Angular Momentum Techniques in Quantum Mechanics

by

V. Devanathan

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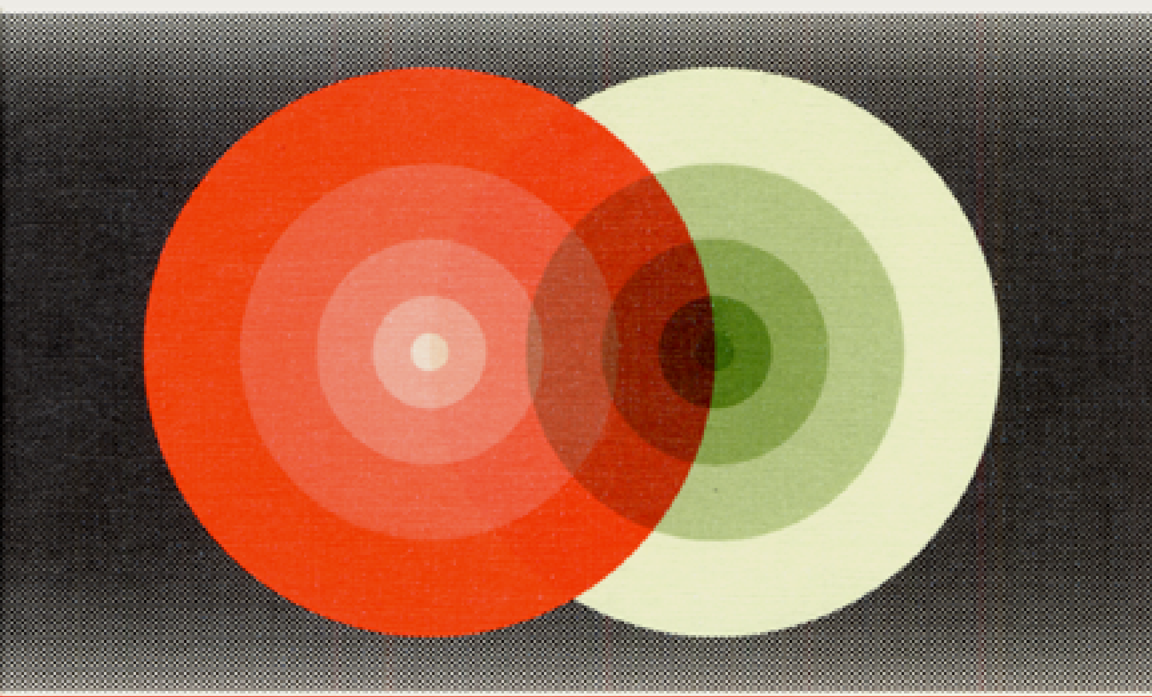

Fundamental Theories of Physics

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To my teacher

Professor Alladi Ramakrishnan

who has inspired me to take to research and teaching

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CONTENTS

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Preface

A course in angular momentum techniques is essential for quantitative study of problems in atomic physics, molecular physics, nuclear physics and solid state physics. This book has grown out of such a course given to the students of the M.Sc. and M.Phil. degree courses at the University of Madras. An elementary knowledge of quantum mechanics is an essential pre-requisite to undertake this course but no knowledge of group theory is assumed on the part of the readers. Although the subject matter has group-theoretic origin, special efforts have been made to avoid the group-theoretical language but place emphasis on the algebraic formalism developed by Racah (1942a, 1942b,1943,1951). How far I am successful in this project is left to the discerning reader to judge.

After the publication of the two classic books, one by Rose and the other by Edmonds on this subject in the year 1957, the application of angular momentum techniques to solve physical problems has become so common that it is found desirable to organize a separate course on this subject to the students of physics. It is to cater to the needs of such students and research workers that this book is written. A large number of questions and problems given at the end of each chapter will enable the reader to have a clearer understanding of the subject. Solutions to selected problems are added so that the students can refer to them in case they are unable to solve those problems by themselves and also seek guidelines for solving other problems.

The angular momentum coefficients,. the rotation matrices, tensor operators, evaluation of matrix elements, the gradient formula, identical particles, the statistical tensors, traces of angular momentum matrices, the helicity formalism and the spin states of the Dirac particles are some of the topics dealt with in this book. These topics cover the entire range of angular momentum techniques that are being widely used in the study of both non-relativistic and relativistic problems in Physics. Application to physical problems that are given in this book are mostly drawn from the author's own experience and hence may appear lop-sided in favour of nuclear and particle physics.

There is a bewildering variety of notations and phase conventions used in the literature and those adopted in this book correspond mostly to those used by Rose with some exceptions. A square bracket has been used for the Clebsch-Gordan coefficient and the author has found this notation more convenient for working out complicated problems involving Clebsch-Gordan coefficients. This notation has been used earlier by a few authors. For the convenience of readers, a list of symbols and notations used in this book is given separately in Appendix G.

The author has originally thought of including computer programs in FORTRAN for the calculation of angular momentum coefficients and for the evaluation of certain important matrix elements but since the usage of computers has become so common and each has his own choice of language, it is felt more appropriate to give general expressions that can be used for computer programming rather than giving the program in any one particular language.

The author has worked extensively on problems involving angular momentum algebra in the early stages of his research career and is indebted to Prof. Alladi Ramakrishnan and Prof. M.E. Rose for having inspired him to take to research in this area. The author has benefited greatly with discussions with his earlier collaborators Prof. M.E. Rose, Prof. H. Überall, Prof. G. Ramachandran, Prof. K. Srinivasa Rao, Prof. R. Parthasarathy, Prof. G. Shanmugam and Prof. N. Arunachalam. The author is grateful to Prof. P.R. Subramanian, Dr. V. Girija, Dr. M. Rajasekaran, Dr. S. Karthiyayini, Dr. G. Janhavi, Dr. K. Ganesamurthy, Dr. S. Ganesa Murthy, Mr. P. Ratna Prasad, Mr. S. Arunagiri for many interesting discussions and careful reading of the manuscript at different stages of writing and to a host of students who have been a source of inspiration. The book had gone through many drafts and the initial drafts were prepared by Mr. L. Thulasidoss with great patience and the final version was prepared with meticulous care with the help of Mr. S. Ganesa Murthy and Ms. D. Sudha and the software support received from Mr. K. Shivaji, Mr. T. Samuel and Dr. G. Subramonium. The author is grateful to the University Grants Commission and to the Tamil Nadu State Council for Science and Technology for sponsoring this book under the book writing scheme and to Professors P. Ramasamy, P.R. Subramanian, R. Ramachandran and K. Subramanian for extending all the facilities for completing the manuscript. The author acknowledges with thanks the facilities offered by the Department of Nuclear Physics of the University of Madras, the Crystal Growth Centre of the Anna University, the Institute of Mathematical Sciences and the Tamil Nadu Academy of Sciences for the preparation of the manuscript. It is a pleasure to thank Professor Krishnaswami Alladi for suggesting the Kluwer Academic Publishers for publication of this book and Mr. D.J. Larner, the Publishing Director and Ms. Margaret Deignan of the Kluwer Academic Publishers for rapidly processing the manuscript and undertaking the publication.

V. Devanathan

CHAPTER 1

ANGULAR MOMENTUM OPERATORS AND THEIR MATRIX ELEMENTS

1.1. Quantum Mechanical Definition

In classical mechanics, the angular momentum vector is defined as the cross product of the position vector $\boldsymbol{r}$ and the momentum vector $\boldsymbol{p}$. i.e.

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{1.1}$$

Both $\boldsymbol{r}$ and $\boldsymbol{p}$ change sign under inversion of co-ordinate system and so they are called Polar Vectors. It is easy to see that $\boldsymbol{L}$ behaves differently and will not change sign under inversion of coordinate system and it is known as a Pseudo-Vector or an Axial Vector.

The transition to quantum mechanics can be made by incorporating the uncertainty principle into the classical definition and $\boldsymbol{L}$ becomes a Hermitian operator. Introducing the uncertainty principle expressed in the form of commutators,

$$[x, p_x] = i\hbar, \quad [y, p_y] = i\hbar, \quad [z, p_z] = i\hbar, \tag{1.2}$$

we obtain the commutation relations (Schiff, 1968) for the components of angular momentum operator.

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y. \tag{1.3}$$

These commutators define the angular momentum in quantum mechanics (Schiff, 1968; Rose, 1957; Ramakrishnan, 1962) and this definition is more general and admits half integral quantum numbers. For this purpose, let us denote the quantum mechanical angular momentum operator by $\boldsymbol{J}$ and also use the convention that the angular momentum is expressed in units of $\hbar$.

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \tag{1.4}$$

In a compact notation, the three Eqs. (1.4) become

$$\boldsymbol{J} \times \boldsymbol{J} = i\boldsymbol{J}. \tag{1.5}$$

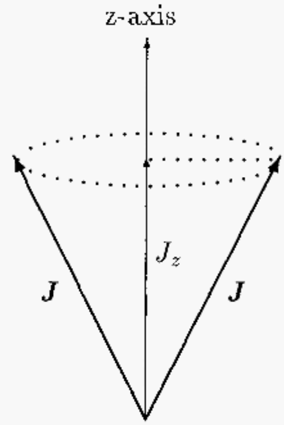

Figure 1.1. The angular momentum vector, for which its projection on the z-axis alone is well defined but not its projection on the x-axis or y-axis.

Equation (1.5) is the starting point of our investigation and our aim is to draw as much information as possible from this definition.

1.2. Physical Interpretation of Angular Momentum Vector

Although the components of the angular momentum operator do not commute among themselves, it is easy to show that the square of the angular momentum operator

$$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2 \tag{1.6}$$

$$[\boldsymbol{J}^2, J_x] = 0, \quad [\boldsymbol{J}^2, J_y] = 0, \quad [\boldsymbol{J}^2, J_z] = 0. \tag{1.7}$$

Equations (1.4) and (1.7) are amenable to simple physical interpretation. It is possible to find the simultaneous eigenvalues of $\boldsymbol{J}^2$ and of one of the components, say J_z alone but it is impossible to find precisely the eigenvalues of J_x and J_y at the same time. Representing the operators by matrices, one can say that $\boldsymbol{J}^2$ and J_z can be diagonalized in the same representation but not the other components J_x and J_y. Physically this means that one can know at the most, the magnitude of the angular momentum vector and its projection on one of the axes. The projections on the other two axes cannot be determined. This is illustrated in Fig. 1.1, in which the angular momentum vector is depicted to be anywhere on the cone. If ψ_{jm} is the eigenfunction of the operators $\boldsymbol{J}^2$ and J_z, then

$$\boldsymbol{J}^2\psi_{jm} = \eta_j\psi_{jm} \tag{1.8}$$

and

$$J_z\psi_{jm} = m\psi_{jm}. \tag{1.9}$$

In the above equations, j and m are the quantum numbers used to define the eigenfunctions and the corresponding eigenvalues of the operators are η_j and m. We are interested in finding the spectrum of values that j and m can take and also the eigenvalue η_j

1.3. Raising and Lowering Operators

Let us define two more operators J_+ and J_- which we shall call raising and lowering operators.

$$J_\pm = J_x \pm iJ_y. \tag{1.10}$$

This nomenclature will become obvious, once their roles are understood. The following commutation relations can be easily obtained.

$$[\boldsymbol{J}^2, J_\pm] = 0, \quad [J_z, J_\pm] = \pm J_\pm. \tag{1.11}$$

Let us now generate a new function Φ by allowing $J_\pm$ to operate on ψ_{jm} and examine whether this new function is an eigenfunction of $\boldsymbol{J}^2$ and J_z operators. If so, what are their eigenvalues ?
Let

$$J_\pm\psi_{jm} = \Phi_\pm. \tag{1.12}$$

Then

$$\begin{aligned} \boldsymbol{J}^2\Phi_\pm &= \boldsymbol{J}^2 J_\pm \psi_{jm} \\ &= J_\pm \boldsymbol{J}^2 \psi_{jm}, \qquad \text{using Eq. (1.11)} \\ &= \eta_j \Phi_\pm, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} J_z\Phi_{jm} &= J_z J_\pm \psi_{jm} \\ &= (J_\pm J_z \pm J_\pm)\psi_{jm}, \qquad \text{using Eq. (1.11)} \\ &= J_\pm(J_z \pm 1)\psi_{jm} \\ &= (m \pm 1)\Phi_\pm. \end{aligned} \tag{1.14}$$

Thus we find that $\Phi_\pm$ is an eigenfunction of $\boldsymbol{J}^2$ and J_z operators. The eigenvalue of the operator $\boldsymbol{J}^2$ remains unchanged but the eigenvalue of the operator J_z is stepped up or stepped down by unity. It is precisely for this reason, the operator $J_\pm$ is called the raising or lowering operator. Sometimes they are also known as ladder operators.

1.4. Spectrum of Eigenvalues

From the above discussion, it is obvious that m can take a spectrum of values differing by unity for a given value of *j*. It is easy to show that the values that *m* can take are bounded for a given *j*. For this, consider the following relation:

$$(J_x^2 + J_y^2)\psi_{jm} = (\boldsymbol{J}^2 - J_z^2)\psi_{jm} = (\eta_j - m^2)\psi_{jm}. \tag{1.15}$$

Since the diagonal elements of the squares of Hermitian operators J_x and J_y are either positive or zero,

$$\eta_j - m^2 \geq 0. \tag{1.16}$$

This means that the values of *m* are bounded for a given value of *j*. Let us denote the lowest value of *m* by m_1 and the highest value of *m* by m_2, the spectrum of values that *m* can take being

$$m_1, m_1 + 1, \ldots, m_2. \tag{1.17}$$

Then it follows that

$$J_+\psi_{jm_2} = 0, \tag{1.18}$$

$$J_-\psi_{jm_1} = 0. \tag{1.19}$$

Operating J_- on the left of Eq. (1.18) and J_+ on the left of Eq. (1.19), we obtain

$$J_-J_+\psi_{jm_2} = 0, \tag{1.20}$$

$$J_+J_-\psi_{jm_1} = 0. \tag{1.21}$$

Since

$$J_-J_+ = \boldsymbol{J}^2 - J_z(J_z + 1), \tag{1.22}$$

$$J_+J_- = \boldsymbol{J}^2 - J_z(J_z - 1), \tag{1.23}$$

we get the following relations:

$$\eta_j - m_2(m_2 + 1) = 0, \tag{1.24}$$

$$\eta_j - m_1(m_1 - 1) = 0. \tag{1.25}$$

From Eqs. (1.24) and (1.25), we obtain

$$m_2(m_2+1) = m_1(m_1-1),$$

i.e.,

$$(m_1+m_2)(m_2-m_1+1) = 0. \tag{1.26}$$

Since m_2 - m_1 is positive by our choice, it follows that $m_1 = -m_2$. If we label the highest value of m by j, then the spectrum of values that m can take can be written down as follows,

$$-j, -j+1, \ldots, j-1, j. \tag{1.27}$$

There are $2j+1$ values of m and hence $2j+1$ should be an integer. That means $2j$ is an integer and hence j can be either an integer or half integer. Using Eq. (1.24) or Eq. (1.25), we obtain the eigenvalue of $\boldsymbol{J}^2$ operator

$$\eta_j = j(j+1). \tag{1.23}$$

Let us now summarize the results so far obtained. Starting from the quantum mechanical definition of angular momentum given by Eq. (1.5), we have shown that the eigenvalue of $\boldsymbol{J}^2$ operator is $j(j+1)$ where j can take integral or half integral values and for a given j, the eigenvalues of J_z operator, viz., m can take a spectrum of values from $-j$ to $+j$ in steps of unity. It is to be emphasized that all these results follow directly from the definition of angular momentum operator (Eq. (1.5)) and no assumption or approximation has been made in deducing these results.

1.5. Matrix Elements

Having deduced the eigenvalues of $\mathbf{J}^2$ and J_z operators, let us now proceed to determine the matrix elements of J_x and J_y operators or equivalently $J_\pm$ in the same representation in which $\boldsymbol{J}^2$ and J_z are diagonal. Then

$$J_x = \frac{1}{2}(J_+ + J_-). \tag{1.29}$$

$$J_y = \frac{1}{2i}(J_+ - J_-). \tag{1.30}$$

Equations (1.13) and (1.14) clearly show that $\Phi_\pm$ is an eigenfunction of $\boldsymbol{J}^2$ and J_z operators with eigenvalues η_j and $m \pm 1$. That means that $\Phi_\pm$ and the normalized function $\psi_{jm\pm 1}$ may differ at the most by a constant factor.

$$\Phi_\pm = |\Gamma_\pm|\, \psi_{j,m\pm 1}. \tag{1.31}$$

Taking the scalar product, we get

$$\langle \Phi_\pm, \Phi_\pm \rangle = |\Gamma_\pm|^2. \tag{1.32}$$

Expanding the left hand side and using Eqs. (1.22) and (1.23), we obtain

$$\begin{aligned}\langle\Phi_\pm,\Phi_\pm\rangle &= \langle J_\pm\psi_{jm}\,|\,J_\pm\psi_{jm}\rangle \\ &= \langle\psi_{jm}\,|\,J_\mp J_\pm\,|\,\psi_{jm}\rangle \\ &= \eta_j - m(m\pm 1).\end{aligned} \tag{1.33}$$

From Eqs. (1.32),(1.33) and (1.28), it follows that

$$|\Gamma_\pm|^2 = j(j+1) - m(m\pm 1) = (j\mp m)(j\pm m+1). \tag{1.34}$$

Taking the square root, we get

$$\Gamma_\pm = [(j\mp m)(j\pm m+1)]^{1/2}. \tag{1.35}$$

There is an uncertainty with respect to the phase factor which is fixed usually by convention. Now we have at our disposal all the required matrix elements.

$$\langle j'm'\,|\,\boldsymbol{J}^2\,|\,jm\rangle = j(j+1)\,\delta_{jj'}\,\delta_{mm'}. \tag{1.36}$$
$$\langle j'm'\,|\,J_z\,|\,jm\rangle = m\,\delta_{jj'}\,\delta_{mm'}. \tag{1.37}$$
$$\langle j'm'\,|\,J_+\,|\,jm\rangle = [(j-m)(j+m+1)]^{1/2}\,\delta_{jj'}\,\delta_{m'\,m+1}. \tag{1.38}$$
$$\langle j'm'\,|\,J_-\,|\,jm\rangle = [(j+m)(j-m+1)]^{1/2}\,\delta_{jj'}\,\delta_{m'\,m-1}. \tag{1.39}$$

The above matrix elements are sufficient to construct all the angular momentum matrices.

1.6. Angular Momentum Matrices

Since all the matrix elements (1.36)–(1.39) connect states with the same j but different m values, it is usual to construct angular momentum matrices for a given j value. It is customary to label the rows by m' values ($m' = j, j-1, \ldots, -j+1, -j$) and the columns by m values ($m = j, j-1, \ldots, -j+1, -j$). So, for a given j, the angular momentum matrices are of dimension $(2j+1) \times (2j+1)$.

For $j = 1/2$, m can take only two values 1/2 and -1/2. Using Eqs. (1.36)–(1.39) and labeling the rows and columns by the eigenvalues m of the J_z operator, we obtain the following matrices for $j = 1/2$.

$$\boldsymbol{J}^2 = \begin{array}{c} m'/m \\ 1/2 \\ -1/2 \end{array}\begin{array}{c} \begin{array}{cc} 1/2 & -1/2 \end{array} \\ \left[\begin{array}{cc} 3/4 & 0 \\ 0 & 3/4 \end{array}\right] \end{array}, \tag{1.40}$$

$$J_z = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad J_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{1.41}$$

From J_+ and J_- matrices, we can obtain J_x and J_y using Eqs.(1.29) and (1.30).

$$J_x = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}. \tag{1.42}$$

We find that the matrices J_x, J_y and J_z for $j = 1/2$ are related to the well-known Pauli spin matrices (σ).

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.43}$$

$$J_x = \frac{1}{2}\sigma_x, \quad J_y = \frac{1}{2}\sigma_y, \quad J_z = \frac{1}{2}\sigma_z. \tag{1.44}$$

In a similar way, we can construct the angular momentum matrices for j = 1.

$$J^2 = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad J_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.45}$$

$$J_+ = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_- = \sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{1.46}$$

$$J_x = \sqrt{\frac{1}{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{1.47}$$

Construction of angular momentum matrices for higher values of j can be done following the same procedure.

Review Questions

1.1 Define angular momentum in classical mechanics and incorporating the uncertainty principle, obtain the quantum mechanical definition of angular momentum as a set of commutation relations.

1.2 Starting from the commutation relations of angular momentum operators, determine the allowed spectrum of eigenvalues of $\boldsymbol{J}^2$ and J_z operators.

1.3 Define the raising and lowering angular momentum operators and obtain their matrix elements between any two angular momentum states. Are these operators Hermitian?

1.4 Assuming the following commutation rules obeyed by the angular momentum operators $[J_x, J_y] = iJ_z$, $[J_y, J_z] = iJ_x$, $[J_z, J_x] = iJ_y$, show that $J_+ = J_x + iJ_y$ is a raising operator for the eigenvalue of J_z.

1.5 Obtain the matrix representation of the angular momentum operators for $j = \frac{1}{2}$. Establish their connection with the Pauli matrices.

Problems

1.1 Given the commutation relations (1.2), obtain the commutation relations (1.3).

1.2 Evaluate the commutators $[J_x^2, J_z]$ and $[J_y^2, J_z]$ and show that

$$[J_x^2, J_z] = -[J_y^2, J_z].$$

1.3 Given that $J \times J = iJ$, obtain the following commutation relations:

$$\text{(a) } [\boldsymbol{J}^2, J_z] = 0, \quad \text{(b) } [J_z, J_\pm] = \pm J_\pm, \quad \text{(c) } [J_+, J_-] = 2J_z,$$

where $J_\pm = J_x \pm iJ_y$.

1.4 Evaluate (a) $J_x \psi_{jm}$ and (b) $\psi_{jm} J_y$

1.5 For $j = \frac{1}{2}$, show that $J_x^2 = J_y^2 = J_z^2 = \frac{1}{4}$.

1.6 For $j = 1$, show that $J_x^3 = J_x$, $J_y^3 = J_y$ and $J_z^3 = J_z$.

1.7 Show that the Pauli matrices obey the following relations:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1; \qquad \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 3.$$

1.8 If σ denotes the Pauli vector and $\boldsymbol{A}$ a vector, write down explicitly $\sigma . \boldsymbol{A}$ in the form of a 2 x 2 matrix.

1.9 If σ denotes the Pauli vector and $\boldsymbol{A}$ and $\boldsymbol{B}$ are polar vectors, show that

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}).$$

1.10 Construct the angular momentum matrices for $j = \frac{3}{2}$.

Solutions to Selected Problems

1.1 The components of angular momentum operator $\boldsymbol{L}$ are

$$L_x = yp_z - zp_y; \quad L_y = zp_x - xp_z; \quad L_z = xp_y - yp_x.$$

The commutator $[L_x, L_y]$ is given by

$$\begin{aligned}
[L_x, L_y] &= L_x L_y - L_y L_x \\
&= (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y) \\
&= yp_x(p_z z - zp_z) + xp_y(zp_z - p_z z) \\
&= -i\hbar yp_x + i\hbar xp_y \\
&= i\hbar L_z.
\end{aligned}$$

Similarly, the other cyclic commutation relations are obtained.

1.2

$$\begin{aligned}
[J_x^2, J_z] &= J_x[J_x, J_z] + [J_x, J_z]J_x \\
&= -iJ_xJ_y - iJ_yJ_x.
\end{aligned}$$

Similarly,

$$[J_y^2, J_z] = J_y[J_y, J_z] + [J_y, J_z]J_y = iJ_yJ_x + iJ_xJ_y.$$

Hence

$$[J_x^2, J_z] = -[J_y^2, J_z].$$

1.8

$$\boldsymbol{\sigma} \cdot \boldsymbol{A} = \sigma_x A_x + \sigma_y A_y + \sigma_z A_z.$$

Using the matrix representation (1.43) for the Pauli operators, we obtain

$$\boldsymbol{\sigma} \cdot \boldsymbol{A} = \begin{bmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{bmatrix}.$$

1.9 The Pauli matrices obey the following relations:

$$\begin{aligned}
&\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1. \\
&\sigma_x\sigma_y = -\sigma_y\sigma_x; \quad \sigma_y\sigma_z = -\sigma_z\sigma_y; \quad \sigma_z\sigma_x = -\sigma_x\sigma_z. \\
&\sigma_x\sigma_y = i\sigma_z; \quad \sigma_y\sigma_z = i\sigma_x; \quad \sigma_z\sigma_x = i\sigma_y.
\end{aligned}$$

Expanding $(\sigma \cdot \boldsymbol{A})(\sigma \cdot \boldsymbol{B})$ in terms of the Cartesian components,

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = (\sigma_x A_x + \sigma_y A_y + \sigma_z A_z)(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z),$$

and using the above relations for the Pauli matrices, the final result

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

is obtained after rearrangement.

COUPLING OF TWO ANGULAR MOMENTA

2.1. The Clebsch-Gordan Coefficients

Problems involving the addition of two angular momenta abound in physics. They may be the angular momenta of the two particles in a system or the orbital and spin angular momenta of a single particle.

If $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the operators corresponding to the two angular momenta, then the resultant angular momentum operator $\boldsymbol{J}$ is obtained by the vector addition

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2. \tag{2.1}$$

It follows that

$$\begin{aligned} J_x &= J_{1x} + J_{2x}, \\ J_y &= J_{1y} + J_{2y}, \\ J_z &= J_{1z} + J_{2z}. \end{aligned} \tag{2.2}$$

Squaring (2.1) we obtain,

$$\boldsymbol{J}^2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2\boldsymbol{J}_1 \cdot \boldsymbol{J}_2. \tag{2.3}$$

Since by our construction $\boldsymbol{J}$ is an angular momentum operator, it should obey the same commutation relations as $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$.

$$[\boldsymbol{J}^2, J_z] = 0; \quad [J_x, J_y] = iJ_z. \tag{2.4}$$

The other cyclic relations follow. The commutation relations (2.4) can be deduced from the commutation relations obeyed by $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$. It is to be noted that $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are two independent operators and hence they should mutually commute. However, it is found that

$$[\boldsymbol{J}^2, J_{1z}] = -[\boldsymbol{J}^2, J_{2z}] \neq 0. \tag{2.5}$$

Thus we have two sets of mutually commuting operators.

$$\begin{aligned} &\text{set I:} \quad && \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, J_{1z}, J_{2z}. \\ &\text{set II:} \quad && \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}^2, J_z. \end{aligned} \tag{2.6}$$

So, it is possible to find the simultaneous eigenvalues of either the first set of operators or the second set but not both. The eigenfunctions, denoted by their quantum numbers $|j_1 j_2 m_1 m_2\rangle$ corresponding to the first set of operators are said to be in the uncoupled representation and the eigenfunctions $|j_1 j_2 j m\rangle$ corresponding to the second set belong to the coupled representation. These representations are connected by a unitary transformation. The functions $|j_1 j_2 j m\rangle$ can be expanded in terms of functions $|j_1 j_2 m_1 m_2\rangle$ and vice versa.

$$|j_1 j_2 j m\rangle = \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 m_1 m_2\rangle. \tag{2.7}$$

The quantity $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$ is the expansion coefficient whose dependence on the quantum numbers is explicitly denoted. This coefficient is known as the Clebsch-Gordan (C.G.) coefficient (Condon and Shortley, 1935) or the vector addition coefficient and it is the unitary transformation coefficient that occurs when one goes from the uncoupled to the coupled representation. Although there is a variety of notations for the Clebsch-Gordan coefficient (Condon and Shortley, 1935; Rose, 1957; Pal, 1982), the author has found the above notation very convenient to work out complicated problems involving C.G. coefficients. From Eq. (2.7), we get

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}. \tag{2.8}$$

This C.G. coefficient can be determined without the phase factor and the standard phase convention is such as to make the C.G. coefficient real. Then taking the complex conjugate of Eq. (2.8), we get

$$\langle j_1 j_2 j m | j_1 j_2 m_1 m_2\rangle = \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}, \tag{2.9}$$

from which the inverse relation of Eq. (2.7) is obtained.

$$|j_1 j_2 m_1 m_2\rangle = \sum_{jm} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 j m\rangle. \tag{2.10}$$

2.2. Some Simple Properties of C.G. Coefficients

It is easy to show that $m_1 + m_2 = m$. Otherwise the C.G. coefficient will vanish. Operating J_z on Eq. (2.7) from the left and using (2.2), we obtain

$$m|j_1 j_2 j m\rangle = \sum_{m_1 m_2} (m_1 + m_2) \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 m_1 m_2\rangle. \tag{2.11}$$

Expanding $|j_1 j_2 j m\rangle$ once again in terms of $|j_1 j_2 m_1 m_2\rangle$ using Eq. (2.7), we get

$$\sum_{m_1 m_2} (m - m_1 - m_2) \left[\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right] |j_1 j_2 m_1 m_2\rangle = 0. \qquad (2.12)$$

Since the functions $|j_1 j_2 m_1 m_2\rangle$ are linearly independent, it follows that each of the coefficients in the summation should be identically zero.

$$(m - m_1 - m_2) \left[\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right] = 0. \qquad (2.13)$$

Thus it is evident that unless $m = m_1 + m_2$, the C.G. coefficient should vanish. There are in total $(2j_1 + 1)(2j_2 + 1)$ linearly independent functions $|j_1 j_2 m_1 m_2\rangle$. Since the total number of linearly independent functions is preserved in any unitary transformation, the number of independent functions $|j_1 j_2 j m\rangle$ in the coupled representation should be the same. Hence,

$$\sum_j (2j + 1) = (2j_1 + 1)(2j_2 + 1). \qquad (2.14)$$

The maximum value of j i.e., j_{max} should be obviously $(j_1 + j_2)$ since the maximum value of m is $j_1 + j_2$, j_1 and j_2 being the maximum values of m_1 and m_2. By simple enumeration, one can find $j_{min} = |j_1 - j_2|$. Thus j can assume a spectrum of values from $|j_1 - j_2|$ to $|j_1 + j_2|$ in steps of unity. Thereby j_1, j_2 and j obey the triangular condition $\Delta\,(j_1 j_2 j)$. Otherwise the C.G. coefficient will vanish.

The distinction between the uncoupled and the coupled representations vanishes if one of the two angular momenta were to vanish and hence the C.G. coefficient which is the element of the unitary transformation becomes unity.

$$\left[\begin{array}{ccc} j & 0 & j \\ m & 0 & m \end{array} \right] = \left[\begin{array}{ccc} 0 & j & j \\ 0 & m & m \end{array} \right] = 1. \qquad (2.15)$$

The functions $|j_1 j_2 m_1 m_2\rangle$ and $|j_1 j_2 j m\rangle$ are orthonormal.

$$\langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2\rangle = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}. \qquad (2.16)$$

$$\langle j_1 j_2 j' m' | j_1 j_2 j m\rangle = \delta_{jj'}\, \delta_{mm'}. \qquad (2.17)$$

Using the expansion (2.7), we obtain

$$\langle j_1 j_2 j' m' | j_1 j_2 j m\rangle = \sum_{m_1 m_2} \sum_{m_1' m_2'} \left[\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right] \left[\begin{array}{ccc} j_1 & j_2 & j' \\ m_1' & m_2' & m' \end{array} \right] \times \langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2\rangle, \qquad (2.18)$$

Application of Eqs. (2.16) and (2.17) yields

$$\sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix} = \delta_{jj'}\,\delta_{mm'}. \tag{2.19}$$

In a similar way, starting from Eq. (2.16) and applying the expansion (2.10) twice,we get one more relation.

$$\sum_{j} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'}. \tag{2.20}$$

The relations (2.19) and (2.20) are known as orthonormality relations of the C.G. coefficients.

It may be observed that in Eqs. (2.7) and (2.19), although there are two summations m_1 and m_2, one is redundant because of the constraint $m_1 + m_2 = m$.

2.3. General Expressions for C.G. Coefficients

General expressions for C.G. coefficients have been derived by Racah using the algebraic methods. Since these derivations are complicated, the reader is referred to the original literature. Here we give only Racah's closed expression (Rose, 1957) for C.G. coefficients, since it is more convenient for writing a computer program for numerical evaluation of C.G. coefficients.

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = \delta_{m\, m_1+m_2}\left[(2j+1)AB\right]^{\frac{1}{2}} \sum_{\nu} \frac{(-1)^{\nu}}{\nu!} C_{\nu}^{-1}, \tag{2.21}$$

with

$$\begin{aligned} A &= \frac{(j_1+j_2-j)!\,(j+j_1-j_2)!\,(j_2+j-j_1)!}{(j_1+j_2+j+1)!}, \\ B &= (j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!, \\ C_{\nu} &= (j_1+j_2-j-\nu)!\,(j_1-m_1-\nu)!\,(j_2+m_2-\nu)! \\ &\quad \times (j-j_2+m_1+\nu)!\,(j-j_1-m_2+\nu)!. \end{aligned}$$

The summation index ν assumes all integer values for which the factorial arguments are not negative. A computer program for the C.G. coefficients based on the above formula can be written and the reader will find it useful for any numerical study of any physical problem involving C.G. coefficients.

Algebraic formulae for particular values of $j_2 (j_2 = \frac{1}{2}, 1)$ are given in Tables B1 and B2 in Appendix B since their occurrence is very common in physical problems. For higher values of J_2, the reader is referred to Condon and Shortley (1935) and Varshalovich et al. (1988). Several numerical tables of C.G. coefficients are also available. But these have become obsolete after the proliferation of fast electronic computers.

2.4. Symmetry Properties of C.G. Coefficients

A study of the general expressions for the C.G. coefficients will reveal the following symmetry properties.

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = (-1)^{j_1+j_2-j} \begin{bmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{bmatrix} \tag{2.22}$$

$$= (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix} \tag{2.23}$$

$$= (-1)^{j_1-m_1} \frac{[j]}{[j_2]} \begin{bmatrix} j_1 & j & j_2 \\ m_1 & -m & -m_2 \end{bmatrix} \tag{2.24}$$

$$= (-1)^{j_2+m_2} \frac{[j]}{[j_1]} \begin{bmatrix} j & j_2 & j_1 \\ -m & m_2 & -m_1 \end{bmatrix}, \tag{2.25}$$

where the symbol $[j]$ is defined by

$$[j] = (2j+1)^{1/2}. \tag{2.26}$$

Relations (2.22)–(2.25) bring out the symmetry properties of the C.G. coefficients under the permutations of any two columns or the reversal of the sign of the projection quantum numbers. Note that when the third column is permuted with the first or the second, there is a reversal of the sign of the projection quantum numbers of the permuted columns. This is essential to preserve the relation $m_1 + m_2 = m$. By using symmetry relation (2.22), one finds

$$\begin{bmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{bmatrix} = (-1)^{j_1+j_2-j} \begin{bmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.27}$$

Thereby one obtains the condition

$$\begin{bmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{2.28}$$

if $j_1 + j_2 - j$ is odd. Moreover) the quantum numbers j_1,j_2 and j should all be integers; otherwise the projection quantum numbers cannot be zero. This special C.G. coefficient is known as parity C.G. coefficient since in physical problems such a coefficient contains the parity selection rule.

The reader is referred to Biedenharn (1970) and Srinivasa Rao and Rajeswari (1993) for further study of symmetry properties of C.G. coefficients.

2.5. Iso-Spin

It is observed that the nuclear forces are charge independent and, as a consequence, it is found advantageous to treat proton and neutron (neglecting the small mass difference between them) as the two charge states of one and the same particle, nucleon. To distinguish the two charge states of the nucleon, a new quantum number[1], iso-spin, has been introduced in analogy with the spin quantum number. The iso-spin is a vector in an hypothetical space known as iso-spin space and its projection on the quantization axis distinguishes the different charge states of a particle. For the nucleon, τ is equal to $\frac{1}{2}$ with two possible projections, $m_\tau = +\frac{1}{2}$ corresponding to the proton and $m_\tau = -\frac{1}{2}$ corresponding to the neutron[2]. The iso-spin wave function of a nucleon can be written in the two-component form, the first component giving the amplitude of probability of finding the nucleon to be a proton and the second component giving the amplitude for finding it to be a neutron.

$$\eta_P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \eta_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.29}$$

Further, in analogy with the three Pauli spin matrices, we introduce three iso-spin operators in iso-spin space.

$$\tau_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \tau_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \tau_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.30}$$

These operators operate on the two-component iso-spin wave function (2.29).

The iso-spin wave function of the two-nucleon system can be constructed in the same way as the spin wave function of a system of two spin-$\frac{1}{2}$ particles.

Since the pion has three charge states, π^+, π^0, π^-, it can be described by giving an iso-spin $\tau = 1$ with projections $m_\tau = 1, 0, -1$. Given the iso-spin projection, the charge q of the pion or the nucleon is given by a simple relation

$$q = m_\tau + \frac{B}{2}, \tag{2.31}$$

[1]It is sometimes called isotopic spin or isobaric spin or simply I-spin

[2]This convention is used in particle physics. In nuclear physics, it is customary to take $m_\tau = +\frac{1}{2}$ for neutron and $m_\tau = -\frac{1}{2}$ for proton since most nuclei contain more neutrons than protons so that the iso-spin projection quantum number for most nuclei will be positive.

where B is the baryon number. For the nucleon, $B = 1$ and for the pion, $B = 0$. If the strange particles are also included in the scheme by introducing another quantum number, called strangeness quantum number S, then Eq. (2.31) can be modified to read

$$q = m_\tau + \frac{B+S}{2}. \tag{2.32}$$

This is known as the Gell-mann–Nishijima relation.

The iso-spin of the pion-nucleon system can be constructed by coupling the iso-spins of the pion and the nucleon in the same way as we do the coupling of two angular momenta by means of C.G. coefficients.

2.6. Notation

Different notations have come into vogue for the C. G. coefficient. Some of the notations commonly used in literature (Condon and Shortley, 1935; Brink and Satchler, 1962; Schiff, 1968; Rose, 1957; Varshalovich et al., 1988) are $\langle j_1 m_1 j_2 m_2 | jm \rangle$, $C(j_1 j_2 j, m_1 m_2 m)$ and $C^{jm}_{j_1 m_1 j_2 m_2}$. The Wigner 3j symbol (Edmonds, 1957), $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$, is related to the C.G. coefficient by the relation

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{[j]} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{bmatrix}. \tag{2.33}$$

The 3j symbol has higher symmetry.

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} \tag{2.34}$$

$$= (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix}. \tag{2.35}$$

The value of 3j is unchanged under an even permutation of the columns.

Review Questions

2.1 (a) In case of coupling of two angular momenta $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$, evaluate the following: commutator brackets $[\boldsymbol{J}^2, J_{1z}]$ and $[\boldsymbol{J}^2, J_z]$.
(b) For a two-particle system, explain why there are two different angular momentum representations. How are the eigenfunctions in the two representations connected?

2.2 (a) Define the Clebsch-Gordan coefficient and discuss their symmetry properties.
(b) Deduce the orthonormality relations of C. G. coefficients.

2.3 (a) Show that the C.G. coefficient $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$ vanishes unless $m_1 + m_2 = m$.

(b) What are the characteristics of the parity C.G. coefficients and why are they so called?

(c) How is the C.G. coefficient related to the Wigner 3j symbol?

Problems

2.1 Using the general properties, determine the values of the following C.G. coefficients:

$$(a)\begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad (b)\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad (c)\begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix},$$

$$(d)\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (e)\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad (f)\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix},$$

$$(g)\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}, \quad (h)\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad (i)\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix},$$

$$(j)\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

2.2 Two spin-$\frac{1}{2}$ particles are in the triplet state (S=1). Construct their coupled spin function in terms of the spin states of the individual particles. Identify the non-vanishing C.G. coefficients and find their values.

2.3 Write down the spin-orbit coupled wave function for a *p*-electron in an atom. Use the table of C.G. coefficients.

2.4 Construct the spin-orbit coupled wave function for a *d*-electron in an atom using the table of C.G. coefficients.

2.5 Obtain the following relations:

$$(a)\ \sum_{m_1}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_2 & j_1 & j' \\ -m_2 & -m_1 & -m' \end{bmatrix} = \delta_{jj'}\,\delta_{mm'}.$$

$$(b)\ \sum_{m_1}(-1)^{j_1+m_1}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1 & j' & j_2 \\ -m_1 & m' & m_2 \end{bmatrix}$$

$$= (-)^{j_1+j_2-j'}\frac{[j_2]}{[j']}\,\delta_{jj'}\,\delta_{mm'}.$$

$$(c)\ \sum_{m}(-1)^{j-m}\begin{bmatrix} j & j & \nu \\ m & -m & 0 \end{bmatrix} = (2j+1)^{1/2}\,\delta_{\nu 0}.$$

$$\text{(d)} \quad \sum_{j,m} (-1)^m \begin{bmatrix} j & j & j \\ m & -m & 0 \end{bmatrix} = 1.$$

2.6 Starting from the spin-orbit coupled wave function of spin-$\frac{1}{2}$ particle

$$\psi(j = \tfrac{3}{2}, m = \tfrac{3}{2}) = Y_1^1 \, \alpha$$

and using the lowering operator

$$J_- = L_- + S_-$$

repeatedly, determine the coupled wave functions,

$$\psi\left(\tfrac{3}{2}, \tfrac{1}{2}\right), \quad \psi\left(\tfrac{3}{2}, -\tfrac{1}{2}\right), \quad \psi\left(\tfrac{3}{2}, -\tfrac{3}{2}\right).$$

Y_1^1 denotes the spherical harmonic and a the spin-up state. Identify the relevant C.G. coefficients.

2.7 Starting from the spin-orbit coupled wave function of spin-$\frac{1}{2}$ particle

$$\psi(j = \tfrac{3}{2}, m = -\tfrac{3}{2}) = Y_1^{-1} \, \beta$$

and using the raising operator

$$J_+ = L_+ + S_+$$

repeatedly, determine the coupled wave functions,

$$\psi(\tfrac{3}{2}, -\tfrac{1}{2}), \quad \psi(\tfrac{3}{2}, \tfrac{1}{2}), \quad \psi(\tfrac{3}{2}, \tfrac{3}{2}).$$

Y_1^{-1} denotes the spherical harmonic and β the spin-down state. Identify the relevant C .G. coefficients.

2.8 Show that a two-particle system with total angular momentum $J = 2j$ is symmetric under exchange. It is given that each particle carries with it an angular momentum j.

2.9 Show that a system of two phonons, each carrying an angular momentum 2 can exist only in the angular momentum states 0, 2 and 4. Explain why the odd angular momenta are excluded.

2.10 Construct the possible spin wave functions of a system consisting of two spin-$\frac{1}{2}$ particles and examine their symmetry under exchange of particles. Find the eigenvalues of the operator σ1 · σ2 for that system and hence construct the spin exchange operator.

2.11 Construct the iso-spin wave function for a system consisting of a proton and π meson.

2.12 Show from iso-spin considerations that the cross-section for the reaction $p + p \rightarrow d + \pi^+$ is twice that of the reaction $n + p \rightarrow d + \pi^0$.

Solutions to Selected Problems

2.1 (a) Using the symmetry property (2.24), we obtain

$$\begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix} = -\frac{1}{[2]}\begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} = -\frac{1}{\sqrt{5}}.$$

(b) It is a stretched case. So, it follows that

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = 1.$$

(c) This C.G. coefficient and another with reversed magnetic quantum numbers alone occur in the expansion of the eigenfunction $|\frac{1}{2}\ \frac{1}{2}\ 1\ 0\rangle$ and hence the sum of their squares should be unity. Hence it follows that

$$\begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}.$$

(d) This is a parity C.G. coefficient and it is zero since $j_1 + j_2 - j$ is odd. It follows from Eq. (2.28) that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

(e) The C.G. coefficients are the expansion coefficients and the sum of their squares should be unity since the eigenfunctions are normalized.

$$\sum_{m_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}^2 = 1.$$

In the present case,

$$\sum_{m_1} \begin{bmatrix} 2 & 1 & 2 \\ m_1 & m_2 & 0 \end{bmatrix}^2 = 1.$$

In the expansion, there are three C.G. coefficients, of which one is the parity C.G. coefficient $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ which is zero. The other two C.G.

coefficients are determined using the above property and the symmetry relation.

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 2 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}.$$

The sign is not uniquely determined.

(f) Since the triangular condition $\Delta(j_1 j_2 j)$ is not obeyed, the C.G. coefficient $\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = 0.$

(g) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} = 0$, since $m_1 > j_1$ is not allowed.

(h) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} = 0$, since $m \neq m_1 + m_2$.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = 1$, since it is a stretched case as (b).

(j) Using the symmetry property (2.24), we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \frac{1}{[1]}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}}.$$

2.2 If α and β denote the spin-up and spin-down states respectively, then

$$\begin{aligned}
|\tfrac{1}{2}\,\tfrac{1}{2}\,1\,1\rangle &= \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix}\alpha(1)\,\alpha(2) = \alpha(1)\,\alpha(2), \\
|\tfrac{1}{2}\,\tfrac{1}{2}\,1\,0\rangle &= \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix}\alpha(1)\,\beta(2) \\
&\quad + \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix}\beta(1)\,\alpha(2) \\
&= \frac{1}{\sqrt{2}}[\alpha(1)\,\beta(2) + \beta(1)\,\alpha(2)]. \\
|\tfrac{1}{2}\,\tfrac{1}{2}\,1\,-1\rangle &= \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & -1/2 & -1 \end{bmatrix}\beta(1)\,\beta(2) = \beta(1)\,\beta(2).
\end{aligned}$$

The non-vanishing C.G. coefficients are

$$\begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & -1/2 & -1 \end{bmatrix} = 1.$$

$$\begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \sqrt{\frac{1}{2}}.$$

2.3 For a p-electron, the allowed values of j are $\frac{3}{2}$ and $\frac{1}{2}$. The spin-orbit coupled wave function is denoted by $|l\ s\ j\ m\rangle$.

$$|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle = Y_1^1(\hat{\boldsymbol{r}})\,\alpha.$$

$$\begin{aligned}|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle &= \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} Y_1^0(\hat{\boldsymbol{r}})\,\alpha \\ &\quad + \begin{bmatrix} 1 & 1/2 & 3/2 \\ 1 & -1/2 & 1/2 \end{bmatrix} Y_1^1(\hat{\boldsymbol{r}})\,\beta \\ &= \sqrt{\frac{2}{3}}\,Y_1^0(\hat{\boldsymbol{r}})\,\alpha + \sqrt{\frac{1}{3}}\,Y_1^1(\hat{\boldsymbol{r}})\,\beta.\end{aligned}$$

$$\begin{aligned}|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle &= \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & -1/2 & -1/2 \end{bmatrix} Y_1^0(\hat{\boldsymbol{r}})\,\beta \\ &\quad + \begin{bmatrix} 1 & 1/2 & 3/2 \\ -1 & 1/2 & -1/2 \end{bmatrix} Y_1^{-1}(\hat{\boldsymbol{r}})\,\alpha \\ &= \sqrt{\frac{2}{3}}\,Y_1^0(\hat{\boldsymbol{r}})\,\beta + \sqrt{\frac{1}{3}}\,Y_1^{-1}(\hat{\boldsymbol{r}})\,\alpha.\end{aligned}$$

$$|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ -\tfrac{3}{2}\rangle = Y_1^{-1}(\hat{\boldsymbol{r}})\,\beta.$$

$$\begin{aligned}|1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle &= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} Y_1^0(\hat{\boldsymbol{r}})\,\alpha \\ &\quad + \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & -1/2 & 1/2 \end{bmatrix} Y_{1,1}(\hat{\boldsymbol{r}})\,\beta \\ &= -\sqrt{\frac{1}{3}}\,Y_1^0(\hat{\boldsymbol{r}})\,\alpha + \sqrt{\frac{2}{3}}\,Y_{1,1}(\hat{\boldsymbol{r}})\,\beta.\end{aligned}$$

$$\begin{aligned}|1\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle &= \begin{bmatrix} 1 & 1/2 & 1/2 \\ -1 & 1/2 & -1/2 \end{bmatrix} Y_1^{-1}(\hat{\boldsymbol{r}})\,\alpha \\ &\quad + \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & -1/2 & -1/2 \end{bmatrix} Y_1^0(\hat{\boldsymbol{r}})\,\beta \\ &= -\sqrt{\frac{2}{3}}\,Y_1^1(\hat{\boldsymbol{r}})\,\alpha + \sqrt{\frac{1}{3}}\,Y_1^0(\hat{\boldsymbol{r}})\,\beta.\end{aligned}$$

2.5 (a)
$$\begin{bmatrix} j_2 & j_1 & j' \\ -m_2 & -m_1 & -m' \end{bmatrix} = \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}.$$

Hence the result follows from Eq. (2.19).

(b)
$$\begin{aligned}\begin{bmatrix} j_1 & j' & j_2 \\ -m_1 & m' & m_2 \end{bmatrix} &= (-1)^{j_1+m_1}\frac{[j_2]}{[j']}\begin{bmatrix} j_1 & j_2 & j' \\ -m_1 & -m_2 & -m' \end{bmatrix} \\ &= (-1)^{j_1+m_1}(-1)^{j_1+j_2-j'}\frac{[j_2]}{[j']}\begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}.\end{aligned}$$

Substituting this and summing over m_1, we obtain the result.

(c) Multiply the L.H.S. by $\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix}$ which is unity.

$$\sum_m (-1)^{j-m} \begin{bmatrix} j & j & \nu \\ m & -m & 0 \end{bmatrix} \begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix}$$
$$= \sum_m \begin{bmatrix} j & j & \nu \\ m & -m & 0 \end{bmatrix} \begin{bmatrix} j & j & 0 \\ m & -m & 0 \end{bmatrix} [j]$$
$$= \sqrt{2j+1}\, \delta_{\nu 0}.$$

(d) Multiply the L.H.S. by $\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix}$ which is unity.

$$\sum_{j,m} (-1)^m \begin{bmatrix} j & j & j \\ m & -m & 0 \end{bmatrix} \begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix}$$
$$= \sum_{j,m} (-1)^m (-1)^{j-m} [j] \begin{bmatrix} j & j & j \\ m & -m & 0 \end{bmatrix} \begin{bmatrix} j & j & 0 \\ m & -m & 0 \end{bmatrix}$$
$$= \sum_j (-1)^j \sqrt{2j+1}\, \delta_{j0} \quad \text{(summing over } m \text{ yields } \delta_{j0})$$
$$= 1.$$

2.10 A system consisting of two spin-$\frac{1}{2}$ particles can exist in triplet spin ($S = 1$) or singlet spin ($S = 0$) state. Denoting the spin-up and spin-down states of the spin-$\frac{1}{2}$ particle by α and β, the spin wave function in the coupled representation can be written as

$$\begin{aligned} |S=1, M_S=1\rangle &= \alpha_1\, \alpha_2, \\ |S=1, M_S=0\rangle &= \sqrt{\frac{1}{2}}(\alpha_1\, \beta_2 + \beta_1\, \alpha_2), \\ |S=1, M_S=-1\rangle &= \beta_1\, \beta_2, \\ |S=0, M_S=0\rangle &= \sqrt{\frac{1}{2}}(\alpha_1\, \beta_2 - \beta_1\, \alpha_2). \end{aligned}$$

From an inspection of the above wave functions, it can be seen that the spin triplet state is symmetric and the spin singlet state is anti-symmetric under exchange. For the construction of the spin exchange operator, first we need to determine the eigenvalues of the operator $\sigma \cdot \sigma$ corresponding to the spin triplet and spin singlet states.

$$\begin{aligned}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 &= \frac{1}{2}[(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)^2 - \boldsymbol{\sigma}_1^2 - \boldsymbol{\sigma}_2^2] \\ &= \frac{1}{2}[4S(S+1) - 3 - 3] \\ &= \begin{cases} 1, & \text{if } S = 1; \\ -3, & \text{if } S = 0. \end{cases}\end{aligned}$$

The spin exchange operator is $P_\sigma = \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ since it yields the eigenvalue +1 for the spin triplet state and −1 for the spin singlet state.

$$\begin{aligned} P_\sigma|S=1, M_S\rangle &= |S=1, M_S\rangle, \\ P_\sigma|S=0, M_S\rangle &= -|S=0, M_S\rangle. \end{aligned}$$

2.11 The iso-spin wave functions of proton, π^+, π^0 and π^- are

$$\begin{aligned} p &= |T=\tfrac{1}{2}, M_T=\tfrac{1}{2}\rangle, \\ \pi^+ &= |T=1, M_T=1\rangle, \\ \pi^0 &= |T=1, M_T=0\rangle, \\ \pi^- &= |T=1, M_T=-1\rangle. \end{aligned}$$

The iso-spin wave functions of $p\pi^+$, $p\pi^0$ and $p\pi^-$ are

$$\begin{aligned} |p\pi^+\rangle &= |\tfrac{1}{2}\ \tfrac{1}{2}\rangle\,|1\ 1\rangle \\ &= |\tfrac{3}{2}\ \tfrac{3}{2}\rangle. \\ |p\pi^0\rangle &= |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ 0\rangle \\ &= \left[\begin{array}{ccc} 1/2 & 1 & 3/2 \\ 1/2 & 0 & 1/2 \end{array}\right]|\tfrac{3}{2}\ \tfrac{1}{2}\rangle \\ &\quad + \left[\begin{array}{ccc} 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/2 \end{array}\right]|\tfrac{1}{2}\ \tfrac{1}{2}\rangle \\ &= \sqrt{\frac{2}{3}}|\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle. \\ |p\pi^-\rangle &= |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|1\ -1\rangle \\ &= \left[\begin{array}{ccc} 1/2 & 1 & 3/2 \\ 1/2 & -1 & -1/2 \end{array}\right]|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle \\ &\quad + \left[\begin{array}{ccc} 1/2 & 1 & 1/2 \\ 1/2 & -1 & -1/2 \end{array}\right]|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \\ &= \sqrt{\frac{1}{3}}|\tfrac{3}{2}\ -\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle. \end{aligned}$$

VECTORS AND TENSORS IN SPHERICAL BASIS

3.1. The Spherical Basis

It is more convenient to describe the vectors and tensors in the spherical basis since they can be easily expressed in their irreducible forms and their law of transformation under rotation also becomes much simpler.

Denoting the unit vectors in the Cartesian basis as e_x , e_y and e_z and in the spherical basis as ϵ_1^1, ϵ_1^0 and ϵ_1^{-1}, we can express any vector $\boldsymbol{A}$ as follows.

$$\begin{aligned} \boldsymbol{A} &= A_x \boldsymbol{e}_x + A_y \boldsymbol{e}_y + A_z \boldsymbol{e}_z \\ &= -A_1^1 \epsilon_1^{-1} - A_1^{-1} \epsilon_1^{+1} + A_1^0 \epsilon_1^0 \\ &= \sum_{\mu=1,0,-1} (-1)^\mu \, A_1^\mu \, \epsilon_1^{-\mu}. \end{aligned} \tag{3.1}$$

where

$$A_1^1 = -\frac{(A_x + iA_y)}{\sqrt{2}}, \quad A_1^0 = A_z, \quad A_1^{-1} = \frac{(A_x - iA_y)}{\sqrt{2}}, \tag{3.2}$$

and

$$\epsilon_1^1 = -\frac{(\boldsymbol{e}_x + i\boldsymbol{e}_y)}{\sqrt{2}}, \quad \epsilon_1^0 = \boldsymbol{e}_z, \quad \epsilon_1^{-1} = \frac{(\boldsymbol{e}_x - i\boldsymbol{e}_y)}{\sqrt{2}}. \tag{3.3}$$

It follows from Eqs. (3.2) that the complex conjugate of A_1^μ is

$$A_1^{\mu *} = (-1)^\mu A_1^{-\mu}.$$

We will have occasions to use later the spherical cornponents of the position vector r in terms of the spherical harmonics of order 1. Using (3.1), we obtain

$$\boldsymbol{r} = -r_1^1 \epsilon_1^{-1} - r_1^{-1} \epsilon_1^1 + r_1^0 \epsilon_1^0, \tag{3.4}$$

where

$$r_1^1 = -(x + iy)/\sqrt{2}, \quad r_1^{-1} = (x - iy)/\sqrt{2}, \quad r_1^0 = z ; \tag{3.5}$$

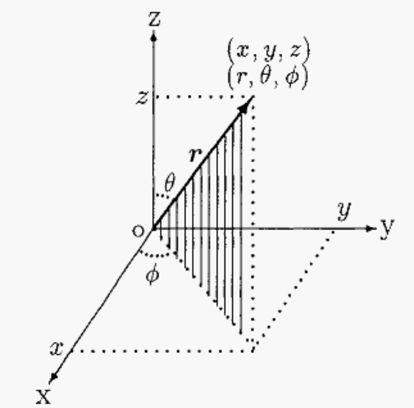

Figure 3.1. The position vector $\boldsymbol{r}$

with

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,. \tag{3.6}$$

Equation (3.6) can be obtained from an inspection of Fig. 3.1. Substituting (3.6) into (3.5), it follows that

$$r_1^1 = -\frac{r}{\sqrt{2}}\sin\theta\, e^{i\phi} = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^1(\hat{\boldsymbol{r}}), \tag{3.7}$$

$$r_1^{-1} = \frac{r}{\sqrt{2}}\sin\theta\, e^{-i\phi} = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^{-1}(\hat{\boldsymbol{r}}), \tag{3.8}$$

$$r_1^0 = r\cos\theta = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0(\hat{\boldsymbol{r}}), \tag{3.9}$$

where $Y_{1\mu}$ are spherical harmonics of order 1.

$$\begin{aligned} Y_1^1(\hat{\boldsymbol{r}}) &= -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}, \\ Y_1^{-1}(\hat{\boldsymbol{r}}) &= \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}, \\ Y_1^0(\hat{\boldsymbol{r}}) &= \sqrt{\frac{3}{4\pi}}\cos\theta. \end{aligned} \tag{3.10}$$

Using the above relations, we finally obtain the position vector r in terms of the spherical harmonics of order 1.

$$\boldsymbol{r} = \sqrt{\frac{4\pi}{3}}\, r \sum_{\mu} (-1)^{\mu}\, Y_1^{\mu}\, \epsilon_1^{-\mu}. \tag{3.11}$$

3.2. Scalar and Vector Products in Spherical Basis

Given any two vectors ***A*** and ***B***, we can construct a scalar or a vector (tensor of rank 1) or a tensor of rank 2. Let us express the scalar product of ***A*** and ***B*** separately in terms of Cartesian and spherical components.

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= \sum_{\mu=-1,0,1} (-1)^{\mu} A_1^{\mu} B_1^{-\mu} \\ &= \sum_{\mu=-1,0,1} A_1^{\mu} B_1^{\mu *}. \end{aligned} \tag{3.12}$$

This follows from the orthogonality of the unit vectors defined in a self consistent way in these two bases.

Cartesian basis: $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$,

Spherical basis: $\boldsymbol{\epsilon}_1^{\mu *} \cdot \boldsymbol{\epsilon}_1^{\nu} = \delta_{\mu\nu}$,

where

$$\epsilon_1^{\mu *} = (-1)^{\mu} \epsilon_1^{-\mu}. \tag{3.13}$$

Let ***C*** denote the vector product of ***A*** and ***B***.

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}. \tag{3.14}$$

Expanding in terms of spherical components

$$\boldsymbol{C} = \sum_{\mu\,\nu} (-1)^{\mu+\nu} A_1^{\mu} B_1^{\nu} (\boldsymbol{\epsilon}_1^{-\mu} \times \boldsymbol{\epsilon}_1^{-\nu}). \tag{3.15}$$

The vector product of any two unit vectors in Cartesian basis is given by

$$e_i \times e_j = e_k, \quad (i,j,k \text{ in cyclic order}) \tag{3.16}$$

and using this, the vector product of any two unit spherical vectors can be obtained.

$$\boldsymbol{\epsilon}_1^{\mu} \times \boldsymbol{\epsilon}_1^{\nu} = i\, S(\mu - \nu)\, \boldsymbol{\epsilon}_1^{\mu+\nu}. \tag{3.17}$$

where $S(\mu - \nu)$ denotes the sign of the quantity $(\mu - \nu)$, if $\mu \neq \nu$ and zero if $\mu = \nu$. From an inspection, it can be seen that the C.G. coefficient $\begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \mu+\nu \end{bmatrix}$ can be used to play the same role as the function $S(\mu - \nu)$.

Incorporating this, we obtain,

$$\epsilon_1^\mu \times \epsilon_1^\nu = i\sqrt{2}\begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \mu+\nu \end{bmatrix}\epsilon_1^\lambda, \quad \lambda = \mu+\nu. \tag{3.18}$$

Introducing this definition to the vector product in Eq. (3.15), we get

$$\begin{aligned} \boldsymbol{C} &= i\sqrt{2}\sum_{\mu\nu}(-1)^\lambda \begin{bmatrix} 1 & 1 & 1 \\ -\mu & -\nu & -\lambda \end{bmatrix} A_1^\mu B_1^\nu \epsilon_1^{-\lambda} \\ &= -i\sqrt{2}\sum_\lambda (-1)^\lambda T_1^\lambda \epsilon_1^{-\lambda}, \end{aligned} \tag{3.19}$$

where

$$T_1^\lambda = \sum_\mu \begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{bmatrix} A_1^\mu B_1^\nu. \tag{3.20}$$

Thus the sherical components of $\boldsymbol{C}$ are given by

$$C_1^\lambda = -i\sqrt{2}\,T_1^\lambda, \qquad (\lambda = 1, 0, -1) \tag{3.21}$$

where T_1^λ is defined by Eq. (3.20). In the discussion to follow, T_1^λ is called a component of the spherical tensor of rank 1 formed by taking the tensor product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ and it is to be noted that this differs by a factor of $-i\sqrt{2}$ from the spherical component of the vector C_1^λ obtained by taking the vector product of $\boldsymbol{A}$ and $\boldsymbol{B}$. From Eq. (3.20), it follows that the complex conjugate of T_1^λ is given by

$$\begin{aligned} T_1^{\lambda *} &= \sum_\mu \begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{bmatrix} A_1^{\mu *} B_1^{\nu *} \\ &= \sum_\mu (-1)^{\mu+\nu} \begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{bmatrix} A_1^{-\mu} B_1^{-\nu} \\ &= -(-1)^\lambda \sum_\mu \begin{bmatrix} 1 & 1 & 1 \\ -\mu & -\nu & -\lambda \end{bmatrix} A_1^{-\mu} B_1^{-\nu} \\ &= -(-1)^\lambda T_1^{-\lambda}. \end{aligned} \tag{3.22}$$

3.3. The Spherical Tensors

Now consider the direct product of the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. The products of their Cartesian components represented below in a matrix form denote the nine components of a Cartesian tensor of rank 2

$$\begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}. \tag{3.23}$$

This is said to be in a reducible form since it is possible to group the linear combinations of these components with different sets which transform among themselves under rotation. The trace

$$S = \sum_i A_i B_i, \tag{3.24}$$

is the scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$ and hence invariant under rotation. The anti-symmetric tensor having three components

$$V_k = \frac{1}{2}(A_i B_j - A_j B_i), \qquad (i, j, k \text{ cyclic}) \tag{3.25}$$

transforms as a vector since

$$\boldsymbol{V} = \frac{1}{2}(\boldsymbol{A} \times \boldsymbol{B}). \tag{3.26}$$

The symmetric tensor with zero trace (traceless symmetric tensor)

$$T_{ij} = \frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3} S\, \delta_{ij}, \tag{3.27}$$

having six components, of which only five are linearly independent because of the constraint of zero trace, transform among themselves under rotation.

Although the quantities *S*, $\boldsymbol{V}$ and *T* are in irreducible forms and transform in the same way as spherical harmonics of order 0, 1 and 2, it is more convenient to express them in the spherical basis rather than in the Cartesian basis. The tensors expressed in the spherical basis are known as spherical tensors. The spherical tensor of rank k has (2k+1) components and they transform under rotation in the same way as ψ_{jm} with j = k.

$$T_k^{\mu}(\mathbf{r}') = \sum_{\nu} D^k_{\nu\mu}(\alpha\beta\gamma)\, T_k^{\nu}(\mathbf{r}). \tag{3.28}$$

The position vector r changes into r' in the rotated coordinate system. The quantities $D^k_{\nu\mu}(\alpha\beta\gamma)$ are the elements of the rotation matrix defined in the next chapter.

3.4. The Tensor Product

Given any two tensors $T_{k_1}^{\mu_1}$ and $T_{k_2}^{\mu_2}$, we can define a tensor product of these two tensors.

$$T_k^{\mu} = \sum_{\mu_1} \begin{bmatrix} k_1 & k_2 & k \\ \mu_1 & \mu_2 & \mu \end{bmatrix} T_{k_1}^{\mu_1} T_{k_2}^{\mu_2}. \tag{3.29}$$

The allowed values of k lie between $|k_1 - k_2|$ and $|k_1 + k_2|$. We also give below the inverse relation which we will have occasion to use later.

$$T_{k_1}^{\mu_1} T_{k_2}^{\mu_2} = \sum_{k} \begin{bmatrix} k_1 & k_2 & k \\ \mu_1 & \mu_2 & \mu \end{bmatrix} T_k^{\mu}. \tag{3.30}$$

Now let us, for illustration, construct spherical tensors of rank 0,1 and 2, given the two vectors ***A*** and ***B***.

$$\begin{aligned} T_0^0 &= \sum_{\mu} \begin{bmatrix} 1 & 1 & 0 \\ \mu & -\mu & 0 \end{bmatrix} A_1^{\mu} B_1^{-\mu} \\ &= -\sqrt{\frac{1}{3}} \sum_{\mu} (-1)^{\mu} A_1^{\mu} B_1^{-\mu} = -\sqrt{\frac{1}{3}} \boldsymbol{A} \cdot \boldsymbol{B}. \end{aligned} \tag{3.31}$$

$$T_1^{\mu} = \sum_{\mu_1} \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu \end{bmatrix} A_1^{\mu_1} B_1^{\mu_2}. \tag{3.32}$$

$$T_2^{\mu} = \sum_{\mu_1} \begin{bmatrix} 1 & 1 & 2 \\ \mu_1 & \mu_2 & \mu \end{bmatrix} A_1^{\mu_1} B_1^{\mu_2}. \tag{3.33}$$

In Table 3.1, we explicitly give the components of T_1^{μ} and T_2^{μ} in terms of the spherical components of the vectors ***A*** and ***B***. Note that

$$T_k^{\mu *} = (-1)^{k+\mu} T_k^{-\mu}, \tag{3.34}$$

when the spherical tensor T_k^{μ} is constructed from any two vectors ***A*** and ***B***. In general, when the spherical tensor T_k^{μ} is constructed by taking a tensor product of two tensors $T_{k_1}^{\mu_1}$ and $T_{k_2}^{\mu_2}$ as illustrated in Eq. (3.29), the complex conjugate of T_k^{μ} is given by

$$T_k^{\mu *} = (-1)^{k_1+k_2-k} (-1)^{\mu} T_k^{-\mu}. \tag{3.35}$$

TABLE 3.1. Components of spherical tensors of rank 1 and 2 constructed from any two given vectors **A** and **B**.

μ	T_1^μ	T_2^μ
2	—	$A_1^1 B_1^1$
1	$\sqrt{\frac{1}{2}}(A_1^1 B_1^0 - A_1^0 B_1^1)$	$\sqrt{\frac{1}{2}}(A_1^1 B_1^0 + A_1^0 B_1^1)$
0	$\sqrt{\frac{1}{2}}(A_1^1 B_1^{-1} - A_1^{-1} B_1^1)$	$\sqrt{\frac{1}{6}}(A_1^1 B_1^{-1} + A_1^{-1} B_1^1 + 2A_1^0 B_1^0)$
−1	$\sqrt{\frac{1}{2}}(A_1^0 B_1^{-1} - A_1^{-1} B_1^0)$	$\sqrt{\frac{1}{2}}(A_1^{-1} B_1^0 + A_1^0 B_1^{-1})$
−2	—	$A_1^{-1} B_1^{-1}$

Review Questions

3.1 (a) Define unit vectors in spherical basis and show that they are orthogonal.
(b) Given any two vectors, construct a scalar, a spherical tensor of rank 1 and a spherical tensor of rank 2.

3.2 Write down the scalar product of two vectors in terms of their cartesian and spherical components.

3.3 If r is the position vector, express it in terms of its spherical components and hence show that

$$\boldsymbol{r} = \sqrt{\frac{4\pi}{3}}\, r \sum_\mu (-1)^\mu Y_1^\mu(\hat{\boldsymbol{r}}) \epsilon_1^{-\mu},$$

where $Y_1^\mu(\hat{\boldsymbol{r}})$ is a spherical harmonic of order 1 and r is the modulus of the vector r.

3.4 Given any two vectors ***A*** and ***B***, construct a vector product and a tensor product of rank 1. How are their spherical components related?

3.5 If ***C*** = ***A*** x ***B*** , show that the spherical component C_1^λ of the vector ***C*** is given by

$$C_1^\lambda = -i\sqrt{2}\, T_1^\lambda,$$

where

$$T_1^\lambda = \sum_\mu \begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{bmatrix} A_1^\mu B_1^\nu .$$

is a component of the spherical tensor of rank 1 formed by taking the tensor product of the two vectors ***A*** and ***B***.

Problems

3.1 Given any two vectors ***A*** and ***B***, find their scalar product and compare it with the tensor of rank 0 constructed by taking their tensor product.

3.2 Given the three vectors ***A***, ***B*** and ***C***, construct a spherical tensor of rank 3.

3.3 Given the three vectors ***A***, ***B*** and ***C***, construct a spherical tensor of rank 0 using all the three vectors.

3.4 Given the three vectors ***A***, ***B*** and ***C***, construct a spherical tensor of rank 2 using all the three vectors.

3.5 Given any two spherical tensors T_{k1} and T_{k2} of rank k_1 and k_2 respectively, construct a spherical tensor T_k of rank k and hence show that the complex conjugate of T_k^μ is given by

$$T_k^{\mu *} = (-1)^{k_1+k_2-k}(-1)^\mu T_k^{-\mu}.$$

3.6 If ***J*** is the angular momentum vector operator, express the spherical components of this vector operator in terms of the J_z operator and the raising and lowering operators J_+ and J_-. Hence determine the effect of J_1^μ with μ = 1,0, −1 operating on the angular momentum state $|j, m\rangle$.

Solutions to Selected Problems

3.1 The scalar product:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_\mu (-1)^\mu A_1^\mu B_1^{-\mu}.$$

The tensor product of rank 0:

$$\begin{aligned} T_0^0 &= \sum_\mu \begin{bmatrix} 1 & 1 & 0 \\ \mu & -\mu & 0 \end{bmatrix} A_1^\mu B_1^{-\mu} = \sum_\mu (-1)^{1-\mu} \frac{1}{\sqrt{3}} A_1^\mu B_1^{-\mu} \\ &= -\frac{1}{\sqrt{3}} \boldsymbol{A} \cdot \boldsymbol{B}. \end{aligned}$$

3.2 To construct a tensor of rank 3, given the three vectors ***A***, ***B*** and ***C***, first construct a tensor T_2^m of rank 2 with vectors ***A*** and ***B*** and then take the tensor product of T_2^m with ***C***.

$$T_2^m = \sum_\mu \begin{bmatrix} 1 & 1 & 2 \\ \mu & \nu & m \end{bmatrix} A_1^\mu B_1^\nu \, \delta_{m\,\mu+\nu},$$

$$
\begin{aligned}
T_3^M &= \sum_m \begin{bmatrix} 2 & 1 & 3 \\ m & \lambda & M \end{bmatrix} T_2^m C_1^\lambda \, \delta_{M\ m+\lambda} \\
&= \sum_{\mu,\nu} \begin{bmatrix} 1 & 1 & 2 \\ \mu & \nu & m \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ m & \lambda & M \end{bmatrix} A_1^\mu B_1^\nu C_1^\lambda \, \delta_{m\ \mu+\nu} \, \delta_{M\ m+\lambda}.
\end{aligned}
$$

Imposing the constraints on the magnetic quantum numbers and finding the values of the C.G. Coefficients from the tables, the tensor components of T_3^M of rank 3 are obtained. The allowed values of M are 3, 2, 1, 0, −1, −2,and −3.

$$
\begin{aligned}
T_3^3 &= A_1^1 B_1^1 C_1^1. \\
T_3^2 &= \sqrt{\frac{1}{3}}(A_1^0 B_1^1 C_1^1 + A_1^1 B_1^0 C_1^1 + A_1^1 B_1^1 C_1^0). \\
T_3^1 &= \sqrt{\frac{1}{15}}\left(A_1^1 B_1^{-1} C_1^1 + A_1^{-1} B_1^1 C_1^1 + A_1^1 B_1^1 C_1^{-1}\right) \\
&\quad + \sqrt{\frac{4}{15}}\left(A_1^0 B_1^0 C_1^1 + A_1^1 B_1^0 C_1^0 + A_1^0 B_1^1 C_1^0\right). \\
T_3^0 &= \sqrt{\frac{1}{10}}\left(A_1^0 B_1^{-1} C_1^1 + A_1^{-1} B_1^0 C_1^1 + A_1^1 B_1^{-1} C_1^0 + A_1^{-1} B_1^1 C_1^0\right) \\
&\quad + \sqrt{\frac{1}{10}}\left(4\, A_1^0 B_1^0 C_1^0 + A_1^1 B_1^0 C_1^{-1} + A_1^0 B_1^1 C_1^{-1}\right). \\
T_3^{-1} &= \sqrt{\frac{1}{15}}\left(A_1^{-1} B_1^{-1} C_1^1 + A_1^1 B_1^{-1} C_1^{-1} + A_1^{-1} B_1^1 C_1^{-1}\right) \\
&\quad + \sqrt{\frac{4}{15}}\left(A_1^0 B_1^{-1} C_1^0 + A_1^{-1} B_1^0 C_1^0 + A_1^0 B_1^0 C_1^{-1} +\right). \\
T_3^{-2} &= \sqrt{\frac{1}{3}}(A_1^{-1} B_1^{-1} C_1^0 + A_1^{-1} B_1^0 C_1^{-1} + A_1^0 B_1^{-1} C_1^{-1}). \\
T_3^{-3} &= A_1^{-1} B_1^{-1} C_1^{-1}.
\end{aligned}
$$

It can easily be verified that for each component of the tensor, the sum of the squares of the coefficients of all the terms is unity. This property can be used to check the correctness of ones calculation.

3.5 The spherical components of the vector operator ***J*** are:

$$J_1^0 = J_z; \quad J_1^1 = -J_+/\sqrt{2}; \quad J_-/\sqrt{2}.$$

From Eqs. (1.37) - (1.39), the effect of operation of J_z, J_+, J_- on the angular momentum state $|j, m\rangle$ are known. Hence it follows that

$$\begin{aligned} J_1^0 \, |j,m\rangle &= m \, |j,m\rangle, \\ J_1^1 \, |j,m\rangle &= -\sqrt{\frac{1}{2}} \, J_+ \, |j,m\rangle \\ &= -\sqrt{\frac{1}{2}} \sqrt{(j-m)(j+m+1)} \, |j,m+1\rangle, \\ J_1^{-1} \, |j,m\rangle &= \sqrt{\frac{1}{2}} \, J_- \, |j,m\rangle \\ &= \sqrt{\frac{1}{2}} \sqrt{(j+m)(j-m+1)} \, |j,m-1\rangle. \end{aligned}$$

ROTATION MATRICES - I

4.1. Definition of Rotation Matrix

The rotation matrices define the transformation properties of angular momentum eigenfunctions ψ_{jm} under rotation of coordinate system.

$$\psi_{jm}(\boldsymbol{r}') = \sum_{m'} D^j_{m'm}(\alpha\beta\gamma)\ \psi_{jm'}(\boldsymbol{r}), \tag{4.1}$$

where $D^j_{m'm}(\alpha,\beta,\gamma)$ denotes an element of the rotation matrix, the rotation being described by a set of three Euler angles α,β,γ. The angular momentum eigenfunctions $\psi_{jm}(\boldsymbol{r}')$ are in the rotated coordinate system S', whereas the functions $\psi_{jm'}(\boldsymbol{r})$ denote the eigenfunctions in the original coordinate system S. Hence these functions should be related by a unitary transformation. For integer values of *j*, it is easy to show that the functions ψ_{jm} transform as the spherical components of an irreducible tensor of rank j. In this chapter, we shall obtain the rotation matrices from a consideration of the transformation properties of a vector (spherical tensor of rank 1) and spherical tensors of higher rank.

4.2. Rotation in terms of Euler Angles

Consider a right handed coordinate system. Any general rotation R in the three dimensional space can be conveniently described in terms of the three Euler angles $\alpha\beta$ and γ ($0 < \alpha < 2\pi$, $0 < \beta < \pi$, $0 < \gamma < 2\pi$).

$$R = R_{Z_2}(\gamma)\ R_{Y_1}(\beta)\ R_Z(\alpha). \tag{4.2}$$

$R_Z(\alpha)$ denotes a rotation through an angle α about the Z axis[1]. This results in the change of the reference frame $XYZ \rightarrow X_1Y_1Z_1$, Z_1 axis coinciding with the Z axis. This is followed by a rotation through an angle β about the Y_1 axis and then through an angle γ about the Z_2 axis. The complete

[1]Normally, lower case letters are used for the suffixes but. in chapters 4 and 5, upper case letters are used for suffixes in certain cases for the purpose of clarity.

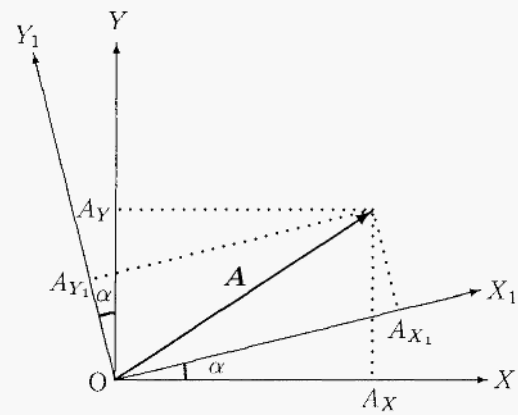

Figure 4.1. Rotation about the Z–axis through an angle α

rotation R can be denoted explicitly in the following sequence.

$$XYZ \xrightarrow[Z\text{ - axis}]{\alpha} X_1Y_1Z_1 \xrightarrow[Y_1\text{ - axis}]{\beta} X_2Y_2Z_2 \xrightarrow[Z_2\text{ - axis}]{\gamma} X'Y'Z'$$

4.3. Transformation of a Spherical Vector under Rotation of Coordinate System

Let us now consider the transformation of the spherical components of a vector $\boldsymbol{A}$ under a general rotation R of the coordinate system and obtain the transformation matrix. This is done in three steps. First let us make a rotation through an angle a about the Z axis as illustrated in Fig. 4.1. The Cartesian components of $\boldsymbol{A}$ transform as follows:

$$\begin{aligned} A_{X_1} &= A_X \cos\alpha + A_Y \sin\alpha, \\ A_{Y_1} &= A_Y \cos\alpha - A_X \sin\alpha, \\ A_{Z_1} &= A_Z. \end{aligned} \tag{4.3}$$

In matrix notation,

$$\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix}. \tag{4.4}$$

To know how the spherical components transform, we need to express the spherical components in terms of the Cartesian components. The transformation of the Cartesian components is already given in Eq. (4.4).

$$\begin{aligned}(A_1^1)_1 &= -\sqrt{\frac{1}{2}}\,(A_{X_1} + iA_{Y_1}) \\ &= -\sqrt{\frac{1}{2}}\,(A_X \cos\alpha + A_Y \sin\alpha + iA_Y \cos\alpha - iA_X \sin\alpha) \\ &= -\sqrt{\frac{1}{2}}\,(A_X\, e^{-i\alpha} + iA_Y\, e^{-i\alpha}) \\ &= A_1^1\, e^{-i\alpha}, \quad \text{since } A_1^1 = -\sqrt{\frac{1}{2}}(A_X + iA_Y). \end{aligned} \tag{4.5}$$

Similarly,

$$(A_1^0)_1 = A_1^0, \tag{4.6}$$

$$(A_1^{-1})_1 = A_1^{-1}\, e^{i\alpha}. \tag{4.7}$$

The transformation of the spherical components can now be conveniently written in a matrix form.

$$\begin{bmatrix} (A_1^1)_1 \\ (A_1^0)_1 \\ (A_1^{-1})_1 \end{bmatrix} = \begin{bmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} \begin{bmatrix} A_1^1 \\ A_1^0 \\ A_1^{-1} \end{bmatrix}. \tag{4.8}$$

In a concise notation,

$$\boldsymbol{A}_1 = M_Z(\alpha)\, \boldsymbol{A}, \tag{4.9}$$

where $M_Z(\alpha)$ is the transformation matrix for rotation about the Z axis through an angle α.

Next let us consider a rotation through an angle β about the Y_1 axis. The Cartesian components A_{X1}, A_{Y1}, A_{Z1} transform into A_{X2}, A_{Y2}, A_{Z2} and the equations of transformation are given below:

$$\begin{aligned} A_{X2} &= A_{X1}\cos\beta - A_{Z1}\sin\beta. \\ A_{Y2} &= A_{Y1}. \\ A_{Z2} &= A_{Z1}\cos\beta + A_{X1}\sin\beta. \end{aligned} \tag{4.10}$$

This transformation can be expressed more elegantly in the matrix form as follows.

$$\begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix}. \tag{4.11}$$

The equations for transformation of the spherical components can be obtained following the same procedure as before.

$$\begin{aligned}(A_1^1)_2 &= -\sqrt{\frac{1}{2}}\{A_{X2} + iA_{Y2}\} \\ &= -\sqrt{\frac{1}{2}}\{A_{X1}\cos\beta - A_{Z1}\sin\beta + iA_{Y1}\} \\ &= \frac{1}{2}\left\{(A_1^1)_1 - (A_1^{-1})_1\right\}\cos\beta + \sqrt{\frac{1}{2}}(A_1^0)_1\sin\beta \\ &\quad + \frac{1}{2}\left\{(A_1^1)_1 + (A_1^{-1})_1\right\} \\ &= \frac{1}{2}(A_1^1)_1(1+\cos\beta) + \sqrt{\frac{1}{2}}(A_1^0)_1\sin\beta \\ &\quad + \frac{1}{2}(A_1^{-1})_1(1-\cos\beta). \qquad (4.12)\end{aligned}$$

Similarly,

$$(A_1^0)_2 = -\sqrt{\frac{1}{2}}\sin\beta\,(A_1^1)_1 + \cos\beta\,(A_1^0)_1 + \sqrt{\frac{1}{2}}\sin\beta\,(A_1^{-1})_1. \qquad (4.13)$$

$$\begin{aligned}(A_1^{-1})_2 &= \frac{1}{2}(1-\cos\beta)\,(A_1^1)_1 - \sqrt{\frac{1}{2}}\sin\beta\,(A_1^0)_1 \\ &\quad + \frac{1}{2}(1+\cos\beta)\,(A_1^{-1})_1. \qquad (4.14)\end{aligned}$$

Denoting the transformation matrix by $M_Y(\beta)$,

$$M_{Y_1}(\beta) = \begin{bmatrix} \frac{1}{2}(1+\cos\beta) & \sqrt{\frac{1}{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ -\sqrt{\frac{1}{2}}\sin\beta & \cos\beta & \sqrt{\frac{1}{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & -\sqrt{\frac{1}{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{bmatrix}, \qquad (4.15)$$

we obtain

$$\begin{aligned}\boldsymbol{A}_2 &= M_{Y_1}(\beta)\,\boldsymbol{A}_1 \qquad (4.16)\\ &= M_{Y_1}(\beta)\,M_Z(\alpha)\,\boldsymbol{A}. \qquad (4.17)\end{aligned}$$

Lastly, we have to perform a rotation through an angle γ about the Z_2 axis. The resulting transformation matrix is the product of the three transformation matrices obtained for rotations through the three Euler angles.

$$M(\alpha,\beta,\gamma) = M_{Z_2}(\gamma)\,M_{Y_1}(\beta)\,M_Z(\alpha), \qquad (4.18)$$

and the transformed vector $\boldsymbol{A'}$ is given by

$$\boldsymbol{A'} = M(\alpha,\beta,\gamma)\,\boldsymbol{A}. \qquad (4.19)$$

4.4. The Rotation Matrix $D^1(\alpha,\beta,\gamma)$

It is to be pointed out that the transformation matrix M is not the rotation matrix defined in this book. According to the law of matrix multiplication, any component of the transformed vector $\boldsymbol{A'}$ +-is given by

$$A'_\mu = \sum_\nu M_{\mu\nu}(\alpha,\beta,\gamma)\, A_\nu, \tag{4.20}$$

whereas the rotation matrix $D^1(\alpha,\beta,\gamma)$ is defined such that

$$A'_\mu = \sum_\nu D^1_{\nu\mu}(\alpha,\beta,\gamma)\, A_\nu. \tag{4.21}$$

Hence the rotation matrix D is the transpose of the transformation matrix M defined in Eq. (4.18). We give below explicitly $D^1(\alpha\beta\gamma)$ in a matrix form

$$D^1(\alpha,\beta,\gamma) = \begin{bmatrix} e^{-i\gamma}\frac{1+\cos\beta}{2}e^{-i\alpha} & -\frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} & e^{i\gamma}\frac{1-\cos\beta}{2}e^{-i\alpha} \\ e^{-i\gamma}\frac{\sin\beta}{\sqrt{2}} & \cos\beta & -e^{i\gamma}\frac{\sin\beta}{\sqrt{2}} \\ e^{-i\gamma}\frac{1-\cos\beta}{2}e^{i\alpha} & \frac{\sin\beta}{\sqrt{2}}e^{i\alpha} & e^{i\gamma}\frac{1+\cos\beta}{2}e^{i\alpha} \end{bmatrix}. \tag{4.22}$$

Above we have shown explicitly how to construct the rotation matrix $D^1(\alpha,\beta,\gamma)$ which defines the transformation properties of a vector (spherical tensor of rank 1). In the same way, we can construct the rotation matrices for spherical tensors of higher rank.

There are in vogue different conventions[2] for the definition of D functions. The convention that is used here is identical with the convention of Rose (1957) and is widely used in elementary particle physics. For instance, Jacob and Wick (1959) use this convention in the formulation of helicity formalism[3] for the description of scattering theory.

4.5. Construction of other Rotation Matrices

In Table 3.1, the spherical components of a spherical tensor of rank 2 are explicitly given in terms of the spherical components of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. Since we know how the spherical components of a vector transform, it is a straight-forward procedure to construct the rotation matrices $D^2(\alpha,\beta,\gamma)$ for the transformation of a spherical tensor of rank 2. Although this procedure is straight forward, it is rather tedious and rarely one will opt for

[2]For the different conventions used by several authors, please refer to Varshalovich et al. (1988), p118.

[3]The helicity formalism is discussed in Chapter 13.

this exercise. Also this method is restricted to the construction of D^j for only integer values of j. However there is an alternative, simple and elegant way of constructing the elements of rotation matrices of higher dimensions from the elements of rotation matrices of lower dimensions using the inverse of the C.G. series (Eq. (5.55)). This latter procedure is applicable for constructing the rotation matrices of both integer and half-integer ranks. For this purpose, we require the rotation matrix $D^{\frac{1}{2}}(\alpha, \beta, \gamma)$ and starting from this all the D^j matrices can be obtained by successive application of the inverse C.G. series (Eq. (5.55)).

Review Questions

4.1 Define the Rotation Matrix and explain how the rotation about an arbitrary axis $\hat{n}$ can be expressed in terms of the Euler angles of rotation.

4.2 Show how the spherical components of a vector transform under rotation and hence obtain the rotation matrix corresponding to a rotation through an angle β about the Y axis.

4.3 Check whether the transformation matrix $M(\beta)$ given by Eq. (4.15) is unitary.

Problems

4.1 Show that a rotation of the coordinate system about an arbitrary axis $\hat{n}$ is equivalent to Euler angles of rotation. Hence obtain a relation between the two sets of rotation parameters.

4.2 Given any two vectors ***A*** and ***B***, construct a spherical tensor T_2^μ of rank 2 and obtain the rotation matrix $D^2(\alpha)$ for a rotation about the Z axis from the known transformation properties of spherical components of vectors ***A*** and ***B*** under rotation.

4.3 Given any two vectors ***A*** and ***B***, construct a spherical tensor T_2^μ of rank 2 and obtain the rotation matrix $D^2(\beta)$ for a rotation about the Y axis from the known transformation properties of spherical components of vectors ***A*** and ***B*** under rotation.

4.4 Given any two vectors ***A*** and ***B***, construct a spherical tensor of rank 2 and study its transformation properties under rotation of coordinate system. Hence obtain the rotation matrix for $j = 2$.

Solutions to Selected Problems

4.1 This problem is dealt with in the Appendix A, to which the reader is referred.

4.2 Components T_2^μ of spherical tensor of rank 2 are constructed from vectors ***A*** and ***B***.

$$T_2^\mu = \sum_{\mu_1} \begin{bmatrix} 1 & 1 & 2 \\ \mu_1 & \mu_2 & \mu \end{bmatrix} A_1^{\mu_1} B_1^{\mu_2}.$$

These components are explicitly given in Table 3.1. If the coordinate system is rotated through an angle α about the Z axis, the components of the second rank tensor are transformed as given below.

$$
\begin{aligned}
(T_2^2)_1 &= (A_1^1)_1 (B_1^1)_1 = A_1^1 \, e^{-i\alpha} \, B_1^1 \, e^{-i\alpha} = T_2^2 \, e^{-2i\alpha}, \\
(T_2^1)_1 &= \frac{1}{\sqrt{2}} \left\{ (A_1^1)_1 (B_1^0)_1 + (A_1^0)_1 (B_1^1)_1 \right\} \\
&= \frac{1}{\sqrt{2}} \left\{ A_1^1 \, B_1^0 \; + \; A_1^0 \, B_1^1 \right\} e^{-i\alpha} = T_2^1 \, e^{-i\alpha}.
\end{aligned}
$$

Similarly,

$$
(T_2^0)_1 = T_2^0, \quad (T_2^{-1})_1 = T_2^{-1} \, e^{i\alpha}, \quad (T_2^{-2})_1 = T_2^{-2} \, e^{2i\alpha}.
$$

Thus, the transformation matrix $M(\alpha)$ for T_2^μ for rotation through an angle α about the Z axis is obtained from the relation

$$
\begin{bmatrix} (T_2^2)_1 \\ (T_2^1)_1 \\ (T_2^0)_1 \\ (T_2^{-1})_1 \\ (T_2^{-2})_1 \end{bmatrix}
=
\begin{bmatrix}
e^{-2i\alpha} & 0 & 0 & 0 & 0 \\
0 & e^{-i\alpha} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & e^{i\alpha} & 0 \\
0 & 0 & 0 & 0 & e^{2i\alpha}
\end{bmatrix}
\begin{bmatrix} T_2^2 \\ T_2^1 \\ T_2^0 \\ T_2^{-1} \\ T_2^{-2} \end{bmatrix}.
$$

The rotation matrix is the transpose of the transformation matrix. Since the transformation matrix is a diagonal matrix, the rotation matrix coincides with transformation matrix for rotation about the Z axis.

4.3 The transformation matrix for T_2^μ for rotation through an angle β about the Y axis is a little more complicated since it is non-diagonal. But the method is essentially the same.

$$
M(\beta) =
$$

$$
\begin{bmatrix}
\frac{1}{4}(1+\cos\beta)^2 & \frac{1}{2}(1+\cos\beta)\sin\beta & \sqrt{\frac{3}{8}}\sin^2\beta & \frac{1}{2}(1-\cos\beta)\sin\beta & \frac{1}{4}(1-\cos\beta)^2 \\
-\frac{1}{2}(1+\cos\beta)\sin\beta & \frac{1}{2}(\cos\beta+\cos 2\beta) & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(\cos\beta-\cos 2\beta) & \frac{1}{2}(1-\cos\beta)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & -\sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(3\cos^2\beta-1) & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
-\frac{1}{2}(1-\cos\beta)\sin\beta & \frac{1}{2}(\cos\beta-\cos 2\beta) & -\sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(\cos\beta+\cos 2\beta) & \frac{1}{2}(1+\cos\beta)\sin\beta \\
\frac{1}{4}(1-\cos\beta)^2 & -\frac{1}{2}(1-\cos\beta)\sin\beta & \sqrt{\frac{3}{8}}\sin^2\beta & -\frac{1}{2}(1+\cos\beta)\sin\beta & \frac{1}{4}(1+\cos\beta)^2
\end{bmatrix}.
$$

The rotation matrix is the transpose of the transformation matrix $M(\beta)$.

4.4 The rotation matrix $D^2(\alpha,\beta,\gamma)$ is the transpose of the transformation matrix $M(\alpha,\beta,\gamma)$.

$$M(\alpha, \beta, \gamma) = M_{Z_2}(\gamma)\, M_{Y_1}(\beta)\, M_Z(\alpha).$$

The transformation matrix $M(\alpha)$ for rotation about the Z axis is worked out in Problem (4.2) and the transformation matrix $M(\beta)$ for rotation about the Y axis is given in Problem (4.3).

CHAPTER 5

ROTATION MATRICES - II

5.1. The Rotation Operator

Let us consider an infinitesimal rotation $\delta\alpha$ about the Z-axis of a right-handed coordinate system and investigate how the wave function transforms.

$$\Psi'(\boldsymbol{r}) = R_Z(\delta\alpha)\,\Psi(\boldsymbol{r}) = \Psi(\boldsymbol{r}'), \tag{5.1}$$

where $R_Z(\delta\alpha)$ is the rotation operator which causes a rotation of the coordinate system S $\rightarrow$ S' through an infinitesimal angle $\delta\alpha$ about the Z-axis. Under rotation,

$$\boldsymbol{r} \longrightarrow \boldsymbol{r}', \tag{5.2}$$

$$\Psi(\boldsymbol{r}) \longrightarrow \Psi(\boldsymbol{r}') = \Psi'(\boldsymbol{r}). \tag{5.3}$$

Under the rotation of coordinate system S $\rightarrow$ S', the coordinates of a physical point changes from r to r' and the function $\Psi(r)$ transforms to $\Psi(r')$, which, in turn, becomes a new function $\Psi'(r)$ when expressed in terms of the old coordinate r.

$$\begin{aligned}\Psi'(\boldsymbol{r}) &= \Psi(\boldsymbol{r}') \\ &= \Psi(x + y\delta\alpha,\, y - x\delta\alpha,\, z) \\ &= \Psi(x,y,z) + \delta\alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\Psi(x,y,z).\end{aligned} \tag{5.4}$$

The last step is obtained by applying the Taylor series expansion and neglecting terms involving higher powers of $\delta\alpha$. Since the Z-component of the orbital angular momentum operator L_Z is given by

$$L_Z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{5.5}$$

we have

$$\Psi'(\boldsymbol{r}) = (1 - i\,\delta\alpha\,L_Z)\Psi(\boldsymbol{r}). \tag{5.6}$$

Let us now generalize the relation (5.6) and replace the operator $\boldsymbol{L}$ by $\boldsymbol{J}$.

$$\Psi'(\boldsymbol{r}) = (1 - i\,\delta\alpha\, J_Z)\Psi(\boldsymbol{r}). \tag{5.7}$$

Equation (5.7) gives the transformation of the function due to an infinitesimal rotation through an angle $\delta\alpha$ about the Z-axis. Making a large number (n) of such infinitesimal rotations, one can obtain a finite rotation α about the Z-axis.

$$R_Z(\alpha)\Psi(\boldsymbol{r}) = (1 - i\,\delta\alpha\, J_Z)^n\Psi(\boldsymbol{r}) = e^{-i\,\alpha\, J_Z}\,\Psi(\boldsymbol{r}), \tag{5.8}$$

where $\alpha = n\,\delta\alpha$. In a similar way, we can find the rotation operator corresponding to a rotation about the Y-axis.

$$R_Y(\beta)\Psi(\boldsymbol{r}) = e^{-i\beta J_Y}\,\Psi(\boldsymbol{r}). \tag{5.9}$$

It is to be noted that $\boldsymbol{J}^2$ commutes with the rotation operators and hence j is a good quantum number under rotation.

Any general rotation can be described in terms of three parameters (Goldstein, 1980; Bohr and Mottelson, 1969). They may be the three Euler angles α,β,γ or they may correspond to a rotation ψ about an axis $\hat{\boldsymbol{n}}$ which is fixed by the two parameters θ and ϕ.

$$R_{\hat{\boldsymbol{n}}}(\psi) = R(\alpha,\beta,\gamma), \tag{5.10}$$

where

$$R_{\hat{\boldsymbol{n}}}(\psi) = e^{-i\psi\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J}}, \tag{5.11}$$

and

$$\begin{aligned} R(\alpha,\beta,\gamma) &= R_{Z_2}(\gamma)\, R_{Y_1}(\beta)\, R_Z(\alpha) \\ &= e^{-i\gamma J_{Z_2}}\, e^{-i\beta J_{Y_1}}\, e^{-i\alpha J_Z}. \end{aligned} \tag{5.12}$$

We have the following relation between the parameters specifying the single rotation and the Euler angles (vide Appendix A).

$$\begin{aligned} \cos\frac{\psi}{2} &= \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}, \\ \sin\frac{\psi}{2}\sin\theta &= \sin\frac{\beta}{2}, \\ \phi &= \frac{\gamma-\alpha}{2} + \frac{\pi}{2}. \end{aligned} \tag{5.13}$$

In the expansion for $R(\alpha,\beta,\gamma)$ given by Eq. (5.12), only the rotation through an angle α is carried out about the Z-axis of the original coordinate system but the rotations β and γ are carried out about the axes Y_1 and Z_2 of the new coordinate systems obtained in successive rotations.

$$XYZ \underset{R_Z(\alpha)}{\longrightarrow} X_1Y_1Z_1 \underset{R_{Y_1}(\beta)}{\longrightarrow} X_2Y_2Z_2 \underset{R_{Z_2}(\gamma)}{\longrightarrow} X'Y'Z'.$$

Since the rotations are unitary transformations, we can subject the operators to unitary transformations successively in order to denote all the rotations with respect to the original coordinate system. For instance,

$$\begin{aligned} e^{-i\gamma J_{Z_2}} &= R_{Y_1}(\beta)\, e^{-i\gamma J_{Z_1}}\, [R_{Y_1}(\beta)]^{-1} \\ &= e^{-i\beta J_{Y_1}}\, e^{-i\gamma J_{Z_1}}\, e^{i\beta J_{Y_1}}. \end{aligned} \tag{5.14}$$

Substituting Eq. (5.14) in Eq. (5.12), we get

$$R(\alpha,\beta,\gamma) = e^{-i\beta J_{Y_1}}\, e^{-i\gamma J_{Z_1}}\, e^{-i\alpha J_Z}. \tag{5.15}$$

Once again, we can subject the operators in the coordinate system $X_1\ Y_1\ Z_1$ to a unitary transformation and obtain the corresponding operators in the coordinate system XYZ.

$$e^{-i\beta J_{Y_1}}\, e^{-i\gamma J_{Z_1}} = e^{-i\alpha J_Z}\, e^{-i\beta J_Y}\, e^{-i\gamma J_Z}\, e^{i\alpha J_Z}. \tag{5.16}$$

Substituting (5.16) into (5.15), we get finally,

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_Z}\, e^{-i\beta J_Y}\, e^{-i\gamma J_Z}. \tag{5.17}$$

In the expression (5.17) for $R(\alpha\beta\gamma)$ all the rotations are carried out in the original coordinate system and its usefulness will be seen in the next section. The rotation operator R is unitary, that is

$$R^\dagger R = RR^\dagger = 1; \quad R^{-1} = R^\dagger. \tag{5.18}$$

5.2. The $d^j_{m'm}(\beta)$ Matrix

The rotation matrix $D^j_{m'm}(\alpha\beta\gamma)$ has been defined in Eq. (4.1) of the previous chapter and now we can express its elements as the matrix elements of the rotation operator $R(\alpha,\beta\gamma)$.

$$\begin{aligned} \Psi_{jm}(\boldsymbol{r}') &= R(\alpha,\beta,\gamma)\,\Psi_{jm}(\boldsymbol{r}) \\ &= \sum_{m'} D^j_{m'm}(\alpha,\beta,\gamma)\,\Psi_{jm'}(\boldsymbol{r}), \end{aligned} \tag{5.19}$$

or

$$D^j_{m'm}(\alpha,\beta,\gamma) = \langle \Psi_{jm'}(\boldsymbol{r}) \,|\, R(\alpha,\beta,\gamma) \,|\, \Psi_{jm}(\boldsymbol{r}) \rangle = \langle jm' \,|\, R(\alpha,\beta,\gamma) \,|\, jm \rangle. \tag{5.20}$$

Using the explicit form (5.17) for $R(\alpha\beta\gamma)$ and remembering that the angular momentum functions are eigenfunctions of J_Z operator, we obtain

$$D^j_{m'm}(\alpha,\beta,\gamma) = \langle jm' \,|\, e^{-i\alpha J_Z}\, e^{-i\beta J_Y}\, e^{-i\gamma J_Z} \,|\, jm \rangle = e^{-i\alpha m'} \langle jm' \,|\, e^{-i\beta J_Y} \,|\, jm \rangle\, e^{-i\gamma m}. \tag{5.21}$$

The last step was obtained by allowing the operator $e^{-i\alpha J_Z}$ to operate on the left state and the operator $e^{-i\gamma J_Z}$ on the right state. This was possible only because both the operators and the states correspond to the same coordinate system.

In our representation, J_Y is purely imaginary and hence the matrix element $\langle jm' \,|\, e^{-i\beta J_Y} \,|\, jm \rangle$ is real. Denoting this matrix element by $d^j_{m'm}(\beta)$, we have

$$D^j_{m'm}(\alpha\beta\gamma) = e^{-i\alpha m'}\, d^j_{m'm}(\beta)\, e^{-i\gamma m}. \tag{5.22}$$

Since $d^j_{m'm}(\beta)$ is unitary and real, the following symmetry relations are satisfied.

$$d^j_{m'm}(\beta) = d^j_{mm'}(-\beta) \tag{5.23}$$

$$= (-1)^{m'-m} d^j_{mm'}(\beta) \tag{5.24}$$

$$= d^j_{-m,-m'}(\beta). \tag{5.25}$$

Once we obtain the matrix $d^j_{m'm}(\beta)$, the construction of the full rotation matrix $D^j_{m'm}(\alpha,\beta,\gamma)$ is simple because of Eq. (5.22). Also, the construction of $d^j_{m'm}(\beta)$ for higher j-values[1] can be done starting from the lower j-values using the coupling rule for rotation matrices (inverse C.G. series) to be discussed in Sec. 5.5

5.3. The Rotation Matrix for Spinors

We shall now obtain the rotation matrix for $j = \frac{1}{2}$. For a rotation about the Y-axis, the rotation operator is given by

$$R_Y(\beta) = e^{-i\beta S_Y}, \tag{5.26}$$

[1]Rotation matrices for j = $\frac{1}{2}$, 1 are given in Eqs. (5.33) and (5.98). For the explicit, forms of the rotation matrices for higher j values, the reader is referred to Varshalovich et al. (1988).

where S_Y is the Y-component of the spin operator. .Expressing it in terms of the Pauli spin operator σ_y, we have

$$R_Y(\beta) = e^{-i\frac{\beta}{2}\sigma_y}. \tag{5.27}$$

Recalling the following series expansions

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, &(5.28)\\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, &(5.29)\\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, &(5.30)\end{aligned}$$

and the property of the Pauli matrices,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1, \tag{5.31}$$

we obtain a simple form for the rotation matrix.

$$\begin{aligned} R_Y(\beta) &= e^{-i\frac{\beta}{2}\sigma_y} \\ &= 1 - i\frac{\beta}{2}\sigma_y - \frac{(\frac{\beta}{2}\sigma_y)^2}{2!} + i\frac{(\frac{\beta}{2}\sigma_y)^3}{3!} + \frac{(\frac{\beta}{2}\sigma_y)^4}{4!} \cdots \\ &= 1 - i\frac{\beta}{2}\sigma_y - \frac{(\frac{\beta}{2})^2}{2!} + i\frac{(\frac{\beta}{2})^3}{3!}\sigma_y + \frac{(\frac{\beta}{2})^4}{4!} \cdots \\ &= \left(1 - \frac{(\frac{\beta}{2})^2}{2!} + \frac{(\frac{\beta}{2})^4}{4!} - \cdots\right) - i\sigma_y\left(\frac{\beta}{2} - \frac{(\frac{\beta}{2})^3}{3!} + \cdots\right) \\ &= \cos\frac{\beta}{2} - i\sigma_y \sin\frac{\beta}{2}. \end{aligned} \tag{5.32}$$

Substituting the matrix elements of σ_y, we obtain the matrix representation for the operator $R_Y(\beta)$ and it is denoted by $d^{1/2}(\beta)$.

$$\begin{aligned} d^{1/2}(\beta) &= R_Y(\beta) = \cos\frac{\beta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin\frac{\beta}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}. \end{aligned} \tag{5.33}$$

In a similar way, we can obtain the rotation matrices for rotations about the X or Z-axis.

$$R_X(\theta) = \cos\frac{\theta}{2} - i\sigma_x \sin\frac{\theta}{2} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}. \tag{5.34}$$

$$R_Z(\theta) = \cos\frac{\theta}{2} - i\sigma_z \sin\frac{\theta}{2} = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}. \tag{5.35}$$

Let us now investigate the effect of rotation of the coordinate system on the eigenfunction Ψ_m. A rotation through an angle β about the Y-axis yields

$$\chi_m = \sum_{m'} d_{m'm}(\beta)\, \Psi_{m'}. \tag{5.36}$$

In Eq. (5.36)) an explicit mention of the quantum number j is omitted but it is understood that $j = \frac{1}{2}$ in the following discussion. If we wish to express the eigenfunctions Ψ and χ as column vectors and *d* as a matrix, and use the usual rule of matrix multiplication, then we find the matrix d^T which is the transpose of the d matrix to be more convenient.

$$\chi_m = \sum_{m'} (d^T(\beta))_{mm'}\, \Psi_{m'}. \tag{5.37}$$

Writing explicitly, we have

$$\begin{bmatrix} \chi_{\frac{1}{2}} \\ \chi_{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} \Psi_{\frac{1}{2}} \\ \Psi_{-\frac{1}{2}} \end{bmatrix}. \tag{5.38}$$

If we start with a pure state $\Psi_{\frac{1}{2}}$ which is a spinor with spin up $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then a rotation through an angle 2π about the Y-axis yields

$$\begin{aligned} \begin{bmatrix} \chi_{\frac{1}{2}} \\ \chi_{-\frac{1}{2}} \end{bmatrix} &= \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= -\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{5.39}$$

This is in contradiction to the case of a vector for which the rotation through an angle 2π leaves the vector undisturbed. In the case of spinor, a rotation through an angle 4π is necessary to get the same spinor. That is why the spinors are sometimes referred to as 'half-vectors'.

Also there is an interesting feature that a spinor exhibits. For a spinor located at the origin of the coordinate system, a rotation through an angle π about the X-axis is not equivalent to a rotation through an angle π about the Y-axis.

$$R_X(\pi)\, \Psi_{\frac{1}{2}} = -i\Psi_{-\frac{1}{2}} = \varphi; \tag{5.40}$$

$$R_Y(\pi)\, \Psi_{\frac{1}{2}} = -\Psi_{-\frac{1}{2}} = \varphi'. \tag{5.41}$$

For a vector located at the origin, these two rotation will invert the vector. But it is not so in the case of spinors. However, it can be shown that the two spinors φ and φ ' differ by a rotation through an angle π about the Z-axis.

$$R_Z(\pi)\varphi = -\Psi_{\frac{1}{2}} = \varphi'; \tag{5.42}$$
$$R_Z(-\pi)\varphi' = -i\Psi_{\frac{1}{2}} = \varphi. \tag{5.43}$$

That is why a spinor can be considered as a vector with a thickness.

5.4. The Clebsch-Gordan Series

In this section, we shall obtain a coupling rule for rotation matrices and it is deduced from the coupling scheme of two angular momenta.

$$|j_1 m_1\rangle\,|j_2 m_2\rangle = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |jm\rangle. \tag{5.44}$$

Rotating the coordinate system through the Euler angles (α,β,γ), we obtain

$$\sum_{\nu_1\nu_2} D^{j_1}_{\nu_1 m_1}(\omega)\, D^{j_2}_{\nu_2 m_2}(\omega)\, |j_1\nu_1\rangle|j_2\nu_2\rangle = \sum_{j\mu} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} D^j_{\mu m}(\omega)|j\mu\rangle, \tag{5.45}$$

where the argument ω of the D matrix stands for the set of Euler angles α,β,γ. The state $|j\mu\rangle$ on the right hand side can be expanded as

$$|j\mu\rangle = \sum_{\mu'_1} \begin{bmatrix} j_1 & j_2 & j \\ \mu'_1 & \mu'_2 & \mu \end{bmatrix} |j_1\mu'_1\rangle\,|j_2\mu'_2\rangle. \tag{5.46}$$

Inserting this into Eq. (5.45) and taking the scalar product with $|j_1\mu_1\rangle|j_2\mu_2\rangle$, we obtain

$$\sum_{\nu_1\nu_2} D^{j_1}_{\nu_1 m_1}(\omega)\, D^{j_2}_{\nu_2 m_2}(\omega)\, \delta_{\mu_1\nu_1}\,\delta_{\mu_2\nu_2} = \sum_{j\mu\mu'_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ \mu'_1 & \mu'_2 & \mu \end{bmatrix} D^j_{\mu m}(\omega)\,\delta_{\mu_1\mu'_1}\,\delta_{\mu_2\mu'_2}. \tag{5.47}$$

The sum over μ on the right-hand side of Eq. (5.47) can be replaced by $\mu_2' = \mu - \mu_1'$. Now, performing the summation over the projection quantum numbers, we obtain

$$D^{j_1}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega) = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^j_{\mu m}(\omega). \quad (5.48)$$

This is known as the Clebsch-Gordan series (C.G. series).

5.5. The Inverse Clebsch-Gordan Series

Starting from the C.G. series (Eq. (5.48)), an inverse series can be obtained using the orthogonality of the C.G. coefficients. Multiplying both sides of Eq. (5.48) by $\begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix}$ and summing over m_1, we obtain

$$\begin{aligned} \sum_{m_1} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix} & D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega) \\ &= \sum_j \delta_{jj'} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^j_{\mu m}(\omega) \\ &= \begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j'}_{\mu m}(\omega). \end{aligned} \quad (5.49)$$

Equation (5.49) was obtained by applying the orthonormality condition (Eq. 2.19) of C.G. coefficients.

Once again, multiplying both sides by $\begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix}$ and summing over μ_1, we obtain

$$\sum_{m_1 \mu_1} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega) = D^{j'}_{\mu m}(\omega). \quad (5.50)$$

This is known as the inverse C.G. series. There is an alternative way of obtaining this series.

The alternative method is to start from the following coupling rule of two angular momenta.

$$|jm\rangle = \sum_{m_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |jm_1\rangle |jm_2\rangle. \quad (5.51)$$

Rotate the coordinate system through the Euler angles α,β,γ. Applying the transformation, we now have

$$\begin{aligned}\sum_{m'} D^j_{m'm}(\omega)|jm'\rangle &= \sum_{m_1\mu_1\mu_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega)|j_1\mu_1\rangle|j_2\mu_2\rangle \\ &= \sum_{m_1\mu_1\mu_2}\sum_{j'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & M' \end{bmatrix} \\ &\quad \times D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega)|j'M'\rangle. \end{aligned} \tag{5.52}$$

Taking the scalar product on both sides with $|J\mu\rangle$, we obtain

$$\begin{aligned}\sum_{m'} D^j_{m'm}(\omega)\delta_{Jj}\delta_{\mu m'} &= \sum_{m_1\mu_1\mu_2}\sum_{j'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & M' \end{bmatrix} \\ &\quad \times D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega)\, \delta_{Jj'}\, \delta_{\mu M'}. \end{aligned} \tag{5.53}$$

Replacing the summation index μ_2 by M' and summing over M' and j' on the right and over m' on the left, we obtain

$$\begin{aligned}D^j_{\mu m}(\omega)\,\delta_{Jj} &= \sum_{m_1\mu_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1 & j_2 & J \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega). \end{aligned} \tag{5.54}$$

Finally, we obtain

$$D^j_{\mu m}(\omega) = \sum_{m_1\mu_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega). \tag{5.55}$$

which is the same as Eq. (5.50). The inverse C.G. series can be used to generate the elements of all the matrices $D_j(\omega)$, $(j\ (j > \frac{1}{2})$, if the rotation matrix $D^{\frac{1}{2}}(\omega)$ is given.

5.6. Unitarity and Symmetry Properties of the Rotation Matrices

Rotation of a coordinate system is equivalent to performing a unitary transformation on the functions.

$$\Psi_{jm}(\boldsymbol{r}') = \sum_{m'} D^j_{m'm}(\omega)\, \Psi_{jm'}(\boldsymbol{r}), \tag{5.56}$$

$$\Psi_{j\mu}(\boldsymbol{r}') = \sum_{\mu'} D^j_{\mu'\mu}(\omega)\,\Psi_{j\mu'}(\boldsymbol{r}). \tag{5.57}$$

Taking their scalar product and summing over *m'*, we obtain

$$\langle\Psi_{j\mu}(\boldsymbol{r}')|\Psi_{jm}(\boldsymbol{r}')\rangle = \sum_{m'\mu'} D^{j\,*}_{\mu'\mu}(\omega)\,D^j_{m'm}(\omega)\,\langle\Psi_{j\mu'}(\boldsymbol{r})|\Psi_{jm'}(\boldsymbol{r})\rangle,$$

$$\delta_{\mu m} = \sum_{\mu'} D^{j\,*}_{\mu'\mu}(\omega)\,D^j_{\mu'm}(\omega). \tag{5.58}$$

The inverse relation of (5.56) is

$$\begin{aligned}\Psi_{jm}(\boldsymbol{r}) &= \sum_{m'} D^{j\,*}_{mm'}(\omega)\,\Psi_{jm'}(\boldsymbol{r}') \\ &= \sum_{m'm''} D^{j\,*}_{mm'}(\omega)\,D^j_{m''m'}(\omega)\,\Psi_{jm''}(\boldsymbol{r}).\end{aligned} \tag{5.59}$$

Taking the scalar product with $\Psi_{j\mu}(\boldsymbol{r})$, on both sides of Eq. (5.59), we obtain

$$\delta_{\mu m} = \sum_{m'} D^{j\,*}_{mm'}(\omega)\,D^j_{\mu m'}(\omega). \tag{5.60}$$

Equations (5.58) and (5.60) are the mathematical expressions denoting the unitarity of the D-matrices.

It is easy to see that two successive rotations through Euler angles ω_1 and ω_2 is equivalent to a single Euler rotation ω. This yields a relationship between the D-matrices.

$$\begin{aligned}\Psi_{jm}(\boldsymbol{r}'') &= \sum_{m'} D^j_{m'm}(\omega_2)\,\Psi_{jm'}(\boldsymbol{r}') \\ &= \sum_{m'm''} D^j_{m'm}(\omega_2)\,D^j_{m''m'}(\omega_1)\,\Psi_{jm''}(\boldsymbol{r}) \\ &= \sum_{m''} D^j_{m''m}(\omega)\,\Psi_{jm''}(\boldsymbol{r}).\end{aligned} \tag{5.61}$$

Hence

$$D^j_{m''m}(\omega) = \sum_{m'} D^j_{m''m'}(\omega_1)\,D^j_{m'm}(\omega_2). \tag{5.62}$$

The D-matrices exhibit the following symmetry properties:

$$D^{j\,*}_{\mu m}(\alpha,\beta,\gamma) = (-1)^{\mu-m}\,D^j_{-\mu-m}(\alpha,\beta,\gamma). \tag{5.63}$$

$$D^j_{\mu m}(-\gamma,-\beta,-\alpha) = D^{j\,*}_{m\mu}(\alpha,\beta,\gamma). \tag{5.64}$$

If ω denotes the Euler angles of rotation (α,β,γ), the inverse rotation ω^{-1} is denoted by the Euler angles $(-\gamma,-\beta,-\alpha)$. The symmetry property (5.64) follows from the unitary nature of the transformation and Eq. (5.63) directly follows from Eqs. (5.22) and (5.24).

Using the group theory, a general expression for $D^j_{\mu m}(\alpha,\beta,\gamma)$ has been obtained by Wigner and it is also given by Rose (1957).

$$\begin{aligned} D^j_{\mu m}(\alpha,\beta,\gamma) &= e^{-i\mu\alpha}\, e^{-im\gamma}\, [(j+m)!(j-m)!(j+\mu)!(j-\mu)!]^{\frac{1}{2}} \\ &\times \sum_x \frac{(-1)^x}{x!(j-x-\mu)!(j+m-x)!(\mu+x-m)!} \\ &\times \left(\cos\frac{\beta}{2}\right)^{2j+m-\mu-2x} \left(-\sin\frac{\beta}{2}\right)^{\mu-m+2x}. \end{aligned} \tag{5.65}$$

The sum over x is over all integer values for which the factorial arguments are greater than or equal to zero.

5.7. The Spherical Harmonic Addition Theorem

Consider any two points P_1 and P_2 on a unit sphere. In a certain coordinate system S, their coordinates are (θ_1,ϕ_1) and (θ_2,ϕ_2). In a rotated coordinate system S', let their coordinates be (θ'_1,ϕ'_1) and (θ'_2,ϕ'_2). (See Table 5.1.) Then we can show that

$$\begin{aligned} \mathcal{I} &= \sum_m Y_l^{m*}(\theta_1,\phi_1)\, Y_l^m(\theta_2,\phi_2) \\ &= \sum_m Y_l^{m*}(\theta'_1,\phi'_1)\, Y_l^m(\theta'_2,\phi'_2). \end{aligned} \tag{5.66}$$

In other words, the quantity $\mathcal{I}$ is invariant under rotation of coordinate system.

To prove this, consider the quantity $\mathcal{I}$ defined in the rotated coordinate system S'

$$\mathcal{I} = \sum_m Y_l^{m*}(\theta'_1,\phi'_1)\, Y_l^m(\theta'_2,\phi'_2). \tag{5.67}$$

The spherical harmonics given in frame S' can be obtained from the spherical harmonics defined in frame S using the rotation matrices (Eq. (5.19)).

$$\mathcal{I} = \sum_{mm_1m_2} D^{l*}_{m_1m}(\omega)\, D^l_{m_2m}(\omega)\, Y_l^{m_1*}(\theta_1,\phi_1)\, Y_l^{m_2}(\theta_2,\phi_2). \tag{5.68}$$

TABLE 5.1. Polar coordinates of points P_1 and P_2 in different coordinate systems S, S′ and S_0

Points	S	S′	S_0
P_1	θ_1, ϕ_1	θ'_1, ϕ'_1	$0, 0$
P_2	θ_2, ϕ_2	θ'_2, ϕ'_2	$\theta, 0$

Summing over m and applying the orthonormality of rotation matrices,

$$\sum_m D^{l*}_{m_1 m}(\omega)\, D^l_{m_2 m}(\omega) = \delta_{m_1 m_2}, \tag{5.69}$$

we obtain

$$\mathcal{I} = \sum_{m_1 m_2} Y_l^{m_1 *}(\theta_1, \phi_1)\, Y_l^{m_2}(\theta_2, \phi_2)\, \delta_{m_1 m_2}, \tag{5.70}$$

thereby proving that $\mathcal{I}$ is invariant under rotation.

Now let us choose a convenient coordinate system S_0, in which P_1 lies on the Z-axis and P_2 in the X-Z plane. Their coordinates in the frame S_0 are (0,0) and (θ,0). The invariant quantity in this frame has a simple structure

$$\begin{aligned} \mathcal{I} &= \sum_m Y_l^{m*}(0,0)\, Y_l^m(\theta, 0) \\ &= \sum_m \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}\, Y_l^m(\theta, 0) \\ &= \sqrt{\frac{2l+1}{4\pi}}\, Y_l^0(\theta, 0). \end{aligned} \tag{5.71}$$

Equating $\mathcal{I}$ in the two frames S_0 and S, we arrive at the well-known theorem known as the spherical harmonic addition theorem.

$$\sqrt{\frac{2l+1}{4\pi}}\, Y_l^0(\theta, 0) = \sum_m Y_l^{m*}(\theta_1, \phi_1)\, Y_l^m(\theta_2, \phi_2),$$

or

$$Y_l^0(\theta, 0) = \sqrt{\frac{4\pi}{2l+1}} \sum_m Y_l^{m*}(\theta_1, \phi_1)\, Y_l^m(\theta_2, \phi_2). \tag{5.72}$$

The angle θ is the angle subtended by the two points P_1 and P_2. Expressing $Y_l^0(\theta,0)$ in terms of Legendre function,

$$Y_l^0(\theta,0)=\sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta), \tag{5.73}$$

we obtain an alternative form for the spherical harmonic addition theorem.

$$P_l(\cos\theta)=\frac{4\pi}{2l+1}\sum_m Y_l^{m*}(\theta_1,\phi_1)\,Y_l^m(\theta_2,\phi_2). \tag{5.74}$$

5.8. The Coupling Rule for the Spherical Harmonics

Now let us consider a rotation of the frame from S to S_0. In the frame S, the coordinates of the points P_1 and P_2 are (θ_1,ϕ_1) and (θ_2,ϕ_2). In the new frame S_0, P_1 lies on the Z-axis and P_2 in the X–Z plane with coordinates $(\theta,0)$. This rotation corresponds to the Euler angles

$$(\alpha,\beta,\gamma)=(\phi_1,\theta_1,0). \tag{5.75}$$

Let us investigate how the spherical harmonic $Y_l^m(\theta_2,\phi_2)$ associated with the point P_2 transforms under this rotation

$$Y_l^0(\theta,0)=\sum_m D^l_{m0}(\phi_1,\theta_1,0)\,Y_l^m(\theta_2,\phi_2). \tag{5.76}$$

Comparing this equation with Eq. (5.72) obtained for the spherical harmonic addition theorem, we get the relation

$$D^l_{m0}(\phi_1,\theta_1,0)=\sqrt{\frac{4\pi}{2l+1}}\,Y_l^{m*}(\theta_1,\phi_1). \tag{5.77}$$

This is a very useful relation giving the connection between the rotation matrices for integral j and the spherical harmonics and this relation can be directly used to obtain a coupling rule for the spherical harmonics with the same arguments.

Consider the Clebsch-Gordan series

$$D^{l_1}_{m_10}(\phi,\theta,0)\,D^{l_2}_{m_20}(\phi,\theta,0) =\sum_l \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{bmatrix} D^l_{m,0}(\phi,\theta,0). \tag{5.78}$$

Replacing the rotation matrices by spherical harmonics using Eq. (5.77), we obtain

$$\sqrt{\frac{4\pi}{2l_1+1}}\sqrt{\frac{4\pi}{2l_2+1}}\,Y_{l_1}^{m_1\,*}(\theta,\phi)\,Y_{l_2}^{m_2\,*}(\theta,\phi)$$
$$=\sum_l \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{bmatrix}\sqrt{\frac{4\pi}{2l+1}}\,Y_l^{m\,*}(\theta,\phi). \quad (5.79)$$

Taking the complex conjugate of the above equation and remembering that the C.G. coefficients are real, we have

$$Y_{l_1}^{m_1}(\theta,\phi)\,Y_{l_2}^{m_2}(\theta,\phi) = \sum_l \left\{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right\}^{\frac{1}{2}}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$
$$\times\begin{bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{bmatrix} Y_l^m(\theta,\phi). \quad (5.80)$$

This is the coupling rule for the spherical harmonics with the same argument. The C.G. coefficient $\begin{bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{bmatrix}$ is the parity C.G. coefficient which is nonvanishing only if $l_1 + l_2 - l$ is even. This implies that the 1 values in the summation take either all even values or all odd values depending upon l_1 and l_2.

The above rule permits an easy evaluation of the integral involving three spherical harmonics,

$$I = \int Y_{l_3}^{m_3\,*}(\theta,\phi)\,Y_{l_2}^{m_2}(\theta,\phi)\,Y_{l_1}^{m_1}(\theta,\phi)\,d\Omega. \quad (5.81)$$

First let us couple the two spherical harmonics $Y_{l_1}^{m_1}(\theta,\phi)\;Y_{l_2}^{m_2}(\theta,\phi)$ and then integrate, applying the orthonormality condition of the spherical harmonics.

$$I = \sum_l \frac{[l_1][l_2]}{\sqrt{4\pi}[l]}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{bmatrix}$$
$$\times\int Y_{l_3}^{m_3\,*}(\theta,\phi)\,Y_l^m(\theta,\phi)\,d\Omega. \quad (5.82)$$

Since the last integral simply yields $\delta_{ll3}\ \delta_{mm3}$, we obtain

$$I = \frac{[l_1][l_2]}{\sqrt{4\pi}[l_3]}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5.83)$$

where the notation

$$[l] = (2l+1)^{\frac{1}{2}} \tag{5.84}$$

is used.

5.9. Orthogonality and Normalization of the Rotation Matrices

In this section, we shall show that the functions $D^j_{m'm}(\omega)$ are orthogonal on the surface of the unit sphere and evaluate the integral

$$I = \int D^{j_1\,*}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega)\, d\omega, \tag{5.85}$$

where

$$\int d\omega = \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma. \tag{5.86}$$

Since

$$D^{j_1\,*}_{\mu_1 m_1}(\omega) = (-1)^{\mu_1 - m_1}\, D^{j_1}_{-\mu_1 -m_1}(\omega), \tag{5.87}$$

and

$$\begin{aligned} D^{j_1}_{-\mu_1 -m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega) &= \sum_j \begin{bmatrix} j_1 & j_2 & j \\ -\mu_1 & \mu_2 & \mu \end{bmatrix} \\ &\times \begin{bmatrix} j_1 & j_2 & j \\ -m_1 & m_2 & m \end{bmatrix} D^j_{\mu m}(\omega), \end{aligned} \tag{5.88}$$

we have

$$I = (-1)^{\mu_1 - m_1} \sum_j \begin{bmatrix} j_1 & j_2 & j \\ -\mu_1 & \mu_2 & \mu \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ -m_1 & m_2 & m \end{bmatrix} \int D^j_{\mu m}(\omega) d\omega. \tag{5.89}$$

We can now evaluate the integral occurring in Eq. (5.89) by expanding $D^j_{\mu m}(\omega)$ in terms of $d^j_{\mu m}(\beta)$.

$$D^j_{\mu m}(\omega) = e^{-i\alpha\mu}\, d^j_{\mu m}(\beta)\, e^{-i\gamma m}. \tag{5.90}$$

$$\begin{aligned} \int D^j_{\mu m}(\omega) d\omega &= \int_0^{2\pi} e^{-i\alpha\mu}\, d\alpha \int_0^{\pi} d^j_{\mu m}(\beta)\, \sin\beta\, d\beta \int_0^{2\pi} e^{-i\gamma m}\, d\gamma \\ &= (2\pi)^2\, \delta_{\mu 0}\, \delta_{m0} \int_0^{\pi} d^j_{\mu m}(\beta)\, \sin\beta\, d\beta\,. \end{aligned} \tag{5.91}$$

Since the projection quantum numbers μ and *m* are zero, *j* can assume only integer values and hence $d^j_{00}(\beta)$ can be expressed in terms of $Y_l^m(\beta, 0)$ using Eq. (5.77).

$$d^j_{00}(\beta) = \frac{\sqrt{4\pi}}{[l]} Y_l^0(\beta, 0)\, \delta_{jl}. \tag{5.92}$$

Now the integration over the angle β can easily be performed.

$$\int_0^\pi Y_l^0(\beta, 0) \sin\beta\, d\beta = \frac{\sqrt{4\pi}}{2\pi}\, \delta_{l0}. \tag{5.93}$$

Thus, we obtain

$$\int D^j_{\mu m}(\omega) d\omega = 8\pi^2\, \delta_{j0}\, \delta_{\mu 0}\, \delta_{m0}. \tag{5.94}$$

Substituting this value of the integral in Eq. (5.89), we obtain

$$\begin{aligned} I &= (-1)^{\mu_1 - m_1} \sum_j \begin{bmatrix} j_1 & j_2 & j \\ -\mu_1 & \mu_2 & \mu \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ -m_1 & m_2 & m \end{bmatrix} \\ &\quad \times 8\pi^2\, \delta_{j0}\, \delta_{\mu 0}\, \delta_{m0}. \end{aligned} \tag{5.95}$$

The summation over j is equivalent to replacing j by 0. Since $\mu = m = 0$, it follows that $\mu_1 = \mu_2$ and $m_1 = m_2$.

$$\begin{aligned} I &= (-1)^{\mu_1 - m_1} \begin{bmatrix} j_1 & j_2 & 0 \\ -\mu_1 & \mu_2 & 0 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & 0 \\ -m_1 & m_2 & 0 \end{bmatrix} \\ &\quad \times 8\pi^2\, \delta_{j_1 j_2}\, \delta_{\mu_1 \mu_2}\, \delta_{m_1 m_2}. \end{aligned} \tag{5.96}$$

Using the symmetry properties of C.G. coefficient, we finally obtain

$$I = \frac{8\pi^2}{2j_2 + 1}\, \delta_{j_1 j_2}\, \delta_{\mu_1 \mu_2}\, \delta_{m_1 m_2}. \tag{5.97}$$

Review Questions

5.1 Construct the rotation operator in terms of Euler angles of rotation and deduce the rotation matrix for $j = \frac{1}{2}$.

5.2 What is a spinor? A spinor is sometimes referred to as ‘half vector’ or ‘vector with thickness’. Explain why?

5.3 Define the rotation matrix and deduce the Clebsch-Gordan series and its inverse. Indicate the significance of the inverse Clebsch-Gordan series.

5.4 Define the rotation matrix and deduce its unitary and symmetry properties.

5.5 Given the rotation matrix for $j = \frac{1}{2}$, explain how the rotation matrices of higher order can be constructed using the inverse Clebsch-Gordan series.

5.6 State and prove the spherical harmonic addition theorem and therefrom deduce .the coupling rule for spherical harmonics. Apply the results so obtained to evaluate the matrix element of a spherical harmonic, $\langle l_f m_f | Y_l^m | l_i m_i \rangle$.

5.7 Evaluate the integral

$$\int D^{j_1 *}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega)\, d\omega,$$

where ω denotes the Euler angles of rotation α,β,γ. Show that the rotation matrices are orthogonal.

Problems

5.1 For $j = 1$, show that $J_Y^3 = J_Y$. Using this relation and the definition of the rotation operator, obtain the rotation matrix $d^1(\beta)$ for rotation through an angle β about the Y axis.

5.2 Using the inverse Clebsch-Gordan series, construct the rotation matrix $D^1(\alpha\beta\gamma)$ given that

$$d^{1/2}(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}.$$

The following C.G. coefficients are given:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \pm\frac{1}{2} & \pm\frac{1}{2} & \pm 1 \end{bmatrix} = 1, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \pm\frac{1}{2} & \mp\frac{1}{2} & 0 \end{bmatrix} = \sqrt{\frac{1}{2}}.$$

5.3 Using the inverse C.G. series, construct the rotation matrix for $j = \frac{3}{2}$, given the rotation matrices for $j = 1$ and $j = \frac{1}{2}$.

5.4 Given the rotation matrix for $j = 1$, construct the rotation matrix for $j = 2$, using the inverse Clebsch Gordan series.

Solutions to Selected Problems

5.1 For $j = 1$, write down explicitly the matrices for J_Y, J_Y^2 and J_Y^3.

$$J_Y = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_Y^2 = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad J_Y^3 = J_Y.$$

$$\begin{aligned}
e^{-i\beta J_Y} &= 1+\sum_{n=1}^{\infty}\frac{(-i\beta J_Y)^n}{n!}\\
&= 1+\sum_{n=0}^{\infty}\frac{(-i\beta J_Y)^{2n+1}}{(2n+1)!}+\sum_{n=0}^{\infty}\frac{(-i\beta J_Y)^{2n+2}}{(2n+2)!}\\
&= 1-iJ_Y\sin\beta+J_Y^2(\cos\beta-1)\\
&= I+\frac{\sin\beta}{\sqrt{2}}\begin{bmatrix}0&-1&0\\1&0&-1\\0&1&0\end{bmatrix}-\frac{\cos\beta-1}{2}\begin{bmatrix}-1&0&1\\0&-2&0\\1&0&-1\end{bmatrix},
\end{aligned}$$

where I denotes the unit matrix. A simple addition of the matrices yields the rotation matrix $d^l(\beta)$.

$$d^1(\beta)=\begin{bmatrix}\frac{1}{2}(1+\cos\beta) & -\sqrt{\frac{1}{2}}\sin\beta & \frac{1}{2}(1-\cos\beta)\\ \sqrt{\frac{1}{2}}\sin\beta & \cos\beta & -\sqrt{\frac{1}{2}}\sin\beta\\ \frac{1}{2}(1-\cos\beta) & \sqrt{\frac{1}{2}}\sin\beta & \frac{1}{2}(1+\cos\beta)\end{bmatrix}. \qquad (5.98)$$

5.2 Using the inverse C.G. series, the elements of the rotation matrix $d^l(\beta)$ can be obtained.

$$\begin{aligned}
d^1_{m\mu}(\beta) &= \sum_{m_1,\mu_1}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&1\\m_1&m_2&m\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&1\\\mu_1&\mu_2&\mu\end{bmatrix}\\
&\quad\times d^{\frac{1}{2}}_{m_1\mu_1}(\beta)d^{\frac{1}{2}}_{m_2\mu_2}(\beta)
\end{aligned}$$

For the elements d^1_{11}, d^1_{1-1}, d^1_{-11} and d^1_{-1-1}, there is only one non-vanishing term in the expansion. Substituting the values of the C.G. coefficients and the elements of the rotation matrix $d^{\frac{1}{2}}(\beta)$, we obtain

$$\begin{aligned}
d^1_{11}(\beta) &= \cos^2\frac{\beta}{2} = \frac{1+\cos\beta}{2};\\
d^1_{1-1}(\beta) &= \sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2};\\
d^1_{-11}(\beta) &= \sin^2\frac{\beta}{2} = \frac{1-\cos\beta}{2};\\
d^1_{-1-1}(\beta) &= \cos^2\frac{\beta}{2} = \frac{1+\cos\beta}{2}.
\end{aligned}$$

For the elements $d^1_{10}, d^1_{01}, d^1_{0-1}$ and d^1_{-10}, there are two terms in the expansion and substituting the values of C.G. coefficients and the elements of the rotation matrix $d^{\frac{1}{2}}(\beta)$, we obtain

$$\begin{aligned}
d^1_{10}(\beta) &= -\frac{2}{\sqrt{2}}\cos\frac{\beta}{2}\sin\frac{\beta}{2} = -\frac{\sin\beta}{\sqrt{2}};\\
d^1_{01}(\beta) &= \frac{2}{\sqrt{2}}\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{\sin\beta}{\sqrt{2}};\\
d^1_{0-1}(\beta) &= -\frac{2}{\sqrt{2}}\cos\frac{\beta}{2}\sin\frac{\beta}{2} = -\frac{\sin\beta}{\sqrt{2}};\\
d^1_{-10}(\beta) &= \frac{2}{\sqrt{2}}\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{\sin\beta}{\sqrt{2}}.
\end{aligned}$$

The element of the rotation matrix $d^1_{00}(\beta)$ has four terms in the expansion. Substituting the values of the C.G. coefficients and the elements of the $d^{\frac{1}{2}}$ rotation matrix, we find

$$d^1_{00}(\beta) = \cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2} = \cos\beta.$$

The calculated elements are exactly the elements of the rotation matrix given in Eq. (5.98).

TENSOR OPERATORS AND REDUCED MATRIX ELEMENTS

6.1. Irreducible Tensor Operators

We have seen that the angular momentum functions, Ψ_{jm} transform as irreducible tensors of rank j. In a similar way, the irreducible tensor operators[1] are defined by their transformation properties under rotation. If U_R is the Unitary transformation operator corresponding to a rotation R of the coordinate system, then the angular momentum functions $\Psi_{jm}(r)$ and the irreducible tensor operators $T_k^{\mu}(\hat{\boldsymbol{r}})$ transform as follows.

$$U_R\,\Psi_{jm}(\boldsymbol{r}) = \sum_{m'} D^{j}_{m'm}(\omega)\,\Psi_{jm'}(\boldsymbol{r}). \tag{6.1}$$

$$U_R\,T_k^{\mu}(\hat{\boldsymbol{r}})\,U_R^{-1} = \sum_{\mu'} D^{k}_{\mu'\mu}(\omega)\,T_k^{\mu'}(\hat{\boldsymbol{r}}). \tag{6.2}$$

The operators $T_k^{\mu}(\hat{\boldsymbol{r}})$ obeying Eq. (6.2) is said to be an irreducible tensor operator of rank k and it has $2k + 1$ components ($\mu = -k, \ldots, 0, \ldots k$). The above definitions are such that the equations involving tensor operators and also the matrix elements of the tensor operators retain the same form under rotation of coordinate system.

The spherical harmonics $Y_l^{m}(\hat{\boldsymbol{r}})$ play a dual role, sometimes as angular momentum eigenfunctions of a particle moving under the influence of a spherically symmetric potential and in many cases they also occur as irreducible tensor operators inducing transitions. Depending upon their role, the spherical harmonics transform according to Eq. (6.1) or Eq. (6.2).

6.2. Racah's Definition

Racah (1942b) defines the irreducible tensor operators in terms of their

[1]For supplementary study of irreducible tensor operators and angular momentum coefficients, the reader is referred to Biedenharn and Van Dam, 1965 and Biedenharn and Louck, 1981.

commutation relations with the angular momentum operators J_z and $J_\pm$.

$$[J_Z, T_k^\mu] = \mu T_k^\mu. \tag{6.3}$$

$$[J_\pm, T_k^\mu] = \{(k \mp \mu)(k \pm \mu + 1)\}^{\frac{1}{2}} T_k^{\mu\pm 1}. \tag{6.4}$$

The equivalence of the two definitions (6.2) and (6.3, 6.4) can be shown by considering an infinitesimal rotation of the coordinate system. For an infinitesimal rotation $\delta\phi$ about the Z-axis, the rotation operator U_R is given by

$$U_R = e^{-i\,\delta\phi\,J_Z}, \tag{6.5}$$

and consequently Eq. (6.2) becomes

$$e^{-i\,\delta\phi\,J_Z}\, T_k^\mu(\hat{r})\, e^{i\,\delta\phi\,J_Z} = \sum_{\mu'} \langle k\mu' \,|\, e^{-i\,\delta\phi\,J_Z} \,|\, k\mu\rangle\, T_k^{\mu'}(\hat{r}), \tag{6.6}$$

where the element of the rotation matrix $D^k_{\mu'\mu}$ is expressed as the matrix element of the rotation operator U_R. Expanding the exponentials and neglecting the second and higher order terms of $\delta\phi$, we obtain (suppressing the argument $\hat{r}$ for the operator T_k^μ hereafter)

$$(1 - i\,\delta\phi\,J_Z)\, T_k^\mu\, (1 + i\,\delta\phi\,J_Z) = \sum_{\mu'} \langle k\mu' |(1 - i\,\delta\phi\,J_Z)| k\mu\rangle\, T_k^{\mu'}. \tag{6.7}$$

Simplifying, we get

$$[J_Z\,, T_k^\mu] = \sum_{\mu'} \langle k\mu' \,|J_Z|\, k\mu\rangle T_k^{\mu'}. \tag{6.8}$$

Since $|k\mu\rangle$ is an eigenstate of the operator J_z with eigenvalue μ, we get at once the relation (6.3) from Eq. (6.8).

Equation (6.8) was obtained by considering an infinitesimal rotation about the Z-axis. We will get similar relations if we consider infinitesimal rotations about the X and Y axes.

$$[J_X\,, T_k^\mu] = \sum_{\mu'} \langle k\mu' \,|J_X|\, k\mu\rangle\, T_k^{\mu'}. \tag{6.9}$$

$$[J_Y\,, T_k^\mu] = \sum_{\mu'} \langle k\mu' \,|J_Y|\, k\mu\rangle\, T_k^{\mu'}. \tag{6.10}$$

Combining Eqs. (6.9) and (6.10), we obtain

$$[J_X \pm iJ_Y\,, T_k^\mu] = \sum_{\mu'} \langle k\mu' \,|J_X \pm iJ_Y|\, k\mu\rangle\, T_k^{\mu'}. \tag{6.11}$$

which in turn, yields Eq. (6.4). Thus we have shown that the two definitions of the irreducible tensor operators are equivalent.

In deriving Eq. (6.9), it was assumed that the element of the rotation matrix corresponding to a rotation $\delta\theta$ about the X-axis is given by

$$D^k_{\mu'\mu} = \langle k\mu' \,| e^{-i\,\delta\theta\, J_X} |\, k\mu\rangle. \tag{6.12}$$

This is of course true, but the usual practice is to express the rotation in terms of the Euler angles α,β,γ. The Euler angles corresponding to an infinitesimal rotation about the X-axis are given by

$$(\alpha,\beta,\gamma) = (-\frac{\pi}{2}, \delta\theta, \frac{\pi}{2}). \tag{6.13}$$

This will yield the rotation matrix

$$D^k_{\mu'\mu}(-\frac{\pi}{2}, \delta\theta, \frac{\pi}{2}) = e^{i\,(\mu'-\mu)\pi/2} \langle k\mu' | e^{-i\,\delta\theta\, J_Y} | k\mu\rangle, \tag{6.14}$$

which when substituted in Eq. (6.2) gives the following relation

$$[J_X, T_k^{\mu'}] = \sum_{\mu'} e^{i\,(\mu'-\mu)\pi/2}\, \langle k\mu' \,|\, J_Y \,|\, k\mu\rangle\, T_k^{\mu'}, \tag{6.15}$$

which' is equivalent to Eq. (6.9).

6.3. The Wigner-Eckart Theorem

The Wigner-Eckart theorem states that the matrix element of an irreducible tensor operator between any two well-defined angular momentum states can be factored out into two parts, one part depending on the magnetic quantum numbers and the other part completely independent of them. The first part contains the entire geometry or the symmetry properties of the system and the second part is concerned with the dynamics of the physical process. The theorem states that the entire dependence of the matrix element on the magnetic quantum numbers can be factored out as a C.G. coefficient and the other factor which is independent of the projection quantum numbers is known as the reduced matrix element or the double-bar matrix element.

$$\langle j_f\, m_f \,|\, T_k^{\mu} \,|\, j_i\, m_i\rangle = \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} \langle j_f \,||\, T_k \,||\, j_i\rangle. \tag{6.16}$$

Equation (6.16) is the mathematical statement of the Wigner-Eckart theorem. Unfortunately there is no uniformity in the precise statement of the Wigner-Eckart theorem and consequently the reduced matrix element differs from one to another. The reduced matrix element as defined in Eq. (6.16) is identical with that of Rose but differs from that of Edmonds (1957) by a factor.

$$\langle j_f || T_k || j_i \rangle_{\text{Edmonds}} = \sqrt{2j_f + 1}\, \langle j_f || T_k || j_i \rangle.$$

It can be easily seen that the first factor viz., the C.G. coefficient depends on the coordinate system that is used to evaluate the matrix element and it also implies the law of conservation of angular momentum. If this factorization is possible in one coordinate system, then it is easy to show that it is possible in every other coordinate system obtained by rotation. The matrix element in the rotated coordinate system (writing the coordinates explicitly) is given by

$$\begin{aligned}\langle \Psi_{j_f m_f}(\boldsymbol{r}') \,|\, T_k^{\mu}(\hat{\boldsymbol{r}}') \,|\, \Psi_{j_i m_i}(\boldsymbol{r}') \rangle &= \sum_{m'_f \mu' m'_i} D^{j_f^*}_{m'_f m_f}(\omega)\, D^{k}_{\mu' \mu}(\omega)\, D^{j_i}_{m'_i m_i}(\omega) \\ &\times \langle \Psi_{j_f m'_f}(\boldsymbol{r}) \,|\, T_k^{\mu'}(\hat{\boldsymbol{r}}) \,|\, \Psi_{j_i m'_i}(\boldsymbol{r}) \rangle. \quad (6.17)\end{aligned}$$

The coordinate r pertains to the original coordinate system and the coordinate r', to the rotated coordinate system. If we assume such a factorization as given in Eq. (6.16) in the original coordinate system, then we have

$$\begin{aligned}\langle \Psi_{j_f m_f}(\boldsymbol{r}') \,|\, T_k^{\mu}(\hat{\boldsymbol{r}}') \,|\, \Psi_{j_i m_i}(\boldsymbol{r}') \rangle &= \sum_{m'_f \mu' m'_i} D^{j_f^*}_{m'_f m_f}(\omega)\, D^{k}_{\mu' \mu}(\omega)\, D^{j_i}_{m'_i m_i}(\omega) \\ &\times \begin{bmatrix} j_i & k & j_f \\ m'_i & \mu' & m'_f \end{bmatrix} \langle j_f || T_k || j_i \rangle. \quad (6.18)\end{aligned}$$

Coupling the two rotation matrices by applying the C.G. series

$$D^{j_i}_{m'_i m_i}(\omega)\, D^{k}_{\mu' \mu}(\omega) = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \begin{bmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{bmatrix} D^{J}_{M' M}(\omega), \quad (6.19)$$

and substituting it into Eq. (6.18), we obtain after summing over m'_i and μ' (Note that $m'_f = m'_i + \mu' = M'$),

$$\begin{aligned}\langle \Psi_{j_f m_f}(\boldsymbol{r}') \,|\, T_k^{\mu}(\hat{\boldsymbol{r}}') \,|\, \Psi_{j_i m_i}(\boldsymbol{r}') \rangle &= \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \sum_{m'_f} D^{j_f^*}_{m'_f m_f}(\omega) \\ &\times D^{J}_{m'_f M}(\omega) \langle j_f || T_k || j_i \rangle\, \delta_{j_f J}. \quad (6.20)\end{aligned}$$

Summing over J is equivalent to replacing J by jf and the summation over m'_f yields $\delta_{m_f M}$ as a result of unitarity of D-matrices. Finally we obtain

$$\langle \Psi_{j_f m_f}(\boldsymbol{r}') \,|\, T_k^{\mu}(\hat{\boldsymbol{r}}') \,|\, \Psi_{j_i m_i}(\boldsymbol{r}') \rangle = \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} \langle j_f \,||\, T_k \,||\, j_i \rangle. \tag{6.21}$$

Thus we have shown that if the matrix element can be written as a product of C.G. coefficients and the reduced matrix element in one coordinate system, then it can be factorized in the same way in every other coordinate system.

The foregoing discussion cannot be considered strictly as the proof of the Wigner-Eckart theorem, although it serves as a consistency check. There are three different proofs of the Wigner-Eckart theorem, one due to Wigner (Brink and Satchler, 1962), another due to Schwinger (Edmonds, 1957) and the third due to Racah (Rose, 1957a). The first of the proofs make use of the definition Eq. (6.2) and the third rests on the commutation relation (6.3) and (6.4).

6.4. Proofs of the Wigner-Eckart Theorem

6.4.1. METHOD I

We shall first write down explicitly the matrix element Q of an irreducible tensor operator of rank k.

$$\begin{aligned} Q &= \langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \,|\, T_k^{\mu}(\hat{\boldsymbol{r}}) \,|\, \Psi_{j_i m_i}(\hat{\boldsymbol{r}}) \rangle, \\ &= \int \Psi^*_{j_f m_f}(\hat{\boldsymbol{r}})\, T_k^{\mu}(\hat{\boldsymbol{r}})\, \Psi_{j_i m_i}(\hat{\boldsymbol{r}})\, d\Omega. \end{aligned} \tag{6.22}$$

The angular integration in Eq. (6.22) can be carried out either by rotating the functions in a fixed coordinate system or by rotating the coordinate system, keeping the functions fixed. We shall opt for the latter method. Let us consider a rotation of the coordinate system through the Euler angles such that the angular coordinate $\hat{\boldsymbol{r}}$ goes from (0,0) to (θ, ϕ) .

$$\begin{aligned} Q &= \int \sum_{m'_f \mu' m'_i} D^{j_f *}_{m'_f m_f}(\Omega)\, D^{k}_{\mu' \mu}(\Omega)\, D^{j_i}_{m'_i m_i}(\Omega) \\ &\quad \times \Psi^*_{j_f m_f}(0,0)\, T_k^{\mu}(0,0)\, \Psi_{j_i m_i}(0,0)\, d\Omega. \end{aligned} \tag{6.23}$$

We shall first couple the two D-matrices using the C.G. series.

$$D^{j_i}_{m'_i m_i}(\Omega)\, D^{k}_{\mu' \mu}(\Omega) = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \begin{bmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{bmatrix} D^{J}_{M' M}(\Omega). \tag{6.24}$$

Substituting this into Eq. (6.23), we obtain

$$Q = \sum_{m_i' \mu' m_f'} \sum_J \left[\begin{array}{ccc} j_i & k & J \\ m_i & \mu & M \end{array} \right] \left[\begin{array}{ccc} j_i & k & J \\ m_i' & \mu' & M' \end{array} \right]$$
$$\times \Psi^*_{j_f m_f'}(0,0)\, T_k^{\mu'}(0,0)\, \Psi_{j_i m_i'}(0,0)$$
$$\times \int D^{j_f^*}_{m_f' m_f}(\Omega)\, D^J_{M'M}(\Omega)\, d\Omega, \qquad (6.25)$$

where

$$d\Omega = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi. \qquad (6.26)$$

The integration over $d\Omega$ can be carried out easily.

$$\int D^{j_f^*}_{m_f' m_f}(\Omega)\, D^J_{M'M}(\Omega)\, d\Omega = \frac{4\pi}{2j_f + 1} \delta_{j_f J}\, \delta_{m_f' M'}\, \delta_{m_f M}. \qquad (6.27)$$

Inserting (6.27) into (6.25) and summing over J and m_f', we obtain

$$Q = \left[\begin{array}{ccc} j_i & k & j_f \\ m_i & \mu & M \end{array} \right] \delta_{m_f M}$$
$$\times \left\{ \frac{4\pi}{2j_f + 1} \sum_{m_i' \mu'} \left[\begin{array}{ccc} j_i & k & j_f \\ m_i' & \mu' & m_f' \end{array} \right] \Psi^*_{j_f m_f'}(0,0)\, T_k^{\mu'}(0,0)\, \Psi_{j_i m_i'}(0,0) \right\}. \qquad (6.28)$$

The quantity within the curly bracket in Eq. (6.28) is independent of the projection quantum numbers because of the summation over m_i' and μ'. Thus the matrix element Q depends on the projection quantum numbers only through the C.G. coefficients. The reduced matrix element is the quantity within the curly bracket and, as we have shown, it is independent of the projection quantum numbers.

$$\langle j_f \,||\, T_k \,||\, j_i \rangle = \frac{4\pi}{2j_f + 1} \sum_{m_i' \mu'} \left[\begin{array}{ccc} j_i & k & j_f \\ m_i' & \mu' & m_f' \end{array} \right]$$
$$\times \Psi^*_{j_f m_f'}(0,0)\, T_k^{\mu'}(0,0)\, \Psi_{j_i m_i'}(0,0). \qquad (6.29)$$

It will be instructive to calculate the reduced matrix element in the special case of the spherical harmonics. From Eq. (6.29), we have

$$\langle l_f \,||\, Y_l \,||\, l_i \rangle = \frac{4\pi}{2l_f + 1} \sum_{m_i' m'} \left[\begin{array}{ccc} l_i & l & l_f \\ m_i' & m' & m_f' \end{array} \right]$$
$$\times Y_{l_f}^{m_f'^*}(0,0)\, Y_l^{m'}(0,0)\, Y_{l_i}^{m_i'}(0,0). \qquad (6.30)$$

Since

$$Y_l^m(0,0) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}, \tag{6.31}$$

Eq. (6.30) simplifies to

$$\langle l_f \,||\, Y_l \,||\, l_i\rangle = \left\{\frac{(2l_i+1)(2l+1)}{4\pi(2l_f+1)}\right\}^{\frac{1}{2}} \left[\begin{array}{ccc} l_i & l & l_f \\ 0 & 0 & 0 \end{array}\right]. \tag{6.32}$$

Thus, according to the Wigner-Eckart theorem, the matrix element of Y_l^m is

$$\langle l_f m_f \,|\, Y_l^m \,|\, l_i m_i\rangle = \left[\begin{array}{ccc} l_i & l & l_f \\ m_i & m & m_f \end{array}\right] \langle l_f \,||\, Y_l \,||\, l_i\rangle, \tag{6.33}$$

where the reduced matrix element is given by Eq. (6.32). This result is identical with the result obtained earlier using the coupling rule of the spherical harmonics.

6.4.2. METHOD II

This proof is originally due to Schwinger and it is also given by Edmonds (1957). First let us consider the effects of operation of an irreducible tensor operator $T_k^\mu(\hat{\boldsymbol{r}})$ on the angular momentum eigenfunction $\Psi_{j_i m_i}(\hat{\boldsymbol{r}})$ and study the transformation property of the resulting function under rotation in order to obtain its structure. Let

$$\Phi(\hat{\boldsymbol{r}}) = T_k^\mu(\hat{\boldsymbol{r}})\,\Psi_{j_i m_i}(\hat{\boldsymbol{r}}). \tag{6.34}$$

Under rotation of the coordinate system, $\Phi(\hat{\boldsymbol{r}})$ changes to $\Phi(\hat{\boldsymbol{r}'})$.

$$\begin{aligned} \Phi(\hat{\boldsymbol{r}'}) &= U_R\, T_k^\mu(\hat{\boldsymbol{r}})\, U_R^{-1}\, U_R\, \Psi_{j_i m_i}(\hat{\boldsymbol{r}}), \\ &= \sum_{\mu' m_i'} D^k_{\mu'\mu}(\omega)\, T_k^{\mu'}(\hat{\boldsymbol{r}})\, D^{j_i}_{m_i' m_i}(\omega)\, \Psi_{j_i m_i'}(\hat{\boldsymbol{r}}). \end{aligned} \tag{6.35}$$

The result (6.35) is obtained using Eqs. (6.1) and (6.2). Using the C.G. series for coupling the two rotation matrices, we obtain

$$\begin{aligned} \Phi(\hat{\mathbf{r}}') &= \sum_{\mu' m_i'} \sum_J \left[\begin{array}{ccc} j_i & k & J \\ m_i & \mu & M \end{array}\right] \left[\begin{array}{ccc} j_i & k & J \\ m_i' & \mu' & M' \end{array}\right] \\ &\quad \times D^J_{M'M}(\omega)\, T_k^{\mu'}(\hat{\boldsymbol{r}})\, \Psi_{j_i m_i'}(\hat{\boldsymbol{r}}). \end{aligned}$$

Rearranging and replacing the summation over μ' by M', we get

$$\Phi(\hat{\mathbf{r}}') = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \sum_{M'} D^J_{M'M}(\omega) \times \left\{ \sum_{m_i'} \begin{bmatrix} j_i & k & J \\ m_i' & \mu' & M' \end{bmatrix} T_k^{\mu'}(\hat{\boldsymbol{r}})\, \Psi_{j_i m_i'}(\hat{\boldsymbol{r}}) \right\}. \tag{6.36}$$

We see that the quantity within the curly bracket transforms under rotation as a tensor of rank J and the function Φ can be expressed as a linear sum of such tensors of rank J, J taking the spectrum of values from $|j_i - k|$ to $j_i + k$. Let

$$\psi_{\eta J M'}(\hat{\boldsymbol{r}}) = \sum_{m_i'} \begin{bmatrix} j_i & k & J \\ m_i' & \mu' & M' \end{bmatrix} T_k^{\mu'}(\hat{\boldsymbol{r}})\, \Psi_{j_i m_i'}(\hat{\boldsymbol{r}}), \tag{6.37}$$

where η denotes the additional quantum numbers j_i and k. Now Eq. (6.36) becomes

$$\Phi(\hat{\mathbf{r}}') = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \sum_{M'} D^J_{M'M}(\omega)\, \psi_{\eta J M'}(\hat{\boldsymbol{r}}), \tag{6.38}$$

and it gives the transformation property of the function under rotation. From this study, we obtain the structure of the function $\Phi(\hat{\boldsymbol{r}})$.

$$\Phi(\hat{\boldsymbol{r}}) = T_k^{\mu}(\hat{\boldsymbol{r}})\, \Psi_{j_i m_i}(\hat{\boldsymbol{r}}) = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \psi_{\eta J M'}(\hat{\boldsymbol{r}}). \tag{6.39}$$

This result can be used to evaluate the matrix element of a tensor operator.

$$\langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \mid T_k^{\mu}(\hat{\boldsymbol{r}}) \mid \Psi_{j_i m_i}(\hat{\boldsymbol{r}}) \rangle = \sum_J \begin{bmatrix} j_i & k & J \\ m_i & \mu & M \end{bmatrix} \times \langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \mid \psi_{\eta J M'}(\hat{\boldsymbol{r}}) \rangle. \tag{6.40}$$

To find the scalar product, $\langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \mid \psi_{\eta J M'}(\hat{\boldsymbol{r}}) \rangle$, let us expand $\Psi_{jm}(\hat{\boldsymbol{r}})$ in terms of the complete set of functions $\psi_{\eta j m}(\hat{\boldsymbol{r}})$.

$$\Psi_{jm}(\hat{\boldsymbol{r}}) = \sum_\eta a_{\eta j m}\, \psi_{\eta j m}(\hat{\boldsymbol{r}}). \tag{6.41}$$

This is because the two functions $\Psi_{jm}(\hat{\boldsymbol{r}})$ and $\psi_{\eta j m}(\hat{\boldsymbol{r}})$ may be in different representation and so they must be connected by a unitary transformation.

The quantities $a_{\eta jm}$ are the coefficients of such unitary transformation and the summation η is over the additional quantum numbers such as j_i and k which define the function $\psi_{\eta jm}(\hat{\boldsymbol{r}})$. First we shall show that the coefficient $a_{\eta jm}$ is independent of the magnetic quantum number m. For this, consider the expansion of the two functions $\Psi_{jm}(\hat{\boldsymbol{r}})$ and $\Psi_{j\,m+1}(\hat{\boldsymbol{r}})$.

$$\Psi_{jm}(\hat{\boldsymbol{r}}) = \sum_{\eta} a_{\eta jm}\, \psi_{\eta jm}(\hat{\boldsymbol{r}}). \tag{6.42}$$

$$\Psi_{j\,m+1}(\hat{\boldsymbol{r}}) = \sum_{\eta} a_{\eta\, j\, m+1}\, \psi_{\eta\, j\, m+1}(\hat{\boldsymbol{r}}). \tag{6.43}$$

Allowing the operator $J_+/\{(j - m)(j + m + 1)\}^{\frac{1}{2}}$ to operate on both sides of Eq. (6.41), we obtain

$$\Psi_{j\,m+1}(\hat{\boldsymbol{r}}) = \sum_{\eta} a_{\eta jm}\, \psi_{\eta\, j\, m+1}(\hat{\boldsymbol{r}}). \tag{6.44}$$

Comparing (6.43) and (6.44), we see that

$$a_{\eta\, j\, m+1} = a_{\eta jm}. \tag{6.45}$$

Therefore the expansion coefficient $a_{\eta jm}$ is independent of m and hence can be simply written as $a_{\eta j}$. Now the scalar product becomes

$$\langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \mid \psi_{\eta JM}(\hat{\boldsymbol{r}}) \rangle = N\, a_{\eta j_f}\, \delta_{j_f J}\, \delta_{m_f M}, \tag{6.46}$$

where N is the normalization constant of the function $\psi_{\eta JM}$.

$$N = \langle \psi_{\eta JM} \mid \psi_{\eta JM} \rangle. \tag{6.47}$$

Using the relation (6.46) in Eq. (6.40), we finally obtain

$$\langle \Psi_{j_f m_f}(\hat{\boldsymbol{r}}) \mid T_k^{\mu}(\hat{\boldsymbol{r}}) \mid \Psi_{j_i m_i}(\hat{\boldsymbol{r}}) \rangle = \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} N\, a_{\eta j_f}, \tag{6.48}$$

where the quantity $N\, a_{\eta j_f}$ is independent of the projection quantum numbers and is known as the reduced matrix element.

6.4.3. METHOD III

This method is originally due to Racah and rests on the commutation relations (6.3) and (6.4). For details, reference may be made to Rose (1957a).

First let us find the matrix elements of the commutators (6.3) and (6.4) between the two angular momentum states $|j_i\, m_i\rangle$ and $|j_f\, m_f\rangle$. From Eq. (6.3), we have

$$\langle j_f\, m_f \,|\, J_Z T_k^\mu - T_k^\mu J_Z \,|\, j_i\, m_i\rangle = \mu \langle j_f\, m_f \,|\, T_k^\mu \,|\, j_i\, m_i\rangle. \tag{6.49}$$

The operator J_z may be allowed to operate on the left or the right state, as the case may be, yielding their eigenvalues. The resulting equation is

$$(m_f - m_i - \mu)\langle j_f\, m_f \,|\, T_k^\mu \,|\, j_i\, m_i\rangle = 0. \tag{6.50}$$

Equation (6.49) simply states that the matrix element of the tensor operator will be non-vanishing only if

$$m_f = m_i + \mu. \tag{6.51}$$

In a similar way, the commutation relation (6.4) will yield another equation for the matrix element.

$$\begin{aligned}\langle j_f\, m_f \,|\, J_\pm T_k^\mu \,|\, j_i\, m_i\rangle &- \langle j_f\, m_f \,|\, T_k^\mu J_\pm \,|\, j_i\, m_i\rangle \\ &= \Gamma_\pm(k,\mu)\langle j_f\, m_f \,|\, T_k^{\mu\pm1} \,|\, j_i\, m_i\rangle,\end{aligned} \tag{6.52}$$

with the notation

$$\Gamma_\pm(k,\mu) = \{(k \mp \mu)(k \pm \mu + 1)\}^{\frac{1}{2}}. \tag{6.53}$$

Remembering that the Hermitian conjugate of J_+ operator is J_- and vice versa and allowing the operator $J_\pm$ to act on the left or the right state, we get

$$\begin{aligned}\Gamma_\mp(j_f, m_f)\,\langle j_f\; m_f \mp 1 \,|\, T_k^\mu \,|\, j_i\; m_i\rangle &- \Gamma_\pm(j_i, m_i)\langle j_f\; m_f \,|\, T_k^\mu \,|\, j_i\; m_i \pm 1\rangle \\ &= \Gamma_\pm(k,\mu)\langle j_f\; m_f \,|\, T_k^{\mu\pm1} \,|\, j_i\; m_i\rangle,\end{aligned} \tag{6.54}$$

where

$$\Gamma_\pm(j,m) = \{(j \mp m)(j \pm m + 1)\}^{\frac{1}{2}}, \tag{6.55}$$

$$\Gamma_\mp(j,m) = \{(j \pm m)(j \mp m + 1)\}^{\frac{1}{2}}. \tag{6.56}$$

It can be shown that the C.G. coefficient obeys the same Eqs. (6.49) and (6.54) obtained by replacing the matrix elements by the corresponding C.G. coefficients and hence we infer that the dependence of the matrix elements on the projection quantum numbers is the same as that of the C.G. coefficients.

To obtain equations similar to Eqs. (6.49) and (6.54) for C.G. coefficients, consider the coupling of two angular momenta $|j_i m_i\rangle$ and $|k\mu\rangle$ to yield the resultant angular momentum $|j_f m_f\rangle$.

$$|j_f m_f\rangle = \sum_{m_i \mu} \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} |j_i\, m_i\rangle\, |k\, \mu\rangle. \tag{6.57}$$

The summation indices m_i and μ are dummy indices and hence it does not matter if these indices are replaced by m_i' and μ' or m_i'' and μ" depending on the convenience. Remembering that

$$J_Z = J_{1Z} + J_{2Z}, \tag{6.58}$$

and allowing them to operate on Eq. (6.57), we get

$$m_f |j_f m_f\rangle = \sum_{m_i' \mu'} (m_i' + \mu') \begin{bmatrix} j_i & k & j_f \\ m_i' & \mu' & m_f \end{bmatrix} |j_i\, m_i'\rangle\, |k\, \mu'\rangle. \tag{6.59}$$

Substituting the expansion (6.57) on the left hand side of Eq. (6.59) and taking the scalar product of both sides with $|k\mu\rangle|j_i m_i\rangle$, we get

$$(m_f - m_i - \mu) \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} = 0.$$

In a similar way, the operators

$$J_{\mp} = J_{1\mp} + J_{2\mp}$$

operating on Eq. (6.57) yield

$$\Gamma_{\mp}(j_f, m_f)\, |j_f\, m_f \mp 1\rangle = \sum_{m_i' \mu'} \begin{bmatrix} j_i & k & j_f \\ m_i' & \mu' & m_f \end{bmatrix} \times \left[\Gamma_{\mp}(j_i, m_i')\, |j_i\, m_i' \mp 1\rangle |k\, \mu'\rangle + \Gamma_{\mp}(k, \mu')\, |j_i\, m_i'\rangle |k\, \mu' \mp 1\rangle \right]. \tag{6.61}$$

Expanding $|j_f\, m_f \mp 1\rangle$ in terms of uncoupled states,

$$|j_f\, m_f \mp 1\rangle = \sum_{m_i'' \mu''} \begin{bmatrix} j_i & k & j_f \\ m_i'' & \mu'' & m_f \mp 1 \end{bmatrix} |j_i\, m_i''\rangle |k\, \mu''\rangle, \tag{6.62}$$

and substituting it in Eq. (6.61), we obtain

$$\Gamma_{\mp}(j_f, m_f) \sum_{m_i'' \mu''} \begin{bmatrix} j_i & k & j_f \\ m_i'' & \mu'' & m_f \mp 1 \end{bmatrix} |j_i\, m_i''\rangle |k\, \mu''\rangle = \sum_{m_i' \mu'} \begin{bmatrix} j_i & k & j_f \\ m_i' & \mu' & m_f \end{bmatrix} \left[\Gamma_{\mp}(j_i, m_i')\, |j_i\, m_i' \mp 1\rangle\, |k\, \mu'\rangle + \Gamma_{\mp}(k, \mu')\, |j_i\, m_i'\rangle\, |k\, \mu' \mp 1\rangle \right]. \tag{6.63}$$

Taking the scalar product with $|j_i m_i\rangle|k\mu\rangle$ on both sides of Eq. (6.63), we get

$$\Gamma_{\mp}(j_f, m_f)\begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \mp 1 \end{bmatrix} = \begin{bmatrix} j_i & k & j_f \\ m_i \pm 1 & \mu & m_f \end{bmatrix}\Gamma_{\pm}(j_i, m_i) + \begin{bmatrix} j_i & k & j_f \\ m_i & \mu \pm 1 & m_f \end{bmatrix}\Gamma_{\pm}(k, \mu), \quad (6.64)$$

since

$$\Gamma_{\mp}(j_i, m_i')\,\delta_{m_i\, m_i' \mp 1} = \Gamma_{\pm}(j_i, m_i), \quad (6.65)$$

$$\Gamma_{\mp}(k, \mu')\,\delta_{\mu\, \mu' \mp 1} = \Gamma_{\pm}(k, \mu). \quad (6.66)$$

Transposing the first term on the right to the left, we get an equation which is similar to Eq. (6.54). This study shows that the matrix elements of tensor operators have the same dependence on projection quantum numbers as the C.G. coefficients. Therefrom it follows that the dependence of a matrix element on projection quantum numbers can be factored out as a C.G. coefficient, and so the remaining factor called the reduced matrix element should be independent of those projection quantum numbers.

6.5. Tensors and Tensor Operators

In this section, we shall discuss some relations involving tensors and tensor operators (Racah, 1942b).

The effect of a tensor operator T_k^μ operating on U_j^m which is a tensor of rank j is to yield a linear sum of tensors $V_\lambda^{m_\lambda}$ of rank λ, λ varying from $|j - k|$ to $j + k$.

$$T_k^\mu U_j^m = \sum_\lambda \begin{bmatrix} j & k & \lambda \\ m & \mu & m_\lambda \end{bmatrix} V_\lambda^{m_\lambda}. \quad (6.67)$$

Just as one can write the tensor product of two tensors $U_{\lambda 1}$ and $U_{\lambda 2}$,

$$(U_{\lambda_1} \times U_{\lambda_2})_\lambda^m = \sum_{m_1} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda \\ m_1 & m_2 & m \end{bmatrix} U_{\lambda_1}^{m_1} U_{\lambda_2}^{m_2}, \quad (6.68)$$

and its inverse relation

$$U_{\lambda_1}^{m_1} U_{\lambda_2}^{m_2} = \sum_\lambda \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda \\ m_1 & m_2 & m \end{bmatrix} (U_{\lambda_1} \times U_{\lambda_2})_\lambda^m, \quad (6.69)$$

we can also write similar relations for tensor operators $T_{k_1}^{\mu_1}$ and $T_{k_2}^{\mu_2}$:

$$(T_{k_1} \times T_{k_2})_k^{\mu} = \sum_{\mu_1} \begin{bmatrix} k_1 & k_2 & k \\ \mu_1 & \mu_2 & \mu \end{bmatrix} T_{k_1}^{\mu_1} T_{k_2}^{\mu_2}, \tag{6.70}$$

and

$$T_{k_1}^{\mu_1} T_{k_2}^{\mu_2} = \sum_{k} \begin{bmatrix} k_1 & k_2 & k \\ \mu_1 & \mu_2 & \mu \end{bmatrix} (T_{k_1} \times T_{k_2})_k^{\mu}. \tag{6.71}$$

The complex conjugate of a tensor U_λ^m is given by (for integer values of λ and U real)

$$(U_\lambda^m)^* = (-1)^m U_\lambda^{-m}, \tag{6.72}$$

and if U is complex, the corresponding relation is

$$(U_\lambda^m)^* = (-1)^m (U^*)_\lambda^{-m}. \tag{6.73}$$

This choice of phase coincides with that for spherical harmonics. However, sometimes in quantum mechanical applications, it is convenient to redefine irreducible tensors as

$$\tilde{U}_j = i^j U_j, \tag{6.74}$$

for which the complex conjugate is given by

$$(\tilde{U}_j^m)^* = (-1)^{j-m} \tilde{U}_j^{-m}. \tag{6.75}$$

The choice of this phase can be used for tensors of integer as well as half-integer rank j.

For a tensor operator of integer rank k, the complex conjugate is given by

$$(T_k^\mu)^* = (-1)^\mu T_k^{-\mu}. \tag{6.76}$$

The scalar product of two tensor operators T_k and S_k of equal rank is given by

$$T_k \cdot S_k = \sum_{\mu} (-1)^\mu T_k^\mu S_k^{-\mu}, \tag{6.77}$$

and it can also be expressed as a zero rank tensor obtained by taking the tensor product; of T_k and S_k.

$$\begin{aligned}(T_k \times S_k)_0^0 &= \sum_\mu \begin{bmatrix} k & k & 0 \\ \mu & -\mu & 0 \end{bmatrix} T_k^\mu S_k^{-\mu} \\ &= \sum_\mu \frac{(-1)^{k-\mu}}{[k]} T_k^\mu S_k^{-\mu} \\ &= \frac{(-1)^k}{[k]} T_k \cdot S_k. \end{aligned} \tag{6.78}$$

In the derivation of the above result, the symmetry property of the C.G. coefficient has been used. The inverse relation is

$$T_k . S_k = (-1)^k [k] (T_k \times S_k)_0^0. \tag{6.79}$$

The concrete examples of spherical tensors are the angular momentum eigenfunctions. The spherical harmonic operator $Y_{l,m}$, the spherical components of the position vector operator r, the momentum operator $p = -i\nabla$ and the Pauli spin operator σ may be cited as examples of spherical tensor operators. For instance, the position vector operator r can be written as

$$\boldsymbol{r} = \sqrt{\frac{4\pi}{3}}\, r \sum_\mu (-1)^\mu Y_1^\mu(\hat{\boldsymbol{r}})\, \epsilon_1^{-\mu}. \tag{6.80}$$

Review Questions

6.1 Define irreducible tensor operators (a) using the transformation properties under rotation and (b) using their commutation relations with angular momentum operators. Establish the equivalence of these two definitions.

6.2 State and prove the Wigner-Eckart theorem. Explain its importance.

6.3 Construct a function $\Phi(\hat{\boldsymbol{r}})$ by operating an irreducible tensor operator $T_k^\mu(\hat{\boldsymbol{r}})$ on the angular momentum eigenfunction $\Psi_{j_i m_i}(\hat{\boldsymbol{r}})$. Study the transformation property of the function $\Phi(\hat{\boldsymbol{r}})$ under rotation of coordinate system and hence deduce the Wigner-Eckart theorem.

6.4 Give Racah's definition of irreducible tensor operators and show that the matrix elements of such tensor operators have the same dependence on projection quantum numbers as that of C.G. Coefficients. Hence deduce the Wigner-Eckart theorem.

6.5 Given any two tensor operators $T_{k_1}^{\mu_1}$ and $T_{k_2}^{\mu_2}$, construct their tensor product. What are the allowed values for the rank of the tensor operator so constructed? If $k_1 = k_2$, construct their tensor product of rank zero and show how it differs from their scalar product.

Problems

6.1 Evaluate the matrix element $\langle j_f\, m_f\, | J_1^\mu |\, j_i\, m_i\rangle$, where J_1^μ is the spherical component of the angular momentum operator J.

6.2 Write down the spherical components of the position vector r regarding them as spherical tensor operators. Evaluate their matrix elements between orbital angular momentum eigenfunctions and deduce the selection rules.

6.3 Evaluate

$$\sum_{m_i\, m_f} \left| \sum_{k\,\mu} \langle j_f\, m_f \,|\, T_k^\mu \,|\, j_i\, m_i\rangle \right|^2 .$$

6.4 Show that the tensor potential

$$S_{12} = \frac{3}{r^2}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{r})\,(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$

of the nucleon-nucleon interaction is a scalar product of two tensor operators, each of rank 2, as given below.

$$S_{12} = \sqrt{\frac{24\pi}{5}}\, Y_2(\hat{\boldsymbol{r}}) \cdot (\sigma_1 \times \sigma_2)_2.$$

Solutions to Selected Problems

6.1 Using the Wigner-Eckart theorem,

$$\begin{aligned}
\langle j_f\, m_f |\, J_1^\mu \,| j_i\, m_i\rangle &= \begin{bmatrix} j_i & 1 & j_f \\ m_i & \mu & m_f \end{bmatrix} \langle j_f \,||\, J \,||\, J_i\rangle\, \delta_{j_i\, j_f}\, \delta_{m_i+\mu\, m_f} \\
&= \begin{bmatrix} j_i & 1 & j_f \\ m_i & \mu & m_f \end{bmatrix} \sqrt{j(j+1)}\, \delta_{j_i\, j_f}\, \delta_{m_i+\mu\, m_f}.
\end{aligned}$$

6.2

$$\begin{aligned}
\boldsymbol{r} &= \sqrt{\frac{4\pi}{3}}\, r \sum_\mu (-1)^\mu\, Y_1^\mu(\hat{\boldsymbol{r}})\, \epsilon_1^{-\mu}. \\
\langle l_f\, m_f \,|\, \boldsymbol{r} \,|\, l_i\, m_i\rangle &= \sqrt{\frac{4\pi}{3}}\, r \sum_\mu (-1)^\mu\, \langle l_f\, m_f \,|\, Y_1^\mu(\hat{\boldsymbol{r}}) \,|\, l_i\, m_i\rangle \epsilon_1^{-\mu}.
\end{aligned}$$

Using the Wigner-Eckart theorem, we obtain

$$\langle l_f\, m_f \,|\, Y_1^\mu(\hat{\boldsymbol{r}}) \,|\, l_i\, m_i\rangle = \begin{bmatrix} l_i & 1 & l_f \\ m_i & \mu & m_f \end{bmatrix} \langle l_f \,||\, Y_1 \,||\, l_i\rangle.$$

The reduced matrix element involves the parity C.G. coefficient

$$\langle l_f \,||\, Y_1 \,||\, l_i \rangle = \begin{bmatrix} l_i & 1 & l_f \\ 0 & 0 & 0 \end{bmatrix} \frac{[l_i][1]}{\sqrt{4\pi}[l_f]}.$$

Parity C.G. coefficient gives the selection rule

$$l_f = l_i \pm 1,$$

since the parity C.G. coefficient vanishes if $l_i = l_f$.

6.3 Using the Wigner-Eckart theorem,

$$\langle j_f\, m_f \,|\, T_k^\mu \,|\, j_i\, m_i \rangle = \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} \langle j_f \,||\, T_k \,||\, j_i \rangle.$$

The square of the matrix element will involve cross terms with indices k and k'. The resulting C.G coefficients can be simplified by performing the summation over the magnetic quantum numbers m_i, m_f.

$$\sum_{m_i, m_f} \begin{bmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{bmatrix} \begin{bmatrix} j_i & k' & j_f \\ m_i & \mu' & m_f \end{bmatrix} = \frac{[j_f]^2}{[k][k']} \delta_{k,k'}\, \delta\mu, \mu'.$$

The summation over μ and μ' are redundant since

$$\mu = \mu' = m_f - m_i.$$

The final result is

$$\sum_{m_i\, m_f} \left| \sum_{k\, \mu} \langle j_f\, m_f \,|\, T_k^\mu \,|\, j_i\, m_i \rangle \right|^2 = \frac{[j_f]^2}{[k]^2} |\langle j_f \,||\, T_k \,||\, j_i \rangle|^2.$$

CHAPTER 7

COUPLING OF THREE ANGULAR MOMENTA

7.1. Definition of the U-Coefficient

When there are three angular momenta, we have six mutually commuting operators

$$J_1^2,\ J_2^2,\ J_3^2,\ J_{1Z},\ J_{2Z},\ J_{3Z}, \tag{7.1}$$

for which one can find simultaneous eigenvalues. We can find a coupled representation by successive addition of two angular momenta. This can be done in more ways that one as shown in Fig. 7.1. For instance

$$\boldsymbol{J}_{12} = \boldsymbol{J}_1 + \boldsymbol{J}_2, \quad \boldsymbol{J} = \boldsymbol{J}_{12} + \boldsymbol{J}_3, \tag{7.2}$$

or alternatively

$$\boldsymbol{J}_{23} = \boldsymbol{J}_2 + \boldsymbol{J}_3, \quad \boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_{23}, \tag{7.3}$$

It is possible to go from the uncoupled representation to any one of the coupled representations by a unitary transformation and in the same way it is possible to go from one coupled representation to another coupled representation by means of a unitary transformation. The commuting operators in the two coupled representations are respectively

$$J_1^2,\ J_2^2,\ J_3^2,\ J_{12}^2,\ J^2,\ J_Z, \tag{7.4}$$

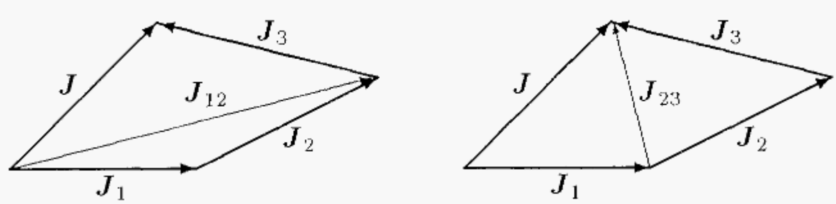

Figure 7.1. Coupling of Three Angular Momenta

and

$$J_1^2,\ J_2^2,\ J_3^2,\ J_{23}^2,\ J^2,\ J_Z, \tag{7.5}$$

for which simultaneous eigenvalues can be determined. Denoting the corresponding eigenstates by $|(j_1, j_2)j_{12}j_3; jm\rangle$ and $|j_1(j_2, j_3)j_{23}; jm\rangle$, we can relate them by a unitary transformation.

$$|(j_1 j_2)j_{12}\, j_3\, ;\, j\, m\rangle = \sum_{j_{23}} U(j_1\, j_2\, j\, j_3; j_{12}\, j_{23})\, |j_1\, (j_2 j_3) j_{23}\, ;\, j\, m\rangle, \tag{7.6}$$

and

$$|j_1(j_2 j_3)j_{23}; jm\rangle = \sum_{j_{12}} U(j_1 j_2 j j_3; j_{12} j_{23})\, |(j_1 j_2)j_{12} j_3; jm\rangle, \tag{7.7}$$

where $U(j_1 j_2 j j_3\, ;\, j_{12} j_{23})$ is a unitary transformation coefficient.

The U-coefficient is the unitary transformation coefficient which enables one to go from one scheme of coupling to another scheme of coupling, and so it is to be anticipated that the U-coefficient should reduce to unity when the two schemes of coupling become identical due to the vanishing of one of the three angular momenta j_1, j_2 and j_3.

$$U(j_1\, j_2\, j\, 0\, ;\, j_{12}\, j_{23}) = \delta_{j_{12}\, j}\, \delta_{j_2\, j_{23}}. \tag{7.8}$$

It is our purpose here to express the U-coefficient as products of C.G. coefficients with a summation over projection quantum numbers. It can be seen that the U-coefficient is independent of the projection quantum numbers. This offers a great advantage. In many problems involving products of a large number of C.G. coefficients, reduction can be made to the U-coefficient which does not involve the projection quantum numbers and hence independent of the choice of the frame of reference.

7.2. The U-Coefficient in terms of C.G. Coefficients

We shall now explicitly write the eigenstates in the two representations in terms of the eigenstates in the original uncoupled representation.

$$\begin{aligned} |(j_1 j_2)j_{12} j_3; jm\rangle &= \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix} \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix} \\ &\quad \times |j_1 m_1\rangle\, |j_2 m_2\rangle\, |j_3 m_3\rangle. \qquad (7.9) \\ |j_1(j_2 j_3)j_{23}; jm\rangle &= \sum_{m_1' m_2'} \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2' & m_3' & m_{23}' \end{bmatrix} \begin{bmatrix} j_1 & j_{23} & j \\ m_1' & m_{23}' & m \end{bmatrix} \\ &\quad \times |j_1 m_1'\rangle\, |j_2 m_2'\rangle\, |j_3 m_3'\rangle. \qquad (7.10) \end{aligned}$$

Substituting Eqs. (7.9) and (7.10) into Eq. (7.6) and taking the scalar product on both sides with $|j_1\mu_1\rangle|j_2\mu_2\rangle|j_3\mu_3\rangle$, we obtain

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ \mu_1 & \mu_2 & \mu_{12} \end{bmatrix}\begin{bmatrix} j_{12} & j_3 & j \\ \mu_{12} & \mu_3 & m \end{bmatrix} = \sum_{j_{23}} U(j_1j_2jj_3;j_{12}j_{23})\begin{bmatrix} j_2 & j_3 & j_{23} \\ \mu_2 & \mu_3 & \mu_{23} \end{bmatrix}\begin{bmatrix} j_1 & j_{23} & j \\ \mu_1 & \mu_{23} & m \end{bmatrix}. \quad (7.11)$$

Equivalent relations can be obtained by using the orthonormality of C.G. coefficients. Multiplying both sides by $\begin{bmatrix} j_2 & j_3 & j'_{23} \\ \mu_2 & \mu_3 & \mu_{23} \end{bmatrix}$ and summing over μ_2, we obtain

$$\sum_{\mu_2}\begin{bmatrix} j_1 & j_2 & j_{12} \\ \mu_1 & \mu_2 & \mu_{12} \end{bmatrix}\begin{bmatrix} j_{12} & j_3 & j \\ \mu_{12} & \mu_3 & m \end{bmatrix}\begin{bmatrix} j_2 & j_3 & j'_{23} \\ \mu_2 & \mu_3 & \mu_{23} \end{bmatrix} = U(j_1j_2jj_3;j_{12}j'_{23})\begin{bmatrix} j_1 & j'_{23} & j \\ \mu_1 & \mu_{23} & m \end{bmatrix}. \quad (7.12)$$

Replace j'_{23} by j_{23} and once again multiply both sides by the C.G. coefficient $\begin{bmatrix} j_1 & j_{23} & j \\ \mu_1 & \mu_{23} & m \end{bmatrix}$ Then sum over μ_1. Using again the orthogonality of C.G. coefficients, we obtain,

$$U(j_1j_2jj_3;j_{12}j_{23}) = \sum_{\mu_1\mu_2}\begin{bmatrix} j_1 & j_2 & j_{12} \\ \mu_1 & \mu_2 & \mu_{12} \end{bmatrix}\begin{bmatrix} j_{12} & j_3 & j \\ \mu_{12} & \mu_3 & m \end{bmatrix} \times \begin{bmatrix} j_2 & j_3 & j_{23} \\ \mu_2 & \mu_3 & \mu_{23} \end{bmatrix}\begin{bmatrix} j_1 & j_{23} & j \\ \mu_1 & \mu_{23} & m \end{bmatrix}. \quad (7.13)$$

Equation (7.13) can be obtained directly from Eq. (7.6) or Eq. (7.7) by expressing the coefficient as a scalar product of the two eigenstates obtained in the two schemes of coupling.

$$U(j_1j_2jj_3;j_{12}j_{23}) = \langle (j_1j_2)j_{12}j_3, jm \,|\, j_1(j_2j_3)j_{23}, jm\rangle. \quad (7.14)$$

Now expanding the two coupled states in terms of uncoupled states using Eq. (7.9) and Eq. (7.10) and applying the orthonormality condition for the uncoupled states, we obtain the relation (7.13).

7.3. Independence of U-Coefficient from Magnetic Quantum Numbers

Out of the six projection quantum numbers $\mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{23}$ and m that occur on the right hand side of Eq. (7.13), m is fixed by the definition of

U-coefficient (Eqs. (7.6), (7.7) and (7.14)) and consequently only two are free variables due to the following three constraints.

$$\mu_{12} = \mu_1 + \mu_2; \quad \mu_{23} = \mu_2 + \mu_3; \quad m = \mu_1 + \mu_2 + \mu_3. \tag{7.15}$$

Since there is a summation over the only two free variables μ_1 and μ_2 in Eq. (7.13), the U-coefficient is independent of the projection quantum numbers.

The independence of the U-coefficients from the projection quantum numbers can also be seen in an alternative way (Ramachandran, 1962). In Eq. (7.14), the U-coefficient is expressed as a scalar product of the two eigenstates obtained in the two coupled representations. The scalar product can also he treated as a matrix element of the unit operator between these states. Applying the Wigner-Eckart theorem, we obtain the reduced matrix element which is independent of projection quantum numbers.

$$\begin{aligned} U(j_1j_2jj_3;j_{12}j_{23}) &= \langle j_1(j_2j_3)j_{23};jm\,|\,1\,|\,(j_1j_2)j_{12}j_3;jm\rangle \\ &= \begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix} \langle j_1(j_2j_3)j_{23};j\,||\,1\,||\,(j_1j_2)j_{12}j_3;j\rangle \\ &= \langle j_1(j_2j_3)j_{23};j\,||\,1\,||\,(j_1j_2)j_{12}j_3;j\rangle. \end{aligned} \tag{7.16}$$

Thus we find that the U-coefficient is in fact the reduced matrix element of the unit operator taken between the eigenstates in the two coupled representations and hence independent of the projection quantum numbers.

7.4. Orthonormality of the U-Coefficients

Each of the coupled states $|(j_1j_2)j_{12}j_3;jm\rangle$ and $|j_1(j_2j_3)j_{23};jm\rangle$ obey the orthonormality property and hence by using Eq. (7.6), we arrive at the orthonormality property of the U-coefficients.

$$\sum_{j_{23}} U(j_1j_2jj_3;j_{12}j_{23})\,U(j_1j_2jj_3;j'_{12}j_{23}) = \delta_{j_{12}j'_{12}}. \tag{7.17}$$

In a similar way, the inverse relation (7.7) yields

$$\sum_{j_{12}} U(j_1j_2jj_3;j_{12}j_{23})\,U(j_1j_2jj_3;j_{12}j'_{23}) = \delta_{j_{23}j'_{23}}. \tag{7.18}$$

7.5. The Racah Coefficient and its Symmetry Properties

The U-coefficient is related to the Racah coefficient W which has some simple symmetry properties[1].

$$U(abcd;ef) = [e][f]\,W(abcd;ef) \tag{7.19}$$

The U-coefficient and the Racah coefficient will vanish if the four triangular conditions $\Delta(abe)$, $\Delta(cde)$, $\Delta(acf)$ and $\Delta(bdf)$ are not satisfied. The parameters *a, b, c, d, e* and *f* in the Racah coefficient can be interchanged as one likes provided these four triangular relations are preserved and the new Racah coefficient thus obtained differs from the old one utmost by a phase factor.

$$\begin{aligned} W(abcd;ef) &= W(cdab;ef) = W(badc;ef), & (7.20)\\ &= W(acbd;fe) = W(bdac:fe), & (7.21)\\ &= (-1)^{a+d-e-f}\,W(ebcf;ad), & (7.22)\\ &= (-1)^{b+c-e-f}\,W(aefd;bc). & (7.23) \end{aligned}$$

Also a new coefficient T(abcd,ef) can be defined such that it is invariant under permutation of any of its arguments provided all the four triangular relations are preserved.

$$T(abcd,ef) = (-1)^{a+b+c+d}\,W(abcd,ef). \tag{7.24}$$

The wigner 6-j symbol is related to the Racah coefficient as follows.

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{a+b+c+d}\,W(abcd,ef). \tag{7.25}$$

Algebraic as well as numerical tables of Racah coefficients are available. Also a closed expression for the Racah coefficient has been deduced by Racah and it is widely used for computer programming.

$$\begin{aligned} W(abcd,ef) &= \Delta(abe)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf)\\ &\times\sum_x \frac{(-1)^{x+a+b+c+d}\,(x+1)!}{(x-a-b-e)!(x-c-d-e)!(x-a-c-f)!(x-b-d-f)!}\\ &\times\frac{1}{(a+b+c+d-x)!(a+d+e+f-x)!(b+c+e+f-x)!}, \end{aligned} \tag{7.26}$$

[1]For a detailed study of the symmetry properties, the reader may refer to Biedenharn et al. (1952, 1965, 1981) and Srinivasa Rao and Rajeswari (1993).

where $\Delta(abc)$ is the triangle coefficient, symmetric in its arguments.

$$\triangle(abc) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{\frac{1}{2}} \tag{7.27}$$

This coefficient vanishes unless the triangle condition in *a,b* and *c* is satisfied. The summation index *x* assumes all integer values for which the factorial arguments are not negative.

7.6. Evaluation of Matrix Elements

The following matrix elements can be evaluated using the concept of U-coefficients.

$$Q_1 = \langle j_1' j_2' j' m' \,|\, T_k^\mu(1) \,|\, j_1 j_2 j m\rangle. \tag{7.28}$$

$$Q_2 = \langle j_1' j_2' j' m' \,|\, T_k^\mu(2) \,|\, j_1 j_2 j m\rangle. \tag{7.29}$$

$$Q_3 = \langle j_1' j_2' j' m' \,|\, T_k(1)\cdot T_k(2) \,|\, j_1 j_2 j m\rangle. \tag{7.30}$$

In a two-particle system, a transition occurs from its initial coupled state $|j_1 j_2 j m\rangle$ to its final coupled state $|j_1' j_2' j' m'\rangle$ due to a tensor operator $T_k^\mu(1)$ operating on particle 1 or $T_k^\mu(2)$ operating on particle 2 or a scalar product of two tensor operators, one acting on particle 1 and the other acting on particle 2.

A straight-forward method is to write down the coupled angular momentum wave functions in the uncoupled representation using the C.G. coefficients and then apply the Wigner-Eckart theorem to obtain the reduced matrix elements. For instance,

$$\begin{aligned} Q_1 &= \sum_{m_1, m_1'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1' & j_2' & j' \\ m_1' & m_2' & m' \end{bmatrix} \\ &\quad \times \langle j_1' m_1' j_2' m_2' \,|\, T_k^\mu(1) \,|\, j_1 m_1 j_2 m_2\rangle, \\ &= \sum_{m_1 m_1'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1' & j_2' & j' \\ m_1' & m_2' & m' \end{bmatrix} \begin{bmatrix} j_1 & k & j_1' \\ m_1 & \mu & m_1' \end{bmatrix} \\ &\quad \times \langle j_1' \,||\, T_k(1) \,||\, j_1\rangle\, \delta_{j_2 j_2'}\, \delta_{m_2 m_2'}. \end{aligned} \tag{7.31}$$

The summation over m_1' is redundant since $m_1' = m_1 + \mu$ and the matrix element Q_1 exists only if $j_2 = j_2'$ and $m_2 = m_2'$. The three C.G. coefficients in Eq. (7.31) can be suitably rearranged using the symmetry properties to yield

$$\begin{aligned}
&\sum_{m_1}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\begin{bmatrix} j_1' & j_2 & j' \\ m_1' & m_2 & m' \end{bmatrix}\begin{bmatrix} j_1 & k & j_1' \\ m_1 & \mu & m_1' \end{bmatrix} \\
&= (-1)^{j_1+k-j_1'}\sum_{m_1}\begin{bmatrix} k & j_1 & j_1' \\ \mu & m_1 & m_1' \end{bmatrix}\begin{bmatrix} j_1' & j_2 & j' \\ m_1' & m_2 & m' \end{bmatrix}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}, \\
&= (-1)^{j_1+k-j_1'}\begin{bmatrix} k & j & j' \\ \mu & m & m' \end{bmatrix} U(kj_1j'j_2, j_1'j), \qquad \text{using Eq. (7.12)}, \\
&= (-1)^{j_1+k-j_1'}(-1)^{k+j-j'}\begin{bmatrix} j & k & j' \\ m & \mu & m' \end{bmatrix} U(kj_1j'j_2, j_1'j). \qquad (7.32)
\end{aligned}$$

Substituting Eq. (7.32) into Eq. (7.31) and simplifying the phase factor, we obtain

$$Q_1 = (-1)^{j_1-j_1'-j+j'}\begin{bmatrix} j & k & j' \\ m & \mu & m' \end{bmatrix} U(j_2j_1j'k, jj_1')\,\langle j_1' \,||\, T_k(1) \,||\, j_1\rangle. \qquad (7.33)$$

Using a similar procedure, we can evaluate the matrix element Q_2.

$$Q_2 = \begin{bmatrix} j & k & j' \\ m & \mu & m' \end{bmatrix} U(j_1j_2j'k, jj_2')\,\langle j_2' \,||\, T_k(1) \,||\, j_2\rangle. \qquad (7.34)$$

To evaluate Q_3, we observe that the transition operator is a scalar in the two-particle space and hence $j' = j$ and $m' = m$.

$$\begin{aligned}
Q_3 &= \langle j_1'j_2'j'm \,|\, T_k(1)\cdot T_k(2) \,|\, j_1j_2jm\rangle \\
&= \sum_{\mu}(-1)^{\mu}\langle j_1'j_2'jm \,|\, T_k^{\mu}(1)\, T_k^{-\mu}(2) \,|\, j_1j_2jm\rangle. \qquad (7.35)
\end{aligned}$$

Expanding the initial and final two-particle states into uncoupled single particle states and applying the Wigner-Eckart theorem, we obtain a product of four C.G. coefficients which when summed over magnetic quantum numbers yield a U-coefficient as shown in Eq. (7.13). After some rearrangement, we finally obtain

$$Q_3 = (-1)^k\frac{[j_2']}{[j_2]} U(j_1kjj_2', j_1'j_2)\,\langle j_1' \,||\, T_k(1) \,||\, j_1\rangle\langle j_2' \,||\, T_k(2) \,||\, j_2\rangle. \qquad (7.36)$$

It is instructive to obtain the matrix element Q_2 by applying the Wigner-Eckart theorem to the combined two-particle space and then use a simple argument with respect to the coupling scheme.

$$Q_2 = \begin{bmatrix} j & k & j' \\ m & \mu & m' \end{bmatrix}\langle j_1'j_2'j' \,||\, T_k(2) \,||\, j_1j_2j\rangle\,\delta_{j_1'j_1}. \qquad (7.37)$$

The kronecker delta in Eq. (7.37) arises since the particle 1 does not undergo any transition because the operator acts only on the particle 2. By a simple argument, it can be shown that the matrix element in the coupled representation can be expressed in terms of the matrix element in uncoupled representation using the U-coefficient.

Coupling scheme adopted in the coupled representation	Coupling scheme adopted in the uncoupled representation
$\boldsymbol{J}_1 + \boldsymbol{J}_2 = \boldsymbol{J}$	$\boldsymbol{J}_2 + \boldsymbol{K} = \boldsymbol{J}_2'$
$\boldsymbol{J} + \boldsymbol{K} = \boldsymbol{J}'$	$\boldsymbol{J}_1 + \boldsymbol{J}_2' = \boldsymbol{J}'$

We at once observe that the above two coupling schemes are exactly the two coupling schemes, we studied earlier in the coupling of three angular momenta

$$\boldsymbol{J}' = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{K}, \tag{7.38}$$

and one can go from one scheme to the other scheme by means of unitary transformation denoted by the U-coefficient. So, it follows that

$$\langle j_1 j_2' j' \,||\, T_k(2) \,||\, j_1 j_2 j\rangle = U(j_1 j_2 j' k, j j_2')\, \langle j_2' \,||\, T_k(2) \,||\, j_2\rangle. \tag{7.39}$$

In a similar way, we can evaluate Q_1.

$$Q_1 = \left[\begin{array}{ccc} j & k & j' \\ m & \mu & m' \end{array}\right] \langle j_1' j_2' j' \,||\, T_k(1) \,||\, j_1 j_2 j\rangle \delta_{j_2' j_2}. \tag{7.40}$$

If we switch the order of coupling of particles 1 and 2, we get

$$\begin{aligned} \langle j_1' j_2 j' \,||\, T_k(1) \,||\, j_1 j_2 j\rangle &= (-1)^{j_1+j_2-j} (-1)^{j_1'+j_2-j'} \\ &\quad \times \langle j_2 j_1' j' \,||\, T_k(1) \,||\, j_2 j_1 j\rangle, \end{aligned} \tag{7.41}$$

using the symmetry properties of C. G. coefficients. The reduced matrix element occurring on the left hand side of Eq. (7.39) is identical with the reduced matrix element occurring on the right hand side of Eq. (7.41) except for the interchange of the particle labels 1 and 2. So from Eq. (7.39), it follows

$$\langle j_2 j_1' j' \,||\, T_k(1) \,||\, j_2 j_1 j\rangle = U(j_2 j_1 j' k, j j_1')\, \langle j_1' \,||\, T_k(1) \,||\, j_1\rangle. \tag{7.42}$$

Combining Eqs. (7.40),(7.41) and (7.42), we obtain the result given in Eq. (7.33).

Review Questions

7.1 (a) In the coupling of three angular momenta, show that there is more than one coupling scheme. Define the unitary transformation coefficient U that connects one coupling scheme with another and express it in terms of C.G. coefficient.
(b) Show that the U-coefficient is independent of the magnetic quantum numbers and deduce the orthonormality of the U coefficients.

7.2 Define the Racah coefficient and state its symmetry properties. How is it related to the 6-j symbol?

7.3 Evaluate the following two-particle matrix elements and express them in terms of single particle matrix elements.

$$\begin{aligned}&(a)\quad \langle j_1' j_2' j' m' \,|\, T_k^{\mu}(2) \,|\, j_1 j_2 j m\rangle.\\&(b)\quad \langle j_1' j_2' j' m' \,|\, T_k^{\mu}(1) \,|\, j_1 j_2 j m\rangle.\\&(c)\quad \langle j_1' j_2' j' m' \,|\, T_k(1)\cdot T_k(2) \,|\, j_1 j_2 j m\rangle.\end{aligned}$$

Problems

7.1 Determine the following Racah coefficients using their general properties:

$$\text{(i) } W(1021,11),\quad \text{(ii) } W(1232,11),\quad \text{(iii) } W(1111,10).$$

7.2 Evaluate the following reduced matrix elements:

$$\begin{aligned}&(a)\quad \langle l_f \tfrac{1}{2} j_f \,\|\, Y_l \,\|\, l_i \tfrac{1}{2} j_i\rangle,\\&(b)\quad \langle l_f \tfrac{1}{2} j_f \,\|\, \sigma\cdot\hat{r} \,\|\, l_i \tfrac{1}{2} j_i\rangle.\end{aligned}$$

7.3 Evaluate the matrix element:

$$\langle l_f \tfrac{1}{2} j_f m_f \,|\, \sigma\cdot L \,|\, l_i \tfrac{1}{2} j_i m_i\rangle.$$

7.4 Find the expectation value of the operator $Y_2^0(\hat{r})$ in the single nucleon state $|l\,\tfrac{1}{2}\,j\,m=j\rangle$. Consider both the possible values of $j(=\ l\ \pm\ \tfrac{1}{2})$.

Solutions to Selected Problems

7.1 (i) $U(1021,\ 11)\ =\ 1$ since the two coupling schemes are identical because one of the angular momenta to be added is zero.
From Eq. (7.19), it follows

$$U(1021,11) = [1][1]\,W(1021,11).$$

Hence the result $W(1021\ ,\ 11) = \frac{1}{3}$ follows.

(ii) $W(1232, 11) = 0$ since the triangular condition is not satisfied in one case.

(iii) $W(1111\ ,\ 10) = -W(1101\ ,\ 11) = -W(1011\ ,\ 11) = -\frac{1}{3}$.

7.2 (a) Using Eqs. (7.28) and (7.33), we obtain

$$\langle l_f \tfrac{1}{2} j_f \,||\, Y_l \,||\, l_i \tfrac{1}{2} j_i\rangle = (-1)^{l_i - l_f - j_i + j_f}\, U(\tfrac{1}{2} l_i j_f l, j_i l_f)\, \langle l_f \,||\, Y_l \,||\, l_i\rangle,$$

where

$$\langle l_f \,||\, Y_l \,||\, l_i\rangle = \frac{[l_i][l]}{\sqrt{4\pi}[l_f]} \begin{bmatrix} l_i & l & l_f \\ 0 & 0 & 0 \end{bmatrix}.$$

The above result can be deduced from Eq. (5.82).

(b) Since $\sigma \cdot \hat{r}$ is a scalar (strictly a pseudoscalar) in the combined space of configuration and spin, j_f should be equal to j_i. So, let us impose the condition $j_i = j_f = j$.

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = \sqrt{\frac{4\pi}{3}}\, \boldsymbol{\sigma} \cdot Y_1(\hat{\boldsymbol{r}}).$$

Applying the result (7.36), we obtain

$$\langle l_f \tfrac{1}{2} j \,||\, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} \,||\, l_i \tfrac{1}{2} j\rangle = -U(l_i 1 j \tfrac{1}{2}, l_f \tfrac{1}{2})\, \langle l_f ||Y_1 \,||\, l_i\rangle \langle \tfrac{1}{2} || \boldsymbol{\sigma} \,||\, \tfrac{1}{2}\rangle,$$

with

$$\begin{aligned} \langle l_f \,||\, Y_1 \,||\, l_i\rangle &= \frac{[l_i][1]}{\sqrt{4\pi}[l_f]} \begin{bmatrix} l_i & 1 & l_f \\ 0 & 0 & 0 \end{bmatrix}, \\ \langle \tfrac{1}{2} \,||\, \boldsymbol{\sigma} \,||\, \tfrac{1}{2}\rangle &= [1]. \end{aligned}$$

Substituting the algebraic expressions for the U-coefficient and the C.G. coefficient and simplifying, we finally obtain

$$\langle l_f \tfrac{1}{2} j \,||\, \boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} \,||\, l_i \tfrac{1}{2} j\rangle = -1.$$

The above result can be obtained from a simple consideration. Since $\sigma \cdot \hat{r}$ is a pseudo-scalar in the j-space, its parity is -1. Therefore

$$l_f = l_i \pm 1.$$

In other words, if $l_i = j + \frac{1}{2}$, then $l_f = j - \frac{1}{2}$ or vice versa.

Since the square of the operator

$$(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \cdot (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}} + i\boldsymbol{\sigma} \cdot (\hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}}) = 1,$$

it follows that

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = -1.$$

remembering that $\sigma \cdot \hat{\boldsymbol{r}}$ is a pseudo-scalar.

COUPLING OF FOUR ANGULAR MOMENTA

8.1. Definition of LS-jj Coupling Coefficient

If there are two particles with spin, then the determination of their resultant angular momentum involves the addition of four angular momenta. Two of them are their orbital angular momenta l_1 and l_2 and the other two, their spin angular momenta s_1 and s_2. Their resultant angular momentum can be found in more than one way. One way is known as the L-S coupling scheme and there is another way called the j-j coupling scheme.

L-S coupling scheme	j-j coupling scheme
$\boldsymbol{l}_1 + \boldsymbol{l}_2 = \boldsymbol{L}$	$\boldsymbol{l}_1 + \boldsymbol{s}_1 = \boldsymbol{j}_1$
$\boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{S}$	$\boldsymbol{l}_2 + \boldsymbol{s}_2 = \boldsymbol{j}_2$
$\boldsymbol{L} + \boldsymbol{S} = \boldsymbol{J}$	$\boldsymbol{j}_1 + \boldsymbol{j}_2 = \boldsymbol{J}$

Just as one can go from the uncoupled representation to anyone of the coupled representation by means of unitary transformation, it is also possible to go from one coupled representation to another coupled representation, by unitary transformation. In each representation, there are a set of eight mutually commuting operators for which simultaneous eigenvalues can be determined and they are given below.

a) Uncoupled representation

$$l_1^2,\ l_2^2,\ s_1^2,\ s_2^2,\ l_{1z},\ l_{2z},\ s_{1z},\ s_{2z}.$$

b) L-S coupled representation

$$l_1^2,\ l_2^2,\ s_1^2,\ s_2^2,\ L^2,\ S^2,\ J^2,\ J_z.$$

c) j-j coupled representation

$$l_1^2,\ l_2^2,\ s_1^2,\ s_2^2,\ j_1^2,\ j_2^2,\ J^2,\ J_z.$$

Let us denote the state in each representation by their quantum num bers, and expand the state in the j-j coupled representation in terms of a complete set of L-S coupled states.

$$|l_1 l_2 s_1 s_2, j_1 j_2 JM\rangle = \sum_{L,S} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} |l_1 l_2 s_1 s_2, LSJM\rangle. \tag{8.1}$$

The coefficients $\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix}$ are the elements of the unitary transformation matrix. They are known as the LS-jj coupling coefficient and can be expressed as the scalar product of the states in the two coupling schemes.

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \langle l_1 l_2 s_1 s_2, LSJM \mid l_1 l_2 s_1 s_2, j_1 j_2 JM\rangle. \tag{8.2}$$

These coefficients can be expressed in terms of products of six C.G. coefficients and, like U-coefficients, these LS-jj coupling coefficients are also independent of projection quantum numbers. Since the C.G. coefficients are chosen to be real, it follows that the LS-jj coupling coefficients are also real. Consequently by taking the complex conjugate of Eq. (8.2) we obtain,

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \langle l_1 l_2 s_1 s_2, j_1 j_2 JM \mid l_1 l_2 s_1 s_2, LSJM\rangle. \tag{8.3}$$

8.2. LS-jj Coupling Coefficient in terms of C.G. Coefficients

The LS-jj coupling coefficients can be expanded in terms of products of six C.G. coefficients using either Eq. (8.2) or Eq. (8.3). For this purpose, each of the coupled states $|l_1 l_2 s_1 s_2, LSJM\rangle$ and $|l_1 l_2 s_1 s_2, j_1 j_2 JM\rangle$ (which hereafter will be referred to simply as $|LSJM\rangle$ and $|j_1 j_2 JM\rangle$) has to be expressed in terms of uncoupled states.

$$|LSJM\rangle = \sum_{m_1,\mu_1,m_L} \begin{bmatrix} l_1 & l_2 & L \\ m_1 & m_2 & m_L \end{bmatrix} \begin{bmatrix} s_1 & s_2 & S \\ \mu_1 & \mu_2 & m_S \end{bmatrix} \begin{bmatrix} L & S & J \\ m_L & m_S & M \end{bmatrix} \times |l_1 m_1\rangle |l_2 m_2\rangle |s_1 \mu_1\rangle |s_2 \mu_2\rangle. \tag{8.4}$$

$$|j_1 j_2 JM\rangle = \sum_{\lambda_1,\lambda_2,M_1} \begin{bmatrix} l_1 & s_1 & j_1 \\ \lambda_1 & \nu_1 & M_1 \end{bmatrix} \begin{bmatrix} l_2 & s_2 & j_2 \\ \lambda_2 & \nu_2 & M_2 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & J \\ M_1 & M_2 & M \end{bmatrix} \times |l_1 \lambda_1\rangle |l_2 \lambda_2\rangle |s_1 \nu_1\rangle |s_2 \nu_2\rangle. \tag{8.5}$$

Substituting Eqs. (8.4) and (8.5) in Eq. (8.2), we obtain

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \sum_{m_1,\mu_1,m_L,\lambda_1,\lambda_2,M_1} \begin{bmatrix} l_1 & l_2 & L \\ m_1 & m_2 & m_L \end{bmatrix} \begin{bmatrix} s_1 & s_2 & S \\ \mu_1 & \mu_2 & m_S \end{bmatrix} \times \begin{bmatrix} L & S & J \\ m_L & m_S & M \end{bmatrix} \begin{bmatrix} l_1 & s_1 & j_1 \\ \lambda_1 & \nu_1 & M_1 \end{bmatrix} \begin{bmatrix} l_2 & s_2 & j_2 \\ \lambda_2 & \nu_2 & M_2 \end{bmatrix} \times \begin{bmatrix} j_1 & j_2 & J \\ M_1 & M_2 & M \end{bmatrix} \langle l_1 m_1 \,|\, l_1 \lambda_1 \rangle \langle l_2 m_2 \,|\, l_2 \lambda_2 \rangle \times \langle s_1 \mu_1 \,|\, s_1 \nu_1 \rangle \langle s_2 \mu_2 \,|\, s_2 \nu_2 \rangle. \quad (8.6)$$

In the above equation, the summation over $m_L = m_1 + m_2$ is equivalent to summation over m_2 and similarly, the summation over $M_1 = \lambda_1 + \nu_1$ is equivalent to summation over ν_1. Therefore

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \sum_{m_1,\mu_1,m_2,\lambda_1,\lambda_2,\nu_1} \begin{bmatrix} l_1 & l_2 & L \\ m_1 & m_2 & m_L \end{bmatrix} \begin{bmatrix} s_1 & s_2 & S \\ \mu_1 & \mu_2 & m_S \end{bmatrix} \times \begin{bmatrix} L & S & J \\ m_L & m_S & M \end{bmatrix} \begin{bmatrix} l_1 & s_1 & j_1 \\ \lambda_1 & \nu_1 & M_1 \end{bmatrix} \begin{bmatrix} l_2 & s_2 & j_2 \\ \lambda_2 & \nu_2 & M_2 \end{bmatrix} \times \begin{bmatrix} j_1 & j_2 & J \\ M_1 & M_2 & M \end{bmatrix} \delta_{m_1\lambda_1}\, \delta_{m_2\lambda_2}\, \delta_{\mu_1\nu_1}\, \delta_{\mu_2\nu_2}. \quad (8.7)$$

Summing over the magnetic quantum numbers λ_1, λ_2 and ν_1 is equivalent to replacing them by m_1, m_2 and μ_1 because of the δ functions. The last δ function in Eq. (8.7) is redundant since

$$M = m_1 + m_2 + \mu_1 + \mu_2 = \lambda_1 + \lambda_2 + \nu_1 + \nu_2.$$

Hence it follows that

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \sum_{m_1,\mu_1,m_2} \begin{bmatrix} l_1 & l_2 & L \\ m_1 & m_2 & m_L \end{bmatrix} \begin{bmatrix} s_1 & s_2 & S \\ \mu_1 & \mu_2 & m_S \end{bmatrix} \times \begin{bmatrix} L & S & J \\ m_L & m_S & M \end{bmatrix} \begin{bmatrix} l_1 & s_1 & j_1 \\ m_1 & \mu_1 & M_1 \end{bmatrix} \times \begin{bmatrix} l_2 & s_2 & j_2 \\ m_2 & \mu_2 & M_2 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & J \\ M_1 & M_2 & M \end{bmatrix}. \quad (8.8)$$

In Eq. (8.8), the LS-jj coupling coefficient is expressed as a product of six C.G. coefficients with a summation over three projection quantum numbers m_1, m_2 and $\mu 1$.

8.3. Independence of the LS-jj Coupling Coefficients from the Magnetic Quantum Numbers

It is easy to see that the LS-jj coupling coefficients will not depend on the projection quantum numbers. In Eq. (8.8) there are altogether 9 projection quantum numbers viz., m_1, m_2, μ_1, μ_2, m_L, m_S, M_1, M_2 and M. Of these, the four quantum numbers m_L, m_S, M_1 and M_2 are not independent since

$$m_L = m_1 + m_2; \quad m_S = \mu_1 + \mu_2; \quad M_1 = m_1 + \mu_1; \quad M_2 = m_2 + \mu_2.$$

Also the projection quantum number M is fixed as per the definition of LS - jj coupling coefficient given by Eq. (8.2). Out of the remaining four quantum numbers m_1, m_2, μ_1 and μ_2, only three are free variables since

$$M = m_1 + m_2 + \mu_1 + \mu_2.$$

On the right hand side of Eq. (8.8), there is a. summation over these three variables, thereby making the LS-jj coupling coefficient independent of the projection quantum numbers.

The independence of the LS-jj coupling coefficient from the projection quantum numbers can also be shown in a more elegant way using the definition (8.2).

$$\begin{aligned} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} &= \langle LSJM \mid j_1 j_2 JM \rangle \\ &= \langle LSJM \mid 1 \mid j_1 j_2 JM \rangle \\ &= \begin{bmatrix} J & 0 & J \\ M & 0 & M \end{bmatrix} \langle LSJ \,\|\, 1 \,\|\, j_1 j_2 J \rangle \\ &= \langle LSJ \,\|\, j_1 j_2 J \rangle. \end{aligned} \tag{8.9}$$

Above, we have considered the scalar product $\langle LSJM \mid j_1 j_2 JM \rangle$ as the matrix element of the unit operator taken between the two coupled states. Recognizing the unit operator as a zero rank tensor and applying the Wigner-Eckart theorem, we obtain the desired result that the LS-jj coupling coefficient is, in fact, a reduced matrix element which is independent of the magnetic quantum numbers.

8.4. Simple Properties

Putting $s_1 = 0$ and $s_2 = s$ in Eq. (8.2), we obtain

$$\begin{bmatrix} l_1 & 0 & j_1 \\ l_2 & s & j_2 \\ L & s & J \end{bmatrix} = \langle l_1 l_2 0 s, LSJM \mid l_1 l_2 0 s, j_1 j_2 JM \rangle$$
$$= \langle (l_1 l_2) Ls, JM \mid l_1 (l_2 s) j_2, JM \rangle$$
$$= U(l_1 l_2 J s, L j_2). \qquad (8.10)$$

Thus we find that the LS-jj coupling coefficient will reduce to a U-coefficient if one of the four angular momenta were to be zero. This is what we should expect since there is effectively only three angular momenta to be coupled.

Using the property of the orthonormality of the functions, one can obtain the orthonormality of the LS-jj coupling coefficients.

$$\sum_{L,S} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} \begin{bmatrix} l_1 & s_1 & j_1' \\ l_2 & s_2 & j_2' \\ L & S & J \end{bmatrix} = \delta_{j_1 j_1'} \, \delta_{j_2 j_2'}. \qquad (8.11)$$

$$\sum_{j_1,j_2} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L' & S' & J \end{bmatrix} = \delta_{LL'} \, \delta_{SS'}. \qquad (8.12)$$

Instead of the LS-jj coupling coefficient, we can define the Wigner 9-j symbol (sometimes referred to as the X-coefficient) which has a better symmetry property under permutation of columns or rows.

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = [j_1][j_2][L][S] \begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix}. \qquad (8.13)$$

The curly bracket in Eq. (8.13) is referred to as the Wigner 9-j symbol or the X-coefficient. The 9-j symbol is invariant under even permutation of rows or columns but odd permutation will introduce a phase factor (-1)T where $T = l_1 + s_1 + j_1 + l_2 + s_2 + j_2 + L + S + J$.

$$\begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix} = (-1)^T \begin{Bmatrix} l_2 & s_2 & j_2 \\ l_1 & s_1 & j_1 \\ L & S & J \end{Bmatrix}. \qquad (8.14)$$

Also an interchange of rows and columns will leave the Wigner 9-j symbol as well as the LS-jj coupling coefficient invariant.

8.5. Expansion of 9-j Symbol into Racah Coefficients

Consider the expression (8.8) for the LS-jj coupling coefficient. There are six C.G. coefficients which can be grouped into three pairs. Applying the relation Eq. (7.10) to each pair, we obtain

$$\begin{bmatrix} l_1 & l_2 & L \\ m_1 & m_2 & m_L \end{bmatrix}\begin{bmatrix} L & S & J \\ m_L & m_S & M \end{bmatrix} = \sum_L U(l_1 l_2 J S; Lt) \underset{(a)}{\begin{bmatrix} l_2 & S & t \\ m_2 & m_S & m_t \end{bmatrix}} \underset{(b)}{\begin{bmatrix} l_1 & t & J \\ m_1 & m_t & M \end{bmatrix}} . \quad (8.15)$$

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ m_1 & \mu_1 & M_1 \end{bmatrix}\begin{bmatrix} j_1 & j_2 & J \\ M_1 & M_2 & M \end{bmatrix} = \sum_u U(l_1 s_1 J j_2; j_1 u) \underset{(c)}{\begin{bmatrix} s_1 & j_2 & u \\ \mu_1 & M_2 & m_u \end{bmatrix}} \underset{(d)}{\begin{bmatrix} l_1 & u & J \\ m_1 & m_u & M \end{bmatrix}} . \quad (8.16)$$

$$\begin{aligned} \begin{bmatrix} s_1 & s_2 & S \\ \mu_1 & \mu_2 & m_S \end{bmatrix}\begin{bmatrix} l_2 & s_2 & j_2 \\ m_2 & \mu_2 & M_2 \end{bmatrix} &= (-1)^{s_1-\mu_1}\frac{[S]}{[s_2]}\begin{bmatrix} s_1 & S & s_2 \\ \mu_1 & -m_S & -\mu_2 \end{bmatrix}\begin{bmatrix} s_2 & l_2 & j_2 \\ \mu_2 & -m_2 & -M_2 \end{bmatrix} \\ &= (-1)^{s_1-\mu_1}\frac{[S]}{[s_2]}\sum_v U(s_1 S j_2 l_2; s_2 v) \\ &\quad \times \underset{(e)}{\begin{bmatrix} S & l_2 & v \\ -m_S & -m_2 & m_v \end{bmatrix}} \underset{(f)}{\begin{bmatrix} s_1 & v & j_2 \\ \mu_1 & m_v & -M_2 \end{bmatrix}} . \end{aligned} \quad (8.17)$$

Substituting Eqs. (8.15), (8.16) and (8.17) into Eq. (8.8), we can now perform the summation over the three projection quantum numbers in the order $m_2, \mu 1$ and m_1. Of the six C.G. coefficients, we now have, (a) and (e) alone depend on m_2. Similarly (c) and (f) alone depend on $\mu 1$. The summation over m_2 and $\mu 1$ are carried out using the symmetry and the orthonormal properties of C.G. coefficients and they yield the δ functions as shown below:

$$\sum_{m_2} \begin{bmatrix} l_2 & S & t \\ m_2 & m_S & m_t \end{bmatrix} \begin{bmatrix} S & l_2 & v \\ -m_S & -m_2 & m_v \end{bmatrix}$$
$$= \sum_{m_2} \begin{bmatrix} l_2 & S & t \\ m_2 & m_S & m_t \end{bmatrix} \begin{bmatrix} l_2 & S & v \\ m_2 & m_S & -m_v \end{bmatrix}$$
$$= \delta_{tv}\, \delta_{m_t\, -m_v}. \tag{8.18}$$

$$\sum_{\mu_1} (-1)^{s_1-\mu_1} \begin{bmatrix} s_1 & j_2 & u \\ \mu_1 & M_2 & m_u \end{bmatrix} \begin{bmatrix} s_1 & v & j_2 \\ \mu_1 & m_v & -M_2 \end{bmatrix}$$
$$= \sum_{\mu_1} \begin{bmatrix} s_1 & j_2 & u \\ \mu_1 & M_2 & m_u \end{bmatrix} \begin{bmatrix} s_1 & j_2 & v \\ \mu_1 & M_2 & -m_v \end{bmatrix} \frac{[j_2]}{[v]}$$
$$= \frac{[j_2]}{[v]} \delta_{uv}\, \delta_{m_u\, -m_v}. \tag{8.19}$$

Thus we obtain

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \sum_{t,u,v} \frac{[S][j_2]}{[s_2][v]}\, U(l_1 l_2 J S, Lt)\, U(l_1 s_1 J j_2, j_1 u)$$
$$\times\, U(s_1 S j_2 l_2, s_2 v) \sum_{m_1} \begin{bmatrix} l_1 & t & J \\ m_1 & m_t & M \end{bmatrix} \begin{bmatrix} l_1 & u & J \\ m_1 & m_u & M \end{bmatrix}$$
$$\times\, \delta_{tv}\, \delta_{m_t\, -m_v}\, \delta_{uv}\, \delta_{m_u\, -m_v}. \tag{8.20}$$

Summing over u and v is equivalent to replacing u and v by t. Then the summation over m_l simply yields unity. It is to be stressed here that the summation over m_l is to be done last since M has a fixed value in the definition (8.2). Hence

$$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} = \sum_{t} \frac{[S][j_2]}{[s_2][t]}\, U(l_1 l_2 J S, Lt)\, U(l_1 s_1 J j_2, j_1 t)$$
$$\times\, U(s_1 S j_2 l_2, s_2 t). \tag{8.21}$$

Replacing the U-coefficient by W-coefficients and the LS-jj coupling coefficient by the 9-j symbol, we finally obtain

$$\begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix} = \sum_{t} (2t+1)\, W(l_1 l_2 J S, Lt)\, W(l_1 s_1 J j_2, j_1 t)$$
$$\times\, W(s_1 S j_2 l_2, s_2 t). \tag{8.22}$$

In Eqs. (8.21) and (8.22), there is a summation over t, the upper and lower bounds of which are determined by the triangular condition to be satisfied by all the three Racah coefficients.

Numerical tables for the 9-j symbol are available. Once we have a computer program for the Racah coefficients, it can be extended to the 9-j symbols using the formula (8.22). It will be useful to write a computer program for the 9-j symbols based on the formula (8.22).

8.6. Evaluation of Matrix Elements

The following matrix element can be evaluated using the concept of LS-jj coupling coefficient.

$$\begin{aligned} Q &= \langle j_1' j_2' j' m' \,|\, [T_{k_1}(1) \times T_{k_2}(2)]_k^\mu \,|\, j_1 j_2 j m\rangle \\ &= \begin{bmatrix} j & k & j' \\ m & \mu & m' \end{bmatrix} \langle j_1' j_2' j' \,||\, [T_{k_1}(1) \times T_{k_2}(2)]_k \,||\, j_1 j_2 j\rangle. \end{aligned} \quad (8.23)$$

In the above reduction, the Wigner-Eckart theorem is used.

The reduced matrix element does not depend on the projection quantum numbers and it can be expressed as a product of two single particle reduced matrix elements using the concept of recoupling scheme that arises in the addition of four angular momenta $j' = j_1 + j_2 + k_1 + k_2$.

Scheme A	Scheme B
$j_1 + j_2 = j$	$j_1 + k_1 = j_1'$
$k_1 + k_2 = k$	$j_2 + k_2 = j_2'$
$j + k = j'$	$j_1' + j_2' = j'$

Scheme A corresponds to the two-particle reduced matrix element given in Eq. (8.23) and scheme B is what we require to express the two-particle reduced matrix element in terms of two single particle reduced matrix elements. One can go from scheme A to scheme B by means of LS-jj coupling coefficient.

$$\begin{aligned} &\langle j_1' j_2' j' \,||\, [T_{k_1}(1) \times T_{k_2}(2)]_k \,||\, j_1 j_2 j\rangle \\ &\qquad = \begin{bmatrix} j_1 & j_2 & j \\ k_1 & k_2 & k \\ j_1' & j_2' & j \end{bmatrix} \langle j_1' \,||\, T_{k_1}(1) \,||\, j_1\rangle \langle j_2' \,||\, T_{k_2}(2) \,||\, j_2\rangle. \end{aligned} \quad (8.24)$$

The matrix elements Q_1, Q_2 and Q_3 denoted by Eqs. (7.28), (7.29) and (7.30) and evaluated in the last chapter using U-coefficients can be considered as special cases (i) $k_2 = 0$, (ii) $k_1 = 0$ and (iii) $k_1 = k_2 = \kappa$ with $k = 0$.

For the purpose of illustration, we shall indicate below how Q_3 can be obtained.

$$\begin{aligned}
[T_\kappa(1)\times T_\kappa(2)]_0^0 &= \sum_\mu \begin{bmatrix} \kappa & \kappa & 0 \\ \mu & -\mu & 0 \end{bmatrix} T_\kappa^\mu(1)\,T_\kappa^{-\mu}(2) \\
&= \sum_\mu \frac{(-1)^{\kappa-\mu}}{[\kappa]} T_\kappa^\mu(1)\,T_\kappa^{-\mu}(2) \\
&= \frac{(-1)^\kappa}{[\kappa]} T_\kappa(1)\cdot T_\kappa(2). \qquad (8.25)
\end{aligned}$$

From Eq. (8.23), it follows that

$$\begin{aligned}
&\langle j_1' j_2' j' \,||\, [T_\kappa(1)\times T_\kappa(2)]_0 \,||\, j_1 j_2 j\rangle \\
&= \begin{bmatrix} j_1 & j_2 & j \\ \kappa & \kappa & 0 \\ j_1' & j_2' & j \end{bmatrix} \langle j_1' \,||\, T_\kappa(1) \,||\, j_1\rangle \langle j_2' \,||\, T_\kappa(2) \,||\, j_2\rangle\, \delta_{j,j'} \\
&= \frac{[j][j_1'][j_2']}{[\kappa][j_2][j_1'][j]} \begin{bmatrix} \kappa & 0 & \kappa \\ j_1 & j & j_2 \\ j_1' & j & j_2' \end{bmatrix} \langle j_1' \,||\, T_\kappa(1) \,||\, j_1\rangle \langle j_2' \,||\, T_\kappa(2) \,||\, j_2\rangle \\
&= \frac{[j_2']}{[\kappa][j_2]} U(k\, j_1\, j_2'\, j,\, j_1'\, j_2) \langle j_1' \,||\, T_\kappa(1) \,||\, j_1\rangle \langle j_2' \,||\, T_\kappa(2) \,||\, j_2\rangle. \qquad (8.26)
\end{aligned}$$

From Eqs. (8.24) and (8.25), we deduce the result given in Eq. (7.36).

Review Questions

8.1 Define the Unitary transformation coefficient that occurs in the coupling of four angular momenta and express it in terms of C.G. coefficients.

8.2 Show that the LS-jj coupling coefficient that occurs in the addition of four angular momenta is independent of the magnetic quantum numbers.

8.3 Define the 9-j symbol and derive an expression for it in terms of Racah coefficients.

Problems

8.1 Evaluate the following matrix element:

$$\langle l_f\, \tfrac{1}{2}\, j_f\, m_f \,|\, (Y_l(\hat{\boldsymbol{r}})\times\sigma)_\lambda^{m_\lambda} \,|\, l_i\, \tfrac{1}{2}\, j_i\, m_i\rangle.$$

8.2 Denoting the angular momentum wave function of the deuteron by $|l\,s\,j\,m\rangle$, show that

$$(a)\ \langle 2\,1\,1\,m\,|\,S_{12}\,|\,0\,1\,1\,m\rangle = \sqrt{8}\,,$$

$$(b)\ \langle 2\,1\,1\,m\,|\,S_{12}\,|\,2\,1\,1\,m\rangle = -2\,,$$

where S_{12} is the tensor potential

$$\begin{aligned} S_{12} &= \frac{3}{r^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \\ &= \sqrt{\frac{24\pi}{5}}Y_2(\hat{\boldsymbol{r}})\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)_2. \end{aligned}$$

8.3 Show that

$$\begin{aligned} &\sum_{m_\lambda}[Y_l(\hat{\boldsymbol{k}})\times K_n]_\lambda^{m_\lambda}\left\{[Y_l(\hat{\boldsymbol{k}})\times K_{n'}]_\lambda^{m_\lambda}\right\}^* \\ &\quad = (-1)^{l'+n'-N}\frac{[\lambda]^2\,[l]\,[l']}{\sqrt{4\pi}\,[N]}\begin{bmatrix} l & l' & N \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad\times W(l\,n\,l'\,n',\,\lambda\,N)\,\{Y_N(\hat{\boldsymbol{k}})\cdot(K_n\times K_{n'}^*)_N\}. \end{aligned}$$

Solutions to Selected Problems

8.1 The matrix element can be evaluated directly by using Eq. (8.23).

$$\begin{aligned} &\langle l_f\,\tfrac{1}{2}\,j_f\,m_f\,|\,(Y_l(\hat{\mathbf{r}})\times\sigma)_\lambda^{m_\lambda}\,|\,l_i\,\tfrac{1}{2}\,j_i\,m_i\rangle \\ &\quad = \begin{bmatrix} j_i & \lambda & j_f \\ m_i & m_\lambda & m_f \end{bmatrix}\begin{bmatrix} l_i & \frac{1}{2} & j_i \\ l & 1 & \lambda \\ l_f & \frac{1}{2} & j_f \end{bmatrix}\langle l_f\,||\,Y_l(\hat{\mathbf{r}})\,||\,l_i\rangle \\ &\quad\times\langle\tfrac{1}{2}\,||\,\sigma\,||\,\tfrac{1}{2}\rangle, \end{aligned}$$

with

$$\begin{aligned} \langle l_f\,||\,Y_l(\hat{\mathbf{r}})\,||\,l_i\rangle &= \begin{bmatrix} l_i & l & l_f \\ 0 & 0 & 0 \end{bmatrix}\frac{[l_i][l]}{\sqrt{4\pi}[l_f]}; \\ \langle\tfrac{1}{2}\,||\,\sigma\,||\,\tfrac{1}{2}\rangle &= [1] = \sqrt{3}. \end{aligned}$$

8.2 Let us denote the matrix elements in (a) and (b) by Q_1 and Q_2.

$$\begin{aligned} Q_1 &= \langle 2\,1\,1\,m\,|\,S_{12}\,|\,0\,1\,1\,m\rangle. \\ Q_2 &= \langle 2\,1\,1\,m\,|\,S_{12}\,|\,2\,1\,1\,m\rangle. \end{aligned}$$

Substituting explicitly the operator S_{12},

$$Q_1 = \sqrt{\frac{24\pi}{5}}\langle 2\,1\,1\,m\,|\,Y_2.(\sigma_1 \times \sigma_2)_2\,|\,0\,1\,1\,m\rangle.$$

This can be written as a product of two matrix elements using the relation (7.36).

$$\begin{aligned} Q_1 &= \sqrt{\frac{24\pi}{5}}U(0211,21)\langle 2\,||\,Y_2\,||\,0\rangle \\ &\quad \times\langle s_1=\tfrac{1}{2}, s_2=\tfrac{1}{2}, s=1\,||\,(\sigma_1\times\sigma_2)_2\,||\,s_1=\tfrac{1}{2}, s_2=\tfrac{1}{2}, s=1\rangle \\ &= \sqrt{\frac{24\pi}{5}}\,\langle 2\,||\,Y_2\,||\,0\rangle\,\langle \tfrac{1}{2},\tfrac{1}{2},1\,||\,(\sigma_1\times\sigma_2)_2\,||\,\tfrac{1}{2},\tfrac{1}{2},1\rangle. \end{aligned}$$

The matrix element $\langle 2\,||\,Y_2\,||\,0\rangle$ can be evaluated using the coupling scheme for spherical harmonics.

$$\langle 2\,||\,Y_2\,||\,0\rangle = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\frac{[0][2]}{\sqrt{4\pi}[2]} = \sqrt{\frac{1}{4\pi}}.$$

The evaluation of the other matrix element involves the LS-jj coupling coefficient.

$$\begin{aligned} &\langle \tfrac{1}{2}\,\tfrac{1}{2}\,1\,||\,(\sigma_1\times\sigma_2)_2\,||\,\tfrac{1}{2}\,\tfrac{1}{2}\,1\rangle \\ &\quad = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}\langle \tfrac{1}{2}\,||\,\sigma_1\,||\,\tfrac{1}{2}\rangle\,\langle \tfrac{1}{2}\,||\,\sigma_2\,||\,\tfrac{1}{2}\rangle. \end{aligned}$$

The LS-jj coupling coefficient can be written in terms of the 9-j symbol.

$$\begin{aligned} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} &= [1][2][\tfrac{1}{2}]^2\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix} \\ &= \sqrt{15}\times 2\times\frac{1}{9}, \end{aligned}$$

substituting the value $\frac{1}{9}$ for the 9-j symbol occurring in the above equation. Each matrix element of the Pauli spin operator yields a value $\sqrt{3}$.

$$\langle \tfrac{1}{2}\,||\,\sigma_1\,||\,\tfrac{1}{2}\rangle = \langle \tfrac{1}{2}\,||\,\sigma_2\,||\,\tfrac{1}{2}\rangle = \sqrt{3}.$$

Thus

$$\langle \tfrac{1}{2}\,\tfrac{1}{2}\,1\,||\,(\sigma_1\times\sigma_2)_2\,||\,\tfrac{1}{2}\,\tfrac{1}{2}\,1\rangle = \sqrt{\frac{20}{3}}.$$

Substituting the values of the matrix elements obtained, we finally get

$$Q_1 = \sqrt{\frac{24\pi}{5}} \cdot \sqrt{\frac{1}{4\pi}} \cdot \sqrt{\frac{20}{3}} = \sqrt{8}.$$

The other matrix element Q_2 can be evaluated by a similar procedure.

$$\begin{aligned} Q_2 &= \sqrt{\frac{24\pi}{5}} \langle 2\,1\,1\,m \,|\, Y_2 \cdot (\sigma_1 \times \sigma_2)_2 \,|\, 2\,1\,1\,m \rangle \\ &= \sqrt{\frac{24\pi}{5}}\, U(2121, 12)\, \langle 2 \,||\, Y_2 \,||\, 2 \rangle \\ &\quad \times \langle \tfrac{1}{2}\,\tfrac{1}{2}\,1 \,||\, (\sigma_1 \times \sigma_2)_2 \,||\, \tfrac{1}{2}\,\tfrac{1}{2}\,1 \rangle. \end{aligned}$$

The reduced matrix element $\langle 2 \,||\, Y_2 \,||\, 2 \rangle$ is given by

$$\begin{aligned} \langle 2 \,||\, Y_2 \,||\, 2 \rangle &= \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \frac{[2][2]}{\sqrt{4\pi}[2]} \\ &= -\sqrt{\frac{5}{14\pi}}. \end{aligned}$$

The numerical value of the Racah coefficient $W(2121,\ 12)$ is obtained from the tables.

$$W(2\,1\,2\,1\,,\,1\,2) = \frac{1}{10}\sqrt{\frac{7}{3}}.$$

Substituting the numerical values, we finally obtain

$$Q_2 = -2.$$

PARTIAL WAVES AND THE GRADIENT FORMULA

9.1. Partial Wave Expansion for a Plane Wave

The plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ can be expanded into partial waves and this expansion is familiarly known as Rayleigh's expansion.

$$\begin{aligned} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} &= e^{ikr\cos\theta} \\ &= \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta) \\ &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l\, j_l(kr)\, Y_l^{m*}(\hat{\boldsymbol{k}})\, Y_l^m(\hat{\boldsymbol{r}}), \end{aligned} \tag{9.1}$$

where $j_l(kr)$ is the spherical Bessel function (vide Appendix E).

To obtain the relation (9.1), we shall seek the solution of the free particle wave equation

$$(\nabla^2 + k^2)\,\psi(\boldsymbol{r}) = 0, \tag{9.2}$$

in the Cartesian coordinate system as well as in the spherical coordinates. In the Cartesian coordinate system, the solution can be written as

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{9.3}$$

In the spherical polar coordinates, the wave equation can be separated by the usual technique and the solution with azimuthal symmetry can be written as a product of the radial function $j_l(kr)$ and the angular function $P_l(\cos\theta)$. The general solution is a linear combination of $j_l(kr)\ Pl(\cos\theta)$. Thus

$$\psi(\boldsymbol{r}) = \sum_l a_l\, j_l(kr)\, P_l(\cos\theta). \tag{9.4}$$

Combining Eqs. (9.3) and (9.4), we obtain

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_l a_l\, j_l(kr)\, P_l(\cos\theta), \tag{9.5}$$

where a_l is a coefficient which has to be determined. For this, multiply both sides of Eq. (9.5) by $P_L(\cos\theta)$ and integrate over the polar angle. Denoting $\cos\theta$ by x and using the orthogonality relation

$$\int_{-1}^{1} P_l(x)\, P_L(x)\, dx = \frac{2}{2l+1}\,\delta_{Ll}\,, \tag{9.6}$$

we obtain

$$\int_{-1}^{1} e^{ikrx}\, P_L(x)\, dx = \frac{2}{2L+1}\, a_L\, j_L(kr)\,. \tag{9.7}$$

Integrating by parts, the left hand side of Eq. (9.7) becomes

$$\int_{-1}^{1} e^{ikrx}\, P_L(x)\, dx = \frac{1}{ikr}\left[e^{ikrx}\, P_L(x)\right]_{-1}^{+1} - \frac{1}{ikr}\int_{-1}^{+1} e^{ikrx}\, P_L'(x)\, dx. \tag{9.8}$$

The first term on the right hand side of Eq. (9.8) can be evaluated remembering that

$$P_L(1) = 1,\;\; P_L(-1) = (-1)^L,\;\; i^L e^{-iL\pi/2} = 1. \tag{9.9}$$

Thus,

$$\begin{aligned}
\frac{1}{ikr}\left[e^{ikrx}\, P_L(x)\right]_{-1}^{+1} &= \frac{1}{ikr}\left\{e^{ikr} - (-1)^L\, e^{-ikr}\right\} \\
&= \frac{i^L}{ikr}\left\{e^{i(kr-\frac{L\pi}{2})} - e^{-i(kr-\frac{L\pi}{2})}\right\} \\
&= \frac{i^L}{kr}\, 2\,\sin(kr - \frac{L\pi}{2}).
\end{aligned} \tag{9.10}$$

The second term on the right hand side of Eq. (9.8) can be further integrated by parts but they yield contributions of order $\frac{1}{r^2}, \frac{1}{r^3}, \cdots$ and as a consequence they are negligible as $r \to \infty$. Asymptotically, Eq. (9.7) reads

$$2\,(i)^L\, \frac{\sin(kr - \frac{L\pi}{2})}{kr} = \frac{2}{2L+1}\, a_L\, \frac{\sin(kr - \frac{L\pi}{2})}{kr}, \tag{9.11}$$

since

$$j_L(kr) \to \frac{\sin(kr - \frac{L\pi}{2})}{kr}, \tag{9.12}$$

as $r \to \infty$. From Eq. (9.11), we obtain

$$a_L = (i)^L\,(2L+1). \tag{9.13}$$

Substituting the value of this coefficient in Eq. (9.5), we get the desired result (9.1).

9.2. Distorted Waves

Now let us consider how a plane wave gets distorted by a spherically symmetric potential.

The Schrödinger equation for the scattering of a particle of mass m by a spherically symmetric potential $V(r)$ is given by

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right] \psi(\boldsymbol{r}) = E\,\psi(\boldsymbol{r}), \tag{9.14}$$

or, equivalently

$$\left[\nabla^2 + k^2\right] \psi(\boldsymbol{r}) = U(r)\,\psi(\boldsymbol{r}), \tag{9.15}$$

where

$$k^2 = \frac{2mE}{\hbar^2}; \quad U(r) = \frac{2mV(r)}{\hbar^2}. \tag{9.16}$$

In the absence of potential (i.e. $V(r) = 0$ for all values of r), the Schrödinger equation (9.15) reduces to

$$\left[\nabla^2 + k^2\right] \psi(\boldsymbol{r}) = 0\,, \tag{9.17}$$

the solution of which is a plane wave

$$\psi(\boldsymbol{r}) = e^{i\,\boldsymbol{k}\,\cdot\,\boldsymbol{r}}\,. \tag{9.18}$$

Denoting the direction of the incident particle to be along the z-axis and expanding the plane wave in terms of angular momentum eigenfunctions (the Rayleigh expansion), we obtain

$$\psi(\boldsymbol{r}) = e^{ikr\cos\theta} = e^{ikz} = \sum_l (2l+1)\, i^l\, j_l(kr)\, P_l(\cos\theta)\,, \tag{9.19}$$

where $j_l(kr)$ is the spherical Bessel function and $P_l(\cos\theta)$ is the Legendre polynomial. Equation (9.19) is known as the partial wave expansion of the plane wave.

A similar partial wave expansion is possible for the solution of the Schrödinger equation (9.15) with the potential. Introducing the spherical coordinates into Eq. (9.15) and separating the variables, it can be seen that the eigenfunction $\psi(\boldsymbol{r})$ is a linear combination of the products of radial and angular momentum eigenfunctions.

$$\begin{aligned}\psi(\boldsymbol{r}) &= \sum_l \alpha_l\, R_l(r)\, Y_l^0(\hat{\boldsymbol{r}})\,,\\ &= \sum_l \beta_l\, R_l(r)\, P_l(\cos\theta)\,,\end{aligned} \tag{9.20}$$

where α_l and β_l are the coefficients in the expansion. Comparing (9.20) with Eq. (9.19), we find that the coefficient

$$\beta_l = (2l+1)\, i^l\,, \tag{9.21}$$

since the radial function $R_l(r)$ should tend to $j_l(kr)$ in the limit of zero potential. In Eq. (9.20), the z-component of angular momentum is taken to be zero because the incident wave is assumed to be along the z-direction and the spherically symmetric potential will not disturb the angular momentum. The radial function $R_l(r)$ is the solution of the radial equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l(r)}{dr}\right) + \left(k^2 - U(r) - \frac{l(l+1)}{r^2}\right) R_l(r) = 0\,. \tag{9.22}$$

If the potential $V(r)$ is zero everywhere, then the radial equation reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l(r)}{dr}\right) + \left(k^2 - \frac{l(l+1)}{r^2}\right) R_l(r) = 0\,, \tag{9.23}$$

whose solution corresponds to the spherical Bessel function,

$$R_l(r) = j_l(kr), \qquad \text{(Potential } V(r) = 0 \text{ everywhere)} \tag{9.24}$$

which is regular at the origin.

Thus we see that the effect of the spherically symmetric potential is only the modification of the radial function. If the potential is not spherically symmetric, it is not possible to separate the equation in the spherical coordinates.

9.3. The Gradient Formula

There are many physical situations in which one needs the effect of the gradient operator ∇ operating on a system described by $\phi(r)Y_L^M(\hat{\boldsymbol{r}})$ where $\phi(r)$ is a radial function and $Y_L^M(\hat{\boldsymbol{r}})$, the spherical harmonic denoting the angular momentum eigenfunction. The multipole fields and the transitions induced by the momentum operator are specific instances which require the use of the gradient formula. So, in this section, we shall derive the following well known gradient formula[1]:

$$\nabla\phi(r)Y_L^M(\hat{\boldsymbol{r}}) = -\sqrt{\frac{L+1}{2L+1}}\, D_-\phi(r)\, \mathbf{Y}_{L,L+1}^M(\hat{\boldsymbol{r}}) + \sqrt{\frac{L}{2L+1}}\, D_+\phi(r)\, \mathbf{Y}_{L,L-1}^M(\hat{\boldsymbol{r}}), \tag{9.25}$$

where

$$D_- = \frac{d}{dr} - \frac{L}{r}, \qquad D_+ = \frac{d}{dr} + \frac{L+1}{r}, \tag{9.26}$$

and $\mathbf{Y}_{L,l}^M(\hat{\boldsymbol{r}})$ is the vector spherical harmonic (Blatt and Weisskopf, 1952; Rose, 1957) defined by

$$\mathbf{Y}_{L,l}^M(\hat{\boldsymbol{r}}) = \sum_m \begin{bmatrix} l & 1 & L \\ m & \mu & M \end{bmatrix} Y_l^m(\hat{\boldsymbol{r}})\,\hat{\epsilon}_1^\mu . \tag{9.27}$$

In some cases, one may require the result of operation of one of the spherical components ∇_1^μ of the gradient operator and we give below this particular case also.

$$\nabla_1^\mu\phi(r)Y_L^M(\hat{\boldsymbol{r}}) = \sqrt{\frac{L+1}{2L+3}} \begin{bmatrix} L & 1 & L+1 \\ M & \mu & M+\mu \end{bmatrix} Y_{L+1}^{M+\mu}(\hat{\boldsymbol{r}})D_-\phi(r) - \sqrt{\frac{L}{2L-1}} \begin{bmatrix} L & 1 & L-1 \\ M & \mu & M+\mu \end{bmatrix} Y_{L-1}^{M+\mu}(\hat{\boldsymbol{r}})D_+\phi(r). \tag{9.28}$$

Equation (9.28) can be thrown into a more general and symmetric form (Devanathan and Girija, 1985) by defining an unit operator ∇_0^0 and allowing the quantum number n to take either the value 0 or 1.

$$\nabla_n^\mu\phi(r)Y_L^M(\hat{\boldsymbol{r}}) = \sum_l \frac{[L]}{[l]} \begin{bmatrix} L & n & l \\ M & \mu & M+\mu \end{bmatrix} \begin{bmatrix} L & n & l \\ 0 & 0 & 0 \end{bmatrix} Y_L^{M+\mu}(\hat{\boldsymbol{r}}) \times \{\delta_{l\,L+1}(D_-\phi(r)) + \delta_{l\,L-1}(D_+\phi(r)) + \delta_{l\,L}\,\phi(r)\}, \tag{9.29}$$

with the notation $[L] = (2L+1)^{\frac{1}{2}}$. For $n = 1$, 1 can take only two values $l = L+1$ and $l = L-1$ because of the parity C.G. Coefficient. Substituting

[1]For several applications of the gradient formula in Nuclear Physics, the reader may refer to Eisenberg and Greiner (1976).

the algebraic value (Eq. (9.49) of the parity C.G. Coefficient in Eq. (9.29), we retrieve Eq. (9.28). On the other hand, for $n = 0{,}1 \equiv L$ and consequently Eq. (9.29) reduces to

$$\nabla_0^0 \left(\phi(r) Y_l^M(\hat{\boldsymbol{r}})\right) = \phi(r)\, Y_L^M(\hat{\boldsymbol{r}})\,. \tag{9.30}$$

9.4. Derivation of the Gradient Formula

To derive the gradient formula (9.25), we require a convenient form for ∇. The direct representation of ∇ in spherical basis,

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\mathbf{e}}_\phi}{r \sin\theta}\frac{\partial}{\partial \phi}, \tag{9.31}$$

is not very convenient but the expansion of the triple vector product,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}), \tag{9.32}$$

suggests a more convenient form

$$\begin{aligned} \boldsymbol{\nabla} &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}.\boldsymbol{\nabla}) - \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{\nabla}) \\ &= \hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \frac{i}{r}(\hat{\boldsymbol{r}} \times \boldsymbol{L}), \end{aligned} \tag{9.33}$$

where $\hat{\boldsymbol{r}} \equiv \hat{\mathbf{e}}_r$ is the unit radius vector and $\boldsymbol{L}$, the orbital angular momentum operator. On operation of ∇ on the wave function, we obtain a decomposition into two parts, one radial and the other tangential.

$$\boldsymbol{\nabla}\, \phi(r)\, Y_L^M(\hat{\boldsymbol{r}}) = \mathbf{A} + \mathbf{B}, \tag{9.34}$$

with

$$\mathbf{A} = \hat{\boldsymbol{r}}\, Y_L^M(\hat{\boldsymbol{r}})\, \frac{d\phi}{dr}, \tag{9.35}$$

and

$$\mathbf{B} = -i\,\frac{\phi}{r}\,(\hat{\boldsymbol{r}} \times \boldsymbol{L})\, Y_L^M(\hat{\boldsymbol{r}}). \tag{9.36}$$

The evaluation of the radial part ($\mathbf{A}$) is simple whereas the evaluation of the tangential part ($\mathbf{B}$) is a bit complicated.

To evaluate $\mathbf{A}$, we need to express $\hat{\boldsymbol{r}}$ in terms of the spherical harmonic of order 1,

$$\hat{\boldsymbol{r}} = \sqrt{\frac{4\pi}{3}} \sum_\mu (-1)^\mu Y_1^\mu(\hat{\boldsymbol{r}}) \hat{\epsilon}_1^{-\mu}, \tag{9.37}$$

and then use the coupling rule for spherical harmonics to obtain

$$\mathbf{A} = \frac{d\phi}{dr}\sqrt{\frac{4\pi}{3}}\sum_{l,\mu}(-1)^{\mu}\begin{bmatrix} L & 1 & l \\ M & \mu & m \end{bmatrix}\begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix} \times \frac{[L][1]}{\sqrt{4\pi}[l]} Y_l^m(\hat{\boldsymbol{r}})\,\hat{\boldsymbol{\epsilon}}_1^{-\mu}. \tag{9.38}$$

The notation $[J] = \sqrt{2J+1}$ is used in the above equation. Using the definition of vector spherical harmonics (Eq. (9.27)), we obtain after simplification

$$\mathbf{A} = -\frac{d\phi}{dr}\sum_{l}\begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix}\mathbf{Y}_{Ll}^{M}(\hat{\boldsymbol{r}}). \tag{9.39}$$

To evaluate B, we need to express the vector product $\hat{\boldsymbol{r}} \times \boldsymbol{L}$ in terms of the spherical tensor $\mathbf{T}_1$ of rank 1 obtained by taking the tensor product of $\boldsymbol{r}$ and $\boldsymbol{L}$.

It is to be emphasized here that the vector obtained by taking the cross product of any two vectors differs by a factor from the spherical tensor of rank 1 constructed by taking the tensor product of these two vectors as shown in Eq. (3.21).

$$\hat{\boldsymbol{r}} \times \boldsymbol{L} = -i\sqrt{2}\,\mathbf{T}_1, \tag{9.40}$$

where

$$\mathbf{T}_1 = \sum_{\lambda}(-1)^{\lambda}\, T_1^{\lambda}\,\hat{\boldsymbol{\epsilon}}_1^{-\lambda}, \tag{9.41}$$

with

$$\begin{aligned} T_1^{\lambda} &= \sqrt{\frac{4\pi}{3}}\,(Y_1(\hat{\boldsymbol{r}}) \times L_1)_1^{\lambda} \\ &= \sqrt{\frac{4\pi}{3}}\sum_{\mu}\begin{bmatrix} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{bmatrix} Y_1^{\mu}(\hat{\boldsymbol{r}}) L_1^{\nu}. \end{aligned} \tag{9.42}$$

The operation of L_1^{ν} on $Y_L^M(\hat{\boldsymbol{r}})$ yields

$$L_1^{\nu} Y_L^M(\hat{\boldsymbol{r}}) = \sqrt{L(L+1)}\begin{bmatrix} L & 1 & L \\ M & \nu & M+\nu \end{bmatrix} Y_L^{M+\nu}(\hat{\boldsymbol{r}}). \tag{9.43}$$

Next we couple the two spherical harmonics $Y_L^{M+\nu}(\hat{\boldsymbol{r}})$ and $Y_1^{\mu}(\hat{\boldsymbol{r}})$ to obtain

$$\mathbf{B} = -\sqrt{2L(L+1)}\,\frac{\phi}{r}\sum_{\lambda,\mu,l}(-1)^{\lambda}\left[\begin{array}{ccc} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{array}\right]\left[\begin{array}{ccc} L & 1 & L \\ M & \nu & M+\nu \end{array}\right] \times\left[\begin{array}{ccc} L & 1 & l \\ M+\nu & \mu & m \end{array}\right]\left[\begin{array}{ccc} L & 1 & l \\ 0 & 0 & 0 \end{array}\right]\frac{[L]}{[l]}Y_l^m(\hat{\boldsymbol{r}})\,\hat{\boldsymbol{\epsilon}}_1^{-\lambda}. \qquad (9.44)$$

The summation over the projections μ and λ can now be performed in succession.

$$\sum_{\mu}(-1)^{\lambda}\left[\begin{array}{ccc} L & 1 & l \\ M+\nu & \mu & m \end{array}\right]\left[\begin{array}{ccc} L & 1 & L \\ M & \nu & M+\nu \end{array}\right]\left[\begin{array}{ccc} 1 & 1 & 1 \\ \mu & \nu & \lambda \end{array}\right] = \frac{[l]}{[L]}U(l1L1,L1)\left[\begin{array}{ccc} l & 1 & L \\ -m & \lambda & -M \end{array}\right]. \qquad (9.45)$$

$$\sum_{\lambda}\left[\begin{array}{ccc} l & 1 & L \\ -m & \lambda & -M \end{array}\right]Y_l^m(\hat{\boldsymbol{r}})\,\hat{\boldsymbol{\epsilon}}_1^{-\lambda} = (-1)^{l+1-L}\,\mathbf{Y}_{Ll}^{M}(\hat{\boldsymbol{r}}). \qquad (9.46)$$

Note that $l + 1 - L$ should be even due to the parity C.G. Coefficient occurring in Eq. (9.44). Expressing the U coefficient in terms of the Racah coefficient, we obtain

$$\mathbf{B} = -\sqrt{2L(L+1)}\,[L][1]\frac{\phi}{r}\sum_{l}W(l1L1,L1)\left[\begin{array}{ccc} L & 1 & l \\ 0 & 0 & 0 \end{array}\right]\mathbf{Y}_{Ll}^{M}(\hat{\boldsymbol{r}}). \qquad (9.47)$$

In the summation over l that occurs in Eqs. (9.39) and (9.47), l can take only two values $|L-1|$ and $L+1$ due to the occurrence of the parity C.G. coefficient. Algebraic expressions are available for both the Racah coefficients and the parity C.G.coefficients and they are given below.

$$\begin{aligned} W(l1L1,L1) &= -\left\{\frac{L}{6(L+1)(2L+1)}\right\}^{\frac{1}{2}}, \quad && l = L+1, \\ &= \left\{\frac{L+1}{6L(2L+1)}\right\}^{\frac{1}{2}}, && l = L-1, \end{aligned} \qquad (9.48)$$

$$\begin{aligned} \left[\begin{array}{ccc} L & 1 & l \\ 0 & 0 & 0 \end{array}\right] &= \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}, \quad && l = L+1, \\ &= -\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}, && l = L-1. \end{aligned} \qquad (9.49)$$

Using these algebraic expressions in Eqs. (9.39) and (9.47), we obtain finally

$$\mathbf{A} = \frac{d\phi}{dr}\left\{\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}\mathbf{Y}^{M}_{LL-1}(\hat{\boldsymbol{r}}) - \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}\mathbf{Y}^{M}_{L,L+1}(\hat{\boldsymbol{r}})\right\}, \qquad (9.50)$$

and

$$\mathbf{B} = \frac{\phi}{r}\left\{\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}(L+1)\mathbf{Y}^{M}_{LL-1}(\hat{\boldsymbol{r}}) + \left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}L\,\mathbf{Y}^{M}_{L,L+1}(\hat{\boldsymbol{r}})\right\}. \qquad (9.51)$$

Adding **A** and **B,** we get the gradient formula given by Eq. (9.25).

Equation (9.29) can be obtained from Eq. (9.25) by taking the dot product of both sides of Eq. (9.25) with the unit spherical vector $\hat{\epsilon}_1^{\mu}$. We have

$$\boldsymbol{\nabla}\cdot\hat{\epsilon}_1^{\mu} = \nabla_1^{\mu}, \qquad (9.52)$$

and

$$\begin{aligned}\mathbf{Y}^{M}_{L,l}(\hat{\boldsymbol{r}})\cdot\hat{\epsilon}_1^{\mu} &= (-1)^{\mu}\begin{bmatrix} l & 1 & L \\ M+\mu & -\mu & M\end{bmatrix} Y_l^{M+\mu}(\hat{\boldsymbol{r}}) \\ &= -(-1)^{l+1-L}\frac{[L]}{[l]}\begin{bmatrix} L & 1 & l \\ M & \mu & M+\mu\end{bmatrix} Y_l^{M+\mu}(\hat{\boldsymbol{r}}). \qquad (9.53)\end{aligned}$$

Using equations (9.52) and (9.53), we get the relation (9.28).

9.5. Matrix Elements Involving the ∇ Operator

In this section, let us evaluate the following two single particle matrix elements which involve the gradient operator.

$$\begin{aligned}Q_1 &= \langle u_{n_fl_f}(r), l_fm_f\,|\,f(r)(Y_l(\hat{\boldsymbol{r}})\times\nabla)^{m_\lambda}_{\lambda}\,|\,u_{n_il_i}(r), l_im_i\rangle. \qquad (9.54)\\ Q_2 &= \langle u_{n_fl_f}(r), l_f\tfrac{1}{2}j_fM_f\,|\,f(r)\,\{(Y_l(\hat{\boldsymbol{r}})\times\nabla)_{\lambda}\times\sigma_N\}^{m_\Lambda}_{\Lambda}\,|\,u_{n_il_i}(r), l_i\tfrac{1}{2}j_iM_i\rangle. \qquad (9.55)\end{aligned}$$

The matrix element (9.54) denotes the transition of a particle from the initial state $|u_{n_il_i}(r), l_im_i\rangle$ to the final state $|u_{n_fl_f}(r), l_fm_f\rangle$. The functions $u_{n_il_i}(r)$ and $u_{n_fl_f}(r)$ are the radial functions and the angular functions $|l_im_i\rangle$ and $|l_fm_f\rangle$ are the spherical harmonics $Y_{l_i}^{m_i}(\hat{\boldsymbol{r}})$ and $Y_{l_f}^{m_f}(\hat{\boldsymbol{r}})$.

To evaluate the matrix element (9.54), first we have to expand the tensor product $(Y_l(\hat{\boldsymbol{r}}) \times \nabla)^{m_\lambda}_\lambda$ and then apply the gradient formula (9.28).

$$Q_1 = \sum_{m \text{ or } \mu} \begin{bmatrix} l & 1 & \lambda \\ m & \mu & m_\lambda \end{bmatrix} \times \langle u_{n_f l_f}(r), l_f m_f \,|\, f(r)\, Y_l^m(\hat{\boldsymbol{r}}) \nabla_1^\mu \,|\, u_{n_i l_i}(r), l_i m_i \rangle. \quad (9.56)$$

$$\nabla_1^\mu u_{n_i l_i}(r) Y_{l_i}^{m_i}(\hat{\boldsymbol{r}}) = \sqrt{\frac{l_i+1}{2l_i+3}} \begin{bmatrix} l_i & 1 & l_i+1 \\ m_i & \mu & m_i+\mu \end{bmatrix} Y_{l_i+1}^{m_i+\mu}(\hat{\boldsymbol{r}}) D_-(l_i) u_{n_i l_i}(r) - \sqrt{\frac{l_i}{2l_i-1}} \begin{bmatrix} l_i & 1 & l_i-1 \\ m_i & \mu & m_i+\mu \end{bmatrix} Y_{l_i-1}^{m_i+\mu}(\hat{\boldsymbol{r}}) D_+(l_i) u_{n_i l_i}(r). \quad (9.57)$$

In Eq. (9.56), the summation can he made either over m or over μ. Substituting (9.57) into (9.56), we have

$$Q_1 = \sum_{m \text{ or } \mu} \begin{bmatrix} l & 1 & \lambda \\ m & \mu & m_\lambda \end{bmatrix} \times \left\{ \sqrt{\frac{l_i+1}{2l_i+3}} \begin{bmatrix} l_i & 1 & l_i+1 \\ m_i & \mu & m_i+\mu \end{bmatrix} \langle l_f m_f \,|\, Y_l^m(\hat{\boldsymbol{r}}) \,|\, l_i+1, m_i+\mu \rangle F_- - \sqrt{\frac{l_i}{2l_i-1}} \begin{bmatrix} l_i & 1 & l_i-1 \\ m_i & \mu & m_i+\mu \end{bmatrix} \langle l_f m_f \,|\, Y_l^m(\hat{\boldsymbol{r}}) \,|\, l_i-1, m_i+\mu \rangle F_+ \right\}, \quad (9.58)$$

where

$$F_- = \int u_{n_f l_f}(r)\, f(r)\, D_-(l_i) u_{n_i l_i}(r) r^2 dr\,, \quad (9.59)$$

and

$$F_+ = \int u_{n_f l_f}(r)\, f(r)\, D_+(l_i) u_{n_i l_i}(r) r^2 dr\,, \quad (9.60)$$

with the notation

$$D_-(l_i) = \frac{d}{dr} - \frac{l_i}{r}\,, \quad (9.61)$$

and

$$D_+(l_i) = \frac{d}{dr} + \frac{l_i+1}{r}\,. \quad (9.62)$$

By the Wigner-Eckart theorem,

$$\langle l_f\, m_f\,|\, Y_l^m(\hat{\boldsymbol{r}})\,|\, l_i+1,\, m_i+\mu\rangle = \begin{bmatrix} l_i+1 & l & l_f \\ m_i+\mu & m & m_f \end{bmatrix} \times \langle l_f\,||\, Y_l(\hat{\boldsymbol{r}})\,||\, l_i+1\rangle\,, \qquad (9.63)$$

and

$$\langle l_f\, m_f\,|\, Y_l^m(\hat{\boldsymbol{r}})\,|\, l_i-1,\, m_i+\mu\rangle = \begin{bmatrix} l_i-1 & l & l_f \\ m_i+\mu & m & m_f \end{bmatrix} \times \langle l_f\,||\, Y_l(\hat{\boldsymbol{r}})\,||\, l_i-1\rangle. \qquad (9.64)$$

Substituting (9.63) and (9.64) into Eq. (9.58)) we find that each term in (9.58) consists of three C.G. coefficients and the summation over μ can be performed as indicated below. Considering only the factors that depend on the projection quantum numbers, we obtain

$$\sum_\mu \begin{bmatrix} l_i & 1 & l_i+1 \\ m_i & \mu & m_i+\mu \end{bmatrix} \begin{bmatrix} l_i+1 & l & l_f \\ m_i+\mu & m & m_f \end{bmatrix} \begin{bmatrix} l & 1 & \lambda \\ m & \mu & m_\lambda \end{bmatrix} = (-1)^{l+1-\lambda}\, U(l_i 1 l_f l, l_i+1\lambda) \begin{bmatrix} l_i & \lambda & l_f \\ m_i & m_\lambda & m_f \end{bmatrix}. \qquad (9.65)$$

Similarly,

$$\sum_\mu \begin{bmatrix} l_i & 1 & l_i-1 \\ m_i & \mu & m_i+\mu \end{bmatrix} \begin{bmatrix} l_i-1 & l & l_f \\ m_i+\mu & m & m_f \end{bmatrix} \begin{bmatrix} l & 1 & \lambda \\ m & \mu & m_\lambda \end{bmatrix} = (-1)^{l+1-\lambda}\, U(l_i 1 l_f l, l_i-1\lambda) \begin{bmatrix} l_i & \lambda & l_f \\ m_i & m_\lambda & m_f \end{bmatrix}. \qquad (9.66)$$

Equations (9.65) and (9.66) are obtained by using the symmetry relation (2.23) for C.G. coefficient and the relation (7.12). Using Eqs. (9.65) and (9.66)) we obtain

$$Q_1 = (-1)^{l+1-\lambda} \begin{bmatrix} l_i & \lambda & l_f \\ m_i & m_\lambda & m_f \end{bmatrix} \left\{ \left(\frac{l_i+1}{2l_i+3}\right)^{\frac{1}{2}} \langle l_f\,||\, Y_l(\hat{\boldsymbol{r}})\,||\, l_i+1\rangle \times U(l_i 1 l_f l, l_i+1\,\lambda)\, F_- - \left(\frac{l_i}{2l_i-1}\right)^{\frac{1}{2}} \langle l_f\,||\, Y_l(\hat{\boldsymbol{r}})\,||\, l_i-1\rangle U(l_i 1 l_f l, l_i-1\,\lambda)\, F_+ \right\}. \qquad (9.67)$$

In the matrix element (9.54), we recognize that the transition operator is a tensor of rank λ and projection m_λ and as a consequence, the Wigner-Eckart theorem can be directly applied to obtain

$$Q_1 = \begin{bmatrix} l_i & \lambda & l_f \\ m_i & m_\lambda & m_f \end{bmatrix} \times \langle u_{n_f l_f}(r), l_f \,||\, f(r)(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \,||\, u_{n_i l_i}(r), l_i \rangle. \quad (9.68)$$

Comparing Eqs. (9.67) and (9.68), we get the relation

$$\langle u_{n_f l_f}(r), l_f \,||\, f(r)(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \,||\, u_{n_i l_i}(r), l_i \rangle = (-1)^{l+1-\lambda} \left\{ \left(\frac{l_i+1}{2l_i+3} \right)^{\frac{1}{2}} \langle l_f \,||\, Y_l(\hat{\boldsymbol{r}}) \,||\, l_i+1 \rangle U(l_i 1 l_f l, l_i+1\lambda) F_- - \left(\frac{l_i}{2l_i-1} \right)^{\frac{1}{2}} \langle l_f \,||\, Y_l(\hat{\boldsymbol{r}}) \,||\, l_i-1 \rangle U(l_i 1 l_f l, l_i-1\lambda) F_+ \right\}, \quad (9.69)$$

where

$$\langle l_f \,||\, Y_l(\hat{\boldsymbol{r}}) \,||\, l_i+1 \rangle = \frac{[l_i+1][l]}{\sqrt{4\pi}[l_f]} \begin{bmatrix} l_i+1 & l & l_f \\ 0 & 0 & 0 \end{bmatrix}, \quad (9.70)$$

and

$$\langle l_f \,||\, Y_l(\hat{\boldsymbol{r}}) \,||\, l_i-1 \rangle = \frac{[l_i-1][l]}{\sqrt{4\pi}[l_f]} \begin{bmatrix} l_i-1 & l & l_f \\ 0 & 0 & 0 \end{bmatrix}. \quad (9.71)$$

Substituting Eqs. (9.70) and (9.71) into Eq. (9.69) and expressing the U-coefficients in terms of the Racah coefficients, we finally obtain

$$\langle u_{n_f l_f}(r), l_f \,||\, f(r)(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \,||\, u_{n_i l_i}(r), l_i \rangle = (-1)^{l+1-\lambda} \frac{[l][\lambda]}{\sqrt{4\pi}[l_f]} \times \left\{ \sqrt{(l_i+1)(2l_i+3)} \begin{bmatrix} l_i+1 & l & l_f \\ 0 & 0 & 0 \end{bmatrix} W(l_i 1 l_f l, l_i+1\lambda) F_- - \sqrt{l_i(2l_i-1)} \begin{bmatrix} l_i-1 & l & l_f \\ 0 & 0 & 0 \end{bmatrix} W(l_i 1 l_f l, l_i-1\lambda) F_+ \right\}. \quad (9.72)$$

The matrix element Q_2 can be evaluated by recognizing the transition operator to be a tensor of rank λ and projection m_λ . The initial and final single particle states are the spin-orbit coupled wave functions. The radial part of the wave function is assumed to depend only on l and not on j as in the case of the oscillator potential. Even if it depends on j as in the case of Wood-Saxon spin-orbit coupled potential, there is very little change in the procedure. Further N can assume only the value 0 or 1. If $N = 0$, then σ_0 is the unit operator which is a zero rank tensor. If $N = 1$, then σ_1 is the Pauli spin operator which is a tensor of rank 1.

Applying the Wigner-Eckaat theorem, we get

$$Q_2 = \begin{bmatrix} j_i & \Lambda & j_f \\ M_i & m_\Lambda & M_f \end{bmatrix} \times \langle u_{n_f l_f}(r), l_f \tfrac{1}{2} j_f \,||\, f(r)\,\{(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \times \sigma_N\}_\Lambda \,||\, u_{n_i l_i}(r), l_i \tfrac{1}{2} j_i \rangle. \tag{9.73}$$

The reduced matrix element occurring in Eq. (9.73) can be separated out into the orbital and spin parts by introducing the LS-jj coupling coefficient discussed in chapter 8. Thus,

$$\begin{aligned} &\langle u_{n_f l_f}(r), l_f \tfrac{1}{2} j_f \,||\, f(r)\,\{(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \times \sigma_N\}_\Lambda \,||\, u_{n_i l_i}(r), l_i \tfrac{1}{2} j_i \rangle \\ &= \begin{bmatrix} l_i & \frac{1}{2} & j_i \\ \lambda & N & \Lambda \\ l_f & \frac{1}{2} & j_f \end{bmatrix} \langle u_{n_f l_f}(r), l_f \,||\, f(r)\,[Y_l(\hat{\boldsymbol{r}}) \times \nabla]_\lambda \,||\, u_{n_i l_i}(r), l_i \rangle \\ &\times \langle \tfrac{1}{2} \,||\, \sigma_N \,||\, \tfrac{1}{2} \rangle. \end{aligned} \tag{9.74}$$

The reduced matrix element

$$\langle u_{n_f l_f}(r), l_f \,||\, f(r)\,[Y_l(\hat{\boldsymbol{r}}) \times \nabla]_\lambda \,||\, u_{n_i l_i}(r), l_i \rangle$$

is given by expression (9.72) whereas

$$\langle \tfrac{1}{2} \,||\, \sigma_N \,||\, \tfrac{1}{2} \rangle = [N]. \tag{9.75}$$

It will be useful to write a computer program to evaluate the reduced matrix element

$$\langle u_{n_f l_f}(r), l_f \tfrac{1}{2} j_f \,||\, f(r)\,\{(Y_l(\hat{\boldsymbol{r}}) \times \nabla)_\lambda \times \sigma_N\}_\Lambda \,||\, u_{n_i l_i}(r), l_i \tfrac{1}{2} j_i \rangle,$$

using Eq. (9.74).

Review Questions

9.1 Expand the plane wave $e^{i\boldsymbol{k} \cdot \boldsymbol{r}}$ into partial waves and deduce Rayleigh's formula.

9.2 Evaluate $(\hat{\boldsymbol{r}} \times \boldsymbol{L}) Y_L^M(\hat{\boldsymbol{r}})$.

9.3 State and prove the gradient formula.

9.4 Evaluate the following matrix elements using the gradient formula:

(a) $\langle u_{n_f l_f}(r), l_f m_f \mid f(r)(Y_l(\hat{\boldsymbol{r}})\times\nabla)^{m_\lambda}_{\lambda} \mid u_{n_i l_i}(r), l_i m_i\rangle.$

(b) $\langle u_{n_f l_f}(r), l_f \tfrac{1}{2} j_f M_f \mid f(r)\,\{(Y_l(\hat{\boldsymbol{r}})\times\nabla)_\lambda \times \sigma_N\}^{m_\Lambda}_{\Lambda} \mid u_{n_i l_i}(r), l_i \tfrac{1}{2} j_i M_i\rangle.$

Problems

9.1 Expand $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ into partial waves and evaluate the integral

$$\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d\Omega_r.$$

Verify your result by direct integration.

9.2 Evaluate the integral $\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\, d\Omega_r$ where $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and hence show that

$$j_0(qr) = \sum_l (2l+1)\, P_l(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)\, j_l(k_1 r)\, j_l(k_2 r). \tag{9.76}$$

9.3 Using the Rayleigh expansion for the plane wave

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{l,m} (i)^l\, j_l(kr)\, Y_l^{m*}(\hat{\boldsymbol{k}})\, Y_l^m(\hat{\boldsymbol{r}}),$$

evaluate the integral I involving the momentum operator P_{op} by applying the gradient formula.

$$I = \int \mathrm{p}_{\mathrm{op}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\Omega_r.$$

The following relations will be useful.

$$D_+ j_1(kr) = k\, j_0(kr). \tag{9.77}$$

$$\boldsymbol{k} = k\sqrt{\frac{4\pi}{3}} \sum_\mu (-1)^\mu\, Y_1^\mu(\hat{\boldsymbol{k}})\, \epsilon_1^{-\mu}. \tag{9.78}$$

Verify your result by direct evaluation.

9.4 Evaluate the integral

$$\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}} Y_L^M(\hat{\boldsymbol{r}})\, d\Omega_r,$$

where $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and obtain the following expression for the spherical Bessel functions $j_L(qr)$ in terms of $j_{l1}(k_1r)$ and $j_{l2}(k_2r)$.

$$\begin{aligned} j_L(qr)\, Y_L^M(\hat{\boldsymbol{q}}) &= 4\pi \sum_{l_1,l_2} (i)^{l_1-l_2-L} (-1)^{l_1+l_2-L} \frac{[l_1][l_2]}{\sqrt{4\pi}[L]} \\ &\times \begin{bmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{bmatrix} j_{l_1}(k_1 r)\, j_{l_2}(k_2 r) \\ &\times \left[Y_{l_1}(\hat{\boldsymbol{k}}_1) \times Y_{l_2}(\hat{\boldsymbol{k}}_2) \right]_L^M . \end{aligned} \tag{9.79}$$

9.5 Using the Rayleigh expansion for the plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ and the gradient formula, show that

$$\mathbf{p}_{\rm op}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

The following useful relations are given[2]:

$$\begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix} = \begin{cases} \sqrt{\frac{L+1}{2L+1}}\, \delta_{l\,L+1} \\ -\sqrt{\frac{L}{2L+1}}\, \delta_{l\,L-1} \end{cases} . \tag{9.80}$$

$$D_- j_l(kr) = -k\, j_{l+1}(kr). \tag{9.81}$$

$$D_+ j_l(kr) = k\, j_{l-1}(kr). \tag{9.82}$$

$$\boldsymbol{k} = k\sqrt{\frac{4\pi}{3}} \sum_{\mu} (-1)^{\mu}\, Y_1^{\mu}(\hat{\boldsymbol{k}})\, \hat{\epsilon}_1^{-\mu}. \tag{9.83}$$

Solutions to Selected Problems

9.1

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{l,m} i^l j_l(kr) Y_l^{m*}(\hat{\boldsymbol{k}}) Y_l^m(\hat{\boldsymbol{r}}).$$

Since

$$\int Y_l^m(\hat{\boldsymbol{r}})\, d\Omega_r = \sqrt{4\pi}\, \delta_{l0}\, \delta_{m,0},$$

[2]The relations (9.81) and (9.82) can be deduced from Eq. (E.10).

and

$$Y_0^0(\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{4\pi}},$$

we obtain

$$\int e^{i\,\boldsymbol{k}\,\cdot\,\boldsymbol{r}}\, d\Omega_r = 4\pi j_0(kr).$$

The same result can be obtained by direct evaluation as follows:

$$\begin{aligned}\int e^{i\,\boldsymbol{k}\,\cdot\,\boldsymbol{r}}\, d\Omega_r &= 2\pi \int_0^{\pi} e^{ikr\cos\theta} \sin\theta d\theta \\ &= 2\pi \int_{-1}^{+1} e^{ikrx} dx \\ &= 4\pi \frac{\sin kr}{kr} \\ &= 4\pi j_0(kr).\end{aligned}$$

9.2 From the previous problem 9.1, we have

$$\int e^{i\,\boldsymbol{q}\cdot\boldsymbol{r}}\, d\Omega_r = 4\pi\, j_0(qr).$$

To obtain the required relation, we need to use the partial wave expansions for exp($i\ \boldsymbol{k}_1 \cdot r$) and exp($-i\ \mathbf{k}_2 \cdot r$) separately and integrate.

$$\begin{aligned}\int e^{i\,\boldsymbol{k}_1\cdot\boldsymbol{r}}\, e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}}\, d\Omega_r &= (4\pi)^2 \sum_{l_1,l_2,m_1,m_2} (i)^{l_1-l_2} j_{l_1}(k_1 r) j_{l_2}(k_2 r) \\ &\quad \times Y_{l_1}^{m_1}(\hat{\boldsymbol{k}}_1) Y_{l_2}^{m_2^*}(\hat{\boldsymbol{k}}_2) \int Y_{l_1}^{m_1^*}(\hat{\boldsymbol{r}}) Y_{l_2}^{m_2}(\hat{\boldsymbol{r}})\, d\Omega_r.\end{aligned}$$

Using the orthonormality of the spherical harmonics

$$\int Y_{l_1}^{m_1^*}(\hat{\boldsymbol{r}}) Y_{l_2}^{m_2}(\hat{\boldsymbol{r}}) d\Omega_r = \delta_{l_1 l_2}\, \delta_{m_1 m_2},$$

and the spherical harmonic addition theorem

$$P_l(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2) = \frac{4\pi}{2l+1} \sum_m Y_l^{m^*}(\hat{\boldsymbol{k}}_1) Y_l^m(\hat{\boldsymbol{k}}_2),$$

we obtain the required result,

$$j_0(qr) = \sum_l (2l+1) P_l(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2)\, j_l(k_1 r)\, j_l(k_2 r).$$

9.3 The momentum operator P_{op} can be written as

$$\mathbf{p}_{\text{op}} = -i\nabla = -i\sum_{\mu}(-1)^{\mu}\nabla^{\mu}\hat{\epsilon}_1^{-\mu}. \tag{9.84}$$

Using the Rayleigh expansion for the plane wave and the above expansion for the momentum operator, we obtain

$$I = 4\pi \sum_{l,m,\mu}(i)^{l-1}Y_l^{m^*}(\hat{\boldsymbol{k}})(-1)^{\mu}\hat{\epsilon}_1^{-\mu}\int \nabla^{\mu}j_l(kr)Y_l^m(\hat{\boldsymbol{r}})\,d\Omega_r. \tag{9.85}$$

Thus the problem reduces to the evaluation of the following integral by the application of the gradient formula.

$$\begin{aligned}
&\int \nabla^{\mu}j_l(kr)Y_l^m(\hat{\boldsymbol{r}})\,d\Omega_r \\
&= \sqrt{\frac{l+1}{2l+3}}\begin{bmatrix} l & 1 & l+1 \\ m & \mu & m+\mu \end{bmatrix}[D_-j_l(kr)]\int Y_{l+1}^{m+\mu}(\hat{\boldsymbol{r}})d\Omega_r \\
&\quad - \sqrt{\frac{l}{2l-1}}\begin{bmatrix} l & 1 & l-1 \\ m & \mu & m+\mu \end{bmatrix}[D_+j_l(kr)]\int Y_{l-1}^{m+\mu}(\hat{\boldsymbol{r}})d\Omega_r.
\end{aligned}$$

The first term on the R.H.S. vanishes because l cannot take negative values. The integral in the second term is given by

$$\int Y_{l-1}^{m+\mu}(\hat{\boldsymbol{r}})\,d\Omega_r = \sqrt{4\pi}\,\delta_{l1}\,\delta_{m\,-\mu}.$$

Hence,

$$\begin{aligned}
&\int \nabla^{\mu}j_l(kr)Y_l^m(\hat{\boldsymbol{r}})\,d\Omega_r \\
&= -\sqrt{4\pi}\begin{bmatrix} l & 1 & 0 \\ m & \mu & 0 \end{bmatrix}[D_+j_l(kr)]\,\delta_{l1}\,\delta_{m\,-\mu} \\
&= (-1)^m\sqrt{\frac{4\pi}{3}}k\,j_0(kr)\,\delta_{l1}\,\delta_{m\,-\mu}.
\end{aligned} \tag{9.86}$$

The last step is obtained by substituting the value of the C.G. coefficient and using the relation $D_{+j1}(kr) = k_{j0}(kr)$.
Substituting (9.86) into (9.85) and recalling that the momentum vector $\boldsymbol{k}$ can be written as (following Eq. (9.78)),

$$\boldsymbol{k} = k\sqrt{\frac{4\pi}{3}}\sum_{\mu}(-1)^{\mu}Y_1^{\mu}(\hat{\boldsymbol{k}})\epsilon_1^{-\mu},$$

we finally obtain

$$I = 4\pi\, \boldsymbol{k}\, j_0(kr). \tag{9.87}$$

The result (9.87) can also be obtained by direct evaluation as shown below:

$$\mathbf{p}_{\rm op}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

Now the integral I becomes

$$\begin{aligned} I &= \boldsymbol{k}\int e^{ikr\cos\theta}\sin\theta\, d\theta\, d\phi \\ &= 2\pi\,\boldsymbol{k}\int_{-1}^{+1} e^{ikrx}\, dx \\ &= 4\pi\,\boldsymbol{k}\,\frac{\sin kr}{kr} = 4\pi\,\boldsymbol{k}\, j_0(kr). \end{aligned}$$

9.5 Using the Rayleigh expansion for the plane wave and the expansion (9.84) for the momentum operator, we obtain

$$\begin{aligned} \mathbf{p}_{\rm op}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} &= -i\sum_{\mu}(-1)^{\mu}\nabla^{\mu}\hat{\epsilon}_1^{-\mu}4\pi\sum_{l,m} i^l j_l(kr) Y_l^{m^*}(\hat{\boldsymbol{k}}) Y_l^m(\hat{\boldsymbol{r}}) \\ &= -4\pi\sum_{l,m,\mu}(i)^{l+1}(-1)^{\mu}\hat{\epsilon}_1^{-\mu} Y_l^{m^*}(\hat{\boldsymbol{k}})\nabla^{\mu} j_l(kr) Y_l^m(\hat{\boldsymbol{r}}). \end{aligned} \tag{9.88}$$

Using the gradient formula, we get

$$\begin{aligned} \nabla^{\mu} j_l(kr) Y_l^m(\hat{\boldsymbol{r}}) &= \sqrt{\frac{l+1}{2l+3}}\begin{bmatrix} l & 1 & l+1 \\ m & \mu & m+\mu \end{bmatrix} Y_{l+1}^{m+\mu}(\hat{\boldsymbol{r}})[D_- j_l(kr)] \\ &\quad - \sqrt{\frac{l}{2l-1}}\begin{bmatrix} l & 1 & l-1 \\ m & \mu & m+\mu \end{bmatrix} Y_{l-1}^{m+\mu}(\hat{\boldsymbol{r}})[D_+ j_l(kr)] \\ &= \sqrt{\frac{l+1}{2l+3}}\begin{bmatrix} l+1 & 1 & l \\ m+\mu & -\mu & m \end{bmatrix}\frac{[l+1]}{[l]} \\ &\quad \times (-1)^{1+\mu} Y_{l+1}^{m+\mu}(\hat{\boldsymbol{r}})\,(-k\, j_{l+1}(kr)) \\ &\quad - \sqrt{\frac{l}{2l-1}}\begin{bmatrix} l-1 & 1 & l \\ m+\mu & -\mu & m \end{bmatrix}\frac{[l-1]}{[l]} \\ &\quad \times (-1)^{1+\mu} Y_{l-1}^{m+\mu}(\hat{\boldsymbol{r}})\,(k\, j_{l-1}(kr)) \\ &= -k\sum_L \begin{bmatrix} L & 1 & l \\ m+\mu & -\mu & m \end{bmatrix}\begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad \times Y_L^{m+\mu}(\hat{\boldsymbol{r}})\frac{[L]}{[l]}(-1)^{\mu} j_L(kr)\left\{\delta_{L\,l+1} - \delta_{L\,l-1}\right\}. \end{aligned} \tag{9.89}$$

In the last step, we have made use of the relations (9.80) - (9.82). Substituting (9.89) into (9.88) replacing $m+\mu$ by m_L and rearranging, we obtain

$$\begin{aligned} \mathbf{p}_{\mathrm{op}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} &= 4\pi k \sum_{m_L,\mu,L} \sqrt{\frac{4\pi}{3}} (i)^L Y_L^{m_L}(\hat{\boldsymbol{r}}) (-1)^{m_L} (-1)^{\mu}\, \hat{\epsilon}_1^{-\mu} j_L(kr) \\ &\times \left\{ \sum_l \begin{bmatrix} L & 1 & l \\ -m_L & \mu & -m \end{bmatrix} \begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix} \right. \\ &\times \left. \frac{[L][1]}{\sqrt{4\pi}[l]} Y_l^{-m}(\hat{\boldsymbol{k}}) \left(\delta_{L\, l+1} + \delta_{L\, l-1}\right) \right\}. \end{aligned} \qquad (9.90)$$

Because of the parity C.G. coefficient, the only two possible values of l are L - 1 and L + 1 and hence the δ functions within the curly brackets are redundant. Summing over l, we obtain

$$\begin{aligned} \sum_l \begin{bmatrix} L & 1 & l \\ -m_L & \mu & -m \end{bmatrix} \begin{bmatrix} L & 1 & l \\ 0 & 0 & 0 \end{bmatrix} \frac{[L][1]}{\sqrt{4\pi}[l]} Y_l^{-m}(\hat{\boldsymbol{k}}) \\ = Y_L^{-m_L}(\hat{\boldsymbol{k}}) Y_1^{\mu}(\hat{\boldsymbol{k}}). \end{aligned} \qquad (9.91)$$

Substituting (9.91) into (9.90), we get

$$\begin{aligned} \mathbf{p}_{\mathrm{op}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} &= \left\{ k\sqrt{\frac{4\pi}{3}} \sum_{\mu} (-1)^{\mu} Y_1^{\mu}(\hat{\boldsymbol{k}}) \hat{\epsilon}_1^{-\mu} \right\} \\ &\times \left\{ 4\pi \sum_{L,m_L} i^L j_L(kr)\, Y_L^{m_L}(\hat{\boldsymbol{r}})\, Y_L^{-m_L}(\hat{\boldsymbol{k}}) (-1)_L^{m} \right\}. \end{aligned} \qquad (9.92)$$

The first curly bracket in (9.92) denotes the vector $\boldsymbol{k}$ and the second curly bracket is the Rayleigh expansion for the plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$. Thus we retrieve the familiar result

$$\mathbf{p}_{\mathrm{op}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

IDENTICAL PARTICLES

10.1. Fermions and Bosons

Particles can be broadly classified into two categories, fermions and bosons. Fermions are particles with half-integral spin quantum numbers and the wave function of a system of identical fermions is antisymmetric with respect to exchange of any two particles. Bosons are particles with integral spin quantum numbers and the wave function of a system of identical bosons are symmetric with respect to exchange of any two of them. Below we shall consider how to construct the wave function of a system of fermions or bosons with appropriate symmetry in the angular momentum coupling scheme (Rose, 1957a; Racah, 1943).

10.2. Two Identical Fermions in j-j Coupling

Consider two fermions in equivalent orbits denoted by the quantum number j. Since their j-values are the same, their magnetic quantum numbers should be different according to the Pauli exclusion principle. The coupled wave function of two identical fermions in equivalent orbits can be written as

$$\psi_{JM}(1,2) = \sum_m \begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} \phi_{jm}(1)\,\phi_{j\,M-m}(2). \tag{10.1}$$

Defining P_{12} as the permutation operator which exchanges the particle indices 1 and 2, we have

$$\begin{aligned} \psi_{JM}(2,1) &= P_{12}\,\psi_{JM}(1,2) \\ &= \sum_m \begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} \phi_{jm}(2)\,\phi_{j\,M-m}(1) \\ &= (-1)^{2j-J} \left\{ \sum_m \begin{bmatrix} j & j & J \\ M-m & m & M \end{bmatrix} \phi_{j\,M-m}(1)\,\phi_{jm}(2) \right\}. \end{aligned} \tag{10.2}$$

Equation (10.2) is obtained using the symmetry property of C.G. coefficients. The quantity within the curly bracket is identically equal to $\psi_{JM}(1,2)$ given in Eq. (10.1) since m is only a dummy summation index. Thus

$$\psi_{JM}(2,1) = (-1)^{2j-J}\,\psi_{JM}(1,2). \tag{10.3}$$

Since $2j$ is odd, J should be even to assure the antisymmetry of the wave function with respect to exchange of particles. Hence in the angular momentum coupling scheme, the antisymmetry of the wave function of two fermions in equivalent orbits is automatically taken into account if the total angular momentum J is restricted to even integers.

10.3. Construction of Three-Fermion Wave Function

Now let us try to extend the foregoing discussion to the construction of three-particle wave function with proper symmetry. This can be done by first coupling the angular momenta of particles 1 and 2 and then adding the resultant angular momentum j_{12} so obtained to the angular momentum of the third particle.

$$\begin{aligned}\psi(j^2(j_{12})j;JM) &= \sum_{m_1,m}\begin{bmatrix} j_{12} & j & J \\ m & m_3 & M\end{bmatrix}\begin{bmatrix} j & j & j_{12} \\ m_1 & m_2 & m\end{bmatrix} \\ &\quad\times\,\phi_{j,m_1}(1)\,\phi_{j,m_2}(2)\,\phi_{j,m_3}(3).\end{aligned} \tag{10.4}$$

The wave function $\psi(j^2(j_{12})j;JM)$ will be antisymmetric with respect to exchange of particles 1 and 2 if j_{12} is even. In general, more than one value of j_{12} is possible and the permitted even integer values of j_{12} can be obtained from the following two conditions.

$$\begin{aligned}0 \le j_{12} &\le 2j-1. \\ |J-j| \le j_{12} &\le J+j.\end{aligned} \tag{10.5}$$

So. a proper three-particle wave function will he a linear sum of wave functions $\psi(j^2(j_{12})j;JM)$ with different j_{12} values. In addition, it is to be ensured that the wave function so constructed is antisymmetric with respect to exchange of particles 2 and 3. If this is done, then the wave function will be automatically antisymmetric with respect to exchange of particles 1 and 3 since the permutation operator P_{13} can be expressed in terms of P_{12} and P_{23}.

$$P_{13} = P_{12}\,P_{23}\,P_{12}. \tag{10.6}$$

Let ψ_{JM} be the properly normalized three-particle wave function which is antisymmetric with respect to exchange of any two particles. Then,

$$\psi_{JM} = \sum_{j_{12}} F_{j_{12}}\, \psi(j^2(j_{12})\, j; JM), \tag{10.7}$$

where the coefficients F_{j12} are known as the coefficients of fractional parentage (c.f.p) with normalization condition

$$\sum_{j_{12}} (F_{j_{12}})^2 = 1. \tag{10.8}$$

We adopt the convention that all the fractional parentage coefficients are real. The customary notation for the three-particle c.f.p. denoted by F_{j12} defined in equation (10.7) is

$$F_{j_{12}} = \langle j^2(j_{12})\, j \,|\}\, j^3\, J\rangle. \tag{10.9}$$

We require that ψ_{JM} should be antisymmetric with respect to exchange of any two particles. Then

$$\psi_{JM}(1,2,3) = -\psi_{JM}(2,1,3) = \psi_{JM}(2,3,1). \tag{10.10}$$

We follow the construction scheme already outlined to obtain the above wave functions.

$$\begin{aligned} \Psi_{JM}(1,2,3) &= \sum_{j_{12}} F_{j_{12}}\, \psi(j^2(j_{12})\, j; J\, M) \\ &= \sum_{j_{12}} F_{j_{12}} \sum_{m,m_1} \begin{bmatrix} j_{12} & j & J \\ m & m_3 & M \end{bmatrix} \begin{bmatrix} j & j & j_{12} \\ m_1 & m_2 & m \end{bmatrix} \\ &\quad \times \phi_{j\,m_1}(1)\, \phi_{j\,m_2}(2)\, \phi_{j\,m_3}(3). \qquad (10.11) \\ \Psi_{JM}(2,3,1) &= \sum_{j_{23}} F_{j_{23}}\, \psi(j^2(j_{23})\, j : J\, M) \\ &= \sum_{j_{23}} F_{j_{23}} \sum_{\mu,\mu_2} \begin{bmatrix} j_{23} & j & J \\ \mu & \mu_1 & M \end{bmatrix} \begin{bmatrix} j & j & j_{23} \\ \mu_2 & \mu_3 & \mu \end{bmatrix} \\ &\quad \times \phi_{j\,\mu_1}(1)\, \phi_{j\,\mu_2}(2)\, \phi_{j\,\mu}(3). \qquad (10.12) \end{aligned}$$

Using the symmetry properties of C.G. coefficients, we can change the order of coupling j_{23} and j to obtain

$$\begin{aligned}\Psi_{JM}(2,3,1) &= \sum_{j_{23}} F_{j_{23}} (-1)^{j_{23}+j-J} \sum_{\mu,\mu_2} \begin{bmatrix} j & j_{23} & J \\ \mu_1 & \mu & M \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j & j_{23} \\ \mu_2 & \mu_3 & \mu \end{bmatrix} \phi_{j\,\mu_1}(1)\,\phi_{j\,\mu_2}(2)\,\phi_{j\,\mu}(3) \\ &= \sum_{j_{23}} F_{j_{23}} (-1)^{j_{23}+j-J} \psi(j\, j^2(j_{23}); JM). \end{aligned} \tag{10.13}$$

Note that j_{23} is even since the wave function $\psi(j^2(j_{23})j; JM)$ is constructed so as to be antisymmetric with respect to exchange of particles 2 and 3. Hence

$$\psi_{JM}(2,3,1) = \sum_{j_{23}} F_{j_{23}} (-1)^{j-J} \psi(j\, j^2(j_{23}); JM). \tag{10.14}$$

According to Eqs. (10.10) (10.11a) and (10.14),

$$\sum_{j_{12}} F_{j_{12}}\, \psi(j^2(j_{12})\, j; JM) = \sum_{j_{23}} F_{j_{23}} (-1)^{j-J}\, \psi(j\, j^2(j_{23}); JM). \tag{10.15}$$

Taking the scalar product with $\psi(j^2(j'_{12})\, j; JM)$ on both sides, we obtain

$$\begin{aligned} F_{j'_{12}} &= \sum_{j_{23}} F_{j_{23}}\, (-1)^{j-J}\, \langle \psi(j^2(j'_{12})\, j; JM) | \psi(j\, j^2(j_{23}); JM) \rangle \\ &= \sum_{j_{23}} F_{j_{23}}\, (-1)^{j-J} U(j\, j\, J\, j, j'_{12}\, j_{23}) \end{aligned} \tag{10.16}$$

The above result is obtained simply because the two coupled wave functions $\psi(j^2(j'_{12})j; JM)$ and $\psi(j\, j^2(j_{23}); JM)$ are obtained following the two different coupling schemes outlined in Chapter 7 and hence the scalar product of these two wave functions is just the recoupling coefficient $U(j\, j\, J\, j, j'_{12}\, j_{23})$. Replacing j'_{12} by j_{12}, Eq. (10.16) can be written as

$$\sum_{j_{23}} F_{j_{23}} \left\{ (-1)^{j-J} U(j\, j\, J\, j, j_{12}\, j_{23}) - \delta_{j_{12}, j_{23}} \right\} = 0. \tag{10.17}$$

Using the notation

$$A_{j_{12}, j_{23}} = (-1)^{j-J} U(j\, j\, J\, j, j_{12}\, j_{23}), \tag{10.18}$$

we can rewrite Eq. (10.17) as

$$\sum_{j_{23}} F_{j_{23}} \left(A_{j_{12}, j_{23}} - \delta_{j_{12}, j_{23}} \right) = 0. \tag{10.19}$$

Equation (10.19) can be solved to obtain the solution for F, only if

$$\det\left(A_{j_{12},j_{23}} - \delta_{j_{12},j_{23}}\right) = 0. \tag{10.20}$$

The condition (10.20) gives the permissible values of J for which the three particle wave function will have the required antisymmetric property under exchange of particles. Once the permitted J values are determined, the fractional parentage coefficients are calculated using Eq. (10.19).

The fractional parentage coefficients can also be calculated following an alternative procedure outlined below.

Starting from Eq. (10.7), it is possible to rewrite the wave function $\psi(j^2(j_{12})j; JM)$ in terms of $\psi(jj^2(j_{23}); JM)$ using the recoupling coefficient $U(jjJj;j_{12}j_{23})$.

$$\psi(j^2(j_{12})\,j; JM) = \sum_{j_{23}} U(j\,j\,J\,j\,; j_{12}\,j_{23})\,\psi(j\,j^2(j_{23}); JM). \tag{10.21}$$

The function $\psi(j\,j^2(j_{23}); JM)$ will be antisymmetric with respect to exchange of particles 2 and 3 only if j_{23} is even. Using Eq. (10.21) in Eq. (10.7), we get

$$\psi_{JM} = \sum_{j_{12}} F_{j_{12}} \sum_{j_{23}} U(j\,j\,J\,j\,; j_{12}\,j_{23})\,\psi(j\,j^2(j_{23}); JM). \tag{10.22}$$

The antisymmetric three fermion wave function ψ_{JM} exists only if both j_{12} and j_{23} are even integers. If j_{23} is odd, then the coefficient of $\psi(jj^2(j_{23}); JM)$ should vanish.

$$\sum_{j_{12}} F_{j_{12}}\, U(j\,j\,J\,j\,; j_{12}\,j_{23}) = 0, \qquad (j_{23} \text{ odd integer}). \tag{10.23}$$

Equation (10.23) along with Eq. (10.8) can also be used to determine F_{j12}.

10.4. Calculation of Fractional Parentage Coefficients

We shall calculate, for the sake of illustration, the fractional parentage coefficients for a simple configuration $(j)^3$ of equivalent particles. In a given state j, the maximum number of particles that can be accommodated is $2j + 1$. The lowest value of j which allows the configuration $(j)^3$ is $j = \frac{3}{2}$. Below we compute the fractional parentage coefficients for the three-particle configuration with $j = \frac{3}{2}$.

The allowed values of j_{12} and j_{23} are 0 and 2. Among the various possible values of J, only $J = \frac{3}{2}$ satisfies the condition (10.20). This can be easily checked with the following values of the Racah coefficients.

$$W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},00) = W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},02) = W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},20) = -\frac{1}{4}.$$
$$W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},22) = \frac{3}{20}. \qquad (10.24)$$

With the help of these coefficients, the quantities $A_{j12,j23}$ are evaluated.

$$A_{00} = -\frac{1}{4}, \quad A_{02} = A_{20} = -\frac{\sqrt{5}}{4}, \quad A_{22} = \frac{3}{4}. \qquad (10.25)$$

They satisfy the determinantal equation (10.20).

$$\begin{vmatrix} A_{00}-1 & A_{02} \\ A_{20} & A_{22}-1 \end{vmatrix} = 0. \qquad (10.26)$$

The c.f.p. can be calculated from the matrix equation

$$\begin{bmatrix} A_{00}-1 & A_{02} \\ A_{20} & A_{22}-1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_2 \end{bmatrix} = 0, \qquad (10.27)$$

and the normalization condition

$$F_0^2 + F_2^2 = 1. \qquad (10.28)$$

The corresponding values of F_0 and F_2 are

$$F_0 = \sqrt{\frac{1}{6}}, \quad F_2 = -\sqrt{\frac{5}{6}}. \qquad (10.29)$$

The same result can be obtained in an alternative way by using Eq. (10.23). For $J = \frac{3}{2}$ and $j_{23} = 1$, we have

$$F_0\, U(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},0\,1) + F_2\, U(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},2\,1) = 0. \qquad (10.30)$$

For $J = \frac{3}{2}$ and $j_{23} = 3$, the corresponding equation is

$$F_0\, U(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},0\,3) + F_2\, U(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},2\,3) = 0. \qquad (10.31)$$

The values of the corresponding Racah coefficients are found from the tables.

$$W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},0\,1) = \frac{1}{4}, \quad W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},2\,1) = \frac{1}{20},$$
$$W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},0\,3) = \frac{1}{4}, \quad W(\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2},2\,3) = \frac{1}{20}. \qquad (10.32)$$

Using these values, Eqs. (10.30) and (10.31) become

$$\frac{\sqrt{3}}{4}F_0 + \frac{\sqrt{15}}{20}F_2 = 0, \qquad (10.33)$$

$$\frac{\sqrt{7}}{4}F_0 + \frac{\sqrt{35}}{20}F_2 = 0, \qquad (10.34)$$

Both these equations yield the relation

$$F_2 = -\sqrt{5}F_0. \qquad (10.35)$$

This relation along with the normalization condition (10.28) yields the same value (10.29) obtained earlier.

For values of J other than $\frac{3}{2}$, $j_{12} = 0$ will not be permitted and hence there exists only one term in the summation of Eq. (10.23).

$$F_2\, U(\tfrac{3}{2}\,\tfrac{3}{2}\,J\,\tfrac{3}{2},2\,1) = 0.$$

$$F_2\, U(\tfrac{3}{2}\,\tfrac{3}{2}\,J\,\tfrac{3}{2},2\,3) = 0.$$

If the U-coefficient vanishes, then the equation becomes trivial. On the other hand, if the U-coefficient is non-vanishing, then we get a trivial result that the c.f.p. is zero.

10.5. The Iso-Spin

Instead of considering separately the protons and neutrons, we can consider the nucleons as identical particles if we include the iso-spin quantum numbers in the description of their states. The proton and the neutron are considered as the two states of one and the same particle called nucleon with iso-spin quantum number $\frac{1}{2}$, the proton having the projection $+\frac{1}{2}$ and the neutron having the projection $-\frac{1}{2}$. (Some authors use the opposite convention $+\frac{1}{2}$ for neutrons and $-\frac{1}{2}$ for protons). The iso-spins are compounded in the same way as angular momenta but it is to be remembered that the iso-spin space is an hypothetical space, different from the space in which the angular momenta are coupled. Using the j - j coupling scheme, the two-nucleon wave function in the coupled representation can be written as

$$\psi_{JM,\tau\mu}(1,2) = \sum_{m,\mu_1} \begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \tau \\ \mu_1 & \mu-\mu_1 & \mu \end{bmatrix} \times \phi_{jm,\frac{1}{2}\mu_1}(1)\, \phi_{j\,M-m,\frac{1}{2}\,\mu-\mu_1}(2), \qquad (10.36)$$

where τ is the total iso-spin quantum number and μ the iso-spin projection. Operating P_{12} on the wave function, we get

$$\begin{aligned} P_{12}\,\psi_{JM,\tau\mu}(1,2) &= \psi_{JM,\tau\mu}(2,1) \\ &= (-1)^{2j-J}(-1)^{1-\tau}\psi_{JM,\tau\mu}(1,2). \end{aligned} \qquad (10.37)$$

Since $2j$ is odd, we get the condition that the two-nucleon wave function is antisymmetric if $J + \tau$ is an odd integer.

10.6. The Bosons

Bosons are particles with integral spins and their total wave function is symmetric with respect to exchange of particles. The analysis made in section (10.2) can be repeated with small modifications. The two-boson spin wave function can be written as

$$\psi_{JM}(1,2) = \sum_{m} \begin{bmatrix} j & j & J \\ m & M-m & M \end{bmatrix} \phi_{jm}(1)\, \phi_{j\,M-m}(2). \tag{10.38}$$

Here j is an integer. Allowing the exchange operator P_{12} to operate on $\psi_{JM}(1,2)$, we get

$$\begin{aligned} P_{12}\, \psi_{JM}(1,2) &= \psi_{JM}(2,1) \\ &= (-1)^{2j-J} \psi_{JM}(1,2). \end{aligned} \tag{10.39}$$

For bosons, $2j$ is an even integer. Since the bosons are symmetric under exchange, J should be even if their spatial wave function is symmetric, and odd, if their spatial wave function is antisymmetric. Since the spatial wave function of the two particles depend upon their relative orbital angular momentum L, it follows that $L + J$ should be even for two bosons. Thus in the case of deuterons with spin 1, we have two possibilities: (i) $J = 0$ or 2 and L is even (ortho-deuterium molecule) (ii) $J = 1$ and L is odd (para-deuterium molecule).

10.7. The m-scheme

Here we follow a simple scheme by which the total angular momentum of a system of particles can be determined by enumerating the possible m-states. In section (10.3), we have seen that only certain values are permitted for the total angular momentum if a certain symmetry is assumed under exchange of particles. The permitted angular momentum of a system of particles can be obtained by enumerating first the m-states in the uncoupled representation and obtaining thereby the m-states in the coupled representation. It is to be emphasized that the total number of states should remain unaltered when we go from one representation to another.

Let us consider two fermions in equivalent states with $j = \frac{3}{2}$. Since their $j-$valuesare the same, they should differ at least in their m-values. In Table 10.1, the possible m-states are enumerated.

TABLE 10.1. Allowed magnetic quantum numbers for a system of two fermions in equivalent state $j = \frac{3}{2}$.

m_1	m_2	$m = m_1 + m_2$
+3/2	+1/2	+2
	−1/2	+1
	−3/2	0
+1/2	−1/2	0
	−3/2	−1
−1/2	−3/2	−2

TABLE 10.2. Allowed magnetic quantum numbers for a system of three fermions in $j = \frac{3}{2}$ state.

m_1	m_2	m_3	$m = m_1 + m_2 + m_3$
+3/2	+1/2	−1/2	+3/2
		−3/2	+1/2
	−1/2	−3/2	−1/2
+1/2	−1/2	−3/2	−3/2

The total number of states is six, of which $J = 2$ will account for 5 states and the remaining one state will correspond to $J = 0$. Thus we find that the permitted values of the total angular momentum of two fermions in $j = \frac{3}{2}$ state are 2 and 0. This result is the same as was obtained earlier but here it is obtained by simple enumeration.

This method can be extended to find the total angular momentum of three or more fermions. Now let us illustrate the method for the case of three fermions in $j = \frac{3}{2}$ state and present the results in Table 10.2. Thus we find that the total angular momentum $J = \frac{3}{2}$ will account for all the four possible m−states in the last column.

This method is equally applicable to bosons. Let us consider in Table 10.3 three bosons, each with angular momentum $j = 1$. All the three single particle states can have the same m value unlike the case of fermions. There are in total 10 states, of which seven of them belong to $J = 3$ and the remaining three belong to $J = 1$. Thus the permitted values of the

TABLE 10.3. Allowed magnetic quantum numbers for a system of three bosons in $j = 1$ state.

m_1	m_2	m_3	$m = m_1 + m_2 + m_3$
+1	+1	+1	+3
		0	+2
		−1	+1
	0	0	+1
		−1	0
	−1	−1	−1
0	0	0	0
		−1	−1
	−1	−1	−2
−1	−1	−1	−3

total angular momentum of a system of three bosons each of $j = 1$, are $J = 3$ and 1.

Thus the m-scheme is applicable to bosons as well as fermions and is useful to obtain the permitted values of total angular momentum by simple enumeration of the m-states.

Review Questions

10.1 Three fermions are in equivalent orbitals, defined by the quantum number j. Construct their antisymmetric wave functions and find the permitted J values.

10.2 What is meant by coefficient of fractional parentage (cfp). Find the cfp values for three particles in equivalent orbitals $j = \frac{3}{2}$.

10.3 How do you explain the existence of two types of deuterium molecule on the basis of the symmetry of the wave functions?

10.4 Using the $m-$scheme,find the permitted values of the total angular momentum for three identical fermions in the equivalent orbital state $j = \frac{3}{2}$.

10.5 Find the permitted values of total angular momentum for three identical bosons with $j = 1$, using the m-scheme.

Problems

10.1 Two identical fermions are in $j = \frac{5}{2}$ state. Using the $m-$scheme, show that the permitted values of the total angular momentum of two-fermion system are $J = 0,2,4$.

10.2 Three identical fermions are in $j = \frac{5}{2}$ state. Using the m – scheme, find the permitted values for the total angular momentum of the three-fermion system.

10.3 Using the m–scheme, find the permitted values of angular momenta for (a) 2-phonon system and (b) 3-phonon system, if each phonon carries an angular momentum of $j = 2$.

Solutions to Selected Problems

10.2 A table similar to Table 10.2 can be constructed for three identical fermions in $j = \frac{5}{2}$ orbitals. From the table, it can be inferred that the permitted values of total angular momentum for the three fermion system are $\frac{9}{2}, \frac{5}{2}, \frac{3}{2}$.

10.3 Constructing tables similar to Table 10.3, we can obtain the permitted values J of angular momenta. For two-phonon state, we obtain $J =$ 4,2,0 and for three-phonon state, $J =$ 6,4,3,2,0.

CHAPTER 11

DENSITY MATRIX AND STATISTICAL TENSORS

11.1. Concept of the Density Matrix

The concept of density matrix is introduced in the study of the behaviour of a system consisting of an aggregate of particles. When we are considering the emission of light by atoms or the emission of γ-rays by nuclei, we are not experimentally investigating the emission by a single particle but by a group of particles. Similarly, when we are investigating the scattering process, we are considering the scattering of a beam of particles on a target having many scattering centres. Thus we are lead inevitably to deal with aggregates of particles and statistical distribution of those particles in different states. Although the concept of density matrix is broad-based, we are more interested in its application to the study of the statistical distribution of particles with spin j into the various magnetic substates denoted by the quantum number m $(m = -j, -j+1, \ldots, +j)$.

The discussion that follows is equally applicable to the consideration of a single particle with probability distribution of its occupation in a complete set of orthonormal states or to an aggregate of particles with a statistical distribution in various states.

An arbitrary wave function Ψ can be expanded in terms of a complete set of orthonormal eigenfunctions ψ_i .

$$\Psi = \sum_i a_i \psi_i. \tag{11.1}$$

Then

$$|\Psi|^2 = \sum_{i,j} a_i \psi_i \, a_j^* \psi_j^* = \sum_{i,j} a_i a_j^* \psi_i \psi_j^*. \tag{11.2}$$

From the expansion coefficients, we can form a matrix ρ known as the density matrix.

$$\rho_{ij} = a_i a_j^*. \tag{11.3}$$

The properties of the density matrix can be deduced. It is a Hermitian matrix and, when diagonalized, the diagonal elements ρ_{ii} gives the probability of finding the system in the eigenstate ψ_i. If the wave function Ψ is normalized, then

$$\begin{aligned}\int \Psi^* \Psi \, d\tau &= \sum_{i,j} a_i^* a_j \int \psi_i^* \psi_j \, d\tau \\ &= \sum_{i,j} \rho_{ji} \delta_{ij} = \sum_i \rho_{ii} = 1. \end{aligned} \tag{11.4}$$

We have obtained the result that

$$\mathrm{Tr}\, \rho = 1. \tag{11.5}$$

Since the trace of a matrix is invariant under unitary transformation, the density matrix is amenable to easy physical interpretation when thrown into a diagonal form by unitary transformation. The diagonal elements correspond to the statistical weights or probability of finding the system Ψ in the various substates ψ_i and the total probability (the trace of the density matrix) adds up to one.

If we make a measurement of some dynamical variable F of the system described by the wave function Ψ, then the expectation value is given by

$$\begin{aligned}\langle F \rangle &= \int \Psi^* F \Psi d\tau \\ &= \sum_{i,j} a_i^* a_j \int \psi_i^* F \psi_j \, d\tau \\ &= \sum_{i,j} F_{ij} \rho_{ji} = \sum_i (F\rho)_{ii} \\ &= \mathrm{Tr}\,(F\rho) = \mathrm{Tr}\,(\rho F). \end{aligned} \tag{11.6}$$

The concept of density matrix will become clearer when we consider its application to a beam of particles with spin $\frac{1}{2}$. The wave function Ψ in this case will consist only of two terms

$$\Psi = a_1 \psi_1 + a_2 \psi_2, \tag{11.7}$$

where ψ_1 and ψ_2 describe the two states of polarization $+\frac{1}{2}$ and $-\frac{1}{2}$.

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{11.8}$$

Thus the wave function of the beam is

$$\Psi = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{11.9}$$

The density matrix

$$\rho = \Psi\,\Psi^\dagger = \begin{bmatrix} a_1 a_1^* & a_1 a_2^* \\ a_2 a_1^* & a_2 a_2^* \end{bmatrix}, \tag{11.10}$$

completely characterizes the beam since the diagonal elements denote the intensities of the polarization states whereas the off-diagonal elements furnish the relative phase.

The expectation values of the unit matrix and the Pauli matrices are

$$\begin{aligned} I = \langle 1 \rangle &= \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= a_1 a_1^* + a_2 a_2^* ; \end{aligned} \tag{11.11}$$

$$\begin{aligned} P_x = \langle \sigma_x \rangle &= \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= a_1 a_2^* + a_2 a_1^* ; \end{aligned} \tag{11.12}$$

$$\begin{aligned} P_y = \langle \sigma_y \rangle &= \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= i(a_1 a_2^* - a_2 a_1^*) ; \end{aligned} \tag{11.13}$$

$$\begin{aligned} P_z = \langle \sigma_z \rangle &= \begin{bmatrix} a_1^* & a_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= a_1 a_1^* - a_2 a_2^* . \end{aligned} \tag{11.14}$$

The quantities P_x, P_y and P_z can be considered as the components of the polarization vector $\boldsymbol{P}$ and the density matrix can be written in terms of $\boldsymbol{P}$.

$$\rho = \frac{1}{2}(I + \boldsymbol{\sigma}.\boldsymbol{P}). \tag{11.15}$$

Writing explicitly,

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + \langle \sigma_z \rangle & \langle \sigma_x \rangle - i\langle \sigma_y \rangle \\ \langle \sigma_x \rangle + i\langle \sigma_y \rangle & 1 - \langle \sigma_z \rangle \end{bmatrix}. \tag{11.16}$$

Substituting the values of $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$and $\langle \sigma_z \rangle$from Eqs. (11.12 - 11.14) into Eq. (11.16), we retrieve the result (11.10). The density matrix is diagonal, only when the polarization vector $\boldsymbol{P}$ is along the z-axis.

Having introduced the concept of density matrix and applied it to the simple case of spin$-\frac{1}{2}$ particles, we are now in a position to construct the density matrix for a general case and apply it to study the oriented systems (Blin-Stoyle and Grace, 1957).

11.2. Construction of the Density Matrix

The density matrix is a Hermitian matrix and it is of dimension n x n where n is the number of basic states. Hence the density matrix can be expressed as a linear combination of n^2 independent matrices S_μ (of order n) which may be chosen suitably.

$$\rho = \sum_{\mu=1}^{n^2} C_\mu S_\mu . \tag{11.17}$$

The linear independence of the base matrices is expressed by the orthogonality relation

$$\mathrm{Tr}\,(S_\mu S_\nu) = n\,\delta_{\mu,\nu} . \tag{11.18}$$

The expectation value of S_μ is

$$\langle S_\mu \rangle = \frac{\mathrm{Tr}\,(\rho S_\mu)}{\mathrm{Tr}\,\rho} = \frac{n C_\mu}{\mathrm{Tr}\,\rho} . \tag{11.19}$$

Then,

$$\rho = \frac{\mathrm{Tr}\,\rho}{n} \sum_{\mu=1}^{n^2} \langle S_\mu \rangle S_\mu . \tag{11.20}$$

One of the S_μ can always be chosen to be the unit matrix I so that

$$\rho = \frac{\mathrm{Tr}\,\rho}{n} \left(I + \sum_{\mu=1}^{n^2-1} \langle S_\mu \rangle S_\mu \right) . \tag{11.21}$$

In the case of spin$-\frac{1}{2}$ particles, the density matrix can be written as

$$\rho = \frac{1}{2}(I + \boldsymbol{\sigma}.\boldsymbol{P}) , \tag{11.22}$$

since Tr ρ = 1 and the polarization vector $\boldsymbol{P} = \langle \boldsymbol{\sigma} \rangle$. This is identical with Eq. (11.15) obtained earlier for the spin-$\frac{1}{2}$ system.

Choosing the base matrices S_μ to represent the components of the spherical tensors $T_k^{m_k}$ $(k = 0,1, \ldots , n - 1)$, the density matrix can be written as

$$\rho = \frac{\mathrm{Tr}\,\rho}{n} \sum_{k=0}^{n-1} \sum_{m_k=-k}^{+k} \langle T_k^{m_k} \rangle^* T_k^{m_k} . \tag{11.23}$$

It is easy to see that

$$\sum_{k=0}^{n-1} (2k+1) = n \times n . \tag{11.24}$$

The spherical tensors T_k^μ can be constructed out of angular momentum operators

$$T_1^\mu = c_1 J^\mu, \quad T_2^\mu = c_2 (J \times J)_2^\mu, \quad T_3^\mu = c_3 \left((J \times J)_2 \times J\right)_3^\mu , \tag{11.25}$$

where c_1, c_2 and c_3 are constants that are determined from the normalization condition

$$\mathrm{Tr}\,(T_k^{m_k \dagger}\, T_{k'}^{m_{k'}}) = n\, \delta_{k,k'}\, \delta_{m_k, m'_k} . \tag{11.26}$$

The spherical tensor parameters $\langle T_k^{m_k} \rangle$ (hereafter called $t_k^{m_k}$) are the expectation values of the tensor operators $T_k^{m_k}$.

$$t_k^{m_k} = \langle T_k^{m_k} \rangle = \frac{\mathrm{Tr}\,(\rho\, T_k^{m_k})}{\mathrm{Tr}\,\rho} . \tag{11.27}$$

The matrix element of the tensor operator $T_k^{m_k}$ is given by

$$\begin{aligned} \langle j\, m' | T_k^{m_k} | j\, m \rangle &= \begin{bmatrix} j & k & j \\ m & m_k & m' \end{bmatrix} \langle j \,||T_k||\, j \rangle \\ &= \begin{bmatrix} j & k & j \\ m & m_k & m' \end{bmatrix} [k] , \end{aligned} \tag{11.28}$$

using the notation $[k] = \sqrt{2k+1}$.

11.3. Fano's Statistical Tensors

In a representation in which the density matrix is diagonal, the nuclear orientation is completely defined by a set of parameters $\langle T_k^0 \rangle$ defined in the last section or equivalently through Fano's statistical tensors $G_k(j)$ (Fano and Racah, 1959; Rose, 1957b).

$$\begin{aligned} G_k(j) &= \sum_m (-1)^{j-m} p_m \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix} \\ &= \sum_m p_m \frac{[k]}{[j]} \begin{bmatrix} j & k & j \\ m & 0 & m \end{bmatrix} \\ &= \sum_m \langle jm|\rho|jm\rangle \langle j||T_k||j\rangle \frac{1}{[j]} \begin{bmatrix} j & k & j \\ m & 0 & m \end{bmatrix} \\ & \qquad \text{since } \langle j||T_k||j\rangle = [k], \\ &= \sum_m \langle jm|\rho|jm\rangle \langle j\, m|T_k^0|j\, m\rangle \frac{1}{[j]} \\ &= \frac{1}{[j]} \mathrm{Tr}\,(T_k^0\, \rho) \\ &= \frac{1}{[j]} \langle T_k^0 \rangle. \end{aligned} \tag{11.29}$$

Instead of the statistical weights p_m, it is found more convenient to use $G_k(j)$ in the study of the effect of the initial emitting state on the angular distribution and polarization of the emitted radiation. It is not the weight factors p_m of the initial nuclear state but certain moments of p_m that are of importance in determining the effect of the anisotropy of the initial state on the emitted radiation. Since p_m is normalized to unity i.e.,

$$\sum_{m=-j}^{j} p_m = 1\,, \tag{11.30}$$

it follows that

$$p_m = \frac{1}{2j+1}\,, \tag{11.31}$$

for unoriented system. Substituting the value of p_m given by Eq. (11.31), we obtain the statistical tensor for the unoriented system.

$$G_k(j) = \frac{1}{2j+1} \sum_m (-1)^{j-m} \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix}. \tag{11.32}$$

Multiplying the right hand side of Eq. (11.32) by the C.G. coefficient $\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix}$ which is unity and simplifying, we obtain

$$\begin{aligned}
G_k(j) &= \frac{1}{2j+1}\sum_m (-1)^{j-m}\begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix}\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix} \\
&= \frac{1}{2j+1}\sum_m (-1)^{j-m}\begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix} \\
&\quad \times \left\{(-1)^{j-m}[j]\begin{bmatrix} j & j & 0 \\ m & -m & 0 \end{bmatrix}\right\} \\
&= \frac{1}{\sqrt{2j+1}}\sum_m \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix}\begin{bmatrix} j & j & 0 \\ m & -m & 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2j+1}}\,\delta_{k,0}. \qquad (11.33)
\end{aligned}$$

It is shown that for unoriented system, the statistical tensor with $k = 0$ alone exists. The oriented system is characterized by the existence of higher rank statistical tensors. But $G_0(j)$ is always equal to $\frac{1}{\sqrt{2j+1}}$, even for oriented systems.

$$\begin{aligned}
G_0(j) &= \sum_m (-1)^{j-m} p_m \begin{bmatrix} j & j & 0 \\ m & -m & 0 \end{bmatrix} \\
&= \sum_m (-1)^{j-m} p_m \frac{(-1)^{j-m}}{[j]}\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix} \\
&= \frac{1}{[j]}\sum_m p_m \\
&= \frac{1}{\sqrt{2j+1}}. \qquad (11.34)
\end{aligned}$$

It can be shown that the higher rank statistical tensors depend upon the higher moments of the statistical weights p_m.

$$\begin{aligned}
G_1(j) &= \sum_m (-1)^{j-m} p_m \begin{bmatrix} j & j & 1 \\ m & -m & 0 \end{bmatrix} \\
&= \sum_m (-1)^{j-m} p_m \sqrt{\frac{3}{2j+1}}(-1)^{j-m}\begin{bmatrix} j & 1 & j \\ m & 0 & m \end{bmatrix}, \qquad (11.35)
\end{aligned}$$

using the symmetry property of the C.G. coefficient. Substituting the value of the C.G. coefficient

$$\begin{bmatrix} j & 1 & j \\ m & 0 & m \end{bmatrix} = \frac{m}{\sqrt{j(j+1)}}, \qquad (11.36)$$

we obtain

$$G_1(j) = \frac{[1]}{[j]} \frac{1}{\sqrt{j(j+1)}} \sum_m m\, p_m\,. \tag{11.37}$$

When the orientation of the nucleus is such that the first moment of p_m is non-zero i.e.,

$$\sum_m m\, p_m \neq 0\,,$$

then

$$G_1(j) \neq 0\,,$$

and the nucleus is said to be polarized with the polarization defined by

$$P_N(j) = \sum_m m\, p_m / j\,, \tag{11.38}$$

such that

$$G_1(j) = \left\{ \frac{3j}{(2j+1)(j+1)} \right\}^{\frac{1}{2}} P_N(j)\,. \tag{11.39}$$

In a similar way, $G_2(j)$ can be shown to depend upon the second moment of the statistical weights p_m.

$$\begin{aligned} G_2(j) &= \sum_m (-1)^{j-m} p_m \left[\begin{array}{ccc} j & j & 2 \\ m & -m & 0 \end{array} \right] \\ &= \sum_m p_m \sqrt{\frac{5}{2j+1}} \left[\begin{array}{ccc} j & 2 & j \\ m & 0 & m \end{array} \right], \end{aligned} \tag{11.40}$$

Substituting the value of the C.G. coefficient

$$\left[\begin{array}{ccc} j & 2 & j \\ m & 0 & m \end{array} \right] = \frac{3m^2 - j(j+1)}{[j(j+1)(2j-1)(2j+3)]^{\frac{1}{2}}}\,, \tag{11.41}$$

we obtain

$$\begin{aligned} G_2(j) &= \left\{ \frac{5}{j(j+1)(2j-1)(2j+1)(2j+3)} \right\}^{\frac{1}{2}} \\ &\quad \times \sum_m p_m \left(3m^2 - j(j+1)\right). \end{aligned} \tag{11.42}$$

When $G_2(j) \neq 0$, the nucleus is said to be aligned.

It can be observed that Fano's statistical tensor $G_k(j)$ and the statistical weights p_m are transforms of each other.

$$G_k(j) = \sum_{m=-j}^{j} (-1)^{j-m} p_m \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix}. \tag{11.43}$$

Multiply both sides by the C.G. coefficient $\begin{bmatrix} j & j & k \\ m' & -m' & 0 \end{bmatrix}$ and do the summation over k first and then over m.

$$\begin{aligned} \sum_k G_k(j) \begin{bmatrix} j & j & k \\ m' & -m' & 0 \end{bmatrix} &= \sum_k \sum_m (-1)^{j-m} p_m \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j & k \\ m' & -m' & 0 \end{bmatrix} \\ &= \sum_m (-1)^{j-m} p_m \, \delta_{m,m'} \\ &= (-1)^{j-m'} p_{m'}. \end{aligned} \tag{11.44}$$

Replacing m' by m, we obtain

$$p_m = \sum_{k=0}^{2j} (-1)^{j-m} G_k(j) \begin{bmatrix} j & j & k \\ m & -m & 0 \end{bmatrix}. \tag{11.45}$$

Thus we find from Eqs. (11.43) and (11.45) that p_m and $G_k(j)$ are transforms of each other.

It can be easily seen that the statistical tensors of odd and even rank correspond to physical situations with different types of symmetry. Oriented systems that can be described by statistical tensors of odd rank are said to be polarized whereas those described by statistical tensors of even rank are said to be aligned. In polarized systems, the positive and negative directions about the axis of symmetry can be distinguished. But alignment is that type of orientation which does not distinguish between the positive and negative directions of the axis of symmetry. Sometimes in the literature, the terms polarization and alignment are used to denote specifically the orientations $G_1(j)$ and $G_2(j)$ respectively. Since for any system, the statistical tensors with $k > 3$ are usually quite small, there is no practical distinction between the two terminologies although the former is preferable since it avoids the need for introducing new names for describing orientations with higher rank statistical tensors.

11.4. Oriented and Non-Oriented Systems

Unoriented, oriented and non-oriented spin systems have been discussed in some detail in preceding sections but it is desirable to stress on their distinguishing features since the nomenclature is not followed uniformly (Ramachandran, 1987; Ramachandran et al., 1984, 1986, 1987a, 1987b).

In an unoriented spin system, the particles are distributed uniformly among the various sub-states with different projection quantum numbers. In other words, the statistical weight p_m is $1/(2j+1)$ for a particle with spin j. The density matrix is a scalar matrix with each of the diagonal elements equal to $1/(2j+1)$. This is often referred to as unpolarized system. A spin zero system is always an unoriented or unpolarized system.

For a oriented system, the density matrix is diagonal and the diagonal elements denote the statistical weights p_m for the occupation of the various magnetic sub-states. It is this system that can be conveniently described in terms of the tensor parameters $\langle T_k^0 \rangle$ or Fano's statistical tensors $G_k(j)$ discussed in Sec. 11.3. It has a unique axis of orientation and if it is chosen as the quantization axis, the density matrix becomes diagonal. A spin$-\frac{1}{2}$ system has always a unique axis of orientation and the density matrix can be diagonalized by a rotation of coordinate system.

For $j \geq 1$, the density matrix ρ (of dimension $n \times n$, $n = 2j + 1$) cannot always be diagonalized by a rotation of coordinate system and such spin systems are known as non-oriented spin systems. The density matrix is, in general, Hermitian and it can always be diagonalized through a unitary transformation but it is only in the case of $n = 2$, this unitary transformation can be identified with a rotation of coordinate system. In all other cases with $n > 2$, the unitary transformations generated by rotations in three-dimensional space constitute only a subset of su(n). Hence, a non-oriented spin system is defined as a system with spin $j \geq 1$ if its density matrix cannot be diagonalized through a rotation of coordinate system or equivalently if the system cannot be characterized by a set of statistical weights p_m. Such a system cannot be described in terms of Fano's statistical tensors $G_k(j)$ alone but requires a complete set of spherical tensor parameters $t_k^{m_k}$ ($k = 1, ..., n - 1$; $m_k = -k, -k+1, ..., +k$).

One can gain some insight by considering a geometrical method of constructing a spherical tensor $t_k^{m_k}$. Given a unit vector, a spherical tensor of rank 1 can be constructed. Given two unit vectors, a spherical tensor of rank 2 can be constructed. In a similar way, to construct a spherical tensor of rank k, a set of k unit vectors are required.

$$t_k^{m_k} = r_k \left(\hat{\boldsymbol{Q}}_1 \times \hat{\boldsymbol{Q}}_2 \times \cdots \times \hat{\boldsymbol{Q}}_k \right)_k^{m_k}, \tag{11.46}$$

where r_k is a scalar and $\hat{\boldsymbol{Q}}_1, \hat{\boldsymbol{Q}}_2, \cdots, \hat{\boldsymbol{Q}}_k$ are unit vectors, each of which can be defined by two parameters (polar angles) θ, ϕ. Thus, we find that a total number of $2k + 1$ parameters are required to construct a tensor $t_k^{m_k}$ of rank k.

For spin-$\frac{1}{2}$ system, the density matrix consists only of spherical tensor of first rank t_1^m and hence the system appears as a uniaxial system. For $j \geq 1$ system, the density matrix includes spherical tensors of second and higher ranks which are built out of two and more number of unit vectors and hence the system appears as a multiaxial system. Unless the multiple axes coincide and the system becomes uniaxial, the density matrix cannot be diagonalized by rotation in three-dimensional space. The multiaxial system with $j \geq 1$ is known as non-oriented system. To make it oriented, the system should be made uniaxial.

11.5. Application to Nuclear Reactions

In any nuclear reaction, the final nucleus will, in general, be oriented even though the initial nucleus may not be oriented. To be specific, let us consider a nuclear reaction in which a nucleus makes a transition from an initial state $|J_i\, M_i\rangle$ to a final state $|J_f\, M_f\rangle$. The density matrix ρ_f for the final nuclear state is defined such that its elements are given by (Devanathan et al., 1972)

$$\begin{aligned}(\rho_f)_{M_f,M_f'} &= \sum_{M_i,M_i'} \langle J_f\, M_f|H_I|J_i\, M_i\rangle\, (\rho_i)_{M_i,M_i'}\, \langle J_f\, M_f'|H_I|J_i\, M_i'\rangle^* \\ &= \langle J_f\, M_f|H_I\, \rho_i\, H_I{}^\dagger|J_f\, M_f'\rangle, \end{aligned} \tag{11.47}$$

where ρ_i is the density matrix describing the initial nucleus and and H_I is the interaction Hamiltonian that causes the nuclear transition. The density matrix ρ_f completely describes the reaction cross section σ and the spin orientation of the final nucleus $\langle T_k^\mu \rangle$.

$$\sigma = C\,\mathrm{Tr}\left(H_I\,\rho_i\,H_I{}^\dagger\right) = C\,\mathrm{Tr}\,\rho_f\,, \tag{11.48}$$

$$\langle T_k^\mu \rangle = \frac{\mathrm{Tr}\,\left(T_k^\mu\,\rho_f\right)}{\mathrm{Tr}\,\rho_f}, \tag{11.49}$$

where C is a constant. If the initial nucleus is unoriented, then

$$(\rho_i)_{M_i,M_i'} = \frac{1}{2J_i + 1}\,\delta_{M_i,M_i'}. \tag{11.50}$$

The interaction Hamiltonian is a, scalar and its general structure should be

$$H_I = \sum_{\lambda, m_\lambda} \left(A_\lambda^{m_\lambda}\right)^* U_\lambda^{m_\lambda}, \tag{11.51}$$

where $U_\lambda^{m_\lambda}$ is a, spherical tensor operator in nuclear coordinates inducing nuclear transition and the spherical tensor component $A_\lambda^{m_\lambda}$ may refer to the radiated field. Substituting Eqs. (11.50) and (11.51) into Eq. (11.47), the elements of the density matrix ρ_f for the final state of the nucleus are obtained.

$$\begin{aligned}(\rho_f)_{M_f,M_f'} &= \frac{1}{2J_i+1}\sum_{M_i}\sum_{\lambda,\lambda',m_\lambda,m_{\lambda'}} \left(A_\lambda^{m_\lambda}\right)^* \left(A_{\lambda'}^{m_{\lambda'}}\right) \\ &\quad \times \langle J_f\, M_f|U_\lambda^{m_\lambda}|J_i\, M_i\rangle\langle J_f\, M_f'|U_{\lambda'}^{m_{\lambda'}}|J_i\, M_i\rangle^*. \end{aligned} \tag{11.52}$$

Given the density matrix ρ_f, we can evaluate the traces, Tr ρ_f and Tr $(T_k^\mu\, \rho_f)$.

$$\begin{aligned}\mathrm{Tr}\,\rho_f &= \frac{1}{2J_i+1}\sum_{M_i,M_f}\sum_{\lambda,\lambda',m_\lambda,m_\lambda'} \left(A_\lambda^{m_\lambda}\right)^* \left(A_{\lambda'}^{m_{\lambda'}}\right) \\ &\quad \times \begin{bmatrix} J_i & \lambda & J_f \\ M_i & m_\lambda & M_f\end{bmatrix}\begin{bmatrix} J_i & \lambda' & J_f \\ M_i & m_\lambda' & M_f\end{bmatrix} \\ &\quad \times \langle J_f||U_\lambda||J_i\rangle\langle J_f\,||U_{\lambda'}||\,J_i\rangle^*. \end{aligned} \tag{11.53}$$

$$\begin{aligned}\mathrm{Tr}\,\left(T_k^\mu\,\rho_f\right) &= \sum_{M_f,M_f'} \left(T_k^\mu\right)_{M_f',M_f} (\rho_f)_{M_f,M_f'} \\ &= \frac{1}{2J_i+1}\sum_{M_i,M_f,M_f'}\sum_{\lambda,\lambda',m_\lambda,m_{\lambda'}} \left(A_\lambda^{m_\lambda}\right)^* \left(A_{\lambda'}^{m_{\lambda'}}\right) \\ &\quad \times \langle J_f\, M_f'|T_k^\mu|J_f\, M_f\rangle\langle J_f\, M_f|U_\lambda^{m_\lambda}|J_i\, M_i\rangle \\ &\quad \times \langle J_f\, M_f'|U_{\lambda'}^{m_{\lambda'}}|J_i\, M_i\rangle^* \\ &= \frac{1}{2J_i+1}\sum_{M_i,M_f,M_f'}\sum_{\lambda,\lambda',m_\lambda,m_{\lambda'}} \left(A_\lambda^{m_\lambda}\right)^* \left(A_{\lambda'}^{m_{\lambda'}}\right) \\ &\quad \times \begin{bmatrix} J_f & k & J_f \\ M_f & \mu & M_f'\end{bmatrix}\begin{bmatrix} J_i & \lambda & J_f \\ M_i & m_\lambda & M_f\end{bmatrix}\begin{bmatrix} J_i & \lambda' & J_f \\ M_i & m_{\lambda'} & M_f'\end{bmatrix} \\ &\quad \times \langle J_f||T_k||J_f\rangle\,\langle J_f||U_\lambda||J_i\rangle\,\langle J_f\,||U_{\lambda'}||\,J_i\rangle^*. \end{aligned} \tag{11.54}$$

First let us sum over M_i. The product of three C.G. Coefficients that occurs in Eq. (11.54) can be expressed as a product of U-Coefficient and a C.G.

Coefficient, using the techniques developed in Chap, 7.

$$\sum_{M_i}\begin{bmatrix} J_f & k & J_f \\ M_f & \mu & M'_f \end{bmatrix}\begin{bmatrix} J_i & \lambda & J_f \\ M_i & m_\lambda & M_f \end{bmatrix}\begin{bmatrix} J_i & \lambda' & J_f \\ M_i & m_{\lambda'} & M'_f \end{bmatrix}$$
$$= (-1)^{\lambda-m_\lambda}\frac{[J_f]^2}{[k][\lambda']}\sum_{M_i}\begin{bmatrix} \lambda & J_i & J_f \\ M_\lambda & M_i & M_f \end{bmatrix}$$
$$\times\begin{bmatrix} J_f & J_f & k \\ M_f & -M'_f & -\mu \end{bmatrix}\begin{bmatrix} J_i & J_f & \lambda' \\ M_i & -M'_f & -m_{\lambda'} \end{bmatrix}$$
$$= (-1)^{\lambda-m_\lambda}\frac{[J_f]^2}{[k][\lambda']}U(\lambda\, J_i\, k\, J_f, J_f\, \lambda')\begin{bmatrix} \lambda & \lambda' & k \\ m_\lambda & -m_{\lambda'} & -\mu \end{bmatrix}. \tag{11.55}$$

The summations over M_f and M'_f in Eq. (11.54) are redundant since $M_f = M_i + m_\lambda$ and $M'_f = M_i + m_{\lambda'}$. Substituting (11.55) into (11.54)) we obtain

$$\begin{aligned}\mathrm{Tr}\,\left(T_k^\mu\,\rho_f\right) &= \frac{1}{2J_i+1}\sum_{\lambda,\lambda',m_\lambda,m_{\lambda'}}\left(A_\lambda^{m_\lambda}\right)^*\left(A_{\lambda'}^{m_{\lambda'}}\right)\\ &\times(-1)^{\lambda-m_\lambda}\frac{[J_f]^2}{[k][\lambda']}U(\lambda\, J_i\, k\, J_f, J_f\, \lambda')\\ &\times\begin{bmatrix} \lambda & \lambda' & k \\ m_\lambda & -m_{\lambda'} & -\mu \end{bmatrix}\langle J_f||T_k||J_f\rangle\,\langle J_f||U_\lambda||J_i\rangle\\ &\times\langle J_f\,||U_{\lambda'}||\,J_i\rangle^*. \end{aligned}\tag{11.56}$$

Let us now perform the summation over m_λ and $m_{\lambda'}$.

$$\sum_{m_\lambda,m_{\lambda'}}(-1)^{\lambda-m_\lambda}\begin{bmatrix} \lambda & \lambda' & k \\ m_\lambda & -m_{\lambda'} & -\mu \end{bmatrix}\left(A_\lambda^{m_\lambda}\right)^*\left(A_{\lambda'}^{m_{\lambda'}}\right)$$
$$= (-1)^{\lambda'-k}\left(A_\lambda{}^*\times A_{\lambda'}\right)_k^\mu. \tag{11.57}$$

Substituting the above result and after rearrangement, we obtain

$$\begin{aligned}\mathrm{Tr}\,\left(T_k^\mu\,\rho_f\right) &= \frac{1}{2J_i+1}\sum_{\lambda,\lambda'}(-1)^{\lambda'-k}\left(A_\lambda{}^*\times A_{\lambda'}\right)_k^\mu\frac{[J_f]^2}{[k][\lambda']}\\ &\times U(\lambda\, J_i\, k\, J_f, J_f\, \lambda')\,\langle J_f||T_k||J_f\rangle\\ &\times\langle J_f||U_\lambda||J_i\rangle\,\langle J_f\,||U_{\lambda'}||\,J_i\rangle^*. \end{aligned}\tag{11.58}$$

Equation(11.58) can be equally used to obtain $\mathrm{Tr}\rho_f$ by substituting $k = 0$.

$$\mathrm{Tr}\,\rho_f = \frac{1}{2J_i+1}\sum_\lambda\frac{[J_f]^2}{[\lambda]^2}\left(A_\lambda{}^*.A_\lambda\right)|\langle J_f||U_\lambda||J_i\rangle|^2. \tag{11.59}$$

As an another example, let us consider a reaction in which the nucleus makes two successive transitions. It is a cascade process (Devanathan and Subramanian, 1975; Racah, 1951; Devons and Goldfarb, 1957) in which the nucleus first makes a transition from an initial spin state J_i to an intermediate spin state J_I due to an interaction H_a and subsequently to a final spin state J_f due to another interaction H_b. Assuming the initial nucleus to be unpolarized, the density matrix ρ_f for the nuclear state is given by

$$(\rho_f)_{M_f,M_f'} = \sum_{J_I,J_I'} \sum_{M_i,M_I,M_I'} \langle J_f\, M_f|H_b|J_I\, M_I\rangle \langle J_I\, M_I|H_a|J_i\, M_i\rangle \times \langle J_f\, M_f'|H_b|J_I'\, M_I'\rangle^* \langle J_I'\, M_I'|H_a\,|J_i\, M_i\rangle^*, \qquad (11.60)$$

The interaction energy is a scalar and hence the interaction Hamiltonians H_a and H_b must have the structure

$$H_a = \sum_{\lambda,m_\lambda} \left(A_\lambda^{m_\lambda}\right)^* U_\lambda^{m_\lambda},$$
$$H_b = \sum_{\nu,m_\nu} (B_\nu^{m_\nu})^*\, V_\nu^{m_\nu}. \qquad (11.61)$$

In the above expansions, $U_\lambda^{m_\lambda}, V_\nu^{m_\nu}$ refer to spherical tensor operators in the nuclear coordinates and the spherical tensor components $A_\lambda^{m_\lambda}, B_\nu^{m_\nu}$ refer to the radiated field.

The tensor moments $\langle T_k^\mu\rangle$ of the spin orientation of the final nuclear state is given once again by

$$\langle T_k^\mu\rangle = \frac{\mathrm{Tr}\,(T_k^\mu\, \rho_f)}{\mathrm{Tr}\,\rho_f}, \qquad (11.62)$$

where

$$\mathrm{Tr}\ \left(T_k^\mu\, \rho_f\right) = \frac{1}{2J_i+1} \sum_S \left(A_\lambda^{m_\lambda}\right)^* \left(A_{\lambda'}^{m_\lambda'}\right) (B_\nu^{m_\nu})^* \left(B_{\nu'}^{m_\nu'}\right) \times\langle J_f\, M_f'|T_k^\mu|J_f\, M_f\rangle \langle J_f\, M_f|V_\nu^{m_\nu}|J_I\, M_I\rangle \times\langle J_I\, M_I|U_\lambda^{m_\lambda}|J_i\, M_i\rangle \langle J_f\, M_f'|V_{\nu'}^{m_\nu'}|J_I'\, M_I'\rangle^* \times\langle J_I'\, M_I'|U_{\lambda'}^{m_\lambda'}|J_i\, M_i\rangle\,. \qquad (11.63)$$

The summation index S stands for a set of 15 variables $\lambda, \lambda', \nu, \nu', J_I, J_I'$, $m_\lambda, m_\lambda', m_\nu, m_\nu', M_I, M_I', M_f, M_f'$ and M_i. To obtain Eq. (11.63), the expansions (11.61) of the interaction Hamiltonians H_a and H_b are used.

There are five matrix elements in Eq. (11.63) and applying the Wigner-Eckart theorem, we obtain five C.G. Coefficients and five reduced matrix elements. After several regroupings of C.G. Coefficients, we obtain three U–Coefficients and the tensor components of the radiated fields A and B are suitably coupled to obtain a tensor of rank k and projection quantum number μ by a judicious summation over magnetic quantum numbers. Here, we only give the final result.

$$\begin{aligned} \mathrm{Tr}\ (T_k^{\mu}\rho_f) &= \frac{1}{2J_i+1}\sum_R [(A_\lambda^* \times A_{\lambda'})_\Lambda \times (B_\nu^* \times B_{\nu'})_\Gamma]_k^\mu \\ &\times(-1)^{J_f-J_i}(-1)^{\nu+\Lambda-\Omega}\frac{[J_f]^2}{[J_i][k]^2}U(\Lambda\,\nu\,k\,\nu', \Omega\,\Gamma) \\ &\times U(J_I\,\lambda\,J_I'\,\lambda', J_i\,\Lambda)\,U(J_f\,\nu\,J_I'\,\Lambda, J_i\,\Omega) \\ &\times U(J_f\,\Omega\,J_f\,\nu', J_I'\,k)\,\langle J_f||T_k||J_f\rangle\,\langle J_f||V_\nu||J_I\rangle \\ &\times\langle J_I||U_\lambda||J_i\rangle\langle J_f||V_{\nu'}||J_I'\rangle^*\langle J_I'||U_{\lambda'}||J_i\rangle^*, \qquad (11.64) \end{aligned}$$

where the summation index R stands for a set of nine variables $J_I, J_I', \nu, \nu', \lambda, \lambda', \Lambda, \Omega$ arid Γ.

Equation (11.64) is very general and can be applied to obtain the nuclear transition probability and the spin orientation in any cascade involving a two-step process. By substituting $k = 0$ and $\mu = 0$, one can obtain Tr ρf which is a measure of the cross section for this cascade process.

Putting $\nu = \nu' = 0$, we obtain the result of the single step process.

$$\begin{aligned} \mathrm{Tr}\ (T_k^{\mu}\rho_f) &= \frac{1}{2J_i+1}\sum_{\lambda,\lambda'}(-1)^{J_f-J_i}(A_\lambda^* \times A_{\lambda'})_k^\mu\frac{[J_f]^2}{[J_i][k]^2} \\ &\times U(J_f\,\lambda\,J_f\,\lambda', J_i\,k)\,\langle J_f||T_k||J_f\rangle \\ &\times\langle J_f||U_\lambda||J_i\rangle\,\langle J_f||U_{\lambda'}||J_i\rangle^*. \qquad (11.65) \end{aligned}$$

This is identical with Eq. (11.58) since

$$U(J_f\,\lambda\,J_f\,\lambda', J_i, k) = (-1)^{J_f+\lambda'-J_i-k}\frac{[k][J_i]}{[J_f][\lambda']}U(\lambda\,J_i\,k\,J_f, J_f\,\lambda'). \qquad (11.66)$$

Review Questions

11.1 Define and explain the concept of the density matrix. For a spin-1/2 system, write down explicitly the density matrix in terms of the polarization parameters P_x, P_y and P_z. Find the expectation value of a dynamical variable F in terms of the density matrix.

11.2 Construct the density matrix for a system of arbitrary angular momentum $\boldsymbol{J}$ and express it in terms of the statistical or multipole parameters $\langle T_k^{\mu} \rangle$.

11.3 What are Fano's statistical tensors and how are they related to the statistical parameters $\langle T_k^{\mu} \rangle$? Express Fano's statistical tensors in terms of the moments of statistical weights p_m which denote the occupancy probabilities of different m states.

11.4 Show that the statistical weights p_m and Fano's statistical tensors $\langle T_k^{\mu} \rangle$ are transforms of each other. Distinguish between polarization and alignment of an oriented system.

11.5 What are unoriented, oriented and non-oriented systems? Explain their distinguishing features.

11.6 A non-oriented system is a multiaxial system and hence its density matrix cannot be diagonalized by a rotation of coordinate system. Explain this statement and deduce the conditions under which it can be diagonalized.

11.7 In a nuclear reaction, the nucleus makes a transition from an initial state of spin J_i to a final state of spin J_f due to an interaction Hamiltonian H_I. Deduce an expression for the cross section and the polarization of the final nucleus if the initial nucleus is unpolarized.

11.8 In a cascade process, the nucleus makes a transition from an initial spin state J_i to an intermediate spin state J_I and then to a final spin state J_f by successive interactions defined by interaction Hamiltonians H_a and H_b. Deduce an expression for the cross section and the polarization of the final nucleus, if the initial nucleus is unpolarized.

Problems

11.1 In a certain coordinate system, a spin$-\frac{1}{2}$ system is in a pure state $|\Psi_{1/2}\rangle$ with spin-up. Find the rotation matrix that transforms the state $|\Psi_{1/2}\rangle$ into $|\alpha\rangle$ that has the polarization vector with polar angles $(\theta, 0)$. Hence obtain the density matrix corresponding to the state $|\alpha\rangle$.

11.2 Construct the density matrix and determine the polarization vector for a system defined by the spinor

$$\chi = \begin{bmatrix} e^{i\alpha} \cos\delta \\ e^{i\beta} \sin\delta \end{bmatrix}.$$

Find the matrix U_R which rotates this state into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

11.3 Construct the simultaneous eigenvectors of J^2 and $\boldsymbol{J} \cdot \hat{\boldsymbol{n}}$ for the states with $j = 1$. Show that if a measurement of J_z is made on a state in which $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ is certainly unity, the eigenvalues 1,0, –1 of J_z are obtained with relative probabilities $\cos^4\frac{\theta}{2}$, $2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$, $\sin^4\frac{\theta}{2}$ respectively where 6 is the angle between $\hat{\mathbf{n}}$ and the z-axis.

11.4 The scattering amplitude for the scattering of a spin-$\frac{1}{2}$ particle is given by $\sigma\cdot\boldsymbol{K}+L$, where $\boldsymbol{K}$ denotes the spin-dependent amplitude and L, the spin-independent amplitude. If the incident particle is unpolarized, show that the scattered particle is polarized with polarization

$$\boldsymbol{P}=\frac{i(K\times K^*)+L^*K+LK^*}{K.K^*+LL^*}.$$

11.5 If the scattering amplitude for the scattering of a spin-$\frac{1}{2}$ particle is given by $\sigma\cdot\boldsymbol{K}+L$, where $\boldsymbol{K}$ denotes the spin-dependent amplitude and L, the spin-independent amplitude, deduce an expression for the scattering cross section if the incident beam is polarized.

11.6 Deduce Eq. (11.64) from Eq. (11.63) by using angular momentum recoupling coefficients.

Solutions to Selected Problems

11.1 Let the frame of reference be X, Y, Z in which the system is in a pure state $|\Psi_{1/2}\rangle$.

$$|\Psi_{1/2}\rangle=\begin{bmatrix}1\\0\end{bmatrix}.$$

This ket has to be rotated through an angle 6 about the Y-axis. Then the polar angles of the polarization vector of this ket will be (θ,0). This is equivalent to rotation of the coordinate system through an angle –θ about the Y-axis. The corresponding rotation matrix is obtained from (5.33).

$$U_R=d^{1/2}(-\theta)=\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}.$$

Then

$$|\alpha\rangle=(U_R)^T|\Psi_{1/2}\rangle=[d^{1/2}(-\theta)]^T|\Psi_{1/2}\rangle.$$

Explicitly,

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}.$$

To obtain the density matrix, one can use either expression (11.10) or (11.15). The third method is to find the density matrix in the rotated coordinate system by using the rotation operator.

Method 1:
Using Eq. (11.10), we obtain the density matrix ρ.

$$\rho = |\alpha\rangle\langle\alpha| = \begin{bmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}.$$

Method 2:
From Eq. (11.15), we have

$$\rho = \frac{1}{2}\begin{bmatrix} 1+P_z & P_x - iP_y \\ P_x + iP_y & 1-P_z \end{bmatrix},$$

where $P_z = \cos\theta$, $P_x = \sin\theta$, $P_y = 0$. Substituting these values, we obtain the density matrix.

$$\begin{aligned} \rho &= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}. \end{aligned}$$

Method 3:
In the *X, Y, Z* frame, let us denote the density matrix as ρ_0.

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

To bring $|\Psi_{1/2}\rangle$ to $|\alpha\rangle$, a rotation of the coordinate system through an angle $-\theta$ is to be made about the Y-axis. ($U_R = [\, d^{1/2}(-\theta)]^T$.)

$$\begin{aligned} \rho &= U_R\,\rho_0\,U_R{}^\dagger \\ &= \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}. \end{aligned}$$

11.2 The density matrix ρ is given by

$$\rho = \chi\chi^\dagger = \begin{bmatrix} \cos^2\delta & e^{i(\alpha-\beta)}\cos\delta\,\sin\delta \\ e^{i(\beta-\alpha)}\cos\delta\sin\delta & \sin^2\delta \end{bmatrix}.$$

If the direction of the polarization vector is given by $\hat{\mathbf{n}}(\theta,\phi)$, then the density matrix can be written as

$$\begin{aligned}\rho &= \frac{1}{2}(1+\boldsymbol{\sigma}.\boldsymbol{P}) \\ &= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & 1-\cos\theta \end{bmatrix}.\end{aligned}$$

Comparing this with the density matrix obtained earlier, we find that

$$\theta = 2\,\delta; \quad \phi = \beta - \alpha.$$

The required matrix U_R that rotates the state χ such that

$$U_R\,\chi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is

$$U_R = \begin{bmatrix} \cos\delta\, e^{i\phi/2} & \sin\delta\, e^{-i\phi/2} \\ -\sin\delta\, e^{i\phi/2} & \cos\delta\, e^{-i\phi/2} \end{bmatrix},$$

with $\phi = \beta - \alpha$.

11.3 Choose the quantization axis as $\hat{\boldsymbol{n}}$. Then the eigenvectors of J^2 and $\boldsymbol{J}\cdot\hat{\boldsymbol{n}}$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The density matrix ρ_n in this basis is diagonal and it is given by

$$\rho_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The rotation matrix U_R that transforms ρ_n into ρ_z to the basis in which the unit vector $\hat{\mathbf{n}}$ makes an angle θ with the z-axis is

$$U_R = \left(d^1(-\theta)\right)^T = \begin{bmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{bmatrix}.$$

The density matrix ρ_z in the new basis is given by

$$\begin{aligned}\rho_z &= U_R\,\rho_n\,U_R{}^\dagger,\\ &= \begin{bmatrix} \cos^4\frac{\theta}{2} & \sqrt{2}\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} \\ \sqrt{2}\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} & 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} & \sqrt{2}\sin^3\frac{\theta}{2}\cos\frac{\theta}{2} \\ \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} & \sqrt{2}\cos\frac{\theta}{2}\sin^3\frac{\theta}{2} & \sin^4\frac{\theta}{2} \end{bmatrix}.\end{aligned}$$

The diagonal elements of the above matrix give the relative probabilities of obtaining the eigenvalues $1, 0, -1$ for J_z in any measurement.

PRODUCTS OF ANGULAR MOMENTUM MATRICES AND THEIR TRACES

12.1. General Properties

Let us first recall the general properties of angular momentum matrices, products of angular momentum matrices and their traces.

1. Of the angular momentum matrices, J^2 is a scalar matrix

$$J^2 = \eta\, I, \tag{12.1}$$

where η denotes the eigenvalue $j(j + 1)$ of J^2 operator and I, the unit matrix. The trace of the matrix J^2 is

$$\operatorname{Tr} J^2 = \eta(2j + 1) = \Omega. \tag{12.2}$$

2. The remaining angular momentum matrices are the Cartesian components J_x, J_y, J_z of angular momentum, the ladder operators J_+, J_- and the spherical components J_1^1, J_1^{-1}, J_1^0 of angular momentum. These are traceless matrices and they obey the following commutation relations:

$$[\,J_x, J_y\,] = i\,J_z, \qquad [J_y, J_z] = i\,J_x, \quad [J_z, J_x] = -i\,J_y. \tag{12.3}$$

$$[J_+, J_-] = 2\,J_z, \qquad [J_z, J_\pm] = \pm J_\pm. \tag{12.4}$$

$$[J_1^1, J_1^{-1}] = -J_1^0, \quad [J_1^0, J_1^1] = J_1^1, \quad [J_1^0, J_1^{-1}] = -J_1^{-1}. \tag{12.5}$$

Although these matrices are traceless, their products can have a non-vanishing trace.

3. The matrices J_x, J_y, J_z are Hermitian matrices. If the matrices J^2 and any one of the components are diagonalized simultaneously, then, in that representation, one of the two remaining matrices has only real elements and the other has only imaginary elements. It is possible to go from one representation in which J^2 and one of components, say J_z are diagonal to another representation in which J^2 and J_x are diagonal by unitary transformation but the trace is invariant under such unitary transformation.

4. The trace of a product of matrices is not changed under cyclic permutation of matrices.

$$\mathrm{Tr}\,(A_1\,A_2\,A_3\,\ldots\,A_{n-1}\,A_n) = \mathrm{Tr}\,(A_n\,A_1\,A_2\,\ldots\,A_{n-1}). \qquad (12.6)$$

5. Consider a product of Cartesian components J_x, J_y, J_z of angular momentum.

$$A = J_a\,J_b\,J_c\,J_d\,\ldots. \qquad (12.7)$$

If J_x occurs α times, J_y occurs β times and J_z occurs γ times in A, then

Tr A is real, if α,β,γ are all even integers,
Tr A is purely imaginary, if α,β,γ are all odd integers,
Tr A is zero, if α,β,γ are of mixed type. (12.8)

The above simple result regarding the nature of the trace is obtained using the property (3) that in a given representation, two of the three matrices consist of real elements whereas the third consists of purely imaginary elements.
If α, β, γ are of mixed type, then, in one representation it is possible to choose the matrix that occurs odd number of times as the one that contains purely imaginary elements. In another representation, one can choose the matrix that occurs even number of times as the one that contains purely imaginary elements. In the first case, TrA is purely imaginary and in the second case, Tr A is real. Since this is in contradiction with the property that the trace is invariant under unitary transformation, Tr A should be identically zero, if α, β, γ are of mixed type.
If α, β, γ are all even (odd) integers, then the matrix that has purely imaginary elements will occur even (odd) number of times in any representation. Since i^{even} (i^{odd}) is real (imaginary), Tr A is real (imaginary).
6. (a) $\mathrm{Tr}\,(J_L^\alpha\,J_M^\beta\,J_N^\gamma)$ is invariant under an interchange of powers of J_L, J_M and J_N which are the Cartesian components of angular momentum.
(b) $\mathrm{Tr}\,(J_L^\alpha\,J_M^\beta\,J_N^\gamma) = (-1)^{\alpha+\beta+\gamma}\,\mathrm{Tr}\,(J_L^\alpha\,J_N^\gamma\,J_M^\beta)$.
7. Consider a product B of angular momentum matrices with J_+, J_- and J_z.

$$B = J_a\,J_b\,J_c\,J_d\,\ldots, \qquad (12.9)$$

with J_a, J_b, J_c, J_d, ... standing for any one of the angular momentum matrices J_+, J_-, J_z. Let J_+ occur p times, J_- occur q times and J_z

occur r times. The J_+ operator steps up the projection quantum number m by 1, the J_- operator steps down the m value by 1 and the J_z operator leaves the m value unaffected. Since Tr B is the sum of the diagonal matrix elements of B,

$$\mathrm{Tr}\, B = \sum_m \langle j\, m|B|j\, m\rangle, \tag{12.10}$$

the trace exists only when $p = q$.

8. Consider a product C of angular momentum matrices in spherical basis.

$$C = J_a\, J_b\, J_c\, J_d\, \ldots, \tag{12.11}$$

where J_a, J_b, J_c, J_d, ... denote any one of the spherical components of angular momentum, J_1^1, J_1^{-1}, J_1^0. Since

$$J_1^1 = -\frac{1}{\sqrt{2}} J_+, \quad J_1^{-1} = \frac{1}{\sqrt{2}} J_-, \quad J_1^0 = J_z,$$

it follows from our earlier consideration that for non-vanishing trace of C, the number of J_1^1 matrices should be equal to the number of J_1^{-1} in the product of matrices.

9. It has been shown by Subramanian and Devanathan (1974) that Tr A, Tr B and Tr C are polynomials in η, the eigenvalue of the J^2 operator and recursion relations for these polynomials have been developed by De Meyer and Vanden Berghe (1978) and Subramanian and Devanathan (1980).

Ambler et al. (1962a, 1962b) have obtained the traces of a limited number of angular momentum matrices for systems of arbitrary spin and the study has been extended by Subramanian and Devanathan (1974, 1979, 1980, 1985), De Meyer and Vanden Berghe (1978a, 1978b), Thakur (1975), Rashid (1979), Kaplan and Zia (1979), Ullah (1980a, 1980b) and Witschel (1971, 1975).

12.2. Evaluation of Tr J_z^{2p}

The evaluation of Tr J_z^{2p} is of fundamental importance[1] since any trace of products of Cartesian components of angular momentum can be expressed in terms of this. Also, the trace is invariant under unitary transformation

[1]Subramanian (1986a,1986b) has evaluated the Tr J_z^{2p} using the Brillouin function and also obtained a generating function for the trace.

and so the same result will be obtained for any other Cartesian component of angular momentum. In other words,

$$\mathrm{Tr}\, J_x^{2p} = \mathrm{Tr}\, J_y^{2p} = \mathrm{Tr}\, J_z^{2p}. \tag{12.12}$$

We need to evaluate the trace for only even powers, since the trace of odd powers vanish as is evident from the property (5) discussed in Sec. 12.1. Accordingly, $\mathrm{Tr}\,(J_z^\gamma) = \mathrm{Tr}\,(J_x^\alpha J_y^\beta J_z^\gamma)$ with $\alpha = \beta = 0$ and γ odd should vanish, since α,β,γ are of even and odd mixture, zero being considered as even integer. Hence,

$$\mathrm{Tr}\, J_x^p = \mathrm{Tr}\, J_y^p = \mathrm{Tr}\, J_z^p = 0 \quad (p = \text{ odd integer}). \tag{12.13}$$

Choose a representation in which J_z is diagonal. Then

$$\begin{aligned}\mathrm{Tr}\,(J_z^{2p}) &= \sum_{m=-j}^{j} \langle jm|J_z^{2p}|jm\rangle \\ &= \sum_{m=-j}^{j} m^{2p} = \frac{2}{2p+1} B_{2p+1}(j+1), \end{aligned} \tag{12.14}$$

where $B_{2p+1}(j+1)$ is the Bernoulli polynomial of the first kind of degree $(2p+1)$ in $(j+1)$. A brief description of the Bernoulli polynomials is given in Appendix F. The reader may refer to Miller (1960) for more details.

We shall illustrate the usefulness of the result obtained above by giving a few examples. From Eq. (12.14), we obtain

$$\begin{aligned}\mathrm{Tr}\,(J_z^2) &= \frac{2}{3} B_3(j+1) \\ &= \frac{2}{3}(j+1)(j+\tfrac{1}{2})j = \frac{1}{3}\Omega, \end{aligned} \tag{12.15}$$

$$\begin{aligned}\mathrm{Tr}\,(J_z^4) &= \frac{2}{5} B_5(j+1) \\ &= \frac{1}{15}\Omega(3\eta - 1). \end{aligned} \tag{12.16}$$

The required Bernoulli polynomials $B_3(x)$ and $B_5(x)$ are taken from Appendix F.

$$B_3(x) = x(x-\tfrac{1}{2})(x-1), \tag{12.17}$$

$$B_5(x) = x(x-\tfrac{1}{2})(x-1)(x^2-x-\tfrac{1}{3}). \tag{12.18}$$

We shall give two more examples, the evaluation of Tr $(J_x J_y J_z)$ and Tr $(J_z^2 J_x^2)$.

$$J_xJ_yJ_z = (J_yJ_x + iJ_z)J_z = J_yJ_xJ_z + iJ_z^2. \tag{12.19}$$

Taking the trace on both sides and using the property (6b) and the result (12.15), $\mathrm{Tr}(J_x J_y J_z)$ is evaluated.

$$\begin{aligned} \mathrm{Tr}(J_xJ_yJ_z) - \mathrm{Tr}(J_yJ_xJ_z) &= i\,\mathrm{Tr}(J_z^2). \\ 2\,\mathrm{Tr}(J_xJ_yJ_z) &= i\frac{1}{3}\Omega. \end{aligned} \tag{12.20}$$

Let us now evaluate the other trace, $\mathrm{Tr}(J_z^2J_x^2)$ which requires a knowledge of $\mathrm{Tr}\,J_z^2$ and $\mathrm{Tr}\,J_z^4$.

$$\begin{aligned} J_z^2J_x^2 &= J_z^2(J^2 - J_y^2 - J_z^2), \\ J_z^2J_x^2 + J_z^2J_y^2 &= J_z^2J^2 - J_z^4, \\ 2\,\mathrm{Tr}(J_z^2J_x^2) &= \eta\,\mathrm{Tr}(J_z^2) - \mathrm{Tr}(J_z^4) \\ &= \frac{1}{3}\eta\Omega - \frac{1}{15}\Omega(3\eta - 1) \\ &= \frac{1}{15}\Omega(2\eta + 1). \end{aligned} \tag{12.21}$$

Thus, we see that to evaluate the trace of any product of Cartesian components of angular momentum, we require only a knowledge of the trace of even powers of J_z.

It has been shown by Subramanian (1974) that the Bernoulli polynomial in s can be expressed as another polynomial in $u = s^2 - s$.

$$B_{2p+1}(s) = s(s-1)(2s-1)F_{p-1}(u). \tag{12.22}$$

In Eq. (12.22), $F_{p-1}(u)$ is a polynomial of degree $(p - 1)$ in u. Hence,

$$\begin{aligned} \mathrm{Tr}(J_z^{2p}) &= \frac{2}{2p-1}B_{2p+1}(j+1) \\ &= \frac{2}{2p+1}(j+1)j(2j+1)F_{p-1}(\eta) \\ &= \frac{2\Omega}{2p+1}F_{p-1}(\eta) = \Omega G_{p-1}(\eta), \end{aligned} \tag{12.23}$$

where $\eta = j(j + 1)$ is the eigenvalue of the J^2 operator, $\Omega = \eta(2j + 1)$ is the trace of the J^2 matrix and $F_{p-1}(\eta)$, $G_{p-1}(\eta) = [2/(2p+1)]F_{p-1}(\eta)$ are polynomials of degree $(p - 1)$ in η. It is an important observation made by Subramanian and Devanathan (1974) that $\mathrm{Tr}\,J_z^{2p}$ is a polynomial[2] in η and

[2] This property has been used by Pearce (1976) for the study of spin correlations in the Heisenberg model.

they obtained a recursion relation for the polynomial $G_p(\eta)$ (Subramanian and Devanathan, 1980).

$$\begin{aligned}\eta(4\eta+1)\frac{d^2}{d\eta^2}G_{p-1}(\eta)+2(7\eta+1)\frac{d}{d\eta}G_{p-1}(\eta)+6\,G_{p-1}(\eta)\\ =\frac{2p(2p-1)\mathrm{Tr}\,(J_\lambda^{2p-2})}{2j+1}\\ =2p(2p-1)\eta\,G_{p-2}(\eta)\,.\end{aligned} \tag{12.24}$$

The last step is obtained using the relation

$$\mathrm{Tr}(J_\lambda^{2p-2})=\Omega\,G_{p-2}(\eta). \tag{12.25}$$

Starting from the lowest order

$$\mathrm{Tr}(J_\lambda^0)=\mathrm{Tr}\,(I)=2j+1, \tag{12.26}$$

traces of higher orders ($2p$) can be obtained successively using the aforesaid differential recurrence relation.

12.3. Evaluation of Tr $(J_-^k\,J_+^k)$

For the trace of the product of J_- J_+ matrices to exist, the power of J_+ should be the same as the power of J_-, since

$$\mathrm{Tr}\,(J_-^k\,J_+^k)=\sum_{m=-j}^{j}\langle jm|J_-^k\,J_+^k|jm\rangle. \tag{12.27}$$

The J_+ operator steps up the m value by unity and the J_- operator steps down the m value by unity.

$$J_\pm|jm\rangle=[(J\mp m)(j\pm m+1)]^{\frac{1}{2}}|j\,m\pm 1\rangle. \tag{12.28}$$

By repeated application of J_+ and J_- operator, we obtain

$$\begin{aligned}\langle j\,m+k|J_+^k|j\,m\rangle &= \{(j-m)(j-m-1)\ldots(j-m-k+1)\}^{\frac{1}{2}}\\ &\quad\times\{(j+m+1)(j+m+2)\ldots(j+m+k)\}^{\frac{1}{2}}\\ &= \left\{\frac{(j-m)!\,(j+m+k)!}{(j-m-k)!\,(j+m)!}\right\}^{\frac{1}{2}}.\end{aligned} \tag{12.29}$$

$$\begin{aligned}\langle j\,m|J_-^k|j\,m+k\rangle &= \{(j+m+k)(j+m+k-1)\ldots(j+m+1)\}^{\frac{1}{2}}\\ &\quad\times\{(j-m-k+1)(j-m-k)\ldots(j-m)\}^{\frac{1}{2}}\\ &= \left\{\frac{(j+m+k)!\,(j-m)!}{(j+m)!\,(j-m-k)!}\right\}^{\frac{1}{2}}.\end{aligned} \tag{12.30}$$

Hence,

$$\mathrm{Tr}\,(J_-^k\, J_+^k) = \sum_{m=-j}^{j} \frac{(j-m)!\,(j+m+k)!}{(j-m-k)!\,(j+m)!}. \tag{12.31}$$

In the above summation, $m > j - k$ will not contribute since $(j - m - k)!$ will become negative which is not allowed. Therefore,

$$\mathrm{Tr}\,(J_-^k\, J_+^k) = \sum_{m=-j}^{j-k} \frac{(j-m)!\,(j+m+k)!}{(j-m-k)!\,(j+m)!}. \tag{12.32}$$

Since m is a dummy index, over which summation is performed, one can replace m by $-m$ in Eq. (12.32).

$$\mathrm{Tr}\,(J_-^k\, J_+^k) = \sum_{m=-j+k}^{j} \frac{(j+m)!\,(j-m+k)!}{(j+m-k)!\,(j-m)!}. \tag{12.33}$$

Let us now define new variables t and u,

$$t = j + m - k, \qquad u = 2j - k, \tag{12.34}$$

such that the summation extends from $t = 0$ to $t = u$. Replacing the variables j and m by the new variables t and u, we get

$$\begin{aligned} \mathrm{Tr}\,(J_-^k\, J_+^k) &= \sum_{t=0}^{u} \frac{(t+k)!\,(u-t+k)!}{t!\,(u-t)!} \\ &= (k!)^2 \sum_{t=0}^{u} \binom{t+k}{t} \binom{u-t+k}{u-t}, \end{aligned} \tag{12.35}$$

where $\binom{a}{b} = a!/\{b!(a-b)!\}$ denotes the binomial coefficient. Using the identity,

$$\begin{aligned} \sum_{t=0}^{u} \binom{t+k}{t} \binom{u-t+k}{u-t} &= \binom{2k+1+u}{u} \\ &= \binom{2k+1+u}{2k+1}, \end{aligned} \tag{12.36}$$

we finally obtain the result

$$\mathrm{Tr}\,(J_-^k\, J_+^k) = \frac{(k!)^2\,(2j+k+1)!}{(2k+1)!\,(2j-k)!}. \tag{12.37}$$

This result was first obtained by Subramanian and Devanathan (1974) using the concept of statistical tensors discussed in Chap. 11. Subsequently, it was rederived by De Meyer and Vanden Berghe (1978) using the above algebraic method. Ullah (1980a,1980b) also obtained Eq. (12.37) using the angular momentum operator identities and rotation operators.

12.4. Recurrence Relations for Tr $(J_-^k J_z^l J_+^k)$

It is possible to express the trace of any product of a definite number of angular momentum operators J_+, J_- and J_z as a sum of traces of the type $\mathrm{Tr}\,(J_-^k J_z^l J_+^k)$. De Meyer and Vanden Berghe (1978) have obtained a recurrence relation for the trace with respect to the number of J_z operators.

It is convenient to introduce the shorthand notation

$$\mathrm{Tr}\,(J_-^k J_z^l J_+^k) = \mathrm{Tr}\,(k, l), \tag{12.38}$$

for developing a recurrence relation starting from $\mathrm{Tr}(k, 0)$ for which an analytical expression (12.37)

$$\mathrm{Tr}\,(k, 0) = \frac{(k!)^2\,(2j+k+1)!}{(2k+1)!\,(2j-k)!},$$

has been obtained. It turns out that a distinction has to be made between even and odd l values, leading to a different type of recurrence relation for either case. Here, we only give the final results.

For odd *l*,

$$\sum_{i=0}^{2n-1} \binom{2n}{i} (-k)^{2n-1}\mathrm{Tr}\,(k, i) = 0. \tag{12.39}$$

$$\sum_{i=0}^{2n-2} \binom{2n-1}{i} (-k)^{2n-1-i}\mathrm{Tr}\,(k, i) = -2\,\mathrm{Tr}\,(k, 2n-1). \tag{12.40}$$

Both Eqs. (12.39) and (12.40) are recurrence relations, either of which can be used to express Tr (k, l) for odd l as a sum of quantities Tr (k, i) with $i < l$. For the purpose of illustration, let us calculate Tr (k, l). From Eq. (12.40), it follows that for $n = 1$

$$-\binom{1}{0} (-k)\,\mathrm{Tr}\,(k, 0) = 2\,\mathrm{Tr}\,(k, 1), \tag{12.41}$$

or

$$\mathrm{Tr}\,(k, 1) = \tfrac{1}{2}k\,\mathrm{Tr}\,(k, 0). \tag{12.42}$$

The same result is obtained using Eq. (12.39) also.

For even *l*,

$$
\begin{aligned}
(2k+l+2)\mathrm{Tr}\,(k,l+1) &= k(k+1)\mathrm{Tr}\,(k,l)+\eta\sum_{i=0}^{l-1}\binom{l}{i}\mathrm{Tr}\,(k,i) \\
&\quad -\sum_{i=0}^{l-1}\binom{l}{i}\mathrm{Tr}\,(k,i+1) \\
&\quad -\sum_{i=0}^{l-2}\binom{l}{i}\mathrm{Tr}\,(k,i+2). \qquad (12.43)
\end{aligned}
$$

The above equation can be used to evaluate Tr (k, l) for even l, in terms of quantities Tr (k, i) with $i < l$.

Let us calculate Tr $(k, 2)$ with the help of Eq. (12.43).

$$(2k+3)\mathrm{Tr}\,(k,2)=k(k+1)\mathrm{Tr}\,(k,1)+\eta\mathrm{Tr}\,(k,0)-\mathrm{Tr}\,(k,1). \qquad (12.44)$$

Since

$$\mathrm{Tr}\,(k,1)=\tfrac{1}{2}k\,\mathrm{Tr}\,(k,0),$$

it follows that

$$\mathrm{Tr}\,(k,2)=\frac{1}{2(2k+3)}(2\eta+k^3+k^2-k)\,\mathrm{Tr}\,(k,0). \qquad (12.45)$$

12.5. Some Simple Applications

12.5.1. STATISTICAL TENSORS

We have seen in Chap. 11 that the final nuclear spin orientation in any nuclear reaction can be completely described by a set of parameters $\langle T_k^{m_k}\rangle$ using the density matrix ρ_f and the statistical tensors $T_k^{m_k}$.

$$\langle T_k^{m_k}\rangle=\frac{\mathrm{Tr}\,(T_k^{m_k}\,\rho_f)}{\mathrm{Tr}\,\rho_f}. \qquad (12.46)$$

The statistical tensor $T_k^{m_k}$ is a spherical tensor of rank k in the spin space of the final nucleus that satisfies the normalization condition

$$\mathrm{Tr}\,(T_k^{m_k\,\dagger}\,T_{k'}^{m_{k'}})=(2j+1)\,\delta_{kk'}\,\delta_{m_k m_{k'}}, \qquad (12.47)$$

subject to the condition $0 \le k \le 2j$, where j is the spin of the final nucleus.

Using the Wigner–Eckart theorem, we have

$$\langle j'm'|T_k^{m_k}|jm\rangle=\begin{bmatrix} j & k & j' \\ m & m_k & m' \end{bmatrix}\langle j'\|T_k\|j\rangle. \qquad (12.48)$$

Since J^2 commutes with $T_k^{m_k}$ i.e., $[J^2,T_k^{m_k}]=0$, $j'=j$. Consequently,

$$\begin{aligned}
\mathrm{Tr}\,(T_k^{m_k\dagger}T_{k'}^{m_{k'}}) &= \sum_{mm'}(-1)^{m_k}\langle jm|T_k^{-m_k}|jm'\rangle\langle jm'|T_{k'}^{m_{k'}}|jm\rangle \\
&= \sum_{mm'}(-1)^{m_k}\left[\begin{array}{ccc} j & k & j \\ m' & -m_k & m \end{array}\right]\left[\begin{array}{ccc} j & k' & j \\ m & m_{k'} & m' \end{array}\right] \\
&\quad \times\langle j||T_k||j\rangle\,\langle j||T_{k'}||j\rangle \\
&= \frac{[j]^2}{[k]^2}\,|\langle j||T_k||j\rangle|^2\,\delta_{kk'}\,\delta_{m_k m_{k'}}. \qquad (12.49)
\end{aligned}$$

The last step was obtained by summing over the magnetic quantum numbers and using the symmetry and orthonormality properties of C.G. coefficients.

$$\begin{aligned}
&\sum_{mm'}(-1)^{m_k}\left[\begin{array}{ccc} j & k & j \\ m' & -m_k & m \end{array}\right]\left[\begin{array}{ccc} j & k' & j \\ m & m_{k'} & m' \end{array}\right] \\
&= \sum_{mm'}(-1)^{m_k}(-1)^{k-m_k}(-1)^{j+k-j}\left[\begin{array}{ccc} j & k & j \\ m & m_k & m' \end{array}\right]\left[\begin{array}{ccc} j & k' & j \\ m & m_{k'} & m' \end{array}\right] \\
&= \sum_{mm'}\frac{[j]^2}{[k][k']}\left[\begin{array}{ccc} j & j & k \\ m & -m' & -m_k \end{array}\right]\left[\begin{array}{ccc} j & j & k' \\ m & -m' & -m_{k'} \end{array}\right] \\
&= \frac{[j]^2}{[k][k']}\,\delta_{kk'}\,\delta_{m_k m_{k'}}. \qquad (12.50)
\end{aligned}$$

Comparing Eqs. (12.47) and (12.49), we get the value of the reduced matrix element $\langle j||T_k||j\rangle$, assuming it to be real and positive.

$$\langle j||T_k||j\rangle = [k]. \qquad (12.51)$$

12.5.2. CONSTRUCTION OF T_K^K

It is possible to construct the spin tensor T_k^k using the angular momentum operator J.

$$T_k^k = G_k(J_1^1)^k = G_k(-1/\sqrt{2})^k(J_+)^k, \qquad (12.52)$$

where Gk is a constant that depends upon k. To determine this constant, let us first find the matrix elements of the spin tensor T_k^k and the ladder operator $(J_+)^k$ between the states $|j, j-k\rangle$ and $|j, j\rangle$.

$$\langle j,j|T_k^k|j,j-k\rangle = \left[\begin{array}{ccc} j & k & j \\ j-k & k & j \end{array}\right]\langle j||T_k||j\rangle. \qquad (12.53)$$

Substituting Racah's expression for the C.G. coefficient,

$$\begin{bmatrix} j & k & j \\ j-k & k & j \end{bmatrix} = (-1)^k \left\{ \frac{(2j+1)!(2k)!}{k!(2j+k+1)!} \right\}^{\frac{1}{2}}, \tag{12.54}$$

and the value of the reduced matrix element from Eq. (12.51), we obtain

$$\langle j,j|T_k^k|j,j-k\rangle = (-1)^k \left\{ \frac{(2j+1)!(2k)!}{k!(2j+k+1)!} \right\}^{\frac{1}{2}} [k]. \tag{12.55}$$

Starting from the matrix element of J_+ operator, we can obtain the matrix element of $(J_+)^k$ by successive operation of J_+, k times.

$$\langle j,m+1|J_+|j,m\rangle = \{(j-m)(j+m+1)\}^{\frac{1}{2}}, \tag{12.56}$$

$$\begin{aligned} \langle j,m|(J_+)^k|j,m-k\rangle &= \{(j-m+k)(j-m+k-1)\cdots(j-m+1) \\ &\quad \times(j+m-k+1)(j+m-k+2)\cdots(j+m)\}^{\frac{1}{2}} \\ &= \left\{ \frac{(j-m+k)!(j+m)!}{(j-m)!(j+m-k)!} \right\}^{\frac{1}{2}}. \end{aligned} \tag{12.57}$$

Putting $m = j$ in the above equation,

$$\langle j,j|(J_+)^k|j,j-k\rangle = \left\{ \frac{k!\,(2j)!}{(2j-k)!} \right\}^{\frac{1}{2}}, \tag{12.58}$$

and substituting Eqs. (12.55) and (12.58) in Eq. (12.52), we finally obtain the value of G_k.

$$G_k = \frac{1}{k!} \left\{ \frac{2^k(2j+1)(2k+1)!(2j-k)!}{(2j+k+1)!} \right\}^{\frac{1}{2}}. \tag{12.59}$$

12.5.3. ANALYTICAL EXPRESSION FOR $\mathrm{Tr}\,(J_-^K J_+^K)$

The spin tensor is normalized according to Eq. (12.47) such that

$$\mathrm{Tr}\,(T_k^{k\dagger} T_k^k) = 2j+1. \tag{12.60}$$

Expressing the spin tensor T_k^k in terms of J_+ using Eq. (12.52), we obtain

$$(G_k)^2(1/2)^k \mathrm{Tr}\,(J_-^k J_+^k) = 2j+1. \tag{12.61}$$

Substituting the value of Gk from Eq. (12.59), we get a compact analytical expression for Tr $(J_-^k J_+^k)$.

$$\begin{aligned} \mathrm{Tr}\,(J_-{}^k J_+{}^k) &= \frac{2^k}{|G_k|^2}(2j+1) \\ &= \frac{(k!)^2\,(2j+k+1)!}{(2k+1)!\,(2j-k)!}. \end{aligned} \tag{12.62}$$

This result was first obtained by Subramanian and Devanathan (1974) using the method outlined above and is identical with Eq. (12.37) obtained using the algebraic method of De Meyer and Vanden Berghe (1978) discussed in Sec. 12.3.

12.5.4. ELASTIC SCATTERING OF PARTICLES OF ARBITRARY SPIN

Let us consider the scattering of particles of arbitrary spin j by a target nucleus of zero spin. The transition operator for this scattering can be of the general form

$$t = (\boldsymbol{J}\cdot\boldsymbol{A})^{2j} + (\boldsymbol{J}\cdot\boldsymbol{B})^{2j-1} + (\boldsymbol{J}\cdot\boldsymbol{C})^{2j-2} + \cdots. \tag{12.63}$$

since a tensor of maximum rank $2j$ is necessary to connect one projection of j to another projection of j.

The density matrix ρ_f of the scattered beam completely describes the spin orientation which can be represented conveniently by a set of parameters $\langle T_k^{m_k}\rangle$ defined by Eq. (12.46). The spherical tensor operator $T_k^{m_k}$ of rank k is in the spin space of the scattered beam and it satisfies the normalization condition (12.47). The differential cross section is given by Tr ρ_f,

$$\mathrm{Tr}\,\rho_f = t\,t^\dagger. \tag{12.64}$$

For the special case,

$$t = (\boldsymbol{J}\cdot\boldsymbol{C}) + D, \tag{12.65}$$

we give below Tr ρ_f and Tr $(\boldsymbol{T}_1\rho_f)$.

$$\mathrm{Tr}\,\rho_f = \frac{2j+1}{3}(\eta\boldsymbol{C}\cdot\boldsymbol{C}^* + 3DD^*), \tag{12.66}$$

$$\begin{aligned} \mathrm{Tr}(\boldsymbol{T}_1\rho_f) &= \sqrt{3/\eta}\,\mathrm{Tr}\,(\boldsymbol{J}\rho_f), \\ &= \sqrt{\eta/12}\,(2j+1)\{i(\boldsymbol{C}\times\boldsymbol{C}^*) + 2\boldsymbol{C}D^* + 2\boldsymbol{D}C^*\}. \end{aligned} \tag{12.67}$$

In a similar way, it is possible to evaluate $\mathrm{Tr}\,(T_2^{\pm 2}\rho_f)$, $\mathrm{Tr}\,(T_2^{\pm 1}\rho_f)$, $\mathrm{Tr}\,(T_2^{0}\rho_f)$ but $\mathrm{Tr}\,(T_k^{m_k}\rho_f) = 0$ for $k \geq 3$.

Review Questions

12.1 What are the properties obeyed by the product of Cartesian components of angular momentum matrices? Find the trace of J_λ^{2p} where λ stands for any one of the Cartesian basis x, y or z and p an integer. Show that the trace is a polynomial in η which is the eigenvalue of the operator J^2.

12.2 Evaluate (a) $\mathrm{Tr}(J_z^2)$, (b) $\mathrm{Tr}(J_z^4)$, (c) $\mathrm{Tr}(J_x J_y J_z)$, (d) $\mathrm{Tr}(J_z^2 J_x^2)$ in terms of the angular momentum quantum number j.

12.3 Find the condition for the trace of a product of angular momentum matrices in spherical basis to be non-vanishing. Evaluate $\mathrm{Tr}(J_-^k J_+^k)$, where J_+ and J_- are the ladder operators and k, an integer.

12.4 Construct the spin tensor operator T_k^k using J_+ operator and hence evaluate $\mathrm{Tr}(J_-^k J_+^k)$. The following C.G. coefficient is given:

$$\begin{bmatrix} j & k & j \\ j-k & k & j \end{bmatrix} = (-1)^k \left\{ \frac{(2j+1)!(2k)!}{k!(2j+k+1)!} \right\}^{\frac{1}{2}} .$$

12.5 Discuss briefly how the elastic scattering of particles with arbitrary spin by a spin zero target nucleus can be investigated using the trace techniques of angular momentum matrices. Assuming the transition operator to be of the form $\boldsymbol{J} \cdot \boldsymbol{C} + D$, obtain expressions for the scattering cross section and the polarization of the scattered beam.

Problems

12.1 Using the general properties of traces of products of angular momentum operators, choose from the following, the products of angular momentum operators whose trace is zero.

$$\text{(a) } J_x J_y, \quad \text{(b) } J_x J_y J_z, \quad \text{(c) } J_x^2 J_y^2, \quad \text{(d) } J_x^2 J_y^2 J_z,$$
$$\text{(e) } J_-^2 J_z J_+, \quad \text{(f) } J_-^2 J_z J_+^2, \quad \text{(g) } J_-^2 J_z^2 J_+ .$$

Give reasons for your answer.

12.2 Evaluate the following traces of products of angular momentum matrices.

$$\text{(a) } \mathrm{Tr}(J_x^3 J_y J_z), \quad \text{(b) } \mathrm{Tr}(J_z)^6, \quad \text{(c) } \mathrm{Tr}(J_x^4 J_y^2).$$

12.3 Evaluate the following traces.

$$\text{(a) } \mathrm{Tr}(J_- J_+), \quad \text{(b) } \mathrm{Tr}(J_- J_z J_+), \quad \text{(c) } \mathrm{Tr}(J_+ J_z J_-),$$
$$\text{(d) } \mathrm{Tr}(J_-^2 J_+^2), \quad \text{(e) } \mathrm{Tr}(J_- J_z^2 J_+), \quad \text{(f) } \mathrm{Tr}(J_+ J_z^2 J_-).$$

12.4 Evaluate the traces of the following:

$$(a)\,(\boldsymbol{J}\cdot\boldsymbol{A}),\ (b)\,\boldsymbol{J}(\boldsymbol{J}\cdot\boldsymbol{A}),\ (c)\,(\boldsymbol{J}\cdot\boldsymbol{A})(\boldsymbol{J}\cdot\boldsymbol{B}),\ (d)\,\boldsymbol{J}(\boldsymbol{J}\cdot\boldsymbol{A})(\boldsymbol{J}\cdot\boldsymbol{B}),$$

where $\boldsymbol{J}$ denotes the angular momentum operator and $\boldsymbol{A}$ and $\boldsymbol{B}$ are ordinary polar vectors.

12.5 Using the trace techniques of angular momentum operators, construct the spin tensor operators T_2^{μ}, $\mu = 2, 1, 0, -1, -2$.

Solutions to Selected Problems

12.1 (a) $\mathrm{Tr}(J_x J_y) = 0$, (d) $\mathrm{Tr}(J_x^2 J_y^2 J_z) = 0$,
since for the non-vanishing trace, the powers of J_x, J_y and J_z should all be even or odd.

(e) $\mathrm{Tr}(J_-^2 J_z J_+) = 0$, (g) $\mathrm{Tr}(J_-^2 J_z^2 J_+) = 0$,
since the powers of J_- should be equal to the power of J_+, for the trace to be non-vanishing.

12.4 (a) $\mathrm{Tr}(\boldsymbol{J}\cdot\boldsymbol{A}) = \mathrm{Tr}(J_x A_x + J_y A_y + J_z A_z) = 0.$

Expanding and retaining only the non-vanishing terms,

$$\begin{aligned}(b)\ \mathrm{Tr}\boldsymbol{J}(\boldsymbol{J}\cdot\boldsymbol{A}) &= \mathrm{Tr}(J_x^2)A_x\hat{\boldsymbol{i}} + \mathrm{Tr}(J_y^2)A_y\hat{\boldsymbol{j}} + \mathrm{Tr}(J_z^2)A_z\hat{\boldsymbol{k}}\\ &= \frac{1}{3}\Omega\,\boldsymbol{A}.\end{aligned}$$

$$\begin{aligned}(c)\ \mathrm{Tr}(\boldsymbol{J}\cdot\boldsymbol{A})(\boldsymbol{J}\cdot\boldsymbol{B}) &= \mathrm{Tr}\{(J_xA_x + J_yA_y + J_zA_z)\\ &\qquad\times(J_xB_x + J_yB_y + J_zB_z)\}\\ &= \mathrm{Tr}(J_x^2)A_xB_x + \mathrm{Tr}(J_y^2)A_yB_y + \mathrm{Tr}(J_z^2)B_z^2\\ &= \frac{1}{3}\Omega\,(\boldsymbol{A}\cdot\boldsymbol{B}).\end{aligned}$$

$$\begin{aligned}(d)\ \mathrm{Tr}\boldsymbol{J}(\boldsymbol{J}\cdot\boldsymbol{A})(\boldsymbol{J}\cdot\boldsymbol{B}) &= \mathrm{Tr}\{J_x(J_xA_x + J_yA_y + J_zA_z)\\ &\qquad\times(J_xB_x + J_yB_y + J_zB_z)\}\,\hat{\boldsymbol{i}} + \cdots\\ &= \mathrm{Tr}\{(J_xJ_yJ_z)A_yB_z + (J_xJ_zJ_y)A_zB_y\}\hat{\boldsymbol{i}}\\ &\quad + \cdots\\ &= i\,\frac{1}{6}\Omega\,(\boldsymbol{A}\times\boldsymbol{B}).\end{aligned}$$

12.5 The spin tensor T_2^{μ} can be constructed from the basic angular momentum operator J in spherical basis.

$$\begin{aligned} T_2^{\mu} &= k\,(J_1 \times J_1)_2^{\mu} \\ &= k \sum_{\mu_1,\mu_2} \begin{bmatrix} 1 & 1 & 2 \\ \mu_1 & \mu_2 & \mu \end{bmatrix} J_1^{\mu_1} J_1^{\mu_2}. \end{aligned}$$

The constant k can be determined from the normalization condition

$$\mathrm{Tr}\,(T_2^{\mu\dagger} T_2^{\mu}) = 2j+1.$$

Let us give below the explicit forms of the spin tensor operators.

$$\begin{aligned} T_2^2 &= k\,J_1^1\,J_1^1 = \frac{k}{2} J_+^2, \\ T_2^1 &= \frac{k}{\sqrt{2}}(J_1^0\,J_1^1 + J_1^1\,J_1^0) = -\frac{k}{2}(J_z\,J_+ + J_+\,J_z), \\ T_2^0 &= \frac{k}{\sqrt{6}}(J_1^1\,J_1^{-1} + J_1^{-1}\,J_1^1 + 2\,J_1^0\,J_1^0) \\ &= \frac{k}{\sqrt{6}}(-\tfrac{1}{2}\,J_+\,J_- - \tfrac{1}{2}\,J_-J_+ + 2\,J_z^2). \end{aligned}$$

In the above equations, the spherical tensor operators J_1^1, J_1^{-1}, J_1^0 are expressed in terms of the ladder operators J_+, J_- and J_z operator. In a similar way, the explicit forms of the other spin tensor operators T_2^{-2}, T_2^{-1} can be given. The constant k that occurs in each of the spin tensor operators is the same and can be determined using the normalization condition of anyone of them. For instance,

$$\begin{aligned} \mathrm{Tr}\,(T_2^{2\dagger}\,T_2^2) &= \frac{k^2}{4}(J_-^2\,J_+^2) \\ &= \frac{k^2}{4}\frac{2}{15}\,\Omega\,(2j-1)(2j+3). \end{aligned}$$

Using the normalization condition,

$$\frac{k^2}{4}\frac{2}{15}\,\Omega\,(2j-1)(2j+3) = 2j+1.$$

Since $\Omega = j(j+1)(2j+1)$, it follows that

$$k = \left(\frac{30}{j(j+1)(2j-1)(2j+3)}\right)^{\frac{1}{2}}.$$

It can be verified that the same result will be obtained if we choose instead the $\mathrm{Tr}\,(T_2^{1\dagger}\,T_2^1)$ or $\mathrm{Tr}\,(T_2^{0\dagger}\,T_2^0)$.

CHAPTER 13

THE HELICITY FORMALISM

13.1. The Helicity States

The component of spin s along the direction of motion of a particle is known as its helicity and the helicity quantum number is usually denoted by the symbol λ. It is also the component of total angular momentum $\boldsymbol{J}$ along the direction of motion since the orbital angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is perpendicular to the direction of motion $\hat{\boldsymbol{p}}$ and consequently its projection m_l on the momentum axis is zero.

The helicity formalism has been developed by Jacob and Wick (1959) for relativistic description of scattering of particles with spin and the decay of particles and resonant states. It is equally applicable to massless particles. The helicity formalism leads to simpler intensity and polarization formula over the conventional method in the study of scattering and reaction of particles. The advantages of using the helicity states are many.

1. There is no need to separate the total angular momentum $\boldsymbol{J}$ into orbital and spin parts and hence avoid the difficulties and complications that arise in the treatment of relativistic particles.
2. The helicity λ is invariant under rotations and so states can be constructed with definite $\boldsymbol{J}$ and helicities.
3. The helicity λ is well defined also for massless particles and so there is no need for separate treatment for massless particles.
4. The helicity states are directly related to individual polarization properties of the particles and hence convenient for the polarization study over the conventional formalism of choosing a reference frame with a fixed quantization axis, say z-axis. In the conventional scheme, one has to shuttle back and forth between two representations, one in which the scattering or reaction is conveniently described and the other in which the states are labeled with individual spin components.

In order to specify the helicity states of a particle of mass m and spin s, it is not necessary to know the relativistic wave equation for such a particle. It is enough to know that such a wave equation exists and their plane wave solutions, representing states of definite linear momentum p

and corresponding positive energy $E = (m^2 + p^2)^{1/2}$, have the following properties:

1. For each p, there are $2s + 1$ linearly independent solutions which can be characterized as states of definite helicity λ.

$$\lambda = s, s-1, \ldots, -s. \tag{13.1}$$

   These states characterized by p and λ form a complete set of orthogonal states for a free particle of mass m. If $m = 0$, the number of independent solutions reduces to two: $\lambda = \pm s$. For example, a photon has only two independent helicity states $\lambda = \pm 1$.
2. In the case of ordinary rotation in three dimensional space, the direction of p changes but the helicity λ remains unchanged.
3. Under space reflection about the origin (i.e. parity operation), the helicity λ of a moving particle changes sign.
4. When a Lorentz transformation is applied in the direction of p, the magnitude of p changes and in some cases, the direction of p also, if $m \neq 0$. If the direction of p is not reversed, the helicity λ remains unchanged under Lorentz transformation.

Let $\psi_{p,\lambda}$ denote the state of a particle with momentum p in the positive z-direction. By Lorentz transformation, all states $\psi_{p,\lambda}$ with fixed λ and variable p can be generated. If $m \neq 0$, it is possible to reach the rest state with $p = 0$ by Lorentz transformation. In the rest state, since the total angular momentum of the particle is equal to its spin, it is possible to obtain the relative phases of the states $\psi_{0,\lambda}$ by the requirement

$$(J_x \pm iJ_y)\psi_{0\lambda} = [(s \mp \lambda)(s \pm \lambda + 1)]^{1/2}\psi_{0\lambda\pm 1}. \tag{13.2}$$

In the above equation, J_x, J_y, J_z are the standard spin matrices. For a massless particle, no finite Lorentz transformation can reduce p to zero. For this, we have only two helicity states with $\lambda = \pm s$ and it is possible to go from one state to another by means of a reflection,

$$Y = \mathrm{e}^{-i\pi J_y}\mathcal{P}, \tag{13.3}$$

where $\mathcal{P}$ denotes the parity operator corresponding to reflection with respect to the origin ($x, y, z \to -x, -y, -z$), the operator $\mathrm{e}^{-i\pi J_y}$ denotes a rotation about the y axis through an angle π and Y, the reflection in the xz plane. The operator Y transforms the state $\psi_{p,s}$ into $\psi_{p,-s}$ apart from a phase factor.

$$Y\psi_{p,s} = \eta\psi_{p,-s}. \tag{13.4}$$

Since Y commutes with a Lorentz transformation in the z direction, η should be independent of p. It is therefore a constant which we shall call the "parity factor" of the particle. For example, the $\lambda = \pm 1$ solutions for a photon are $A_\pm = \mp\sqrt{2}(e_x \pm ie_y)\exp(ipz)$ such that $YA_\pm = -A_\mp$. Comparing this with Eq. (13.4), we obtain $\eta = -1$.

It is instructive to check the consistency of Eq. (13.4) with Eq. (13.2) for $m \neq 0$. In this case, $\mathcal{P}$ transforms $\psi_{0\lambda}$ into itself apart from a phase-factor which must be independent of λ ($\mathcal{P}$ commutes with J). Hence

$$\mathcal{P}\,\psi_{0\lambda} = \eta\,\psi_{0\lambda}. \tag{13.5}$$

Furthermore

$$e^{-i\pi J_y}\,\psi_{0\lambda} = \sum_{\lambda'} d^s_{\lambda',\lambda}(\pi)\,\psi_{0\lambda'}, \tag{13.6}$$

where the matrix element $d^s_{\lambda',\lambda}(\pi)$ is given by

$$d^s_{\lambda',\lambda}(\pi) = (-1)^{s-\lambda}\delta_{\lambda',-\lambda}. \tag{13.7}$$

Comparing Eqs. (13.5) and (13.6) and applying a Lorentz transformation in the z direction on both sides, we get

$$Y\,\psi_{p\lambda} = \eta(-1)^{s-\lambda}\psi_{p,-\lambda}, \tag{13.8}$$

which for $\lambda = s$ reduces to (13.4).

If $\psi_{p\lambda}$ denotes a state with momentum in the positive z direction, how can we define a state $\chi_{p\lambda}$ with momentum in the negative z direction? We will have occasion to use the state $\chi_{p,\lambda}$ in the treatment of two-particle scattering in centre of momentum frame wherein one particle moves in the positive direction while the other particle moves in the negative direction. A rotation through an angle π about the y axis corresponds to a transformation $x, y, z \rightarrow -x, y, -z$ and hence

$$\chi_{p\lambda} = (-1)^{s-\lambda}\,e^{-i\pi J_y}\,\psi_{p\lambda}. \tag{13.9}$$

The phase factor $(-1)^{s-\lambda}$ is introduced such that

$$\chi_{0\lambda} = \psi_{0,-\lambda}. \tag{13.10}$$

The result (13.10) is obtained from Eqs. (13.6) and (13.7).

It is possible to generate states $|p\,\theta\,\phi;\lambda\rangle$ with momentum $\boldsymbol{p}'(=p\,\theta\,\phi)$ in an arbitrary direction specified by polar angles $\theta\phi$ by means of a suitable

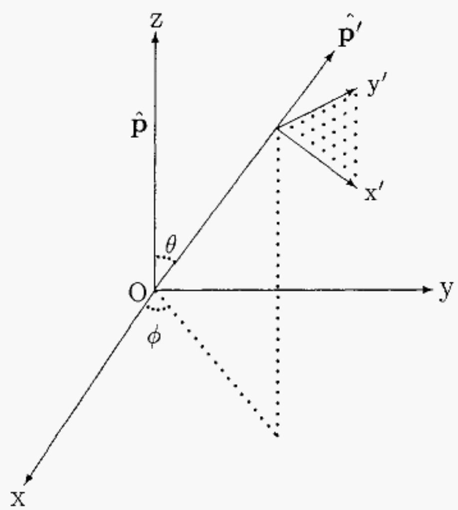

Figure 13.1. The fixed frame of reference x, y, z and the helicity frame x',y',z' (z' coinciding with the direction $\hat{\mathbf{p}}'$).

rotation $R(\alpha,\beta,\gamma)$ applied to states $\psi_{p\lambda}$ having a momentum $\boldsymbol{p}$ in the positive z-direction.

$$|p\,\theta\,\phi;\lambda\rangle = R(\alpha,\beta,\gamma)\,\psi_{p,\lambda}. \tag{13.11}$$

In the present notation, the state $\psi_{p\lambda}$ can be equivalently denoted as $|p\,0\,0;\lambda\rangle$. Two different conventions are in vogue for the choice of angles of rotation in R. Jacob and Wick (1959) used $\alpha=\phi,\beta=\theta,\gamma=-\phi$, corresponding to a rotation through an angle θ about the normal to the plane containing $\boldsymbol{p}$ and $\boldsymbol{p}'$. It is found more convenient to adopt the convention of Jacob (1964) and choose $\alpha=\phi,\beta=\theta,\gamma=0$. In this case, the x' and y' axes to be associated with the helicity direction $\hat{\boldsymbol{p}}'$ as z' axis are as indicated in Fig. 13.1. The positive x' direction is along the direction $(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}')\times\hat{\boldsymbol{p}}'$ and the positive y' direction coincides with the unit vector $(\hat{\boldsymbol{p}}\times\hat{\boldsymbol{p}}')$.

The state $|p\,0\,0;\lambda\rangle\ (=\psi_{p\lambda})$ is a plane wave state with momentum p in the direction of z-axis (chosen coordinate system) and it can be expanded in terms of states $|p\,j\,m;\lambda\rangle$ of definite angular momentum j and projection m. In the chosen coordinate system, $m=\lambda$ for all j

$$|p\,0\,0;\lambda\rangle = \sum_j C_j\,|p\,j\,\lambda;\lambda\rangle, \tag{13.12}$$

where C_j are the coefficients of expansion. Applying a rotation operator $R(\phi,\theta,0)$ on both sides, we obtain

$$|p\,\theta\,\phi,\lambda\rangle = \sum_{jm} C_j\, D^j_{m\lambda}(\phi,\theta,0)\,|p\,j\,m;\lambda\rangle. \tag{13.13}$$

The expansion coefficients C_j are determined by specifying the normalizations of the plane wave states $|p\,\theta\,\phi,\lambda\rangle$ and the angular momentum eigenstates $|p\,j\,m;\lambda\rangle$ and by using the orthogonality relations of the rotation matrices. The plane wave state $|p\,\theta\,\phi,\lambda\rangle$ is normalized such that

$$\langle p'\,\theta'\,\phi';\lambda'\,|\,p\,\theta\,\phi;\lambda\rangle = \delta_{p,p'}\,\delta_2(\theta\phi,\theta'\phi')\,\delta_{\lambda,\lambda'}, \tag{13.14}$$

where $\delta_2(\theta\phi,\theta'\phi')$ stands for

$$\delta_2(\theta\phi,\theta'\phi') = \delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi'). \tag{13.15}$$

The eigenstates of total angular momentum obey the normalization

$$\langle p'\,j'\,m';\lambda'\,|\,p\,j\,m;\lambda\rangle = \delta_{pp'}\,\delta_{jj'}\,\delta_{mm'}\,\delta_{\lambda\lambda'}. \tag{13.16}$$

The orthogonality relations of d-matrices are given by

$$\int_0^\pi d^j_{m\mu}(\beta)\,d^{j'}_{m\mu}(\beta)\,\sin\beta\,d\beta = \frac{2}{2j+1}\delta_{jj'}, \tag{13.17}$$

$$\frac{1}{2}\sum_j (2j+1)d^j_{m\mu}(\beta)\,d^j_{m\mu}(\beta') = \delta(\cos\beta - \cos\beta'). \tag{13.18}$$

Using the normalizations (13.14) and (13.16) of the plane wave states and the angular momentum states and the orthogonality of d-matrices (13.18), we obtain the expansion coefficient C_j.

$$C_j = \sqrt{\frac{2j+1}{4\pi}} \tag{13.19}$$

Thus, we obtain the important result of the expansion of the plane wave state as a sum of angular momentum states for a particle of arbitrary spin s.

$$|p\,\theta\,\phi;\lambda\rangle = \sum_{jm}\sqrt{\frac{2j+1}{4\pi}}D^j_{m\lambda}(\theta,\phi,0)|p\,j\,m;\lambda\rangle. \tag{13.20}$$

Since total angular momentum of the particle and its helicity are invariant under rotation, it is possible to obtain the inverse relation which enables us to project states of definite total angular momentum and helicity from the plane wave state.

$$|p\,j\,m\,;\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}\int D^{j*}_{m\lambda}(\phi,\theta,0)|p\,\theta\,\phi;\lambda\rangle\,d\Omega, \tag{13.21}$$

where

$$d\Omega = \sin\theta\,d\theta\,d\phi. \tag{13.22}$$

Equivalently, the transformation matrix that corresponds to a transition from the angular momentum state to the plane wave state is

$$\langle p\,j\,m;\lambda\,|\,p\,\theta\,\phi;\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}D^{j}_{m\lambda}(\phi,\theta,0). \tag{13.23}$$

It is easy to verify that the normalizations (13.14) and (13.16) are consistent with the definitions (13.20) and (13.21), using the orthogonality relations of d-matrices. From Eq. (13.20), we find

$$\begin{aligned}\langle p'\,\theta'\,\phi';\lambda'|p\,\theta\,\phi;\lambda\rangle &= \sum_{jm}\sum_{j'm'}\frac{[j'][j]}{4\pi}D^{j'*}_{m'\lambda'}(\phi',\theta',0)D^{j}_{m\lambda}(\phi,\theta,0)\\ &\quad\times\langle p'\,j'\,m';\lambda'\,|\,p\,j\,m;\lambda\rangle\\ &= \delta_{pp'}\sum_{jm}\frac{2j+1}{4\pi}D^{j*}_{m\lambda}(\phi'\,\theta'\,0)D^{j}_{m\lambda}(\phi,\theta,0)\\ &= \delta_{pp'}\,\delta_2(\theta\,\phi,\theta'\,\phi'),\end{aligned} \tag{13.24}$$

using the normalization (13.16) and the orthogonality relation (13.18) of the d-matrices. Similarly, starting with Eq. (13.21) and using the normalization (13.14) and the orthogonality relation (13.17), we obtain

$$\begin{aligned}\langle p'\,j'\,m';\lambda'|p\,j\,m;\lambda\rangle &= \frac{[j'][j]}{4\pi}\int D^{j*}_{m\lambda}(\phi,\theta,0)\,D^{j'}_{m'\lambda'}(\phi',\theta',0)\\ &\quad\times\langle p'\,\theta'\,\phi';\lambda'|p\,\theta\,\phi;\lambda\rangle d\Omega\,d\Omega'\\ &= \delta_{pp'}\,\delta_{jj'}\,\delta_{mm'}.\end{aligned} \tag{13.25}$$

Equation (13.20) is the expansion of the angular function of a plane wave. It may be noted that the angular dependence of the wave function is given by a D-function instead of a spherical harmonic function which occurs in the case of spin-zero particle. For spin-zero particle,

$$\lambda = 0;\quad j\to l;\quad D^{j}_{m0}(\phi,\theta,0)\to\sqrt{\frac{4\pi}{2l+1}}Y^{m*}_{l}(\theta,\phi). \tag{13.26}$$

Hence, for spin-zero particle, Eqs. (13.20), (13.21) and (13.23) reduce to

$$|p\,\theta\,\phi;\lambda=0\rangle \;\rightarrow\; \sum_{lm} Y_l^{m*}(\theta,\phi)\,|lm\rangle = \sum_{lm} Y_l^{m*}(\hat{\boldsymbol{p}})\,Y_l^m(\hat{\boldsymbol{r}}), \tag{13.27}$$

$$|p\,j\,m;\lambda\rangle \;\rightarrow\; Y_l^m(\hat{\boldsymbol{r}}), \tag{13.28}$$

$$\langle p\,j\,m;\lambda\,|\,p\,\theta\,\phi;\lambda\rangle \;\rightarrow\; Y_l^{m*}(\theta,\phi) = Y_l^{m*}(\hat{\boldsymbol{p}}). \tag{13.29}$$

13.2. Two-Particle Helicity States

In the two-body scattering such as $a + b \rightarrow c + d$, the initial and final states are two-particle states. A non-interacting two-particle plane wave state with helicities λ_1 and λ_2 can be written as a direct product of two one-particle states (Martin and Spearman, 1970; Jacob, 1964).

$$|\boldsymbol{p}_1\,\boldsymbol{p}_2;\lambda_1\,\lambda_2\rangle = |\boldsymbol{p}_1;\lambda_1\rangle \otimes |\boldsymbol{p}_2;\lambda_2\rangle. \tag{13.30}$$

It is advantageous to go to the centre of momentum (c.m) frame and analyse the wave function in terms of centre of mass motion and relative motion in c.m. system.

$$|\boldsymbol{p}_1\,\boldsymbol{p}_2;\lambda_1\,\lambda_2\rangle = |\boldsymbol{P}\rangle \otimes |\boldsymbol{p};\lambda_1\,\lambda_2\rangle, \tag{13.31}$$

where $|\boldsymbol{P}\rangle$ is the state vector denoting the c.m. motion and $|\boldsymbol{p};\lambda_1\,\lambda_2\rangle$, the relative motion of the two-particle system.

In any physical problem, we are concerned only with the wave function denoting the relative motion in c.m. system and our aim is to construct the two-particle helicity states of definite total angular momentum.

To start with, let us consider the relative motion of the two particles to be along the z-axis, one particle moving along the positive z-axis and the other particle moving with the same momentum p along the negative z-axis. Then

$$|\boldsymbol{p};\lambda_1\,\lambda_2\rangle = |p,\,\theta=0,\,\phi=0;\lambda_1\,\lambda_2\rangle = \psi_{p\lambda_1}\,\chi_{p\lambda_2}, \tag{13.32}$$

where $\psi_{p\lambda_1}$ denotes the one-particle state with momentum p along the positive z-axis and helicity λ_1, and $\chi_{p\lambda 2}$ as defined in Eq. (13.9), denotes the state of the other particle with momentum p along the negative z-axis and helicity λ_2. The resultant helicity λ of the two-particle system is

$$\lambda = \lambda_1 - \lambda_2. \tag{13.33}$$

The two-particle state vectors $|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle$, representing relative motion along any arbitrary direction can be generated by a suitable rotation $R(\phi,\theta,0)$.

$$|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle = R(\phi,\theta,0)|p\,0\,0\,;\,\lambda_1\,\lambda_2\rangle. \tag{13.34}$$

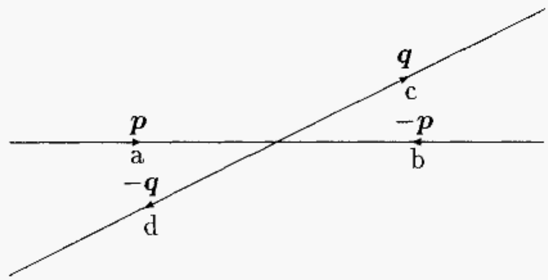

Figure 13.2. The two-body scattering in c.m. system

The plane wave state is a sum over all angular momentum eigenstates and conversely an angular momentum eigenstate can be obtained by angular momentum projection of plane wave state. Using the procedure followed in Sec. 13.1, expressions for two-particle plane wave state and angular momentum eigenfunctions are obtained.

$$|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle = \sum_j \sqrt{\frac{2j+1}{4\pi}}\,D^j_{m\lambda}(\phi,\theta,0)\,|p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle, \tag{13.35}$$

$$|p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle = \sqrt{\frac{2j+1}{4\pi}}\int D^{j\,*}_{m\lambda}(\phi,\theta,0)\,|p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle d\Omega. \tag{13.36}$$

The normalizations of the state vectors in the two representations are given by

$$\langle p\,\theta'\,\phi'\,;\,\lambda_1'\,\lambda_2'\,|\,p\,\theta\,\phi\,;\,\lambda_1\,\lambda_2\rangle = \delta_2(\theta\phi,\theta'\phi')\,\delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'}, \tag{13.37}$$

$$\langle p\,j'\,m'\,;\,\lambda_1'\,\lambda_2'\,|\,p\,j\,m\,;\,\lambda_1\,\lambda_2\rangle = \delta_{jj'}\,\delta_{m\,m'}\,\delta_{\lambda_1\lambda_1'}\,\delta_{\lambda_2\lambda_2'}. \tag{13.38}$$

13.3. Scattering of Particles with Spin

13.3.1. SCATTERING CROSS SECTION

Consider a two-body scattering of particles with spin

$$a + b \rightarrow c + d \tag{13.39}$$

in the c.m. system as described in Fig. 13.2. The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{p}\right)^2 |\langle q\,\theta\,\phi\,;\,\lambda_c\,\lambda_d|\,T(W)\,|p\,0\,0\,;\,\lambda_a\,\lambda_b\rangle|^2, \tag{13.40}$$

where p denotes the relative momentum of the two particles along the z-axis in the initial state and q denotes the relative momentum of the scattered particles in the final state making an angle ϕ,θ with the incident direction in the c.m. frame. The total energy in the c.m. system is denoted by W and it is conserved in any reaction.

$$W = (p^2 + m_a^2)^{\frac{1}{2}} + (p^2 + m_b^2)^{\frac{1}{2}} = (q^2 + m_c^2)^{\frac{1}{2}} + (q^2 + m_d^2)^{\frac{1}{2}}. \quad (13.41)$$

For evaluating the T-matrix, it is transformed to jm representation.

$$\langle \theta\, \phi; \lambda_c\, \lambda_d \,|\, T(W)\, |0\, 0\, ; \lambda_a\, \lambda_b\rangle = \sum_{jm} \sum_{j'm'} \langle \theta\, \phi; \lambda_c\, \lambda_d \,|\, j'\, m'; \lambda_c\, \lambda_d\rangle$$
$$\times \langle j'\, m'; \lambda_c\, \lambda_d \,|\, T(W)\, |j\, m\, ; \lambda_a\, \lambda_b\rangle \langle j\, m; \lambda_a\, \lambda_b\, |0\, 0\, ; \lambda_a\, \lambda_b\rangle. \quad (13.42)$$

The rotational invariance implies the conservation of angular momentum and hence j is a good quantum number.

$$\langle j'\, m'; \lambda_c\, \lambda_d \,|\, T(W)\, |j\, m\, ; \lambda_a\, \lambda_b\rangle = \delta_{jj'}\, \delta_{mm'}\, \langle \lambda_c\, \lambda_d \,|\, T^j(W)\, |\lambda_a\, \lambda_b\rangle. \quad (13.43)$$

Using Eqs. (13.35), Eq. (13.42) becomes

$$\langle \theta\, \phi; \lambda_c\, \lambda_d \,|\, T(W)\, |0\, 0\, ; \lambda_a\, \lambda_b\rangle = \sum_{jm} \frac{2j+1}{4\pi} D^{j*}_{m\lambda_f}(\phi, \theta, 0)$$
$$\times \langle \lambda_c\, \lambda_d \,|\, T^j(W)\, |\lambda_a\, \lambda_b\rangle D^j_{m\lambda_i}(0, 0, 0), \quad (13.44)$$

with $\lambda_i = \lambda_a - \lambda_b$ and $\lambda_f = \lambda_c - \lambda_d$. Since

$$D_{m\lambda_i}(0, 0, 0) = \delta_{m\lambda_i}, \quad (13.45)$$

we obtain

$$\langle \theta\, \phi; \lambda_c\, \lambda_d \,|\, T(W)\, |0\, 0\, ; \lambda_a\, \lambda_b\rangle = \sum_{j} \frac{2j+1}{4\pi} D^{j*}_{\lambda_i\lambda_f}(\phi, \theta, 0)$$
$$\times \langle \lambda_c\, \lambda_d \,|\, T^j(W)\, |\lambda_a\, \lambda_b\rangle. \quad (13.46)$$

Denoting the scattering amplitude in the helicity basis by $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta, \phi)$, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = |f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta, \phi)|^2. \quad (13.47)$$

From Eqs. (13.40),(13.46) and (13.47), we find

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta, \phi) = \sum_j (2j+1) D^{j*}_{\lambda_i\lambda_f}(\phi, \theta, 0)\, f^j_{\lambda_f\lambda_i}(W), \quad (13.48)$$

with

$$f^j_{\lambda_f\lambda_i}(W) = \frac{1}{2p}\langle\lambda_c\lambda_d\,|\,T^j(W)\,|\lambda_a\lambda_b\rangle. \tag{13.49}$$

For scattering of spinless particles,

$$j \to l; \qquad \lambda_a, \lambda_b, \lambda_c, \lambda_d \to 0; \qquad \lambda_i, \lambda_f \to 0. \tag{13.50}$$

$$D^{j^*}_{\lambda_i\lambda_f}(\phi,\theta,0) \to D^{l^*}_{00}(\phi,\theta,0) = \left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}} Y^0_l(\theta,\phi) = P_l(\cos\theta). \tag{13.51}$$

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\phi) \to \sum_l (2l+1)P_l(\cos\theta)f_l(W). \tag{13.52}$$

The amplitude $f_l(W)$ $(= T_l(W)/2p)$ is known as the partial wave scattering amplitude for spinless particles. When the particles considered have spin, the total angular momentum j is a good quantum number and for each j, there are several scattering amplitudes which depend on helicity states but the number of independent amplitudes get reduced by invoking parity and time reversal invariance.

Equations (13.47) and (13.48) are general expressions applicable for scattering of particles with arbitrary spin. These formulae are relativistically correct and they are applicable equally well to massless particles and to particles without spin. It is found that the D-functions that occur for particles with spin reduce to Legendre functions for particles without spin.

Let us now explicitly square the scattering amplitude (13.48) and obtain an expression for the differential cross section and total cross section.

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \rho_{\lambda_a\lambda_b}\sum_{jj'}(2j+1)(2j'+1)f^j_{\lambda_f\lambda_i}(W)f^{j'^*}_{\lambda_f\lambda_i}(W) \\ &\quad\times D^{j^*}_{\lambda_i\lambda_f}(\phi,\theta,0)\,D^{j'}_{\lambda_i\lambda_f}(\phi,\theta,0),\end{aligned} \tag{13.53}$$

where $\rho_{\lambda_a\lambda_b}$ denotes the density matrix that describes the initial state. Using the symmetry property of the D-functions and using the C.G. series (5.48), we obtain

$$\begin{aligned}&D^{j^*}_{\lambda_i\lambda_f}(\phi,\theta,0)\,D^{j'}_{\lambda_i\lambda_f}(\phi,\theta,0) \\ &\quad= (-1)^{\lambda_i-\lambda_f}D^j_{-\lambda_i,-\lambda_f}(\phi,\theta,0)\,D^{j'}_{\lambda_i\lambda_f}(\phi,\theta,0) \\ &\quad= (-1)^{\lambda_i-\lambda_f}\sum_l\begin{bmatrix} j & j' & l \\ -\lambda_i & \lambda_i & 0\end{bmatrix}\begin{bmatrix} j & j' & l \\ -\lambda_f & \lambda_f & 0\end{bmatrix}D^l_{00}(\phi,\theta,0).\end{aligned} \tag{13.54}$$

Note that

$$D^l_{00}(\phi,\theta,0)=\sqrt{\frac{4\pi}{2l+1}}Y_l^0(\theta,\phi)=P_l(\cos\theta). \tag{13.55}$$

If the incident and the target particles are not polarized and if the polarization of the final particles are not observed, we need to sum over λ_c and λ_d and average over λ_a and λ_b.

$$\frac{d\sigma}{d\Omega}=\frac{1}{(2s_a+1)(2s_b+1)}\sum_{(\lambda)}\sum_{jj'}(2j+1)(2j'+1)\mathrm{Re}\{f^j_{\lambda_f\lambda_i}f^{j'*}_{\lambda_f\lambda_i}\}$$
$$\times\sum_l(-1)^{\lambda_i-\lambda_f}\begin{bmatrix} j & j' & l\\ -\lambda_i & \lambda_i & 0\end{bmatrix}\begin{bmatrix} j & j' & l\\ -\lambda_f & \lambda_f & 0\end{bmatrix}P_l(\cos\theta), \tag{13.56}$$

where the summation index (λ) stands for helicities λ_a, λ_b, λ_c, λ_d of all incident and scattered particles and Re stands for real part of $\{f^j_{\lambda_f\lambda_i}f^{j'*}_{\lambda_f\lambda_i}\}$ In the above formula, the statistical weight $(2s+1)$ has to be replaced by 2 for a massless particle.

Integrating (13.56) over the solid angle, we obtain the total cross section

$$\sigma=\frac{4\pi}{(2s_a+1)(2s_b+1)}\sum_{(\lambda)}\sum_j(2j+1)\,|f^j_{\lambda_f\lambda_i}|^2, \tag{13.57}$$

using the following relations:

$$\int P_l(\cos\theta)\,d\Omega = 4\,\pi\,\delta_{l0}, \tag{13.58}$$

$$\begin{bmatrix} j & j' & l\\ -\lambda & \lambda & 0\end{bmatrix}\delta_{l0} = \frac{(-1)^{j+\lambda}}{[j]}\,\delta_{jj'}\,. \tag{13.59}$$

13.3.2. INVARIANCE UNDER PARITY AND TIME REVERSAL

From Eq. (13.56), we find that, for each value of j, there are in total $(2s_a+1)(2s_b+1)(2s_c+1)(2s_d+1)$ helicity amplitudes. Invariance under parity and time reversal reduces the number of independent amplitudes.

The helicity defined by $\boldsymbol{J}\cdot\hat{\boldsymbol{p}}$ changes sign under space inversion. A state with helicity λ is transformed into a state with helicity $-\lambda$. If $\mathcal{P}$ is the parity operator,

$$\mathcal{P}\,|jm;\lambda_a\,\lambda_b\rangle=\eta_a\eta_b(-1)^{j-s_a-s_b}|jm;-\lambda_a,-\lambda_b\rangle, \tag{13.60}$$

where η_a, η_b denote the intrinsic parities of the two particles with spin s_a and s_b. $\mathcal{P}$ is a unitary operator and invariance of the S-matrix under parity implies that $\mathcal{P}^\dagger S \mathcal{P} = \mathcal{S}$. Since $S = 1 + iT$, it follows that $\mathcal{P}^\dagger T \mathcal{P} = T$.

$$\begin{aligned} \langle \lambda_c, \lambda_d | T^j(W) | \lambda_a, \lambda_b \rangle &= \langle \lambda_c, \lambda_d | \mathcal{P}^\dagger T^j(W) \mathcal{P} | \lambda_a, \lambda_b \rangle \\ &= \eta_a \eta_b \eta_c \eta_d (-1)^{s_c+s_d-s_a-s_b} \\ &\quad \times \langle -\lambda_c, -\lambda_d | T^j(W) | -\lambda_a, -\lambda_b \rangle. \end{aligned} \tag{13.61}$$

Under time reversal, both $\boldsymbol{J}$ and $\boldsymbol{p}$ change sign and hence the helicity does not change. By applying the time reversal operator T to the state $|j\, m : \lambda_a \lambda_b\rangle$, we obtain a new state with the same angular momentum and helicities but with an opposite eigenvalue of J_z. With the phase conventions of Jacob and Wick (1959),

$$\mathcal{T} |j\, m; \lambda_a\, \lambda_b\rangle = (-1)^{j-m} |j\, -m; \lambda_a\, \lambda_b\rangle. \tag{13.62}$$

The operator T is antiunitary and hence the invariance under time reversal implies $\mathcal{T}^\dagger S^\dagger \mathcal{T} = S$.

$$\begin{aligned} \langle \lambda_c\, \lambda_d | T^j(W) | \lambda_a\, \lambda_b \rangle &= \langle jm; \lambda_c\, \lambda_d | T^j(W) | jm; \lambda_a\, \lambda_b \rangle \\ &= \langle jm; \lambda_c\, \lambda_d | \mathcal{T}^\dagger T^{j\dagger}(W) \mathcal{T} | jm; \lambda_a\, \lambda_b \rangle \\ &= (-1)^{j-m} \langle jm; \lambda_c\, \lambda_d | \mathcal{T}^\dagger T^{j\dagger}(W) | j\, -m; \lambda_a \lambda_b \rangle \\ &= (-1)^{j-m} \langle j\, -m; \lambda_a\, \lambda_b | T^j(W) \mathcal{T} | j\, m; \lambda_c \lambda_d \rangle \\ &= \langle j\, -m; \lambda_a\, \lambda_b | T^j(W) | j\, -m; \lambda_c \lambda_d \rangle \\ &= \langle \lambda_a \lambda_b | T^j(W) | \lambda_c \lambda_d \rangle. \end{aligned} \tag{13.63}$$

This yields the familiar result that under time reversal invariance, the transition $a + b \rightarrow c + d$ is equal to the inverse transition $c + d \rightarrow a + b$.

For identical particles, we have a further relation.

$$\langle \lambda_c\, \lambda_d | T^j(W) | \lambda_a\, \lambda_b \rangle = \langle \lambda_d\, \lambda_c | T^j(W) | \lambda_b\, \lambda_a \rangle. \tag{13.64}$$

13.3.3. POLARIZATION STUDIES

Since the polarizations of the particles are considered separately, formulas giving polarizations take a simple form in the Helicity Formalism. The longitudinal polarization can obviously be introduced by giving different weights to the positive and negative helicity amplitudes in Eq. (13.56). However, it is the angular distribution of the transverse polarization that is more informative.

Transverse polarization is usually defined by means of the expectation value of a transverse component of the spin. The definition of transverse components of spin is somewhat arbitrary in the relativistic case and for a massless particle, the transverse component cannot be defined at all. So, in what follows, we consider only the transverse polarization of a particle with finite mass, for which one can go to the rest frame by Lorentz transformation. The helicity remains unchanged in Lorentz transformation and so also the density matrix in helicity basis. Using the known non-relativistic form for spin matrices, we obtain after simplification that (the reader is referred to solved problem 13.1 for derivation)

$$\begin{aligned}\langle s_y \rangle &= \mathrm{Tr}\,(s_y\,\rho) \\ &= \sum_{\lambda} \{(s+\lambda)(s-\lambda+1)\}^{1/2}\,\mathrm{Im}(\rho_{\lambda-1,\lambda}),\end{aligned} \tag{13.65}$$

where Im(...) denotes the imaginary part of the quantity within the bracket. Using the algebraic form of C.G. coefficient,

$$\begin{aligned}\begin{bmatrix} s & 1 & s \\ \lambda & -1 & \lambda-1 \end{bmatrix} &= \left\{\frac{(s+\lambda)(s-\lambda+1)}{2s(1+s)}\right\}^{1/2} \\ &= -\begin{bmatrix} s & 1 & s \\ \lambda-1 & 1 & \lambda \end{bmatrix},\end{aligned} \tag{13.66}$$

Equation (13.65) can be rewritten as

$$\langle s_y \rangle = -\{2s(1+s)\}^{1/2} \sum_{\lambda} \begin{bmatrix} s & 1 & s \\ \lambda-1 & 1 & \lambda \end{bmatrix} \mathrm{Im}(\rho_{\lambda-1,\lambda}). \tag{13.67}$$

We shall consider two specific cases. 1. The incident particle a is transversely polarized with the polarization $\langle s_{ay} \rangle$. What is the "polarized cross section" i.e., the part of the cross section $d\sigma/d\Omega$ which is proportional to $\langle s_{ay} \rangle$? 2. The incident and target particles are unpolarized. What is the transverse polarization $\langle s_{cy'} \rangle$ of the outgoing particle c in the reaction?

Case 1

If the incident particle a has transverse polarization $\langle s_{ay} \rangle$, then its spin density matrix can be written as (the reader is referred to solved problem 13.2 for derivation)

$$\rho_a = \frac{1}{2s_a+1}\left[1 + \frac{3}{s_a(s_a+1)}\langle s_{ay}\rangle s_{ay} + \cdots\right]. \tag{13.68}$$

If we restrict our consideration to vector polarization and neglect higher order tensor contributions, the density matrix for the initial system is

$$\rho_i = \rho_a\rho_b = \frac{1}{(2s_a+1)(2s_b+1)}\left[1 + \frac{3}{s_a(s_a+1)}\langle s_{ay}\rangle\, s_{ay}\right]. \tag{13.69}$$

The cross section depends on the density matrix for the final state which is evaluated if the scattering amplitude f and the density matrix of the initial state ρ_i are known.

$$\begin{aligned}\operatorname{Tr}\rho_f &= \operatorname{Tr}(f\rho_i f^\dagger) = \operatorname{Tr}(\rho_i f^\dagger f)\\ &= \sum_{\lambda_i\lambda_i'}(\rho_i)_{\lambda_i'\lambda_i}(f^\dagger f)_{\lambda_i\lambda_i'},\end{aligned} \tag{13.70}$$

where $(f^\dagger f)$ can be considered as the density matrix ρ_f^0 corresponding to the final state when the incident particles are unpolarized. Using Eq. (13.69) for the density matrix for the initial system, the polarized cross section $\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}}$ that is propotional to $\langle s_{ay}\rangle$ is obtained from (13.70).

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}} &= \frac{3}{(2s_a+1)(2s_b+1)s_a(s_a+1)}\langle s_{ay}\rangle\\ &\quad\times\sum_{\lambda_i}(s_{ay})_{\lambda_i-1,\lambda_i}\operatorname{Im}(f^\dagger f)_{\lambda_i,\lambda_i-1}.\end{aligned} \tag{13.71}$$

Expanding $(f^\dagger f)_{\lambda_i,\lambda_i-1}$ as $\sum_{\lambda_f}(f^\dagger)_{\lambda_i,\lambda_f}(f)_{\lambda_f,\lambda_i-1}$ and substituting the expansion (13.48) for the scattering amplitude f, we obtain

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}} &= \frac{3}{(2s_a+1)(2s_b+1)s_a(s_a+1)}\langle s_{ay}\rangle\sum_{jj'}\sum_{\lambda_f\lambda_i}(2j+1)(2j'+1)\\ &\quad\times\operatorname{Im}\left\{D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'*}_{\lambda_i-1,\lambda_f}(\phi,\theta,0)\,f^{j*}_{\lambda_f,\lambda_i}\,f^{j'}_{\lambda_f,\lambda_i-1}\right\}\\ &\quad\times(s_{ay})_{\lambda_i-1,\lambda_i}.\end{aligned} \tag{13.72}$$

Equation (13.72) can be simplified by coupling the two D-matrices by using C.G. series (5.48).

$$\begin{aligned}&D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'*}_{\lambda_i-1,\lambda_f}(\phi,\theta,0)\\ &\quad=(-1)^{\lambda_i-1-\lambda_f}D^j_{\lambda_i,\lambda_f}(\phi,\theta,0)\,D^{j'}_{1-\lambda_i,-\lambda_f}(\phi,\theta,0)\\ &\quad=(-1)^{\lambda_i-1-\lambda_f}\sum_L\begin{bmatrix} j & j' & L\\ \lambda_i & 1-\lambda_i & 1\end{bmatrix}\begin{bmatrix} j & j' & L\\ \lambda_f & -\lambda_f & 0\end{bmatrix}D^L_{10},\end{aligned} \tag{13.73}$$

with

$$\begin{aligned}D^L_{10}(\phi,\theta,0) &= \mathrm{e}^{-i\phi}\,d^L_{10}(\theta)\\ &= \mathrm{e}^{-i\phi}\left[-\{L(L+1)\}^{-\frac{1}{2}}\sin\theta\,P_L'(\cos\theta)\right].\end{aligned} \tag{13.74}$$

Using Eq. (13.66), we obtain the matrix element of s_{ay}.

$$\begin{aligned}(s_{ay})_{\lambda_i-1,\lambda_i} &= \{(s_a+\lambda_i)(s_a-\lambda_i+1)\}^{1/2} \\ &= -\{2s_a(1+s_a)\}^{1/2}\begin{bmatrix} s_a & 1 & s_a \\ \lambda_i-1 & 1 & \lambda_i \end{bmatrix}. \end{aligned} \tag{13.75}$$

Substituting Eqs. (13.73) - (13.75) into Eq. (13.72), we obtain the polarized cross section arising from the transverse polarization $\langle s_{ay}\rangle$ of particle *a.*

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{p}} &= \frac{3}{(2s_a+1)(2s_b+1)}\sqrt{\frac{2}{s_a(1+s_a)}}\langle s_{ay}\rangle \\ &\times \sum_{jj'}\sum_{L}\sum_{\lambda_i\lambda_f}(2j+1)(2j'+1)\mathrm{Im}\left\{f^{j\,*}_{\lambda_f,\lambda_i}\, f^{j'}_{\lambda_f,\lambda_i-1}\,\mathrm{e}^{-i\phi}\right\} \\ &\times(-1)^{\lambda_i-\lambda_f}\begin{bmatrix} j & j' & L \\ \lambda_i & 1-\lambda_i & 1\end{bmatrix}\begin{bmatrix} j & j' & L \\ \lambda_f & -\lambda_f & 0\end{bmatrix} \\ &\times\begin{bmatrix} s_a & 1 & s_a \\ \lambda_i & -1 & \lambda_i-1\end{bmatrix}\{L(L+1)\}^{-\frac{1}{2}}\sin\theta\, P_L'(\cos\theta). \end{aligned} \tag{13.76}$$

Case 2

Let us now consider the transverse polarization of one final particle, say *c,* when the initial particles are not polarized and when the polarization of the other final particle *d* is not observed. The polarization of particle *c* of spin s_c normal to the production plane is

$$\langle s_{cy'}\rangle = \frac{\mathrm{Tr}\,(s_{cy'}\rho_f)}{\mathrm{Tr}\,\rho_f}, \tag{13.77}$$

where $\mathrm{Tr}\rho_f$ is just the differential cross section $d\sigma/d\Omega$. So,

$$\langle s_{cy'}\rangle\frac{d\sigma}{d\Omega} = \mathrm{Tr}\,(s_{cy'}\rho_f). \tag{13.78}$$

Using Eqs. (13.65) and (13.67), we obtain

$$\begin{aligned}\mathrm{Tr}\,(s_{cy'}\rho_f) &= \sum_{\lambda_c}\{(s_c+\lambda_c)(s_c-\lambda_c+1)\}^{\frac{1}{2}}\,\mathrm{Im}\,(\rho_f)_{\lambda_c-1,\lambda_c} \\ &= -\{2s_c(1+s_c)\}^{1/2}\sum_{\lambda_c}\begin{bmatrix} s_c & 1 & s_c \\ \lambda_c-1 & 1 & \lambda_c\end{bmatrix} \\ &\quad\times\mathrm{Im}\,(\rho_f)_{\lambda_c-1,\lambda_c}. \end{aligned} \tag{13.79}$$

Since the particles in the initial state are not polarized, the elements of the spin density matrix of the final state is given by

$$(\rho_f)_{\lambda'_f,\lambda_f} = \frac{1}{(2s_a+1)(2s_b+1)} f_{\lambda'_f,\lambda_i} f^*_{\lambda_f,\lambda_i} \tag{13.80}$$

where, for brevity, single helicity quantum number is used to denote a two-particle helicity state as shown below.

$$\lambda_i = \{\lambda_a, \lambda_b\}; \;\; \lambda_f = \{\lambda_c, \lambda_d\}; \;\; \lambda'_f = \{\lambda_c - 1, \lambda_d\}. \tag{13.81}$$

Substituting Eq. (13.48) for the helicity amplitudes $f_{\lambda'_f,\lambda_i}, f^*_{\lambda_f,\lambda_i}$, we obtain

$$\begin{aligned}(\rho_f)_{\lambda'_f,\lambda_f} &= \sum_{jj'} \frac{(2j+1)(2j'+1)}{(2s_a+1)(2s_b+1)} D^{j*}_{\lambda_i\lambda'_f}(\phi,\theta,0)\, D^{j'}_{\lambda_i\lambda_f}(\phi,\theta,0) \\ &\quad \times f^j_{\lambda'_f,\lambda_i} f^{j'*}_{\lambda_f,\lambda_i}. \end{aligned} \tag{13.82}$$

Coupling the two rotation matrices using C.G. series (5.48) and using Eq. (13.79), we obtain

$$\begin{aligned}\langle s_{cy'}\rangle \frac{d\sigma}{d\Omega} &= \frac{1}{(2s_a+1)(2s_b+1)} \left[-\{2s_c(1+s_c)\}^{\frac{1}{2}}\right] \begin{bmatrix} s_c & 1 & s_c \\ \lambda_c - 1 & 1 & \lambda_c \end{bmatrix} \\ &\quad \times \sum_{(\lambda)} \sum_{jj'L} (2j+1)(2j'+1)(-1)^{\lambda_i - \lambda'_f} \begin{bmatrix} j & j' & L \\ -\lambda_i & \lambda_i & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j' & L \\ -\lambda'_f & \lambda_f & 1 \end{bmatrix} D^L_{0,1}(\phi,\theta,0)\, f^j_{\lambda'_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i}. \end{aligned} \tag{13.83}$$

Using the analytical expression for the rotation matrix,

$$D^L_{01}(\phi,\theta,0) = d^L_{01}(\theta) = \{L(L+1)\}^{-\frac{1}{2}} \sin\theta\, P'_L(\cos\theta), \tag{13.84}$$

we finally obtain

$$\begin{aligned}\langle s_{cy'}\rangle \frac{d\sigma}{d\Omega} &= \frac{1}{(2s_a+1)(2s_b+1)} \left[-\{2s_c(1+s_c)\}^{\frac{1}{2}}\right] \begin{bmatrix} s_c & 1 & s_c \\ \lambda_c - 1 & 1 & \lambda_c \end{bmatrix} \\ &\quad \times \sum_{(\lambda)} \sum_{jj'L} (2j+1)(2j'+1)(-1)^{\lambda_i - \lambda'_f} \begin{bmatrix} j & j' & L \\ -\lambda_i & \lambda_i & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} j & j' & L \\ -\lambda'_f & \lambda_f & 1 \end{bmatrix} \frac{\sin\theta\, P'_L(\cos\theta)}{\sqrt{L(L+1)}} \operatorname{Im}\left\{ f^j_{\lambda'_f\lambda_i} f^{j'*}_{\lambda_f\lambda_i} \right\}. \end{aligned} \tag{13.85}$$

A similar formula may be obtained for $\langle s_{cx'}\rangle$ and may be shown to vanish, as one expects, if the scattering matrix satisfies the symmetry condition for parity conservation discussed in Sec. 13.3.2.

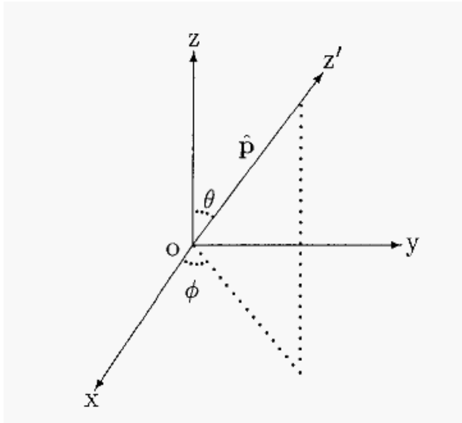

Figure 13.3. The unprimed coordinate system is the rest frame of γ and the primed coordinate system is the helicity frame for the decay products α and β.

13.4. Two-Body Decay

Let us now investigate the two-body decay of an unstable resonance or, more generally, of a system of definite angular momentum and parity (Lee and Yang, 1958; Byers and Fenster, 1963; Jackson, 1965). The observables are the intensity and polarization of the angular distributions of the decay products. There are two main objectives. 1. One is to obtain information on the mechanism of production of a resonance. In this case, it is better to work in terms of the density matrix elements themselves since they give direct information on the population of the angular momentum substates. 2. The other is to determine the spin and parity of the resonance by studying various moments of angular distributions. For this, it is often convenient to express the density matrix in terms of multipole parameters.

To be specific, we choose the rest frame of γ with a fixed z axis (quantization axis) to describe its two-body decay into α and β (vide Fig. 13.3). If $\boldsymbol{p}$ and $-\boldsymbol{p}$ are the momenta of α and β in this frame, then the state vector of the two particles containing the angular and helicity information is denoted by $|p\,\theta\,\phi;\lambda_\alpha\lambda_\beta\rangle$ which can be expanded in terms of angular momentum eigenstates.

$$|p\,\theta\,\phi;\lambda_\alpha\lambda_\beta\rangle = \sum_{jm}\sqrt{\frac{2j+1}{4\pi}}\,D^j_{m\lambda}(\phi,\theta,0)\,|jm;\lambda_\alpha\lambda_\beta\rangle, \tag{13.86}$$

with

$$\lambda = \lambda_\alpha - \lambda_\beta. \tag{13.87}$$

The amplitude for the decay $\psi \to \alpha + \beta$ from a definite state $|jm\rangle$ of γ is given by (suppressing the label p hereafter)

$$\begin{aligned} f_{\lambda m}(\lambda_\alpha \lambda_\beta) &= \langle \theta\,\phi : \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle \\ &= \sqrt{\frac{2j+1}{4\pi}} \, D^{j*}_{m\lambda}(\phi, \theta, 0) \langle jm; \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle. \end{aligned} \tag{13.88}$$

Since the interaction Hamiltonian H is a scalar under rotation, its matrix element depends on λ_α and λ_β but not on m. So, let us denote the matrix element by $H(\lambda_\alpha, \lambda_\beta)$.

If the resonant state γ is denoted by the density matrix ρ_i, then the density matrix ρ_f corresponding to the final state is given by

$$\begin{aligned} (\rho_f)_{\lambda_f, \lambda_{f'}} &= \sum_{mm'} f_{\lambda_f m} \, \rho_{mm'} \, f^*_{\lambda_{f'} m'} \\ &= \sum_{m,m'} \frac{2j+1}{4\pi} \, D^{j\,*}_{m\lambda_f}(\phi, \theta, 0) \, D^{j}_{m'\lambda_{f'}}(\phi, \theta, 0) \langle jm; \lambda_\alpha \lambda_\beta | \, H \, | jm \rangle \\ &\quad \times \langle jm | \, \rho_i \, | jm' \rangle \, \langle jm; \lambda'_\alpha \lambda'_\beta | \, H \, | jm' \rangle^* \end{aligned} \tag{13.89}$$

with $\lambda_{f'} = \lambda'_\alpha - \lambda'_\beta$ and $\lambda_f = \lambda_\alpha - \lambda_\beta$.

The angular distribution $I(\theta, \phi)$ of the decay particles is obtained by taking the trace of ρ_f.

$$\begin{aligned} I(\theta, \phi) &= \mathrm{Tr}\, \rho_f \\ &= \sum_{\lambda_\alpha \lambda_\beta} \sum_{mm'} \frac{2j+1}{4\pi} \, D^{j*}_{m\lambda}(\phi, \theta, 0) D^{j}_{m'\lambda}(\phi, \theta, 0) \\ &\quad \times H(\lambda_\alpha, \lambda_\beta) \, H^*(\lambda_\alpha, \lambda_\beta) (\rho_i)_{mm'}, \end{aligned} \tag{13.90}$$

with the notation

$$\lambda = \lambda_\alpha - \lambda_\beta; \quad H(\lambda_\alpha, \lambda_\beta) = \langle jm; \lambda_\alpha \, \lambda_\beta | \, H \, | j\, m \rangle. \tag{13.91}$$

Separating the terms that depend on m and m', we get

$$\begin{aligned} &\sum_{mm'} D^{j*}_{m\lambda}(\phi, \theta, 0) \, D^{j}_{m'\lambda}(\phi, \theta, 0) (\rho_i)_{mm'} \\ &\qquad = \sum_{mm'} (-1)^{m-\lambda} \, D^{j}_{-m,-\lambda}(\phi, \theta, 0) \, D^{j}_{m'\lambda}(\phi, \theta, 0) \, (\rho_i)_{mm'} \\ &\qquad = \sum_{mm'} (-1)^{m-\lambda} \, e^{i(m-m')\phi} \, d^{j}_{-m,-\lambda}(\theta) \, d^{j}_{m'\lambda}(\theta) \, (\rho_i)_{mm'}. \end{aligned} \tag{13.92}$$

The rotation matrices $d^j(\theta)$ are known and hence the angular distribution can be obtained in terms of the density matrix of the initial system. The normalized angular distribution is given by

$$\frac{I(\theta,\phi)}{\int I(\theta,\phi)\,d\Omega}. \tag{13.93}$$

It is easy to show that

$$\int I(\theta,\phi)\,d\Omega = \sum_{\lambda_\alpha,\lambda_\beta} |H(\lambda_\alpha,\lambda_\beta)|^2, \tag{13.94}$$

since

$$\int D^{j*}_{m\lambda}(\phi,\theta,0)\,D^{j}_{m'\lambda}(\phi,\theta,0)\,d\Omega = \frac{4\pi}{2j+1}\,\delta_{mm'}, \tag{13.95}$$

and

$$\sum_{mm'}(\rho_i)_{mm'}\,\delta_{mm'} = \mathrm{Tr}\,\rho_i = 1. \tag{13.96}$$

Let us now illustrate the above discussion by considering the decay of a spin-1 system into two spin-zero particles. For this, there is only one helicity matrix element $H(0,0)$ since $\lambda_\alpha = \lambda_\beta = 0$. Since $j = 1$ and $\lambda = 0$, the required d^j matrix elements are

$$d^1_{10} = -\frac{1}{\sqrt{2}}\sin\theta; \quad d^1_{00} = \cos\theta; \quad d^1_{-1,0} = \frac{1}{\sqrt{2}}\sin\theta. \tag{13.97}$$

Substituting these values of d^1 matrix elements, the normalized angular distribution of the decay particle is obtained in terms of the spin density matrix of the parent system.

$$\begin{aligned}\frac{I(\theta,\phi)}{\int I(\theta,\phi)\,d\Omega} &= \frac{3}{4\pi}\Big[\cos^2\theta\,\rho_{0,0} + \tfrac{1}{2}\sin^2\theta\,(\rho_{1,1}+\rho_{-1,-1}) \\ &\quad - \sin^2\theta\,\mathrm{Re}(\mathrm{e}^{2i\phi}\,\rho_{1,-1}) \\ &\quad - \sqrt{\tfrac{1}{2}}\sin 2\theta\,\mathrm{Re}(\mathrm{e}^{i\phi}\,\rho_{1,0} - \mathrm{e}^{-i\phi}\,\rho_{-1,0})\Big].\end{aligned} \tag{13.98}$$

As discussed in Sec. 11.2, the density matrix can be expanded in terms of spherical tensor parameters which are also known as multipole parameters.

Using Eq. (11.23), the elements of the density matrix can be written as

$$\begin{aligned}(\rho_i)_{mm'} &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^*\langle jm|\,T_k^{m_k}\,|jm'\rangle \\ &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^*\begin{bmatrix} j & k & j \\ m' & m_k & m\end{bmatrix}\langle j||\,T_k^{m_k}\,||j\rangle \\ &= \frac{1}{2j+1}\sum_{k=0}^{2j}\sum_{m_k=-k}^{+k}\langle T_k^{m_k}\rangle^*\begin{bmatrix} j & k & j \\ m' & m_k & m\end{bmatrix}[k]. \qquad (13.99)\end{aligned}$$

The product of two rotation matrices that occur in Eq. (13.90) can be simplified using the formula (5.48), familiarly known as the C.G. series.

$$\begin{aligned}&D^{j^*}_{m\lambda}(\phi,\theta,0)\,D^{j}_{m'\lambda}(\phi,\theta,0) \\ &\quad= (-1)^{m-\lambda}\,D^{j}_{-m,-\lambda}(\phi,\theta,0)\,D^{j}_{m'\lambda}(\phi,\theta,0) \\ &\quad= (-1)^{m-\lambda}\sum_L\begin{bmatrix} j & j & L \\ -m & m' & M\end{bmatrix}\begin{bmatrix} j & j & L \\ -\lambda & \lambda & 0\end{bmatrix}D^L_{M0}(\phi,\theta,0). \qquad (13.100)\end{aligned}$$

The resulting rotation matrix $D^L_{M0}(\phi,\theta,0)$ can have only integer values for L and it can be expressed as a spherical harmonic using Eq. (5.76).

$$D^L_{M0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2L+1}}\,Y_L^{M*}(\theta,\phi). \qquad (13.101)$$

Substituting Eqs. (13.99 - 13.101) into Eq. (13.90), we obtain

$$\begin{aligned}I(\theta,\phi) &= \sum_{\lambda_\alpha\lambda_\beta}\sum_{mm'}\sum_{L}\{4\pi(2L+1)\}^{-1/2}(-1)^{m-\lambda} \\ &\quad\times\begin{bmatrix} j & j & L \\ -m & m' & M\end{bmatrix}\begin{bmatrix} j & j & L \\ -\lambda & \lambda & 0\end{bmatrix}Y_L^{M*}(\theta,\phi)\,|H(\lambda_\alpha,\lambda_\beta)|^2 \\ &\quad\times\sum_{k\,m_k}\begin{bmatrix} j & k & j \\ m' & m_k & m\end{bmatrix}[k]\,\langle T_k^{m_k}\rangle^*. \qquad (13.102)\end{aligned}$$

Equation (13.102) is simplified by performing first the summation over *m* and then replacing the summation over *m'* by *M*.

$$\begin{aligned}&\sum_{mm'}(-1)^{m-\lambda}\begin{bmatrix} j & j & L \\ -m & m' & M\end{bmatrix}\begin{bmatrix} j & k & j \\ m' & m_k & m\end{bmatrix} \\ &\qquad= \sum_M(-1)^M(-1)^{j-k+\lambda}\,\frac{[j]}{[k]}\,\delta_{Lk}\,\delta_{M,-m_k}. \qquad (13.103)\end{aligned}$$

Also

$$\sum_L (-1)^M Y_L^{M*} \langle T_L^{-M} \rangle^* = \sum_M Y_L^{M*} \langle T_L^M \rangle. \tag{13.104}$$

Substituting these results in Eq. (13.102) and replacing k and m_k by L and $-M$ because of the delta functions, we finally obtain

$$\begin{aligned} I(\theta,\phi) &= \sum_{\lambda_\alpha \lambda_\beta} \sum_L (-1)^{j-\lambda} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \lambda & -\lambda & 0 \end{bmatrix} \\ &\quad \times |H(\lambda_\alpha, \lambda_\beta)|^2 \sum_M \langle T_L^M \rangle Y_L^{M*}(\theta,\phi). \end{aligned} \tag{13.105}$$

Integrating over the solid angle and using the following identities

$$\int Y_L^{M*}(\theta,\phi)\, d\Omega = \sqrt{4\pi}\, \delta_{L0}\, \delta_{M0}, \tag{13.106}$$

$$\begin{bmatrix} j & j & 0 \\ \lambda & -\lambda & 0 \end{bmatrix} = \frac{(-1)^j}{[j]}, \tag{13.107}$$

$$\langle T_0^0 \rangle = 1, \tag{13.108}$$

we retrieve the result (13.94).

$$\int I(\theta,\phi)\, d\Omega = \sum_{\lambda_\alpha, \lambda_\beta} |H(\lambda_\alpha, \lambda_\beta)|^2.$$

By inspection of Eq. (13.105), it is seen that the statistical tensors $\langle T_L^M \rangle$ are related to the spherical harmonic moments of $I(\theta,\phi)$.

$$\begin{aligned} \int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega &= \sum_{\lambda_\alpha \lambda_\beta} (-1)^{j-\lambda} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \lambda & -\lambda & 0 \end{bmatrix} \\ &\quad \times |H(\lambda_\alpha, \lambda_\beta)|^2 \langle T_L^M \rangle. \end{aligned} \tag{13.109}$$

Case 1: Decay into two spinless particles
In the case of decay into two spinless particles,

$$\lambda_\alpha = \lambda_\beta = \lambda = 0.$$

Equation (13.109) now reduces to

$$\begin{aligned} \int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega &= \sum_{\lambda_\alpha\lambda_\beta} (-1)^j \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad \times |H(0,0)|^2 \langle T_L^M \rangle. \end{aligned} \qquad (13.110)$$

Here j is an integer and L should be even because of the parity C.G. coefficient. Since

$$\int I(\theta,\phi)\, d\Omega = |H(0,0)|^2, \qquad (13.111)$$

it follows that the normalized spherical harmonic moments of angular distribution is

$$\frac{\int I(\theta,\phi)\, Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = \sum_{\lambda_\alpha\lambda_\beta} (-1)^j \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ 0 & 0 & 0 \end{bmatrix} \langle T_L^M \rangle. \qquad (13.112)$$

Case 2: Decay into a spin-$\frac{1}{2}$ and a spin-zero particle
From parity considerations, the two amplitudes $H(\frac{1}{2},0)$ and $H(-\frac{1}{2},0)$ are related.

$$H(-\tfrac{1}{2},0) = \epsilon\, H(\tfrac{1}{2},0), \qquad (13.113)$$

where $\epsilon = \pm 1$. From (13.60), it follows that

$$\epsilon = \eta_\alpha\, \eta_\beta\, \eta_\gamma\, (-1)^{j-\frac{1}{2}}. \qquad (13.114)$$

If parity is conserved in the decay, then $\epsilon = \pm 1$ corresponding to the orbital angular momentum $(l = j \mp \frac{1}{2})$ of the ab system. The conservation of parity requires that the product of intrinsic parities $\eta_\alpha\eta_\beta\eta_\gamma = (-1)^l$. Thus determines the intrinsic parity of the γ resonance. However

$$|H(-\tfrac{1}{2},0)|^2 = |H(\tfrac{1}{2},0)|^2. \qquad (13.115)$$

Consequently,

$$\int I(\theta,\phi)\, d\Omega = |H(\tfrac{1}{2},0)|^2 + |H(-\tfrac{1}{2},0)|^2 = 2\,|H(\tfrac{1}{2},0)|^2, \qquad (13.116)$$

and the normalized angular distribution is given by

$$\frac{\int I(\theta,\phi)\,Y_L^M(\theta,\phi)\,d\Omega}{\int I(\theta,\phi)\,d\Omega} = (-1)^{j-\frac{1}{2}}\frac{[j]}{\sqrt{4\pi}[L]}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \times\langle T_L^M\rangle\left\{\frac{1+(-1)^L}{2}\right\}. \qquad (13.117)$$

The C.G. coefficient ensures that the spherical harmonic moments with L > $2j$ vanish, and so the observation of a statistically significant non-vanishing average value of Y_L^M means that the spin of the γ resonance is at least $\frac{1}{2}L$.

The distribution of the longitudinal polarization of the spin-$\frac{1}{2}$ particle that comes from the decay is

$$P_l(\theta,\phi) = \frac{I_{\frac{1}{2}}(\theta,\phi) - I_{-\frac{1}{2}}(\theta,\phi)}{I_{\frac{1}{2}}(\theta,\phi) + I_{-\frac{1}{2}}(\theta,\phi)}. \qquad (13.118)$$

The denominator $I_{\frac{1}{2}}(\theta,\phi) + I_{-\frac{1}{2}}(\theta,\phi)$ is just equal to $I(\theta,\phi)$. Hence

$$P_l(\theta,\phi)\,I(\theta,\phi) = I_{\frac{1}{2}}(\theta,\phi) - I_{-\frac{1}{2}}(\theta,\phi). \qquad (13.119)$$

Using Eq. (13.105), we obtain the helicity distributions.

$$\begin{aligned} P_l(\theta,\phi)\,I(\theta,\phi) &= \sum_{LM}\frac{[j]}{\sqrt{4\pi}[L]}\langle T_L^M\rangle Y_L^{M*}(\theta,\phi)\left\{(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\right. \\ &\quad \left. -(-1)^{j+\frac{1}{2}}\begin{bmatrix} j & j & L \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\right\}|H(\tfrac{1}{2},0)|^2 \\ &= \sum_{LM}\frac{[j]}{\sqrt{4\pi}[L]}\langle T_L^M\rangle Y_L^{M*}(\theta,\phi)(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \\ &\quad \times\{1-(-1)^L\}\,|H(\tfrac{1}{2},0)|^2. \end{aligned} \qquad (13.120)$$

After normalization, the longitudinal polarization of the angular distribution is

$$\begin{aligned} &\frac{\int P_l(\theta,\phi)\,I(\theta,\phi)\,Y_L^M(\theta,\phi)\,d\Omega}{\int I(\theta,\phi)\,d\Omega} \\ &= \frac{[j]}{\sqrt{4\pi}[L]}(-1)^{j-\frac{1}{2}}\begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\langle T_L^M\rangle\left\{\frac{1-(-1)^L}{2}\right\}. \end{aligned} \qquad (13.121)$$

It is observed that the longitudinal polarization yields information about odd L multipole parameters while the particle distribution gives information about even L multipole parameters. These studies do not throw any light on the parity of the resonant (parent) state. Only the study of the transverse polarization of the decay products gives valuable information on the parity of the parent state.

Since we are considering two-body decay, of which one particle has spin-$\frac{1}{2}$ and the other spin zero, we need to consider only the transverse polarization of the spin-$\frac{1}{2}$ particle. The transverse polarization is the expectation value of σ_x or σ_y operator. Let us illustrate the method by calculating the x component of polarization.

$$P_x = \langle \sigma_x \rangle = \frac{\mathrm{Tr}\,(\sigma_x \rho_f)}{\mathrm{Tr}\,\rho_f}. \tag{13.122}$$

Equivalently,

$$P_x\, I(\theta, \phi) = \mathrm{Tr}\,(\sigma_x \rho_f). \tag{13.123}$$

To evaluate Tr $(\sigma_x \rho_f)$ we proceed in steps. First let us show that Tr $(\sigma_x \rho_f)$ is just the real part of the spin density matrix element $(\rho_f)_{\frac{1}{2},-\frac{1}{2}}$.

$$\begin{aligned}
\mathrm{Tr}\,(\sigma_x\, \rho_f) &= \tfrac{1}{2}\mathrm{Tr}\,\{(\sigma_+ + \sigma_-)\,\rho_f\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda\lambda'}(\sigma_+)_{\lambda\lambda'}(\rho_f)_{\lambda'\lambda} + \sum_{\lambda\lambda'}(\sigma_-)_{\lambda\lambda'}\,(\rho_f)_{\lambda'\lambda}\right\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda\lambda'}\delta_{\lambda',\lambda-1}(\rho_f)_{\lambda'\lambda} + \sum_{\lambda\lambda'}\delta_{\lambda'-1,\lambda}\,(\rho_f)_{\lambda'\lambda}\right\} \\
&= \tfrac{1}{2}\left\{\sum_{\lambda}(\rho_f)_{\lambda-1,\lambda} + \sum_{\lambda'}(\rho_f)_{\lambda',\lambda'-1}\right\} \\
&= \tfrac{1}{2}\sum_{\lambda}\{(\rho_f)_{\lambda-1,\lambda} + (\rho_f)_{\lambda,\lambda-1}\} \\
&= \mathrm{Re}\sum_{\lambda}(\rho_f)_{\lambda,\lambda-1}.
\end{aligned} \tag{13.124}$$

The last step is obtained by invoking the Hermitian property of the density matrix. For the spin-$\frac{1}{2}$ particle, the helicity can assume only two values $+\frac{1}{2}$ and $-\frac{1}{2}$ and hence λ in the above expression can take only one value $\frac{1}{2}$. Hence we obtain a simple result that

$$\mathrm{Tr}\,(\sigma_x\, \rho_f) = \mathrm{Re}\,(\rho_f)_{\frac{1}{2},-\frac{1}{2}}. \tag{13.125}$$

Using Eq. (13.69), we obtain (suppressing for the present the Euler angles of rotation (ϕ,θ,0) in the rotation matrix)

$$\begin{aligned}\mathrm{Tr}\,(\sigma_x\rho_f) &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}D^{j^*}_{m,\frac{1}{2}}\,D^{j}_{m',-\frac{1}{2}}H(\tfrac{1}{2},0)\,H(-\tfrac{1}{2},0)^*(\rho_i)_{mm'}\\ &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}(-1)^{m-\frac{1}{2}}\,D^{j}_{-m,-\frac{1}{2}}D^{j}_{m',-\frac{1}{2}}\,\epsilon\,|H(\tfrac{1}{2},0)|^2\,(\rho_i)_{mm'}\\ &= \mathrm{Re}\sum_{mm'}\frac{2j+1}{4\pi}(-1)^{m-\frac{1}{2}}\sum_{L}\left[\begin{array}{ccc} j & j & L\\ -m & m' & M\end{array}\right]\\ &\quad\times\left[\begin{array}{ccc} j & j & L\\ -\frac{1}{2} & -\frac{1}{2} & -1\end{array}\right]D^{L}_{M,-1}\,\epsilon\,|H(\tfrac{1}{2},0)|^2\,(\rho_i)_{mm'}. \qquad (13.126)\end{aligned}$$

The above result is obtained using the C.G. series for the coupling of the rotation matrices and the relation between the helicity amplitudes, viz., $H(-\frac{1}{2},0)=\epsilon\,H(\frac{1}{2},0)$ Expressing the density matrix of the initial resonant state in terms of the multipole parameters as given in Eq. (13.99),

$$(\rho_i)_{mm'}=\frac{1}{2j+1}\sum_{km_k}\langle T_k^{m_k}\rangle^*\left[\begin{array}{ccc} j & k & j\\ m' & m_k & m\end{array}\right][k],$$

it will be convenient to separate the terms that depend upon *m* and *m′* and perform the summation over *m* and replace the summation over *m′* by *M*.

$$\begin{aligned}&\sum_{mm'}(-1)^{m-\frac{1}{2}}\left[\begin{array}{ccc} j & j & L\\ -m & m' & M\end{array}\right]\left[\begin{array}{ccc} j & k & j\\ m' & m_k & m\end{array}\right]\\ &\qquad=(-1)^{2j-k}(-1)^{j-\frac{1}{2}}\frac{[j]}{[k]}\sum_{M}(-1)^M\,\delta_{L,k}\,\delta_{M,-m_k}. \qquad (13.127)\end{aligned}$$

Substituting the above result, we get after simplification

$$\begin{aligned}P_x\,I(\theta,\phi)=\mathrm{Tr}\,(\sigma_x\,\rho_f) &= \mathrm{Re}\sum_{LM}\frac{[j]}{4\pi}(-1)^{j-\frac{1}{2}}(-1)^M\left[\begin{array}{ccc} j & j & L\\ \frac{1}{2} & \frac{1}{2} & 1\end{array}\right]\\ &\quad\times D^{L}_{M,-1}(\phi,\theta,0)\,\langle T_L^{-M}\rangle^*\,\epsilon\,|H(\tfrac{1}{2},0)|^2. \quad (13.128)\end{aligned}$$

In a similar way, one can calculate the transverse polarization P_y.

$$P_y\,I(\theta,\phi)=\mathrm{Tr}\,(\sigma_y\,\rho_f)=\frac{1}{2i}\mathrm{Tr}\,\{(\sigma_+-\sigma_-)\,\rho_f\}. \qquad (13.129)$$

Following the same procedure as before, we can show that

$$\begin{aligned} P_y I(\theta,\phi) &= -\mathrm{Im}\,(\rho_f)_{\frac{1}{2},-\frac{1}{2}} \\ &= -\mathrm{Im}\sum_{LM}\frac{[j]}{4\pi}(-1)^{j-\frac{1}{2}}(-1)^M \begin{bmatrix} j & j & L \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \\ &\quad \times D^L_{M,-1}(\phi,\theta,0)\,\langle T_L^{-M}\rangle^* \,\epsilon\, |H(\tfrac{1}{2},0)|^2. \end{aligned} \tag{13.130}$$

Thus the study of the transverse polarization, which depends on ϵ, will yield the parity of the resonant state.

Hitherto, we have considered only the parity conserving two-body decay. For parity non-conserving weak decay such as the decay of hyperons, only small modifications are necessary. The interaction Hamiltonian, in this case, is a sum of two terms, one scalar H_e and the other pseudoscalar H_o.

$$H = H_{\mathrm{e}} + H_{\mathrm{o}}. \tag{13.131}$$

Under parity operation,

$$\mathcal{P}^{-1} H_{\mathrm{e}} \mathcal{P} = H_{\mathrm{e}}; \quad \mathcal{P}^{-1} H_{\mathrm{o}} \mathcal{P} = -H_{\mathrm{o}}. \tag{13.132}$$

This means that the matrix element is a sum of two terms,

$$H_{\mathrm{e}}(\lambda_\alpha,\lambda_\beta) + H_{\mathrm{o}}(\lambda_\alpha,\lambda_\beta), \tag{13.133}$$

where H_e and H_o obeys the following relations:

$$H_{\mathrm{e}}(-\lambda_\alpha,-\lambda_\beta) = \eta_\alpha\eta_\beta\eta_\gamma(-1)^{j-s_\alpha-s_\beta}; \tag{13.134}$$

$$H_{\mathrm{o}}(-\lambda_\alpha,-\lambda_\beta) = -\eta_\alpha\eta_\beta\eta_\gamma(-1)^{j-s_\alpha-s_\beta}. \tag{13.135}$$

To be specific, let us consider a weak decay of a hyperon into a baryon of spin-$\frac{1}{2}$ and a meson of spin zero. The various distributions involve the following combinations of $H_{\mathrm{e}}(\frac{1}{2},0)$ and $H_{\mathrm{o}}(\frac{1}{2},0)$:

$$\begin{aligned} a &= \frac{2\,\mathrm{Re}(H_{\mathrm{e}}(\frac{1}{2},0)\,H_{\mathrm{o}}^*(\frac{1}{2},0))}{|H_{\mathrm{e}}(\frac{1}{2},0)|^2 + |H_{\mathrm{o}}(\frac{1}{2},0)|^2}, \\ b &= \frac{2\,\mathrm{Im}\,(H_{\mathrm{e}}(\frac{1}{2},0)\,H_{\mathrm{o}}^*(\frac{1}{2},0))}{|H_{\mathrm{e}}(\frac{1}{2},0)|^2 + |H_{\mathrm{o}}(\frac{1}{2},0)|^2}, \\ c &= \frac{|H_{\mathrm{e}}(\frac{1}{2},0)|^2 - |H_{\mathrm{o}}(\frac{1}{2},0)|^2}{|H_{\mathrm{e}}(\frac{1}{2},0)|^2 + |H_{\mathrm{o}}(\frac{1}{2},0)|^2}. \end{aligned} \tag{13.136}$$

It is easy to observe that $a^2 + b^2 + c^2 = 1$. The various changes that occur in our earlier study of parity conserving two-body decay can easily be determined and we only quote the final results for normalized angular distribution and the longitudinal polarization. Equations (13.117) and (13.121) get modified to yield

$$\frac{\int I(\theta,\phi) Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = (-1)^{j-\frac{1}{2}} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \langle T_L^M \rangle \times \left\{ \frac{1+(-1)^L}{2} + \frac{a(1-(-1)^L)}{2} \right\}, \tag{13.137}$$

$$\frac{\int P_l(\theta,\phi)\, I(\theta,\phi) Y_L^M(\theta,\phi)\, d\Omega}{\int I(\theta,\phi)\, d\Omega} = (-1)^{j-\frac{1}{2}} \frac{[j]}{\sqrt{4\pi}[L]} \begin{bmatrix} j & j & L \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \langle T_L^M \rangle \times \left\{ \frac{1-(-1)^L}{2} + \frac{a(1+(-1)^L)}{2} \right\}. \tag{13.138}$$

The reader may note the interchange of the roles played by even and odd L in the above equations.

For the relativistic treatment of angular momentum states for three-body system and for the three-body decay, the reader is referred to Wick (1962) and Berman and Jacob (1965).

13.5. Muon Capture

We shall now apply the helicity formalism to discuss the capture of muon by spin-zero target nucleus,

$$\mu^- + A(j_i = 0) \to B(j_f \geq 1) + \nu_\mu, \tag{13.139}$$

and investigate the asymmetry in the angular distribution of the recoil nucleus B and its polarization.

The usual source of muon is from π decay and it is polarized in the direction of its flight. When it is incident on a target, it is slowed down and caught in Bohr orbits. It cascades down to lower orbits emitting X-rays known as muonic X-rays and ultimately reaches the 1s orbit before it is captured by the nucleus through weak interaction. It is observed that depolarization takes place during the process of slowing down and cascading, but yet there is a residual polarization of order 15 to 20% in the 1s orbit at the time of capture by spin zero nucleus.

The muon polarization which coincides with the direction of incident muon is assumed as the z-axis of the rest frame of the initial system as shown in Fig. 13.4. This corresponds in the final state, to the centre of momentum system, with the recoil momentum $\boldsymbol{p} = -\nu$, making an angle

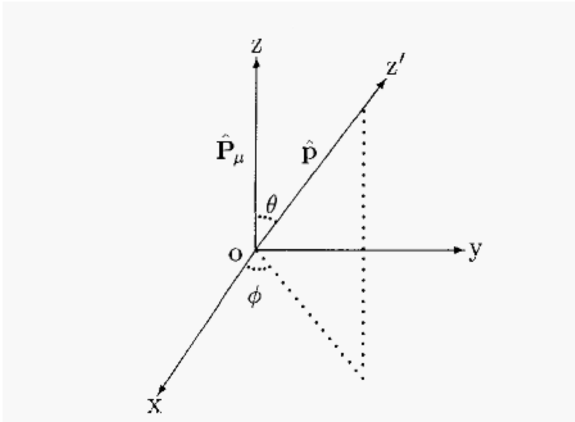

Figure 13.4. The muon polarization is along the z-axis of the rest frame of the muon-nucleon system and the momentum of the recoiling nucleus is along the z'-axis of the rotating frame which is otherwise called the helicity frame.

θ,ϕ with the z-axis. For describing this process, we have two frames of reference, one is the fixed frame of reference with z-axis in the direction of muon polarization and the other, the rotating frame of reference with z'-axis coinciding with the direction of recoiling nucleus. The latter frame of reference is obtained from the former by rotation through Euler angles (ϕ,θ,0)

Since the target nucleus is of zero spin, the total angular momentum of the initial system (μ- + A) is $\frac{1}{2}$ and is described by the state vector $|\frac{1}{2}\, m\rangle$. The final state is the recoiling nucleus B with spin j_f and helicity 1f, and the muon neutrino ν_μ with spin-$\frac{1}{2}$ and helicity $-\frac{1}{2}$. Expanding the final state in terms of definite angular momentum following Eq. (13.35),

$$|\theta, \phi; \lambda_f, -\tfrac{1}{2}\rangle = \sum_{jM} \sqrt{\frac{2j+1}{4\pi}}\, D^j_{M,\lambda}(\phi, \theta, 0)|jM; \lambda_f, -\tfrac{1}{2}\rangle, \qquad (13.140)$$

the transition amplitude can be obtained in the helicity basis.

$$\begin{aligned} f_{\lambda m} &= \langle \theta\phi; \lambda_f, -\tfrac{1}{2}\, |H|\, \tfrac{1}{2} m\rangle \\ &= \sum_{jM} \sqrt{\frac{2j+1}{4\pi}}\, D^{j\,*}_{M\lambda}(\phi, \theta, 0)\, \langle jM; \lambda_f, -\tfrac{1}{2}\, |H|\, \tfrac{1}{2} m\rangle. \end{aligned} \qquad (13.141)$$

Since H is a scalar under rotation, $j = \frac{1}{2}$ and $M = m$, there can be only two partial wave helicity amplitudes $\langle \frac{1}{2} M; \lambda_f, -\frac{1}{2} |H| \frac{1}{2} m\rangle$ corresponding to the total angular momentum $\frac{1}{2}$. These partial wave helicity amplitudes will hereafter be represented by H_λ where $\lambda = \lambda_f + \frac{1}{2} = \pm\frac{1}{2}$. Thus,

$$f_{\lambda m} = \sqrt{\frac{1}{2\pi}}\, D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, H_\lambda. \tag{13.142}$$

The elements of the density matrix for the final system is given by

$$(\rho_f)_{\lambda,\lambda'} = \sum_m f_{\lambda m}\,(\rho_i)_{mm'}\, f^*_{\lambda' m'}, \tag{13.143}$$

where ρ_i denotes the density matrix for the initial system which is taken to be in the diagonal form in the rest frame.

$$\rho_i = \tfrac{1}{2}(1 + \boldsymbol{\sigma}\cdot\boldsymbol{P}_\mu) = \tfrac{1}{2}(1 + 2\,\boldsymbol{s}\cdot\boldsymbol{P}_\mu), \tag{13.144}$$

where σ denotes the Pauli spin operator, $\boldsymbol{s}$ the spin of the muon and $\boldsymbol{P}_\mu$ the polarization of the muon which is in the z direction. Substituting the eigenvalue of s_z in the density matrix of the initial state,

$$(\rho_f)_{\lambda,\lambda'} = \tfrac{1}{2}\sum_m f_{\lambda m}\,(1 + 2mP_\mu)\, f^*_{\lambda' m}. \tag{13.145}$$

Using Eq. (13.142) and the explicit form of $D^{\frac{1}{2}}$ rotation matrices, we obtain the following results:

$$\begin{aligned}
\sum_m f_{\lambda m}\, f^*_{\lambda' m} &= \frac{1}{2\pi}\sum_m D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, D^{\frac{1}{2}}_{m\lambda'}(\phi,\theta,0)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{2\pi}\sum_m d^{\frac{1}{2}\,\dagger}_{\lambda m}(\theta)\, d^{\frac{1}{2}}_{m\lambda'}(\theta)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{2\pi}\,|H_\lambda|^2\,\delta_{\lambda,\lambda'}. \qquad (13.146)\\
\sum_m f_{\lambda m}\, m\, f^*_{\lambda' m} &= \frac{1}{2\pi}\sum_m D^{\frac{1}{2}\,*}_{m\lambda}(\phi,\theta,0)\, m\, D^{\frac{1}{2}}_{m\lambda'}(\phi,\theta,0)\, H_\lambda\, H^*_{\lambda'} \\
&= \frac{1}{4\pi}\Big\{2\,\lambda\,|H_\lambda|^2\cos\theta\,\delta_{\lambda,\lambda'} - H_{\frac{1}{2}}\, H^*_{-\frac{1}{2}}\sin\theta\,\delta_{\lambda-\lambda',1} \\
&\qquad - H_{-\frac{1}{2}}\, H^*_{\frac{1}{2}}\sin\theta\,\delta_{\lambda-\lambda',-1}\Big\}. \qquad (13.147)
\end{aligned}$$

Consolidating the above results, we obtain the angular distribution of the recoil nucleus $I(\theta,\phi)$.

$$I(\theta,\phi) = \mathrm{Tr}\,\rho_f = \frac{1}{4\pi}\sum_\lambda |H_\lambda|^2 + \frac{1}{2\pi}\sum_\lambda \lambda |H_\lambda|^2\, P_\mu\cos\theta. \tag{13.148}$$

Writing it in a more compact form,

$$I(\theta,\phi) = \frac{\Gamma}{4\pi}\,\Lambda(\theta), \tag{13.149}$$

with

$$\Gamma = \sum_\lambda |H_\lambda|^2, \quad \Lambda(\theta) = 1 + \alpha\, P_\mu \cos\theta, \tag{13.150}$$

we find the asymmetry coefficient of the recoil angular distribution to be

$$\alpha = 2\,\frac{\sum_\lambda \lambda |H_\lambda|^2}{\sum_\lambda |H_\lambda|^2}. \tag{13.151}$$

The quantity Γ represents the capture rate.

The longitudinal polarization of the recoil nucleus is

$$\begin{aligned} P_L &= \frac{\mathrm{Tr}(\boldsymbol{J}\cdot\boldsymbol{p})\,\rho_f}{\mathrm{Tr}\,\rho_f} \\ &= \frac{\sum_{\lambda,\lambda'} (\boldsymbol{J}\cdot\boldsymbol{p})_{\lambda,\lambda'}\,(\rho_f)_{\lambda',\lambda}}{\sum_\lambda (\rho_f)_{\lambda,\lambda}}. \end{aligned} \tag{13.152}$$

Since

$$(\boldsymbol{J}\cdot\boldsymbol{p})_{\lambda,\lambda'} = \lambda_f\,\delta_{\lambda,\lambda'} = (\lambda - \tfrac{1}{2})\,\delta_{\lambda,\lambda'}, \tag{13.153}$$

the longitudinal polarization becomes

$$P_L = \frac{\sum_\lambda (\lambda - \frac{1}{2})\,(\rho_f)_{\lambda,\lambda}}{\sum_\lambda (\rho_f)_{\lambda,\lambda}}. \tag{13.154}$$

In the absence of muon polarization ($P_\mu = 0$),

$$P_L^0 = \frac{\sum_\lambda (\lambda - \frac{1}{2})|H_\lambda|^2}{\sum_\lambda |H_\lambda|^2} = \frac{\alpha}{2} - \frac{1}{2}. \tag{13.155}$$

Thus we arrive at a well known relation for the observables in muon capture.

$$\alpha - 2\,P_L^0 = 1. \tag{13.156}$$

Since the muon capture process is completely described by two helicity amplitudes $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$, all the observables in muon capture can be expressed in terms of these amplitudes and their relative phase. Hence it follows that there cannot be more than three independent observables in muon capture. For further details of helicity formalism as applied to muon capture, the reader is referred to Bernabeu (1975) and Subramanian et al. (1976, 1979).

Review Questions

13.1 (a) Write down the non-interacting two-particle wave function in terms of the plane wave helicity basis and the angular momentum basis and obtain the transformation from one basis to the other.
(b) Discuss the advantages of using the helicity formalism for the study of two-particle scattering and obtain expressions for the angular distributions and polarization of the scattered particles.

13.2 (a) Consider the two-body decay of a resonant state and deduce an expression for the angular distribution of the decay products in terms of the decay products in terms of the statistical parameters $\langle T_k^\mu \rangle$ defining the initial system. Also find the spherical harmonic moments of the angular distribution.
(b) Apply the above consideration to the decay of a resonant state into (i) two spinless particles and (ii) one spin-$\frac{1}{2}$ and the other spin-zero particle.

13.3 Discuss how is it possible to determine the spin and parity of a resonant state by observing the angular distributions and polarization of the decay products. Restrict your considerations to the decay into two particles.

13.4 Consider muon capture by a spin-zero target nucleus and show that the asymmetry in the angular distribution of the final nucleus with respect to the polarization vector of the initial muon is related to longitudinal polarization of the final nucleus by a simple relation $\alpha - 2P_L^0 = 1$, where α denotes the asymmetry coefficient and P_L^0 denotes the longitudinal polarization of the final nucleus for muon polarization zero.

Problems

13.1 If a particle with spin j has transverse polarization, show that

$$\begin{aligned} \langle J_x \rangle &= \{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}} \operatorname{Re} \rho_{\lambda-1,\lambda}, \\ \langle J_y \rangle &= \{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}} \operatorname{Im} \rho_{\lambda-1,\lambda}. \end{aligned}$$

13.2 A particle with spin s is transversally polarized. If the transverse polarization is denoted by $\langle s_y \rangle$, then show that its spin density matrix is given by

$$\rho = \frac{1}{2s+1}\left(1 + \sqrt{\frac{3}{s(s+1)}} \langle s_y \rangle \, s_y\right).$$

Show that the density matrix reduces to the familiar formula

$$\rho = \tfrac{1}{2}(1 + \boldsymbol{\sigma} \cdot \boldsymbol{P})$$

for the spin-$\frac{1}{2}$ particle with vector polarization $\boldsymbol{P}$.

13.3 Discuss the pion-nucleon and nucleon-nucleon scattering using the helicity formalism and enumerate the number of independent scattering amplitudes in each case.

13.4 Discuss the following decays

$$\begin{array}{lll} \text{(a)}\ Y^*(1385\,\text{MeV}) & \to & \Lambda\,\pi, \\ \text{(b)}\ \Xi^*(1530\,\text{MeV}) & \to & \Xi\,\pi, \end{array}$$

and explain how you will determine the spin and parity of the parent systems. (These are parity conserving decays through strong interaction. The spin of the hyperons Λ and Ξ is $\frac{1}{2}$ and the spin of π is zero.)

Solutions to Selected Problems

13.1 The transverse polarization of a particle with spin j is the expectation value of the operators J_x and J_y.

$$\begin{aligned}
\langle J_x \rangle &= \text{Tr}\,(J_x\,\rho) = \frac{1}{2}\text{Tr}\,\{(J_+ + J_-)\,\rho\} \\
&= \frac{1}{2}\left\{\sum_{\lambda,\lambda'}(J_+)_{\lambda,\lambda'}(\rho)_{\lambda',\lambda} + \sum_{\lambda,\lambda'}(J_-)_{\lambda',\lambda}\,(\rho)_{\lambda,\lambda'}\right\} \\
&= \frac{1}{2}\left\{\sum_{\lambda,\lambda'}\{(j-\lambda')(j+\lambda'+1)\}^{\frac{1}{2}}\,\delta_{\lambda,\lambda'+1}(\rho)_{\lambda',\lambda}\right. \\
&\qquad \left. + \sum_{\lambda,\lambda'}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\,\delta_{\lambda',\lambda-1}\,(\rho)_{\lambda,\lambda'}\right\} \\
&= \frac{1}{2}\sum_{\lambda}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\,\{(\rho)_{\lambda-1,\lambda} + (\rho)_{\lambda,\lambda-1}\}\,.
\end{aligned}$$

Since ρ is a Hermitian matrix, it follows that

$$\langle J_x \rangle = \sum_{\lambda}\{(j+\lambda)(j-\lambda)+1\}^{\frac{1}{2}}\text{Re}\,(\rho)_{\lambda-1,\lambda}.$$

Since $J_y = \frac{1}{2i}(J_+ - J_-)$, it can be shown in a similar manner that

$$\langle J_y \rangle = \sum_{\lambda}\{(j+\lambda)(j-\lambda+1)\}^{\frac{1}{2}}\text{Im}\,(\rho)_{\lambda-1,\lambda}.$$

13.2 Retaining only the first order term and neglecting higher order tensor orientations, the density matrix can be written as

$$\rho = \frac{1}{2s+1}\left(1 + \langle T_1^\mu \rangle\, T_1^\mu\right),$$

where the tensor operator T_1^μ is normalized such that

$$\mathrm{Tr}\,(T_1^{\mu\dagger}\, T_1^{\mu'}) = (2s+1)\,\delta_{\mu,\mu'}.$$

The normalized T_1^μ operator is

$$T_1^\mu = \sqrt{\frac{3}{s(s+1)}}\; s_1^\mu .$$

Substituting it in the expression for ρ, we get

$$\rho = \frac{1}{2s+1}\left(1 + \frac{3}{s(s+1)}\,\langle s_1^\mu \rangle\, s_1^\mu\right).$$

For spin-$\frac{1}{2}$ particle, the density matrix reduces to

$$\begin{aligned}\rho &= \tfrac{1}{2}(1 + 4s^\mu\langle s^\mu\rangle)\\ &= \tfrac{1}{2}(1 + \sigma^\mu\langle\sigma^\mu\rangle)\\ &= \tfrac{1}{2}(1 + \boldsymbol{\sigma}\cdot\boldsymbol{P})\end{aligned}$$

13.3 For each partial wave scattering amplitude, the number of helicity amplitudes is $(2s_a + 1)(2s_b + 1)(2s_c + 1)(2s_d + 1)$. But by the application of invariance and symmetry principles, the number of independent amplitudes is considerably reduced.

For pion-nucleon scattering, the number of helicity amplitudes is 4, since the pion spin is zero and the nucleon spin is $\frac{1}{2}$. Explicitly, the amplitudes are

$$\begin{aligned}&\text{(i)}\ \ \langle 0, \tfrac{1}{2}|T|0, \tfrac{1}{2}\rangle, &&\text{(ii)}\ \ \langle 0, \tfrac{1}{2}|T|0, -\tfrac{1}{2}\rangle,\\ &\text{(iii)}\ \ \langle 0, -\tfrac{1}{2}|T|0, \tfrac{1}{2}\rangle, &&\text{(iv)}\ \ \langle 0, -\tfrac{1}{2}|T|0, -\tfrac{1}{2}\rangle.\end{aligned}$$

By application of parity conservation, the helicity amplitudes (i) and (iv) are equal and (ii) and (iii) are equal. The application of time reversal invariance implies that amplitudes (ii) and (iii) are equal and so it does not give any new relation. Hence the number of independent amplitudes required for describing the pion-nucleon scattering is only two.

For describing the nucleon-nucleon scattering, the total number of helicity amplitudes required is 16, since the nucleon has spin-$\frac{1}{2}$. The parity invariance reduces the number of independent helicity amplitudes from 16 to 8 and the time reversal invariance reduces further the number of independent helicity amplitudes from 8 to 6. By invoking the relation for the identical particles, the number is further reduced to 5. The five independent partial wave helicity amplitudes are given below in a matrix form.

	$++$	$+-$	$-+$	$--$
$++$	f_1^J	f_5^J	f_5^J	f_2^J
$+-$	f_5^J	f_3^J	f_4^J	f_5^J
$-+$	f_5^J	f_4^J	f_3^J	f_5^J
$--$	f_2^J	f_5^J	f_5^J	f_1^J

The rows and columns denote the helicity states of the final and initial systems, using for brevity + for $+\frac{1}{2}$ and - for $-\frac{1}{2}$ helicity states. For instance, in the table, f_1^J denotes the helicity amplitude

$$f_1^j = \frac{1}{2p}\langle +\tfrac{1}{2}, +\tfrac{1}{2} | T^j | +\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \frac{1}{2p}\langle -\tfrac{1}{2}, -\tfrac{1}{2} | T^j | -\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$

THE SPIN STATES OF DIRAC PARTICLES

14.1. The Dirac Equation

Starting with the relativistic equation for the energy–momentum of a particle

$$E^2 = p^2c^2 + m^2c^4, \tag{14.1}$$

we obtain the Dirac Hamiltonian (Schiff, 1968; Ramakrishnan, 1962) for a free particle by linearizing the energy-momentum relation.

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2, \tag{14.2}$$

where α and β, known as Dirac matrices, obey the following conditions

$$\begin{aligned} &\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I, \\ &\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad (i \neq j;\ \ i, j = x, y, z), \\ &\beta\alpha_i + \alpha_i\beta = 0 \quad (i = x, y, z), \end{aligned} \tag{14.3}$$

so that the relation (14.1) is satisfied. In Eq. (14.3), I denotes the unit matrix and α_x, α_y, α_z and β are 4 x 4 matrices which can be conveniently written in the 2 x 2 form using the Pauli matrices[1].

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix}; \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}. \tag{14.4}$$

Using natural units $(\hbar = c = 1)$, the Dirac equation can be written as

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)\psi = E\psi. \tag{14.5}$$

Writing it in a more simplified form $A\psi = 0$, a non-trivial solution for ψ can be obtained by imposing the condition, det $A = 0$.

$$\det A = \det(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m - EI) = \begin{vmatrix} m - E & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -(m+E) \end{vmatrix} = 0. \tag{14.6}$$

[1]The transition from Pauli to Dirac matrices is investigated by Ramakrishnan (1967a, 1967b, 1972) in a series of papers known as the L-matrix theory by developing a grammar of anti-commuting matrices and extending the formalism to a more general ω commutation relations involving the roots of unity.

This leads to the relativistic relation $E^2 = p^2 + m^2$, i.e., the Eq. (14.1) in natural units. This guarantees that the Dirac equation satisfies the relativistic relation (14.1) and it is the linearized form of the relativistic energy–momentum relation in operator formalism. It follows that the Dirac Hamiltonian has two eigenvalues $\pm E$.

Since the Dirac Hamiltonian has two eigenvalues $+E$ and $-E$ with $E = (p^2 + m^2)^{\frac{1}{2}}$, we need to find the eigenfunctions corresponding to these two eigenvalues. The Dirac Hamiltonian is a 4 x 4 matrix and consequently the eigenfunction is a four-component column vector. It is found more convenient to write the solution in the two-component form $\psi = \begin{bmatrix} u \\ v \end{bmatrix}$ and write the Dirac Eq. (14.5) using the Pauli matrices.

$$\begin{bmatrix} m & \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -m \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = E \begin{bmatrix} u \\ v \end{bmatrix}. \tag{14.7}$$

This leads to two coupled equations, from which the ratio v/u can be determined.

$$m\,u + \boldsymbol{\sigma} \cdot \boldsymbol{p}\, v = E\, u; \qquad \frac{v}{u} = \frac{E - m}{\boldsymbol{\sigma} \cdot \boldsymbol{p}}. \tag{14.8}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{p}\, u - m\, v = E\, v; \qquad \frac{v}{u} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + m}. \tag{14.9}$$

Since the Dirac equation gives only the ratio, one is free to choose either u or v as $\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Choosing $u = \chi_\pm$ in Eq. (14.9) and $v = \chi_\pm$ in Eq. (14.8), we obtain two sets of solutions for positive energy states.

$$\begin{array}{cc} \text{Set I} & \text{Set II} \\ \psi_1 = \begin{bmatrix} \chi_+ \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi_+ \end{bmatrix} & \psi_1' = \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E-m} \chi_+ \\ \chi_+ \end{bmatrix} \\ \psi_2 = \begin{bmatrix} \chi_- \\ \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E+m} \chi_- \end{bmatrix} & \psi_2' = \begin{bmatrix} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E-m} \chi_- \\ \chi_- \end{bmatrix} \end{array} \tag{14.10}$$

The first set is the conventional one and the second set is identical with the negative energy solutions if E is taken as negative. For positive energy, the second set becomes indeterminate in the limit $E \to m$ when $\boldsymbol{p} \to 0$.

In a similar way, we can find the solution for the negative energy states of the Dirac equation.

$$\begin{bmatrix} m & \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -m \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = -|E| \begin{bmatrix} u \\ v \end{bmatrix}, \tag{14.11}$$

which, in turn, leads to the following two coupled equations:

$$m\,u + \boldsymbol{\sigma}\cdot\boldsymbol{p}\,v = -|E|\,u; \qquad \frac{u}{v} = -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}, \tag{14.12}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,u - m\,v = -|E|\,v; \qquad \frac{u}{v} = -\frac{|E|-m}{\boldsymbol{\sigma}\cdot\boldsymbol{p}}. \tag{14.13}$$

These coupled equations, in a similar way, give two sets of solutions for the negative energy states.

$$\begin{array}{cc} \text{Set I} & \text{Set II} \\ \psi_3 = \begin{bmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}\chi_+ \\ \chi_+ \end{bmatrix} & \psi_3' = \begin{bmatrix} \chi_+ \\ -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|-m}\chi_+ \end{bmatrix} \\ \psi_4 = \begin{bmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}\chi_- \\ \chi_- \end{bmatrix} & \psi_4' = \begin{bmatrix} \chi_- \\ -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|-m}\chi_- \end{bmatrix} \end{array} \tag{14.14}$$

For negative energy states, the first set of solutions is to be taken, the second set becoming indeterminate in the limit $|E| \to m$ as $\boldsymbol{p} \to 0$.

14.2. Orthogonal and Closure Properties

It can be easily verified that the solutions, $\psi_1, \psi_2, \psi_3, \psi_4$, of Dirac equation given in Eqs. (14.10) and (14.14) are orthogonal but they are not normalized. Using the conventional normalization as in non-relativistic quantum mechanics, we have

$$\psi_i^\dagger \psi_j = \delta_{ij}, \tag{14.15}$$

which yields a normalization factor

$$N = \sqrt{\frac{|E|+m}{2|E|}}. \tag{14.16}$$

The solutions given in Eqs. (14.10) and (14.14) should be multiplied by N to obtain normalized solutions for the positive and negative energy states. It may, however, be noted that $E = |E|$ for positive energy solutions and $E = -|E|$ for negative energy solutions. The normalized solutions of the Dirac equation are

$$\begin{aligned}\psi_{1,2} &= \sqrt{\frac{E+m}{2E}}\begin{bmatrix}\chi_\pm \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_\pm\end{bmatrix} \\ &= \{2E(E+m)\}^{-\frac{1}{2}}\begin{bmatrix}(E+m)\chi_\pm \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi_\pm\end{bmatrix},\end{aligned} \tag{14.17}$$

$$\begin{aligned}\psi_{3,4} &= \sqrt{\frac{|E|+m}{2|E|}}\begin{bmatrix}-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}\chi_\pm \\ \chi_\pm\end{bmatrix} \\ &= \{2|E|(|E|+m)\}^{-\frac{1}{2}}\begin{bmatrix}-\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi_\pm \\ (|E|+m)\,\chi_\pm\end{bmatrix}.\end{aligned} \tag{14.18}$$

Above, each element of a column vector is itself a two-component column vector. For instance,

$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi_+ &= (\sigma_x p_x + \sigma_y p_y + \sigma_z p_z)\chi_+ \\ &= \begin{bmatrix}p_z & p_x - ip_y \\ p_x + ip_y & -p_z\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}p_z \\ p_x + ip_y\end{bmatrix}.\end{aligned} \tag{14.19}$$

Similarly,

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi_- = \begin{bmatrix}p_x - ip_y \\ -p_z\end{bmatrix}. \tag{14.20}$$

Using the above results, we can write down the normalized solutions of the Dirac equation in the four-component form.

$$\begin{aligned}\psi_1 &= N_0\begin{bmatrix}E+m \\ 0 \\ p_z \\ p_x+ip_y\end{bmatrix}, \qquad \psi_2 = N_0\begin{bmatrix}0 \\ E+m \\ p_x-ip_y \\ -p_z\end{bmatrix}, \\ \psi_3 &= N_0\begin{bmatrix}-p_z \\ -(p_x+ip_y) \\ |E|+m \\ 0\end{bmatrix}, \qquad \psi_4 = N_0\begin{bmatrix}-(p_x-ip_y) \\ p_z \\ 0 \\ |E|+m\end{bmatrix},\end{aligned} \tag{14.21}$$

with $N_0 = \{2|E|(|E|+m)\}^{-\frac{1}{2}}$.

The χ functions obey the following orthonormal and closure properties:

$$\langle\chi_i|\chi_j\rangle = \chi_i^\dagger\chi_j = \delta_{ij}, \qquad i,j = +,-. \tag{14.22}$$

$$\sum_i |\chi_i\rangle\langle\chi_i| = \sum_i \chi_i\chi_i^\dagger = I, \qquad i = +,-. \tag{14.23}$$

The normalized ψ functions, in a similar way, obey the orthonormality condition and satisfy the closure relation.

$$\langle\psi_i|\psi_j\rangle = \psi_i^\dagger\,\psi_j = \delta_{ij}, \qquad i,j = 1,2,3,4. \tag{14.24}$$

$$\sum_{i=1}^{4}|\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{4}\psi_i\,\psi_i^\dagger = I. \tag{14.25}$$

Instead of summing over all the four states in Eq. (14.25), a partial sum can be made either over positive energy states or negative energy states to yield (refer solved problems (14.1) and (14.2) for derivation)

$$\sum_{i=1}^{2}|\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{2}\psi_i\,\psi_i^\dagger = \frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m}{E}\right). \tag{14.26}$$

$$\sum_{i=3}^{4}|\psi_i\rangle\langle\psi_i| = \sum_{i=3}^{4}\psi_i\,\psi_i^\dagger = \frac{1}{2}\left(I - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m}{|E|}\right). \tag{14.27}$$

Summing Eqs. (14.26) and (14.27), we obtain once again Eq. (14.25). The operators $\frac{1}{2}\left(I \pm \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m}{|E|}\right)$ in Eqs. (14.26) and (14.27) are sometimes referred to as projection operators for positive and negative energy states (Rose, 1961) since

$$\frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m}{E}\right)\psi_i = \begin{cases}\psi_i, & i = 1,2;\\ 0, & i = 3,4.\end{cases} \tag{14.28}$$

$$\frac{1}{2}\left(I - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m}{|E|}\right)\psi_i = \begin{cases}0, & i = 1,2;\\ \psi_i, & i = 3,4.\end{cases} \tag{14.29}$$

14.3. Sum Over Spin States

We are now in a position to treat the scattering of Dirac particles when the spins of both the incident and scattered particles are not observed. To be specific, we shall consider the Coulomb scattering of electrons by a nucleus of charge *ze* but the formalism given below is sufficiently general and is applicable to any problem since each problem differs from the rest only in the choice of transition operator $\mathcal{O}$.

The transition matrix element T_{fi} is given by

$$T_{fi} = (\tilde{\psi}_f\,\mathcal{O}\,\psi_i), \tag{14.30}$$

where $\tilde{\psi}_f = \psi_f^\dagger$ g0 and $\gamma_0 \equiv \beta$. The square of the matrix element is obtained by summing over the final spin states and averaging over the initial spin states.

$$|T_{fi}|^2 = \frac{1}{2}\sum_{i,f}(\tilde{\psi}_f\, \mathcal{O}\, \psi_i)(\tilde{\psi}_f\, \mathcal{O}\, \psi_i)^\dagger = \frac{1}{2}\sum_{i,f}(\psi_f^\dagger \gamma_0 \mathcal{O} \psi_i)(\psi_i^\dagger \mathcal{O}^\dagger \gamma_0 \psi_f), \tag{14.31}$$

where the summation indices *i, f* are over the two spin states denoted by ψ_1 and ψ_2, corresponding to positive energy states only. Using the algebra of matrix multiplication)

$$\begin{aligned} |T_{fi}|^2 &= \frac{1}{2}\sum_{i,f}\sum_{\rho,\lambda,\rho',\lambda'} \{(\psi_f^\dagger)_\rho\,(\gamma_0\mathcal{O})_{\rho\lambda}\,(\psi_i)_\lambda\}\{(\psi_i^\dagger)_{\rho'}\,(\mathcal{O}^\dagger\gamma_0)_{\rho'\lambda'}(\psi_f)_{\lambda'}\} \\ &= \frac{1}{2}\sum_{i,f}\sum_{\rho,\lambda,\rho',\lambda'} (\psi_f^\dagger)_\rho\,(\gamma_0\mathcal{O})_{\rho\lambda}\,(\psi_i\psi_i^\dagger)_{\lambda\rho'}\,(\mathcal{O}^\dagger\gamma_0)_{\rho'\lambda'}(\psi_f)_{\lambda'} \\ &= \frac{1}{2}\sum_{i,f}\sum_{\rho,\lambda,\rho',\lambda'} (\gamma_0\mathcal{O})_{\rho\lambda}\,(\psi_i\psi_i^\dagger)_{\lambda\rho'}\,(\mathcal{O}^\dagger\gamma_0)_{\rho'\lambda'}(\psi_f\psi_f^\dagger)_{\lambda'\rho} \\ &= \frac{1}{2}\sum_{\rho,\lambda,\rho',\lambda'} (\gamma_0\mathcal{O})_{\rho\lambda}\,(\Lambda_i)_{\lambda\rho'}\,(\mathcal{O}^\dagger\gamma_0)_{\rho'\lambda'}(\Lambda_f)_{\lambda'\rho} \\ &= \frac{1}{2}\mathrm{Tr}\,(\gamma_0\,\mathcal{O}\,\Lambda_i\,\mathcal{O}^\dagger\,\gamma_0\,\Lambda_f). \end{aligned} \tag{14.32}$$

In the above equation, Λ_i and Λ_f are the projection operators obtained after summing over the two spin states corresponding to the positive energy state.

$$\Lambda_i = \frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_i + \beta m}{E_i}\right), \tag{14.33}$$

$$\Lambda_f = \frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}_f + \beta m}{E_f}\right). \tag{14.34}$$

For Coulomb scattering of electrons on nuclei of charge *ze,* the relevant transition operator is

$$\mathcal{O} = \frac{4\pi z e^2}{q^2}\gamma_0, \tag{14.35}$$

where $q^2 = (\boldsymbol{p}_f - \boldsymbol{p}_i)^2$ is the three-momentum transfer. Substituting $\mathcal{O}$ in Eq. (14.32) and remembering that $\gamma_0^2 = 1$, we obtain

$$|T_{fi}|^2 = \frac{1}{2}\left(\frac{4\pi z e^2}{q^2}\right)^2\,\mathrm{Tr}\,(\Lambda_i\,\Lambda_f). \tag{14.36}$$

Substituting expressions (14.33) and (14.34) for the projection operators Λ_i and Λ_f, we get

$$\begin{aligned}
\mathrm{Tr}\,(\Lambda_i\Lambda_f) &= \frac{1}{4E_iE_f}\mathrm{Tr}\,\{(E_i+\boldsymbol{\alpha}\cdot\boldsymbol{p}_i+\beta m)(E_f+\boldsymbol{\alpha}\cdot\boldsymbol{p}_f+\beta m)\} \\
&= \frac{1}{4E_iE_f}\mathrm{Tr}\,(E_iE_f+m^2+\boldsymbol{\alpha}\cdot\boldsymbol{p}_i\,\boldsymbol{\alpha}\cdot\boldsymbol{p}_f) \\
&= \frac{1}{4E_iE_f}\mathrm{Tr}\,(E_iE_f+m^2+\boldsymbol{p}_i\cdot\boldsymbol{p}_f) \\
&= \frac{1}{4E_iE_f}(4E_iE_f+4m^2+4\,\boldsymbol{p}_i\cdot\boldsymbol{p}_f). \qquad (14.37)
\end{aligned}$$

In deriving Eq. (14.37), the following relations were utilized.

$$\begin{aligned}
&\mathrm{Tr}\,\beta=0;\quad \mathrm{Tr}\,\alpha_x=\mathrm{Tr}\,\alpha_y=\mathrm{Tr}\,\alpha_z=0; \\
&\mathrm{Tr}\,\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})=\mathrm{Tr}\,(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta=0;\quad \mathrm{Tr}\,(\boldsymbol{\alpha}\cdot\boldsymbol{p}_i)(\boldsymbol{\alpha}\cdot\boldsymbol{p}_f)=4\,\boldsymbol{p}_i\cdot\boldsymbol{p}_f.
\end{aligned} \qquad (14.38)$$

For elastic scattering, $E_i = E_f$ and $|\mathbf{p_i}| = |\mathbf{p_f}|$ in c.m. frame. If θ is the scattering angle,

$$\begin{aligned}
\mathrm{Tr}\,(\Lambda_i\Lambda_f) &= \frac{1}{4E^2}(4E^2+4m^2+4p^2\cos\theta) \\
&= \frac{1}{4E^2}\{4E^2+4(E^2-p^2)+4p^2\cos\theta\} \\
&= \frac{1}{4E^2}\{8E^2+4p^2(\cos\theta-1)\} \\
&= 2\left(1-\frac{p^2}{E^2}\sin^2\frac{\theta}{2}\right) \\
&= 2\left(1-v^2\sin^2\frac{\theta}{2}\right), \qquad (14.39)
\end{aligned}$$

where $v = p/E$ is the velocity of the electron. The matrix element square now becomes

$$|T_{fi}|^2=\left(\frac{4\pi ze^2}{q^2}\right)^2\left(1-v^2\sin^2\frac{\theta}{2}\right), \qquad (14.40)$$

with $q^2 = (\boldsymbol{p}_f - \boldsymbol{p}_i)^2 = 4p^2\sin^2(\theta/2)$.

The scattering cross section is given by

$$d\sigma=\frac{2\pi}{v}|T_{fi}|^2\rho_f, \qquad (14.41)$$

where ρ_f is the density of final states.

$$\rho_f = \frac{dn}{dE_f} = \frac{d^3p_f}{(2\pi)^3 dE_f} = \frac{p_f^2 dp_f d\Omega}{(2\pi)^3 dE_f} = \frac{p_f E_f d\Omega}{(2\pi)^3}. \tag{14.42}$$

Substituting the density of final states, we obtain the differential cross section after simplification.

$$\frac{d\sigma}{d\Omega} = \frac{z^2 e^4}{4p^2 v^2 \sin^4(\theta/2)} \left(1 - v^2 \sin^2 \frac{\theta}{2}\right). \tag{14.43}$$

14.4. In Feynman's Notation

Multiply the Dirac equation (14.5) by β from the left

$$(\beta \boldsymbol{\alpha} \cdot \boldsymbol{p} + m)\psi = \beta E \psi. \tag{14.44}$$

Introducing γ matrices

$$\gamma_0 = \beta, \qquad \boldsymbol{\gamma} = \beta \boldsymbol{\alpha}, \tag{14.45}$$

we can rewrite the Dirac equation after rearrangement in the form

$$\not{p}\, \psi = m\, \psi, \qquad \not{p} = \gamma_0 E - \boldsymbol{\gamma} \cdot \boldsymbol{p}. \tag{14.46}$$

Writing the characteristic equation for the matrix $\not{p}$,

$$\begin{vmatrix} E - \lambda & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -E - \lambda \end{vmatrix} = 0, \tag{14.47}$$

one finds the eigenvalues of the matrix $\not{p}$.

$$\lambda^2 = E^2 - p^2 = m^2; \quad \lambda = \pm m. \tag{14.48}$$

This leads to two equations, one for 'positive eigenvalue' state and the other for 'negative eigenvalue' state.

$$\not{p}\, \psi_p = m\, \psi_p, \tag{14.49}$$

$$\not{p}\, \psi_n = -m\, \psi_n, \tag{14.50}$$

where ψ_p and ψ_n denote the positive and negative eigenvalue states. The equation for ψ_n is obtained by reversing the sign of energy and momentum so that $\not{p}$ is changed into $-\not{p}$. The state ψ_n which represents the negative energy electron with momentum $-\boldsymbol{p}$ is to be associated with the state of a positron with positive energy and momentum ***+p,*** according to the Dirac hole theory.

Earlier, we have normalized $\psi^\dagger\psi$ to 1 but this normalization is not relativistically invariant. Since $\psi^\dagger\psi$ (which is the fourth component of a four-vector current) transforms as the fourth component of a four-vector, it is possible to make a relativistically invariant normalization by setting it equal to the fourth component of a suitable four-vector, say, energy-momentum four-vector. Feynman (1962) has chosen the normalization[2]

$$\psi_p^\dagger\psi_p = 2E \text{ or equivalently } \tilde{\psi}_p\psi_p = 2m, \tag{14.51}$$

for positive eigenvalue solutions. The normalized solutions ψ_p are

$$\psi_p = \sqrt{E+m}\begin{bmatrix} \chi_\pm \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_\pm \end{bmatrix}. \tag{14.52}$$

It can be easily verified that

$$\sum_{\text{spins}} \psi_p\tilde{\psi}_p = \not{p} + m. \tag{14.53}$$

The normalized negative eigenvalue solutions are

$$\psi_n = (E+m)^{\frac{1}{2}}\begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_\pm \\ \chi_\pm \end{bmatrix}, \tag{14.54}$$

the normalization being

$$\tilde{\psi}_n\psi_n = -2m. \tag{14.55j}$$

It can be easily verified by matrix multiplication that

$$\sum_{\text{spins}} \psi_n\tilde{\psi}_n = \not{p} - m. \tag{14.56}$$

It can be verified that

$$\sum_{\text{spins}} \{\psi_p\tilde{\psi}_p - \psi_n\tilde{\psi}_n\} = 2mI. \tag{14.57}$$

[2]Schweber et al. (1956) choose a slightly different normalization $\tilde{\psi}_p\psi_p = 1$ and $\tilde{\psi}_n\psi_n = -1$ such that

$$\sum_{\text{spins}}(\tilde{\psi}_p\psi_p - \tilde{\psi}_n\psi_n) = 4 \quad\text{and}\quad \sum_{\text{spins}}(\psi_p\tilde{\psi}_p - \psi_n\tilde{\psi}_n) = I.$$

The operators $\not{p} + m$ and $\not{p} - m$ are the projection operators for the positive and negative eigenvalue states.

$$\begin{aligned}
(\not{p} + m)\psi_p &= 2m\psi_p, \\
(\not{p} + m)\psi_n &= 0, \\
(\not{p} - m)\psi_p &= 0, \\
(\not{p} - m)\psi_n &= -2m\psi_n.
\end{aligned} \tag{14.58}$$

It may be observed that the positive energy spinors of Dirac coincide with the positive eigenvalue spinors of Feynman, except for normalization factor, whereas the negative energy spinors of Dirac differ from the negative eigenvalue spinors of Feynman with respect to the sign of the momentum vector ***p***. The source of this discrepancy can easily be traced. The negative energy solutions of the Dirac equation are obtained by changing the sign of energy alone and not momentum, whereas in Feynman's negative eigenvalue equation, the signs of both energy and momentum are reversed. An electron with energy $-|E|$ and momentum ***−p*** is equivalent to a positron with energy $|E|$ and momentum ***p***.

Let us now reconsider the problem of summing over spin states using Feynman's notation. The square of the matrix element (14.31) can be evaluated using Feynman's projection operator for positive energy states.

$$\begin{aligned}
|T_{fi}|^2 &= \frac{1}{2}\sum_{if}(\tilde{\psi}_f O\psi_i)(\tilde{\psi}_f O\psi_i)^\dagger \\
&= \frac{1}{2}\sum_{if}(\tilde{\psi}_f O\psi_i)(\psi_i^\dagger O^\dagger \tilde{\psi}_f^\dagger) \\
&= \frac{1}{2}\sum_{if}(\tilde{\psi}_f O\psi_i)(\tilde{\psi}_i \gamma_0 O^\dagger \gamma_0 \psi_f) \\
&= \frac{1}{2}\sum_{if}(\tilde{\psi}_f O\psi_i)(\tilde{\psi}_i \tilde{O}\psi_f),
\end{aligned} \tag{14.59}$$

where $\sum_{if}$ denotes the summation over the positive energy spin states of the incident and scattered particle and $\tilde{O}$ stands for

$$\tilde{O} = \gamma_0 O^\dagger \gamma_0. \tag{14.60}$$

Replacing $\sum_i \psi_i\tilde{\psi}_i$ by $(\not{p}_i + m)$ which is the projection operator for positive energy states, we obtain

$$|T_{fi}|^2 = \frac{1}{2}\sum_{f}(\tilde{\psi}_f O(\not{p}_i + m)\tilde{O}\psi_f). \tag{14.61}$$

Using the algebra of matrix multiplication, we get

$$\begin{aligned} |T_{fi}|^2 &= \frac{1}{2}\sum_f \sum_{\rho,\lambda,\rho',\lambda'} (\tilde{\psi}_f)_\rho (\mathcal{O})_{\rho\lambda}(\not{p}_i + m)_{\lambda\rho'}(\tilde{\mathcal{O}})_{\rho'\lambda'}(\psi_f)_{\lambda'} \\ &= \frac{1}{2}\sum_f \sum_{\rho,\lambda,\rho',\lambda'} (\mathcal{O})_{\rho\lambda}(\not{p}_i + m)_{\lambda\rho'}(\tilde{\mathcal{O}})_{\rho'\lambda'}(\psi_f\tilde{\psi}_f)_{\lambda'\rho} \\ &= \frac{1}{2}\mathrm{Tr}\left(\mathcal{O}(\not{p}_i + m)\,\tilde{\mathcal{O}}\,(\not{p}_f + m)\right). \qquad (14.62) \end{aligned}$$

The transition probability per unit time is given by Fermi's golden rule

$$\text{Transition rate} = 2\pi(\Pi N)^{-1}|T_{fi}|^2\rho_f, \qquad (14.63)$$

where ΠN denotes the normalization factor $2E$ for each of the initial and final particles and ρ_f is the density of states for the final particle. The cross section is the transition rate per unit incident flux.

14.5. A Consistency Check

We have deduced two different expressions (14.32) and (14.62), for the square of the transition amplitude $|T_{fi}|^2$, one using the Dirac matrices and the other using Feynman's notation. They must be equivalent. To show this, let us start with the projection operator Λ for positive energy states.

$$\begin{aligned} \Lambda &= \frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m}{E}\right) \\ &= \frac{1}{2E}(E + \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m). \qquad (14.64) \end{aligned}$$

Multiply by $\beta^2 = I$ from the right to obtain

$$\begin{aligned} \Lambda &= \frac{1}{2E}(\beta E + \boldsymbol{\alpha}\cdot\mathbf{p}\beta + m)\beta \\ &= \frac{1}{2E}(\beta E - \beta\boldsymbol{\alpha}\cdot\mathbf{p} + m)\beta \\ &= \frac{1}{2E}(\not{p} + m)\beta. \qquad (14.65) \end{aligned}$$

Substituting the expression (14.65) for Λ, into Eq. (14.32) and remembering that $\beta \equiv \gamma_0$ and $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, we get

$$\begin{aligned} |T_{fi}|^2 &= \frac{1}{8E_iE_f}\mathrm{Tr}\left(\gamma_0\mathcal{O}(\not{p}_i + m)\gamma_0\mathcal{O}^\dagger\gamma_0(\not{p}_f + m)\gamma_0\right) \\ &= \frac{1}{8E_iE_f}\mathrm{Tr}\left(\mathcal{O}(\not{p}_i + m)\tilde{\mathcal{O}}(\not{p}_f + m)\right). \qquad (14.66) \end{aligned}$$

This is identical with Eq. (14.62) except for the additional factor $1/(4E_iE_f)$ which we include, in Feynman's formalism, as normalization factor $(\Pi N)^{-1}$, as indicated in Eq. (14.63).

14.6. Algebra of γ Matrices

The square of the transition matrix element given by Eq. (14.62) involves the trace of a product of γ matrices. So, it will be fruitful to study the algebra of γ matrices (Feynman, 1962; Ramakrishnan, 1962) for evaluating $|T_{fi}|^2$. The γ matrices obey the following relations:

$$\begin{aligned} &\gamma_0^2 = 1, \quad \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1, \\ &\gamma_0\gamma_{x,y,z} + \gamma_{x,y,z}\gamma_0 = 0, \\ &\gamma_x\gamma_y + \gamma_y\gamma_x = 0, \quad \gamma_y\gamma_z + \gamma_z\gamma_y = 0, \quad \gamma_z\gamma_x + \gamma_x\gamma_z = 0. \end{aligned} \tag{14.67}$$

Using a unified notation, Eq. (14.67) can be written as

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \tag{14.68}$$

where $g_{\mu\nu}$ is a metric defined by

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ +1, & \mu = \nu = 0 \\ -1, & \mu, \nu = x, y, z. \end{cases} \tag{14.69}$$

Besides, the matrix γ_0 is Hermitian whereas the matrices γ_x, γ_y, γ_z are anti-Hermitian.

$$\gamma_0^\dagger = \gamma_0; \quad \gamma_k^\dagger = -\gamma_k, \; k = x, y, z. \tag{14.70}$$

It is convenient to define a matrix γ_5 which occurs frequently.

$$\gamma_5 = \gamma_x\gamma_y\gamma_z\gamma_0 = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{14.71}$$

It is easy to verify that

$$\gamma_5^\dagger = -\gamma_5; \quad \gamma_5^2 = -1; \quad \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0. \tag{14.72}$$

Following Feynman, we can define $\not{a}$ as follows:

$$\not{a} = a_0\gamma_0 - a_x\gamma_x - a_y\gamma_y - a_z\gamma_z. \tag{14.73}$$

It can be shown that

$$\not a \gamma_5 = -\gamma_5 \not a; \tag{14.74}$$

$$\not a \not b = -\not b \not a + 2\, a \cdot b \quad (a \cdot b = a_\mu b_\mu); \tag{14.75}$$

$$\gamma_x \not a \gamma_x = \not a + 2 a_x \gamma_x; \tag{14.76}$$

$$\gamma_\mu \not a \gamma_\mu = -2 \not a; \tag{14.77}$$

$$\gamma_\mu \not a \not b \gamma_\mu = 4\, a \cdot b; \tag{14.78}$$

$$\gamma_\mu \not a \not b \not c \gamma_\mu = -2 \not c \not b \not a. \tag{14.79}$$

It is important to recall the elementary properties of traces,

$$\begin{aligned} &\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(BCA) = \mathrm{Tr}\,(CAB), \\ &\mathrm{Tr}\,(A + B) = \mathrm{Tr}\, A + \mathrm{Tr}\, B, \end{aligned} \tag{14.80}$$

for evaluating the traces involving a product of γ matrices. It is known that the trace of a γ matrix is zero.

$$\mathrm{Tr}\,\gamma_\mu = 0 \qquad (\mu = 0, x, y, z). \tag{14.81}$$

Also the trace of an odd number of γ matrices vanishes. To prove this, we start with the relation (14.72) which is equivalent to

$$\gamma_5 \gamma_\mu (\gamma_5)^{-1} = -\gamma_\mu. \tag{14.82}$$

It follows that

$$\gamma_5 \gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_n} (\gamma_5)^{-1} = (-1)^n \gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_n}. \tag{14.83}$$

Taking the trace of both sides of Eq. (14.83) and using the elementary property of the trace that Tr (ABC) = Tr (BCA), we obtain immediately that

$$(-1)^n \mathrm{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}) = \mathrm{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}). \tag{14.84}$$

Equation (14.84) implies that the the trace of an odd number of gamma matrices vanishes.

If n is even, it is always possible to reduce it to n - 2 factors. For example,

$$\begin{aligned} \mathrm{Tr}(\gamma_\mu \gamma_\nu) &= \mathrm{Tr}\,(\gamma_\nu \gamma_\mu) \\ &= \tfrac{1}{2}\,\mathrm{Tr}\,(\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu), && \text{since Tr(AB)=Tr(BA)} \\ &= g_{\mu\nu}\,\mathrm{Tr}\, I, && \text{using Eq. (14.68)} \\ &= 4\, g_{\mu\nu}. \end{aligned} \tag{14.85}$$

In a similar way, it can be shown that

$$\text{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda) = 4g_{\mu\lambda}g_{\nu\rho} - 4g_{\lambda\nu}g_{\rho\mu} + 4g_{\lambda\rho}g_{\mu\nu}. \tag{14.86}$$

The following traces which occur frequently are given.

$$\begin{aligned} \text{Tr}\,(\not{a}\not{b}) &= \tfrac{1}{2}\text{Tr}\,(\not{a}\not{b} + \not{b}\not{a}) = \text{Tr}\,(a \cdot b) = 4\,a \cdot b, \\ \text{Tr}\,(\not{a}\not{b}\not{c}) &= 0. \end{aligned} \tag{14.87}$$

We shall illustrate the foregoing discussion by evaluating the trace in Eq. (14.62). The relevant operator for Coulomb scattering of an electron by a nucleus of charge ze is

$$\mathcal{O} = \frac{4\pi z e^2}{q^2}\gamma_0. \tag{14.88}$$

Substituting the operator $\mathcal{O}$ in Eq. (14.62), we obtain

$$|T_{fi}|^2 = \frac{1}{2}\left(\frac{4\pi z e^2}{q^2}\right)^2 \text{Tr}\left(\gamma_0(\not{p}_i + m)\,\gamma_0\,(\not{p}_f + m)\right), \tag{14.89}$$

where $\not{p} = \gamma_0 - \boldsymbol{\gamma} \cdot \boldsymbol{p}$. Since the trace of a product of an odd number of γ matrices vanishes,

$$\begin{aligned} \text{Tr}\left(\gamma_0(\not{p}_i + m)\,\gamma_0\,(\not{p}_f + m)\right) &= \text{Tr}\left(\gamma_0\,\not{p}_i\,\gamma_0\,\not{p}_f + m^2\right) \\ &= \text{Tr}\left((-\not{p}_i\gamma_0 + 2E_i)\gamma_0\not{p}_f + m^2\right) \\ &= \text{Tr}\left(-\not{p}_i\not{p}_f + 2E_i\gamma_0\not{p}_f + m^2\right) \\ &= -4\,p_i \cdot p_f + 8E_iE_f + 4m^2. \end{aligned} \tag{14.90}$$

Equations (14.85) and (14.87) have been used in deducing the last step in the above equation. Expanding the scalar product of the four-vectors $p_i \cdot p_f = E_iE_f - \boldsymbol{p}_i \cdot \boldsymbol{p}_f$ and rearranging, we get

$$\text{Tr}\left(\gamma_0(\not{p}_i + m)\,\gamma_0\,(\not{p}_f + m)\right) = 4E_iE_f + 4p_ip_f\cos\theta + 4m^2, \tag{14.91}$$

where θ denotes the angle of scattering. Equation (14.91) is the same as Eq. (14.37), deduced earlier except for a factor that is accounted in Feynman's formulation as the normalization factor as indicated in Eq. (14.63).

Review Questions

14.1 Write down the Dirac equation for a free particle and obtain its solutions. How many solutions are there and how are they interpreted? Discuss the orthogonal and closure properties of such solutions.

14.2 Obtain the projection operators for the positive and negative energy states of the Dirac Hamiltonian. How are they constructed and why are they called projection operators?

14.3 Obtain the Dirac equation in Feynman's notation and obtain its solution. Show that the negative eigenvalue solutions of Feynman differ from the negative energy solutions of Dirac. How will you account for this discrepancy?

14.4 In the case of scattering of Dirac particle, find the transition rate if the initial and the final spin states are not observed. Assume the transition operator to be $\mathcal{O}$.

14.5 In the case of Coulomb scattering of electron by a nucleus, deduce an expression for the cross section.

Problems

14.1 Given the positive energy solutions ψ_1 and ψ_2 of free particle Dirac equation, find $\sum_{i=1,2} \psi_i \psi_i^\dagger$ and show that it can be considered as the projection operator for positive energy solutions.

14.2 Given the negative energy solutions ψ_3 and ψ_4 of free particle Dirac equation, find $\sum_{i=3,4} \psi_i \psi_i^\dagger$ and show that it can be considered as the projection operator for negative energy solutions.

14.3 Using the algebra of γ matrices, deduce Eqs. (14.73) - (14.78) in Feynman's notation.

14.4 Given the transition operator $\mathcal{O} = \gamma \cdot \boldsymbol{A}$, where $\boldsymbol{A}$ is a vector but not an operator, calculate the square of the matrix element for the transition of an electron if the initial and final spin states are not observed.

14.5 Given the transition operator $\mathcal{O} = \gamma_\mu J_\mu$, where J_μ is a four-vector current, calculate the square of the matrix element for the transition of a Dirac particle if the initial and final spin states are not observed.

Solutions to Selected Problems

14.1 Using normalized wave functions ψ_1 and ψ_2 for positive energy Dirac particle as given in Eq. (14.17), the normalization being $\psi_i^\dagger \psi_j = \delta_{ij}$, we find

$$\sum_{i=1,2} \psi_{1,2}\psi_{1,2}^\dagger = \sum_{i=+,-} \frac{E+m}{2E} \begin{bmatrix} \chi_i \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_i \end{bmatrix} \begin{bmatrix} \chi_i & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_i \end{bmatrix}$$
$$= \frac{1}{2E} \begin{bmatrix} E+m & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & \frac{p^2}{E+m} \end{bmatrix}.$$

Since $p^2 = E^2 - m^2$, it follows that $p^2/(E+m) = E - m$. The resulting matrix can be written in terms of the Dirac matrices.

$$\Lambda_+ = \sum_{i=1,2} \psi_i\psi_i^\dagger = \frac{1}{2E}(EI + \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{p})$$
$$= \frac{1}{2}\left(I + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m}{E}\right).$$

Λ_+ is called the projection operator for positive energy states since

$$\Lambda_+\psi_i = \begin{cases} \psi_i, & i = 1,2; \\ 0, & i = 3,4. \end{cases}$$

14.2 The normalized wave functions ψ_3 and ψ_4, corresponding to the negative energy eigenstates are given in Eq. (14.18). Use them and follow the same procedure as in Problem (14.1). Only the final result is given.

$$\Lambda_- = \sum_{i=3,4} \psi_i\psi_i^\dagger = \frac{1}{2|E|}(|E|I - \beta m - \boldsymbol{\alpha}\cdot\boldsymbol{p})$$
$$= \frac{1}{2}\left(I - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m}{|E|}\right).$$

Λ_- is called the projection operator for negative energy states since

$$\Lambda_-\psi_i = \begin{cases} 0, & i = 1,2; \\ \psi_i, & i = 3,4. \end{cases}$$

14.4 From Eq. (14.62), we have

$$|T_{fi}|^2 = \frac{1}{2}\mathrm{Tr}\left\{\mathcal{O}\left(\not{p}_i + m\right)\tilde{\mathcal{O}}\left(\not{p}_f + m\right)\right\},$$

where

$$\tilde{\mathcal{O}} = \gamma_0\mathcal{O}\gamma_0.$$

The transition operator $\mathcal{O}$, in the present case is $\gamma \cdot \boldsymbol{A}$. Substituting it, we shall write down the product of operators $\{...\}$.

$$\begin{aligned}\{\cdots\} &= \gamma_k A_k\left(\gamma_\mu (p_i)_\mu + m\right)\gamma_0\gamma_l^\dagger A_l^* \gamma_0\left(\gamma_\nu (p_f)_\nu + m\right)\\ &= \gamma_k A_k\left(\gamma_\mu (p_i)_\mu + m\right)\gamma_l A_l^*\left(\gamma_\nu (p_f)_\nu + m\right).\end{aligned}$$

We have used above the relations $\gamma_l^\dagger = -\gamma_l$, $\gamma_0\gamma_k = -\gamma_k\gamma_0$ and $\gamma_0^2 = 1$. Since the trace of a product of odd number of γ matrices is zero,

$$\mathrm{Tr}\{\cdots\} = \mathrm{Tr}\{\gamma_k\gamma_\mu\gamma_l\gamma_\nu A_k A_l^*(p_i)_\mu(p_f)_\nu + \gamma_k\gamma_l A_k A_l^* m^2\}.$$

The indices k and l denote the components of a three vector and the indices μ and ν denote the components of a four-vector. We have earlier evaluated the traces of even number of γ matrices.

$$\begin{aligned}&\mathrm{Tr}(\gamma_k\gamma_\mu\gamma_l\gamma_\nu) = 4g_{k\mu}g_{l\nu} - 4g_{kl}g_{\mu\nu} + 4g_{k\nu}g_{\mu l};\\ &\mathrm{Tr}(\gamma_k\gamma_l) = 4g_{kl}.\end{aligned}$$

Using the above results,

$$\begin{aligned}|T_{fi}|^2 &= \frac{1}{2}\{4g_{k\mu}g_{l\nu}(p_i)_\mu A_k(p_f)_\nu A_l^* - 4g_{kl}g_{\mu\nu}A_k A_l^*(p_i)_\mu(p_f)_\nu\\ &\quad +4g_{k\nu}g_{\mu l}(p_f)_\nu A_k(p_i)_\mu A_l^* + 4g_{kl}A_k A_l^* m^2\}\\ &= \frac{1}{2}\{4(\boldsymbol{p}_i\cdot\boldsymbol{A})(\boldsymbol{p}_f\cdot\boldsymbol{A}^*) + 4\boldsymbol{A}\cdot\boldsymbol{A}^*(E_iE_f - \boldsymbol{p}_i\cdot\boldsymbol{p}_f)\\ &\quad +4(\boldsymbol{p}_f\cdot\boldsymbol{A})(\boldsymbol{p}_i\cdot\boldsymbol{A}^*) - 4m^2\boldsymbol{A}\cdot\boldsymbol{A}^*\}\\ &= 2\left\{(E_iE_f - \boldsymbol{p}_i\cdot\boldsymbol{p}_f - m^2)\boldsymbol{A}\cdot\boldsymbol{A}^* + (\boldsymbol{p}_i\cdot\boldsymbol{A})(\boldsymbol{p}_f\cdot\boldsymbol{A}^*)\right.\\ &\quad \left.+(\boldsymbol{p}_f\cdot\boldsymbol{A})(\boldsymbol{p}_i\cdot\boldsymbol{A}^*)\right\}.\end{aligned}$$

EQUIVALENCE OF ROTATION ABOUT AN ARBITRARY AXIS TO EULER ANGLES OF ROTATION

Rotation of the coordinate system through an angle ψ about an arbitrary axis denoted by the unit vector $\hat{n}$ (θ,ϕ) is equivalent to successive rotations through the Euler angles α,β,γ about the z-axis, the new y-axis and the new z-axis respectively. In what follows, we shall try to obtain a relation between the two sets of three parameters θ,ϕ, ψ and α,β,γ describing the rotation.

The procedure is outlined below. First, we consider the rotation about an arbitrary axis $\hat{n}$ and obtain the transformation matrix $M(e_0, \boldsymbol{e})$ in terms of certain parameters e_0, $\boldsymbol{e}$, known as the Euler parameters which are related to the rotation parameters θ,ϕ, ψ. Then we consider unitary transformations in a two-dimensional complex space, which is equivalent to a rotation in the three-dimensional real space. The unitary transformation matrix Q in complex space when expressed in terms of certain parameters yields the same transformation matrix for three-dimensional rotation obtained earlier in terms of Euler parameters for rotation about an arbitrary axis $\hat{n}$. Thus, we identify the parameters used in the description of unitary matrix Q in complex space with the Euler parameters. Since the parameters describing the complex unitary matrix Q are related to Euler angles of rotation α,β,γ, we deduce the required relation between the two sets of rotation parameters θ,ϕ, ψ and α,β,γ.

It is also possible to obtain the transformation matrix $M(\alpha,\beta,\gamma)$ due to Euler angles of rotation directly, and comparing this with the transformation matrix obtained by rotation through an angle ψ about an axis denoted by the unit vector $\hat{n}(\theta,\phi)$, we deduce the required relations for their equivalence.

A.1. Rotation About an Arbitrary Axis

Rotation of the coordinate system through an angle ψ about an axis in the anticlockwise direction is equivalent to a clockwise rotation of the object through the same angle about the same axis in the fixed coordinate system.

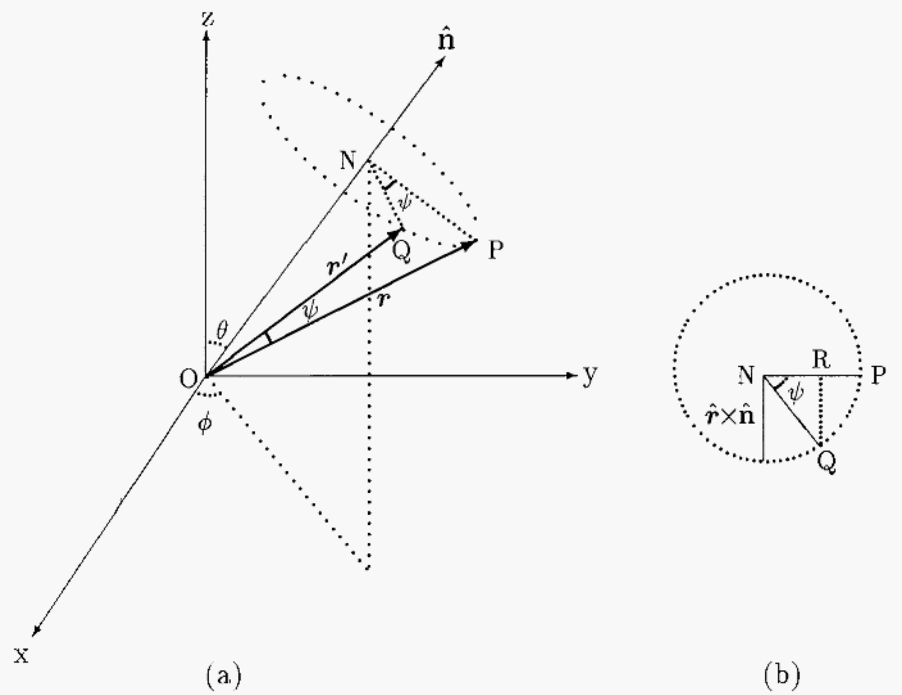

Figure A.1. (a) Effect of rotation about an arbitrary axis $\hat{\boldsymbol{n}}$ through an angle ψ. The point P moves to point Q and the vector $\boldsymbol{r} \to \boldsymbol{r}'$. (b) The section normal to the plane is shown separately.

Consider a vector $\boldsymbol{r}$ denoted by **OP** rotated in the clockwise direction through an angle ψ about the axis $\hat{\boldsymbol{n}}$. The new vector $\boldsymbol{r}'$ is denoted by **OQ.** From Figure A.1, we obtain the following relations.

$$\boldsymbol{r} = \mathbf{OP} = \mathbf{ON} + \mathbf{NP}; \tag{A.1}$$

$$\boldsymbol{r}' = \mathbf{OQ} = \mathbf{ON} + \mathbf{NQ} = \mathbf{ON} + \mathbf{NR} + \mathbf{RQ}; \tag{A.2}$$

$$|\mathbf{OP}| = |\mathbf{OQ}|; \quad |\mathbf{NP}| = |\mathbf{NQ}|; \tag{A.3}$$

$$|\mathbf{NP}| = |\boldsymbol{r} \times \hat{\boldsymbol{n}}|; \quad |\mathbf{NR}| = \mathrm{NQ}\cos\psi; \quad |\mathbf{RQ}| = \mathrm{NQ}\sin\psi; \tag{A.4}$$

$$\mathbf{NP} = \mathbf{OP} - \mathbf{ON} = \boldsymbol{r} - (\boldsymbol{r}.\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}; \tag{A.5}$$

$$\mathbf{NR} = (\boldsymbol{r} - (\boldsymbol{r}.\hat{\boldsymbol{n}})\hat{\boldsymbol{n}})\cos\psi; \tag{A.6}$$

$$\mathbf{RQ} = (\boldsymbol{r} \times \hat{\boldsymbol{n}})\sin\psi. \tag{A.7}$$

Hence

$$\begin{aligned} \boldsymbol{r}' &= (\boldsymbol{r}.\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + (\boldsymbol{r} - (\boldsymbol{r}.\hat{\boldsymbol{n}})\hat{\boldsymbol{n}})\cos\psi + (\boldsymbol{r} \times \hat{\boldsymbol{n}})\sin\psi \\ &= \boldsymbol{r}\cos\psi + (\boldsymbol{r}.\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}(1 - \cos\psi) + (\boldsymbol{r} \times \hat{\boldsymbol{n}})\sin\psi. \end{aligned} \tag{A.8}$$

Let us now introduce the Euler parameters e_0, e_1, e_2, e_3 and express the vector $\boldsymbol{r'}$ in terms of $\boldsymbol{r}$ and the Euler parameters.

$$e_0 = \cos\frac{\psi}{2}, \quad \boldsymbol{e} = \hat{\boldsymbol{n}}\sin\frac{\psi}{2}, \tag{A.9}$$

such that

$$e_0^2 + \boldsymbol{e}^2 = 1, \tag{A.10}$$

or equivalently

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1. \tag{A.11}$$

It follows that

$$\cos\psi = 1 - 2\sin^2\frac{\psi}{2} = 2\cos^2\frac{\psi}{2} - 1 = 2e_0^2 - 1, \tag{A.12}$$

$$\hat{\boldsymbol{n}}\sin\psi = 2\hat{\boldsymbol{n}}\sin\frac{\psi}{2}\cos\frac{\psi}{2} = 2e_0\boldsymbol{e}. \tag{A.13}$$

Using the above relations, $\boldsymbol{r'}$ can be written as a function of $\boldsymbol{r}$ and the Euler parameters e_0 and $\boldsymbol{e}$.

$$\boldsymbol{r}' = \boldsymbol{r}(2e_0^2 - 1) + 2(\boldsymbol{r}.\boldsymbol{e})\boldsymbol{e} + 2(\boldsymbol{r}\times\boldsymbol{e})e_0. \tag{A.14}$$

Using Eq. (A.14), the components x' y', z' can be explicitly written in terms of x, y, z.

$$x' = x(2e_0^2 - 1) + 2e_1(xe_1 + ye_2 + ze_3) + 2e_0(ye_3 - ze_2), \tag{A.15}$$

$$y' = y(2e_0^2 - 1) + 2e_2(xe_1 + ye_2 + ze_3) + 2e_0(ze_1 - xe_3), \tag{A.16}$$

$$z' = z(2e_0^2 - 1) + 2e_3(xe_1 + ye_2 + ze_3) + 2e_0(xe_2 - ye_1). \tag{A.17}$$

Writing more elegantly in the matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M(e_0, \boldsymbol{e}) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{A.18}$$

we obtain the transformation matrix $M(e_0, \boldsymbol{e})$,

$$M(e_0, \boldsymbol{e}) = \begin{bmatrix} 2(e_0^2 + e_1^2) - 1 & 2(e_1e_2 + e_3e_0) & 2(e_3e_1 - e_2e_0) \\ 2(e_1e_2 - e_3e_0) & 2(e_0^2 + e_2^2) - 1 & 2(e_3e_2 + e_1e_0) \\ 2(e_3e_1 + e_2e_0) & 2(e_2e_3 - e_1e_0) & 2(e_0^2 + e_3^2) - 1 \end{bmatrix}, \tag{A.19}$$

with the Euler parameters expressed in terms of the rotation parameters

$$e_0 = \cos\tfrac{\psi}{2}, \qquad e_1 = \sin\theta\cos\phi\sin\tfrac{\psi}{2},$$
$$e_2 = \sin\theta\sin\phi\sin\tfrac{\psi}{2}, \qquad e_3 = \cos\theta\sin\tfrac{\psi}{2}. \tag{A.20}$$

A.2. The Euler Angles of Rotation

Rotation in a two dimensional complex space is equivalent to a rotation in the three-dimensional real space.

Choose a matrix operator P

$$P = \boldsymbol{\sigma} \cdot \boldsymbol{r} = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}, \tag{A.21}$$

and perform unitary transformation Q on it.

$$P' = Q\, P\, Q^{\dagger}. \tag{A.22}$$

Since Q is a unitary unimodular operator, one can obtain certain conditions between the elements of this unitary matrix in the two-dimensional complex space.

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad Q^{\dagger} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}, \tag{A.23}$$

$$Q\, Q^{\dagger} = Q^{\dagger}\, Q = 1, \quad \det Q = 1. \tag{A.24}$$

The conditions (A.24) yields the relations

$$c = -b^*, \quad d = a^*. \tag{A.25}$$

and the matrix

$$Q = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \tag{A.26}$$

has only four parameters, of which only three are independent because of the unimodular condition ($\det Q = 1$).

The transformed operator P' is given by

$$\begin{aligned} P' &= \begin{bmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}. \end{aligned} \tag{A.27}$$

From Eq. (A.27), one can obtain the following relations:

$$x' + iy' = -a^* b^* z + a^{*^2}(x + iy) - b^{*^2}(x - iy) - a^* b^* z, \tag{A.28}$$

$$x' - iy' = -abz + a^2(x - iy) - b^2(x + iy) - abz, \tag{A.29}$$

$$z' = (ab^* + a^* b)x + iy(ba^* - ab^*) + (aa^* - bb^*)z, \tag{A.30}$$

from which one can deduce the transformation matrix $M(a,b)$ in the three dimensional real space.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M(a,b) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{A.31}$$

where,

$$M(a,b) = \begin{bmatrix} \frac{1}{2}(a^2 - b^2 + a^{*^2} - b^{*^2}) & \frac{i}{2}(a^{*^2} + b^{*^2} - a^2 - b^2) & -(ab + a^*b^*) \\ \frac{i}{2}(a^2 - b^2 - a^{*^2} + b^{*^2}) & \frac{1}{2}(a^2 + b^2 + a^{*^2} + b^{*^2}) & i(a^*b^* - ab) \\ ab^* + a^*b & i(a^*b - ab^*) & aa^* - bb^* \end{bmatrix}. \tag{A.32}$$

The parameters a and b occurring in Eq. (A.32) are complex quantities and let us define them in terms of the real parameters e_0, e_1, e_2 and e_3.

$$a = e_0 + ie_3; \quad b = e_2 + ie_1. \tag{A.33}$$

The transformation matrix $M(a,b)$ can be rewritten in terms of e_0, e_1, e_2 and e_3 using the definition (A.33) of a and b. The transformation matrix M(e0,e1,e2,e3) so obtained is identical with the transformation matrix given in Eq. (A.19) describing the rotation about an arbitrary axis $\hat{\boldsymbol{n}}$ and the quantities e_0, e_1, e_2, e_3 defined in Eq. (A.33) are identical with the Euler parameters introduced earlier in Eq. (A.9) or Eq. (A.20).

For Euler angles of rotation, the unitary matrix Q can be written as a product of three unitary matrices

$$Q = Q_\gamma Q_\beta Q_\alpha. \tag{A.34}$$

The unitary matrices Q_α, Q_β, and Q_γ can be deduced from the known properties of coordinate transformation under rotation. It is found on inspection that the matrices Q_α, Q_β and Q_γ can be written in a compact form using the Pauli matrices. The matrices so obtained are

$$Q_\alpha = \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix} = \cos\frac{\alpha}{2} + i\sigma_3 \sin\frac{\alpha}{2}. \tag{A.35}$$

$$Q_\beta = \begin{bmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} = \cos\frac{\beta}{2} + i\sigma_2 \sin\frac{\beta}{2}. \tag{A.36}$$

$$Q_\gamma = \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix} = \cos\frac{\gamma}{2} + i\sigma_3 \sin\frac{\gamma}{2}. \tag{A.37}$$

The product of these matrices is denoted by Q.

$$Q = Q_\gamma Q_\beta Q_\alpha = \begin{bmatrix} e^{+i(\gamma+\alpha)/2} \cos\frac{\beta}{2} & e^{+i(\gamma-\alpha)/2} \sin\frac{\beta}{2} \\ -e^{-i(\gamma-\alpha)/2} \sin\frac{\beta}{2} & e^{-i(\gamma+\alpha)/2} \cos\frac{\beta}{2} \end{bmatrix}. \tag{A.38}$$

Comparing the unitary matrix (A.38) with (A.26) and expressing the complex elements in terms of the real parameters e_0, e_1, e_2, e_3 defined in Eq. (A.33), we obtain

$$\begin{aligned} e_0 &= \cos\tfrac{\gamma+\alpha}{2} \cos\tfrac{\beta}{2}, \quad & e_1 &= \sin\tfrac{\gamma-\alpha}{2} \sin\tfrac{\beta}{2}, \\ e_2 &= \cos\tfrac{\gamma-\alpha}{2} \sin\tfrac{\beta}{2}, \quad & e_3 &= \sin\tfrac{\gamma+\alpha}{2} \cos\tfrac{\beta}{2}. \end{aligned} \tag{A.39}$$

From Eqs. (A.39) and (A.20), the required relations between the two sets of rotation parameters, $\theta,\phi,\ \psi$ and α,β,γ are deduced.

$$\cos\frac{\psi}{2} = \cos\frac{\gamma+\alpha}{2} \cos\frac{\beta}{2}, \tag{A.40}$$

$$\sin\theta \cos\phi \sin\frac{\psi}{2} = \sin\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} \tag{A.41}$$

$$\sin\theta \sin\phi \sin\frac{\psi}{2} = \cos\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2}. \tag{A.42}$$

$$\cos\theta \sin\frac{\psi}{2} = \sin\frac{\gamma+\alpha}{2} \cos\frac{\beta}{2}. \tag{A.43}$$

Squaring Eqs. (A.41) and (A.42) and adding, we obtain

$$\sin^2\theta \sin^2\frac{\psi}{2} = \sin^2\frac{\beta}{2}, \tag{A.44}$$

the square root of which yields the relation

$$\sin\theta \sin\frac{\psi}{2} = \sin\frac{\beta}{2}. \tag{A.45}$$

Substituting this in Eq. (A.42), we get

$$\sin\phi = \cos\frac{\gamma-\alpha}{2} = \sin\left(\frac{\pi}{2} + \frac{\gamma-\alpha}{2}\right). \tag{A.46}$$

Thus

$$\phi = \frac{\pi}{2} + \frac{\gamma-\alpha}{2}. \tag{A.47}$$

Equations (A.40), (A.45) and (A.47) relate the two sets of rotation parameters.

A.3. Direct method of obtaining the transformation matrix

The vector $\boldsymbol{r}$ in the original coordinate system becomes vector $\boldsymbol{r}'$ in the rotated coordinate system obtained by successive Euler angles of rotation α,β,γ.

$$x\,y\,z \;\xrightarrow[z-\text{axis}]{\alpha}\; x_1\,y_1\,z_1 \;\xrightarrow[y_1-\text{axis}]{\beta}\; x_2\,y_2\,z_2 \;\xrightarrow[z_2-\text{axis}]{\gamma}\; x'\,y'\,z'$$

The transformation matrix for these rotations are discussed in Chap.4.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = M(\alpha,\beta,\gamma) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{A.48}$$

where

$$\begin{aligned} M(\alpha,\beta,\gamma) &= M_{z_2}(\gamma)\,M_{y_1}(\beta)\,M_z(\alpha) \\ &= \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \\ &\quad \times \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{A.49}$$

By direct matrix multiplication, the matrix elements of $M(\alpha,\beta,\gamma)$ are obtained as given below:

$$M_{11} = \cos\gamma\cos\alpha\cos\beta - \sin\gamma\sin\alpha \tag{A.50}$$
$$M_{12} = \cos\gamma\sin\alpha\cos\beta + \sin\gamma\cos\alpha \tag{A.51}$$
$$M_{13} = -\cos\gamma\sin\beta \tag{A.52}$$
$$M_{21} = -\sin\gamma\cos\alpha\cos\beta - \cos\gamma\sin\alpha \tag{A.53}$$
$$M_{22} = -\sin\gamma\sin\alpha\cos\beta + \cos\gamma\cos\alpha \tag{A.54}$$
$$M_{23} = \sin\gamma\sin\beta \tag{A.55}$$
$$M_{31} = \cos\alpha\sin\beta \tag{A.56}$$
$$M_{32} = \sin\alpha\sin\beta \tag{A.57}$$
$$M_{33} = \cos\beta \tag{A.58}$$

Using the trignometric relations,

$$\cos A\cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}, \tag{A.59}$$

$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}, \tag{A.60}$$

$$\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}, \tag{A.61}$$

$$\cos A \sin B = \frac{1}{2}\{\sin(A+B) - \sin(A-B)\}, \tag{A.62}$$

$$\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}, \tag{A.63}$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}, \tag{A.64}$$

and defining the parameters as given in Eq. (A.39), it can be shown that the transformation matrix $M(\alpha,\beta,\gamma)$ is identical with the transformation matrix $M(e_0, \boldsymbol{e})$ defined in Eq. (A.19). For the purpose of illustration, let us choose the matrix element $M_{11}(\alpha,\beta,\gamma)$.

$$\begin{aligned} M_{11} &= \cos\gamma\cos\alpha\cos\beta - \sin\gamma\sin\alpha \\ &= \frac{1}{2}\{\cos(\gamma+\alpha) + \cos(\gamma-\alpha)\}\cos\beta \\ &\quad -\frac{1}{2}\{\cos(\gamma-\alpha) - \cos(\gamma+\alpha)\} \\ &= \frac{1}{2}\cos(\gamma+\alpha)\{\cos\beta + 1\} + \frac{1}{2}\cos(\gamma-\alpha)\{\cos\beta - 1\} \\ &= \cos(\gamma+\alpha)\cos^2\frac{\beta}{2} - \cos(\gamma-\alpha)\sin^2\frac{\beta}{2} \\ &= \left\{2\cos^2\frac{\gamma+\alpha}{2} - 1\right\}\cos^2\frac{\beta}{2} - \left\{1 - 2\sin^2\frac{\gamma-\alpha}{2}\right\}\sin^2\frac{\beta}{2} \\ &= 2\cos^2\frac{\gamma+\alpha}{2}\cos^2\frac{\beta}{2} + 2\sin^2\frac{\gamma-\alpha}{2}\sin^2\frac{\beta}{2} - 1 \\ &= 2(e_0^2 + e_1^2) - 1. \end{aligned} \tag{A.65}$$

In a similar way, all the other matrix elements can be expressed in terms of the parameters e_0, e_1, e_2, e_3 and the resulting transformation matrix $M(\alpha,\beta,\gamma)$ is identical with the matrix (A.19).

APPENDIX B

TABLES OF CLEBSCH-GORDAN COEFFICIENTS

TABLE B1. $\begin{bmatrix} j_1 & \frac{1}{2} & j \\ m_1 & m_2 & m \end{bmatrix}$

j	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\left[\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$	$\left[\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$
$j_1 - \frac{1}{2}$	$-\left[\frac{j_1-m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$	$\left[\frac{j_1+m+\frac{1}{2}}{2j_1+1}\right]^{\frac{1}{2}}$

TABLE B2. $\begin{bmatrix} j_1 & 1 & j \\ m_1 & m_2 & m \end{bmatrix}$

j	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\left[\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right]^{\frac{1}{2}}$
j_1	$-\left[\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}\right]^{\frac{1}{2}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\left[\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}\right]^{\frac{1}{2}}$
$j_1 - 1$	$\left[\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1+m)(j_1+m+1)}{2j_1(2j_1+1)}\right]^{\frac{1}{2}}$

APPENDIX C

TABLES OF RACAH COEFFICIENTS

TABLE C1. $W(a\,b\,c\,d;\tfrac{1}{2}\,f)$

	$c = d + \frac{1}{2}$
$a = b + \frac{1}{2}$	$(-1)^{b+d-f}\left[\frac{(b+d+f+2)(b+d-f+1)}{(2b+1)(2b+2)(2d+1)(2d+2)}\right]^{\frac{1}{2}}$
$a = b - \frac{1}{2}$	$(-1)^{b+d-f}\left[\frac{(f-b+d+1)(f+b-d)}{2b(2b+1)(2d+1)(2d+2)}\right]^{\frac{1}{2}}$
	$c = d - \frac{1}{2}$
$a = b + \frac{1}{2}$	$(-1)^{b+d-f}\left[\frac{(f+b-d+1)(f-b+d)}{(2b+1)(2b+2)2d(2d+1)}\right]^{\frac{1}{2}}$
$a = b - \frac{1}{2}$	$(-1)^{b+d-f-1}\left[\frac{(b+d+f+1)(b+d-f)}{2b(2b+1)2d(2d+1)}\right]^{\frac{1}{2}}$

TABLE C2. $W(a\,b\,c\,d; 1\,f)$

	$c = d + 1$
$a = b + 1$	$(-1)^{b+d-f}\left[\frac{(b+d+f+3)(b+d+f+2)(b+d-f+2)(b+d-f+1)}{4(2b+1)(2b+3)(b+1)(2d+1)(2d+3)(d+1)}\right]^{\frac{1}{2}}$
$a = b$	$(-1)^{b+d-f}\left[\frac{(b+d+f+2)(b+d-f+1)(-b+d+f+1)(b-d+f)}{4b(2b+1)(b+1)(2d+1)(d+1)(2d+3)}\right]^{\frac{1}{2}}$
$a = b - 1$	$(-1)^{b+d-f}\left[\frac{(b-d+f)(b-d+f-1)(-b+d+f+2)(-b+d+f+1)}{4(2b+1)(2b-1)b(d+1)(2d+1)(2d+3)}\right]^{\frac{1}{2}}$

	$c = d$
$a = b + 1$	$(-1)^{b+d-f}\left[\frac{(b+d+f+2)(b-d+f+1)(b+d-f+1)(-b+d+f)}{4(2b+1)(2b+3)(b+1)(2d+1)d(d+1)}\right]^{\frac{1}{2}}$
$a = b$	$(-1)^{b+d-f}\left[\frac{b(b+1)+d(d+1)-f(f+1)}{4b(2b+1)(b+1)(2d+1)d(d+1)}\right]^{\frac{1}{2}}$
$a = b - 1$	$(-1)^{b+d-f-1}\left[\frac{(b+d+f+1)(b+d-f)(b-d+f)(-b+d+f+1)}{4(2b+1)(2b-1)b(d+1)d(2d+1)}\right]^{\frac{1}{2}}$

	$c = d - 1$
$a = b + 1$	$(-1)^{b+d-f}\left[\frac{(-b+d+f)(-b+d+f-1)(b-d+f+2)(b-d+f+1)}{4d(2b+1)(2b+3)(b+1)(2d-1)(2d+1)}\right]^{\frac{1}{2}}$
$a = b$	$(-1)^{b+d-f-1}\left[\frac{(b+d+f+1)(b-d+f+1)(-b+d+f)(b+d-f)}{4bd(2b+1)(b+1)(2d+1)(2d-1)}\right]^{\frac{1}{2}}$
$a = b - 1$	$(-1)^{b+d-f}\left[\frac{(b+d+f+1)(b+d+f)(b+d-f)(b+d-f-1)}{4bd(2b+1)(2b-1)(2d+1)(2d-1)}\right]^{\frac{1}{2}}$

THE SPHERICAL HARMONICS

The spherical harmonics are the solutions of the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+l(l+1)\right]Y_l^m(\theta,\phi)=0, \quad \text{(D.1)}$$

and they can be expressed in terms of the associated Legendre functions $P_l^m(x)$.

$$\begin{aligned} Y_l^m(\theta,\phi) &= (-1)^{\frac{m+|m|}{2}}\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{\frac{1}{2}} P_l^{|m|}(x)\,e^{im\phi}, \\ &\quad (l=0,1,2,\cdots;m=-l,-l+1,\cdots,l-1,l.) \end{aligned} \quad \text{(D.2)}$$

where $x = \cos\theta$ and the associated Legendre functions $P_l^m(x)$ are the derivatives of the Legendre function $P_l(x)$

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \quad m \geq 0. \quad \text{(D.3)}$$

Note that $P_l^0(x) = P_l(x)$ and the Legendre functions are defined by

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l. \quad \text{(D.4)}$$

Substituting the expression for the associated Legendre functions, a general expression for the spherical harmonics is obtained.

$$\begin{aligned} Y_l^m(\theta,\phi) &= \frac{1}{2^l l!}(-1)^{(m+|m|)/2}\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{\frac{1}{2}} e^{im\phi} \\ &\quad \times (1-x^2)^{|m|/2}\frac{d^{l+|m|}}{dx^{l+|m|}}(x^2-1)^l. \end{aligned} \quad \text{(D.5)}$$

The spherical harmonics satisfy the relation

$$Y_l^{m*} = (-1)^m Y_l^{-m}, \quad \text{(D.6)}$$

and they are normalized such that

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_{l'}^{m'*}(\theta,\phi)\, Y_l^m(\theta,\phi) = \delta_{l,l'}\,\delta_{m,m'}. \tag{D.7}$$

The completeness relation for the spherical harmonics is given by

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^{m*}(\theta,\phi) Y_l^m(\theta',\phi') = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta'). \tag{D.8}$$

Let us list some of the relations involving sums over m but with fixed l for spherical harmonics.

$$\sum_{m=-l}^{l} |Y_l^m(\theta,\phi)|^2 = \frac{2l+1}{4\pi}, \tag{D.9}$$

$$\sum_{m=-l}^{l} m|Y_l^m(\theta,\phi)|^2 = 0, \tag{D.10}$$

$$\sum_{m=-l}^{l} m^2|Y_l^m(\theta,\phi)|^2 = \frac{l(l+1)(2l+1)}{8\pi}\sin^2\theta. \tag{D.11}$$

The spherical harmonics have an inversion symmetry property of great importance. The direction opposite to (θ,ϕ) is $(\pi - \theta,\phi + \pi)$. From an examination of Eq. (D.5), we obtain a relation

$$Y_l^m(\pi-\theta,\phi+\theta) = (-1)^l\, Y_l^m(\theta,\phi), \tag{D.12}$$

which means that the spherical harmonics have positive parity for even l and negative parity for odd l.

It can be easily verified that

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^0(\theta,\phi). \tag{D.13}$$

The first four Legendre polynomials $P_l(x)$ are given in Table D1 and the explicit forms of the spherical harmonics for l = 0, 1, 2, 3, 4 are presented in Table D2.

TABLE D1. Legendre Polynomials $P_l(x)$

l	$P_l(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$

TABLE D2. Normalized spherical harmonics $Y_l^m(\theta, \phi)$

l	m	$Y_l^m(\theta, \phi)$
0	0	$\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}$
1	0	$\left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\theta$
	± 1	$\mp \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\, e^{\pm i\phi}$
2	0	$\left(\frac{5}{16\pi}\right)^{\frac{1}{2}} (3\cos^2\theta - 1)$
	± 1	$\mp \left(\frac{15}{8\pi}\right)^{\frac{1}{2}} \cos\theta \sin\theta\, e^{\pm i\phi}$
	± 2	$\left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \sin^2\theta\, e^{\pm 2i\phi}$
3	0	$\left(\frac{7}{16\pi}\right)^{\frac{1}{2}} \cos\theta\,(5\cos^2\theta - 3)$
	± 1	$\mp \left(\frac{21}{64\pi}\right)^{\frac{1}{2}} \sin\theta\,(5\cos^2\theta - 1)\, e^{\pm i\phi}$
	± 2	$\left(\frac{105}{32\pi}\right)^{\frac{1}{2}} \cos\theta \sin^2\theta\, e^{\pm 2i\phi}$
	± 3	$\mp \left(\frac{35}{64\pi}\right)^{\frac{1}{2}} \sin^3\theta\, e^{\pm 3i\phi}$
4	0	$\frac{3}{16\sqrt{\pi}}(35\cos^4\theta - 30\cos^2\theta + 3)$
	± 1	$\mp\frac{3}{8}\left(\frac{5}{\pi}\right)^{\frac{1}{2}} \sin\theta\,(7\cos^3\theta - 3\cos\theta)\, e^{\pm i\phi}$
	± 2	$\frac{3}{8}\left(\frac{5}{2\pi}\right)^{\frac{1}{2}} \sin^2\theta\,(7\cos^2\theta - 1)\, e^{\pm 2i\phi}$
	± 3	$\mp\frac{3}{8}\left(\frac{35}{\pi}\right)^{\frac{1}{2}} \sin^3\theta \cos\theta\, e^{\pm 3i\phi}$
	± 4	$\frac{3}{16}\left(\frac{35}{2\pi}\right)^{\frac{1}{2}} \sin^4\theta\, e^{\pm 4i\phi}$

THE SPHERICAL BESSEL AND NEUMANN FUNCTIONS

The spherical Bessel function $j_l(x)$ and the spherical Neumann function $n_l(x)$ are defined in terms of the ordinary Bessel functions of odd-half-integer order.

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(x), \tag{E.1}$$

$$n_l(x) = (-1)^{l+1} \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{-l-\frac{1}{2}}(x), \tag{E.2}$$

where l is an integer. The spherical Bessel and Neumann functions are the solutions of the differential equation

$$x^2 \frac{d^2R}{dx^2} + 2x\frac{dR}{dx} + \left[x^2 - l(l+1)\right] R = 0. \tag{E.3}$$

Explicit expressions for the fist few spherical Bessel and Neumann functions are given below:

$$\begin{aligned} j_0(x) &= \tfrac{\sin x}{x}, & n_0(x) &= -\tfrac{\cos x}{x}, \\ j_1(x) &= \tfrac{\sin x}{x^2} - \tfrac{\cos x}{x}, & n_1(x) &= -\tfrac{\cos x}{x^2} - \tfrac{\sin x}{x}, \\ j_2(x) &= \left(\tfrac{3}{x^3} - \tfrac{1}{x}\right)\sin x - \tfrac{3}{x^2}\cos x, & n_2(x) &= -\left(\tfrac{3}{x^3} - \tfrac{1}{x}\right)\cos x - \tfrac{3}{x^2}\sin x. \end{aligned} \tag{E.4}$$

The spherical Bessel and Neumann functions take simple forms in the limiting cases. As $x \to 0$,

$$j_l(x) \longrightarrow \frac{x^l}{(2l+1)!!}; \quad n_l(x) \longrightarrow -\frac{(2l-1)!!}{x^{l+1}}; \tag{E.5}$$

with

$$(2l+1)!! \equiv 1 \times 3 \times 5 \times \cdots \times (2l+1). \tag{E.6}$$

In the asymptotic limit i.e., as $x \to \infty$,

$$j_l(x) \longrightarrow \frac{1}{x}\sin(x - \tfrac{1}{2}l\pi); \quad n_l(x) \longrightarrow -\frac{1}{x}\cos(x - \tfrac{1}{2}l\pi). \tag{E.7}$$

We give below some useful recurrence relations for $j_l(x)$:

$$j_{l-1}(x) + j_{l+1}(x) = \frac{2l+1}{x} j_l(x), \tag{E.8}$$

$$\frac{d}{dx} j_l(x) = \frac{1}{2l+1}[l\, j_{l-1}(x) - (l+1)\, j_{l+1}(x)] \tag{E.9}$$

$$= j_{l-1}(x) - \frac{l+1}{x} j_l(x). \tag{E.10}$$

$$\frac{d}{dx}[x^{l+1}\, j_l(x)] = x^{l+1}\, j_{l-1}(x). \tag{E.11}$$

$$\frac{d}{dx}[x^{-l}\, j_l(x)] = -x^{-l}\, j_{l+1}(x). \tag{E.12}$$

The same recurrence relations are obeyed by the spherical Neumann functions $n_l(x)$ also. The reader may find the following integarl formulas involving spherical Bessel functions useful.

$$\int j_1(x)\, dx = -j_0(x). \tag{E.13}$$

$$\int j_0(x)\, x^2\, dx = x^2\, j_1(x). \tag{E.14}$$

$$\int j_l^2(x)\, x^2 dx = \tfrac{1}{2} x^3 [j_l^2(x) - j_{l-1}(x)\, j_{l+1}(x)]. \tag{E.15}$$

$$\int_0^\infty j_l^2(x)\, dx = \frac{\pi}{2} \frac{1}{2l+1}. \tag{E.16}$$

THE BERNOULLI POLYNOMIALS

The Bernoulli polynomial $B_s(x)$ is defined by

$$\frac{t\, e^{xt}}{e^t - 1} = \sum_{s=0}^{\infty} B_s(x) \frac{t^s}{s!}, \qquad (|t| < 2\pi). \tag{F.1}$$

$B_s(0)$ are called Bernoulli numbers B_s.

$$\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} B_s \frac{t^s}{s!}. \tag{F.2}$$

With the exception of B_1, all odd Bernoulli numbers vanish. From Eqs. (A.1) and (A.2), we get

$$e^{xt} \sum_{s=0}^{\infty} B_s \frac{t^s}{s!} = \sum_{s=0}^{\infty} B_s(x) \frac{t^s}{s!}. \tag{F.3}$$

Equating coefficients of equal powers oft, we get

$$B_s(x) = \sum_{j=0}^{s} \binom{s}{j} B_{s-j}\, x^j. \tag{F.4}$$

The first few Bernoulli numbers and Bernoulli polynomials (Miller, 1960; Subramanian, 1974) are listed below:

Bernoulli numbers

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \\ B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}. \tag{F.5}$$

Bernoulli polynomials

$$
\begin{aligned}
B_0(x) &= 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x = x(x - \tfrac{1}{2})(x - 1), \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x = x(x - \tfrac{1}{2})(x - 1)(x^2 - x - \tfrac{1}{3}), \\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}. \qquad \text{(F.6)}
\end{aligned}
$$

LIST OF SYMBOLS AND NOTATION

Angular Momentum Coupling Coefficients

$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$	Clebsch-Gordan coefficient or C.G. coefficient
$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$	3-j symbol
$U(j_1 j_2 j j_3, j_{12} j_{23})$	U-coefficient (Unitary transformation coefficient for the coupling of three angular momenta)
$W(j_1 j_2 j j_3, j_{12} j_{23})$	Racah coefficient
$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$	6-j symbol
$\begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix}$	LS-jj coupling coefficient
$\begin{Bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{Bmatrix}$	9-j symbol

Angular Momentum Eigenstates and Operators

$\|jm\rangle$	Angular momentum eigenstate
$\|j_1 j_2 m_1 m_2\rangle$	Eigenstate of two angular momenta in uncoupled representation
$\|j_1 j_2 j m\rangle$	Eigenstate of two angular momenta in coupled representation

$\|(j_1j_2)j_{12}j_3;jm\rangle$	Eigenstate of three angular momenta in one coupled representation
$\|j_1(j_2j_3)j_{23};jm\rangle$	Eigenstate of three angular momenta in another coupled representation
$\|l_1l_2s_1s_2,LSJM\rangle$	Eigenstate of four angular momenta in L-S coupled representation
$\|l_1l_2s_1s_2,j_1j_2JM\rangle$	Eigenstate of four angular momenta in j-j coupled represent at ion
χ_+	Spin-up state for the Dirac particle
χ_-	Spin-down state for the Dirac particle
$\|p\theta\phi,\lambda\rangle$	Helicity state of a particle moving with momentum $\boldsymbol{p}$ (= p, θ,ϕ) and helicity λ
$\|p\theta\phi,\lambda_1\lambda_2\rangle$	Two-particle helicity state moving with relative momentum $\boldsymbol{p}$ and helicities λ_1 and λ_2 as described in centre of momentum frame
$\boldsymbol{J}^2$	Square of the angular momentum operator
J_x, J_y, J_z	Components of angular momentum operator in Cartesian basis
J_+, J_-	Raising and lowering angular momentum operators (Ladder operators)
J_1^1, J_1^0, J_1^{-1}	Components of angular momentum operator in spherical basis
$\sigma_x, \sigma_y, \sigma_z$	Pauli spin operators
τ_x, τ_y, τ_z	Iso-spin operators for nucleons

Rotation Operator and Rotation Matrices

$\omega = \alpha, \beta, \gamma$	Euler angles of rotation
$R(\alpha,\beta,\gamma)$, U_R	Rotation operators
$D^j_{m'm}(\omega)$	Rotation matrix $\langle jm'\|R(\omega)\|jm\rangle$
$d^j_{m'm}(\beta)$	Rotation matrix for rotation about the y-axis $\langle jm'\|R_y(\beta)\|jm\rangle$
$M(\alpha,\beta,\gamma)$	Transformation matrix

Special Functions

B_s	Bernoulli number
$Bs(x)$	Bernoulli polynomial
$J_v(x)$	Bessel function of order v
$N_v(x)$	Neumann function of order v
$j_l(x)$	Spherical Bessel function $j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x)$
$n_l(x)$	Spherical Neumann function $n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$
$P_l(x)$	Legendre function
$P_l^m(x)$	Associated Legendre function

Vectors, Tensors and Tensor Operators

$e_x,\ e_y,\ e_z$	Unit vectors in Cartesian basis
$\epsilon_1^{+1}, \epsilon_1^0, \epsilon_1^{-1}$	Unit vectors in spherical basis
$\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi$	Polar basis vectors
$\hat{\boldsymbol{n}}(\theta, \phi)$	Unit vector with polar angles θ,ϕ
$A_x,\ A_y,\ A_z$	Cartesian components of a vector ***A***
A_1^1, A_1^0, A_1^{-1}	Spherical components of a vector ***A.***
$T_k^\mu(\hat{\boldsymbol{r}})$	Spherical tensor operator of rank *k* and projection μ
$Y_l^m(\hat{\boldsymbol{r}})$	Spherical harmonic
$\boldsymbol{Y}_{Ll}^M(\hat{\boldsymbol{r}})$	Vector spherical harmonic $\boldsymbol{Y}_{Ll}^M(\hat{\boldsymbol{r}}) = \sum_m \begin{bmatrix} l & 1 & L \\ m & \mu & M \end{bmatrix} Y_l^m(\hat{\boldsymbol{r}})\epsilon_1^\mu$
$G_k(j)$	Fano's statistical tensor
$t_k^{m_k}, \langle T_k^{m_k} \rangle$	Spherical tensor parameter $t_k^{m_k} = \langle T_k^{m_k} \rangle = \mathrm{Tr}\,(\rho T_k^{m_k})/\mathrm{Tr}\,\rho$
ρ	Density matrix

Miscellaneous Symbols

δ_{ij}	Kronecker δ-symbol
$d\Omega$	Element of solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$
$[j]$	$(2j+1)^{\frac{1}{2}}$
$\binom{m}{n}$	Binomial coefficient $\binom{m}{n} = \frac{m!}{n!(m-n)!}$
$\alpha_x,\ \alpha_y,\ \alpha_z,\ \beta$	Dirac matrices
$\gamma_x,\ \gamma_y,\ \gamma_z,\ \gamma_0$	Gamma matrices
$\not{a}$	$\alpha_0\gamma_0 - \alpha_x\gamma_x - \alpha_y\gamma_y - \alpha_z\gamma_z$
$\langle j_f m_f \vert T_k^\mu \vert j_i m_i \rangle$	Matrix element of a tensor operator T_k^μ
$\langle j_f \vert\vert T_k \vert\vert j_i \rangle$	Reduced matrix element of a tensor operator of rank k
$\langle j^2(j_{12})j \vert\} j^3 J\rangle$	Coefficient of fractional parentage (c.f.p.)

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