toward a constriction in a tube. The flow velocity increases in inverse proportion to the tube area

Then with the definition $0! \equiv 1$, we can write the Taylor Series as

$$f(x) = \sum_{\mathrm{n}=0}^{\infty} \frac{f^{(\mathrm{n})}}{n!} (x-a)^{\mathrm{n}} . \tag{2.41}$$

### *2.4.2 Analyticity*

The definition of analyticity is (see [8, p. 59])

**Definition 2.1. Analytic function**. A function is called analytic in a domain $D$ when it can be expanded in a Taylor Series[11] at any point within $D$ and is convergent in some neighborhood of that point.

## 2.5 Vector Calculus

### *2.5.1 Field Quantities*

In mathematical terms a vector field has components which are functions of the position vector **r**. As an example of a vector field, in Fig. 2.3 we have drawn the velocity vector in a fluid flowing toward a constriction in a tube. We have indicated the flow velocity vector by arrows at each location in the tube. For an incompressible fluid mass conservation requires that the flow velocity is inversely proportional to the cross sectional area of the tube. Therefore the velocity increases toward the throat of the constriction and is greatest at the throat. The direction of the flow velocity also changes with the position in the tube.

We have then a vector, the fluid velocity **v**, which is a function of each point, designated by a position vector **r**. In a general three dimensional situation

$$\begin{aligned}\mathbf{v} &= \hat{e}_1 v_1(\mathbf{r}) + \hat{e}_2 v_2(\mathbf{r}) + \hat{e}_3 v_3(\mathbf{r}) \\ &= \hat{e}_\mu v_\mu(\mathbf{r}) .\end{aligned}$$

[11] We have used Taylor Series in place of power series.

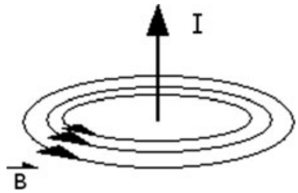



**Fig. 2.4** Magnetic field around a wire with constant current $I$ forms closed loops around the wire

We are primarily interested in electric and magnetic field vectors. The electric field vector near a stationary electric charge has the same form as the gravitational field vector discussed in Chap. 1. That is the magnitude of the field decreases as $1/r^2$, where $r$ is the distance from the charge. The direction of the electric field vector is either away from the charge (positive charge) or toward the charge (negative charge). The magnetic field from a steady electrical current has the form of circular loops in the plane perpendicular to the current. The magnitude of the magnetic field decreases in inverse proportion to the radius of the loop, i.e. as $1/r$. We have illustrated this situation in Fig. 2.4.

Because field quantities depend on general coordinates they can be differentiated with respect to those quantities. There are three differential operators in the vector calculus. These are the gradient, the divergence, and the curl. We shall consider these in detail here.

### 2.5.2 *The Gradient*

The gradient is a differential operator that operates on a scalar function of position to produce a vector field. In the rectangular Cartesian basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ the gradient is defined as

$$\begin{aligned}\operatorname{grad}\Phi &= \hat{e}_1\frac{\partial\Phi}{\partial x_1} + \hat{e}_2\frac{\partial\Phi}{\partial x_2} + \hat{e}_3\frac{\partial\Phi}{\partial x_3} \\ &= \hat{e}_\mu\frac{\partial\Phi}{\partial x_\mu},\end{aligned} \tag{2.42}$$

using the summation convention.

For any function $\Phi(\mathbf{r})$ there are surfaces for which $\Phi(\mathbf{r}) = \text{constant}$. For example, if $\Phi(\mathbf{r})$ is the temperature at a point in a system, we can identify the surfaces of constant $\Phi(\mathbf{r})$ by measurement of temperature. Geometrically the vector $\operatorname{grad}\Phi$ is perpendicular to the surfaces $\Phi(\mathbf{r})$ (see exercises). In Fig. 2.5 we have drawn a set of $\Phi(\mathbf{r}) = \text{constant}$ surfaces and the direction of $\operatorname{grad}\Phi$ at three points on the surfaces. If $\Phi(\mathbf{r})$ is a temperature then $-\operatorname{grad}\Phi$ is the direction of the heat flow (vector). The negative sign indicates that heat flow is from high to low temperature.

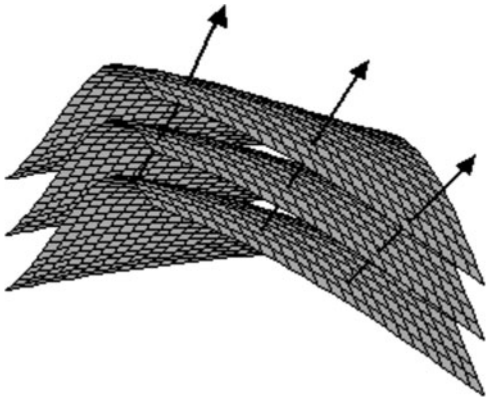


**Fig. 2.5** Three surfaces on which $\varphi(\mathbf{r})$ is constant. The direction of grad $\varphi$ is shown by the arrows. This direction is a function of the particular $\varphi(\mathbf{r}) =$ constant surface and the location on that surface

We can obtain a more complete understanding of the gradient if we recall that the Pfaffian for the differential change in rectangular Cartesian coordinates, with no time dependence, is

$$\begin{aligned} \mathrm{d}\Phi &= \frac{\partial \Phi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \Phi}{\partial x_2}\mathrm{d}x_2 + \frac{\partial \Phi}{\partial x_3}\mathrm{d}x_3 \\ &= \frac{\partial \Phi}{\partial x_\alpha}\mathrm{d}x_\alpha . \end{aligned} \tag{2.43}$$

From our discussion in Sect. 2.3 we know that (2.43) is the differential change in $\Phi$ along the differential displacement

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \hat{e}_1\mathrm{d}x_1 + \hat{e}_2\mathrm{d}x_2 + \hat{e}_3\mathrm{d}x_3 \\ &= \hat{e}_\beta \mathrm{d}x_\beta . \end{aligned} \tag{2.44}$$

And we see that if we take the scalar product of (2.42) with (2.44) we obtain (2.43). The differential change in a general function $\Phi$ in a specified direction d$\mathbf{r}$ is then given by the scalar product of the gradient with that direction.

### *2.5.3 The Divergence*

The *divergence* is a differential operator which operates on a vector field to produce a scalar. In rectangular Cartesian coordinates the divergence of the field vector

$$\mathbf{F} = \hat{e}_\mu F_\mu \tag{2.45}$$

is

$$\mathrm{div}\,\mathbf{F} = \frac{\partial}{\partial x_\mu} F_\mu . \tag{2.46}$$

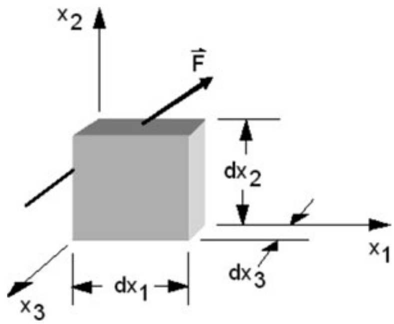



**Fig. 2.6** A vector field **F** penetrating a differential region of space

using the summation convention. Equation (2.46) is the form of div **F** only for rectangular Cartesian coordinates. It is not the form for cylindrical or spherical coordinates.

Because vectors are independent of coordinates chosen for their representation, we may choose to base proofs of propositions on a representation in rectangular Cartesian coordinates. Therefore (2.46) is a very practical form for the divergence. To understand the general form and the physical meaning of the divergence we must, however, turn to Gauss' Theorem.

#### 2.5.3.1 Gauss' Theorem

In our development of *Gauss'*[12] *Theorem* (also known as the *divergence theorem*) we assume that the general vector field (2.45) has continuous components $F_\mu$ with continuous first derivatives, which are functions of the spatial coordinates. This is a less stringent condition than analyticity.

We consider a differential volume, $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ which is penetrated by the field **F**. That is $\vec{F}$ passes through this differential volume. We have drawn the situation in Fig. 2.6. The reference point $(x_1, x_2, x_3)$ is at the back, lower left hand corner of the differential volume. The three components of the vector field **F** have the values $F_1$, $F_2$ and $F_3$ at the point $(x_1, x_2, x_3)$. The values of each of the components of **F** change in the intervals $(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3)$ to

$$F_1 + \left(\frac{\partial F_1}{\partial x_1}\right)\mathrm{d}x_1, \tag{2.47}$$

$$F_2 + \left(\frac{\partial F_2}{\partial x_2}\right)\mathrm{d}x_2 \tag{2.48}$$

[12]Carl Friedrich Gauss (1777–1855) was a German mathematician, astronomer and physicist. From 1807 Gauß was director of the Göttingen observatory and professor at the University of Göttingen.

and

$$F_3 + \left(\frac{\partial F_3}{\partial x_3}\right) \mathrm{d}x_3. \tag{2.49}$$

The partial derivatives in (2.47)–(2.49) are all evaluated at the reference point $(x_1, x_2, x_3)$.

We now define the *differential vector area* for each of the faces of the differential volume in Fig. 2.6 as the product of the differential facial area and the unit vector in the direction perpendicular to the face. The positive direction chosen for the unit vector is *pointed outward* from the differential volume.

For example the differential vector area facing us in Fig. 2.6 is $\mathrm{d}x_1\mathrm{d}x_2\hat{e}_3$.

We shall designate each of these vector areas as d**S**.

We then define the differential flux of **F** out of the differential volume $\mathrm{d}V$ through a particular face as the scalar product **F**·d**S** for that face. Then the differential flux out of the left face of area $\mathrm{d}\mathbf{S} = -\mathrm{d}x_2\mathrm{d}x_3\hat{e}_1$, i.e. that face nearest the reference point $(x_1, x_2, x_3)$, is

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = -F_1\mathrm{d}x_2\mathrm{d}x_3, \tag{2.50}$$

with the minus sign appearing because we have represented the vector **F** as having a positive $x_1$-component $F_1$ on this face. We do not consider any variation of $F_1$ over the area $\mathrm{d}x_2\mathrm{d}x_3$, since this will only introduce fourth order differentials in our final expression, which we would then drop.

The differential flux out of the face of area $\mathrm{d}\mathbf{S} = \mathrm{d}x_2\mathrm{d}x_3\hat{e}_1$ on the right, i.e. farthest from the reference point, using (2.47), is

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = \left[F_1 + \left(\frac{\partial F_1}{\partial x_1}\right) \mathrm{d}x_1\right] \mathrm{d}x_2\mathrm{d}x_3. \tag{2.51}$$

We again do not consider any variation of $F_1$or $\partial F_1/\partial x_1$ over the area $\mathrm{d}x_2\mathrm{d}x_3$.

The sum of the fluxes out of these two faces, which we designate as $\mathrm{d}\Phi_{\mathrm{Sx}}$ is the sum of (2.50) and (2.51), or

$$\mathrm{d}\Phi_{\mathrm{Sx}} = \left(\frac{\partial F_1}{\partial x_1}\right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \tag{2.52}$$

Similarly for the remaining fluxes out of the differential volume $\mathrm{d}V$,

$$\mathrm{d}\Phi_{\mathrm{Sy}} = \left(\frac{\partial F_2}{\partial x_2}\right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{2.53}$$

and

$$\mathrm{d}\Phi_{\mathrm{Sz}} = \left(\frac{\partial F_3}{\partial x_3}\right) \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \tag{2.54}$$

The total differential flux out of the differential volume $\mathrm{d}V$ is the sum of (2.52)–(2.54). This we designate as

$$\mathrm{d}\Phi_{\text{out}} = \sum_{\text{all faces i}} \mathrm{d}\Phi_{\text{Si}} = \sum_{\text{all faces}} \mathbf{F}\cdot\mathrm{d}\mathbf{S}. \tag{2.55}$$

Carrying of the summation over (2.52)–(2.54) we have

$$\mathrm{d}\Phi_{\text{out}} = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.$$

From (2.46) we recognize the term in the brackets ( ) as div **F**. Then

$$\mathrm{d}\Phi_{\text{out}} = \operatorname{div}\mathbf{F}\mathrm{d}V. \tag{2.56}$$

Equating $\mathrm{d}\Phi_{\text{out}}$ in (2.55) and (2.56) we have

$$\operatorname{div}\mathbf{F}\mathrm{d}V = \sum_{\text{all faces of d}V} \mathbf{F}\cdot\mathrm{d}\mathbf{S}. \tag{2.57}$$

To extend (2.57) to finite volumes we construct the finite volume from differential volumes. In doing this we must ask what happens to the flux at the boundaries between our differential volumes. In Fig. 2.7 we illustrate the situation between neighboring differential volumes. The flux out of the differential volume (A) through the face $\mathrm{d}\mathbf{S}_{\text{A}}$ is equal to the flux entering the differential volume (B) through the face $\mathrm{d}\mathbf{S}_{\text{B}}$. Because the differential volumes (A) and (B) fit together, $\mathrm{d}\mathbf{S}_{\text{A}} = -\mathrm{d}\mathbf{S}_{\text{B}}$. Upon summation over the differential blocks making up the finite volume, then, the fluxes over all faces that are joined in this fashion to another differential face cancel. Only the faces forming the outside surface of the closed volume contribute to the result of the summation. Then (2.57) results in

$$\sum_{\mathrm{d}V\text{ in }V} (\operatorname{div}\mathbf{F})\,\mathrm{d}V = \int_{\text{V}} \operatorname{div}\mathbf{F}\mathrm{d}V = \sum_{\mathrm{d}V\text{ in }V}\left(\sum_{\text{all faces of d}V} \mathbf{F}\cdot\mathrm{d}\mathbf{S}\right)$$
$$= \oint_{\text{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S}, \tag{2.58}$$

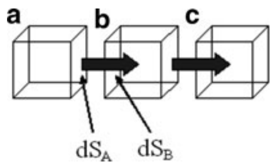



**Fig. 2.7** Cancellation of fluxes in adjoining differential volumes

where the symbol $\oint_S$ indicates closed integration over the entire surface containing $V$. Writing only the equality of the two integrals in (2.58) we have

$$\boxed{\int_V \operatorname{div} \mathbf{F} \mathrm{d}V = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.} \tag{2.59}$$

Equation (2.59) is *Gauss' theorem*. Gauss' Theorem is valid for a general vector field **F** for which the components $F_\mu$ and their first derivatives are continuous functions of the spatial coordinates.

As an example of the application of Gauss' Theorem to a quantity for which there are no sources, we consider the conservation of mass in a flowing fluid.

*Example 2.1.* **Fluid Mass Conservation.** We consider a flowing fluid with mass density $\rho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$, which vary with position and time. The mass flux in $\mathrm{kg\,m^{-2}\,s^{-1}}$ is $\rho v$. The net flow of mass out of an arbitrary volume due to the flux of fluid out of the volume is equal to the decrease of mass in the volume because mass is conserved. That is

$$\oint_S (\rho\mathbf{v}) \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathrm{d}V.$$

We choose the volume $V$ to be fixed in space. We may then bring the time derivative inside the integral as a partial derivative.

$$\oint_S (\rho\mathbf{v}) \cdot \mathrm{d}\mathbf{S} = \int_V \left(\frac{\partial\rho}{\partial t}\right) \mathrm{d}V.$$

Gauss' Theorem requires

$$\int_V \operatorname{div}(\rho\mathbf{v})\,\mathrm{d}V = -\int_V \left(\frac{\partial\rho}{\partial t}\right) \mathrm{d}V,$$

or

$$\int_V dV \left[\operatorname{div}(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t}\right] = 0.$$

Since $V$ is an arbitrary volume, the vanishing of the integral implies that the integrand [ ] must vanish and, hence

$$\frac{\partial\rho}{\partial t} = -\operatorname{div}(\rho\mathbf{v}). \tag{2.60}$$

Equation (2.60) is the general mass conservation equation for a fluid.

#### 2.5.3.2 Divergence: General Definition

From (2.57), which is the differential form of Gauss' Theorem, we obtain a general formulation of the divergence of a vector field. This is

$$\operatorname{div}\mathbf{F} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_{\text{all faces }\Delta S\text{ of }\Delta V}\mathbf{F}\cdot\Delta\mathbf{S}. \tag{2.61}$$

Here the faces $\Delta S$ are differential areas normal (perpendicular) to the basis vectors of the system chosen.

As an example we consider cylindrical coordinates (see Sect. 2.3).

*Example 2.2.* **Divergence in Cylindrical Coordinates.** In the cylindrical basis the vector field is

$$\mathbf{F} = F_{\mathrm{r}}\hat{e}_{\mathrm{r}} + F_{\vartheta}\hat{e}_{\vartheta} + F_{\mathrm{z}}\hat{e}_{\mathrm{z}}.$$

From Fig. 2.1 we notice that the face area $\perp \hat{e}_{\mathrm{r}}$ changes from a distance $r$ to the distance $r+\mathrm{d}r$ from the $z$-axis. At $r$ the area of the face is $r\mathrm{d}\vartheta\mathrm{d}z$ and at $r+\mathrm{d}r$ it is $(r+\mathrm{d}r)\mathrm{d}\vartheta\mathrm{d}z$. The other facial areas do not change from one side of the differential volume to the other. The face area $\perp \hat{e}_{\vartheta}$ is $\mathrm{d}r\mathrm{d}z$ on both sides of $\mathrm{d}V$. And the face area $\perp \hat{e}_{\mathrm{z}}$ is $r\mathrm{d}\vartheta\mathrm{d}r$ on the top and bottom of $\mathrm{d}V$. Then

$$\begin{aligned}\lim_{\Delta V\to 0}\sum_{\text{all faces }\Delta S\text{ of }\Delta V}\mathbf{F}\cdot\Delta\mathbf{S} &= -F_{\mathrm{r}}\,r\mathrm{d}\vartheta\mathrm{d}z + \left(F_{\mathrm{r}} + \frac{\partial F_{\mathrm{r}}}{\partial r}\mathrm{d}r\right)(r+\mathrm{d}r)\,\mathrm{d}\vartheta\mathrm{d}z\\ &\quad + \frac{\partial F_{\vartheta}}{\partial\vartheta}\mathrm{d}\vartheta\mathrm{d}r\mathrm{d}z + \frac{\partial F_{\mathrm{z}}}{\partial z}r\mathrm{d}z\mathrm{d}r\mathrm{d}\vartheta\\ &= F_{\mathrm{r}}\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}z + \frac{\partial F_{\mathrm{r}}}{\partial r}r\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}z + \frac{\partial F_{\vartheta}}{\partial\vartheta}\mathrm{d}\vartheta\mathrm{d}r\mathrm{d}z\\ &\quad + \frac{\partial F_{\mathrm{z}}}{\partial z}r\mathrm{d}z\mathrm{d}r\mathrm{d}\vartheta.\end{aligned}$$

The differential volume is (2.39). Then

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \lim_{\Delta V\to 0}\frac{1}{\Delta V}\sum_{\text{all faces }\Delta S\text{ of }\Delta V}\mathbf{F}\cdot\Delta\mathbf{S}\\ &= \frac{1}{r}F_{\mathrm{r}} + \frac{\partial F_{\mathrm{r}}}{\partial r} + \frac{1}{r}\frac{\partial F_{\vartheta}}{\partial\vartheta} + \frac{\partial F_{\mathrm{z}}}{\partial z}\\ &= \frac{1}{r}\frac{\partial\left(rF_{\mathrm{r}}\right)}{\partial r} + \frac{1}{r}\frac{\partial F_{\vartheta}}{\partial\vartheta} + \frac{\partial F_{\mathrm{z}}}{\partial z}.\end{aligned}$$

## 2.5.4 *The Curl*

The curl is a differential operator that operates on a field quantity to produce another field quantity. In rectangular Cartesian coordinates the curl of the field quantity (2.45) is

$$\begin{aligned}\text{curl}\,\mathbf{F} &= \varepsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x_\alpha}F_\beta\hat{e}_\gamma \\ &= \det\begin{bmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ F_1 & F_2 & F_3\end{bmatrix} \\ &= \left(\frac{\partial F_3}{\partial x_2}-\frac{\partial F_2}{\partial x_3}\right)\hat{e}_1-\left(\frac{\partial F_3}{\partial x_1}-\frac{\partial F_1}{\partial x_3}\right)\hat{e}_2 \\ &\quad+\left(\frac{\partial F_2}{\partial x_1}-\frac{\partial F_1}{\partial x_2}\right)\hat{e}_3. \end{aligned} \tag{2.62}$$

In rectangular Cartesian coordinates the curl has the form of a cross product.

### 2.5.4.1 Rotation

To obtain a physical understanding of the curl and to formulate a general differential expression for the curl we borrow a term known as the rotation or vorticity from fluid mechanics (see [81, p. 88] and [57, p. 16]). In fluid mechanics the rotation is a measure of the swirling or vortex motion of the fluid velocity field. The field quantities of interest to us here are not velocities. But the intuition we have for swirling fluids can be helpful in gaining an understanding of the curl.

The *rotation* **R** of a vector field quantity is a vector, which is defined in terms of an imaginary, small *open area*. An open area must be distinguished from the closed area we introduced with the divergence operator in Sect. 2.5.3. An open area is an area bounded by a *closed contour*, which may be a single closed curve or formed from interconnected segments of curves. A closed area, which encloses a volume, cannot be bounded by a closed contour.

For the sake of simplicity we pick the open area $\Delta a = \Delta x_1 \Delta x_2$ in the $(x_1, x_2)$ plane perpendicular to the unit vector $\hat{e}_3$. We have drawn this open area and the contour bounding it in Fig. 2.8. The reference point for which the coordinate values are $x_1$, and $x_2$ is the intersection of the three unit vectors at the back left hand corner of the open area $\Delta a$. The closed contour, which we designate as $\Delta\mathbf{C}$, bounding $\Delta a$ is the vector sum

$$\begin{aligned}\Delta\mathbf{C} &= \Delta x_1\hat{e}_1 + \Delta x_2\hat{e}_2 - \Delta x_1\hat{e}_1 - \Delta x_2\hat{e}_2 \\ &= \Delta x_\alpha\hat{e}_\alpha \end{aligned} \tag{2.63}$$

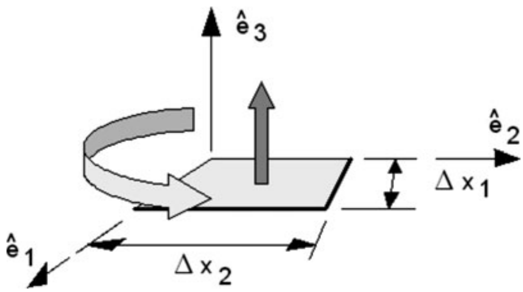



**Fig. 2.8** Rotation of a field quantity

of the segments around the open area beginning and ending at the reference point $(x_1, x_2)$. We have indicated the direction for the contour $\Delta\mathbf{C}$ in (2.63) by the curved arrow in Fig. 2.8.

For the open area $\Delta a$ in Fig. 2.8 the magnitude of the rotation $|\mathbf{R}_{\Delta \mathrm{a}}|$ is defined by

$$|\mathbf{R}_{\Delta \mathrm{a}}|\ \Delta a = \sum_{\text{bounding contour}} \mathbf{F}\cdot\Delta x_{\mathrm{i}}\hat{e}_{\mathrm{i}}, \tag{2.64}$$

where $\Delta x_{\mathrm{i}}\hat{e}_{\mathrm{i}}$ is one of the set of vector segments $\Delta x_1\hat{e}_1$, $\Delta x_2\hat{e}_2$, $-\Delta x_1\hat{e}_1$, $-\Delta x_2\hat{e}_2$ forming $\Delta\mathbf{C}$. In the limit as $\Delta a \to 0$ the area $\Delta a$ shrinks down to the reference point $(x_1, x_2)$ and the term on the left hand side of (2.64) is the value of the rotation at the reference point. We designate the magnitude of the rotation at the point $(x_1, x_2)$ simply as $|\mathbf{R}|$. That is

$$\lim_{\Delta a\to 0}|\mathbf{R}_{\Delta \mathrm{a}}|\ \Delta a = |\mathbf{R}|\ \mathrm{d}a = \lim_{\Delta a\to 0}\sum_{\text{bounding contour}} \mathbf{F}\cdot\Delta x_{\mathrm{i}}\hat{e}_{\mathrm{i}}. \tag{2.65}$$

The vector direction of the rotation is defined by the direction around which we choose to follow the contour $\Delta\mathbf{C}$ (i.e. the curved arrow in Fig. 2.8) when evaluating the sum in (2.64) and the *right hand rule*. If we curl the fingers of our right hand in the direction of the curved arrow, the thumb of our right hand will point in the direction indicated by the vertical arrow in the drawing. This is the vector direction of the rotation. In Fig. 2.8 the direction of the rotation is the unit vector $\hat{e}_3$.

We use the direction of the contour $\Delta\mathbf{C}$ to define a vector differential open area d**a**. The magnitude of d**a** is the limit of the open area $\Delta a$, i.e. d$a$, and the direction is given by the right hand rule from the direction chosen for $\Delta\mathbf{C}$. The directions of d**a** and **R** are then the same. Therefore

$$|\mathbf{R}|\,\mathrm{d}a = \mathbf{R}\cdot\mathrm{d}\mathbf{a}$$

and (2.65) becomes

$$\mathbf{R}\cdot\mathrm{d}\mathbf{a} = \lim_{\Delta a\to 0}\sum_{\text{bounding contour}} \mathbf{F}\cdot\Delta x_{\mathrm{i}}\hat{e}_{\mathrm{i}}. \tag{2.66}$$

Because we have chosen the area $\Delta a$ perpendicular to $\hat{e}_3$ *only* the components of **F** in the $\hat{e}_1$ and $\hat{e}_2$ directions have non-zero scalar products with the segments of $\Delta\mathbf{C}$. At the reference point $(x_1, x_2)$ in Fig. 2.8 these components have the values $F_1$ and $F_2$. At the distances $\Delta x_2$ and $\Delta x_1$ from the reference point the values of these components are $F_1 + (\partial F_1/\partial x_2)\,\Delta x_2$ and $F_2 + (\partial F_2/\partial x_1)\,\Delta x_1$. The partial derivatives are all evaluated at the reference point.

Then the scalar product $\mathbf{F}\cdot\Delta x_i\hat{e}_i$ along the first segment of $\Delta\mathbf{C}$ is

$$+F_1\Delta x_1, \tag{2.67}$$

and along the third segment of $\Delta\mathbf{C}$ the scalar product $\mathbf{F}\cdot\Delta x_i\hat{e}_i$ is

$$-\left(F_1 + \frac{\partial F_1}{\partial x_2}\Delta x_2\right)\Delta x_1. \tag{2.68}$$

Similarly the scalar product $\mathbf{F}\cdot\Delta x_i\hat{e}_i$ along the second segment of $\Delta\mathbf{C}$ is

$$+\left(F_2 + \frac{\partial F_2}{\partial x_1}\Delta x_1\right)\Delta x_2, \tag{2.69}$$

and along the fourth segment of $\Delta\mathbf{C}$ is

$$-F_2\Delta x_2. \tag{2.70}$$

Adding (2.67)–(2.70) and taking the limit as $\Delta a \to 0$ results in the (magnitude) of the rotation at the reference point

$$\begin{aligned} |\mathbf{R}| &= \lim_{\Delta a\to 0}\frac{\Delta x_1\Delta x_2}{\Delta a}\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \\ &= \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right). \end{aligned} \tag{2.71}$$

From (2.62) we see that (2.71) is the $\hat{e}_3$ component of the curl in rectangular Cartesian coordinates.

We have then shown that if we pick an imaginary area in space and evaluate the rotation about that area as the area shrinks down to a single point we have the magnitude of the component of curl **F** at that point in the direction found by the right hand rule. But rather than picking first the area we can first evaluate curl **F** at a point in space. We know then that curl **F**, in both magnitude and direction, is equal to the rotation (vector) **R** about that point from an infinitesimal contour in a plane perpendicular to the vector direction of curl **F**. That is, from (2.66), we have

$$\mathrm{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{a} = \lim_{\Delta a\to 0}\sum_{\text{contour bounding }\Delta \mathrm{a}}\mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta \mathrm{a}}, \tag{2.72}$$

where $\Delta\boldsymbol{\ell}_{\Delta \mathrm{a}}$ is a segment of the contour around the open area $\Delta a$.

We have then a physical understanding of the curl of a vector field **F** in terms of the rotation of **F**. We have also found a method of calculating curl $F$ in (2.72) that is analogous to our method of finding the divergence in (2.61).

#### 2.5.4.2 Stokes' Theorem

For the sake of visual simplicity we first used an open area with contour segments oriented along axes. But in the limit taken in (2.71) the area $\Delta a$ may have any arbitrary form. The right hand side of (2.72) is then

$$\lim_{\Delta a \to 0} \sum_{\text{contour bounding } \Delta \mathrm{a}} \mathbf{F} \cdot \Delta \boldsymbol{\ell}_{\Delta \mathrm{a}} = \lim_{\Delta \mathrm{a} \to 0} \oint_{\mathrm{C}(\Delta \mathrm{a})} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{2.73}$$

where $\mathrm{d}\ell$ is a differential along the contour $\mathrm{C}_{\Delta \mathrm{a}}$ defining $\Delta a$ and the integral is a contour integral. Then (2.72) becomes

$$\operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{a} = \lim_{\Delta \mathrm{a} \to 0} \oint_{\mathrm{C}(\Delta \mathrm{a})} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} \tag{2.74}$$

for each open area $\Delta a$.

What is the result if we construct an arbitrary (finite) open area $a$ from small, and eventually infinitesimal areas $\Delta a$? To answer this question we have drawn two small areas neighboring one another in Fig. 2.9. The scalar products along the adjoining segments cancel one another where areas join. The only contours that do not cancel are those on the outside of the finite area $a$. These form the bounding contour of the finite area $a$.

This is true also for areas that are not planar, but are of *arbitrary shapes*. We may construct any arbitrary finite open area $a$ as a summation of the open areas $\Delta a$ in the limit as these areas go to zero (i.e. become differential open areas d$a$).

In Fig. 2.10 we have drawn a surface constructed from small open areas $\Delta a$ (d$a$) to illustrate the principle. The contributions from the neighboring areas cancel as in Fig. 2.9 leaving only the exterior contour, as we have shown by the arrow marked $C$ in Fig. 2.10.

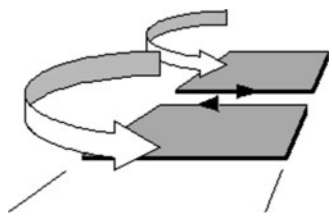


**Fig. 2.9** Cancelation of neighboring components in a general area

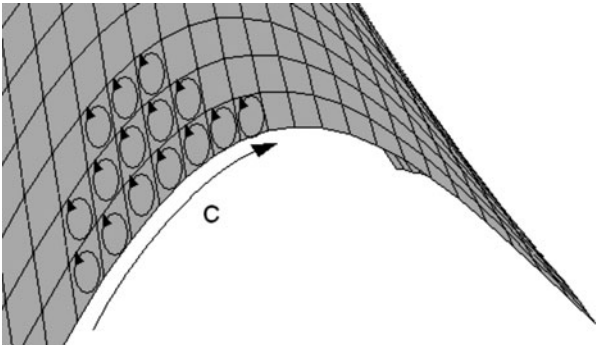



**Fig. 2.10** An arbitrary area constructed from small open areas $\Delta a$. The small contours $\Delta\mathbf{C}$ used to calculate rotations $\mathbf{R}_{\Delta a}$ are indicated by small circles with arrows. The contributions to the total rotation from neighboring segments cancel

If we sum (2.74) over d$a$ we have a mathematical representation of what we have pictured in Fig. 2.10.

$$\sum_{\mathrm{d}a} \mathrm{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{a} = \sum_{\mathrm{d}a} \lim_{\Delta a \to 0} \oint_{C(\Delta a)} \mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell}. \tag{2.75}$$

The summation on the left hand side of (2.75) is the integral over the open area of curl $\mathbf{F}\cdot\mathrm{d}\mathbf{a}$, i.e.

$$\int_a \mathrm{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{a} = \sum_{\mathrm{d}a} \mathrm{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{a}. \tag{2.76}$$

Because the integrals over neighboring contours cancel, as we saw in Figs. 2.9 and 2.10 the integral on the right hand side of (2.75) is the contour integral around the closed contour defining the arbitrary area $a$,

$$\oint_C \mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell} = \sum_{\mathrm{d}a} \lim_{\Delta a \to 0} \oint_{C(\Delta a)} \mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell}. \tag{2.77}$$

Combining (2.76) and (2.77) into (2.75) we have

$$\boxed{\int_a \mathrm{curl}\,\mathbf{F}\cdot \mathrm{d}\mathbf{a} = \oint_C \mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell}.} \tag{2.78}$$

Equation (2.78) is *Stokes' Theorem*.[13] Stokes' Theorem is the integral form of the differential expression in (2.74).

In the construction in Fig. 2.10 we saw that the area $a$ was completely arbitrary. The only requirement is that the area $a$ is bounded by the contour $C$. We can think of this in terms of a wire that we have bent into a loop to form a contour. We dip this wire into soapy water to form a film as the area bounded by the loop. As we move the loop through the air the film takes on arbitrary shapes all of which are defined by the wire loop. Any of these film areas is appropriate for Stokes' Theorem.

*Example 2.3.* **Conservative Force Field.** As an example of the application of Stokes' Theorem, we consider a force field $\mathbf{F}$. The scalar product $\mathbf{F}\cdot \mathrm{d}\boldsymbol{\ell}$ is the work done by the force field in moving a body the distance $\mathrm{d}\boldsymbol{\ell}$. If the force field is conservative there is no work done in moving a particle around a closed contour or path. That is for a conservative force field

$$\oint_{\mathrm{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{\ell} = 0.$$

Therefore, from Stokes' Theorem (2.78),

$$\int_{\mathrm{a}} \mathrm{d}\mathbf{a} \cdot \operatorname{curl} \mathbf{F} = 0.$$

Because the area $a$ in Stokes' Theorem is arbitrary this means that

$$\operatorname{curl} \mathbf{F} = \mathbf{0}$$

everywhere. Since (see exercises)

$$\operatorname{curl} \operatorname{grad} \varphi = 0$$

for arbitrary $\varphi$, $\operatorname{curl} \mathbf{F} = \mathbf{0}$ is satisfied if

$$\mathbf{F} = -\operatorname{grad} \varphi.$$

The negative sign is by convention. A conservative force field can then always be written as the negative gradient of a scalar potential.

[13] George Gabriel Stokes (1819–1903) was an English mathematician and physicist. He was appointed to the Lucasian Chair in Mathematics at Cambridge in 1849. He did fundamental work in fluid dynamics and optics.

#### 2.5.4.3 Curl: General Definition

From the differential form of Stokes' Theorem (2.72) we obtain a general formulation of the curl of a vector field. This is

$$\operatorname{curl}\mathbf{F}\cdot\Delta\mathbf{a} = \sum_{\text{contour bounding }\Delta\text{a}} \mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta\text{a}}. \tag{2.79}$$

We may use this formulation to obtain each component of curl $\mathbf{F}$ from the corresponding open areas $\Delta a$ perpendicular to the components.

As an example we consider cylindrical coordinates (see Sect. 2.3).

*Example 2.4.* **Curl in Cylindrical Coordinates.** We shall evaluate curl $\mathbf{F}$ at the point $(r, \vartheta, z)$ (see Fig. 2.1). In the cylindrical basis the vector field is

$$\mathbf{F} = F_{\text{r}}\hat{e}_{\text{r}} + F_{\vartheta}\hat{e}_{\vartheta} + F_{\text{z}}\hat{e}_{\text{z}}.$$

For the $\hat{e}_{\text{r}}$ component of curl $\mathbf{F}$ the area perpendicular to $\hat{e}_{\text{r}}$ is $\Delta a = r\mathrm{d}\vartheta\mathrm{d}z$ and the sum around the contour of $\mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta\text{a}}$ in a direction determined by the right hand rule for the direction $\hat{e}_{\text{r}}$ is

$$\begin{aligned}\sum_{\text{contour bounding }\Delta\text{a}} \mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta\text{a}} &= F_{\vartheta}r\mathrm{d}\vartheta + \left(F_{\text{z}} + \frac{\partial F_{\text{z}}}{\partial\vartheta}\mathrm{d}\vartheta\right)\mathrm{d}z \\ &\quad -\left(F_{\vartheta} + \frac{\partial F_{\vartheta}}{\partial z}\mathrm{d}z\right)r\mathrm{d}\vartheta - F_{\text{z}}\mathrm{d}z \\ &= \left(\frac{1}{r}\frac{\partial F_{\text{z}}}{\partial\vartheta} - \frac{\partial F_{\vartheta}}{\partial z}\right)r\mathrm{d}\vartheta\mathrm{d}z.\end{aligned}$$

The $\hat{e}_{\text{r}}$ component of curl $\mathbf{F}$ is then

$$[\operatorname{curl}\mathbf{F}]_{\text{r}} = \frac{1}{r}\frac{\partial F_{\text{z}}}{\partial\vartheta} - \frac{\partial F_{\vartheta}}{\partial z}.$$

For the $\hat{e}_{\vartheta}$ component of curl $\mathbf{F}$ the area is $\Delta a = \mathrm{d}r\mathrm{d}z$ and the sum around the contour is

$$\begin{aligned}\sum_{\text{contour bounding }\Delta\text{a}} \mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta\text{a}} &= F_{\text{z}}\mathrm{d}z + \left(F_{\text{r}} + \frac{\partial F_{\text{r}}}{\partial z}\mathrm{d}z\right)\mathrm{d}r \\ &\quad -\left(F_{\text{z}} + \frac{\partial F_{\text{z}}}{\partial r}\mathrm{d}r\right)\mathrm{d}z - F_{\text{r}}\mathrm{d}r \\ &= \left(\frac{\partial F_{\text{r}}}{\partial z} - \frac{\partial F_{\text{z}}}{\partial r}\right)\mathrm{d}r\mathrm{d}z.\end{aligned}$$

The $\hat{e}_\vartheta$ component of curl **F** is then

$$[\text{curl}\,\mathbf{F}]_\vartheta = \frac{\partial F_\text{r}}{\partial z} - \frac{\partial F_\text{z}}{\partial r}.$$

For the $\hat{e}_\text{z}$ component of curl **F** the area is $\Delta a = \text{d}r\text{d}z$ and the sum around the contour is

$$\begin{aligned}\sum_{\text{contour bounding } \Delta\text{a}} \mathbf{F}\cdot\Delta\boldsymbol{\ell}_{\Delta\text{a}} &= F_\text{r}\text{d}r + \left(F_\vartheta + \frac{\partial F_\vartheta}{\partial r}\text{d}r\right)(r+\text{d}r)\,\text{d}\vartheta \\ &\quad - \left(F_\text{r} + \frac{\partial F_\text{r}}{\partial\vartheta}\text{d}\vartheta\right)\text{d}r - F_\vartheta r\text{d}\vartheta \\ &= \left(\frac{1}{r}F_\vartheta + \frac{\partial F_\vartheta}{\partial r} - \frac{1}{r}\frac{\partial F_\text{r}}{\partial\vartheta}\right)r\text{d}r\text{d}\vartheta \\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\vartheta) - \frac{\partial F_\text{r}}{\partial\vartheta}\right]r\text{d}r\text{d}\vartheta.\end{aligned}$$

The $\hat{e}_\text{z}$ component of curl **F** is then

$$[\text{curl}\,\mathbf{F}]_\text{z} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\vartheta) - \frac{\partial F_\text{r}}{\partial\vartheta}\right].$$

The final result for the curl **F** in cylindrical coordinates is

$$\text{curl}\,\mathbf{F} = \hat{e}_\text{r}\left(\frac{1}{r}\frac{\partial F_\text{z}}{\partial\vartheta} - \frac{\partial F_\vartheta}{\partial z}\right) + \hat{e}_\vartheta\left(\frac{\partial F_\text{r}}{\partial z} - \frac{\partial F_\text{z}}{\partial r}\right) + \hat{e}_\text{z}\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\vartheta) - \frac{\partial F_\text{r}}{\partial\vartheta}\right].$$

### 2.5.5 The Laplacian Operator

In some circumstances the laws for a vector field will result in a vanishing curl and a nonvanishing divergence. An example is the electrostatic field. For such a field

$$\text{div}\,\mathbf{F} = g,$$

where $g\,(\mathbf{r})$ is some scalar function of spatial coordinates.

If the curl of a vector field **F** vanishes then the vector field can be written as

$$\mathbf{F} = -\,\text{grad}\;\Phi,$$

where $\Phi$ is a scalar function of the spatial coordinates.

Then for a vector field for which $\operatorname{div}\mathbf{F} = g$ and $\operatorname{curl}\mathbf{F} = \mathbf{0}$,

$$\operatorname{div}\operatorname{grad}\Phi \equiv \nabla^2\Phi = -g\left(\mathbf{r}\right),$$

where $\nabla^2$ is the *Laplacian Operator*, or simply the *Laplacian*, and

$$\nabla^2\Phi = -g\left(\mathbf{r}\right) \tag{2.80}$$

is *Poisson's Equation*.[14] Poisson first published this equation in the *Bulletin de la société philomatique* in 1813.

If $g\left(\mathbf{r}\right) = 0$,

$$\nabla^2\Phi = 0, \tag{2.81}$$

which is *Laplace's equation*.

The actual forms taken by the Laplacian for the various Cartesian systems are

- Rectangular

$$\nabla^2\Phi = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\Phi. \tag{2.82}$$

- Cylindrical

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\Phi}{\partial\vartheta^2}\right) + \frac{\partial^2\Phi}{\partial z^2}. \tag{2.83}$$

- Spherical

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial^2\Phi}{\partial\vartheta^2}\right). \tag{2.84}$$

## 2.6 Differential Equations

Maxwell's Equations will lead us to two potentials from which the fields can be most easily obtained. At each step in the development we will find that these two potentials are solutions to the same nonhomogeneous (or inhomogeneous, see footnote 2.1) partial differential equation with either charge or current densities appearing as sources. These partial differential equations for the potentials will then

[14] Siméon-Denis Poisson (1781–1840), was a French mathematician and physicist.

be central to the theory. Therefore it is absolutely critical that we know we can solve these equations and that the solutions we obtain are unique.

In time independent, static situations the equations for the potentials are Poisson's equation (2.80) and Laplace's Equation (2.81). Time dependence introduces wave phenomena and the partial differential equation will be Helmholtz' Equation

$$\left(\nabla^2 + K^2\right) \Phi = -h\left(\mathbf{r}\right). \tag{2.85}$$

The properties of Laplace', Poisson's, and Helmholtz' Equations are contained in a set of theorems, which we establish in Appendices E, F, and G. We have placed the proof of these theorems in the Appendices, rather than in the present chapter, to keep the discussion fluid. From the set of theorems for these equations we have that

- The solution of Laplace's Equation exists and is unique in a spatial region $V$ if the value of the potential is specified on the boundary surface $S$ of $V$.
- The solutions of Poisson's and Helmholtz' Equations exist and are unique for specific functions $g\left(\mathbf{r}\right)$ and $h\left(\mathbf{r}\right)$ in a spatial region $V$ if the value of the potential is specified on the boundary surface $S$ of $V$.
- Laplace's, Poisson's, and Helmholtz' Equations are linear. Therefore the solution to Poisson's or Helmholtz' Equation for a sum of functions $g_{\mathrm{i}}\left(\mathbf{r}\right)$ or $h_{\mathrm{i}}\left(\mathbf{r}\right)$ is the sum of the solutions for each $g_{\mathrm{i}}\left(\mathbf{r}\right)$ or $h_{\mathrm{i}}\left(\mathbf{r}\right)$. And a solution to Laplace's equation can always be added to a solution of Poisson's equation.

The final theorem for Poisson's equation F.3 establishes that the particular solution to Poisson's equation in a region $V$ (2.80) is

$$\Phi\left(\mathbf{r}\right) = \frac{1}{4\pi}\int_{\mathrm{V}} \frac{g\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\mathrm{d}V', \tag{2.86}$$

and the final theorem for Helmholtz' Equation establishes that the particular solution to Helmholtz' Equation in a region $V$ (2.85) is

$$\Phi\left(\mathbf{r}\right) = \frac{1}{4\pi}\int_{\mathrm{V}} \frac{h\left(\mathbf{r}'\right)\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\mathrm{d}V' \tag{2.87}$$

independent of the value of $\Phi$ on the surface $S$ bounding the region $V$.

In the text we will develop (2.86) and (2.87) as the general solution to Poisson's equation using *Green's Functions*. This approach will differ from the method used in Appendices F and G to obtain (2.86). In the appendices we base the proof of our theorems on Green's Theorem (see Appendix D). The Green's Function approach is more modern and indispensable as a general approach to the solution of any nonhomogeneous differential equation. For these reasons we will devote more space in subsequent chapters to a study of Green's Functions.

### 2.6.1 Helmholtz' Theorem

The equations we will identify as *the* field equations will be those for the divergence and the curl of the vector fields. This will provide a definitive structure for our study. But the reason that these are the field equations is deeper than our desire for structured simplicity. Helmholtz' Theorem guarantees that the field is completely determined if the divergence and the curl of the field are known.

Helmholtz' Theorem and proof are presented here. The proof requires the fact that the solution to Poisson's partial differential equation is unique, which is proved in Appendix F, as was discussed in Sect. 2.6.

**Theorem 2.1.** ***(Helmholtz' Theorem. )*** *Any vector field,* $\mathbf{F}$*, which is finite, uniform, and continuous, may be expressed as*

$$\mathbf{F} = -\,\text{grad}\ \varphi + \text{curl}\ \mathbf{A}, \tag{2.88}$$

*where*

$\varphi$ *is any scalar function of* $\mathbf{r}$

$\mathbf{A}$ *is any vector function of* $\mathbf{r}$

div $\mathbf{A}$ *is some specified function.*

*Proof.* To prove this theorem, we must show that $\varphi$ and $\mathbf{A}$ are uniquely determined by $\mathbf{F}$ as provided in (2.88). Because div curl $\equiv 0$ and curl grad $\equiv \mathbf{0}$ (see exercises) we can obtain separate equations for $\varphi$ and $\mathbf{A}$ by operating on $\mathbf{F}$ with the div and the curl.

$$\begin{aligned} \text{div}\ \mathbf{F} &= -\,\text{div}\,\text{grad}\ \varphi + \text{div}\,\text{curl}\,\mathbf{A} \\ &= -\nabla^2\varphi \end{aligned} \tag{2.89}$$

and

$$\begin{aligned} \text{curl}\,\mathbf{F} &= -\,\text{curl}\,\text{grad}\ \varphi + \ \text{curl}\,\text{curl}\,\mathbf{A} \\ &= -\nabla^2\mathbf{A} + \ \text{grad}\,\text{div}\ \mathbf{A}. \end{aligned} \tag{2.90}$$

By hypothesis $\vec{F}$ and div $\mathbf{A}$ are known. Since $F$ is known, we know div $\mathbf{F}$ and curl $\mathbf{F}$. We shall identify these as

$$\Psi\,(\mathbf{r}) = \text{div}\ \mathbf{F}$$

and

$$\boldsymbol{\Lambda}\,(\mathbf{r}) = \text{curl}\ \mathbf{F}.$$

And we shall identify div **A** as

$$f\,(\mathbf{r}) = \text{div}\ \mathbf{A}$$

Then, using (2.89) we have

$$\nabla^2\varphi = -\Psi\,(\mathbf{r})\,. \tag{2.91}$$

And using (2.90) we have

$$\nabla^2\mathbf{A} = \text{grad}\ f\,(\mathbf{r}) - \mathbf{\Lambda}\,(\mathbf{r})\,. \tag{2.92}$$

Both (2.91) and (2.92) are Poisson's equations. Equation (2.92) is a vector Poisson's equation. It is a collection of three Poisson's equations; one for each component. As we know from Sect. 2.6 the solution of Poisson's equation is unique and for (2.91) is

$$\varphi\,(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{\Psi\,(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'. \tag{2.93}$$

For (2.92) the solution is

$$\mathbf{A}\,(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{[\mathbf{\Lambda}\,(\mathbf{r}') - \text{grad}\ f\,(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'. \tag{2.94}$$

Therefore $\varphi$ and **A** are uniquely determined by **F**. This establishes Helmholtz' Theorem. □

The corollary to Helmholtz' Theorem provides the claim that the divergence and curl specify the field uniquely.

**Corollary 2.1.** *If* $\Psi$ (= div **F**)*,* $\mathbf{\Lambda}$ (= curl **F**)*, and* $f\left(= -\text{div}\ \mathbf{A}\right)$ *are known then the vector field is completely specified.*

That is, if we know $\Psi$, $\mathbf{\Lambda}$, and $f$ we know $\varphi$ and **A** and, therefore, **F**. Noting the definitions of $\Psi$, $\mathbf{\Lambda}$, and $f$ it follows that $F$ is completely determined provided we also specify div **A**.

### *2.6.2 The Del Operator*

Many authors use the operator $\nabla$ (*del*), which is defined so that the gradient of a scalar function $\varphi$ may be written as

$$\text{grad}\,\varphi = \nabla\varphi = \hat{e}_\mu \frac{\partial}{\partial x_\mu}\varphi. \tag{2.95}$$

The divergence and the curl of a vector field **F** are then written as

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} \tag{2.96}$$

and

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}. \tag{2.97}$$

The dot and cross product forms of these differential operators are appropriate if the coordinate system is rectangular Cartesian. However, as we have indicated in detail in Sects. 2.5.3.2 and 2.5.4.3, the formulation is not a simple dot or cross product if the systems are either cylindrical or spherical.

Some authors have formulated the curl operations in cylindrical and spherical coordinates in a manner that allows them to be written as determinants of matrices (e.g. [17, p. 314]). This requires, however, a first row of the matrices that are not sets of basis vectors.

We have chosen to use the left rather than the right hand sides of (2.95)–(2.97). We do this primarily because the physical meaning of the operators is better conveyed in the written form.

### 2.6.3 Dirac Delta Function

In his monograph on the quantum theory [21], first published in 1930, Dirac developed a transformation theory among vector spaces capable of handling both discrete and continuous vectors. A product that produced a Kronecker delta for the discrete vector case resulted in a very interesting situation for the case of continuous vectors. Dirac pointed out that we had to relax the expectation that the scalar products of vectors would be finite or the theory "would be too weak for most practical problems [21, p. 39]."

In a later section [21, p. 58] Dirac introduces a function $\delta(x)$ with the property that

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases}$$

and that

$$\int_{-\infty}^{+\infty} \delta(x)\, \mathrm{d}x = 1.$$

The function $\delta(x)$ he says is so large in the domain $\varepsilon$ around the origin as to make the integral finite. He said that the exact shape of $\delta(x)$ in this domain "does not matter, provided there are no unnecessarily wild variations..." But he believed that "$\delta(x)$ is not a function of $x$ according to the usual mathematical definition of a function, which requires a function to have a definite value for each point in its domain, but is something more general, which we may call an 'improper function' to show up its difference from a function defined by the usual definition." This is the *Dirac Delta Function*, which we shall refer to simply as the *δ-function*.

However, more recently we have been able to show that $\delta(x)$ can be obtained as the limit of a sequence of what are known as *test functions* [89, pp. 97–129]. Test functions are continuous and infinitely differentiable. Such sequences are known as *delta sequences*.

An example of a delta sequence is

$$y_{\mathrm{n}}(x) = \frac{1}{\pi}\frac{\sin(nx)}{x}.$$

Each of the functions $y_{\mathrm{n}}(x)$ is continuous and infinitely differentiable. The function $\delta(x)$ is obtained in the limit

$$\delta(x) = \lim_{\mathrm{n}\to\infty}\frac{1}{\pi}\frac{\sin(nx)}{x}.$$

In Fig. 2.11 we plot $y_{\mathrm{n}}(x)$ for $n = 1, 3, 5$ and 10.

We may shift the location of the point of the infinity of $\delta(x)$ by writing $\delta(x - x')$. The infinity is still at $\delta(0)$, but now the argument becomes zero when $x = x'$. We then speak of $\delta(x - x')$.

If the function $f(x)$ is analytic in the region in which $\delta(x - x')$ goes to infinity, then

$$\int_{-\infty}^{+\infty} f(x)\,\delta\left(x - x'\right)\mathrm{d}x = f\left(x'\right)\int_{-\infty}^{+\infty}\delta\left(x - x'\right)\mathrm{d}x$$
$$= f\left(x'\right).$$

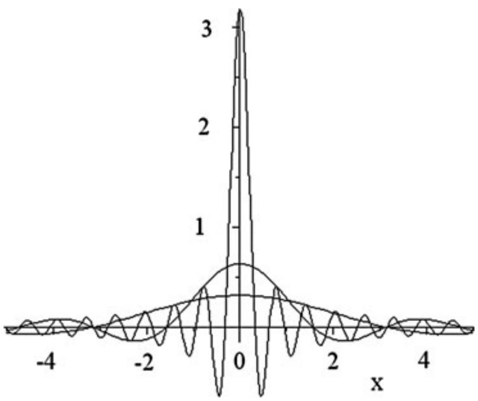



**Fig. 2.11** Plot of the function $y_{\mathrm{n}} = (1/\pi)\sin(nx)/x$ for $n = 1, 3, 5$, and 10

We may also reduce the limits of integration to include only the region containing $x'$. That is if $a < x' < b$ then

$$\boxed{\int_a^b f(x)\,\delta\,(x - x')\,\mathrm{d}x = f\,(x').} \tag{2.98}$$

We may then take

$$\boxed{\delta\,(x - x') = \begin{cases} 0 \text{ if } x \neq x' \\ \infty \text{ if } x = x' \end{cases}} \tag{2.99}$$

and (2.98) as the two statements defining the $\delta$-function. To establish that a particular sequence of test functions is a $\delta$-sequence we must show that both (2.99) and (2.98) are satisfied.

The property (2.98) is the mapping property of the $\delta$-function. The $\delta$-function $\delta\,(x - x')$ is a generalized function that maps $f\,(x)$ into its value at the point $x{=}x'$. We may use the mapping property of the $\delta$-function to prove general properties of the $\delta$-function.

*Example 2.5.* **Evenness of the $\delta$-function.** We can prove that the $\delta$-function is an even function, i.e. that $\delta\,(x - x') = \delta\,(x' - x)$, by showing that

$$\begin{aligned} \int_a^b f\,(x)\,\delta\left(x - x'\right)\mathrm{d}x &= f\left(x'\right) \\ &= \int_a^b f\,(x)\,\delta\left(x' - x\right)\mathrm{d}x, \end{aligned} \tag{2.100}$$

where $x'$ is a point on the interval $a \to b$. The even property of the $\delta$-function has been established if we can show that the last integral on the right hand side results in $f\,(x')$. We first substitute $\xi = x' - x$ in (2.100) with the result

$$\begin{aligned} \int_a^b f\,(x)\,\delta\left(x' - x\right)\mathrm{d}x &= -\int_{x'-a}^{x'-b} f\left(x' - \xi\right)\delta\,(\xi)\,\mathrm{d}\xi \\ &= \int_{x'-b}^{x'-a} f\left(x' - \xi\right)\delta\,(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{2.101}$$

We note, since $b > x' > a$, that $x' - b < 0$ and $x' - a > 0$. The range of integration in the second line of (2.101) then includes the origin of $\xi$. Therefore

$$\int_{x'-b}^{x'-a} f\left(x' - \xi\right)\delta\,(\xi)\,\mathrm{d}\xi = f\left(x'\right)$$

by (2.98). We have then established the even nature of the $\delta$-function.

We may extend the concept of a $\delta$-function to three dimensions by requiring that $\delta(\mathbf{r}-\mathbf{r}')$ is zero everywhere in space except at the point $\mathbf{r}=\mathbf{r}'$ at which point it is infinite. The infinity is such that

$$\int_V \delta\left(\mathbf{r}-\mathbf{r}'\right) f(\mathbf{r})\,\mathrm{d}V = f\left(\mathbf{r}'\right).$$

We see, with $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, that

$$\delta\left(\mathbf{r}-\mathbf{r}'\right) = \delta\left(x-x'\right)\delta\left(y-y'\right)\delta\left(z-z'\right). \tag{2.102}$$

Then, from the example 2.5 we have

$$\delta\left(\mathbf{r}-\mathbf{r}'\right) = \delta\left(\mathbf{r}'-\mathbf{r}\right).$$

*Example 2.6.* **Derivative of the $\delta$-function.** Using (2.98) we can show that the derivative of the $\delta$-function is

$$\frac{\mathrm{d}}{\mathrm{d}x}\delta\left(x-x'\right) = -\frac{1}{\left(x-x'\right)}\delta\left(x-x'\right).$$

We do this by showing that, for a function $f(x)$, which is analytic in a small region containing $x'$, that

$$\int_{-\infty}^{\infty}\mathrm{d}x\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta\left(x-x'\right)\right]f(x) = \int_{-\infty}^{\infty}\mathrm{d}x\left[-\frac{1}{\left(x-x'\right)}\delta\left(x-x'\right)\right]f(x). \tag{2.103}$$

Using the product rule, the integral on the left of (2.103) is

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}x\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta\left(x-x'\right)\right]f(x)\\
&= \int_{-\infty}^{\infty}\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\left[\delta\left(x-x'\right)f(x)\right] - \int_{-\infty}^{\infty}\mathrm{d}x\delta\left(x-x'\right)\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\
&= -\int_{-\infty}^{\infty}\mathrm{d}x\delta\left(x-x'\right)\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\
&= -\left.\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]_{x=x'},
\end{aligned} \tag{2.104}$$

since the $\delta$-function vanishes except at $x = x'$. Because the function $f(x)$ is analytic at $x = x'$, we can write it as a Taylor Series

$$f(x) = f(x') + \frac{d}{dx} f(x)\Big]_{x=x'} (x - x') + \frac{1}{2!}\frac{d^2}{dx^2} f(x)\Big]_{x=x'} (x - x')^2 + \cdots .$$

The integral on the right hand side of (2.103) can then be written as

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx \left[-\frac{1}{(x-x')}\delta(x-x')\right] f(x) \\
&= -\int_{-\infty}^{\infty} dx \left[\frac{1}{(x-x')}\delta(x-x')\right] f(x') \\
&\quad -\int_{-\infty}^{\infty} dx \left[\delta(x-x')\right] \frac{d}{dx} f(x)\Big]_{x=x'} \\
&\quad -\int_{-\infty}^{\infty} dx \left[\delta(x-x')\right] \frac{1}{2!}\frac{d^2}{dx^2} f(x)\Big]_{x=x'} (x-x') + \cdots .
\end{aligned} \tag{2.105}$$

Because the $\delta$-function is even and $(x - x')$ is odd the first integral on the right hand side of (2.105) vanishes. All integrals on the right hand side of (2.105) beyond the second also vanish because they are of the form

$$\int_{-\infty}^{\infty} dx \delta(x-x')(x-x')^{\mathrm{n}} \text{ with } n \geq 1.$$

Only the second integral on the right hand side of (2.105) remains, so

$$\int_{-\infty}^{\infty} dx \left[-\frac{1}{(x-x')}\delta(x-x')\right] f(x) = -\frac{d}{dx} f(x)\Big]_{x=x'} . \tag{2.106}$$

The equality of (2.106) and (2.104) establishes the identity.

There are two representations of the $\delta$-function that will be used extensively in our study of classical field theory. The first of these is

$$\delta(\mathbf{r} - \mathbf{r}') = -\nabla^2 \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.107}$$

In the exercises we show that (2.107) satisfies (2.99) by direct calculation. We can show that (2.98) is also satisfied if we use the solution to Poisson's equation (2.86). If we operate on both sides of (2.86) with the Laplacian operator, we have

$$\nabla^2 \Phi(\mathbf{r}) = \int_{\mathrm{V}} g(\mathbf{r}') \left[\nabla^2 \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}\right] dV. \tag{2.108}$$

Since we began with the general solution to Poisson's equation, (2.108) must result in

$$\int_{\mathrm{V}} g\left(\mathbf{r}'\right)\left[\nabla^2 \frac{1}{4\pi}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right]\mathrm{d}V = -g\left(\mathbf{r}\right), \tag{2.109}$$

which requires that

$$\nabla^2 \frac{1}{4\pi}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = -\delta\left(\mathbf{r}-\mathbf{r}'\right). \tag{2.110}$$

The second representation of the $\delta$-function that we will use extensively is

$$\delta\left(x-x'\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[ik\left(x-x'\right)\right]\mathrm{d}k. \tag{2.111}$$

This may be extended to three dimensions using (2.102).

$$\begin{aligned}
\delta\left(\mathbf{r}-\mathbf{r}'\right) &= \delta\left(x-x'\right)\delta\left(y-y'\right)\delta\left(z-z'\right)\\
&= \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{+\infty}\mathrm{d}k_{\mathrm{x}}\int_{-\infty}^{+\infty}\mathrm{d}k_{\mathrm{y}}\int_{-\infty}^{+\infty}\mathrm{d}k_{\mathrm{z}}\cdots\\
&\quad\cdots\exp\left(ik_{\mathrm{x}}\left(x-x'\right)\right)\exp\left(ik_{\mathrm{y}}\left(y-y'\right)\right)\exp\left(ik_{\mathrm{z}}\left(z-z'\right)\right)\\
&= \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{+\infty}\mathrm{d}^3\mathbf{k}\exp\left[\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)\right],
\end{aligned} \tag{2.112}$$

where $\mathrm{d}^3\mathbf{k} = \mathrm{d}k_{\mathrm{x}}\mathrm{d}k_{\mathrm{y}}\mathrm{d}k_{\mathrm{z}}$ and $\mathbf{k} = \hat{e}_\alpha k_\alpha$.

## 2.7 Summary

In this chapter we have presented the basic mathematical principles employed in the study of classical field theory. We have been particularly careful in our treatment of the vector calculus because the reader should have a good understanding of the physical meaning of the differential operators and of the coordinate systems frequently employed in classical field theory.

Helmholtz' Theorem, provides the guiding principle we shall follow in our development of Maxwell's Equations. For each field in both the static and dynamic (time dependent) cases we shall seek the divergence and the curl equations. These are the field equations, as required by Helmholtz' Theorem.

We ended the chapter with a study of the Dirac $\delta$-function.

Additional mathematical principles and techniques will be developed as we need them in the rest of the text.

## Exercises

**2.1.** Show that the even permutations of 1, 2 and 3 are $(1,2,3)$, $(2,3,1)$ and $(3,1,2)$. Odd permutations are $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ by carrying out the permutation steps.

**2.2.** Show that the general definition of the scalar product $\mathbf{a} \cdot \mathbf{b} = \delta_{\mu\nu} a_\mu b_\nu$ is equivalent to the elementary definition $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \cos \vartheta_{\mathrm{ab}}$ where $\vartheta_{\mathrm{ab}}$ is the angle between **a** and **b**.

**2.3.** Show that the general definition of the vector or cross product

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \varepsilon_{\alpha\beta\gamma} \hat{e}_\alpha a_\beta b_\gamma \\ &= \hat{e}_1 \left(a_2 b_3 - b_2 a_3\right) + \hat{e}_2 \left(b_1 a_3 - a_1 b_3\right) + \hat{e}_3 \left(a_1 b_2 - a_2 b_1\right)\end{aligned}$$

is equivalent to the elementary definition $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| \, |\mathbf{b}| \sin \vartheta_{\mathrm{ab}}$ where $\vartheta_{\mathrm{ab}}$ is the angle between **a** and **b**.

**2.4.** Show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \, (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \, (\mathbf{a} \cdot \mathbf{b})$$

for three arbitrary vectors. This is called the "$bac - cab$" rule and is very useful. It is simplest to show this if you use the Levi-Civita density formulation for the cross product.

**2.5.** Show that $\mathbf{a} \times \mathbf{b}$ is numerically equal to the area of the parallelogram formed by the vectors **a** and **b**, that is the parallelogram whose sides are **a** and **b**.

**2.6.** The triple scalar product is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \, .$$

Show that this is the volume of the parallelepiped with base formed by **b** and **c** and slant height **a**.

**2.7.** Show that $\operatorname{grad} \varphi$ is always perpendicular to the surface $\varphi(\mathbf{r}) = \text{constant}$.
*Hint:* consider a general displacement

$$\mathrm{d}\boldsymbol{\ell} = \hat{e}_\alpha \mathrm{d}x_\alpha$$

and show that $\operatorname{grad} \varphi \cdot \mathrm{d}\boldsymbol{\ell} = 0$ if $\mathrm{d}\boldsymbol{\ell}$ is on a surface $\varphi(\mathbf{r}) = \text{constant}$.

**2.8.** Consider a plane containing the points $(x_0, y_0, z_0)$ and $(x, y, z)$. Let the unit vector perpendicular to the plane be

$$\hat{n} = n_\mu \hat{e}_\mu .$$

Show that the points $(x, y, z)$ satisfy the equation

$$(x - x_0)\, n_x + (y - y_0)\, n_y + (z - z_0)\, n_z = 0,$$

which is then the equation for the plane.

**2.9.** If **A** is a constant vector and **r** is the vector from the origin to the point $(x, y, z)$, show that

$$(\mathbf{r} - \mathbf{A}) \cdot \mathbf{A} = 0$$

is the equation of a plane.

**2.10.** If **A** is a constant vector and **r** is the vector from the origin to the point $(x, y, z)$, show that

$$(\mathbf{r} - \mathbf{A}) \cdot \mathbf{r} = 0$$

is the equation of a sphere.

**2.11.** Using the dot product, find the cosine of the angle between the body diagonal of a unit cube and one of the cube edges.

**2.12.** Show that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}.$$

The dot and cross products may then be exchanged in the scalar triple product.

**2.13.** If **A**, **B**, **C** are vectors from the origin to the points $A, B, C$, show that the vector **D** defined by the expression

$$\mathbf{D} = (\mathbf{A} \times \mathbf{B}) + (\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A})$$

is perpendicular to the plane $ABC$.

**2.14.** (a) Show that **A**, **B**, and **C** are not linearly independent if

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$$

(b) Are the following vectors linearly independent?

$$\mathbf{A} = \hat{e}_y + 3\hat{e}_z$$
$$\mathbf{B} = \hat{e}_x - 2\hat{e}_z$$
$$\mathbf{C} = \hat{e}_x + \hat{e}_y + \hat{e}_z$$

**2.15.** Is curl **F** necessarily perpendicular to **F** for every vector function **F**? Justify your answer.

**2.16.** For the scalar functions $\varphi$ and $\psi$ show that

$$\nabla^2 \varphi\psi = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\,\mathrm{grad}\,\varphi \cdot \mathrm{grad}\,\psi.$$

**2.17.** Let $r$ be the magnitude of the vector from the origin to the point $(x, y, z)$ and let $f(r)$ be an arbitrary function of $r$. Show that

$$\mathrm{grad}\, f(r) = \frac{\mathbf{r}}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r},$$
$$\mathrm{curl}\,[\mathbf{r} f(r)] = \mathbf{0}.$$

**2.18.** The vector field

$$\mathbf{F} = f(r)\,\hat{e}_\vartheta$$

is a vector field oriented circularly around the $z$-axis. Such a field may be a magnetic field, as we shall see, arising from a current density along the $z$-axis.

Show that Stokes' and Gauss' Theorems are valid for this field. Carry out the calculations.

**2.19.** Show that

$$\mathrm{curl}\,\mathrm{grad}\,\Phi = \mathbf{0}$$

for any scalar function $\Phi$.

**2.20.** Show that

$$\mathrm{div}\,\mathrm{curl}\,\mathbf{F} = 0$$

for any vector field $\mathbf{F}$.

**2.21.** Show that

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = \mathrm{grad}\,\mathrm{div}\,\mathbf{F} - \nabla^2\mathbf{F}.$$

This is most easily done using the subscript notation in Cartesian Coordinates.

**2.22.** Show that

$$\text{(a)}\ \mathrm{div}\,\mathbf{r} = 3,$$
$$\text{(b)}\ \mathrm{curl}\,\mathbf{r} = \mathbf{0},$$
$$\text{(c)}\ \mathrm{grad}\,\mathbf{r} = \hat{e}_\mathrm{x}\hat{e}_\mathrm{x} + \hat{e}_\mathrm{y}\hat{e}_\mathrm{y} + \hat{e}_\mathrm{z}\hat{e}_\mathrm{z},$$
$$\text{(d) and, therefore that } \mathbf{G}\cdot\mathrm{grad}\ \mathbf{r} = \mathbf{G},$$
$$\text{(e)}\ \nabla^2\mathbf{r} = 0,$$

where

$$\mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z.$$

**2.23.** Obtain $\nabla'^2 (1/|\mathbf{r} - \mathbf{r}'|)$ in rectangular Cartesian coordinates. Show first that

$$\text{grad}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\,\text{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

and then that

$$\nabla'^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \begin{cases} 0 \text{ if } \mathbf{r} \neq \mathbf{r}' \\ \infty \text{ if } \mathbf{r} = \mathbf{r}' \end{cases}.$$

**2.24.** Assume that you have measured the components of a conservative field in a region of space to be of the form

$$F_x = 6x + y,$$
$$F_y = x.$$

Out of curiosity you want to know the potential that might have resulted in this field.

(a) How would you go about finding this potential?
(b) Find it.

**2.25.** Which (if any) of the following force fields is/are conservative?

(a)

$$F_x = -4\exp\left(-y^2\right)$$
$$F_y = 8xy\exp\left(-y^2\right)$$
$$F_z = 0$$

(b)

$$F_x = -8xyz$$
$$F_y = -4xz$$
$$F_z = -4x^2y$$

(c)

$$F_x = -2xy^2z^2 - y^2z + 2xy$$
$$F_y = -2x^2yz^2 - 2xyz + x^2$$
$$F_z = -2x^2y^2z - xy^2$$

(d)

$$F_{\mathrm{x}} = -yz\exp\left(-x^2y^2\right) + 2x^2yz\exp\left(-x^2y^2\right)$$
$$F_{\mathrm{y}} = -xyz\exp\left(-x^2y^2\right) + 2x^3y^2z\exp\left(-x^2y^2\right)$$
$$F_{\mathrm{z}} = -xy\exp\left(-x^2y^2\right)$$

(e)

$$F_{\mathrm{x}} = -\frac{z}{x}$$
$$F_{\mathrm{y}} = -\frac{z}{y}$$
$$F_{\mathrm{z}} = -\ln(xy)$$

**2.26.** The Laws of physics are generally formulated as integral equations. Our experiments, which are the basis on which those laws rest, are conducted on large systems such as glasses of water or blocks of metal.

Two such laws familiar to the students of elementary physics are Gauss' and Ampère's laws of electromagnetism in integral form

$$\oint_{\mathrm{S}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_{\mathrm{V}} \rho \mathrm{d}V$$

and

$$\oint_{\mathrm{C}} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I = \mu_0 \int_{\mathrm{a}} \mathbf{J}\cdot \mathrm{d}\mathbf{a}.$$

In these integral expressions $\rho$ is the charge density defined such that the integral of the charge density over the volume contained in the surface **S** produces the total charge, $Q$ and **J** is the current density defined such that the integral of the differential flux of the current density ($\mathbf{J}\cdot\mathrm{d}\mathbf{a}$) over the open area defined by the loop, $C$, is the current passing through that area. That is

$$Q = \int_{\mathrm{V}} \rho \mathrm{d}V$$

and

$$I = \int_{\mathrm{a}} \mathbf{J}\cdot \mathrm{d}\mathbf{a}.$$

From Gauss' and Ampère's laws of electromagnetism obtain differential laws for the electric and magnetic fields using Gauss' and Stokes' Theorems of the vector calculus.

**2.27.** Evaluate the following integrals:

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta (x-1) \exp \left(-\alpha x^2 + \beta x\right)$$

$$\int_{0}^{\infty} \mathrm{d}x \delta (x+1) \exp \left(-\alpha x^2 + \beta x\right)$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta (x+\lambda) \cos \left(\frac{2\pi}{\lambda} x\right) \exp \left(-\frac{x^2}{\lambda^2}\right)$$

$$\int_{0}^{10} \mathrm{d}x \delta (x+5) \left(6x^2 + 2x - 3\right)$$

$$\int_{-\infty}^{0} \mathrm{d}x \delta (x+5) \left(6x^2 + 2x - 3\right)$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta (x+\lambda) \sin \left(\frac{2\pi}{\lambda} x\right) \exp \left(-\frac{x^2}{\lambda^2}\right)$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta (x-1) J_{\mathrm{n}} (x) \qquad \text{where } J_{\mathrm{n}} (x) \text{ is a Bessel Function of order } n.$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \delta (x) \operatorname{erf} (x) \qquad \text{where } \operatorname{erf} (x) \text{ is the Error Function.}$$

**2.28.** Begin with the formula we have for the $\delta$-function,

$$\delta \left(x - x'\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \exp \left(ik \left(x - x'\right)\right)$$

and show that the $\delta$-function is even. That is, show that

$$\delta \left(x - x'\right) = \delta \left(x' - x\right).$$

**2.29.** Using the conventional manner of proof, show that

$$\delta (ax) = \frac{1}{a} \delta (x).$$

**2.30.** Establish that

$$\operatorname{div} \operatorname{grad} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -4\pi \delta \left(\mathbf{r} - \mathbf{r}'\right).$$

You should be able to show that the result vanishes unless the points $\mathbf{r}$ and $\mathbf{r}'$ are identical. The integral property requires that

$$\int_V dV \left(-\frac{1}{4\pi}\right) \text{div grad} \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = 1.$$

To show this, choose the (arbitrary) point $\mathbf{r}'$ to be the origin and integrate over a vanishingly small sphere centered on the origin.

**2.31.** Show that

$$\lim_{\alpha \to 0} \left(\frac{1}{\alpha\sqrt{\pi}}\right) \exp\left(-x^2/\alpha^2\right) = \delta\,(x)\,.$$

# Chapter 3
# Electrostatics

*Experiment is the sole source of truth. It alone can teach us something new; it alone can give us certainty.*

*Jules Henri Poincaré*

## 3.1 Introduction

In this chapter we obtain the mathematical equations which completely describe the electrostatic field from the results of a single experiment: that of Coulomb.

Because of the mathematical form of Coulomb's Law the formulation of the electrostatic field will emerge in a natural fashion, although it is a philosophical step. We will introduce a superposition Ansatz to cast the results of Coulomb's Experiment in terms of distributions of charges. And from these we will be able to find the divergence and the curl equations required to specify the field according to Helmholtz' Theorem (see Sect. 2.6.1).

Using Gauss' and Stokes' theorems we will be able to also obtain the field equations in integral form.

## 3.2 Coulomb's Law

### *3.2.1 Coulomb's Experiment*

Coulomb's experiment to determine the force between two electric charges was the first truly quantitative experiment in the study of electricity. We discussed the details of the experiment in Chap. 1, Sect. 1.6.

The situation that Coulomb studied is illustrated in Fig. 3.1. In Fig. 3.1 we have removed Coulomb's apparatus and represented the charges $q_1$ and $q_2$ with their centers located at two arbitrary positions in space defined by the position vectors

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,

DOI 10.1007/978-3-642-23205-3_3, 

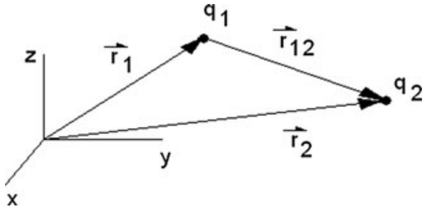



**Fig. 3.1** Coulomb's Experiment. Charge $q_1$ is located by the vector $\mathbf{r}_1$ and charge $q_2$ is located by the vector $\mathbf{r}_2$. The vector $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ is between centers of the charges

$\mathbf{r}_1$ and $\mathbf{r}_2$. The vector between the centers of the charges $q_1$ and $q_2$ is $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$. which is directed from $q_1$ to $q_2$. Coulomb found that the force between two charges is directly proportional to the product of the magnitudes of the charges $q_1 q_2$ and inversely proportional to the square of the distance between the charge centers $|\mathbf{r}_{12}| = |\mathbf{r}_2 - \mathbf{r}_1|$. In mathematical terms the force law is

$$\mathbf{F}_{12} = K_e \frac{q_1 q_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|}, \tag{3.1}$$

where $K_e > 0$ is an empirical constant. Equation (3.1) is the empirical equation[1] describing the experiment and is known as *Coulomb's Law*.

We note that if the value of the force $\mathbf{F}_{12}$ is positive then the force is directed from $q_1$ toward $q_2$ and is repulsive. If $\mathbf{F}_{12}$ is negative then the force is attractive. Because $K_e$ is positive, the algebraic sign of the product $q_1 q_2$ determines whether the value of the force $F_{12}$ is positive or negative. If the two charges are alike in sign then the product $q_1 q_2$ is positive and the force $\mathbf{F}_{12}$ is repulsive. If the two charges are opposite in sign then the product $q_1 q_2$ is negative and the force $\mathbf{F}_{12}$ is attractive.

In Chap. 1 we pointed out that du Fay considered that there were two types of electric charge, while Franklin believed that there was but one. Coulomb was very clearly a believer in the two fluid theory. The natural state of matter, according to Coulomb, was that in which both fluids were present in an equal amount ([97], p. 59).

Although we can understand Coulomb's experiment in terms of a single fluid, provided the charged bodies are sufficiently large for the density of charges to be considered as a fluid density, the point of view Coulomb adopted is more natural. Using Coulomb's position we can interpret the results of the experiment in terms of single charges. In that case our development of a modern theory of classical fields, which acknowledges elementary charges, and which can be used in modern applications, is unhindered. So we shall adopt the two fluid picture and assume that the fluids are composed of charged particles.

---

[1]Empirical results are those obtained from laboratory measurements. The mathematical equation the experimenter obtains to represent the data in terms of known or defined and measurable quantities is the *empirical equation* describing the experiment.

### 3.2.2 Units

The value of the constant $K_e$ in (3.1) depends on the units chosen. In this text we shall use the *Système International d'Unités* (SI) system of units. This is the most common system of units for experimental work in the sciences and engineering, which is the basis for our choice. In this system

$$K_e = 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \tag{3.2}$$

and the charge unit C is called the *coulomb*. By convention $K_e$ is written as

$$K_e = \frac{1}{4\pi\varepsilon_0}$$

where

$$\varepsilon_0 = \frac{1}{36\pi \times 10^9} \text{ N}^{-1} \text{ m}^{-2} \text{ C}^2 \tag{3.3}$$

is the *permittivity of free space***.**

In the SI system Coulomb's Law takes the form

$$\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|}, \tag{3.4}$$

The Gaussian system of units is an alternative to the SI system. The Gaussian system is favored by theoreticians because the equations written in this system have a cleaner look to them.

The SI was formerly known as the MKSA (Meter, Kilogram, Second, Ampere) system. Here the ampere is the unit of electrical current. One ampere of current flows in a wire if the rate at which charge flows past a point on the wire is one coulomb per second.

## 3.3 Superposition

In (3.4) we have an experimental law that we have chosen to cast in terms of charged particles. That we can do that is not a priori self evident. We are using the well-established results of Coulomb's work to write the law for point charges because it suits our purpose to do so.

The point particles we are considering are not necessarily protons or electrons. It is better not to be so specific. The localization of protons and electrons, as well as claims that they are at rest, are issues that involve a quantum description. We shall rather consider the classical charged particle to be a very small sphere of charged matter with a radius that is infinitesimal compared to any measurable dimensions for any situation.

Using this concept of point charge we can construct any sort of charge distribution we choose provided we first make an *Ansatz*[2] regarding the interaction of a single charge with a number of other charges. Our Ansatz will be that the interaction of any single charge with any other single charge is independent of all other charges that are in the vicinity of that charge. The total force of interaction of any single charge with a collection of charges is equal then to the sum over two particle coulomb force terms.

This is known as the principle of *superposition*. It is an Ansatz because we have no direct proof, at this stage in our development, of the validity of this principle. We may later infer its validity from the fact that our final field equations, which will be based on this Ansatz, are correct in that they provide a complete understanding of Classical Fields, which agrees with experimental results.

## 3.4 Distributions of Charges

We may produce distributions of charges by transferring charge from rubbed amber or an electrostatic generator to a solid body. As a result the charges become distributed in and on the solid body. The distribution of charge on or throughout the body depends on the geometrical shape and the chemical composition of the body.

We can also create a distribution of charge just outside of a solid body, such as the filament of an incandescent lamp, by heating the body. The electric current flowing through the filament heats the filament. As the temperature of the filament increases the average energy of the electrons increases, and some electrons will overcome the binding force holding them to the filament. There will then be a distribution of electrons in the region outside of the filament and a corresponding distribution of positive charge within the filament.

In a fluorescent tube light is emitted by atoms which have been struck by free electrons moving through the tube. The density of the electrons varies as a function of the distance down the tube.

At very high temperatures, such as that within stars, matter becomes a plasma, which is he fourth state of matter beyond the solid, liquid, and gas. Generally a plasma consists of free electrons, ions, and atoms. At extreme temperatures no atoms or partially ionized atoms remain and the plasma consists of free electrons and nuclei.

[2]The German term *Ansatz*, which has entered American physics and mathematics, means that we are going to consider a certain construct or point of view to be correct and shall seek results consistent with this construct. In effect the *Ansatz* defines the world we consider. Lack of agreement with experiment would force us to consider a new *Ansatz*.

Primarily because of the present interest in the energy that can be obtained from fusion reactions taking place in plasmas, which occur when bare nuclei collide with sufficient energy to fuse, the study of the physics of plasmas is a major research area.

In any case we may consider that spatial regions containing densities of charges are not unusual. Since these charge densities result from electron or ion concentrations, which in either case are particles, we shall model these charge densities as clusters of the sorts of charged particles we introduced in Sect. 3.3 and use the results of Coulomb's experiment in the form (3.4), with the caveat that Coulomb's experiment only holds for time-independent conditions.

### *3.4.1 Distribution of Point Charges*

With our Ansatz (superposition) we may generalize Coulomb's Law (3.4). We consider that in a specific volume in space $V$ there is a number of point charges $\{q_j\}$ located at the points $\{\mathbf{r}_j\}$. And we focus our attention on a charge $q_i$ located at the point $\mathbf{r}_i$, which is outside of the volume $V$. This situation is shown in Fig. 3.2. In this case our Ansatz results in the total force on the charge $q_i$ given by

$$\mathbf{F}_i = \frac{q_i}{4\pi\varepsilon_0}\sum_{j \text{ in } V'} \frac{q_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2} \frac{\left(\mathbf{r}_i - \mathbf{r}_j\right)}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}. \tag{3.5}$$

If the charge $q_i$ is one of the set of charges $\{q_j\}$ in the volume $V$ we may remove it to the outside of the volume $V$ with a geometrical construction. We construct a small sphere centered on the point $\mathbf{r}_i$ encompassing the charge $q_i$. We then connect this small sphere to the outside of the volume $V$ by constructing a small tunnel, which passes between all the charges $\{q_j\}$. This situation is shown in Fig. 3.3. Consistent with our Ansatz the total force on $q_i$ for the situation in Fig. 3.3 is again (3.5). Equation (3.5) is then the total *electrostatic* force on a point charge $q_i$ located at the point $\mathbf{r}_i$ from a collection of discrete point charges $\{q_j\}$ located at points $\{\mathbf{r}_j\}$ in an identifiable region of space $V$ whether $\mathbf{r}_i$ is outside or inside of the region $V$.

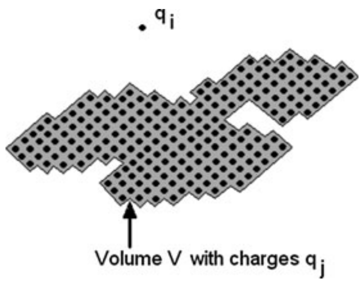



**Fig. 3.2** The charge $q_i$ is at a distance from the volume $V$ containing the charges $q_j$

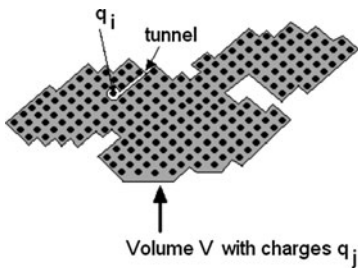



**Fig. 3.3** The charge $q_i$ is one of the charges $\{q_j\}$ contained within volume $V$

We have introduced the designation electrostatic because Coulomb's experiment was performed on static, or stationary charges. This designation is not crucial here. But it will become particularly important when we consider dynamic or time-dependent fields.

### *3.4.2 Volume Charge Density*

If our classical point charges in the volume $V$ are brought close enough together that it is no longer possible to consider them discrete, then the charge densities in Figs. 3.2 and 3.3 become continuous charge densities. Instead of considering a set of discrete points $\{q_j\}$ in the volume $V$, let us divide the volume $V$ into a large number $N$ of small volumes $\Delta V'$ located at the points $\mathbf{r}'$. Each of these small volumes will contain an amount of charge $\Delta q'$. We say that the charge distribution in $V$ is a continuous distribution if

$$\rho\left(\mathbf{r}'\right) \equiv \lim_{\Delta V' \to 0} \frac{\Delta q\left(\mathbf{r}'\right)}{\Delta V'} \tag{3.6}$$

is a continuous function of $\mathbf{r}'$. The function $\rho\left(\mathbf{r}'\right)$ defined by (3.6) is then the continuous charge density in the volume $V$.

We now ask for the force on a charge $q$ located at the point $\mathbf{r}$ at a distance from the continuous distribution in the volume $V$. The spatial or geometrical relationship between the point charge $q$ at $\mathbf{r}$ and the element of charge density $\Delta q' = \rho\left(\mathbf{r}'\right)\Delta V'$ at $\mathbf{r}'$ is shown in Fig. 3.4. The description of the situation is the same as that in the case of discrete charges if we replace the discrete charges $q_j$ with $\Delta q' = \rho\left(\mathbf{r}'\right)\Delta V'$ and sum over $\mathbf{r}'$. Consistent with our Ansatz, the total force on $q$ at $\mathbf{r}$ is then

$$\mathbf{F}_q\left(\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0} \sum_{\text{all } \mathbf{r}' \text{ in } V} \frac{\rho\left(\mathbf{r}'\right)\Delta V'}{\left|\mathbf{r}-\mathbf{r}'\right|^2} \frac{\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{3.7}$$

Each $\mathbf{r}'$ locates a small volume $\Delta V'$ so we may consider the summation in (3.7) to be over the volume elements $\Delta V'$, which we now label as $\Delta V_j'$. As the size

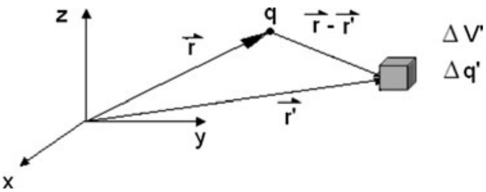



**Fig. 3.4** The spatial (geometrical) relationship between the charge $q$ at $\mathbf{r}$ and the element of charge density $\Delta q' = \rho(\mathbf{r}')\,\Delta V'$ at $\mathbf{r}'$

of the volume elements goes to zero and the number of volume elements goes to infinity the summation in (3.7) goes over into a Riemann integral over the volume $V$ containing the charge density. Here, and in most situations, we extend the region of integration to an arbitrary volume $V$, which will include at least the volume containing the charges. That is

$$\mathbf{F}_q(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \lim_{\Delta V_j'\to 0 \text{ and } N\to\infty} \sum_{\Delta V_j'}^{N} \frac{\rho(\mathbf{r}')\,\Delta V_j'}{|\mathbf{r}-\mathbf{r}'|^2} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{q}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \left(\mathbf{r}-\mathbf{r}'\right) \mathrm{d}V' \tag{3.8}$$

With (3.8) we introduce a notation that we will maintain throughout the text. We designate source coordinates, which locate the charges or charge densities, as $\mathbf{r}'$ and the coordinates where the force, or the field, is to be evaluated as $\mathbf{r}$.

### *3.4.3 Surface Charge Density*

If we distribute charge on a conductor it will, in the static situation, be confined to the surface of the conductor. We find the reason for this in the physical properties of a conductor.

A conductor is a material in which (some of) the electrons are free to move. This is a result of the properties of the atoms and the regular array (crystal) they form in the solid state. If we consider an isolated atom of the conductor quantum mechanics tells us that the electrons are in discrete bound energy states. But when the atoms are linked together in the solid state, quantum mechanics tells us that the energies of the (outer) electrons are no longer discrete, but continuous. The electrons belong to the entire crystal rather than to individual atoms. If there were a net charge at some point within the conducting solid these conduction electrons would experience an electrostatic force and would move. The situation in the conductor would no longer be static. Therefore there can be no free charge within a conductor under static conditions. All charge must be on the surface of the conductor.

There can also be no electrostatic force tangential to the surface of a conductor because a tangential force would result in charge motion on the surface. Under static conditions, then, the charge density on the surface of a conductor will be such that the electrostatic force vector is perpendicular to the conductor surface.

There is then a real physical reason to consider surface charge densities. Because even our classical point charges are vanishingly small we may consider that the surface (of the conductor) on which the charges are located has no thickness. It is a two dimensional area covering the volume of the conductor. The manner in which the charge becomes distributed on the surface of the conductor depends on the geometrical shape of the surface.

We divide the surface into small areas $\Delta S'$ located at the tips of the vectors $\mathbf{r}'$. Each area $\Delta S'$ at $\mathbf{r}'$ will contain an amount of charge $\Delta q'(\mathbf{r}')$. Just as in the case of the continuous (volume) distribution, we say that there is a continuous surface distribution of charge if the function

$$\sigma\left(\mathbf{r}'\right) = \lim_{\Delta S' \to 0} \frac{\Delta q\left(\mathbf{r}'\right)}{\Delta S'} \tag{3.9}$$

is a continuous function of $\mathbf{r}'$. The function $\sigma(\mathbf{r}')$ is the surface density of the charge.

We now consider a classical point charge $q$ located at $\mathbf{r}$, which is distinct from all the points $\mathbf{r}'$ on the surface of the conductor. We have shown the spatial relationship between the point charge $q$ at $\mathbf{r}$ and the element of charge density $\Delta q' = \sigma(\mathbf{r}')\,\Delta S'$ at $\mathbf{r}'$ in Fig. 3.5. The mathematical description of this situation is the same as that in the case of discrete charges if we replace the discrete charges $q_{\mathrm{j}}$ with $\Delta q' = \sigma(\mathbf{r}')\,\Delta S'$ and sum over all $\mathbf{r}'$ on the surface of the conductor. Consistent with our Ansatz, the total force on $q$ at $\mathbf{r}$ is then

$$\mathbf{F}_{\mathrm{q}}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \sum_{\text{all } \mathbf{r}' \text{ on } S} \frac{\sigma(\mathbf{r}')\,\Delta S'}{\left|\mathbf{r}-\mathbf{r}'\right|^2} \frac{(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{3.10}$$

Each $\mathbf{r}'$ locates an element of area on the surface $\Delta S'$. We have designated the total surface area of the conductor as $S$. We may, therefore, consider the summation in (3.10) to be a summation over the $N$ area elements $\Delta S'_{\mathrm{j}}$ on the surface $S$. As the size of the area elements go to zero and the number of these elements goes to infinity, the summation in (3.10) becomes a Riemann integral over the surface $S$. That is

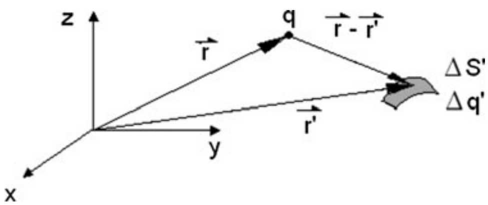



**Fig. 3.5** The spatial relationship between a classical point charge $q$ at $\mathbf{r}$ and the element of surface charge density $\Delta q' = \sigma(\mathbf{r}')\,\Delta a'$ at $\mathbf{r}'$

$$\mathbf{F}_{\mathrm{q}}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \lim_{\Delta S'_{\mathrm{j}}\to 0 \text{ and } \mathrm{N}\to\infty} \sum_{\Delta a'_{\mathrm{j}}}^{\mathrm{N}} \frac{\sigma(\mathbf{r}')\,\Delta S'_{\mathrm{j}}}{|\mathbf{r}-\mathbf{r}'|^2} \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{q}{4\pi\varepsilon_0} \oint_{\mathrm{S}} \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \left(\mathbf{r}-\mathbf{r}'\right) \mathrm{d}S' \tag{3.11}$$

The integral symbol $\oint_{\mathrm{S}}$ indicates that the integration is to be over the entire surface enclosing the volume of the conductor.

As in (3.8) we again designate the source coordinates as $\mathbf{r}'$ and the coordinates where the force, or the field, is to be evaluated as $\mathbf{r}$.

We now have three formulations for the force on a charge $q$ located at a point $\mathbf{r}$ under static, time independent conditions based on our Ansatz of superposition. These are equation (3.5) for the case in which the charges can still be treated as individual points, equation (3.8) for the case in which the charges are distributed in such a way that the charge density is a continuous function of the coordinates within a volume, and equation (3.11) for the case in which the charges are distributed on the surface of a (conducting) body in such a way that the charge density on that surface is a continuous function of the coordinates of the surface.

## 3.5 The Field Concept

In (3.5), (3.8), and (3.11) we have an "action at a distance" formulation for the force on a charge $q$. This is completely in keeping with a Newtonian experimental and mathematical philosophy. Coulomb's experimental results allow us to say no more. If we remove $q$ there is no longer a force.

In Chap. 1, Sect. 1.2 we discussed the ideas of Faraday, which were not legitimately based on any principle that could, at the time, be considered scientific.

To Faraday, and to Thomson and Maxwell, it seemed reasonable that space was affected by the presence of charge distributions. The separate charge $q$ only responded to that reality.

In effect, then, the charge $q$ is our measuring instrument. And we attempt to minimize the effect of our measuring instrument on the object of the measurement. The orbit of Mars, for example, was a good measuring instrument for the gravitational force of the sun because the mass of Mars is so small compared to that of the sun.

We then choose to measure the force on a test charge $q_{\text{test}}$, which we introduce at the point $\mathbf{r}$, for numerous decreasing values of the magnitude of $q_{\text{test}}$. The force will decrease as the magnitude of $q_{\text{test}}$ decreases, so we divide the result of each measurement by the magnitude of the test charge. If we obtain a non-zero result in the limit $q_{\text{test}} \to 0$, we shall then have a measurement of the force per unit charge resulting from a distribution of charges alone.

We shall call this force per unit charge, defined in the limit

$$\boxed{\mathbf{E}(\mathbf{r}) = \lim_{q\to 0} (1/q)\, \mathbf{F}_q(\mathbf{r}),} \tag{3.12}$$

the *electric field* $\mathbf{E}(\mathbf{r})$ at the point $\mathbf{r}$ resulting from the sources at the points $\mathbf{r}'$. In the present situation this is the *electrostatic field*.

The actual measurement would be difficult in the extreme. But, fortunately, because of the form of the force equations (3.5), (3.8), and ((3.11) we have theoretical expressions for the electric field $\mathbf{E}(\mathbf{r})$ without needing to actually make the measurement. These are for

- **Discrete Charges:**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{N} \frac{q_j}{\left|\mathbf{r}-\mathbf{r}_j\right|^3} \left(\mathbf{r}-\mathbf{r}_j\right), \tag{3.13}$$

- **Continuous volume density:**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \left(\mathbf{r}-\mathbf{r}'\right) dV', \tag{3.14}$$

- **Continuous surface density:**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \left(\mathbf{r}-\mathbf{r}'\right) dS'. \tag{3.15}$$

We must, however, realize that in writing equations (3.13)-(3.15) we have stepped outside of the confines of Newtonian experimental and mathematical philosophy into a new area that we may correctly call *field theory*. We are claiming that the electrostatic field exists based on the way in which we have chosen to interpret Coulomb's experiment.

There is nothing wrong with our mathematics. The hard core Newtonian may still claim that all we are doing is using a formula for the action at a distance force that will be experienced by a charge if one is placed at the point $\mathbf{r}$ and that any claim regarding the reality of the field is purely a metaphysical choice we have made.

We will be able to eventually establish the reality of the fields. But we cannot do that yet. As scientists and engineers, we must, at this point. consider the field to be either an hypothesis or a convenient mathematical crutch.

Our (3.13)–(3.15) should still be considered empirical equations, with the additional interpretation of the force in terms of a field. Equations (3.13)–(3.15) are, however, not yet field equations. From Helmholtz' Theorem we realize that the field equations are equations for the divergence and curl of the field. We must now derive the field equations.

## 3.6 Divergence and Curl of E

We can obtain the field equations for **E** by directly taking the divergence and the curl of the empirical equations (3.13)–(3.15).

We note first that each of the equations (3.13)-(3.15) contains a form of the term $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3$, which we recall is

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\,\mathrm{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\} \tag{3.16}$$

(see exercises Chap. 2). And we know the form of the divergence and the curl of $\mathrm{grad}\{1/|\mathbf{r}-\mathbf{r}'|\}$.

The curl of $\mathrm{grad}\{1/|\mathbf{r}-\mathbf{r}'|\}$ vanishes because

$$\boxed{\mathrm{curl}\,\mathrm{grad} \equiv 0} \tag{3.17}$$

(see exercises Chap. 2). And in Sect. 2.6.3 we showed that

$$\mathrm{div}\,\mathrm{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\} = -4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right). \tag{3.18}$$

Using (3.17) the curl of the electrostatic field from a set of discrete charges (3.13) becomes

- **Discrete Charges:**

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\sum_{\mathrm{j}=1} q_{\mathrm{j}}\,\mathrm{curl}\,\mathrm{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_{\mathrm{j}}|}\right\} = \mathbf{0}. \tag{3.19}$$

In our treatment of the fields arising from continuous volume and surface densities in (3.14) and (3.15) we recognize that the curl operates only on the unprimed coordinates, while in each case the integration is over the primed coordinates and the domain of integration is fixed. So we may bring the curl operator inside the volume and the surface integrals. The results are

- **Continuous Charge Density:**

1. **Volume density:**

$$\mathrm{curl}\,\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\int_{\mathrm{V}}\rho\left(\mathbf{r}'\right)\mathrm{curl}\,\mathrm{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\}\mathrm{d}V' = \mathbf{0} \tag{3.20}$$

2. **Surface density:**

$$\operatorname{curl}\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\oint_S \sigma\left(\mathbf{r}'\right)\operatorname{curl}\operatorname{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\}dS' = \mathbf{0} \tag{3.21}$$

The curl of the electrostatic field is then zero in all situations.

The divergence of the electrostatic field is found in a similar fashion. Using (3.18) the divergence of the electrostatic field from a set of discrete charges (3.13) becomes

- **Discrete Charges:**

$$\begin{aligned}\operatorname{div}\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\sum_j q_j \operatorname{div}\operatorname{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}_j|}\right\}\\ &= \frac{1}{\varepsilon_0}\sum_j q_j\delta\left(\mathbf{r}-\mathbf{r}_j\right)\end{aligned} \tag{3.22}$$

- **Continuous Charges Density**

  1. **Volume density**:

Because the divergence operates on the unprimed coordinates, while the integration in (3.14) and (3.15) is over the primed coordinates and a fixed domain, we may bring the divergence inside the integrals in (3.14) and (3.15). Then, using (3.18) as in the derivation of 3.22, we obtain the divergence of the electrostatic field from a continuous distribution of charges (3.14) as

$$\begin{aligned}\operatorname{div}\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\int_{V'}\rho\left(\mathbf{r}'\right)\operatorname{div}\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}dV'\\ &= \frac{1}{\varepsilon_0}\int_{V'}\rho\left(\mathbf{r}'\right)\delta\left(\mathbf{r}-\mathbf{r}'\right)dV'\\ &= \frac{1}{\varepsilon_0}\rho(\mathbf{r})\,.\end{aligned} \tag{3.23}$$

2 **Surface density**

We cannot perform the same mathematical operations on (3.15) because the $\delta-$functions resulting from $\operatorname{div}\operatorname{grad}\left(1/|\mathbf{r}-\mathbf{r}'|\right)$ are infinite over the entire surface of integration. We then no longer have the integral condition on the $\delta-$function (2.98). The surface charge density $\sigma(\mathbf{r})$ in (3.15) is, however, nonzero only over a (two dimensional) surface. We may then replace the surface integral in (3.15) by a volume integral over the volume formed by adding thin sheets of thickness $\varepsilon$ on both sides of the surface on which $\sigma(\mathbf{r}) \neq 0$. Because the charge density $\sigma(\mathbf{r}) \equiv 0$ in the interior of each of these sheets,

$$\operatorname{div} \oint_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,\mathrm{d}S' = \operatorname{div} \int_{\mathrm{V(sheet)}} \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V'$$
$$= -\int_{\mathrm{V(sheet)}} \sigma(\mathbf{r}')\operatorname{div}\operatorname{grad}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\}\mathrm{d}V'$$
$$= 4\pi \int_{\mathrm{V(sheet)}} \sigma(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')\,\mathrm{d}V'. \tag{3.24}$$

We can now perform the integral in (3.24) over the volume V(sheet) using the integral condition for the $\delta-$ function (2.98). The resulting field equation is

$$\boxed{\operatorname{div} \mathbf{E}(\mathbf{r}) = (1/\varepsilon_0)\,\rho_{\mathrm{surface}}(\mathbf{r}).} \tag{3.25}$$

In (3.25) we have written the charge density as $\rho_{\mathrm{surface}}(\mathbf{r})$ rather than $\sigma(\mathbf{r})$ to emphasize that the mathematical form is identical to (3.23), with $\rho(\mathbf{r}) \neq 0$ only on the surface of the volume.

The right hand side of (3.22) is a rather strange mathematical function made up of $\delta-$ functions. However, if we integrate this function over the entire volume shown in Fig. 3.2 we have

$$\int_V \sum_j q_j \delta(\mathbf{r}-\mathbf{r}_j)\,\mathrm{d}V' = \sum_j q_j,$$

which is the total charge in the volume $V$.

Therefore we may consider all of the divergence equations (3.22)–(3.25) to be of the same form, i.e.

$$\boxed{\operatorname{div}\ \mathbf{E}(\mathbf{r}) = 1/\varepsilon_0\,\rho(\mathbf{r})\ .} \tag{3.26}$$

Equation (3.26) is known as *Gauss' Law*.

The two field equations for the electrostatic field are then (3.26) and

$$\boxed{\operatorname{curl} \mathbf{E}(\mathbf{r}) = \mathbf{0}.} \tag{3.27}$$

The source of the electrostatic field is a density of charges $\rho(\mathbf{r})$, which may consist of discrete charges, or a continuous charge density and filling a volume or existing only on a surface.

## 3.7 Integral Electrostatic Field Equations

From (3.26) and (3.27) we can obtain integral equations using Gauss' and Stokes' Theorems (see Sect. 2.5.3.1 (2.59) and Sect. 2.5.4.2 (2.78)).

### 3.7.1 Gauss' Theorem

If we integrate (3.26) over any arbitrary volume $V$, which contains the charges, and apply Gauss' Theorem we have

$$\int_{\mathrm{V}} \mathrm{d}V \ \operatorname{div} \mathbf{E} = \oint_{\mathrm{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\mathrm{V}} \rho\left(\mathbf{r}\right) \mathrm{d}V,$$

or

$$\boxed{\oint_{\mathrm{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \left(1/\varepsilon_0\right) \int_{\mathrm{V}} \rho\left(\mathbf{r}\right) \mathrm{dV},} \tag{3.28}$$

where the surface $S$ contains the volume $V$. Equation (3.28) is *Gauss' Law in integral form*, which is the form of Gauss' Law found in most beginning texts on physics.

We can apply (3.28) for the calculation of the electric (here electrostatic) field if we can construct some imaginary surface $S$ in the region of interest, which is everywhere perpendicular to $\mathbf{E}$ and on which $\mathbf{E}$ is constant in magnitude. Then $\mathbf{E} \cdot \mathrm{d}\mathbf{S} = E \mathrm{d}S$ with $E =$constant and

$$\oint_{\mathrm{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E\left(S\right) \oint_{\mathrm{S}} \mathrm{d}S = E\left(S\right) A_{\mathrm{S}},$$

where $E\left(S\right)$ is the constant value of $E$ on the surface $S$ and $A_{\mathrm{S}}$ is the area of the surface $S$. These situations of high geometrical symmetry result when $\rho\left(\mathbf{r}\right)$ also has this geometrical symmetry.

*Example 3.1.* **Spherical Charge Distribution.** If $\rho\left(\mathbf{r}\right)$ is spherically symmetric around the origin of coordinates the electrostatic field will be directed radially outward from the origin and will be a function of radial distance alone. The electrostatic field is then constant in magnitude and perpendicular to any spherical surface we may construct that is centered on the origin. For a spherical charge distribution with radius $R$ we then have for $r > R$

$$\oint_{\mathrm{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi r^2 E\left(r\right),$$

and

$$\frac{1}{\varepsilon_0} \int_{\mathrm{V}} \rho\left(r\right) \mathrm{d}V = \frac{q}{\varepsilon_0}.$$

The electrostatic field is then

$$\mathbf{E}\left(r\right) = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{e}_{\mathrm{r}}. \tag{3.29}$$

Cylindrically symmetric and planar charge densities produce electrostatic fields perpendicular to, and constant on cylindrical and planar surfaces. We can then apply

Gauss' Law in integral form to these charge densities as well. With these three examples we have, however, exhausted the possible applications of Gauss' Law in integral form.

We will not hesitate to use Gauss' Law in integral form when symmetry permits. For example a geometrical surface appears planar at points an infinitesimal distance from the surface. We may then apply Gauss' Law in integral form to find the electrostatic field infinitesimal distances from a surface of any geometrical form by constructing an infinitesimal cylindrical volume with axis perpendicular to the surface. These infinitesimal surfaces are often called Gaussian pillboxes (see exercises).

### *3.7.2 Stokes' Theorem*

If we integrate (3.27) over any arbitrary area $A$ and apply Stokes' Theorem we have

$$\int_{\mathrm{A}} \mathrm{curl}\, \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \oint_{\mathrm{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0,$$

or

$$\boxed{\oint_{\mathrm{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0,} \tag{3.30}$$

where $C$ is the contour that defines $a$ in the sense that $a$ is any area whose boundary is $C$ and d**a** is defined by the right hand rule from $C$ (see Sect. 2.5.4).

In summary we have, then the differential field (3.26) and (3.27) and the equivalent integral field (3.28) and (3.30) for the electrostatic field.

## 3.8 Summary

This chapter dealt almost exclusively with the results of Coulomb's experiment. We introduced a superposition Ansatz, which allowed us to obtain more general empirical equations for forces from charge distributions. And we introduced the field concept in mathematical form, which permitted us to obtain empirical equations for the fields. Because of the mathematical form of the empirical equations for the electrostatic field we could rather easily find equations for the divergence and the curl of the electrostatic field. These are the electrostatic field equations.

This is generally the approach we will use with the introduction of each field. The mathematical form of the empirical results from each experiment will determine the exact steps we will follow. In most cases the task will become simpler as we gain understanding of the structure of the fields. The goal will always be, however, the field equations in the context of Helmholtz' Theorem.

At the beginning of the 21st century the reality of the field is not speculation. We also know that our superposition Ansatz is valid.

## Exercises

**3.1.** You have placed an arbitrarily shaped conductor. in a region in which there is an electrostatic field. The electrostatic field will induce a surface charge density $\sigma(\mathbf{r})$ on the surface of the conductor sufficient to result in an electrostatic field that is everywhere perpendicular to the surface of the conductor and vanishes within the conductor. In Fig. 3.6 we have drawn an infinitesimal Gaussian pillbox extending inside and outside of a conductor.

Use this infinitesimal Gaussian pillbox encompassing an area $\mathrm{d}S_{\mathrm{i}}$ of the conductor surface located at point $\mathbf{r}_{\mathrm{i}}$ on the surface of the conductor to find the relationship between the electrostatic field just outside the conductor surface and the surface charge density $\sigma_{\mathrm{i}}$ at $\mathbf{r}_{\mathrm{i}}$.

[answer: $E(\mathbf{r}_{\mathrm{i}}) = \sigma_{\mathrm{i}}/\varepsilon_0$]

**3.2.** There is a region in space in which the electric field is in one direction. Choose one of the Cartesian axes, say $x$ to be the axis along which the field is oriented.

(a) Show that the electric field cannot depend on the other two coordinates $(y, z)$ in this region.
(b) If the electrostatic field in this region is also constant what is the charge density in the region?
(c) How would it be possible to generate such a field?

*Answer:*

(a) The curl of the electrostatic field vanishes. Then

$$\begin{aligned}\operatorname{curl}\mathbf{E} &= \mathbf{0}\\ &= \hat{e}_{\mathrm{x}}\left(\frac{\partial E_{\mathrm{z}}}{\partial y} - \frac{\partial E_{\mathrm{y}}}{\partial z}\right) + \hat{e}_{\mathrm{y}}\left(\frac{\partial E_{\mathrm{x}}}{\partial z} - \frac{\partial E_{\mathrm{z}}}{\partial x}\right)\\ &\quad + \hat{e}_{\mathrm{z}}\left(\frac{\partial E_{\mathrm{y}}}{\partial x} - \frac{\partial E_{\mathrm{x}}}{\partial y}\right).\end{aligned}$$

If $E_{\mathrm{y}} = E_{\mathrm{z}} = 0$ then

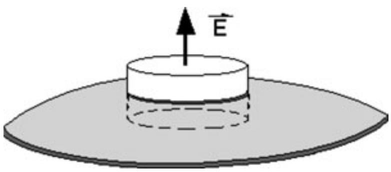



**Fig. 3.6** Infinitesimal Gaussian pillbox with area $\mathrm{d}S_{\mathrm{i}}$ extending inside and outside of the surface of a conductor

$$\mathbf{0} = \hat{e}_{\mathrm{y}} \left( \frac{\partial E_{\mathrm{x}}}{\partial z} \right) - \hat{e}_{\mathrm{z}} \left( \frac{\partial E_{\mathrm{x}}}{\partial y} \right).$$

Both vector components must independently vanish. Therefore

$$\left( \frac{\partial E_{\mathrm{x}}}{\partial z} \right) = \left( \frac{\partial E_{\mathrm{x}}}{\partial y} \right).$$

(b) If $E_{\mathrm{x}} =$ constant, then Gauss' Law is

$$\operatorname{div} \mathbf{E} = \frac{\partial}{\partial x} E_{\mathrm{x}} = 0 = \frac{1}{\varepsilon_0} \rho.$$

Therefore $\rho = 0$.

(c) This electrostatic field can only be generated by charges outside of the region. A charged, flat conducting plate produces an electrostatic field that is constant in the region above the plate. So two charged, flat conducting plates arranged parallel to one another, with a positive charge on one and a negative charge on the other will produce this electrostatic field.

**3.3.** In problems of spherical and cylindrical symmetry we will find it useful to define charge densities for distributions that can be written as $\delta-$functions. The *functional form* of the charge distribution must be such that

$$\int_{\text{all space}} \rho\,(r)\,\mathrm{d}V = Q,$$

where $Q$ is the total charge of the density. It is probably not a simple $\delta-$function.

For a single charge located at the origin we are tempted to write

$$\rho\,(r) = a_{\text{spherical}} \delta\,(r)\,,$$

where we must determine the function $\alpha\,(r)$ such that the total charge is $Q$. But the integral

$$\int_{\mathrm{V}} a_{\text{spherical}} \delta\,(r)\,\mathrm{d}V = \int_0^\infty a_{\text{spherical}} \delta\,(r)\,4\pi r^2 \mathrm{d}r$$

vanishes.

We can find the correct representation of a point charge at the origin by considering a tiny spherical shell of charge

$$\rho\,(r) = a_{\text{spherical}} \delta\,(r - \varepsilon)$$

and take the limit as $\varepsilon \to 0$. Carry this out to show that

$$a_{\text{spherical}}(r) = \frac{Q}{4\pi r^2}$$

**3.4.** In the preceding exercise you found the representation of a point charge at the origin. The problem is similar if we seek to represent a line charge along the $z-$axis using a $\delta-$function. Consider now a cylindrical shell of charge with radius $\varepsilon$. Using the reasoning as in the preceding exercise show that the representation of a charge density along the $z-$axis is

$$\rho(r) = \frac{\lambda}{2\pi r}\delta(r-\varepsilon),$$

where $\lambda$ is the charge per unit length.

**3.5.** For the two concentric spherical shells in the example of the spherical capacitor find

(a) The electrostatic field inside the inner shell.
(b) The charge density on each shell and the *location* of the charge density.
(c) The electrostatic field for the region $r > b$.

**3.6.** Suppose the exponent in the Coulomb field was not exactly 2, but $2-\delta$, where $\delta \ll 1$. The Coulomb field would then be

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^{2-\delta}}$$

Calculate

$$\int_V \text{div}\,\mathbf{E}\mathrm{d}V$$

over a spherical volume of radius $R$ centered on the charge $q$ for this field. Is Gauss' Law still valid for this field?

**3.7.** A very long, nonconducting circular rod of length $L$ and radius $R$ contains a charge density

$$\rho = \left(\frac{\rho_0}{R^2}\right)r^2.$$

Find the electrostatic field near the center of the rod for $r > R$ and $R < r$. Limit your calculation outside the rod to values of $r \ll L$ so that the rod length appears infinite.

**3.8.** A conducting object has an arbitrarily shaped hollow cavity in its interior. If a point charge $q$ is introduced into the cavity, prove that the charge $-q$ is induced on the surface of the cavity.

**3.9.** The electric field at the earth's surface and in the atmosphere can be measured. It is approximately 150 V/ m and pointed downward toward the ground at the earth's surface. At an elevation of 30 km this field has dropped to about 1 V/ m. We may expect that the electric field varies as $1/r^2$ above the earth's surface. For very small vertical distances $y$ above the earth's surface this implies a linear variation of the electric field with $y$.

(a) What is the approximate charge density in the earth's atmosphere between the earth's surface and 30 km?
(b) What is the charge density on the earth?
(c) What is the total charge on the earth (radius = 6, 371 km)?
(d) From where do you suppose the charge came?
(e) If you suppose electrical storms you are correct. What then must be the charge of the lower part of the clouds in an electrical storm?

**3.10.** You have a nonconducting sphere of radius $a$ in which there is a charge density, which is a function only of the radial distance from the center of the sphere. Around this nonconducting sphere you have a spherical conducting shell of inner radius $a$ and outer radius $b$. If

$$\rho(r) = \frac{\rho_0}{a} r^2$$

for $r > 0$,

(a) Plot the electrostatic field as a function of $r$ from $r = 0$ to a value $> b$.
(b) What is the (surface) charge density on the inner and outer surfaces of the conducting shell?

**3.11.** In a certain region of space you have found that there is an electrostatic field only along the $x-$axis. Prove that there can be no dependence of this electrostatic field on either the $y-$ and $z-$coordinates in this region. If there is no charge in this region, prove that the field is also independent of $x$.

**3.12.** In a certain region of pace you have measured a spherically symmetric electrostatic field. The field depends on the radial coordinate as

$$E_\mathrm{r}(r) = E_0 r \exp(-\alpha r)$$

Where $E_0$ and $\alpha$ are positive constant. What is the charge density that produced this electrostatic field?

# Chapter 4
# The Scalar Potential

*He who seeks for methods without having a definite problem in mind seeks for the most part in vain.*

*David Hilbert*

## 4.1 Introduction

In this chapter we introduce and develop the properties of the electrostatic scalar potential. This is the first of two potentials in classical field theory both of which appeared in the original work by Maxwell. We will find that these potentials are central to the theory replacing the fields in advanced topics. The fact that a scalar potential exists follows immediately from the second electrostatic field equation, that for the curl of **E**. The electrostatic force is conservative and is, therefore, obtainable from a potential energy.

From Gauss' Law, the first of the field equations, we will find that the electrostatic potential satisfies Poisson's Equation, for which we know the solution. With the electrostatic potential we then have the basis for calculating the electrostatic field in realistic situations.

Our final step will be a formulation of the energy in space resulting from charge densities.

## 4.2 Potential Energy

Because $\text{curl}\,\text{grad}\, f(\mathbf{r}) = 0$ for any scalar function $f(\mathbf{r})$ (see exercises Chap. 2), the second electrostatic field (3.27), which is

$$\text{curl}\,\mathbf{E} = \mathbf{0}, \tag{4.1}$$

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,

DOI 10.1007/978-3-642-23205-3_4, 

results in the fact that we can write the electrostatic field as

$$\mathbf{E} = -\operatorname{grad}\varphi(\mathbf{r}). \tag{4.2}$$

The function $\varphi(\mathbf{r})$ is the *electrostatic (scalar) potential*. The negative sign is convention resulting from our physical understanding of $\varphi(\mathbf{r})$.

In Example 2.3 we showed that the electrostatic field is conservative because it satisfies (4.1). The work done by the electrostatic field on a charge in a differential distance $\mathrm{d}\boldsymbol{\ell} = \hat{e}_\mu \mathrm{d}x_\mu$ is then equal to a decrease in potential energy of the charge. The work is done at the expense of the potential energy. That is

$$\begin{aligned} q\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} &= -q\operatorname{grad}\varphi(\mathbf{r})\cdot\hat{e}_\mu\mathrm{d}x_\mu \\ &= -q\delta_{\nu\mu}\frac{\partial\varphi}{\partial x_\nu}\mathrm{d}x_\mu \\ &= -q\mathrm{d}\varphi, \end{aligned} \tag{4.3}$$

since $\mathrm{d}\varphi = \partial\varphi/\partial x_\mu \mathrm{d}x_\mu$ is the Pfaffian for $\varphi$. The electrostatic potential is then the potential energy in the electrostatic field per unit charge and has the units of *volt*[1] (V).

From Newton's Second Law for a charged particle of charge $q$ and mass $m$ we have

$$\begin{aligned} q\mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} &= m\mathrm{d}v_\nu\frac{\mathrm{d}\ell_\nu}{\mathrm{d}t} = m\mathrm{d}\left(\frac{\mathrm{d}\ell_\nu}{\mathrm{d}t}\right)\frac{\mathrm{d}\ell_\nu}{\mathrm{d}t} \\ &= \mathrm{d}\left(\frac{1}{2}mv^2\right), \end{aligned} \tag{4.4}$$

since the velocity in the direction $\mathrm{d}\boldsymbol{\ell}$ is $\mathbf{v} = \mathrm{d}\boldsymbol{\ell}/\mathrm{d}t$. Equating (4.3) and (4.4) we have

$$\mathrm{d}\left(\frac{1}{2}mv^2 + q\varphi\right) = 0. \tag{4.5}$$

That is the quantity

$$\mathcal{H} = \frac{1}{2}mv^2 + q\varphi$$

is a constant for the motion of a charge $q$ of mass $m$ in an electrostatic field. $\mathcal{H}$ is the total energy of the charge[2] in the electrostatic field.

[1]The unit of the volt honors Count Alessandro Volta who developed the first electrochemical cell (see Sect. 1.8).

[2]We have designated the total mechanical energy of the charge as $\mathcal{H}$ rather than $E$ to avoid confusion with the designation for the field, and because $\mathcal{H}$ is the standard designation of the Hamiltonian, which, for conservative systems, is the total energy.

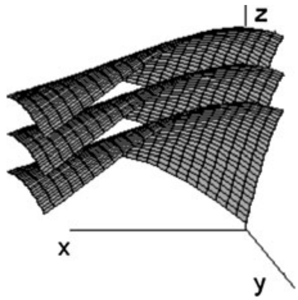



**Fig. 4.1** Three of a family of equipotential surfaces in space. Each surface is defined by $\varphi(x, y, z) = \text{constant}$. The electrostatic field is everywhere perpendicular to this family of surfaces

## 4.3 Potential Surfaces

The scalar potential function $\varphi(x, y, z)$ is a family of surfaces in $(x, y, z) -$ space, which, because of (4.2), are perpendicular to the electric field vector at each point. In Fig. 4.1 we have drawn a set of general surfaces in $(x, y, z) -$ space to illustrate the situation. The surfaces in Fig. 4.1 are called equipotential surfaces because on each of them $\varphi(x, y, z) = \text{constant}$. Work is done on or by an electric charge in moving from one equipotential surface to the other. But no work is done on a charge moving on an equipotential surface.

## 4.4 Poisson's Equation

By inserting (4.2) into the first electrostatic field (3.26), which is

$$\operatorname{div} \mathbf{E} = \frac{1}{\varepsilon_0}\rho, \tag{4.6}$$

we obtain a differential equation for the scalar potential

$$\nabla^2\varphi = -\frac{1}{\varepsilon_0}\rho, \tag{4.7}$$

Equation (4.7) is *Poisson's Equation*. We first encountered Poisson's Equation in Sect. 2.6. From (2.86) (see also Appendix F, Theorem F.3) the particular solution to (4.7) is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\mathrm{d}V'. \tag{4.8}$$

To illustrate the use of (4.8) we present two examples. The first of these is the potential from a point charge, for which we already know the answer.

*Example 4.1.* **The point charge.** We consider a point charge of magnitude $q$ located at the point $x = a$, $y = b$, $z = c$. The charge density is

$$\rho\left(x', y', z'\right) = q\delta\left(x' - a\right)\delta\left(y' - b\right)\delta\left(z' - c\right),$$

which is a function of the source coordinates $(x', y', z')$. We note that

$$\begin{aligned}\int_{\mathrm{V}} \rho\left(\mathbf{r}'\right) \mathrm{d}V' &= q \int_{\mathrm{V}'} \delta\left(x' - a\right)\delta\left(y' - b\right)\delta\left(z' - c\right) \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\ &= q.\end{aligned}$$

We must also have a formulation of the distance $|\mathbf{r} - \mathbf{r}\,'|$, which is

$$\left|\mathbf{r} - \mathbf{r}\,'\right| = \left\{\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right\}^{\frac{1}{2}}.$$

Then (4.8) becomes

$$\varphi\left(\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0} \int_{\mathrm{V}} \frac{\delta\left(x' - a\right)\delta\left(y' - b\right)\delta\left(z' - c\right)}{\left\{(x - x')^2 + (y - y')^2 + (z - z')^2\right\}^{\frac{1}{2}}} \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'.$$

The integration over the $\delta-$ functions simply replaces the variables $x'$, $y'$, and $z'$ with $a$, $b$, and $c$. That is

$$\varphi\left(\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left\{(x - a)^2 + (y - b)^2 + (z - c)^2\right\}^{\frac{1}{2}}} \tag{4.9}$$

for a single point charge located at $(a, b, c)$.

From the potential for a single charge and our superposition Ansatz we can obtain the potential for an electric dipole. The electric dipole is an arrangement of two opposite charges separated by a distance $\ell$. The electric dipole moment is $p_{\mathrm{d}} = q\ell$. We show this in the drawing in Fig. 4.2. In Fig. 4.2 we have drawn the two charges as finite for illustrative purposes. We located the center of the dipole at the point $(a, b, c)$. The two charges are then located at $(a, b, c + \ell/2)$ and $(a, b, c - \ell/2)$. Using (4.9) for each charge the potential for the dipole is

$$\begin{aligned}\varphi\left(\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0 R} \Bigg[ &\left\{1 - \frac{\ell\left(z - c\right)}{R^2} + \frac{\ell^2}{4R^2}\right\}^{-1/2} \\ &- \left\{1 + \frac{\ell\left(z - c\right)}{R^2} + \frac{\ell^2}{4R^2}\right\}^{-1/2}\Bigg],\end{aligned} \tag{4.10}$$

where $R = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$. Expanding (4.10) in terms of $\ell/R$, i.e. for small dipole spacing compared to observation distance,

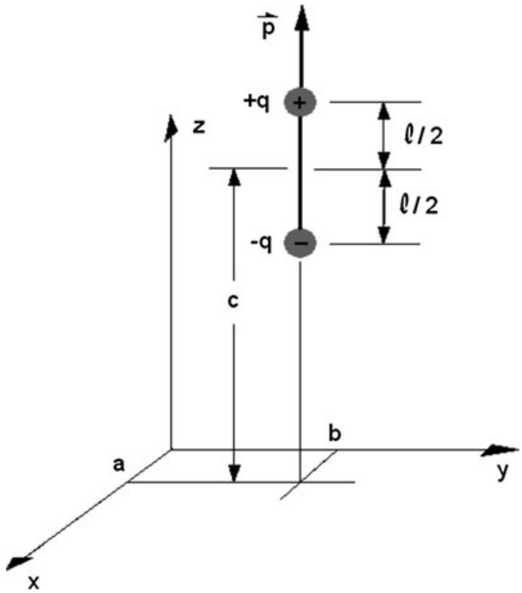



**Fig. 4.2** Electric dipole $\mathbf{p}_\mathrm{d}$

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 R}\left[\left(\frac{\ell}{R}\right)\frac{(z-c)}{R}\right] + O\left[\left(\frac{\ell}{R}\right)^3\right]$$
$$= \frac{p_\mathrm{d}}{4\pi\varepsilon_0 R^3}(z-c) + O\left[\left(\frac{\ell}{R}\right)^3\right]. \tag{4.11}$$

If we place the center of the dipole at the origin and introduce spherical coordinates with $z = r\cos\phi$, where $r$ replaces $R$ as the distance to the field or observation point, then (4.11) becomes

$$\varphi(\mathbf{r}) = \frac{p_\mathrm{d}}{4\pi\varepsilon_0}\frac{\cos\phi}{r^2}, \tag{4.12}$$

The electrostatic field is

$$\mathbf{E} = -\,\mathrm{grad}\,\frac{p_\mathrm{d}}{4\pi\varepsilon_0}\frac{\cos\phi}{r^2}, \tag{4.13}$$

which, using (A.9), is

$$\mathbf{E} = \frac{p_\mathrm{d}}{4\pi\varepsilon_0 r^3}\left[\hat{e}_\mathrm{r} 2\cos\phi + \hat{e}_\phi \sin\phi\right]. \tag{4.14}$$

As our second example we consider a ring of charge. We ask first only for the electrostatic potential along the $z$–axis. There will then be no difficulty in

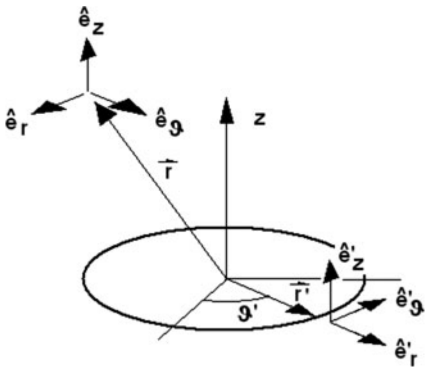



**Fig. 4.3** Charged ring with total charge $Q_{\text{ring}}$. We have included the unit vector triads $(\hat{e}_{\text{r}}, \hat{e}_{\vartheta}, \hat{e}_{\text{z}})$ and $\left(\hat{e}'_{\text{r}}, \hat{e}'_{\vartheta}, \hat{e}'_{\text{z}}\right)$ at the tips of the field and source vectors

the integration. We must, however, be careful about the formulation of the charge density for the ring.

*Example 4.2.* **The ring of charge.** In Fig. 4.3 we have drawn a ring of charge of radius $a$. The total charge on the ring is $Q_{\text{ring}}$ C. Because the charge density is located only on the ring $\rho(\mathbf{r}')$ is proportional to $\delta(r'-a)\,\delta(z')$. But $\rho(\mathbf{r}')$ might depend also on some function of the magnitude of $r'$. So we write

$$\rho\left(\mathbf{r}'\right) = g\left(r'\right)\delta\left(r'-a\right)\delta\left(z'\right).$$

We find $g(r')$ from the requirement that $\int \rho(\mathbf{r}')\mathrm{d}V' = Q_{\text{ring}}$. In cylindrical coordinates $\mathrm{d}V' = r'\mathrm{d}r'\mathrm{d}\vartheta'\mathrm{d}z'$. Then

$$\begin{aligned} Q_{\text{ring}} &= \int_{\vartheta=0}^{2\pi} \mathrm{d}\vartheta' \int_{\text{z}=-\infty}^{+\infty} \mathrm{d}z' \int_{\text{r}=0}^{+\infty} r'\mathrm{d}r' g\left(r'\right)\delta\left(r'-a\right)\delta\left(z'\right) \\ &= 2\pi g(a)\,a. \end{aligned}$$

That is

$$g\left(r'\right) = \frac{Q_{\text{ring}}}{2\pi r'}$$

and

$$\rho\left(\mathbf{r}'\right) = \frac{Q_{\text{ring}}}{2\pi r'}\delta\left(r'-a\right)\delta\left(z'\right).$$

The source vector is

$$\mathbf{r}' = a\hat{e}'_{\text{r}},$$

where $\hat{e}'_{\text{r}}$ is the unit vector which locates the source point. If we only seek the electrostatic potential on the $z-$axis the field point is

$$\mathbf{r} = z\hat{e}_{\text{z}}.$$

The magnitude of the vector difference

$$\mathbf{r} - \mathbf{r}' = z\hat{e}_{\mathrm{z}} - a\hat{e}'_{\mathrm{r}}$$

is

$$\left|z\hat{e}_{\mathrm{z}} - a\hat{e}'_{\mathrm{r}}\right| = \sqrt{z^2 + a^2}.$$

The scalar potential (4.8) is then

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_{\vartheta'=0}^{2\pi}\mathrm{d}\vartheta'\int_{z'=-\infty}^{+\infty}\mathrm{d}z'\int_{r'=0}^{+\infty}r'\mathrm{d}r'\,\frac{Q_{\mathrm{ring}}\delta(r'-a)\,\delta(z')}{2\pi r'\sqrt{z^2+a^2}}\\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q_{\mathrm{ring}}}{\sqrt{z^2+a^2}}.\end{aligned}$$

In the preceding example we see that the integrations over $r'$ and $z'$ leave us with an integral over $\vartheta'$ of a charge $\mathrm{d}q' = \lambda a \mathrm{d}\vartheta'$ on an infinitesimal length of the ring where $\lambda = Q_{\mathrm{ring}}/2\pi a$ is the linear charge density in the ring in units of $\mathrm{C\ m^{-1}}$.

With what is becoming common access to sophisticated mathematical software packages, we can extend our study of the potential in the space around a ring of charge. In doing so we will gain some insight into the electrostatic potential. Or we may simply have our insight corroborated.

If we want to know the general form of the potential in the space surrounding the charged ring we must use a general form for the field point vector in Fig. 4.3

$$\mathbf{r} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

Because of symmetry around the $z-$ axis the electrostatic potential will only depend on $(r, z)$ and will be independent of $\vartheta$. We may then choose the orientation of the unit field vector $\hat{e}_{\mathrm{r}}$ to be fixed. It is convenient to choose the angle orienting $\hat{e}_{\mathrm{r}}$ to be $\vartheta = 0$.

The source point vector is still $\mathbf{r}' = a\hat{e}'_{\mathrm{r}}$, where now $\hat{e}'_{\mathrm{r}} = \hat{e}_{\mathrm{r}}\cos\vartheta' + \hat{e}_{\vartheta}\sin\vartheta'$, as we can see from Fig. 4.4 where the unit vectors have been placed in a unit circle.

The vector difference $\mathbf{r} - \mathbf{r}\,'$ is then

$$\mathbf{r} - \mathbf{r}\,' = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}} - a\left(\hat{e}_{\mathrm{r}}\cos\vartheta' + \hat{e}_{\vartheta}\sin\vartheta'\right).$$

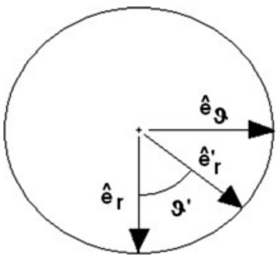



**Fig. 4.4** Unit circle with unit vectors

Then the magnitude of the difference between the field and source vectors is

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{r^2+a^2+z^2-2ar\cos\vartheta'}.$$

And the scalar potential (8.11) is

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_{\vartheta'=0}^{2\pi} d\vartheta' \int_{z'=-\infty}^{+\infty} dz' \int_{r'=0}^{+\infty} r'dr' \frac{Q_{\text{ring}}\delta(r'-a)\,\delta(z')}{2\pi r'\sqrt{r^2+a^2+z^2-2ar\cos\vartheta'}} \\ &= \frac{\lambda a}{4\pi\varepsilon_0}\int_{\vartheta'=0}^{2\pi} \frac{\mathrm{d}\vartheta'}{\sqrt{r^2+a^2+z^2-2ar\cos\vartheta'}} \end{aligned} \tag{4.15}$$

where $\lambda = Q_{\text{ring}}/2\pi a$ is the linear charge density on the ring.

The integral in (4.15) is known in terms of tabulated functions. The result for the scalar potential is

$$\varphi(\mathbf{r}) = \frac{\lambda a}{\pi\varepsilon_0}\sqrt{\frac{1}{a^2+r^2+z^2-2ar}}K\left(2\sqrt{-\frac{ar}{a^2+r^2+z^2-2ar}}\right), \tag{4.16}$$

where $K(k)$ is the complete elliptic integral of the first kind defined as

$$K(k) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}, \tag{4.17}$$

with no restrictions on $k$. In Fig. 4.5 we have plotted the potential (4.16) for radial positions $r$ with the vertical distance above the plane of the ring $z$ as a parameter.

Lengths in these graphs are in units of the ring radius $a$ and potentials are in units of $\lambda/4\pi\varepsilon_0$. All graphs of the potential have 100 data points. The values of $z$ for the potentials shown in Fig. 4.5 are 0.1 – 0.5, 1.0, and 2.0. The values of $z$ are indicated on the plot.

The peak at the radial distance of 1.0 is from the ring. This is more evident as $z$ decreases.

In Fig. 4.6 we have plotted the single scalar potential for $z = 0.1$. On the plot we have indicated the values of $(\partial\varphi/\partial r)_z$ ($z$ held constant) and included arrows to indicate the direction of the radial field.

The radial field vanishes on the $z-$ axis (radial distance $= 0$). The radial field decreases as the radial distance becomes larger, which we have indicated by decreasing the size of the arrow.

In Fig. 4.7 we have plotted both the scalar potential and the radial component of the electrostatic field in units of $\lambda/4\pi a\varepsilon_0$.

Inside the ring the radial field is negative (points toward the axis). Outside it is positive (points away from the axis).

We note that the radial field outside of the ring decreases slowly. The electrostatic field is long range. This makes it necessary to use a shielded potential in kinetic theoretical studies of charged particles (see exercises).

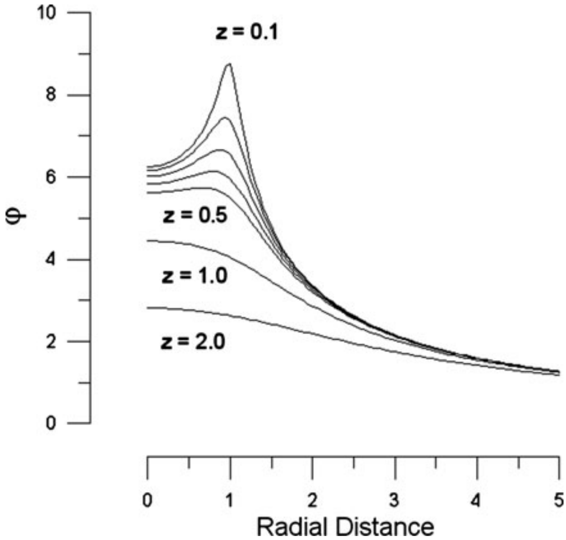



**Fig. 4.5** Scalar potential $\varphi$ from a charged ring. Radial dependence of potential is plotted with vertical distance above the plane of the ring as a parameter. Each graph has 100 data points

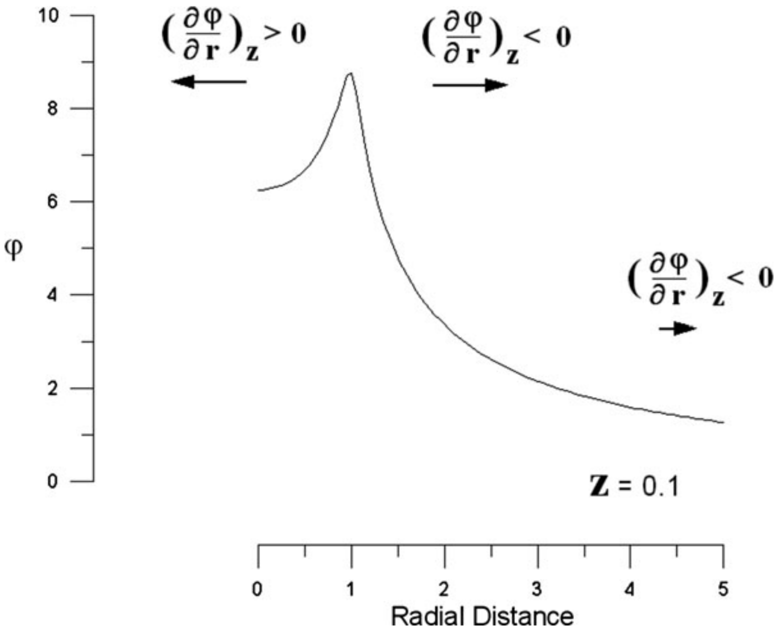



**Fig. 4.6** Scalar potential $\varphi$ from a charged ring for a vertcal height above the plane of the ring $z = 0.1$. Values of $(\partial\varphi/\partial r)_z$ and direction of electric field are indicated

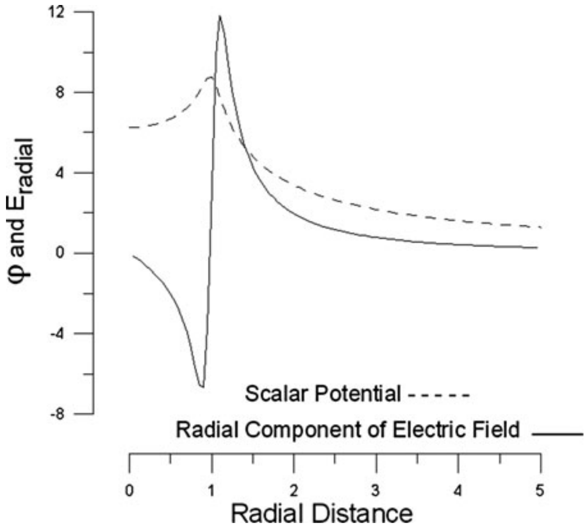



**Fig. 4.7** Scalar potential $\varphi$ and radial component of electrostatic field $E_{\text{radial}}$ from a charged ring for a vertical distance $z = 0.1$. The electric field was obtained numerically from the potential by three point differentiation

Our more complete study of the potential from a ring of charge has then provided graphical results that we can understand intuitively.

We have elected not to complete this study with parametric graphs for $\varphi(z, r = \text{constant})$.

## 4.5 Multipole Expansion

In our study of the potential from the charged ring we chose to evaluate the integral (4.15) numerically rather than expanding the integrand, as we did for the dipole, to obtain an approximate analytical result. Moderate effort using a commercially available software package (Maple) gained us graphical insight into the form of the electrostatic potential near the ring.

But this was a textbook example. And the charge distribution was simple and symmetric. In applications we will normally not know the charge distribution $\rho(\mathbf{r}')$ and may have only measurements made at relatively large distances from the distribution. Most applications are to charge densities of molecular or smaller dimensions. Our objective then will not be to find the electrostatic potential, but to infer the form of the charge distribution from measurements of the electrostatic field made at great distances from the distribution.

So we seek the general form of the electrostatic potential at distances from a charge distribution which are large compared to the dimensions of the distribution. Particularly we seek characteristics of the charge density that can be identified in the form of the electrostatic field at large distances from the density.

We begin with the solution to Poisson's Equation (4.8) and place the charge distribution at the origin. The magnitude $r'$ of the vector to the source point $\mathbf{r}'$ is then very small compared to the magnitude $r$ of the vector to the field point $\mathbf{r}$. And we may expand $1/|\mathbf{r}-\mathbf{r}'|$ in terms of $r'/r$. The result is

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r}+\frac{\mathbf{r}}{r^2}\cdot\frac{\mathbf{r}'}{r}\\ &\quad+\frac{1}{2}\frac{1}{r^5}\left[\left(\left(3x^2-r^2\right)x'x'+3xyx'y'+3xzx'z'\right)\right.\\ &\quad+\left(\left(3y^2-r^2\right)y'y'+3xyx'y'+3yzy'z'\right)\\ &\quad\left.+\left(\left(3z^2-r^2\right)z'z'+3xzx'z'+3yzy'z'\right)\right]\\ &=\frac{1}{r}+\frac{\mathbf{r}}{r^2}\cdot\frac{\mathbf{r}'}{r}+\frac{1}{2}\frac{1}{r^5}x_\mu x_\nu\left\{3x'_\mu x'_\nu-\left(r'\right)^2\delta_{\mu\nu}\right\},\end{aligned} \tag{4.18}$$

using subscript notation and the Einstein sum convention. The next term in the expansion is $O\left[\left(r'/r\right)^3\right]$, which we drop.

With (4.18) the solution to Poisson's Equation is

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0 r}\int_V \rho\left(\mathbf{r}'\right)\mathrm{d}V'+\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\cdot\int_V \mathbf{r}'\rho\left(\mathbf{r}'\right)\mathrm{d}V'\\ &\quad+\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{1}{r^5}x_\mu x_\nu\int_V\left\{3x'_\mu x'_\nu-\left(r'\right)^2\delta_{\mu\nu}\right\}\rho\left(\mathbf{r}'\right)\mathrm{d}V'.\end{aligned} \tag{4.19}$$

In (4.18) we identify the terms characterizing the charge distribution as the *total charge*

$$Q=\int_V \rho\left(\mathbf{r}'\right)\mathrm{d}V', \tag{4.20}$$

the *electrostatic dipole moment*

$$\mathbf{p}_{\mathrm{d}}=\int_V \mathbf{r}'\rho\left(\mathbf{r}'\right)\mathrm{d}V', \tag{4.21}$$

and the *electrostatic quadrupole moment*[3]

$$\mathbf{Q}_{\mu\nu}=\int_V\left\{3x'_\mu x'_\nu-\left(r'\right)^2\delta_{\mu\nu}\right\}\rho\left(\mathbf{r}'\right)\mathrm{d}V'. \tag{4.22}$$

Our series solution to Poisson's equation is then (cf. [48], p. 138)

$$\varphi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}+\frac{\mathbf{r}\cdot\mathbf{p}_{\mathrm{d}}}{r^3}+\frac{1}{2}\frac{1}{r^5}x_\mu x_\nu\mathbf{Q}_{\mu\nu}\right]. \tag{4.23}$$

---

[3]The quadrupole moment is a tensor.

Measurements of the electrostatic field at (comparitively) great distances from the charge density can then reveal total charge in the volume $V$ and information about the distribution of the charge in terms of dipole and possibly quadrupole moments of the charge density.

## 4.6 Energy Storage

### *4.6.1 Electrostatic Energy Density*

Charges located at an infinite distance from one another experience no force of interaction. To assemble charges into a charge density, or densities, we must do work against the Coulomb force between the charges. Any density of charge in a region of space then results in energy stored in that region. We now seek a mathematical expression for this stored energy.

From (4.8) we see that the electrostatic potential in a region of space depends solely[4] upon the distribution of charges in space $\rho(\mathbf{r})$. The energy required to bring an additional infinitesimal amount of charge into a region in which $\varphi(\mathbf{r}) \neq 0$ is, therefore, dependent only on the distribution of charge already present. And, because the electrostatic field is conservative, it makes no difference how the final charge density is assembled.

The mathematical development is simplified if we consider that the charges are assembled in such a way that the charge density at each step is proportional to the final density $\rho(\mathbf{r})$. We may then identify a parameter $0 \leq \eta \leq 1$ and require that during the charging process the charge density is $\eta\rho(\mathbf{r})$. A step in the charging process increases $\eta$ by $\mathrm{d}\eta$. In this step the charge in the infinitesimal volume $\mathrm{d}V$, centered on the point $\mathbf{r}$, is increased by an amount $\rho(r)\mathrm{d}\eta\mathrm{d}V$.

Because the integrand in (4.8) is directly proportional to $\rho(\mathbf{r})$ the electrostatic potential $\varphi(\mathbf{r})$ is also directly proportional to $\eta$ at each step in the charging process. The increase in potential energy which results from bringing the infinitesimal charge $\rho(\mathbf{r})\mathrm{d}\eta\mathrm{d}V$ from an infinite distance away to the point $\mathbf{r}$ is then $\mathrm{d}U_{\mathrm{E}} = \rho(\mathbf{r})\varphi(\mathbf{r})\eta\mathrm{d}\eta\mathrm{d}V$. The total potential energy of the charge distribution $\rho(\mathbf{r})$ is the integral of $\mathrm{d}U_{\mathrm{E}}$ over the charging process, which is

$$\begin{aligned} U_{\mathrm{E}} &= \int_{\eta=0}^{1} \eta\mathrm{d}\eta \int_{\mathrm{V}} \rho(\mathbf{r})\varphi(\mathbf{r})\,\mathrm{d}V \\ &= \frac{1}{2}\int_{\mathrm{V}} \rho(\mathbf{r})\varphi(\mathbf{r})\,\mathrm{d}V, \end{aligned} \tag{4.24}$$

where $V$ is a spatial volume containing all the charge density.

[4]The solution for $\varphi$ in (4.8) is unique (see Appendices).

From Gauss' Law, (3.26), the charge density is

$$\rho(\mathbf{r}) = \varepsilon_0 \operatorname{div} \mathbf{E}. \tag{4.25}$$

With the vector identity (A.19) we can write (4.25) as

$$\begin{aligned} \rho(\mathbf{r})\, \varphi(\mathbf{r}) &= \varepsilon_0 \varphi \operatorname{div} \mathbf{E} \\ &= \varepsilon_0 \operatorname{div}(\varphi \mathbf{E}) - \varepsilon_0 (\operatorname{grad} \varphi) \cdot \mathbf{E}. \end{aligned} \tag{4.26}$$

Then, using (4.26), (4.24) becomes

$$U_{\mathrm{E}} = \frac{1}{2} \varepsilon_0 \int_{\mathrm{V}} \left[\operatorname{div}(\varphi \mathbf{E}) - (\operatorname{grad} \varphi) \cdot \mathbf{E}\right] \mathrm{d}V. \tag{4.27}$$

If we apply Gauss' Theorem (2.59) to the first term in the integrand in (4.27) we have

$$\int_{\mathrm{V}} \operatorname{div}(\varphi \mathbf{E})\, \mathrm{d}V = \oint_{\mathrm{S}} \varphi \mathbf{E} \cdot \mathrm{d}\mathbf{S}. \tag{4.28}$$

The charge density occupies a finite spatial volume. And the electrostatic potential, as well as the electrostatic field, vanishes at infinity. The volume $V$ in (4.27) is arbitrary. We may then take it to be all space with the surface $S$ at infinity. The integral over the surface $S$ in (4.28) then vanishes. If we use (4.2) in (4.27) we have

$$U_{\mathrm{E}} = \int_{\mathrm{V}} \frac{1}{2} \varepsilon_0 E^2 \mathrm{d}V. \tag{4.29}$$

The charge density $\rho(\mathbf{r})$ then results in an energy density

$$u_{\mathrm{E}} = \frac{1}{2} \varepsilon_0 E^2 \tag{4.30}$$

in space.

The potential energy is not confined only to the regions where $\rho(\mathbf{r}) \neq 0$, as would be the case in an action at a distance picture. Our result in equation (4.30) is then fundamentally different from any result based on action at a distance. Our result identifies an *electrostatic field energy density*.

We realize, based on our discussion of Hertz' work, that electric and magnetic fields are real. They can be detached from matter and propagate through empty space. With our result we realize, even before we identify the energy in a propagating wave, that there is an energy density in the electrostatic field.

### *4.6.2 Energy of a Set of Conductors*

The charge densities we considered in Sect. 4.6.1 may be on a set of conductors arranged in some arbitrary fashion in a region of space. This arrangement may have a geometrical symmetry as in two parallel plates, a plate and a sphere, or simply a sphere. But we require no sort of symmetry.

In Sect. 3.4.3 we outlined the requirements for the distribution of surface charge density on a conductor, which may result from an excess of charge or may be induced by an external electrostatic field. The requirement that there is no electrostatic force (field) within or on the surface of the conductor is satisfied if the conductor is at a uniform, constant potential. The fact that the electrostatic field must be perpendicular to the surface of the conductor is satisfied if the gradient the electrostatic potential is

$$[\mathrm{grad}\,\varphi\,(\mathbf{r}_{\mathrm{i}})]_{\mathrm{n}} = -\sigma_{\mathrm{i}}\,(\mathbf{r}_{\mathrm{i}})\,/\varepsilon_0 \tag{4.31}$$

at all points $\mathbf{r}_{\mathrm{i}}$ on the surface of the conductor (see exercises). These conditions on the value of the electrostatic potential and its gradient on the surface of the conductor are the boundary conditions required for the solution to Laplace's Equation in a region bounded by surfaces of constant potential.

We now ask for the electrostatic energy that can be stored on an arbitrarily arranged set of conductors.

#### 4.6.2.1 Coefficients of Potential

We place an excess charge on one of the conductors, which we label as $j$. There are no free charges anywhere, except on the conductor $j$. The result will be an electrostatic field in the region of space we are considering. And this field will induce charge densities on all the other conductors.

The solutions to Laplace's Equation are additive. The electrostatic potential $\varphi\,(\mathbf{r})$ at a point $\mathbf{r}$ is then a linear sum of the contributions from each of the conductors. We designate the contribution from the conductor $i$ at the point $\mathbf{r}$ as $\varphi_{\mathrm{i}}\,(\mathbf{r})$. If there are $N$ conductors the potential at the point $\mathbf{r}$ is then

$$\varphi\,(\mathbf{r}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \alpha_{\mathrm{i}}\varphi_{\mathrm{i}}\,(\mathbf{r})\,. \tag{4.32}$$

If we multiply the potential (4.32) by a constant $\kappa$ (kappa) the result

$$\kappa\varphi\,(\mathbf{r}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \alpha_{\mathrm{i}}\kappa\varphi_{\mathrm{i}}\,(\mathbf{r}) \tag{4.33}$$

is also a solution to Laplace's Equation in the region we are considering. In this solution the contribution from each conductor to the total potential is also multiplied by $\kappa$, as we see in (4.33). Then $\mathrm{grad}\,\varphi_{\mathrm{i}}\,(\mathbf{r}_{\mathrm{i}})$ on the surface of each conductor is multiplied by $\kappa$, which, from (4.31) means that the value of the surface charge density on the $i$th conductor is multiplied by $\kappa$ as well. This includes the $j$th conductor, which is the only one carrying a net charge.

We conclude then that increasing the net charge on the $j$th conductor by a factor $\kappa$ results in an increase in the potential at each point in space by a factor $\kappa$ and an increase in the contribution to this potential by each of the other conductors by the same factor $\kappa$. That is, whatever the net charge on the $j$th conductor may be, the contribution to the potential $\varphi_{\mathrm{i}}(\mathbf{r})$ at any point $r$ in space may be written as

$$\varphi_{\mathrm{i}}(\mathbf{r}) = p_{\mathrm{ij}} Q_{\mathrm{j}}, \tag{4.34}$$

where $Q_{\mathrm{j}}$ is the net charge we have placed on the conductor $j$. The *coefficients of potential* $p_{\mathrm{ij}}$ depend only on the geometry of the $i$th conductor and its location (cf. [83], p. 76, [48], p. 48).

We may pursue the same argument if we add a charge to the $k$th conductor as well. Our conclusion will be that the potential $\varphi_{\mathrm{i}}(\mathbf{r})$ at any point $r$ in space may be written as

$$\varphi_{\mathrm{i}}(\mathbf{r}) = p_{\mathrm{ij}} Q_{\mathrm{j}} + p_{\mathrm{ik}} Q_{\mathrm{k}}, \tag{4.35}$$

where $Q_{\mathrm{j}}$ is the net charge we have placed on the conductor $j$ and $Q_{\mathrm{k}}$ is the net charge we have placed on the conductor $k$. Continuing in this fashion we have

$$\varphi_{\mathrm{j}} = \sum_{\mathrm{i}}^{\mathrm{N}} p_{\mathrm{ji}} Q_{\mathrm{i}}. \tag{4.36}$$

The coefficients of potential are dependent only on the geometrical shape and relative location of the conductors.

#### 4.6.2.2 Capacitance

For a set of conductors carrying charges the integration of (4.24) is straightforward. The charge density is only nonzero on the surfaces of the conductors where the electrostatic potential is constant. Therefore (4.24) results in a sum of integrals over the surfaces of the $N$ conductors

$$\begin{aligned} U_{\mathrm{E}} &= \frac{1}{2} \sum_{\mathrm{j}}^{\mathrm{N}} \int_{\mathrm{a}} \sigma_{\mathrm{j}}(\mathbf{r})\, \varphi_{\mathrm{j}}(\mathbf{r})\, \mathrm{d}a \\ &= \frac{1}{2} \sum_{\mathrm{j}}^{\mathrm{N}} Q_{\mathrm{j}} \varphi_{\mathrm{j}}. \end{aligned} \tag{4.37}$$

Using (4.36) equation (4.37) becomes

$$U_E = \frac{1}{2}\sum_j^N \sum_i^N Q_j p_{ji} Q_i. \tag{4.38}$$

Because the order of partial differentiation is immaterial $p_{ij} = p_{ji}$ (see exercises).

If we consider two initially uncharged conductors and provide a conduction pathway and an electromotive force so that charge may be transferred from one conductor to the other we will have a system of two conductors one of which carries a charge $+Q$ and the other of which carries a charge $-Q$. We shall designate $Q_1 = +Q$ and $Q_2 = -Q$. Then from (4.38) the stored energy is

$$U_E = \frac{1}{2}\left(p_{11} - 2p_{12} + p_{22}\right) Q^2. \tag{4.39}$$

From (4.36) the difference in the electrostatic potentials of the two conductors is

$$V_{12} = \varphi_1 - \varphi_2 = \left(p_{11} - 2p_{12} + p_{22}\right) Q. \tag{4.40}$$

Combining (4.39) and (4.40) we have

$$U_E = \frac{1}{2}\left(p_{11} - 2p_{12} + p_{22}\right)^{-1} V_{12}^2. \tag{4.41}$$

Equation (4.41) is a completely general result for two conductors with equal charges of opposite signs. And the term $\left(p_{11} - 2p_{12} + p_{22}\right)^{-1}$ depends only on the geometry and relative location of the two conductors. It is independent of the electrical charge. We define this term as the *capacitance* $C$ of the two conductors.

From the energy storage (4.41) we have what we may consider to be the fundamental definition of capacitance

$$U_E = \frac{1}{2} C V^2. \tag{4.42}$$

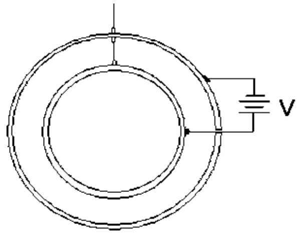
And from (4.40) we have what we may consider

$$C V = Q. \tag{4.43}$$

to be a *working definition of capacitance*. Equation (4.43) is much easier to use in calculations of capacitance.

*Example 4.3.* **Spherical Capacitor**. As an example we consider two thin spherical conducting shells of radii $a$ and $b$, with $a < b$. Each shell can be separated into hemispheres so that they can be mounted concentrically by a very thin nonconducting cord as we have drawn in Fig. 4.8. A voltage source (electromotive force) removes charge from the outer shell and deposits it on the inner shell. To find the capacitance of these spherical shells we must relate the potential difference

**Fig. 4.8** Concentric spherical shells of radii $a$ and $b$




between the shells to the charge transferred. The electrostatic field in the space between the inner and outer spherical shells we found in Example 3.1. The result is (3.29). From (4.2) we have a differential equation for the electrostatic potential $\varphi$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}. \tag{4.44}$$

Choosing the scalar potential to be $V_{\mathrm{ab}}$ at $r = a$ and 0 at $r = b$ we may integrate (4.44) to obtain

$$V_{\mathrm{ab}} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right), \tag{4.45}$$

or

$$4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)V_{\mathrm{ab}} = q.$$

The capacitance is then

$$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

Because energy is required in the charging process and because a field permeates the space around it, a single charged conductor also stores electrostatic energy and will have a capacitance. As an example we consider a positive charge $q$ distributed on a spherical conductor of radius $R$.

*Example 4.4.* **Single Sphere Capacitor.** The electrostatic field in the space surrounding a spherical conductor with a charge $q$ is (see (3.29))

$$E(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

for $r > R$. And from the integral of (4.30) the electrostatic energy density in the space surrounding the conductor is

$$u_{\mathrm{E}} = \frac{1}{2}\frac{q^2}{(4\pi)^2\varepsilon_0}\frac{1}{r^4}. \tag{4.46}$$

The total electrostatic energy is the integral of (4.46) over all space surrounding the spherical conductor

$$
\begin{aligned}
U_{\mathrm{E}} &= \frac{1}{2}\frac{q^2}{(4\pi)^2\,\varepsilon_0}\int_{\mathrm{r=R}}^{\infty}\frac{1}{r^4}\left(4\pi r^2\right)\mathrm{d}r \\
&= \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{R}.
\end{aligned} \tag{4.47}
$$

To find the electrostatic potential of the sphere we integrate (4.44) from $r$ to $\infty$, with $\varphi(\infty) = 0$. The result is

$$
\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}. \tag{4.48}
$$

The electrostatic potential on the surface of the sphere is then

$$
V_{\mathrm{R}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R}. \tag{4.49}
$$

We may use (4.49) and (4.43) to obtain the capacitance, or we may calculate the total energy in the space surrounding the sphere and use (4.42). For illustrative purposes we choose the latter approach. Using (4.42) we can write (4.47) as

$$
\begin{aligned}
U_{\mathrm{E}} &= \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{R} \\
&= \frac{1}{2}C\,V_{\mathrm{R}}^2 \\
&= \frac{1}{2}C\left(\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\right)^2.
\end{aligned}
$$

The capacitance $C$ is then

$$
C = 4\pi\varepsilon_0 R.
$$

## 4.7 Summary

We devoted this chapter to the scalar electrostatic potential alone because it is a vehicle for the study of some properties of the electrostatic field, which are not as easily accessible in terms of the field equations alone.

The electrostatic scalar potential satisfies Poisson's Equation, which is a great simplification over any attempt to apply Coulomb's Law directly in the actually calculation of the electrostatic field. The properties of Poisson's Equation are thoroughly understood (see Appendix F) permitting us practical access to the behavior of the electrostatic potential. And the calculation of a scalar quantity is always easier than the calculation of a vector quantity.

The electrostatic potential is then the route of choice in any study of electrostatic problems beyond the limits of the integral form of Gauss' Law.

The electrostatic potential  was also the basis of our study of energy storage and our discovery of an energy density in the electrostatic field. The practical energy storage element, the capacitor, emerged naturally from this study.

This does not yet establish the reality of the electrostatic field. We can still claim that the stored energy is only the result of separation of charges and not of the presence of a field. The identification of a static field energy is, however, a major step toward demonstrating the reality of fields.

## Exercises

**4.1.** Why is the electrostatic field always perpendicular to the electrostatic potential surfaces $\varphi =$ constant?

**4.2.** In Fig. 4.1 the potential surfaces are

$$\varphi\,(x,y,z) = z - \exp\left(-x^2\right)\sin\,(x+y)\,.$$

What is the charge distribution required for this potential?

**4.3.** You have a spherical copper shell made of two hemispheres of outer radius $R$ that can be connected or disconnected from one another. You plan to mount a voltage sensor within the sphere and then close it. The wires from the sensor you will wrap around a nonconducting cord from which the sphere is suspended. You have also mounted a second voltage sensor on a nonconducting rod that can be moved to any location a distance $r > R$ from the center of the sphere.

You plan to charge the sphere to $Q$ C and then to measure the potential inside and outside the sphere.

You first do the theory to decide what you expect to measure before you go into the laboratory.

(a) What will be the potential inside the sphere, i.e. $r < R$? [Consult the theorems on Poisson's Equation in the Appendices.]
(b) What will be the potential as a function of $r$ for $r > R$? [Outside of the sphere $\rho = 0$ and the potential must satisfy Laplace's Equation $\nabla^2\varphi = 0$. Use the Laplacian for spherical coordinates in Appendix A.1, equation A.12. There is spherical symmetry so partial derivatives of the potential $\varphi$ with respect to $\phi$ and $\vartheta$ vanish. Find the radial dependence of $\varphi$. This will be the form of the potential.]

**4.4.** Because the surface of a conductor is an equipotential surface, $\varphi\,(\mathbf{r})$ an infinitesimal distance from the surface of a conductor has the geometry of the conductor surface.

Consider that a conductor has a spikelike projection as we have drawn in Fig. 4.9. The surface of the conductor is $f = f\,(x,y,z)$ which is also $\varphi\,(x,y,z) =$ constant. An infinitesimal distance from the surface of the conductor the potential satisfies

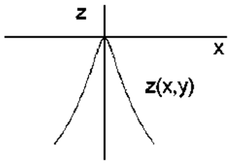



**Fig. 4.9** Spike on the surface of a conductor

Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{4.50}$$

At the peak the partial derivatives $\partial^2\varphi/\partial x^2$ and $\partial^2\varphi/\partial y^2$ are inversely proportional to the radius of curvature $R_C$ of the peak ([15], vol 1, p. 282-3), which is very small if the peak is pointed. Use this fact and the requirement that $\varphi$ satisfies (4.50) to show that there will be a concentration of charge at the peak of the spikelike projection.

Note that at the peak of the spike there is only an electrostatic field $E_z$.

**4.5.** You have a very thin circular copper disk of radius $R$, which you have suspended by nonconducting threads so that the plane of the disk is parallel to the wooden laboratory floor. You intend to charge the disk with $Q$ C of charge. You have also mounted a nonconducting support coming down from the ceiling to a point very close to the surface of the disk, which will serve as a track for a probe measuring the electrostatic potential. The tip of the probe will always be on the disk axis.

Assume that the charge is uniformly distributed over the disk surfaces.

What do you expect to measure as $\varphi(z)$ along the axis of the disk?

In an example we found the electrostatic potential on the axis of a ring of charge. You may consider that your disk is made up of rings of charge and integrate this result. But you should also consider the integral solution of Poisson's Equation directly realizing that the charge is distributed on the disk surface (it is copper). If both approaches are equivalent you should get the same answer.

**4.6.** What is the electric field $E_z(z)$ along the axis of the disk in the preceding exercise?

**4.7.** You have a right circular copper cylinder of radius $R$ and length $\ell$, which you have suspended by nonconducting threads so that ends of the cylinder are parallel to the wooden laboratory floor. You intend to charge the cylinder with $Q$ C of charge. You have also mounted a nonconducting support coming down from the ceiling to a point very close to the surface of the disk, which will serve as a track for a probe measuring the electrostatic potential. The tip of the probe will always be on the cylinder axis.

Assume that the charge is uniformly distributed over the surfaces of the cylinder. It is $\sigma_e$ on the ends of the cylinder and $\sigma_s$ on the side.

What do you expect to measure as $\varphi(z)$ along the axis of the cylinder?

In an example we found the electrostatic potential on the axis of a ring of charge. And in a previous exercise we found the electrostatic potential from a plate. You may consider that your cylinder is made up of rings of charge and end plates to find the result. But you should also consider the integral solution of Poisson's Equation directly realizing that the charge is distributed on the cylinder surface (it is copper). If both approaches are equivalent you should get the same answer.

**4.8.** What is the electrostatic field along the axis of the cylinder in the preceding exercise, but outside of the cylinder, for points $z \pm \ell/2 \gg R$ and $z \gg \ell$? Comment on the answer referring to the Coulomb field from a point charge.
[Answer: $\frac{2\pi R\ell\sigma_s}{4\pi\varepsilon_0}\frac{1}{z^2}$]

**4.9.** The general form of Poisson's Equation for the electrostatic potential in three dimensions is

$$\nabla^2\varphi = -\frac{1}{\varepsilon_0}\rho(\mathbf{r}).$$

You have shown that

$$\text{div grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right).$$

Using this, show that the general solution to Poisson's Equation in three dimensions is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \mathrm{d}V'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},$$

where the integration is over the volume containing all sources.

**4.10.** The *screened Coulomb potential*

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{\exp(-r/\lambda)}{r} \text{ for } r > 0$$

is appropriate for a charge $q$ (at the origin) in a semiconducting medium, where it is termed Thomas-Fermi shielding, or in plasmas, where it is termed Debye shielding.

(a) What is the electrostatic field in the region $r > 0$? Compare your result to the Coulomb field.
(b) What is the charge density for this potential in the region $r > 0$?
(c) Interpret physically what you have discovered in the charge density for $r > 0$.

Notice in your analysis that the origin must be avoided as a mathematical point. The charge density, the potential, and the electrostatic field are all infinite at the origin.

**4.11.** Lise Meitner analyzed the experiments of Otto Hahn in Berlin. Meitner was then a refugee from (Nazi) Germany (She was an Austrian. Germany had annexed Austria.) in Sweden. Hahn had discovered that after ${}^{238}_{92}\mathrm{U}$ was bombarded by neutrons (zero charge) Barium (Ba) could be found in the products. Meitner realized that charge conservation meant that Krypton (Kr) was also there, although Hahn's radiochemistry technique did not pick up Krypton. Meitner imagined that the nucleus had split. She knew the difference in masses of ${}^{238}_{92}\mathrm{U}$ and ${}^{137}_{56}\mathrm{Ba}+{}^{83}_{46}\mathrm{Kr}$.

And she knew Einstein's mass-energy relationship $E = \Delta mc^2$, where $\Delta m$, is the mass lost. The result was (about) 200 MeV. She then asked if this was the potential energy of ${}^{137}_{56}\mathrm{Ba}$ and ${}^{83}_{46}\mathrm{Kr}$ nuclei located at twice the nuclear radius apart, which would be the potential energy just after splitting. If this potential energy was 200 MeV, she understood fission (nuclear splitting). What would be the approximate nuclear radius (assume the same for Ba and Kr) for her theory to work? The nuclear charge on the ${}^{137}_{56}\mathrm{Ba}$ nucleus is $56\left(1.60217733 \times 10^{-19}\ \mathrm{C}\right) = 8.9722 \times 10^{-18}$ C and on the ${}^{83}_{46}\mathrm{Kr}$ nucleus is $46\left(1.60217733 \times 10^{-19}\ \mathrm{C}\right) = 7.37 \times 10^{-18}$ C. [Answer: nuclear radius $R_{\mathrm{nuclear}} \approx 9.286 \times 10^{-15}$ m]

[Meitner did this calculation sitting on a log in a snowy woods at Kungälv, Sweden, with her Nephew Otto Frisch. Frisch would later be part of the Los Alamos team of the Manhattan Project. Meitner would declare that she wanted nothing to do with weapons.]

**4.12.** Using the general expression for the electrostatic dipole moment

$$\mathbf{p}_{\mathrm{d}} = \int_{\mathrm{V}} \mathbf{r}' \rho\left(\mathbf{r}'\right) dV'$$

show that the electrostatic dipole moment of the charge density (centered at the origin)

$$\rho = Q\delta(x)\delta(y)\delta\left(z - \frac{\ell}{2}\right) \\ -Q\delta(x)\delta(y)\delta\left(z + \frac{\ell}{2}\right)$$

is $\mathbf{p}_{\mathrm{d}} = \hat{e}_{\mathrm{z}} Q\ell$.

**4.13.** Using the general expression for the electrostatic quadrupole moment

$$\mathbf{Q}_{\mu\nu} = \int_{\mathrm{V}} \left\{3x'_{\mu}x'_{\nu} - \left(r'\right)^2 \delta_{\mu\nu}\right\} \rho\left(\mathbf{r}'\right) \mathrm{d}V'.$$

and the charge density (centered at the origin)

$$\rho = Q\delta(x)\delta(y)\delta\left(z - \frac{\ell}{2}\right) \\ -Q\delta(x)\delta(y)\delta\left(z + \frac{\ell}{2}\right)$$

to find the electrostatic quadrupole moment of this density.

[Answer: $\mathbf{Q}_{\mu\nu} = 0 \forall \mu, \nu$]

**4.14.** Consider two conducting plates separated by a distance $d$, which very small compared to the plate dimensions. We extract a charge $q$ from one plate, which we choose to have the reference electrostatic potential of $V = 0$, and deposit this charge on the other plate, which will now have the electrostatic potential $V$. Except for a (relatively) small curvature at the plate boundary (fringing), the electrostatic field is uniform between the two plates.

What is the capacitance of this pair of plates?

**4.15.** Using the fact that the order of partial differentiation is immaterial in (4.38) show that $p_{\text{ij}} = p_{\text{ji}}$.

**4.16.** You have a very thin, nonconducting circular disk of radius $R$ has a uniform charge density, which you may consider to be a surface charge $\sigma$. Find the electrostatic potential and the electrostatic field at a point on the axis of the disk at a distance $z$ from the plane of the disk.

**4.17.** You have a nonconducting, right circular cylinder of radius $R$ and height $L$. Inside this nonconducting cylinder is a nonuniform charge density $\rho(z) = \rho_0 + \beta z$, where $z$ is the axial coordinate and $z = 0$ at the center of the cylinder and $\rho_0$ and $\beta$ are constants. Accordingly you choose to use cylindrical coordinates with origin at the center of the nonconducting cylinder and $z-$axis along the cylinder axis.

Using the electrostatic field you have obtained in the preceding exercise, find the electrostatic field at the center of the nonconducting cylinder.

**4.18.** You have a nonconducting right circular cylinder of radius $R$ and length $L$, which you have charged to a uniform charge density $\rho$ and hung by a thread from the ceiling. The thread is well-insulated so that there is no loss of charge from the cylinder.

Choosing the origin of your coordinates to be the base of the cylinder, find the electrostatic potential at a point on the cylinder axis but external to the distribution.

**4.19.** You have a nonconducting sphere of radius $R$ in which there is a uniform charge density $\rho_0$. Integrate the energy density over all space to obtain what is called the self-energy of this charge distribution. Assume that you can use $\varepsilon_0$ inside the nonconductor.

**4.20.** Biological membranes are lipid bilayers approximately 50 Å thick. In a nerve cell the resting potential varies slightly from cell to cell. But in a healthy cell it is about 100 mV. Experiments are conducted on planar lipid bilayers between 150 $\mu m-$300 $\mu m$ in diameter. These planar bilayers are models for cell membranes.

(a) What is the capacitance of a bilayer with a diameter of 200 ¯am?
(b) What is the energy within the bilayer when the potential is 100 mV?
(c) What is the electrostatic field energy density?
(d) What is the electrostatic field?
(e) How does this compare with the electrostatic field required to breakdown air, which is $1.181\,1 \times 10^6$ V/ m.

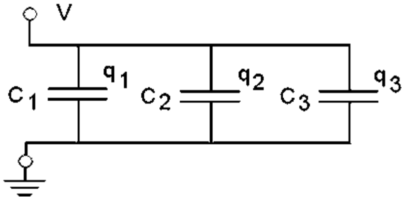



Fig. 4.10 Capacitors in parallel

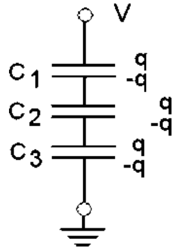



Fig. 4.11 Capacitors in series

**4.21.** (a) What is the capacity of a capacitor that can store 1.0 J at 100 V?

(b) Assuming the capacitor has parallel plates separated by $10^{-5}$ m what is the necessary area of the plates? Comment on the result.

**4.22.** You want to store a large electrostatic field energy and then use this energy in an experiment requiring a rapid discharge. Your idea is to arrange a group of parallel plate capacitors in a parallel connection as we have drawn in Fig. 4.10. You plan to charge this arrangement, reasoning that you have added the energies in each capacitor, and then to discharge the bank of capacitors across your apparatus.

(a) Show that the stored energy can be written in terms of an electrostatic potential $V$ applied to a single capacitor of magnitude $C_T = C_1 + C_2 + C_3$.

(b) Show that this addition of capacitances will result if you use our working definition of capacitance $CV = q$ on the arrangement in Fig. 4.10.
An alternative arrangement would have been the series combination we have drawn in Fig. 4.11.

(c) Show that the energy stored in this arrangement of capacitors can be written in terms of an electrostatic potential $V$ applied to a single capacitor of magnitude $C_T$ where $C_T^{-1} = C_1^{-1} + C_2^{-1} + C_3^{-1}$.

(d) Show that the equivalent single capacitor for a parallel combination of capacitors is always greater than that for a series combination.

# Chapter 5
# Magnetostatics

*[Of] three people walking together, at least one can be my teacher*

*Confucius, 551–478 B.C.E.*

## 5.1 Introduction

In this chapter we will introduce the properties of the time independent magnetostatic field. The magnetostatic field is a result of electrical current densities similarly to the manner in which the electrostatic field is the result of charge densities. And so we must understand the flow of electrical current as well.

Two classic experiments lie behind our understanding of magnetism. These are the experiments of Oersted and of Ampère. We will combine these to obtain a formulation of the magnetic field resulting from moving charges and of the magnetic force on moving charges.

We will again make a superposition Ansatz to formulate the fields and forces arising from general charge densities. The result will be an equivalent, although slightly more complex, integral formulation for the magnetostatic field than that obtained from Coulomb's Law for the electrostatic field. This is the Biot–Savart Law, presented to the French Academy 6 weeks after Ampère presented the results of his experiments.

But we will not follow the method of deriving the field equations that we used for the electrostatic case. The direct use of the Biot–Savart Law to obtain the field equations becomes tedious because of a cross product. But the result of Oersted's experiment introduces the concept of a vector potential. Basing our derivation of the magnetostatic field equations on this vector potential simplifies our work immensely.

Helmholtz' Theorem requires that we know both the divergence and the curl of the vector potential. This introduces the important concept of gauge and gauge transformation into the formulation of field theory. We will discuss this carefully.

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_5, 

We will end the chapter with a formulation of the field equations for the electrostatic and magnetostatic fields.

## 5.2 Current

### *5.2.1 Current Density*

Electrical current results from the transport of electrical charge. The flow of a fluid made up of charged particles, rather than molecules, produces an electrical current, because the flow results in the transport of charge. This is analogous to the flow of an ordinary fluid that results in the transport of mass.

There must be more than one species of charged particles present in the fluid in order to preserve electrical neutrality. It is possible that only one species is actually flowing while the other(s) is/are fixed in space, as will be the case in solids. But we consider the general case here with more than one species in motion.

We shall approximate the motion of each species of charged particles by defining an average velocity vector $\mathbf{v}_{\mathrm{ave}}^{(\alpha)}[\,\mathrm{m\,s}^{-1}]$ and an average particle density $n^{(\alpha)}[\,\mathrm{m}^{-3}]$ for the $\alpha$th species of charged particles with the charge $q_\alpha[\,\mathrm{C}]$.[1] The current density vector for the $\alpha^{\mathrm{th}}$ species is

$$\mathbf{J}_\alpha = n^{(\alpha)} q_\alpha \mathbf{v}_{\mathrm{ave}}^{(\alpha)} \tag{5.1}$$

in units of $[\,\mathrm{C\,m}^{-2}\,\mathrm{s}^{-1}]$. And the total current density in the fluid is

$$\boxed{\mathbf{J} = \sum_\alpha \mathrm{n}^{(\alpha)} \mathrm{q}_\alpha \mathbf{v}_{\mathrm{ave}}^{(\alpha)}.} \tag{5.2}$$

From (5.2) we see that $\mathbf{J}_\alpha$ is in the direction of $\mathbf{v}_{\mathrm{ave}}^{(\alpha)}$ if $q_\alpha$ is positive and in the direction of $-\mathbf{v}_{\mathrm{ave}}^{(\alpha)}$ if $q_\alpha$ is negative. The current density vector is then in the *direction* of the flow of positive charge whether the charge carriers are positive or negative, as Franklin proposed.

We have illustrated this in Fig. 5.1, where we have drawn a section of conductor through which the current flows. The vector **J** is tangent to the more darkly shaded conductor boundary in Fig. 5.1. The current flows through the lightly shaded end caps of the conductor and does not cross the darkly shaded boundary.

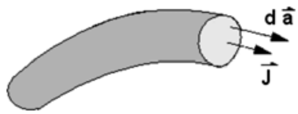



**Fig. 5.1** Current density vector **J**

[1] All particles of a particular species do not move at the average velocity. But we shall not conduct a kinetic theoretical treatment here.

We may associate a differential vector area d**a** with the end cap, as we have shown in Fig. 5.1. The rate at which charge passes through the differential end cap area d**a** is then the scalar product **J**·d**a** [ C s$^{-1}$]. The electrical current is the total charge passing through the end cap, which is the integral

$$\boxed{\mathrm{I} = \int_{\text{end cap}} \mathbf{J} \cdot \mathrm{d}\mathbf{a}.} \tag{5.3}$$

If the amount of charge passing through the end cap area in time d$t$ is d$Q$, then the current can also be designated as

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}, \tag{5.4}$$

which is is the definition of current used in electrical circuits.

### *5.2.2 Charge Conservation*

We now realize that Franklin's supposition that charge is conserved is one of the foundational physical laws. It shows no variation down to the level of elementary particles. Our task here is to obtain a mathematical statement of charge conservation that we can use at a macroscopic level.

We consider an arbitrary closed volume $V$ with a surface area $S$. There is a charge density $\rho(\mathbf{r}, t)$ [ C m$^{-3}$] in $V$ and a current density $\mathbf{J}(\mathbf{r}, t)$ [ C m$^{-2}$ s$^{-1}$] passes through the surface of the volume. At any instant the total charge contained in $V$ is

$$Q(t) = \int_{\mathrm{V}} \rho(\mathbf{r}, t)\, \mathrm{d}V. \tag{5.5}$$

Since d**S** points *out of the volume*, the total rate at which charge is transported *out* of the volume $V$ in a time interval d$t$ by the current density is

$$\left.\frac{\mathrm{d}Q(t)}{\mathrm{d}t}\right]_{\text{out}} = \oint_{\mathrm{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \tag{5.6}$$

Using (5.5) the rate of loss of the charge in $V$ is

$$\left.\frac{\mathrm{d}Q(t)}{\mathrm{d}t}\right]_{\text{loss}} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{V}} \rho(\mathbf{r}, t)\, \mathrm{d}V. \tag{5.7}$$

If $V$ does not change, i.e. if $V$ is fixed, we can bring the time derivative inside the integral where it becomes a partial derivative.

$$\left.\frac{\mathrm{d}Q\left(t\right)}{\mathrm{d}t}\right]_{\text{loss}} = -\int_{\mathrm{V}} \frac{\partial\rho\left(\mathbf{r},t\right)}{\partial t}\mathrm{d}V. \tag{5.8}$$

Since charge is conserved , the charge in the volume $V$ (5.5) can only change by transport across the boundary. Therefore (5.8) must be equal to (5.6). That is

$$\boxed{-\int_{\mathrm{V}} \partial\rho\left(\mathbf{r},t\right)/\partial t\mathrm{dV} = \oint_{\mathrm{S}} \mathbf{J}\left(\mathbf{r},t\right)\cdot\mathrm{d}\mathbf{S}.} \tag{5.9}$$

Equation (5.9) is an integral statement of charge conservation. We need a differential statement as well.

Application of Gauss' Theorem to (5.9) yields

$$\int_{\mathrm{V}} \left\{\frac{\partial\rho\left(\mathbf{r},t\right)}{\partial t} + \operatorname{div}\mathbf{J}\left(\mathbf{r},t\right)\right\}\mathrm{d}V = 0 \tag{5.10}$$

for any *arbitrary* volume $V$. In (5.10) we have, an integral that always vanishes for any arbitrarily chosen volume. This can only be true if the integrand vanishes everywhere. Therefore

$$\boxed{\partial\rho\left(\mathbf{r},t\right)/\partial\mathrm{t} + \operatorname{div}\mathbf{J}\left(\mathbf{r},t\right) = 0.} \tag{5.11}$$

Equation (5.11) is the differential form of the equation of charge conservation.[2]

In the time independent case, (5.11) becomes

$$\operatorname{div}\mathbf{J}\left(\mathbf{r}\right) = 0. \tag{5.12}$$

We get (5.12) from (5.6) if the surface $S$ allows no charge to pass through it so that

$$\oint_{\mathrm{S}} \mathbf{J}\cdot\mathrm{d}\mathbf{S} = 0.$$

Such a surface is the dark shaded outside of the wire in Fig. 5.1. The current then must flow in closed wire loops.

## 5.3 Oersted's Experiment

We discussed Oersted's experiments in a historical context in Sect. 1.9.1 of Chap. 1. And we illustrated Oersted's observation in Fig. 1.4. Our goal is now to cast these experiments into mathematical language.

[2]This same equation, with $\mathbf{J} = \rho_{\text{mass}}\mathbf{v}_{\text{ave}}$, is that of mass conservation in fluid mechanics.

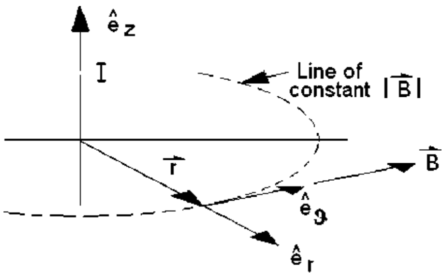



**Fig. 5.2** Cylindrical coordinate representation of the results of Oersted's experiment

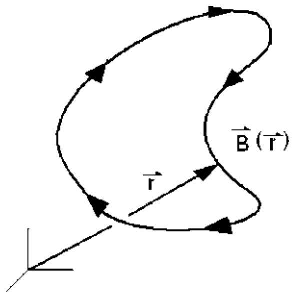



**Fig. 5.3** The Magnetic Field lines form closed loops

In Fig. 5.2 we have drawn a cylindrical coordinate system $(r, \vartheta, z)$ in which we may most easily represent the magnetic field lines Oersted observed. The vector **B** is the *magnetic field induction* (see Sect. 1.11.2.2). The wire, and hence the current $I$, lies along the $z-$ axis with current flowing in the positive $\hat{e}_z$ direction. For this case the lines along which $|\mathbf{B}| = B$ is constant are concentric circles and the magnetic field induction vector **B** has only an $\hat{e}_\vartheta$ component. That is

$$\mathbf{B}(\mathbf{r}) = B(r)\,\hat{e}_\vartheta . \tag{5.13}$$

We may produce magnetic fields with arbitrary geometrical forms by appropriately arranging wires and selecting the currents passing through them. This is because magnetic fields resulting from currents satisfy a superposition principle as do electric fields. We discuss this in detail in Sect. 5.6.

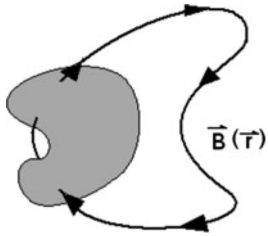
For example a magnetic field induction vector could have a geometrical form such as we have drawn in Fig. 5.3.

The lines of magnetic field induction, or simply magnetic field lines, must, however, always form *closed contours*, as we have shown in Fig. 5.3. This is the result of Oersted's experiment.

If we place a closed volume, real or imaginary, in the region shown in Fig. 5.3 the magnetic field line will then pass through the closed volume as we have shown in Fig. 5.4.

Corresponding to every entry point of the field line there must be an exit point for any arbitrary closed volume. Then the surface integral of the differential magnetic flux $d\Phi_B = \mathbf{B} \cdot d\mathbf{S}$ must vanish

**Fig. 5.4** Magnetic field line passing through a closed volume




$$\boxed{\oint_S \mathbf{B}\cdot d\mathbf{S} = 0.} \tag{5.14}$$

Using Gauss' Theorem (5.14) becomes

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{B}\, dV = 0. \tag{5.15}$$

That is the integral of div **B** over any *arbitrary volume* must always vanish. This can only be true if

$$\boxed{\operatorname{div} \mathbf{B} = 0.} \tag{5.16}$$

Equation (5.16) is *Oersted's Result* and is the first of the magnetostatic field equations.

Since

$$\operatorname{div} \operatorname{curl} \equiv 0,$$

an immediate consequence of (5.16) is the fact that **B** can always be written as

$$\boxed{\mathbf{B} = \operatorname{curl} \mathbf{A},} \tag{5.17}$$

where **A** is the *vector potential*. In the magnetostatic case **A** is a function only of the spatial coordinates.

Equation (5.16) also means that there can be no magnetic monopoles. A magnetic monopole would produce magnetic field lines that are not closed contours.

The existence of magnetic monopoles is a part of the Dirac theory of quantum fields (cf. [28], p. 431). But these were never detected during Dirac's lifetime.

Magnetic monopoles have possibly been recently detected in spin ices (2009) [73]. But the detected monopoles are not those predicted by Dirac. These monopoles seem to be an emergent (quasiparticle) phenomenon. The spin ice state is well described by networks of aligned dipoles resembling solenoidal tubes. At the ends of these tubes the defects appear as magnetic monopoles. The detection was by diffuse neutron scattering with application of a magnetic field to manipulate the density and orientation of the strings. The specific heat measurements in these spin ices, near the absolute zero of thermodynamic temperature, are describable as a gas of magnetic monopoles interacting by a Coulomb interaction.

It is too early to speculate on what implications, if any, this may have for classical field theory. At this point we shall continue to assume an absolute validity of (5.16).

## 5.4 Ampère's Experiment

In Chap. 1 we pointed out that within a week of hearing Arago's report on Oersted's experiment to the French Academy of Science Ampère presented a paper on magnetism to the Academy in which he demonstrated the force between two wires through which electric currents were passed. We presented the details of the experiment in Fig. 1.5.

In Fig. 5.5 we present a vector diagram of the experiment.

The unit vectors $\hat{\ell}_{1,2}$ indicate the directions of the currents $I_{1,2}$, which are carried by thin wires. The unit vector $\hat{r}_{12}$ indicates the direction from wire 1 to wire 2 and $r_{12}$ is the distance between the wires. The force $\mathbf{F}_{12}$ is the force between the two wires, which, in Fig. 5.5, is in the direction $-\hat{r}_{12}$.

From Ampère's experiment the empirical equation for the force between two parallel wires of length $\ell$ carrying currents $I_1$ and $I_2$ and separated by a distance $r_{12}$, as we have illustrated in Fig. 5.5, is

$$\mathbf{F}_{12} = -K_{\text{m}} \frac{I_1 I_2 \ell}{r_{12}} \hat{r}_{12}, \tag{5.18}$$

where $K_{\text{m}} > 0$ is an empirical constant. The force $\mathbf{F}_{12}$ is attractive if the currents in the wires are parallel ($I_1 I_2 > 0$ in Fig. 5.5) and repulsive if the currents are antiparallel ($I_1 I_2 < 0$ in Fig. 5.5).

Equation (5.18) is an empirical result that can be demonstrated with moderate care in an undergraduate laboratory if the wires are rigid, thin,[3] and their separation is not too great.

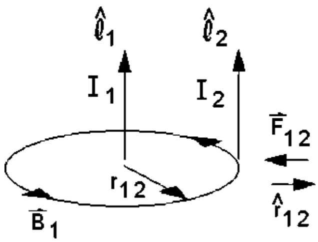



**Fig. 5.5** Vector diagram of Ampère's experiment

[3]Thin is a relative term. The currents normally used are large and the wire must have a low resistance to prevent melting.

The result (5.18) is valid for very long thin conductors. We will implicitly assume the wires to be very long in our analysis of the next sections. And we will implicitly ignore the fact that the wires must form closed circuits.

In the field picture we claim that the current $I_1$ is the source of a magnetic field with induction $\mathbf{B}_1$, which we indicate in Fig. 5.5. From Oersted's experiments we know the form and orientation of this magnetic field. We then interpret Ampère's empirical result (5.18) as the force on a current $I_2$ in a wire of length $\ell$ caused by a magnetic field $\mathbf{B}_1$ produced by current $I_1$.

The terms on the right hand side of (5.18) that refer to the current $I_2$ and the wire carrying that current are contained solely in the product $I_2\ell$. In the field picture we then interpret the force as a product of $(I_2\ell)$ and the magnitude of the magnetic field induction $\mathbf{B}_1$. Because we know, from Oersted's experiments, that the field $\mathbf{B}_1$ is in the *azimuthal direction* $\hat{e}_{\vartheta,1}$ around wire $I_1$, we can write

$$\mathbf{B}_1 = |\mathbf{B}_1|\,\hat{e}_{\vartheta,1} = K_\mathrm{m}\frac{I_1}{r_{12}}\hat{e}_{\vartheta,1} \tag{5.19}$$

In the field picture the force $\mathbf{F}_{12}$ is then

$$\mathbf{F}_{12} = -I_2\ell\,|\mathbf{B}_1|\,\hat{r}_{12}, \tag{5.20}$$

which is perpendicular to both the magnetic field induction vector $\mathbf{B}_1$ and the direction $\hat{\ell}_2$ of the current $I_2$. We must now resolve the direction of $\mathbf{F}_{12}$ in terms of $\hat{\ell}_1$, $\hat{\ell}_2$, and $\hat{r}_{12}$.

### *5.4.1 Direction of the Force*

From Fig. 5.5 we see that we can write the unit vector $-\hat{r}_{12}$ as

$$-\hat{r}_{12} = \hat{\ell}_2 \times \hat{e}_{\vartheta,1}. \tag{5.21}$$

And the unit vector $\hat{e}_{\vartheta,1}$ is in the direction

$$\hat{e}_{\vartheta,1} = \hat{\ell}_1 \times \hat{r}_{12}. \tag{5.22}$$

Then (5.21) becomes the triple vector product

$$-\hat{r}_{12} = \hat{\ell}_2 \times \left(\hat{\ell}_1 \times \hat{r}_{12}\right). \tag{5.23}$$

With (5.23) the empirical equation for the force $\mathbf{F}_{12}$ in (5.18) can be written as

$$\mathbf{F}_{12} = K_\mathrm{m}\frac{I_1 I_2}{r_{12}}\ell\left[\hat{\ell}_2 \times \left(\hat{\ell}_1 \times \hat{r}_{12}\right)\right]. \tag{5.24}$$

If we use the result in (5.22) in (5.19) we find that the magnetic field induction from current $I_1$, which flows in the direction $\hat{\ell}_1$, is

$$\mathbf{B}_1 = K_\mathrm{m} \frac{I_1}{r_{12}} \left( \hat{\ell}_1 \times \hat{r}_{12} \right). \tag{5.25}$$

Then with (5.25) and (5.24) we see that the force of a magnetic field on a wire of length $\ell$ in which there is a current $I$ is

$$\mathbf{F} = \left( I \ell \hat{\ell} \right) \times \mathbf{B} \tag{5.26}$$

where $\hat{\ell}$ is the direction of current flow.

Although we used the field picture in our discussion, the vector terms on the right hand side of (5.24) are only unit vectors in the directions of the currents and between the wires. None of these actually refers to a field.

### 5.4.2 The Constant

The constant of proportionality $K_\mathrm{m}$ appearing in (5.18) is can be obtained from the definition of the unit of electrical current. The International Bureau of Weights and Measures (*Le Bureau international des poids et mesures* (*BIPM*)) has defined the *ampere* (unit of electric current).

> The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7}$, N of force per meter of length.

Then, with $I_1 = I_2 = 1$ A, $r_{12} = 1$ m, and $\ell = 1$ m, we have

$$F_{12} = K_\mathrm{m} = 2 \times 10^{-7} \ \mathrm{N\ A^{-2}}. \tag{5.27}$$

The constant $K_\mathrm{m}$ is normally written as

$$K_\mathrm{m} = \frac{\mu_0}{2\pi}, \tag{5.28}$$

where $\mu_0$ is the *permeability of free space*, which, from (5.28) has the numerical value

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N\ A^{-2}}. \tag{5.29}$$

Then (5.24) becomes

$$\mathbf{F}_{12} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{r_{12}} \ell \left[ \hat{\ell}_2 \times \left( \hat{\ell}_1 \times \hat{r}_{12} \right) \right]. \tag{5.30}$$

Equation (5.30) is the final vector form of Ampère's empirical result. It appears formidable because of the triple vector product in the brackets [ ].

Some authors abandon hope of obtaining detailed information from this empirical equation (e.g. [12], p. 153) and some avoid the issue by beginning with the Biot–Savart law without reference directly to the experimental details (e.g. [83], p. 197).

Because we are developing field theory based on experiment, we shall work with (5.30) as our fundamental experimental result and extract from it the force on a charged particle, the form of the magnetic field from a current, and the Biot–Savart Law for a magnetic field from a general current loop. The effort will not be mathematically difficult and will leave us with a result that does not have glaring gaps between experiment and theory.

## 5.5 Consequences of Ampère's Experiment

### *5.5.1 Force on a Charge*

We begin with (5.26), which gives the force on a straight wire of length $\ell$ carrying a current $I$ in the presence of a magnetic field with induction **B**.

Using (5.3) we can write

$$I\ell\hat{\ell} = \hat{\ell}\int_{\mathrm{a}} \mathbf{J}\cdot\ell\mathrm{d}\mathbf{a} = \int_{\mathrm{V}} \mathbf{J}\mathrm{d}V, \tag{5.31}$$

where $\mathrm{d}V = \ell\,|\mathrm{d}\mathbf{a}|$ and $\hat{\ell}\,|\mathbf{J}| = \mathbf{J}$ for a straight wire. With (5.31) (5.26) becomes

$$\mathbf{F} = \int_{\mathrm{V}} \mathbf{J}\times\mathbf{B}\mathrm{d}V. \tag{5.32}$$

From (5.32) we interpret

$$\mathbf{f}_{\mathrm{mag}} = \mathbf{J}\times\mathbf{B} \tag{5.33}$$

as the magnetic force per unit volume inside the straight wire.

If we now use our particle based picture of current from equation (5.2) in (5.33) we can find the form of the force on each charged particle.

If we consider that the particles of each species $\alpha$ are point particles we may *represent* them by $\delta-$functions. Each of these point particles $(i)$ has a position, which we designate as $\mathbf{r}_{\mathrm{i}}^{(\alpha)}$ and is moving with a velocity $\mathbf{v}_{\mathrm{i}}^{(\alpha)}$, which may be distinct for each particle. Then in a very small volume $\Delta V$, which still contains a large number of point particles,[4] we have the particle number density

---

[4]The volume $\Delta V$ is infinitesimal compared to macroscopic dimensions but still contains an enormous number of classical point particles.

$$\delta n^{(\alpha)} = \frac{1}{\Delta V} \sum_{\text{all } i \text{ in } \Delta V} \delta\left(\mathbf{r} - \mathbf{r}_{\mathrm{i}}^{(\alpha)}\right). \tag{5.34}$$

We have designated this form of the number density as $\delta n^{(\alpha)}$ to indicate that $\delta$−functions are used for the particles. By the property of the $\delta$−function (2.98), the integral of $\delta n^{(\alpha)}$ over the volume $\Delta V$ is the number of particles of species $\alpha$ in the volume $\Delta V$ divided by $\Delta V$, which is density $n^{(\alpha)}$ at the location of $\Delta V$.

From (5.2) and (5.34) we then *represent* the current density in $\Delta V$ as

$$\delta \mathbf{J} = \frac{1}{\Delta V} \sum_{\alpha} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\mathrm{i}}^{(\alpha)} \delta\left(\mathbf{r} - \mathbf{r}_{\mathrm{i}}^{(\alpha)}\right), \tag{5.35}$$

and we *represent* the magnetic force density in $\Delta V$ as

$$\delta \mathbf{f}_{\mathrm{mag}} = \frac{1}{\Delta V} \sum_{\alpha} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\mathrm{i}}^{(\alpha)} \delta\left(\mathbf{r} - \mathbf{r}_{\mathrm{i}}^{(\alpha)}\right) \times \mathbf{B}. \tag{5.36}$$

The force density in the volume $\Delta V$ is then the integral of (5.36) over $\Delta V$. Using the integral property of the $\delta$−function (2.98) the force density is then

$$\begin{aligned} \mathbf{f}_{\mathrm{mag}}(\Delta V) &= \frac{1}{\Delta V} \sum_{\alpha} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\mathrm{i}}^{(\alpha)} \left[\int_{\Delta V} \delta\left(\mathbf{r} - \mathbf{r}_{\mathrm{i}}^{(\alpha)}\right) \mathrm{d}V'\right] \times \mathbf{B} \\ &= \frac{1}{\Delta V} \sum_{\alpha} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\mathrm{i}}^{(\alpha)} \times \mathbf{B}, \end{aligned} \tag{5.37}$$

which is the sum of the forces on the individual particles in the volume $\Delta V$ divided by the volume $\Delta V$. Therefore from (5.37) we see that the magnetic force on an individual charge $q$ with velocity $\mathbf{v}$ is $q\mathbf{v} \times \mathbf{B}$.

### 5.5.2 *Field from a Straight Wire*

From (5.19), with (5.28) and (5.3) the magnetic field induction from a long straight wire at a distance $r$ from the wire is

$$\mathbf{B} = \frac{\mu_0}{2\pi} \frac{1}{r} \hat{e}_\vartheta \int_{\mathrm{a}} \mathbf{J} \cdot \mathrm{d}\mathbf{a}. \tag{5.38}$$

Because the magnitude of $\mathbf{B}$ in (5.38) depends only on $r$ we can write

$$2\pi r \left|\mathbf{B}\right| = \oint_{\mathrm{C}} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{5.39}$$

where $C$ is the closed circular contour of radius $r$ around the origin, i.e. $\mathrm{d}\boldsymbol{\ell} = \hat{e}_\vartheta r \mathrm{d}\vartheta$. Combining (5.38) and (5.39) we see that

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a} \tag{5.40}$$

Equation (5.40) is *Ampère's Circuital Law* written for a straight wire. It is not yet the final form of Ampère's Circuital Law.

Using Stokes' Theorem, (5.40) becomes

$$\int_a (\mathrm{curl}\,\mathbf{B} - \mu_0 \mathbf{J}) \cdot \mathrm{d}\mathbf{a} = 0. \tag{5.41}$$

Because we have obtained (5.41) for long, thin, straight wires, we cannot claim that the integrand must always vanish for the integral to vanish. So this is not an adequate derivation of the second magnetostatic field equation.

We shall nevertheless pursue the consequences of equation (5.41), accepting the limitation to long straight wires. This will bring us to the Biot–Savart Law as it was originally presented by Biot and Savart. And then we will generalize it.

### 5.5.3 Biot–Savart Law

If we limit considerations to the case of the magnetic field around a long, thin, straight wire, we have from 5.41

$$\mathrm{curl}\,\mathbf{B} = \mu_0 \mathbf{J}. \tag{5.42}$$

Using (5.17) and (A.16) in (5.42) we have

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{A} = \mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{5.43}$$

If we assume that $\mathrm{div}\,\mathbf{A} = 0$ (5.43) becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{5.44}$$

which is a *vector Poisson's Equation*.

A vector Poisson's Equation is three Poisson's Equations, one for each of the components of $\mathbf{A}$, i.e.

$$\nabla^2 A_\mu = -\mu_0 J_\mu \tag{5.45}$$

for $\mu = 1 - 3$. Each of these involves only the corresponding component of the current density vector. If we can establish this result in general there will be a considerable simplification in our work. We will then have a single equation to solve for all electrostatic and magnetostatic problems with sources.

In Sect. 5.9 we will discuss the choice of $\operatorname{div} \mathbf{A} = 0$. Here we simply introduce our choice as an assumption.

We solved Poisson's equation in Sect. 2.6. The solution of (5.44) is found from the separate solutions of (5.45) and summing over components. The result is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}'}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V', \tag{5.46}$$

where we have written $\mathbf{J}' = \mathbf{J}(\mathbf{r}')$. At this stage in our development the volume $V$ in (5.46) includes only the central conductor carrying the current density $\mathbf{J}'$.

We obtain the magnetic field from (5.46) using (5.17). Because the integral is over primed (source) coordinates and the curl operates only on unprimed coordinates, we can take the curl operator inside the integral in (5.46). Then using the vector identity $\operatorname{curl}(\varphi \mathbf{F}) = \operatorname{grad}\varphi \times \mathbf{F} + \varphi \operatorname{curl} \mathbf{F}$ (see(A.21)), and noting that $J'$ is a function only of the primed coordinates, we have

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_V \operatorname{grad}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) \times \mathbf{J}' \mathrm{d}V' \\ &= \frac{\mu_0}{4\pi} \int_V \mathbf{J}' \times \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}V'. \end{aligned} \tag{5.47}$$

The volume $V$ is defined by the region of space in which $\mathbf{J}' \neq \mathbf{0}$, which is the interior of a long thin conductor. The differential volume of this conductor is $\mathrm{d}V' = \mathrm{d}\ell' \mathrm{d}a'$, where $\mathrm{d}\ell'$ is a differential length along the conductor and $\mathrm{d}a'$ is the differential area of the conductor cross section perpendicular to the vector $\mathrm{d}\boldsymbol{\ell}'$. Since $\mathbf{J}' = J'(\mathrm{d}\boldsymbol{\ell}'/\mathrm{d}\ell')$, we have

$$\mathbf{J}' dV' = J' \mathrm{d}a' \mathrm{d}\boldsymbol{\ell}'. \tag{5.48}$$

Then, using (5.3), we can integrate over the area of the conductor in (5.47) to obtain the current $I$ in the conductor. Then (5.47) becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\boldsymbol{\ell}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \tag{5.49}$$

where $C$ is the contour defined by the (thin) wire conductor. Similarly (5.46) becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{\mathrm{d}\boldsymbol{\ell}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{5.50}$$

In the integrals (5.49) and (5.46) the contours are not closed. This is because we have limited our treatment to long straight wires, as were used in Ampère's 1820 experiment.

Recall that we have implictly ignored the fact that someplace the circuit must be closed. We assume that this closing of the circuit is sufficiently far from the region we are considering that we can ignore any stray magnetic fields.

Equation (5.49), which is the curl with respect to the field coordinates of (5.50), is the *Biot–Savart Law* for the magnetic field induction from a straight line conductor. Biot and Savart first presented this as the differential magnetic field (induction) at the point $\mathbf{r}$ resulting from the differential element of current $I\,\mathrm{d}\boldsymbol{\ell}'$ (see Chap. 1). From (5.49) this differential field is

$$\mathrm{d}\mathbf{B}\left(\mathbf{r}\right)=\frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\boldsymbol{\ell}'\times\left(\mathbf{r}-\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \tag{5.51}$$

Equation (5.51) is then the original form of the Biot–Savart Law as it was originally presented to the French Academy of Science in 1820.

The original presentation by Biot and Savart to the French Academy was also limited to long straight wires ([97], p. 86). We can obtain the presently accepted form of the Biot–Savart Law by introducing a superposition Ansatz as we did in our development of the electrostatics field equations (see Sect. 3.3).

## 5.6 Superposition

A superposition Ansatz was introduced in later work by Biot, Savart, and Pierre Simon Marquis de Laplace. The final mathematical formulation was due to Laplace ([36], p. 187).

The superposition Ansatz is also a physical necessity. We pointed out that for current confined to flow in conductors, such as wires in a laboratory, the current density vectors $\mathbf{J}$ must form closed loops. In the general time independent case $\partial\rho/\partial t = 0$. Charge conservation (5.11) then requires that $\mathrm{div}\,\mathbf{J} = 0$ for current densities in more general regions of space. If $\mathrm{div}\,\mathbf{J} = 0$ the current density vectors must form closed loops in these more general regions, if the situation is time independent. Isolated lines of current simply cannot exist in the time independent case. Therefore we must have a theory that includes curved conductors.

Basically the superposition Ansatz for the magnetostatic field is the same as that for the electrostatic field. In both instances the Ansatz results in a field from a sum of sources, which is equal to the sum of fields arising from each of the sources separately. The sources of the magnetic field are, however, interconnected differential lengths of conductors (wires) carrying currents rather than differential volumes of charged matter. The visual pictures, therefore, differ.

In Fig. 5.6 we have an illustration of the differential magnetic field induction $\mathrm{d}\mathbf{B}$ resulting from a differential length of wire $\mathrm{d}\boldsymbol{\ell}'$ carrying a current $I$. This is the situation described by (5.51). And in 5.7 we have drawn the contributions to the magnetic field induction $\mathrm{d}\mathbf{B}_1$ and $\mathrm{d}\mathbf{B}_2$ resulting from two infinitesimal lengths of a curved conductor $\mathrm{d}\boldsymbol{\ell}_1'$ and $\mathrm{d}\boldsymbol{\ell}_2'$.

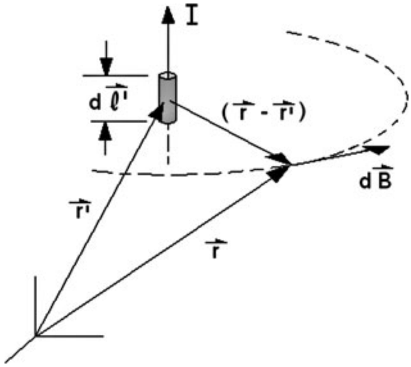



**Fig. 5.6** The differential magnetic field d**B** produced by the differential length d$\boldsymbol{\ell}'$ of a straight wire. The plane in which d**B** lies is perpendicular to d$\boldsymbol{\ell}'$

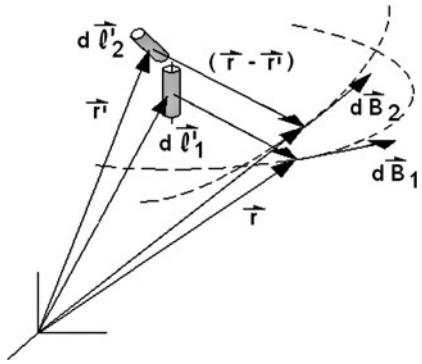



**Fig. 5.7** Two differential magnetic fields d$\mathbf{B}_{1,2}$ produced by two differential lengths d$\boldsymbol{\ell}_{1,2}$ of a curved wire carrying a current. To avoid cluttering the drawing we have not labeled the separate position vectors

According to the superposition Ansatz the magnetic field induction from $\mathrm{d}\boldsymbol{\ell}_1 + \mathrm{d}\boldsymbol{\ell}_2$ is the vector sum $\mathrm{d}\mathbf{B}_1 + \mathrm{d}\mathbf{B}_2$. We may use superposition in the same way to provide a formulation for the vector potential from a general current density vector.

The drawing in Fig. 5.7 is purely illustrative. We have separated the fields $\mathrm{d}\mathbf{B}_1$ and $\mathrm{d}\mathbf{B}_2$ slightly because they lie in planes perpendicular to the segments $\mathrm{d}\boldsymbol{\ell}_1$ and $\mathrm{d}\boldsymbol{\ell}_2$ of the conducting wire. The segments are, however, infinitesimal and changes are not abrupt.

We have also made no attempt to draw the field resulting from superposition in Fig. 5.7. Nor have we attempted to label all of the position vectors.

Using (5.51) the superposition Ansatz finally allows us to write

$$\mathbf{B}(r) = \lim_{\mathrm{d}\mathbf{B}_\mathrm{j} \to \mathbf{0} \text{ and } \mathrm{N} \to \infty} \sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{d}\mathbf{B}_\mathrm{j}, \tag{5.52}$$

which is the Riemann integral

$$\mathbf{B}(r) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathrm{d}\boldsymbol{\ell}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{5.53}$$

for the entire magnetic field induction in the region of space near the now arbitrarily curved wire. The integral in (5.53), which is the presently accepted form of the Biot–Savart Law, is a closed contour integral.

The superposition Ansatz has allowed us to reform the volume $V$ in (5.47) for the very thin, straight wire of Ampère's apparatus into the volume of a conductor with arbitrary shape and dimensions without changing the basic mathematical form of (5.47). In the same way (5.46) for the vector potential is also unchanged in mathematical form, while the volume over which the integration is performed takes on a general form. We now have

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\mathbf{J}'}{\left|\mathbf{r}-\mathbf{r}\,'\right|}\mathrm{d}V' \tag{5.54}$$

for any *arbitrary volume* $V$.

## 5.7 Multipole Expansion

In Sect. 4.5 we carried out a multipole expansion of the electrostatic field at a large distance from a localized distribution of charge. There we pointed out that in applications we are often not interested in calculating the potential from known charge distributions. Rather we may have experimental measurements of dipole and quadrupole moments from which we attempt to estimate the charge density. The same is true here.

We may be able to measure magnetic fields at relatively large distances from localized current densities in regions of possibly molecular size. In these situations our interest is in finding the actual form of the current density from the field measurements.

In the case of the electrostatic field we expanded the solution to Poisson's Equation for the scalar potential in terms of the size of the charge distribution compared to the distance to the observation point $\mathbf{r}'/r$. Here we will expand the vector potential, which also satisfies Poisson's Equation. The expansion of (5.54) is then the same form as that we considered in Sect. 4.5. In the case of the vector potential, however, we carry out the expansion of $1/\left|\mathbf{r}-\mathbf{r}'\right|$ only to *first order* in $\mathbf{r}'/r$, which is (see (4.19))

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = \frac{1}{r} + \frac{\mathbf{r}}{r^2}\cdot\frac{\mathbf{r}'}{r}. \tag{5.55}$$

The multipole expansion for the vector potential (5.54) is then

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi r}\int_{\mathrm{V}} \mathbf{J}'\mathrm{d}V' + \frac{\mu_0}{4\pi r^3}\mathbf{r}\cdot\int_{\mathrm{V}} \mathbf{r}'\mathbf{J}'\mathrm{d}V'. \tag{5.56}$$

We now wish to convert the integrals on the right hand side of (5.56) to simpler forms. Our general strategy will be to use Gauss' Theorem and the fact that $\operatorname{div}\mathbf{J} = 0$ in the time independent case. We will then write the integrands in terms of a divergence.

The first integrand on the right hand side of (5.56) is simply $\mathbf{J}'$, which can be written as

$$\begin{aligned}\hat{e}_\mu \operatorname{div}'\left(x'_\mu \mathbf{J}'\right) &= \hat{e}_\mu \operatorname{grad}' x'_\mu \cdot \mathbf{J}' + \hat{e}_\mu x'_\mu \operatorname{div}' \mathbf{J}' \\ &= \hat{e}_\mu J'_\mu = \mathbf{J}' \end{aligned} \tag{5.57}$$

since $\operatorname{div}'\mathbf{J}' = 0$. Then, with (5.57) and Gauss' Theorem, the first integral on the right hand side of (5.56) is

$$\begin{aligned}\int_{\mathrm{V}} \mathbf{J}' \mathrm{d}V' &= \hat{e}_\mu \int_{\mathrm{V}} \operatorname{div}'\left(x'_\mu \mathbf{J}'\right) \mathrm{d}V' \\ &= \hat{e}_\mu \oint_{\mathrm{S}} x'_\mu \mathbf{J}' \cdot \mathrm{d}\mathbf{S}' \\ &= \mathbf{0}, \end{aligned} \tag{5.58}$$

since any current on the surface of the volume $V$ must be parallel to the surface, i.e. perpendicular to $\mathrm{d}\mathbf{S}$. The first term on the right hand side of (5.56) then vanishes.

The second integral on the right hand side of (5.56) we shall write as

$$\mathbf{r} \cdot \int_{\mathrm{V}} \mathbf{r}'\mathbf{J}' \mathrm{d}V' = \int_{\mathrm{V}} \mathbf{r} \cdot \mathbf{r}'\mathbf{J}' \mathrm{d}V',$$

since $\mathbf{r}$ is a constant as far as the integration is concerned. In subscript notation the dyadic product is

$$\mathbf{r}'\mathbf{J}' = \hat{e}_\mu \hat{e}_\nu x'_\mu J'_\nu.$$

And the scalar product of $\mathbf{r}$ with $\mathbf{r}'\mathbf{J}'$ is

$$\begin{aligned}\mathbf{r} \cdot \mathbf{r}'\mathbf{J}' &= (\hat{e}_\lambda x_\lambda) \cdot \hat{e}_\mu \hat{e}_\nu x'_\mu J'_\nu \\ &= (\hat{e}_\nu)\, \delta_{\lambda\mu} x_\lambda x'_\mu J'_\nu. \end{aligned} \tag{5.59}$$

With (5.59) The second integral then becomes

$$\int_{\mathrm{V}} \mathbf{r} \cdot \mathbf{r}'\mathbf{J}' \mathrm{d}V' = (\hat{e}_\nu)\, \delta_{\lambda\mu} x_\lambda \int_{\mathrm{V}} x'_\mu J'_\nu \mathrm{d}V'. \tag{5.60}$$

We are interested then in the integral of the product $x'_\mu J'_\nu$ over $V'$. This we can obtain with rather foresightful use of Gauss' Theorem.

First we note that (A.19), with $\operatorname{div}'\mathbf{J}' = 0$, is

$$\mathrm{div}'\left(x'_\mu x'_\nu \mathbf{J}'\right) = \left[\mathrm{grad}'\left(x'_\mu x'_\nu\right)\right]\cdot\mathbf{J}'. \tag{5.61}$$

And, since

$$\mathrm{grad}'\left(x'_\mu x'_\nu\right) = \hat{e}_\mu x'_\nu + x'_\mu \hat{e}_\nu, \tag{5.62}$$

we have

$$\left[\mathrm{grad}'\left(x'_\mu x'_\nu\right)\right]\cdot\mathbf{J}' = x'_\nu J'_\mu + x'_\mu J'_\nu. \tag{5.63}$$

Now from (5.61) and (5.63) and using Gauss' Theorem we have

$$\begin{aligned}\int_{\mathrm{V}} \mathrm{div}'\left(x'_\mu x'_\nu \mathbf{J}'\right)\mathrm{d}V' &= \int_{\mathrm{V}}\left(x'_\nu J'_\mu + x'_\mu J'_\nu\right)\mathrm{d}V' \\ &= \oint_{\mathrm{S}} x'_\mu x'_\nu \mathbf{J}'\cdot\mathrm{d}S' = 0.\end{aligned} \tag{5.64}$$

since the current density is only parallel to the surface of the volume $V$.

Therefore

$$\int_{\mathrm{V}} x'_\nu J'_\mu \mathrm{d}V' = -\int_{\mathrm{V}} x'_\mu J'_\nu \mathrm{d}V'. \tag{5.65}$$

Equation (5.65) is the little gem we wanted from our foresightful use of Gauss' Theorem.

With (5.65) we can write (5.60) as

$$\int_{\mathrm{V}} \mathbf{r}\cdot\mathbf{r}'\mathbf{J}'\mathrm{d}V' = \frac{1}{2}\left(\hat{e}_\nu\right)\delta_{\lambda\mu}x_\lambda \int_{\mathrm{V}}\left(x'_\mu J'_\nu - x'_\nu J'_\mu\right)\mathrm{d}V'. \tag{5.66}$$

And, using the *bac* − *cab* rule,

$$\begin{aligned}\frac{1}{2}\left(\hat{e}_\nu\right)\delta_{\lambda\mu}x_\lambda\left(x'_\mu J'_\nu - x'_\nu J'_\mu\right) &= \frac{1}{2}\hat{e}_\nu\left(x_\mu x'_\mu J'_\nu - x'_\nu x_\mu J'_\mu\right) \\ &= \frac{1}{2}\left[\mathbf{J}'\left(\mathbf{r}\cdot\mathbf{r}'\right) - \mathbf{r}'\left(\mathbf{r}\cdot\mathbf{J}'\right)\right] \\ &= -\frac{1}{2}\mathbf{r}\times\left(\mathbf{r}'\times\mathbf{J}'\right).\end{aligned} \tag{5.67}$$

Therefore

$$\mathbf{r}\cdot\int_{\mathrm{V}}\mathbf{r}'\mathbf{J}'\mathrm{d}V' = -\frac{1}{2}\mathbf{r}\times\int_{\mathrm{V}}\left(\mathbf{r}'\times\mathbf{J}'\right)\mathrm{d}V'. \tag{5.68}$$

The general definition of the magnetic moment of a current density is

$$\boxed{\mathbf{m} = (1/2)\int_{\mathrm{V}}(\mathbf{r}'\times\mathbf{J}')\mathrm{dV}'.} \tag{5.69}$$

And we may identify the magnetic moment density in the volume $V$ as

$$\mathbf{M} = \frac{1}{2}\left(\mathbf{r}' \times \mathbf{J}'\right). \tag{5.70}$$

This magnetic moment density in (5.70) is a result of the distribution of current in the volume $V$.

With (5.58) and (5.68) our multipole expansion for the vector potential (5.56) becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{5.71}$$

The magnetic field induction from the vector potential in (5.71) potential is (see exercises)

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[3\left(\frac{\mathbf{m} \cdot \mathbf{r}}{r^5}\right)\mathbf{r} - \frac{\mathbf{m}}{r^3}\right]. \tag{5.72}$$

## 5.8 Divergence and Curl of B

We could derive the magnetic field equations by taking the divergence and the curl of **B** in (5.53). This approach would be mathematically the same as that used in our study of the electrostatic field. However, because of its mathematical structure, obtaining equations for the divergence and the curl of (5.53) requires tedious and unnecessary vector manipulations. With the results of Sect. 5.6 we have an avenue to a more elegant derivation using the vector potential in (5.54).

The first field equation

$$\operatorname{div}\mathbf{B} = 0 \tag{5.73}$$

(see (5.16)) remains unaltered. The reasons for its validity have not been changed in the discussion of Ampère's results.

Equation (5.73) guarantees that **B** can be written as

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{5.74}$$

where **A** is a vector field. According to Helmholtz' Theorem we must also specify div **A** as well. In Sect. 5.9 we show that the choice of div **A** is arbitrary. Once we have specified div **A** the vector potential becomes a completely specified vector field and we may work entirely with **A** as equivalent to **B**.

Because the solution of Poisson's Equation is unique (5.54) is equivalent to the statement

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}, \tag{5.75}$$

where now the current density **J** is a *general* (divergenceless[5]) vector field quantity. Using the vector identity (A.16) in (5.75) and choosing $\operatorname{div}\mathbf{A} = 0$ we have

$$\nabla^2\mathbf{A} = -\operatorname{curl}\operatorname{curl}\mathbf{A} = -\mu_0\mathbf{J}. \tag{5.76}$$

[5] A vector field for which the divergence is zero is called a *solenoidal* vector field.

Then with (5.74) equation (5.76) becomes

$$\boxed{\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J}} \tag{5.77}$$

Equation (5.77), which is *Ampère's Law*, is the second magnetostatic field equation. That (5.77) is identical to (5.42) for a long straight wire follows, since the current density for a long straight wire is a special form of **J** in (5.77).

There has been no lack of mathematical rigor in our choice to use the vector potential rather than the magnetic field induction in our derivation of (5.77). For the interested reader, however, the field (5.73) and (5.77) are obtained from (5.47) in Appendix C.

## 5.9 Gauge Transformation

The specific value we choose for div **A** is called the *gauge*. In Sect. 5.5.3 we chose div $\mathbf{A} = 0$. To keep our development of the magnetic field fluid we elected not to discuss this choice there. We only noted that the choice div $\mathbf{A} = 0$ resulted in Poisson Equations for both $\varphi$ and **A**, which was a considerable simplification because we have a solution to Poisson's Equation. We shall now consider this question in detail.

The choice of div **A** is not a trivial issue. Fortunately most authors treat this choice carefully (e.g. [87], p. 102, [58], p. 54, and [48], p. 176). We shall be careful here as well.

The issue is one of complete determination of vector fields. We know from Helmholtz' Theorem that the magnetic field induction **B** is completely determined by the field (5.73) and (5.77). And Oersted's Result, expressed in (5.73), allows us to identify **B** as the curl of the vector potential **A**. If we had no further interest in the vector potential, however, and worked exclusively with **B** then **A** could be ignored.

But we have already used **A** in our derivation of the second magnetostatic field equation. And at a later point we will consider the wave equations for the scalar and vector potentials to be a complete representation of Maxwell's Equations. So we must consider the complete determination of **A** to be as critical as that of **B**.

The difficulty is that curl $\mathbf{A} = \mathbf{B}$ is only one of the two field equations required for the complete determination of **A**. We must have an equation for div **A** in order to specify **A** completely.

Because curl grad $f = \mathbf{0}$ for any scalar function $f$ whatsoever, knowing the curl of **A** leaves **A** indeterminate to within the gradient of a scalar function. That is if

$$\mathbf{A}_1 = \mathbf{A} + \operatorname{grad} f, \tag{5.78}$$

then,

$$\begin{aligned}\mathbf{B} &= \operatorname{curl}\mathbf{A}_1 = \operatorname{curl}\mathbf{A} + \operatorname{curl}\operatorname{grad} f \\ &= \operatorname{curl}\mathbf{A}\end{aligned}$$

We have then an infinity of possible choices for **A** each differing by the gradient of a scalar potential $f$. And our question becomes one of specifying $f$. This question will be resolved by our choice for div **A**.

We are dealing at this point in our development with only spatial dependencies. The divergence of **A** is then at most a function of spatial coordinates. We specify this function as $g(\mathbf{r})$. Then, since $\operatorname{div}\mathbf{A}_1 = \operatorname{div}\mathbf{A}$, from (5.78) we have

$$\begin{aligned}\operatorname{div}\mathbf{A}_1 &= g(\mathbf{r}) \\ &= g(\mathbf{r}) + \nabla^2 f.\end{aligned} \tag{5.79}$$

Therefore

$$\nabla^2 f = 0 \tag{5.80}$$

and $f$ satisfies Laplace's Equation regardless of the choice of $g(\mathbf{r})$. Because the solution to Laplace's Equation is unique, the same result for $f$ is obtained for any and all choices of $g(\mathbf{r})$. And only $f$ can have any effect on the value of **A**.

We have then complete freedom in our choice of $g(\mathbf{r}) = \operatorname{div}\mathbf{A}$. We make this choice, therefore, on the basis of mathematical simplification of the equations for the scalar and vector potentials. Here we have chosen what is called the Coulomb Gauge

$$\operatorname{div}\mathbf{A} = 0, \tag{5.81}$$

and the components of **A** satisfy Poisson's Equation, as does $\varphi$.

## 5.10 The Static Field Equations

In this chapter we have finished our derivation of the static field equations, which, in differential form, are

$$\boxed{\begin{array}{ll}\operatorname{div}\mathbf{E} = \rho/\varepsilon_0 & \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{E} = \mathbf{0} & \operatorname{curl}\mathbf{B} = \mu_0\mathbf{J},\end{array}} \tag{5.82}$$

together with charge conservation for the static case

$$\boxed{\operatorname{div}\mathbf{J} = 0.} \tag{5.83}$$

We may integrate the divergence equations over a volume and the curl equations over an open area defined by a contour and then apply Gauss' and Stoke's Theorems to obtain the integral form of (5.82) and (5.83) as

$$\boxed{\begin{array}{cc} \oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = (1/\varepsilon_0)\int_V \rho \mathrm{d}V & \oint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0 \\ \oint_C \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = 0 & \oint_C \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_a \mathbf{J}\cdot \mathrm{d}\mathbf{a}, \end{array}} \tag{5.84}$$

together with charge conservation for the static case

$$\boxed{\oint_S \mathbf{J}\cdot\mathrm{d}\mathbf{S} = 0.} \tag{5.85}$$

The second magnetostatic field equation

$$\oint_C \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_a \mathbf{J}\cdot \mathrm{d}\mathbf{a} \tag{5.86}$$

is the final form of *Ampère's Circuital Law*. The contour and the area are now arbitrary.

The curl $\mathbf{E}$ and div $\mathbf{B}$ equations have resulted in the identification of scalar and vector potentials $\varphi$ and $\mathbf{A}$ which satisfy Poisson's Equations

$$\nabla^2\varphi = -\frac{1}{\varepsilon_0}\rho \tag{5.87}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{5.88}$$

provided we introduce the Coulomb gauge

$$\mathrm{div}\,\mathbf{A} = 0. \tag{5.89}$$

The Poisson Equations for the potentials have the solutions

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' \tag{5.90}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'. \tag{5.91}$$

From these potentials we can calculate the electrostatic and electromagnetic fields as

$$E = -\operatorname{grad}\varphi \tag{5.92}$$

$$B = \operatorname{curl}\mathbf{A}. \tag{5.93}$$

## 5.11 Summary

The magnetostatic field has a structural, and, hence, mathematical depth not present in the case of the electrostatic field. If we chose we could still describe the force between the current and the magnetic needle in Oersted's experiment and between the wires in Ampère's experiment in terms of action at a distance. But the peculiar direction of the force on the magnetic needle and of the force on a moving charge $q\mathbf{v} \times \mathbf{B}$ makes the introduction of the magnetostatic field logical rather than just convenient.

Because of the physics of the magnetic field we have found reasoning with the vector potential to be much simpler than direct use of the magnetic field itself. The magnetic force on currents or charges is based on the magnetic field induction. And our intuition will be based on the magnetic field induction and not on the vector potential. However, the vector potential will play an increasingly important role in our development.

We introduced the Coulomb Gauge $\operatorname{div}\mathbf{A} = 0$ to guarantee that the potentials each satisfied Poisson's Equation. And we devoted the final section of the chapter to an explanation of gauge and gauge transformation. We will change the choice of gauge as we study time dependent fields and waves. But the concept will remain.

## Exercises

**5.1.** In Chap. 1 we noted that Maxwell was able to calculate the speed of the waves he had predicted from data obtained in the electromagnetic experiments of Wilhelm Weber and Friedrich Kohlrausch. These data were for $\varepsilon_0$ and $\mu_0$. Maxwell's prediction was for a wave passing through the aether with a speed of $1/\sqrt{\varepsilon_0\mu_0}$. You now have values for $\varepsilon_0$ and $\mu_0$ and can perform the calculation Maxwell performed, although with probably less enthusiastic anticipation. Obtain a numerical value for the product $\varepsilon_0\mu_0$ and compare it to the value for the value of $1/c^2$, where $c$ is the speed of light. The value obtained by Hippolyte Fizeau in air was ($3.14858 \times 10^8$ m s$^{-1}$), the more accurate value found by Léon Foucault was ($3.08 \times 10^8$ m s$^{-1}$), and the present experimental value is $c = 2.99792458 \times 10^8$ m s$^{-1}$.

**5.2.** What is the magnetic induction resulting from the vector potential

$$\mathbf{A} = -\hat{e}_\mathrm{x}\frac{B}{2}y + \hat{e}_\mathrm{y}\frac{B}{2}x?$$

**5.3.** Show that the magnetic induction resulting from the vector potential

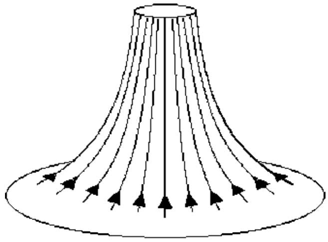


**Fig. 5.8** Magnetic induction with $z-$dependence

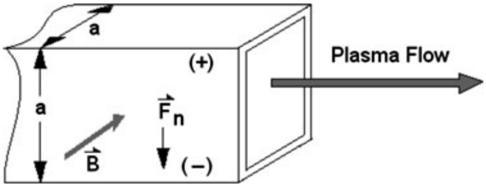



**Fig. 5.9** MHD generator

$$\mathbf{A} = -\hat{e}_x y \frac{B}{2} \exp(+\beta z) + \hat{e}_y x \frac{B}{2} \exp(+\beta z)$$

has the form shown in Fig. 5.8.

To discover the form of the magnetic field induction as a function of $z$ you will need to consider the geometrical form taken by $B(x, y)$ when $z$ =constant. And then you will need to ask for the *size* of this geometrical form for a *constant value* of $|\mathbf{B}|$ as $z$ *increases*.

**5.4.** In the 1960s and 1970s we were exploring many ideas for efficient and clean energy conversion. One of these was a magnetohydrodynamic (MHD) generator that worked on the Hall effect (see Sect. 1.9.3). We accelerated a gas thermodynamically to high velocity and then (partially) ionized it to form a low density plasma. We then passed this high velocity plasma into a region in which there was a uniform magnetic field of induction **B**. We have drawn the situation in Fig. 5.9.

We may ignore the deflection of the ions by the magnetic field because they are so massive compared to the electrons. You know the drift velocity $\mathbf{v}_e$ of the electrons. This is the flow velocity of the gas. And you know the electron density $n_e$. The channel width is $a$ and the channel length in the region in which $B \neq 0$ is $\ell$.

As in Hall's experiment to determine the sign of the charges flowing in a conductor, the magnetic force acts on the electrons, which is a non-electrostatic force. This non-electrostatic force does work on the charges moving them to one or the other side of the channel resulting in a potential difference. The electrical current density in a plasma is related to the electric field by $\mathbf{J} = \sigma \mathbf{E}$, where $\sigma$ is the conductivity of the plasma. It is proportional to the electron density. You may assume that you know $\sigma$.

(a) What is the Hall voltage, which is due to the charge separation in the direction of the non-electrostatic force?

(b) What is the Hall current, which is the current driven by the non-electrostatic force?
(c) What is the power output per unit volume of the MHD generator? This is the product of Hall current and Hall voltage. Show that the units are correct.

The channel size is limited by practical considerations. The thermodynamic acceleration of the flow (before ionization) is produced in a nozzle after the gas is heated. The flow will then be supersonic. The magnetic field is produced by coils with an iron core. How would you increase MHD power? On what does the output depend?

**5.5.** We can find a $\delta$−function representation of a line current source along the $z$−axis beginning with a cylinder with current density on the surface of radius $\varepsilon$, which is proportional to $\delta\,(r-\varepsilon)$, and then finding the limit as $\varepsilon \to 0$. Then we can use this representation in the evaluation of the vector potential from the solution of Poisson's Equation

$$\mathbf{A}\,(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\mathbf{J}'}{|\mathbf{r}-\mathbf{r}\,'|}\mathrm{d}V'.$$

(a) Write

$$\mathbf{J} = \lim_{\varepsilon\to 0} I_0\alpha\,(r)\,\delta\,(r-\varepsilon)\,\hat{e}_{\mathrm{z}}$$

for the line current source and show that

$$\alpha\,(r) = \frac{1}{2\pi r}.$$

(b) Then use this result to show that

$$\mathbf{A} = \hat{e}_{\mathrm{z}}A_{\mathrm{z}} = \hat{e}_{\mathrm{z}}\left[\frac{I_0\mu_0}{2\pi}\ln\,(r) + \text{constant}\right]$$

for a very thin wire carrying a current $I_0$.

**5.6.** Show that curl $\mathbf{A}$ with

$$\mathbf{A} = \left(\mu_0/\left(4\pi r^2\right)\right)\mathbf{m}\times\hat{e}_{\mathrm{r}}$$

yields

$$\mathbf{B} = \left(m\mu_0/\left(4\pi r^3\right)\right)\left[\hat{e}_{\mathrm{r}}2\cos\phi + \hat{e}_{\phi}\sin\phi\right].$$

**5.7.** Show that

$$\int_{\mathrm{a}}\mathbf{B}\cdot\mathrm{d}\mathbf{a} = \oint_{\mathrm{C}}\mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell},$$

where $\mathbf{A}$ is the vector potential and $\mathbf{B}$ is the magnetic field induction.

**5.8.** Use the general definition of the magnetic moment

$$\mathbf{m} = \frac{1}{2}\int_{V} \left(\mathbf{r}' \times \mathbf{J}'\right) dV'$$

and the current density for the ring of wire

$$J = I_0 \frac{1}{r} \delta\left(r - R\right) \delta\left(\phi - \frac{\pi}{2}\right) \hat{e}_{\vartheta}$$

to show that magnetic moment of the ring of current is $\pi R^2 I_0$ in the direction perpendicular to the plane of the wire.

[You will want to perform the calculation in spherical coordinates.]

**5.9.** The result of the multipole expansion of the vector potential was

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}.$$

Find the general form of the magnetic field induction from this vector potential.

{Answer: $\mathbf{B} = \left(\mu_0/4\pi\right)\left[3\left(\mathbf{m}\cdot\mathbf{rr}/r^5\right) - \left(\mathbf{m}/r^3\right)\right]$}

**5.10.** The long cylindrical solenoid is an example of a situation that is traditionally considered to be very simple, but which is not if considered in detail.

(a) Show that application of Ampère's Circuital Law to a long cylindrical solenoid with $N_\lambda$ turns of wire per unit length carrying a current $I_0$ yields an axial magnetic field induction inside the solenoid of $B_z = \mu_0 N_\lambda I_0$ if we neglect the magnetic field induction outside and close to the surface of the solenoid and assume that the magnetic field induction is constant over the cross section of the solenoid. .

(b) Show that this magnetic field induction requires a vector potential

$$\mathbf{A} = \hat{e}_{\vartheta}\frac{\mu_0 N_\lambda I_0}{2} r.$$

(c) Show that the vector potential above requires a current density within the solenoid of

$$\mathbf{J} = -\hat{e}_{\vartheta}\frac{N_\lambda I_0}{2}\frac{1}{r}.$$

The difficulty must be in the assumptions made in part (a). Comment on this.

**5.11.** The cylindrical solenoid is symmetric in azimuthal angle. So we must require that $\partial/\partial\vartheta = 0$ and that there is no magnetic field induction in the $\hat{e}_{\vartheta}$.

(a) What requirements do these limitations place on $\partial A_{\vartheta}/\partial z$, $\partial A_{r}/\partial z$, and $\partial A_{z}/\partial r$?

(b) Show that $\operatorname{div}\mathbf{B} = 0$ requires that $\partial A_{\vartheta}/\partial z \neq 0$.

# Chapter 6
# Applications of Magnetostatics

*Thunder in the hands of nature is electricity in the hands of physicists.*

*Francois Arago*

## 6.1 Introduction

This chapter we shall devote entirely to applications of the Biot–Savart Law and the solution of Poisson's Equation for the vector potential. All that we will do here depends entirely on the results of the preceding chapter in which we developed the basic theory of magnetostatic fields. We have chosen to separate this chapter from the theoretical development primarily to keep the chapter lengths reasonable. We hope that this will benefit the reader.

The integrals that appear in the Biot–Savart Law are complicated by the cross product. This requires some care, but does not necessarily make them any more formidable. The integrals required in the calculation of the vector potential are, in principle, simpler than those in the Biot–Savart Law. However, we cannot stop with the calculation of the vector potential. We must take the curl of the vector potential to obtain the magnetic field induction. Because derivatives are sensitive to variations in a function, this may limit some of the approximations that can be used in the calculation of the vector potential.

The examples we have selected for this chapter are intended only to provide an introduction into the calculation of magnetic fields and the use of current arrangements to create fields of practical form.

## 6.2 Biot–Savart Law

In spite of the awkward appearance of the vector cross product in the integrand, it is often not difficult to use the Biot–Savart Law directly to calculate the magnetic field induction. This is particularly the case when the geometry of the current

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_6, 

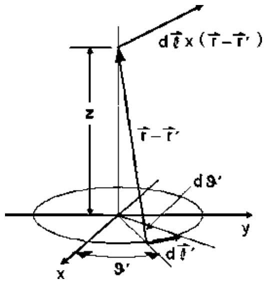


**Fig. 6.1** ire loop of radius $R$ carrying a constant current $I_0$

arrangement is simple and the general spatial dependence of the induction is not required.

One example of a simple geometry is a ring of wire carrying a current. This is particularly simple if we ask only for the magnetic field induction on the axis of the ring.

*Example 6.1.* **Magnetic Field Induction from a Wire Loop**. As an example we consider a circular loop of wire with radius $R$ carrying a constant current $I_0$ and ask for the magnetic induction $\mathbf{B}$ at points on the axis of the loop. We have drawn the situation in Fig. 6.1.The position vector to the source, $\mathbf{r}'$, is

$$\mathbf{r}' = \hat{e}_{\mathrm{x}}\left(R\cos\vartheta'\right) + \hat{e}_{\mathrm{y}}\left(R\sin\vartheta'\right)$$

and the vector to the field point is

$$\mathbf{r} = \hat{e}_{\mathrm{z}} z.$$

Then

$$(\mathbf{r}-\mathbf{r}') = \hat{e}_{\mathrm{z}} z - \hat{e}_{\mathrm{x}}\left(R\cos\vartheta'\right) - \hat{e}_{\mathrm{y}}\left(R\sin\vartheta'\right)$$

and

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{z^2+R^2}.$$

The differential length along the wire is

$$\mathrm{d}\boldsymbol{\ell}' = R\mathrm{d}\vartheta'\left(-\hat{e}_{\mathrm{x}}\sin\vartheta' + \hat{e}_{\mathrm{y}}\cos\vartheta'\right).$$

So

$$\mathrm{d}\boldsymbol{\ell}' \times (\mathbf{r}-\mathbf{r}') = \hat{e}_{\mathrm{x}} z\cos\vartheta' + \hat{e}_{\mathrm{y}} z\sin\vartheta' + \hat{e}_{\mathrm{z}} R$$

Then the Biot–Savart law Chap. 5 equation (5.53) is

$$\mathbf{B}(z) = \frac{\mu_0 I_0 R}{4\pi}\int_0^{2\pi}\mathrm{d}\vartheta'\frac{\left(\hat{e}_{\mathrm{x}} z\cos\vartheta' + \hat{e}_{\mathrm{y}} z\sin\vartheta' + \hat{e}_{\mathrm{z}} R\right)}{\left(z^2+R^2\right)^{3/2}}$$
$$= \frac{\mu_0 I_0}{2}\frac{R^2}{\left(z^2+R^2\right)^{3/2}}\hat{e}_{\mathrm{z}}. \qquad (6.1)$$

For the planar current loop in Example 6.1 the *magnetic moment*

$$\mathbf{m} = \pi R^2 I_0 \hat{e}_z. \tag{6.2}$$

is the product of the current in the loop $I_0$, the area of the loop $\pi R^2$, and the unit vector $\hat{e}_z$ of the current loop. The sense of the magnetic moment is provided by the right hand rule with respect to the direction of the current flow.

We can then write the magnetic field induction (6.1) along the $z-$axis as

$$\mathbf{B}(z) = \frac{\mu_0 \mathbf{m}}{2\pi} \left(z^2 + R^2\right)^{-3/2}.$$

The magnetic moment of the planar loop (6.2) is a special case of a more general formulation of the magnetic moment

$$\mathbf{m} = \frac{1}{2} \int_V \left(\mathbf{r}' \times \mathbf{J}'\right) \mathrm{d}V', \tag{6.3}$$

which emerged from our multipole expansion in Chap. 5. And in the exercises in Chap. 5 we found that the identification we have made here also results from an application of (6.3) to a current loop.

## 6.3 Vector Potential

As an example of the practical use of the magnetic vector potential we ask for the vector potential for the circular wire ring of the preceding example.

Because we do not need to calculate the cross product $\mathrm{d}\boldsymbol{\ell}' \times (\mathbf{r} - \mathbf{r}')$ before performing the calculation, the integration appears less cumbersome. We require, however, the curl of the result to find the magnetic field induction. So we must, therefore, obtain a more complete form of the vector potential in order to use the result to calculate the magnetic field induction. Derivatives of a function are sensitive to any approximations we may make in evaluating the function.

If our goal is to find the magnetic field induction for a particular arrangement of currents, basing the calculation on the vector potential is not necessarily simpler. The role of the vector potential will, however, be increasingly important in our development of field theory. We are not then faced with simply a personal preference on which to base the calculation of a static magnetic field induction. We must gain familiarity with the vector potential.

*Example 6.2.* **Vector potential from a wire loop.** We consider again a circular loop of wire with radius $R$ carrying a constant current $I$ and ask for the vector potential $\mathbf{A}$ in the space surrounding the loop. Because of symmetry we locate the center of the wire loop at the origin of our coordinate system. We have drawn the arrangement in Fig. 6.2. The current density vector is

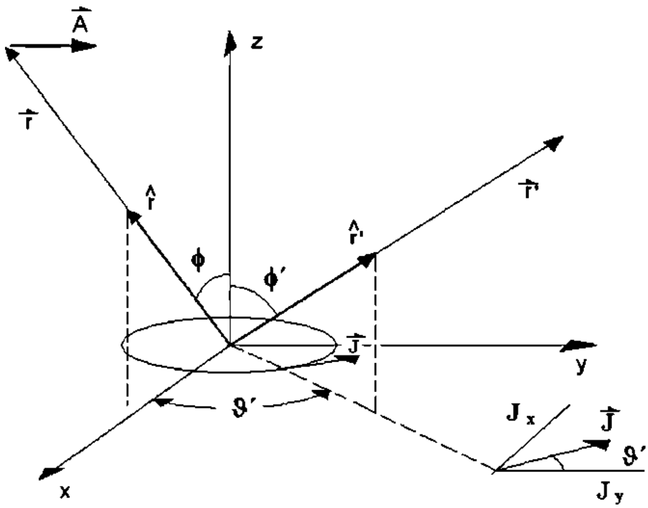



**Fig. 6.2** Wire loop of radius $R$ carrying a constant current $I_0$

$$\mathbf{J} = I_0 \alpha(r)\,\delta(r-R)\,\delta\left(\phi - \frac{\pi}{2}\right)\hat{e}_\vartheta$$

in a spherical coordinate system. We have included $\alpha(r)$ in the definition of $J$ to account for any possible dependence on the radial coordinate. The differential area in a plane perpendicular to $\hat{e}_\vartheta$ is

$$\mathrm{d}\mathbf{a} = r\mathrm{d}r\mathrm{d}\phi\hat{e}_\vartheta,$$

so the current in the loop is

$$\begin{aligned} I_0 &= I_0 \int_{\mathrm{r}=0}^{+\infty} \int_{\phi=0}^{2\pi} \alpha(r)\,\delta(r-R)\,\delta\left(\phi - \frac{\pi}{2}\right) r\mathrm{d}r\mathrm{d}\phi \\ &= I_0 R\alpha(R)\,. \end{aligned}$$

Therefore $\alpha(r) = 1/r$ and the current density in the loop is

$$\mathbf{J} = I_0 \frac{1}{r}\delta(r-R)\,\delta\left(\phi - \frac{\pi}{2}\right)\hat{e}_\vartheta.$$

For clarity and ease of representation we shall write the vectors to the source and field points in rectangular coordinates. These are the $(x, y, z)$ axes shown in Fig. 6.2.

Because of the rotational symmetry about the $z-$ axis we need only consider field points in a single plane containing the $z-$ axis. We choose this to be the $(x, z)$ plane. In this plane the field points are

$$\mathbf{r} = \hat{e}_\mathrm{x} r \sin\phi + \hat{e}_\mathrm{z} r \cos\phi.$$

The vector to the source points is the general vector

$$\mathbf{r}' = \hat{e}_x r' \sin\phi' \cos\vartheta' + \hat{e}_y r' \sin\phi' \sin\vartheta' + \hat{e}_z r' \cos\phi'.$$

Then

$$\begin{aligned}\mathbf{r} - \mathbf{r}' = {} & \hat{e}_x \left(r \sin\phi - r' \sin\phi' \cos\vartheta'\right) - \hat{e}_y r' \sin\phi' \sin\vartheta' \\ & + \hat{e}_z \left(r \cos\phi - r' \cos\phi'\right)\end{aligned}$$

The form of the current density vector will limit the range of source point variables $(r', \vartheta', \phi')$.

The distance between field and source point is

$$\begin{aligned}\left|\mathbf{r} - \mathbf{r}'\right| = {} & \Big[\left(r \sin\phi - r' \sin\phi' \cos\vartheta'\right)^2 \\ & + \left(-r' \sin\phi' \sin\vartheta'\right)^2 + \left(r \cos\phi - r' \cos\phi'\right)^2\Big]^{\frac{1}{2}} \\ = {} & \big[(r')^2 + r^2 - 2rr' \cos\phi \cos\phi' \\ & - 2rr' \cos\vartheta' \sin\phi \sin\phi'\big]^{\frac{1}{2}}.\end{aligned}$$

We note that the current density vector has no component in the $z$–direction. The vector potential will then have no component in the $z$–direction.

From the diagram of the current density vector $\mathbf{J}$ and the components $J_x$ and $J_y$ in the lower right hand corner of Fig. 6.2 we see that the current density represented in rectangular coordinates is

$$\mathbf{J} = -\hat{e}_x \left|\mathbf{J}\right| \sin\vartheta' + \hat{e}_y \left|\mathbf{J}\right| \cos\vartheta'.$$

The rectangular components of the vector potential are then

$$\begin{aligned}A_x &= -\frac{I_0\mu_0}{4\pi} \int_V \frac{\delta\left(r' - R\right) \delta\left(\phi' - \frac{\pi}{2}\right) \sin\vartheta' \left(r'\right)^2 \sin\phi' \mathrm{d}r' \mathrm{d}\phi' \mathrm{d}\vartheta'}{r' \left[(r')^2 + r^2 - 2rr' \cos\phi \cos\phi' - 2rr' \cos\vartheta' \sin\phi \sin\phi'\right]^{\frac{1}{2}}} \\ &= -\frac{I_0\mu_0}{4\pi} R \int_{\vartheta'=0}^{2\pi} \frac{\sin\vartheta' \mathrm{d}\vartheta'}{\left\{R^2 + r^2 - 2rR \cos\vartheta' \sin\phi\right\}^{\frac{1}{2}}} = 0, \qquad (6.4)\end{aligned}$$

and

$$\begin{aligned}A_y &= \frac{I_0\mu_0}{4\pi} \int_{V'} \frac{\delta\left(r' - R\right) \delta\left(\phi' - \frac{\pi}{2}\right) \cos\vartheta' \left(r'\right)^2 \sin\phi' \mathrm{d}r' \mathrm{d}\phi' \mathrm{d}\vartheta'}{r' \left[(r')^2 + r^2 - 2rr' \cos\phi \cos\phi' - 2rr' \cos\vartheta' \sin\phi \sin\phi'\right]^{\frac{1}{2}}} \\ &= \frac{I_0\mu_0}{4\pi} \frac{R}{r} \int_{\vartheta'=0}^{2\pi} \frac{\cos\vartheta' \mathrm{d}\vartheta'}{\left\{1 + (R/r)^2 - 2\left(R/r\right) \cos\vartheta' \sin\phi\right\}^{\frac{1}{2}}}. \qquad (6.5)\end{aligned}$$

The integral in (6.5) is not easy to perform in general.[1] We can, however, obtain approximations for large distances from the current loop compared to the loop radius, i.e. for $R/r \ll 1$.

Expanding part of the integrand of (6.5) we have

$$\left\{(1+\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)\cos\vartheta'\sin\phi\right\}^{-\frac{1}{2}}$$
$$=1+\frac{R}{r}\cos\vartheta'\sin\phi+\left(\frac{R}{r}\right)^2\left(\frac{3}{2}\cos^2\vartheta'\sin^2\phi-\frac{1}{2}\right)+O\left[\left(\frac{R}{r}\right)^3\right].$$

Then

$$A_{\mathrm{y}}=\frac{I_0\mu_0}{4\pi}\frac{R}{r}\left\{\int_{\vartheta'=0}^{2\pi}\cos\vartheta'\mathrm{d}\vartheta'+\frac{R}{r}\sin\phi\int_{\vartheta'=0}^{2\pi}\cos^2\vartheta'\mathrm{d}\vartheta'\right.$$
$$\left.+\left(\frac{R}{r}\right)^2\int_{\vartheta'=0}^{2\pi}\left(\frac{3}{2}\cos^3\vartheta'\sin^2\phi-\frac{1}{2}\cos\vartheta'\right)\mathrm{d}\vartheta'+O\left[\left(\frac{R}{r}\right)^3\right]\right\}$$
$$=\frac{I_0\mu_0}{4}\left(\frac{R}{r}\right)^2\sin\phi+O\left[\left(\frac{R}{r}\right)^4\right] \tag{6.6}$$

Recalling that we have used the symmetry of the problem to evaluate the vector potential only in the plane $(x,z)$, we realize that this $A_{\mathrm{y}}$ is actually the component $A_\vartheta$ in spherical coordinates. So

$$\mathbf{A}=\hat{e}_\vartheta\frac{I_0\mu_0}{4}\left(\frac{R}{r}\right)^2\sin\phi \tag{6.7}$$

to within $O\left[(R/r)^4\right]$.

We then find the magnetic field induction as

$$\mathbf{B}=\mathrm{curl}\,\mathbf{A}=\hat{e}_{\mathrm{r}}\frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(A_\vartheta\sin\phi\right)-\hat{e}_\phi\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\vartheta\right)$$
$$=\frac{I_0\mu_0}{4}\left[\hat{e}_{\mathrm{r}}\left(\frac{R}{r}\right)^2\frac{1}{r\sin\phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin^2\phi\right)-\hat{e}_\phi R^2\sin\phi\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\right]$$
$$=\frac{I_0R^2\mu_0}{4r^3}\left[\hat{e}_{\mathrm{r}}2\cos\phi+\hat{e}_\phi\sin\phi\right]$$
$$=\frac{m\mu_0}{4\pi r^3}\left[\hat{e}_{\mathrm{r}}2\cos\phi+\hat{e}_\phi\sin\phi\right] \tag{6.8}$$

[1] Gradshteyn and Ryzhik (2.571, 7.) [34]. The integral is tabulated as a sum of a generalized hypergeometric series and an elliptic integral of the thrid kind, provided $r > 2a$.

in spherical coordinates, where $m = I_0 \pi R^2$ is the magnetic moment of the current loop.

From Example 6.2 we retrieve the result found in Example 6.1 if we set $\phi$ equal to zero.

In terms of the magnetic moment **m** the vector potential in 6.7 is

$$\mathbf{A} = \left(\frac{\mu_0}{4\pi r^2}\right) \mathbf{m} \times \hat{e}_{\mathrm{r}}. \tag{6.9}$$

And from (6.8) the magnetic induction **B** from a current loop, written in terms of the magnitude of the magnetic moment $m$ of the loop, is

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3} \left[\hat{e}_{\mathrm{r}} 2\cos\phi + \hat{e}_\phi \sin\phi\right]. \tag{6.10}$$

Equation (6.10) bears some resemblance to the electric field arising from an electric dipole $\mathbf{p}_{\mathrm{d}}$, Chap. 4 (4.14), which we repeat here for continuity

$$\mathbf{E} = \frac{p_{\mathrm{d}}}{4\pi \varepsilon_0 r^3} \left[\hat{e}_{\mathrm{r}} 2\cos\phi + \hat{e}_\phi \sin\phi\right]. \tag{6.11}$$

This resemblance between the magnetic moment and the electric dipole moment has no practical implication at this point. When we discuss electric and magnetic fields in matter in Chap. 15 this similarity will, perhaps, help our understanding as we consider atomic properties. There we will construct our understanding of the electric and magnetic properties of matter in part on the properties of the atoms making up the matter.

Our present interest, however, is in constructing magnetic fields in space from macroscopic elements. The ring of current we studied in Examples 6.1 and 6.2 is such a macroscopic element we may use to construct magnetic fields of more general forms. The Helmholtz Coil is an arrangement of two current rings which produces a uniform magnetic field in a region of space, which is accessible for experiments.

*Example 6.3.* **Helmholtz Coil**. We form 2 wire coils by winding wire $N$ times around hoops of radius $R$. This results in a more practical wire loop than a single thick wire. We then place these coils along a single axis $z$ spacing them a distance $2a$ apart and, using an external power supply, cause a current $I_0$ to flow in the wire in each coil. The current in each coil is then $NI$. We have drawn this with the lines of magnetic induction in Fig. 6.3. This arrangement is called a Helmholtz coil (Hermann von Helmholtz).

The magnetic induction adds vectorially in the region between the two coils. The induction from a single coil located at the origin is (6.1). For the coil at $z = +a$ we have

$$B_{\mathrm{z}}^{(+)}(z) = \frac{\mu_0 \left(\pi R^2 N I_0\right)}{2\pi} \frac{1}{\left(R^2 + (z-a)^2\right)^{3/2}} \tag{6.12}$$

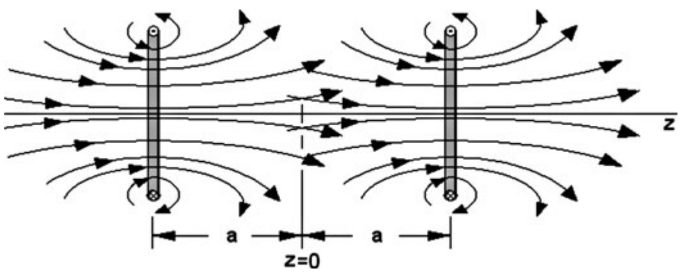



**Fig. 6.3** Helmholtz coils

and for the coil at $z = -a$,

$$B_z^{(-)}(z) = \frac{\mu_0 \left(\pi R^2 N I_0\right)}{2\pi} \frac{1}{\left(R^2 + (z+a)^2\right)^{3/2}}. \tag{6.13}$$

The total magnetic induction is the sum of (6.12) and (6.13), or

$$B_z(z) = \frac{\mu_0 \left(\pi R^2 N_{\text{coils}} I_0\right)}{4\pi} \left[\frac{1}{\left(R^2 + (z+a)^2\right)^{3/2}} + \frac{1}{\left(R^2 + (z-a)^2\right)^{3/2}}\right] \tag{6.14}$$

In Fig. 6.4 we have plotted (6.12), (6.13) and (6.14) for selected values of $\alpha = a/R$. In each plot the top curve is the plot of (6.14) and the lower (crossed) plots are for (6.12) and (6.13). The abscissa in each plot covers the distance between the two coils.

From these plots we see that a high and relatively uniform induction is obtained for $\alpha = 0.5$. That is we have solved the problem of creating a uniform and constant magnetic field in a region accessible to an experimenter.

Another macroscopic element for constructing magnetic fields in space is the line source of current. In our next example we calculate the vector potential from a line current source.

*Example 6.4.* **Vector potential from a long, thin wire.** We call the constant current $I_0$ and choose coordinates such that the wire is parallel to the $z-$ axis and passes through the horizontal plane at the point $(x_0, y_0)$. We have drawn the situation in Fig. 6.5.

The current density vector is then

$$\mathbf{J} = I_0 \alpha(x) \beta(y) \delta(x - x_0) \delta(y - y_0) \hat{e}_z.$$

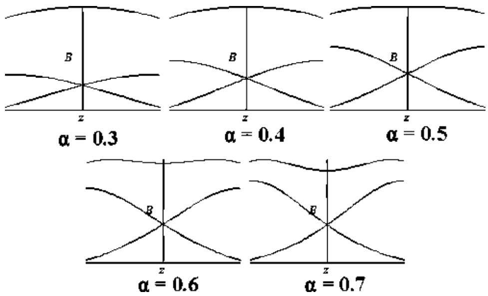



**Fig. 6.4** Plots of magnetic induction near the center of a Helmholtz Coil for values of $\alpha = a/R$, where $2a =$ coil separation and $R =$ coil radius

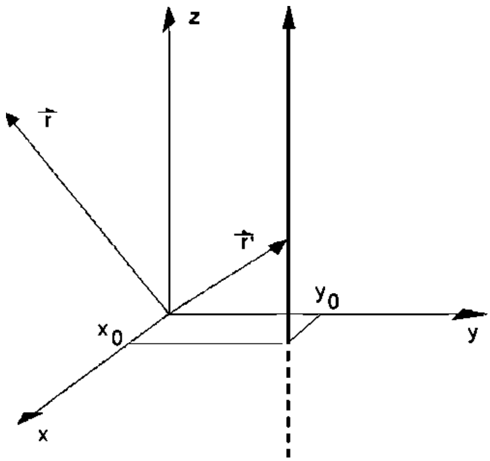



**Fig. 6.5** Straight thin wire with constant current

And the current flowing in the wire is

$$\begin{aligned} I_0 &= \int \mathrm{d}x\mathrm{d}y I_0 \alpha\left(x\right) \beta\left(y\right) \delta\left(x - x_0\right) \delta\left(y - y_0\right) \\ &= I_0 \alpha\left(x_0\right) \beta\left(y_0\right), \end{aligned}$$

from which we see that $\alpha = \beta = 1$.

The current density is then

$$\mathbf{J} = I_0 \delta\left(x - x_0\right) \delta\left(y - y_0\right) \hat{e}_{\mathrm{z}}.$$

The field point is anywhere in space

$$\mathbf{r} = x\hat{e}_{\mathrm{x}} + y\hat{e}_{\mathrm{y}} + z\hat{e}_{\mathrm{z}}$$

and the source point

$$\mathbf{r}' = x'\hat{e}_{\mathrm{x}} + y'\hat{e}_{\mathrm{y}} + z'\hat{e}_{\mathrm{z}}$$

is also anywhere in space, but it will be limited to points along the wire by the current density.

The distance between the field and source points is

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

Using Chap. 5 (5.54) the vector potential is

$$\begin{aligned}\mathbf{A} &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\int_V \frac{\delta(x'-x_0)\,\delta(y'-y_0)\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'}{\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{\frac{1}{2}}}\\ &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}z'}{\left\{R^2+(z-z')^2\right\}^{\frac{1}{2}}},\end{aligned}$$

where

$$R^2 = (x-x_0)^2+(y-y_0)^2\,.$$

Because the wire is infinitely long the vector potential is independent of $z$. We may then set $z$ equal to zero and

$$\begin{aligned}\mathbf{A} &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\lim_{L\to\infty}\int_{-L/2}^{+L/2}\frac{\mathrm{d}z'}{\{R^2+z'^2\}^{\frac{1}{2}}}\\ &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\lim_{L\to\infty}\ln\left(z+\sqrt{R^2+z'^2}\right)\Big]_{-L/2}^{+L/2}\\ &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\lim_{L\to\infty}\ln\frac{\left(L/2+\sqrt{R^2+(L/2)^2}\right)}{\left(-L/2+\sqrt{R^2+(L/2)^2}\right)}\\ &= \hat{e}_z \frac{I_0\mu_0}{4\pi}\lim_{L\to\infty}\ln\left(\frac{L^2+R^2}{R^2}\right)\\ &= -\hat{e}_z \frac{I_0\mu_0}{2\pi}\ln(R)+\hat{e}_z\frac{I_0\mu_0}{2\pi}\lim_{L\to\infty}\ln(L)\,.\end{aligned}$$

The second term here is a constant, even though it is very large. Since we only require the curl of the vector potential to obtain the magnetic field, we may then drop this constant as of no importance. The vector potential from a very long wire passing through the point $(x_0, y_0)$ is then

$$\mathbf{A} = -\hat{e}_z \frac{I_0\mu_0}{4\pi}\ln\left[(x-x_0)^2+(y-y_0)^2\right]. \tag{6.15}$$

In our next example we obtain the magnetostatic field induction from the vector potential (6.15).

*Example 6.5.* **Magnetic induction from a long, thin wire**. For simplicity, since we have a single wire, we choose $x_0 = y_0 = 0$, so that the wire passes through the origin. From the preceding example the vector potential is then

$$\mathbf{A} = -\hat{e}_{\text{z}} \frac{I_0 \mu_0}{2\pi} \ln(r) .$$

And the magnetic field is

$$\begin{aligned}\mathbf{B} &= \operatorname{curl} \mathbf{A} \\ &= \hat{e}_\vartheta \frac{I_0 \mu_0}{2\pi} \frac{\mathrm{d}}{\mathrm{d}r} \ln(r) \\ &= \hat{e}_\vartheta \frac{I_0 \mu_0}{2\pi r} . \end{aligned} \tag{6.16}$$

Equation (6.16) is identical to what we originally deduced from Ampère's experiment in Chap. 5 (5.39).

## 6.4 Summary

In this chapter we have provided an introduction to the use of the Biot–Savart Law and the solution of Poisson's Equation for the vector potential. We have not intended that our treatment be exhaustive. We have considered two important building blocks for the construction of magnetic fields in space. And we have considered the use of current rings to form the Helmholtz Coil.

The importance of this chapter is primarily in what the reader will gain in familiarity with magnetic fields and their geometry.

## Exercises

**6.1.** Consider a thin strip of metal of width $w$ and very long. The current in the strip is uniform and directed along its length; the total current is $I$. Find the vector potential using the result of the example in the text for the vector potential from a thin wire with a constant current. Construct the metal strip of thin wires. You will need to define a current per unit width of the strip and integrate.

**6.2.** Using the vector potential found for the metal strip in the preceding exercise obtain the magnetic induction arising from the metal strip with a constant current.

Show that when $x, y \gg w$ that the magnetic field becomes circular.

**6.3.** Use the Biot–Savart result for the magnetic field on the axis of a loop of wire with a constant current to obtain the magnetic induction near the center of a long solenoid of radius $R$ and length $L$ with $N$ windings and a current $I_0$. For a long solenoid $R/L \ll 1$.

[Answer: $\hat{e}_z \mu_0 N_\lambda I_0$ if $R/L \ll 1$]

**6.4.** In elementary texts it is shown that the magnetic field induction inside a long cylindrical solenoid is

$$\mathbf{B} = \hat{e}_z \mu_0 N_\lambda I_0,$$

where $N_\lambda$ is the number of wire windings per unit length around the solenoid and $I_0$ is the current in the wire. Using the integral relationship you found in the preceding exercise obtain the vector potential inside the solenoid required for this magnetic field induction. The inner radius of the cylindrical solenoid is $R$.

Show that the curl of your result for $\mathbf{A}$ does produce the required constant magnetic field induction above.

[Answer: $\mathbf{A} = \hat{e}_\vartheta \left(\mu_0 N_\lambda I_0/2\right) r$]

**6.5.** From Poisson's Equation find the current density that results in the vector potential you found in the preceding exercise.

Comment on the relationship between this current density and that in the solenoid.

**6.6.** We can find the way in which the magnetic field induction varies with small axial displacements near the center by integrating d$\mathbf{B}$ in exercise 6.3 from a length of $-L_1$ to $+L_2$. The solenoid will then have a length of $L = L_1 + L_2$ and the lengths $L_1$ and $L_2$ will measured from a point on the $z-$axis slightly displaced from the center of the solenoid as shown in Fig. 6.6.

Find the magnetic field induction as a function of $\cos\vartheta_2$ and $\cos\vartheta_1$.

[Answer: $\mathbf{B} = \hat{e}_z \frac{\mu_0 N_\lambda I_0}{2} \left(\cos\vartheta_2 + \cos\vartheta_1\right)$]

**6.7.** Since $L_1 = L - z$ and $L_2 = L + z$, the result from the preceding exercise can be used to obtain the axial magnetic field induction as a function of $z$ near the center of the solenoid, i.e. for small $z$.

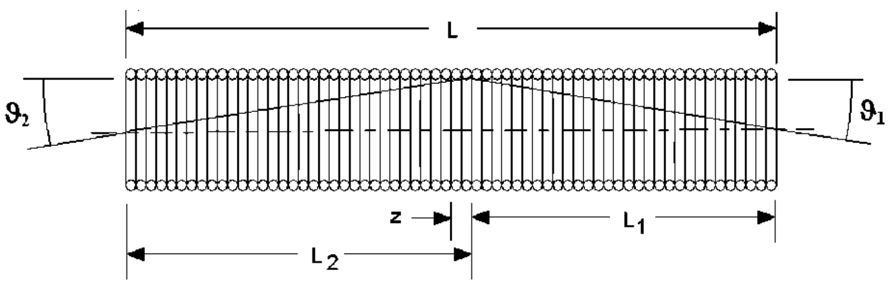



**Fig. 6.6** Long cylindrical solenoid

[Answer: $\mathbf{B} = \hat{e}_{\mathrm{z}}\mu_0 N_\lambda I_0 \cos\vartheta_0 \left(1 - \frac{3}{2}R^2z^2/\left(L^2+R^2\right)^2\right)$ where $\cos\vartheta_0 = L/\sqrt{L^2+R^2}$]

**6.8.** Use the result from the preceding exercise and the fact that $\operatorname{div}\mathbf{B} = 0$ to obtain the radial component of the magnetic field induction near the center of the solenoid.

[Answer: $B_{\mathrm{r}} = \frac{3}{2}\mu_0 N_\lambda I_0 \cos\vartheta_0 R^2 zr/\left(L^2+R^2\right)^2$]

**6.9.** From the results of the preceding the magnetic field induction within a long solenoid with $N_\lambda$ windings per unit length carrying a current $I_0$ is

$$\mathbf{B} = \hat{e}_{\mathrm{z}}\left(B_0 - B_1 z^2\right) + \hat{e}_{\mathrm{r}}\left(B_1 zr\right)$$

where

$$B_0 = \mu_0 N_\lambda I_0 \cos\vartheta_0$$

$$B_1 = \frac{3}{2}\mu_0 N_\lambda I_0 \cos\vartheta_0 \frac{R^2}{\left(L^2+R^2\right)^2}.$$

Although we would normally calculate the magnetic field induction from the vector potential, we may also find the vector potential, if we so choose, from the magnetic field induction. That is from $B = \operatorname{curl} A$. For the case of cylindrical symmetry, the curl results in

$$\operatorname{curl}\mathbf{A} = -\hat{e}_{\mathrm{r}}\left(\frac{\partial A_\vartheta}{\partial z}\right) + \hat{e}_{\mathrm{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\vartheta\right)\right].$$

Then

$$\frac{\partial A_\vartheta}{\partial z} = -B_1 zr$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\vartheta\right) = B_0 - B_1 z^2.$$

These two equations can be integrated to obtain $A_\vartheta$. We must only realize that integration of partial differentials results in arbitrary functions of $r$ and $z$. Perform the integration to find $A_\vartheta$.

[Answer: $A_\vartheta = \frac{1}{2}B_0 r - \frac{1}{2}B_1 z^2 r$]

**6.10.** You have a hollow plastic ball of radius $a$, which can be separated into two hemispheres. Around each hemisphere you have wrapped insulated wire as tightly as possible. At the equatorial end of each wire you have plugs. You can then fasten the two hemispheres together and plug the wires together to have a continuous wrapping around the plastic ball.

You have been careful in the wire winding so that when the hemispheres are joined the wire winding is continuous between the hemispheres, i.e. the winding does not reverse direction. There are $N$ total windings around the hollow plastic ball completely covering its surface.

In Fig. 6.7 we have drawn the hollow ball with a quadrant cut away. The cross sections of the wires are represented by the dots. We have not drawn the plugs at the

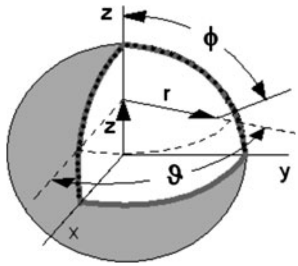



**Fig. 6.7** Hollow plastic ball with wire winding

equator nor the wires from the top and bottom to the power supply for the current. The current in the winding is $I_0$.

Using the magnetic induction from a single loop determine the magnetic induction at the center of the sphere.

**6.11.** Use the fact that div $\mathbf{B} = 0$ to get an approximate expression for $B_r$ (the radial component of the magnetic induction) that is valid for points very near the axis of a wire loop of radius $a$ and a constant current $I_0$.

**6.12.** Use the vector potential for the wire loop

$$\mathbf{A} = \left(\frac{\mu_0}{4\pi r^2}\right) \mathbf{m} \times \hat{e}_r$$

to obtain the magnetic induction $\mathbf{B}$ for the loop, written in terms of the magnitude of the magnetic moment $m$ of the loop, as

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3} \left[\hat{e}_r 2\cos\phi + \hat{e}_\phi \sin\phi\right].$$

It will be easiest to first find the the product $\mathbf{m} \times \hat{e}_r$ in spherical coordinates and use the curl in spherical coordinates in the Appendices.

**6.13.** In an example in the text we found the vector potential for a long, straight wire carrying a current $I_0$. You place two wires parallel to each other and a distance $2a$ apart. You then connect the two wires in series in a circuit with a single battery so that the same current, again $I_0$ flows through each wire, but in opposite directions. What is the magnetic vector potential for this arrangement of wires in terms of the distances $r_2$ and $r_1$ from the field point to the wires in a plane perpendicular to the wires?

**6.14.** Laboratory measurements are of magnetic field induction and not of the vector potential. The results of the preceding exercise provided the vector potential for an arrangement of two parallel wires carrying currents of equal magnitude in opposite directions. Find the magnetic field induction in a plane perpendicular to these wires when the current in each is $I_0$.

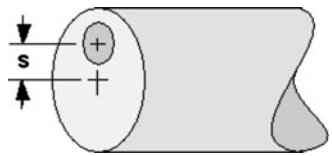



**Fig. 6.8** Soft conductor with hole for rigid mounting rod

In the laboratory you cannot make measurements an infinite distance from infinite parallel wires. But your mathematical result can be used to predict what such a measurement would yield. Measurements made an infinite distance from the parallel wires will be those obtained in the limit as $a \to 0$ in your expression for **B**. What is this result? Does this make logical sense?

**6.15.** In an experiment you must mount a cylindrical conductor with radius $b$, that is not negligibly small, horizontally. The conductor will be made of a soft material and will bend. You are unable, because of the experimental design, to support the soft conductor using external mountings. Your solution to the structural problem is to form the cylindrical conductor with a rigid rod of nonconducting material of radius $a$ running axially down the conductor with center a distance $s$ from the center of the conductor. We have drawn the soft conductor in Fig. 6.8. What effect will this solution have on the magnetic field geometry? In your experiment you are, fortunately, only interested in the field directly below the conductor. So you only need to calculate the field at a point $r > b$ along the radius.

[Note: This is an exercise in the use of superposition.]

# Chapter 7
# Particle Motion

*I regarded as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of application that one must study them; one judges their utility there and appraises the manner of making use of them.*

*Joseph Louis Lagrange*

## 7.1 Introduction

The motion of charged particles in electric and magnetic fields is never appropriately separated from a study of the fields. Plasma physics, with applications in astrophysics and thermonuclear fusion, is an integral part of modern physics. In this chapter we will introduce the treatment of the motion of charged particles in the presence of electric and magnetic fields as a branch of analytical mechanics.

## 7.2 Analytical Mechanics

### *7.2.1 Euler–Lagrange Formulation*

Joseph Louis Lagrange published the formulation which we shall use in 1788 under the title of *Analytical Mechanics* (*Mécanique Analitique*) [56]. To emphasize that he had abandoned the awkward methods of geometry used by Newton, Lagrange wrote that "this book contains no diagrams". The formulation of the laws of mechanics appeared in the form of a *variational principle*.

The roots of this formulation lie in the ideas of Pierre-Louis Moreau de Maupertuis (1698–1759) and Leonard Euler (1707–1783) [26].

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_7, 

The variational principle states that the functional

$$\boxed{S = \int_{t_1}^{t_2} dt L(\{q\}, \{\dot{q}\}, t),} \tag{7.1}$$

has an extreme value when the (generalized[1]) coordinates $\{q\}$ and velocities $\{\dot{q}\}$ satisfy the so-called *Euler–Lagrange Equations* [31], which are equivalent to Newton's laws.

*Remark 7.1.* Because generalized coordinates are traditionally designated as $q$, and the letter $q$ is normally used to indicate the charge on a particle, throughout this chapter we will use $Q$ as the designation of charge on a particle.

Dots will be used throughout this chapter to indicate differentiation with respect to time. This is a standard notation.

The function

$$L(\{q\}, \{\dot{q}\}, t) = T(\{\dot{q}\}) - V(\{q\})$$

is the *Lagrangian*. It is equal to the difference between the kinetic ($T$) and potential ($V$) energies of the particle. Because the kinetic and potential energies are usually easy to write down, the Euler–Lagrange formulation is much simpler than a direct application of Newton's laws. The Euler–Lagrange formulation is also that appropriate to *relativistic mechanics*.

The Euler–Lagrange equations are the differential equations

$$\left[\frac{\partial}{\partial q_{\mathrm{r}}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_{\mathrm{r}}}\right] L(\{q\}, \{\dot{q}\}, t) = 0 \tag{7.2}$$

for each coordinate $q_{\mathrm{r}}$. For a single particle moving in a potential $V(q)$, the Lagrangian is $L = (1/2)\, m\dot{q}^2 - V(q)$. The Euler–Lagrange equation for this single particle is then

$$-\frac{\partial V}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} m\dot{q} = 0,$$

which is Newton's Second Law.

From here it is a short mathematical step to the formulation of (Sir) William Rowan Hamilton, which results in the canonical, or fundamental equations. We will base our treatment on the *canonical equations*.

### 7.2.2 Hamiltonian Formulation

The mathematical step is to perform a *Legendre Transformation* ([40], p. 51) of the Lagrangian. The Legendre Transformation exchanges one set of independent

[1] Generalized here implies that the coordinates may include constraints.

variables for another, which is obtained from a derivative of the original function. The transformation then preserves information and adds nothing extraneous.

In the transformation of the Lagrangian to the Hamiltonian the dependence on the velocities $\{\dot{q}\}$ is exchanged for a dependence on the *canonical momenta* $\{p\}$ defined as $p_{\mathrm{r}} = \partial L/\partial \dot{q}_{\mathrm{r}}$. In many applications the canonical momenta are simply $p_{\mathrm{r}} = m\dot{q}_{\mathrm{r}}$. This is not, however, the case for the motion of a charged particle in an electromagnetic field.

The *Hamiltonian* is defined as

$$\mathcal{H}(\{q\},\{p\},t) = \frac{\partial L}{\partial \dot{q}_\mu}\dot{q}_\mu - L$$

or

$$\boxed{\mathcal{H}(\{q\},\{p\},t) = p_\mu \dot{\mathrm{q}}_\mu - \mathrm{L},} \tag{7.3}$$

where we use the Einstein summation convention.

The canonical equations are

$$\boxed{\dot{\mathrm{q}}_\mu = \partial \mathcal{H}/\partial \mathrm{p}_\mu} \tag{7.4}$$

and

$$\boxed{\dot{\mathrm{p}}_\mu = -\partial \mathcal{H}/\partial \mathrm{q}_\mu.} \tag{7.5}$$

We see that these are first order differential equations in the time $t$. This is a major advantage particularly in the study of particle motion in an electromagnetic field.

## 7.3 Electrodynamics

### *7.3.1 The Langrangian*

The simplest approach is to claim that the Lagrangian for a charged particle of mass $m$ and charge $Q$[2] moving in a electromagnetic field with a scalar potential $\varphi$ and a vector potential $\mathbf{A}$ is

$$\boxed{L = (1/2)\mathrm{m}\dot{\mathrm{q}}_\mu \dot{\mathrm{q}}_\mu - Q\varphi + QA_\mu \dot{\mathrm{q}}_\mu,} \tag{7.6}$$

where $\varphi$ is the electrical (scalar) potential and $A_\mu$ is the $\mu$th component of the vector potential. We recognize the first two terms here as the standard kinetic and electric potential energies. The third term is the source of the magnetic field force.

[2]Our use of capital $Q$ to designate charge, rather than lower case $q$, is a result of the fact that the latter is traditionally used to designate generalized coordinate, as we do here.

We can convince ourselves that (7.6) is the correct Lagrangian if we obtain the Euler–Lagrange equations from this Lagrangian and compare them with Newton's Second Law for a charge moving in a region in which there are constant electric and magnetic fields.

By partially differentiating the Lagrangian (7.6) with respect to $\dot{q}_\mu$ we find that the $\mu$th component of the canonical momentum is

$$\boxed{\mathrm{p}_\mu = \partial \mathrm{L}/\partial \dot{\mathrm{q}}_\mu = \mathrm{m}\dot{\mathrm{q}}_\mu + \mathrm{Q}\mathrm{A}_\mu.} \tag{7.7}$$

Since the $A_\mu$ are independent of time, the time derivative of $p_\mu$, defined in (7.7), is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} p_\mu &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{q}_\mu + QA_\mu\right) \\ &= m\ddot{q}_\mu. \end{aligned} \tag{7.8}$$

Partially differentiating the Lagrangian (7.6) with respect to the coordinate $q_\mu$ we have

$$\frac{\partial L}{\partial q_\mu} = -Q\frac{\partial \varphi}{\partial q_\mu} + Q\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu. \tag{7.9}$$

The Euler–Lagrange equations are then

$$\begin{aligned} m\ddot{q}_\mu &= \frac{\partial L}{\partial q_\mu} \\ &= -Q\frac{\partial \varphi}{\partial q_\mu} + Q\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu. \end{aligned} \tag{7.10}$$

The first term on the right hand side of (7.10) is the product of the electric charge and the $\mu$th component of the electric field. To show that

$$Q\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu = Q\left(\mathbf{v}\times\mathbf{B}\right)_\mu$$

requires some vector analysis, but is straightforward (see exercises). The Euler–Lagrange equations (7.10) are then

$$\begin{aligned} m\ddot{q}_\mu &= -Q\frac{\partial \varphi}{\partial q_\mu} + Q\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu \\ &= QE_\mu + Q\left(\mathbf{v}\times\mathbf{B}\right)_\mu. \end{aligned} \tag{7.11}$$

The right hand side of (7.11) is the $\mu$th component of the *Lorentz Force*, which is the force acting on a charged particle in an electromagnetic field. Therefore our Lagrangian has produced the accepted form of the equations of motion of a charged particle in the presence of combined electric and magnetic fields.

### 7.3.2 The Hamiltonian

With the canonical momenta (7.7), the Hamiltonian (7.3) for the charged particle in the electromagnetic field is

$$\boxed{\mathcal{H} = (1/2m)\left(p_\mu - QA_\mu\right)^2 + Q\varphi,} \tag{7.12}$$

which is a function only of the coordinates, through $\varphi$ and $A_\mu$, and the canonical momenta.

The canonical equations are

$$\begin{aligned}\dot{q}_\mu &= \frac{\partial \mathcal{H}}{\partial p_\mu} \\ &= \frac{1}{m}\left(p_\mu - QA_\mu\right),\end{aligned} \tag{7.13}$$

which is simply (7.7), and

$$\begin{aligned}\dot{p}_\mu &= -\frac{\partial \mathcal{H}}{\partial q_\mu} \\ &= -\frac{1}{m}\left(p_\mu - Q\frac{\mathrm{d}A_\mu}{\mathrm{d}q_\mu}\right) - Q\frac{\partial \varphi}{\partial q_\mu}.\end{aligned} \tag{7.14}$$

The canonical (7.13) and (7.14) will be those on which we will base our study of the motion of charged particles in an electromagnetic field. These are coupled first order differential equations, which are easier to deal with than the second order equations resulting from Newton's Second Law.

## 7.4 Particle Motion

We will consider the motion of single charged particles in magnetic and combinations of electric and magnetic fields by means of examples. Some examples we will be able to treat analytically, but some of the more interesting will require numerical solution. The analytic solutions will be simplified through the use of complex valued functions.

### 7.4.1 Magnetic Fields

The first example we consider is the motion of a charged particle in the presence of a constant and uniform magnetic field with induction **B**. We shall assume that the initial particle velocity is perpendicular to the magnetic field so that motion is in a plane perpendicular to **B**.

*Example 7.1.* **Motion in a Uniform Magnetic Field**. We consider a particle of charge $Q$ and mass $m$ in a uniform magnetic field of induction $B$, which we choose to be oriented along the $z-$axis. We consider that the particle is released from the point $x = R$, $y = 0$ with a velocity in the negative $\hat{e}_{\mathrm{y}}$ direction.

In the exercises of Chap. 5 we showed that a uniform constant magnetic field with induction $\mathbf{B} = \hat{e}_{\mathrm{z}} B$ results from the vector potential

$$\mathbf{A} = -\hat{e}_{\mathrm{x}} \frac{B}{2} y + \hat{e}_{\mathrm{y}} \frac{B}{2} x. \tag{7.15}$$

The Hamiltonian for a particle with charge $Q$ and mass $m$ moving in this uniform magnetic field is then

$$\mathcal{H} = \frac{1}{2m} \left( p_{\mathrm{x}} + m \frac{\Omega}{2} y \right)^2 + \frac{1}{2m} \left( p_{\mathrm{y}} - m \frac{\Omega}{2} x \right)^2, \tag{7.16}$$

where $\Omega = QB/m$ is the cyclotron frequency.[3]

The canonical equations are

$$\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p_{\mathrm{x}}} = \frac{1}{m} \left( p_{\mathrm{x}} + m \frac{\Omega}{2} y \right), \\
\dot{y} &= \frac{\partial \mathcal{H}}{\partial p_{\mathrm{y}}} = \frac{1}{m} \left( p_{\mathrm{y}} - m \frac{\Omega}{2} x \right), \\
\dot{p}_{\mathrm{x}} &= -\frac{\partial \mathcal{H}}{\partial x} = \frac{\Omega}{2} \left( p_{\mathrm{y}} - m \frac{\Omega}{2} x \right), \\
\dot{p}_{\mathrm{y}} &= -\frac{\partial \mathcal{H}}{\partial y} = -\frac{\Omega}{2} \left( p_{\mathrm{x}} + m \frac{\Omega}{2} y \right).
\end{aligned} \tag{7.17}$$

We can simplify the (7.17) by defining the complex valued functions

$$Z = x + iy \tag{7.18}$$

and

$$P_{\mathrm{Z}} = p_{\mathrm{x}} + i p_{\mathrm{y}}. \tag{7.19}$$

Then, using (7.17), we obtain differential equations for $Z$

$$\begin{aligned}
\dot{Z} &= \frac{1}{m} \left( p_{\mathrm{x}} + i p_{\mathrm{y}} \right) - \frac{1}{2} \Omega \left( ix - y \right) \\
&= \frac{1}{m} P_{\mathrm{Z}} - \frac{1}{2} \Omega i Z
\end{aligned} \tag{7.20}$$

[3] $\Omega = QB/m$ is the frequency of gyration of a charge $Q$ with mass $m$ in a cyclotron.

and for $P_Z$

$$\dot{P}_Z = -\frac{1}{2}\Omega\left(ip_x - p_y\right) - m\left(\frac{1}{2}\Omega\right)^2(x+iy)$$
$$= -\frac{1}{2}\Omega iP_Z - m\left(\frac{1}{2}\Omega\right)^2 Z. \tag{7.21}$$

These equations are linear, first order differential equations with constant coefficients. Such equations are solved by exponentials.

We then choose as solutions

$$Z = \tilde{Z}\exp(\alpha t)$$
$$P_Z = \tilde{P}_Z\exp(\alpha t). \tag{7.22}$$

This choice is an Ansatz. We choose our solutions to be of the form (7.22) and then ask for the requirements imposed by (7.20) and (7.21) for this Ansatz to be valid.

With the solutions (7.22) (7.20) and (7.21) become

$$\alpha\tilde{Z} = \left(\frac{1}{m}\right)\tilde{P}_Z - \frac{1}{2}\Omega i\tilde{Z} \tag{7.23}$$

and

$$\alpha\tilde{P}_Z = -\frac{1}{2}\Omega i\,\tilde{P}_Z - m\left(\frac{1}{2}\Omega\right)^2\tilde{Z}. \tag{7.24}$$

Equations (7.23) and (7.24) are linear algebraic equations for $\tilde{Z}$ and $\tilde{P}_Z$. In matrix form (7.23) and (7.24) are

$$\begin{bmatrix} -\left(\frac{1}{2}\Omega i+\alpha\right) & \frac{1}{m} \\ -m\left(\frac{1}{2}\Omega\right)^2 & -\left(\frac{1}{2}\Omega i+\alpha\right)\end{bmatrix}\begin{bmatrix}\tilde{Z}\\ \tilde{P}_Z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{7.25}$$

The linear (7.25) have non-trivial solutions only when the determinant of the coefficients vanishes. That is

$$\det\begin{bmatrix} -\left(\frac{1}{2}\Omega i+\alpha\right) & \frac{1}{m} \\ -m\left(\frac{1}{2}\Omega\right)^2 & -\left(\frac{1}{2}\Omega i+\alpha\right)\end{bmatrix} = i\Omega\alpha + \alpha^2 = 0. \tag{7.26}$$

This is the condition on $\alpha$ that results from our Ansatz. That is, we must have either $\alpha = 0$, in which case the charge is not moving, or

$$\alpha = -i\Omega.$$

We choose the latter as the only truly nontrivial solution.

Our functions $Z$ and $P_Z$ are then

$$\begin{aligned} Z &= \tilde{Z}\exp(-i\Omega t) \\ &= \tilde{Z}\left(\cos(\Omega t) - i\sin(\Omega t)\right) \end{aligned} \tag{7.27}$$

and

$$\begin{aligned} P_Z &= \tilde{P}_Z\exp(-i\Omega t) \\ &= \tilde{P}_Z\left(\cos(\Omega t) - i\sin(\Omega t)\right), \end{aligned} \tag{7.28}$$

where we have used Euler's Identity[4] (see exercises)

$$\exp(i\vartheta) = \cos\vartheta + i\sin\vartheta. \tag{7.29}$$

If we put the value we have for $\alpha\,(= -i\Omega)$ into (7.25) we can obtain the relationship between $\tilde{Z}$ and $\tilde{P}_Z$. That is

$$\begin{bmatrix} \frac{1}{2}\Omega i & \frac{1}{m} \\ -m\left(\frac{1}{2}\Omega\right)^2 & \frac{1}{2}\Omega i \end{bmatrix}\begin{bmatrix} \tilde{Z} \\ \tilde{P}_Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7.30}$$

from which we find that

$$\tilde{P}_Z = -i\frac{m\Omega}{2}\tilde{Z}. \tag{7.31}$$

To obtain the value of $\tilde{Z}$ we use the initial conditions.

At time $t = 0$ we have

$$x(t = 0) = \operatorname{Re} Z(t = 0) = R,$$

and

$$y(t = 0) = \operatorname{Im} Z(t = 0) = 0.$$

From (7.27) we have

$$Z(t = 0) = \tilde{Z}.$$

[4]Leonhard Euler (1707–1783) was a Swiss mathematician who spent most of his life in Germany and Russia. He is known as the "Mozart of Mathematics."

Therefore Re $\tilde{Z} = R$ and Im $\tilde{Z} = 0$. That is $\tilde{Z} = R$, and is real. The solution (7.27) is then

$$Z(t) = R(\cos(\Omega t) - i\sin(\Omega t)),$$

And, from (7.18), the coordinates are

$$\begin{aligned} x(t) &= \mathrm{Re}(Z) = R\cos(\Omega t) \\ y(t) &= \mathrm{Im}(Z) = -R\sin(\Omega t), \end{aligned} \tag{7.32}$$

From (7.31) and $\tilde{Z} = R$ we have

$$\tilde{P}_{\mathrm{Z}} = -i\frac{m\Omega}{2}R,$$

which is imaginary. The solution (7.28) is then

$$P_{\mathrm{Z}}(t) = -\frac{m\Omega}{2}R\sin(\Omega t) - i\frac{m\Omega}{2}R\cos(\Omega t).$$

And, from (7.19), the canonical momenta are

$$\begin{aligned} p_{\mathrm{x}}(t) &= -\frac{m\Omega}{2}R\sin(\Omega t) \\ p_{\mathrm{y}}(t) &= -\frac{m\Omega}{2}R\cos(\Omega t). \end{aligned} \tag{7.33}$$

Recall from (7.7) that the definitions of the canonical momenta include the vector potential and are not simply products of mass and velocity.

The resultant motion is circular at a constant angular velocity $-\hat{e}_{\mathrm{z}}\Omega$. This motion is clockwise around the $z$−axis if $Q$ is positive. The Lorentz Force is

$$\begin{aligned} \mathbf{F} &= Q\mathbf{v}\times\mathbf{B} \\ &= -QBv\hat{e}_{\vartheta}\times\hat{e}_{\mathrm{z}} = -QBv\hat{e}_{\mathrm{r}}, \end{aligned}$$

which is toward the center of the circular orbit.

In the next example we consider a magnetic field that varies in the $z$−direction. This will result in a mirror-like effect on the motion of the charge.

*Example 7.2.* **Motion in a Non-uniform Magnetic Field**. We consider the motion of a charge in a region of space in which there is a non-homogeneous magnetic field[5] with

$$\mathbf{A} = -\hat{e}_{\mathrm{x}}y\frac{B}{2}\exp(az) + \hat{e}_{\mathrm{y}}x\frac{B}{2}\exp(az) \tag{7.34}$$

[5]The vector potential must have zero divergence in the static case, i.e. $div\,\mathbf{A} = 0$. This is true for the potential given.

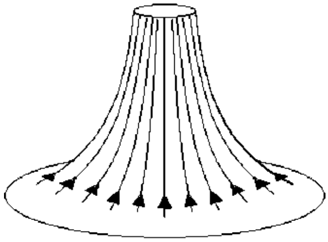


**Fig. 7.1** Magnetic field induction **B** from the vector potential $\mathbf{A} = -\hat{e}_x y \frac{B}{2} \exp(az) + \hat{e}_y x \frac{B}{2} \exp(az)$

The magnetic induction for this vector potential is found in the exercises in Chap. 5. The result is

$$\begin{aligned} B_x &= -x \left( a \frac{B}{2} \right) \exp(az) \\ B_y &= -y \left( a \frac{B}{2} \right) \exp(az) \\ B_z &= B \exp(az) \, . \end{aligned}$$

This magnetic field induction has the form shown in Fig. 7.1.

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left[ \left( p_x + \frac{QB}{2} y \exp(az) \right)^2 + \left( p_y - \frac{QB}{2} x \exp(az) \right)^2 \right] + \frac{1}{2m} p_z^2 .$$

With $\Omega = QB/m$ the canonical equations are

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{m} p_x + \frac{1}{2} \Omega y \exp(az) \\ \dot{y} &= \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{m} p_y - \frac{1}{2} \Omega x \exp(az) \\ \dot{z} &= \frac{\partial \mathcal{H}}{\partial p_z} = \frac{1}{m} p_z \\ \dot{p}_x &= -\frac{\partial \mathcal{H}}{\partial x} = \frac{1}{2} \Omega \left( p_y - \frac{1}{2} m \Omega x \exp(az) \right) \exp(az) \\ \dot{p}_y &= -\frac{\partial \mathcal{H}}{\partial y} = -\frac{1}{2} \Omega \left( p_x + \frac{1}{2} m \Omega y \exp(az) \right) \exp(az) \\ \dot{p}_z &= -\frac{\partial \mathcal{H}}{\partial z} = -\frac{1}{2} \Omega a \left[ \left( p_x + \frac{1}{2} m \Omega y \exp(az) \right) y \right. \\ &\quad \left. - \left( py - \frac{1}{2} m \Omega x \exp(az) \right) x \right] \exp(az) \, . \end{aligned} \tag{7.35}$$

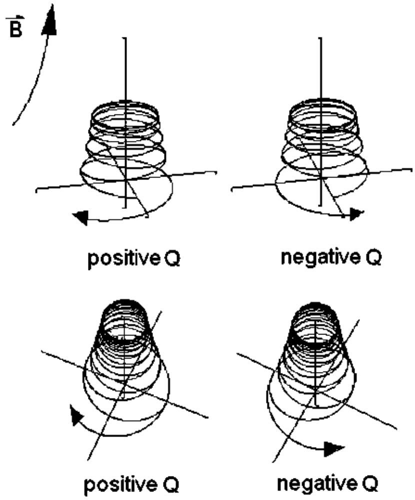



**Fig. 7.2** Motion of a charge in a spatially varying magnetic field. Trajectory on the left is for *positive charge*, on the right is for *negative charge*

This set of six first order equations is complicated by the presence of the $z-$dependence in the magnetic field. The solution must be obtained numerically. But the numerical solution of (7.35) is not difficult with the mathematical packages that are presently available.

The advantage of the first order canonical equations is that a *Runge-Kutta* algorithm[6] can be applied directly. We obtained the particle trajectory from a numerical integration of the canonical (7.35) using a Runge-Kutta algorithm on Maple 12. In the numerical solution we released the charged particle on the $x-$axis at $x = 1$ with a momentum in the $y-$and $z-$directions. The result was the trajectory shown in Fig. 7.2.

In Fig. 7.2 we have plotted results for both positive and for negative charges. The charges spiral along the magnetic field lines moving in the positive $z-$direction until they are deflected and then they spiral out with growing radius along the negative $z-$direction. The top images are for a small initial momentum and the bottom for a larger initial momentum.

We see a reflection for both charges. The larger momentum makes the spiral of the charge more evident. These results show that we can create a magnetic mirror which will reflect the charges of either sign.

*Example 7.3.* **Magnetic Bottle.** If we arrange two magnetic mirrors opposed to one another we can create a *magnetic bottle* as we have shown in Fig. 7.3. In the left panel of Fig. 7.3 we show the magnetic field required to produce the magnetic bottle and in the right panel we show the motion of a charged particle in the magnetic

[6]These very important numerical techniques for the solution of first order differential equations were developed around 1900 by the German mathematicians C. Runge and M.W. Kutta.

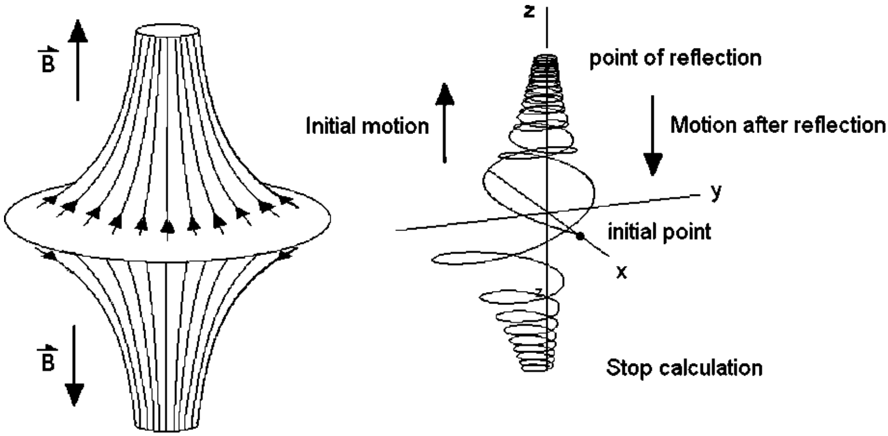



**Fig. 7.3** Magnetic bottle to trap charges in a region of space. The magnetic field arrangement is shown in the left panel and the motion of the charged particle is shown in the right panel

bottle. The charged particle undergoes a spiral motion and is reflected at each end of the magnetic bottle.

It is not difficult to create any magnetic field geometry we choose by the external arrangement of coils carrying constant currents. A magnetic field with the spatial structure of Fig. 7.3 can be produced. Magnetic bottles of this basic type were used in some early experiments on magnetic confinement of fusion plasmas. Problems of plasma leakage and instabilities have made these types of bottles impractical. Most modern magnetic confinement uses toroidal geometries.

### *7.4.2 Electric and Magnetic Fields*

If we add an electric field perpendicular to the magnetic field both the electric and the magnetic forces will be in the same plane. The result must then be a distortion of the circular motion we found for the uniform magnetic field.

*Example 7.4.* **Motion in Perpendicular Fields**. We consider the motion of a charged particle in a region of space in which there is a uniform magnetic field with induction $\mathbf{B} = \hat{e}_z B$ and a uniform electric field $\mathbf{E} = \hat{e}_y E$.

For a static magnetic field with induction $\mathbf{B} = \hat{e}_z B$ the vector potential is (7.15). And for an electric field $\mathbf{E} = \hat{e}_y E$ the electrostatic potential is

$$\varphi = -Ey.$$

The Hamiltonian is then

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\left(p_{\mathrm{x}} - QA_{\mathrm{x}}\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - QA_{\mathrm{y}}\right)^2 - QEy \\
&= \frac{1}{2m}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right)^2 - QEy. \qquad (7.36)
\end{aligned}
$$

where $\Omega = QB/m$.

Motion is then entirely in the $(x, y)$ plane.

The canonical equations are

$$
\begin{aligned}
\dot{x} &= \frac{1}{m}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right) \\
\dot{y} &= \frac{1}{m}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right) \\
\dot{p}_{\mathrm{x}} &= \frac{\Omega}{2}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right) \\
\dot{p}_{\mathrm{y}} &= -\frac{\Omega}{2}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right) + QE \qquad (7.37)
\end{aligned}
$$

We again simplify the problem if we introduce the complex variables $Z$ and $P_{\mathrm{z}}$ as defined in (7.18) and (7.19). We can then combine the Canonical equations to give

$$
\begin{aligned}
\dot{Z} &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - i\frac{1}{2}\Omega\left(x + iy\right) \\
&= \frac{1}{m}P_{\mathrm{Z}} - i\frac{1}{2}\Omega Z. \qquad (7.38)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{P}_{\mathrm{Z}} &= -\frac{1}{4}m\Omega^2\left(x + iy\right) - i\frac{1}{2}\Omega\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) + iQE \\
&= -\frac{1}{4}m\Omega^2 Z - i\frac{1}{2}\Omega P_{\mathrm{Z}} + iQE, \qquad (7.39)
\end{aligned}
$$

We must then solve (7.38) and (7.39) simultaneously.

In mathematical terms we now have a set of nonhomogeneous equations because of the presence of the electric field term $+iQE$ in (7.39).

We see from the set of canonical (7.37) that the term $QE$ appears in the equation for $\dot{p}_{\mathrm{y}}$, which can be written as

$$
\dot{p}_{\mathrm{y}} = -\frac{\Omega}{2}m\dot{x} + QE.
$$

It seems then that $QE$ is related to a velocity in the $\hat{e}_x$ direction. The $x-$coordinate is the real part of $Z$. We, therefore, choose to write

$$Z = Z' + v_0 t, \tag{7.40}$$

where $Z'$ is the solution for motion in only the magnetic field. Using (7.38) the function $P_Z$ becomes

$$\begin{aligned} P_Z &= m\dot{Z} + im\left(\frac{\Omega}{2}\right)Z \\ &= \left(m\dot{Z}' + im\left(\frac{\Omega}{2}\right)Z'\right) + mv_0 + im\left(\frac{\Omega}{2}\right)v_0 t. \end{aligned} \tag{7.41}$$

From the first two of the canonical (7.37) we see that

$$P_Z = m\dot{Z} + im\left(\frac{\Omega}{2}\right)Z, \tag{7.42}$$

whether or not an electric field is present. Therefore, with (7.42), we may write (7.41) as

$$P_Z = P'_Z + mv_0 + im\left(\frac{\Omega}{2}\right)v_0 t. \tag{7.43}$$

Equations (7.40) and (7.43) are then our proposed solutions, i.e. our Ansatz. Putting (7.40) and (7.43) into (7.38) and (7.39) we have

$$\dot{Z}' = \frac{1}{m}P'_Z - i\frac{\Omega}{2}Z'. \tag{7.44}$$

and

$$\dot{P}'_Z = -m\left(\frac{\Omega}{2}\right)^2 Z' - i\frac{\Omega}{2}P'_Z - i\Omega m v_0 + iQE. \tag{7.45}$$

Since we are requiring that $Z'$ and $P'_Z$ satisfy the equations for motion in a uniform magnetic field alone, (7.44) is an identity, and (7.45) becomes

$$-i\Omega m v_0 + iQE = 0. \tag{7.46}$$

For the equations of motion in a magnetic field alone see Example 7.1 (7.20) and (7.21).

From ((7.46) we have

$$v_0 = \frac{QE}{m\Omega} = \frac{E}{B}. \tag{7.47}$$

Our Ansatz has then provided a correct solution. The motion is a combination of circular motion and a uniform translation in a direction perpendicular to both the electric and the magnetic fields. The translation velocity is $v_0 = E/B$. Specifically

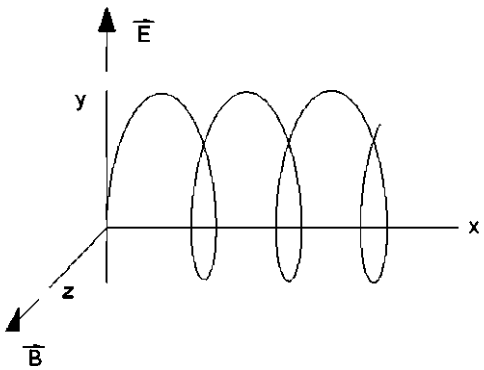



**Fig. 7.4** Trajectory of a charge in crossed electric **E** and magnetic **B** fields. Trajectory is cycloidal

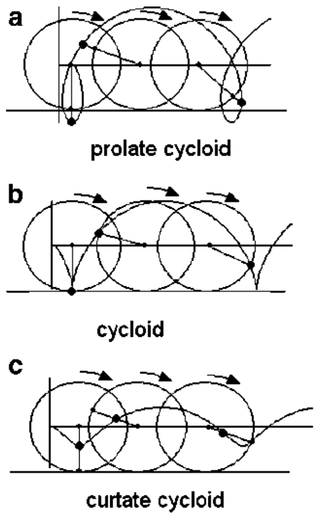



**Fig. 7.5** Three cycloidal forms the trajectory can take. The tracing spot is emphasized on each radial arm

$$x(t) = \mathrm{Re}(Z) = R\cos(\Omega t) + \frac{E}{B}t$$
$$y(t) = \mathrm{Im}(Z) = -R\sin(\Omega t)$$

This motion is cycloidal as we have shown in Fig. 7.4.

There are three cycloidal forms which the trajectory can take. These are plotted in Fig. 7.5. Each of these cycloidal forms is traced by a spot on the radial arm of a rolling disk. The prolate cycloid is traced by a spot on the radial arm at a distance greater than the radius from the center of the disk. The cycloid is traced by a spot at the circumference of the disk. And the curtate cycloid is traced by a spot at a distance less than the radius from the center of the disk.

Each of these three forms of the trajectory will result depending upon the magnitude of the translational velocity $v_0 = E/B$ (see exercises). The drift is in the direction of $\mathbf{E} \times \mathbf{B}$.

## 7.5 Plasmas

A plasma is an ionized gas. Low temperature plasmas, with temperatures of the order of $10^3$ K, consist of ions, electrons, and neutral atoms. The greatest interest in plasmas at the end of the twentieth and the beginning of the twenty-first centuries has been in thermonuclear fusion.

Thermonuclear fusion is the process by which hydrogen is converted into atoms of higher atomic number in stars. The hope is to produce a controlled thermonuclear fusion reactor on the earth's surface.

One of the most promising fusion energy sources is from the deuterium-tritium (D-T) fuel cycle. Deuterium ($^2_1$D) and tritium ($^3_1$T) are isotopes of hydrogen with one and two neutrons respectively. These are sometimes referred to as hydrogen-2 and hydrogen-3. The D-T fusion reaction is

$$^2_1\mathrm{D} +^3_1 \mathrm{T} \rightarrow^4_2 \mathrm{He}\ \ (3.5\ \mathrm{MeV}) +^1_0 \mathrm{n}\ \ (14.1\ \mathrm{MeV})\,. \tag{7.48}$$

In (7.48) we have indicated the kinetic energies of the products in parentheses. At the temperatures required for the D-T reaction (about $800 \times 10^6$ K) the plasma is fully ionized.

A detailed description of the plasma begins with an approach developed by Yurii Klimontovich ([53], pp. 409–488). In the Klimontovich approach the plasma density is represented by a sum of $\delta-$functions. For the $N$ particles of the species $\alpha$, for example,

$$N_\alpha\left(\mathbf{r},\mathbf{p},t\right) = \sum_{\mathrm{i=1}}^{\mathrm{N}\alpha''} \delta\left(\mathbf{r}-\mathbf{r}_{\mathrm{i}\alpha}\left(t\right)\right)\delta\left(\mathbf{p}-\mathbf{p}_{\mathrm{i}\alpha}\left(t\right)\right), \tag{7.49}$$

where $N\alpha''$ is the number of particles of the species $\alpha$ and the trajectory of the $i$th particle of the species $\alpha$ is $\{\mathbf{r}_{\mathrm{i}\alpha}\left(t\right), \mathbf{p}_{\mathrm{i}\alpha}\left(t\right)\}$. The trajectories of each of these individual charged particles are determined by the electric and magnetic fields at the instantaneous location of the particle. These fields are in turn determined by the charged particles themselves through the field equations. No approximations are made regarding interactions among particles.

The equation of motion for the Klimontovich density (7.49) is

$$\frac{\partial}{\partial t}N_\alpha + \frac{1}{m}\mathbf{p}\cdot \mathrm{grad}\, N_\alpha + \mathbf{F}_\mathrm{L}\cdot \mathrm{grad}_\mathrm{p}\, N_\alpha = 0. \tag{7.50}$$

where $\mathbf{F}_L$ is the Lorentz Force

$$\mathbf{F}_L = Q\left[\mathbf{E} + \frac{1}{m}\mathbf{p} \times \mathbf{B}\right]. \tag{7.51}$$

The fields **E** and **B** include externally imposed fields as well as those resulting from the particles. The Klimontovich description contains then a complete (classical) description of the plasma in which the particles are represented by $\delta-$functions.

We may obtain a higher level description in terms of measured quantities by ensemble averaging (7.50) and the field equations ([40], pp. 159–160). If we consider the Lorentz Force to result only from ensemble averaged fields (a mean-field approximation) then the ensemble average of (7.50) results in what is called the Vlasov Equation, which has the appearance of a "collisionless" kinetic equation.[7]

When the plasma phenomena of interest occur on time scales larger than the period of rotation of the particles about the magnetic field lines (i.e. the frequency of interest is smaller than the gyro-frequency), and when the particle gyro-radius is small compared with the spatial variation of the background magnetic field, then the actual gyro-motion of a charged particle as it moves along the field line, which we have shown in Fig. 7.6, can be approximated by the motion of a charged ring, which we have shown in Fig. 7.7. This simple picture, which ignores the detailed gyro-motion, signals the beginning of the modern gyrokinetic theory, which has had profound influence in both analytical and numerical understanding of the behavior of magnetized fusion plasmas (see e.g. [55, 60–62]).

We discuss these modern advances in plasma physics to show the importance of an understanding of individual charged particle motion to the most advanced topics in plasma physics. The gyrokinetic approach to the physics of fusion plasmas replaces the very complex equations of the general Klimontovich theory with a theory based on the Vlasov and Poisson equations. At the time of this writing the gyrokinetic theory is being applied to studies of plasma turbulence.

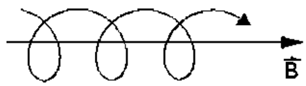



**Fig. 7.6** Gyromotion of a charged plasma particle around a magnetic field line

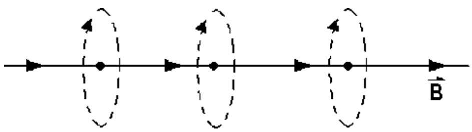



**Fig. 7.7** Gyrokinetic theory. The representation of the particle is the moving charged ring

[7]The field-particle correlation functions are neglected.

## 7.6 Summary

In this brief chapter we have outlined the basic mechanics of the motion of charged particles in the presence of magnetic and combined electric and magnetic fields. We elected to treat the mechanics of the particle motion in terms of the canonical equations of Hamilton, which are first order in the time, rather than the formulation in terms of second order equations ($\mathbf{F} = m\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$), which may be more familiar to some readers.

The canonical equations provide a more natural basis for analytical studies of magnetic field forces and can be treated directly by Runge-Kutta algorithms in more complex situations.

Because the Hamiltonian formulation may be new to the reader we showed that the Euler–Lagrange (7.11) are identical to the second order equations for the motion of charged particles. And then we showed the transition from the Euler–Lagrange approach to that of Hamilton.

With examples in the chapter and the exercises we have outlined the basic forms of the motion of charged particles in magnetic and combined electric and magnetic fields. The motion of charged particles in fields should not, however, be considered as ends in themselves. We, therefore, presented a section connecting the most modern approach in the study of fusion plasmas to the motion of individual particles in the fields.

The gyrokinetic approach to the study of plasmas does not result from an approximation imposed on the plasma, but from an ordering which emerged from the plasma.

## Exercises

**7.1.** In the text we claimed that to show

$$\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu = (\mathbf{v}\times\mathbf{B})_\mu \tag{7.52}$$

required some steps in vector algebra. These steps are always easier if we use subscript notation for cross products. It is also often easier to work with what we anticipate as a final form and show that this arises from the initial form. In the final form we have a cross product, which we write as

$$(\mathbf{v}\times\mathbf{B})_\mu = \varepsilon_{\mu\nu\gamma}\dot{q}_\nu B_\gamma$$

and a curl, which relates **B** to **A**. This is

$$B_\gamma = \varepsilon_{\gamma\alpha\beta}\frac{\partial A_\beta}{\partial q_\alpha}.$$

For time independent magnetic fields we also know that

$$\frac{\mathrm{d}}{\mathrm{d}t}A_\mu = \frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu = 0.$$

Put these together to obtain (7.52).

**7.2.** Establish Euler's identity $\exp(i\vartheta) = \cos\vartheta + i\sin\vartheta$ using the expansions for $\exp(i\vartheta)$, $\cos\vartheta$ and $\sin\vartheta$.

**7.3.** In example 7.1 we found solutions for the coordinates as (7.32) and for the canonical momenta as (7.33). These were solutions to the canonical (7.17).

(a) From the first two canonical (7.17) solve algebraically for the canonical momenta in terms of the coordinates to see that the canonical momenta are not products of mass and velocity.

(b) Show by direct substitution that (7.32) and (7.33) satisfy these first two canonical equations.

**7.4.** Write Newton's Second Law in terms of acceleration $\mathbf{a} = \mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$ and the Lorentz Force $Q\mathrm{d}\mathbf{x}/\mathrm{d}t \times \mathbf{B}$ with $\mathbf{B} = B\hat{e}_z$ and initial velocity in the plane $\perp\hat{e}_z$. Obtain a solution by introducing the complex variable $Z = x + iy$.

**7.5.** Show that for the vector potential and the magnetic field in example 7.2 $\operatorname{div}\mathbf{A} = \operatorname{div}\mathbf{B} = 0$.

**7.6.** Show that the motion of a charged particle with charge $Q$ and mass $m$ in a uniform magnetic field with induction $\mathbf{B} = B\hat{e}_z$, that is with vector potential $\mathbf{A} = -\hat{e}_x\frac{B}{2}y + \hat{e}_y\frac{B}{2}x$, and initial momentum in the $z-$direction is as we have shown in Fig. 7.8.

[This is a slight extension of Example 7.1. You will need in addition the $\dot{z}$ and $\dot{p}_z$ equations.]

**7.7.** Show that the motion of a charged particle with charge $Q$ and mass $m$ in a uniform magnetic field with induction $\mathbf{B} = B\hat{e}_z$ and a uniform electric field $\mathbf{E} = \hat{e}_z E$ is as we have shown in Fig. 7.9.

[This is a slight extension of the preceding exercise.]

**7.8.** Consider a charged particle moving in a region in which there is both a magnetic induction (field) and an electric field present, with the electric field at

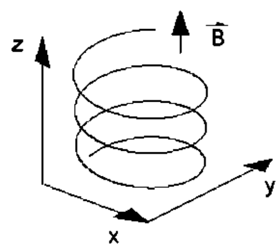



**Fig. 7.8** Motion of a positive charge in a uniform magnetic induction field with an initial momentum $p_z$

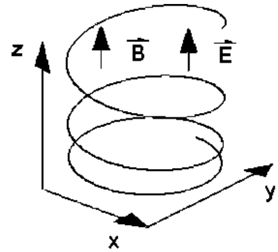



**Fig. 7.9** Motion of a charged particle in parallel magnetic induction and electric field oriented along the $z-$axis

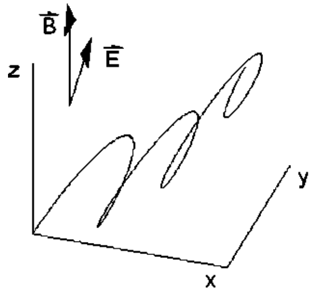



**Fig. 7.10** Motion of a charged particle in a region with magnetic induction and electric field that are neither parallel nor at right angles

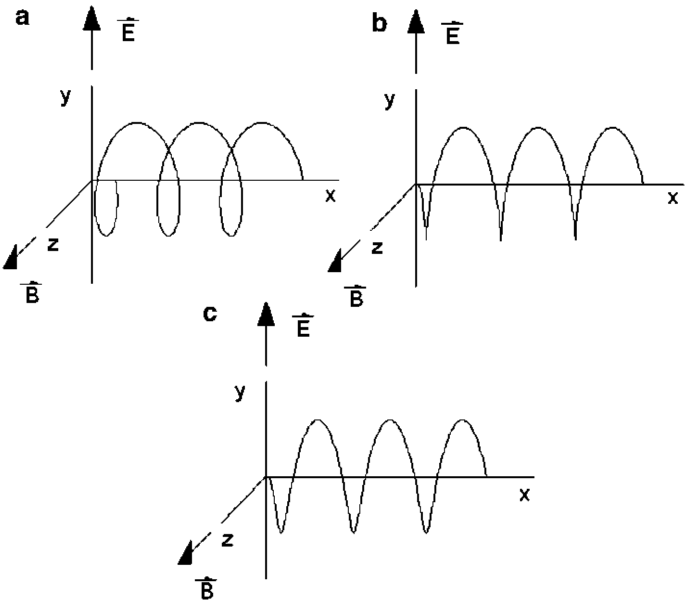



**Fig. 7.11** Possible cycloidal orbits for a charged particle moving in uniform crossed electrostatic and magnetic fields

an angle $\neq \pi/2$ with respect to the magnetic induction. Show that the motion of the charged particle is of the form shown in Fig. 7.10. Recall that for a static magnetic field along the direction $\hat{e}_z$ the vector potential is

$$\mathbf{A} = -\hat{e}_{\mathrm{x}}\frac{B}{2}y + \hat{e}_{\mathrm{y}}\frac{B}{2}x,$$

and for an electrostatic field with components along the directions $\hat{e}_{\mathrm{y}}$ and $\hat{e}_{\mathrm{z}}$ the electrostatic potential is

$$\varphi = -E_{\mathrm{y}}y - E_{\mathrm{z}}z.$$

**7.9.** In Example 7.4 we found that the motion of a charged particle in perpendicular electrostatic and uniform magnetic fields to be a combination of circular motion at a constant angular velocity and linear motion along the $x-$axis at a constant linear velocity, as we illustrated in Fig. 7.4. This motion is cycloidal. In Fig. 7.11 a, b, and c we have plotted three possible basic forms that the orbit of the charged particle may take.

The form of the orbit depends on the relationship among the magnitude of the electrostatic field $E$, the magnitude of the magnetic field induction $B$, the charge to mass ratio for the particle $Q/m$, and the radius of the orbit $R$. What is the relationship among $E$, $B$, $Q/m$, and $R$ in each case?

# Chapter 8
# Green's Functions

> *A mathematician may say anything he pleases, but a physicist must be at least partially sane.*
>
> *Josiah Willard Gibbs*

## 8.1 Introduction

This chapter is devoted to the solution of Poisson's Equation using a Green's Function.

We have already solved Poisson's equation in Sect. 4.4 (see (4.8)). That solution was the last of a set of theorems for Poisson's Equation, which we proved in Appendix F. Since the solution to Poisson's equation is unique, the solutions we obtain here will be identical to those that we would have obtained from (4.8). The difference will only be in approach.

The Green's Function method is, however, applicable beyond Poisson's Equation. It is a systematic method for solving nonhomogeneous (or inhomogeneous, see footnote 2.1) differential equations. Our treatment here will, therefore, be of a more general nature. Our results will be applicable to any nonhomogeneous differential equation.

The theorems we proved in Appendix F for the solution of Poisson's Equation were based on Green's Theorem, which was published in Green's Essay in 1828. The Green's Function solution is based on the Dirac $\delta-$ function. As we pointed out in Sect. 2.6.3 the $\delta-$function was initially proposed by Dirac in the first edition of his monograph on quantum theory in 1928 [21]. Our present study of the Green's Function relies then on at least 100 years of mathematical development. The result is a far more intuitive understanding of the solution. The more modern Green's Function approach will also prove indispensable when we treat radiation from moving charges.

This chapter and the subsequent Chap. 9 on solutions of Laplace's Equation are strictly mathematical. In these chapters we treat solutions to the potential equations

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_8, 

for static fields. These chapters are integral to learning how to solve the equations of static fields. They are not, however, absolutely necessary for an understanding of the theory of classical fields.

## 8.2 General Formulation

We may write a general, nonhomogeneous linear differential equation in the form

$$L^{(n)}(\mathbf{r})\, f(\mathbf{r}) = g(\mathbf{r}), \tag{8.1}$$

where $g(\mathbf{r})$ is the nonhomogeneous or *source term*. In the case of Poisson's Equation $g(\mathbf{r})$ is the charge density multiplied by $(-1/\varepsilon_0)$. The general linear differential operator $L^{(n)}(\mathbf{r})$ contains partial derivatives up to order $n$.

Because $L^{(n)}(\mathbf{r})$ is linear the general solution is a sum of individual solutions. If there are two source terms $g_1$ and $g_2$ for which the solutions are $f_1$ and $f_2$, then the solution for the source $g = g_1 + g_2$ is $f = f_1 + f_2$. And if $g(\mathbf{r}) = 0$, (8.1) will have a homogeneous solution $f^{(h)}$ such that

$$L^{(n)}(\mathbf{r})\, f^{(h)}(\mathbf{r}) = 0. \tag{8.2}$$

The general solution to (8.1) will then be the sum of the nonhomogeneous solution specific to $g(\mathbf{r})$ and the homogeneous solution.

The solution to the nonhomogeneous equation specific to $g(\mathbf{r})$ is called the particular solution. We will designate the particular solution as $f^{(p)}$. The general solution is then $f = f^{(p)} + f^{(h)}$. It is this general solution that must satisfy the boundary conditions imposed by the physical situation.

We will obtain the particular solution to (8.1) using a *Green's Function*. The Green's Function $G(\mathbf{r};\mathbf{r}')$ is dependent on two points $\mathbf{r}$ and $\mathbf{r}'$ and is a solution to (8.1) when the source is a $\delta-$ function with singularity at $r'$. That is, the Green's Function satisfies

$$\boxed{\mathrm{L}^{(n)}(\mathbf{r})\mathrm{G}(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}').} \tag{8.3}$$

In terms of the Green's Function we claim that the particular solution to the nonhomogeneous equation (8.1) is

$$\boxed{\mathrm{f}^{(p)}(\mathbf{r}) = \int_{\mathrm{V}} \mathrm{G}(\mathbf{r};\mathbf{r}')\mathrm{g}(\mathbf{r}')\mathrm{dV}',} \tag{8.4}$$

where the integration is over all *source coordinates* $\mathbf{r}'$. To show that (8.4) is the particular solution for the source $g(\mathbf{r})$ we need only show that $f^{(p)}(\mathbf{r})$ solves (8.1). We can do this by operating on $f^{(p)}(\mathbf{r})$ in (8.4) with $L^{(n)}(\mathbf{r})$.

Because the linear differential operator $L^{(n)}(\mathbf{r})$ operates only on the coordinates $\mathbf{r}$, and because the volume $V$ in (8.4) is fixed, we can bring the operator $L^{(n)}(\mathbf{r})$

inside the integral. Then

$$L^{(n)}(\mathbf{r})\, f^{(p)}(\mathbf{r}) = \int_V L^{(n)}(\mathbf{r})\, G\left(\mathbf{r};\mathbf{r}'\right) g\left(\mathbf{r}'\right) \mathrm{d}V'. \tag{8.5}$$

The only term inside the integral in (8.5) that depends on the coordinates $\mathbf{r}$ is the Green's Function $G(\mathbf{r};\mathbf{r}')$. Therefore $L^{(n)}(\mathbf{r})$ only operates on $G(\mathbf{r};\mathbf{r}')$ producing $\delta(\mathbf{r}-\mathbf{r}')$, according to (8.3). Then (8.5) becomes

$$\begin{aligned} L^{(n)}(\mathbf{r})\, f^{(p)}(\mathbf{r}) &= \int_{V'} \delta\left(\mathbf{r}-\mathbf{r}'\right) g\left(\mathbf{r}'\right) \mathrm{d}V' \\ &= g(\mathbf{r}), \end{aligned} \tag{8.6}$$

which is (8.1).

We then conclude that (8.4) is a general solution to (8.1) with the Green's Function defined as a solution to (8.3).

The Green's Function solution (8.4) has a particularly appealing form for physicists and engineers. The Green's Function $G(\mathbf{r};\mathbf{r}')$ is the response of a linear system, represented by the operator $L^{(n)}(\mathbf{r})$, to a discrete unit source, represented by $\delta(\mathbf{r}-\mathbf{r}')$. The total solution is the sum over these sources weighted by the density of sources $g(\mathbf{r}')$. P.M. Morse and H. Feshbach very clearly point this out in their classic text on theoretical physics ([75], pp. 791–793).

## 8.3 Poisson's Equation

We will now apply these general ideas to find the Green's Function for *Poisson's Equation*. The Green's Function $G(\mathbf{r};\mathbf{r}')$ we seek will be a solution to Poisson's Equation when the source $\rho(\mathbf{r})$ is the $\delta-$ function $\delta(\mathbf{r}-\mathbf{r}')$. That is

$$\nabla^2 G\left(\mathbf{r};\mathbf{r}'\right) = -\frac{1}{\varepsilon_0}\delta\left(\mathbf{r}-\mathbf{r}'\right). \tag{8.7}$$

In Sect. 2.6.3 we found that (see (2.107))

$$\nabla^2 \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\delta\left(\mathbf{r}-\mathbf{r}'\right). \tag{8.8}$$

Comparing (8.8) and (8.7) we see that

$$G\left(\mathbf{r};\mathbf{r}'\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{8.9}$$

is the Green's Function for Poisson's Equation.

The particular solution to Poisson's Equation, corresponding to (8.4), is then

$$\varphi^{(\mathrm{p})}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathrm{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'. \tag{8.10}$$

To this particular solution we must add the homogeneous solution to Poisson's Equation, which is a solution to Laplace's Equation. The sum must satisfy the boundary conditions.

If we are considering an infinite spatial region the value of the potential at infinity vanishes. The solution of Laplace's equation that vanishes on the boundary is $\varphi^{(\mathrm{h})}(\mathbf{r}) = 0$ (see Appendix E Theorem E.2). Therefore, in the infinite region the total solution to Poisson's Equation is the Green's Function solution

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathrm{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'. \tag{8.11}$$

## 8.4 Green's Function in One Dimension

We now seek the solution to the $n$th order linear differential equation

$$L^{(\mathrm{n})}(x)\, f(x) = g(x) \tag{8.12}$$

in the region $a \leq x \leq b$ with boundary conditions on $f(x)$ specified at the end points $x = a$ and $x = b$. In what we will refer to as *standard form* the operator $L^{(\mathrm{n})}(x)$ in (8.12) is the $n$th order linear differential operator

$$L^{(\mathrm{n})}(x) = a_0\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}x^{\mathrm{n}}} + a_1\frac{\mathrm{d}^{\mathrm{n-1}}}{\mathrm{d}x^{\mathrm{n-1}}} + \cdots + a_{\mathrm{n}}, \tag{8.13}$$

and the coefficients $a_{\mathrm{q}}$ are generally functions of the independent variable, i.e. $a_{\mathrm{q}} = a_{\mathrm{q}}(x)$.

The $n$th order homogeneous equation

$$L^{(\mathrm{n})}(x)\, h(x) = 0. \tag{8.14}$$

has $n$ linearly independent solutions $\{h_{\mathrm{k}}\}_{\mathrm{k}=1}^{\mathrm{n}}$ ([2], p. 191) and the general solution to the homogeneous linear differential equation is a sum of these, i.e.

$$f^{(\mathrm{h})}(x) = \sum_{\mathrm{k}=1}^{\mathrm{n}} \alpha_{\mathrm{k}} h_{\mathrm{k}}(x),$$

where $\alpha_{\mathrm{k}}$ are constants.

We require that the one dimensional Green's Function satisfies

$$L^{(n)}(x)\,G\left(x;x'\right)=\delta\left(x-x'\right). \tag{8.15}$$

The particular solution to (8.12) is then

$$f^{(p)}(x)=\int_{a}^{b}dx'G\left(x;x'\right)g\left(x'\right). \tag{8.16}$$

We recognize that according to (8.15) $G(x;x')$ satisfies the homogeneous (8.14) everywhere except at the point $x=x'$. We must, therefore, be able to construct the Green's Function from the set $\{h_k\}_{k=1}^{n}$ of solutions to the homogeneous differential (8.14). The Green's Function we construct will also satisfy the boundary conditions at each end of the interval $[a,b]$.[1]

In Fig. 8.1 we illustrate the interval and the location of the point $x'$. We have left the region around the point $x=x'$ open in Fig. 8.1 because we have not yet specified conditions that must hold at that point.

We begin by constructing functions $U_1(x)$ and $U_2(x)$ from linear sums of functions in the set $\{h_k\}_{k=1}^{n}$ of solutions to the homogeneous (8.14). We construct these functions such that $U_1(x)$ satisfies the boundary conditions at $x=a$ and $U_2(x)$ satisfies the boundary conditions at $x=b$. We do not require that $U_1(x)$ satisfies the boundary conditions at $x=b$ or that $U_2(x)$ satisfies the boundary conditions at $x=a$. We then write the Green's Function in two separate regions as

$$\boxed{\mathrm{G}(x;x')=\begin{cases}A(x')\,U_1(x) & \text{for } a\le x<x'\\ B(x')\,U_2(x) & \text{for } x'<x\le b\end{cases},} \tag{8.17}$$

and do not define the Green's function at the point $x'$. The functions $U_1(x)$ and $U_2(x)$ are, however, well behaved at $x=x'$. The functions $A(x')$ and $B(x')$ depend solely on the movable point $x'$, while $U_1(x)$ and $U_2(x)$ are independent of $x'$. We will choose the functions $A(x')$ and $B(x')$ based on the requirement that the Green's Function (8.17) satisfies (8.15).

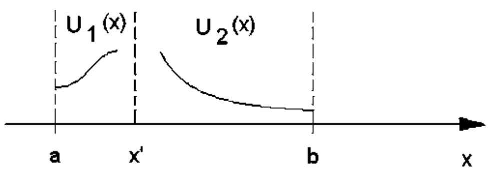



**Fig. 8.1** The interval $[a,b]$ on the $x-$ axis. The point of discontinuity is $x'$. The function $U_1(x)$ solves (8.14) and satisfies the boundary conditions at $x=a$. The function $U_2(x)$ solves (8.14) and satisfies the boundary conditions at $x=b$

[1]These may be general, such as finite values of potentials, rather than specific.

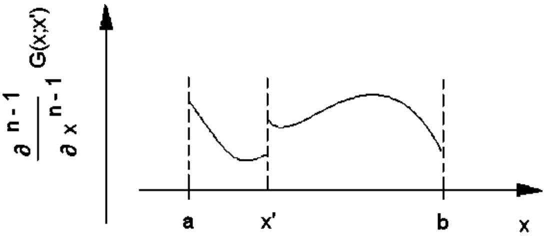



**Fig. 8.2** Discontinuity in the derivative $\partial^{n-1} G(x;x') / \partial x^{n-1}$ at the point $x = x'$

*Remark 8.1.* The choice of the functions $U_1(x)$ and $U_2(x)$ is usually not difficult. We must only recall that the boundary conditions we have are normally expressed for the electrostatic field, which is the negative gradient of the electrostatic scalar potential for which we seek the Green's Function.

A $\delta-$ function results from the derivative of a discontinuity ([89], p. 109). Therefore we want all partial derivatives of the Green's Function $G(x;x')$ with respect to $x$ up to and including the $n-1^{\text{st}}$ partial derivative to be finite in the immediate neighborhood of the point $x = x'$. But we require a finite jump in the $n-1^{\text{st}}$ partial derivative between $x = x' - \varepsilon$ and $x = x' + \varepsilon$, where $\varepsilon$ is an infinitesimal. We have drawn this requirement in Fig. 8.2.

The partial derivative $\partial^{n} G(x;x') / \partial x^{n}$ then results in a $\delta-$function at the point $x = x'$. In this way we can produce the apparently wicked behavior of (8.3) without resorting to any unusual properties of the Green's Function.

To discover the form of the discontinuity in the partial derivative $\partial^{n-1} G(x;x') / \partial x^{n-1}$ at the point $x = x'$ we turn to the requirement that $G(x;x')$ satisfies the (8.15). Integrating both sides of (8.15) with respect to $x$ over the interval $[a,b]$ we obtain

$$\int_{a}^{b} dx \left[ a_0 \frac{\partial^{n}}{\partial x^{n}} + a_1 \frac{\partial^{n-1}}{\partial x^{n-1}} + \cdots + a_n \right] G(x;x') = \int_{a}^{b} dx \delta(x - x') = 1 \tag{8.18}$$

Since $G(x;x')$ satisfies the homogeneous (8.14) everywhere except at $x'$, the integrand on the left hand side of (8.18) vanishes everywhere except at $x = x'$. So we only need to consider the integral on the left hand side of (8.18) over the two $\varepsilon$ intervals centered on $x'$. Therefore the (8.18) can be written as

$$\lim_{\varepsilon \to 0} \int_{x=x'-\varepsilon}^{x=x'+\varepsilon} dx \left[ a_0 \frac{\partial^{n}}{\partial x^{n}} + a_1 \frac{\partial^{n-1}}{\partial x^{n-1}} + \cdots + a_n \right] G(x;x') = 1 \tag{8.19}$$

We require that $G(x;x')$ is continuous over this $2\varepsilon$ interval in the limit as $\varepsilon \to 0$. That is we require that $G(x;x')$ has the same value on both sides of the point $x = x'$. Then

$$\lim_{\varepsilon \to 0} \int_{x=x'-\varepsilon}^{x=x'+\varepsilon} dx a_n G(x;x') = \left[ a_n(x') G(x';x') \right] \lim_{\varepsilon \to 0} \int_{x=x'-\varepsilon}^{x=x'+\varepsilon} dx = 0, \tag{8.20}$$

since

$$\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x = 0. \tag{8.21}$$

All the functions $a_{\mathrm{m}}(x)$ are continuous over the interval $[a, b]$. And all the partial derivatives $\partial^{\mathrm{m}} G(x, x') / \partial x^{\mathrm{m}}$ from $m = 1$ to $m = n - 1$ are continuous over the two separate $\varepsilon-$ intervals $x' - \varepsilon \le x \le x'$ and $x' \le x \le x' + \varepsilon$ in the limit as $\varepsilon \to 0$. Therefore for each $m$ from $m = 1$ to $m = n - 1$ we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x a_{\text{n-m}}(x)\,\frac{\partial^{\mathrm{m}}}{\partial x^{\mathrm{m}}} G\left(x; x'\right)\\
&= a_{\text{n-m}}\left(x'\right)\lim_{\varepsilon\to 0}\left[\frac{\partial^{\mathrm{m}}}{\partial x^{\mathrm{m}}} G\left(x' - \varepsilon; x'\right)\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x'}} \mathrm{d}x\right.\\
&\quad\left. + \frac{\partial^{\mathrm{m}}}{\partial x^{\mathrm{m}}} G\left(x' + \varepsilon; x'\right)\int_{\mathrm{x'}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x\right]\\
&= 0,
\end{aligned} \tag{8.22}$$

since

$$\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x'}} \mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathrm{x'}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x = 0. \tag{8.23}$$

The *finite* discontinuity in the partial derivative $\partial^{\mathrm{n}-1} G(x, x') / \partial x^{\mathrm{n}-1}$ over the $2\varepsilon-$ interval $x' - \varepsilon \le x \le x' + \varepsilon$ in the limit as $\varepsilon \to 0$ causes no difficulty because the product of a finite number and the limits of the integrals in (8.23) still vanishes. Equation (8.19) is then reduced to

$$\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x a_0 \frac{\partial^{\mathrm{n}}}{\partial x^{\mathrm{n}}} G\left(x; x'\right) = 1. \tag{8.24}$$

We cannot treat the integral in (8.24) as we treated integrals of lower order derivatives in (8.22) because the partial derivative $\partial^{\mathrm{n}} G(x; x') / \partial x^{\mathrm{n}}$ becomes infinite on the interval of integration in (8.24). We can, however, convert (8.24) to integrals involving lower order partial derivatives by partial integration. That is

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x a_0(x)\,\frac{\partial^{\mathrm{n}}}{\partial x^{\mathrm{n}}} G\left(x; x'\right)\\
&= \lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x \frac{\partial}{\partial x}\left[a_0(x)\,\frac{\partial^{\mathrm{n}-1}}{\partial x^{\mathrm{n}-1}} G\left(x; x'\right)\right]\\
&\quad - \lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x \left[\frac{\mathrm{d}a_0(x)}{\mathrm{d}x}\frac{\partial^{\text{n-1}}}{\partial x^{\text{n-1}}} G\left(x; x'\right)\right].
\end{aligned} \tag{8.25}$$

Since $\mathrm{d}a_0/\mathrm{d}x$ is a continuous function, the second integral on the right hand side of (8.25) is of the same form as the integrals in (8.22) and, therefore, vanishes. Combining (8.25) with (8.24) we have the form of our requirement (8.19) as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x a_0(x)\frac{\partial^{\mathrm{n}}}{\partial x^{\mathrm{n}}}G\left(x;x'\right)\\
&=\lim_{\varepsilon\to 0}\int_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}} \mathrm{d}x\frac{\partial}{\partial x}\left[a_0(x)\frac{\partial^{\mathrm{n\text{-}1}}}{\partial x^{\mathrm{n\text{-}1}}}G\left(x;x'\right)\right]\\
&=a_0\left(x'\right)\lim_{\varepsilon\to 0}\left[\frac{\partial^{\mathrm{n\text{-}1}}}{\partial x^{\mathrm{n\text{-}1}}}G\left(x;x'\right)\right]_{\mathrm{x=x'-\varepsilon}}^{\mathrm{x=x'+\varepsilon}}\\
&=1. \qquad (8.26)
\end{aligned}$$

From (8.26) we have the final requirement on the $n-1^{\mathrm{st}}$ order partial derivative of the Green's Function as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left[\frac{\partial^{\mathrm{n}-1}}{\partial x^{\mathrm{n}-1}}G\left(x;x'\right)\right]_{\mathrm{x=x='-\varepsilon}}^{\mathrm{x=x'+\varepsilon}}\\
&=\left[\frac{\partial^{\mathrm{n\text{-}1}}}{\partial x^{\mathrm{n}-1}}G\left(x;x'\right)\right]_{\mathrm{x=x'+}}-\left[\frac{\partial^{\mathrm{n}-1}}{\partial x^{\mathrm{n}-1}}G\left(x;x'\right)\right]_{\mathrm{x=x'-}}\\
&=\frac{1}{a_0\left(x'\right)} \qquad (8.27)
\end{aligned}$$

In (8.27) the subscripts $x=x'+$ and $x=x'-$ indicate that the partial derivatives are to be evaluated just to the right and just to the left of the point $x'$. That is $x=x'+$ and $x=x'-$ are values of $x$ just to the right and just to the left of the point $x'$.

The $n-1^{\mathrm{st}}$ order partial derivative of the Green's Function then has a discontinuity equal to $1/a_0\left(x'\right)$, which is the value of the coefficient of the highest order derivative in the linear differential operator (8.13) evaluated at the point $x'$. This is called the *jump condition* on the $n-1^{\mathrm{st}}$ order partial derivative of the Green's Function.

If we agree to use the *shorthand notation*

$$G^{(\mathrm{q})}\left(x';x'\right)\equiv\lim_{\varepsilon\to 0}\left[\frac{\partial^{\mathrm{q}}}{\partial x^{\mathrm{q}}}G\left(x;x'\right)\right]_{\mathrm{x'-\varepsilon}}^{\mathrm{x'+\varepsilon}}, \qquad (8.28)$$

our conditions on the Green's Function can be written systematically.

1. The functions $U_1(x)$ and $U_2(x)$ are linear sums of the homogeneous solutions that satisfy the boundary conditions for $x<x'$ and $x'<x$ respectively.
2. The Green's Function is

$$G\left(x;x'\right)=\begin{cases}A\left(x'\right)U_1\left(x\right) & x<x'\\ B\left(x'\right)U_2\left(x\right) & x>x'\end{cases}$$

with

3. $G^{(\mathrm{m})}\left(x';x'\right)=0$ for $m=0,1,\ldots n-2$ and
4. $G^{(\mathrm{n\text{-}1})}\left(x';x'\right)=1/a_0\left(x'\right)$.

The conditions (3) and (4) we use to obtain linear algebraic equations for the functions $A\left(x'\right)$ and $B\left(x'\right)$.

We are then able to obtain the Green's Function for any differential equation for which we can find the homogeneous solution(s).

The Green's Function solution is the particular solution of the nonhomogeneous differential equation. The complete solution is the sum of the particular solution and a sum of the homogeneous solutions. That is

$$\varphi\left(x\right)=\varphi^{(\mathrm{p})}\left(x\right)+\sum_{\mathrm{k}=1}^{\mathrm{n}}\alpha_{\mathrm{k}}h_{\mathrm{k}}\left(x\right), \qquad (8.29)$$

where the constants $\alpha_{\mathrm{k}}$ are chosen to satisfy the boundary conditions imposed on the problem.

*Example 8.1.* **Charged Spherical Shell.** We consider a spherical shell of radius $a$ on which a total charge $Q$ is uniformly distributed, which we have drawn in Fig. 8.3. We are interested in the electrostatic scalar potential for the regions $r<a$ and $r>a$. Because of the symmetry we use spherical coordinates $(r,\vartheta,\phi)$.

Since the charge density is uniform we have no $\vartheta$ or $\phi$ dependence and Poisson's Equation is (see Appendix A.1 equation(A.12))

$$\nabla^2\varphi=\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\varphi\right)=-\frac{1}{\varepsilon_0}\rho\left(r\right).$$

Writing the differential operator in the standard form of (8.13) we have

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\varphi\right)=\frac{\mathrm{d}^2}{\mathrm{d}r^2}\varphi+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\varphi.$$

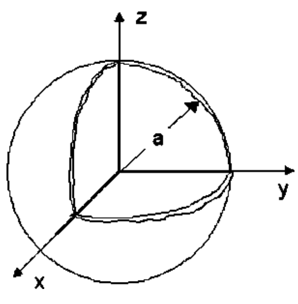



**Fig. 8.3** Thin charged spherical shell of radius $a$

Identifying the coefficients,

$$a_0 = 1, a_1 = \frac{2}{r}, \text{ and } a_2 = 0.$$

The (general) boundary conditions are

$$\varphi(0) = \text{constant} < \infty$$
$$\varphi(\infty) = 0.$$

The homogeneous equation is

$$\nabla^2 h = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}h\right) = 0,$$

and the homogeneous solutions are

$$h_1 = \text{constant}$$

and

$$h_2 = \frac{\text{constant}}{r}.$$

Of these solutions only $h_1$ satisfies the condition at the origin and only $h_2$ satisfies the condition at infinity. That is $U_1(r) =$constant and $U_2(r) =$constant$/r$ and our Green's Function has the form

$$G\left(r;r'\right) = \begin{cases} A\left(r'\right) \text{ for } 0 \le r < r' \\ B\left(r'\right)/r \text{ for } r' < r < \infty. \end{cases}$$

The constants are absorbed into the functions $A(r')$ and $B(r')$.

The differential operator is second order. The $n-1^{\text{st}}$ derivative is then $\partial G(r;r')/\partial r$. The conditions (3) and (4), i.e. the continuity and jump conditions on the Green's Function, are then $G^{(0)}(r';r') = 0$ and $G^{(1)}(r';r') = 1/a_0 = 1$. The first partial derivative of the Green's Function is

$$\partial G\left(r;r'\right)/\partial r = \begin{cases} 0 \text{ for } 0 \le r < r' \\ -B\left(r'\right)/r^2 \text{ for } r' < r < \infty. \end{cases}.$$

Then our condition of continuity of the Green's Function is

$$A\left(r'\right) = \frac{B\left(r'\right)}{r'} \tag{8.30}$$

and the jump condition on the first partial derivative is

$$-\frac{B(r')}{(r')^2} - 0 = 1. \tag{8.31}$$

Solving (8.30) and (8.31) for $A(r')$ and $B(r')$ we obtain the Green's Function as

$$G(r;r') = \begin{cases} -r' \text{ for } 0 \leq r < r' \\ -\frac{(r')^2}{r} \text{ for } r' < r < \infty. \end{cases} \tag{8.32}$$

We now need the form of $\rho(r)$.

If we consider that the shell has an infinitesimal thickness then $\rho(r)$ will be proportional to a $\delta-$ function. The form of $\rho(r)$, which upon integration over all space produces a total charge $Q$, is

$$\rho(r) = \frac{Q}{4\pi r^2}\delta(r-a). \tag{8.33}$$

The particular solution using our Green's Function is found from an integration over the source coordinates $r'$. This corresponds to the integration in (8.11), which was over the source coordinates. The integral here is, however, only over the single coordinate $r'$, not over a volume. The integration extends over the entire region $0 \leq r' < \infty$.

$$\varphi^{(p)}(r) = -\frac{1}{\varepsilon_0}\int_{r'=0}^{+\infty} dr' G(r;r')\,\rho(r'). \tag{8.34}$$

Because the Green's Function is not defined at the point $r' = r$, we remove this from the integration and (8.34) becomes

$$\begin{aligned}\varphi^{(p)}(r) = &-\frac{1}{\varepsilon_0}\int_{r'=0}^{r} dr' G(r;r')\,\rho(r') \\ &-\frac{1}{\varepsilon_0}\int_{r'=r}^{+\infty} dr' G(r;r')\,\rho(r').\end{aligned} \tag{8.35}$$

With the charge density (8.33) (8.35) is

$$\begin{aligned}\varphi^{(p)}(r) = &-\frac{1}{\varepsilon_0}\int_{r'=0}^{r} dr' G(r;r')\frac{Q}{4\pi(r')^2}\delta(r'-a) \\ &-\frac{1}{\varepsilon_0}\int_{r'=r}^{+\infty} dr' G(r;r')\frac{Q}{4\pi(r')^2}\delta(r'-a).\end{aligned} \tag{8.36}$$

Now we must decide upon which part of the Green's Function (8.32) is to be used in which integral in (8.36).

The first integral in (8.36) is over $0 \leq r' < r$. In this range $G(r;r') = -(r')^2/r$, which is the lower line in (8.32). The second integral is over $r < r' < \infty$. In this range $G(r;r') = -r'$, which is the top line in (8.32). Then (8.36) becomes

$$\varphi^{(p)}(r) = \frac{1}{\varepsilon_0}\int_{r'=0}^{r} dr' \frac{(r')^2}{r}\frac{Q}{4\pi (r')^2}\delta(r'-a) + \frac{1}{\varepsilon_0}\int_{r'=r}^{\infty} dr' (r') \frac{Q}{4\pi (r')^2}\delta(r'-a). \quad (8.37)$$

To go farther we must decide if we are interested in $r$ inside or outside the spherical shell.

Inside the sphere $r < a$ and the first integral in (8.37) vanishes because the $\delta-$ function vanishes inside the sphere. The second integral in (8.37) produces

$$\varphi^{(p)}(r) = \frac{1}{\varepsilon_0}\int_{r'=r}^{\infty} dr' (r') \frac{Q}{4\pi (r')^2}\delta(r'-a) = \frac{Q}{4\pi\varepsilon_0 a}. \quad (8.38)$$

Outside the sphere $r > a$ and the second integral in (8.37) vanishes because the $\delta-$ function vanishes outside the sphere. The first integral in (8.37) produces

$$\varphi^{(p)}(r) = \frac{1}{\varepsilon_0}\int_{r'=0}^{r} dr' \frac{1}{r}\frac{Q}{4\pi}\delta(r'-a) = \frac{Q}{4\pi\varepsilon_0 r}. \quad (8.39)$$

In this example we have carefully noted the forms of the Green's Function in each of the final integrals and the location of the spherical shell for each region of $r'$. We caution the reader to be as slow and methodical at this step as we have been in this example.

Our example has shown us that the entire empty region inside the sphere ($r < a$) is at a uniform potential. This is essentially what Priestly observed in his experiment and from which he claimed that the potential for the electrostatic field must have the same form as that of the gravitational field (see Sect. 1.7). We have then, without necessarily intending to do so, shown that Priestly was correct. There is no evidence that Priestly did anything resembling the calculation we have here. His insight was, however, remarkable.

*Example 8.2.* **Nonconducting Cylinder.** We consider a long nonconducting cylinder with a charge density $\rho(r)$ as we have drawn in Fig. 8.4. The charge density is symmetric around the $z$-axis. If we assume that the cylinder is very long and

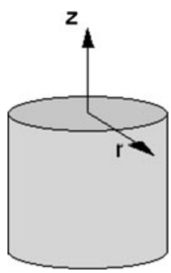



**Fig. 8.4** Charged nonconducting cylinder

that we are interested only in the electrostatic potential near the midpoint of the cylinder, we may neglect any end effects and treat this as a problem with only a radial dependence.

Poisson's Equation is then

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = -\frac{1}{\varepsilon_0}\rho\left(r\right) \tag{8.40}$$

(see Appendix A.1 (A.8)). In standard form (8.40) is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{1}{\varepsilon_0}\rho\left(r\right), \tag{8.41}$$

from which we can identify

$$a_0 = 1. \tag{8.42}$$

The homogeneous equation is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}h\right) = 0. \tag{8.43}$$

which has solutions $h_1 = \text{constant}$ and $h_2 = \ln\left(r\right)$.

The only homogeneous solution that will yield a finite result at the origin is $h_1 =$ constant. Then $U_1\left(r\right) =$constant. In the region $r > r'$ the function $U_2\left(r\right)$ will then be $\ln\left(r\right)$. This does not satisfy the boundary condition as $r \to \infty$. So our solution will not be correct as $r \to \infty$. This is acceptable, since the assumption that the cylinder is very long can only hold for finite values of $r$.

The Green's Function is then

$$G\left(r;r'\right) = \begin{cases} A\left(r'\right) \text{ for } 0 \leq r < r' \\ B\left(r'\right)\ln\left(r\right) \text{ for } r' < r \end{cases}. \tag{8.44}$$

And the first derivative of the Green's Function is

$$G'\left(r;r'\right) = \begin{cases} 0 \text{ for } 0 \leq r < r' \\ B\left(r'\right)\dfrac{1}{r} \text{ for } r' < r \end{cases}. \tag{8.45}$$

The requirements on the Green's Function are continuity at $r = r'$

$$A\left(r'\right) = B\left(r'\right)\ln\left(r'\right),$$

and discontinuity of the first derivative at $r = r'$

$$B\left(r'\right)\frac{1}{r'} - 0 = 1.$$

Then $B(r') = r'$, $A(r') = r' \ln(r')$, and the Green's Function is

$$G\left(r;r'\right) = \begin{cases} r' \ln(r') & \text{for } 0 \le r < r' \\ r' \ln(r) & \text{for } r' < r \end{cases} . \tag{8.46}$$

The electrostatic potential is obtained from

$$\varphi(r) = -\frac{1}{\varepsilon_0}\int_{\mathrm{r}'=0}^{\mathrm{r}} \mathrm{d}r' r' \ln(r)\, \rho\left(r'\right) - \frac{1}{\varepsilon_0}\int_{\mathrm{r}'=\mathrm{r}}^{\infty} \mathrm{d}r' r' \ln\left(r'\right) \rho\left(r'\right) \tag{8.47}$$

## 8.5 Vector Potential

Because the vector potential also satisfies Poisson's Equation for the magnetostatic field, we can also find a Green's Function solution in one dimension for problems involving the vector potential. We must only remember that the quantity measured is the magnetic field induction and not the vector potential. Therefore boundary conditions will be formulated in terms of magnetic field induction.

## 8.6 Summary

In this chapter we have presented a general method for solving linear, nonhomogeneous differential equations. There are other methods we can use, such as expansion in eigenfunctions. These methods, however, often suffer from slow convergence, as Morse and Feshbach point out ([75], p. 791). The Green's Function provides an alternative and completely general method of solution.

The generality lies in the fact that the Green's Function, defined by the equation $L^{(\mathrm{n})}(\mathbf{r})\, G(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$, is the right inverse of the linear differential operator $L^{(\mathrm{n})}(\mathbf{r})$ ([8], p. 39, [75], pp. 869–886). If the nonhomogeneous differential equation has a unique solution then that solution is the Green's Function solution.

If we can find the solutions to the homogeneous differential equation the we can construct the Green's Function. We repeat the steps here for the reader's convenient reference.

1. The functions $U_1(x)$ and $U_2(x)$ are linear sums of the homogeneous solutions that satisfy the boundary conditions for $x < x'$ and $x' < x$ respectively.
2. The Green's Function is

$$G\left(x;x'\right) = \begin{cases} A(x')\, U_1(x) & x < x' \\ B(x')\, U_2(x) & x > x' \end{cases}$$

with

3. $G^{(m)}(x'; x') = 0$ for $m = 0, 1, \ldots n - 2$ and
4. $G^{(n\text{-}1)}(x'; x') = 1/a_0(x')$.

## Exercises

**8.1.** In an example in the chapter we found the Green's Function for a long nonconducting cylinder in which there is a charge density dependent only on the radial coordinate. For the Green's Function found in that example, find the potential inside the nonconductor ($r < R$) for the charge densities

(a) $\rho(r) = \rho_0$ =constant
(b) $\rho(r) = \alpha r^2$

In each case check to see that Poisson's Equation is satisfied by your solution.

**8.2.** For the two long nonconducting cylinders with the charge densities

(a) $\rho(r) = \rho_0$ =constant
(b) $\rho(r) = \alpha r^2$

considered in the preceding exercise find the electrostatic potential and the electrostatic field outside of the cylinder in each case. Use the same Green's Function.

Cast both answers in terms of charge density per unit length along the nonconductors.

In each case check to see that Poisson's Equation is satisfied by your solution.

**8.3.** You have a very long thin wire, which you have mounted in the laboratory between two points separated vertically by a distance $L$. The mounting clamps are insulators and we may assume that the humidity is low enough in the laboratory that the charge leakage from the wire will be minimal. Your intention is to charge the thin wire to a total charge of $Q$ C. You realize that the total charge will reside on the surface in this static case. So the charge density is

$$\rho(r) = \frac{\lambda}{2\pi r}\delta(r - R),$$

where $R$ is the radius of the thin wire and $\lambda = Q/L$. You intend to measure the electric potential in the region around the wire near the middle of the wire and to reference that to the potential of the wire.

Using a one dimensional Green's Function obtain a theoretical prediction for the result.

Can the same Green's Function used in the preceding exercise be used for a long cylindrical conductor?

**8.4.** You have a small conducting sphere of radius $a$ concentrically enclosed by a nonconducting sphere of radius $R$. The nonconducting sphere has a charge density $\rho(r) = A/r$ with $A$ a constant for $a < r \leq R$. The total charge in the nonconductor is $Q$. Use a one dimensional Green's Function to obtain the potential for $a < r \leq R$ and for $r > R$.

Choose the electrostatic potential, and hence the Green's Function to be constant at the origin. Is the Green's Function from the example in the text then appropriate for this exercise?

**8.5.** In the preceding exercise we required that the value of the Green's Function at the origin was constant. This is the choice of a boundary condition. We may choose that constant to be zero. And we may argue that this is more legitimate for the situation considered with no charge in the region $0 < r < a$.

Show that requiring that the electrostatic potential and the Green's Function vanish at the origin produces the Green's Function

$$G\left(r; r'\right) = \begin{cases} 0 & 0 \leq r < r' \\ r' - \frac{(r')^2}{r} & r' < r \leq \infty. \end{cases}.$$

**8.6.** Use the Green's Function obtained in the preceding exercise to obtain the potential for the charge density charge density $\rho(r) = A/r$ with $A$ a constant for $a < r \leq R$. Compare this potential with that obtained from the boundary condition requiring that the Green's Function was a constant at the origin.

**8.7.** In the exercises in Chap. 6 we found that the magnetic field induction inside a long solenoid (length $L$ and internal radius $R$) is not a constant, even near the center, but is (approximately)

$$\mathbf{B} = \hat{e}_{\mathrm{z}}\left(B_0 - B_1 z^2\right) + \hat{e}_{\mathrm{r}}\left(B_1 z r\right)$$

where

$$B_0 = \mu_0 N_\lambda I_0 \cos\vartheta_0,$$

$$B_1 = \frac{3}{2}\mu_0 N_\lambda I_0 \cos\vartheta_0 \frac{R^2}{\left(L^2 + R^2\right)^2},$$

$N_\lambda$ is the number of wire turns per unit length around the solenoid, $I_0$ is the current in the wire, and $\cos\vartheta_0 = L/\sqrt{L^2 + R^2}$.

So we cannot legitimately claim that obtaining the vector potential near the center of the solenoid can be reduced to a one dimensional problem. We even calculated the vector potential in the exercises of Chap. 5 and found that

$$\mathbf{A} = \hat{e}_\vartheta A_\vartheta = \hat{e}\left(\frac{1}{2}B_0 r - \frac{1}{2}B_1 z^2 r\right).$$

The vector potential is, however, independent of the azimuthal angle $\vartheta$. And the dependence on $z$ near the center will be weak. It may, then, be interesting to investigate the consequences of treating the vector potential as dependent only on $r$.

If we assume dependence of the vector potential only on $r$ is it again possible to use the Green's Function that we used in the preceding problems?

Find the vector potential at the center of a cylindrical solenoid using a Green's Function. From this obtain the magnetic field inside the solenoid. Compare this with the value obtained from Ampère's Circuital Law.

**8.8.** Consider a very large, flat, thin conducting plate of thickness $2a$ which you have charged. You choose coordinates such that the $x$−axis is perpendicular to the plate and paces through the center of the plate. The point $x = a$ is the surface of the plate. There is no external electrostatic field so you realize that the surface charge density on both sides of the sheet will be the same and equal to $\sigma_0$ C m$^{-2}$.

For small values of $x$ you may consider that the electrostatic potential depends only on the coordinate $x$. Poisson's Equation for the potential in the region $x > 0$ is then

$$\frac{\mathrm{d}^2}{x^2}\varphi(x) = -\frac{1}{\varepsilon_0}\rho.$$

For the surface charge density

$$\rho(x) = \sigma_0\delta(x-a).$$

Obtain the one dimensional Green's Function for Poisson's Equation and from that Green's Function obtain the electrostatic potential and the electrostatic field near the plate.

**8.9.** Consider two concentric thin spherical conducting shells of radii $a$ and $b$, with $a < b$. The thickness of each is $\varepsilon \ll a$. The shells carry charges $Q_\mathrm{a}$ and $Q_\mathrm{b}$. Using a Green's Function find the potential in the regions $r < a$, $(a+\varepsilon) < r < b$, $b < r < (b+\varepsilon)$, and $b < r$.

We should not expect to know the charge on each of the shells. But we can set the potentials. If these are $V_\mathrm{a}$ and $V_\mathrm{b}$, what are the charges $Q_\mathrm{a}$ and $Q_\mathrm{b}$?

**8.10.** Consider a long hollow conducting cylinder of outer radius $b$ and inner radius $a$ with a uniform current $I_0$ flowing through it in an axial direction. That is the current density is

$$\mathbf{J}(r) = \hat{e}_\mathrm{z}\begin{cases} J_0 \text{ for } a \leq r \leq b \\ \quad 0 \text{ otherwise}\end{cases}$$

Use a one dimensional Green's Function to find the vector potential and then the magnetostatic field induction for $0 \leq r < a$, $a \leq r \leq b$ and $r > R$.

Note that only the component of the vector potential $A_\mathrm{z}$ is nonzero. Find the form this must take at the origin to decide on form of the Green's Function near the origin.

# Chapter 9
# Laplace's Equation

*TRUTH! JUSTICE! Those are the immutable laws.*

*Pierre Simon, Marquis de Laplace*

## 9.1 Introduction

In this chapter we will consider the solution of *Laplace's Equation*

$$\nabla^2 \Phi = 0 \tag{9.1}$$

using *separation of variables*. We can claim that this chapter is required by the preceding Chap. 8, which indicated the importance of the homogeneous solution to Poisson's Equation. However, as with Chap. 8 it is not integral to a study of the theory of classical fields. The reader already familiar with the use of separation of variables to solve Laplace's Equation may skip this chapter without any loss in continuity.

In Appendix E we have outlined the fundamental theorems for the solution of Laplace's Equation.

In each of the coordinate systems considered, rectangular, cylindrical and spherical, we will show that the (unique) solution of Laplace's Equation is a product of functions dependent solely on the independent variables of each respective system. That is in the *rectangular system* we have

$$\Phi = X_{\text{Rect}}(x)\, Y_{\text{Rect}}(y)\, Z_{\text{Rect}}(z)\,,$$

in the *cylindrical system*

$$\Phi = R_{\text{Cyl}}(r)\, \Theta_{\text{Cyl}}(\vartheta)\, Z_{\text{Cyl}}(z)$$

and in the *spherical system*

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_9, © Springer-Verlag Berlin Heidelberg 2012

$$\Phi = R_{\mathrm{Sph}}(r)\,\Theta_{\mathrm{Sph}}(\vartheta)\,\Phi_{\mathrm{Sph}}(\phi)\,.$$

For the case of cylindrical coordinates we shall limit our considerations to situations in which there is no dependence on $z$. This will simplify the solution of the equation for $R_{\mathrm{Cyl}}(r)$ considerably. If we include a dependence on $z$ the equation for $R_{\mathrm{Cyl}}(r)$ becomes *Bessel's Equation* ([48], p. 103). The solutions of Bessel's Equation are known and produce no difficulty in themselves. But the resulting summations are more complicated in appearance. Since our objective here is not a detailed study of Laplace's equation it is prudent to limit the complexity.

For the case of spherical coordinates we will consider only situations with no azimuthal dependence, i.e. symmetry about an axis. This limits the polar angle solutions $\Phi_{\mathrm{Sph}}(\phi)$ to the *Legendre Polynomials* rather than requiring the more complicated *associated Legendre Functions*.

These properties of Laplace's equation provide at least potentially interesting situations. And a detailed study of the solutions of Laplace's equation, we can claim, is a legitimate part of the study of classical fields. Our objective in this text is, however, to concentrate first on the derivation of the complete mathematical description of the fields, which is contained in Maxwell's Equations. So our discussion of Laplace's equation will be brief.

We will provide examples of the use of our separation of variables solutions. However, in keeping with our objective in the text, we provide no exercises following this chapter. Exercises which use the separation of variables solutions directly may be found in Chap. 15.

## 9.2 Forms of Laplace's Equation

We provided the forms of the *Laplacian Operator* in various coordinate systems in Sect. 2.5.5. These are

- *Rectangular*

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial z_3^2}\right)\Phi = 0, \tag{9.2}$$

- *Cylindrical*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2 \Phi}{\partial \vartheta^2}\right)+\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{9.3}$$

- *Spherical*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial \Phi}{\partial \phi}\right)+\frac{1}{r^2\sin^2\phi}\left(\frac{\partial^2 \Phi}{\partial \vartheta^2}\right) = 0. \tag{9.4}$$

We shall now show that each of these equations is solved by a separation of variables of the form indicated in Sect. 9.1.

## 9.3 Rectangular Coordinates

We make the Ansatz that the function $\Phi\,(x, y, z)$ in (9.2) can be written as

$$\Phi = X_{\text{Rect}}\,(x)\,Y_{\text{Rect}}\,(y)\,Z_{\text{Rect}}\,(z)\,, \tag{9.5}$$

with $X_{\text{Rect}}Y_{\text{Rect}}Z_{\text{Rect}} \neq 0$ and inquire into the conditions under which this is possible. Inserting (9.5) into (9.2) and dividing through by $X_{\text{Rect}}Y_{\text{Rect}}Z_{\text{Rect}} \neq 0$, we have

$$\frac{X''_{\text{Rect}}\,(x)}{X_{\text{Rect}}\,(x)} + \frac{Y''_{\text{Rect}}\,(y)}{Y_{\text{Rect}}\,(y)} + \frac{Z''_{\text{Rect}}\,(z)}{Z_{\text{Rect}}\,(z)} = 0, \tag{9.6}$$

where we have used the standard primed notation for derivatives. Each of the terms in (9.6) is dependent only on one of the three independent variables $x$, $y$ or $z$. We have emphasized this by including the dependence explicitly.

If we write (9.6) as

$$\frac{X''_{\text{Rect}}\,(x)}{X_{\text{Rect}}\,(x)} = -\frac{Y''_{\text{Rect}}\,(y)}{Y_{\text{Rect}}\,(y)} - \frac{Z''_{\text{Rect}}\,(z)}{Z_{\text{Rect}}\,(z)}, \tag{9.7}$$

we have on the left hand side a function only of the independent variable $x$ and on the right hand side a function of only the independent variables $(y, z)$. A function of the independent variable $x$ can be equal to a function of the independent variables $(y, z)$ only if both functions are equal to a constant. That is

$$\frac{X''_{\text{Rect}}\,(x)}{X_{\text{Rect}}\,(x)} = -k_{\text{x}}^2, \tag{9.8}$$

Proceeding in this fashion we conclude that our separation Ansatz (9.5) is possible if and only if each of the individual terms $X''_{\text{Rect}}\,(x)\,/\,X_{\text{Rect}}\,(x)$, $Y''_{\text{Rect}}\,(y)\,/\,Y_{\text{Rect}}\,(y)$ and $Z''_{\text{Rect}}\,(z)\,/\,Z_{\text{Rect}}\,(z)$ is independently equal to a constant.

We choose

$$\frac{Y''_{\text{Rect}}\,(y)}{Y_{\text{Rect}}\,(y)} = -k_{\text{y}}^2, \tag{9.9}$$

$$\frac{Z''_{\text{Rect}}\,(z)}{Z_{\text{Rect}}\,(z)} = k_{\text{x}}^2 + k_{\text{y}}^2. \tag{9.10}$$

The (9.8)–(9.10) are then the requirements for our separation Ansatz. The constants $k_{\text{x}}$ and $k_{\text{y}}$ are, at this point, arbitrary and will be determined by the boundary conditions.

### *9.3.1 Eigenvalue Problems*

The (9.8)–(9.10) are of a special type. The second derivative is a *linear* mathematical operator. In each of the (9.8)–(9.10) we have the statement that action of this linear mathematical operator on a function, from a particular set of functions, results in a product of that function and a certain constant. That is

$$\mathfrak{L} f_\lambda = \lambda f_\lambda \tag{9.11}$$

where $\mathfrak{L}$ is the linear operator, the function $f_\lambda$ is the function from the set $\{f_\lambda\}$ and $\lambda$ is the constant.

To find this set of functions and the constants is to solve what is called an *eigenvalue problem*. In German *eigen* means unique or singular. The term eigenvalue is a partial Anglicization of the German *Eigenwert*. The solution to (9.11) is a unique set of functions corresponding to the unique set of values for $\lambda$. These are known as *eigenfunctions*.

Because (9.11) is a linear equation the general solution is a sum over the solutions, which are eigenfunctions. The solutions to (9.8) and (9.9) are sines and cosines. We are then representing our solution in terms of sines and cosines. This is known as a Fourier series representation of the solution. The solution itself is neither a sine nor a cosine. But we can add the sines and cosines together to obtain a representation of the solution.

Rather than sines and cosines we may also choose to represent our solutions as complex exponentials.

*Example 9.1.* **Representation of a Square Wave**. As an example we consider an attempt to represent a square wave by three sinusoidal functions. We show the result graphically in Fig. 9.1. In the top panel of Fig. 9.1 we have shown the three sinusoids separately as $V_{1,2,3}$ and in the bottom panel we have shown the sum of these three sinusoids as $V$. Before we had digital electronics this was the way square wave forms were produced in the laboratory. But more than only three sinusoids were used.

The expansion of a solution in eigenfunctions is a completely general approach to the solution of differential and partial differential equations. The basic idea is very physical. The eigenfunctions represent the natural types of motion characteristic of the system. The physical argument is that the general motion is then a summation of these characteristic motions. The problem becomes then one of adjusting the

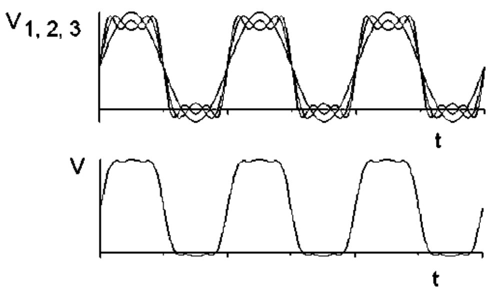



**Fig. 9.1** Fourier series representation of a square wave. The independent variable is the time $t$. The representation is in terms of three sinusoidal functions. These are plotted separately in the top panel and as a summation in the bottom panel

summation so that it fits the requirements (boundary and/or initial conditions) imposed on the system.

The fact that the solution is a linear sum is a consequence of the linearity of the operator. Our representation here of the square wave in terms of sinusoidal functions is based on the fact that the circuit equations are linear and have sinusoidal solutions.

The separation in cylindrical and spherical coordinates will produce eigenvalue problems as well.

In the late 1920s we discovered that the quantum theory was based on an eigenvalue problem This was the key that opened a door. Our study here is not so dramatic. We have only found that we are able to formulate a solution to Laplace's Equation in terms of eigenfunctions.

*Example 9.2.* As an example we consider a rectangular box as we have drawn in Fig. 9.2. The box has metal sides and base, which are all grounded, i.e. the electrostatic potential on the four sides and the base is equal to zero. The top is a nonconductor which can contain any charge distribution we may choose. The electrostatic potential on the top will then have some value, which is generally a function of $(x, y)$. We call this $V(x, y)$.

The solutions for $X_{\text{Rect}}$ and $Y_{\text{Rect}}$ are

$$X_{\text{Rect, k}_\text{x}} = \exp\left(\pm i k_\text{x} x\right) \text{ or } \cos\left(k_\text{x} x\right), \ \sin\left(k_\text{x} x\right), \tag{9.12}$$

and

$$Y_{\text{Rect, k}_\text{y}} = \exp\left(\pm i k_\text{y} y\right) \text{ or } \cos\left(k_\text{y} y\right), \ \sin\left(k_\text{y} y\right), \tag{9.13}$$

and the solution for $Z_{\text{Rect}}$ is

$$Z_{\text{Rect, k}_\text{x}\text{, k}_\text{y}} = \exp\left(\pm z\sqrt{k_\text{x}^2 + k_\text{y}^2}\right). \tag{9.14}$$

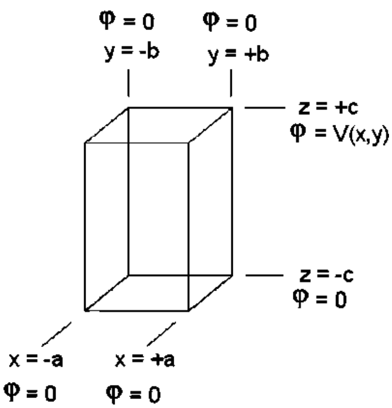



**Fig. 9.2** Rectangular box with metal sides and a nonconducting top. The sides are all grounded at electrostatic potential equal to zero and there is a charge distribution on the top, which results in a potential $V(x, y)$

To satisfy the boundary conditions that $\varphi(\pm a) = \varphi(\pm b) = 0$ the functions $X_{\mathrm{Rect,\,k_x}}$ and $Y_{\mathrm{Rect,\,k_y}}$ must be even, i.e. cosines, with

$$k_{\mathrm{x}} = k_{\mathrm{n}} = \frac{(2n-1)\,\pi}{2a} \quad \text{and } k_{\mathrm{y}} = k_{\mathrm{m}} = \frac{(2m-1)\,\pi}{2b}\,. \tag{9.15}$$

We then have

$$X_{\mathrm{Rect,\,n}} = \cos(k_{\mathrm{n}} x) \tag{9.16}$$

and

$$Y_{\mathrm{Rect,\,m}} = \cos(k_{\mathrm{m}} y) \tag{9.17}$$

Then

$$Z_{\mathrm{Rect,\,nm}} = \exp(\pm k_{\mathrm{nm}} z) \tag{9.18}$$

with

$$k_{\mathrm{nm}} = \pm\sqrt{k_{\mathrm{n}}^2 + k_{\mathrm{m}}^2} = \pm\frac{\pi}{2}\sqrt{\frac{(2n-1)^2}{a} + \frac{(2n-1)^2}{b}}. \tag{9.19}$$

The function $Z_{\mathrm{Rect}}$ is a sum of the solutions (9.18). We then write

$$Z_{\mathrm{Rect,\,nm}} = A_{\mathrm{nm}} \exp(k_{\mathrm{nm}} z) + B_{\mathrm{nm}} \exp(-k_{\mathrm{nm}} z)\,. \tag{9.20}$$

the boundary condition at $z = -c$ requires that

$$A_{\mathrm{nm}} = -B_{\mathrm{nm}} \exp(2k_{\mathrm{nm}} c)\,,$$

and that

$$\begin{aligned} Z_{\mathrm{Rect,\,nm}} &= -B_{\mathrm{nm}} \exp(2k_{\mathrm{nm}} c) \exp(k_{\mathrm{nm}} z) + B_{\mathrm{nm}} \exp(-k_{\mathrm{nm}} z) \\ &= \Phi_{\mathrm{nm}} \sinh(k_{\mathrm{nm}}(z + c))\,, \end{aligned} \tag{9.21}$$

where $\Phi_{\mathrm{nm}}$ is, at this point, completely arbitrary. The general solution to our problem is then

$$\varphi(x, y, z) = \sum_{\mathrm{n,m}=1}^{\infty} \Phi_{\mathrm{nm}} \cos(k_{\mathrm{n}} x) \cos(k_{\mathrm{m}} y) \sinh(k_{\mathrm{nm}}(z + c))\,. \tag{9.22}$$

The coefficients $\Phi_{\mathrm{nm}}$ are chosen to satisfy the boundary condition at $z = +c$. Then

$$V(x, y) = \sum_{\mathrm{n,m}=1}^{\infty} \Phi_{\mathrm{nm}} \cos(k_{\mathrm{n}} x) \cos(k_{\mathrm{m}} y) \sinh(2k_{\mathrm{nm}} c)\,. \tag{9.23}$$

Because

$$\int_{-a}^{+a} \cos\left(\frac{(2p-1)\,\pi}{2a} x\right) \cos\left(\frac{(2q-1)\,\pi}{2a} x\right) dx = a\delta_{\mathrm{pq}}, \tag{9.24}$$

with $p$ and $q$ integer, we have

$$\begin{aligned}\int_{-a}^{+a} \mathrm{d}x \int_{-b}^{+b} \mathrm{d}y V(x,y)\cos(k_{\mathrm{r}}x)\cos(k_{\mathrm{s}}y) &= \sum_{\mathrm{n,m}=1}^{\infty} \Phi_{\mathrm{nm}} \sinh(2k_{\mathrm{nm}}c)\cdots \\ &\quad \cdots \int_{-a}^{+a} \mathrm{d}x \cos(k_{\mathrm{n}}x)\cos(k_{\mathrm{r}}x)\cdots \\ &\quad \cdots \int_{-b}^{+b} \mathrm{d}y \cos(k_{\mathrm{m}}y)\cos(k_{\mathrm{s}}y) \\ &= ab \sum_{\mathrm{n,m}=1}^{\infty} \Phi_{\mathrm{nm}} \sinh(2k_{\mathrm{nm}}c)\,\delta_{\mathrm{nr}}\delta_{\mathrm{ms}} \\ &= ab\Phi_{\mathrm{rs}} \sinh(2k_{\mathrm{rs}}c)\,. \end{aligned} \tag{9.25}$$

Then

$$\Phi_{\mathrm{nm}} = \frac{1}{ab\sinh(2k_{\mathrm{nm}}c)} \int_{-a}^{+a} \mathrm{d}x \int_{-b}^{+b} \mathrm{d}y V(x,y)\cos(k_{\mathrm{n}}x)\cos(k_{\mathrm{m}}y) \tag{9.26}$$

## 9.4 Cylindrical Coordinates

We will limit our treatment of Laplace's Equation in cylindrical coordinates to situations in which there is no $z-$dependence. Then Laplace's (9.3) becomes

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\vartheta^2} = 0. \tag{9.27}$$

We make the Ansatz that the function $\Phi(r,\vartheta)$, which is a solution to (9.27), can be written as

$$\Phi = R_{\mathrm{Cyl}}(r)\,\Theta_{\mathrm{Cyl}}(\vartheta)\,, \tag{9.28}$$

with $R_{\mathrm{Cyl}}(r)\,\Theta_{\mathrm{Cyl}}(\vartheta) \neq 0$. Inserting (9.28) into (9.27) and dividing through by $R_{\mathrm{Cyl}}(r)\,\Theta_{\mathrm{Cyl}}(\vartheta)$, we have

$$\frac{1}{R_{\mathrm{Cyl}}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}\right) + \frac{1}{\Theta_{\mathrm{Cyl}}r^2}\frac{\mathrm{d}^2\Theta_{\mathrm{Cyl}}}{\mathrm{d}\vartheta^2} = 0, \tag{9.29}$$

or

$$\frac{r}{R_{\mathrm{Cyl}}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}\right) = -\frac{1}{\Theta_{\mathrm{Cyl}}}\frac{\mathrm{d}^2\Theta_{\mathrm{Cyl}}}{\mathrm{d}\vartheta^2} \tag{9.30}$$

The left hand side of (9.30) is a function only of the independent coordinate $r$ and the right hand side of (9.30) is a function only of the independent coordinate $\vartheta$.

These must then both be equal to a constant, which we choose to be $\alpha^2$.

$$\frac{r}{R_{\mathrm{Cyl}}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}\right)=-\frac{1}{\Theta_{\mathrm{Cyl}}}\frac{\mathrm{d}^2\Theta_{\mathrm{Cyl}}}{\mathrm{d}\vartheta^2}=\alpha^2. \tag{9.31}$$

We have chosen the constant in (9.31) as a square because of the second derivatives. And we have chosen a positive quantity because we desire sinusoidal solutions in $\vartheta$. This is done with foresight. But the boundary conditions will finally fix the values of the constants.

The solutions to the differential equation for $\Theta_{\mathrm{Cyl}}$ in (9.31) are

$$\Theta_{\mathrm{Cyl},\,\alpha}=\cos\left(\alpha\vartheta\right),\ \sin\left(\alpha\vartheta\right), \tag{9.32}$$

or a complex exponential. Regardless of the form of the potential $\Phi$ it will be a periodic function of $\vartheta$ with period $2\pi$. Then (9.32) must be periodic under a $2\pi$ rotation. This will be sthe case if

$$\alpha=n\ \ n=0,\pm1,\pm2,\pm3,.... \tag{9.33}$$

For a particular value of $n$ the solution to (9.31) is then the sum

$$\Theta_{\mathrm{Cyl,\,n}}=A_{\mathrm{n}}\cos\left(n\vartheta\right)+B_{\mathrm{n}}\sin\left(n\vartheta\right). \tag{9.34}$$

We note that if $n=0$ the function $\Theta_{\mathrm{Cyl},\,0}=A_0$, a constant.

The differential equation for $R_{\mathrm{Cyl}}$, from (9.31), is then

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}\right)=n^2R_{\mathrm{Cyl}}. \tag{9.35}$$

And if $n=0$ we have

$$r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}=\text{constant or }0, \tag{9.36}$$

which results in

$$R_{\mathrm{Cyl},\,0}=C_0\ln\left(r\right)+D_0, \tag{9.37}$$

where $D_0$ is a constant.

For $n\neq0$ (9.35) is

$$r^2\frac{\mathrm{d}^2R_{\mathrm{Cyl}}}{\mathrm{d}r^2}+r\frac{\mathrm{d}R_{\mathrm{Cyl}}}{\mathrm{d}r}=-n^2R_{\mathrm{Cyl}}, \tag{9.38}$$

which is solved by

$$R_{\mathrm{Cyl},\,\beta}=r^{\pm\beta}. \tag{9.39}$$

That is, putting (9.39) into (9.38) we have

$$r^2\left(\pm\beta\right)\left(\pm\beta-1\right)r^{\pm\beta-2}+r\left(\pm\beta\right)r^{\pm\beta-1}=n^2r^{\pm\beta}, \tag{9.40}$$

or

$$r^{\pm\beta}\left[(\pm\beta)(\pm\beta-1)+(\pm\beta)-n^2\right]=0. \tag{9.41}$$

from which we find that $\pm\beta=\pm n$. That is the solutions to (9.38) are

$$R_{\text{Cyl, n}}=r^{\text{n}}, \tag{9.42}$$

where $n$ may take on all positive and negative integer values excluding 0, for which the solution is (9.37).

Combining (9.42) with (9.34) we have the functions

$$\Phi_0(r,\vartheta)=C_0\ln(r)+D_0, \tag{9.43}$$

where we have incorporated $A_0$ into the final values of $C_0$ and $D_0$, and

$$\begin{aligned}\Phi_{\text{n}}(r,\vartheta)&=R_{\text{Cyl, n}}\Theta_{\text{Cyl, n}}\\&=A_{\text{n}}r^{\text{n}}\cos(n\vartheta)+r^{\text{n}}B_{\text{n}}\sin(n\vartheta),\end{aligned} \tag{9.44}$$

where $n$ takes on all positive and negative integer values. The general solution for $\Phi(r,\vartheta)$ is then

$$\Phi(r,\vartheta)=C_0\ln(r)+D_0+\sum_{n=-\infty}^{\infty}A_{\text{n}}r^{\text{n}}\cos(n\vartheta)+B_{\text{n}}r^{\text{n}}\sin(n\vartheta). \tag{9.45}$$

*Example 9.3.* As an example we ask for the potential inside a nonconducting cylindrical shell of radius $a$ on which the potential is $V(\vartheta)$.

The potential within the cylinder cannot depend on negative powers of $r$ or on $\ln(r)$ because at the origin these are infinite. Then the potential within the cylindrical shell is of the form

$$\Phi(r,\vartheta)=\Phi_0+\sum_{n=1}^{\infty}A_{\text{n}}r^{\text{n}}\cos(n\vartheta)+B_{\text{n}}r^{\text{n}}\sin(n\vartheta)$$

where $\Phi_0=A_0+D_0$.

We evaluate the constants $\Phi_0$, $A_{\text{n}}$ and $B_{\text{n}}$ from the potential $V(\vartheta)$ on the nonconducting shell

$$V(\vartheta)=\Phi_0+\sum_{n=1}^{\infty}A_{\text{n}}a^{\text{n}}\cos(n\vartheta)+B_{\text{n}}a^{\text{n}}\sin(n\vartheta).$$

Here $\Phi_0$is equal to any constant term in $V(\vartheta)$.

Because

$$\int_0^{2\pi}\cos(n\vartheta)\cos(m\vartheta)\,\text{d}\vartheta=\pi\delta_{\text{nm}}, \tag{9.46}$$

$$\int_0^{2\pi} \sin(n\vartheta)\sin(m\vartheta)\,d\vartheta = \pi\delta_{nm} \tag{9.47}$$

and

$$\int_0^{2\pi} \sin(n\vartheta)\cos(m\vartheta)\,d\vartheta = 0$$

the constants $A_n$ and $B_n$ are

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} V(\vartheta)\cos(n\vartheta)\,d\vartheta.$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} V(\vartheta)\sin(n\vartheta)\,d\vartheta.$$

## 9.5 Spherical Coordinates

We make the Ansatz that the function $\Phi(r,\phi,\vartheta)$, which is a solution to (9.4) can be written as

$$\Phi = R_{Sph}(r)\,\Phi_{Sph}(\phi)\,\Theta_{Sph}(\vartheta), \tag{9.48}$$

with $R_{Sph}(r)\,\Phi_{Sph}(\phi)\,\Theta_{Sph}(\vartheta) \neq 0$. This will require two separations. The first separation is

$$\Phi = R_{Sph}(r)\,Y_{Sph}(\vartheta,\varphi). \tag{9.49}$$

Then (9.4) results in

$$\frac{1}{R_{Sph}}\frac{d}{dr}\left(r^2\frac{dR_{Sph}}{dr}\right) + \frac{1}{Y_{Sph}}\left\{\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_{Sph}}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 Y_{Sph}}{\partial\vartheta^2}\right\} = 0. \tag{9.50}$$

The term involving $R_{Sph}$ depends only on the independent variable $r$ and the term involving $Y_{Sph}$ depends only on the independent coordinates $(\vartheta,\phi)$, Therefore each of these terms must be equal to a constant.

$$\frac{1}{R_{Sph}}\frac{d}{dr}\left(r^2\frac{dR_{Sph}}{dr}\right) = \alpha^2 \tag{9.51}$$

$$\frac{1}{Y_{Sph}}\left\{\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y_{Sph}}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 Y_{Sph}}{\partial\vartheta^2}\right\} = -\alpha^2. \tag{9.52}$$

We now try a separation of (9.52) with the Ansatz that the function $Y_{Sph}(\vartheta,\varphi)$ can be written as

$$Y_{Sph}(\vartheta,\varphi) = \Theta_{Sph}(\vartheta)\,\Phi_{Sph}(\varphi). \tag{9.53}$$

Inserting (9.53) into (9.52) and multiplying through by $\sin^2\phi$ we have

$$\frac{1}{\Phi_{\mathrm{Sph}}}\sin\phi\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}\Phi_{\mathrm{Sph}}}{\mathrm{d}\phi}\right)+\alpha^2\sin^2\phi=-\frac{1}{\Theta_{\mathrm{Sph}}}\frac{\mathrm{d}^2\Theta_{\mathrm{Sph}}}{\mathrm{d}\vartheta^2}. \tag{9.54}$$

The term on the left hand side of (9.54) is a function only of the independent coordinate $\phi$ and the term on the right hand side is a function only of the independent coordinate $\vartheta$. The equality of these two terms in (9.54) can only hold if they are equal to the same constant. As in the case of cylindrical coordinates, regardless of the form of the potential $\Phi$, the solution will be periodic in $\vartheta$ with period $2\pi$.

We, therefore, choose the constant to be the square of a number $m$. Then

$$\Theta_{\mathrm{Sph}}=\cos(m\vartheta),\ \sin(m\vartheta), \tag{9.55}$$

or a complex exponential. And

$$\frac{1}{\Phi_{\mathrm{Sph}}}\sin\phi\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\sin\phi\frac{\mathrm{d}\Phi_{\mathrm{Sph}}}{\mathrm{d}\phi}\right)+\alpha^2\sin^2\phi=m^2 \tag{9.56}$$

If we introduce

$$x=\cos\phi \tag{9.57}$$

we have

$$\sin\phi\frac{\mathrm{d}}{\mathrm{d}\phi}=\sin\phi\frac{\mathrm{d}x}{\mathrm{d}\phi}\frac{\mathrm{d}}{\mathrm{d}x}=\left(x^2-1\right)\frac{\mathrm{d}}{\mathrm{d}x}$$

and (9.56) becomes

$$\left(x^2-1\right)\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-1\right)\frac{\mathrm{d}\Phi_{\mathrm{Sph}}}{\mathrm{d}x}\right)-\alpha^2\Phi_{\mathrm{Sph}}\right]=m^2\Phi_{\mathrm{Sph}} \tag{9.58}$$

We achieve a considerable simplification, and yet retain the ability to consider interesting problems, if we limit our consideration to situations for which there is no azimuthal dependence. That is

$$\frac{1}{\Theta_{\mathrm{Sph}}}\frac{\mathrm{d}^2\Theta_{\mathrm{Sph}}}{\mathrm{d}\vartheta^2}=m^2=0. \tag{9.59}$$

With $m^2=0$, (9.58) becomes, since $\left(x^2-1\right)\neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-1\right)\frac{\mathrm{d}\Phi_{\mathrm{Sph}}}{\mathrm{d}x}\right)-\alpha^2\Phi_{\mathrm{Sph}}=0, \tag{9.60}$$

or

$$\left(x^2-1\right)\frac{\mathrm{d}^2\Phi_{\mathrm{Sph}}}{\mathrm{d}x^2}+2x\frac{\mathrm{d}\Phi_{\mathrm{Sph}}}{\mathrm{d}x}-\alpha^2\Phi_{\mathrm{Sph}}=0. \tag{9.61}$$

This is *Legendre's Equation*. It is an eigenvalue problem

$$\mathfrak{L}_{\text{Legendre}}\Phi_{\text{Sph}} = \alpha^2\Phi_{\text{Sph}} \tag{9.62}$$

with

$$\mathfrak{L}_{\text{Legendre}} = \left(x^2 - 1\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{9.63}$$

The eigenvalues for this problem are

$$\alpha^2 = n\,(n+1) \text{ with } n = 0, 1, 2, \ldots. \tag{9.64}$$

And the eigenfunctions are polynomials

$$P_{\text{n}}(x) = \frac{1}{2^{\text{n}}n!}\frac{\mathrm{d}^{\text{n}}}{\mathrm{d}x^{\text{n}}}\left(x^2 - 1\right)^{\text{n}}. \tag{9.65}$$

We show this in Appendix H. The first few Legendre Polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \\ &\vdots \end{aligned} \tag{9.66}$$

The Legendre Polynomials are orthogonal, as a result of the properties of Legendre's Equation ([20], p. 169, 170). But they are not normalizable. Specifically

$$\int_{-1}^{+1} P_{\text{q}}(x)\,P_{\text{p}}(x)\,\mathrm{d}x = \frac{2}{2q+1}\delta_{\text{qp}}. \tag{9.67}$$

From Legendre's Equation we can directly show that

$$\int_{-1}^{+1} P_{\text{n}}(x)\,\mathrm{d}x = 0. \tag{9.68}$$

In the event that there is a dependence on $\vartheta$ and $m \neq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1 - x^2\right)\frac{\mathrm{d}\Phi_{\text{Sph}}}{\mathrm{d}x}\right] + \left[n\,(n+1) - \frac{m^2}{\left(1 - x^2\right)}\right]\Phi_{\text{Sph}} = 0, \tag{9.69}$$

which is the *associated Legendre Equation*. The solutions are the *associated Legendre Functions*. The associated Legendre Functions are ([20], p171)

$$P_{\mathrm{n}}^{\mathrm{m}}(x) = \left(1 - x^2\right)^{\frac{\mathrm{m}}{2}} \frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{d}x^{\mathrm{m}}} P_{\mathrm{n}}(x), \tag{9.70}$$

with the additional requirement that

$$-n \leq m \leq +n. \tag{9.71}$$

The functions $Y_{\mathrm{Sph}}(\vartheta, \varphi) = \Theta_{\mathrm{Sph}}(\vartheta)\, \Phi_{\mathrm{Sph}}(\varphi)$ originally introduced in (9.53), with $m \neq 0$, are termed the *Spherical Harmonics*. These are

$$Y_{\mathrm{Sph}}(\vartheta, \phi) = P_{\mathrm{n}}^{\mathrm{m}}(\cos\phi)\left[A_{\mathrm{m}}\cos(m\vartheta) + B_{\mathrm{m}}\sin(m\vartheta)\right] \tag{9.72}$$

The differential (9.51) for $R_{\mathrm{Sph}}$ is now, with $\alpha^2 = n(n+1)$,

$$r^2 \frac{\mathrm{d}^2 R_{\mathrm{Sph}}}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}R_{\mathrm{Sph}}}{\mathrm{d}r} = n(n+1)\, R_{\mathrm{Sph}}. \tag{9.73}$$

We try the solution

$$R_{\mathrm{Sph}} = r^{\mathrm{q}}.$$

Substituting this into (9.73) we have

$$q(q+1) = n(n+1),$$

or $q = n$, and $-(n+1)$. The solution for $R_{\mathrm{Sph}}$ is then

$$R_{\mathrm{Sph}} = D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)}. \tag{9.74}$$

If there is no azimuthal ($\vartheta$) dependence, i.e. if there is symmetry about an axis, the solution to Laplace's Equation in spherical coordinates is then a sum of the solutions

$$\begin{aligned} \Phi_{\mathrm{n}}(r, \phi) &= R_{\mathrm{Sph}}(r)\, \Phi_{\mathrm{Sph}}(\phi) \\ &= \left[D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)}\right] P_{\mathrm{n}}(\cos\phi). \end{aligned} \tag{9.75}$$

If there is azimuthal dependence, i.e. there is no axis of symmetry, the solution to Laplace's Equation in spherical coordinates becomes

$$\begin{aligned} \Phi_{\mathrm{nm}}(r, \phi, \vartheta) &= R_{\mathrm{Sph}}(r)\, \Phi_{\mathrm{Sph}}(\phi)\, \Theta_{\mathrm{Sph}}(\vartheta) \\ &= \left[D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)}\right] \cdots \\ &\quad \cdots P_{\mathrm{n}}^{\mathrm{m}}(\cos\phi)\left[A_{\mathrm{m}}\cos(m\vartheta) + B_{\mathrm{m}}\sin(m\vartheta)\right]. \end{aligned} \tag{9.76}$$

*Example 9.4.* As an example of the application of (9.75) we consider a conducting sphere of radius $a$ in a uniform electric field oriented along the polar direction $\phi = 0$.

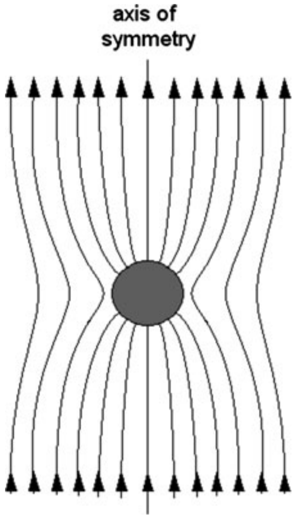



**Fig. 9.3** Conducting sphere of radius $a$ in a uniform electric field

We have drawn the situation in Fig. 9.3. The value of the coordinate along the polar axis is $z = r\cos\phi$ and the potential for the uniform electric field at great distances from the conducting sphere is

$$\begin{aligned}\lim_{z\to\infty} \Phi &= -E_0 z \\ &= -E_0 r\cos\phi.\end{aligned}$$

On the surface of the sphere the potential is a constant, which we shall call $\Phi_C$.

The general solution is a sum over all solutions of the form (9.75)

$$\begin{aligned}\Phi(r,\phi) &= \sum_{n=0}^{\infty}\left[D_n r^n + G_n r^{-(n+1)}\right] P_n(\cos\phi) \\ &= \left[D_0 + G_0\frac{1}{r}\right] + \left[D_1 r + G_1\frac{1}{r^2}\right]\cos\phi \\ &\quad + \sum_{n=2}^{\infty}\left[D_n r^n + G_n r^{-(n+1)}\right] P_n(\cos\phi).\end{aligned}$$

In the limit of large $r$ this is

$$\begin{aligned}\lim_{r\to\infty}\Phi(r,\phi) &= D_0 + D_1 r\cos\phi + \sum_{n=2}^{\infty} D_n r^n P_n(\cos\phi) \\ &= -E_0 r\cos\phi.\end{aligned}$$

Then $D_0 = 0$, $D_1 = -E_0$, and $D_{\mathrm{n}} = 0$ for all $\geqslant 2$. On the surface of the sphere we have

$$\begin{aligned}\Phi\,(a,\phi) &= \Phi_{\mathrm{C}} \\ &= -E_0 a P_1\,(\cos\phi) + \sum_{\mathrm{n}=0}^{\infty} G_{\mathrm{n}} a^{-(n+1)} P_{\mathrm{n}}\,(\cos\phi)\,,\end{aligned}$$

identifying $P_1\,(\cos\phi) = \cos\phi$.

We may now use the properties (9.67) and (9.68) to obtain equations for the coefficients $G_{\mathrm{n}}$. Equating first the constant terms,

$$G_0 = a\Phi_{\mathrm{C}}.$$

Multiplying through by $P_1\,(x)$ and integrating we have

$$\begin{aligned}E_0 a \int_{-1}^{+1} P_1^2\,(x)\,\mathrm{d}x &= \sum_{\mathrm{n}=0}^{\infty} G_{\mathrm{n}} a^{-(n+1)} \int_{-1}^{+1} P_{\mathrm{n}}\,(x)\,P_1\,(x)\,\mathrm{d}x \\ &= G_1 a^{-2} \int_{-1}^{+1} P_1^2\,(x)\,\mathrm{d}x,\end{aligned}$$

or

$$G_1 = E_0 a^3.$$

Multiplying through by $P_{\mathrm{m}}\,(x)$ and integrating results in $G_{\mathrm{m}} = 0$ for all $m \geqslant 2$. Then

$$\Phi\,(r,\phi) = \Phi_{\mathrm{C}} \frac{a}{r} + E_0 \left( a^3 \frac{1}{r^2} - r \right) \cos\phi.$$

The value of the potential $\Phi_{\mathrm{C}}$ of the conducting sphere is arbitrary. Choosing $\Phi_{\mathrm{C}} = 0$ we have

$$\Phi\,(r,\phi) = E_0 \left( a^3 \frac{1}{r^2} - r \right) \cos\phi.$$

## 9.6 Summary

In this chapter we have presented separation of variables solutions to Laplace's Equation for each of the three coordinate systems that are important to our study of classical fields. The electrostatic scalar potential solves Laplace's Equation when there are no free charge densities except on the surfaces, where they produce boundary conditions. These are then the homogeneous solutions to Poisson's Equation and could form the basis for Green's Functions in complex situations.

In the examples we worked through, the reader can see the general method to be used in working with these solutions of Laplace's Equation. We fit the general solutions to the boundary conditions of the particular problem of interest. This is no specific methodology to follow. Experience and intuition help.

A primary interest in this text is, however, in the dynamical behavior of electric and magnetic fields. We will, therefore, not use the results of this chapter until we study the behavior of fields in matter in Chap. 15, when we encounter polarization and magnetization.

# Chapter 10
# Time Dependence

*What is Maxwell's theory? Maxwell's theory is the system of Maxwell's equations.*

*Heinrich Hertz*

## 10.1 Introduction

In Chap. 1 Sect. 1.10 we discussed the history of Faraday's discovery that a time rate of change in the magnetic field induction causes an electric field.

In his experiments Faraday measured electrical current. The electrostatic field is conservative ($\text{curl}\,\mathbf{E} = \mathbf{0}$) and unable to move charges through a circular wire. The electrodynamic field that that is induced by a sudden change in the magnetic field induction is, therefore, of a different character than that resulting from charge densities.

In this chapter we will transform the laboratory results Faraday obtained into the mathematical language of classical field theory. And we will introduce the displacement current proposed by Maxwell, which is mathematically the inverse of Faraday's Law. With Faraday's Law and the displacement current our previous static field equations will be transformed into Maxwell's Equations and the prediction of the electromagnetic wave.

## 10.2 Faraday's Law

An *electromotive force* (*emf*) is required to drive charges around a wire loop producing a current. Electromotive force was not a clearly defined term in the 19th century ([97], p. 192). And in modern terminology the emf is not a force at all, but the work done per unit charge in a circuit. The emf, which we designate here as $\mathcal{E}$ is defined as

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_10, © Springer-Verlag Berlin Heidelberg 2012

$$\mathcal{E} = \oint_C \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell}, \tag{10.1}$$

where $C$ is the contour (wire loop) in which the current resulting from the work done by the field **E** flows.

In 1832 Faraday showed that the induced electromotive force in a wire loop is independent of the nature of the wire and that the induced electric field is simply proportional to the change in the number of magnetic lines of force intersecting area bounded by the contour of *any* wire loop ([97], pp. 191–192).

From (10.1) this means that the component of the electric field along the contour of the wire loop is dependent solely on the change in the magnetic field induction penetrating the area of defined by the loop. This is the *flux* of magnetic field induction through the area defined by the wire loop.

In Fig. 10.1 we have drawn an *arbitrary contour* $C$ with the area defined by the contour penetrated by a magnetic field induction **B**.

For visual clarity we have drawn the contour in a single plane and have drawn the magnetic field induction lines perpendicular to that plane. The contour need not be in a plane and the open area bounded by the contour is also arbitrary, as is the direction and possible curvature of the lines of magnetic induction. The sense of the differential area d**a** is determined by the right hand rule (see Sect. 2.5.4) relatively to the direction around the contour.

The differential flux of magnetic field induction is $\mathrm{d}\Phi_\mathrm{B} = \mathbf{B}\cdot\mathrm{d}\mathbf{a}$. The total flux of magnetic field induction $\Phi_\mathrm{B}$ through the open area bounded by $C$ is then

$$\Phi_\mathrm{B} = \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}. \tag{10.2}$$

Expressed mathematically[1] Faraday's discovery (*Faraday's Law*) is then that the emf (10.1) is proportional to the change in $\Phi_\mathrm{B}$. This change took place over the time it took Faraday to throw the switch. His later experiments in which he pushed a

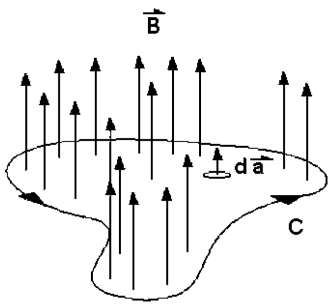



**Fig. 10.1** Magnetic field penetrating the area defined by an arbitrary contour $C$

[1]Faraday did not speak the language of mathematics.

permanent magnet into a cardboard tube with a coil wrapped around it gave him the understanding that it was the rate at which $\Phi_B$ changed, and not only the change, that was important.

Equating (10.1) to the time derivative of (10.2) we have the mathematical form of Faraday's Law based strictly on laboratory results

$$\oint_C \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}. \tag{10.3}$$

The negative sign in (10.3) results from our choice of the sense of the differential area d**a** from the direction of the contour $C$ according to the right hand rule. Equation (10.3) is Faraday's Law in integral form.

Because the area $a$ in (10.3) is fixed, i.e. independent of the time, we can bring the time derivative inside the integral as a partial derivative resulting in

$$\boxed{\oint_C \mathbf{E}\cdot\mathrm{d}\boldsymbol{\ell} = -\int_a \partial\mathbf{B}/\partial\mathrm{t}\cdot\mathrm{d}\mathbf{a}.} \tag{10.4}$$

If we now apply Stokes' Theorem (2.78) to the contour integral in (10.4) we have

$$\int_a \left[\mathrm{curl}\,\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B}\right] \cdot \mathrm{d}\mathbf{a} = \mathbf{0}. \tag{10.5}$$

Since the area $a$ is arbitrary, the integral will always vanish if and only if the integrand vanishes. That is

$$\boxed{\mathrm{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial\mathrm{t}.} \tag{10.6}$$

This is Faraday's Law in differential form.

Since it is an equation for the curl of a field **E**, (10.6) is a field equation. It replaces (3.27) if the fields depend on the time.

The physical picture of the orientation of the induced electric field relative to an increasing magnetic field is shown in Fig. 10.2.

The direction of the induced field is in accordance with (10.3) and a qualitative law formulated by (Heinrich) Emil Lenz (1804–1865) published shortly after

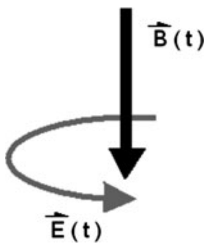



**Fig. 10.2** Electric field induced by varying magnetic field

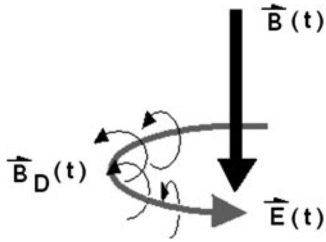



**Fig. 10.3** Magnetic field $\mathbf{B}_D(t)$ (possibly) induced by a time dependent electric field. The original magnetic filed $B(t)$ and the induced electric field $E(t)$ are both increasing in the direction shown

Faraday's discovery [63]. Lenz claimed that when a conducting circuit is moved in a magnetic field the induced current flows in such a direction that the ponderomotive forces on it tend to oppose the motion ([19], p. 45; [97], p. 222).

If in Fig. 10.2 we consider that a loop of wire is placed at the location of the induced electric field an electric current will flow in the direction of the field. This, by Ampère's Law (5.77), will result in a circular magnetic field around the current that will oppose the direction of the original time-dependent magnetic field.

What happens if we elect not to place a wire at the location of the induced electric field? We have drawn the situation if a magnetic field induction, which we labeled as $\mathbf{B}_D(t)$ is induced by the electric field $\mathbf{E}(t)$ alone in Fig. 10.3.

In Fig. 10.3 there is no current. If what is pictured in Fig. 10.3 actually occurs then charges are not necessary for the existence of the fields in the time dependent case. The original field $\mathbf{B}(t)$ was, in Faraday's experiments, produced by moving charges. But his experiments also indicated that the induced field $\mathbf{E}(t)$ was independent of matter. If the time rate of change of this electric filed produces also a (time dependent) magnetic field time dependent fields will fill the space in the immediate vicinity of the initial magnetic field.

We should then ask whether or not $\partial\mathbf{E}(t)/\partial t$ acts like a current.

## 10.3 Displacement Current

As we saw in Sect. 1.11.2.2, Maxwell proposed the *displacement current*

$$\boxed{\mathbf{J}_D \equiv \varepsilon_0 \partial\mathbf{E}/\partial \mathrm{t}} \tag{10.7}$$

in his third installment of "Physical Lines." If we introduce this idea here we find an answer to our question. The induced field in Fig. 10.2 varies with the time in accordance with Faraday's Law (10.6). If we claim that the rate of change of the electric field is equivalent to a current, as in (10.7), then a time-dependent electrical field alone will produce a magnetic field in accordance with Lenz' Law.

This approach may attribute more credence to Lenz' qualitative law than we wish. There is, however, more reason to add the displacement current than a desire to retain Lenz' Law. If we neglect the displacement current and take the divergence of Ampère's Law (5.77) we have

$$\operatorname{div}\operatorname{curl}\mathbf{B} = 0 = \mu_0 \operatorname{div}\mathbf{J}, \tag{10.8}$$

which is in violation of charge conservation (5.11) for the time-dependent case.

But if we add the displacement current to Ampère's Law obtaining

$$\operatorname{curl} B = \mu_0 \left( J + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E} \right), \tag{10.9}$$

we arrive at mathematical consistency with charge conservation.

Taking the divergence of (10.9) and using Gauss' Law (3.26) we have

$$\begin{aligned}\operatorname{div}\operatorname{curl}\mathbf{B} = 0 &= \mu_0 \left( \operatorname{div}\mathbf{J} + \frac{\partial}{\partial t}\varepsilon_0 \operatorname{div}\mathbf{E} \right) \\ &= \mu_0 \left( \operatorname{div}\mathbf{J} + \frac{\partial}{\partial t}\rho \right).\end{aligned} \tag{10.10}$$

The final bracket ( ) on the right hand side in (10.10) vanishes by charge conservation (5.11).

To demand charge conservation as the requirement for the introduction of the displacement current places us on surer footing than requiring that Lenz' Law be satisfied. But the argument based on charge conservation fails in empty space. In empty space $\rho = 0$ and $\mathbf{J} = \mathbf{0}$ and neither of the sets of equations

$$\boxed{\begin{array}{cc} \operatorname{div}\mathbf{E} = 0 & \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t & \operatorname{curl}\mathbf{B} = \mu_0\varepsilon_0\partial\mathbf{E}/\partial t, \end{array}} \tag{10.11}$$

nor

$$\boxed{\begin{array}{cc} \operatorname{div}\mathbf{E} = 0 & \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t & \operatorname{curl}\mathbf{B} = 0, \end{array}} \tag{10.12}$$

is mathematically inconsistent. So our argument for preferring the set (10.11) over the set (10.12) must be based on another criterion in empty space.

Maxwell also did not claim that the reason for introducing the displacement current was necessary for charge conservation. If this was his reason he certainly would have mentioned it in print. But he did not. When he was writing "Physical Lines" Maxwell was interested in what he called *molecular vortices* and a mechanical picture based on elastic strain of the aether ([86], pp. 112–113).

The symmetry of the set (10.11) is more beautiful than the asymmetry of the set (10.12). This may be sufficient reason for a theoretical physicist to choose (10.11) over (10.12). But that is a metaphysical and not a scientific reason for the choice.

The scientific reason is that (10.11) admits of wave solutions while (10.12) does not. After Hertz' experiments of 1887 [45], no equations that did not admit of a wave solution in empty space could be considered.

## 10.4 Magnetostatic Energy

In no preceding chapter did we consider magnetostatic field energy. The reason for this omission is that the magnetic field transfers no energy to charged particles in motion. That left us no obvious way to compute the energy transfer in the static case. Faraday's Law, however, allows us to compute the energy required to establish a magnetic field.

We consider a toroidal solenoid with a rectangular cross section, as we have drawn in Fig. 10.4. We have chosen a rectangular cross section to simplify the integration. The current to the toroidal solenoid is supplied by an external source.

We may choose the external source to vary (increase) very slowly so that at any instant in the experiment the displacement current in Ampère's Law may be neglected compared to the current density **J**. At any time in the experiment we may then use Ampère's Circuital Law to obtain the magnetic field induction in the solenoid.

Ampère's Circuital Law shows us that the magnetic field is contained entirely within the closed volume of the toroidal solenoid. Application of Ampère's Circuital Law to the contour $C$ in Fig. 10.4 results in a magnetic field induction

$$B(r) = \frac{N_\ell \mu_0 I}{2\pi r}, \tag{10.13}$$

where $N_\ell$ is the number of wire windings in the solenoid, $I$ is the current in the wire, and $R < r < R + w$ is the radius of the contour $C$.

The flux of the magnetic field induction inside the solenoid is the integral of (10.13) over the cross section of the solenoid multiplied by the number of windings in the solenoid

$$\Phi_{\mathrm{B}} = \frac{N_\ell^2 \mu_0 I}{2\pi} h \int_{\mathrm{R}}^{\mathrm{R+w}} \frac{\mathrm{d}r}{r} = \frac{N_\ell \mu_0 I}{2\pi} h \ln\left(\frac{R+w}{R}\right). \tag{10.14}$$

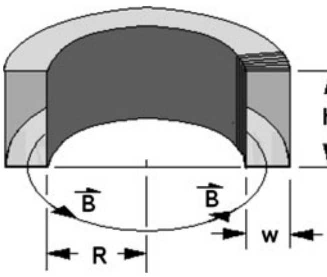



**Fig. 10.4** Toroidal solenoid with a rectangular cross section

The rate of change of $\Phi_{\mathrm{B}}$ is equal to the emf $\mathcal{E}$ required to increase the current in the solenoid

$$\mathcal{E} = \frac{N_\ell^2 \mu_0}{2\pi} h \ln\left(\frac{R+w}{R}\right) \frac{\mathrm{d}I}{\mathrm{d}t}. \tag{10.15}$$

The emf in (10.15) results in an incremental work $\delta W$ done on each increment of charge $\delta Q$ charge passing through the solenoid, which is

$$\delta W = \mathcal{E}\delta Q. \tag{10.16}$$

The rate at which work is done to produce the magnetic field in the solenoid is then

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t} &= \mathcal{E} I \\ &= \frac{N_\ell^2 \mu_0}{2\pi} h \ln\left(\frac{R+w}{R}\right) \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} I^2\right).\end{aligned} \tag{10.17}$$

We may now integrate (10.17) over the time from the beginning of the experiment when $I = 0$ to the final time when we have a current $I$ in the solenoid. The result is the total work done to increase the magnetic field induction in the solenoid from $B = 0$ to a final value of $B$. This total work done is the magnetic energy in the solenoid

$$U_{\mathrm{B}} = \frac{1}{2} \frac{N_\ell^2 \mu_0}{2\pi} h \ln\left(\frac{R+w}{R}\right) I^2. \tag{10.18}$$

We now ask if we can formulate this total energy in terms of the magnetic field induction. From (10.13) the square of the magnetic field induction along the contour $C$ is

$$\left(\frac{N_\ell^2 \mu_0^2}{4\pi^2 r^2}\right) I^2 = B^2(r) \tag{10.19}$$

If we integrate (10.19) over the volume of the solenoid we have

$$\begin{aligned}\left(\frac{N_\ell^2 \mu_0^2}{4\pi^2}\right) I^2 h \int_0^{2\pi} \mathrm{d}\vartheta \int_{\mathrm{R}}^{\mathrm{R+w}} \frac{\mathrm{d}r}{r} &= \left(\frac{N_\ell^2 \mu_0^2}{2\pi}\right) I^2 h \ln\left(\frac{R+w}{R}\right) \\ &= h \int_0^{2\pi} \mathrm{d}\vartheta \int_{\mathrm{a}}^{\mathrm{a+h}} r^2 \mathrm{d}r B^2(r)\end{aligned} \tag{10.20}$$

Combining (10.18) and (10.20) we see that the total magnetic field energy in the solenoid is

$$U_{\mathrm{B}} = h \int_0^{2\pi} \mathrm{d}\vartheta \int_{\mathrm{a}}^{\mathrm{a+h}} r^2 \mathrm{d}r \left[\frac{1}{2}\frac{1}{\mu_0} B^2(r)\right]. \tag{10.21}$$

The density of the magnetic field energy $u_{\mathrm{B}}$ per unit volume is then

$$\boxed{\mathrm{u_B} = (1/2)(1/\mu_0)\mathrm{B}^2.} \tag{10.22}$$

This is the *magnetostatic field energy* in the toroidal solenoid.

This magnetostatic energy density bears mathematical resemblance to the electrostatic field energy density

$$u_{\mathrm{E}} = (1/2)\,\varepsilon_0 E^2,$$

which we obtained in Sect. 4.6.1. At this point in our development the resemblance is only incidental. The basis for this resemblance will be revealed as we study wave motion and energy transport by waves.

## 10.5 Maxwell's Equations

For classical electromagnetic fields the set of (10.11) is the valid set of equations for free space with charge and current densities absent. These are the equations referred to, in modern terminology, as *Maxwell's Equations*.

In what follows we have a list of Maxwell's Equations in which we identify the individual field equations by the names normally associated with them and provide the experimental evidence on which they are based. We have done this to emphasize the fact that classical field theory has the structure which Newton and Barrow decided was correct for an experimental and mathematical philosophy.

For the electric field we have

- *Gauss' Law*
$$\operatorname{div}\mathbf{E} = \frac{1}{\varepsilon_0}\rho \tag{10.23}$$
is a result of *Coulomb's Experiment* in which the force between two charges was measured.
- *Faraday's Law*
$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{10.24}$$
is a result of *Faraday's Experiment* in which the induction of an electric field by a changing magnetic field was discovered.

and for the magnetic field we have

- *Oersted's Result*
$$\operatorname{div}\mathbf{B} = 0 \tag{10.25}$$
is a result of *Oersted's Experiment* identifying electric current as the source of magnetic fields and the fact that the geometrical form of the magnetic field is a closed loop.
- *Ampère's Law*
$$\operatorname{curl}\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}\right) \tag{10.26}$$

is a result of *Ampère's Experiment* in which the force between two parallel wires carrying currents was measured. Here we have added Maxwell's *displacement current* to the original form of Ampère's Law.

and then for the charges and currents we have

- *Charge conservation*

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\mathbf{J} = 0 \tag{10.27}$$

is a mathematical expression of *Franklin's proposal* that charge is conserved.

Using Gauss' Theorem and Stokes' Theorem Maxwell's Equations may be written in integral form.

- *Gauss' Law*

$$\oint_{\mathrm{S}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_{\mathrm{V}} \rho \mathrm{d}V \tag{10.28}$$

- *Faraday's Law*

$$\oint_{\mathrm{C}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\int_{\mathrm{a}} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{a} \tag{10.29}$$

- *Oersted's Result*

$$\oint_{\mathrm{S}} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0 \tag{10.30}$$

- *Ampère's Law*

$$\oint_{\mathrm{C}} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \int_{\mathrm{a}} \left(\mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}\right)\cdot \mathrm{d}\mathbf{a} \tag{10.31}$$

- *Charge conservation*

$$\oint_{\mathrm{S}} \mathbf{J}\cdot \mathrm{d}\mathbf{S} = -\int_{\mathrm{V}} \frac{\partial}{\partial t}\rho \mathrm{d}V. \tag{10.32}$$

The integral form of Maxwell's Equations is useful in applications to situations for which the integrals may be replaced by algebraic expressions. That is the fields must be constant along the contours or over the surfaces of integration.

Maxwell's equations do not, however, form a mechanical picture, such as Maxwell sought, based on the aether (see Chap. 1). His friend Thomson saw no way for the earth to move freely through the elastic aether ([79], p. 274). And, now, after the failure to measure the motion of the earth through the aether and Einstein's development of a theory in which aether plays no role (see Sect. 1.14), we realize that the hope of 19$^{\text{th}}$ century British physicists for a mechanical picture of electromagnetic fields failed.

Hertz recognized the philosophical difficulty. He posed the question: "What is Maxwell's theory?" and answered that: "Maxwell's theory is Maxwell's system of equations ([46], p. 21)." Once we remove all of the architecture of the aether, that

Maxwell considered important, we are left with the same theory and the same results that are tested in the laboratory without any reference to the aether. Hertz even said that his experiments were not guided by Maxwell's ideas ([46], p. 20).

Einstein's answer to the asymmetry in our understanding of Faraday's Law does not alter the fact that in a stationary laboratory we observe an induced electric field from a change in a magnetic field. And we know that the existence of an electromagnetic wave in that stationary laboratory is evidence that what we have called the displacement current is the result of a real physical law. We know that Maxwell's Equations are valid. But we cannot explain them in terms of a mechanical picture.

## 10.6 Summary

As we began this chapter we had the complete static field equations. At the end of the chapter we had the complete Maxwell Equations.

Faraday's experimental discovery of electromagnetic induction as a time dependent phenomenon was the crucial initial step. This was then followed by Maxwell's proposal that a time varying electric field is equivalent to a current in the sense that the time rate of change of the electric field induces a magnetic field. The combination of Faraday's Law and Maxwell's displacement current completed the theory.

Based on Faraday's Law we were, at last, able to logically introduce magnetic field energy. Before we inderstood Faraday's Law we could not speak of the work required to produce a magnetic field.

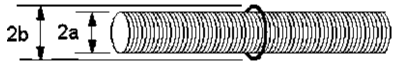
We devoted some space to an analysis of Maxwell's displacement current. If charges and currents are present the displacement current is required by charge conservation. But in empty space we are left with only the proposal that the field equations must be symmetric, which is a proposal waiting verification.

At the end of the chapter we have Maxwell's Equations as a theory waiting for verification or falsification.

## Exercises

**10.1.** Simplifying the arrangement Faraday used, you have placed a wire loop of radius $b$ coaxial with and at the center of a long straight solenoid of a radius $a$. The solenoid has $N_\lambda$ turns per unit length and a length of $L_\mathrm{S}$. We have drawn the arrangement in Fig. 10.5. You have made $b$ only slightly larger than $a$ so that you may neglect any complications from the magnetic field external to the solenoid. You also assume that the magnetic field induction inside the solenoid is uniform and can be found from Ampère's Circuital Law. You will close a switch on a battery (with emf of $V$ V) connected to the solenoid to produce a current. After you close the switch the emf produced in the solenoid opposes the flow of current.

**Fig. 10.5** Solenoid with coaxial wire loop




(a) Using Ampère's Circuital Law, and assuming that the magnetic field induction is uniform within the entire solenoid, show that the emf $\mathcal{E}$ produced in the solenoid is related to the changing solenoid current $I$ by $\mathcal{E} = L\mathrm{d}I/\mathrm{d}t$ where $L = \pi a^2 \mu_0 N_\lambda^2 L_S$. This is called the self inductance of the solenoid.
(b) If there is a resistance $R\,\Omega$ in the battery circuit supplying the current to the solenoid, then $I$ will satisfy the differential equation

$$V = RI + L\frac{\mathrm{d}I}{\mathrm{d}t}.$$

Show that the current

$$I(t) = \frac{V}{R}\left[1 - \exp\left(-\frac{R}{L}t\right)\right]$$

solves this differential equation.
(c) You obtain a recording of the current in the wire loop $I_b(t)$. What do you expect the result to be if the loop has resistance $R_b\,\Omega$?

**10.2.** In the preceding exercise you used a loop with radius $b$ not much larger that $a$. What happens if you increase the loop radius $b$ considerably? For example, assume that you have a solenoid of inner radius 0.5 cm and a length of 10 cm and you try an outer loop of radius 50 cm.

**10.3.** In a region of space we have a uniform magnetic field of induction $B$. We can produce this field between closely spaced, parallel poles of a magnet. We then drag a rectangular wire loop through this region as we have illustrated in Fig. 10.6. The motion of the loop is uniform, i.e. the velocity of the loop is a constant, which we identify as $v$. The resistance of the loop is $R\,\Omega$ The current measured by the galvanometer shown in the drawing is then $\mathcal{E}/R$, where $\mathcal{E}$ is the emf.

(a) Using Faraday's Law find the emf $\mathcal{E}$ in the loop.
(b) The charges in the part of the wire loop still in the magnetic field are moving in the magnetic field and will experience a force. From this force calculate the emf that will result from this effect.

**10.4.** In the preceding exercise you calculated the emf appearing in the loop using two distinct methods and got the same result. Is this just an interesting result, or is this an intolerable situation?

**10.5.** An industrial generator, driven by a turbine, consists of an $N$-turn coil of area $A$, which rotates in a magnetic field with induction $B$ about an axis perpendicular to the field, with a frequency of rotation $\nu$. Find the emf in the coil.

**10.6.** You have arranged the system in Fig. 10.7.

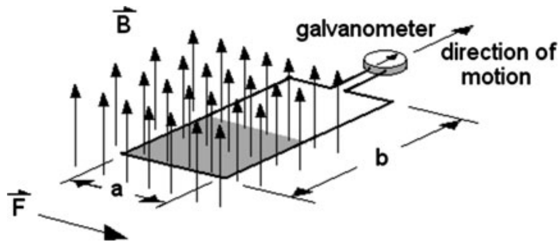



**Fig. 10.6** Moving a rectangular loop through a magnetic field

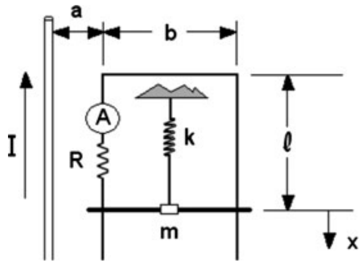



**Fig. 10.7** Test of Faraday's Law with SHM

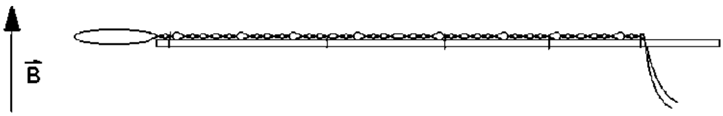



**Fig. 10.8** Flip-coil magnetometer

In the vertical wire you have a *dc* current $I$. To the right of this wire you have a vertical wire loop with a recording ammeter $A$. The resistance $R$ is that internal to the ammeter. You have then suspended a copper bar of mass $m$ by a spring with constant $k$ mounted to a solid support. By tilting the wires in your loop slightly forward you assure that the copper bar always contacts the loop.

In equilibrium the bar hangs at a distance $\ell$ from the top of your wire loop. You can then pull it down a distance $x_0$ and release it and the bar will oscillate. The oscillation coordinate you call $x$. You consider only simple harmonic motion of your copper bar.

What do you expect for the record from the ammeter? Find the current as a function of time and any other parameters. You may neglect losses.

**10.7.** A typical flip-coil magnetometer consists of a (usually small) wire loop attached to the end of a rod. The leads from the loop are twisted together to avoid any further emf other than from the loop. We position the loop at the point we want to measure the magnetic field with the field perpendicular to the plane of the flip coil. Then, using the rod, we flip the coil. There will be an emf in the coil from Faraday's Law that we can measure by connecting the leads to a meter. We have illustrated the flip-coil magnetometer in Fig. 10.8.

The coil radius is $a$ and there are $n$ turns in the coil.

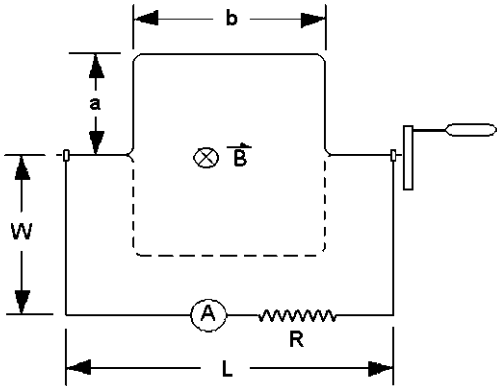



**Fig. 10.9** Hand-operated generator

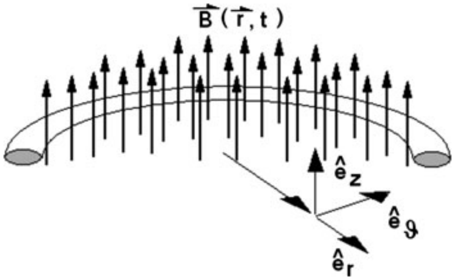



**Fig. 10.10** Betatron toroidal accelerating chamber showing Magnetic induction inside the torus

During the flip of the coil you record the emf as a function of time so you can perform a numerical integration. Assume that the coil rotates at a constant rate during the flip.

(a) What is the emf you record as a function of time?
(b) What is the magnetic induction?

**10.8.** You have built a hand-operated generator, which we have drawn in Fig. 10.9. There is a uniform magnetic field of induction **B** into the paper.

You crank at as constant a rate as possible $\omega$ ( rad s$^{-1}$). The only resistance in the loop is the internal resistance of the ammeter $R$.

Find the current that the ammeter measures.

**10.9.** A betatron is an accelerator for electrons. The cyclotron does not work for electrons because the mass of the electron increases as the velocity approaches that of light. As a consequence the cyclotron frequency changes by a considerable amount making acceleration impossible. The betatron is an ingenious answer to this problem. In the betatron there is no natural frequency that must be maintained constant. The accelerating field is produced by induction through a change in the strength of a magnetic field outside of the betatron.

The betatron is shown schematically in Fig. 10.10. The electrons are accelerated in a toroidal vacuum chamber of mean radius $R$. There is only a $z$–component of the magnetic field induction $\mathbf{B} = \hat{e}_z B_z(r)$, which is a function of the radial distance $r$ from the axis. In Fig. 10.10 we have only drawn the magnetic field induction lines inside the torus. The chamber is also immersed in the magnetic field.

(a) Show that the electron's tangential velocity is $v_t = qB_z(R)R/m$, where $q$ is the electron charge.
(b) If the magnetic field is slowly increased in magnitude, show that the emf induced around the electron's orbit is such as to accelerate the electron.
(c) Show that in order for the electron to stay in a single orbit, the radial variation of $B_z$ inside the orbit must be such that the spatial average of the increase in $B_z(r)$ (averaged over the area enclosed by the orbit) is equal to twice the increase in $B_z(R)$ during the same time interval.

Your analysis will be based on Faraday's and Newton's Laws. It will be easiest to write Newton's Second Law in simple vector form, rather than using the canonical equations, since you are looking for velocities.

**10.10.** Faraday's homopolar generator consists of a metal disk that rotates in a uniform magnetic field perpendicular to the plane of the disk. Show that the potential difference produced between the center of the disk and its periphery is $V = \nu\Phi_B$ where $\Phi_B$ is the flux through the disk and $\nu$ is its frequency of rotation.

**10.11.** You have a long solenoid wrapped with thick copper wire so that large currents can be carried. There are $N_\lambda$ windings per unit length. In the center of the solenoid you have mounted a cylindrical nonmagnetic conductor of length $L$ and radius $R$ with axis coinciding with that of the solenoid. The mountings are of a nonconducting material with a high melting point. You intend to heat the conductor by inducing a current in it. If you pass an alternating electrical current through the wire of the solenoid you expect to obtain an alternating electrical current in the conductor, which will result in internally heating the conductor.

Assume that near the center of the solenoid the magnetic field induction is spatially uniform and varies only with the time.

(a) What is the electric field inside the conductor?
(b) Assuming Ohm's Law so that the current density is $\mathbf{J} = \sigma\mathbf{E}$, what is the current in the conductor?

**10.12.** In a prior exercise you obtained the self inductance of a cylindrical solenoid based on the assumptions that the magnetostatic field induction was vanishingly small immediately outside the solenoid and uniform inside. But you know that the field inside the solenoid is not uniform. The form is closer to that shown in Fig. 10.11.

In Fig. 10.11 we have also drawn the differential cross sectional area of the solenoid d$\mathbf{a}$. The number of turns of wire per unit length of the solenoid is $N_\ell$ and the current in the wire is $I$.

For the more realistic situation shown in Fig. 10.11 that the flux of the magnetic induction inside the solenoid is still of the form

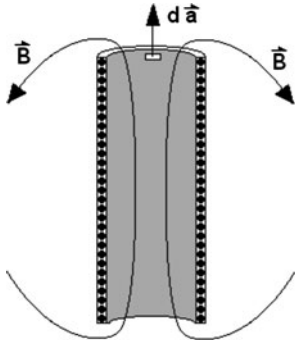



**Fig. 10.11** Solenoid with magnetostatic field lines drawn

$$\Phi_{\mathrm{B}} = LI$$

so that the emf is still

$$\mathcal{E} = L\frac{\mathrm{d}I}{\mathrm{d}t}$$

and the energy present in the solenoid is still

$$U_{\mathrm{B}} = \frac{1}{2}LI^2.$$

Show that the inductance is

$$L = \left(N_\ell^2 \mu_0/4\pi\right) \int_{\mathrm{Vs}} \Bigg[ (1/r) \int_{\mathrm{VJ}} \left(1/\left|\mathbf{r}-\mathbf{r}'\right|\right) \mathrm{d}V' + \int_{\mathrm{VJ}} \partial \left(1/\left|\mathbf{r}-\mathbf{r}'\right|\right)/\partial r \mathrm{d}V' \Bigg] dV.$$

**10.13.** The preceding exercise shows that we may still define an inductance for a cylindrical solenoid in the actual situation encountered in electrical circuits. As a consequence we also find that the expression for the energy in the magnetic field, in terms of the inductance, was identical to that obtained for the toroidal solenoid, even though the magnetic field is confined in the toroidal case and permeates the space around the cylindrical solenoid.

Comment on this apparent paradox.

**10.14.** In his first experiments Faraday formed sets of solenoids by wrapping layered helices of wire concentrically around a wooden dowel (Sect. 1.10). The result was a pair of magnetically coupled, concentric solenoids. In Fig. 10.12 we have drawn a simplified version of Faraday's solenoids. Here we have reduced the windings to the two separate solenoids that were actually there. We shall consider that the solenoids have the same total length $\ell$ and have radii $R_1 < R_2$. The number of windings in the inner solenoid we chose to be $N_1$ and that in the outer solenoid $N_2$.

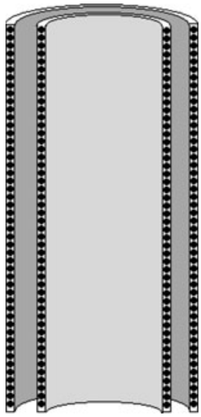


**Fig. 10.12** Concentric cylindrical solenoids

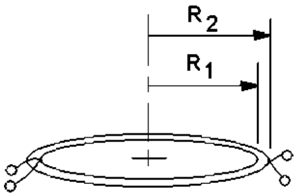



**Fig. 10.13** Two loops of approximately equal radii, but different numbers of turns

Show that the emf in the solenoid $i\ (= 1, 2)$ coil is

$$\mathcal{E}_{\rm i} = -\sum_{\rm j=1}^{2} M_{\rm ij}\frac{{\rm d}I_{\rm j}}{{\rm d}t}$$

where $M_{\rm ii} = L_{\rm i}$ is the self inductance of solenoid $i$ and $M_{12} = M_{21}$ is the mutual inductance between the solenoids. The relationship $M_{12} = M_{21}$ is an example of Neumann's[2] Formula.

For simplicity assume, as in a prior exercise, that the magnetic field induction produced by currents in each solenoid is uniform throughout that solenoid and that the external magnetic field may be neglected.

**10.15.** In Fig. 10.13 we have drawn two circular wire loops, which we have placed concentrically on the laboratory table.

The radii $R_1$ and $R_2$ of the loops are almost the same, i.e. $R_1 \approx R_2$, although $R_1 < R_2$. There are $N_1$ turns in the loop of radius $R_1$ and $N_2$ turns in the loop of radius $R_2$. You want to know the self inductance of the loop of radius $R_1$ and the mutual inductance of the two loops.

[2]Franz Ernst Neumann (1798–1895) was a German mathematician and physicist. He became professor of mineralogy and physics at the University of Königsberg in 1829.

The magnetic field induction for all points within the area of a single loop is not easy to calculate, as we saw in examples. Symmetry tells us, however, that the lines of magnetic field in the plane of the loop must be perpendicular to that plane. The magnetic field induction will vary as a function of the radial distance from the center. But we can define an average induction $\bar{B}$ such that the flux through the area of a loop of radius $R$ is

$$\Phi_{\mathrm{B}} = N\left(\pi R^2\right)\bar{B},$$

where $N$ is the number of turns in the loop.

We also know that the magnetic field induction in the plane of the loop increases with current in the loop $I$ and with the number of turns. We assume that the relation is linear and write

$$\bar{B} = \mu_0 \frac{NI}{2\pi R} g\left(R\right),$$

where $g\left(R\right)$ is some function that may vary with $R$.

Using these ideas show that Neumann's Formula requires that $g\left(R\right) \propto R$.

# Chapter 11
# Electromagnetic Waves

*God runs electromagnetics on Monday, Wednesday, and Friday by the wave theory, and the devil runs it by quantum theory on Tuesday, Thursday, and Saturday.*

*William Lawrence Bragg*

## 11.1 Introduction

In Chap. 10 we obtained the full Maxwell Equations in the presence of charges and currents as sources. And in Chap. 1 Sect. 1.12.1 we encountered the experiments of Hertz, which identified the electromagnetic waves predicted by Maxwell. In this chapter we will solve Maxwell's Equations in free space, without the presence of charges or currents as sources. This solution will provide us with a detailed picture of the structure of the electromagnetic waves that are possible in the context of classical electromagnetic field theory.

Until Hertz' discovery of electromagnetic waves in the laboratory we could legitimately consider the field concept to be a useful mathematical construct with no reality beyond the vision of Faraday and Maxwell. We could also have been profoundly skeptical of the field concept, as many scientists were in the latter part of the 19th century (see e.g. [18], p. 164). But Hertz' discovery was of a reality that was transported from one part of the laboratory to another. And with our realization that the elastic material aether of the 19th century does not exist, we acknowledge Hertz' discovery as verification that electromagnetic fields could be transported across empty space.

In this chapter we will find that, in the context of Maxwell's Equations, the simplest propagating waves are plane waves. These are mathematical functions of a single spatial coordinate and the time. We will discover that the electric and magnetic field vectors of these plane waves must be perpendicular to the direction of wave motion making them transverse waves like water waves. But we will also

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
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find that the phenomenon known as polarization results in a rotation of these field vectors around the direction of propagation.

The quantum mechanical impossibility of obtaining single frequency plane waves means that we must consider a superposition, which will lead us to the Fourier transformation and a more general formulation of wave propagation.

We will find that with the Lorentz Gauge both the scalar and vector potentials satisfy wave equations with sources.

## 11.2 Wave Equations

Maxwell's Equations (see Sect. 10.5) are

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \rho/\varepsilon_0 & \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t & \operatorname{curl}\mathbf{B} &= \mu_0\left(\mathbf{J}+\varepsilon_0\partial\mathbf{E}/\partial t\right), \end{aligned} \tag{11.1}$$

Taking the curl of Faraday's Law and of Ampère's Law and using (A.16) we have

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{E} &= \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \\ &= -\operatorname{curl}\frac{\partial}{\partial t}\mathbf{B}. \end{aligned} \tag{11.2}$$

and

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{B} &= \operatorname{grad}\operatorname{div}\mathbf{B} - \nabla^2\mathbf{B} \\ &= \operatorname{curl}\mu_0\left(\mathbf{J}+\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}\right). \end{aligned} \tag{11.3}$$

Since curl is an operator containing partial derivatives and the order of partial differentiation makes no difference[1]

$$\operatorname{curl}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\operatorname{curl}. \tag{11.4}$$

With (11.4), and using Gauss' and Ampère's Laws, (11.2) becomes

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{1}{\varepsilon_0}\operatorname{grad}\rho + \mu_0\frac{\partial}{\partial t}\mathbf{J}. \tag{11.5}$$

With (11.4), and using Oersted's Result and Faraday's Law, (11.3) becomes

[1]A proof of this property of partial derivatives may be found in any text on multivariate calculus (e.g. [15], volume II, pp. 55–56).

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \,\text{curl}\, \mathbf{J}. \tag{11.6}$$

In (11.5) and (11.6) we have used the fact that $\varepsilon_0\mu_0 = 1/c^2$.

Equations (11.5) and (11.6) are the electromagnetic wave equations with sources. The sources are the charge density $\rho(\mathbf{r}, t)$ and the current density $\mathbf{J}(\mathbf{r}, t)$. We will study the production of wave fields from the motion of charged particles in a subsequent chapter.

If we consider a region of space in which there are neither currents nor charges, i.e. a region of space in which $\mathbf{J} = \mathbf{0}$ and $\rho = 0$, then (11.5) and (11.6) become

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \mathbf{0}, \tag{11.7}$$

and

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = \mathbf{0}. \tag{11.8}$$

Before proceeding we recall that if we choose empty space with $\mathbf{J} = \mathbf{0}$ and $\rho = 0$ we are dealing with vector fields $\mathbf{E}$ and $\mathbf{B}$ for which the divergence vanishes. Such vector fields are termed *solenoidal*. This is the situation considered by Maxwell for the light wave in 1868, with the exception that he kept $\mathbf{J}$ in Ampère's Law (see Sect. 1.11.2.2). We, therefore, try *plane wave solutions* to (11.7) and (11.8).

## 11.3 Plane Waves

The functions

$$\sin(\omega t - k_x x) \tag{11.9}$$

and

$$\cos(\omega t - k_x x) \tag{11.10}$$

represent propagating sinusoidal (oscillatory) disturbances around ambient conditions. These disturbances may be, for example, waves on the surface of water or sound waves in air. The disturbance we measure is the rising and falling of the water surface or the oscillating pressure of the sound wave. If we are sufficiently far from the source of the disturbance, such as a pebble dropped into a calm pond or a plucked guitar string, the water or sound waves passing us appear to be going in one direction, which we have called $x$ in (11.9) and (11.10).

In Fig. 11.1 we have plotted (11.10) for two different times. At the initial time the function (disturbance) is that represented by the dashed line and at the later time the disturbance is represented by the solid line. In the case of either the water or sound waves the disturbance propagates at a constant velocity to the right along the $x-$axis.

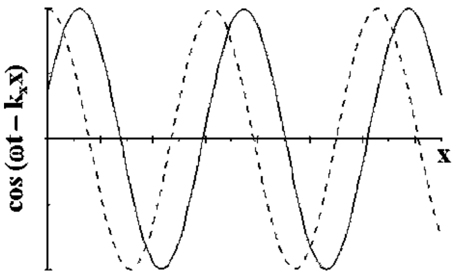


**Fig. 11.1** Sinusoidal disturbance $\cos(\omega t - k_x x)$ moving along the $x$−axis. The dashed line represents the distrurbance (*a plane wave*) at a certain time and the solid line is at a later time

We can simplify our mathematical treatment of general oscillatory disturbances in space by introducing Euler's Identity

$$\exp(i\vartheta) = \cos\vartheta + i\sin\vartheta. \tag{11.11}$$

That is

$$\sin(\omega t - k_x x) = \mathrm{Im}\left[\exp(i\omega t - ik_x x)\right] \tag{11.12}$$

and

$$\cos(\omega t - k_x x) = \mathrm{Re}\left[\exp(i\omega t - ik_x x)\right] \tag{11.13}$$

We see the mathematical simplification when we differentiate $\exp(i\omega t - ik_x x)$. The first partial derivatives of the complex exponential representation are

$$\frac{\partial}{\partial t}\exp(i\omega t - ik_x x) = i\omega\exp(i\omega t - ik_x x), \tag{11.14}$$

and

$$\frac{\partial}{\partial x}\exp(i\omega t - ik_x x) = -ik_x\exp(i\omega t - ik_x x). \tag{11.15}$$

And the second partial derivatives are

$$\frac{\partial^2}{\partial t^2}\exp(i\omega t - ik_x x) = -\omega^2\exp(i\omega t - ik_x x), \tag{11.16}$$

and

$$\frac{\partial^2}{\partial x^2}\exp(i\omega t - ik_x x) = -k_x^2\exp(i\omega t - ik_x x). \tag{11.17}$$

With (11.16) and (11.17) we have

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\exp(i\omega t - ik_x x) = -\left(k_x^2 - \frac{\omega^2}{c^2}\right)\exp(i\omega t - ik_x x) \tag{11.18}$$

and we see that $\exp(i\omega t - ik_x x)$ satisfies the one-dimensional form of (11.7) or (11.8) provided $c = \omega/k_x$.

When we choose to represent the disturbances by complex exponentials rather than the real functions sine and cosine we have greater simplicity, which will be indispensable in our discussions. We do, however, then elect to work in the complex plane. This presents no problem if we are studying the characteristics of wave motion, which involve only permitted values for frequency $\omega$ and wave vector $k$. If we need actual values of electric and magnetic fields we must then find the real parts of our complex valued functions.

In the complex plane we *represent* the wave solutions as

$$\mathbf{E}(\mathbf{k}, \omega) \exp(i\omega t - i\mathbf{k} \cdot \mathbf{r}) \tag{11.19}$$

$$\mathbf{B}(\mathbf{k}, \omega) \exp(i\omega t - i\mathbf{k} \cdot \mathbf{r}), \tag{11.20}$$

The real parts of (11.19) and (11.20) are

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = \frac{1}{2} [&\mathbf{E}(\mathbf{k}, \omega) \exp(i\omega t - i\mathbf{k} \cdot \mathbf{r}) \\ &+ \mathbf{E}^* \left(\mathbf{k}^*, \omega^*\right) \exp\left(-i\omega^* t + i\mathbf{k}^* \cdot \mathbf{r}\right)\big] \end{aligned} \tag{11.21}$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) = \frac{1}{2} [&\mathbf{B}(\mathbf{k}, \omega) \exp(i\omega t - i\mathbf{k} \cdot \mathbf{r}) \\ &+ \mathbf{B}^* \left(\mathbf{k}^*, \omega^*\right) \exp\left(-i\omega^* t + i\mathbf{k}^* \cdot \mathbf{r}\right)\big], \end{aligned} \tag{11.22}$$

which are the actual physical solutions. If the waves are not damped (do not decrease with time at a point in space) or dispersed (do not decrease spatially at an instant in time) $\omega$ and $\mathbf{k}$ will be real.

If we substitute (11.19) and (11.20) into (11.7) and (11.8) we have

$$\left(k^2 - \frac{\omega^2}{c^2}\right) [\mathbf{E}(\mathbf{k}, \omega) \text{ or } \mathbf{B}(\mathbf{k}, \omega)] \exp(i\omega t - i\mathbf{k} \cdot \mathbf{r}) = \mathbf{0}, \tag{11.23}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{11.24}$$

is real. That is our complex exponential solutions are valid provided the relationship between $\omega$ and $\mathbf{k}$ is

$$\omega(\mathbf{k}) = c\sqrt{k_x^2 + k_y^2 + k_z^2}, \tag{11.25}$$

which is real as well. There is no damping or dispersion of plane waves in empty space. From (11.25) we see that the wave velocity is $c = \omega/|\mathbf{k}|$, where $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$, which is the velocity we found for the one-dimensional wave.

Equation (11.25) is the *dispersion relation* for plane electromagnetic waves in empty space. In general a dispersion relation provides the frequency $\omega$ as a function of the wave vector $k$ that is required for the waves to propagate. Since the dispersion relation (in empty space) requires that $\omega$ and $\mathbf{k}$ are real, taking the complex conjugate of (11.23) we have

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\left[\mathbf{E}^*(\mathbf{k},\omega) \text{ or } \mathbf{B}^*(\mathbf{k},\omega)\right]\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r}) = \mathbf{0}. \tag{11.26}$$

But we can also obtain (11.26) from (11.23) by replacing $(\mathbf{k},\omega)$ by $(-\mathbf{k},-\omega)$. Therefore

$$\mathbf{E}^*(\mathbf{k},\omega) = \mathbf{E}(-\mathbf{k},-\omega)$$

and

$$\mathbf{B}^*(\mathbf{k},\omega) = \mathbf{B}(-\mathbf{k},-\omega)$$

for the wave in empty space. And the electric and magnetic waves fields in empty space (11.21) and (11.22) are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = \frac{1}{2}&[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) \\ &+ \mathbf{E}(-\mathbf{k},-\omega)\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})]\end{aligned} \tag{11.27}$$

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) = \frac{1}{2}&[\mathbf{B}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) \\ &+ \mathbf{B}(-\mathbf{k},-\omega)\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})],\end{aligned} \tag{11.28}$$

## 11.4 Plane Wave Structure

We begin our investigation of the structure of plane waves in empty space with Maxwell's Equations in the absence of charges and currents (10.11), which are

$$\begin{aligned}\operatorname{div}\mathbf{E} &= 0 & \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} & \operatorname{curl}\mathbf{B} &= \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}.\end{aligned} \tag{11.29}$$

For the plane wave solutions (11.19) and (11.20) we have

$$\begin{aligned}&\operatorname{div}[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})] \\ &= -i\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}),\end{aligned} \tag{11.30}$$

$$\begin{aligned}&\text{curl}\left[\mathbf{E}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right)\right]\\&=-i\mathbf{k}\times\mathbf{E}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right),\end{aligned}\tag{11.31}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\mathbf{E}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right)\right]\\&=i\omega\cdot\mathbf{E}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right)\end{aligned}\tag{11.32}$$

and corresponding expressions for the magnetic field. Then Maxwell's Equations (11.29) for waves in empty space become

$$\begin{aligned}\mathbf{k}\cdot\mathbf{E}&=0 &\qquad \mathbf{k}\cdot\mathbf{B}&=0\\\mathbf{k}\times\mathbf{E}&=\omega\mathbf{B} &\qquad \mathbf{k}\times\mathbf{B}&=-\frac{\omega}{c^2}\mathbf{E}.\end{aligned}\tag{11.33}$$

From Gauss' Law ($\mathbf{k}\cdot\mathbf{E}=0$) and from Oersted's Result ($\mathbf{k}\cdot\mathbf{B}=0$) in (11.33) we see that the wave vector $\mathbf{k}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

From Faraday's Law, or from Ampère's Law, we have

$$\mathbf{E}\cdot(\mathbf{k}\times\mathbf{E})=0=\mathbf{E}\cdot(\omega\,\mathbf{B}),$$

or

$$\mathbf{B}\cdot(\mathbf{k}\times\mathbf{B})=0=-\mathbf{B}\cdot\left(\frac{\omega}{c^2}\,\mathbf{E}\right).$$

So $\mathbf{E}$ is perpendicular to $\mathbf{B}$. Then $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular (orthogonal).

If we cross $\mathbf{E}$ into Faraday's Law and use the $bac-cab$ rule with $\mathbf{E}\cdot\mathbf{k}=0$ we have

$$\mathbf{E}\times(\mathbf{k}\times\mathbf{E})=\mathbf{k}\left(E^2\right)=\omega\mathbf{E}\times\mathbf{B}.\tag{11.34}$$

The wave vector $k$ is then in the direction of $\mathbf{E}\times\mathbf{B}$. We may obtain the same result from Ampère's Law.

Since $k/\omega=1/c$, (11.34) is

$$\frac{1}{c}E^2=|\mathbf{E}|\,|\mathbf{B}|\,,$$

or

$$|\mathbf{B}|=\frac{1}{c}\,|\mathbf{E}|\tag{11.35}$$

Therefore the electromagnetic waves in empty space are transverse waves, i.e. the fields are perpendicular to the direction of wave motion (the wave vector $\mathbf{k}$). The orientation of the wave vector with respect to the fields is given by $\mathbf{E}\times\mathbf{B}$ and, in the SI system of units the magnitudes of the fields are related by $|\mathbf{B}|=|\mathbf{E}|/c$.

### 11.4.1 Polarization

The vector functions $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{B}(\mathbf{k},\omega)$ are complex. If we choose $\mathbf{k}$ to be oriented along the $z$−axis $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{B}(\mathbf{k},\omega)$ have complex vector components in the $\hat{e}_{\mathrm{x}}$ and $\hat{e}_{\mathrm{y}}$ directions $E_{\mathrm{x}}(\mathbf{k},\omega)$, $E_{\mathrm{y}}(\mathbf{k},\omega)$ and $B_{\mathrm{x}}(\mathbf{k},\omega)$, $B_{\mathrm{y}}(\mathbf{k},\omega)$. We may always write a complex number as the product of a real number and the exponential of a phase angle. We then write

$$\begin{aligned}
E_{\mathrm{x}}(\mathbf{k},\omega) &= E_{\mathrm{x}}^{(\mathrm{r})}\exp(i\varphi_{\mathrm{x}}) \\
E_{\mathrm{y}}(\mathbf{k},\omega) &= E_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\varphi_{\mathrm{y}}\right) \\
B_{\mathrm{x}}(\mathbf{k},\omega) &= B_{\mathrm{x}}^{(\mathrm{r})}\exp(i\psi_{\mathrm{x}}) \\
B_{\mathrm{y}}(\mathbf{k},\omega) &= B_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\psi_{\mathrm{y}}\right),
\end{aligned}$$

where the superscript (r) indictaes a real quantity. The complex vector quantities $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{B}(\mathbf{k},\omega)$ then become

$$\mathbf{E}(\mathbf{k},\omega) = \hat{e}_{\mathrm{x}}E_{\mathrm{x}}^{(\mathrm{r})}\exp(i\varphi_{\mathrm{x}}) + \hat{e}_{\mathrm{y}}E_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\varphi_{\mathrm{y}}\right) \tag{11.36}$$

and

$$\mathbf{B}(\mathbf{k},\omega) = \hat{e}_{\mathrm{x}}B_{\mathrm{x}}^{(\mathrm{r})}\exp(i\psi_{\mathrm{x}}) + \hat{e}_{\mathrm{y}}B_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\psi_{\mathrm{y}}\right). \tag{11.37}$$

In (11.36) and (11.37) the angles $\varphi_{\mathrm{x,y}}$ and $\psi_{\mathrm{x,y}}$ are real. Then (11.19) and (11.20) become

$$\begin{aligned}
&\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - ikz) \\
&= \hat{e}_{\mathrm{x}}E_{\mathrm{x}}^{(\mathrm{r})}\exp(i\omega t - ikz + i\varphi_{\mathrm{x}}) \\
&\quad + \hat{e}_{\mathrm{y}}E_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\omega t - ikz + i\varphi_{\mathrm{y}}\right)
\end{aligned} \tag{11.38}$$

and

$$\begin{aligned}
&\mathbf{B}(\mathbf{k},\omega)\exp(i\omega t - ikz) \\
&= \hat{e}_{\mathrm{x}}B_{\mathrm{x}}^{(\mathrm{r})}\exp(i\omega t - ikz + i\psi_{\mathrm{x}}) \\
&\quad + \hat{e}_{\mathrm{y}}B_{\mathrm{y}}^{(\mathrm{r})}\exp\left(i\omega t - ikz + i\psi_{\mathrm{y}}\right),
\end{aligned} \tag{11.39}$$

The real vector fields $\mathbf{E}(z,t)$ and $\mathbf{B}(z,t)$ associated with the wave are the real parts of (11.38) and (11.39). Since the only complex quantities on the right hand sides of (11.38) and (11.39) are the exponentials, the real vector fields are

$$\begin{aligned}
\mathbf{E}(z,t) = {} & \hat{e}_{\mathrm{x}}E_{\mathrm{x}}^{(\mathrm{r})}\cos(\omega t - kz + \varphi_{\mathrm{x}}) \\
& + \hat{e}_{\mathrm{y}}E_{\mathrm{y}}^{(\mathrm{r})}\cos\left(\omega t - kz + \varphi_{\mathrm{y}}\right)
\end{aligned} \tag{11.40}$$

and

$$\mathbf{B}(z,t) = \hat{e}_x B_x^{(r)} \cos(\omega t - kz + \psi_x) + \hat{e}_y B_y^{(r)} \cos\left(\omega t - kz + \psi_y\right) \quad (11.41)$$

In general the phase angles $\varphi_{x,y}$ and $\psi_{x,y}$ are all distinct from one another and may take on any values. From Sect. 11.4 we know, however, that $\mathbf{E}(z,t)$ and $\mathbf{B}(z,t)$ are always perpendicular to one another and are related in magnitudes by (11.35). Therefore we need only analyze either $\mathbf{E}(z,t)$ or $\mathbf{B}(z,t)$ in detail. We choose to consider $\mathbf{E}(z,t)$.

We now choose a point on the $z-$axis, along which the wave propagates, to be the origin with $z = 0$. At the origin the electric field associated with the plane wave is then

$$\mathbf{E}(z=0,t) = \hat{e}_x E_x^{(r)} \cos(\omega t + \varphi_x) + \hat{e}_y E_y^{(r)} \cos\left(\omega t + \varphi_y\right). \quad (11.42)$$

Equation (11.42) provides the components of the field in the two directions $\hat{e}_x$ and $\hat{e}_y$.

The relationship between the phase angles $\varphi_x$ and $\varphi_y$ determines the form of the wave. We may choose one of the phase angles to orient the wave at time $t = 0$ and then consider the other to be a parameter.

We shall choose $\varphi_x = \pi/2$ so that at time $t = 0$ the electric field is oriented along the $y-$axis.

We then divide the wave period into a number of time intervals and calculate the values of the $x-$ and $y-$components of the electric field from (11.42) for various values of the second phase angle $\varphi_y$. A plot of the electric field vector for these time steps provides a visual picture of the form of the electric field vector over a wave period.

We divided the wave period into 25 equal parts so that the plot of the result will clearly show the circular and elliptical forms traced by the tip of the electric field vector. We also plotted only 20 of the wave vectors so that the pattern does not close and the direction of rotation of the vector can be readily seen.

We plot the results of our calculations in Fig. 11.2.

In each of the panels of Fig. 11.2 we are looking into the oncoming wave. Our first observation is of the wave with only a $y-$component. In panels (a)–(c) this first observation is in the positive $y-$direction. In panels (d)–(f) it is in the negative $y-$direction. The rotation is counterclockwise if $\varphi_x > \varphi_y$ (panels (a–c)) or clockwise if $\varphi_y > \varphi_x$ (panels (d)–(f)). The polarization of the wave is elliptical in panels (b), (c), (e) and (f) and circular in panels (a) and (d).

If $\varphi_x = \varphi_y$ or $\varphi_y \pm \pi$ the polarization is linear. Panels (c) and (f) are close to linear polarization. We chose to show conditions close to linear polarization because only a line appears at the linear condition.

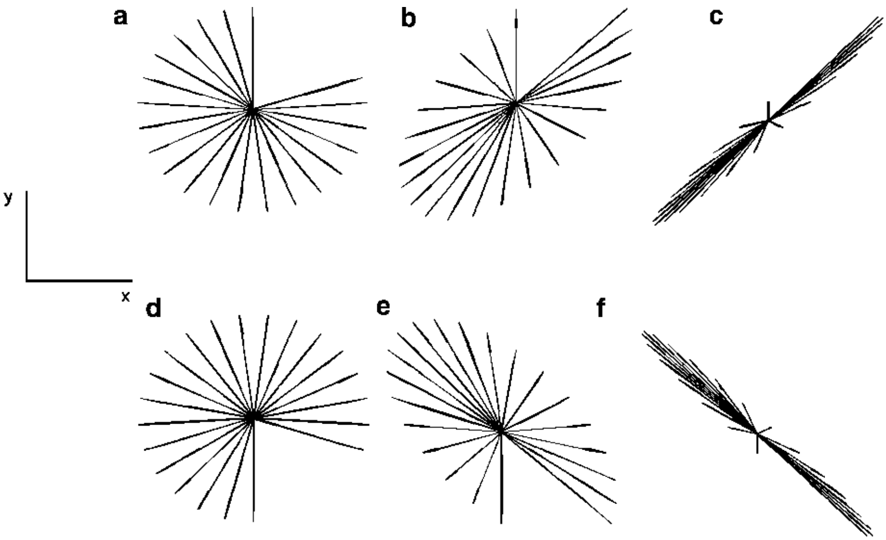


**Fig. 11.2** Electric field vector of polarized electromagnetic wave. Polarization is ellliptical in panels (**b**), (**c**), (**e**) and (**f**) and circular in panels (**a**) and (**d**). Rotation is counterclockwise if $\varphi_x > \varphi_y$ (panels (**a**–**c**)) or clockwise if $\varphi_y > \varphi_x$ (panels (**d**–**f**)). If $\varphi_x = \varphi_y$ or $\varphi_y \pm \pi$ the polarization is linear. Panels (**c**) and (**f**) are close to linear polarization

In Fig. 11.3 we have drawn a more representative picture of the rotation of an elliptically polarized electric field vector.

We chose the polarized electric field of Fig. 11.2b as our example. The wave is travelling at a velocity $c$ down the $z-$axis. The inset in the upper right hand corner of Fig. 11.3 repeats panel (b) from Fig. 11.2 for clarification. We have also indicated the rotation direction in this inset.

The electric field vector indicated as (1) in Fig. 11.3 is the first observation at time $t = 0$, which is followed by observations (2), … (20) with (20) as the observation that has "just been made" in Fig. 11.3.

The polarization we have shown in Fig. 11.3 is called *left-hand elliptical polarization* ([83], p. 420; [48], p. 274). In the terminology of modern physics this is called *positive helicity*. The original terminology comes from the fact that the electric field vector in Fig. 11.3 forms a left-handed screw.

In the first time step the electric field vector (2) moves to the point on the $z-$axis occupied by field vector (1). This requires a counterclockwise rotation around the $z-$axis, which would be the advance of a left-handed screw.

The term positive helicity indicates that if the fingers of the right hand are permitted to rotate in the (counterclockwise) direction of rotation of the electric field then the thumb of the right hand points in the direction of propagation of the wave.

The wave is still a plane wave. The spatial disturbance is still along the $z-$axis. The electric field still varies sinusoidally along the $z-$axis at any time. and the vectors **E**, **B**, and **k** are still mutually orthogonal at any instant of time. But we

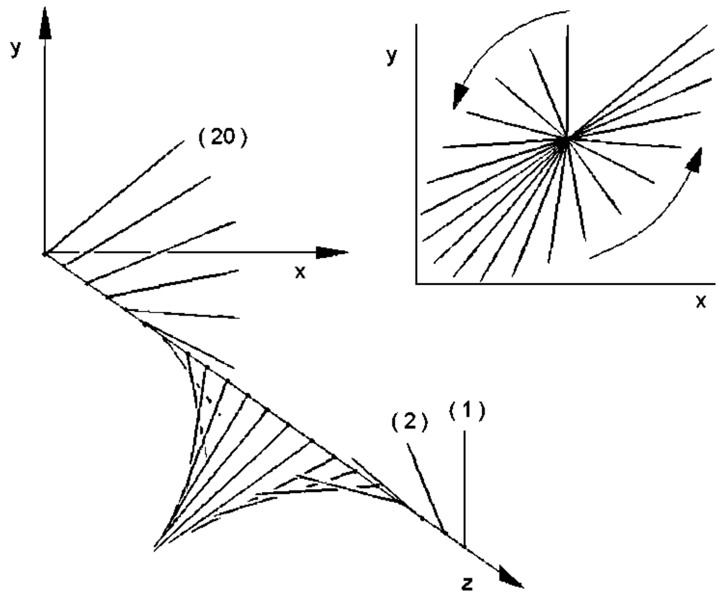


**Fig. 11.3** Polarized wave moving along the $z$—axis. The field vector is plotted at equal time intervals. The inset is a plot of the vectors projected onto the $(x, y)$—plane. The direction of rotation is indicated by the arrows in the insert. The angles are $\varphi_x = \pi/2$ and $\varphi_y = 0.2\pi$

have discovered that the electromagnetic components **E** and **B** of the disturbance are not generally simply oriented in single spatial directions. They rotate around the propagation vector **k**. Only if the wave is linearly polarized are **E** and **B** fixed in spatial orientation. The magnitudes of the vectors **E** and **B** also generally change during the rotation (elliptical polarization). The ratio of the magnitudes remains, however, always $|\mathbf{B}| = |\mathbf{E}|/c$ as required by (11.35).

It is the solenoidal character of the electric and magnetic fields (see Sect. 11.2) that requires that the **E** and **B** fields are always perpendicular to the wave vector **k**. The magnetic field is always solenoidal. And the electric field is solenoidal as long as there is no free electric charge to be taken into account. We will consider the possibility of longitudinal waves, those for which the electric field vector is not solenoidal, when we consider transport in dispersive media.

## 11.5 General Wave Solutions

### *11.5.1 Spread of Waves*

We have considered a plane wave solution to Maxwell's Equations in empty space with a particular wave **k** vector and frequency $\omega$. Such a wave is called monochromatic (single color or frequency). Maxwell's Equations tell us that the only propagating solutions in empty space are these monochromatic waves. But a monochromatic wave is an idealization unattainable in reality.

Light from a laser is very close to monochromatic as are the components of the spectrum in a gas discharge. But the limitations of quantum mechanical uncertainty between the lifetime and energy of a quantum state deny the possibility of a wave with a single frequency ([70], pp. 499–500). A formula for the quantum limited line width (Full Width at Half Maximum – FWHM) in a laser was first obtained by Arthur Schawlow and Charles Townes before the experimental demonstration of the laser [84].

Even the limit in which the only broadening is quantum mechanical is difficult to reach in practice. In gas discharges collisions of the emitting atom with particularly electrons cause shifts in the energy levels (perturbations) resulting in collisional or pressure broadening of the emitted spectral line.

### *11.5.2 Representation in Plane Waves*

To reconcile the apparent difficulty between the mathematical requirement that only monochromatic waves are allowed by our theory and the experimental fact that there exist no monochromatic waves, we have the linearity of Maxwell's Equations. We can always construct a general solution to Maxwell's Equations from a sum over monochromatic plane wave solutions.

Mathematically we can represent a function $g(x)$ in a space if we have a *complete* set of functions $\left\{\psi_{\mathrm{j}}(x)\right\}$ that span that space. This set of functions is the *basis* for representation of functions in the space. That is

$$g(x) = \sum_{\mathrm{j}} g_{\mathrm{j}} \psi_{\mathrm{j}}(x) \,.$$

If the index $j$ varies continuously, as in our case when the index is a wave vector $\mathbf{k}$, the sum becomes an integral.

### *11.5.3 Fourier Transform*

We are using complex exponentials for the plane wave. So our mathematical problem is to show that the complex exponential functions $\{\psi(k)\} = \left\{\exp(\pm ikx)/\sqrt{2\pi}\right\}$, dependent on the continuous index $k$, form a complete basis in which we can represent the sorts of functions we may expect to encounter as electromagnetic field components of waves.

The field components of waves will be finite continuous functions with continuous derivatives.[2] We will integrate over all values of the index $k$ from positive to negative infinity to obtain the complete representation of the wave field. And we will integrate over all values of the spatial coordinates to obtain the coefficients in the representation. We must, therefore, place requirements on the integrability of the field components.

The fields will be produced in a finite, often small, region of space. With no dispersion or damping the intensity of the electromagnetic wave, which we will find is proportional to the square of the wave field, decreases at a rate proportional to $1/\left(\text{distance}\right)^2$. We may then realistically claim that the field components of waves produced in a finite region of space vanish at infinity, and that the wave fields are absolutely integrable, i.e. $\int_{-\infty}^{+\infty}|f(x)|\mathrm{d}x < \infty$, and square integrable, i.e. $\int_{-\infty}^{+\infty}|f(x)|^2\mathrm{d}x < \infty$. These are the mathematical requirements for the representation we seek ([20], p. 267), which is

$$\boxed{\mathrm{f}(x) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{+\infty}\mathrm{f}(k)\exp\left(-ikx\right)\mathrm{dk}} \tag{11.43}$$

This representation is exact and the set $\{\psi(k)\}$ is complete if the right hand side of (11.43) is identical to $f(x)$. This is true if the function $f(k)$ is

$$\boxed{\mathrm{f}(k) = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{+\infty}\mathrm{f}(x)\exp\left(ikx\right)\mathrm{dx},} \tag{11.44}$$

since then, substituting $f(k)$ from (11.44), written as an integral over $x'$, into (11.43), we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\left(x'\right)\exp\left(ik\left(x'-x\right)\right)\mathrm{d}x'\mathrm{d}k \\ &= \int_{-\infty}^{+\infty} f\left(x'\right)\delta\left(x'-x\right)\mathrm{d}x' \\ &= f(x)\,. \end{aligned} \tag{11.45}$$

where we have used (2.111), which we repeat here for reference.

$$\delta\left(x'-x\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(ik\left(x'-x\right)\right)\mathrm{d}k. \tag{11.46}$$

[2]Functions satisfying Maxwell's Equations must have continuous first derivatives and those satisfying the wave equation must have continuous second derivatives.

The representation of the $\delta$−function in (11.46) is a consequence of the *completeness* of the set of functions $\left\{\exp\left(\pm ikx\right)/\sqrt{2\pi}\right\}$. If (11.46) holds then the functions $\left\{\exp\left(\pm ikx\right)/\sqrt{2\pi}\right\}$ are a complete set. We shall refer to this as the *completeness relation* for the set $\left\{\exp\left(\pm ikx\right)/\sqrt{2\pi}\right\}$.[3]

The integral in (11.44) results in a unique value $f(k)$ for each $f(x)$. Because $k$ takes on continuous values this results in a continuous mapping of the function $f(x)$ into $f(k)$. This is a *Fourier Transform* and the two (11.44) and (11.43) are referred to as a *Fourier Transform pair*.

We can represent a general spatial function $f(x,y,z) = f(\mathbf{r})$, which is absolutely integrable and square integrable over all space, in terms of integrals over three separate sets of basis functions. That is

$$\boxed{\mathrm{f}(\mathbf{r}) = (2\pi)^{-3/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z\mathrm{f}(\mathbf{k})\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right).} \tag{11.47}$$

The coefficients $f(\mathbf{k})$ are then

$$\boxed{\mathrm{f}(\mathbf{k}) = (2\pi)^{-3/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{f}(\mathbf{r})\exp\left(i\mathbf{k}\cdot\mathbf{r}\right).} \tag{11.48}$$

Equations (11.47) and (11.48) are also a Fourier Transform pair.

For shorthand we introduce the notation

$$\int \mathrm{d}^3\mathbf{k} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z, \tag{11.49}$$

and

$$\int \mathrm{d}^3\mathbf{r} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{11.50}$$

This notation simplifies the appearance of our expressions without sacrificing clarity.

We can demonstrate completeness by substituting $f(\mathbf{k})$ from (11.48), written as an integral over $\mathbf{r}'$, into (11.47). That is

$$\begin{aligned} f(\mathbf{r}) &= \left(\frac{1}{2\pi}\right)^3\int \mathrm{d}^3\mathbf{r}'\mathrm{d}^3\mathbf{k}\, f\left(\mathbf{r}'\right)\exp\left[i\mathbf{k}\cdot\left(\mathbf{r}'-\mathbf{r}\right)\right] \\ &= \left(\frac{1}{2\pi}\right)^3\int \mathrm{d}^3\mathbf{r}'\, f\left(\mathbf{r}'\right)\delta\left(\mathbf{r}'-\mathbf{r}\right) \\ &= f(\mathbf{r}), \end{aligned} \tag{11.51}$$

[3] This is a specific form of the general requirement for the completeness of a set of continuous vectors first proven by Dirac [21].

where we have used (2.112), which we repeat here for reference.

$$\boxed{\delta\left(\mathbf{r}'-\mathbf{r}\right)=(1/2\pi)^3\int \mathrm{d}^3\mathbf{k}\exp\left[i\mathbf{k}\cdot\left(\mathbf{r}'-\mathbf{r}\right)\right].} \tag{11.52}$$

Equation (11.52) is the completeness relation for the set of functions

$$\left\{\frac{1}{\left(\sqrt{2\pi}\right)^3}\exp\left(\pm i\mathbf{k}\cdot\mathbf{r}\right)\right\}.$$

Similarly we may also perform a Fourier Transform over the time as

$$\boxed{\mathrm{f}(\omega)=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{+\infty}\mathrm{f}(t)\exp\left(-i\omega t\right)\mathrm{d}t,} \tag{11.53}$$

with the inverse

$$\boxed{\mathrm{f}(t)=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{+\infty}\mathrm{f}(\omega)\exp\left(i\omega t\right)\mathrm{d}\omega.} \tag{11.54}$$

The completeness relation is

$$\boxed{\delta\left(t-t'\right)=(1/2\pi)\int_{-\infty}^{+\infty}\exp\left(i\omega\left(t-t'\right)\right)\mathrm{d}\omega.} \tag{11.55}$$

Therefore, in the language of the Fourier Transform, our general wave solutions in empty space are

$$\mathbf{E}\left(\mathbf{r},t\right)=\left(\frac{1}{2\pi}\right)^2\int\mathbf{E}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right)\mathrm{d}\omega\mathrm{d}^3\mathbf{k} \tag{11.56}$$

$$\mathbf{B}\left(\mathbf{r},t\right)=\left(\frac{1}{2\pi}\right)^2\int\mathbf{B}\left(\mathbf{k},\omega\right)\exp\left(i\omega t-i\mathbf{k}\cdot\mathbf{r}\right)\mathrm{d}\omega\mathrm{d}^3\mathbf{k}. \tag{11.57}$$

The vector field components $\mathbf{E}\left(\mathbf{k},\omega\right)$ and $\mathbf{B}\left(\mathbf{k},\omega\right)$ are then

$$\mathbf{E}\left(\mathbf{k},\omega\right)=\left(\frac{1}{2\pi}\right)^2\int\mathbf{E}\left(\mathbf{r},t\right)\exp\left(-i\omega t+i\mathbf{k}\cdot\mathbf{r}\right)\mathrm{d}t\mathrm{d}^3\mathbf{r} \tag{11.58}$$

$$\mathbf{B}\left(\mathbf{k},\omega\right)=\left(\frac{1}{2\pi}\right)^2\int\mathbf{B}\left(\mathbf{r},t\right)\exp\left(-i\omega t+i\mathbf{k}\cdot\mathbf{r}\right)\mathrm{d}t\mathrm{d}^3\mathbf{r}. \tag{11.59}$$

In (11.56) and (11.57) we have representations of propagating wave forms in empty space, which are not themselves plane waves. The plane waves out of which we have constructed the wave forms, however, satisfy the dispersion relation for propagation in empty space. The wave forms are then carried by propagating plane waves.

## 11.6 Fourier Transformed Equations

The Fourier transform replaces derivatives by algebraic expressions. We can see this using (11.56) as an example. Because the integrals are over all values of $\mathbf{k}$ and $\omega$ we may bring partial derivatives with respect to space and time inside the integrals. The only part of the integrand dependent on space and time is the complex exponential $\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})$, and (see exercises)

$$\begin{aligned}\frac{\partial}{\partial t}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) &= i\omega\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\\ \mathrm{div}\left[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right] &= -i\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\\ \mathrm{curl}\left[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right] &= -i\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) &= \left(\frac{1}{2\pi}\right)^2\int i\omega\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}\omega\mathrm{d}^3\mathbf{k}\\ \mathrm{div}\,\mathbf{E}(\mathbf{r},t) &= \left(\frac{1}{2\pi}\right)^2\int(-i\mathbf{k})\cdot\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}\omega\mathrm{d}^3\mathbf{k}\\ \mathrm{curl}\,\mathbf{E}(\mathbf{r},t) &= \left(\frac{1}{2\pi}\right)^2\int(-i\mathbf{k})\times\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}\omega\mathrm{d}^3\mathbf{k}.\end{aligned}$$

The Fourier Transform of Maxwell's Equations results then in the set of algebraic equations

$$\boxed{\begin{array}{ll}-i\mathbf{k}\cdot\mathbf{E} = \rho/\varepsilon_0 & \mathbf{k}\cdot\mathbf{B} = 0\\ \mathbf{k}\times\mathbf{E} = \omega\mathbf{B} & -i\mathbf{k}\times\mathbf{B} = \mu_0\left(\mathbf{J} + i\omega\varepsilon_0\mathbf{E}\right),\end{array}} \tag{11.60}$$

where all dependent variables, including the charge and current densities, are functions of $(\mathbf{k},\omega)$.

The set of (11.60) is fundamental for the study of wave propagation in empty space, even in the presence of charges and currents. The quantities $\mathbf{E} = \mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{B} = \mathbf{B}(\mathbf{k},\omega)$ are the field components of the general wave form in a continuous (complete) basis of plane waves.

If we set the charge and current densities to zero in (11.60) the result is (11.33), as we expect. In that case both sets of equations are for the field components of plane waves.

In a plasma the charges and currents result from ions and electrons, which satisfy a set of particle equations (see Sect. 7.5). The basic form of the (11.60) for the electromagnetic wave components remains, however, unchanged.

## 11.7 Scalar and Vector Potentials

Oersted's Result that the divergence of the magnetic field induction vanishes is unaffected by any time dependent variations in the fields. Therefore, the magnetic field induction is still equal to the curl of a vector potential.

But in the time dependent case the curl of the electric field is not zero. So the electric field is no longer simply the negative gradient of a scalar potential. If we use $\mathbf{B} = \operatorname{curl}\mathbf{A}$ in Faraday's Law, however, we have

$$\operatorname{curl}\left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}\right) = \mathbf{0}. \quad (11.61)$$

Since $\operatorname{curl}\operatorname{grad} \equiv 0$, (11.61) indicates that

$$\mathbf{E} = -\operatorname{grad}\varphi - \frac{\partial}{\partial t}\mathbf{A}, \quad (11.62)$$

where now $\varphi = \varphi(\mathbf{r}, t)$. And the electric field is found from the negative gradient of a scalar potential and the time rate of change of the vector potential.

To find an equation for the scalar potential we use the electric field in (11.62) with Gauss' Law. The result is

$$\nabla^2\varphi = -\frac{1}{\varepsilon_0}\rho - \frac{\partial}{\partial t}\operatorname{div}\mathbf{A}. \quad (11.63)$$

And if we use $\mathbf{B} = \operatorname{curl}\mathbf{A}$, which is equivalent to Oersted's Result, in Ampère's Law we get

$$\nabla^2\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right). \quad (11.64)$$

If we now introduce the electric field from (11.62) into (11.64) we obtain

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} - \operatorname{grad}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi\right), \quad (11.65)$$

which is a wave equation with sources for the vector potential.

We also get a wave equation with sources for the scalar potential if we subtract $\left(1/c^2\right)\partial^2\varphi/\partial t^2$ from both sides of (11.63) .

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{1}{\varepsilon_0}\rho + \frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right). \quad (11.66)$$

If we choose

$$\boxed{\operatorname{div}\mathbf{A} + \left(1/c^2\right)\partial\varphi/\partial t = 0} \quad (11.67)$$

the equations (11.65) and (11.66) are identical in form.

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{1}{\varepsilon_0}\rho \tag{11.68}$$

and

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{11.69}$$

But can we choose the divergence of **A** arbitrarily in the time dependent case?

We introduced the concept of gauge and of gauge transformation in Sect. 5.9. Our argument there involved *only* Oersted's Result and the requirements of Helmholtz' Theorem. Neither Oersted's result nor Helmholtz' Theorem has been affected by time dependence of the fields. Therefore the results of Sect. 5.9 are valid here as well. We then have complete freedom in our choice of div **A**. Therefore the choice of div **A** specified by (11.67) is completely legitimate in the time dependent case.

Equation (11.67) is the Lorentz Gauge. And (11.68) and (11.69) are the wave equations for the scalar potential $\varphi\,(\mathbf{r},t)$ and the vector potential $A\,(\mathbf{r},t)$.

The form of (11.68) and (11.69) is particularly convenient because the wave equation for $\varphi\,(\mathbf{r},t)$ has only $\rho\,(\mathbf{r},t)$ as a source while the wave equation for $A\,(\mathbf{r},t)$ has only $\mathbf{J}\,(\mathbf{r},t)$ as a source.

The operator on the left hand side of the (11.68) and (11.69) is often called the *d'Alembertian* after Jean-Baptiste le Rond d'Alembert.[4] The designation of this wave operator takes on various forms. The form we will use in our discussion of special relativity is

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

because we will designate the time coordinate the first of the four coordinates in Minkowski space. This is also the designation used by Jackson [48]. There is, however, no particular reason to introduce this notation here.

## 11.8 Summary

In this chapter we have shown that the solutions to Maxwell's Equations in empty space is a set of plane waves moving at the speed of light. These are transverse waves in which the electric and magnetic field components are perpendicular to one another and both are perpendicular to the direction of motion of the wave, defined by the wave vector **k**.

[4]Jean-Baptiste le Rond d'Alembert (1717–1783) was a French mathematician, mechanician, physicist, philosopher, and music theorist.

We found that the electric and magnetic field components of the wave generally rotated around the direction of propagation resulting in circular and elliptically polarized waves. In a linearly polarized wave the fields do not rotate.

Quantum uncertainty rules out the existence of monochromatic plane waves. We showed how more general (real) waves could be constructed using a Fourier Transformation. The Fourier transformed form of Maxwell's Equations gave us a set of equations which we can use to study more general wave propagation, including that in plasmas.

In the final section we introduced the Lorentz Gauge and obtained wave equations for both the scalar and vector potential. In these equations the source terms are separated. The source term for the scalar potential is the charge density and the source term for the vector potential is the current density. This form of the field equations will become particularly useful in some of the subsequent chapters.

## Exercises

**11.1.** Show that the functions (11.9) and (11.10) are both sinusoidal functions that move to the right along the $x-$axis with undiminished amplitude at a velocity $v = \omega/k$. Do this by picking a point where each function is constant and showing that the point moves to the right with this velocity.

**11.2.** Consider that an electromagnetic wave in empty space has a magnetic field component

$$\mathbf{B} = \hat{e}_{\mathrm{x}} B \sin(\omega t + ky)$$

with no magnetic components in either the $y-$ or $z-$directions.

(a) What is the direction of propagation of this electromagnetic wave?
(b) For **B** with this orientation what is the orientation of **E**?

**11.3.** Begin with the equations for curl curl **E** and curl curl **B**. Carry out the details of the derivation of both (11.5) and (11.6).

**11.4.** You may always orient your coordinates so that the propagation of a disturbance or wave form in empty space is along a rectangular Cartesian axis. Designating this axis to be $z$ the general equation to be satisfied by the electric or magnetic fields, which we designate here generally as $f$, of an electromagnetic wave form in empty space is

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial z^2} = 0.$$

(a) Show that this wave equation is satisfied by any arbitrary $f = f(p)$ where $p = (\omega t \pm kz)$ provided $c = \pm\omega/k$.
(b) In the laboratory you have an electronic flashlamp that produces a flash of very short duration, which you collimate to produce a directed pulse. You assume

that the shape of the pulse is Gaussian, i.e. $f = K\exp\left(-t^2/\tau^2\right)$, at the location of the lamp. Does your result in part (a) indicate that this pulse will propagate to the target in your laboratory?

**11.5.** Show that for any vector field function $\mathbf{F}(\mathbf{r})$

$$\operatorname{div}\mathbf{F}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^{3/2}\int(-i\mathbf{k})\cdot\mathbf{F}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{k}$$

$$\operatorname{curl}\mathbf{F}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^{3/2}\int(-i\mathbf{k})\times\mathbf{F}(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{k}$$

by bringing the divergence and curl inside of the integrals.

[You will need the differential operator relations in the Appendix.]

**11.6.** Show that

$$\begin{aligned}&\operatorname{div}\left[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right]\\&= -i\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}),\end{aligned}$$

$$\begin{aligned}&\operatorname{curl}\left[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right]\\&= -i\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right]\\&= i\omega\cdot\mathbf{E}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\end{aligned}$$

for plane waves. [You will need to use the differential identities in the Appendix.]

**11.7.** Show that for any vector field function $\mathbf{F}(\mathbf{r})$ that vanishes at infinity and satisfies the conditions of absolute and square integrability the Fourier Transform of $\operatorname{div}\mathbf{F}(\mathbf{r})$ is

$$\left(\frac{1}{2\pi}\right)^{3/2}\int\left[\operatorname{div}\mathbf{F}(\mathbf{r})\right]\exp(i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{r} = -i\mathbf{k}\cdot\mathbf{F}(\mathbf{k})$$

by Fourier Transforming $\operatorname{div}\mathbf{F}(\mathbf{r})$ directly, i.e. by evaluating the integral directly.

This exercise will require use of Gauss' Theorem. You will also need the differential operator relations in the Appendix.

**11.8.** Show that for any vector field function $\mathbf{F}(\mathbf{r})$ that vanishes at infinity and satisfies the conditions of absolute and square integrability the Fourier Transform of $\operatorname{curl}\mathbf{F}(\mathbf{r})$ is

$$\left(\frac{1}{2\pi}\right)^{3/2}\int[\mathrm{curl}\,\mathbf{F}(\mathbf{r})]\exp(i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{r}=-i\mathbf{k}\times\mathbf{F}(\mathbf{k})$$

by Fourier Transforming curl $\mathbf{F}(\mathbf{r})$ directly, i.e. by evaluating the integral directly.

This can be shown by writing out the curl in rectangular Cartesian coordinates and integrating over separate coordinates.

**11.9.** In the preceding exercise we showed that Maxwell's Equations allow the propagation of disturbances that are not distinctly wavelike in nature. Yet we have found in the chapter that the general solution of Maxwell's Equations in empty space is a plane wave in which $E$, $B$, and $k$ are mutually orthogonal.

Supposedly we reconciled this in terms of a Fourier Transformation. And we found that the Fourier Transformed Maxwell Equations in empty space are

$$\begin{array}{ll}\mathbf{k}\cdot\mathbf{E}=0 & \mathbf{k}\cdot\mathbf{B}=0\\ \mathbf{k}\times\mathbf{E}=\omega\mathbf{B} & -\mathbf{k}\times\mathbf{B}=\left(\omega/c^2\right)\mathbf{E}.\end{array}$$

These equations then determine which solutions are allowed.

Explain how the reality of the propagating very short light pulse is reconciled by the Fourier Transformation.

**11.10.** If charges and currents are present we must deal with the full Maxwell Equations. In the chapter we showed that the Fourier Transform of these full Maxwell Equations is

$$\begin{array}{ll}-i\mathbf{k}\cdot\mathbf{E}=\rho/\varepsilon_0 & \mathbf{k}\cdot\mathbf{B}=0\\ \mathbf{k}\times\mathbf{E}=\omega\mathbf{B} & -i\mathbf{k}\times\mathbf{B}=\mu_0\left(\mathbf{J}+i\omega\varepsilon_0\mathbf{E}\right).\end{array}$$

Assume that the current density is related to the electric field by

$$\mathbf{J}=\sigma\mathbf{E},$$

which is a general form of Ohm's Law.

Show that the field vectors $\mathbf{E}$ and $\mathbf{B}$ must satisfy

$$\left[i\omega\mu_0\sigma\mathbf{1}+\left(k^2-\omega^2/c^2\right)\mathbf{1}-\mathbf{kk}\right]\cdot(\mathbf{E}\text{ or }\mathbf{B})=\mathbf{0}.$$

**11.11.** Begin with the Fourier Transform of the full Maxwell Equations in a conducting medium ($\sigma\neq 0$)

$$\begin{array}{ll}-i\mathbf{k}\cdot\mathbf{E}=\rho/\varepsilon_0 & \mathbf{k}\cdot\mathbf{B}=0\\ \mathbf{k}\times\mathbf{E}=\omega\mathbf{B} & \mathbf{k}\times\mathbf{B}=\left(i\mu_0\sigma-\omega/c^2\right)\mathbf{E}.\end{array}$$

and show that for transverse waves to propagate in a conducting medium it is necessary that

$$i\omega\mu_0\sigma+\left(k^2-\omega^2/c^2\right)=0.$$

**11.12** Begin with the Fourier Transform of the full Maxwell Equations in a conducting medium ($\sigma \neq 0$)

$$-i\mathbf{k}\cdot\mathbf{E} = \rho/\varepsilon_0 \qquad \mathbf{k}\cdot\mathbf{B} = 0$$
$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B} \qquad \mathbf{k}\times\mathbf{B} = \left(i\mu_0\sigma - \omega/c^2\right)\mathbf{E}.$$

and show that for longitudinal waves to propagate in a conducting medium it is necessary that

$$i\mu_0\sigma - \omega/c^2 = 0$$

and that there is no magnetic field associated with the longitudinal wave.

# Chapter 12
# Energy and Momentum

*It is not that we propose a theory and Nature may shout NO; rather, we propose a maze of theories and Nature may shout INCONSISTENT.*

*Imre Lakatos*

## 12.1 Introduction

In the preceding chapter we saw that the propagation of electromagnetic waves in empty space is a mathematical consequence of Maxwell's Equations. And in Sect. 1.12.2 we saw that Maxwell and Hertz were aware that the waves would transport energy and momentum. Hertz also calculated the energy in the waves he observed using the theory of Poynting. And final experimental confirmation that light waves carry momentum was in 1899 ([97], p. 307). In this chapter we will develop a consistent formulation of field energy and momentum based on the fact that the reality of the fields must be reflected in general energy and momentum theorems.

Because we are considering the propagation of waves in vacuum, we shall require that all matter is point like. The charged particles we consider are then free electrons and ions.

Energy and momentum are collective properties of a system made up of particles interacting with one another through the fields that result from the densities and currents of those particles. From the First Law of Thermodynamics we know that the energy of an isolated system is conserved ([40], p. 6). And from the laws of mechanics we know that the momentum of an isolated system is conserved ([32], p. 55). Our goal is now to extend these fundamental principles to combinations of fields and particles and finally to obtain general equations for field quantities alone.

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3_12, 

We begin our discussion with a theorem for a system property that can be transported and has sources (and sinks). We will then apply this theorem to the energy and momentum exchange represented by the basic dynamics of the field-particle interaction.

## 12.2 Transport Theorem

We consider a general system property, which we designate by the subscript $\nu$. We require only that this property be identifiable in terms of particle quantities such as velocities, densities, and charges or field quantities such as **E** and **B**.

We designate the density of the quantity $\nu$ at the point **r** at the time $t$ as $\rho_\nu(\mathbf{r}, t)$. The amount $Q_\nu$ of $\nu$ in an arbitrary volume $V$ at time $t$ is then

$$Q_\nu = \int_V \rho_\nu(\mathbf{r}, t)\, \mathrm{d}V. \tag{12.1}$$

If the quantity $\nu$ is a particle property it will be transported by the particles as they move from one point to another. If it is a field quantity it will be transported by the propagating waveform. We designate the flux density of $\nu$ at the point **r** and the time $t$ as $\boldsymbol{\lambda}_\nu(\mathbf{r}, t)$. The rate of flow of $\nu$ out of the arbitrary volume $V$ is then

$$\dot{\Phi}_\nu(t) = \oint_S \boldsymbol{\lambda}_\nu(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{S}, \tag{12.2}$$

where $S$ encloses $V$.

In general the quantity $\nu$ has sources. Specifically we may speak of the rate of production of $\nu$ per unit volume at the point **r** and the time $t$, which we shall designate as $\dot{w}_\nu(\mathbf{r}, t)$. We shall consider this to be a net production term and shall not attempt to distinguish between losses and gains. Then the rate at which the amount $Q_\nu$ increases in the arbitrary volume $V$ is the difference between the total production rate of $\nu$ in the volume and the loss of $\nu$ due to flux out of the volume. That is

$$\int_V \frac{\partial}{\partial t}\rho_\nu(\mathbf{r}, t)\, \mathrm{d}V = \int_V \dot{w}_\nu(\mathbf{r}, t)\, \mathrm{d}V - \oint_S \boldsymbol{\lambda}_\nu(\mathbf{r}, t) \cdot \mathrm{d}S. \tag{12.3}$$

Applying Gauss' Theorem to the integral over $S$ on the right hand side of (12.3) and collecting terms into a single integrand we have

$$\int_V \left[\frac{\partial}{\partial t}\rho_\nu(\mathbf{r}, t) + \operatorname{div}\boldsymbol{\lambda}_\nu(\mathbf{r}, t) - \dot{w}_\nu(\mathbf{r}, t)\right] \mathrm{d}V = 0. \tag{12.4}$$

Equation (12.4) is the integral equation for the production and transport of the quantity $\nu$. Because the volume $V$ is arbitrary the integral in (12.4) can only vanish if the integrand vanishes. That is

$$\boxed{\partial\rho_{\nu}\left(\mathbf{r},t\right)/\partial\mathrm{t}+\operatorname{div}\boldsymbol{\lambda}_{\nu}\left(\mathbf{r},t\right)=\dot{w}_{\nu}\left(\mathbf{r},t\right)} \tag{12.5}$$

Equation (12.5) is then a general differential equation for a physical quantity that has a source and is transported within the system. This equation will be key to the identification of terms.

## 12.3 Electromagnetic Field Energy

We begin our discussion of the electromagnetic field energy by considering an arbitrary closed, isolated system containing electromagnetic fields and matter. The total energy of this system must be constant by the First Law of thermodynamics. Energy will be transferred between the fields and the particles but no energy is lost in the transfer.

We require that the matter is particulate. And, as we did in Sect. 5.5.1, we represent the number density ( $\mathrm{m}^{-3}$) of the species $\alpha$ of particles in a small volume $\Delta V$ of the system as

$$\delta n^{(\alpha)}=\frac{1}{\Delta V}\sum_{\text{all } i \text{ in } \Delta V}\delta\left(\mathbf{r}-\mathbf{r}_{\mathrm{i}}^{(\alpha)}\left(t\right)\right), \tag{12.6}$$

using the notation $\delta n^{(\alpha)}$ to indicate that we are representing the particles by $\delta-$functions. The vector $\mathbf{r}_{\mathrm{i}}^{(\alpha)}\left(t\right)$ is the trajectory of the $i$th particle of the $\alpha$th species. By the property of the $\delta-$function (2.98), the integral of $\delta n^{(\alpha)}$ over the volume $\Delta V$ is the number of particles of species $\alpha$ in the volume $\Delta V$ divided by $\Delta V$, which is density $n^{(\alpha)}$ of the particles of species $\alpha$ at the location of $\Delta V$.

We consider that the particles of the species $\alpha$ have a charge $q_{\alpha}$. We ignore uncharged particles. In this model we are explicitly neglecting the atomic structure of matter. Atoms, and molecules, absorb and emit electromagnetic energy (light) at certain frequencies corresponding to the differences between quantum energy levels.

In gas discharges, which are low temperature (electron temperature $\sim 10^4$ K) and low density plasmas, only the atomic spectrum is detected. The probability of exciting electron levels in the ions by collisions with free electrons is negligibly small. Because they are unaffected by the fields the atoms in the discharge are at room temperature and can be considered stationary with respect to the electrons. We neglect these inelastic electron-atom collisions and the spectral emission.

To be absolutely consistent in our picture, we must also claim that the energy of interaction between particles and fields is conserved in any small volume of the system. Logically this is no problem for volumes that are small compared to macroscopic variations in system properties. The very small size of electrons and ions essentially assures the validity of this claim.

The Lorentz force acting on each particle of the $\alpha^{\text{th}}$ species is

$$\mathbf{F}_{\text{i}}^{(\alpha)} = q_\alpha \left( \mathbf{E} + \mathbf{v}_{\text{i}}^{(\alpha)} \times \mathbf{B} \right).$$

We may then identify the electromagnetic force per unit volume acting on particles of the $\alpha$th species in the volume $\Delta V$ as

$$\delta \mathbf{f}_{\text{em}}^{(\alpha)} = \frac{1}{\Delta V} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \left( \mathbf{E} + \mathbf{v}_{\text{i}}^{(\alpha)} \times \mathbf{B} \right) \delta \left( \mathbf{r} - \mathbf{r}_{\text{i}}^{(\alpha)}(t) \right). \tag{12.7}$$

The rate at which work is done by the electromagnetic fields on these particles is

$$\begin{aligned} \delta \dot{w}_{\text{part}}^{(\alpha)} &= \frac{1}{\Delta V} \sum_{\text{all } i \text{ in } \Delta V} \left[ q_\alpha \left( \mathbf{E} + \mathbf{v}_{\text{i}}^{(\alpha)} \times \mathbf{B} \right) \cdot \mathbf{v}_{\text{i}}^{(\alpha)} \right] \delta \left( \mathbf{r} - \mathbf{r}_{\text{i}}^{(\alpha)}(t) \right) \\ &= \mathbf{E} \cdot \left[ \frac{1}{\Delta V} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\text{i}}^{(\alpha)} \delta \left( \mathbf{r} - \mathbf{r}_{\text{i}}^{(\alpha)}(t) \right) \right], \end{aligned} \tag{12.8}$$

since

$$\left( \mathbf{v}_{\text{i}}^{(\alpha)} \times \mathbf{B} \right) \cdot \mathbf{v}_{\text{i}}^{(\alpha)} = 0.$$

The current density (5.1), carried by the $\alpha$th species of particle in the volume $\Delta V$ is

$$\delta \mathbf{J}^{(\alpha)} = \frac{1}{\Delta V} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\text{i}}^{(\alpha)} \delta \left( \mathbf{r} - \mathbf{r}_{\text{i}}^{(\alpha)} \right), \tag{12.9}$$

where the velocity of the $i$th particle is $\mathbf{v}_{\text{i}}^{(\alpha)}$. Then, with (12.9) (12.8) becomes

$$\delta \dot{w}_{\text{part}}^{(\alpha)} = \mathbf{E} \cdot \delta \mathbf{J}^{(\alpha)}. \tag{12.10}$$

The total rate at which work is done by electromagnetic fields on the all the particles in $\Delta V$ is then a summation of (12.10) over $\alpha$, which is

$$\delta \dot{w}_{\text{part}} = \mathbf{E} \cdot \delta \mathbf{J}, \tag{12.11}$$

where

$$\delta \mathbf{J} = \frac{1}{\Delta V} \sum_{\alpha} \sum_{\text{all } i \text{ in } \Delta V} q_\alpha \mathbf{v}_{\text{i}}^{(\alpha)} \delta \left( \mathbf{r} - \mathbf{r}_{\text{i}}^{(\alpha)} \right) \tag{12.12}$$

is the total current in $\Delta V$.

If we integrate (12.11) over $\Delta V$ we have an equation for the rate at which work is done by the electromagnetic fields on all of the particles in the volume $\Delta V$ divided by the volume $\Delta V$. By the integral property of the $\delta-$function this results in

$$\dot{w}_{\mathrm{part}} = \mathbf{J} \cdot \mathbf{E}, \tag{12.13}$$

which is the total rate at which work is done per unit volume on the particles at the location of the small volume $\Delta V$ at a particular time. This is the rate at which energy is transferred from the fields to the particles (per unit volume). In our system this is the negative of the rate at which particles transfer energy to the fields (per unit volume). That is the rate of increase in the electromagnetic field energy (per unit volume), at the position of the volume $\Delta V$ is

$$\dot{w}_{\mathrm{em}} = -\mathbf{J} \cdot \mathbf{E} \tag{12.14}$$

We now require an equation that provides $-\mathbf{J} \cdot \mathbf{E}$ and contains all the information in Maxwell's Equations

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \rho/\varepsilon_0 & \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t & \operatorname{curl}\mathbf{B} &= \mu_0\left(\mathbf{J} + \varepsilon_0 \partial\mathbf{E}/\partial t\right). \end{aligned} \tag{12.15}$$

From Faraday's Law we have

$$-\frac{1}{\mu_0}\mathbf{B} \cdot \operatorname{curl}\mathbf{E} = \frac{\partial}{\partial t}\frac{1}{2}\frac{1}{\mu_0}B^2. \tag{12.16}$$

And from Ampère's Law

$$\frac{1}{\mu_0}\mathbf{E} \cdot \operatorname{curl}\mathbf{B} = \mathbf{J} \cdot \mathbf{E} + \frac{\partial}{\partial t}\frac{1}{2}\varepsilon_0 E^2 \tag{12.17}$$

Adding (12.16) and (12.17) and using (A.20) we have an equation for $-\mathbf{J} \cdot \mathbf{E}$

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) + \operatorname{div}\left(\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}\right) = -\mathbf{J} \cdot \mathbf{E}, \tag{12.18}$$

which is the equation we sought. If we compare (12.18) with (12.5) we can identify

$$\boxed{\mathcal{E}_{\mathrm{em}} = (1/2)\left(\varepsilon_0 E^2 + (1/\mu_0)\, B^2\right)} \tag{12.19}$$

as the electromagnetic field energy density and

$$\boxed{\mathbf{S} = (1/\mu_0)\,\mathbf{E} \times \mathbf{B}} \tag{12.20}$$

as the flux vector for the electromagnetic field energy. The vector **S** defined in (12.20) is the *Poynting Vector* (see Sect. 1.12.2). We then have a general differential form of the electromagnetic energy equation as

$$\boxed{\partial\mathcal{E}_{\mathrm{em}}/\partial t + \operatorname{div}\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}.} \tag{12.21}$$

We will not pursue the particle energy equation at this point. To correctly treat the interaction of a system of particles with the electromagnetic field we would have to pursue the development we began in Sect. 7.5 with (7.50).

## 12.4 Electromagnetic Field Momentum

We will develop an equation for the field momentum from the same point of view as that used in the preceding section. We consider an arbitrary isoltated system in which the only important forces are electromagnetic. Specifically we ignore all gravitational field forces as insignificant compared to the Lorentz Force. The total momentum of this system is conserved and in a small volume $\Delta V$ the gain in momentum by a particle is transferred from the fields. This will allow us to identify the rate of change of field momentum (per unit volume) as we did the rate of change of field energy in Sect. 12.3.

Newton's Second Law tells us that the Lorentz Force acting on the particles of the $\alpha$th species (12.7) is the rate of increase of momentum of these particles (per unit volume). Summing equation (12.7) over all species of particles $\alpha$ and integrating over the volume $\Delta V$ we have the total force (per unit volume) of the fields on the particles at the location of the small volume $\Delta V$.

$$\mathbf{f}_{\mathrm{em}} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \tag{12.22}$$

We now turn to Maxwell's Equations to obtain $\rho\mathbf{E}$ and $\mathbf{J}\times\mathbf{B}$.

To obtain an equation for $\rho\mathbf{E}$ we combine Gauss' and Faraday's Laws. The result is

$$\varepsilon_0\left(\operatorname{div}\mathbf{E}\right)\mathbf{E} + \varepsilon_0\left(\operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t}\right)\times\mathbf{E} = \rho\mathbf{E}. \tag{12.23}$$

We can obtain an equation for $\mathbf{J}\times\mathbf{B}$ from Ampère's Law alone. To include the information of Oersted's result and to symmetrize our equation with (12.23), however, we add $\left(\operatorname{div}\mathbf{B}\right)\mathbf{B}/\mu_0 = \mathbf{0}$ obtaining

$$\frac{1}{\mu_0}\left(\operatorname{div}\mathbf{B}\right)\mathbf{B} + \left(\frac{1}{\mu_0}\operatorname{curl}\mathbf{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\times\mathbf{B} = \mathbf{J}\times\mathbf{B}. \tag{12.24}$$

With (12.23) and (12.24) (12.22) becomes

$$\frac{\partial}{\partial t}\left(\varepsilon_0 \mathbf{E}\times\mathbf{B}\right)-\left[\varepsilon_0\left(\operatorname{div}\mathbf{E}\right)\mathbf{E}+\frac{1}{\mu_0}\left(\operatorname{div}\mathbf{B}\right)\mathbf{B}+\varepsilon_0\left(\operatorname{curl}\mathbf{E}\right)\times\mathbf{E}+\frac{1}{\mu_0}\left(\operatorname{curl}\mathbf{B}\right)\times\mathbf{B}\right]$$
$$=-\mathbf{f}_{\mathrm{em}}. \tag{12.25}$$

Since $-\mathbf{f}_{\mathrm{em}}$ is the source term for the electromagnetic field momentum density, we may compare (12.25) with (12.5). On doing so we see that the first term on the left hand side of (12.25) is the partial time derivative of the electromagnetic field momentum density. That is the electromagnetic field momentum density is

$$\boxed{\mathbf{P}_{\mathrm{em}}=\varepsilon_0\vec{E}\times\vec{B}.} \tag{12.26}$$

According to (12.5), the remaining term $(-[\cdots])$ on the left hand side of (12.25) is the divergence of the momentum flux. We must now convert this term to a divergence in order to identify the momentum flux. Since this divergence term is itself a vector, the momentum flux must be a tensor with two indices, rather than a vector with one index.

We shall work here in a rectangular Cartesian basis for simplicity. The results will be valid for all systems (see Sect. 2.2.2).

Writing the components of the momentum flux density as $T_{\mu\nu}$, the divergence of the momentum flux is then

$$\delta_{\lambda\mu}\frac{\partial}{\partial x_\lambda}T_{\mu\nu}=\frac{\partial}{\partial x_\mu}T_{\mu\nu},$$

using the Einstein summation convention. The $\nu$th components of the two terms making up the second term on the left hand side of (12.25) are

$$-\left[\varepsilon_0\left(\operatorname{div}\,\mathbf{E}\right)\mathbf{E}\,+\frac{1}{\mu_0}\left(\operatorname{div}\mathbf{B}\right)\mathbf{B}\right]_\nu=-\left(\varepsilon_0\frac{\partial E_\mu}{\partial x_\mu}E_\nu+\frac{1}{\mu_0}\frac{\partial B_\mu}{\partial x_\mu}B_\nu\right). \tag{12.27}$$

and

$$-\left[\frac{1}{\mu_0}\left(\operatorname{curl}\,\mathbf{B}\right)\times\mathbf{B}+\varepsilon_0\left(\operatorname{curl}\mathbf{E}\right)\times\mathbf{E}\right]_\nu$$
$$=-\frac{1}{\mu_0}\varepsilon_{\gamma\mu\nu}\varepsilon_{\alpha\beta\gamma}\frac{\partial B_\beta}{\partial x_\alpha}B_\mu-\varepsilon_0\varepsilon_{\gamma\mu\nu}\varepsilon_{\alpha\beta\gamma}\frac{\partial E_\beta}{\partial x_\alpha}E_\mu, \tag{12.28}$$

where $\varepsilon_{\gamma\mu\nu}$ and $\varepsilon_{\alpha\beta\gamma}$ are Levi-Civita densities (see Sect. 2.2.4). Since $\varepsilon_{\alpha\beta\gamma}=-\varepsilon_{\gamma\beta\alpha}$, (12.28) becomes

$$-\left[\frac{1}{\mu_0}\left(\operatorname{curl}\,\mathbf{B}\right)\times\mathbf{B}+\varepsilon_0\left(\operatorname{curl}\mathbf{E}\right)\times\mathbf{E}\right]_\nu$$
$$=\frac{1}{\mu_0}\varepsilon_{\gamma\mu\nu}\varepsilon_{\gamma\beta\alpha}\frac{\partial B_\beta}{\partial x_\alpha}B_\mu+\varepsilon_0\varepsilon_{\gamma\mu\nu}\varepsilon_{\gamma\beta\alpha}\frac{\partial E_\beta}{\partial x_\alpha}E_\mu \tag{12.29}$$

In (12.29) we can have $\beta = \mu$ and $\alpha = \nu$ with a positive result, because then $\varepsilon_{\gamma\mu\nu} = \varepsilon_{\gamma\beta\alpha}$, or $\beta = \nu$ and $\alpha = \mu$ with a negative result, because then $\varepsilon_{\gamma\mu\nu} = -\varepsilon_{\gamma\beta\alpha}$. Then (12.29) is

$$-\left[\frac{1}{\mu_0}(\text{curl }\mathbf{B}) \times \mathbf{B} + \varepsilon_0\,(\text{curl }\mathbf{E}) \times \mathbf{E}\right]_\nu = \frac{1}{\mu_0}\frac{\partial B_\mu}{\partial x_\nu}B_\mu + \varepsilon_0\frac{\partial E_\mu}{\partial x_\nu}E_\mu$$
$$-\frac{1}{\mu_0}\frac{\partial B_\nu}{\partial x_\mu}B_\mu - \varepsilon_0\frac{\partial E_\nu}{\partial x_\mu}E_\mu \qquad (12.30)$$

Combining the terms from (12.27) and (12.30) we have

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu} = -\left(\varepsilon_0\frac{\partial E_\mu}{\partial x_\mu}E_\nu + \frac{1}{\mu_0}\frac{\partial B_\mu}{\partial x_\mu}B_\nu\right)$$
$$+\frac{1}{\mu_0}\frac{\partial B_\mu}{\partial x_\nu}B_\mu + \varepsilon_0\frac{\partial E_\mu}{\partial x_\nu}E_\mu$$
$$-\frac{1}{\mu_0}\frac{\partial B_\nu}{\partial x_\mu}B_\mu - \varepsilon_0\frac{\partial E_\nu}{\partial x_\mu}E_\mu. \qquad (12.31)$$

Now

$$\varepsilon_0\frac{\partial E_\mu}{\partial x_\mu}E_\nu = \varepsilon_0\frac{\partial}{\partial x_\mu}E_\mu E_\nu - \varepsilon_0 E_\mu\frac{\partial E_\nu}{\partial x_\mu}$$

and

$$\frac{1}{\mu_0}\frac{\partial B_\mu}{\partial x_\mu}B_\nu = \frac{1}{\mu_0}\frac{\partial}{\partial x_\mu}B_\mu B_\nu - \frac{1}{\mu_0}B_\mu\frac{\partial B_\nu}{\partial x_\mu}$$

Then (12.31) becomes

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial x_\nu}\delta_{\mu\nu}\left(\frac{1}{\mu_0}B_\alpha B_\alpha - \varepsilon_0 E_\alpha E_\alpha\right)$$
$$-\left(\varepsilon_0\frac{\partial}{\partial x_\mu}E_\mu E_\nu + \frac{1}{\mu_0}\frac{\partial}{\partial x_\mu}B_\mu B_\nu\right) \qquad (12.32)$$

And the stress tensor we sought is

$$T_{\mu\nu} = \delta_{\mu\nu}\frac{1}{2}\left(\frac{1}{\mu_0}B^2 + \varepsilon_0 E^2\right) - \varepsilon_0 E_\mu E_\nu - \frac{1}{\mu_0}B_\mu B_\nu. \qquad (12.33)$$

In dyadic notation this is ([85], p. 23; [48], p. 239; [97], p. 303)

$$\boxed{\boldsymbol{T} = (1/2)\left(\varepsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right)\mathbf{1} - \varepsilon_0\mathbf{EE} - (1/\mu_0)\,\mathbf{BB}.} \qquad (12.34)$$

Combining (12.26) and (12.34) in (12.25) our final equation for field momentum is

$$\boxed{\partial \mathbf{P}_{\mathrm{em}}/\partial \mathrm{t} + \operatorname{div} \boldsymbol{T} = -\rho \mathbf{E} - \mathbf{J} \times \mathbf{B}} \tag{12.35}$$

There is some ambiguity regarding the algebraic sign that appears on the stress tensor. Our definitions here agree with that of Schwinger, et al. ([85], p. 23). But Jackson's definition is the negative of ours ([48], p. 239). The difference lies in the placement of the divergence term in the conservation theorem (12.5). Schwinger, et al. place the term as we do, while Jackson places it on the other side of the equation. The result is a difference in the algebraic sign. The final equations written for transport of the field momentum are, however, identical.

Equations (12.33) and (12.34) are forms of *Maxwell's Stress Tensor*. Maxwell's understanding of the detailed dynamics of the interaction of, for example, a light wave with a material surface differed from ours ([97], p. 307). In the end, however, the force from an electromagnetic wave on a material surface is the result of the flux of momentum to the surface of the material. And that is the integral of the scalar product of the stress tensor (momentum flux tensor) with the surface area. Maxwell's prediction of light pressure on a material surface was the same as ours would be with a different understanding, but with the same mathematical equations.

We may relate the momentum density (vector) $\mathbf{P}_{\mathrm{em}}$ in (12.26) to the energy flux vector (Poynting Vector) $\mathbf{S}$ in (12.20). Using $\varepsilon_0 \mu_0 = 1/c^2$ we have

$$\mathbf{P}_{\mathrm{em}} = \frac{1}{c^2} \mathbf{S} \tag{12.36}$$

This result differs by a factor of two from the relationship obtained for a beam of material particles (see exercises).We do, however, obtain this result for a relativistic beam of massless particles.

## 12.5 Static Field Energies

In Sects. 4.6.1 and 10.4 we obtained equations for the electrostatic and magnetostatic energy densities. These were (4.31)

$$u_{\mathrm{E}} = \frac{1}{2} \varepsilon_0 E^2$$

for the electrostatic energy density and (10.22)

$$u_{\mathrm{B}} = \frac{1}{2} \frac{1}{\mu_0} B^2$$

for the magnetostatic energy density. These are the same equations as those we obtained in this chapter for the energy densities for the electromagnetic (dynamic) fields.

There is no particular mystery in the fact that these equations are identical. In both Sect. 4.6.1 and in the present chapter we based our derivation on the transfer of energy from the fields to the particles by the action of the electric field. The electric field energy densities should logically then be the same.

In this chapter the magnetic field energy appeared because we included Faraday's Law and the displacement current, which govern the field-flield interaction in our derivation. In our derivation of the magnetostatic field in Sect. 10.4 we also used Faraday's Law and the work done by the induced emf on the charged particles. We only slowed the time scale in order to be able to ignore the displacement current.

So the derivation of the magnetic field energy was actually the same in both cases as well.

## 12.6 Summary

In this chapter we have identified the energy and momentum carried by electromagnetic waves. If the waves in fact represent a transport of electromagnetic fields from one spatial point to another, then there must be a description, consistent with the theory of these fields, for the transport of energy and momentum.

We were able to obtain energy and momentum densities and fluxes based solely on Maxwell's Equations. That is the field picture in Maxwell's Equations has, inherent within it, a consistent description of energy and momentum transport within the waves it predicts. Although the laboratory detection of waves assures us of their reality, the theory would not be complete if an understanding of the energy and momentum transport did not emerge naturally from it.

## Exercises

**12.1.** Consider a stream of material particles, with a density $n$ m$^{-3}$, all moving in a single direction with a velocity $\mathcal{V}$. The energy flux density is the rate at which particle kinetic energy is transported past an area perpendicular to the stream. This has units $\left(\text{kg m}^2\ \text{s}^{-2}\right)\ \text{m}^{-2}\ \text{s}^{-1}$. The momentum density in the stream of particles is the momentum per particle multiplied by the density of particles in the stream. This has units $\left(\text{kg m s}^{-1}\right)\ \text{m}^{-3}$. Find the algebraic relationship between the momentum density and the energy flux density for this stream of material particles.

Your stream of particles is basically the Newtonian picture of light. Compare your relationship to that for an electromagnetic wave from Maxwell's Equations in (12.36). How would you decide experimentally between the two pictures based on a momentum/energy measurement?

**12.2.** Show that the term $\dot{w}_{\text{em}}$ in the equation

$$\dot{w}_{\text{em}} = \mathbf{E} \cdot \mathbf{J}$$

has the units of work per unit time per unit volume.

**12.3.** Show that the Lorentz Force density

$$\mathbf{f}_{\text{em}} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

has the units of momentum per unit time per unit volume.

**12.4.** A straight cylindrical metal wire of conductivity $\sigma$ and radius $R$ carries an axial current density $\mathbf{J} = J\hat{e}_z$, which is constant across the cross section and does not vary in time. Assume Ohm's Law is valid in the wire

$$\mathbf{J} = \sigma \mathbf{E}.$$

Find the Poynting vector at the surface of the wire and the energy flux across the surface for a length $L$ of the wire. How does this compare to the so-called Joule heating rate in the wire $I^2 R_{\text{es}}$, where here $R_{\text{es}}$ is the resistance of the wire and is

$$R_{\text{es}} = \frac{L}{\sigma \pi R^2}$$

**12.5.** In a certain region of space there is an electrostatic field and also a magnetostatic field. There are no charges or currents in the region. Show that although the Poynting vector may be nonzero, the surface integral of $\mathbf{S} \cdot \hat{n}$ vanishes over an arbitrary closed surface inside the region.

**12.6.** The electric field component of an electromagnetic wave is

$$\mathbf{E} = \hat{e}_x E_0 \cos \omega \left( \sqrt{\epsilon_0 \mu_0} z - t \right) + \hat{e}_y E_0 \sin \omega \left( \sqrt{\epsilon_0 \mu_0} z - t \right)$$

where $E_0$ is a constant and $\sqrt{\epsilon_0 \mu_0} = 1/c$. Find the Poynting vector.

**12.7.** Consider a spherical shell of charge with radius $R$ and uniform surface charge density $\sigma_0$. Determine the self-energy of the distribution by integration over the field energy density, $\frac{1}{2}\varepsilon_0 \int_V \mathbf{E} \cdot \mathbf{E} \, dV$.

**12.8.** Show that the energy flux (energy per unit area) of an electromagnetic wave emitted from a small (point) source decreases as $1/r^2$, where $r$ is the distance from the source, provided there is no damping or dispersion of the wave. That is provided there is no loss of wave energy as a function of time or distance. This is the requirement for energy conservation.

If there is damping and/or dispersion of an electromagnetic wave there must be a path for the energy. Can you describe a path for energy transport from an electromagnetic wave in empty space?

# Chapter 13
# Special Relativity

*Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a union of the two will preserve an independent reality.*

*Hermann Minkowski*

## 13.1 Introduction

Prior to this point we have accepted a Euclidean geometry and a Newtonian concept of space and time. Newton said, "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable." And regarding time he said, "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external ..." ([77], p. 6). And we have allowed ourselves to imagine that we can observe all frames of reference from some separate position, perhaps at rest in the universe.

Albert Einstein realized that our concept of time in this picture was flawed and, as a consequence, so was our concept of space. It is simply not possible to occupy a separate position in the universe and observe the occurrences in separate frames. We are present in a frame of reference and our measurements, which are the basis of our science, are dependent on that fact.

We will begin this chapter with an outline of Einstein's ideas in his 1905 paper on special relativity entitled *On the Electrodynamics of Moving Bodies*. Then we will formalize these ideas using the four dimensional framework of Hermann Minkowski. Minkowski's four dimensional union of space and time provides a simplifying structure in which we can cast Einstein's ideas.

The laws of physics must be independent of coordinate system. This Ansatz led Einstein to base his considerations of general relativity (1916) on tensors. Because of their importance in the modern relativistic treatment of electromagnetic fields we will provide an introduction to some of the basic properties of tensors in this chapter.

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics, DOI 10.1007/978-3-642-23205-3_13, 

We will then be able to show that the laws of electrodynamics are covariant under any transformation consistent with Einstein's concepts of time and space.

Conventions and notations in relativity are changing. This is fortunate because the notation is becoming simpler and more understandable. But this is also unfortunate because not all authors use the same notation. We have chosen here to use the basic notation of J.D. Jackson [48] in our development.

## 13.2 The New Kinematics

We do not completely understand Einstein's thinking before 1905. Pais has outlined what we know ([78], pp. 130–133). There we can find what was understood by the European scientific community before 1905 and what was not.

The Michelson and Morley result was disturbing to physicists. The Irish physicist George Francis FitzGerald wrote a single paragraph paper "The Ether and the Earth's Atmosphere", which was published in the American journal *Science* in 1889. Pais publishes this paper in full ([78], p. 122). FitzGerald believed that the only hypothesis that could reconcile the results of the Michelson-Morley experiment was one which claimed there was a shortening of the length of material bodies as they moved in the aether. He suggested that it was "not improbable" that the electric molecular forces are affected by motion through the aether.

Lorentz cites an exchange with FitzGerald in his 1895 paper "Michelson's Interference Experiment" ([24], p. 4) and again when he presented what are now known as the Lorentz or Lorentz-FitzGerald transformation equations in 1904 ([24], pp. 11–34). Lorentz also suggested the same origin for the shortening in the molecular interactions is transmitted through the aether. For Lorentz as well as for FitzGerald the problem was one of mechanics and of interactions. It was not one of time and space. Pais wrote that Lorentz never fully made the transition from the old dynamics to the new kinematics ([78], p. 167). Einstein was not familiar with the work of Lorentz beyond 1895 ([78], p. 125).

The great French mathematician and physicist Henri Poincaré understood the difficulties of time and simultaneity. He wrote that, "... we have not even direct intuition of the simultaneity of two events occurring in different places ..." [Pais, p. 133] Einstein and his friends in their Akademie Olympia[1] studied Poincaré in detail. But Poincaré did not carry this idea farther, as Einstein did. Rather Poincaré suggested that the difficulty might lie in Newtonian mechanics and added the hypothesis of FitzGerald and Lorentz to the mechanics ([78], p. 128).

[1]The Akademie Olympia was a small group of three friends: Maurice Solovine, Konrad Habicht, and Einstein, who "met regularly to discuss philosophy, physics, and literature, from Plato to Dickens. They solemnly constituted themselves as founders and sole members of the 'Akademie Olympia,' dined together, typically on sausage, cheese, fruit, and tea, and generally had a wonderul time. " [Pais, p. 47].

Einstein presented the new kinematics in the first part of his June of 1905 paper *On the Electrodynamics of Moving Bodies*. In the second part of the paper he treated electrodynamics.

Einstein based his theory on only two postulates ([24], pp. 37–38)

1. The same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. (These are inertial frames.)
2. Light is always propagated in empty space with a definite velocity $c$ which is independent of the of the state of motion of the emitting body.

The first postulate Einstein called the "Principle of Relativity." This is the postulate that the laws of electromagnetism require no modification to account for uniform motion. This will become part of the postulate that the laws of physics are covariant. As we shall see the second postulate is inescapable, since we need to synchronize clocks or timepieces.

The step that Einstein brings to the discussion is the formulation of the concept of time. Time is not as Newton claimed, an absolute quantity. Time is defined by the interval between events as determined by measurement. The problem was then "to evaluate the times of events occurring at places remote from the" measuring instrument[2] ([24], p. 39).

As we will find in our treatment, all that is actually necessary is the new concept of time, that Einstein called "the step." Einstein made this claim in a review paper in 1907 ([78], p. 141). That is the Principle of Relativity is a statement about the meaning of time. So we shall begin our discussion of (special) relativity as Einstein did with the concept of time.

### *13.2.1 Time*

Einstein said that all judgements in which time plays a part are judgements of *simultaneous events*. An event occurs at a certain time $t_{\mathrm{e}}$ if the occurrence of that event and the event that the time $t_{\mathrm{e}}$ appears on our timepiece are simultaneous. We can then define the time at the location of the timepiece to be what is registered by the timepiece. This is satisfactory if we only need a definition of time in the immediate vicinity of the timepiece. It fails when we try to define time at a remote location.

For example someone with a timepiece may be at some point, which we call $A$. And someone else, with an identical timepiece, may be at another point $B$ a distance from $A$. We can then define the time at $A$ and the time at $B$. But we cannot define a universal time unless we synchronize the timepieces at $A$ and $B$.

[2] Einstein used the term "watch" here. The reader in the 21st century may, however, not as readily think in terms of the hands of a watch as did Einstein's readers. So we use the term "timepiece," which is probably digital.

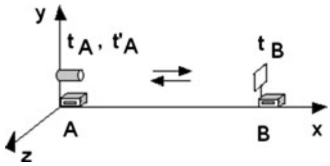


**Fig. 13.1** Synchronization of the timepieces at points *A* and *B*

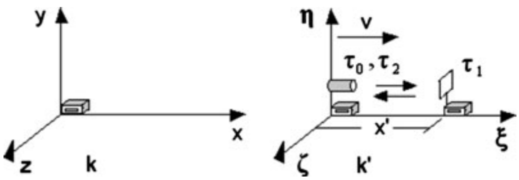


**Fig. 13.2** Timepiece synchronization experiment conducted in the moving inertial system $k'$ and observed from the stationary system $k$

By Einstein's second postulate we know that the time required for a light pulse to travel from $A$ to $B$ is always the same as the time required for a pulse to travel from $B$ to $A$. We consider a light pulse emitted from $A$ at time $t_{\rm A}$. This pulse is then reflected from point $B$ at time $t_{\rm B}$, and arrives back at point $A$ at the time $t'_{\rm A}$. We have drawn a picture of the synchronization process in Fig. 13.1. The timepieces are synchronized if

$$\boxed{{\rm t_B}-{\rm t_A} = {\rm t'_A}\text{-}{\rm t_B}} \tag{13.1}$$

In this fashion we may synchronize all timepieces in a single inertial frame in which all points $A$ and $B$ are stationary with respect to one another. Furthermore if timepiece $A$ synchronizes with timepiece $B$ and with timepiece $C$ then timepiece $B$ synchronizes with timepiece $C$. The time of an event is then the time noted on a timepiece in the vicinity of the event and, because all timepieces in the frame are synchronized, we can speak about simultaneity of events in a particular inertial frame.

To connect the times measured in two inertial frames, Einstein devised a thought experiment.[3] He asked how a time synchronization experiment, conducted by a person in a moving inertial frame, would appear if observed by a person[4] in a stationary inertial frame.

We designate the stationary frame as $k$ and the moving frame as $k'$. Frame $k$ has coordinates $(x, y, z)$ and the time $t$ and the moving frame $k'$ has coordinates $(\xi, \eta, \zeta)$ and the time $\tau$. We choose the axes $x$ and $\xi$ to be aligned with one another and with the velocity $v$. We have drawn the inertial frames and the synchronization experiment in Fig. 13.2. The the light source is located at the origin of frame $k'$. A person in frame $k$ measures the distance between the light source and the reflector as

[3]From the German *Gedankenexperiment*. In a thought experiment it must be possible to construct the required apparatus and to perform all the measurements. A thought experiment is not fanciful.

[4]The standard term is "observer" for the German *Beobachter*. The use of person seems less awkward here.

With modern timepieces a single person can gather the data.

$$x' = x - vt. \tag{13.2}$$

At time $\tau_0$, registered on a timepiece located at the origin of $k'$, a light pulse is sent from the light source down the $\xi-$axis. This pulse is reflected from a mirror at a point on the $\xi-$axis. A timepiece at this point registers the time $\tau_1$. The pulse returns to the origin arriving at time $\tau_2$. The synchronization (13.1) requires that

$$\tau_1 = \frac{1}{2}(\tau_0 + \tau_2) \tag{13.3}$$

The person in frame $k$ seeks a functional relationship between the time $\tau$ of the moving frame $k'$ in terms of measurements made in frame $k$. In general this will be

$$\tau = \tau\left(x', y, z, t\right). \tag{13.4}$$

Because of the situation being considered here, the $x-$coordinate is replaced by a point $x'$, which is at rest in frame $k'$. Because space and time are homogeneous, this relationship, Einstein claimed, will be linear.

For the experiment $y = z = 0$. The person in frame $k$ records a time $t$ for the beginning of the experiment, and observes that the light pulse moves down the $x-$axis at a velocity $c - v$ relatively to the apparatus in $k'$arriving at the reflector in $k'$ at time $t + x'/(c - v)$. This person in $k$ then observes that the returning pulse moves at a velocity $c + v$ relatively to the apparatus in $k'$ arriving at the origin of $k'$ at time $t + x'/(c - v) + x'/(c + v)$. The experimental data recorded by the person in frame $k$ result in three values for the function $\tau$. These are

$$\begin{aligned}
\tau_0 &= \tau(0, 0, 0, t) \\
\tau_1 &= \tau\left(x', 0, 0, t + \frac{x'}{c - v}\right) = \tau_0 + x'\frac{\partial \tau}{\partial x'} + \frac{x'}{c - v}\frac{\partial \tau}{\partial t} \\
\tau_2 &= \tau\left(0, 0, 0, t + \frac{x'}{c - v} + \frac{x'}{c + v}\right) = \tau_0 + \left(\frac{x'}{c - v} + \frac{x'}{c + v}\right)\frac{\partial \tau}{\partial t}.
\end{aligned}$$

With Einstein, we now choose $x'$ to be infinitesimal. Then (13.3) becomes

$$\boxed{\partial\tau/\partial \mathrm{x}' + \left[v/\left(c^2 - v^2\right)\right]\partial\tau/\partial \mathrm{t}=0} \tag{13.5}$$

This is a linear partial differential equation with constant coefficients. Since $\tau$ is a linear function, for a specific $v$ the solution of (13.5) is

$$\tau = a\left(t - \frac{v}{c^2 - v^2}x'\right), \tag{13.6}$$

where $a$ is a function of the velocity $v$. Equation (13.6) is the functional relationship (13.4).

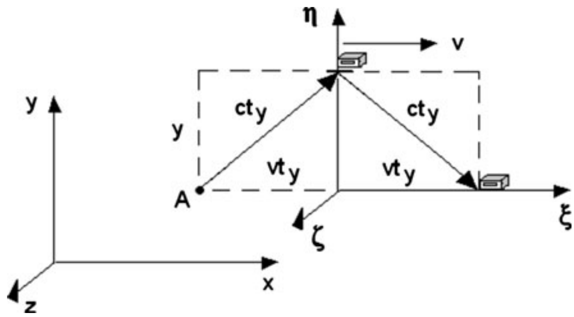


**Fig. 13.3** Time synchronization experiment conducted by someone in the $k'$—frame along the $\eta$—axis and observed by someone in the $k$—frame

To find the dependence of $\tau$ on the coordinates $y$ and $z$ of frame $k$ we consider that the person in frame $k$ observes time synchronization experiments carried out by the person in frame $k'$ in which a light pulses are sent down the $\eta$ and then down the $\zeta$ axis. In Fig. 13.3 we have drawn a picture of the time synchronization experiment on the $\eta$—axis, which shows the path of the light pulse from the perspective of the person in frame $k$. The time $t_y$ is that recorded by the person in frame $k$.

From Fig. 13.3 we can show that

$$\frac{\partial \tau}{\partial y} = 0$$

(see exercises). By symmetry we also have

$$\frac{\partial \tau}{\partial z} = 0.$$

After we determine the function $a$ (13.6) will be the general form of the relation between the time $\tau$ measured in frame $k'$ and the time $t$ measured in frame $k$.

### *13.2.2 Space*

Einstein also used thought experiments to obtain functional relationships among spatial coordinates. In these experiments he considered that light pulses were again sent down axes in frame $k'$ and observed in frame $k$ as well. He used the (13.6) to relate the observations in the two frames.

In one such thought experiment a person in the inertial frame $k'$ sends a light pulse down the $\xi$—axis. The pulse covers a distance $\xi = c\tau$ in the time $\tau$. If a person in $k$ measures this distance as $x'$ then, using (13.6), we have the description of the experiment in terms used by the person in $k$ as

$$\begin{aligned} \xi &= c\tau \\ &= ca\left(t - \frac{v}{c^2 - v^2}x'\right). \end{aligned} \tag{13.7}$$

Because of the constancy of the velocity of light, the person in $k$ observes that the light pulse has a velocity $c - v$ relatively to the $\xi$−axis. Then, for the person in $k$, the time duration of the experiment is

$$t = \frac{x'}{c - v}. \tag{13.8}$$

With (13.8) (13.7) becomes

$$\xi = a\left(\frac{c^2}{c^2 - v^2}x'\right). \tag{13.9}$$

And then with (13.2) (13.9) results in

$$\xi = a\frac{1}{1 - v^2/c^2}\left(x - vt\right). \tag{13.10}$$

For a thought experiment in which a person in frame $k'$ sends a light pulse down the $\eta$−axis the picture, as seen by someone in frame $k$, we have drawn in Fig. 13.4. A person in frame $k$ observes that a light pulse sent from the origin of frame $k'$ down the $\eta$−axis propagates at the velocity $\sqrt{c^2 - v^2}$ (see exercises) and records a time

$$t = \frac{y}{\sqrt{c^2 - v^2}} \tag{13.11}$$

for the duration of the experiment.

The result of the experiment as recorded in frame $k'$ is

$$\eta = c\tau. \tag{13.12}$$

With (13.6) (13.12) becomes

$$\eta = ca\left(t - \frac{v}{c^2 - v^2}x'\right). \tag{13.13}$$

Since the pulse originates at the origin of $k'$, we have $x' = 0$. Then, using (13.11), equation (13.13) is

$$\eta = a\frac{c}{\sqrt{c^2 - v^2}}y. \tag{13.14}$$

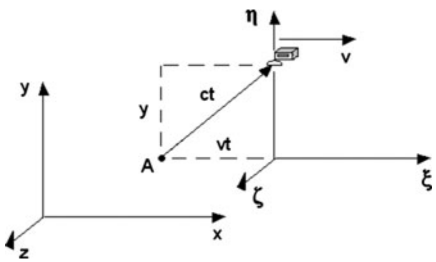


**Fig. 13.4** Thought experiment in which a person in frame $k'$sends a light pulse down the $\eta$−axis

By symmetry we also have

$$\zeta = a\frac{c}{\sqrt{c^2 - v^2}}z, \tag{13.15}$$

By considering a transformation from frame $k$ to frame $k'$ and back to frame $k$ Einstein was able to show that the quantity

$$\phi(v) = a\frac{c}{\sqrt{c^2 - v^2}} \tag{13.16}$$

is equal to unity (i.e. $\phi(v) = \phi(-v) = 1$). Then

$$a(v) = \sqrt{1 - \frac{v^2}{c^2}} \tag{13.17}$$

### 13.2.3 Lorentz Transformation

From the preceding section we have the complete transformation equations for the time and spatial coordinates between the inertial frames $k$ and $k'$. These are the *Lorentz Transformation* equations as Einstein presented them.

We shall, however, replace $(\tau, \xi, \eta, \zeta)$ with $(t', x', y', z')$ to obtain a more modern representation.

$$\begin{aligned} t' &= \gamma\left(t - \frac{\beta x}{c}\right) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z, \end{aligned} \tag{13.18}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \tag{13.19}$$

and $\beta = v/c$.

## 13.3 Minkowski Space-Time

### 13.3.1 Four Dimensions

The space of four coordinates, in which time is treated on an equal footing with spatial coordinates, is referred to as *Minkowski Space* because it was first proposed by Hermann Minkowski[5] (1864–1909). Minkowski's paper *Space and Time*, delivered to the 80th Assembly of German Natural Scientists and Physicians, at Cologne, September, 1908, is printed in translated form in ([24], pp. 75–91). There Minkowski claimed that

> Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

Minkowski's formalism is a great simplification to special relativity. Initially, however, Einstein was unimpressed. He called this "superfluous learnedness" ([78], p. 152). But later, Einstein adopted the Minkowski formalism. And in 1916 he acknowledged his debt to Minkowski. The formalism was instrumental in the transition from special to general relativity.

We define the coordinates of Minkowski Space using a scheme which preserves $x^1$, $x^2$, and $x^3$ for the spatial coordinates and identifies $x^0$ as the time coordinate, which is $ct$. Specifically

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z. \end{aligned} \tag{13.20}$$

We then have the four dimensional position vectors

$$\mathbf{x} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \text{ and } \mathbf{x}' = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} \tag{13.21}$$

for points in four dimensional inertial frames $k$ and $k'$.

Points in this four dimensional space are called *world points*. In three dimensional terms a world point joins or associates a spatial point $(x, y, z)$ with a temporal point $ct$ registered on a timepiece. The world point is then an event. World points are connected by *world lines*. For example the first part of a time synchronization experiment consists of the events 1) light pulse leaves point $A$ at time $t_{\mathrm{A}}$ and 2) light pulse arrives at point $B$ at time $t_{\mathrm{B}}$. The world line connects these two events.

[5]Hermann Minkowski was a German mathematician of Lithuanian Jewish descent. He was one of Einstein's professors at the Eidgenössische Polytechnikum in Zürich.

### 13.3.2 Four Vectors

The vectors in (13.21) are called *four vectors* or 4−*vectors*[6]. This designation is not simply because they have four dimensions. It is a result of the way they transform under Lorentz Transformation. We discuss this in Sect. 13.6.

In this terminology the Lorentz Transformation (13.18) becomes

$$\boxed{\begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^1\right) \\ x'^1 &= \gamma\left(x^1 - \beta x^0\right) \\ x'^2 &= x^2 \\ x'^3 &= x^3. \end{aligned}} \tag{13.22}$$

The inverse of the Lorentz Transformation may be found by simply replacing $\beta$ with $-\beta$ and exchanging the $k$ and $k'$ coordinates, since a person in frame $k'$ sees $k$ receding in the $x'^1$ direction at a velocity $v$.

$$\boxed{\begin{aligned} x^0 &= \gamma\left(x'^0 + \beta x'^1\right) \\ x^1 &= \gamma\left(x'^1 + \beta x'^0\right) \\ x^2 &= x'^2 \\ x^3 &= x'^3. \end{aligned}} \tag{13.23}$$

### 13.3.3 The Minkowski Axiom

From the Lorentz Transformation (13.22) we find that the differential lengths of a world line, observed from two inertial frames, are related as

$$\begin{aligned} &\pm\left[\left(\mathrm{d}x^0\right)^2 - \left(\mathrm{d}x^1\right)^2 - \left(\mathrm{d}x^2\right)^2 - \left(\mathrm{d}x^3\right)^2\right] \\ &= \pm\left[\left(\mathrm{d}x'^0\right)^2 - \left(\mathrm{d}x'^1\right)^2 - \left(\mathrm{d}x'^2\right)^2 - \left(\mathrm{d}x'^3\right)^2\right] \end{aligned} \tag{13.24}$$

(see exercises). The equality in (13.24) holds regardless of the sign we may attach to the square bracket. Minkowski introduced a fundamental axiom, which we shall refer to as the *Minkowski Axiom*, that requires the positive sign to be chosen. Minkowski said

> The substance at any world point may always, with the appropriate determination of space and time, be looked upon as at rest ([24], p. 80).

[6] The designation 4-vector is that used by Jackson. We choose it here as well.

That is in *some* inertial frame $k'$ we will have $\mathrm{d}x' = \mathrm{d}y' = \mathrm{d}z' = 0$ for a substantive, material body. Then, since $c^2 \mathrm{d}t'^2 > 0$, we realize that the square of the differential world line in frame $k'$ is $\mathrm{d}\mathbf{s}'^2 = c^2 \mathrm{d}t'^2 > 0$. But from (13.24) we know that $\mathrm{d}\mathbf{s}^2 = \mathrm{d}\mathbf{s}'^2$. That is $\mathrm{d}\mathbf{s}^2$ is an *invariant scalar* on Lorentz Transformation between inertial frames, which we must take to be positive for a material body.

For the world line of a light beam we can never have $\mathrm{d}x' = \mathrm{d}y' = \mathrm{d}z' = 0$ in any frame whatsoever. For the world line of a light beam

$$dx'^2 + dy'^2 + dz'^2 = c^2 dt'^2 \tag{13.25}$$

and $\mathrm{d}s'^2 = 0$. We may then write the mathematical form of Minkowski's Axiom as

$$\boxed{\mathrm{d}\mathbf{s}^2 = \left(\mathrm{d}x^0\right)^2 - \left(\mathrm{d}x^1\right)^2 - \left(\mathrm{d}x^2\right)^2 - \left(\mathrm{d}x^3\right)^2 \geq 0.} \tag{13.26}$$

Using (13.20), the inequality (13.26) requires that

$$c^2 \geq \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 .$$

for any material body. Therefore, according to the Minkowski Axiom, the velocity $c$ of light is a *limiting velocity* for material bodies.[7] And the limiting case $\mathrm{d}s^2 = 0$ holds only for light.

This limit on velocity can be retrieved from results we shall develop. The Minkowski Axiom is, however, the foundational statement of this limitation.

### *13.3.4 The Light Cone*

We cannot picture Minkowski four dimensional space. We can, however, picture the Minkowski space representation of the motion of a material particle and a light pulse in a two dimensional spatial plane. We choose the motion to be in the $x^1$, $x^2$ plane and construct the time axis $x^0$ of our Minkowski space perpendicular to this plane.

In two dimensional Cartesian space the wave front of a light pulse emitted from the origin forms an expanding circle of radius $ct$. In our limited Minkowski space the wave front of the light pulse emitted from the origin $\mathbf{0} = \left(x^0, x^1, x^2\right) = (0, 0, 0)$ is represented by a circular cone with axis $x^0$. We call this the *light cone*. The Minkowski Axiom requires that the world line of a material particle passing through the origin must lie within the light cone. If we extend the light cone into the *past*

[7] Separate inertial frames must contain (material) measuring instruments, i.e. rods and timepieces.

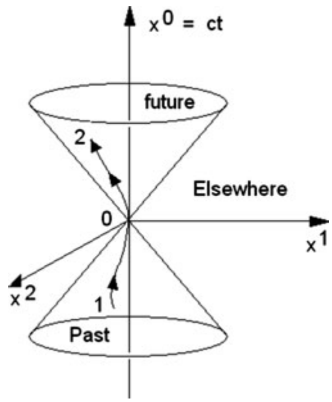


**Fig. 13.5** Minkowski space with two spatial dimensions. The third dimension is $ct$. The world line for a material particle is that from 1 to 2

$\left(x^0 < 0\right)$ the earlier world line of the material particle must lie within this extension of the light cone.[8]

In Fig. 13.5 we have drawn this limited, three dimensional Minkowski space, and have drawn a representative world line of a material particle within the light cone passing from point 1 in the past, through the origin, to point 2 in the future.

Intervals on the light cone, for which $\mathrm{d}s^2 = 0$, are accessible only by light. We call these *lightlike intervals*. If $\mathrm{d}s^2 > 0$ we call the interval a *timelike interval*. Timelike intervals satisfy the Minkowski Axiom and lie inside the light cone in Fig. 13.5. World points on a timelike interval are possible *future* world points for the particle. All points within the extension of the light cone along the negative $x^0$ axis are possible *past* world points for a particle. If $\mathrm{d}s^2 < 0$ we call the interval a *spacelike interval*. Spacelike intervals violate the Minkowski Axiom and are not accessible to material particles. We refer to these world points collectively as *elsewhere* ([48], p. 519).

## 13.4 Formal Lorentz Transform

We may write the general Lorentz Transformation of the differential of a world line d**x** from one inertial frame into its form d**x**′in another as

$$\boxed{\mathrm{d}\mathbf{x}' = \mathbf{A}\cdot\mathrm{d}\mathbf{x},} \tag{13.27}$$

where **A** is the *dyadic form* of the (tensor) transformation operator. Equation (13.27) is a general differential (a Pfaffian) for d**x**′. That is, for example, the differential of the time coordinate $\mathrm{d}x'^0$ in the frame $k'$ is

$$\mathrm{d}x'^0 = \frac{\partial x'^0}{\partial x^\mu}\mathrm{d}x^\mu$$

[8]The geometrical definition of a cone includes both $x^0 > 0$ and $x^0 < 0$.

using the Einstein sum convention for repeated (Greek) indices $\mu = 0, \ldots, 4$.

The elements of the Lorentz Transformation matrix are then the partial derivatives $\partial x'^{\alpha}/\partial x^{\beta}$. Because the Lorentz Transformation is linear, the partial derivatives $\partial x'^{\alpha}/\partial x^{\beta}$ are constants, dependent only on the relative velocity of the inertial frames. We obtain the elements of **A** and **A**$^{-1}$ from the Lorentz transform (13.22) and the inverse Lorentz transform (13.23). We write these elements as

$$(\mathbf{A})^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \tag{13.28}$$

and

$$\left(\mathbf{A}^{-1}\right)^{\alpha}_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\beta}}. \tag{13.29}$$

In this notation the *Kronecker delta* is

$$\delta^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\lambda}}\frac{\partial x^{\lambda}}{\partial x'^{\beta}} = \frac{\partial x^{\alpha}}{\partial x'^{\lambda}}\frac{\partial x'^{\lambda}}{\partial x^{\beta}}. \tag{13.30}$$

For translation of frame $k'$ along the axis $x^1$ of frame $k$ the matrices **A** and **A**$^{-1}$ are

$$\boxed{\mathbf{A}=\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \tag{13.31}$$

and

$$\boxed{\mathbf{A}^{-1}=\begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.} \tag{13.32}$$

The choice of the spatial axis along which we orient the relative velocity of the inertial frames is arbitrary. We will stay with Einstein's original choice of the $x-$axis, as does Jackson.

## 13.5 Time and Space

With our formalization of the Lorentz Transformation we are now in a position to discover the differences between specific world lines observed in different inertial frames. In this section we will consider two fairly simple world lines that will provide an understanding of the measurements of time and length in different inertial frames. But our goal is also to show the use of the Lorentz Transformation matrix.

### 13.5.1 Time Dilation

We consider two inertial frames $k$ and $k'$ such as pictured in Fig. 13.2. The times $t$ and $t'$ are measured by stationary timepieces in frames $k$ and $k'$. These times are appropriate to either of those frames and may be called *local times*. We can most easily find the relationship between these local times from the invariance of $\mathrm{d}\mathbf{s}^2$.

The people in frame $k'$ observe an event occurring at, or in the immediate vicinity of the origin of $k'$ as having a duration $\mathrm{d}t'$ measured by the timepiece at the origin. According to the people in $k'$ the differential world line of the event is

$$\mathrm{d}\mathbf{s}' = \begin{bmatrix} c\mathrm{d}t' \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13.33}$$

The people in $k$ observe the same event. They measure the duration of this event to be $\mathrm{d}t$ on their timepiece at the origin of $k$ and determine that, during the event in question, the origin of $k'$ has moved a spatial distance $\hat{e}_{\mathrm{x}}\mathrm{d}x + \hat{e}_{\mathrm{y}}0 + \hat{e}_{\mathrm{z}}0$. The event at the origin of $k'$ then has the differential world line

$$\mathrm{d}\mathbf{s} = \begin{bmatrix} c\mathrm{d}t \\ \mathrm{d}x \\ 0 \\ 0 \end{bmatrix}. \tag{13.34}$$

The invariance of $\mathrm{d}s^2$ requires that

$$c^2\mathrm{d}t'^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2. \tag{13.35}$$

Then

$$\boxed{\mathrm{dt}' = \mathrm{dt}\sqrt{1-\beta^2},} \tag{13.36}$$

is the relationship between the differential local times measured in the two inertial frames. This is the relationship obtained by Einstein ([24], p. 49). Since (13.36) requires that $\mathrm{d}t' < \mathrm{d}t$, the timepiece in $k'$ is slower that the timepiece in $k$. This is referred to as *time dilation*.

We can use the Lorentz Transformation matrix (13.31) to analyze this experiment as well. Here frame $k'$ moves at a velocity $v$ in the direction of the $x-$axis. Then $\mathrm{d}y = \mathrm{d}z = 0$ and $\mathrm{d}x = v\mathrm{d}t$, which is the distance that the origin of $k'$ moves in the time $\mathrm{d}t$. From (13.34) the differential world line in $k$ is then

$$\mathrm{d}\mathbf{s} = \begin{bmatrix} c\mathrm{d}t \\ v\mathrm{d}t \\ 0 \\ 0 \end{bmatrix}. \tag{13.37}$$

This differential world line is transformed into the differential world line $d\mathbf{s}'$ in $k'$ by

$$d\mathbf{s}' = \mathbf{A} \cdot d\mathbf{s}. \tag{13.38}$$

Then using (13.31) and (13.37) in (13.38) we have

$$d\mathbf{s}' = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\mathrm{d}t \\ v\mathrm{d}t \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma c\mathrm{d}t - \gamma\beta v\mathrm{d}t \\ \gamma v\mathrm{d}t - \gamma\beta c\mathrm{d}t \\ 0 \\ 0 \end{bmatrix}. \tag{13.39}$$

That is

$$c\mathrm{d}t' = \gamma c\mathrm{d}t - \gamma\beta v\mathrm{d}t. \tag{13.40}$$

And, since the timepiece is at the origin in $k'$,

$$0 = \gamma v\mathrm{d}t - \gamma\beta c\mathrm{d}t. \tag{13.41}$$

Equation (13.41) is an identity. And (13.40) is identical to (13.36).

The term *local time* was first used by Lorentz ([24], p. 15). However, as Minkowski points out, Einstein first recognized that the times $t$ and $t'$ are equivalent ([24], p. 82). Minkowski then defines

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1-\beta^2} \tag{13.42}$$

as the *proper time* of the world point along the world line in Fig. 13.5. This is the time indicated by a timepiece at rest with respect to the material particle on the world line. The time interval $\mathrm{d}t$ is the corresponding time measured in an inertial frame considered to be at rest.

### *13.5.2 Space Contraction*

To discover the effect of motion on the dimensions of a body we consider that in frame $k'$ there is a rod of length $L_0$ lying along the $x'$ axis. We consider that the rod is measured by someone in frame $k'$ and also by someone in frame $k$. For simplicity we assume that the measurements begin when the origins of $k$ and $k'$ coincide.

To make the measurement the person in frame $k'$ sends a pulse of light from the origin of $k'$ down the rod and records the time $\mathrm{d}t'$ taken for the pulse to reach the end of the rod. In frame $k'$ the rod length is $L_0 = c\mathrm{d}t'$. A person in frame $k$ observes that light pulse traversing a rod of length $L = (c - v)\mathrm{d}t$ in a time $\mathrm{d}t$, since the person in frame $k$ observes the light pulse to move at a velocity $c - v$ relatively to the rod.

In frame $k'$ the differential world line that results from the light pulse moving from the origin to a point at the end of the rod of length $L_0$ is

$$\mathrm{d}\mathbf{s}' = \begin{bmatrix} c\mathrm{d}t' \\ L_0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_0 \\ L_0 \\ 0 \\ 0 \end{bmatrix}. \tag{13.43}$$

For the person in frame $k$ the differential world line is that of the light pulse moving from the origin to a point at the end of the rod of length $L$. At the end of the measurement the end of the rod is at the point $v\mathrm{d}t + L$. The differential world line in $k$ is then

$$\mathrm{d}\mathbf{s} = \begin{bmatrix} c\mathrm{d}t \\ v\mathrm{d}t + L \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L/(1-\beta) \\ L/(1-\beta) \\ 0 \\ 0 \end{bmatrix}, \tag{13.44}$$

where we have used $c\mathrm{d}t = L/(1-\beta)$ for the time the person in frame $k$ measures for the light to traverse the distance $L$.

In this case the differential world line is lightlike. For a lightlike line $\mathrm{d}\mathbf{s}^2 = 0$, as we can see from (13.43) and (13.44). So we cannot use the invariance of $\mathrm{d}\mathbf{s}^2$ to find a relation between $L$ and $L_0$. We, therefore, turn directly to the Lorentz transform (13.22). The equation $x'^0 = \gamma x^0 - \gamma\beta x^1$ is, with (13.43) and (13.44),

$$L_0 = \gamma \frac{L}{1-\beta} - \gamma\beta \frac{L}{1-\beta} = \gamma L. \tag{13.45}$$

And the equation $x'^1 = \gamma x^1 - \gamma\beta x^0$ is, with (13.43) and (13.44), identical to (13.45). Therefore the relationship between the length of the rod as seen by people in frames $k$ and $k'$ is

$$\boxed{\mathrm{L} = \mathrm{L}_0\sqrt{1-\beta^2}} \tag{13.46}$$

The length of the moving rod appears shorter to the person in frame $k$ than it does to the person moving with the rod. The dimensions of the body in the directions perpendicular to the relative velocity are not affected. This is referred to as *length contraction*.

We can also analyze the measurement using the Lorentz Transformation matrix (13.31). Using (13.31) and (13.44) in (13.38) we have

$$\mathrm{d}\mathbf{s}' = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L/(1-\beta) \\ L/(1-\beta) \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} L\beta\gamma/(\beta-1) - L\gamma/(\beta-1) \\ L\beta\gamma/(\beta-1) - L\gamma/(\beta-1) \\ 0 \\ 0 \end{bmatrix}, \tag{13.47}$$

which, by comparing the second line of (13.47) with (13.43), results again in (13.45).

Equation (13.46) is the *FitzGerald-Lorentz contraction*, which we discussed in Sect. 13.2. Minkowski correctly considered this hypothesis ungrounded, claiming it had been introduced "as a gift from above ([24], p. 81)." The resolution is in Einstein's idea regarding time.

### *13.5.3 Velocities*

The velocity of a particle is defined in terms of displacement and time. We then expect that the velocity of a moving particle will be seen differently in different inertial frames.

Let us consider that we are in the stationary inertial frame $k$. Someone in the inertial frame $k'$ moving at the constant velocity $\mathbf{v} = v\hat{e}_{\mathrm{x}}$ with respect to us observes a particle moving with a velocity $\mathbf{u}'$ with components $\left(u'_{\mathrm{x}}, u'_{\mathrm{y}}, u'_{\mathrm{z}}\right)$. In a short time $\mathrm{d}t'$ that person observes the differential world line $\mathrm{d}\mathbf{s}'$ of the particle to be

$$\mathrm{d}\mathbf{s}' = \begin{bmatrix} c\,\mathrm{d}t' \\ u'_{\mathrm{x}}\mathrm{d}t' \\ u'_{\mathrm{y}}\mathrm{d}t' \\ u'_{\mathrm{z}}\mathrm{d}t' \end{bmatrix}. \tag{13.48}$$

We observe that the differential world line $\mathrm{d}\mathbf{s}$ of this particle as

$$\mathrm{d}\mathbf{s} = \begin{bmatrix} c\,\mathrm{d}t \\ u_{\mathrm{x}}\mathrm{d}t \\ u_{\mathrm{y}}\mathrm{d}t \\ u_{\mathrm{z}}\mathrm{d}t \end{bmatrix}. \tag{13.49}$$

The displacements (13.48) and (13.49) are related by the Lorentz Transformation

$$\mathrm{d}\mathbf{s} = \mathbf{A}^{-1} \cdot \mathrm{d}\mathbf{s}'. \tag{13.50}$$

Carrying out the matrix multiplication we have

$$\mathbf{ds} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\mathrm{d}t' \\ u'_{\mathrm{x}}\mathrm{d}t' \\ u'_{\mathrm{y}}\mathrm{d}t' \\ u'_{\mathrm{z}}\mathrm{d}t' \end{bmatrix} = \begin{bmatrix} \gamma\left(1+\beta\beta'_{\mathrm{x}}\right)c\mathrm{d}t' \\ \gamma\left(\beta'_{\mathrm{x}}+\beta\right)c\mathrm{d}t' \\ \beta'_{\mathrm{y}}c\mathrm{d}t' \\ \beta'_{\mathrm{z}}c\mathrm{d}t' \end{bmatrix}, \tag{13.51}$$

where we have introduced $\beta_{\mathrm{j}} = u_{\mathrm{j}}/c$ for $j = 1, 2, 3$. Then

$$\mathbf{ds} = \begin{bmatrix} c\mathrm{d}t \\ \beta_{\mathrm{x}}c\mathrm{d}t \\ \beta_{\mathrm{y}}c\mathrm{d}t \\ \beta_{\mathrm{z}}c\mathrm{d}t \end{bmatrix} = \begin{bmatrix} \gamma\left(1+\beta\beta'_{\mathrm{x}}\right)c\mathrm{d}t' \\ \gamma\left(\beta'_{\mathrm{x}}+\beta\right)c\mathrm{d}t' \\ \beta'_{\mathrm{y}}c\mathrm{d}t' \\ \beta'_{\mathrm{z}}c\mathrm{d}t' \end{bmatrix} \tag{13.52}$$

By equating components in (13.52) and solving for the velocity components in the second inertial frame ($k$) we have

$$u_{\mathrm{x}} = \frac{u'_{\mathrm{x}} + c\beta}{1 + \beta u'_{\mathrm{x}}/c}, \tag{13.53}$$

$$u_{\mathrm{y}} = \frac{u'_{\mathrm{y}}}{\gamma\left(1 + \beta u'_{\mathrm{x}}/c\right)}, \tag{13.54}$$

and

$$u_{\mathrm{z}} = \frac{u'_{\mathrm{z}}}{\gamma\left(1 + \beta u'_{\mathrm{x}}/c\right)}. \tag{13.55}$$

We realize that the Minkowski Axiom requires that particle velocities are always less than the speed of light. Using (13.53) we can show that if the inertial frame $k'$ has a velocity $v < c$ and if the particle moving in $k'$ also has a velocity $u_{\mathrm{x}} < c$, the velocity of the particle as measured in $k$ is also $< c$ regardless of how close $v$ and $u_{\mathrm{x}}$ are to $c$. We have drawn the situation in Fig. 13.6.

If we choose $\beta'_{\mathrm{x}} = 1 - \kappa$ and $\beta = 1 - \lambda$ then (13.53) always results in

$$\beta_{\mathrm{x}} = \frac{\beta'_{\mathrm{x}} + \beta}{1 + \beta\beta'_{\mathrm{x}}} = \frac{2 - \kappa - \lambda}{2 - \kappa - \lambda + \kappa\lambda} < 1. \tag{13.56}$$

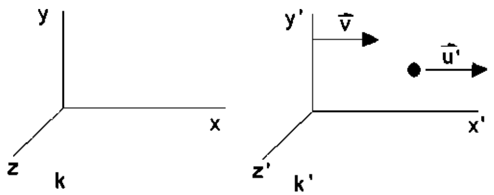


**Fig. 13.6** A body moving with a velocity $\mathbf{u}' = u'_{\mathrm{x}}\hat{e}_{\mathrm{x}}$ relatively to the inertial frame $k'$, which has a velocity $\mathbf{v} = v\hat{e}_{\mathrm{x}}$ relatively to inertial frame $k$

This is the argument that Einstein provided ([24], pp. 50–51).

If we choose $\beta'_{\mathrm{x}} = 1$, so that we are considering a light pulse in frame $k'$ rather than a particle. Then $\kappa = 0$ and (13.56) results in $\beta_{\mathrm{x}} = 1$, which is Einstein's second postulate.

## 13.6 Tensors

If we take seriously Minkowski's claim that we need a unity of space and time as a representation of reality then we should be prepared to accept an expanded geometry. This will also simplify our mathematical treatment of relativity.

In the Minkowski Axiom we have a statement that begins our new understanding of geometry. The square of the distance along a world line is, in analytic geometry, a scalar or dot product between two vectors. We are then led to consider a scalar product between general 4−vectors **A** and **B**. We write this as

$$\mathbf{A} \cdot \mathbf{B} = A_\alpha B^\alpha, \tag{13.57}$$

In (13.57) we have introduced a superscript on the elements of **B** in keeping with the notation for the four dimensional world point (13.21). The use of a subscript on the elements of **A** is because we anticipate (correctly) that the form of this vector will be different from that of **B**.

We are interested in the conditions which make $\mathbf{A} \cdot \mathbf{B}$ an invariant under Lorentz Transformation. That is

$$A_\alpha B^\alpha = A'_\nu B'^\nu \tag{13.58}$$

We require first that the elements of the vector **B** transform in the same way that the coordinates transform, which is

$$\mathrm{d}x'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} \mathrm{d}x^\beta, \tag{13.59}$$

with the inverse

$$\mathrm{d}x^\beta = \frac{\partial x^\beta}{\partial x'^\alpha} \mathrm{d}x'^\alpha, \tag{13.60}$$

That is we require that

$$B'^\sigma = \frac{\partial x'^\sigma}{\partial x^\nu} B^\nu \tag{13.61}$$

with the inverse

$$B^\nu = \frac{\partial x^\nu}{\partial x'^\sigma} B'^\sigma. \tag{13.62}$$

Then (13.58) becomes

$$A_\alpha B^\alpha = A_\nu \frac{\partial x^\nu}{\partial x'^\sigma} B'^\sigma. \tag{13.63}$$

The invariance expressed in (13.58) results provided the vector **A** transforms as

$$A'_{\sigma} = \frac{\partial x^{\nu}}{\partial x'^{\sigma}} A_{\nu}. \tag{13.64}$$

The vectors **A** and **B** must then transform differently in order for the scalar product to be an invariant. This is a distinguishing characteristic of Minkowski space.

Vectors that transform according to (13.61) are *contravariant vectors*. The indices on a contravariant vector are superscripts. Vectors that transform according to (13.64) are *covariant vectors*. The indices on a covariant vector are subscripts.

The character of vectors in Minkowski space is defined by the way in which they transform from one inertial frame to another. The invariance of the scalar d$\mathbf{s}^2$ is also defined by the properties of the Lorentz Transformation. Both the vectors and the (invariant) scalars we have encountered in the theory of relativity are defined more restrictively than what we may have previously considered to be (general) vectors and scalars. Such quantities are called *tensors*.

An invariant scalar is a tensor of rank zero. A contravariant or covariant vector is a tensor of rank one. The rank of the tensor is the number of indices required in its definition. Accordingly we have *contravariant* and *covariant tensors* of rank two, which transform as

$$A'^{\sigma\tau} = \frac{\partial x'^{\sigma}}{\partial x^{\mu}} A^{\mu\nu} \frac{\partial x'^{\tau}}{\partial x^{\nu}} \tag{13.65}$$

and

$$A'_{\sigma\tau} = \frac{\partial x^{\mu}}{\partial x'^{\sigma}} A_{\mu\nu} \frac{\partial x^{\nu}}{\partial x'^{\tau}} \tag{13.66}$$

respectively.

Tensors may also be of higher rank and we may have mixed tensors, which transform as, for example

$$A'^{\tau}_{\sigma} = \frac{\partial x'^{\tau}}{\partial x^{\nu}} A^{\nu}_{\mu} \frac{\partial x^{\mu}}{\partial x'^{\sigma}}.$$

Because vectors are a subset of tensors, we have moved beyond our previous treatment in terms of the vector calculus. Einstein realized that this move was necessary while he was working on the general theory of relativity. In 1912, at the time of his call (back) to Zürich, he realized that he needed a mathematics beyond what he understood at that time. His friend, Marcel Grossmann, Dean of the mathematics and physics section of the *Eidgenössische Technische Hochschule* (Swiss Federal Institute of Technology), the ETH, and the one who had called him to Zürich, introduced Einstein to tensors ([78], p. 212).

The overriding question was the form which the laws of physics take in order to make them independent of coordinate system. Special relativity provides a kinematics that guarantees the invariance of the laws of mechanics and electrodynamics in inertial frames. But the question goes beyond inertial frames. It also goes beyond uniformly translating frames ([80], p. 149). Euclidean geometry has to be abandoned ([24], p. 116) and we must think of coordinates as associated with the world points in a unique and continuous manner ([80], p. 149). Such coordinate frames are called *Gaussian*.

What sort of form do the general laws of physics take? Einstein said that

> The general laws of nature are to be expressed in equations which hold good for all systems of coordinates, that is are covariant with respect to any substitutions whatever (generally covariant).

Pauli points out that we cannot prove that this claim is valid. It must be made as an Ansatz or postulate ([80], p. 149).

There seems to be little gained in attempting to stay within the confines of the ordinary vector calculus as we consider the general implications of relativity. We expect that the reader will eventually go beyond our development in this book. We will, therefore, introduce the basic ideas and concepts in a manner that will facilitate an easy transition to more advanced study of electrodynamics and mechanics.

## 13.7 Metric Space

To form the invariant quantity $d\mathbf{s}^2$ (see (13.26)) from the differentials of (13.21) we write

$$\boxed{d\mathbf{s}^2 = dx^{\alpha} g_{\alpha\beta} dx^{\beta}.} \tag{13.67}$$

Here $g_{\alpha\beta}$ are the elements of the *metric tensor* for Minkowski space.

The form of $d\mathbf{s}^2$ in (13.67) is forced upon us by the Lorentz Transformation and the Minkowski Axiom, unless we choose to introduce the imaginary quantity $i$ into the time components of the vectors (13.21). We have chosen here to keep the components of the vectors real and to introduce the metric tensor.

Einstein introduced the metric or *fundamental tensor* in the paper on the general theory of relativity in 1916 ([24], pp. 127–131). From our introduction of the metric tensor in (13.67) we see that the values of the elements of $g_{\alpha\beta}$ will provide the structure to be taken on by the four dimensional space. This is integral to the general theory of relativity.

The space of the special theory is what is termed *flat*. For flat Minkowski space the metric tensor is

$$\boxed{\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.} \tag{13.68}$$

From (13.57) we know that a scalar product is a product of a covariant and a contravariant vector. Then, from (13.67) we see that

$$dx_{\alpha} = g_{\alpha\beta} dx^{\beta} \tag{13.69}$$

and

$$dx^{\beta} = g^{\beta\alpha} dx_{\alpha}, \tag{13.70}$$

where $g^{\beta\alpha}$ is the inverse of $g_{\alpha\beta}$. That is

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu}. \tag{13.71}$$

The (13.69) and (13.70) show us, then, that $g^{\alpha\beta}$ raises and $g_{\alpha\beta}$ lowers the index of a vector in Minkowski space. The metric tensor then transforms the contravariant to the covariant form of a vector. And the inverse of the metric tensor transforms the covariant to the contravariant form of a vector.

## 13.8 Four-Velocity

In Sect. 13.5.3 we considered the transformation of the velocities

$$u^{\mu} = \frac{dx^{\mu}}{dt} \tag{13.72}$$

in the inertial frame $k$, with $\mu = 1, 2, 3$. These velocities have meaning for someone in a particular inertial frame. Equation (13.72) is not, however, a reasonable definition of velocity for use in relativistic mechanics.

If we use the proper time (13.42) in place of $dt$ in (13.72) we have a 4−*velocity* (vector)

$$\begin{aligned} U^{\mu} &\equiv \frac{dx^{\mu}}{d\tau} \\ &= \gamma_{u} \begin{bmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} \end{aligned} \tag{13.73}$$

where $\mu = 0, 1, 2, 3$ and

$$\begin{aligned} \gamma_{u} &= \left(1 - \frac{u_{x}^{2} + u_{y}^{2} + u_{z}^{2}}{c^{2}}\right)^{-1/2} \\ &= \left(1 - \beta_{u}^{2}\right)^{-1/2}. \end{aligned} \tag{13.74}$$

The square of the magnitude of the 4−velocity is

$$U^{\mu} U_{\mu} = \frac{dx^{\mu}}{d\tau} g_{\mu\nu} \frac{dx^{\nu}}{d\tau}$$

$$= \gamma_{\mathrm{u}}^2 \left( c^2 - u_{\mathrm{x}}^2 - u_{\mathrm{y}}^2 - u_{\mathrm{z}}^2 \right)$$
$$= c^2, \tag{13.75}$$

which is an *invariant*.

## 13.9 Mass, Momentum, and Energy

### *13.9.1 Mass*

Einstein treated mass in the last section of his paper on special relativity. There he analyzed the slow acceleration of a charged particle, which he considered to be an electron, in an electric field, and related the observations of the motion of the electron in the inertial frames $k$ and $k'$.

From the requirements of Newton's Second law, and the transformation of the electric field, he concluded that the mass of the electron was velocity dependent ([24], pp. 61–63). The longitudinal mass, for motion along the axis of translation of $k'$, Einstein found to be $\gamma^3 m$, and the transverse mass, for motion perpendicular to the axis of translation of $k'$, he found to be $\gamma^2 m$. This peculiarity he noted was a result of the "definition of force and acceleration" he had chosen. That the mass is velocity dependent is, nevertheless, fundamental.

Max Jammer points out that Gilbert N. Lewis and Richard C. Tolman picked up the discussion of relativistic mass in 1909 ([49], p. 161; [64]). Rather than Newton's Second Law, Lewis and Tolman based their discussion on conservation of momentum and concluded that the relativistic mass is generally $\gamma m$.

Conservation of momentum is the appropriate covariant law. Conservation of momentum is an integral part of the Euler–Lagrange formulation of analytical mechanics, on which any complete consideration of relativistic mechanics must be based. So we may accept the result of Lewis and Tolman.

Wolfgang Rindler has a particularly lucid treatment of the relativistic mass ([82], pp. 70–72). He considers a totally inelastic collision as viewed in two inertial frames. We follow Rindler's presentation here.

We consider that in frame $k$, which is at rest, there is a particle with mass $m$ moving at a velocity of $-u\hat{e}_{\mathrm{y}}$ and an identical particle coming toward the first with velocity $\mathbf{w} = \tilde{u}\hat{e}_{\mathrm{y}} + v\hat{e}_{\mathrm{x}}$. The particles collide and stick together. The resultant velocity of the combined particle is in the $\hat{e}_{\mathrm{x}}$ direction. There is no motion in the $\hat{e}_{\mathrm{y}}$ direction after the collision.

This collision is seen by someone in the inertial frame $k'$ moving with a velocity $\mathbf{v} = v\hat{e}_{\mathrm{x}}$ relatively to the inertial frame $k$. As seen by the person in frame $k'$ the second particle is moving only in the $+\hat{e}'_{\mathrm{y}}$ direction and the collision appears as a mirror image of the collision seen by the person in frame $k$. Specifically the person in frame $k'$ measures the velocity of the particle moving in the $+\hat{e}'_{\mathrm{y}}$ direction as $u\hat{e}'_{\mathrm{y}}$.

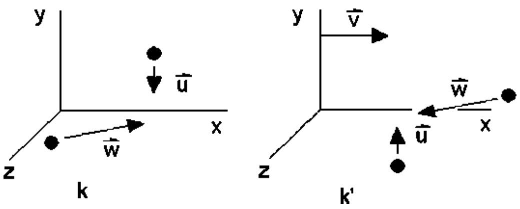


**Fig. 13.7** Collision between two particles of equal rest mass observed by people in two inertial frames

We have drawn the collision as seen by people in the two frames in Fig. 13.7. After the collision there is no motion in the $\hat{e}_{\mathrm{y}}$ or $\hat{e}'_{\mathrm{y}}$ directions, as viewed from either frame.

In the frame $k'$ the particle moving in the $+\hat{e}'_{\mathrm{y}}$ direction has no component of velocity in the $\hat{e}'_{\mathrm{x}}$ direction. Then, using (13.54) we see that a person in frame $k$ measures the $\hat{e}_{\mathrm{y}}$ component velocity of the particle moving in the $+\hat{e}_{\mathrm{y}}$ direction to be $\tilde{u} = u/\gamma_{\mathrm{v}}$. Therefore, momentum conservation in the direction $\hat{e}_{\mathrm{y}}$, as required by the person in frame $k$, is

$$0 = m\,(w)\,\frac{u}{\gamma_{\mathrm{v}}} - m\,(u)\,u. \tag{13.76}$$

We have included a possible velocity dependence of the masses of the particles by writing $m\,(w)$ and $m\,(u)$.

For arbitrary $u \neq 0$, (13.76) requires that

$$m\,(w)\,\frac{1}{\gamma_{\mathrm{v}}} = m\,(u)\,. \tag{13.77}$$

Equation (13.77) holds for all velocities $u$. As $u$ becomes small, but not zero, we may drop the velocity dependence in $m\,(u)$ and $m\,(w) \to m\,(v)$. Then (13.77) becomes

$$m\,(v) = m\gamma_{\mathrm{v}} = \frac{m}{\sqrt{1-\beta_{\mathrm{v}}^2}}. \tag{13.78}$$

This is the Lewis and Tolman result.

Equation (13.78) is the dependence of mass on velocity. The mass $m$ is the mass of the particle as measured by someone at rest with respect to the particle. This is sometimes referred to as the *rest mass* of the particle. We see from (13.78) that the mass of the particle becomes infinite as the velocity approaches that of light.

### *13.9.2 Four-Momentum*

The 4−momentum is defined logically in terms of the 4−velocity (13.73) as

$$P^{\mu} \equiv mU^{\mu}$$

$$= m\gamma_{\mathrm{u}} \begin{bmatrix} c \\ u_{\mathrm{x}} \\ u_{\mathrm{y}} \\ u_{\mathrm{z}} \end{bmatrix}, \tag{13.79}$$

This is sometimes called the *energy-momentum* 4−*vector* ([37], p.510; [49], p. 164).

Because the rest mass and the square of the 4−velocity $U^{\mu}$ are invariants, the square of the 4−momentum $P^{\mu}$ is as well. Specifically, using (13.75) we find that

$$\begin{aligned} P^{\mu}P_{\mu} &= m^2 U^{\mu}U_{\mu} \\ &= m^2\gamma_{\mathrm{u}}^2\left(c^2 - u_{\mathrm{x}}^2 - u_{\mathrm{y}}^2 - u_{\mathrm{z}}^2\right) \\ &= m^2c^2, \end{aligned} \tag{13.80}$$

or

$$\sqrt{\left(P^{\mu}P_{\mu}\right)c^2} = mc^2 \tag{13.81}$$

is an invariant under Lorentz Transformation. From (13.81) and the second line of (13.80) we find

$$\left(P^{\mu}P_{\mu}\right)c^2 = m^2c^4 = \left(m\gamma_{\mathrm{u}}c^2\right)^2 - p^2c^2 \tag{13.82}$$

where

$$p^2 = p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2 + p_{\mathrm{z}}^2 \tag{13.83}$$

is the square of the three spatial components of the momentum vector. Using (13.78) the momenta $p_{\mathrm{x,y,z}}$ are

$$p_{\mathrm{x,y,z}} = m\left(u\right)u_{\mathrm{x,y,z}} = m\gamma_{\mathrm{u}}u_{\mathrm{x,y,z}}. \tag{13.84}$$

### *13.9.3 Energy*

Einstein introduced the relationship between mass and energy in September of 1905 with the publication entitled *Does the Inertia of a Body Depend on its Energy Content* ([24], pp. 69–71). This appeared three months after the paper on special relativity, which was published in June. In this three page paper he showed that if the energy of a body changes through the emission of electromagnetic radiation there is a proportional loss in the inertial mass of the body.

In the June paper Einstein found equations for shifts in frequency and energy of a light wave as observed by a person in a moving frame. Then, in September, he considered what would happen if electromagnetic waves were emitted from a body stationary in frame $k$, as we have illustrated in Fig. 13.8. The two light waves propagate in opposite directions but at an *arbitrary angle* $\phi$ relatively to the $x$−axis.

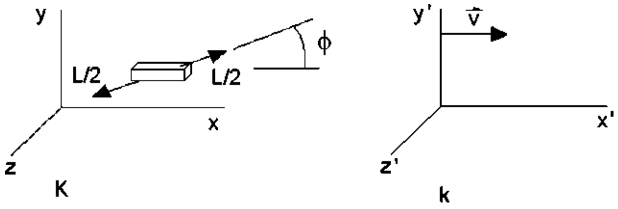


**Fig. 13.8** Electromagnetic waves emitted in opposite directions by a body at rest in the (*stationary*) inertial frame $K$

Each wave carries an energy $\frac{1}{2}L$ so that the total energy emitted by the body is $L$. With the results from the June paper, Einstein showed that the kinetic energy of the body as measured in frame $k$ is diminished in this process by an amount

$$L\left(\frac{1}{\sqrt{1-v^2/c^2}}-1\right)\approx\frac{1}{2}\left(\frac{L}{c^2}\right)v^2. \tag{13.85}$$

That is if a body gives off the energy $L$ in the form of radiation its mass diminishes by $\Delta m = L/c^2$. This is not yet $E = mc^2$. There were also no data in 1905 on which to test the theory. Einstein suggested radium salts as a source of data. But actual experimental data were 34 years in the future[9] [69].

Here we shall follow Pauli ([80], pp. 116–117) to obtain the Einstein mass-energy relation we seek.

We begin with Newton's Second Law

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_{\mathrm{u}}\mathbf{u}\right)=\mathbf{F}, \tag{13.86}$$

where $\mathbf{F}$ is the Lorentz Force. Performing the derivative indicated in (13.86) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_{\mathrm{u}}\mathbf{u}\right)=m\gamma_{\mathrm{u}}c\left(\gamma_{\mathrm{u}}^2\beta_{\mathrm{u}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\beta_{\mathrm{u}}\right)\beta_{\mathrm{u}}+m\gamma_{\mathrm{u}}c\frac{\mathrm{d}}{\mathrm{d}t}\beta_{\mathrm{u}}. \tag{13.87}$$

The rate at which work is done on a particle of (relativistic) mass $m(u) = m\gamma_{\mathrm{u}}$ moving with velocity $\mathbf{u}$ is $\mathbf{F}\cdot\mathbf{u}$. Taking the scalar product of (13.86) with $\mathbf{u}$ and using (13.87)

$$\begin{aligned}\mathbf{u}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_{\mathrm{u}}\mathbf{u}\right)&=m\gamma_{\mathrm{u}}^3c^2\beta_{\mathrm{u}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\beta_{\mathrm{u}}\\&=\mathbf{F}\cdot\mathbf{u}.\end{aligned} \tag{13.88}$$

[9] Lise Meitner's analysis of the Hahn and Strassmann experiments used Einstein's mass-energy relationship to show that nuclear fission had occurred.

From straightforward differentiation we also find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_{\mathrm{u}}c^2\right) = m\gamma_{\mathrm{u}}^3 c^2 \beta_{\mathrm{u}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\beta_{\mathrm{u}}. \tag{13.89}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_{\mathrm{u}}c^2\right) = \mathbf{F} \cdot \mathbf{u}. \tag{13.90}$$

The kinetic energy of the particle is then

$$E_{\mathrm{kin}} = m\gamma_{\mathrm{u}}c^2 + \text{ constant}. \tag{13.91}$$

To identify the constant we expand $\gamma_{\mathrm{u}}$ in powers of $\beta_{\mathrm{u}}$. Carrying the expansion to second order in $\beta_{\mathrm{u}}$ we have

$$E_{\mathrm{kin}} \approx mc^2 + \frac{1}{2}mu^2 + \text{ constant}. \tag{13.92}$$

We then retrieve the known classical result for the kinetic energy if we choose the constant to be $-mc^2$. This we term the *rest energy* (rest mass was defined in Sect. 13.9.1) of the particle. This is the energy present in the particle when at rest.

Equation (13.91) then becomes

$$E_{\mathrm{kin}} = m\gamma_{\mathrm{u}}c^2 - mc^2\,. \tag{13.93}$$

If we identify

$$\boxed{\mathrm{E{=}m}\gamma_{\mathrm{u}}\mathrm{c}^2} \tag{13.94}$$

as the *total energy* of the particle then (13.93) indicates that the kinetic energy is the difference between the total energy and the rest energy.

We are now able to identify the component $P^0$ in the 4−momentum (13.79). It is

$$P^0 = m\gamma_{\mathrm{u}}c = \frac{E}{c}. \tag{13.95}$$

The condition of invariance of $P^{\mu}P_{\mu}$ (13.82) then becomes

$$\boxed{\mathrm{E}^2 = \mathrm{p}^2\mathrm{c}^2 + \mathrm{m}^2\mathrm{c}^4,} \tag{13.96}$$

which is a general relationship between momentum and energy. From (13.80) we also see that

$$P^{\mu}P_{\mu} = \frac{m^2c^4}{c^2} = \frac{E_0^2}{c^2},$$

where $E_0 = mc^2$ is the particle rest energy.

## 13.10 Electrodynamics

### *13.10.1 Field Equations*

We recall (Sect. 11.7) that if we choose the Lorentz Gauge

$$\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0, \tag{13.97}$$

the equations for the vector and scalar potential take on the form

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J}, \tag{13.98}$$

and

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{1}{\varepsilon_0}\rho. \tag{13.99}$$

The electric and magnetic fields are then

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \operatorname{grad}\varphi, \tag{13.100}$$

and

$$\mathbf{B} = \operatorname{curl}\mathbf{A}. \tag{13.101}$$

Equations (13.98) and (13.99) together with the Lorentz Gauge (13.97) we may consider to be the general form of the Maxwell field equations. We then obtain the electric and magnetic fields from the vector and scalar potentials. In this picture it appears that the electric and magnetic fields, with which we began our study, have become secondary quantities. This is not actually the case, however. We are simply adopting a simpler and, perhaps, more elegant approach. If we can show that the (13.97–13.99) are of the same form in all inertial frames we have established the covariance of Maxwell's theory.

Einstein showed the covariance of the Maxwell Equations in detail using the Maxwell, or Maxwell-Hertz Equations. His development was quite clear with the caveat that the notation in 1905 was very awkward. Here we will show the covariance of Maxwell's equations using the same basic approach Einstein used, in the modern notation. But first we need some derivative relationships and some new 4−vectors.

### *13.10.2 Derivatives*

To write the (13.97–13.99) in the terms of tensors and Minkowski space coordinates we must have Minkowski space forms for the partial derivatives.

Using the chain rule we obtain

$$\frac{\partial \Phi}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial \Phi}{\partial x^{\beta}} \tag{13.102}$$

for the elements of the partial derivative of a function $\Phi$ with respect to the contravariant coordinate $x'^{\alpha}$. Comparing (13.102) with (13.64) we see that $\partial \Phi / \partial x^{\beta}$ is a covariant vector. Differentiation with respect to *contravariant component* transforms as a *covariant vector operator*. That is

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} \tag{13.103}$$

is a *covariant differential operator*. Some authors (see e.g. [48], pp. 535–6) choose to write the covariant differential operator in shorthand as

$$\begin{aligned} \partial_{\alpha} \equiv \frac{\partial}{\partial x^{\alpha}} &= \begin{bmatrix} \partial/\partial x^0 \\ \partial/\partial x^1 \\ \partial/\partial x^2 \\ \partial/\partial x^3 \end{bmatrix} \\ &= \left( \frac{\partial}{c\partial t}, \text{grad} \right), \end{aligned} \tag{13.104}$$

which then transforms as (13.103).

Using the chain rule we write the *divergence in Minkowski space* as

$$\frac{\partial A'^{\mu}}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} A'^{\mu}. \tag{13.105}$$

Since the elements $\partial x^{\nu} / \partial x'^{\mu}$ are independent of coordinates, (13.105) can be written as

$$\frac{\partial A'^{\mu}}{\partial x'^{\mu}} = \frac{\partial}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} A'^{\mu}. \tag{13.106}$$

Using (13.62) (13.106) becomes

$$\frac{\partial A'^{\mu}}{\partial x'^{\mu}} = \frac{\partial A^{\nu}}{\partial x^{\nu}}. \tag{13.107}$$

The divergence in Minkowski space is then invariant under Lorentz Transformation.

In the inertial frame $k$ the four dimensional Laplacian is the operator is what we have called the *d'Alembertian*. In Minkowski space we designate this operator as

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

$$= \frac{\partial^2}{\partial x^0 \partial x^0} - \frac{\partial^2}{\partial x^1 \partial x^1} - \frac{\partial^2}{\partial x^2 \partial x^2} - \frac{\partial^2}{\partial x^3 \partial x^3}. \tag{13.108}$$

We obtain this operator as the square of the differential operator (13.103). Although it may appear unnecessarily pedantic with our present understanding of the Lorentz Transformation, we can show, by carrying out the partial derivatives and using the chain rule, that the partial derivative operator $\Box$ is an invariant (see exercises).

We also observe that formally we may write

$$\Box = \frac{\partial}{\partial x^\mu} g^{\mu\nu} \frac{\partial}{\partial x^\nu} = \frac{\partial^2}{\partial x^0 \partial x^0} - \nabla^2, \tag{13.109}$$

which, in matrix form, is

$$\begin{bmatrix} \partial/\partial x^0 & \partial/\partial x^1 & \partial/\partial x^2 & \partial/\partial x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \partial/\partial x^0 \\ \partial/\partial x^1 \\ \partial/\partial x^2 \\ \partial/\partial x^3 \end{bmatrix}$$

$$= \frac{\partial^2}{\partial x^0 \partial x^0} - \nabla^2. \tag{13.110}$$

If we use the shorthand notation of (13.104) we may write (13.109) as

$$\Box = \partial_\mu g^{\mu\nu} \partial_\nu = \partial_\mu \partial^\mu \tag{13.111}$$

where

$$\begin{aligned} \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu \\ &= \begin{bmatrix} \partial/\partial x^0 \\ -\partial/\partial x^1 \\ -\partial/\partial x^2 \\ -\partial/\partial x^3 \end{bmatrix} \\ &= \left( \frac{\partial}{c\partial t}, -\,\mathrm{grad} \right) \end{aligned} \tag{13.112}$$

is the *contravariant differential operator*. The use of $\partial^\mu$ and $\partial_\nu$ is only a simplifying notation. It is convenient and common. But there is no new physics here.

### *13.10.3 Current and Potential Vectors*

We can make our treatment more systematic if we define two new 4−vectors in Minkowski space. These are the *current (density)* 4−*vector*

$$\boxed{\mathbf{J}=\begin{bmatrix} c\rho \\ J^1 \\ J^2 \\ J^3 \end{bmatrix},} \tag{13.113}$$

and the *4-potential vector*

$$\boxed{\mathbf{A}=\begin{bmatrix} \varphi/c \\ A^1 \\ A^2 \\ A^3 \end{bmatrix}.} \tag{13.114}$$

Here $\rho$ is the charge density, $J^\alpha$ are the components of the current density, $\varphi$ is the electric potential, and $A^\alpha$ are the components of the vector potential.

### *13.10.4 Electrodynamic Covariance*

With (13.104) and (13.114) we see that the Lorentz Gauge (13.97) is

$$\partial_\alpha A^\alpha = \left(\frac{\partial \varphi}{c^2 \partial t}, \frac{\partial A^\mu}{\partial x^\mu}\right) = 0, \tag{13.115}$$

which, from (13.107) we see is invariant under Lorentz Transformation. We then have invariance of the Lorentz Gauge.

Using (13.104) and (13.113) we see that charge conservation is

$$\partial_\alpha J^\alpha = \left(\frac{\partial \rho}{\partial t}, \frac{\partial J^\mu}{\partial x^\mu}\right) = 0. \tag{13.116}$$

Therefore, charge conservation is also invariant under Lorentz Transformation.

Now we can see the reason that we elected to treat the potentials as primary. Using (13.108) and (13.114) we see that the combination of (13.98) and (13.99) into 4−vector form can be written as

$$\boxed{\Box \mathbf{A}=\mu_0 \mathbf{J}.} \tag{13.117}$$

We know that the d'Alembertian $\Box$ is an invariant. And Lorentz transforming the current 4−vector and 4−potential vector we find that the second and third components of the current density and the potential vectors are $J'^2 = J^2$, $J'^3 = J^3$, $A'^2 = A^2$, $A'^3 = A^3$. Therefore the equations

$$\Box A^\alpha = \mu_0 J^\alpha \tag{13.118}$$

for $\alpha = 2, 3$ transform to

$$\Box A'^{\alpha} = \mu_0 J'^{\alpha}. \tag{13.119}$$

That is the form of the second and third components of (13.117) is unaltered by Lorentz transform.

The zeroth and first elements of **A** and **J** transform as

$$\varphi' = \gamma \left(\varphi - c\beta A^1\right), \tag{13.120}$$

$$A'^1 = \gamma \left(A^1 - \frac{1}{c}\beta\varphi\right), \tag{13.121}$$

$$\rho' = \gamma \left(\rho - \frac{1}{c}\beta J^1\right), \tag{13.122}$$

and

$$J'^1 = \gamma \left(J^1 - c\beta\rho\right). \tag{13.123}$$

If the form of the zeroth and first components of (13.117) is unaltered by Lorentz Transformation then

$$\Box \frac{\varphi'}{c} = \mu_0 c \rho' \tag{13.124}$$

and

$$\Box A'^1 = \mu_0 J'^1. \tag{13.125}$$

Using (13.120)-(13.123) and the fact that

$$\Box \frac{\varphi}{c} = \mu_0 c \rho \tag{13.126}$$

and

$$\Box A^1 = \mu_0 J^1, \tag{13.127}$$

the validity of (13.124) and (13.125) follows.

We have then established that the form of Maxwell's Equations is invariant under Lorentz Transformation. That is Maxwell's Equations are covariant.

### *13.10.5 Field Strength Tensor*

We obtain the electric and magnetic fields from the potentials **A** and $\varphi$ using (13.100) and (13.101). The spatial derivatives in the grad and curl appearing in (13.100) and (13.101) are found in the first through third elements of the contravariant derivative $\partial^{\mu}$ in (13.112) and the time derivative $\partial/\partial t$ is found in the zeroth element of $\partial^{\mu}$. Specifically if we define the *antisymmetric contravariant tensor*

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} = -F^{\alpha\beta}, \tag{13.128}$$

we find, for example, that

$$F^{10} = \partial^1 A^0 - \partial^0 A^1 = -\frac{\partial\varphi}{c\partial x} - \frac{\partial A_x}{c\partial t} = \frac{E_x}{c} \tag{13.129}$$

and

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \tag{13.130}$$

This is the *field strength tensor*.

Evaluating the elements (see exercises) we can represent the field strength tensor as the matrix

$$\boxed{\mathbf{F} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}} \tag{13.131}$$

This contravariant tensor transforms as (13.65), which we repeat here for the sake of continuity.

$$F'^{\sigma\tau} = \frac{\partial x'^{\sigma}}{\partial x^{\mu}} F^{\mu\nu} \frac{\partial x'^{\tau}}{\partial x^{\nu}}. \tag{13.132}$$

We may evaluate each of the terms in the matrix representation of $F'^{\sigma\tau}$ (see exercises) or we may transform $\mathbf{F}$ using the Lorentz Transformation matrix (13.31). To carry this out we first recall that $(\mathbf{A})^{\alpha}_{\beta} = \partial x'^{\alpha}/\partial x^{\beta}$ are the elements of the Lorentz Transformation matrix (see (13.28)). Then, in matrix form, (13.132) is

$$\mathbf{F}' = \mathbf{AFA}. \tag{13.133}$$

Because $\delta^{\alpha}_{\beta} = \left(\partial x'^{\alpha}/\partial x^{\lambda}\right)\left(\partial x^{\lambda}/\partial x'^{\beta}\right) = \left(\partial x^{\alpha}/\partial x'^{\lambda}\right)\left(\partial x'^{\lambda}/\partial x^{\beta}\right)$ (see (13.30)), we have

$$\begin{aligned} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} F'^{\sigma\tau} \frac{\partial x^{\beta}}{\partial x'^{\tau}} &= \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x'^{\sigma}}{\partial x^{\mu}} F^{\mu\nu} \frac{\partial x'^{\tau}}{\partial x^{\nu}} \frac{\partial x^{\lambda\beta}}{\partial x'^{\tau}} \\ &= \delta^{\alpha}_{\mu} F^{\mu\nu} \delta^{\beta}_{\nu} \\ &= F^{\alpha\beta} \end{aligned} \tag{13.134}$$

We now recall that $\left(\mathbf{A}^{-1}\right)^{\alpha}_{\beta} = \partial x^{\alpha}/\partial x'^{\beta}$ are the elements of the inverse of the Lorentz Transformation matrix (see (13.29)). Then, in matrix form, (13.132) is

$$\mathbf{F} = \mathbf{A}^{-1}\mathbf{F}'\mathbf{A}^{-1}. \tag{13.135}$$

We may now carry out the transformation of the Field Strength Tensor using matrices. The result is

$$\mathbf{F}' = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13.136}$$

$$= \begin{bmatrix} 0 & -E_x/c & -(\gamma/c)\left(E_y - c\beta B_z\right) & -(\gamma/c)\left(E_z + c\beta B_y\right) \\ \frac{1}{c}E_x & 0 & -\gamma\left(B_z - (\beta/c)\,E_y\right) & \gamma\left(B_y + (\beta/c)\,E_z\right) \\ (\gamma/c)\left(E_y - c\beta B_z\right) & \gamma\left(B_z - (\beta/c)\,E_y\right) & 0 & -B_x \\ (\gamma/c)\left(E_z + c\beta B_y\right) & -\gamma\left(B_y + (\beta/c)\,E_z\right) & B_x & 0 \end{bmatrix}.$$

Although the same laws of electrodynamics and optics are valid for all inertial frames of reference, the electric and magnetic fields a person measures depend on the relative state of motion of the person. This we see in (13.136). For example, if in the stationary inertial frame $k$ we have only an electrical field $\mathbf{E} = E_z\hat{e}_z$ then in an inertial frame $k'$ moving with a velocity $\mathbf{v} = v\hat{e}_x$ a person will detect an electric field $\mathbf{E}' = \gamma E_z\hat{e}_z$ and a magnetic field induction $\mathbf{B}' = (\beta/c)\,\gamma E_z\hat{e}_y$. We may identify these fields by comparing the form of $\mathbf{F}'$ that we have in final matrix in equation (13.136) with $\mathbf{F}$ in (13.131).

Similarly if we have only a magnetic field $\mathbf{B} = B_y\hat{e}_y$ in the stationary frame $k$ then in the moving frame $k'$ a person will detect a magnetic field with induction $\mathbf{B}' = \gamma B_y\hat{e}_y$ and an electric field $\mathbf{E}' = c\beta B_y\hat{e}_z$.

## 13.11 Moving Charges

With what we now know about the effect of motion on the electric and magnetic fields we can consider some aspects of particle motion. We shall later treat the emission of electromagnetic radiation from accelerated charges. Here we treat only uniform motion. The fields measured by someone in an inertial frame $k'$ in which the charge is at rest, as we may expect from the discussion in the preceding section, will differ from those measured by a person in a stationary frame $k$.

*Example 13.1.* Two stationary, identical classical point charges located a distance from one another only interact via a Coulomb force. But if these two charges move uniformly in a direction perpendicular to the distance between them we may expect the charges to be subject to a magnetic force as well as the Coulomb force. In Fig. 13.9 we place the two charges in the inertial frame $k'$ moving at a velocity $v$ along the $x-$axis of the stationary frame $k$. In the moving inertial frame $k'$ the electrostatic field from the two charges penetrates the space between them. But at the location of each charge the electrostatic force is repulsive and directed between the centers. At the point $(0,0,a)$ the electrostatic field is $+E'_z$ and at the point $(0,0,-a)$ the electrostatic field is $-E'_z$. We can calculate the magnitude of the electrostatic field $E'_z$ in frame $k'$ from Coulomb's Law. The result is

$$E'_z = \frac{Q}{4\pi\varepsilon_0}\frac{1}{(2a)^2}. \tag{13.137}$$

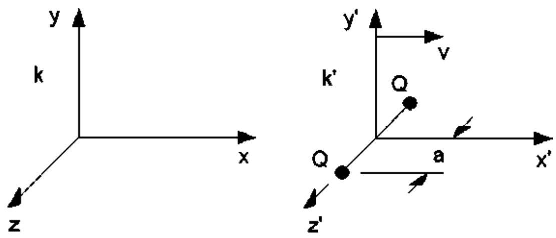


**Fig. 13.9** Two moving charges observed in two frames

The field strength tensor in $k'$ at the location of each point charge is

$$\mathbf{F}'(0,0,\pm a) = \begin{bmatrix} 0 & 0 & 0 & \mp E_z'/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm E_z'/c & 0 & 0 & 0 \end{bmatrix}. \tag{13.138}$$

Since lengths in the $y$ and $z$–directions are unaffected by the motion, we evaluate the field strength tensor at the points $(0,0,\pm a)$ as measured by someone in the $k$ frame. The result is

$$\begin{aligned} \mathbf{F}(vt,0,\pm a) &= \mathbf{A}^{-1}\mathbf{F}'(0,0,\pm a)\,\mathbf{A}^{-1} \\ &= \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \mp E_z'/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm E_z'/c & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & \mp(\gamma/c)\,E_z' \\ 0 & 0 & 0 & \mp\beta(\gamma/c)\,E_z' \\ 0 & 0 & 0 & 0 \\ \pm(\gamma/c)\,E_z' & \pm\beta(\gamma/c)\,E_z' & 0 & 0 \end{bmatrix}. \end{aligned} \tag{13.139}$$

Someone in frame $k$ then measures both a magnetic and an electric field. At the point $(vt,0,+a)$ the electric field is in the positive $z$–direction and increased by a factor $\gamma$ and the magnetic field induction is

$$B_y = -\frac{1}{c}\beta\gamma E_z'.$$

The results for the fields are the same in magnitude, but with reversed directions at point $-a$. As measured by someone in the inertial frame $k$ the force on the charge at $(vt,0,+a)$ is the Lorentz Force

$$F_{+a} = Q\gamma E_z' - Qc\beta B_y$$

$$= Q\gamma\left(1-\beta^2\right)E'_z$$
$$= Q\sqrt{1-\beta^2}\,E'_z,$$

and the force on the charge at $(vt, 0, -a)$ is the Lorentz Force

$$F_{\text{-a}} = -Q\gamma E'_z + Qc\beta B_y$$
$$= -Q\sqrt{1-\beta^2}\,E'_z.$$

The person in frame $k$ then observes that the repulsive force between the charges is diminished by a magnetic attractive force.

We may also ask for the field resulting from the motion of a single charged particle moving in empty space with no external fields acting on it. We consider the charge to be positive and the motion to be relativistic. The charge does not radiate as it moves with a constant velocity and our detectors will not register energy lost from the charge in the form of radiated electromagnetic (wave) fields.

*Example 13.2.* We have drawn a picture of the experiment in Fig. 13.10. A person in the inertial frame $k'$, stationary relatively to the charged particle, will only detect a Coulomb field

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{(r')^3}. \tag{13.140}$$

in the space surrounding the charge. The field strength tensor in $k'$ is

$$\mathbf{F}' = \begin{bmatrix} 0 & -E'_x/c & -E'_y/c & -E'_z/c \\ E'_x/c & 0 & 0 & 0 \\ E'_y/c & 0 & 0 & 0 \\ E'_z/c & 0 & 0 & 0 \end{bmatrix} \tag{13.141}$$

The field strength tensor in frame $k$, which we find from matrix multiplication, is

$$\mathbf{F} = \mathbf{A}^{-1}\mathbf{F}'\mathbf{A}^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -E'_x/c & -E'_y/c & -E'_z/c \\ E'_x/c & 0 & 0 & 0 \\ E'_y/c & 0 & 0 & 0 \\ E'_z/c & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

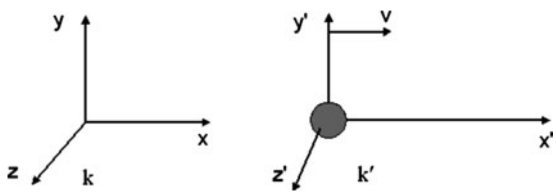


**Fig. 13.10** Single moving charge

$$= \begin{bmatrix} 0 & -\frac{1}{c}E'_x & -(\gamma/c)\,E'_y & -(\gamma/c)\,E'_z \\ -\frac{1}{c}E'_x & 0 & -\beta\,(\gamma/c)\,E'_y & -\beta\,(\gamma/c)\,E'_z \\ (\gamma/c)\,E'_y & \beta\,(\gamma/c)\,E'_y & 0 & 0 \\ (\gamma/c)\,E'_z & \beta\,(\gamma/c)\,E'_z & 0 & 0 \end{bmatrix}. \tag{13.142}$$

And **F** has the general form

$$\mathbf{F} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & 0 \\ E_z/c & -B_y & 0 & 0 \end{bmatrix}. \tag{13.143}$$

We can then identify the fields as a person in frame $k$ will see them.

The coordinates $\mathbf{r}'$ are transformed according to (13.22). In frame $k$ the distances $(x', y', z')$ become $[\gamma\,(x - \beta ct)\,, y, z]$. The magnitude of the transformed $\mathbf{r}'$ is

$$r = \sqrt{\gamma^2\,(x - \beta ct)^2 + y^2 + z^2} \tag{13.144}$$

The components of the electric field, as seen by someone in frame $k$, are then

$$E_x = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma\,(x - \beta ct)}{r^3}, \tag{13.145}$$

$$E_y = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma y}{r^3}, \tag{13.146}$$

and

$$E_z = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma z}{r^3}. \tag{13.147}$$

And the components of the magnetic field induction, as seen by someone in frame $k$, are

$$B_y = -\beta\gamma^2\frac{Q}{4\pi c\varepsilon_0}\frac{z}{r^3} \tag{13.148}$$

and

$$B_z = \beta\gamma^2\frac{Q}{4\pi c\varepsilon_0}\frac{y}{r^3}. \tag{13.149}$$

For simplicity we shall consider that measurements are made by the person in frame $k$ with detectors set up at the origin of $k$. Then measurements are made for only small values of $(x, y, z)$. We shall also set the time $t = 0$ when the charge crosses the plane at $x = 0$.

The velocity of the charge is close to $c$. Therefore, as the charge approaches and passes the origin, i.e. for small values of $(x, y, z)$, we have $|\beta ct| \gg |x|$, $|y|$, and $|z|$.

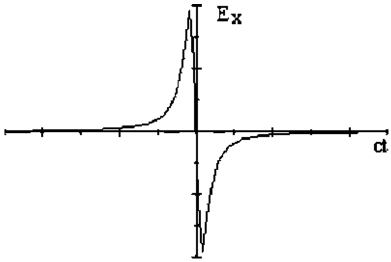


**Fig. 13.11** Electric field $E_x$ from positive charge passing the origin at relativistic velocity

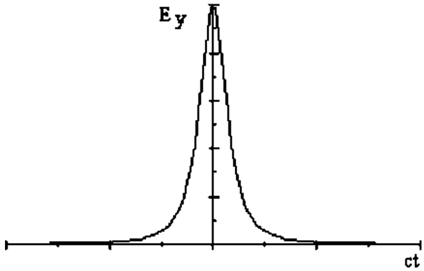


**Fig. 13.12** Electric field $E_y$ from positive charge passing the origin at relativistic velocity

Then $x - \beta ct \approx -\beta ct$ and from (13.145) the $x-$component of the field is

$$E_x = -\frac{Q}{4\pi\varepsilon_0}\frac{\gamma\beta ct}{\left[\gamma^2(\beta ct)^2 + y^2 + z^2\right]^{3/2}}. \tag{13.150}$$

As the charge passes the origin the instrument measuring $E_x$ then records first a large field in the positive $x-$direction ($t < 0$ during the approach) and then a large field in the negative $x-$direction ($t > 0$). We have plotted this in Fig. 13.11.

From (13.146) the $y-$component of the electric field is

$$E_y = \frac{Q}{4\pi\varepsilon_0}\gamma\frac{y}{\left[\gamma^2(\beta ct)^2 + y^2 + z^2\right]^{3/2}}$$

This is the $y-$component of a *Coulomb field* for each $x \approx \gamma\beta ct$. The instrument measuring $E_y$ at the origin will record data for a Coulomb field for each value of the time $t$. The value of $E_y$ will, however, change very rapidly as the charge passes the origin at a velocity close to $c$. So the instrument measuring $E_y$ will register a single positive pulse in the direction of $+\hat{e}_y$ for $y > 0$ and a symmetric single negative pulse for $y < 0$. We have plotted the positive pulse in Fig. 13.12.

The instrument measuring $E_z$ will register a similar result, as we see by comparing (13.146) and (13.147).

We see from the spatial similarity of the magnetic field induction components (13.148) and (13.149) to the electric field components (13.146) and (13.147) that the instruments at the origin of $k$ measuring the magnetic field induction components will register a rapid pulse, similar to that in Fig. 13.12, as the charge passes. We can deduce the basic geometrical form of the magnetic field induction from equations (13.148) and (13.149). At the time $t = 0$ when the charge passes the origin the magnetic induction is a maximum. At $t = 0$,

$$B_{\mathrm{y}} = -\frac{\beta}{1-\beta^2}\frac{Q}{4\pi c\varepsilon_0}\frac{z}{\left[y^2+z^2\right]^{3/2}} \tag{13.151}$$

and

$$B_{\mathrm{z}} = \frac{\beta}{1-\beta^2}\frac{Q}{4\pi c\varepsilon_0}\frac{y}{\left[y^2+z^2\right]^{3/2}}. \tag{13.152}$$

Then,

$$B_{\mathrm{y}}^2 + B_{\mathrm{z}}^2 = \left(\frac{\beta}{1-\beta^2}\right)^2\left(\frac{Q}{4\pi c\varepsilon_0}\right)^2\frac{1}{\left[y^2+z^2\right]^2}. \tag{13.153}$$

That is $B = \sqrt{B_{\mathrm{y}}^2 + B_{\mathrm{z}}^2}$ is a constant when the radial distance from the origin of $k$ is a constant. From (13.151) and (13.152) we can see that the direction of the magnetic field induction obeys the right hand rule. That is the single charge moving at a velocity close to $c$ behaves as a single charge current. The dependence of $B$ on the radius is, however, not the same.

## 13.12 Summary

We cannot speak of relativity without encountering the ideas of Einstein. In this chapter we elected to present those ideas as Einstein presented them. As a result the beginning section approximates an outline of Einstein's June, 1905, paper. And where we lift out examples that Einstein also treated we have indicated similarities. It is our hope that in this way the reader will receive a more concrete understanding of the theory of relativity.

We have taken a similar approach to the melding of space and time that Minkowski brought about. Minkowski understood Einstein's ideas completely, as well as the efforts, successes, and failures of those who preceded Einstein. As Einstein, Minkowski is present throughout the chapter.

The Lorentz or FitzGerald-Lorentz Transformation is central to the mathematical discussion. In 1905 Einstein was unaware of the existence of the Lorentz Transformation. And the path Einstein took to obtain the transformation is fundamentally different than that taken by Lorentz. FitzGerald and Lorentz introduced a mechanical postulate, while Einstein had discovered something about the meaning of time that no one had seen.

We pursued tensors for the simplicity they bring to the subject. With tensors we were able to show the covariance of Maxwell's Equations on Lorentz Transformation. This was what Einstein originally wanted in 1905, and what he had shown in a rather more awkward form. We ended the chapter with the form that the fields take in different inertial frames. And then we applied these to evaluating the fields from moving charges. This is not radiation. Radiation from accelerating charges will be the subject of a subsequent chapter.

## Exercises

**13.1.** For the time synchronization experiment in Fig. 13.3 define the times for the emission of light from the origin of frame $k'$, reflection from the point $\eta$, and return of the light to the origin of frame $k'$ as $\tau_0$, $\tau_1$, and $\tau_2$ respectively. Using equation (13.3) and the approach used to find $\partial\tau/\partial x'$, show that $\partial\tau/\partial y = 0$ and $\partial\tau/\partial z = 0$.

**13.2.** The Lorentz Transformation equations for the differential coordinates of a world line are

$$\begin{aligned} \mathrm{d}x'^0 &= \gamma\left(\mathrm{d}x^0 - \beta\mathrm{d}x^1\right) \\ \mathrm{d}x'^1 &= \gamma\left(\mathrm{d}x^1 - \beta\mathrm{d}x^0\right) \\ \mathrm{d}x'^2 &= \mathrm{d}x^2 \\ \mathrm{d}x'^3 &= \mathrm{d}x^3. \end{aligned}$$

Using this these equations to show that

$$\begin{aligned} &c^2\left(\mathrm{d}t\right)^2 - \left(\mathrm{d}x\right)^2 - \left(\mathrm{d}y\right)^2 - \left(\mathrm{d}z\right)^2 \\ &= c^2\left(\mathrm{d}t'\right)^2 - \left(\mathrm{d}x'\right)^2 - \left(\mathrm{d}y'\right)^2 - \left(\mathrm{d}z'\right)^2. \end{aligned}$$

**13.3.** Events which are measured as occurring simultaneously in one inertial frame are not necessarily simultaneous in another inertial frame. Consider, for example, that someone in frame $k'$ places a flash lamp at the origin and detectors at points $\xi = +L$ and $\xi = -L$ on the $\xi-$axis. For clarity we identify two events. Event (1) is the arrival of the light pulse at the $\xi = +L$ detector and event (2) is the arrival of the light pulse at the $\xi = -L$ detector. In frame $k'$ events (1) and (2) are simultaneous.

Show that the events (1) and (2) are not simultaneous in frame $k$.

**13.4.** Show by direct use of the matrix product with (13.31) and (13.32) that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1},$$

where **1** is the identity, which in matrix form is

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**13.5.** Show, using a matrix product, that the covariant vector with elements $x_\alpha = g_{\alpha\beta}x^\beta$ is

$$x_\alpha = \begin{bmatrix} ct \\ -x \\ -y \\ -z \end{bmatrix}.$$

**13.6.** Show, using a matrix product, that the contravariant vector with elements $x^\beta$ can be obtained from $x_\alpha g^{\alpha\beta}$.

**13.7.** Show that the inverse of the metric tensor is

$$\mathbf{g}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

by demonstrating that $\mathbf{g}\mathbf{g}^{-1} = \mathbf{g}^{-1}\mathbf{g} = \mathbf{1}$.

**13.8.** Beginning with the equation for the addition of velocities in the direction of motion of the inertial frame $k'$, show that

(a) If $w_x = c$ then $u_x = c$ and indicate why this demonstrates the validity of Einstein's second postulate.
(b) If $w_x < c$ then $u_x < c$.

**13.9.** In the Princeton Tokamak Fusion Test Reactor (TFTR) an electron temperature of $T_e = 6.5 \times 10^3$ eV was measured in a neutral beam injection experiment in 1987 [91]. Statistical mechanics teaches us that the kinetic energy of the electrons is $(3/2)\,k_B T_e = (3/2)\,T_e\,(\text{ eV})$, where $T_e\,(\text{eV})$ is the electron temperature in eV. The rest energy of an electron is $mc^2 = 0.51099906$ MeV. What was the relativistic mass of the electrons in this experiment? What was the electron thermal velocity as a factor of the speed of light? (i.e. what was $\beta_u$?)

**13.10.** Obtain the Lorentz Transformation of the current 4−vector and 4−potential vector to show that the second and third components of the current density and the potential vectors are $J'^2 = J^2$, $J'^3 = J^3$, $A'^2 = A^2$, $A'^3 = A^3$.

**13.11.** In the text we said that evaluating the elements of the *field strength tensor*

$$F^{\beta\alpha} = \partial^\beta A^\alpha - \partial^\alpha A^\beta = -F^{\alpha\beta}.$$

we can write $F^{\beta\alpha}$ as the matrix

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

The first two elements, for example, are

$$F^{00} = \partial^0 A^0 - \partial^0 A^0 = 0$$

and

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 \\ &= \frac{\partial A_{\mathrm{x}}}{c\partial t} + \frac{\partial \varphi}{c\partial x} = -\frac{E_{\mathrm{x}}}{c}. \end{aligned}$$

Carry out the evaluation of the remaining terms.

**13.12.** Using the Lorentz transform in the form (13.22) and the inverse (13.23) show that the four dimensional Laplacian in Minkowski space

$$\begin{aligned} \Box &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \\ &= \frac{\partial^2}{\partial x^0 \partial x^0} - \frac{\partial^2}{\partial x^1 \partial x^1} - \frac{\partial^2}{\partial x^2 \partial x^2} - \frac{\partial^2}{\partial x^3 \partial x^3}. \end{aligned}$$

is an invariant.

In this you will need to recognize that $x^0$ and $x^1$ are functions of $x'^0$ and of $x'^1$. The chain rule requires then that the partial derivatives are

$$\frac{\partial}{\partial x^0} = \frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^0}\frac{\partial}{\partial x'^1}$$

$$\frac{\partial}{\partial x^1} = \frac{\partial x'^0}{\partial x^1}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^1}\frac{\partial}{\partial x'^1},$$

with

$$\frac{\partial x'^0}{\partial x^0} = \gamma,$$

and so forth. The second partial derivatives are then

$$\frac{\partial^2}{\partial x^0 \partial x^0} = \left(\frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^0}\frac{\partial}{\partial x'^1}\right)\left(\frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^0}\frac{\partial}{\partial x'^1}\right)$$

$$\frac{\partial^2}{\partial x^1 \partial x^1} = \left(\frac{\partial x'^0}{\partial x^1}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^1}\frac{\partial}{\partial x'^1}\right)\left(\frac{\partial x'^0}{\partial x^1}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^1}\frac{\partial}{\partial x'^1}\right),$$

**13.13.** We consider that in frame $k$ (stationary) we have a only a uniform and constant magnetic field (induction) $\mathbf{B} = B_{\mathrm{z}}\hat{e}_{\mathrm{z}}$ in a certain area of the $(x, y)$ −plane. We move a square loop of wire through this region. When the area of the loop is only partially penetrated by the magnetic field induction, as we have shown in Fig. 13.13, we measure a current on the galvanometer.

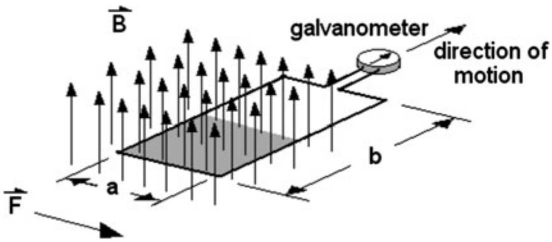


**Fig. 13.13** Rectangular loop of wire being moved through a region in which there is a constant and uniform magnetic field with induction $\mathbf{B} = B_z\hat{e}_z$. A current appears in the wire if the loop area is partially penetrated by the field

Prior to 1905 this experiment was described either by using Faraday's Law or by claiming that the charges in the wire experienced a force in the direction $-\hat{e}_y$ by the magnetic force $q\mathbf{v} \times \mathbf{B}$ in the direction shown. For a person in frame $k'$ moving with the wire, the magnetic force argument fails because the charges are stationary. There is no magnetic force in frame $k'$.

(a) Using the transformation properties for the fields, what is the force acting on the charges in the length $a$ of the wire?
(b) Is the emf generated in the loop by this force identical to that of Faraday's Law?
(c) Comment on this whether or not this experiment demonstrates a resolution to the asymmetry Einstein pointed out.

# Chapter 14
# Radiation

*The scientist, if he is to be more than a plodding gatherer of bits of information, needs to exercise an active imagniation.*

*Linus Pauling*

## 14.1 Introduction

In some of the preceding chapters we have considered the structure and transport of waves without speaking directly to the origin of those waves. That the origin of the waves is ultimately in the charges and their motion was, perhaps, evident in our treatment of the electromagnetic energy, as well as in the form of the wave equations for the electric and magnetic fields in the presence of $\rho$ and $\mathbf{J}$. But we have not yet considered the details of the process of the emission of electromagnetic energy by moving charges. This will be the subject of the present chapter.

We will base our treatment in this chapter on the wave equations for the vector and scalar potentials with sources $\rho$ and $\mathbf{J}$. We can obtain general solutions to these equations using a Green's Function. These solutions are known in the literature as the *Liénard–Wiechert potentials*. We shall then consider the use the Liénard–Wiechert potentials in the study of specific examples.

## 14.2 Waves from Sources

As in Sect. 13.10.1 we will again base our discussion on the Lorentz Gauge

$$\operatorname{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \tag{14.1}$$

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
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and wave equations for the scalar and vector potentials

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi = -\frac{\rho}{\varepsilon_0} \tag{14.2}$$

and

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A} = -\mu_0\,\mathbf{J}\,. \tag{14.3}$$

Equations (14.2) and (14.3) are a consequence of the full Maxwell Equations and are, therefore, an equivalent expression of the full Maxwell Equations. We use a different order for the time and spatial derivatives in (14.1–14.3) than that of Sect. 13.10.1. In Minkowski Space the time coordinate precedes the spatial coordinates. In our work here the placing the spatial variables first will make the solution of these equations appear more orderly.

Equations (14.2) and (14.3) are partial differential equations in the three spatial variables and the time. A Fourier Transform with respect to the time will eliminate the time derivatives in (14.2) and (14.3) and will introduce a dependence on the angular frequency $\omega$. We will then have equations in the spatial coordinates alone, which we will be able to solve using a Green's Function.

For continuity we repeat here the Fourier transform pair for the time. The Fourier Transform is

$$f_\omega(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}t'\, f\left(\mathbf{r},t'\right)\exp\left(i\omega t'\right) \tag{14.4}$$

and the inverse is

$$f(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\omega f_\omega(\mathbf{r})\exp(-i\omega t)\,. \tag{14.5}$$

Noting that

$$\frac{\partial^2}{\partial t^2} f(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\omega\left(-\omega^2\right) f_\omega(\mathbf{r})\exp(-i\omega t)\,,$$

and using (14.5) (14.2) and (14.3) become

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\omega\left\{\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\varphi_\omega(\mathbf{r}) + \frac{\rho_\omega(\mathbf{r})}{\varepsilon_0}\right\}\exp(-i\omega t) = 0 \tag{14.6}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\omega\left\{\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\mathbf{A}_\omega(\mathbf{r}) + \mu_0\mathbf{J}_\omega(\mathbf{r})\right\}\exp(-i\omega t) = \mathbf{0} \tag{14.7}$$

Because the integral is over all $\omega$, the integrals in (14.6) and (14.7) vanish if and only if the integrands vanish. We have, therefore, the Fourier transformed form of the wave equations for the potentials

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\varphi_\omega(\mathbf{r}) = -\frac{1}{\varepsilon_0}\rho_\omega(\mathbf{r}) \tag{14.8}$$

and

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\mathbf{A}_\omega(\mathbf{r}) = -\mu_0\mathbf{J}_\omega(\mathbf{r}), \tag{14.9}$$

where the Fourier transform of the charge density $\rho_\omega(\mathbf{r})$ is

$$\rho_\omega(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}t'\rho\left(\mathbf{r}, t'\right)\exp\left(i\omega t'\right) \tag{14.10}$$

and of the current density $\mathbf{J}_\omega(\mathbf{r})$ is

$$\mathbf{J}_\omega(\mathbf{r}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}t'\mathbf{J}\left(\mathbf{r}, t'\right)\exp\left(i\omega t'\right) \tag{14.11}$$

The partial differential (14.8) and (14.9) are *Helmholtz Equations*. The general form of the Helmholtz Equation is

$$\left[\nabla^2 + K^2\right]\Psi(\mathbf{r}) = -f(\mathbf{r}). \tag{14.12}$$

In the exercises we show that

$$\operatorname{div}\operatorname{grad}\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} = -K^2\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

if $\mathbf{r} \neq \mathbf{r}'$. That is

$$F_{\mathrm{H}}\left(\mathbf{r}, \mathbf{r}'\right) = \exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right) / \left|\mathbf{r}-\mathbf{r}'\right|$$

solves the homogeneous Helmholtz Equation when $\mathbf{r} \neq \mathbf{r}'$.

If we can now show that

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to 0}\left[\nabla^2 + K^2\right]\left(\frac{1}{4\pi}\right)F_{\mathrm{H}}\left(\mathbf{r}, \mathbf{r}'\right) = -\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

we will have demonstrated that

$$\left(\frac{1}{4\pi}\right)F_{\mathrm{H}}\left(\mathbf{r}, \mathbf{r}'\right) = \exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right) / \left(4\pi\left|\mathbf{r}-\mathbf{r}'\right|\right)$$

is a Green's Function for the Helmholtz Equation. Because we have shown that $F_{\mathrm{H}}\left(\mathbf{r},\mathbf{r}'\right)$ satisfies the homogeneous Helmholtz Equation we need only consider the limit as $\left|\mathbf{r}-\mathbf{r}'\right| \to 0$ for this final step.

Euler's identity results in

$$\lim_{\left|\mathbf{r}-\mathbf{r}'\right|\to 0} \exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right) = \lim_{\mathrm{R}\to 0}\cos\left(KR\right) = 1.$$

Then

$$\lim_{\left|\mathbf{r}-\mathbf{r}'\right|\to 0}\left(\operatorname{div}\operatorname{grad}\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} + K^2\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\right)$$
$$= \lim_{\left|\mathbf{r}-\mathbf{r}'\right|\to 0}\left(\operatorname{div}\operatorname{grad}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} + K^2\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right)$$

Using (2.107), which is

$$\delta\left(\mathbf{r}-\mathbf{r}'\right) = -\nabla^2\frac{1}{4\pi}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|},$$

and neglecting $\lim_{\left|\mathbf{r}-\mathbf{r}'\right|\to 0} K^2\left(1/\left|\mathbf{r}-\mathbf{r}'\right|\right)$ compared to the $\delta-$function, we have the result

$$\left(\operatorname{div}\operatorname{grad} + K^2\right)\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} = -4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right).$$

Therefore

$$\left(\frac{1}{4\pi}\right)F_{\mathrm{H}}\left(\mathbf{r},\mathbf{r}'\right) = \frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{4\pi\left|\mathbf{r}-\mathbf{r}\,'\right|}, \tag{14.13}$$

is the Green's Function for the Helmholtz Equation.

With the Green's Function (14.13) we may immediately write down the solutions of (14.6) and (14.7) for the potentials as

$$\varphi_{\omega}\left(\mathbf{r}\right) = \frac{1}{4\pi\varepsilon_0}\int\limits_{\mathrm{V\,(all\ sources)}} \mathrm{d}V'\frac{\rho_{\omega}\left(\mathbf{r}\,'\right)\exp\left(\pm i\omega\left|\mathbf{r}-\mathbf{r}'\right|/c\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|} \tag{14.14}$$

and

$$\mathbf{A}_{\omega}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi}\int\limits_{\mathrm{V\,(all\ sources)}} \mathrm{d}V'\frac{\mathbf{J}_{\omega}\left(\mathbf{r}\,'\right)\exp\left(\pm i\omega\left|\mathbf{r}-\mathbf{r}'\right|/c\right)}{\left|\mathbf{r}-\mathbf{r}\right|}, \tag{14.15}$$

Here we have reintroduced $K = \omega/c$ from (14.6) and (14.7).

By inverting the Fourier transform using (14.5) we obtain the potentials as functions of $\left(\mathbf{r},t\right)$. For the scalar potential

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{2\pi}}\right)\int\limits_{\text{V (all sources)}} \mathrm{d}V' \frac{\rho_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdots$$
$$\cdots\int_{-\infty}^{+\infty}\mathrm{d}\omega\exp\left[i\omega\left(-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right] \tag{14.16}$$

and similarly for $\mathbf{A}(\mathbf{r},t)$.

With (14.10) (14.16) becomes

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{2\pi}\right)\int\limits_{\text{V (all sources)}} \mathrm{d}V' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}\mathrm{d}t'\rho\left(\mathbf{r}',t'\right)\cdots$$
$$\cdots\int_{-\infty}^{+\infty}\mathrm{d}\omega\exp\left[i\omega\left(t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right]. \tag{14.17}$$

We note the representation of the $\delta$−function

$$\left(\frac{1}{2\pi}\right)\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega\exp\left\{i\omega\left[t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right\} = \delta\left(t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right). \tag{14.18}$$

(see (B.1)) that appears in (14.17). Upon integrating over the $\delta$−function equation (14.17) becomes

$$\begin{aligned}\varphi(\mathbf{r},t) &= \frac{1}{4\pi\varepsilon_0}\int\limits_{\text{V (all sources)}} dV' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{+\infty}dt'\rho\left(\mathbf{r}',t'\right)\delta\left(t'-t\pm\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\int\limits_{\text{V (all sources)}} dV' \frac{\rho\left(\mathbf{r}',t\pm|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|}\end{aligned} \tag{14.19}$$

The time $t$ is the time at which we measure the field from this potential at the point $\mathbf{r}$. The principle of causality requires that only the motion of the charges at the point $\mathbf{r}'$ that occur at the earlier time $t-|\mathbf{r}-\mathbf{r}'|/c < t$ can affect the field measured at the field point $\mathbf{r}$ at the time $t$. For historical reasons this is termed the retarded time. The advanced time $t+|\mathbf{r}-\mathbf{r}'|/c$ is of no interest to us here, because this time is later than the time of observation of the wave. We then drop the $+$ sign in (14.19) as not physically realistic.

The physically realistic solutions for the potentials are

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\limits_{\text{V(all sources)}} \mathrm{d}V' \frac{\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{|\mathbf{r}-\mathbf{r}'|} \tag{14.20}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int\limits_{\mathrm{V(all\,sources)}} \mathrm{d}V' \frac{\mathbf{J}\left(\mathbf{r}\,',t - \left|\mathbf{r}-\mathbf{r}\,'\right|/c\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|}. \tag{14.21}$$

Equations (14.20) and (14.21) are the scalar and vector potentials for which the charge and current densities appearing in the integrals are sources. These equations include the familiar solutions to Poisson's Equation if $\left|\mathbf{r}-\mathbf{r}\,'\right|/c \ll t$, which is the case if the field point is close to the source point. This reduction to the solution to Poisson's equation must be the case.

Equations (14.20) and (14.21), however, must also include the fields that may be observed at great distances from the sources. These are the radiated waves generated by the motion of the charged particles.

Wave solutions are those whose sinusoidal periodicity may be identified over a large number of wavelengths, which is true if the damping is small. And waves must carry energy. So radiative wave solutions are those for which there is an identifiable loss of energy from the moving charges.

To obtain workable forms for (14.20) and (14.21) we must integrate over the charge and current densities. The equations we will obtain were first developed by Alfred-Marie Liénard in 1898 and independently by Emil Wiechert in 1900 with work continuing into the early 1900s.

## 14.3 Liénard–Wiechert Potentials

In this section we shall obtain the form of (14.20) and (14.21) which can be applied directly to specific forms of charged particle motion. These will be the *Liénard–Wiechert potentials*.

Because we are interested only in the behavior of charges at the retarded time $t' = t - \left|\mathbf{r}-\mathbf{r}\,'\right|/c$, we write (14.20) and (14.21) as

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0} \int\limits_{\mathrm{V(all\,sources)}} \mathrm{d}V' \frac{\rho\left(\mathbf{r}\,',t'\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|} \tag{14.22}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int\limits_{\mathrm{V(all\,sources)}} \mathrm{d}V' \frac{\mathbf{J}\left(\mathbf{r}\,',t'\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|}, \tag{14.23}$$

We also consider that the volume containing the radiating charges is small compared to the distance from the charges to the field point $\left|\mathbf{r}-\mathbf{r}\,'\right|$. The point $\mathbf{r}\,'$ locates a (charged) particle in the small volume $V$ (all sources). If we choose some representative point $\mathbf{r}_1$ in the volume $V$ (all sources) we may write

$$\left|\mathbf{r}-\mathbf{r}\,'\right| \approx \left|\mathbf{r}-\mathbf{r}_1\right| = R, \tag{14.24}$$

and bring this factor outside of the integrals in (14.22) and (14.23).

But we still face a difficulty in the evaluation of the integrals in (14.22) and (14.23). The difficulty is that the charges are moving and the volume $V$(all sources) in (14.22) and (14.23) is a function of the retarded time $t'$. The motion of the charged particles is subjected to the fields at their locations. These fields are, in turn, caused by the charged particles. So the difficulty cannot be easily removed.

Some authors discuss the integration problem at length. These discussions are good for clarifying the physical origin of the problem (see e.g. [37], pp. 429–33). We shall, however, resolve the problem mathematically (see e.g. [83], pp. 545–7), which seems, finally, to be the simplest approach.

In (14.24) we introduced a representative point $\mathbf{r}_1$ located within the volume $V$ (all sources), which covers the moving charges at the time $t'$. We can remove the dependence of the integration on the time $t'$ if we consider a volume $V_1$ centered on the point $\mathbf{r}_1$ which covers the trajectories of the charged particles over the time interval say $\Delta t'$ of interest for the calculation. We then finally introduce a representative time $t_1$ to replace the times $t'$ in the interval $\Delta t'$. This scheme will allow us to write the difficult integrals over $V$ (all sources) by integrals independent of the time. That is

$$\int\limits_{\mathrm{V(all\,sources)}} \mathrm{d}V' \frac{\rho\left(\mathbf{r}\,',t'\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|} \Longrightarrow \frac{1}{\left|\mathbf{r}-\mathbf{r}_{\,1}\right|}\int\limits_{\mathrm{V}_1} \mathrm{d}V_1 \rho\left(\mathbf{r}_{\,1},t_1\right) \tag{14.25}$$

and

$$\int\limits_{\mathrm{V(all\,sources)}} \mathrm{d}V' \frac{\mathbf{J}\left(\mathbf{r}\,',t'\right)}{\left|\mathbf{r}-\mathbf{r}\,'\right|} \Longrightarrow \frac{1}{\left|\mathbf{r}-\mathbf{r}_{\,1}\right|}\int\limits_{\mathrm{V}_1} \mathrm{d}V_1 \mathbf{J}\left(\mathbf{r}_{\,1},t_1\right). \tag{14.26}$$

Our problem is then to find the dependence of $(\mathbf{r}_1, t_1)$ on $(\mathbf{r}', t')$ and the dependence of the differential volume $\mathrm{d}V_1$ on $\mathrm{d}V'$. The dependence of $(\mathbf{r}_1, t_1)$ on $(\mathbf{r}', t')$ we obtain from a Taylor expansion involving the trajectories of the particles. And we obtain the relationship between the differential volumes from the Jacobian for the transformation between $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}', t')$.

The point $\mathbf{r}_1$ is close to $\mathbf{r}'$ and the time $t_1$ is close to $t'$. Because the spatial point $\mathbf{r}_1$ and the time $t_1$ are fixed we perform the Taylor expansion for the spatial point $\mathbf{r}'$ about $\mathbf{r}_1$ with the time difference $(t' - t_1)$ as the expansion parameter. That is

$$\begin{aligned}\mathbf{r}'\left(t'\right) &= \mathbf{r}_1 + \left.\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t'}\right]_{t_1}\left(t'-t_1\right) + \frac{1}{2}\left.\frac{\mathrm{d}^2\mathbf{r}'}{\mathrm{d}^2t'}\right]_{t_1}\left(t'-t_1\right)^2 + \cdots \\ &= \mathbf{r}_1 + \mathbf{v}_1\left(t'-t_1\right) + \frac{1}{2}\mathbf{a}_1\left(t'-t_1\right)^2 + \cdots,\end{aligned} \tag{14.27}$$

where $\mathbf{v}_1$ and $\mathbf{a}_1$ are the representative velocity and acceleration of the charges at the representative time $t_1$.

Equation (14.27) is our transformation of the trajectory $\mathbf{r}'(t')$ to the representative point $\mathbf{r}_1$. That the transformation involves representations of the velocity and acceleration should come as no surprise to us.

We now turn to the problem of transforming the volume differential $\mathrm{d}V'$. This problem is not difficult. The technique is, however, often only treated in pure mathematics courses, which are not necessarily taken by physics and engineering students. We have included a brief, but sufficient discussion of the technique in Appendix I. There we show how we may convert variables of integration using what is called the Jacobian determinant, or simply the Jacobian.[1]

The differential volumes $\mathrm{d}V' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$ and $\mathrm{d}V_1 = \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}z_1$ are related by

$$\mathrm{d}V_1 = J\,\mathrm{d}V', \tag{14.28}$$

where the Jacobian $J$ is defined by

$$\begin{aligned} J &= \frac{\partial (x_1, y_1, z_1)}{\partial (x', y', z')} \\ &= \det \begin{bmatrix} \partial x_1/\partial x' & \partial x_1/\partial y' & \partial x_1/\partial z' \\ \partial y_1/\partial x' & \partial y_1/\partial y' & \partial y_1/\partial z' \\ \partial z_1/\partial x' & \partial z_1/\partial y' & \partial z_1/\partial z' \end{bmatrix}. \end{aligned} \tag{14.29}$$

We can compute the partial derivatives appearing in the Jacobian (14.29) from (14.27) written for $\mathbf{r}_1$ in terms of the actual trajectory $\mathbf{r}'(t')$. From the Taylor series (14.27) we obtain $\mathbf{r}_1$ in terms of the actual trajectory $\mathbf{r}'(t')$ and the representative velocity $\mathbf{v}_1$ and acceleration $\mathbf{a}_1$ as

$$\mathbf{r}_1 = \mathbf{r}'\left(t'\right) - \mathbf{v}_1\left(t' - t_1\right) - \frac{1}{2}\mathbf{a}_1\left(t' - t_1\right)^2 - \cdots. \tag{14.30}$$

The derivatives appearing in (14.29) are then

$$\frac{\partial x_1}{\partial x'} = 1 - v_{\mathrm{x},1}\frac{\partial t'}{\partial x'} - a_{\mathrm{x},1}\left(t' - t_1\right)\frac{\partial t'}{\partial x'} - \cdots, \tag{14.31}$$

$$\frac{\partial x_1}{\partial y'} = 1 - v_{\mathrm{x},1}\frac{\partial t'}{\partial y'} - a_{\mathrm{x},1}\left(t' - t_1\right)\frac{\partial t'}{\partial y'} - \cdots, \tag{14.32}$$

$$\frac{\partial x_1}{\partial z'} = 1 - v_{\mathrm{x},1}\frac{\partial t'}{\partial z'} - a_{\mathrm{x},1}\left(t' - t_1\right)\frac{\partial t'}{\partial z'} - \cdots, \tag{14.33}$$

with similar equations for $\partial y_1/\partial x'$, $\partial y_1/\partial y'$, $\cdots$. We evaluate the partial derivatives of the time $t'$ with respect to the coordinates, such as $\partial t'/\partial x'$, from the general equation expressing $t'$ in terms of $t$, which is

[1]Carl Gustav Jacob Jacobi (1804–1851) was a German mathematician and inspiring teacher.

$$t' = t - \frac{|\mathbf{r} - \mathbf{r}\,'|}{c}. \tag{14.34}$$

we have, for example

$$\frac{\partial t'}{\partial x'} = \frac{\hat{e}'_{\mathrm{Rx}}}{c}, \tag{14.35}$$

where $\hat{e}'_{\mathrm{Rx}}$ is the $x-$component of the unit vector in the direction of $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

The evaluation of the Jacobian is then not difficult, although tedious. The result is

$$J = 1 - \frac{1}{c}\mathbf{v}' \cdot \hat{e}'_{\mathrm{R}} - \frac{1}{c}\frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} \cdot \hat{e}'_{\mathrm{R}}\left(t' - t\right) + \cdots \tag{14.36}$$

(see [83], p. 547). In (14.36) we have written $\mathbf{v}' \approx \mathbf{v}_1$. If we neglect the third term on the right hand side of (14.36) as small compared to 1 we have

$$\mathrm{d}V_1 = J\mathrm{d}V' = \left(1 - \frac{1}{c}\mathbf{v}' \cdot \hat{e}'_{\mathrm{R}}\right)\mathrm{d}V', \tag{14.37}$$

and the potentials (14.22) and (14.23) become

$$\varphi\left(\mathbf{r}, t\right) = \frac{1}{4\pi\varepsilon_0 R}\frac{1}{\left(1 - \mathbf{v}' \cdot \hat{e}'_{\mathrm{R}}/c\right)}\int_{\mathrm{V}_1}\rho\left(\mathbf{r}_1, t_1\right)\mathrm{d}V_1 \tag{14.38}$$

$$\mathbf{A}\left(\mathbf{r}, t\right) = \frac{\mu_0}{4\pi\ R}\frac{1}{\left(1 - \mathbf{v}' \cdot \hat{e}'_{\mathrm{R}}/c\right)}\int_{\mathrm{V}_1}\mathbf{J}\left(\mathbf{r}_1, t_1\right)\mathrm{d}V_1. \tag{14.39}$$

We can now perform the integrals in (14.38) and (14.39) obtaining the total charge in the distribution,

$$\int_{\mathrm{V(all\ sources)}}\rho\left(\mathbf{r}_1, t_1\right)\mathrm{d}V_1 = \sum Q \tag{14.40}$$

and the total current density of the distribution

$$\int_{\mathrm{V(all\ sources)}}\mathbf{J}\left(\mathbf{r}_1, t_1\right)\mathrm{d}V_1 = \sum Q\mathbf{v}'. \tag{14.41}$$

We have written a general summation sign in each of the (14.40) and (14.41) to indicate that there may be a distribution of charges. We may readily also limit these to a single charge.

For a single charge the Liénard–Wiechert potentials are then

$$\boxed{\varphi\left(\mathbf{r}, t\right) = \left[Q/\left(4\pi\varepsilon_0\right)\right]\left[1/\left(R - \mathbf{v} \cdot \mathbf{R}/c\right)\right]} \tag{14.42}$$

$$\boxed{\mathbf{A}\left(\mathbf{r},t\right)=\left[\mu_0 Q/\left(4\pi\right)\right]\left[\mathbf{v}/\left(R-\mathbf{v}\cdot\mathbf{R}/c\right)\right],} \tag{14.43}$$

where we have dropped the prime on the velocity of the charge to obtain the most familiar form these take in the literature (cf. [58], p. 186).

## 14.4 Plane Waves

We may consider that the emitted electromagnetic radiation is in the form of plane waves if the field point is a great distance from the radiating charged particles producing the potentials (14.42) and (14.43). Far from the sources the potentials satisfy wave equations without sources,

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi=0, \tag{14.44}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}=\mathbf{0}, \tag{14.45}$$

and the Lorentz Gauge (14.1).

The divergence of the magnetic field induction $\mathbf{B}$ always vanishes, so we must have

$$\mathbf{B}=\operatorname{curl}\mathbf{A}. \tag{14.46}$$

There is, however, an ambiguity in the potentials for the plane wave. For plane waves we have the relationships among the fields, wave vector $\mathbf{k}$ and angular frequency $\omega$ as

$$\mathbf{k}\times\mathbf{E}=\omega\mathbf{B} \tag{14.47}$$

and

$$\mathbf{k}\times\mathbf{B}=-\frac{\omega}{c^2}\mathbf{E} \tag{14.48}$$

(see Sect. 11.4).

The electric field may always be obtained from the magnetic field for plane waves. A separate calculation of the scalar potential is then, for plane waves, superfluous. And it is possible to choose the potentials such that the scalar potential vanishes (see e.g. [57], p. 55). Therefore for plane waves a distance from the charges generating them we are able to simply choose $\varphi = 0$. The Lorentz Gauge then requires that

$$\operatorname{div}\mathbf{A}=0. \tag{14.49}$$

This is the Coulomb Gauge, which we have now obtained from the Lorentz Gauge for plane waves. The Coulomb Gauge is also called the radiation gauge ([85], p. 342) (see exercises).

With (14.46) we can write Faraday's Law as

$$\operatorname{curl}\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}=-\frac{\partial}{\partial t}\operatorname{curl}\mathbf{A}. \tag{14.50}$$

Then, neglecting arbitrary constants,

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}. \tag{14.51}$$

Using (14.46) and (14.51) we can solve for **B** and **E** if we have **A**.

From (14.47) and (14.48) and using the dispersion relation for plane waves $c = \omega/k$ (see (11.25)) we find that

$$\mathbf{E} \times \mathbf{B} = \hat{k}\left(\frac{1}{c}E^2\right), \tag{14.52}$$

and

$$\mathbf{E} \times \mathbf{B} = \hat{k}\,(c)\left(B^2\right), \tag{14.53}$$

where $\hat{k}$ is the unit vector in the direction of wave propagation.

The Poynting Vector is then either

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \\ &= \hat{k}c\frac{1}{\mu_0 c^2}\left(E^2\right).\end{aligned} \tag{14.54}$$

or

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \\ &= \hat{k}c\frac{1}{\mu_0}\left(B^2\right)\end{aligned} \tag{14.55}$$

We may combine (14.54) and (14.55) with $c^2 = 1/\varepsilon_0\mu_0$ to give

$$\mathbf{S} = \hat{k}cu_{\text{em}} \tag{14.56}$$

where $u_{\text{em}} = (1/2)\left(\varepsilon_0 E^2 + (1/\mu_0)\,B^2\right)$ is the electromagnetic energy density in the wave.

Because we are basing our treatment on the vector potential, we choose to work with the magnetic field. We then identify the electromagnetic energy flux as (14.55).

## 14.5 Sources

We continue to consider field points that are far from the source points, i.e. the charged particles emitting the radiation. The emitted radiation is, however, not necessarily uniform in all directions. The motion of the charged particles may be of any arbitrary form depending on the fields at their location and on their identities.

We consider a spherical surface with radius equal to the distance between the source and field points and specify the direction of the unit vector $\hat{n}$ in terms of the spherical azimuthal and polar angles (see Sect. 2.3.3). In order that we can identify the amount of energy radiated into a differential area of the sphere, we choose the direction of $\hat{n}$ to lie within a cone specified by the limits $\sin\phi \mathrm{d}\phi \mathrm{d}\vartheta$. This we define to be the solid angle $\mathrm{d}\Theta$. That is

$$\mathrm{d}\Theta = \sin\phi \mathrm{d}\phi \mathrm{d}\vartheta, \tag{14.57}$$

and we note that

$$\int_{\text{sphere}} \mathrm{d}\Theta = 4\pi, \tag{14.58}$$

The differential area of a sphere of radius $R$ is then

$$\mathrm{d}S_{\text{sphere}} = R^2 \mathrm{d}\Theta. \tag{14.59}$$

Using (14.55) for the Poynting vector we then have the rate at which electromagnetic energy is radiated into the area subtended by the solid angle $\mathrm{d}\Theta$ as

$$\mathrm{d}\dot{U}_{\text{em}} = \frac{c}{\mu_0} \left(B^2\right) R^2 \mathrm{d}\Theta \tag{14.60}$$

From (14.47) and (14.51) we obtain the magnetic field induction from the vector potential as

$$\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times E = -\left(\frac{1}{c}\right) \hat{k} \times \frac{\partial}{\partial t} \mathbf{A}, \tag{14.61}$$

where the vector potential is the Liénard–Wiechert potential (14.43). For the non-relativistic case this is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi R} \sum Q_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}. \tag{14.62}$$

We may now consider specific types of possible motion of the charges emitting the radiation.

### 14.5.1 Dipole Radiation

We consider a group of charges $\{Q_{\mathrm{i}}\}$ located at the points $\{\mathbf{r}_{\mathrm{i}}\}$. The dipole moment of this group of charges is

$$\mathbf{p}_{\mathrm{d}} = \sum Q_{\mathrm{i}} \mathbf{r}_{\mathrm{i}}. \tag{14.63}$$

Assuming that the magnitude of each charge is constant, the summation appearing on the right hand side of (14.62) is

$$\sum Q_{\mathrm{i}}\mathbf{v}_{\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}t}\sum Q_{\mathrm{i}}\mathbf{r}_{\mathrm{i}} = \dot{p}_{\mathrm{d}}\hat{e}_{\mathrm{v}}, \tag{14.64}$$

where $\hat{e}_{\mathrm{v}}$ is a unit vector in the direction of $\mathrm{d}\mathbf{p}_{\mathrm{d}}/\mathrm{d}t$. Then (14.62) is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi R}\dot{p}_{\mathrm{d}}\hat{e}_{\mathrm{v}}. \tag{14.65}$$

From (14.65) we obtain the partial time derivative of the vector potential as

$$\frac{\partial}{\partial t}\mathbf{A} = \frac{\mu_0}{4\pi R}\ddot{p}_{\mathrm{d}}\hat{e}_{\mathrm{a}}, \tag{14.66}$$

where $\hat{e}_{\mathrm{a}}$ is the unit vector in the direction of $\mathrm{d}^2\mathbf{p}_{\mathrm{d}}/\mathrm{d}t^2$. From (14.61) the magnetic field induction in the plane wave at the field point is then

$$\mathbf{B} = \frac{\mu_0\ddot{p}_{\mathrm{d}}}{4\pi Rc}\hat{e}_{\mathrm{a}} \times \hat{n}. \tag{14.67}$$

Combining (14.60) and (14.67) the energy radiated by the moving group of charges into the solid angle $\mathrm{d}\Theta$ is

$$\mathrm{d}\dot{U}_{\mathrm{em}} = \frac{\mu_0\ddot{p}_{\mathrm{d}}^2}{(4\pi)^2 c}(\hat{e}_{\mathrm{a}} \times \hat{n})^2\,\mathrm{d}\Theta \tag{14.68}$$

We are free to orient the axes of our reference frame in any way we choose. We orient them so that $\hat{e}_{\mathrm{a}}$ lies along the polar axis and $\hat{e}_{\mathrm{a}} \times \hat{n} = \sin\phi$. With (14.57) (14.68) then becomes

$$\mathrm{d}\dot{U}_{\mathrm{em}} = \frac{\mu_0\ddot{p}_{\mathrm{d}}^2}{(4\pi)^2 c}\sin^3\phi\,\mathrm{d}\phi\,\mathrm{d}\vartheta. \tag{14.69}$$

Integrating (14.69) over the azimuthal angle $\vartheta$ from 0 to $2\pi$ and over the polar angle $\phi$ from 0 to $\pi$ we have the total rate at which energy is lost by the charged particles to radiation. This is

$$\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t} = \frac{\mu_0\ddot{p}_{\mathrm{d}}^2}{6\pi c}. \tag{14.70}$$

We can now specialize this result to a single charged particle by identifying

$$\ddot{p}_{\mathrm{d}} = Q^2(\ddot{r})^2.$$

Then (14.70) becomes

$$\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t} = \frac{\mu_0 Q^2(\ddot{r})^2}{6\pi c}. \tag{14.71}$$

The (14.70) and (14.71) are general expressions of the rate at which energy is emitted by electromagnetic radiation from a group of charged particles or a single charged particle. We have restricted the permittivity and permeability to be $\varepsilon_0$

and $\mu_0$, which means we have neglected dielectric and magnetic media. We then exclude Cherenkov radiation, which can result from the motion of a charged particle through a dielectric medium. Cherenkov radiation is emitted if a charged particle has a velocity exceeding the phase velocity, not the group velocity, of light in the dielectric medium (see e.g. [48], p. 638).

The rather clear message in both of the (14.70) and (14.71) is that the charge radiates only when it is accelerated (cf. [58], p. 200).

## *14.5.2 Charge in a Magnetic Field*

A charged particle moving in a circle in a magnetic field is accelerated. The acceleration in this case is $(Q/m)\,\mathbf{v}\times\mathbf{B}$ (see Chap. 7, Sect. 7.4.1). Equation (14.71) then becomes

$$\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t}=\frac{\mu_0 Q^4 v^2 B^2}{6\pi m^2 c}. \tag{14.72}$$

If we are interested also in the direction of the radiation we cannot integrate over the solid angle we did to obtain (14.70). The acceleration of the charge is toward the center of the circle and cannot be considered to be along the polar axis. For clarity we have drawn the situation in Fig. 14.1. The magnetic field and the direction from the center of the particle orbit to the observation point, which is the direction of the unit vector $\hat{n}$, are now separated by the polar angle $\phi$ and not the azimuthal angle $\vartheta$. The calculation, which is lengthy, although no more difficult than before, may be found in the text by Landau and Lifshitz ([58], p. 227). The ratio of the radiation emitted into the plane of the orbit to that emitted in the direction of the external magnetic field (i.e. perpendicular to the particle orbit), for the relativistic case, is

$$\frac{\mathrm{d}\dot{U}_{\mathrm{em}}\big]_{\phi=\pi/2}}{\mathrm{d}\dot{U}_{\mathrm{em}}\big]_{\phi=0}}=\frac{4+3\beta^2}{8\left(1-\beta^2\right)^{5/2}} \tag{14.73}$$

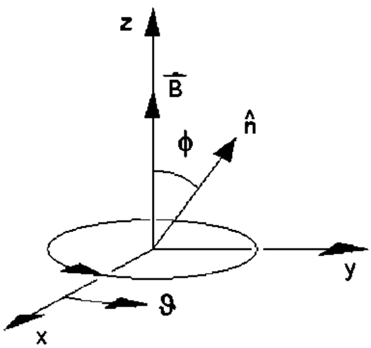


**Fig. 14.1** Motion of a charged particle in a constant and uniform magnetic field. We have oriented the coordinates so that the magnetic field is along the polar axis $z$ and motion is in the $(x, y)$ −plane

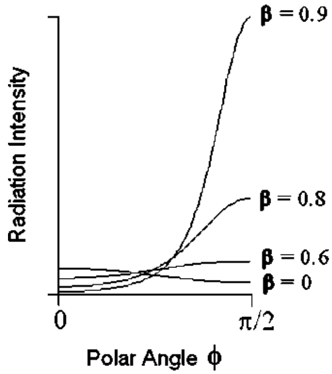


**Fig. 14.2** Radiation intensity emitted by a charge moving in a uniform magnetic field ([58], p. 227 (74.4))

with $\beta = v/c$. We see that as $\beta \to 0$, the ratio in (14.73) approaches $1/2$. In the nonrelativistic case most of the radiation is then emitted perpendicular to the plane of motion. As $\beta$ increases toward unity the radiation intensity becomes concentrated in the plane of the orbit. In Fig. 14.2 we have plotted the radiation intensity as a function of the polar angle $\phi$ for an assortment of values of the parameter $\beta$. In the extreme relativistic case ($\beta \to 1$) we may consider that the radiation is entirely in the plane of the orbit.

## 14.6 Summary

In this chapter we have developed the theory to deal with the emission of radiation from moving charges and applied the theory to a general and then a specific example. We began with a complete electrodynamic description, which the Lorentz Gauge allows us to cast in terms of wave equations for the scalar and vector potentials. For situations in which the field point is at a large distance from the source point we obtained the Liénard–Wiechert potentials for a connection between the wave fields and the motion of the charges causing them. At great distances from the sources the potentials also result in approximately plane electromagnetic waves, for which we can express the energy flux (Poynting) vector in terms of the square of either the magnetic field induction or the electric field. Because we can also choose the scalar potential to be zero for plane waves we based our development on the magnetic field.

Our final equations for the radiated energy showed rather clearly that the radiated energy resulted only from the acceleration of the charges. We applied these equations to the study of radiation from a time varying electric dipole and from a charge moving in a uniform magnetic field. Our use of the permittivity and permeability of free space has prevented any discussion of Cherenkov radiation. This is, however, not a limitation for many very practical applications.

Because we have relied on the results of selected preceding chapters, this treatment is not self-contained. We also used a Jacobian determinant for the transformation of an integral in our development of the Liénard–Wiechert potentials. Because this may not be familiar to the reader we have included a brief, but sufficient, discussion of the Jacobian in the appendices.

## Exercises

**14.1.** Show that

$$\mathrm{div}\left(\mathbf{r}-\mathbf{r}'\right) = 3$$

$$\mathrm{grad}\left|\mathbf{r}-\mathbf{r}'\right| = \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

$$\mathrm{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^{\mathrm{n}}}\right) = -n\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^{\mathrm{n+2}}},$$

With these find grad and grad$'$ of

$$\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

[Answer: $\mathrm{grad}\left(\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)/\left|\mathbf{r}-\mathbf{r}'\right|\right) = \left(\mathbf{r}-\mathbf{r}'\right)\left[-\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \pm iK\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^2}\right]$ $\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)$]

**14.2.** Using the result of the preceding exercise, show that

$$\mathrm{div\,grad}\,\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} = -K^2\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

and that

$$\mathrm{div\,grad}\,\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} = \mathrm{div}'\,\mathrm{grad}'\,\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}.$$

**14.3.** We claimed that for plane waves the scalar potential could be chosen to be zero, which resulted in the radiation gauge $\mathrm{div}\,\mathbf{A} = 0$. Our point was that for plane waves we required no separate calculation of the electric field. Only the magnetic field was required.

Go back to Maxwell's Equations. Choose $\varphi$ =constant. Show then that

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}$$

and that

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \operatorname{grad}\operatorname{div}\mathbf{A}$$

so that we can, indeed, choose $\varphi$ =constant= 0 and div $\mathbf{A} = 0$ for the case of the plane wave.

**14.4.** Someplace in the preceding exercise you may have neglected something. This depends on the point at which you set $\varphi$ =constant. The radiation gauge is appropriate for the study of plane waves radiated from moving charges a distance from the observation point. At what point did you either implicitly or explicitly introduce this restriction? Look at the general equation for $\varphi$.

**14.5.** In the last section of this chapter we considered dipole radiation from either a group of charges or from a single dipole. We recall from our multipole expansions that a general charge density may have a net dipole moment

$$\mathbf{p}_{\mathrm{d}} = \sum Q_{\mathrm{i}} \mathbf{r}_{\mathrm{i}}.$$

We found that it is only accelerating charges that radiate electromagnetic energy.

At which point in our discussion did we decide that acceleration was what was required? Do not simply give an equation number. Discuss what happened.

**14.6.** We considered as well the energy radiated from a charge moving in a uniform magnetic field. We studied this motion in the chapter on particle motion and realize that the frequency of the radiation is the cyclotron frequency $\Omega = QB/m$. Cyclotron radiation is an important source of energy loss in magnetically confined fusion plasmas. In the exercises in the preceding chapter you calculated the velocity of the electrons in the Princeton Tokamak Fusion Test Reactor (TFTR).

Would you expect the cyclotron radiation to be emitted radially outward from the Tokamak (perpendicular to the magnetic field) or axially (along the magnetic field)?

**14.7.** In 1912 Niels Bohr was a post doctoral student with the Ernest Rutherford group in Manchester, England. Rutherford. Rutherford had discovered the nucleus based on experimental studies by Geiger and Marsden of the scatter of $\alpha-$particles from gold foil. This meant that the elaborate atomic model of J.J. Thomson, in which the electrons were immersed in a positive fluid was untenable. But how then could the electrons be located around the nucleus?

What is wrong with the planetary model?

# Chapter 15
# Fields in Matter

*If there is anything more wonderful than matter in the sheer versatility of its behavior, I have yet to hear tell of it.*

*Sir Fred Hoyle*

## 15.1 Introduction

Up to this point in our development we have considered matter only in the form of point particles in a predominately empty space. This treatment is appropriate to gases and to plasmas, but is not appropriate to solid or liquid matter. In this chapter we will discuss the effect that polarizable and magnetizable matter have on electromagnetic fields.

In Maxwell's Equations electric and magnetic fields result from two separate sources. The fields may be produced by matter in the form of charge or current densities (Chaps. 4 and 5). In this case the fields produced may be slowly varying fields. And we may obtain the fields from Poisson's Equations for the scalar and vector potentials. Or they may be wave fields resulting from acceleration of particles (Chap. 14). The fields may also be obtained from one another according to Faraday's Law and the displacement current in Ampère's Law.

This latter electromagnetic interaction between the fields is independent of the presence of matter, except for the effect matter may have on the speed of light appearing as a coefficient of the displacement current in Ampère's Law.

If we ignore radiation fields, which will propagate as waves, we may consider the effect of matter on the fields as a slowly varying phenomena, which is described by Poisson's Equation.

Any detailed modern treatment of matter must be quantum mechanical. And the transition from the microscopic to the macroscopic requires statistical mechanics ([40], Chaps. 9 and 10; [47]). We will discuss the quantum and statistical mechanical results. But we will not work through any of the details.

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## 15.2 Experiments

Direct measurement of fields in solid matter is not possible. But we can measure the effects of these fields indirectly. Our problem is then to identify properties of matter that determine the magnitude of the fields inside matter and can be found by measurements made outside of the matter.

In Maxwell's Equations the empirical constants $\varepsilon_0$ and $\mu_0$ determine the electric and magnetic fields resulting from charge and current densities. We may anticipate, then, that these constants will change in the presence of matter. The result would be a change in magnitudes of the fields resulting from charge and current densities. So our goal is to design an experiment that will measure any change in these constants resulting from the presence of matter.

We introduced the spherical capacitor and toroidal solenoid inductor in our discussions of electrostatic and magnetostatic field energies. In these, as well as in capacitors and inductors generally, the capacitance and inductance are proportional to $\varepsilon_0$ and $\mu_0$ in the absence of matter. We may then expect that filling a capacitor or inductor with matter will allow us to measure the new constants by determining the new values of capacitance and self inductance.

Faraday conducted experiments on parallel plate capacitors with various dielectrics [72]. We may then attribute the first investigations of the effect of using dielectric material in capacitors to Faraday. As we discussed in Sect. 1.10 Faraday also conducted experiments on wood and iron core inductors. We may then also attribute the first investigations of the effect of magnetic material on self inductance to Faraday.

For more accurate experiments, however, we may turn to more modern experiments in which bridge circuits are used. Bridge circuits are still used at the National Institute of Standards and Technology (NIST) for precision measurements [25]. Alternating current bridges can be used to accurately measure capacitance and inductance.

So we have experimental measurements of the constants $\varepsilon$ and $\mu$, that replace $\varepsilon_0$ and $\mu_0$ when dielectric or magnetic material is introduced into capacitors or inductors. The capacitance increases with the addition of a dielectric material between the conductors and the inductance increases with the addition of magnetic material into the volume of the solenoid. From these experimental results we conclude that $\varepsilon > \varepsilon_0$ and $\mu > \mu_0$.

We may now ask for some understanding of the behavior of matter that would result in these experimental results.

### *15.2.1 Dielectrics in Capacitors*

From our working definition of capacitance $C = Q/V$ we know that an increase in capacitance will be the result of a decrease in the potential $V$ if we hold the

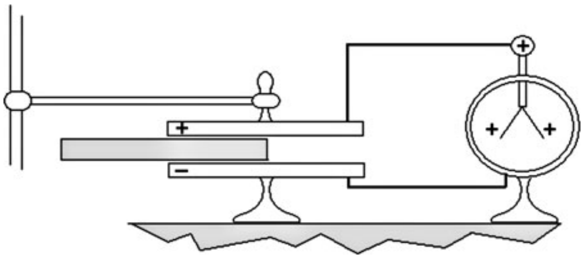


**Fig. 15.1** Charged capacitor, dielectric and gold leaf electroscope

charge on the capacitor constant. In Fig. 15.1 we have drawn a picture of a simple bench top (Faraday vintage) experiment to demonstrate the effect of a *dielectric* (nonconductor) on the capacitor voltage.

We assume that we have carefully prepared a sheet of homogeneous, isotropic dielectric (nonconducting) material for the experiment. To guarantee a constant charge on the capacitor plates we slide the dielectric in place between them without allowing it to touch either plate.

For the demonstration we use a gold leaf electroscope to indicate a change in voltage between the capacitor plates. A decrease in voltage is registered by a decrease in the angle between the gold leaves of the electroscope, which is what we observe.

Since the dielectric is homogeneous and isotropic, the electrostatic field within the dielectric, when it is fully inserted between the plates, will be constant and directed vertically downward in Fig. 15.1. Because the potential has decreased, this constant electrostatic field within the dielectric will be less than the electrostatic field between the plates without the dielectric. From an application of Gauss' Law in integral form we know that this will result from a uniform negative surface charge density on the top of the dielectric and an equal positive surface charge density on the bottom of the dielectric.

We may then ask for the property of the dielectric that would result in these charge densities.

### *15.2.2 Solid Dielectrics*

We will consider only solid dielectrics. Most solids are crystals characterized by atoms ordered in a regular (Bravais[1]) lattice. The chemical bonding, which determines the form of the crystal lattice occurs because the energy of the bonded

[1]The Bravais lattice is named for Auguste Bravais (1811–1863) who was a French physicist.

atoms in a regular array is less than that of the separate atoms. In solid dielectrics the electrons are not free to move throughout the crystal.

Although we may understand some aspects of the chemical bonding among atoms in semi-classical terms, the reason that the energy is lowered by bonding is quantum mechanical (see e.g. [47], pp. 3–12). The quantum states of the electrons are determined by the symmetry of the crystal and the charges on the nuclei. The crystal structure is then also a part of the energy minimization problem.

We can, however, find the crystal structure separately by x-ray diffraction. And we are left with a problem of using a linear combination of atomic orbitals (LCAO) to minimize the energy.

The electronic states of isolated atoms are associated with specific quantum numbers. In a crystal these distinct states become energy bands (see e.g. [70], pp. 166–176).

In somewhat general terms the spatial extension of the electron wave function distinguishes between conductors and dielectrics. When the electron wave function has an extension beyond the neighboring atoms in the crystal then the exact location of the nearest neighbor atoms becomes unimportant and packing density becomes a determining characteristic of the crystal. This is metallic bonding. We will neglect metallic bonding here.

Bonds which localize the wave functions of the electrons are particularly the covalent and the ionic bonds. We can understand the covalent bond only quantum mechanically. The ionic bond is, however, more dependent on coulomb forces and we may understand much of the ionic bond in classical terms. In general the bonding is not exclusively of covalent or ionic, but sometimes may be considered a hybrid between these extremes.

The covalent crystal has an electronic band structure that is actually similar to that of a conductor. The difference is that there is a large energy gap between the valence band and the conduction band so that considerable energy is required to transport electrons ([3], p. 376).

If we apply electric fields to covalent crystals the electrons will move relatively to the nucleus in each atom polarizing the electric charge of the atom. This motion of the electrons on an atom is termed *atomic polarization*. In an ionic crystal, in addition to atomic polarization, there will be an opposing motion of the sublattices, such as, for example, the Na and Cl sublattices of the ionic crystal NaCl. The opposing motion of the sublattices is termed *displacement polarization* ([3], pp. 542–547). Both atomic and displacement polarization contribute to the polarization of the ionic crystal.

We may think of this in terms of a simplified atomic or molecular basis in which atoms or molecules in the crystal become polarized upon application of a field. In Fig. 15.2 we have drawn this picture.

Each of the polarized atomic or molecular units is identical. If we consider a small volume of length $d$ and area $A$ from within the bulk dielectric, as we have drawn in the upper left hand corner of Fig. 15.2, the small volume will be a miniature model of the whole. We may then speak of a dipole moment per unit volume for the crystal.

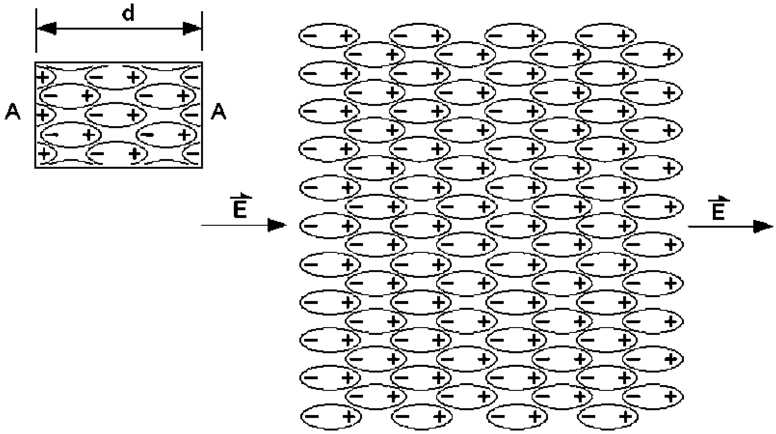


**Fig. 15.2** Polarization of a dielectric

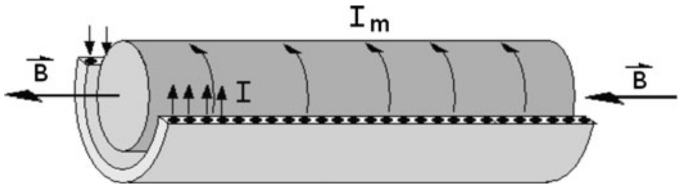


**Fig. 15.3** Solenoid with a bar of magnetic material inserted

### *15.2.3 Magnetic Cores in Inductors*

Faraday found that by using a magnetic material instead of wood he increased the induced electric field in the secondary loop of a transformer (see Fig. 1.6). This will occur if the magnetic field induction is increased by the magnetic material.

In Fig. 15.3 we have drawn a cylindrical solenoid with a core of magnetic material inserted.

The magnetic core in In Fig. 15.3 results in an increase in the self inductance of the solenoid. We can understand this in terms of Ampère's Law if there is a magnetization current $I_m$ circulating around the bar of magnetic material, as we have drawn in Fig. 15.3. That is the external magnetic field induction from the solenoid results in a current around the magnetic core, which increases the magnetic field induction within the core above what would be present without the core.

This effect is in agreement with Ampère's original idea about the cause of permanent magnetism ([97], p. 92).

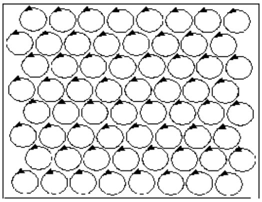

**Fig. 15.4** Ampère's theory of permanent magnetism

## *15.2.4 Magnetism in Solids*

According to Ampère permanent magnetism is caused by microscopic current loops within a magnetic material. This was more than simply an idea. It was a developed theory. Ampère introduced the concept of a magnetic molecule with a current flowing perpetually within it. The current was the primary cause of the magnetism, which Ampère considered to be an electrical phenomenon. Magnetic matter then contained magnetic molecules, which could be oriented by an applied field.

The total magnetization would be zero if the magnetic molecules were randomly oriented. In that case the net current would also vanish. In the magnetized condition the magnetic molecules would align in a particular direction as we have illustrated in Fig. 15.4.

In this magnetized state the net current within the material would be zero because of the cancellation of all of the molecular currents when molecules are adjacent to one another. On the boundary of the magnetized sample, however, there would be a net current oriented around the sample. This net electrical current on the boundary was the sum of the electrical currents from the magnetic molecules. The theory was a beautiful appreciation of the atomic basis of matter with the implication that even atoms possessed an internal structure.

As atomic theory developed at the beginning of the 20th century we believed that we could identify these currents as a result of the orbital motion of electrons. But the path to truth is not constructed solely on simple arguments.

Pierre Curie studied the magnetization as a function of the magnetic field intensity **H** finding them proportional ([65], p. 18). The proportionality constant $\chi$, which is the *magnetic susceptibility*, Curie found was inversely proportional to the thermodynamic temperature $T$. In diamagnetic substances $\chi$ is negative. In paramagnetic and ferromagnetic substances it is positive.

With the statistical mechanics of Maxwell, Rudolf Clausius, Ludwig Boltzmann, and J. Willard Gibbs we had a method to treat large numbers of particles (atoms or molecules) and obtain thermodynamic properties (see e.g. [40]). Paul Langevin conducted a statistical mechanical analysis of the magnetized state. He considered that in a diamagnetic material Lenz' Law holds at an atomic level producing a microscopic magnetic field that opposed the external field. In a paramagnetic material the atoms had permanent magnetic moments, which were free to rotate and were aligned by the external magnetic field. The result was an internal, microscopic field that added to the external field. The resulting expression Langevin obtained for $\chi$ yielded Curie's result from a Taylor expansion ([65], p. 19).

Pierre Weiss then proposed that the interactions among the microscopic magnetic moments in a paramagnetic material could be described by a molecular magnetic field. The result was the so-called Curie-Weiss Law, which predicted a nonzero Curie temperature at which the ratio of $\chi$ to the magnetic field intensity becomes infinite ([65], p. 20). An explanation of ferromagnetism was farther in the future and required an understanding of the quantum description of the electron.

The electron presented intellectual problems because it is an elementary particle and cannot be understood in classical terms. In 1921 Arthur Compton proposed that the electron possessed a spin and a magnetic moment [13]. These were intrinsic properties of the electron and were independent of any orbital motion of the electron.

Based on the experimental evidence available to them, George Uhlenbeck and Samuel Goudsmit showed that internal angular momentum of the electron, the spin, had the value $\hbar/2$ [92]. This was half the quantum of orbital angular momentum.

The intrinsic magnetic moment of the electron was twice the value expected for this angular momentum. This apparent problem resolved by Dirac in his 1928 paper *The Quantum Theory of the Electron* [22]. The electron spin emerged naturally from Dirac's theory ([28], p. 143).

Ferromagnetism is a phenomenon resulting from the spin of the electron and does not exist as a classical phenomenon ([65], p. 39). In classical physics the magnetic interaction is between two current loops and was described by Ampère. This is a dipole interaction. The interaction between spin magnetic moments is not a dipole interaction. It is an exchange interaction. The exchange interaction can be understood in terms of wave function overlap and the Pauli exclusion principle ([65], p. 39).

The exchange interaction among a collection of atoms in which the $j$th atom is located at the point $\mathbf{r}_\mathrm{j}$ and has the total spin operator $\mathbf{S}_\mathrm{j}$ is described by the Heisenberg, or Heisenberg-Dirac Hamiltonian operator

$$\boxed{\mathcal{H}_\mathrm{spin} = -\sum\nolimits_{\mathrm{i}\neq\mathrm{j}} \mathrm{J}_\mathrm{ij}\mathbf{S}_\mathrm{i}\cdot\mathbf{S}_\mathrm{j}.} \tag{15.1}$$

The quantities $J_\mathrm{ij}$ are the exchange coupling constants, or exchange parameters, or exchange coefficients ([43], [3], p. 681; [65], p. 139). For ferromagnetic systems $J_\mathrm{ij} > 0$.

In general we must also consider that the collection of atoms is in an external magnetic field with an intensity $\mathbf{H}$. In this case there is an additional contribution to the Hamiltonian

$$-g\mu_\mathrm{B}\mathbf{H}\cdot\sum_\mathrm{i}\mathbf{S}_\mathrm{i}, \tag{15.2}$$

where $g$ is the Landé $g-$factor and $\mu_\mathrm{B}$ is the Bohr magneton, which is the magnetic moment of the lowest state in the Bohr model of the hydrogen atom ([3], pp. 646, 654). The spin operator $\mathbf{S}_\mathrm{j}$ is a vector operator with two quantum states.

The Heisenberg Hamiltonian (15.1) results in a magnetization of the material that does not vanish even after an external magnetic field is removed. At thermal equilibrium there are domains in a ferromagnetic material with spins aligned as a result of the exchange interaction. An external magnetic field results in the alignment of these according to (15.2). Removal of the field does not return all of the domains to a random orientation.

Because it is nonlinear in the spins the Heisenberg Hamiltonian (15.1) presents difficulties in any theoretical treatment.

The quantum mechanical picture of paramagnetism results in a spin contribution to the magnetic susceptibility. The susceptibility has both a diamagnetic contribution and a spin contribution, that accounts for what is known as Van Vleck paramagnetism ([3], p. 653; [93]). Paramagnetism is then also a phenomenon that results from the electron spin and is quantum mechanical. Paramagnetism, however, does not persist if the magnetic field is removed.

We now have a basic understanding of the response of solid matter to electric and magnetic fields in terms of atomic structure.

The atoms occupying the crystal sites of a dielectric will be polarized by the application of an external electric field. This polarization will result from motion of the electrons on the individual atoms and from the motion of sublattices depending on the ionic character of the chemical bonds in the crystal.

The atoms occupying the crystal sites in a magnetic material will have net spins, which come from the addition of unpaired electron spins. These materials will be paramagnetic or ferromagnetic depending ultimately on the overlap of electron wave functions between atoms.

In either the dielectric or the magnetic material we may then consider that the atoms have either an electrical dipole moment or a magnetic moment.

## 15.3 Potentials from Slowly Varying Fields

In slowly varying fields the scalar and vector potential satisfy Poisson's Equation. The solutions of Poisson's Equation for the potentials are

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathrm{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' \tag{15.3}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' \tag{15.4}$$

In (15.3) and (15.4) we integrate over the volume containing the matter. Classically this is the matter containing all charge and all current densities.

### *15.3.1 Atoms and Multipole Expansions*

In the preceding sections we identified polarization and magnetism in solids as atomic and electronic phenomena. Our integration of (15.3) and (15.4) must take

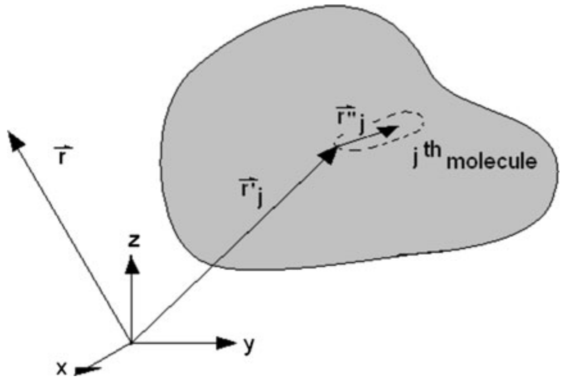


**Fig. 15.5** Integration over bulk matter considered as summation over molecular integrals

this into account. To accomplish this we write the integrals in (15.3) and (15.4) as sums of integrals over individual atoms or molecules depending upon the form of the crystal. We have shown the situation pictorially in Fig. 15.5.

We locate the $j$th atomic site with the vector $\mathbf{r}_j'$. Then we define a vector $\mathbf{r}_j''$ from $\mathbf{r}_j'$ to points within the atom or molecule. For example if we are dealing with an ionic crystal such as NaCl our integration will be over the molecule NaCl and our vector $\mathbf{r}_j'$ will be to the lattice site of either the Na or the Cl sublattice. The position vector $\mathbf{r}_j'$ is constant during the integration of $\mathbf{r}_j''$ over the volume $V_j$ of the $j$th atom. For each atom or molecule identified by $\mathbf{r}_j'$ we have then

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \Longrightarrow \frac{1}{\left|\mathbf{r}-\mathbf{r}_j'-\mathbf{r}_j''\right|}. \tag{15.5}$$

In Sects. 4.5 and 5.7 we carried out multipole expansions for both the charge and the current density. The expansion of (15.5) around the point $\mathbf{r}-\mathbf{r}_j'$ follows from these multipole expansions. The result is

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}_j'-\mathbf{r}_j''\right|} = \frac{1}{\left|\mathbf{r}-\mathbf{r}_j'\right|} + \frac{\left(\mathbf{r}-\mathbf{r}_j'\right)\cdot\mathbf{r}_j''}{\left|\mathbf{r}-\mathbf{r}_j'\right|^3} + \frac{1}{2}\left\{\frac{3\left[\left(\mathbf{r}-\mathbf{r}_j'\right)\cdot\mathbf{r}_j''\right]^2}{\left|\mathbf{r}-\mathbf{r}_j'\right|^5} - \frac{\left|\mathbf{r}_j''\right|^2}{\left|\mathbf{r}-\mathbf{r}_j'\right|^3}\right\} + \cdots \tag{15.6}$$

We will drop the third term on the right hand side of (15.6) as small, since $\left|\mathbf{r}_j''\right|$ is of atomic dimensions. Then the multipole expansions for the potentials are

$$
\begin{aligned}
\varphi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_j \int_{V_j} \frac{\rho\left(\mathbf{r}_j{}''\right)}{\left|\mathbf{r}-\mathbf{r}_j'-\mathbf{r}_j{}''\right|} \mathrm{d}V_j'' \\
&= \frac{1}{4\pi\varepsilon_0}\sum_j \frac{1}{\left|\mathbf{r}-\mathbf{r}_j'\right|}\int_{V_j} \rho\left(\mathbf{r}_j''\right)\mathrm{d}V'' \\
&+ \frac{1}{4\pi\varepsilon_0}\sum_j \frac{\left(\mathbf{r}-\mathbf{r}_j'\right)}{\left|\mathbf{r}-r_j'\right|^3}\cdot\int_{V_j} \mathbf{r}_j''\rho\left(\mathbf{r}_j''\right)\mathrm{d}V_j''
\end{aligned}
\quad (15.7)
$$

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\sum_j \int_{V_j} \frac{\mathbf{J}\left(\mathbf{r}''\right)}{\left|\mathbf{r}-\mathbf{r}_j'-\mathbf{r}_j{}''\right|} \mathrm{d}V_j'' \\
&= \frac{\mu_0}{4\pi}\sum_j \frac{1}{\left|\mathbf{r}-\mathbf{r}_j'\right|}\int_{V_j} \mathbf{J}\left(\mathbf{r}_j{}''\right)\mathrm{d}V_j'' \\
&+ \frac{\mu_0}{4\pi}\sum_j \frac{\left(\mathbf{r}-\mathbf{r}_j'\right)}{\left|\mathbf{r}-\mathbf{r}_j'\right|^3}\cdot\int_{V_j} \mathbf{r}_j{}''\mathbf{J}\left(\mathbf{r}_j{}''\right)\mathrm{d}V_j''
\end{aligned}
\quad (15.8)
$$

Within the $j$th atom or molecule the charge density is $\rho\left(\mathbf{r}_j''\right)$ and the current density is $\mathbf{J}\left(\mathbf{r}_j''\right)$. The charge density must finally be determined by the quantum mechanical density of the electrons and the separation of the nuclei. In this sense the quantum mechanical charge density has a classical analog and we may think of the charge density in classical terms.

In (15.7) then

$$
\int_{V_j} \rho\left(\mathbf{r}_j''\right)\mathrm{d}V'' = Q_j = 0 \quad (15.9)
$$

is the total charge on the $j$th atom or molecule, which is zero. And

$$
\int_{V_j} \mathbf{r}_j''\rho\left(\mathbf{r}_j''\right)\mathrm{d}V_j'' = \mathbf{p}_{d,j}^{(a)} \quad (15.10)
$$

is the electric dipole moment for the $j$th atom or molecule.

We must treat the vector potential in (15.4) with more caution.

The susceptibility $\chi$ is a macroscopic thermodynamic property of the material. The calculation of $\chi$ must then be based on statistical mechanics. Because the Hamiltonian is a sum of diamagnetic, paramagnetic, and possibly ferromagnetic contributions we can still, however, speak of these as separate contributions to what we can refer to as a molecular magnetic moment.

There are lucid treatments of these magnetic effects by H. Ibach and H. Lüth [47] and by N.W. Ashcroft and N.D. Mermin [3] based on quantum statistical mechanics. In both of these the authors particularly consider the difficulties associated with the Heisenberg Hamiltonian (15.1), which is nonlinear in the spins.

In (15.8) then

$$\int_{\mathrm{V_j}} \mathbf{J}\left(\mathbf{r}_{\mathrm{j}}''\right) \mathrm{d}V_{\mathrm{j}}'' = \mathbf{0} \tag{15.11}$$

form charge conservation, as we showed in Sect. 5.7. And

$$\frac{\left(\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right)}{\left|\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right|^{3}} \cdot \int_{\mathrm{V_j}} \mathbf{r}_{\mathrm{j}}'' \mathbf{J}\left(\mathbf{r}_{\mathrm{j}}''\right) \mathrm{d}V_{\mathrm{j}}'' = \mathbf{m}_{\mathrm{j}}^{(\mathrm{a})} \times \frac{\left(\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right)}{\left|\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right|^{3}}, \tag{15.12}$$

where $\mathbf{m}_{\mathrm{j}}^{(\mathrm{a})}$ is the magnetic moment of the $j^{\mathrm{th}}$ atom or molecule, which includes diamagnetic, paramagnetic, and possibly ferromagnetic contributions.

The (15.7) and (15.8) are then

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\mathrm{j}} \frac{\left(\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right)}{\left|\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right|^{3}} \cdot \mathbf{p}_{\mathrm{d,j}}^{(\mathrm{a})} \tag{15.13}$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{\mathrm{j}} \mathbf{m}_{\mathrm{j}}^{(\mathrm{a})} \times \frac{\left(\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right)}{\left|\mathbf{r}-\mathbf{r}_{\mathrm{j}}'\right|^{3}} \tag{15.14}$$

### *15.3.2 Polarization and Magnetization Densities*

The summations in (15.13) and (15.14) extend over all atoms or molecules. For mathematical convenience we want to write these as integrals. To do this we consider an infinitesimal volume element $\Delta V_{\mathrm{j}}$ centered on $\mathbf{r}_{\mathrm{j}}'$. This volume element is infinitesimal with respect to all macroscopic dimensions or spatial variations in the system. But it is sufficiently large to contain a vast number of molecules (dimension $\sim 10^{-8}$ m). Then in place of the quantities $\mathbf{p}_{\mathrm{d,j}}^{(\mathrm{a})}$ and $\mathbf{m}_{\mathrm{j}}^{(\mathrm{a})}$ evaluated for molecules located at $\mathbf{r}_j$, we write

$$\begin{aligned} \mathbf{p}_{\mathrm{d,j}}^{(\mathrm{a})}\left(\mathbf{r}_{\mathrm{j}}'\right) &\to \mathbf{P}\left(\mathbf{r}_{\mathrm{j}}'\right) \Delta V_{\mathrm{j}} \\ \mathbf{m}_{\mathrm{j}}^{(\mathrm{a})}\left(\mathbf{r}_{\mathrm{j}}'\right) &\to \mathbf{M}\left(\mathbf{r}_{\mathrm{j}}'\right) \Delta V_{\mathrm{j}}, \end{aligned} \tag{15.15}$$

in which $\mathbf{P}\left(\mathbf{r}_j'\right)$ and $\mathbf{M}\left(\mathbf{r}_j'\right)$ are the dipole moment and magnetic moment densities at the point $\mathbf{r}_j'$.

Then (15.13) and (15.14) can be written as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_j \frac{\left(\mathbf{r}-\mathbf{r}_j'\right)}{\left|\mathbf{r}-\mathbf{r}_j'\right|^3}\cdot\mathbf{P}\left(\mathbf{r}_j'\right)\Delta V_j \tag{15.16}$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\sum_j \mathbf{M}\left(\mathbf{r}_j'\right)\times\frac{\left(\mathbf{r}-\mathbf{r}_j'\right)}{\left|\mathbf{r}-\mathbf{r}_j'\right|^3}\Delta V_j. \tag{15.17}$$

As $\Delta V_j \to 0$ and $j \to \infty$, these become Riemann integrals.

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \mathbf{P}\left(\mathbf{r}'\right)\cdot \operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' \tag{15.18}$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathbf{M}\left(\mathbf{r}'\right)\times \operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V', \tag{15.19}$$

where we have used

$$\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{15.20}$$

From the vector relationship (A.19) we have

$$\operatorname{div}'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{P}\left(\mathbf{r}'\right)\right) = \mathbf{P}\left(\mathbf{r}'\right)\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}'\mathbf{P}\left(\mathbf{r}'\right). \tag{15.21}$$

Then using (15.21) in (15.18) and applying Gauss' Theorem the scalar potential becomes

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{P}\left(\mathbf{r}'\right)\cdot\mathrm{d}\mathbf{S}' - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}'\mathbf{P}\left(\mathbf{r}'\right)\mathrm{d}V'. \tag{15.22}$$

From the vector relationship (A.21) we have

$$\operatorname{curl}'\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{M}\left(\mathbf{r}'\right)\right] = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{curl}'\mathbf{M}\left(\mathbf{r}'\right) + \left(\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{M}\left(\mathbf{r}'\right). \tag{15.23}$$

Then using (15.23) in (15.19) and the integral relationship

$$\int_{\mathrm{V}} \operatorname{curl} f(\mathbf{r})\,\mathbf{v}(\mathbf{r})\,\mathrm{d}V = -\oint_{\mathrm{S}} f(\mathbf{r})\,\mathbf{v}(\mathbf{r}) \times \mathrm{d}\mathbf{S} \tag{15.24}$$

(see exercises) the vector potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_{\mathrm{S}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{M}(\mathbf{r}') \times \mathrm{d}\mathbf{S}' + \frac{\mu_0}{4\pi} \int_{\mathrm{V}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{curl}' \mathbf{M}(\mathbf{r}')\,\mathrm{d}V', \tag{15.25}$$

### *15.3.3 Polarization Charges and Magnetization Currents*

Comparing (15.22) with (15.3) we see that

$$\boxed{\sigma_{\mathrm{p}} = \mathbf{P}(\mathbf{r}) \cdot \hat{\mathrm{n}},} \tag{15.26}$$

where $\hat{n}$ is the unit vector normal to the surface of the matter, is a surface polarization charge density and

$$\boxed{\rho_{\mathrm{p}} = -\operatorname{div} \mathbf{P}(\mathbf{r})} \tag{15.27}$$

is a volume polarization charge density.

As Fig. 15.2 indicates there will always be a surface polarization charge density in dielectrics. There will be a volume polarization charge density, however, only if the polarization varies inside the dielectric.

Comparing (15.25) with (15.4) we see that

$$\boxed{\mathbf{J}_{\mathrm{M}}^{(\mathrm{s})} = \mathbf{M}(\mathbf{r}) \times \hat{\mathrm{n}}} \tag{15.28}$$

is a surface magnetization current density and

$$\boxed{\mathbf{J}_{\mathrm{M}} = \operatorname{curl} \mathbf{M}(\mathbf{r})} \tag{15.29}$$

is a volume magnetization current density.

For diamagnetism, which we have illustrated in Fig. 15.4, we can show that $\mathbf{J}_{\mathrm{M}}^{(\mathrm{s})}$ is a current per unit length appearing only on the surface of the matter. This is Ampère's idea that the current loops produce a net current only on the surface of matter.

For paramagnetic and ferromagnetic materials the classical argument for the surface current density cannot be used. However, $\mathbf{J}_{\mathrm{M}}^{(\mathrm{s})}$ still takes on the role *mathematically* of a current per unit length on the surface of the matter. And, for the case in which the magnetization varies inside the material, $\mathbf{J}_{\mathrm{M}}$ still takes on the role of current density within the material.

With the volume polarization charge density $\rho_{\mathrm{p}}$ and the volume magnetization current density $\mathbf{J}_{\mathrm{M}}$ we identify the charge density in a dielectric as

$$\rho(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}) - \operatorname{div}\mathbf{P}(\mathbf{r}), \tag{15.30}$$

and the current density in a magnetizable material as

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_{\mathrm{f}}(\mathbf{r}) + \operatorname{curl}\mathbf{M}(\mathbf{r}) \tag{15.31}$$

where $\rho_{\mathrm{f}}(\mathbf{r})$ and $\mathbf{J}_{\mathrm{f}}(\mathbf{r})$ are the free charge and current densities.

And with $\rho_{\mathrm{p}}$ and $\mathbf{J}_{\mathrm{M}}$ the corresponding (static) field equations are

$$\operatorname{div}\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho_{\mathrm{f}}(\mathbf{r}) - \frac{1}{\varepsilon_0}\operatorname{div}\mathbf{P}(\mathbf{r}) \tag{15.32}$$

and

$$\operatorname{curl}\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}_{\mathrm{f}}(\mathbf{r}) + \mu_0\operatorname{curl}\mathbf{M}(\mathbf{r}), \tag{15.33}$$

or

$$\operatorname{div}\left[\varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right] = \rho_{\mathrm{f}}(\mathbf{r}) \tag{15.34}$$

and

$$\operatorname{curl}\left[\frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r})\right] = \mathbf{J}_{\mathrm{f}}(\mathbf{r}). \tag{15.35}$$

In empty space the displacement is $\mathbf{D} = \varepsilon_0\mathbf{E}$. And in empty space (or in air) Gauss' Law can be written as

$$\operatorname{div}\mathbf{D}(\mathbf{r}) = \rho_{\mathrm{f}}(\mathbf{r}). \tag{15.36}$$

Therefore, comparing (15.36) and (15.34) leads us to identify

$$\mathbf{D}(\mathbf{r}) = \varepsilon_0\mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) \tag{15.37}$$

as the displacement vector in a dielectric.

We also define the magnetic field intensity vector $\mathbf{H}$ as

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}), \tag{15.38}$$

so that (15.35) becomes

$$\operatorname{curl}\mathbf{H}(\mathbf{r}) = \mathbf{J}_{\mathrm{f}}(\mathbf{r}). \tag{15.39}$$

## 15.4 Interaction of Fields

In the preceding section we limited our treatment to slowly varying fields. We could then use Poisson's Equation for the potentials. We now wish to extend our treatment to the time dependent case.

In the introduction to this chapter we pointed out that fields may be obtained from one another according to Faraday's Law and the displacement current in Ampère's Law. This field–field interaction is time dependent and takes place independently of matter.

We must, however, also consider the possibility that an additional, apparent field–field interaction will result within matter employing the matter as an intermediary. This would be the case if the time rate of change in the polarization charge density, caused by a time dependent electric field, results in a current density. Such a current density would cause a magnetic field according to Ampère's Law and the result would be indistinguishable from a field–field interaction.

Or this would be the case if the magnetization current density, caused by a time dependent magnetic field, results in a time dependent charge density. Such a charge density would cause a time dependent electric field according to Gauss' Law and the result would be indistinguishable from a field–field interaction.

To investigate these possibilities we turn to charge conservation, which we have already shown is a fundamental concept invariant under Lorentz Transformation.

If the polarization charge density $\rho_{\mathrm{p}} = -\operatorname{div}\mathbf{P}(\mathbf{r})$ results in a polarization current density $J_{\mathrm{p}}$ then

$$\begin{aligned} \operatorname{div}\mathbf{J}_{\mathrm{p}} &= -\frac{\partial}{\partial t}\rho_{\mathrm{p}} = \frac{\partial}{\partial t}\operatorname{div}\mathbf{P} \\ &= \operatorname{div}\frac{\partial}{\partial t}\mathbf{P}. \end{aligned} \tag{15.40}$$

That is

$$\boxed{\mathbf{J}_{\mathrm{p}} = \partial\mathbf{P}/\partial t} \tag{15.41}$$

and the polarization of matter does result in a polarization current density provided the electric field depends on time.

At the molecular level this can be understood in the terms of time dependent motion of electrons on molecules. This differs from Faraday's and Maxwell's original understanding of the displacement current in matter in that our treatment here is based on a modern picture of matter and has no reference to an aether. Maxwell was an atomist (see e.g. [40], Chap. 8). But his interest, in this case, was in the aether.

The charge, $\rho_{\mathrm{M}}$, produced by a magnetization current density

$$\mathbf{J}_{\mathrm{M}} = \operatorname{curl}\mathbf{M} \tag{15.42}$$

must satisfy

$$\frac{\partial \rho_{\mathrm{M}}}{\partial t} = -\operatorname{div}\mathbf{J}_{\mathrm{M}} = -\operatorname{div}\operatorname{curl}\mathbf{M}$$
$$= 0, \tag{15.43}$$

since div curl $= 0$. The magnetization of the material, then, produces no charge density.

We must then include a polarization contribution to the current appearing in Ampère's Law. But no magnetization term must be added to Faraday's Law.

## 15.5 Maxwell's Equations in Matter

In the presence of matter, therefore, the charge and current densities are

$$\rho\left(\mathbf{r},t\right) = \rho_{\mathrm{f}}\left(\mathbf{r},t\right) - \operatorname{div}\mathbf{P}\left(\mathbf{r},t\right) \tag{15.44}$$

and

$$\mathbf{J}\left(\mathbf{r},t\right) = \mathbf{J}_{\mathrm{f}}\left(\mathbf{r},t\right) + \operatorname{curl}\,\mathbf{M}\left(\mathbf{r},t\right) + \frac{\partial}{\partial t}\mathbf{P}\left(\mathbf{r},t\right). \tag{15.45}$$

Then, with (15.44) and (15.37), Gauss' Law becomes

$$\boxed{\operatorname{div}\mathbf{D}\left(\mathbf{r},t\right) = \rho_{\mathrm{f}}\left(\mathbf{r},t\right).} \tag{15.46}$$

And, with (15.45), Ampere's Law becomes

$$\operatorname{curl}\left[\frac{1}{\mu_0}\mathbf{B}\left(\mathbf{r},t\right) - \mathbf{M}\left(\mathbf{r},t\right)\right] = \mathbf{J}_{\mathrm{f}}\left(\mathbf{r},t\right) + \frac{\partial}{\partial t}\left[\varepsilon_0\mathbf{E}\left(\mathbf{r},t\right) + \mathbf{P}\left(\mathbf{r},t\right)\right]. \tag{15.47}$$

Then, including (15.38) and (15.37), equation (15.47) is

$$\boxed{\operatorname{curl}\mathbf{H}\left(\mathbf{r},t\right) = \mathbf{J}_{\mathrm{f}}\left(\mathbf{r},t\right) + \partial\mathbf{D}\left(\mathbf{r},t\right)/\partial\mathrm{t}.} \tag{15.48}$$

Faraday's Law and Oersted's Result are unchanged by the presence of matter.

In summary, then Maxwell's Equations in the presence of matter are

$$\boxed{\begin{array}{ll} \operatorname{div}\mathbf{D} = \rho_{\mathrm{f}} & \operatorname{div}\mathbf{B} = 0 \\ \operatorname{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t & \operatorname{curl}\,\mathbf{H} = \mathbf{J}_{\mathrm{f}} + \partial\mathbf{D}/\partial t. \end{array}} \tag{15.49}$$

In integral form these are

$$\boxed{\begin{array}{ll} \oint_S \mathbf{D}\cdot \mathrm{d}\mathbf{S} = \int_V \rho_f \mathrm{d}V & \oint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0 \\ \oint_C \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\int_a \partial \mathbf{B}/\partial t \cdot \mathrm{d}\mathbf{a} & \oint_C \mathbf{H}\cdot \mathrm{d}\boldsymbol{\ell} = \int_V (\mathbf{J}_f + \partial \mathbf{D}/\partial t)\cdot \mathrm{d}\mathbf{a} \end{array}} \tag{15.50}$$

These forms of Maxwell's Equations contain only free charges and currents. The polarization and magnetization are contained in the definitions of **D** and **H**.

## 15.6 Constitutive Equations

### *15.6.1 Polarization*

The polarization of a dielectric $\mathbf{P}(\mathbf{r})$ is related to the electric field inside the dielectric. From our brief discussion of polarization in Sect. 15.2.2 we realize that in a general dielectric crystal the relation may not be simple. The direction of polarization resulting from an electric field will be dependent on the orientation of the axes of the crystal with the electric field. And the relation of the polarization to the electric field may not be linear. In general there is a *constitutive equation* of the form

$$\boxed{\mathbf{P}(\mathbf{r}) = \varepsilon_0 \boldsymbol{\chi}(E)\cdot \mathbf{E},} \tag{15.51}$$

where $\boldsymbol{\chi}(E)$ is the electric susceptibility[2]. The electrical susceptibility is generally a tensor of rank two (see Sect. 13.6). Here we will assume that the dielectric is linear so that $\boldsymbol{\chi}$ is not a function of the electric field. We will also assume that the dielectric is isotropic, so that $\boldsymbol{\chi} = \chi$ is a tensor of rank zero, i.e. a scalar. And we will consider only homogeneous dielectrics and neglect any dependence of $\chi$ on position.

### *15.6.2 Magnetization*

In a similar fashion the magnetization $\mathbf{M}(\mathbf{r})$ will depend on the magnetic field intensity **H** in the magnetic material. Based on our discussion of magnetization in Sect. 15.2.4 we should not expect the dependence of the magnetization on the magnetic field intensity to be simple either. The magnetization will depend on the crystal orientation in the magnetic field. And, in the ferromagnetic case, the

[2] There is some discrepancy in this definition. Some authors write polarization as $\boldsymbol{\chi}\cdot\mathbf{E}$ [83] and others as $\varepsilon_0\boldsymbol{\chi}\cdot\mathbf{E}$ [76]. We choose the latter in keeping with the AIP Handbook [35].

dependence of magnetization on magnetic field intensity is strong and nonlinear. The constitutive equation in magnetizable matter is

$$\boxed{\mathbf{M}(\mathbf{r}) = \boldsymbol{\chi}_\mathrm{M}(H) \cdot \mathbf{H},} \tag{15.52}$$

where $\boldsymbol{\chi}_\mathrm{M}(H)$ is the magnetic susceptibility. We shall assume that the diamagnetic and paramagnetic materials we treat here are isotropic and homogeneous and that $\boldsymbol{\chi}_\mathrm{M}(H) = \chi_\mathrm{M}$ is a constant scalar. We will consider ferromagnetic systems separately below.

### *15.6.3 Permittivity and Permeability*

The susceptibilities are not convenient in themselves. More convenient parameters are the permittivity

$$\varepsilon = \varepsilon_0 (1 + \chi) \tag{15.53}$$

in the dielectric and the permeability

$$\mu = \mu_0 (1 + \chi_\mathrm{M}) \tag{15.54}$$

in magnetizable matter.

Then

$$\mathbf{D} = \varepsilon \mathbf{E}. \tag{15.55}$$

and

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} \tag{15.56}$$

In each case we define dimensionless quantities to indicate the relative polarization and magnetization of materials upon application of electric and magnetic fields. These are the dielectric constant

$$K = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi \tag{15.57}$$

and the relative permeability

$$K_\mathrm{M} = \frac{\mu}{\mu_0} = 1 + \chi_\mathrm{M}. \tag{15.58}$$

## 15.7 Boundary Conditions on Fields

We cannot measure the fields **E**, **D**, **B**, and **H** inside matter. We can obtain these fields from external measurements, however, if we know the conditions that govern the changes in these fields as the boundary from free space (air) into the solid matter is crossed. We can find the general boundary conditions that must be satisfied by the vector fields **E**, **D**, **B**, and **H** upon transition from one substance into another by applying Maxwell's Equations in integral form to infinitesimal Gaussian pillboxes or contours constructed on the boundaries between the substances. One of these substances may be air.

### *15.7.1 Electric Field*

From (15.50) the integral field equations for **D** and **E** are

$$\oint_{\mathrm{S}} \mathbf{D}\cdot \mathrm{d}\mathbf{S} = \int_{\mathrm{V}} \rho_{\mathrm{f}} \mathrm{d}V \tag{15.59}$$

and

$$\oint_{\mathrm{C}} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\int_{\mathrm{a}} \frac{\partial \mathbf{B}}{\partial t}\cdot \mathrm{d}\mathbf{a}. \tag{15.60}$$

In Fig. 15.6a and b we have drawn the boundary between two dielectrics.

On the boundary we have constructed an infinitesimal Gaussian pillbox (panel (a)) and a contour (panel (b)). The pillbox is a right circular cylinder with end caps parallel to the surface. And the contour is a rectangle with horizontal sides parallel (tangential) to the surface. The height of the pillbox and the length of the sections of the contour perpendicular to the surface are vanishingly small.

We may also choose the dimensions of the end caps of the Gaussian pillbox and the sections of the contour parallel to the surface arbitrarily small compared to macroscopic dimensions, such as surface curvatures. Then we may consider that, as far as the pillbox or contour are concerned, the boundary surface is planar.

As we shrink the axial length of the Gaussian pillbox to zero only the surface charge remains within the pillbox and

$$\int_{\mathrm{V}} \rho_{\mathrm{f}} \mathrm{d}V = \sigma_{\mathrm{f}} a_{\mathrm{cap}},$$

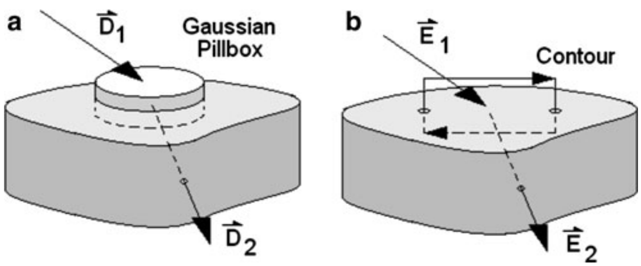


**Fig. 15.6** The surface of a dielectric with (**a**) a Gaussian pillbox having ends within and without the dielectric (**b**) a rectangular contour having sides within and without the dielectric

where $\sigma_f$ is the (net) surface charge density on dielectric boundary and $a_{cap}$ is the area of the end cap of the pillbox.

Applying (15.59) to the Gaussian pillbox in Fig. 15.6 panel (a) we have then

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = (D_{2,n} - D_{1,n})\, a_{cap} = \sigma_f a_{cap}, \tag{15.61}$$

where $D_{2,n}$ and $D_{1,n}$ are the components of the displacement vector normal (perpendicular) to the surface of the end caps. From (15.61) we see that

$$\boxed{D_{2,n} - D_{1,n} = \sigma_f.} \tag{15.62}$$

The difference in the normal components of the displacement vector on two sides of a dielectric is equal to the density of the free surface charge on the boundary. If there is no free charge on the dielectric surface then the normal component of the displacement vector is continuous across a boundary between two dielectrics or between free space and a dielectric.

Since the legs of the contour in Fig. 15.6 panel (b) are vanishingly small, the area enclosed by the contour becomes zero as we shrink the sections of the contour perpendicular to the surface to zero. Then

$$\lim_{a_C\to 0}\int_{a_C} \frac{\partial}{\partial t}\mathbf{B}\cdot \mathbf{da} = \lim_{a_C\to 0} \frac{\partial}{\partial t}\mathbf{B}\cdot n_C a_C = 0 \tag{15.63}$$

where $n_C$ is the unit vector perpendicular to the plane of the contour and $a_C$ is the area enclosed by the contour.

Therefore, even in the presence of a time dependent magnetic field, (15.60) is

$$\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = (E_{2,T} - E_{1,T})\, L_T = 0, \tag{15.64}$$

where $E_{2,T}$ and $E_{1,T}$ are the components of the electric field tangential to the surface and $L_T$ is the length of the tangential sections of the contour. Then

$$\boxed{E_{2,T} = E_{1,T}} \tag{15.65}$$

and the tangential component of the electric field vector is continuous across a boundary between two dielectrics or between free space and a dielectric.

The boundary conditions on the component of the electric field tangent to the surface between dielectrics results in a boundary condition on the electrostatic scalar potential $\varphi$. In Fig. 15.7 we have labeled the end points of the contour of Fig. 15.6 panel (b).

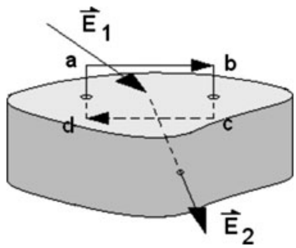


**Fig. 15.7** Boundary conditions on the scalar potential

For arbitrarily small $L_T$

$$E_{1,T}L_T = -(\text{grad}\,\varphi)\,L_T$$
$$= -\frac{\varphi_b - \varphi_a}{ab}L_T = \varphi_a - \varphi_b \tag{15.66}$$

and, similarly,

$$E_{1,T}L_T = \varphi_d - \varphi_c. \tag{15.67}$$

With (15.66) and (15.67) then (15.64) is the requirement that

$$\varphi_a - \varphi_b = \varphi_d - \varphi_c. \tag{15.68}$$

In the limit as the lengths of the contour perpendicular to the surface go to zero the points $a$ and $d$ become a single point $ad$ and $b$ and $c$ become a point $bc$. The equality in (15.68), for any locations of the points $ad$ and $bc$, then requires that $\varphi_a = \varphi_d$ and that $\varphi_b = \varphi_c$.

The electrostatic potential is then constant across a boundary between two dielectrics.

### *15.7.2 Magnetic Field*

From (15.50) the integral field equations for **B** and **H** are

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0 \tag{15.69}$$

and

$$\oint_C \mathbf{H}(\mathbf{r})\cdot d\boldsymbol{\ell} = \int_{a_C} \mathbf{J}_f(\mathbf{r})\cdot d\mathbf{a}, \tag{15.70}$$

where, as above, $a_C$ is the area enclosed by the contour $C$.

In Fig. 15.8a and b we have drawn the boundary between two magnetic materials.

On the boundary we have again constructed an infinitesimal Gaussian pillbox (panel (a)) or contour (panel (b)). As before the pillbox is a right circular cylinder

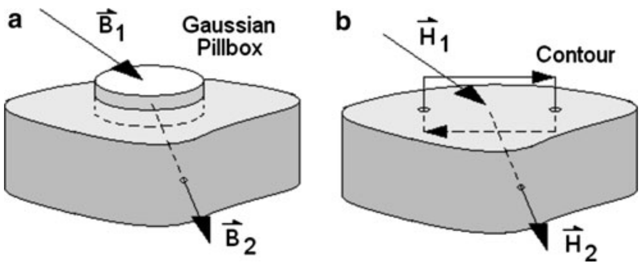


**Fig. 15.8** Boundary conditions on **B** and **H**

with end caps parallel to the surface. And the contour is a rectangle with horizontal sides parallel (tangential) to the surface.

The height of the pillbox and the length of the sections of the contour perpendicular to the surface are vanishingly small. The pillbox and the contour are also very small compared to curvatures in the boundary, as in the case of the dielectric.

Applying (15.69) to the Gaussian pillbox in Fig. 15.8 panel (a) we have

$$\oint_{S} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = (B_{2,\mathrm{n}} - B_{1,\mathrm{n}})\, a_{\mathrm{cap}} = 0, \tag{15.71}$$

where $B_{2,\mathrm{n}}$ and $B_{1,\mathrm{n}}$ are the components of the magnetic field induction vector normal (perpendicular) to the surface and $a_{\mathrm{cap}}$ is the area of the pillbox end cap. From (15.71) we have

$$\boxed{\mathrm{B}_{2,\mathrm{n}} - \mathrm{B}_{1,\mathrm{n}} = 0.} \tag{15.72}$$

Then the normal component of the magnetic field induction vector is continuous across a boundary between two magnetic materials or between free space and a magnetic material.

Since the legs of the contour in Fig. 15.8 panel (b) are vanishingly small, the area enclosed by the contour becomes zero as we shrink the sections of the contour perpendicular to the surface to zero. Then

$$\lim_{\mathrm{a_C}\to 0} \int_{\mathrm{a_C}} \mathbf{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{a} = J_{\mathrm{f,S}} L_{\mathrm{T}} \tag{15.73}$$

where $J_{\mathrm{f,S}}$ is the free surface current density on the boundary and $L_{\mathrm{T}}$ is the length of the section of the contour parallel to the boundary.
Therefore (15.70) is

$$\oint_{C} \mathbf{H}\cdot \mathrm{d}\boldsymbol{\ell} = (H_{2,\mathrm{T}} - H_{1,\mathrm{T}})\, L_{\mathrm{T}} = J_{\mathrm{f,S}} L_{\mathrm{T}}, \tag{15.74}$$

where $H_{2,\mathrm{T}}$ and $H_{1,\mathrm{T}}$ are the components of the magnetic field intensity tangential to the boundary, or

$$\boxed{\mathrm{H}_{2,\mathrm{T}} - \mathrm{H}_{1,\mathrm{T}} = \mathrm{J}_{\mathrm{f,S}},} \tag{15.75}$$

in the presence of a surface current on the boundary.
In the absence of a surface current

$$H_{2,\mathrm{T}} = H_{1,\mathrm{T}} \tag{15.76}$$

and the tangential component of the magnetic field intensity vector is continuous across a boundary between two magnetic materials or between free space and a magnetic material.

We may write the boundary condition (15.75) in vector form. Choosing $\hat{n}_2$ as the normal to the boundary and pointing in the direction of $\mathbf{H}_2$ the vector relationship is

$$\hat{n}_2 \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_{\mathrm{f,S}}. \tag{15.77}$$

## 15.8 Ferromagnetism

A straightforward discussion of the classification of magnetic systems may be found in the text *Physics of Magnetism* by Soshin Chikazumi ([11], pp. 6–19). For the sake of simplicity we have elected to distinguish only among diamagnetic, paramagnetic, and ferromagnetic systems.

Ferromagnetic systems are characterized by spontaneous magnetization, which is a result of the Heisenberg exchange interaction between or among spins. The presence of an exchange interaction is, however, not limited to ferromagnetic materials. The magnetic interactions among the spins of atoms in diluted magnetic semiconductors (DMS) are based on exchange (see e.g. [88]).

In ferromagnetic materials there are domains in which the spins of atoms are aligned in a certain direction (see e.g. [96], p. 317; [76], p. 298; [11], p. 10). In the nonmagnetized state these domains are randomly aligned. When a magnetic field is applied to a ferromagnetic material the domains line up with the field as a result of the contribution (15.2) to the Hamiltonian.

The energy of the interaction among spins at the domain boundaries provides an additional contribution to the Hamiltonian that is very complicated and depends on domain boundary structures.

We can obtain the functional dependence of the magnetic field induction on the magnetic field intensity in a ferromagnetic material experimentally using a Rowland Ring.[3] The Rowland Ring is a toroidal solenoid in which the wire is tightly wrapped

[3]The Rowland Ring was developed by Henry Augustus Rowland (1848–1901).

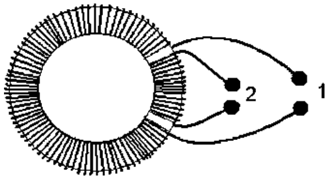


**Fig. 15.9** Rowland Ring. The connections (1) are to the primary coil, which produces the magnetic field in the ring. The connections (2) are to the secondary coil, in which the current is a result of Farady's Law

around a ring made of the material under investigation. In Fig. 15.9 we have drawn the basic form of the Rowland Ring.

In Fig. 15.9 the primary coil is (1) and the secondary coil is (2). The current in the primary coil produces the magnetic field in the ring and the current in the secondary coil results from Faraday's Law. If the radius of the ring is considerably larger than the radius of the cross section we can consider the magnetic field inside the material making up the ring to be uniform.

There is no free charge current density $\mathbf{J}_{\mathrm{f}}$ on the skin of the Rowland ring. And the magnetic field at the outer surface of the ferromagnetic material of the Rowland Ring, produced by the current in the wound wire, is parallel to the surface of the ring. Therefore, from the boundary condition on the magnetic field intensity (15.76), the intensity produced by the primary coil outside of the ferromagnetic material of the ring

$$\mathbf{H}_{\mathrm{out}} = \frac{1}{\mu_0}\mathbf{B}_{\mathrm{out}} \tag{15.78}$$

is equal to the magnetic field intensity $\mathbf{H}_{\mathrm{in}}$ inside the ring.

The magnetic field induction within the ring is, therefore,

$$\begin{aligned}\mathbf{B}_{\mathrm{in}} &= \mu\mathbf{H}_{\mathrm{in}} = \mu\mathbf{H}_{\mathrm{out}} \\ &= \frac{\mu}{\mu_0}\mathbf{B}_{\mathrm{out}} = K_{\mathrm{M}}\mathbf{B}_{\mathrm{out}}.\end{aligned} \tag{15.79}$$

We can, then, measure directly the dependence of the magnetic field induction on magnetic field intensity in the material from which the ring is made. And from this we can find the dependence of the relative permeability $K_{\mathrm{M}}$ on the magnetic field intensity for the material.

In Fig. 15.10 we have plotted the general form of the functional dependence of the magnetic field induction on the magnetic field intensity in a ferromagnetic material.

Fig. 15.10 is the basic form of the magnetization curve for annealed iron.

We can find the permeability of the material graphically using (15.56) and the magnetization curve. In Fig. 15.11 we have plotted the results of a graphical determination of the permeability and the magnetization curve together against the magnetic field intensity.

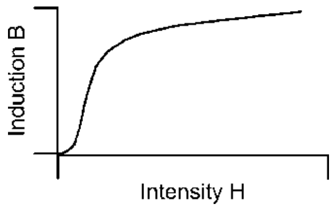


**Fig. 15.10** Magnetization curve of a ferromagnetic material

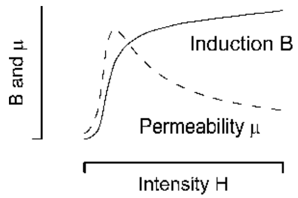


**Fig. 15.11** Magnetic field induction and permeability as functions of magnetic field intensity in a ferromagnetic material

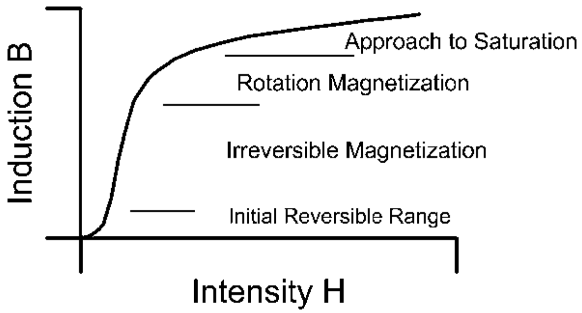


**Fig. 15.12** Magnetization curve of a ferromagnetic material with regions labeled

Chikazumi devotes four chapters of his text to detailed discussions of the magnetization of ferromagnetic systems ([11], pp. 245–302). There are two mechanisms that determine the form of the magnetization curve of a ferromagnetic material. The spins in a particular magnetic domain will rotate in response to the magnetic field intensity. And the domain boundaries will be displaced as the magnetic field intensity is increased. Here we provide only an overview of how these mechanisms affect the form of the magnetization curve.

In Fig. 15.12 we have labeled the regions of the magnetization curve of Fig. 15.10 that can be characterized by the importance of certain mechanisms (see e.g. [11], pp. 245, 6).

The magnetization curve is reversible for low values of the intensity. In this *initial reversible range* the magnetic domains rotate reversibly from the stable directions and the domain boundaries are reversibly displaced. This is the first region indicated in Fig. 15.12.

This initial reversible range is followed by a region in which the slope of the magnetization curve is very steep. This is the *irreversible magnetization range*. In this region the domain boundaries undergo irreversible displacement and there is an irreversible domain rotation, which varies with domain sizes and heterogeneity of the sample. Here there is considerable energy transfer from the magnetic field to the sample. Depending on the rate of magnetization, there may be a rise in temperature of the sample.

In the region of *rotation magnetization* the magnetization is primarily from rotation of spins within domains.

The region indicated as *approach to saturation* is a region also of spin rotation.

When saturation is attained there is only a gradual increase in magnetization with magnetic field intensity and the permeability becomes constant.

### 15.8.1 Hysteresis

In Fig. 15.13 we show the magnetization curve for a ferromagnetic material beginning in an initially demagnetized state and the subsequent demagnetization curve from the saturated state. The arrows on the curves indicate magnetization and demagnetization.

The demagnetization curve will only follow the magnetization curve if we demagnetize the sample in the initial reversible range. If we apply a sufficient magnetic field intensity to the sample to produce irreversible domain boundary displacements and domain rotations then a magnetization of the sample will remain even after we decrease the external magnetic field intensity to zero.

From (15.38) we have

$$\mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{M}(\mathbf{r}) \tag{15.80}$$

when $\mathbf{H} = \mathbf{0}$. Therefore the nonzero value of $\mathbf{B}$ at $\mathbf{H} = \mathbf{0}$ indicates a *residual magnetization* in the sample.

In Fig. 15.13 we have designated the point known as *retentivity* or *remanence* as $r$. At this point the residual magnetization is $B/\mu_0$. This is *the* characteristic property of ferromagnetic materials.

In Fig. 15.13 we have indicated the magnetic field intensity required to bring the magnetic induction in the sample back to zero as $c$. This value of the magnetic field intensity is the *coercive force* or *coercivity*. From (15.38) we see that at the point $c$ we have

$$\mathbf{H}(\mathbf{r}) = -\mathbf{M}(\mathbf{r}) \tag{15.81}$$

when $\mathbf{B} = \mathbf{0}$.

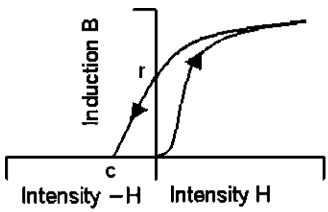


**Fig. 15.13** Hysteresis in a ferromagnetic material. The point $r$ is the remanence and $c$ is the coercivity

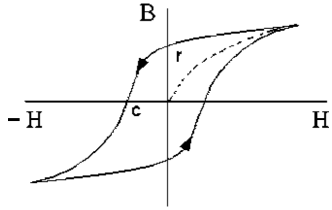


**Fig. 15.14** Hysteresis loop for ferromagnetic material

If we plot the general form of the magnetic field induction $B$ as a function of magnetic field intensity $H$ we have a graph as in Fig. 15.14. In Fig. 15.14 we have shown the initial magnetization curve as a dashed line from $B = H = 0$.

The fact that the sample does not follow the initial magnetization curve as we decrease the magnetic field intensity is called hysteresis. The term hysteresis is from the Greek word $\upsilon\sigma\tau\varepsilon\rho\eta\sigma\iota\zeta$ meaning "deficiency" or "lagging behind."

Based on our discussion in the preceding section we have a fairly solid understanding of the physics of hysteresis in ferromagnets.

### *15.8.2 Modern Directions*

Some important engineering applications, such as in the design of efficient electric motors in vehicles and the production of electric power from wind turbines, depend on permanent magnets of increasing remanence [51].

Magnets are ranked in terms of their *energy product* in $\mathrm{kJ\ m^{-3}}$, which is a combination of how easily the material is magnetized (the magnetization curve) and resistance to demagnetization. Modern high energy product magnets are made of alloys such as iron-aluminum-nickel-cobalt (Alnico) and neodymium-iron-boron (NIB).

The physics of the problem involve the same considerations as our discussion in the preceding section: spin rotation and domain boundary effects. Both of these mechanisms are dependent on crystal structure.

Nanoparticles are becoming important in the research at the time of this writing. The use of nanoparticles alters the boundary effects that we saw are very important in magnetization and in demagnetization.

## 15.9 Summary

In this chapter we have investigated the behavior of electric and magnetic fields in matter and deduced the form of Maxwell's Equations in matter. The field–field interaction, expressed in Faraday's Law and the displacement current are unaffected by the presence of matter. This fact allowed us to introduce the effects of matter based on multipole expansions of the integral solutions to Poisson's Equation into which we introduced an atomic picture of matter.

We based our initial discussion on the macroscopic, experimentally observed effects of introducing dielectric and magnetic matter into capacitors and inductors. We introduced the modern understanding of the microscopic, atomic and molecular picture of matter to produce the final forms of our multipole expansions. This served to clarify the sources of polarization and magnetization in matter.

In the final section of the chapter we discussed ferromagnetism. There we coupled experimental observations with an understanding of the origins of those observations in terms of spin rotations and domain boundary mechanics. The phenomenon of hysteresis is dependent on both of these.

## Exercises

**15.1.** Begin with

$$\mathbf{a} \cdot \int_{\mathrm{V}} \operatorname{curl} \mathbf{v} \mathrm{d}V = \int_{\mathrm{V}} \mathbf{a} \cdot \operatorname{curl} \mathbf{v} \mathrm{d}V$$

for $\mathbf{a} =$ constant vector and

$$\operatorname{div}(\mathbf{v} \times a) = \mathbf{a} \cdot \operatorname{curl} \mathbf{v} - \mathbf{v} \cdot \operatorname{curl} \mathbf{a}$$

to show that

$$\int_{\mathrm{V}} \operatorname{curl} \mathbf{v} \mathrm{d}V = -\oint_{\mathrm{S}} \mathbf{v} \times \mathrm{d}\mathbf{S}.$$

**15.2.** In a dielectric experiment you have a carefully prepared dielectric (permittivity $\varepsilon$) sheet with thickness equal to half the separation you have set for a parallel plate capacitor. The sheet area and the plate area of the capacitor are the same. We have shown the situation in Fig. 15.15.

What is the ratio of the capacitance with and without the dielectric?

**15.3.** You have another sheet of dielectric with permittivity $\varepsilon$ and thickness $y$ that just fits between the plates of your parallel plate capacitor. But then you decide to only push it in halfway. We show this is Fig. 15.16.

You then charge this capacitor.

(a) How is the charge distributed over each half of the plates?
(b) What are the electric fields in the dielectric and in the empty space between the plates?
(c) What is the total capacitance?

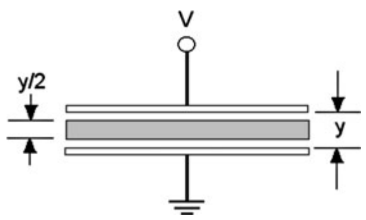


**Fig. 15.15** Capacitor with dielectric sheet

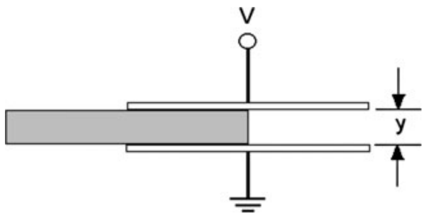


**Fig. 15.16** Capacitor with dielectric inserted halfway

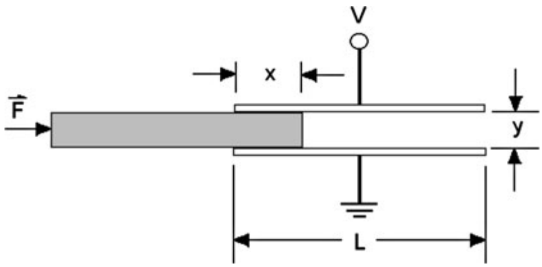


**Fig. 15.17** Inserting a dielectric into a parallel plate capacitor

**15.4.** The capacitance of a parallel plate air capacitor increases if we insert a dielectric with dielectric constant $K$ between the plates. If we *maintain a constant potential* between the plates of the parallel plate capacitor, the energy stored is then increased upon insertion of the dielectric. According to the First Law of Thermodynamics we must then do work on the system to insert the dielectric. We have drawn the situation in Fig. 15.17. The area of the square capacitor pates is $L^2$. The dielectric is a solid rectangular block of thickness $y$ and length and width equal to $L$. In Fig. 15.17 the dielectric block has been inserted a distance $x$ into the space between the capacitor plates. When completely inserted the dielectric block will totally occupy the volume between the capacitor plates.

(a) What is the stored energy in the capacitor as a function of $x$?
(b) If we neglect frictional forces and any temperature change in the dielectric, what is the force required to insert the dielectric as a function of $x$?

**15.5.** Consider a solid dielectric cylinder of radius $a$ and length $L$ made of a material with permittivity $\varepsilon$. You have chosen to define the $z-$axis of your coordinate system to be aligned with the cylinder axis. Let us assume that you are able to establish an electric field oriented in the $\hat{e}_z$ direction which varies as a function of $z$ in such a way that the polarization in the dielectric cylinder is

$$\mathbf{P} = \left(\alpha z^2 + \beta\right) \hat{e}_z.$$

(a) What is the volume polarization charge density within the cylinder?
(b) What is the surface polarization charge density on all surfaces?
(c) What is the total polarization charge in and on the cylinder?

**15.6.** You have a spherical conductor of radius $a$, which you have contained within a spherical dielectric shell. The dielectric material making up the shell has a dielectric constant $K$. You have provided a thin wire to the conductor so that you may charge it to any desired potential. That is you may place an arbitrary free charge on the conducting sphere.

You charge the conductor to $Q$ C.

(a) What is the polarization in the dielectric?
(b) What is the polarization charge density?
(c) What is the surface polarization charge density on the inner and outer surfaces of the dielectric shell?
(d) What is the total polarization charge on the shell?

**15.7.** Assume that in the previous exercise the dielectric shell around the conducting sphere has an electrical susceptibility that is a function of the radial coordinate $\chi = \alpha r$. You again charge the conducting sphere to $Q$ C.

(a) What is then the polarization charge density?
(b) What is the surface polarization charge density on inner and outer surfaces?
(c) What is the total polarization charge in the dielectric?
(d) Does your answer to (c) make physical sense?

**15.8.** You have a sphere of solid dielectric material with dielectric constant $K$ and radius $a$. The sphere is polarized by an external electrostatic field and the resultant polarization inside the sphere is $\mathbf{P} = \mathcal{P}_0 \hat{e}_z$.

(a) What is the polarization charge density inside the sphere?
(b) What is the surface polarization charge density on the sphere?
(c) Show that the total charge on the sphere is zero.

**15.9.** Using the definitions of polarization charge density

$$\rho_p = -\operatorname{div} \mathbf{P}$$

and

$$\sigma_p = \mathbf{P} \cdot \hat{n}$$

show that in general the total polarization charge within and on any closed dielectric body must vanish.

**15.10.** You have hung a small uncharged dielectric sphere of radius $a$ on a thread between two plates of a parallel plate capacitor. When you charge the plates the small dielectric sphere will be suspended in a uniform electric field

$$\mathbf{E} = E_0 \hat{e}_z$$

oriented vertically. the dielectric constant of the material from which the sphere is made is $K$.

You wonder what the polarization charge density is within and on the sphere. You can calculate the polarization charge density from the constitutive relationships in Sect. 15.6 once you have the electric field within the sphere. You can obtain the electric field from the negative gradient of the electrostatic scalar potential, which satisfies Laplace's Equation inside and outside of the sphere.

Fortunately we have already obtained the general solution to Laplace's Equation for spherical geometry in Chap. 9. The result is

$$\varphi(r,\phi,\vartheta) = \sum_{\mathrm{n,m}} \left[D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)}\right] \cdots$$
$$\cdots P_{\mathrm{n}}^{\mathrm{m}}(\cos\phi)\left[A_{\mathrm{m}}\cos(m\vartheta) + B_{\mathrm{m}}\sin(m\vartheta)\right],$$

where $P_{\mathrm{n}}^{\mathrm{m}}(\cos\phi)$ are the *associated Legendre Functions*. For your suspended sphere there will be no dependence on the azimuthal angle $\vartheta$. Then

$$\varphi(r,\phi,\vartheta) = \sum_{\mathrm{n}=0}^{\infty} \left[D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)}\right] P_{\mathrm{n}}(\cos\phi), \tag{15.82}$$

where $P_{\mathrm{n}}(\cos\phi)$ are the Legendre Polynomials. The first few Legendre Polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right) \\ &\vdots \end{aligned}$$

So all you must do is use the limiting requirements and the boundary conditions for the electrostatic scalar potential to find the constants in (15.82).

You may consider that the electric field is equal to $E_0\hat{e}_z$ at $z = \pm\infty$. And you want the potential to remain finite at the origin. In Sect. 15.7 we showed that the electrostatic scalar potential was continuous across boundaries between dielectrics. These observations will give you sufficient information to restrict the number of terms in (15.82) that you need for the solution and to solve for those constants.

What are the terms in (15.82) that you will need for the electrostatic scalar potentials $\varphi_{\mathrm{in}}$ within and $\varphi_{\mathrm{out}}$ in the space surrounding the dielectric sphere?

[Answer: $\varphi_{\mathrm{in}} = D_0^{(\mathrm{in})} + D_1^{(\mathrm{in})} r\cos\phi$, $\varphi_{\mathrm{out}} = D_0^{(\mathrm{out})} + \left[D_1^{(\mathrm{out})} r + G_1^{(\mathrm{out})}\left(\frac{1}{r^2}\right)\right]\cos\phi$]

**15.11.** For the functions you obtained for $\varphi_{\mathrm{in}}$ and $\varphi_{\mathrm{out}}$ in the preceding exercise use the condition that

$$\lim_{\mathrm{r}\to\infty}\varphi = \text{constant} - E_0 z,$$

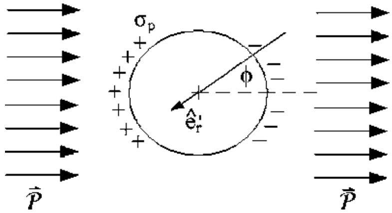


**Fig. 15.18** Cavity left by removal of an atom from a crystal site in a polarized dielectric

the continuity of $\varphi_{\text{in}}$ and $\varphi_{\text{out}}$ on the surface of the sphere, and the continuity of the normal component of the displacement vector to obtain $D_1^{(\text{in})}$, $D_1^{(\text{out})}$, and $G_1^{(\text{out})}$.

Write the potentials.

[Answer: $\varphi_{\text{in}} = V_0 - 3\frac{E_0}{K+2} r\cos\phi$, $\varphi_{\text{out}} = V_0 + \left[-E_0 r + \frac{K-1}{K+2} a^3 E_0 \left(\frac{1}{r^2}\right)\right]\cos\phi$]

**15.12.** To obtain some understanding of the polarization process we consider a simplified model. We assume that if we remove a single atom or molecule from a crystal site in a polarized dielectric that a spherical cavity is left. The polarization of the dielectric is **P**. We assume further that there will be a surface polarization charge on the surface of the cavity resulting from the polarization of the sites surrounding the now empty site. We have drawn the cavity in Fig. 15.18. The angle $\phi$ is measured from the horizontal (dashed) line. In Fig. 15.18 the charge on the infinitesimal spherical surface area d$S$ through which the vector in the direction $\hat{e}_\text{r}'$ passes contains the charge $\sigma_\text{p}\text{d}S$. The differential electrostatic field in the direction $\hat{e}_\text{r}'$ resulting from the surface polarization charge is

$$\text{d}\mathbf{E}_\text{p} = -\hat{e}_\text{r}' \frac{\mathcal{P}\cos\phi}{4\pi\varepsilon_0 a^2}\text{d}S, \tag{15.83}$$

where $a$ is the radius of the spherical cavity.

(a) Show that the sign of the polarization surface charge density charge density is correct as we have shown in Fig. 15.18.
(b) Show that the total field at the empty atomic site (center of the sphere) from the surface polarization is $\mathbf{E}_\text{p} = \mathbf{P}/(3\varepsilon_0)$.
(c) If the externally applied field at the empty atomic site is **E**, what is the total microscopic field at the site?

**15.13.** The polarization of the atom occupying the site atom is proportional to the microscopic field. That is the dipole moment of an atom is

$$\mathbf{p}_\text{d}^{(\text{a})} = \alpha\mathbf{E}_\text{m},$$

where $\alpha$ is the proportionality factor. If there are $n$ atoms per unit volume then the polarization is

$$\mathbf{P} = n\alpha\mathbf{E}_\text{m}.$$

In the preceding exercise we found that

$$\mathbf{E}_{\mathrm{m}} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}.$$

(a) obtain an equation for $\alpha$ as a function of the macroscopic parameters $\varepsilon_0$, $n$, and the dielectric constant $K$ for the dielectric.
(b) Obtain the equation for the electric susceptibility $\chi$ as a function of $\varepsilon_0$, $n$, and $\alpha$. This is called the *Clausius-Mossotti* equation.

**15.14.** A material for which $\mathbf{P} \neq \mathbf{0}$ when the applied field $\mathbf{E} = \mathbf{0}$ is a ferroelectric material. Using

$$\mathbf{P} = n\alpha\mathbf{E}_{\mathrm{m}}.$$

and

$$\mathbf{E}_{\mathrm{m}} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}.$$

(a) Find the condition which, in the context of our simplified theory, $\mathbf{P} \neq \mathbf{0}$ when the applied field $\mathbf{E} = \mathbf{0}$.
(b) Accept that the result you have is approximate. What does the Clausius-Mossotti equation

$$\chi = \frac{\alpha n/\varepsilon_0}{1 - n\alpha/\left(3\varepsilon_0\right)}$$

predict for the electrical susceptibility of a ferroelectric material?

**15.15.** You have a cylindrical shell made of a ferroelectric material. The inner radius of the shell is $a$ and the outer is $b$. the cylinder has a length of $L$. You have chosen a coordinate system with the $z$–axis along the central axis of the cylinder. Because of the form you used for the polarizing electric field the polarization after the removal of the field is

$$\mathbf{P} = \hat{e}_{\mathrm{r}} A \frac{1}{r} + \hat{e}_{\mathrm{z}} B.$$

(a) What is the polarization charge density within the cylinder?
(b) Show that the total polarization charge density on the curved surfaces vanishes.
(c) What is the dipole moment of the cylinder?

**15.16.** You have a long straight wire of radius $a$. You have insulated the wire with a very thin insulating material, the thickness of which you neglect. You have a large number of rings made of ferromagnetic material. The rings have inner radius slightly greater than $a$, to accommodate the insulation on the wire, and outer radius $b$. These rings stack very nicely on top of one another with the long insulated wire down the center. You plan to pass a large current through the central wire, magnetizing the rings. The result will be a collection of Permanent magnetic rings. See Fig. 15.19.

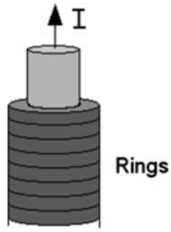


**Fig. 15.19** Magnetizing of rings

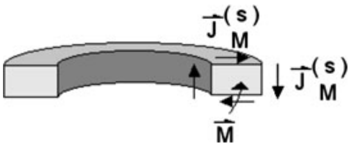

**Fig. 15.20** Cross section of a magnetized ring

The total current in the wire, during the magnetization of the rings, is to be $I$. The rings form a sheath around the wire.

(a) During the magnetization process what is the magnetization current within the sheath made up of the rings?
(b) What is the surface current density on the inner and outer curved surfaces of the sheath?
(c) After the magnetizing current is switched off the magnetization in each ring decreases to a factor of $\lambda$ times what it was during the magnetization process. Find the current densities on the surfaces of the magnetic rings. See Fig. 15.20. Comment on this result and Ampère's theory of permanent magnetism.

**15.17.** If we have a time independent situation in which there are no real current densities then Ampère's Law requires that

$$\operatorname{curl}\mathbf{H} = \operatorname{curl}\mathbf{B} = \mathbf{0}.$$

(a) Show that this means that **H** and **B** can be written as

$$\mathbf{H} = -\operatorname{grad}\varphi_{\mathrm{m}}$$

$$\mathbf{B} = -\mu\operatorname{grad}\varphi_{\mathrm{m}},$$

where $\varphi_{\mathrm{m}}$ is a scalar function that satisfies Laplace's Equation. The function $\varphi_{\mathrm{m}}$ is known as the magnetic scalar potential.
(b) Show that $\varphi_{\mathrm{m}}$ satisfies Laplace's Equation.
(c) Show that $\varphi_{\mathrm{m}}$ is continuous across the boundary between two magnetizable media.

Because the solutions of Laplace's Equation are known (see Chap. 9) this is a useful formulation for treating magnetized systems with no free currents. But we should emphasize that there is only a scalar magnetic potential when free charge current densities are absent.

There is another cautionary note. The fact that $\varphi_{\mathrm{m}}$ satisfies Laplace's equation depends on the spatial independence of the permeability $\mu$, or equivalently on the constancy of the magnetic susceptibility $\chi_{\mathrm{M}}$. The magnetic scalar potential is not a generally useful concept.

**15.18.** As a, perhaps, cautionary example we consider the function

$$\varphi_{\mathrm{m}} = -\frac{I}{2\pi}\vartheta \tag{15.84}$$

in the space surrounding a vertical wire oriented along the $z$–axis in which we have a current $I$. If we consider only the space surrounding the wire we have no current density in the region of interest. The limitations for the applicability of a magnetic scalar potential will, then, be satisfied.

(a) Show that $\mathbf{B} = -\,\mathrm{grad}\,\varphi_{\mathrm{m}}$ results in the correct magnetic field induction in the space around the wire.
(b) Does $\varphi_{\mathrm{m}}$ satisfy Laplace's Equation?
(c) Is equation (15.84) a form of

$$\Phi\left(r,\vartheta\right) = C_0 \ln\left(r\right) + D_0 + \sum_{\mathrm{n=-}\infty}^{\infty} A_{\mathrm{n}} r^{\mathrm{n}} \cos\left(n\vartheta\right) + B_{\mathrm{n}} r^{\mathrm{n}} \sin\left(n\vartheta\right),$$

which in Chap. 9 we showed solved Laplace's Equation?
(d) Comment on the apparent paradox we have here. The function $\varphi_{\mathrm{m}}$ that we proposed works for calculating the magnetic field induction, but it is not a cylindrical harmonic, which is the general solution. What is wrong? You will have to consider carefully our requirements on the separation of variables in Sect. 9.4. We required that

$$-\frac{1}{\Theta_{\mathrm{Cyl}}}\frac{\mathrm{d}^2\Theta_{\mathrm{Cyl}}}{\mathrm{d}\vartheta^2} = \alpha^2 \tag{15.85}$$

where $\alpha$ is a constant. Is this true for (15.84)?

In considering the mathematical aspect of the answer to (d) recall from Sect. 2.6 that the solution of Laplace's Equation is unique.

**15.19.** The magnetic scalar potential promises to be useful particularly in situations involving permanent magnetized material with no free current. We can be more careful in our search for a magnetic scalar potential if we begin with the equation we have for the vector potential in the presence of magnetized material.

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \mathbf{M}'\left(\mathbf{r}\,'\right) \times \mathrm{grad}'\,\frac{1}{\left|\mathbf{r}-\mathbf{r}\,'\right|}\mathrm{d}V', \tag{15.86}$$

where

$$\mathbf{M}' = \mathbf{M}\left(\mathbf{r}\,'\right)$$

Then the magnetic field induction is

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int_{\mathrm{V}} \operatorname{curl}\left[\mathbf{M}' \times \operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] \mathrm{d}V',$$

where we have used

$$\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Then

(a) Using (A.20) show that

$$\operatorname{curl}\left[\mathbf{M}' \times \operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] = \left[\mathbf{M}'\cdot\operatorname{grad}\right]\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{M}'\operatorname{div}\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

(b) And that

$$\left[\mathbf{M}'\cdot\operatorname{grad}\right]\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \operatorname{grad}\left[\mathbf{M}'\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]$$

(c) And finally that

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= -\operatorname{grad}\frac{\mu_0}{4\pi}\int_{\mathrm{V}}\left[\mathbf{M}'\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\mathrm{d}V' \\ &\quad -\frac{\mu_0}{4\pi}\int_{\mathrm{V}}\mathbf{M}'\operatorname{div}\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' \\ &= -\mu_0\operatorname{grad}\frac{1}{4\pi}\int_{\mathrm{V}}\left[\mathbf{M}'\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\mathrm{d}V' \\ &\quad +\mu_0\mathbf{M}\end{aligned}$$

[Hint: seek a $\delta-$function]

This establishes that we can generally write

$$\mathbf{B}(\mathbf{r}) = -\mu_0\operatorname{grad}\varphi_{\mathrm{m}} + \mu_0\mathbf{M}, \tag{15.87}$$

where

$$\varphi_{\mathrm{m}} = \frac{1}{4\pi}\int_{\mathrm{V}}\left[\mathbf{M}'\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]\mathrm{d}V'$$

**15.20.** Using Oersted's Result and the result of the preceding exercise (15.87) show that

$$\nabla^2 \varphi_{\mathrm{m}} = \operatorname{div} \mathbf{M}$$

and that, therefore,

$$\varphi_{\mathrm{m}} = -\frac{1}{4\pi} \int_{\mathrm{V}} \frac{\operatorname{div}' \mathbf{M}'}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'. \tag{15.88}$$

**15.21.** Show that (15.88) becomes

$$\varphi_{\mathrm{m}} = \frac{1}{4\pi} \int_{\mathrm{V}} \left[ \mathbf{M}' \cdot \operatorname{grad}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] \mathrm{d}V'$$

for the space outside of a magnetized region.

**15.22.** As an example that satisfies the requirements for the use of a magnetic scalar potential we choose a uniformly magnetized sphere of radius $a$ in empty space. The magnetization is

$$\begin{aligned} \mathbf{M} &= \mathcal{M}_0 \hat{e}_{\mathrm{z}} \\ &= e_{\mathrm{r}} \mathcal{M}_0 \cos\phi - e_{\phi} \mathcal{M}_0 \sin\phi, \end{aligned}$$

and

$$\operatorname{div} \mathbf{M} = 0.$$

We seek a solution to

$$\nabla^2 \varphi_{\mathrm{m}} = 0$$

in spherical in spherical coordinates with no dependence on the azimuthal angle $\vartheta$.

The general solution is

$$\begin{aligned} \varphi\left(r, \phi, \vartheta\right) &= \sum_{\mathrm{n}=0}^{\infty} \left[ D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)} \right] P_{\mathrm{n}}\left(\cos\phi\right) \\ &= D_0 + G_0 \frac{1}{r} + D_1 r \cos\phi + G_1 \frac{1}{r^2} \cos\phi \\ &\quad + \sum_{\mathrm{n}=2}^{\infty} \left[ D_{\mathrm{n}} r^{\mathrm{n}} + G_{\mathrm{n}} r^{-(\mathrm{n}+1)} \right] P_{\mathrm{n}}\left(\cos\phi\right), \end{aligned}$$

where $P_{\mathrm{n}}\left(\cos\phi\right)$ are the Legendre Polynomials. The first few Legendre Polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \end{aligned}$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$\vdots$$

(a) By considering requirements (e.g. finiteness at the origin and vanishing at infinity) argue the potentials inside and outside of the magnetized sphere must be

$$\varphi_{\mathrm{m}}^{(\mathrm{in})} = D_1^{(\mathrm{in})} r \cos\phi$$

$$\varphi_{\mathrm{m}}^{(\mathrm{out})} = G_1^{(\mathrm{out})} \frac{1}{r^2} \cos\phi.$$

(b) Use the continuity of $\varphi_{\mathrm{m}}$ on the boundary between inside and out to obtain the condition

$$D_1^{(\mathrm{in})} a = G_1^{(\mathrm{out})} \frac{1}{a^2}.$$

(c) Use the continuity of the magnetic field induction across the boundary to obtain the condition [recall (15.87) in magnetized matter]

$$-D_1^{(\mathrm{in})} + \mathcal{M}_0 = 2G_1^{(\mathrm{out})} \frac{1}{a^3}.$$

(d) Solve for $D_1^{(\mathrm{in})}$ and $G_1^{(\mathrm{out})}$ to obtain

$$\varphi_{\mathrm{m}}^{(\mathrm{in})} = \frac{1}{3} M_0 r \cos\phi$$

and

$$\varphi_{\mathrm{m}}^{(\mathrm{out})} = \frac{1}{3} a^3 M_0 \frac{1}{r^2} \cos\phi.$$

(e) Show that the magnetic field intensities are

$$\mathbf{H}_{\mathrm{in}} = -\frac{1}{3} M_0 \hat{e}_{\mathrm{z}}$$

and

$$\mathbf{H}_{\mathrm{out}} = \frac{1}{3} M_0 \frac{a^3}{r^3} \left(\hat{e}_{\mathrm{r}} 2\cos\phi + \hat{e}_\phi \sin\phi\right).$$

And that the magnetic field induction is

$$\mathbf{B}_{\mathrm{in}} = \frac{2}{3} \mu_0 M_0 \hat{e}_{\mathrm{z}}$$

and

$$\mathbf{B}_{\mathrm{out}} = \frac{1}{3} \mu_0 M_0 \frac{a^3}{r^3} \left(\hat{e}_{\mathrm{r}} 2\cos\phi + \hat{e}_\phi \sin\phi\right).$$

# Chapter 16
# Waves in Dispersive Media

*Available energy is energy which we can direct into any desired channel. Dissipated energy is energy which we cannot lay hold of and direct at pleasure, such as the confused agitation of molecules which we call heat.*

*James Clerk Maxwell*

## 16.1 Introduction

In this chapter we consider the propagation of slightly damped waves in nonmagnetic dispersive media. In a dispersive medium electromagnetic wave fields interact with charges and the energy originally present in the electromagnetic fields flows, i.e. is dispersed, to the medium.

The first step in this dispersion process is the transfer of energy from the electromagnetic fields to the kinetic energy of particles moving coherently with the wave. The total wavelike disturbance propagating in the medium then consists of electromagnetic and particle kinetic components. The loss of energy from the particles moving coherently with the wave to the bulk medium causes the dispersion of the wave energy.

Max von Laue first recognized that any description of the total energy of a wave in a medium must consider the motion of particles in 1905 [94]. Specifically von Laue pointed out that the wave energy must involve the energy of the oscillators, which, in 1905, were considered to be the source of electromagnetic energy emitted by matter (see e.g. [54]). The discussion has been taken up in more detail by numerous authors [1,4,5,23,41,42,59,90], and [7]. Bekefi, for example, has shown that neglect of particle energies leads, in some cases, to nonsensical results, such as negative values of total energy in thermal equilibrium.

The majority of these contributions deal solely with the consequences of Maxwell's Equations. The result is a mathematical formula for the total wave energy, proof that this total energy is propagated at the group velocity of the wave,

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and the loss term for the wave energy. To identify the forms of the particle energies, which are coherent with the wave, and the loss mechanism requires a consideration of the particle kinetics.

Except for our verbal description of the basis and some of the recent results of plasma kinetic theory in Chap. 7, we have avoided the kinetic description of matter. Here we shall again only outline the results verbally.

## 16.2 Waves in Matter

In empty space a propagating electromagnetic wave is a transverse plane wave and the dispersion relation is $\omega/k = c$. The dispersion relation in a dispersive medium depends on the capability of the medium to sustain free charge and polarization currents. There are also separate dispersion relations for transverse and longitudinal waves.

### *16.2.1 Representation of Waves*

In Sect. 11.5 we considered waves that were not monochromatic because we realized that we could not produce monochromatic waves, even with lasers, in the laboratory. We are now encountering a situation in which the waves are damped and energy is finally lost to the medium through which the waves pass. These waves are farther from monochromatic than any we can produce in empty space. Therefore we must again resort to a representation in terms of integrals over a continuum of plane waves of the form

$$[\mathbf{E}(\mathbf{k},\omega) \text{ or } \mathbf{B}(\mathbf{k},\omega)] \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,. \tag{16.1}$$

And we have again the representation of the wave fields as Fourier Transforms, which are

$$\mathbf{E}(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^2 \int \mathbf{E}(\mathbf{k},\omega) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\, \mathrm{d}\omega \mathrm{d}^3\mathbf{k} \tag{16.2}$$

and

$$\mathbf{B}(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^2 \int \mathbf{B}(\mathbf{k},\omega) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\, \mathrm{d}\omega \mathrm{d}^3\mathbf{k} \tag{16.3}$$

(see (11.56) and (11.57)).

We may also Fourier Transform the charge and current densities as

$$\rho(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^2 \int \rho(\mathbf{k},\omega) \exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\, \mathrm{d}\omega \mathrm{d}^3\mathbf{k} \tag{16.4}$$

and

$$\mathbf{J}(\mathbf{r},t) = \left(\frac{1}{2\pi}\right)^2 \int \mathbf{J}(\mathbf{k},\omega)\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}\omega\mathrm{d}^3\mathbf{k}. \tag{16.5}$$

### 16.2.2 Dispersion Relation in Matter

In Sect. 11.5 we also showed that the electric and magnetic field components of the plane waves $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{B}(\mathbf{k},\omega)$ satisfy the Fourier Transformed form of Maxwell's Equations, i.e. (11.60), which we repeat here as well for continuity

$$\begin{array}{cc} -i\mathbf{k}\cdot\mathbf{E} = \rho/\varepsilon & \mathbf{k}\cdot\mathbf{B} = 0 \\ \mathbf{k}\times\mathbf{E} = \omega\mathbf{B} & -i\mathbf{k}\times\mathbf{B} = \mu_0(\mathbf{J} + i\omega\varepsilon\mathbf{E}). \end{array} \tag{16.6}$$

In (16.6) the terms $\rho$ and $\mathbf{J}$ are $\rho(\mathbf{k},\omega)$ and $\mathbf{J}(\mathbf{k},\omega)$ in (16.4) and (16.5).

Replacing $\varepsilon_0$ with $\varepsilon$ makes the equations (16.6) applicable to polarizable matter and keeping $\mu_0$ unchanged is appropriate for nonmagnetic matter. If we also assume that the matter being considered satisfies Ohm's Law, i.e.

$$\mathbf{J}(\mathbf{k},\omega) = \sigma(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega), \tag{16.7}$$

then equations (16.6) become

$$\begin{array}{cc} -i\mathbf{k}\cdot\mathbf{E} = \rho/\varepsilon & \mathbf{k}\cdot\mathbf{B} = 0 \\ \mathbf{k}\times\mathbf{E} = \omega\mathbf{B} & -i\mathbf{k}\times\mathbf{B} = \mu_0(\sigma + i\omega\varepsilon)\mathbf{E}, \end{array} \tag{16.8}$$

which are the Fourier Transformed Maxwell Equations in polarizable, nonmagnetic matter in which there are charges that are free to move.

The cross product of $\mathbf{k}$ with either Faraday's or Ampère's Law results, after a few steps of vector algebra, in

$$\boldsymbol{D}(\mathbf{k},\omega)\cdot(\mathbf{E}\text{ or }\mathbf{B}) = \mathbf{0}, \tag{16.9}$$

where

$$\boxed{\boldsymbol{D}(\mathbf{k},\omega)=\left(k^2 - \omega^2 K/c^2\right)\mathbf{1}\text{-}\mathbf{kk}\text{+i}\omega\mu_0\sigma\mathbf{1}.} \tag{16.10}$$

In (16.10) $K$ is the dielectric constant of the matter.

Equation (16.9) is the requirement imposed by Maxwell's Equations on the wave fields. This is the *wave equation* in Fourier $(\mathbf{k},\omega)$ space.

Choosing the propagation direction to be the $z$–axis, we can represent $\boldsymbol{D}(\mathbf{k},\omega)$ in matrix form as

$$\boxed{\boldsymbol{D}\left(\mathbf{k},\omega\right)=\begin{bmatrix} i\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right) & 0 & 0 \\ 0 & i\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right) & 0 \\ 0 & 0 & i\omega\mu_0\sigma-\omega^2K/c^2 \end{bmatrix}} \tag{16.11}$$

Equation (16.9) has a nonzero solution for **E** and **B** only if the determinant of $\boldsymbol{D}\left(\mathbf{k},\omega\right)$ vanishes (see e.g. [20], p. 18).

The determinant of $\boldsymbol{D}\left(\mathbf{k},\omega\right)$ is

$$\det \boldsymbol{D}\left(\mathbf{k},\omega\right)=\left[i\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right)\right]^2\left(i\omega\mu_0\sigma-\omega^2K/c^2\right). \tag{16.12}$$

The general *dispersion relation* is then

$$\left[i\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right)\right]^2\left(i\omega\mu_0\sigma-\omega^2K/c^2\right)=0. \tag{16.13}$$

This dispersion relation specifies $\omega=\omega\left(k\right)$ that must hold for the propagating wave in a nonmagnetic medium.

### *16.2.3 Transverse and Longitudinal Waves*

Only transverse waves are possible in empty space. In matter, however, we have the possibility also of longitudinal waves for which the wave vector **k** and the electric field vector **E** are parallel. For longitudinal waves Faraday's Law requires that $\mathbf{B}=\mathbf{0}$ and there is no magnetic field component associated with a longitudinal wave.

From the Fourier Transform of Maxwell's Equations in (16.8) we see that Gauss' Law forbids the association of any charge density $\rho$ with a transverse wave in matter, for which

$$\mathbf{k}\cdot\left(\mathbf{E}\text{ or }\mathbf{B}\right)=0. \tag{16.14}$$

The assumption of Ohm's Law, however, results in a current density associated with a transverse wave.

Gauss' Law requires a charge density associated with a longitudinal wave in matter, for which

$$\mathbf{k}\times\mathbf{E}=\mathbf{0}. \tag{16.15}$$

and $\mathbf{k}\cdot\mathbf{E}=kE$. And Ampère's Law, or the assumption of Ohm's Law, requires a current density associated with a longitudinal wave in matter.

For transverse waves (16.9) becomes

$$\boldsymbol{D}\left(\mathbf{k},\omega\right)\cdot\left(\mathbf{E}\text{ or }\mathbf{B}\right)=\left[i\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right)\right]\left(\mathbf{E}\text{ or }\mathbf{B}\right)=\mathbf{0} \tag{16.16}$$

and the condition for nonzero **E** and **B** is

$$\boxed{\mathrm{i}\omega\mu_0\sigma+\left(k^2-\omega^2K/c^2\right)=0.} \tag{16.17}$$

For longitudinal waves (16.9) becomes

$$\boldsymbol{D}\,(\mathbf{k},\omega)\cdot\mathbf{E}\ = \omega\left(i\mu_0\sigma - \omega K/c^2\right)\mathbf{E}\ = \mathbf{0}, \tag{16.18}$$

which is simply the product of $\omega$ and Ampère's Law when $\mathbf{B} = \mathbf{0}$. The condition for nonzero $\mathbf{E}$ for the longitudinal wave is then

$$\boxed{\mathrm{i}\mu_0\sigma - \omega\mathrm{K/c}^2 = 0.} \tag{16.19}$$

Equations (16.17) and (16.19) are the requirements that first or the second of the terms in the product in (16.13) vanishes. We can understand the general dispersion relation in equation (16.13) as a product of the dispersion relation for transverse waves (16.17) and the dispersion relation for longitudinal waves (16.19) in nonmagnetic matter. One or the other of these terms must vanish for the wave to propagate.

### *16.2.4 Wave Conductivity*

From (16.17) we have

$$\omega^2 = \frac{c^2}{K}\left(i\omega\mu_0\sigma + k^2\right). \tag{16.20}$$

The wave is undamped and $\omega$ is real if the conductivity is a purely imaginary function of $(\mathbf{k},\omega)$.

From (16.19) we have

$$\omega = i\frac{c^2}{K}\mu_0\sigma. \tag{16.21}$$

And again the wave is undamped if the conductivity is a purely imaginary function of $(\mathbf{k},\omega)$.

We shall designate

$$\sigma\,(\mathbf{k},\omega) = i\sigma_0\,(\mathbf{k},\omega)\,, \tag{16.22}$$

where $\sigma_0\,(\mathbf{k},\omega)$ is a real valued function of $\mathbf{k}$ and $\omega$, as the form of $\sigma\,(\mathbf{k},\omega)$ for both transverse and longitudinal waves. The value of the function $\sigma_0\,(\mathbf{k},\omega)$ is determined by the range of $\mathbf{k}$ and $\omega$ appropriate to either the transverse or longitudinal form of the propagating wave.

### *16.2.5 Wave Energy*

We know from our discussions of energy and momentum in waves (Chap. 12) that the general field energy equation is

$$\frac{\partial}{\partial t}\mathcal{E}_{\mathrm{em}}\,(\mathbf{r},t) + \operatorname{div}\mathbf{S}\,(\mathbf{r},t) = -\mathbf{J}\,(\mathbf{r},t)\cdot\mathbf{E}\,(\mathbf{r},t)\,. \tag{16.23}$$

in which the field energy density in nonmagnetic matter is

$$\mathcal{E}_{\text{em}}(\mathbf{r},t) = \frac{1}{2}\left(\varepsilon E^2(\mathbf{r},t) + \frac{1}{\mu_0}B^2(\mathbf{r},t)\right) \tag{16.24}$$

and the Poynting vector is

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\,. \tag{16.25}$$

Equation (16.23) indicates that the rate of loss of energy from the wave is contained in the term $-\mathbf{J}\cdot\mathbf{E}$, which is the rate of transfer of energy from the fields to the matter.

We shall now consider the consequences of (16.23) when we apply it to waves in dispersive matter. Our goal is to gain an understanding of the total energy of a wave in dispersive matter and the details of the energy transport from the waves to the particles.

## 16.3 Nearly Monochromatic Waves

### *16.3.1 Dispersion of Monochromatic Waves*

We have represented the general wave in a dispersive medium as an integral over monochromatic waves, each of which must satisfy the dispersion relation for the medium. The monochromatic wave is a mathematical fiction in that we cannot produce it in the laboratory. But it is the basis of our representation of the actual laboratory wave. We may then study the energy transport from a single monochromatic wave to obtain a general understanding of the physics.

In our study we will assume that each monochromatic wave is slightly damped and slightly dispersed. That is we assume that both the angular frequency $\omega$ and the wave vector $\mathbf{k}$ have small imaginary parts so that the wave exists in the general form of (16.1) for a large number of wave periods $\tau_{\text{wave}} = 2\pi/\omega_{\text{r}}$ or wave lengths $\lambda_{\text{wave}} = 2\pi/k_{\text{r}}$, where $\omega_{\text{r}}$ is the real part of the angular frequency and $k_{\text{r}}$ is the magnitude of the real part of the wave vector.

Specifically we consider that $\omega_{\text{i}} \ll \omega_{\text{r}} = \operatorname{Re}\omega$ and $|\mathbf{k}_{\text{i}}| \ll k_{\text{r}} = |\operatorname{Re}\mathbf{k}|$. If this were not the case we could not logically consider the disturbance to be a wave.

The real parts of $\omega$ and $\mathbf{k}$ corresponding to the undamped and undispersed wave will satisfy the general dispersion relation (16.13) with $\sigma = i\sigma_0$. These may be either transverse or longitudinal waves with the distinction based on whether the real values of $\omega$ and $\mathbf{k}$ satisfy (16.20) or (16.21).

To illustrate the situation we are considering we have drawn a wave for which $|\mathbf{k}_{\text{i}}| = 0.01\,|\mathbf{k}_{\text{r}}|$ in Fig. 16.1.

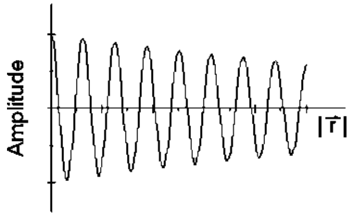


**Fig. 16.1** Slightly dispersed wave with $|\mathbf{k}_\mathrm{i}| = 0.01\,|\mathbf{k}|$

The dispersion factor $k_\mathrm{i} = 0.01 k_\mathrm{r}$, which does not satisfy $k_\mathrm{i} \ll k_\mathrm{r}$, has been chosen only so that the exponential envelope can be easily seen in Fig. 16.1.

The electric and magnetic field components of the (almost) monochromatic wave will have the general forms

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{E}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) + \mathbf{E}^*\exp\left(-i\omega^* t + i\mathbf{k}^*\cdot\mathbf{r}\right)\right] \tag{16.26}$$

and

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{B}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r}) + \mathbf{B}^*\exp\left(-i\omega^* t + i\mathbf{k}^*\cdot\mathbf{r}\right)\right], \tag{16.27}$$

or $\mathbf{B}(\mathbf{r},t) = \mathbf{0}$ for the longitudinal wave. Ohm's Law results in the same space and time dependence for the current density and Gauss' Law results in the same space and time dependence for the charge density.

In the Fourier Transforms (16.2–16.5) the terms $\mathbf{E}(\mathbf{k},\omega)\ldots\mathbf{J}(\mathbf{k},\omega)$ have definite magnitudes determined by the requirement that the functions $\mathbf{E}(\mathbf{r},t)\ldots\mathbf{J}(\mathbf{r},t)$ are well represented. That is, for example, $\mathbf{E}(\mathbf{k},\omega)$ is actually a function of $\mathbf{k}$ and $\omega$.

When we are considering plane waves, such as those in (16.26) and (16.27), however, the vectors $\mathbf{E}$ and $\mathbf{B}$ of the right hand sides of (16.26) and (16.27) have magnitudes determined by $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, but they are not functions of $(\mathbf{k},\omega)$.

### *16.3.2 Time and Space Averages*

To study energy damping and dispersion we want to obtain the form of (16.23) that is dependent on the imaginary parts of the wave vector $\mathbf{k}$ and the angular frequency $\omega$. Because the terms in (16.23) are quadratic in the vectors, this is most easily accomplished by averaging (16.23) over a time $T$ equal to an arbitrary number of wave periods $\tau_\mathrm{wave}$ and over a length $L$ equal to a number of wavelengths $\lambda_\mathrm{wave}$

along the direction $\hat{k}$ of propagation of the wave. We are interested then in terms such as

$$\left\langle \frac{\partial}{\partial t} E\left(\mathbf{r}, t\right)^2 \right\rangle_{\mathrm{T,L}} = \frac{1}{L}\frac{1}{T}\int_0^{\mathrm{T}} \mathrm{d}t \int_0^{\mathrm{L}} \mathrm{d}r_{\mathrm{k}} \frac{\partial}{\partial t} E\left(\mathbf{r}, t\right)^2, \tag{16.28}$$

where $r_{\mathrm{k}}$ is the spatial coordinate in the direction of wave propagation.

This time and space averaging is simply a mathematical device to eliminate the oscillatory behavior of the equations. We can, however, argue that this approach is logical physically.

The wave period is normally very short compared to the variation of any thermodynamic properties of the matter. And measurements we make of absorption and dispersion of wave energy are normally performed over times long compared to the period using probes with dimensions large compared to the wavelength of the wave. The actual experimental measurements are then time and space averages.

Using (16.26) we have

$$\begin{aligned} \frac{\partial}{\partial t} E\left(\mathbf{r}, t\right)^2 = \frac{2i\omega}{4} & \left[\mathbf{E}\cdot\mathbf{E}\exp\left(2i\omega t - 2i\mathbf{k}\cdot\mathbf{r}\right)\right. \\ & \left. - \mathbf{E}^*\cdot\mathbf{E}^*\exp\left(-2i\omega^* t + 2i\mathbf{k}^*\cdot\mathbf{r}\right)\right] \\ & -\frac{2\omega_{\mathrm{i}}}{4}\left(\mathbf{E}^*\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{E}^*\right)\exp\left(-2\omega_{\mathrm{i}} t - 2\mathbf{k}_{\mathrm{i}}\cdot\mathbf{r}\right). \end{aligned} \tag{16.29}$$

The time average involves only the exponential terms. For the first two terms on the right hand side of (16.29) we have

$$\begin{aligned} & \left\langle \exp\left(2i\omega t - 2i\mathbf{k}\cdot\mathbf{r}\right)\right\rangle_{\mathrm{T,L}} \text{ or } \left\langle \exp\left(-2i\omega^* t + 2i\mathbf{k}^*\cdot\mathbf{r}\right)\right\rangle_{\mathrm{T,L}} \\ & = \frac{1}{LT}\int_0^{\mathrm{T}} \mathrm{d}t \int_0^{\mathrm{L}} \mathrm{d}r_{\mathrm{k}} \exp\left(2i\omega t - 2i\mathbf{k}\cdot\mathbf{r}\right) \text{ or } \exp\left(-2i\omega^* t + 2i\mathbf{k}^*\cdot\mathbf{r}\right) \\ & = 0 \end{aligned}$$

for $\omega_{\mathrm{i}}/\omega_{\mathrm{r}} \ll 1$ and $\left|\mathbf{k}_{\mathrm{i}}\right| / \left|\mathbf{k}_{\mathrm{r}}\right| \ll 1$.

We will designate the time and space average of the last term on the right hand side of (16.29) as

$$\begin{aligned} \left\langle \exp\left(-2\omega_{\mathrm{i}} t - 2\mathbf{k}_{\mathrm{i}}\cdot\mathbf{r}\right)\right\rangle_{\mathrm{T,L}} &= \frac{1}{LT}\int_0^{\mathrm{T}} \mathrm{d}t \int_0^{\mathrm{L}} \mathrm{d}r_{\mathrm{k}} \exp\left(-2\omega_{\mathrm{i}} t - 2\mathbf{k}_{\mathrm{i}}\cdot\mathbf{r}\right) \\ &= \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \end{aligned} \tag{16.30}$$

Then

$$\left\langle \frac{\partial}{\partial t} E(\mathbf{r}, t)^2 \right\rangle_{\mathrm{T,L}} = -2\omega_{\mathrm{i}} \frac{1}{4} \left(2E^2\right) \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \tag{16.31}$$

Similarly

$$\left\langle \frac{\partial}{\partial t} B(\mathbf{r}, t)^2 \right\rangle_{\mathrm{T,L}} = -2\omega_{\mathrm{i}} \frac{1}{4} \left(2B^2\right) \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \tag{16.32}$$

In (16.31) $E^2 = \mathbf{E} \cdot \mathbf{E}^*$ and in (16.32) $B^2 = \mathbf{B} \cdot \mathbf{B}^*$.

In a like manner the part of the cross product in the Poynting Vector that will survive the time and space average is

$$\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{4}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) \exp\left[-2\omega_{\mathrm{i}} t + 2\mathbf{k}_{\mathrm{i}} \cdot \mathbf{r}\right]. \tag{16.33}$$

The divergence of (16.33) is

$$\operatorname{div}(\mathbf{E} \times \mathbf{B}) = 2\mathbf{k}_{\mathrm{i}} \cdot \frac{1}{4}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) \exp\left[-2\omega_{\mathrm{i}} t + 2\mathbf{k}_{\mathrm{i}} \cdot \mathbf{r}\right]. \tag{16.34}$$

Performing the time and space average of (16.34)

$$\langle \operatorname{div}(\mathbf{E} \times \mathbf{B}) \rangle_{\mathrm{T,L}} = 2\mathbf{k}_{\mathrm{i}} \cdot \frac{1}{4}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \tag{16.35}$$

And the time and space average of the term $\mathbf{J} \cdot \mathbf{E}$ is

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle_{\mathrm{T,L}} = \frac{1}{4}\left(\mathbf{J}^* \cdot \mathbf{E} + \mathbf{J} \cdot \mathbf{E}^*\right) \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \tag{16.36}$$

With Ohm's Law (16.36) becomes

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle_{\mathrm{T,L}} = \frac{1}{4}\left(\sigma^* + \sigma\right) \mathbf{E} \cdot \mathbf{E}^* \Phi\left(\omega_{\mathrm{i}}, k_{\mathrm{i}}, T\right). \tag{16.37}$$

where

$$\sigma = \sigma(\mathbf{k}, \omega) \text{ and } \sigma^* = \sigma^*\left(\mathbf{k}^*, \omega^*\right)$$

In the succeeding discussion we will simply write $\omega_{\mathrm{r}} = \omega$ and $\mathbf{k}_{\mathrm{r}} = \mathbf{k}$.

### *16.3.3 Field Energy*

With (16.31), (16.32), (16.35), and (16.37) the time and space average of the energy transport (16.23) becomes

$$-2\omega_{\mathrm{i}}\left[\varepsilon E^2 + \frac{1}{\mu_0} B^2\right] + 2\vec{k}_{\mathrm{i}} \cdot \left[\frac{1}{\mu_0}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right)\right]$$
$$= -\left(\sigma^* + \sigma\right) E^2, \tag{16.38}$$

since the common factor

$$\frac{1}{4}\Phi\left(\omega_{\rm i}, k_{\rm i}, T\right) \neq 0.$$

Equation (16.38) is the time and space averaged field energy equation for the wave.

In the presence of slight damping we may expand $\sigma\ (\mathbf{k}, \omega)$ around the real values of $\mathbf{k}$ and $\omega$ and the form of the conductivity $\sigma = i\sigma_0$ at propagation. The result is

$$\sigma = (i\sigma_0) + i\omega_{\rm i}\frac{\partial}{\partial\omega}(i\sigma_0) + i\mathbf{k}_{\rm i}\cdot\frac{\partial}{\partial\mathbf{k}}(i\sigma_0) + \delta\sigma, \tag{16.39}$$

where we have used the shorthand notation $\text{grad}_{\mathbf{k}} = \partial/\partial\mathbf{k}$ and all the derivatives are evaluated at the undamped condition. The term $\delta\sigma$ is the change in the structure of $\sigma$ due to damping and dispersion. The form of this change is dependent upon the kinetic description of the matter.

With (16.39) equation (16.38) becomes

$$\sigma^* + \sigma = -2\omega_{\rm i}\frac{\partial}{\partial\omega}(\sigma_0) - 2\mathbf{k}_{\rm i}\cdot\frac{\partial}{\partial\mathbf{k}}(\sigma_0) + 2\delta\sigma. \tag{16.40}$$

Then using (16.40) in equation (16.38) results in

$$\begin{aligned} &-2\omega_{\rm i}\left[\varepsilon E^2 + \frac{1}{\mu_0}B^2\right] \\ &+2\mathbf{k}_{\rm i}\cdot\left[\frac{1}{\mu_0}\left(\mathbf{E}^*\times\mathbf{B} + \mathbf{E}\times\mathbf{B}^*\right)\right] \\ &= 2\omega_{\rm i}\frac{\partial\sigma_0}{\partial\omega}E^2 + 2\mathbf{k}_{\rm i}\cdot\frac{\partial\sigma_0}{\partial\mathbf{k}}E^2 - 2\delta\sigma E^2. \end{aligned} \tag{16.41}$$

Combining the terms in (16.41) we have

$$\begin{aligned} &-2\omega_{\rm i}\left[\varepsilon E^2 + \frac{1}{\mu_0}B^2 + \frac{\partial\sigma_0}{\partial\omega}E^2\right] \\ &+2\,\mathbf{k}_{\rm i}\cdot\left[\frac{1}{\mu_0}\left(\mathbf{E}^*\mathbf{B} + \mathbf{E}\times\mathbf{B}^*\right) - \frac{\partial\sigma_0}{\partial\mathbf{k}}E^2\right] \\ &= -2\delta\sigma E^2 \end{aligned} \tag{16.42}$$

We will able to simplify (16.42) if we find equations for $B^2$ and $\mathbf{E}^*\times\mathbf{B}+\mathbf{E}\times\mathbf{B}^*$, which involve only the electric field vector $\mathbf{E}$.

From Faraday's law in (16.8) we have

$$\mathbf{B}^*\cdot(\mathbf{k}\times\mathbf{E}) = \omega B^2. \tag{16.43}$$

And from Ampère's law, at the propagation condition (real $\mathbf{k}$ and $\omega$) we have

$$\mathbf{k} \times \mathbf{B}^* = -\mu_0 \sigma_0 \mathbf{E}^* - K \frac{\omega}{c^2} \mathbf{E}^* \tag{16.44}$$

Then

$$\mathbf{E} \cdot \left( \mathbf{k} \times \mathbf{B}^* \right) = -\mu_0 \sigma_0 E^2 - K \frac{\omega}{c^2} E^2. \tag{16.45}$$

Exchanging the dot and the cross in the scalar triple product in (16.45) results in

$$\begin{aligned} \mathbf{E} \cdot \left( \mathbf{k} \times \mathbf{B}^* \right) &= (\mathbf{E} \times \mathbf{k}) \cdot \mathbf{B}^* \\ &= -\mathbf{B}^* \cdot (\mathbf{k} \times \mathbf{E}) . \end{aligned}$$

Then (16.45) becomes

$$\mathbf{B}^* \cdot (\mathbf{k} \times \mathbf{E}) = \mu_0 \sigma_0 E^2 + K \frac{\omega}{c^2} E^2. \tag{16.46}$$

Using (16.43) in (16.46) we have

$$B^2 = \left( \frac{\mu_0}{\omega} \sigma_0 + \frac{K}{c^2} \right) E^2. \tag{16.47}$$

Then equation (16.42) becomes

$$\begin{aligned} &-2\omega_{\mathrm{i}} \left[ 2K\varepsilon_0 + \frac{\sigma_0}{\omega} + \frac{\partial \sigma_0}{\partial \omega} \right] E^2 \\ &+2\,\mathbf{k}_{\mathrm{i}} \cdot \left[ \frac{1}{\mu_0} \left( \mathbf{E}^* \mathbf{B} + \mathbf{E} \times \mathbf{B}^* \right) - \frac{\partial \sigma_0}{\partial \mathbf{k}} E^2 \right] \\ = &-2\delta\sigma E^2 \end{aligned} \tag{16.48}$$

To obtain the second term on the left hand side of (16.48) in a form involving only $E^2$ is more involved. From Faraday's Law

$$\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^* = \frac{1}{\omega} \left[ 2\mathbf{k}E^2 - \left( \mathbf{E}^* \cdot \mathbf{k} \right) \mathbf{E} - (\mathbf{E} \cdot \mathbf{k})\, \mathbf{E}^* \right]. \tag{16.49}$$

Since $E^2$ is not a function of $\mathbf{k}$,

$$\frac{\partial}{\partial \mathbf{k}} \left( E^2 k^2 \right) = 2E^2 \mathbf{k} \tag{16.50}$$

and

$$\frac{\partial}{\partial \mathbf{k}} (\mathbf{E} \cdot \mathbf{k}) \left( \mathbf{E}^* \cdot \mathbf{k} \right) = \left( \mathbf{E}^* \cdot \mathbf{k} \right) \mathbf{E} + (\mathbf{E} \cdot \mathbf{k})\, \mathbf{E}^*. \tag{16.51}$$

Then

$$\frac{1}{\mu_0}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) = \frac{1}{\mu_0\omega}\frac{\partial}{\partial\,\mathbf{k}}\left[E^2k^2 - (\mathbf{k}\cdot\mathbf{E})\left(\mathbf{E}^*\cdot\mathbf{k}\right)\right]$$
$$= \frac{1}{\mu_0\omega}\frac{\partial}{\partial\mathbf{k}}\left[\mathbf{E}\cdot\left(\mathbf{1}k^2 - \mathbf{k}\,\mathbf{k}\right)\cdot\mathbf{E}^*\right]. \qquad (16.52)$$

From (16.10) we have

$$k^2\mathbf{1} - \mathbf{kk} = \boldsymbol{D}\,(\mathbf{k},\omega) + \left(\omega\mu_0\sigma_0 + \frac{\omega^2}{c^2}K\right)\mathbf{1}. \qquad (16.53)$$

Then, using (16.9), equation (16.52) becomes

$$\frac{1}{\mu_0}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) = \frac{1}{\mu_0\omega}\frac{\partial}{\partial\,\mathbf{k}}\left[\left(\omega\mu_0\sigma_0 + \frac{\omega^2}{c^2}K\right)E^2\right] \qquad (16.54)$$

Since $\omega$ depends on $\mathbf{k}$ and properties of the medium, carrying out the $\mathbf{k}-$gradient in (16.54) results in

$$\frac{1}{\mu_0}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) = \left[\left(2K\varepsilon_0 + \frac{\sigma_0}{\omega} + \frac{\partial\sigma_0}{\partial\omega}\right)\frac{\partial\omega}{\partial\mathbf{k}} + \frac{\partial\sigma_0}{\partial\mathbf{k}}\right]E^2. \qquad (16.55)$$

Then the second line in (16.48) becomes

$$2\,\mathbf{k}_\mathrm{i}\cdot\left[\frac{1}{\mu_0}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right) - \frac{\partial\sigma_0}{\partial\,\mathbf{k}}E^2\right]$$
$$= 2\,\mathbf{k}_\mathrm{i}\cdot\left[\left(2K\varepsilon_0 + \frac{\sigma_0}{\omega} + \frac{\partial\sigma_0}{\partial\omega}\right)E^2\right]\frac{\partial\omega}{\partial\mathbf{k}}, \qquad (16.56)$$

With (16.56) equation (16.48) becomes

$$-2\omega_\mathrm{i}\left[2K\varepsilon_0 + \frac{\sigma_0}{\omega} + \frac{\partial\sigma_0}{\partial\omega}\right]E^2$$
$$+2\,\mathbf{k}_\mathrm{i}\cdot\left[\left(2K\varepsilon_0 + \frac{\sigma_0}{\omega} + \frac{\partial\sigma_0}{\partial\omega}\right)E^2\right]\frac{\partial\omega}{\partial\mathbf{k}}$$
$$= -2\delta\sigma E^2. \qquad (16.57)$$

From (16.31), (16.32), and (16.35) we know that the factors $(-2\omega_\mathrm{i})$ and $(2\,\mathbf{k}_\mathrm{i})$ result respectively from the time and space average of the rate of change and divergence of quadratic wave quantities.

The first line in (16.57) is then $\langle\partial\mathcal{E}_\mathrm{wave}/\partial t\rangle_\mathrm{T,L}$, where

$$\boxed{\langle \mathcal{E}_{\text{wave}} \rangle_{\text{T,L}} = [2K\varepsilon_0 + \sigma_0/\omega + \partial\sigma_0/\partial\omega]\text{E}^2} \tag{16.58}$$

is the time and space average of the wave energy $\mathcal{E}_{\text{wave}}$ in matter.

The second line in (16.57) is $\langle \text{div}\,\mathbf{S}_{\text{wave}} \rangle_{\text{T,L}}$, where

$$\boxed{\langle \mathbf{S}_{\text{wave}} \rangle_{\text{T,L}} = \left[(2K\varepsilon_0 + \sigma_0/\omega + \partial\sigma_0/\partial\omega)\, E^2\right] \partial\omega/\partial\mathbf{k}} \tag{16.59}$$

is the time and space average of the flux of $\mathcal{E}_{\text{wave}}$.

And the third line in (16.57) is then the time and space average of the loss of $\mathcal{E}_{\text{wave}}$ to another form of the energy. Because total energy (electromagnetic field plus particle energy) is conserved the loss term $-2\delta\sigma E^2$ must logically represent the transport of wave energy to the particles.

From (16.59) we see that

$$\boxed{\langle \mathbf{S}_{\text{wave}} \rangle_{\text{T,L}} = \langle \mathcal{E}_{\text{wave}} \rangle_{\text{T,L}}\, \partial\omega/\partial\mathbf{k}.} \tag{16.60}$$

The energy $\mathcal{E}_{\text{wave}}$ is then transported at a velocity

$$\boxed{\text{v}_{\text{group}} = \partial\omega/\partial\mathbf{k},} \tag{16.61}$$

which is the *group velocity* of the wave.

From our derivation of (16.57) we realize that only $(2K\varepsilon_0 + \sigma_0/\omega)\, E^2$ is the field energy in the wave. We have, however, no understanding yet of the energy $(\partial\sigma_0/\partial\omega)\, E^2$.

Traditionally this is identified as the particle energy coherent with the wave. That must be true because this term is quadratic in the space and time dependence of the wave and it vanishes if $\sigma_0 = 0$. But recognizing this does not serve to identify the portion of the particle energy involved.

We also must ask for the meaning of the loss term $-2\delta\sigma E^2$. How is the energy transported to the particles?

These questions are considered in detail in [42] for a plasma.

As we pointed out in Chap. 15 a treatment of the motion of charged particles in condensed matter requires quantum mechanics. We shall, therefore, content ourselves with the treatment of a plasma.

### *16.3.4 Particle Energy*

In Chap. 7 Sect. 7.5 we introduced a Klimontovich level description of a plasma. If we ensemble average the Klimontovich level equations we obtain a kinetic description of the plasma [23].

At the ensemble averaged level we can identify macroscopic quantities such as particle densities $\langle N^{(\alpha)}\rangle$, particle velocities $\langle \mathbf{v}^{(\alpha)}\rangle$, and particle kinetic energies $\langle T^{(\alpha)}\rangle$, for the $\alpha$th species of particle in the plasma. The brackets $\langle\cdots\rangle$ with no subscripts indicate ensemble average. These macroscopic, ensemble averaged quantities are functions of spatial coordinates and the time.

In kinetic theory we traditionally separate the velocity $\mathbf{v}$ of particles of the $\alpha$th species into what is called the peculiar velocity $\mathbf{V}^{(\alpha)}$ and the average velocity $\langle \mathbf{v}^{(\alpha)}\rangle$ according to

$$\mathbf{V}^{(\alpha)} = \mathbf{v} - \left\langle \mathbf{v}^{(\alpha)}\right\rangle. \tag{16.62}$$

We can then separate the kinetic energy into a hydrodynamic kinetic energy of the $\alpha^{\text{th}}$ species of particle, defined as

$$T_{\text{hydro}}^{(\alpha)} = \frac{1}{2} m^{(\alpha)} \left\langle N^{(\alpha)}\right\rangle \left\langle \mathbf{v}^{(\alpha)}\right\rangle \cdot \left\langle \mathbf{v}^{(\alpha)}\right\rangle \tag{16.63}$$

and a thermal kinetic energy, or simply thermal energy,

$$T_{\text{thermal}}^{(\alpha)} = \frac{1}{2} m^{(\alpha)} \left\langle N^{(\alpha)}\right\rangle \left\langle \mathbf{V}^{(\alpha)} \cdot \mathbf{V}^{(\alpha)}\right\rangle. \tag{16.64}$$

When the average velocity results from a wave perturbation on the background plasma, with properties which vary on longer time and space scales than the waves, then both $T_{\text{hydro}}^{(\alpha)}$ and $T_{\text{thermal}}^{(\alpha)}$ contain contributions that are second order in wave quantities.

There will then be contributions to $\left\langle \partial T_{\text{hydro}}^{(\alpha)}/\partial t\right\rangle_{\text{T,L}}$ and $\left\langle \partial T_{\text{thermal}}^{(\alpha)}/\partial t\right\rangle_{\text{T,L}}$ that result in functions proportional to $\Phi\left(\omega_{\text{i}}, k_{\text{i}}, T\right)$, as we found in our treatment of the field equations.

We must simply acknowledge that the traditional separation of the particle kinetic energy into $T_{\text{hydro}}^{(\alpha)}$ and $T_{\text{thermal}}^{(\alpha)}$ does not neatly separate wavelike contributions from background contributions.

Our time and space average does, however, permit us to separate particle kinetic energy terms from changes in the state of the background plasma, which involve longer space and time scales. The result is that the coherent particle energy term appearing in $\mathcal{E}_{\text{wave}}$ is

$$\frac{\partial \sigma_0}{\partial \omega} E^2 = \sum_{\alpha} \left( \left\langle T_{\text{hydro}}^{(\alpha)}\right\rangle_{\text{T,L}} + \left\langle T_{\text{thermal}}^{(\alpha)}\right\rangle_{\text{T,L}} \right). \tag{16.65}$$

We then identify $\mathcal{E}_{\text{wave}}$ as the total energy in the (almost) monochromatic wave, which is the sum of the electromagnetic field energy and the coherent particle energy.

The changes in the state of the background plasma can be separated into contributions to the particle density and to the contribution of the $\alpha^{\text{th}}$ particle species to the electrical conductivity $\sigma^{(\alpha)}$, which vary on time and space scales long compared to those of the wave. We can finally identify the loss term $-2\delta\sigma E^2$ in (16.57) as a change in the state of the background plasma.

The absorption process of the electromagnetic wave energy in a plasma then follows a logical path beginning with the excitation of a particle motion coherent with the wave, which is not separable into traditional hydrodynamic and thermal contributions. This energy then passes to the background plasma, changing the particle density and temperature.

The transport of the coherent particle energy to the heating of the background matter results as this coherence is lost. This is a result of shearing forces in the plasma [42].

## 16.4 Note on Group Velocity

We consider an electromagnetic pulse moving in a dispersive medium. The pulse, which we take to have a finite spatial extent, is not a plane wave, but we may use a Fourier representation of the pulse in terms of waves of the form

$$\exp\left[i\omega\left(k\right)t - ikx\right]. \tag{16.66}$$

We will write the wave pulse in terms of waves (16.66) for a range of wave vectors ranging between $k_0 - \Delta k$ and $k_0 + \Delta k$ where $k_0$ is real and $\Delta k \ll k_0$. We define $\omega_0 = \omega\left(k_0\right)$. The properties of the dispersive medium and the form (transverse or longitudinal) of the wave will determine the function $\omega\left(k\right)$.

We then represent the electric field vector as the real part of

$$E\left(x,t\right) = \frac{1}{\sqrt{2\pi}}\int_{\mathrm{k_0}\text{-}\Delta\mathrm{k}}^{\mathrm{k_0}+\Delta\mathrm{k}} \mathrm{d}k C\left(k\right)\exp\left[i\omega\left(k\right)t - ikx\right]. \tag{16.67}$$

The amplitudes of the waves will vary only slightly if the dispersion is small. So we may take $C\left(k\right) \approx C\left(k_0\right)$.

Because the frequency differs only slightly from $\omega_0$, we write a Taylor series for $\omega\left(k\right)$ and hold only first order terms.

$$\omega\left(k\right) \approx \omega_0 + \left.\frac{\partial\omega}{\partial k}\right|_0 \left(k - k_0\right), \tag{16.68}$$

where the derivative in (16.68) is evaluated at $k = k_0$. Then the waves in (16.66) can be written as

$$\begin{aligned} &\exp\left[i\omega\left(k\right)t - ikx\right] \\ &\approx \exp\left[i\omega_0 t - ik_0 x\right]\exp\left[i\left.\frac{\partial\omega}{\partial k}\right|_0 \left(k - k_0\right) - i\left(k - k_0\right)x\right]. \end{aligned} \tag{16.69}$$

We began with a set of carrier waves, which are the plane waves in the integrand in (16.67). The approximation in (16.69) limits us, however, to a single carrier wave, whose complex exponential expression is $\exp[i\omega_0 t - ik_0 x]$. This will make our final result more easily understood. But we must acknowledge the approximation.

The integral for the electric field in (16.67) then has the form

$$E(x,t) \approx \operatorname{Re} C(k_0) \exp[i\omega_0 t - ik_0 x] \cdots$$
$$\cdots \int_{-\Delta k}^{\Delta k} d\xi \exp\left[i v_{\text{group}}^{(0)} t - ix\right] \xi, \tag{16.70}$$

where we have introduced the dummy of integration $\xi = k - k_0$ and

$$v_{\text{group}}^{(0)} = \left. \frac{\partial \omega}{\partial k} \right|_0 \tag{16.71}$$

is the *group velocity* for the pulse we are considering. The group velocity is a consequence of the dispersion relation for plane waves propagating in the medium. That is, it is a property of the medium and of the character (transverse or longitudinal) of the propagating plane waves.

The integral in (16.70) results in

$$\int_{-\Delta k}^{\Delta k} d\xi \exp\left[i v_{\text{group}}^{(0)} t - ix\right] \xi$$
$$= 2\Delta k \frac{1}{\Delta k \left[v_{\text{group}}^{(0)} t - x\right]} \sin\left[\Delta k \left(v_{\text{group}}^{(0)} t - x\right)\right]. \tag{16.72}$$

With (16.72) our representation of the electric field (16.70) is

$$E(x,t) \approx \frac{2}{\left[v_{\text{group}}^{(0)} t - x\right]} \sin\left[\Delta k \left(v_{\text{group}}^{(0)} t - x\right)\right] \cdots$$
$$\cdots \operatorname{Re} C(k_0) \exp\left[k_0 \left(i v_{\text{phase}}^{(0)} t - ix\right)\right], \tag{16.73}$$

where

$$v_{\text{phase}}^{(0)} = \frac{\omega_0}{k_0}$$

is the *phase velocity* of the carrier wave.

In Fig. 16.2 we have plotted (16.73) for the initial time $t = 0$ and a subsequent time $t > 0$.

From Fig. 16.2 we see that both the pulse and the carrier wave move to the right. The phase velocity $v_{\text{phase}}^{(0)}$ of the carrier wave exceeds the pulse velocity $v_{\text{group}}^{(0)}$. But the carrier wave itself does not transport the energy.

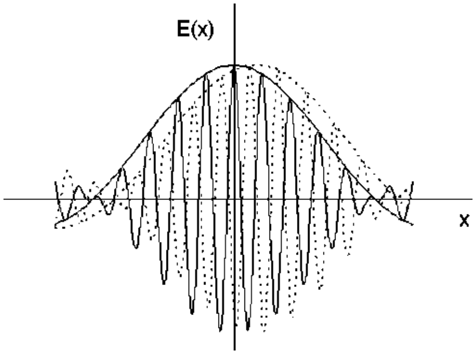


**Fig. 16.2** The envelope and carrier wave for a finite wave pulse in a dispersive medium plotted for time $t = 0$ (*solid*) and $t > 0$ (*dotted*)

## 16.5 Application

To apply what we have learned in this chapter to the motion of waves and the transport of energy in dispersive media we must identify the electrical conductivity for the medium. This requires having a kinetic description for the medium from which we may calculate the velocities of the charge carriers in the presence of a field associated with a propagating electromagnetic wave.

When we have the (wavelike) velocities of the charge carriers $\mathbf{v}^{(\alpha)}$ we can obtain the electrical current from each species of charge carrier as

$$\mathbf{J}^{(\alpha)} = N^{(\alpha)} Q_\alpha \mathbf{v}^{(\alpha)} + n^{(\alpha)} Q_\alpha \mathbf{u}^{(\alpha)} \tag{16.74}$$

In (16.74) the quantities $\mathbf{J}^{(\alpha)}$, $v^{(\alpha)}$, and the particle density $n^{(\alpha)}$ are wave quantities with time and space dependence of the form $\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})$. The particle density $N^{(\alpha)}$ is the background, unperturbed density of the particle species $\alpha$ and $\mathbf{u}^{(\alpha)}$ is a possible streaming velocity of the particles of species $\alpha$, which is not a wavelike quantity.

The current density in (16.74) is then the first order contribution in wavelike quantities arising from the presence of a propagating wave in the medium. The first order wavelike current density in (16.74) will be proportional to the wave fields and can be written, using Maxwell's Equations, as proportional to the electric field of the wave. That is

$$\mathbf{J}^{(\alpha)} = \sigma^{(\alpha)} \mathbf{E} \tag{16.75}$$

At the propagation condition we have relationships between the wave vector $\mathbf{k}$ and the electric field component $\mathbf{E}$ for either longitudinal or transverse waves and for their orientation with respect to streaming velocities and possible magnetic fields. We can then obtain the appropriate form of the conductivity for the undamped, propagating condition for a particular wave. We identify this as $i\sigma_0^{(\alpha)}$ for the species $\alpha$. For the plasma

$$\sigma_0 = \sum_\alpha \sigma_0^{(\alpha)}$$

With the electrical conductivity we are then in a position to calculate the field and particle energies in the wave. With the help of (16.58) we have the field energy

$$\mathcal{E}_{\text{field}} = \left(2\varepsilon + \frac{\sigma_0}{\omega}\right) E^2, \tag{16.76}$$

in the transverse wave or

$$\mathcal{E}_{\text{field}} = (\varepsilon)\, E^2 \tag{16.77}$$

in the longitudinal wave. And we have the particle energy

$$\mathcal{E}_{\text{particle}} = \left[\frac{\partial \sigma_0}{\partial \omega}\right] E^2 \tag{16.78}$$

in the wave.

We have limited our discussion to a plasma in order to avoid entering into discussions of the quantum theory. These would simply take us beyond the scope of this text.

We can simplify our treatment of the plasma in order to obtain equations that are tractable for exercises by considering what is called a cold plasma. In the cold plasma the ions are consider stationary (their temperature is then 0 K).

The electrons are then the only charge carriers free to move.We then drop the designation $\alpha$ for the separate species. We consider further that the electrons have a dominant background density, which is uniform in space and Maxwellian in velocities. The background electron density is $N$ and the perturbed, wavelike contribution to the density is $n$.

We assume that there is also a constant external magnetic field $\mathbf{B}_0$ present in the plasma and define a cyclotron frequency vector

$$\boldsymbol{\Omega} = \frac{Q\mathbf{B}_0}{m}. \tag{16.79}$$

And we assume that the electrons have a streaming velocity $\mathbf{u}$.

We consider that the plasma is perturbed by an almost monochromatic, propagating wave. Electric field associated with the propagating wave is of the form (16.26), which serves to define the vector $\mathbf{E}$.

The wavelike component of the perturbed electron velocity, which has the time and space dependence of the wave, is ([74], p. 120)

$$\begin{aligned}\mathbf{v} = -\frac{Q}{m}\frac{1}{\Omega^2 - (\omega - \mathbf{k}\cdot\mathbf{u})^2}&\left\{-i\,(\omega - \mathbf{k}\cdot\mathbf{u})\left[\mathbf{E} + \frac{1}{\omega}\mathbf{u}\times(\mathbf{k}\times\mathbf{E})\right]\right.\\ &+\boldsymbol{\Omega}\times\left[\mathbf{E} + \frac{1}{\omega}\mathbf{u}\times(\mathbf{k}\times\mathbf{E})\right]\\ &\left.+i\frac{\boldsymbol{\Omega}\boldsymbol{\Omega}}{(\omega - \mathbf{k}\cdot\mathbf{u})}\cdot\left[\mathbf{E} + \frac{1}{\omega}\mathbf{u}\times(\mathbf{k}\times\mathbf{E})\right]\right\}.\end{aligned} \tag{16.80}$$

where $m$ is the electron mass and the expansion has been carried out to first order in the wave quantities.

Conservation of charge in this perturbed, cold plasma is

$$i\omega n - iN\mathbf{k}\cdot\mathbf{v} - in\mathbf{k}\cdot\mathbf{u} = 0$$

from which the wavelike number density is

$$n = \frac{N\mathbf{k}\cdot\mathbf{v}}{(\omega - \mathbf{k}\cdot\mathbf{u})} \tag{16.81}$$

From (16.74) is

$$\mathbf{J} = NQ\mathbf{v} + Q\frac{N\mathbf{k}\cdot\mathbf{v}}{(\omega - \mathbf{k}\cdot\mathbf{u})}\mathbf{u}. \tag{16.82}$$

Using (16.75) we can identify the electrical conductivity and calculate the field and particle energies in the wave.

The character of the wave we consider, whether longitudinal or transverse, and the properties of the plasma supporting the wave will enter the dispersion relation for the wave, equation (16.20) or (16.21).

A useful parameter in considering electron waves in a plasma is the electron plasma frequency

$$\omega_{\mathrm{p,e}} = \sqrt{\frac{NQ^2}{\varepsilon_0 m}}. \tag{16.83}$$

If the ions are allowed to move in the model, then there will be a corresponding ion plasma frequency with ion variables replacing the electron variables in (16.83).

## 16.6 Summary

In this chapter we have taken up a detailed picture of the transport of electromagnetic energy in dispersive media. To avoid the quantum mechanical treatment of the charge motion that would be necessary in the solid state of matter, we conducted our final treatment for a plasma. This provided us insight into the form of the coherent particle energy associated with the wave and the mechanism by which the coherent energy is finally lost to the background plasma.

The mathematical form of the expression for the total wave energy, however, emerges from Maxwell's Equations and requires no particle kinetic description. Therefore, we can still claim that there is a coherent particle kinetic energy associated with a wave moving through solid matter. We are simply lacking an understanding of what part of the particle kinetic energy is the coherent energy.

We recognize that in a solid the loss of energy from a wave to the background state is a loss to the vibrational energy of the crystal. This is an electron-phonon scattering (see e.g. [47], p. 175).

In Appendix J we present a more general derivation of the total wave energy density and transport. The derivation there does not consider an explicit time and space average and holds the tensor character of the conductivity. The physics remains, however, unchanged for the plasma.

## Exercises

**16.1.** We consider a simplification of the cold plasma. We set $\mathbf{\Omega} = \mathbf{u} = \mathbf{0}$. Under these circumstances,

(a) What is the electrical conductivity of the plasma?
(b) For the longitudinal wave find the wave frequency, the coherent particle energy associated with the wave, and the total wave energy.
(c) For the transverse wave find the wave frequency, the group velocity (and show it is $< c$), the coherent particle energy associated with the wave, and the total wave energy.

**16.2.** Consider now a cold plasma with $\mathbf{u} \neq \mathbf{0}$ but still with $\mathbf{\Omega} = \mathbf{0}$. And choose the direction of propagation to be along the stream velocity $\mathbf{u}$. Then $\mathbf{k} \cdot \mathbf{u} = ku$.

(a) What is the electrical conductivity of the plasma for the longitudinal wave? For the transverse wave?
(b) For the longitudinal wave find the wave frequency, The coherent particle energy associated with the wave, and the total wave energy.
(c) For the transverse wave find the wave frequency, the group velocity (and show it is $< c$), the coherent particle energy associated with the wave, and the total wave energy.

**16.3.** Consider now a cold plasma with $\mathbf{u} \neq \mathbf{0}$ and with $\mathbf{\Omega} \neq \mathbf{0}$. Choose $\mathbf{u} \| \mathbf{\Omega}$ and the direction of propagation of the wave to be along the stream velocity $\mathbf{u}$. Then $\mathbf{k} \cdot \mathbf{u} = ku$. Consider only longitudinal waves.

a) What is the electrical conductivity of the plasma for the longitudinal wave?
b) For the longitudinal wave find the wave frequency, the coherent particle energy associated with the wave, and the total wave energy.

# Appendix A
# Vector Calculus

## A.1 Differential Operators

Rectangular Coordinates

$$\mathrm{grad}\,\Phi = \hat{e}_{\mathrm{x}}\frac{\partial \Phi}{\partial x} + \hat{e}_{\mathrm{y}}\frac{\partial \Phi}{\partial y} + \hat{e}_{\mathrm{z}}\frac{\partial \Phi}{\partial z} \tag{A.1}$$

$$\mathrm{div}\,\mathbf{F} = \frac{\partial F_{\mathrm{x}}}{\partial x} + \frac{\partial F_{\mathrm{y}}}{\partial y} + \frac{\partial F_{\mathrm{z}}}{\partial z} \tag{A.2}$$

$$\begin{aligned}\mathrm{curl}\,\mathbf{F} &= \hat{e}_{\mathrm{x}}\left(\frac{\partial F_{\mathrm{z}}}{\partial y} - \frac{\partial F_{\mathrm{y}}}{\partial z}\right) + \hat{e}_{\mathrm{y}}\left(\frac{\partial F_{\mathrm{x}}}{\partial z} - \frac{\partial F_{\mathrm{z}}}{\partial x}\right) \\ &\quad + \hat{e}_{\mathrm{z}}\left(\frac{\partial F_{\mathrm{y}}}{\partial x} - \frac{\partial F_{\mathrm{x}}}{\partial y}\right)\end{aligned} \tag{A.3}$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}. \tag{A.4}$$

Cylindrical Coordinates

$$\mathrm{grad}\,\Phi = \hat{e}_{\mathrm{r}}\frac{\partial \Phi}{\partial r} + \hat{e}_{\vartheta}\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta} + \hat{e}_{\mathrm{z}}\frac{\partial \Phi}{\partial z} \tag{A.5}$$

$$\mathrm{div}\,\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_{\mathrm{r}}\right) + \frac{1}{r}\frac{\partial F_{\vartheta}}{\partial \vartheta} + \frac{\partial F_{\mathrm{z}}}{\partial z} \tag{A.6}$$

$$\begin{aligned}\mathrm{curl}\,\mathbf{F} &= \hat{e}_{\mathrm{r}}\left[\frac{1}{r}\frac{\partial F_{\mathrm{z}}}{\partial \vartheta} - \frac{\partial F_{\vartheta}}{\partial z}\right] + \hat{e}_{\vartheta}\left[\frac{\partial F_{\mathrm{r}}}{\partial z} - \frac{\partial F_{\mathrm{z}}}{\partial r}\right] \\ &\quad + \hat{e}_{\mathrm{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_{\vartheta}\right) - \frac{\partial F_{\mathrm{r}}}{\partial \vartheta}\right]\end{aligned} \tag{A.7}$$

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3, © Springer-Verlag Berlin Heidelberg 2012

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\vartheta^2} + \frac{\partial^2\Phi}{\partial z^2}. \tag{A.8}$$

Spherical Coordinates

$$\operatorname{grad}\Phi = \hat{e}_\mathrm{r}\frac{\partial\Phi}{\partial r} + \hat{e}_\vartheta\frac{1}{r\sin\phi}\frac{\partial\Phi}{\partial\vartheta} + \hat{e}_\phi\frac{1}{r}\frac{\partial\Phi}{\partial\phi} \tag{A.9}$$

$$\operatorname{div}\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_\mathrm{r}\right) + \frac{1}{r\sin\phi}\frac{\partial F_\vartheta}{\partial\vartheta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(F_\phi\sin\phi\right) \tag{A.10}$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \hat{e}_\mathrm{r}\frac{1}{r\sin\phi}\left[\frac{\partial}{\partial\phi}(F_\vartheta\sin\phi) - \frac{\partial F_\phi}{\partial\vartheta}\right] \\ &+ \hat{e}_\vartheta\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\phi\right) - \frac{\partial F_\mathrm{r}}{\partial\phi}\right] \\ &+ \hat{e}_\phi\frac{1}{r}\left[\frac{1}{\sin\phi}\frac{\partial F_\mathrm{r}}{\partial\vartheta} - \frac{\partial}{\partial r}(rF_\vartheta)\right]\end{aligned} \tag{A.11}$$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2\Phi}{\partial\vartheta^2} \\ &+ \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\Phi}{\partial\phi}\right).\end{aligned} \tag{A.12}$$

## A.2 Differential Operator Identities

$$\operatorname{div}\operatorname{grad}\Phi = \nabla^2\Phi \tag{A.13}$$

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0 \tag{A.14}$$

$$\operatorname{curl}\operatorname{grad}\Phi = \mathbf{0} \tag{A.15}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F} \tag{A.16}$$

$$\operatorname{grad}(\Phi\Psi) = \Psi\operatorname{grad}\Phi + \Phi\operatorname{grad}\Psi \tag{A.17}$$

$$\begin{aligned}\operatorname{grad}(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\operatorname{grad})\,\mathbf{G} + \mathbf{F}\times(\operatorname{curl}\mathbf{G}) \\ &+ (\mathbf{G}\cdot\operatorname{grad})\,\mathbf{F} + \mathbf{G}\times(\operatorname{curl}\mathbf{F})\end{aligned} \tag{A.18}$$

$$\operatorname{div}(\Phi\mathbf{F}) = (\operatorname{grad}\Phi)\cdot\mathbf{F} + \Phi\operatorname{div}\mathbf{F} \tag{A.19}$$

$$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = (\operatorname{curl}\mathbf{F})\cdot\mathbf{G} - (\operatorname{curl}\mathbf{G})\cdot\mathbf{F} \tag{A.20}$$

$$\operatorname{curl}(\Phi\mathbf{F}) = (\operatorname{grad}\Phi)\times\mathbf{F} + \Phi\operatorname{curl}\mathbf{F} \tag{A.21}$$

$$\begin{aligned}\operatorname{curl}(\mathbf{F}\times\mathbf{G}) &= (\operatorname{div}\mathbf{G})\,\mathbf{F} - (\operatorname{div}\mathbf{F})\,\mathbf{G} \\ &+ (\mathbf{G}\cdot\operatorname{grad})\,\mathbf{F} - (\mathbf{F}\cdot\operatorname{grad})\,\mathbf{G}\end{aligned} \tag{A.22}$$

# Appendix B
# Dirac Delta Sequences

Here are some examples of $\delta-$ sequences. The most useful in physics are the first three.

(a.)

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \exp\left[ik\left(x-x'\right)\right] = \delta\left(x-x'\right) \tag{B.1}$$

(b.)

$$\left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}^3\mathbf{k} \exp\left[i\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)\right] = \delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{B.2}$$

(c.)

$$\operatorname{div}\operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right) = -4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{B.3}$$

(d.)

$$\lim_{a\to 0}\left(\frac{1}{\pi^{1/2}\alpha}\right)e^{-\frac{x^2}{a^2}} = \delta\left(x\right) \tag{B.4}$$

(e.)

$$\lim_{a\to 0}\left(\frac{\alpha}{\pi}\right)\frac{\sin^2\left(\frac{x}{\alpha}\right)}{x^2} = \delta\left(x\right) \tag{B.5}$$

(f.)

$$\lim_{a\to 0}\left(\frac{\alpha}{\pi}\right)\frac{1}{\left(x^2+\alpha^2\right)} = \delta\left(x\right) \tag{B.6}$$

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# Appendix C
# Divergence and Curl of B

We begin with the integral form of the magnetic field (5.47)

$$\mathbf{B}\left(\mathbf{r}\right) = \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}\,'\right|}\right) \times \mathbf{J}'\mathrm{d}V' \tag{C.1}$$

Using (A.20) the divergence of (C.1) is

$$\begin{aligned} \operatorname{div}\mathbf{B}\left(\mathbf{r}\right) &= \frac{\mu_0}{4\pi}\int_{\mathrm{V}} \mathbf{J}' \cdot \operatorname{curl}\operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right)\mathrm{d}V' \\ &= 0, \end{aligned} \tag{C.2}$$

since $\operatorname{curl}\operatorname{grad} \equiv \mathbf{0}$. Therefore

$$\operatorname{div}\mathbf{B} = 0 \tag{C.3}$$

Taking the curl of $\mathbf{B}$ with respect to the field coordinates, we have

$$\operatorname{curl}\mathbf{B}\left(\mathbf{r}\right) = -\frac{\mu_0}{4\pi}\int_{\mathrm{V},} d\,V'\operatorname{curl}\left[\mathbf{J}' \times \operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right)\right] \tag{C.4}$$

Since $\mathbf{J}'$ is independent of $\mathbf{r}$, using (A.22) we have

$$\begin{aligned} &-\operatorname{curl}\left[\mathbf{J}' \times \operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right)\right] \\ &= -\mathbf{J}'\operatorname{div}\operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right) + \left(\mathbf{J}'\cdot\operatorname{grad}\right)\operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right) \\ &= 4\pi\mathbf{J}'\delta\left(\mathbf{r}-\mathbf{r}'\right) - \left(\mathbf{J}'\cdot\operatorname{grad}'\right)\operatorname{grad}\left(\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\right). \end{aligned} \tag{C.5}$$

Where we have used (2.110) and the fact that $\operatorname{grad} f\left(\mathbf{r}-\mathbf{r}'\right) = -\operatorname{grad}' f\left(\mathbf{r}-\mathbf{r}'\right)$.

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Now, we note that

$$\begin{aligned}\mathrm{div}'\left[\mathbf{J}'\,\mathrm{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] &= \left(\mathrm{div}'\,\mathbf{J}'\right)\mathrm{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\\ &\quad+\left(\mathbf{J}'\cdot\mathrm{grad}'\right)\mathrm{grad}\left(\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\right)\\ &= \left(\mathbf{J}'\cdot\mathrm{grad}'\right)\mathrm{grad}\left(\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\right),\end{aligned} \tag{C.6}$$

since, in the static situation

$$\mathrm{div}'\,\mathbf{J}' = 0. \tag{C.7}$$

Then (C.5) becomes

$$-\mathrm{curl}\left[\mathbf{J}'\times\mathrm{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] = 4\pi\mathbf{J}'\delta\left(\mathbf{r}-\mathbf{r}'\right)-\mathrm{div}'\left[\mathbf{J}'\,\mathrm{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \tag{C.8}$$

and

$$\begin{aligned}\mathrm{curl}\;\mathbf{B}\left(\mathbf{r}\right) &= \mu_0\int_{\mathrm{V}}\mathbf{J}'\delta\left(\mathbf{r}'-\mathbf{r}\right)\mathrm{d}V' - \frac{\mu_0}{4\pi}\int_{\mathrm{V}}\mathrm{div}'\left[\mathbf{J}'\,\mathrm{grad}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right]\mathrm{d}V'\\ &= \mu_0\mathbf{J}\left(\mathbf{r}\right) - \frac{\mu_0}{4\pi}\sum_{\alpha=1}^{3}\hat{e}_\alpha\int_{\mathrm{V}}\mathrm{div}'\left[\mathbf{J}'G_\alpha\left(\mathbf{r},\mathbf{r}'\right)\right]\mathrm{d}V'.\end{aligned} \tag{C.9}$$

where

$$G_\alpha\left(\mathbf{r},\mathbf{r}'\right) = \frac{\partial}{\partial x_\alpha}\left(\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\right) \tag{C.10}$$

is a scalar. Applying Gauss' Theorem,

$$\int_{\mathrm{V}}\mathrm{div}'\left[\mathbf{J}'G_\alpha\left(\mathbf{r},\mathbf{r}'\right)\right]\mathrm{d}V' = \oint_{\mathrm{S}}\mathbf{J}'G_\alpha\left(\mathbf{r},\mathbf{r}'\right)\cdot\mathrm{d}\mathbf{S}'. \tag{C.11}$$

But here $S$ is the surface of the conductor carrying the current density $\mathbf{J}'$. The current density vector is parallel to this surface everywhere. Therefore $\mathbf{J}'G_\alpha\left(\mathbf{r},\mathbf{r}'\right)\cdot\mathrm{d}\mathbf{S}' = 0$ everywhere and

$$\mathrm{curl}\;\mathbf{B}\left(\mathbf{r}\right) = \mu_0\mathbf{J}\left(\mathbf{r}\right). \tag{C.12}$$

# Appendix D
# Green's Theorem

Green's Theorem is a valuable integral theorem involving analytic functions.

**Theorem D.1.** ***Green's Theorem***. *If $\Phi$ and $\Psi$ are analytic everywhere within $V$ then*

$$\int_V \left[\Phi \bigtriangledown^2 \Psi - \Psi \bigtriangledown^2 \Phi\right] dV = \oint_S \left[\Phi \operatorname{grad} \Psi - \Psi \operatorname{grad} \Phi\right] \cdot d\mathbf{S}$$

*Proof.* The divergence of the product $\Phi\mathbf{F}$ is

$$\operatorname{div} \Phi\mathbf{F} = \Phi \operatorname{div} \mathbf{F} + \operatorname{grad} \Phi \cdot \mathbf{F}. \tag{D.1}$$

With $\mathbf{F} = \operatorname{grad} \Psi$ we then have

$$\Phi \bigtriangledown^2 \Psi = \operatorname{div}(\Phi \operatorname{grad} \Psi) - \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi,$$

and

$$\Psi \bigtriangledown^2 \Phi = \operatorname{div}(\Psi \operatorname{grad} \Phi) - \operatorname{grad} \Psi \cdot \operatorname{grad} \Phi.$$

Then, using Gauss' Theorem

$$\begin{aligned} \int_{\mathrm{V}} \left[\Phi \bigtriangledown^2 \Psi - \Psi \bigtriangledown^2 \Phi\right] \mathrm{d}V &= \int_{\mathrm{V}} \operatorname{div}\left[\Phi \operatorname{grad} \Psi - \Psi \operatorname{grad} \Phi\right] \mathrm{d}V \\ &= \oint_{\mathrm{S}} \left[\Phi \operatorname{grad} \Psi - \Psi \operatorname{grad} \Phi\right] \cdot \mathrm{d}\mathbf{S}. \end{aligned}$$

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# Appendix E
# Laplace's Equation

**Theorem E.1.** *If $\Phi_1$ and $\Phi_2$ are solutions of Laplace's equation, then $(a\Phi_1 + b\Phi_2)$ is as well.*

*Proof.* Since the Laplacian is a linear operator

$$\begin{aligned} \nabla^2 (a\Phi_1 + b\Phi_2) &= a \nabla^2 \Phi_1 + b \nabla^2 \Phi_2 \\ &= 0. \end{aligned}$$

**Theorem E.2.** *If $\nabla^2\Phi = 0$ in a region $V$ and $\Phi = 0$ on the surface $S$ of $V$, then $\Phi = 0$ everywhere in $V$.*

*Proof.* Since $\nabla^2\Phi = 0$,

$$\Phi \nabla^2 \Phi = 0,$$

and

$$\int_V \Phi \nabla^2 \Phi \mathrm{d}V = 0.$$

From (D.1)

$$\Phi \nabla^2 \Phi = \mathrm{div}\,(\Phi\, \mathrm{grad}\, \Phi) - \mathrm{grad}\, \Phi \cdot \mathrm{grad}\, \Phi = 0.$$

by hypothesis. Then

$$\begin{aligned} 0 &= \int_V [\mathrm{div}\,(\Phi\, \mathrm{grad}\, \Phi) - \mathrm{grad}\, \Phi \cdot \mathrm{grad}\, \Phi]\, \mathrm{d}V \\ &= \oint_S (\Phi\, \mathrm{grad}\, \Phi) \cdot \mathrm{d}\mathbf{S} - \int_V |\mathrm{grad}\, \Phi|^2\, \mathrm{d}V \qquad (E.1) \end{aligned}$$

using Gauss' Theorem.

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Since $\Phi = 0$ on $S$,

$$\oint_S (\Phi \operatorname{grad} \Phi) \cdot \mathrm{d}\mathbf{S} = 0,$$

and

$$\int_V |\operatorname{grad} \Phi|^2 \, \mathrm{d}V = 0 \tag{E.2}$$

Now $|\operatorname{grad} \Phi|^2 \geq 0$. Therefore, in order for the volume integral (E.2) to vanish,

$$\operatorname{grad} \Phi = 0$$

in $V$. That is $\Phi =$ constant in $V$. But $\Phi = 0$ on the boundary. Therefore $\Phi = 0$ in $V$.

**Corollary E.1.** *If* $\nabla^2 \Phi = 0$ *in a region* $V$ *and* $\partial\Phi/\partial n = 0$ *on the surface* $S$ *of* $V$, *then* $\Phi =$ *constant in* $V$.

*Proof.* As in the proof of *Theorem* E.2,

$$0 = \oint_S (\Phi \operatorname{grad} \Phi) \cdot \mathrm{d}\mathbf{S} - \int_V |\operatorname{grad} \Phi|^2 \, \mathrm{d}V.$$

Since $\partial\Phi/\partial n = \hat{n} \cdot \operatorname{grad} \Phi = 0$, $\operatorname{grad} \Phi = \mathbf{0}$ on $S$. Therefore

$$\oint_S (\Phi \operatorname{grad} \Phi) \cdot \mathrm{d}\mathbf{S} = 0,$$

and

$$\int_V |\operatorname{grad} \Phi|^2 \, \mathrm{d}V = 0.$$

So $\Phi =$ constant in $V$.

**Corollary E.2.** *If* $\nabla^2 \Phi = 0$ *in all space and* $r\Phi(r) \to$ *function of* $(\vartheta, \phi)$ *alone as* $r \to \infty$ *then* $\Phi = 0$ *everywhere.*

*Proof.* By hypothesis

$$\Phi = \frac{f(\vartheta, \phi)}{r}$$

on $S$. Then

$$\frac{\partial \Phi}{\partial r} = -\frac{f(\vartheta, \phi)}{r^2}$$

on $S$. On the surface at infinity

$$\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial n} = \hat{n} \cdot \operatorname{grad} \Phi$$

and

$$\hat{n} \cdot \text{grad}\,\Phi \text{d}S = \text{grad}\,\Phi \cdot \text{d}\mathbf{S}.$$

Therefore

$$\oint_S \Phi \,\text{grad}\,\Phi \cdot \text{d}\mathbf{S} \propto -\int_\Omega \frac{f(\vartheta,\phi)}{r}\frac{f(\vartheta,\phi)}{r^2} r^2 \text{d}\Omega.$$

Here we have written the differential surface area as $r^2\text{d}\Omega$ where $\Omega$ is the solid angle with $\text{d}\Omega = \sin\phi \text{d}\vartheta \text{d}\phi$. Then

$$\oint_S \Phi \,\text{grad}\,\Phi \cdot \text{d}\mathbf{S} \propto -\frac{1}{r}\int_\Omega \frac{|f(\vartheta,\phi)|^2}{r} \text{d}\Omega,$$

and

$$\lim_{r\to\infty} \frac{1}{r}\int_\Omega \frac{|f(\vartheta,\phi)|^2}{r} \text{d}\Omega = 0.$$

Therefore

$$\oint_S \Phi \,\text{grad}\,\Phi \cdot \text{d}\mathbf{S} = 0.$$

Then, using (E.1), which is valid if $\Phi$ satisfies Laplace's Equation, we have

$$\int_V |\text{grad}\,\Phi|^2 \,\text{d}V = 0$$

and, as a consequence,

$$\text{grad}\,\Phi = 0$$

and

$$\Phi = \text{constant}$$

in $V$. Then, since we require also that $\lim_{r\to\infty} r\Phi$ is independent of $r$, the only constant value of $\Phi$ that is possible is $\Phi = 0$.

**Theorem E.3.** *If $\nabla^2\Phi = 0$ in $V$ and $\Phi$ takes on specified values on the surface $S$ bounding $V$, then if a solution exists for $\Phi$ it is unique.*

*Proof.* Assume $\Phi_1$ and $\Phi_2$ are two distinct solutions. Define

$$\Phi = \Phi_1 - \Phi_2.$$

Then

$$\nabla^2\Phi = \nabla^2\Phi_1 - \nabla^2\Phi_2 = 0.$$

Since $\Phi_1$ and $\Phi_2$ are solutions to the same problem they have the same values on the boundary. Therefore $\Phi = 0$ on $S$. Then from *Theorem* E.2 $\Phi = 0$ everywhere in $V$. This is true if and only if $\Phi_1 = \Phi_2$.

**Theorem E.4.** *If* $\mathbf{r}$ *and* $\mathbf{r}'$ *are position vectors from the origin in* $V$, *then*

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = 0$$

*if* $\mathbf{r} \neq \mathbf{r}'$.

*Proof.* See exercises in Chap. 2.

**Theorem E.5.** *If* $\Phi$ *is continuous and has continuous first derivatives at* $\mathbf{r} = \mathbf{r}'$ *and* $S_R$ *is the surface of a sphere of radius* $R$ *centered at* $\mathbf{r}'$ *then*

$$\lim_{R\to 0} \oint_{S_R} \left[ \Phi \operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \operatorname{grad} \Phi \right] \cdot d\mathbf{S} = -4\pi \Phi\left(\mathbf{r}'\right)$$

*Proof.* Since $\Phi$ is continuous and has continuous first derivatives, $R$ can be chosen so small that $\Phi$ is a constant (or deviates from a constant value by an amount $< \varepsilon$) over $S$. Then the integral above is

$$\Phi\left(\mathbf{r}'\right) \oint_{\mathrm{S_R}} \operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{S} - \operatorname{grad} \Phi\left(\mathbf{r}'\right) \cdot \oint_{\mathrm{S_R}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}\mathbf{S}.$$

Here $\Phi\left(\mathbf{r}'\right)$ is the value of $\Phi$ at $\mathbf{r}$ and, therefore, the value of $\Phi$ on $S_{\mathrm{R}}$.
Now (see exercises in Chap. 2)

$$\operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3}.$$

And, since d$\mathbf{S}$ points outward from the volume centered on $\mathbf{r}'$, and the point $\mathbf{r}$ is the location of d$\mathbf{S}$,

$$\mathrm{d}\mathbf{S} = \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}S.$$

Then

$$\operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{S} = -\frac{1}{R^2} \mathrm{d}S = -\mathrm{d}\Omega,$$

the differential solid angle (see proof of *corollary* E.2), and the first integral becomes

$$\Phi\left(\mathbf{r}'\right) \oint_{\mathrm{S_R}} \operatorname{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cdot \mathrm{d}\mathbf{S} = -\Phi\left(\mathbf{r}'\right) \oint_{\mathrm{S_R}} \mathrm{d}\Omega$$
$$= -4\pi \Phi\left(\mathbf{r}'\right).$$

The second integral is

$$\operatorname{grad}\Phi\left(\mathbf{r}'\right)\cdot\oint_{S_R}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}\mathbf{S} = \operatorname{grad}\Phi\left(\mathbf{r}'\right)\cdot\oint_{S_R}\frac{1}{R}\frac{\mathbf{r}'-\mathbf{r}}{R}\mathrm{d}S$$

$$= \operatorname{grad}\Phi\left(\mathbf{r}'\right)\cdot\oint_{S_R}\left(\mathbf{r}'-\mathbf{r}\right)\mathrm{d}\Omega.$$

This integral vanishes by symmetry. For every $\mathbf{r}$ on the surface there is another point diametrically opposed, which produces a contribution which is the negative of that from $\mathbf{r}$.
Therefore

$$\lim_{R\to 0}\oint_{S_R}\left[\Phi\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{grad}\Phi\right]\cdot\mathrm{d}\mathbf{S} = -4\pi\Phi\left(\mathbf{r}'\right)$$

**Theorem E.6.** *If* $\nabla^2\Phi = 0$ *in* $V$ *and* $\mathbf{r}'$ *is a point in* $V$*, then*

$$\Phi\left(\mathbf{r}'\right) = -\frac{1}{4\pi}\oint_{S}\left[\Phi\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{grad}\Phi\right]\cdot d\mathbf{S},$$

*where* $S$ *bounds* $V$.

*Proof.* We divide $V$ into the regions $V-O$ and $O$, where the region $O$ is a very small spherical region centered on $\mathbf{r}'$ that we will shrink to zero. Green's Theorem, written for the region $V-O$ is

$$\int_{V\text{-}O}\left[\Phi\,\nabla^2\,\Psi-\Psi\,\nabla^2\,\Phi\right]\mathrm{d}V = \oint_{S+O}\left[\Phi\operatorname{grad}\Psi-\Psi\operatorname{grad}\Phi\right]\cdot\mathrm{d}\mathbf{S}.$$

The region $V-O$ is bounded by the original boundary $S$ and the surface surrounding $\mathbf{r}'$.
We now choose

$$\Psi = \frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Since the small sphere surrounding the point $\mathbf{r}'$ has been eliminated, from *Theorem* E.4 we have then

$$\nabla^2\Psi = 0.$$

And, because $\nabla^2\Phi = 0$,

$$\int_{V\text{-}O}\left[\Phi\,\nabla^2\,\Psi-\Psi\,\nabla^2\,\Phi\right]\mathrm{d}V = 0.$$

Then

$$0 = \oint_S \left[ \Phi \operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \operatorname{grad} \Phi \right] \cdot \mathrm{d}\mathbf{S}$$
$$- \oint_O \left[ \Phi \operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \operatorname{grad} \Phi \right] \cdot \mathrm{d}\mathbf{S},$$

the negative sign coming from the the convention on d**S** as pointing out of the original volume, i.e. into $O$. From *Theorem* E.5 the integral over $O$ is $-4\pi \Phi\left(\mathbf{r}'\right)$. Then

$$\Phi\left(\mathbf{r}'\right) = -\frac{1}{4\pi} \oint_S \left[ \Phi \operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \operatorname{grad} \Phi \right] \cdot \mathrm{d}\mathbf{S}$$

# Appendix F
# Poisson's Equation

**Theorem F.1.** *If* $\nabla^2 \Phi_1 = -g_1$ *and* $\nabla^2 \Phi_2 = -g_2$ *then*

$$\nabla^2 (\Phi_1 + \Phi_2) = -(g_1 + g_2).$$

*Proof.* The proof is obvious.

**Theorem F.2.** *If* $\nabla^2 \Phi = -g$ *and* $\Phi$ *takes on specified values on the surface* $S$ *of* $V$*, then* $\Phi$ *is uniquely determined in* $V$*.*

*Proof.* Assume that there are two separate solutions $\Phi_1$ and $\Phi_2$ satisfying

$$\nabla^2 \Phi_1 = -g$$

and

$$\nabla^2 \Phi_2 = -g$$

in $V$. If we define

$$\Phi = \Phi_1 - \Phi_2,$$

then

$$\nabla^2 \Phi = 0$$

from *Theorem* F.1. Because $\Phi_1$ and $\Phi_2$ have the same values on the surface $S$, $\Phi = 0$ on the surface $S$. Therefore, from *Theorem* E.2 $\Phi = 0$ throughout $V$.

**Theorem F.3.** *If* $\nabla^2 \Phi = -g$ *in* $V$*, then*

$$\Phi(\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{g(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

*is a particular solution.*

*Proof.* We again exclude the point $\mathbf{r}'$ from the volume $V$ by enclosing $\mathbf{r}'$ within a small volume $O$. Then, if the theorem is valid we have

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$$\nabla^2 \Phi(\mathbf{r}) = \frac{1}{4\pi} \int_{\text{V-O}} g(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} dV' + \frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} dV'.$$

Since the integral is over the primed coordinates, $\nabla^2$ can be brought inside the integral where it does not operate on $g(\mathbf{r}')$. The first integral on the right hand side is zero from *Theorem* E.4, since throughout the volume $V - O$ we have $\mathbf{r} \neq \mathbf{r}'$. Since

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \nabla'^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

(see exercises Chap. 2), we can write the second integral on the right hand side as

$$\frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}V' = \frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}') \nabla'^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} dV'.$$

Since this integral vanishes everywhere except when $\mathbf{r} = \mathbf{r}'$, we can exchange the primed and unprimed variables in the integration and

$$\frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}V' = \frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}) \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} dV.$$

The volume $O$ is a very small (infinitesimal) volume, and, since $\rho$ is a continuous function of spatial coordinates, we may consider it to be constant throughout the volume $O$. Gauss' Theorem then results in

$$\frac{1}{4\pi} \int_{\text{O}} g(\mathbf{r}) \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} dV = \frac{1}{4\pi} g(\mathbf{r}) \oint_{\text{S}_0} \text{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cdot d\mathbf{S}$$

where $S_{\text{O}}$ is the surface around the volume $O$. Since

$$\text{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \cdot d\mathbf{S} = -d\Omega$$

(see *Theorem* E.5), the integral becomes

$$\frac{1}{4\pi} g(\mathbf{r}) \oint_{\text{S}_0} \text{grad} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathrm{d}\mathbf{S} = -g(\mathbf{r}).$$

And the result is equal to $\nabla^2 \Phi(\mathbf{r})$. This establishes the theorem.

# Appendix G
# Helmholtz' Equation

**Theorem G.1.** *If* $\left(\nabla^2 + K^2\right)\Phi_1 = -h_1$ *and* $\left(\nabla^2 + K^2\right)\Phi_2 = -h_2$ *then*

$$\left(\nabla^2 + K^2\right)(\Phi_1 + \Phi_2) = -(h_1 + h_2).$$

*Proof.* The proof is obvious.

**Theorem G.2.** *If* $\left(\nabla^2 + K^2\right)\Phi = -h$ *and* $\Phi$ *takes on specified values on the surface* $S$ *of* $V$*, then* $\Phi$ *is uniquely determined in* $V$*.*

*Proof.* Assume that there are two separate solutions $\Phi_1$ and $\Phi_2$ satisfying

$$\left(\nabla^2 + K^2\right)\Phi_1 = -h$$

and

$$\left(\nabla^2 + K^2\right)\Phi_2 = -h$$

in $V$. If we define

$$\Phi = \Phi_1 - \Phi_2,$$

then

$$\left(\nabla^2 + K^2\right)\Phi = 0$$

from *Theorem* G.1. Because $\Phi_1$ and $\Phi_2$ have the same values on the surface $S$, $\Phi = 0$ on the surface $S$. Therefore, from *Theorem* E.2 $\Phi = 0$ throughout $V$.

**Theorem G.3.** *If* $\left(\nabla^2 + K^2\right)\Phi = -h$ *in* $V$*, then*

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{h(\mathbf{r}')\exp(\pm iK|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}dV'$$

*is a particular solution.*

*Proof.* We again exclude the point $\mathbf{r}'$ from the volume $V$ by enclosing $\mathbf{r}'$ within a small volume $O$. Then, if the theorem is valid we have

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$$\left(\nabla^2 + K^2\right) \Phi (\mathbf{r}) = \frac{1}{4\pi} \int_{V-O} h\left(\mathbf{r}'\right) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV'$$
$$+ \frac{1}{4\pi} \int_{\mathrm{O}} h\left(\mathbf{r}'\right) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV'.$$

Since the integral is over the primed coordinates, $\nabla^2$ can be brought inside the integral where it does not operate on $h\left(\mathbf{r}'\right)$. The first integral on the right hand side is zero from *Theorem* E.4, since throughout the volume $V - O$ we have $\mathbf{r} \neq \mathbf{r}'$. Since

$$\left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} = \left(\nabla'^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

(see exercises Chap. 2), we can write the second integral on the right hand side as

$$\frac{1}{4\pi} \int_{\mathrm{O}} h\left(\mathbf{r}'\right) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV'$$
$$= \frac{1}{4\pi} \int_{\mathrm{O}} h\left(\mathbf{r}'\right) \left(\nabla'^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV'.$$

Since this integral vanishes everywhere except when $\mathbf{r} = \mathbf{r}'$, we can exchange the primed and unprimed variables in the integration and

$$\frac{1}{4\pi} \int_{\mathrm{O}} h\left(\mathbf{r}'\right) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV'$$
$$= \frac{1}{4\pi} \int_{\mathrm{O}} h(\mathbf{r}) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV.$$

The volume $O$ is a very small (infinitesimal) volume, and, since $\rho$ is a continuous function of spatial coordinates, we may consider it to be constant throughout the volume $O$. Gauss' Theorem then results in

$$\frac{1}{4\pi} \int_{\mathrm{O}} h(\mathbf{r}) \left(\nabla^2 + K^2\right) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV$$
$$= \frac{1}{4\pi} h(\mathbf{r}) \oint_{\mathrm{S_O}} \operatorname{grad} \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} \cdot d\mathbf{S}$$
$$+ \frac{K^2}{4\pi} \int_{\mathrm{O}} h(\mathbf{r}) \frac{\exp\left(\pm i K \left|\mathbf{r} - \mathbf{r}'\right|\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} dV \qquad \text{(G.1)}$$

where $S_{\mathrm{O}}$ is the surface around the volume $O$. As the volume $O$ shrinks to zero the distance between the points $\left|\mathbf{r} - \mathbf{r}'\right|$ also shrinks to zero. Then, as the volume $O$ shrinks to zero

$$\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)=1, \tag{G.2}$$

and the first integral on the right hand side of (G.1) becomes

$$\frac{1}{4\pi}h\left(\mathbf{r}\right)\oint_{S_0}\text{grad}\,\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\cdot d\mathbf{S}=\frac{1}{4\pi}h\left(\mathbf{r}\right)\oint_{S_0}\text{grad}\,\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}d\mathbf{S}. \tag{G.3}$$

Since

$$\text{grad}\,\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\cdot \mathrm{d}\mathbf{S}=-d\Omega$$

(see *Theorem* E.5), the integral on the right hand side of (G.3) is

$$\frac{1}{4\pi}h\left(\mathbf{r}\right)\oint_{S_0}\text{grad}\,\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}d\mathbf{S}=-h\left(\mathbf{r}\right). \tag{G.4}$$

With (G.2) the second integral on the right hand side of (G.1) becomes

$$\frac{K^2}{4\pi}\int_O h\left(\mathbf{r}\right)\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}dV=\frac{K^2}{4\pi}\int_O h\left(\mathbf{r}\right)\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}dV. \tag{G.5}$$

For infinitesimal $O$ and analytic $h\left(\mathbf{r}\right)$ the integral on the right hand side of (G.5) is

$$\frac{K^2}{4\pi}\int_O h\left(\mathbf{r}\right)\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\mathrm{d}V=K^2h\left(\mathbf{r}\right)\lim_{R\to 0}\int_O R\mathrm{d}R=0.$$

Therefore using (G.4), (G.1) is

$$\left(\nabla^2+K^2\right)\left\{\frac{1}{4\pi}\int_O h\left(\mathbf{r}\right)\frac{\exp\left(\pm iK\left|\mathbf{r}-\mathbf{r}'\right|\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}dV\right\}=-h\left(\mathbf{r}\right),$$

which establishes the theorem.

# Appendix H
# Legendre's Equation

The differential equation

$$\frac{d}{dx}\left[\left(x^2-1\right)\frac{d}{dx}P_n\right]-n\,(n+1)\,P_n=0 \tag{H.1}$$

or

$$\left(x^2-1\right)\frac{d^2}{dx^2}P_n+2x\frac{d}{dx}P_n-n\,(n+1)\,P_n=0 \tag{H.2}$$

is *Legendre's Equation*. It is solved by the polynomials

$$P_n\,(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{H.3}$$

Which are the *Legendre Polynomials*. Our proof of this will follow the suggested approach in ([16], p. 86).

We define the function

$$u\,(x)=\left(x^2-1\right)^n, \tag{H.4}$$

for which we have the identity

$$\left(x^2-1\right)\frac{du}{dx}=2nx\left(x^2-1\right)^n=2nxu. \tag{H.5}$$

We shall now show that the $(n+1)$ order derivative of (H.5) produces Legendre's equation (H.2) for the polynomial

$$y_n\,(x)=\frac{d^n u}{dx^n}=\frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{H.6}$$

which differs from the $n$th order Legendre polynomial (H.3) by the constant factor $1/\left(2^n n!\right)$, which has no effect on the solution.

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By carrying out the derivatives we see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x^2-1\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = 2x\frac{\mathrm{d}u}{\mathrm{d}x} + \left(x^2-1\right)\frac{\mathrm{d}^2u}{\mathrm{d}x^2},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\left(x^2-1\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = 2\frac{\mathrm{d}u}{\mathrm{d}x} + 4x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \left(x^2-1\right)\frac{\mathrm{d}^3u}{\mathrm{d}x^3},$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\left(x^2-1\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = 6\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 6x\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \left(x^2-1\right)\frac{\mathrm{d}^4u}{\mathrm{d}x^4},$$

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left[\left(x^2-1\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = 12\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + 8x\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + \left(x^2-1\right)\frac{\mathrm{d}^5u}{\mathrm{d}x^5},$$

$$\vdots$$

$$\frac{\mathrm{d}^{\mathrm{n}+1}}{\mathrm{d}x^{\mathrm{n}+1}}\left[\left(x^2-1\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = n\left(n+1\right)\frac{\mathrm{d}^{\mathrm{n}}u}{\mathrm{d}x^{\mathrm{n}}} + 2\left(n+1\right)x\frac{\mathrm{d}^{\mathrm{n}+1}u}{\mathrm{d}x^{\mathrm{n}+1}} + \left(x^2-1\right)\frac{\mathrm{d}^{\mathrm{n}+2}u}{\mathrm{d}x^{\mathrm{n}+2}}. \tag{H.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2nxu\right) = 2nu + 2nx\frac{\mathrm{d}u}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(2nxu\right) = 4n\frac{\mathrm{d}u}{\mathrm{d}x} + 2nx\frac{\mathrm{d}^2u}{\mathrm{d}x^2},$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(2nxu\right) = 6n\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2nx\frac{\mathrm{d}^3u}{\mathrm{d}x^3},$$

$$\vdots$$

$$\frac{\mathrm{d}^{\mathrm{n}+1}}{\mathrm{d}x^{\mathrm{n}+1}}\left(2nxu\right) = 2n\left(n+1\right)\frac{\mathrm{d}^{\mathrm{n}}u}{\mathrm{d}x^{\mathrm{n}}} + 2nx\frac{\mathrm{d}^{\mathrm{n}+1}u}{\mathrm{d}x^{\mathrm{n}+1}}. \tag{H.8}$$

Then, equating the identities (H.7) and (H.8), we have

$$\left(x^2-1\right)\frac{\mathrm{d}^{\mathrm{n}+2}u}{\mathrm{d}x^{\mathrm{n}+2}} + 2x\frac{\mathrm{d}^{\mathrm{n}+1}u}{\mathrm{d}x^{\mathrm{n}+1}} - n\left(n+1\right)\frac{\mathrm{d}^{\mathrm{n}}u}{\mathrm{d}x^{\mathrm{n}}} = 0. \tag{H.9}$$

with (H.6) equation (H.9) becomes

$$\left(x^2-1\right)\frac{\mathrm{d}^2y_{\mathrm{n}}}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y_{\mathrm{n}}}{\mathrm{d}x} - n\left(n+1\right)y_{\mathrm{n}} = 0. \tag{H.10}$$

Multiplying (H.10) by $1/\left(2^{\mathrm{n}}n!\right)$, we have

$$\left(x^2 - 1\right) \frac{\mathrm{d}^2 P_\mathrm{n}}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}P_\mathrm{n}}{\mathrm{d}x} - n\,(n+1)\,P_\mathrm{n} = 0, \qquad \text{(H.11)}$$

which is Legendre's Equation (H.2).

# Appendix I
# Jacobians

The Jacobian or Jacobian determinant is used to transform the integral

$$\int\int_{\mathrm{R}} f\,(x,y)\,\mathrm{d}x\mathrm{d}y$$

performed over $x$ and $y$ in a region $R$ in the $(x, y)$ plane into

$$\int\int_{\mathrm{R}'} f\,(u,v)\,\mathrm{d}u\mathrm{d}v$$

over $u$ an $v$ in a region $R'$ defined by those variables.

Here we will perform the transformation in two steps for the sake of clarity. The first step will transform only the $y$ and the second only the $x$. We designate the first transformation as from $R$ to $B$ and the second as from $B$ to $R'$.

We write the first transformation step as

$$\begin{aligned} x &= x \\ y &= \Phi\,(v,x)\,. \end{aligned} \tag{I.1}$$

We have illustrated this in the Fig. I.1, where we have drawn cells $\Delta R_{\mathrm{ij}}$ and $\Delta B_{\mathrm{ij}}$ in the two regions $R$ and $B$ indicated in panels (a) and (b) of Fig. I.1.

In panel (a) we have the rectangular Cartesian grid of the $(x, y)$ plane. In panel (b) we have the transformed grid. In this we assume that the partial derivative $\partial\Phi/\partial v = \Phi_v \neq 0$ everywhere.

The Riemann integral is

$$\int\int_{\mathrm{R}} f\,(x,y)\,\mathrm{d}x\mathrm{d}y = \lim_{N\to\infty,\Delta R_{\mathrm{ij}}\to 0} \sum_{\mathrm{i,j}}^{\mathrm{N}} f_{\mathrm{ij}}\Delta R_{\mathrm{ij}} \tag{I.2}$$

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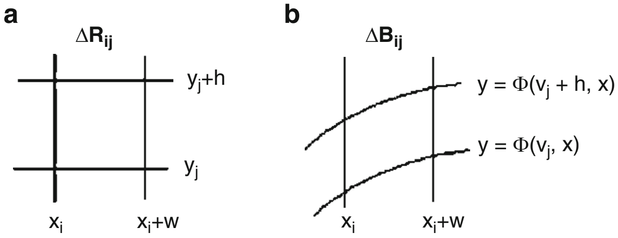


**Fig. I.1** Cells (**a**) $\Delta R_{ij}$ and (**b**) $\Delta B_{ij}$ in the regions $R$ and $B$

$$= \lim_{N\to\infty, \Delta B_{ij}\to 0} \sum_{i,j}^{N} f_{ij} \Delta B_{ij} \tag{I.3}$$

where

$$f_{ij} = f\left(x_i, y_j\right) \tag{I.4}$$

and

$$\Delta B_{ij} = \int_{x_i}^{x_i+w} \left[\Phi\left(v_j, x\right) - \Phi\left(v_j + h, x\right)\right] \mathrm{d}x. \tag{I.5}$$

Now

$$\lim_{h\to 0} \frac{\Phi(v, x) - \Phi(v+h, x)}{h} = \frac{\partial \Phi}{\partial v} = \Phi_v(v, x), \tag{I.6}$$

which, from (I.1) is

$$\Phi_v(v, x) = \frac{\partial y}{\partial v}. \tag{I.7}$$

Using (I.6) (I.5) becomes

$$\Delta B_{ij} = h \int_{x_i}^{x_i+w} \Phi_v\left(\bar{v}_j, x\right) \mathrm{d}x, \tag{I.8}$$

in which $\bar{v}_j$ is a point within the range of $v$ chosen to best approximate the area. Integrating over $x$ we have

$$\Delta B_{ij} = hw\Phi_v\left(\bar{v}_j, \bar{x}_i\right), \tag{I.9}$$

where $\bar{x}_i$ provides the best approximation to the area. We may recognize that (I.9) is the *central limit theorem*. With (I.9) equation (I.3) becomes

$$\int\int_R f(x, y)\, dxdy = \lim_{N\to\infty; h,w\to 0} \sum_{i,j}^{N} f\left(\bar{x}_i, \Phi_v\left(\bar{v}_j, \bar{x}_i\right)\right) hw\Phi_v\left(\bar{v}_j, \bar{x}_i\right), \tag{I.10}$$

since $\bar{y}_j = \Phi_v\left(\bar{v}_j, \bar{x}_i\right)$. Taking the limits specified in (I.10) we have

$$\int\int_{\mathrm{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int\int_{\mathrm{B}} f(x,v)\,\Phi_{\mathrm{v}}(v,x)\,\mathrm{d}x\mathrm{d}v. \tag{I.11}$$

We can now transform $x$ in the same fashion. We define this step by

$$\begin{aligned} v &= v \\ x &= \Psi(u,v)\,. \end{aligned} \tag{I.12}$$

The result is

$$\int\int_{\mathrm{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int\int_{\mathrm{R}'} f(u,v)\,\Phi_{\mathrm{v}}\Psi_{\mathrm{u}}\mathrm{d}u\mathrm{d}v, \tag{I.13}$$

where

$$\Psi_{\mathrm{u}} = \frac{\partial x}{\partial u}. \tag{I.14}$$

The Jacobian determinant is defined as

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \\ &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}. \end{aligned} \tag{I.15}$$

Then

$$\begin{aligned} \frac{\partial(x,y)}{\partial(x,v)} &= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 1 & \frac{\partial x}{\partial v} \\ 0 & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \frac{\partial y}{\partial v}, \end{aligned} \tag{I.16}$$

since $x$ and $y$ are independent variables. Likewise

$$\frac{\partial(x,v)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ 0 & 1 \end{vmatrix} = \frac{\partial x}{\partial u}, \tag{I.17}$$

since $u$ and $v$ are independent variables.

Jacobians obey a chain rule of the form

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(u,v)}. \tag{I.18}$$

To establish the validity of (I.18) we consider the transformations

$$\xi = \phi(x,y)\,;\, \eta = \psi(x,y)$$

and

$$u = \Phi(\xi,\eta)\,;\, v = \Psi(\xi,\eta)\,.$$

The partial derivatives in the Jacobian are then

$$\frac{\partial u}{\partial x} = \frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial x} - \frac{\partial \Phi}{\partial \eta}\frac{\partial \eta}{\partial x} = \Phi_\xi \phi_x + \Phi_\eta \psi_x,$$

$$\frac{\partial u}{\partial y} = \Phi_\xi \phi_y + \Phi_\eta \psi_y,$$

$$\frac{\partial v}{\partial x} = \Psi_\xi \phi_x + \Psi_\eta \psi_x,$$

and

$$\frac{\partial v}{\partial y} = \Psi_\xi \phi_y + \Psi_\eta \psi_y.$$

The Jacobian is then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$
\begin{aligned}
&= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\
&= \left(\Phi_\xi \phi_\mathrm{x} + \Phi_\eta \psi_\mathrm{x}\right)\left(\Psi_\xi \phi_\mathrm{y} + \Psi_\eta \psi_\mathrm{y}\right) \\
&\quad - \left(\Phi_\xi \phi_\mathrm{y} + \Phi_\eta \psi_\mathrm{y}\right)\left(\Psi_\xi \phi_\mathrm{x} + \Psi_\eta \psi_\mathrm{x}\right) \\
&= \left(\Phi_\xi \Psi_\eta - \Phi_\eta \Psi_\xi\right)\left(\phi_\mathrm{x} \psi_\mathrm{y} - \phi_\mathrm{y} \psi_\mathrm{x}\right) \\
&= \begin{vmatrix} \Phi_\xi & \Phi_\eta \\ \Psi_\xi & \Psi_\eta \end{vmatrix} \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} \\
&= \frac{\partial (u, v)}{\partial (\xi, \eta)} \frac{\partial (\xi, \eta)}{\partial (x, y)},
\end{aligned}
$$

which establishes the chain rule for Jacobians.

Combining (I.7), (I.14), (I.16), (I.14), with (I.18) we then have

$$
\begin{aligned}
\frac{\partial (x, y)}{\partial (u, v)} &= \frac{\partial (x, y)}{\partial (x, v)} \frac{\partial (x, v)}{\partial (u, v)} \\
&= \Phi_v \Psi_u. \qquad \text{(I.19)}
\end{aligned}
$$

With (I.19) (I.13) becomes

$$
\int\int_\mathrm{R} f\,(x, y)\, dx dy = \int\int_{\mathrm{R}'} f\,(u, v) \frac{\partial (x, y)}{\partial (u, v)} du dv. \qquad \text{(I.20)}
$$

Equation (I.20) provides a transformation from one integral into another.

This result can be extended in the same fashion to any number of variables. That is

$$
dx_1 dx_2 \cdots dx_n = \frac{\partial (x_1, x_2, \cdots, x_n)}{\partial (\xi_1, \xi_2, \cdots, \xi_n)} d\xi_1 d\xi_2 \cdots d\xi_n.
$$

# Appendix J
# Dispersion

This treatment of the energy in a damped and dispersed wave in a nonmagnetic medium will be more general than the treatment in Chap. 16 Here we will consider the tensor character of the conductivity and conduct an expansion of the wave equation in Fourier space. Our discussion parallels that of [4] and [90]. Abraham Bers has also provided a short treatment of this general situation in [7].

Our treatment here may be considered to be more elegant than that in Chap. 16. But the results are fundamentally unchanged.

As in our treatment in Chap. 16, we will again consider waves

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{E}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right.\\ \left. + \mathbf{E}^{*}\exp\left(-i\omega^{*}t + i\mathbf{k}^{*}\cdot\mathbf{r}\right)\right]\end{aligned} \tag{J.1}$$

and

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{B}\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})\right.\\ \left. + \mathbf{B}^{*}\exp\left(-i\omega^{*}t + i\mathbf{k}^{*}\cdot\mathbf{r}\right)\right],\end{aligned} \tag{J.2}$$

that are very nearly monochromatic. As we pointed out in Chap. 16 the field vectors **E** and **B** in (J.1) and (J.2), which are not functions of $(\mathbf{k},\omega)$.

We accept Ohm's Law

$$\mathbf{J}(\mathbf{k},\omega) = \boldsymbol{\sigma}(\mathbf{k},\omega)\cdot\mathbf{E} \tag{J.3}$$

as valid and base our treatment on the Fourier transformed form of Maxwell's Equations in (16.8). The wave vectors **E** or **B** must then satisfy

$$\boldsymbol{D}(\mathbf{k},\omega)\cdot(\mathbf{E}\text{ or }\mathbf{B}) = \mathbf{0}, \tag{J.4}$$

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where

$$\boldsymbol{D}\,(\mathbf{k},\omega) = \left(k^2 - \frac{\omega^2}{c^2}K\right)\mathbf{1} - \mathbf{kk} + i\omega\mu_0\boldsymbol{\sigma}, \tag{J.5}$$

where $K$ is the dielectric constant for the matter.

Equation (J.4) is the wave equation in Fourier space. Written for the electric field in Einstein subscript notation the wave (J.4) is

$$\left[\left(k^2 - \frac{\omega^2}{c^2}K\right)\boldsymbol{\delta}_{\alpha\beta} - \mathbf{k}_\alpha\mathbf{k}_\beta + i\omega\mu_0\sigma_{\alpha\beta}\right]E_\beta = 0 \tag{J.6}$$

For the undamped and undispersed case, $\mathbf{k}$ and $\omega$ are real. Taking the Hermitian conjugate (adjoint) of (J.6) for real $\mathbf{k}$ and $\omega$ we have

$$E_\beta\left[\left(k^2 - \frac{\omega^2}{c^2}K\right)\boldsymbol{\delta}_{\beta\alpha} - \mathbf{k}_\beta\mathbf{k}_\alpha - i\omega\mu_0\boldsymbol{\sigma}^*_{\beta\alpha}\right] = 0 \tag{J.7}$$

The wave (J.6) and (J.7) are identical if

$$\boldsymbol{\sigma}^*_{\beta\alpha} = -\sigma_{\alpha\beta}, \tag{J.8}$$

which is the requirement that the conductivity tensor is antihermitian at the undamped condition.

The dispersion relation for undamped and undispersed plane waves is

$$\det\boldsymbol{D} = 0. \tag{J.9}$$

We are interested in wavelike solutions that differ only slightly from the undamped and undispersed solutions. For these waves there will be imaginary contributions to $\mathbf{k}$ and $\omega$, which we shall simply identify as $\Delta\mathbf{k}$ and $\Delta\omega$. There will also be a change in the conductivity tensor $\boldsymbol{\sigma}$ resulting from the imaginary additions to $\mathbf{k}$ and $\omega$ as well as the addition of a small structural change to the form of $\boldsymbol{\sigma}$. The wave (J.6), for the slightly damped wave is

$$\mathbf{0} = \left\{\left[(\mathbf{k}+\Delta\mathbf{k})^2 - \frac{(\omega+\Delta\omega)^2}{c^2}K\right]\mathbf{1} - (\mathbf{k}+\Delta\mathbf{k})\,(\mathbf{k}+\Delta\mathbf{k}) + i\,(\omega+\Delta\omega)\,\mu_0\,(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})\right\}\cdot\mathbf{E}. \tag{J.10}$$

If we multiply (J.10) on the left by $\mathbf{E}^*$ and hold terms to first order in $\Delta$ we have

$$\mathbf{0} = \mathbf{E}^*\cdot\left[\left(2\mathbf{k}\cdot\Delta\mathbf{k} - \frac{2\omega\Delta\omega}{c^2}K\right)\mathbf{1} - \mathbf{k}\Delta\mathbf{k} - \Delta\mathbf{kk} + i\Delta\omega\mu_0\,\boldsymbol{\sigma} + \omega\mu_0\Delta\boldsymbol{\sigma}\right]\cdot\mathbf{E}. \tag{J.11}$$

From the Fourier Transformed form of Maxwell's Equations (16.8) we find that

$$\omega \Delta \mathbf{k} \cdot \left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right)$$
$$= 2\Delta \mathbf{k} \cdot \mathbf{k} E^2 - \mathbf{E}^* \cdot \mathbf{k} \Delta \mathbf{k} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{k} \Delta \mathbf{k} \cdot \mathbf{E}^*, \tag{J.12}$$

and

$$\omega \left(\varepsilon E^2 - \frac{1}{\mu_0} B^2\right) = i \mathbf{E}^* \cdot \boldsymbol{\sigma} \cdot \mathbf{E} \tag{J.13}$$

for real $\mathbf{k}$ and $\omega$.

With (J.12) and (J.13) equation (J.11) becomes

$$-\Delta\omega \left(\varepsilon E^2 + \frac{1}{\mu_0} B^2\right) + \Delta \mathbf{k} \cdot \frac{1}{\mu_0} \left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right)$$
$$= -i \mathbf{E}^* \cdot \Delta\boldsymbol{\sigma} \cdot \mathbf{E} \tag{J.14}$$

We now identify

$$\Delta\omega = i\omega_{\mathrm{i}} \text{ and } \Delta \mathbf{k} = i \mathbf{k}_{\mathrm{i}}. \tag{J.15}$$

Then equation (J.14) is

$$-\omega_{\mathrm{i}} \left(\varepsilon E^2 + \frac{1}{\mu_0} B^2\right) + \mathbf{k}_{\mathrm{i}} \cdot \frac{1}{\mu_0} \left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right)$$
$$= -\mathbf{E}^* \cdot \Delta\boldsymbol{\sigma} \cdot \mathbf{E}. \tag{J.16}$$

If we add (J.16) to its complex conjugate we obtain

$$-2\omega_{\mathrm{i}} \left(\varepsilon E^2 + \frac{1}{\mu_0} B^2\right) + 2\mathbf{k}_{\mathrm{i}} \cdot \frac{1}{\mu_0} \left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right)$$
$$= -\mathbf{E} \cdot (\Delta\, \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}^{+}) \cdot \mathbf{E}^*, \tag{J.17}$$

where $\Delta\boldsymbol{\sigma}^{+}$ is the Hermitian conjugate of $\Delta\boldsymbol{\sigma}$.

Upon comparing (J.17) with (16.38) from Chap. 16 we see that (J.17) is the field energy equation for waves in dispersive media.

The perturbation of the conductivity tensor $\boldsymbol{\sigma}$ is

$$\Delta\boldsymbol{\sigma} = \Delta\omega \frac{\partial}{\partial\omega}\boldsymbol{\sigma} + \Delta\mathbf{k} \cdot \frac{\partial}{\partial\mathbf{k}}\boldsymbol{\sigma} + \Delta'\boldsymbol{\sigma}$$
$$= i\omega_{\mathrm{i}} \frac{\partial}{\partial\omega}\boldsymbol{\sigma} + i\mathbf{k}_{\mathrm{i}} \cdot \frac{\partial}{\partial\mathbf{k}}\boldsymbol{\sigma} + \Delta'\boldsymbol{\sigma}, \tag{J.18}$$

where we have used (J.15). In (J.18) we have designated the slight structural change in $\boldsymbol{\sigma}$ that results at the damped condition as $\Delta'\boldsymbol{\sigma}$. In (J.18) we have evaluated $\Delta'\boldsymbol{\sigma}$

using the undamped, real values of $\mathbf{k}$ and $\omega$. With (J.18) we find that

$$\begin{aligned}\Delta\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}^{+} &= i\omega_{\mathrm{i}}\frac{\partial}{\partial\omega}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{+}\right) + i\mathbf{k}_{\mathrm{i}}\cdot\frac{\partial}{\partial\mathbf{k}}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{+}\right) + \left(\Delta'\boldsymbol{\sigma} + \Delta'\boldsymbol{\sigma}^{+}\right) \\ &= -2\omega_{\mathrm{i}}\frac{\partial}{\partial\omega}\boldsymbol{\sigma} - 2\mathbf{k}_{\mathrm{i}}\cdot\frac{\partial}{\partial\mathbf{k}}\boldsymbol{\sigma}^{(\mathrm{A})} + 2\Delta'\boldsymbol{\sigma}^{(\mathrm{H})}.\end{aligned} \tag{J.19}$$

The antihermitian and Hermitian parts of the tensors $\boldsymbol{\sigma}$ and $\Delta'\boldsymbol{\sigma}$ are defined by

$$\boldsymbol{\sigma}^{(\mathrm{A})} = \frac{1}{2i}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{+}\right) \tag{J.20}$$

and

$$\Delta'\boldsymbol{\sigma}^{(\mathrm{H})} = \frac{1}{2}\left(\Delta'\boldsymbol{\sigma} + \Delta'\boldsymbol{\sigma}^{+}\right). \tag{J.21}$$

We recall that the conductivity tensor is antihermitian at the undamped condition. And we see that the Hermitian part of the structural change in $\boldsymbol{\sigma}$ enters the energy equation as a loss term. With (J.19) the energy (J.17) becomes

$$\begin{aligned}&-2\omega_{\mathrm{i}}\left(\varepsilon E^2 + \frac{1}{\mu_0}B^2 + \mathbf{E}\cdot\frac{\partial}{\partial\omega}\boldsymbol{\sigma}^{(\mathrm{A})}\cdot\mathbf{E}^*\right) \\ &+2\mathbf{k}_{\mathrm{i}}\cdot\left[\frac{1}{\mu_0}\left(\mathbf{E}^*\times\mathbf{B} + \mathbf{E}\times\mathbf{B}^*\right) - \mathbf{E}\cdot\frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\mathbf{k}}\cdot\mathbf{E}^*\right] \\ &\quad = -2\mathbf{E}^*\cdot\Delta'\boldsymbol{\sigma}^{(\mathrm{H})}\cdot\mathbf{E},\end{aligned} \tag{J.22}$$

Using Faraday's Law we can show that

$$\begin{aligned}\frac{1}{\mu_0}\left(\mathbf{E}^*\times\mathbf{B} + \mathbf{E}\times\mathbf{B}^*\right) &= \frac{1}{\mu_0\omega}\frac{\partial}{\partial\,\mathbf{k}}\left[E^2k^2 - (\mathbf{k}\cdot\mathbf{E})\left(\mathbf{E}^*\cdot\mathbf{k}\right)\right] \\ &= \frac{1}{\mu_0\omega}\frac{\partial}{\partial\mathbf{k}}\left[\mathbf{E}\cdot\left(\mathbf{1}k^2 - \mathbf{k}\,\mathbf{k}\right)\cdot\mathbf{E}^*\right]\end{aligned} \tag{J.23}$$

(see equation (16.52)). Then using (J.5) at the propagation condition, we find that (J.23) becomes

$$\begin{aligned}&\frac{1}{\mu_0}\left(\mathbf{E}^*\times\mathbf{B} + \mathbf{E}\times\mathbf{B}^*\right) \\ &\quad = \frac{1}{\omega}\frac{\partial}{\partial\mathbf{k}}\left[\mathbf{E}\cdot\left(\omega\boldsymbol{\sigma}^{(\mathrm{A})} + \omega^2\varepsilon\mathbf{1}\right)\cdot\mathbf{E}^*\right] \\ &\quad = \left[\left(\mathbf{E}\cdot\frac{\boldsymbol{\sigma}^{(\mathrm{A})}}{\omega}\cdot\mathbf{E}^* + \mathbf{E}\cdot\frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\omega}\cdot\mathbf{E}^* + 2\varepsilon\mathbf{1}\colon\mathbf{E}^*\mathbf{E}\right)\frac{\partial\omega}{\partial\mathbf{k}} \right. \\ &\qquad \left. + \,\mathbf{E}\cdot\frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\mathbf{k}}\cdot\mathbf{E}^*\right]\end{aligned} \tag{J.24}$$

Then, using Faraday's and Ampère's Laws, we find that we can obtain $B^2$ in terms of the electric field as

$$\omega B^2 = -\mathbf{E}^* \cdot (i\mu_0 \boldsymbol{\sigma} - \omega\mu_0\varepsilon\mathbf{1}) \cdot \mathbf{E},$$

which, after some lines of algebra, becomes

$$\frac{1}{\mu_0} B^2 = \mathbf{E} \cdot \frac{1}{\omega} \boldsymbol{\sigma}^{(\mathrm{A})} \cdot \mathbf{E}^* + \varepsilon E^2. \tag{J.25}$$

With (J.24) and (J.25) the energy (J.22) becomes

$$\begin{aligned} &-2\omega_{\mathrm{i}} \left( 2\varepsilon E^2 + \mathbf{E} \cdot \frac{1}{\omega} \boldsymbol{\sigma}^{(\mathrm{A})} \cdot \mathbf{E}^* + \mathbf{E} \cdot \frac{\partial}{\partial\omega} \boldsymbol{\sigma}^{(\mathrm{A})} \cdot \mathbf{E}^* \right) \\ &+ 2\mathbf{k}_{\mathrm{i}} \cdot \frac{\partial\omega}{\partial\mathbf{k}} \left( 2\varepsilon E^2 + \mathbf{E} \cdot \frac{\boldsymbol{\sigma}^{(\mathrm{A})}}{\omega} \cdot \mathbf{E}^* + \mathbf{E} \cdot \frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\omega} \cdot \mathbf{E}^* \right) \\ &\quad = -2\mathbf{E}^* \cdot \Delta' \boldsymbol{\sigma}^{(\mathrm{H})} \cdot \mathbf{E}. \end{aligned} \tag{J.26}$$

Upon comparison with the treatment in Chap. 16 we see that (J.26) is the general form of (16.57) for slightly damped waves in nonmagnetic matter.

We can then identify the total energy in the damped and dispersed wave as

$$\langle \mathcal{E}_{\mathrm{wave}} \rangle_{\mathrm{T,L}} = 2\varepsilon E^2 + \mathbf{E} \cdot \frac{1}{\omega} \boldsymbol{\sigma}^{(\mathrm{A})} \cdot \mathbf{E}^* + \mathbf{E} \cdot \frac{\partial}{\partial\omega} \boldsymbol{\sigma}^{(\mathrm{A})} \cdot \mathbf{E}^* \tag{J.27}$$

and the total Poynting Vector as

$$\langle \mathbf{S}_{\mathrm{wave}} \rangle_{\mathrm{T,L}} = \frac{\partial\omega}{\partial\mathbf{k}} \left( 2\varepsilon E^2 + \mathbf{E} \cdot \frac{\boldsymbol{\sigma}^{(\mathrm{A})}}{\omega} \cdot \mathbf{E}^* + \mathbf{E} \cdot \frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\omega} \cdot \mathbf{E}^* \right). \tag{J.28}$$

By comparing (J.27) and (J.28) that the total Poynting Vector is equal to the total energy multiplied by the group velocity, i.e.

$$\langle \mathbf{S}_{\mathrm{wave}} \rangle_{\mathrm{T,L}} = \langle \mathcal{E}_{\mathrm{wave}} \rangle_{\mathrm{T,L}} \frac{\partial\omega}{\partial\mathbf{k}}. \tag{J.29}$$

Although we have not used a time and space average in the derivation of (J.26), it is identical to the equation we obtain from the time and space average. And, as we point out in Chap. 16, the time and space average has an experimental meaning. We, therefore, include the subscript T,L notation here.

The understanding of

$$\mathbf{E} \cdot \frac{\partial\boldsymbol{\sigma}^{(\mathrm{A})}}{\partial\omega} \cdot \mathbf{E}^* = \sum_{\alpha} \left( \left\langle T_{\mathrm{hydro}}^{(\alpha)} \right\rangle_{\mathrm{T,L}} + \left\langle T_{\mathrm{thermal}}^{(\alpha)} \right\rangle_{\mathrm{T,L}} \right) \tag{J.30}$$

the coherent particle energy is unchanged. And the loss term $-2\mathbf{E}^* \cdot \Delta' \boldsymbol{\sigma}^{(\mathrm{H})} \cdot \mathbf{E}$ in (J.26) is the transport of the coherent particle energy to the heating of the background matter results as the coherence is lost remains unchanged.

# Appendix K
# Answers to Selected Exercises

2.11 $\cos\alpha = 1/\sqrt{3}$
2.14 (b) These vectors are not linearly independent.
2.15 $\mathbf{F}$ is generally not perpendicular to curl $\mathbf{F}$.
2.19 Note that the order of partial differentiation makes no difference.
2.20 Note that the order of partial differentiation makes no difference.
2.24 (a) Perform a contour integration between two distinct points. The result is contour independent. So choose the contour to make the integration as simple as possible.
b) $\varphi(x, y) = -3x^2 - xy$
2.25 (a), (c), and (e) are conservative.
2.26 $\operatorname{div}\mathbf{E} = \rho/\varepsilon_0$, $\operatorname{curl}\mathbf{B} = \mu_0\mathbf{J}$
2.27 $\int_{-\infty}^{\infty} \mathrm{d}x\delta(x-1)\exp\left(-\alpha x^2 + \beta x\right) = \exp(\beta - \alpha)$,

$$\int_0^{\infty} \mathrm{d}x\delta(x+1)\exp\left(-\alpha x^2 + \beta x\right) = 0,$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\delta(x+\lambda)\cos(2\pi x/\lambda)\exp\left(-x^2/\lambda^2\right) = \exp(-1),$$

$$\int_0^{10} \mathrm{d}x\delta(x+5)\left(6x^2 + 2x - 3\right) = 0,$$

$$\int_{-\infty}^{0} \mathrm{d}x\delta(x+5)\left(6x^2 + 2x - 3\right) = 137,$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\delta(x+\lambda)\sin(2\pi x/\lambda)\exp\left(-x^2/\lambda^2\right) = 0,$$

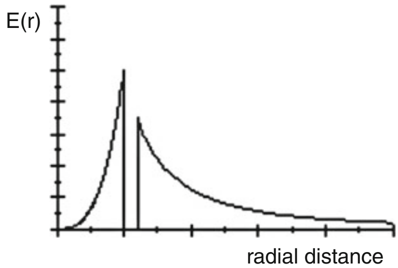
$$\int_{-\infty}^{\infty} \mathrm{d}x\delta(x-1)J_{\mathrm{n}}(x) = J_{\mathrm{n}}(1),$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\delta(x)\operatorname{erf}(x) = 0 \qquad \text{since } \operatorname{erf}(0) = 0$$

3.2 (b) $\rho = 0$ (c) two charged, flat conducting plates arranged parallel to one another, with a positive charge on one and a negative charge on the other will produce this electrostatic field.
3.5 (a) zero (b) $\sigma_{\mathrm{a}} = q/\left(4\pi a^2\right) = (\varepsilon_0 V/a)(b/(b-a))$, $\sigma_{\mathrm{b}} = Q_{\mathrm{b}}/\left(4\pi b^2\right) = -(\varepsilon_0 V/b)(a/(b-a))$ c) electrostatic field is equal to zero for $r > b$.
3.6 $\int_V \operatorname{div}\mathbf{E}\mathrm{d}V = \frac{q}{\varepsilon_0}R^{\delta}$ Gauss' Law is then no longer valid.
3.7 $E(r) = \rho_0 R^2/(4\varepsilon_0 r)$

C.S. Helrich, *The Classical Theory of Fields*, Graduate Texts in Physics,
DOI 10.1007/978-3-642-23205-3, © Springer-Verlag Berlin Heidelberg 2012

**Fig. K.1**




3.8 The charge induced on the surface of the small hollow in the conductor must be $-q$ to counterbalance the $q$ inserted.

3.9 (a) $4.3976 \times 10^{-14}$ C m$^{-3}$ b) $-1.3281 \times 10^{-9}$ C m$^{-2}$ c) $-6.7742 \times 10^{5}$ C. d) This charge comes from lightning striking the earth. In a plasma we can consider the massive ions to be immovable and the electrons to be the charge carriers. So in a lightning bolt electrons carry the negative charge to the earth. Positive and negative lightning can exist, but positive is rarer. In the negative case the upper part of a cloud becomes positively charged and the lower negative. When the negative charge on the cloud is high enough lightning is formed providing a flow of negative charge to the earth.

3.10 (a)

b) $\sigma_{\mathrm{i}} = (\rho_0/5)\,a^2$, $\sigma_{\mathrm{o}} = (\rho_0/5)\left(a^4/b^2\right)$

3.12 $\varepsilon_0 E_0\,[3 - \alpha r]\exp(-\alpha r)$

4.1 If the differential distance $\mathrm{d}\boldsymbol{\ell}$ is on the surface of constant potential then $\mathrm{d}\varphi = 0$ and $q\mathbf{E}{\cdot}\mathrm{d}\boldsymbol{\ell} = 0$. If $\mathbf{E}{\cdot}\mathrm{d}\boldsymbol{\ell} = 0$ then $\mathbf{E}\perp\mathrm{d}\boldsymbol{\ell}$.

4.2 $-4\varepsilon_0 e^{-x^2}\left(\sin(x+y) + x\cos(x+y) - x^2\sin(x+y)\right)$

4.3 (a) Laplace's Equation, Theorem II applies and the potential is zero inside the sphere. (b) $-V_{\mathrm{R}}R/r$

4.5 $\left(Q/\left(2\pi\varepsilon_0 R^2\right)\right)\left(\sqrt{z^2+R^2} - z\right)$

4.6 $\left(Q/\left(2\pi\varepsilon_0 R^2\right)\right)\left(1 - z/\sqrt{z^2+\mathrm{R}^2}\right)$

4.7
$$(\sigma_{\mathrm{s}}R/(2\varepsilon_0))\ln\left(\left[2(z+\ell/2) + 2\sqrt{R^2+(z+\ell/2)^2}\right]/\left[2(z-\ell/2) + 2\sqrt{R^2+(z-\ell/2)^2}\right]\right) + (\sigma_{\mathrm{e}}/(2\varepsilon_0))\left(\sqrt{(z-\ell/2)^2+R^2} + \sqrt{(z+\ell/2)^2+R^2} - (2z)\right)$$

4.8 This is a Coulomb electrostatic field for a small very thin cylinder with negligible charge on the end plates, which is consistent with $z \pm \ell/2 \gg R$.

4.10 (a) $E_{\mathrm{r}} = (q/(4\pi\varepsilon_0))\left(1/(\lambda r) + 1/r^2\right)\exp(-r/\lambda)$ (b) $\rho = -\left(q/\left(4\pi\lambda^2 r\right)\right)\exp(-r/\lambda)$ c) c) This is a negative charge density, which deceases rapidly away from the origin. The positive charge then has attracted negative charges to its vicinity. These negative charges shield the positive charge.

4.14 $C = \varepsilon_0 A/d$

4.16 $\mathbf{E}(z) = \hat{e}_z(\rho\varepsilon/(2\varepsilon_0))\left(1 - z\left(z^2 + R^2\right)^{-1/2}\right)$ for $z > \varepsilon/2$ and $\mathbf{E}(z) = -\hat{e}_z(\rho\varepsilon/(2\varepsilon_0))\left(1 + z\left(z^2 + R^2\right)^{-1/2}\right)$ for $z < -\varepsilon/2$

4.17 $\mathbf{E}_0 = \hat{e}_z(\beta/(8\varepsilon_0))\Big[-L^2 + L\sqrt{L^2 + 4R^2} + 2R^2\ln\left(\left(-L + \sqrt{L^2 + 4R^2}\right)/\left(L + \sqrt{L^2 + 4R^2}\right)\right)\Big]$

4.18 $\varphi = -(\rho/(4\varepsilon_0))\Big\{\Big[(z-L)\sqrt{R^2 + (z-L)^2} + R^2\ln\left(\left(z - L + \sqrt{R^2 + (z-L)^2}\right)/\left(z + \sqrt{R^2 + (z)^2}\right)\right)\Big] \mp (z-L)^2 - \left[(z)\sqrt{R^2 + (z)^2}\right] \pm z^2\Big\}$

4.19 $U_E = (1.2/2)\left(Q^2/(4\pi\varepsilon_0 R)\right)$

4.20 (a) $C = 55.6$ pF b) $U_E = 2.7817 \times 10^{-13}$ J c) $\frac{1}{2}\varepsilon_0 E^2 = 442.73\frac{\mathrm{J}}{\mathrm{m}^3}$ d) $E = 1.0 \times 10^7\frac{\mathrm{V}}{\mathrm{m}}$ e) $E/E_{\mathrm{breakdown}} = 8.47$

4.21 (a) $C = 200$ ¯F (b) $A = 225.88\ \mathrm{m}^2$ c) This is somewhat large for microcircuits.

4.22 (d) $C_T^{(s)}/C_T^{(p)} = C_1C_2/(C_1 + C_2)^2 = C_1C_2/\left(C_1^2 + C_2^2 + 2C_1C_2\right) < 1$

5.1 $\left(1/9 \times 10^{16}\right)\ \mathrm{m}^{-2}\ \mathrm{s}^2$

5.2 $\mathbf{B} = \hat{e}_z B$

5.3 $B_x = -x\beta\frac{B}{2}\exp(+\beta z)$, $B_y = -y\beta\frac{B}{2}\exp(+\beta z)$, $B_z = B\exp(+\beta z)$

5.4 (a) $V_{\mathrm{Hall}} = av_e B$ b) $I_{\mathrm{Hall}} = \sigma v_e B a\ell$ c) $P_{\mathrm{MHD}} = \sigma\ell a^2 v_e^2 B^2$

5.11 (a) $\partial A_\vartheta/\partial z$ is small but $\neq 0$, $\partial A_r/\partial z = \partial A_z/\partial r$ b) $-\frac{1}{r}(\partial/\partial r)(r\partial A_\vartheta/\partial z) + (1/r)(\partial/\partial r)(r\partial A_\vartheta/\partial z) = 0$

6.1 $\mathbf{A} = \hat{e}_z\frac{I_\lambda\mu_0}{4\pi}\Big\{w + x\ln\left[\left((x - w/2)^2 + y^2\right)/\left((x + w/2)^2 + y^2\right)\right] - \frac{w}{2}\ln\left[\left((x - w/2)^2 + y^2\right)\left((x + w/2)^2 + y^2\right)\right] + 2y\arctan[(x - w/2)/y] - 2y\arctan[(x + w/2)/y]\Big\}$

6.2 $\mathbf{B} = (I_0\mu_0/(2\pi r))\hat{e}_\vartheta$ for $x, y \gg w$

6.3 $\mathbf{B} = \hat{e}_z\mu_0 N_\lambda I_0\left(1 + 4(L/R)^2\right)^{-1/2} \approx \hat{e}_z\mu_0 N_\lambda I_0$ if $R/L \ll 1$

6.5 $\mathbf{J} = -\hat{e}_\vartheta(\mu_0 N_\lambda I_0/(2r))$

6.10 $\mathbf{B} = (\mu_0 I_0 N/(2a))\hat{e}_z$.

6.11 $B_r = \frac{3}{4}a^2\mu_0 I_0\left(a^2 + z^2\right)^{-5/2} zr$

6.13 $\mathbf{A} = -\hat{e}_z(I_0\mu_0/(2\pi))\{\ln[r_1/r_2]\}$

6.14 $\mathbf{B} = -(I_0\mu_0 a/\pi)\left[a^4 - 2a^2\left(x^2 - y^2\right) + \left(x^2 + y^2\right)^2\right]^{-1}\left\{\hat{e}_x 2xy + \hat{e}_y\left[a^2 - \left(x^2 - y^2\right)\right]\right\}$ As $a \to 0$ this becomes zero for all values of $(x, y)$. In the limit of small $a$. This is directly proportional to $a$. The magnetic field induction should become zero as the wires coalesce into a single wire.

6.15 $\mathbf{B}(r) = (\mu_0 J/2)\left[b^2/r - a^2/(r+s)\right]\hat{e}_\vartheta$

7.3 (a) he canonical momenta are $p_x = m\dot{x} - m(\Omega/2)y$ and $p_y = m\dot{y} + m(\Omega/2)x$

7.6 The canonical equations are
$\dot{x} = (1/m)(p_x + m(\Omega/2)y)$,
$\dot{y} = (1/m)\left(p_y - m(\Omega/2)x\right)$,
$\dot{z} = (1/m)\,p_z$,
$\dot{p}_x = (\Omega/2)\left(p_y - m(\Omega/2)x\right)$,
$\dot{p}_y = -(\Omega/2)(p_x + m(\Omega/2)y)$,
$\dot{p}_z = 0$
The result is motion in the $z-$direction

7.7 The canonical equations are
$\dot{x} = (1/m)(p_x + m(\Omega/2)y)$
$\dot{y} = (1/m)\left(p_y - m(\Omega/2)x\right)$
$\dot{z} = (1/m)\,p_z$
$\dot{p}_x = (\Omega/2)\left(p_y - m(\Omega/2)x\right)$
$\dot{p}_y = -(\Omega/2)(p_x + m(\Omega/2)y)$
$\dot{p}_z = QE.$

7.8 The canonical equations are
$\dot{x} = (1/m)(p_x + m(\Omega/2)y)$
$\dot{y} = (1/m)\left(p_y - m(\Omega/2)x\right)$
$\dot{z} = \frac{p_z}{m}$
$\dot{p}_x = (\Omega/2)\left(p_y - m(\Omega/2)x\right)$
$\dot{p}_y = -(\Omega/2)(p_x + m(\Omega/2)y) + QE_y$
$p_z = QE_z$

7.9 (a) $(E/B) < R(QB/m)$ (b) $(E/B) = R(QB/m)$ c) $(E/B) > R(QB/m)$

8.1 (a) $\varphi = -(\rho_0/(4\varepsilon_0))\left(2R^2\ln R - R^2 + r^2\right)$ (b) $\varphi = -(\alpha/(16\varepsilon_0))\left(4R^4\ln R - R^4 + r^4\right)$

8.2 (a) $\varphi = -(\rho_0/(2\varepsilon_0))\,R^2\ln r$ b) $\varphi = -(\alpha/(4\varepsilon_0))\,R^4\ln r$

8.3 $\varphi = -(\lambda/(2\pi\varepsilon_0))\ln(r/R)$

8.4 For $a < r < R$, $\varphi = (A/(2\varepsilon_0 r))\left(r^2 - a^2\right) + (A/\varepsilon_0)(R - r)$ For $r > R$, $\varphi = (A/(2\varepsilon_0 r))\left(R^2 - a^2\right)$

8.6 For $a < r < R$, $\varphi = -(A/\varepsilon_0)(r - a) + (A/(2\varepsilon_0 r))\left(r^2 - a^2\right)$ For $R < r$, $\varphi = -(A/\varepsilon_0)(R - a) + (A/(2\varepsilon_0 r))\left(R^2 - a^2\right)$

8.7 $A_\vartheta = -\mu_0 N_\lambda I_0 R\ln(R)$, $\mathbf{B} = -e_z\mu_0 N_\lambda I_0 R\ln(R)\,\frac{1}{r}$

8.8 $\varphi = -(\sigma_0/\varepsilon_0)\,x$ and $\mathbf{E} = \hat{e}_x(\sigma_0/\varepsilon_0)$

8.9 For $(a+\varepsilon) < r < b$, $\varphi = -(Q_a/(4\pi\varepsilon_0 r)) - (Q_b/(4\pi\varepsilon_0 b))$, For $b < r < (b+\varepsilon)$, $\varphi = -(Q_b + Q_a)/(4\pi\varepsilon_0 b)$, For $b < r$, $\varphi = -(Q_a + Q_b)/(4\pi\varepsilon_0 r)$, and $Q_a = 4\pi\varepsilon_0(V_a - V_b)(ab/(a-b))$, $Q_b = -4\pi\varepsilon_0(b/(a-b))(aV_a - bV_b)$

8.10 For $r < a$ $A_z = -\mu_0 J_0\left[\frac{1}{2}b^2\ln b - \frac{1}{2}a^2\ln a - \frac{1}{4}\left(b^2 - a^2\right)\right]$, For $a \le r \le b$ $A_z = \mu_0 J_0\left(a^2/2\right)\ln(r) - \mu_0 J_0(1/4)\,r^2 - \mu_0 J_0\left((1/2)\,b^2\ln b - (1/4)\,b^2\right)$, For $b < r$ $A_z = -\mu_0 J_0(1/2)\left(b^2 - a^2\right)\ln(r)$, For $r < a$

$\mathbf{B} = -\hat{e}_\vartheta \mathrm{d}A_z/\mathrm{d}r = \mathbf{0}$, For $a \leq r \leq b$ $\mathbf{B} = \hat{e}_\vartheta (\mu_0 J_0/2)\left(\left(r^2 - a^2\right)/r\right)$, For $b < r$ $\mathbf{B} = -\hat{e}_\vartheta \mu_0 J_0 (1/2)\left(b^2 - a^2\right)(1/r)$

10.1 (a) $\mathcal{E} = \pi a^2 \mu_0 N_\lambda^2 L_S \mathrm{d}I/\mathrm{d}t$ c) $I_b(t) = \pi a^2 \mu_0 (V/(R_b L)) \exp(-Rt/L)$
10.2 the current will decrease.
10.3 (a) $\mathcal{E} = Bav$ (b) $\mathcal{E} = Bav$
10.4 If Faraday's Law is actually a Law of physics, which it is, then this is an intolerable situation.
10.5 $E = BA\omega \sin \omega t$
10.6 $I_\ell = (\mu_0 I/(4\pi R))\, \omega \ln(b/a)\, x_0 \sin(\omega t)$
10.7 (a) $\mathcal{E}(t) = n\pi a^2 B\omega \sin \omega t$ b) $B = \left(1/\left(2n\pi a^2\right)\right) \int_0^{\tau/2} \mathcal{E}(t)\mathrm{d}t$
10.8 $I = (Bab\omega/R) \sin(\omega t)$
10.11 (a) $E_\vartheta = (1/2)\, r\mu_0 N_\lambda I_0 \sin \omega t$ (b) $I_{\text{conductor}} = (\sigma/4)\, R^2 L \mu_0 N_\lambda I_0 \sin \omega t$
10.13 In the case of the cylindrical solenoid there is an energy density. But the field energy is spread over larger regions of space and a density cannot be easily calculated.

The fact that we can, in each case, identify an inductance $L$, which is a function only of the geometric properties of the solenoid is a result of Faraday's Law and the Biot–Savart Law.

11.2 (a) $y-$axis and move in the *negative* direction b) $\hat{E} = -\hat{e}_z$
11.4 (b) The Gaussian pulse is a laboratory reality. Our result in (a) indicates that it will propagate at the speed $c$ into the space beyond the lamp as a function of $p = \omega t - kz$.
11.9 The Fourier Transformation is a representation of the disturbance in terms of a particular set of basis functions, the complex exponentials $\exp(i\omega t \pm i\mathbf{k}\cdot\mathbf{r})$. In physical terms these are propagating plane waves.
12.1 Momentum density is $\rho_P = 2\lambda_{\mathcal{E}}/\mathcal{V}^2$ where $\lambda_E =$ kinetic energy flux.
12.4 $\mathbf{S} = -(1/\mu_0)(J/\sigma)(\mu_0 JR/2)\,\hat{e}_r$
12.6 $\mathbf{S} = \hat{e}_z E_0^2 \sqrt{\epsilon_0 \mu_0}$
12.7 $Q^2/24\pi\varepsilon_0 R$
13.9 $m\gamma_u c^2 = 0.520\,75$ MeV, $\beta_u = 0.192\,70$
13.13 (a) $\mathbf{F} = -qE'_y \mathbf{e}_y$ b) $\mathcal{E} = \beta a B'_z$ Faraday's Law then predicts the same emf as that resulting from the electric field $E_y$.
14.4 If we are in a region of space close to the moving charges the radiation gauge cannot be used and we must use the Lorentz Gauge.
14.5 It happened when $\partial\mathbf{A}/\partial t = (\mu_0/(4\pi R))\, \ddot{p}_d \hat{e}_a$
14.6 The dominant emission is axial.
14.7 The orbiting electron will radiate energy and will eventually fall into the nucleus. The electron falling into the nucleus will radiate energy at the frequency with which it traverses the orbit. This will change continuously. There can be no line spectrum.
15.2 $(C_T/C) = 2(\varepsilon/(\varepsilon + \varepsilon_0))$
15.3 (a) $\sigma_1 = (2Q/A)\,\varepsilon_0/(\varepsilon_0 + \varepsilon)$, $\sigma_2 = (2Q/A)\,\varepsilon/(\varepsilon_0 + \varepsilon)$ b) $E_1 = E_2 = (2Q/A)\, 1/(\varepsilon_0 + \varepsilon)$ c) $C_T = (A/(2y))(\varepsilon_0 + \varepsilon)$
15.4 (a) $U = \varepsilon_0 L/(2y)\, V^2 [(K-1)x + L]$ (b) $F = \varepsilon_0 L/(2y)\, V^2 (K-1)$

15.5 (a) $\rho_\mathrm{p} = -2\alpha z$ b) $\sigma_\mathrm{p}(z=0) = -\beta$, $\sigma_\mathrm{p}(z=L) = \alpha L^2 + \beta$ c) $Q_\mathrm{p} = Q_\mathrm{p}(\text{inside}) + Q_\mathrm{p}(\text{end caps}) = 0$

15.6 (a) $\mathbf{P}(\mathbf{r}) = (K-1)\,Q/\left(K4\pi r^2\right)\hat{e}_\mathrm{r}$ b) $\rho_\mathrm{p} = 0$ c) $\sigma_\mathrm{p}(a) = -(K-1)\,Q/\left(4\pi K a^2\right)$, $\sigma_\mathrm{p}(b) = (K-1)\,Q/\left(4\pi K b^2\right)$ d) $Q_\mathrm{p,\,total} = 4\pi a^2\sigma_\mathrm{p}(a) + 4\pi b^2\sigma_\mathrm{p}(b) = 0$

15.7 (a) $\rho_\mathrm{p} = -\left(Q/\left(4\pi r^2\right)\right)\left(\alpha/(\alpha r+1)^2\right)$ (b) $\sigma_\mathrm{p}(a) = -(\alpha a/(1+\alpha a))\left(Q/\left(4\pi a^2\right)\right)$, $\sigma_\mathrm{p}(b) = (\alpha b/(1+\alpha b))\left(Q/\left(4\pi b^2\right)\right)$ c) $Q_\mathrm{p,\,total} = 0$ (d) There has been no real charge transferred to the dielectric.

15.8 (a) $\rho_\mathrm{p} = 0$ b) $\sigma_\mathrm{p} = \mathcal{P}_0\cos\phi$ c) $Q_\mathrm{s} = 0$

15.10 $\varphi_\mathrm{in} = D_0^{(\mathrm{in})} + D_1^{(\mathrm{in})} r\cos\phi$, $\varphi_\mathrm{out} = D_0^{(\mathrm{out})} + \left[D_1^{(\mathrm{out})} r + G_1^{(\mathrm{out})}\left(1/r^2\right)\right]\cos\phi$

15.12 $\mathbf{E}_\mathrm{m} = \mathbf{E} + \mathbf{P}/(3\varepsilon_0)$

15.13 (a) $\alpha = (3\varepsilon_0/n)(K-1)/(K+2)$ b) $\chi = (\alpha n/\varepsilon_0)(1 - n\alpha/(3\varepsilon_0))$

15.14 (a) $n\alpha = 3\varepsilon_0$ b) $\chi = (\alpha n/\varepsilon_0)/\xi$

15.15 (a) $\rho_\mathrm{p} = 0$ c) $BL\pi\left(a^2 - b^2\right)$

15.16 (a) $\mathbf{J}_\mathrm{M} = \mathbf{0}$ b) $\mathbf{J}_\mathrm{M}^{(\mathrm{s})} = -\hat{e}_\mathrm{z}(K_\mathrm{M}-1)(I/2\pi b)$ for the outer surface and $\mathbf{J}_\mathrm{M}^{(\mathrm{s})} = +\hat{e}_\mathrm{z}(K_\mathrm{M}-1)(I/2\pi a)$ for the inner surface c) $\mathbf{J}_\mathrm{M}^{(\mathrm{s})}\hat{e}_\mathrm{r}\lambda(K_\mathrm{M}-1)(I/2\pi r)$for the top surface and $\mathbf{J}_\mathrm{M}^{(\mathrm{s})} = -\hat{e}_\mathrm{r}\lambda(K_\mathrm{M}-1)(I/2\pi r)$ for the bottom surface

15.18 (b) yes (c) no

16.1 (a) $\sigma = -iNQ^2/(m\omega)$ (b) $\omega = \omega_\mathrm{p,e}$, $\mathcal{E}_\mathrm{particle} = \varepsilon_0\left(\omega_\mathrm{p,e}^2/\omega^2\right)E^2$, $\mathcal{E}_\mathrm{wave} = \varepsilon_0\left(1+\left(\omega_\mathrm{p,e}^2/\omega^2\right)\right)E^2$ c) $\omega = \sqrt{\omega_\mathrm{p,e}^2 + k^2c^2}$, $\partial\omega/\partial k = kc^2/\sqrt{\omega_\mathrm{p,e}^2 + k^2c^2} < c$, $\mathcal{E}_\mathrm{particle} = \varepsilon_0\left(\omega_\mathrm{p,e}^2/\omega^2\right)E^2$, $\mathcal{E}_\mathrm{wave} = 2\varepsilon_0 E^2$

16.2 (a) longitudinal: $\sigma = -i\varepsilon_0\omega_\mathrm{p,e}^2\omega/(\omega-ku)^2$, transverse: $\sigma = -i\varepsilon_0\omega_\mathrm{p,e}^2/(\omega-ku)$ b) $\omega = ku \pm \omega_\mathrm{p,e}$, $\mathcal{E}_\mathrm{particle} = \varepsilon_0\omega_\mathrm{p,e}^2\left((\omega+ku)/(\omega-ku)^3\right)E^2$, $\mathcal{E}_\mathrm{wave} = \varepsilon_0\left(1+\omega_\mathrm{p,e}^2(\omega+ku)/(\omega-ku)^3\right)E^2$ c) $\omega^2 = \omega_\mathrm{p,e}^2 + k^2c^2$as $u \to 0$, $\partial\omega/\partial k = kc^2/\omega$ as $u \to 0$, $\mathcal{E}_\mathrm{particle} = \varepsilon_0\left(\omega_\mathrm{p,e}^2/(\omega-ku)^2\right)E^2$, $\mathcal{E}_\mathrm{wave} = \left(2\varepsilon_0 + \varepsilon_0\omega_\mathrm{p,e}^2(ku)/\left(\omega(\omega-ku)^2\right)\right)E^2$

16.3 (a) $\sigma = -i\varepsilon_0\omega_\mathrm{p,e}^2\omega/(\omega-ku)^2$ b) $\omega = ku \pm \omega_\mathrm{p,e}$, $\mathcal{E}_\mathrm{particle} = \varepsilon_0\omega_\mathrm{p,e}^2\left((\omega+ku)/(\omega-ku)^3\right)E^2$, $\mathcal{E}_\mathrm{wave} = \varepsilon_0\left(1+\omega_\mathrm{p,e}^2(\omega+ku)/(\omega-ku)^3\right)E^2$

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