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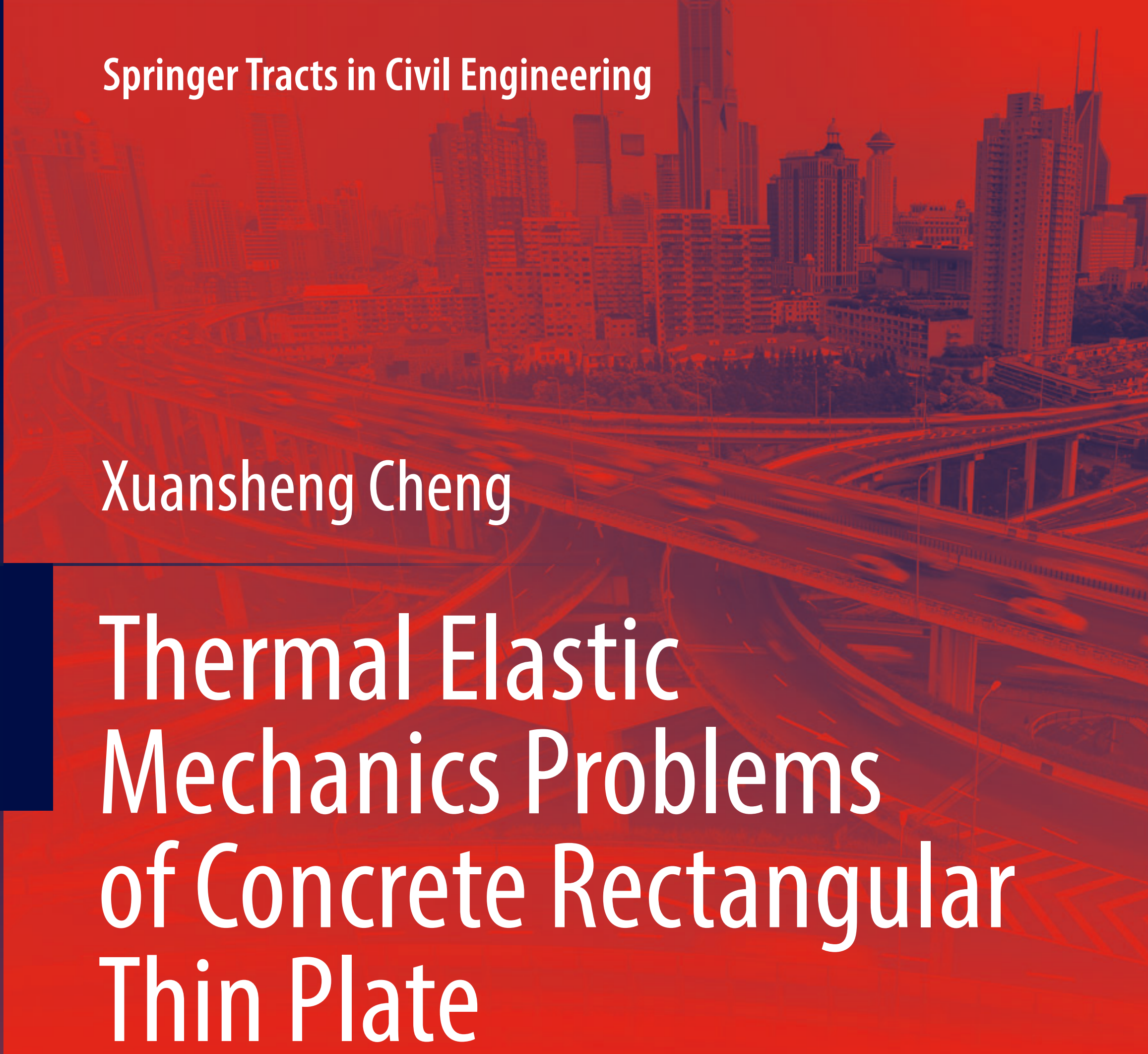

Xuansheng Cheng

Thermal Elastic Mechanics Problems of Concrete Rectangular Thin Plate

Science Press
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Preface

The concrete rectangular thin plate is widely used in civil engineering field, such as rectangular liquid-storage structure, shear wall, concrete roof plate, airport runway and concrete rigid pavement. According to the boundary of the rectangular thin plate, the rectangular thin plate can be generally divided into two types, with four edges supported and with free edges. First, due to the thermal inertia of concrete material itself, concrete thin plate under the effect of non-uniform temperature, a larger temperature difference will be formed in the internal structure so that temperature stress cannot be ignored. Due to the low tensile strength of concrete, if effective measures are not taken to eliminate or resist the temperature stress, the cracks in concrete structures will be caused after the structures are shortly used, and normal use of the structure will be affected. Seriously, structural safety accidents will happen. Second, the instability of concrete thin plate structure can be caused by in-plane compression load, also can be caused by thermal load, if the internal temperature of concrete thin plate structure is too high, instability and failure of concrete structures will be led to. Finally, the vibration problem of concrete structure under the action of mechanical load is drawing more attention nowadays, and few people pay attention to the vibration because of thermal load. In fact, if the existence of the thermal environment vibration is ignored, the calculation error of structure natural frequency and deformation can be caused, so that frequency and the deformation in structure design can overestimated or underestimated.

Since 1998, I have taken lots of design tasks of the rectangular thin plate structure, and found that the cracks of the concrete plate will appear in different degrees under the action of temperature. Therefore, this book describes the thermal bending, thermal buckling, and thermal vibration of thin plates, which have important engineering significance. This book introduces the thermal bending of rectangular thin plate with four edges supported and with free boundary rectangular thin plate, the thermal buckling of concrete rectangular thin plate and thermal vibration with four edges supported, which is the sublimation and summary of my research results about thin plate structure for many years, the publication of the book is bound to have important theoretical significance and engineering practice effect in the civil engineering and mechanical engineering, etc.

The whole book is divided into five chapters. Chapter 1 is the introduction, which mainly introduces the basic situation and the necessity of the research on the thermodynamics of the rectangular thin plate; Chap. 2 is the thermal bending of the rectangular thin plate with four edges supported. According to the common rectangular thin plate with four edges supported in engineering, the concrete rectangular thin plate is divided into six types, which is four edges simply supported, four edges clamped, three edges clamped and one edge simply supported, one edge clamped and three edges simply supported, two adjacent edges clamped and two adjacent edges simply supported, two opposite edges clamped and two opposite edges simply supported, and the thermal bending of the rectangular thin plate is introduced in detail; Chap. 3 describes the thermal bending of rectangular thin plate with free boundary. The thermal bending problem about six types of concrete rectangular thin plate are considered, namely, three edges simply supported and one edge free, three edges clamped and one edge free, two opposite edges clamped one edge simply supported and one edge free, two adjacent edges clamped one edge simply supported and one edge free, two opposite edges simply supported one edge clamped and one edge free, two adjacent edges simply supported one edge clamped and one edge free; in Chap. 4, the thermal buckling of concrete rectangular thin plate is introduced. The thermal buckling of concrete rectangular thin plate with four sides simply supported is discussed. Chapter 5 is about the thermal vibration of concrete rectangular thin plate structure, and the free and forced vibration of the rectangular thin plate with four sides simply supported is introduced.

For people engaging in scientific research, engineering design and construction technology, this book can provide important mechanics concepts, theoretical calculation method and calculation table when analyzing the crack, deformation, stability, comfort design of the concrete rectangular thin plate structure. For the relevant professional researchers (including undergraduates and graduates) in universities and in research institutes, this book can be used as a reference material for concrete structures, thin plate elastic mechanics.

Before the book will imminently be published, the author would like to extend sincere thanks to people who support and care related research projects and the organization workers of publishing. Specially, thanks to National Natural Science Foundation Committee! Thanks to selfless care of Prof. Yong feng Du in Lanzhou University of Technology! Thanks to Dr. Jing Wei and Xinhai Zhou, Masters Xiao yan Zhang, Jia Chen, De Li, Bo Liu and Liang Ma! They have made an indispensable contribution for publication and compilation of this book.

Although the book has made perfect scientific research achievements in the thermodynamic theory problems, for the thermal bending, thermal vibration and thermal buckling with free boundary of the concrete thin rectangular plates, the results need further validation, and the experimental study on thermal vibration and thermal buckling problems still need to be further designed. In order to make the research results widely applicable in design and construction of thin plate structures in civil engineering, the support from professional and technical personnel of civil engineering, engineering mechanics and mechanical engineering is vigorously needed. The book will inevitably have some defects in the theoretical analysis, or could even have some mistakes; criticism and comments from the researchers and readers will be appreciated and please send your suggestions to my e-mail chengxuansheng@gmail.com. The author will very appreciate your help.

Lanzhou, China Xuansheng Cheng
November 2016

Contents

Chapter 1
Introduction

Abstract According to rectangular thin plates with or without the free boundary, the rectangular thin plate can be divided into the rectangular thin plates supported on four sides and the rectangular thin plates with free boundary. The research progress of thermal bending, thermal buckling and thermal vibration of rectangular thin plate is introduced in this chapter.

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1.1 The Basic Status of Rectangular Thin Plate Thermal Problems

The study of temperature effect problem began in 1835 [1]. When giving a speech in French Academy of Science, the French Du Hammel pointed out that for the first time: when the temperature changes, one part of the object will be subject to the constraints of another part, and the object inside can produce thermal stress. The stress is superposed by the two parts, one part of stress is pressure proportional to the temperature change and equal in all directions, and the other part of stress is produced by strain while the temperature is constant. The derived linear thermal stress theory was firstly advocated by the German Neumann in his book in 1841. After the Second World War, the rapid development of thermal power, nuclear power, machinery manufacturing, chemical industry, aircraft, spacecraft, rocket technology and other modern science and technology greatly promoted the research and application of the thermal stress theory [2]. After more than one hundred years of research, thermal elastic mechanics has grown from 1960s to 1970s. Today, there are many literatures about the thermal bending, thermal buckling and thermal vibration [1−3].

1.1.1 Thermal Bending of Rectangular Thin Plate

The temperature internal force and deflection calculation formula of the rectangular thin plate with four edges simply supported and four edges clamped were given by Ugural in their researches [1−9], Jane and Hong analyzed thermal bending of orthogonal anisotropic laminated plate with four edges simply supported by using the Generalized Differential Quadrature (GDQ) method [10]. Shen analyzed non-linear thermal bending response of the functionally graded rectangular plate with four edges simply supported under lateral load [11]. Zenkour obtained the analytical solution of the orthogonal laminated plates with four edges simply supported under the thermal mechanical load [12]. Liu et al. gave the internal force calculation tables of the rectangular thin plate with four edges simply supported, three edges clamped and one edge simply supported or free in their literature [13]. To rectangular thin plates with other boundary conditions, there are no reports in the existing literature.

To calculate the stress and deformation of the rectangular thin plate under the thermal load, temperature field is firstly analyzed. Strictly speaking, the distribution of temperature field is very complicated, and it is a function of three-dimensional coordinates. But for thin plate, as the thickness of the plate is very small in size compared with the other two directions, so for the sake of simplicity, temperature is thought to change along the thickness direction only. People had initially taken the uniform temperature distribution field as the calculation basis, then, began to consider the temperature gradient of concrete structure (temperature difference) for the constant crack damage of concrete structure. At first, it was thought that the temperature distribution was linear. Later, with the progress of the experimental research, people realized that the concrete structure of the temperature internal distribution was nonlinear [14]. Therefore, in engineering calculation, the British Stephenson analyzed the temperature distribution of the concrete structure along the direction of wallboard thickness by using exponential function $T_x = A_0 \mathrm{e}^{-ax}$ based on surface temperature amplitude, where A_0 is the surface temperature fluctuation. New Zealander Priestley also obtained the nonlinear distribution rules by the model test study of Auckland new market viaduct, and his expression is $T_x = T_0 \mathrm{e}^{-ax}$, where T_0 is the temperature difference between the inside and outside surface, and index a is chosen as 10. German scholars Fritz Leonhardt and Kehlbeck et al. also identified the nonlinear temperature field distribution rules in their works. Guo and Shi researched reinforced concrete plate temperature field, and proved that the temperature changes along the plate thickness was a nonlinear change but not much [15]. The author had simplified the three-dimensional heat conduction equation to one-dimensional heat conduction equation, and obtained the nonlinear parabolic temperature field distribution rules combining the boundary condition, and analyzed and discussed the temperature stress of statically indeterminate structures [16−18].

1.1.2 Thermal Buckling of Rectangular Thin Plate

Gossard et al. studied the thermal buckling problems of rectangular thin plate [19]. Klosner and Forray studied the buckling of temperature field under the condition of absolutely uniformly distributed in space [20]. Prabhu and Durvasula researched the thermal buckling problems of rectangular plate with two opposite edges clamped using Galerkin method [21]. Uemura studied the buckling behavior of non-uniform temperature field [22]. Sadovský analyzed the thermal buckling of compressed square plate under non-uniform temperature field with simply supported [23]. Shen and Lin studied the post buckling behavior of thin rectangular plate [24]. Murphy and Ferreira analyzed thermal buckling of aluminum plate with four edges clamped using the energy principle and the test [25]. Wu et al. studied the buckling behavior of functionally gradient rectangular plate, and obtained the calculating formula of the critical buckling temperature [26]. Gong calculated buckling uniform temperature field of thin rectangular plate [27]. Jones analyzed thermal buckling of the uniaxial symmetry orthogonal fiber-reinforced laminates with four edges simply supported under uniform heating [28]. Morimoto et al. analyzed thermal buckling of a functionally graded rectangular plate under partially thermal [29]. Kabir et al. analyzed thermal buckling response of skew symmetric laminated plates with four edges clamped [30].

1.1.3 Thermal Vibration of Rectangular Thin Plate

At present, with the rapid development of science and technology and wide application in engineering, many studies on the thermal vibration behavior of thin plate have been carried out. Chang et al. analyzed the nonlinear free vibration of heated orthotropic anisotropic rectangular thin plate [31, 32]. Ding et al. studied free vibration of cross isotropic rectangular thin plate under thermal environment with simply supported edges [33]. Huang et al. analyzed the vibration characteristics of Functionally Grated Materials plate under thermal environment [34−37]. Hong and Jane studied the shear deformation of vibration under temperature load by using the Generalized Differential Quadrature (GDQ) method [38]. He et al. made theoretical analysis on concrete plate dynamic response under blast loading, and obtained the theoretical calculation formula of large plastic deformation of concrete square plates with edges clamped [39]. Niu analyzed the coupled vibration of elastic thin plate under thermal environment [40].

1.2 The Necessity of Concrete Rectangular Thin Plate

When concrete rectangular thin plates are under the solar radiation, sudden cooling (such as sudden cold at night and cold current lowering temperature) or temperature difference effect caused by other temperature, the temperature of the structure surface rises of falls rapidly. Due to the thermal inertia of the concrete material itself, most of structure internal region is still in the state of original temperature, thus a large temperature gradient (hereinafter referred to as the “temperature difference”) in the thickness direction of the plate is formed [41]. The deformation caused by temperature difference effects is restricted by the redundant internal constraints of concrete thin plate structure [42], so the temperature stress cannot be ignored. Generally speaking, when the engineering structure is statically determinate structure or free body, the temperature difference cannot cause temperature stress. But rectangular thin plate structures are general statically indeterminate structure, therefore, the temperature will inevitably lead to temperature tension on the lower temperature side of the plate. So, under the effect of the temperature or sunshine, cracks will occur soon after use due to the low tensile strength of concrete, and the normal use of the structure will be affected; or even safe incidents of the structure will occur.

Strictly speaking, concrete is an anisotropic and heterogeneous composite material [43]. But in fact, the actual size of the member is more than four times the maximum aggregate particle size, so the book assumes that concrete is isotropic material. Moreover, the proportion of steel in reinforced concrete thin plate is very

Fig. 1.1 Mucilage mixing tank

small, and the heat conduction coefficient of steel is very large, so the reinforced effect can be ignored. Therefore, for simplicity and engineering practical reasons, the reinforced concrete plate is considered in isotropic conditions.

In 1998, the author undertook the design task of 15,000 tons/year mucilage mixing tank (Fig. 1.1) and coagulant configuration tank (Fig. 1.2) in NBR device. The height of the mucilage mixing tank is 7.3 m, the length is 20.7 m, the width is 13.2 m, and the tank liquid temperature is 60 °C; the height of the coagulant configuration tank is 4.5 m, the length is 12.5 m, the width is 9 m, the tank liquid temperature is 45 °C. According to the survey, the plate thickness and reinforcement amount were large in similar project design, and the theory of thermal elasticity showed that: the bigger plate thickness is, the bigger bending stiffness is. So the amount of reinforcement caused by temperature difference is greater. The concrete rectangular liquid storage structure is composed of rectangular plates, so the book discusses thermal bending of concrete rectangular thin plate with four edges clamped and with free boundary, and discusses the thermal vibration and thermal buckling of concrete rectangular plate with simply supported.

Fig. 1.2 Coagulant configuration tank

1.3 The Main Contents of the Book

1.3.1 Thermal Bending of Concrete Rectangular Thin Plate

According to the boundary condition of the rectangular thin plate, it can be divided into two types, which are four edges supported and with free boundary rectangular thin plate. Four edges supported rectangular thin plate can be divided into six types, which are the rectangular thin plates with four edges simply supported, four edges clamped, three edges clamped and one edge simply supported, one edge clamped and three edges simply supported, two adjacent edges clamped and two adjacent edges simply supported, two opposite edges clamped and two opposite edges simply supported. Rectangular thin plate with free boundary (this book only discuss the situation with one free boundary) also can be divided into six types, which are the rectangular thin plates with three edges simply supported and one edge free, three edges clamped and one edge free, two opposite edges clamped one edge simply supported and one edge free, two adjacent edges clamped one edge simply supported and one edge free, two opposite edges simply supported one edge clamped and one edge free, two adjacent edges simply supported one edge clamped and one edge free.

Usually, temperature effect is not considered when calculating the thin plate. But if the temperature of one side is higher than the other, in order to guarantee the normal use of structure, it is necessary to calculate the temperature stress and deformation of the plate structure. For example, for the chemical liquid-storage structure when it stores high temperature liquid, or from the perspective of the use of liquid-storage structure, because of the changes of temperature, many cracks of concrete liquid-storage structure are to appear due to temperature stress, which affects the normal use of liquid-storage structure. For rectangular thin plate made of isotropic materials under the temperature action, only the rectangular thin plate with four edges simply supported and rectangular thin plate with four edges clamped are derived in the existing literature, and analytical solutions are obtained. For rectangular thin plate with four edges simply supported, the deflection equation satisfies the boundary condition of $w = 0$ only, but does not satisfy the boundary condition that bending moment is zero. And so it remains to be further researched whether other deflection function can be obtained. For rectangular thin plate with four edges clamped, according to the existing research results, the rectangular thin plate is statically indeterminate structure, the lower temperature side should be in tension and the higher temperature side should be in compression. Moreover, due to the ubiquitous continuation of the upper and lower sides of the thin plate and the existence of material elastic modulus, it remains to be further researched that whether it is suitable to take the deflection function $w = 0$ directly in the literature existed, and whether there is other deflection function. Though Liu et al. showed the calculation table of rectangular thin plate with three edges clamped and one edge simply supported in their works [13], the horizontal bending moment in the situation of clamped edges were greater than solutions obtained by $w = 0$. For temperature effect calculation of rectangular thin plate with other four edges supported, there have been no reports in the literature. Therefore, various deflection equations and internal force analytic solution of rectangular thin plate under thermal load with four edges supported and with free boundary are derived in this book based on small deflection theory of thin plate.

1.3.2 Thermal Buckling of Concrete Rectangular Thin Plate

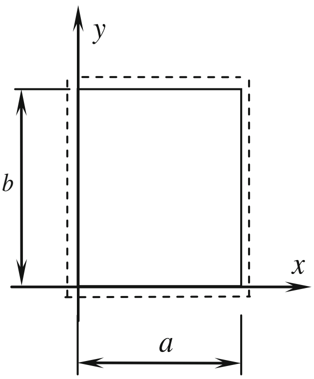
Concrete rectangular thin plate buckling problem can be caused by in-plane compression load, and also can be caused by thermal load. Therefore, there should be a full understanding about thermal buckling behavior of rectangular thin plate. According to domestic and foreign researches about present situation, a lot of research work about the thermal buckling of rectangular thin plate has been made, which laid a solid foundation for the thermal buckling analysis of engineering structure. But for reinforced concrete structure, due to the particularity of the concrete material itself, the above conclusions cannot be applied very well. Due to obvious brittleness of concrete material, the thermal buckling analysis of the concrete material should be based on the classical theory of small deformation. In addition, the constitutive relation of concrete compression suggested by the American Hognestad [44], German Rusch [45] and specifications are the quadratic function. So in this book, based on the theory of small deflection, the quadratic double parameters model is adopted considering the nonlinear effect of concrete material, the balance equation and stability equation of rectangular thin plate under the thermal load are derived, and the buckling behavior of the concrete rectangular thin plates when temperature changes uniformly is researched. Thus the closed-form solution of the critical buckling temperature changes of concrete rectangular thin plate in a uniform temperature changes is obtained, and the effect of material constant, length-width ratio, bedding coefficient and relative thickness of the thin plate on the critical buckling temperature changes is discussed.

1.3.3 Thermal Vibration of Concrete Rectangular Thin Plate

In recent years, many studies have been made on vibration of the nonlinear plate with different geometric features extensively. The content involved the influence of the geometric non-linearity, material non-linearity, anisotropy, shear deformation, moment of inertia, deformation of static load on the thin plate. At present, due to the rapid development of science and technology and its wide use in engineering, a lot of researches about vibration behavior of heating thin plate have been made. For example, Li and Zhou analyzed vibration of heating ring plate, and did few researches on concrete material in studies [46–48]. He et al. analyzed dynamic response of concrete plate under the action of the explosion load, but assumed concrete as ideal rigid-plastic material. These studies laid a solid foundation for the vibration of the thin plate analysis. But due to the particularity of concrete material, the current research results cannot be applied well. Therefore, based on the theory of small deflection, using the quadratic double parameters model, the dynamic equation of thermal elastic problem about concrete rectangular thin plate is derived in this book. Using the Galerkin method and Progression method, the natural frequency and the deflection function of forced vibration of concrete rectangular thin plate under the thermal environment is derived. For convenience of engineering design, concrete rectangular thin plate natural frequency in transverse temperature and uniform temperature change, and the deflection function under the action of uniformly distributed load are given, and the influence of material elastic constants, length-width ratio, relative thickness and temperature of thin plate on natural frequency and deflection function of concrete thin plate is discussed.

Chapter 2
Thermal Bending of Concrete Rectangular Thin Plate with Four Supported Edges

Abstract The deflection equation and the internal force analytical solution of the rectangular thin plate supported on four sides (four edges simply supported, four edges clamped, three edges clamped and one edge simply supported, one edge clamped and three edges simply supported, two adjacent edges clamped and two adjacent edges simply supported, two opposite edges clamped and two opposite edges simply supported) under temperature difference is systematically introduced in this chapter. In order to facilitate the engineering application, the tables for deflection and internal force coefficient calculation based on concrete material are made.



2.1 Introduction

The rectangular thin plate with four supported edges can be classified into six types: rectangular thin plate with four simply supported edges; rectangular thin plate with four clamped edges; rectangular thin plate with three clamped edges and one simply supported edge; rectangular thin plate with three simply supported edges and one clamped edge; rectangular thin plate with two adjacent simply supported edges and two adjacent clamped edges, rectangular thin plate with two opposite simply supported edges and two clamped opposite edges.

X. Cheng, *Thermal Elastic Mechanics Problems of Concrete Rectangular Thin Plate*, Springer Tracts in Civil Engineering,
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For the calculation of the temperature effect of the rectangular thin plate, the existing literature gives the analytical solution for the rectangular thin plate with four simply supported edges and the rectangular thin plate with four clamped edges for the isotropic materials. For example, to the rectangular thin plate with four simply supported edges under temperature disparity, the literature [4] gave the calculation formulas of deflection with any temperature change, namely

$$w(x,y) = \frac{1}{(1-\mu)\pi^2 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{2.1}$$

where $a_{mn} = \frac{4}{ab}\int_0^a\int_0^b M_T(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}y\mathrm{d}x$, $D = \frac{Eh^3}{12(1-\mu^2)}$.

The literature [3] gave the calculation formulas of deflection with temperature change along the thickness change, namely

$$w(x,y) = \frac{16M^*}{(1-\mu)\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}(m,n=1,3,5\ldots) \tag{2.2}$$

where, $M^* = E\alpha\int_{-\frac{h}{2}}^{\frac{h}{2}}(\Delta T)z\mathrm{d}z$.

The literature [49] also gave the approximate calculation formulas of bending moment with temperature change along the thickness change, namely

$$M_{xT}(M_{yT}) = k_{xt}(k_{yt})M^T \tag{2.3}$$

in above equation, M^T stands for the distribution moment of rectangular thin plate with four clamped edges caused by lateral temperature disparity; ΔT stands for the lateral temperature disparity; α stands for the thermal expansion coefficient of material; E stands for elastic modulus of material; μ stands for Poisson's ratio of materials; h stands for the thickness of the thin plate; $k_{xt}(k_{yt})$ stands for the coefficient of bending moment; M_{xT} stands for the distribution moment in the x direction caused by lateral temperature disparity; M_{yT} stands for the distribution moment in the y direction caused by lateral temperature disparity.

It is easy to see that (2.2) is the special form of (2.1), and (2.3) is an approximation of (2.2). Therefore, here only (2.1) will be described. By calculating, (2.1) only satisfies the edge condition of $w = 0$, but does not satisfy the edge condition of moment being equal to zero, and it requires further research whether there are other deflection functions.

Although there are solutions of rectangular thin plate with four clamped edges under the action of the temperature and widely used in engineering [18], such as:

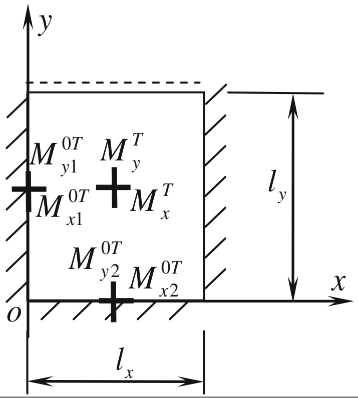
$$M_x = M_y = M^T = \frac{\alpha\Delta TEh^2}{12(1-\mu)} \tag{2.4}$$

(2.4) is obtained by the deflection function satisfying the boundary conditions of $w = 0$. But for statically indeterminate structure under the action of temperature, the lower temperature side is in tension, the higher temperature side is in compression. And it is continuously everywhere on the upper and lower surface. Due to the existence of elastic modulus, whether the deflection function $w = 0$ is appropriate in the existing literature needs further verification, as well as the deflection function should be taken as $w = w\ (x,\ y)$.

Though Liu et al. showed the calculating table of rectangular thin plate with three clamped edges and one simply supported edge (see Table 2.1) in their work [13], transversal moments on the clamped edges were greater than the solution obtained by $w = 0$. For the calculation of temperature effects about other rectangular thin plate with four supported edges, current literatures have not been reported.

Therefore, this chapter is based on the small deflection plate theory. Firstly, through assuming deflection function that meets equilibrium differential equation and some of boundary condition, using the Levy method, the analytical solution of deflection and internal force of isotropic rectangular thin plate with four simply supported edges is derived. Then according to the conclusion of rectangular thin plate with four simply supported edges and boundary conditions of other

Table 2.1 Bending calculation coefficient with three edges clamped and one edge free under temperature disparity



$\mu = \frac{1}{6}$
$M_x^T = k_x^T \alpha \Delta T E h^2 \eta_{re1}$
$M_y^T = k_y^T \alpha \Delta T E h^2 \eta_{re1}$
η_{re1} is reduction factor of considering concrete creep

l_x/l_y	k_{x1}^T	k_{y1}^T	k_{x2}^T	k_{y2}^T	k_x	k_y
0.50	0.1045	0.0987	0.0972	0.1000	0.0973	0.0998
0.75	0.1139	0.0999	0.0982	0.1021	0.0926	0.1003
1.00	0.1233	0.1008	0.0981	0.1094	0.0885	0.0961
1.25	0.1288	0.1011	0.0993	0.1175	0.0869	0.0917
1.50	0.1344	0.1016	0.1008	0.1286	0.0853	0.0873
1.75	0.1329	0.1013	0.1014	0.1344	0.0877	0.0829
2.00	0.1324	0.1008	0.1019	0.1402	0.0901	0.0784

rectangular thin plate with four supported edges, by applying the virtual displacement principle and superposition principle, the deflection equation and analytical solution of internal force of the other rectangular thin plate with four supported edges under the action of lateral temperature disparity is derived, which provides a theoretical basis for later engineering calculation.

2.2 The Basic Equation for the Thermal Elastic Problem of Rectangular Thin Plate

2.2.1 Calculation Assumption

(1) Straight-line which is perpendicular to the mid-plane before deformation is still perpendicular to the deformed mid-plane, and the length has no change;
(2) The stress σ_z, τ_{xz} and τ_{yz} are far less than the other three stresses (σ_x, σ_y and τ_{xy}), so the strain caused by these stress can be neglected;
(3) Each point in mid-plane has not displacement which is parallel to mid-plane, namely $u|_{z=0} = 0$, $v|_{z=0} = 0$.

2.2.2 Basic Equation of Thermal Elasticity

The existing literature [3–5] showed that the geometry equation for the thermal elastic problem of rectangular thin plate is

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\partial^2 w}{\partial x^2} z \\ \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\partial^2 w}{\partial y^2} z \\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2\frac{\partial^2 w}{\partial x \partial y} z \end{cases} \tag{2.5}$$

where u, v stand for the displacements in x, y directions, respectively; w stands for the deflection of any point on the surface of the thin plate; ε_x, ε_y and γ_{xy} stand for the strains of any point on the surface of the thin plate, respectively.

Because of neglecting strain caused by the stress σ_z, physical equation can be written as

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right) + \alpha T \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right) + \alpha T \\ \gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy} \end{cases} \tag{2.6}$$

where $T = T(x, y, z)$ stands for temperature disparity of any point in the thin plate.

Stresses can be written as

$$\begin{cases} \sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{E\alpha T}{1-\mu} \\ \sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{E\alpha T}{1-\mu} \\ \tau_{xy} = \tau_{yx} = -\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.7}$$

As shown in Fig. 2.1, M_x and M_y stand for moments of unit width on the cross section, respectively; M_{xy} stands for torment of unit width on the cross section, hence

$$\begin{cases} M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x \mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{1-\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(T)T\alpha(T)z\mathrm{d}z \\ M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_y \mathrm{d}z = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{1}{1-\mu}\int_{-\frac{h}{2}}^{\frac{h}{2}} E(T)T\alpha(T)z\mathrm{d}z \\ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xy} \mathrm{d}z = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.8}$$

The thickness of the plate compared with the size of the other two directions is very small, for the purpose of engineering application, assuming that the temperature changes along the thickness direction only, namely, we only consider the situation of lateral temperature change. That is to say, in this paper, the analytical solution of lateral deflection and internal forces that is studied is for the specific condition of lateral temperature change; in addition, due to the thin plate, before the plate structure is normally used, the heat release W of concrete condensation sclerosis tends to zero as the change of pouring time. So the original parabolic nonlinear temperature distribution rule becomes the linear situation, namely [17]

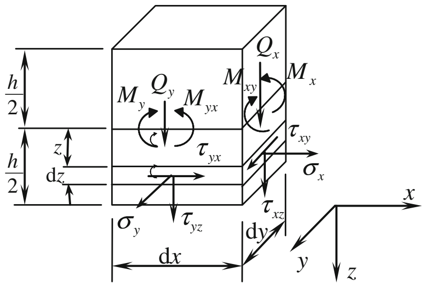


Fig. 2.1 Element forces and stresses sketch map

$$T = \frac{T_2 + T_1}{2} - \frac{(T_2 - T_1)}{h} z \tag{2.9}$$

where T_1 and T_2 stand for the temperature on two surfaces of the thin plate, respectively.

According to the existing literature [15], the relationship between concrete elastic modulus under any temperature normal temperature can be determined by using the following equations:

$$\begin{cases} E(T) = E & T \leq 60\,^\circ\mathrm{C} \\ E(T) = 0.88E \sim 0.94E & 60\,^\circ\mathrm{C} < T \leq 100\,^\circ\mathrm{C} \\ E(T) = 0.95E \sim 1.08E & 100\,^\circ\mathrm{C} < T \leq 300\,^\circ\mathrm{C} \\ E(T) = \left[1 + 18\left(\frac{T}{1000}\right)^{5.1}\right]^{-1} E & T > 300\,^\circ\mathrm{C} \end{cases} \tag{2.10.1}$$

Under the action of temperature, the linear expansion coefficient $\alpha(T)$ is determined by using the following equation:

$$\alpha(T) = 28\left(\frac{T}{1000}\right) \times 10^{-6} \tag{2.10.2}$$

As can be seen, the temperature only slightly affects the elastic modulus of concrete and the linear expansion coefficient under normal temperature. Thus, the approximate values are as follows:

$$E(T) = E,\ \alpha(T) = \alpha \tag{2.11}$$

By substituting (2.9) and (2.11) into (2.8) obtains the following:

$$\begin{cases} M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) - M^T \\ M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) - M^T \\ M_{xy} = -D(1 - \mu)\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{2.12}$$

where $M^T = \frac{E\alpha \Delta T h^2}{12(1-\mu)}$; ΔT stands for the lateral temperature disparity [16, 50].

As it is known that the equilibrium differential equations of elastic surface about thin plate with same thickness is:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

As there is no external load, only temperature, (2.12) is substituted into the above mentioned equation, there is:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w=0. \tag{2.13}$$

2.3 Thermal Bending of Rectangular Thin Plate with Four Edges Simply Supported

2.3.1 Boundary Conditions

In Fig. 2.2, the boundary conditions to clamped edges are:

$$w\Big|_{\substack{x=0\\x=a}}=0,\ M_x\Big|_{\substack{x=0\\x=a}}=0$$

$$w\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}}=0,\ M_y\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}}=0$$

To simply supported edges, due to w = 0 on whole edge, according to (2.12), the above formulas are (Fig. 2.2):

$$w\Big|_{\substack{x=0\\x=a}}=0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}}=-\frac{M^T}{D} \tag{2.14}$$

$$w\Big|_{\substack{y=-\frac{b}{2}\\y=\frac{b}{2}}}=0,\ \frac{\partial^2 w}{\partial y^2}\Big|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}}=-\frac{M^T}{D} \tag{2.15}$$

2.3.2 The Analytical Solution of the Thermal Elastic Problem

According to (2.13), ordering

$$w=\sum_{m=1}^{\infty}X_mY_m-\frac{M^T}{2D}(x-a)x$$

where X_m is only a function about x; Y_m is a function about y only.

According to the edge conditions (2.14), ordering $X_m = \sin\frac{m\pi x}{a}$, so the deflection function w can be written as

$$w = \sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.16}$$

By substituting (2.16) into the differential (2.13), hence:

$$Y_m^{(4)} - 2\frac{m^2\pi^2}{a^2}Y''_m + \frac{m^4\pi^4}{a^4}Y_m = 0$$

Solution of this equation can be written as follows [51]:

$$Y_m = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + C_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}$$

Due to the temperature and the plate are symmetrical about the x-axis, so Y_m must be an even function, thereby $A_m = D_m = 0$, by substituting them into (2.16), hence

$$w = \sum_{m=1}^{\infty}\left(B_m \cosh\frac{m\pi y}{a} + C_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi y}{a} - \frac{M^T}{2D}(x-a)x \tag{2.17}$$

Ordering $\frac{m\pi b}{2a} = \alpha_m$, by substituting (2.17) into the boundary condition (2.15), hence

$$\sum_{m=1}^{\infty}(B_m \cosh\alpha_m + C_m\alpha_m \sinh\alpha_m)\sin\frac{m\pi x}{a} = \frac{M^T}{2D}(x-a)x \tag{2.18}$$

$$\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}[(B_m + 2C_m)\cosh\alpha_m + C_m\alpha_m\sinh\alpha_m]\sin\frac{m\pi x}{a} = -\frac{M^T}{D} \tag{2.19}$$

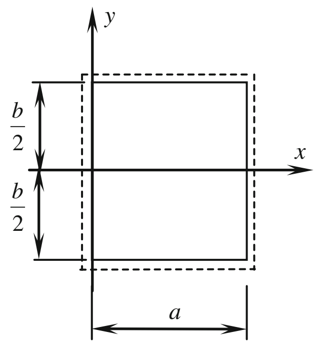


Fig. 2.2 Four edges simply supported

The right side of (2.18) is expanded into a single triangular series, namely

$$\frac{M^T}{2D}(x-a)x = \sum_{m=1}^{\infty}\left[\frac{2}{a}\int_0^a \frac{M^T}{2D}(x-a)x\,\sin\frac{m\pi x}{a}\mathrm{d}x\right]\sin\frac{m\pi x}{a}$$
$$= \sum_{m=1}^{\infty}\frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi - 1)\,\sin\frac{m\pi x}{a}$$

Then (2.18) becomes

$$B_m\,\cosh\alpha_m + C_m\alpha_m\,\sinh\alpha_m = \frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi\ - 1) \tag{2.20}$$

Similarly, the right side of (2.19) is expanded into a single triangular series, namely

$$-\frac{M^T}{D} = \frac{2M_T}{\pi D}\sum_{m=1}^{\infty}\frac{\cos m\pi - 1}{m}\sin\frac{m\pi x}{a}$$

Then the (2.19) becomes

$$(B_m + 2C_m)\,\cosh\alpha_m + C_m\alpha_m\,\sinh\alpha_m = \frac{2a^2M^T}{Dm^3\pi^3}(\cos m\pi - 1) \tag{2.21}$$

By (2.20) and (2.21), there is

$$\begin{cases} B_m = \frac{2a^2M^T}{D\pi^3m^3\,\cos\mathrm{h}\alpha_m}(\cos m\pi - 1) \\ C_m = 0 \end{cases}$$

By substituting B_m and C_m into (2.17), hence

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.22}$$

Since (2.22) is made from satisfying the equilibrium differential Eqs. (2.13) and all boundary condition (2.14) and (2.15), so (2.22) is the deflection function of rectangular thin plate with simply supported edges under transverse temperature disparity. Substituting (2.22) into (2.12), internal force calculation formula is obtained.

Because

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} = \frac{4M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a} - \frac{M^T}{D} \\ \frac{\partial^2 w}{\partial y^2} = -\frac{2M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\cosh\frac{2\alpha_m y}{b}\sin\frac{m\pi x}{a} \\ \frac{\partial^2 w}{\partial x\partial y} = -\frac{4M^T}{D\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\,\cosh\alpha_m}\sinh\frac{2\alpha_m y}{b}\cos\frac{m\pi x}{a} \end{cases}$$

Therefore, there is

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m}\cosh\frac{m\pi y}{a}\sin\frac{m\pi x}{a} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m}\cosh\frac{m\pi y}{a}\sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m}\sinh\frac{m\pi y}{a}\cos\frac{m\pi x}{a} \end{cases} \tag{2.23}$$

(2.23) is internal force analytical solution of rectangular thin plate with simply supported edges under transverse temperature disparity.

2.3.3 Results Analysis

MATLAB software is used to test the accuracy of Eqs. (2.22) and (2.23). The results show that for deflection function w, when taking $m = n = 5$, the result has converged to the exact solution; for the bending moment of unit width, when taking $m = n = 7$, the result has converged to the exact solution; for the bidirectional plate in engineering with any length-width ratio, its internal force solutions are equal to the results with existing literature (Because the existing literature has not given deflection calculation coefficient, it did not make deflection comparison). Because the existing literature has not given the deflection calculation coefficient, so for the convenience and engineering application, supplementing deflection calculation coefficient, thermal bending calculation results of concrete rectangular thin plate with four simply supported sides are made (see Table A.1).

2.3.4 Engineering Design

For a concrete rectangular thin plate with four edges simply supported under temperature variation which is perpendicular to surface, according to (2.10.1) and (2.10.2), E and α are obtained. And then M^T is given, namely $M^T = \frac{E\alpha\Delta Th^2}{12(1-\mu)}$.

According to Table A.1, k_x, k_y and f can be obtained by $\frac{l_x}{l_y}$. Then M_x^T, which is $M_x^T = k_x M^T$, is gotten as well as w_1, and $M_y^T = k_y M^T$. After the bending moment is gotten, the steel bars due to temperature can be designed according to the knowledge of the reinforced concrete. Namely [44−45]

$$\begin{cases} \alpha_{sx} = \frac{M_x^T}{1000\alpha_1 f_c h_0^2} \\ \alpha_{sy} = \frac{M_y^T}{1000\alpha_1 f_c h_0^2} \end{cases} \tag{2.24.1}$$

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) \end{cases} \tag{2.24.2}$$

$$\begin{cases} A_{sx1} = \frac{M_x^T}{f_y \gamma_s h_0} \\ A_{sy1} = \frac{M_y^T}{f_y \gamma_s h_0} \end{cases} \tag{2.24.3}$$

where, M_x^T is the bending moment design value in x direction, and ${M^T}_y$ is the bending moment design value in y direction, and A_{sx1} is the steel section area per meter width in x direction, and A_{sy1} is the steel section area per meter width in y direction, and f_y is the tensile strength design value of the steel, and α_{sx} and α_{sy} are the coefficient of section resistance moment in x and y directions, and α_1 is the equivalent rectangular stress diagram coefficient of concrete compressive zone, and γ_s is the internal force arm coefficient of section, and h_0 is the effective height of the section, $h_0 = h - c$ (c is the thickness of the concrete protective layer), and f_c is the compressive strength design value of the concrete.

If the deflection is w_2 and the area of steel bar is $A_{sx2}(A_{sy2})$ per unit width in $x(y)$ direction caused by other factors except for the temperature are known, there are

$$\begin{cases} A_{sx} = A_{sx1} + A_{sx2} \\ A_{sy} = A_{sy1} + A_{sy2} \\ w = w_1 + w_2 \end{cases} \tag{2.25}$$

Pay attention to that, the formulas of this chapter are based on thin plate structure that is homogeneous elastic body, which does not accord with the concrete. Especially, the creep and cracks of concrete reduce component stiffness, and result in thermal stress relaxation. Therefore, according to Table A.1, calculating bending moment should multiply the reduction factor of 0.65, and bearing capacity calculation should multiply the partial coefficient.

2.3.5 Numerical Example

Example: Taking the liquid storage structure with top plate as an example, the numerical analysis is carried out by a reasonable calculation. The length l_x and width l_y of the plate are both 6 m. The thickness h of the plate is 180 mm. The temperature difference ΔT between the upper and lower surface of plate is 60 °C. The live load p is 1 kN/m^2. The bulk density of concrete is 26 kN/m^3. The value of concrete strength is 30 MPa. The value of steel strength is 360 MPa.

Solution: In view of the fact that the stiffness of wallboard is far greater than the stiffness of top plate in general, the top plate can be regarded as the four edges simply supported. According to the literature [45], the linear expansion coefficient α of concrete is 1×10^{-5} °C. The Poisson's ratio μ of concrete is 1/6. The protective layer thickness of concrete is 10 mm. The elastic modulus E of concrete is 3×10^7 kN/m^2. The design value of compressive strength f_c for concrete is 14.3 N/mm^2. The partial coefficients of the dead load and live load are taken as 1.2 and 1.4, respectively.

Dead load: $g = 0.18 \times 26 = 4.68$ kN/m^2
Live load: $p = 1$ kN/m^2
Design load: $q = 1.4p + 1.2g = 7.02$ kN/m^2.

According to the initial assumption that the diameter of the steel is 10 mm, the distance from the center of the steel in x direction to the down surface of concrete plate, $c_x = c + 10/2$, is 15 mm and the distance from the center of the steel in y direction to the down surface of concrete plate, $c_y = c + 10 + 10/2$, is 25 mm. The distance from the center of the steel in x direction to the top surface of concrete plate, $h_{0x} = h - c_x$, is 165 mm and the distance from the center of the steel in y direction to the top surface of concrete plate, $h_{0y} = h - c_y$, is 155 mm.

1. **Temperature Action**

Taking $E = 3 \times 10^7$ kN/m^2, $\alpha = 1 \times 10^{-5}$ °C, $\Delta T = 60$ °C, $h = 180$ mm and $\mu = 1/6$ into the (2.1), $D = \frac{Eh^3}{12(1-\mu^2)}$, and (2.4), the following results can be gotten.

$$D = \frac{Eh^3}{12(1-\mu^2)} = \frac{3 \times 10^7 \times 0.18^3}{12\left(1 - \frac{1}{6^2}\right)} = 14996.57\,\text{kN} \cdot \text{m}$$

$$M^T = \frac{\alpha \Delta T E h^2}{12(1-\mu)} = \frac{1 \times 10^{-5} \times 60 \times 3 \times 10^7 \times 0.18^2}{12\left(1 - \frac{1}{6}\right)} = 58.32\,\text{kN}$$

From the Table A.1 in the Appendix A, there are

$$f = 0.0737, k_x = 0.4167, k_y = 0.4167$$

$$w_1 = f\frac{l_x^2 M^T}{D} = 0.0737 \times \frac{6 \times 58.32}{14996.57} = 0.0103\,\text{m}$$

$$M_x^T = k_x M^T = 0.4167 \times 58.42 = 24.30\,\text{kN} \cdot \text{m}$$

$$M_y^T = k_y M^T = 0.4167 \times 58.42 = 24.30\,\text{kN} \cdot \text{m}$$

According to the literature [45], $\alpha_1 = 1$, assuming that $h_0 = h_{0x}$, and taking M_x^T, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x^T}{1000\alpha_1 f_c h_0^2} = \frac{24.3 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0624$$

Assuming that $h_0 = h_{0y}$, taking M_y^T, f_c, h_0, and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y^T}{1000\alpha_1 f_c h_0^2} = \frac{24.3 \times 10^6}{1000 \times 1 \times 14.3 \times 155^2} = 0.071$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0624}\right) = 0.9678 \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.071}\right) = 0.9631 \end{cases}$$

Taking γ_{sx} and γ_{sy} into (2.24.3), there is

$$\begin{cases} A_{sx1} = \dfrac{M_x^T}{f_y \gamma_s h_0} = \dfrac{24.3 \times 10^6}{360 \times 0.9678 \times 165} = 422.7\,\text{mm}^2 \\ A_{sy1} = \dfrac{M_y^T}{f_y \gamma_s h_0} = \dfrac{24.3 \times 10^6}{360 \times 0.9631 \times 155} = 452.2\,\text{mm}^2 \end{cases}$$

2. **Load Action**

From the literature [13, 52], $w_2 = f\frac{ql^4}{D}$, $M_x = k_x q l^2$ and $M_y = k_y q l^2$ can be obtained. The value l is the minimum [l_x, l_y].

According to the literature [13, 52], there are

$$f = 0.00406,\, k_x = 0.0368 \,\text{and}\, k_y = 0.0368$$

Taking f, q, l and D into $w_2 = f\frac{ql^4}{D}$, there is

$$w_2 = f\frac{ql^4}{D} = 0.00406 \times \frac{7.02 \times 6^4}{14996.57} = 0.0025$$

Taking k_x, k_y into $M_x = k_x q l^2$ and $M_y = k_y q l^2$ respectively, there are

$$M_x = k_x q l^2 = 0.0368 \times 7.02 \times 6^2 = 9.80\,\text{kN} \cdot \text{m}$$

$$M_y = k_y q l^2 = 0.0368 \times 7.02 \times 6^2 = 9.80\,\text{kN} \cdot \text{m}$$

Assuming that $M_x = M_x^T$ and $h_0 = h_{0x}$, and taking M_x, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x}{1000\alpha_1 f_c h_0^2} = \frac{9.8 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0252$$

Assuming that $M_y = M_y^T$ and $h_0 = h_{0y}$, and taking M_y, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y}{1000\alpha_1 f_c h_0^2} = \frac{9.8 \times 10^6}{1000 \times 1 \times 14.3 \times 155^2} = 0.0285$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0252}\right) = 0.9872 \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0285}\right) = 0.9855 \end{cases}$$

Assuming that $M_x = M_x^T$, $\gamma_s = \gamma_{sx}$ and $h_0 = h_{0x}$, and taking M_x, f_y and γ_s into (2.24.3), there is

$$A_{sx2} = \frac{M_x}{f_y \gamma_s h_0} = \frac{9.8 \times 10^6}{360 \times 0.9872 \times 165} = 167.1\,\text{mm}^2$$

Assuming that $M_y = M_y^T$, $\gamma_s = \gamma_{sy}$ and $h_0 = h_{0y}$, and taking M_y, f_y and γ_s into (2.24.3), there is

$$A_{sy2} = \frac{M_y}{f_y \gamma_s h_0} = \frac{9.8 \times 10^6}{360 \times 0.9855 \times 155} = 178.2\,\text{mm}^2$$

In summary, the analysis results can be obtained under the action of temperature and load. That is

$$w = w_1 + w_2 = 0.0103 + 0.0025 = 0.0128\,\text{m}$$

$$A_{sx} = A_{sx1} + A_{sx2} = 422.7 + 167.1 = 589.8\,\text{mm}^2$$

$$A_{sy} = A_{sy1} + A_{sy2} = 452.2 + 178.2 = 630.4\,\text{mm}^2$$

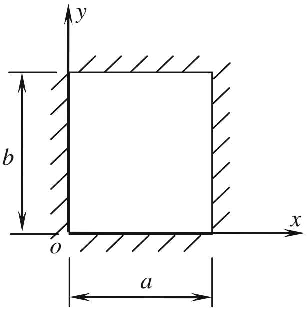
From the above results, the total deflection at the midspan point of the plate is 12.8 mm. The reinforcement area per meter at the center point of the thin plate in the x direction is 589.8 mm^2 and the reinforcement area per meter at the center point of the thin plate in the y direction is 630.4 mm^2.

2.4 Thermal Bending of Rectangular Thin Plate with Four Edges Clamped

2.4.1 Boundary Conditions

In Fig. 2.3, the boundary conditions to the clamped edges are:

Fig. 2.3 Four edges clamped



$$w\Big|_{\substack{x=0\\x=a}}=0,\ \frac{\partial w}{\partial x}\Big|_{\substack{x=0\\x=a}}=0 \tag{2.26}$$

$$w\Big|_{\substack{y=0\\y=b}}=0,\ \frac{\partial w}{\partial x}\Big|_{\substack{y=0\\y=b}}=0 \tag{2.27}$$

2.4.2 Analytical Solution of Thermal Elastic Problem

To satisfy the balance differential Eq. (2.13) and the boundary conditions (2.26) and (2.27), it is apparently that $w = 0$, but on the boundary according to the (2.12), it is known that

$$M_x|_{y=0,b}= M_y|_{x=0,a}= -M^T$$

Rectangular thin plate with four clamped edges under the action of lateral variable temperature disparity is regarded as a superposition of the rectangular thin plate with four simply supported edges under the action of bending moment $M_T^-(M_T^- = -M^T)$ on four edges and rectangular thin plate with four edges simply supported under the action of temperature disparity ΔT.

1. Bending Deformation Energy of Thin Plate

As shown in Fig. 2.1, ignoring the work done by shearing force, $-\frac{1}{2}M_x\frac{\partial^2 w}{\partial x^2}\mathrm{d}x\mathrm{d}y$ is the work done by bending moment $M_x\mathrm{d}y$; $-\frac{1}{2}M_y\frac{\partial^2 w}{\partial y^2}\mathrm{d}x\mathrm{d}y$ is work done by the bending moment $M_y\mathrm{d}x$; $\frac{1}{2}M_{xy}\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}x\mathrm{d}y$ is work done by torque $M_{xy}\mathrm{d}x$; also, because the work done by the torque and the work done by the bending moment are not coupled, deformation energy of differential body is

$$\mathrm{d}V = -\frac{1}{2}\left[M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} - 2M_{xy} \frac{\partial^2 w}{\partial x \partial y}\right]\mathrm{d}x\mathrm{d}y$$

Substituting (2.12) into above equation, and letting $M^T = 0$ in (2.12), there is

$$\mathrm{d}V = \frac{1}{2}D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

To the whole plate, deformation energy in bending plate is

$$\mathrm{d}V = \frac{1}{2}D\iint\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

In above equation, the second item of integrand function is transformed using Green's theorem, there is

$$\begin{aligned}\iint\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y &= \iint\left[\frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)}{\partial x} - \frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right)}{\partial y}\right]\mathrm{d}x\mathrm{d}y \\ &= \int\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\mathrm{d}x + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\mathrm{d}y\right)\end{aligned} \tag{2.28}$$

The line integral of (2.28) is along the whole edge of rectangular thin plate. Because the thin plate is with four edges simply supported, x is constant on the boundary, $\mathrm{d}x = 0$ and $\frac{\partial^2 w}{\partial y^2} = 0$; on the boundary, y is constant, $\mathrm{d}y = 0$ and $\frac{\partial w}{\partial x} = 0$, so (2.28) is simplified to

$$V = \frac{1}{2}D\iint\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 \mathrm{d}x\mathrm{d}y \tag{2.29}$$

2. The Analytic Solution Under Uniform Bending Moment M_T^- on Four Edges

According to the edge condition of rectangular thin plate with four simply supported edges and equilibrium differential equation for the elastic curved surface of identical thickness plate, deflection function may be supposed as

$$w = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} A_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi x}{b} \tag{2.30}$$

By substituting (2.30) into (2.29), deformation energy of plate is

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.31}$$

The slope of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \left. \dfrac{\partial w}{\partial x} \right|_{\substack{x=0 \\ x=a}} = \dfrac{\pi}{a} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \dfrac{j\pi y}{b} \\ \left. \dfrac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \dfrac{\pi}{b} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \dfrac{i\pi x}{a} \end{cases}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \left. \dfrac{\partial w}{\partial x} \right|_{\substack{x=0 \\ x=a}} = \pm \dfrac{\pi}{a} i \sin \dfrac{j\pi y}{b} \delta A_{ij} \\ \delta \left. \dfrac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \dfrac{\pi}{b} j \sin \dfrac{i\pi x}{a} \delta A_{ij} \end{cases}$$

The work done by bending moment along edges of plate is

$$2 \int_0^a M_T^- \frac{\pi}{b} n \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + 2 \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}y \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{a}{2} E_i \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}y = \frac{b}{2} F_j \end{cases}$$

where $E_i = \frac{4M_T^-}{i\pi}$; $F_j = \frac{4M_T^-}{j\pi}$.

The work done of moment is

$$j \frac{\pi a}{b} E_i \delta A_{ij} + i \frac{\pi b}{a} F_j \delta A_{ij}$$

According to (2.31), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{4}{\pi^4 Dab}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(j\frac{\pi\, a}{b}E_i + i\frac{\pi\, b}{a}F_j\right) = \frac{16M_T^-}{\pi^4 D}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}$$

By substituting the above formula into (2.30), hence

$$w(x,y) = \frac{16M_T^-}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{2.32}$$

Substitute (2.32) into (2.12) (where $M^T = 0$), there is

$$\begin{cases} M_x = \frac{16M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{i^2b^2+\mu j^2a^2}{ij(i^2b^2+j^2a^2)}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \frac{16M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j^2a^2+\mu i^2b^2}{ij(i^2b^2+j^2a^2)}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = \frac{16(\mu-1)abM_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{i^2b^2+j^2a^2}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.33}$$

3. The Analytical Solution of Rectangular Thin Plate with Four Simply Supported Edges Under Temperature Disparity ΔT

For easy superposition, in (2.22) and (2.23), x axis can be moved to the $y = -\frac{b}{2}$, so there is

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \tag{2.34}$$

$$\begin{cases} M_x = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ M_y = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} + (\mu-1)M^T \\ M_{xy} = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \end{cases} \tag{2.35}$$

4. The Analytical Solution of Rectangular Thin Plate with Four Simply Supported Edges Under Thermal Load

By superposing (2.32) and (2.34), hence

$$\begin{aligned} w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\ & -\frac{16M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \end{aligned} \tag{2.36}$$

(2.36) is the deflection formula of rectangular thin plate with four clamped edges under lateral temperature disparity.

Superpose (2.33) and (2.35), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad -\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad +(\mu-1)M^T-\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\ \qquad +(1-\mu)\frac{16M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.37}$$

(2.37) is internal force solution of rectangular thin plate with four clamped edges under lateral temperature disparity.

2.4.3 Result Analysis

To test that formulas (2.36) and (2.37) are correct, the software MATLAB is used to program the formulas. The results show that: for deflection function w, when taking $m = n = 69$, the result has converged to exact solution; for the bending moment M_x of unit width, when taking $m = n = 7999$, result has converged to exact solution; for the bending moment M_y of unit width, when taking $m = n = 10999$, the result basically has converged to the exact solution; for clamped concrete rectangular plate with arbitrary length-width ratio, the internal force can be seen in Table 2, and it is identical to the existing literature.

Engineering application is seen in Sect. 2.3.4.

2.5 Thermal Bending of Rectangular Thin Plate with One Edge Simply Supported and Three Edges Clamped

2.5.1 Boundary Conditions

In Fig. 2.4, the edge conditions for the clamped edge are:

$$w\big|_{y=0} = 0,\ \frac{\partial w}{\partial y}\big|_{y=0} = 0 \tag{2.38}$$

$$w\bigg|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial w}{\partial x}\bigg|_{\substack{x=0\\x=a}} = 0 \tag{2.39}$$

To simply supported edges, due to the deflection w = 0 on the whole boundary, by (2.12), the above equation becomes (Fig. 2.4)

$$w\big|_{y=b} = 0,\ \frac{\partial^2 w}{\partial y^2}\big|_{y=b} = -\frac{M^T}{D} \tag{2.40}$$

2.5.2 Analytical Solution for Thermal Elastic Problems

On edges, w = 0, according to (2.12), (2.38), (2.39) and (2.40), hence

$$M_y\big|_{y=b} = 0,\ M_x\bigg|_{\substack{x=0\\x=a}} = -M^T,\ M_y\big|_{y=0} = -M^T$$

Now, Rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness direction is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and rectangular thin plate with four simply

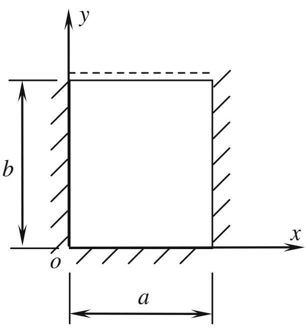


Fig. 2.4 Three edges clamped and one edge simply supported

supported edges under the uniform bending moment M_T^- ($M_T^- = -M^T$) on three adjacent edges.

1. The Analytic Solution Under the Uniform Bending Moment M_T^- on Three Adjacent Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.41}$$

The slope of every point along $x = 0, x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \frac{\partial w}{\partial y}\Big|_{y=0} = \pm \frac{\pi}{b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \frac{i\pi x}{a} \\ \frac{\partial w}{\partial x}\Big|_{\substack{x=0 \\ x=a}} = \pm \frac{\pi}{a} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \frac{j\pi y}{b} \end{cases}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$, $x = a$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \frac{\partial w}{\partial y}\Big|_{y=0} = \pm \frac{\pi}{b} j \sin \frac{i\pi x}{a} \delta A_{ij} \\ \delta \frac{\partial w}{\partial x}\Big|_{\substack{x=0 \\ x=a}} = \pm \frac{\pi}{a} i \sin \frac{j\pi y}{b} \delta A_{ij} \end{cases}$$

The work done by moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + 2 \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}x \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{2aM^T}{i\pi} \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}x = \frac{2bM^T}{j\pi} \end{cases}$$

The work done by moment is

$$2 \left(\frac{ja}{ib} + \frac{2ib}{ja} \right) M_T^- \delta A_{ij}$$

According to Eq. (2.41), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{8M_T^-}{\pi^4 Dij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)$$

By substituting the above formula into the (2.30), hence

$$w(x,y) = \frac{8M_T^-}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \quad (2.42)$$

Substituting (2.42) into (2.12) (where $M^T = 0$), the internal force calculation formula of the thin plate with three simply supported edges and one clamped edges under temperature disparity along thickness is obtained.

$$\begin{cases} M_x = \frac{8M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \frac{8M_T^-}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = \frac{8(\mu-1)M_T^-}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \quad (2.43)$$

2. The Analytic Solution of the Rectangular Thin Plate with Three Clamped Edges and One Simply Supported Edge Under Heat Load

By superposing (2.34) and (2.42), hence

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x - \frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \quad (2.44)$$

(2.44) is just the deflection formula of rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness direction.

Superposing (2.35) and (2.43), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh \alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{i^2}{a^2} + \mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh \alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ \qquad - \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\mu\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m \cosh \alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \cos\frac{m\pi x}{a} \\ \qquad + \frac{8(1-\mu)M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{2i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.45}$$

(2.45) is the internal force calculation formula of the rectangular thin plate with three clamped edges and one simply supported edge under temperature disparity along thickness.

2.5.3 Result Analysis

To test the formulas (2.44) and (2.45) are correct, the software MATLAB is used to program the formulas. The results show that: when taking $m = n = 39$, the result has converged to exact solution; for the bending moment M_x of unit width, when taking $m = n = 7999$, result has converged to exact solution; for the bending moment M_y of unit width, when taking $m = n = 1999$, the result has basically converged to the exact solution at this time, and when taking $m = n = 2001$, the error is only 1/10,000. For the convenience and engineering practical reasons, according to the length-width ratio of the plate, the thermal bending result of the concrete rectangular thin plate with three edges clamped and one simply supported is tabulated (see Table A.3).

Engineering application is seen in Sect. 2.3.4.

2.6 Thermal Bending of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Clamped

2.6.1 Boundary Conditions

In Fig. 2.5, the edge conditions for the clamped edge are:

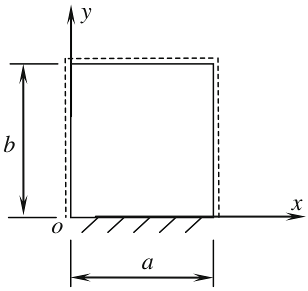


Fig. 2.5 One edge clamped and three edges simply supported

$$w\big|_{y=0} = 0, \frac{\partial w}{\partial y}\big|_{y=0} = 0 \tag{2.46}$$

To simply supported edges, due to the deflection $w = 0$ on the whole boundary, by (2.12), the above equation becomes

$$w\bigg|_{\substack{x=0\\x=a}} = 0, \frac{\partial^2 w}{\partial x^2}\bigg|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{2.47}$$

$$w\big|_{y=b} = 0, \frac{\partial^2 w}{\partial y^2}\big|_{y=b} = -\frac{M^T}{D} \tag{2.48}$$

2.6.2 Analytical Solution for Thermal Elastic Problems

On edges, $w = 0$, according to (2.12), (2.46), (2.47) and (2.48), it is known that

$$M_x\bigg|_{\substack{x=0\\x=a}} = 0, M_y\big|_{y=0} = -M^T, M_y\big|_{y=b} = 0$$

Now, rectangular thin plate with one clamped edges and three simply edge under temperature disparity along thickness is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under the action of bending moment M_T^- on edge $y = 0$.

1. The Analytical Solution of the Uniform Bending Moment M_T^- of One Edge

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.49}$$

The slope of every point along $y = 0$ on bending plane of plate is

$$\left.\frac{\partial w}{\partial y}\right|_{y=0} = \pm \frac{\pi}{b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \frac{i\pi x}{a}$$

When A_{ij} increases to $A_{ij} + \delta A_{ij}$, slope increment of every point along $y = 0$ on bending plane of plate is

$$\delta \left.\frac{\partial w}{\partial y}\right|_{y=0} = \pm \frac{\pi}{b} j \sin \frac{i\pi x}{a} \delta A_{ij}$$

The work done by bending moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij}$$

Because

$$\int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{2aM_T^-}{i\pi}$$

The work done by moment is

$$2\frac{ja}{ib} M_T^- \delta A_{ij}$$

According to (2.49), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 A_{ij} \delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{8}{\pi^4 Db^2} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \frac{jM_T^-}{i}$$

By substituting the above formula into the (2.30), hence

$$w(x,y) = \frac{8M_T^-}{\pi^4 b^2 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{2.50}$$

By substituting (2.50) into (2.12) (where $M^T = 0$) that is the internal force calculation formula of the rectangular thin plate with four simply supported edges under the bending moment M_T^- on one edge.

$$\begin{cases} M_x = \frac{8M_T^-}{\pi^2 b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2} \right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y = \frac{8M_T^-}{\pi^2 b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2} \right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{i\pi y}{b} \\ M_{xy} = -(1-\mu) \frac{8M_T^-}{\pi^2 a b^3} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} j^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.51}$$

2. Analytic Solution of the Rectangular Thin Plate with One Clamped Edge and Three Simply Supported Edges Under Thermal Load

By superposing (2.34) and Eq. (2.50), hence

$$\begin{aligned} w = & -\frac{4a^2 M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh\alpha_m} \cosh\left(\frac{2\alpha_m y}{b} - \alpha_m \right) \sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \\ & - \frac{8M^T}{\pi^4 D b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \end{aligned} \tag{2.52}$$

(2.52) is just the deflection formula of rectangular thin plate with one clamped edge and three simply supported edges under temperature disparity along thickness.

By superposing (2.35) and (2.43), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1) \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m \right) \sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2 b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2} \right) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu) \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} + \alpha_m \right) \sin\frac{m\pi x}{a} + (\mu-1)M^T \\ \qquad - \frac{8M^T}{\pi^2 b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2} \right) \sin\frac{i\pi x}{a} \sin\frac{i\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu) \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} + \alpha_m \right) \cos\frac{m\pi x}{a} \\ \qquad -(\mu-1) \frac{8M^T}{\pi^2 a b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} j^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.53}$$

(2.53) is the internal force calculation formula of the rectangular thin plate with three simply supported edges and one clamped edge under temperature disparity along thickness.

2.6.3 Results Analysis

To test (2.52) and (2.53) are correct, the software MATLAB is used to program the formulas. The results show that: for the deflection w, when taking $m = n = 39$, the result has converged to exact solution. For the bending moment M_x^T of unit width, when taking $m = n = 1999$, result has basically converged to exact solution, and has an error of only 1/10000 in comparison with the result when taking $m = n = 2001$. For the bending moment M_y^T of unit width, when taking $m = n = 7001$, result has basically converged to exact solution, and has an error of only 1/10000 in comparison with the result when taking $m = n = 7003$. For the convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending result of concrete rectangular thin plate with the one edges clamped and three simply supported is tabulated (see Table A.4).

Engineering application is seen in Sect. 2.3.4.

2.7 Thermal Bending of Rectangular Thin Plate with Two Adjacent Edges Simply Supported and Two Opposite Edges Clamped

2.7.1 Boundary Conditions

In Fig. 2.6, to clamped edges, there is:

$$w|_{y=0} = 0, \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0 \tag{2.54}$$

$$w|_{x=0} = 0, \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \tag{2.55}$$

To simply supported edges, due to the deflection w in the whole edge is zero, by (2.12), there is

$$w|_{x=a} = 0, \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a} = -\frac{M^T}{D} \tag{2.56}$$

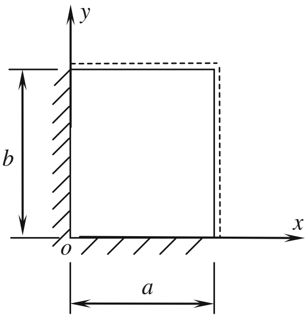


Fig. 2.6 Two adjacent edges clamped and two adjacent edges simply supported

$$w\big|_{y=b} = 0, \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b} = -\frac{M^T}{D} \tag{2.57}$$

2.7.2 Analytical Solution for Thermal Elastic Problems

On edges, w = 0, according to (2.12), (2.54), (2.55), (2.56) and (2.57), there is

$$M_x\big|_{x=a} = 0,\ M_y\big|_{y=b} = 0,\ M_x\big|_{x=0} = -M^T\ M_y\big|_{y=0} = -M^T$$

Now, Rectangular thin plate with two adjacent edges clamped and two adjacent edges simply supported under temperature disparity along thickness is regarded as a superposition of rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under uniform bending moment M_T^- on two adjacent edges.

1. The Solution of the Uniform Bending Moment M_T^- on the Two Adjacent Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^2 \tag{2.58}$$

The slope of every point along x = 0 and y = 0 on bending plane of plate is

$$\begin{cases} \left.\dfrac{\partial w}{\partial y}\right|_{y=0} = \pm\dfrac{\pi}{b} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin\frac{i\pi x}{a} \\ \left.\dfrac{\partial w}{\partial x}\right|_{x=0} = \pm\dfrac{\pi}{a} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin\frac{j\pi y}{b} \end{cases}$$

When A_{ij} increased to $A_{ij} + \delta A_{ij}$, slope increment of every point along $x = 0$ and $y = 0$ on bending plane of plate is

$$\begin{cases} \delta \left.\dfrac{\partial w}{\partial y}\right|_{y=0} = \pm \dfrac{\pi}{b} j \sin \dfrac{i\pi x}{a} \delta A_{ij} \\ \delta \left.\dfrac{\partial w}{\partial x}\right|_{x=0} = \pm \dfrac{\pi}{a} i \sin \dfrac{j\pi y}{b} \delta A_{ij} \end{cases}$$

The work done by bending moment along edges of plate is

$$\int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij} + \int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}x \delta A_{ij}$$

Because

$$\begin{cases} \int_0^a M_T^- \sin \frac{i\pi x}{a} \mathrm{d}x = \frac{2aM^T}{i\pi} \\ \int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}x = \frac{2bM^T}{j\pi} \end{cases}$$

The work done by bending moment is

$$\frac{2ab}{ij} \left(\frac{j^2}{b^2} + \frac{i^2}{a^2} \right) M_T \delta A_{ij}$$

According to (2.58), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 A_{ij} \delta A_{ij}$$

According to the virtual displacement principle, there is

$$A_{ij} = \frac{8}{\pi^4 D} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-1} \frac{M_T^-}{ij}$$

By substituting the above formula into (2.30), hence

$$w(x, y) = \frac{8M_T^-}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-1} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{2.59}$$

Substituting (2.59) into (2.12) (where $M^T = 0$), the internal force calculation formula of the plate with four simply supported edges under the bending moment M_T^- on the two adjacent edges is obtained.

$$\begin{cases} M_x = \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y = \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy} = -(1-\mu)\frac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.60}$$

2. Analytic Solution of Rectangular Thin Plate with Two Adjacent Simply Supported Edge and Two Adjacent Clamped Edges

By superposing (2.34) and (2.59), hence

$$\begin{aligned} w = &-\frac{4a^2M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3\cosh\alpha_m} \cosh\left(\frac{2\alpha_m y}{b} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ &-\frac{M^T}{2D}(x-a)x - \frac{8M^T}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \end{aligned} \tag{2.61}$$

(2.61) is just the deflection formula of rectangular thin plate with two adjacent simply supported edge and two adjacent clamped edges under temperature disparity along thickness.

By superposing (2.35) and (2.61), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad - \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \sin\frac{m\pi x}{a} \\ \qquad -(\mu-1)M^T + \frac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu) \sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\cosh\alpha_m} \cosh\left(\frac{m\pi y}{a} - \alpha_m\right) \cos\frac{m\pi x}{a} \\ \qquad -(\mu-1)\frac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases} \tag{2.62}$$

(2.62) is the internal force calculation formula of rectangular thin plate with two adjacent simply supported edge and two adjacent clamped edges under temperature disparity along thickness.

2.7.3 Results Analysis

To test (2.61) and (2.62) are correct, software MATLAB is used to calculate. The results show that: for deflection function w, when taking $m = n = 69$, the result has converged to exact solution; for the bending moment of M_x^T unit width, when taking $m = n = 5999$, result has converged to exact solution; for the bending moment M_y^T of unit width, when taking $m = n = 5001$, the result has basically converge to the exact solution at this time, and the error is only 1/10000 in comparison with the result when taking $m = n = 4999$. For the convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending results of concrete rectangular plate with two adjacent edges clamped and two adjacent edges simply supported is tabulated (see Table 5).

Engineering application is seen in Sect. 2.3.4.

2.8 Thermal Bending of Rectangular Thin Plate with Two Opposite Edges Simply Supported and Two Opposite Edges Clamped

2.8.1 Boundary Conditions

In Fig. 2.7, to simply supported edges, there is:

$$w\Big|_{\substack{x=0\\x=a}} = 0, \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{2.63}$$

To clamped edges, there is

$$w\Bigg|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}} = 0, \frac{\partial w}{\partial y}\Bigg|_{\substack{y=\frac{b}{2}\\y=-\frac{b}{2}}} = 0 \tag{2.64}$$

2.8.2 Analytical Solution for Thermal Elastic Problems

On edges, $w = 0$, according to (2.12), (2.63) and (2.64), hence

$$M_x\Big|_{\substack{x=0\\x=a}} = 0, M_y\Big|_{\substack{y=0\\y=b}} = -M^T$$

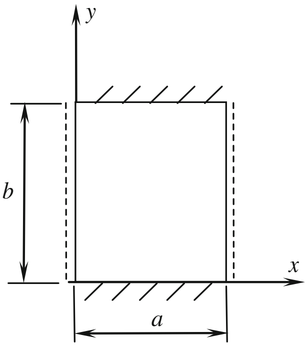


Fig. 2.7 Two opposite edges clamped and two opposite edges simply supported

Now, rectangular thin plate with two opposite edges clamped and two opposite edges simply supported under temperature disparity along thickness is regarded as a superposition of the rectangular thin plate with four simply supported edges under the action of the temperature difference ΔT and the rectangular thin plate with four simply supported edges under the bending moment M_T^- on two opposite edges.

1. The Solution of the Uniform Bending Moment M_T^- on the Two Opposite Edges

The deformation energy of plate is equal to (2.31), namely

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{2.65}$$

The slope of every point along $y = b$ and $y = 0$ on bending plane of plate is

$$\left. \frac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \frac{\pi}{b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} jA_{ij} \sin \frac{i\pi x}{a}$$

When A_{ij} increased to $A_{ij} + \delta A_{ij}$, slope increment of every point along $y = b$ and $y = 0$ on bending plane of plate is

$$\delta \left. \frac{\partial w}{\partial y} \right|_{\substack{y=0 \\ y=b}} = \pm \frac{\pi}{b} j \sin \frac{i\pi x}{a} \delta A_{ij}$$

The work done by moment along edges of plate is

$$2 \int_0^a M_T^- \frac{\pi}{b} j \sin \frac{i\pi x}{a} \mathrm{d}x \delta A_{ij}$$

Because

$$2\int_0^a M_T^- \sin\frac{i\pi x}{a}\mathrm{d}x = \frac{4aM_T^-}{i\pi}$$

The work done by bending moment is

$$\frac{4aM_T^-}{b}\frac{j}{i}\delta A_{ij}$$

According to (2.65), the increment of deformation energy is

$$\delta V = \frac{\pi^4 abD}{4}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^2 A_{ij}\delta A_{ij}$$

According to the virtual displacement principle, hence

$$A_{ij} = \frac{16M_T^-}{\pi^4 b^2 D}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}$$

By substituting the above formula into (2.30), hence

$$w(x,y) = \frac{16M_T^-}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{2.66}$$

By substituting (2.66) into (2.12) (where $M^T = 0$) that is the internal force calculation formula of the plate with four simply supported edges under the bending moment M_T^- on the two opposite edges

$$\begin{cases} M_x = \dfrac{16M_T^-}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y = \dfrac{16M_T^-}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy} = -(1-\mu)\dfrac{16M_T^-}{\pi^2 ab^3}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.67}$$

2. Analytic Solution of Rectangular Thin Plate with Two Opposite Edges Simply Supported and Two Opposite Edges Clamped

By superposing (2.34) and (2.66), hence

$$\begin{aligned} w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ & -\frac{M^T}{2D}(x-a)x-\frac{16M^T}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \end{aligned} \tag{2.68}$$

(2.68) is just the deflection formula of rectangular thin plate with two opposite edges simply supported and two opposite edges clamped under temperature disparity along thickness.

By superposing (2.35) and (2.67), hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \quad -\frac{16M^T}{\pi^2 b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \quad -(\mu-1)M^T-\frac{16M^T}{\pi^2 b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\ \quad -(\mu-1)\frac{16M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases} \tag{2.69}$$

(2.69) is the internal force calculation formula of rectangular thin plate with two opposite edges simply supported and two opposite edges clamped under temperature disparity along thickness.

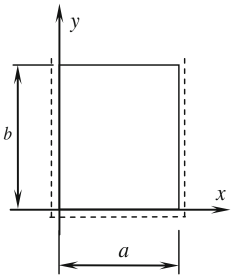
2.8.3 Results Analysis

To test (2.68) and (2.69) are correct, the software MATLAB is used to program the formulas. The results show that: for the deflection fuction w, when taking $m = n = 39$, the result has converged to exact solution; for the bending moment M_x^T of unit width, when taking $m = n = 5999$, result has converged to exact solution; for the bending moment M_y^T of unit width, when taking $m = n = 10{,}001$, the result has basically converge to the exact solution at this time, and the error is only 1/10000 in comparison with the result when taking $m = n = 9999$. For convenience and engineering practical reasons, according to the length-width ratio of the rectangular thin plate, the thermal bending results of concrete rectangular plate with two opposite edges clamped and two opposite edges simply supported is tabulated (see Table A.6).

Engineering application is seen in Sect. 2.3.4.

Chapter 3
Thermal Bending of Concrete Rectangular Thin Plate with Free Boundary

Abstract The deflection equation and the internal force analytical solution of the rectangular thin plate with free boundary (three simply supported edges and one free side, three clamped sides and one free side, two opposite edges clamped and one edge simply supported and one edge free, two adjacent edges clamped and one edge simply supported and one edge free, two opposite edges simply supported and one edge clamped and one edge free, and two adjacent edges simply supported and one edge clamped and one edge free) under temperature difference is systematically introduced in this chapter. In order to facilitate the engineering application, the coefficient calculation table for deflection and internal force based on concrete material is made.



3.1 Introduction

For concrete rectangular thin plate with free boundary under temperature disparity, the existing literature [49, 53] lists the calculation table (Table 3.1) of concrete rectangular thin plate with three edges clamped and one edge free, but the values of transverse bending moment on clamped edges are greater than the solution obtained when $w = 0$. As for other rectangular thin plate with free boundary under temperature disparity, there is no relevant report at present in the existing literatures. Therefore, in this chapter, based on the small deflection thin plate theory and superposition principle, considering temperature variation which is perpendicular to surface, the analytical solution of rectangular thin plate is deduced with three edges

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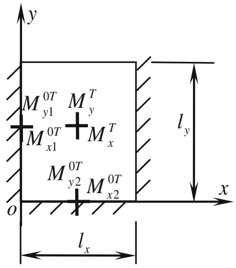
Table 3.1 Bending moment coefficient about the rectangular thin plate with three edges clamped and one edge free under temperature disparity



$$\mu = \frac{1}{6},\ M_x^T = k_x^T \alpha \Delta T E h^2$$
$$M_y^T = k_y^T \alpha \Delta T E h^2$$
$$M_{x1}^{0T} = k_{x1} \alpha \Delta T E h^2$$
$$M_{y1}^{0T} = k_{y1} \alpha \Delta T E h^2$$
$$M_{x2}^{0T} = k_{x2} \alpha \Delta T E h^2$$
$$M_{y2}^{0T} = k_{y2} \alpha \Delta T E h^2$$

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_x	k_y
0.50	0.1018	0.0983	0.0973	0.0975	0.0948	0.0974
0.75	0.1057	0.0980	0.0973	0.1004	0.0925	0.0913
1.00	0.1085	0.0968	0.0974	0.1050	0.0919	0.0851
1.25	0.1072	0.0957	0.0979	0.1085	0.0931	0.0768
1.50	0.1006	0.0965	0.0983	0.1091	0.0951	0.0696
1.75	0.0997	0.0943	0.0975	0.1013	0.0969	0.0633
2.00	0.0981	0.0933	0.0963	0.0957	0.0985	0.0570

simply supported and one edge free, three edges clamped and one edge free, two opposite edges simply supported and one edge clamped and one edge free, two adjacent edges simply supported and one edge clamped and one edge free, two opposite edges clamped and one edge simply supported and one edge free, and two adjacent edges clamped and one edge simply supported and one edge free. Then numerical examples are calculated based on the concrete material and the software MATLAB is used to prove the validity of the solution.

In this chapter, based on the small deflection thin plate theory and superposition principle, considering temperature variation which is perpendicular to surface, the rectangular thin plate with one free edge under temperature disparity is regarded as the superposition of two types of rectangular thin plate, namely, the rectangular thin plate with three simply supported edges and one free edge under the temperature disparity and under the bending moment on different edges. Firstly, by supposing deflection function which has undetermined parameter at free edge, and adopting Levy method, the analytic solution of the rectangular thin plate with three simply supported edges and one free edge under the action of the free boundary deflection function is obtained. Secondly, the analytic solution of the rectangular thin plate with three simply supported edges and one free edge under temperature disparity is obtained. Thirdly, using the solution of rectangular thin plate with four simply supported edges under the bending moment on different edges, the solution of rectangular thin plate with three simply supported edges and one free edge under the bending moment on different edges is obtained. Finally, adopting the superposition principle, the analytical solution of the deflection and bending moment of the rectangular thin plate with one free edge under the transverse temperature variation is acquired, and the calculation coefficient table of the concrete rectangular thin plate with one free edge under transverse temperature disparity is obtained by using MATLAB. Thus it can provide a theoretical basis for the design and calculation of the rectangular thin plate with one free edge under the thermal environment.

3.2 Thermal Bending of the Concrete Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free

3.2.1 Boundary Conditions

As is shown in Fig. 3.1, the boundary conditions are:

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ M_x\Big|_{\substack{x=0\\x=a}} = 0 \tag{3.1}$$

$$w\big|_{y=0} = 0,\ M_y\big|_{y=0} = 0 \tag{3.2}$$

$$M_y\big|_{y=0} = 0,\ M_{xy}\big|_{y=b} = 0,\ F_{Qy}\big|_{y=b} = 0 \tag{3.3}$$

3.2.2 Analytic Solution for the Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Deflection $w_1\big|_{y=b}$

As is shown in Fig. 3.1, let

$$w_1\big|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.4}$$

According to (2.12), the other boundary conditions are:

$$w_1\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial^2 w_1}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = 0 \tag{3.5}$$

$$w_1\big|_{y=0} = 0,\ \frac{\partial^2 w_1}{\partial y^2}\big|_{y=0} = 0 \tag{3.6}$$

$$\frac{\partial^2 w_1}{\partial y^2} + \mu \frac{\partial^2 w_1}{\partial x^2}\Big|_{y=b} = 0 \tag{3.7}$$

According to (2.13), let

$$w_1 = -\sum_{m=1}^{\infty} X_m Y_m \tag{3.8}$$

According to the boundary conditions (3.5), ordering $X_m = \sin\frac{m\pi x}{a}$, so the deflection function w_1 can be written as

$$w_1 = -\sum_{m=1}^{\infty} Y_m \sin\frac{m\pi x}{a} \tag{3.9}$$

Substituting (3.9) into differential equation (2.13), it is obtained as follows

$$Y_m^{(4)} - 2\frac{m^2\pi^2}{a^2} Y_m'' + \frac{m^4\pi^4}{a^4} Y_m = 0$$

The solution of this equation can be written as follows

$$Y_m = A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + C_m \frac{m\pi y}{a} \sinh\frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cosh\frac{m\pi y}{a} \tag{3.10}$$

That is

$$w_1 = \sum_{m=1}^{\infty} \left(A_m \sinh\frac{m\pi y}{a} + B_m \cosh\frac{m\pi y}{a} + C_m \frac{m\pi y}{a} \sinh\frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cosh\frac{m\pi y}{a} \right) \sin\frac{m\pi x}{a} \tag{3.11}$$

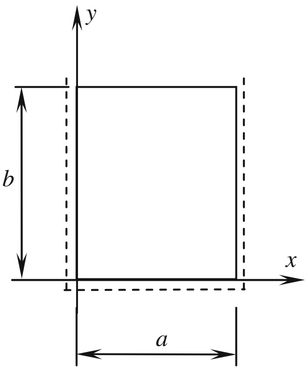


Fig. 3.1 Three supported edges and one free edge

Substituting it into (3.5), the boundary conditions are satisfied naturally, then substituting it into (3.6) yields

$$B_m = C_m = 0$$

Substituting it into (3.4) and (3.7) and letting $\frac{m\pi b}{a} = \beta_m$ there is

$$\begin{cases} A_m \sinh \beta_m + D_m \beta_m \cosh \beta_m = a_m \\ A_m (1-\mu) \sinh \beta_m + D_m [(1-\mu) \beta_m \cosh \beta_m + 2 \sinh \beta_m] = 0 \end{cases} \tag{3.12}$$

Thus

$$\begin{cases} D_m = (\mu - 1) \dfrac{a_m}{2 \sinh \beta_m} \\ A_m = \dfrac{a_m}{2 \sinh \beta_m} [2 + (1-\mu)\beta_m \coth \beta_m] \end{cases}$$

Therefore, the deflection expression of the rectangular thin plate of three simply supported edges and one free edge under the deflection of free edge is

$$w_1 = -\sum_{m=1,3,\ldots}^{\infty} \frac{a_m(1-\mu)}{2\sinh\beta_m} \left[\left(\frac{2}{1-\mu} + \beta_m \coth\beta_m \right) \sinh \frac{m\pi y}{a} - \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a} \tag{3.13}$$

Letting M^T be zero and substituting it into (2.12), there is

$$\begin{cases} M_{x1} = -\dfrac{D}{2a^2} (1-\mu)^2 \pi^2 \displaystyle\sum_{m=1,3,\ldots}^{\infty} \frac{a_m m^2}{\sinh \beta_m} \left[\left(\beta_m \coth \beta_m + 2\frac{1+\mu}{1-\mu} \right) \sinh \frac{m\pi y}{a} - \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a} \\ M_{y1} = -\dfrac{D}{2a^2} (1-\mu)^2 \pi^2 \displaystyle\sum_{m=1,3,\ldots}^{\infty} \frac{a_m m^2}{\sinh \beta_m} \left(\frac{m\pi y}{a} \cosh \frac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a} \end{cases} \tag{3.14}$$

3.2.3 Analytical Solution of the Rectangular Thin Plate With Three Edges Supported and One Edge Free Under the Action of ΔT

For simply supported edges, given that the deflection w_2 is equal to zero along the whole boundary, so according to (2.12), (3.1), (3.2) and (3.3) are turned into

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{3.15}$$

$$w_{y=0} = 0,\ \frac{\partial^2 w}{\partial y^2}\big|_{y=0} = -\frac{M^T}{D} \tag{3.16}$$

$$\begin{cases} \frac{\partial^2 w}{\partial y^2} + \mu\, \frac{\partial^2 w}{\partial x^2}\Big|_{y=b} = 0 \\ \frac{\partial^3 w}{\partial y^3} + (2-\mu)\, \frac{\partial^3 w}{\partial x^2 \partial y}\Big|_{y=b} = 0 \end{cases} \tag{3.17}$$

Assuming that the distribution share force on the free edge is $\overline{F}_{Qy}$, (3.17) is merged into

$$\overline{F}_{Qy}\big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\Bigg|_{y=b} = 0 \tag{3.18}$$

According to (2.23), there are

$$w_2 = -\frac{4a^2 M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3 \cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b} - \alpha_m\right)\sin\frac{m\pi x}{a}\right] - \frac{M^T}{2D}(x-a)x \tag{3.19}$$

$$\begin{cases} M_{x2} = \frac{4M^T}{\pi}(\mu - 1)\sum\limits_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a}\right] \\ M_{y2} = \frac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a}\right] + (\mu - 1)M^T \end{cases} \tag{3.20}$$

Substituting (3.13) into (3.18) yields

$$\overline{F}^1_{Qy}\Big|_{y=b} = -\frac{D\pi^3(1-\mu)^2}{2a^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{a_m m^3}{\sinh^2\beta_m}\left[\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2} + \beta_m\right]\sin\frac{m\pi x}{a}\right\}$$

Substituting (3.19) into (3.18) yields

$$\overline{F}^2_{Qy}\Big|_{y=b} = -\frac{4(3-2\mu)M^T}{\pi b}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{\alpha_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b} - \alpha_m\right)\sin\frac{m\pi x}{a}\right]$$

On the free edge, according to (3.18), hence

$$\overline{F}_{Qy}^{1}\Big|_{y=b}+\overline{F}_{Qy}^{2}\Big|_{y=b}=0$$

That is

$$a_m=-\frac{8a^3(3-2\mu)M^T\ \sinh^2\beta_m}{bD\pi^4(1-\mu)^2}\frac{\frac{\alpha_m}{m^4\cosh\alpha_m}\ \sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\ \frac{\sinh 2\beta_m}{2}+\beta_m}$$

Substituting it into (3.13) and (3.14), it is obtained as follows

$$w_1=\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\ \sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\ \frac{\sinh 2\beta_m}{2}+\beta_m}\left[\left(\frac{2}{1-\mu}+\beta_m\ \coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \tag{3.21}$$

$$\begin{cases} M_{x1}=\dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\dfrac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\ \sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\ \frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\left(\beta_m\ \coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\right\} \\ M_{y1}=\dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left[\dfrac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\ \sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\ \frac{\sinh 2\beta_m}{2}+\beta_m}\times\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\ \coth\beta_m\ \sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a}\right] \end{cases} \tag{3.22}$$

Superimposing (3.19) and (3.21), it is obtained as follows

$$\begin{aligned} w=&-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right]-\frac{M^T}{2D}(x-a)x \\ &+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)} \\ &\times\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\ \sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\ \frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\left(\frac{2}{1-\mu}+\beta_m\ \coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \end{aligned} \tag{3.23}$$

Superimposing (3.20) and (3.22), there are

$$M_x = \frac{4M^T}{\pi}(\mu - 1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b} - \alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2} + \beta_m}\left[\begin{pmatrix}\beta_m\coth\beta_m + \\ 2\dfrac{1+\mu}{1-\mu}\end{pmatrix}\sinh\frac{m\pi y}{a} - \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \tag{3.24}$$

$$M_y = \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} + (\mu - 1)M^T + \frac{2(3-2\mu)M^T}{\pi} \times \sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b} - \alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2} + \beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} - \beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \tag{3.25}$$

3.2.4 Results Analysis

To test the accuracy of (3.23), (3.24) and (3.25), the software MATLAB is used to program the formulas, and the results show that: the deflection w has converged to exact solution when taking $m = n = 9$; for the bending moment M_x of unit width, the result has converged to exact solution when taking $m = n = 17$; for the bending moment M_y of unit width, the result has converged to exact solution when taking $m = n = 1999$, and the error is only 1/10,000 compared with the result when taking $m = n = 1999$, for the convenience and engineering application, according to the length-breadth ratio of the thin plate, the calculated result of the rectangular thin plate with three simply supported edges and one free edge is tabulated (Table A.7).

The engineering application is the same with Sect. 2.3.4.

3.3 Thermal Bending of the Concrete Rectangular Thin Plate with Three Edges Clamped and One Edge Free

3.3.1 Boundary Conditions

As is shown in Fig. 3.2, for clamped edges, the boundary conditions are (Fig. 3.2):

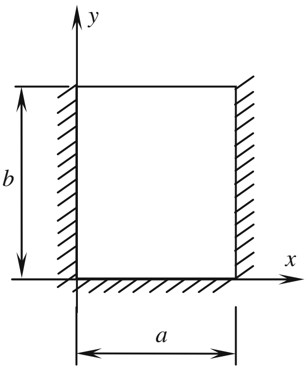


Fig. 3.2 Three clamped edges and one free edge

$$w\Big|_{\substack{x=0\\x=a}} = 0, \frac{\partial w}{\partial x}\Big|_{\substack{x=0\\x=a}} = 0 \tag{3.26}$$

$$w\big|_{y=0} = 0, \frac{\partial w}{\partial y}\big|_{y=0} = 0 \tag{3.27}$$

For the free edge, the bending moment M_y, torque M_{yx}, and transverse shear force F_{Qy} are equal to zero. Assuming that the deflection is expressed by the Sine series on the boundary of $y = b$, so

$$\begin{cases} M_y\big|_{y=b} = 0 \\ M_{yx}\big|_{y=b} = 0 \\ F_{Qy}\big|_{y=b} = 0 \end{cases} \tag{3.28}$$

$$w\big|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.29}$$

Through (2.12), the first part of (3.28) becomes

$$\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\bigg|_{y=b} = -\frac{M^T}{D} \tag{3.30}$$

The distributed shear force on the free boundary is F_{Qy}. Thus, the second and third parts of (3.28) are merged into

$$\overline{F}_{Qy}\big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\bigg|_{y=b} = 0 \tag{3.31}$$

3.3.2 Analytical Solution for the Thermal Elastic Problem

The deflection function that completely satisfies the boundary condition (3.26), (3.27), (3.29), and (3.30) is difficult to be obtained. Thus, the superposition principle is adopted to solve this problem.

Giving that $w = 0$ on the clamped boundary condition, according to the (2.12) (3.26) and (3.27), there is

$$M_x\bigg|_{\substack{x=0\\x=a}} = -M^T,\ M_y\big|_{y=0} = -M^T$$

Thus, the rectangular thin plate with three edges clamped and one edge free under temperature variation that is perpendicular to the surface can be considered as the superposition of two kinds of rectangular thin plate: one with three simply supported edges and one free edge under temperature difference ΔT and another with three simply supported edges and one free edge under the bending moment M_T^- on the three adjacent edges.

1. Analytic Solution of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Action of ΔT

According to (3.23), (3.24) and (3.25), analytical solution of rectangular thin plate with three edges simply supported and one edge free under the action of ΔT is

$$w_1 = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right]-\frac{M^T}{2D}(x-a)x+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}$$
$$\times\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \tag{3.32}$$

$$\begin{cases} M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+\frac{2(3-2\mu)M^T}{\pi} \\ \displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{pmatrix}\beta_m\coth\beta_m\\+2\dfrac{1+\mu}{1-\mu}\end{pmatrix}\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \\ M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T+\frac{2(3-2\mu)M^T}{\pi}\times \\ \displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \end{cases} \tag{3.33}$$

2. Solution of the Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free with the Bending Moment M_T^- on the Three Adjacent Edges

Using the boundary condition (3.29), the rectangular thin plate with three simply supported edges and another free edge under the action of bending moment M_T^- on the three adjacent edges can be considered as the superposition of two kinds of rectangular thin plate: one with four simply supported edges under the bending moment M_T^- on the three adjacent edges, and another with three simply supported edges and one free edge under the deflection $w\big|_{y=b}$ on the free boundary.

According to (2.42) and (2.43), expressions of deflection and bending moment of the rectangular thin plate with four simply supported edges under the bending moment M_T^- on three adjacent edges are as follows:

$$w_2(x,y) = \frac{8M_T^-}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{3.34}$$

$$\begin{cases} M_{x2} = \dfrac{8M_T^-}{\pi^2} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \left\{\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right\} \\ M_{y2} = \dfrac{8M_T^-}{\pi^2} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \left\{\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right\} \end{cases} \tag{3.35}$$

Substituting it into (3.34) and (3.31) yields

$$\overline{F}_{Qy}^3\Big|_{y=b} = -\frac{8M_T^-}{\pi b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \left\{\frac{1}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\left[\frac{j^2}{b^2}+(2-\mu)\frac{i^2}{a^2}\right]\sin\frac{i\pi x}{a}\right\}$$

Letting $a_m = \overline{\overline{a}}_m$ on the free boundary by employing (3.31), it is obtained as follows:

$$\overline{F}_{Qy}^1\Big|_{y=b} + \overline{F}_{Qy}^3\Big|_{y=b} = 0$$

That is

$$\overline{\overline{a}}_m = -\frac{16M_T^- a^3 \sinh^2\beta_m}{\pi^4 Db(1-\mu)^2} \frac{\sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}$$

Substituting it into (3.13) and (3.14), there is

$$w_3 = -\frac{8M^T a^3}{\pi^4 Db(1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \left\{ \frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]\right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m \coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a} \end{array} \right\} \tag{3.36}$$

$$M_{x3} = -\frac{8M^T a}{\pi^2 b} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\beta_m\coth\beta_m + 2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a} \end{array} \right\} \tag{3.37}$$

$$M_{y3} = -\frac{8M^T a}{\pi^2 b} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a} \end{array} \right\} \tag{3.38}$$

Superimposing (3.32), (3.34) and (3.36), there is

$$\begin{aligned} w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\ & +\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\begin{array}{l}\left(\begin{array}{c}\dfrac{2}{1-\mu} \\ +\beta_m\coth\beta_m\end{array}\right)\sinh\dfrac{m\pi y}{a} \\ -\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\} \\ & -\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ & -\frac{8M^T a^3}{\pi^4 Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \end{aligned} \tag{3.39}$$

By superimposing (3.33), (3.35) and (3.37), there is

$$
\begin{aligned}
M_x = {} & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\begin{array}{l}\beta_m\coth\beta_m \\ +2\frac{1+\mu}{1-\mu}\end{array}\right) \\ \times\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
& - \frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\times\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.40}
$$

Superimposing (3.33), (3.35) and (3.38), there is

$$
\begin{aligned}
M_y = {} & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{l}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \\ -\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{array}\right)\sin\frac{m\pi x}{a} \\
& - \frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.41}
$$

3.3.3 Results Analysis

To test the accuracy of (3.39), (3.40) and (3.41), the MATLAB software is used to program the formulas, and the results show that: the deflection w has converged to exact solution when taking $m = n = 9$; for the bending moment M_x of the unit width, the result has converged to exact solution when taking $m = n = 17$; for the bending moment M_y of the unit width, the result has converged to exact solution when taking $m = n = 1999$, and the error is only 1/10,000 compared with the result when taking $m = n = 2001$. For the convenience and engineering utility, according to the length-breadth ratio of the thin plate, the calculated results of the rectangular thin plate with three clamped edges and one free edge are made into the form (Table A.8).

3.3.4 Numerical Example

Example Taking the concrete rectangular thin plate with three edges clamped and one edge simply supported as an example, the calculation process is carried out. The length and width of the plate, l_x and l_y, are both 3.5 m. The thickness of the plate, h, is 100 mm. The temperature difference between the upper and lower surface of the plate, ΔT, is 30 °C. The live load p is 2.0 kN/m^2. The bulk density of concrete is 25 kN/m^3. The concrete strength is 30 MPa. The steel strength is 360 MPa.

Solution According to the literature [45], the linear expansion coefficient of concrete, α, is 1×10^{-5} °C. The Poisson's ratio of concrete, μ, is 1/6. The protective layer thickness of concrete is 10 mm. The elastic modulus of concrete is $E = 3\times10^7$ kN/m^2. The design value of compressive strength for concrete, f_c, is 14.3 N/mm^2. The partial coefficients of the dead load and live load are taken as 1.2 and 1.4 respectively.

$$\text{Dead load: g} = 0.10 \times 25 = 2.5\,\text{kN/m}^2$$

$$\text{Live load:}\, p = 2\text{kN/m}^2$$

$$\text{Design load:}\, q = 1.4p + 1.2g = 5.8\,\text{kN/m}^2$$

According to the initial assumption that the diameter of the steel is 10 mm, the distance from the center of the steel in x direction to the down surface of concrete plate, $c_x = c + 10/2$, is 15 mm and the distance from the center of the steel in y direction to the down surface of concrete plate, $c_y = c + 10 + 10/2$, is 25 mm. The distance from the center of the steel in x direction to the top surface of the concrete plate, $h_{0x} = h - c_x$, is 165 mm and the distance from the center of the steel in y direction to the top surface of the concrete plate, $h_{0y} = h - c_y$, is 155 mm.

1. **Temperature Action**

Taking $E = 3\times10^7$ kN/m^2, $\alpha = 1\times10^{-5}$ °C, $\Delta T = 30$ °C, $h = 100$ mm and $\mu = 1/6$ into (2.1), $D = \frac{Eh^3}{12(1-\mu^2)}$, the following results can be gotten.

$$D = \frac{Eh^3}{12(1-\mu^2)} = \frac{3\times10^7\times0.10^3}{12\left(1-\frac{1}{6^2}\right)} = 2571.43\text{kN}\cdot\text{m}$$

$$M^T = \frac{\alpha\Delta TEh^2}{12(1-\mu)} = \frac{1\times10^{-5}\times30\times3\times10^7\times0.10^2}{12\left(1-\frac{1}{6}\right)} = 9.0\text{kN}$$

From the Table A.8 in the Appendix A, there are

$$f = 0.0777,\ k_x = 0.1893,\ k_y = 0.7676$$

$$w_1 = f\frac{l_x^2M^T}{D} = 0.0777\times\frac{3.5^2\times9}{2571.43} = 0.00333\,\text{m}$$

$$M_x^T = k_xM^T = 0.1893\times9 = 1.704\,\text{kN}\cdot\text{m}$$

$$M_y^T = k_yM^T = 0.7676\times9 = 6.908\,\text{kN}\cdot\text{m}$$

According to the literature [45], $\alpha_1 = 1$, assuming that $h_0 = h_{0x}$, and taking M_x^T, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x^T}{1000\alpha_1f_ch_0^2} = \frac{1.704\times10^6}{1000\times1\times14.3\times165^2} = 0.00438$$

Assuming that $h_0 = h_{0y}$, taking M_y^T, f_c, h_0, and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y^T}{1000\alpha_1f_ch_0^2} = \frac{6.908\times10^6}{1000\times1\times14.3\times155^2} = 0.0201$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases}\gamma_{sx} = 0.5\left(1+\sqrt{1-2\alpha_{sx}}\right) = 0.5\left(1+\sqrt{1-2\times0.00438}\right) = 0.9956\\ \gamma_{sy} = 0.5\left(1+\sqrt{1-2\alpha_{sy}}\right) = 0.5\left(1+\sqrt{1-2\times0.0201}\right) = 0.9898\end{cases}$$

Taking γ_{sx} and γ_{sy} into (2.24.3), there is

$$\begin{cases} A_{sx1} = \dfrac{M_x^T}{f_y \gamma_s h_0} = \dfrac{1.704 \times 10^6}{360 \times 0.9956 \times 165} = 28.8\,\text{mm}^2 \\ A_{sy1} = \dfrac{M_y^T}{f_y \gamma_s h_0} = \dfrac{6.908 \times 10^6}{360 \times 0.9898 \times 155} = 125.1\,\text{mm}^2 \end{cases}$$

2. **Load Action**

From the literature [13,52], $w_2 = f\frac{ql^4}{D}$, $M_x = k_x ql^2$ and $M_y = k_y ql^2$ can be obtained. The value of l is the minimum $[l_x, l_y]$.

According to the literature [13,52], there are

$$f = 0.00189,\ k_x = 0.0304 \text{ and } k_y = 0.0133$$

Taking f, q, l and D into $w_2 = f\frac{ql^4}{D}$, there is

$$w_2 = f\frac{ql^4}{D} = 0.00189 \times \frac{5.8 \times 3.5^4}{2571.43} = 0.00064$$

Taking k_x, k_y into $M_x = k_x ql^2$ and $M_y = k_y ql^2$ respectively, there are

$$M_x = k_x ql^2 = 0.0304 \times 5.8 \times 3.5^2 = 2.16\,\text{kN}\cdot\text{m}$$

$$M_y = k_y ql^2 = 0.0133 \times 5.8 \times 3.5^2 = 0.945\,\text{kN}\cdot\text{m}$$

Assuming that $M_x = M_x^T$ and $h_0 = h_{0x}$, taking M_x, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sx} = \frac{M_x}{1000\alpha_1 f_c h_0^2} = \frac{2.16 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0055$$

Assuming that $M_y = M_y^T$ and $h_0 = h_{0y}$, taking M_y, f_c, h_0 and α_1 into (2.24.1), there is

$$\alpha_{sy} = \frac{M_y}{1000\alpha_1 f_c h_0^2} = \frac{0.945 \times 10^6}{1000 \times 1 \times 14.3 \times 165^2} = 0.0028$$

Taking α_{sx} and α_{sy} into (2.24.2), there is

$$\begin{cases} \gamma_{sx} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sx}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0055}\right) = 0.997 \\ \gamma_{sy} = 0.5\left(1 + \sqrt{1 - 2\alpha_{sy}}\right) = 0.5\left(1 + \sqrt{1 - 2 \times 0.0028}\right) = 0.999 \end{cases}$$

Assuming that $M_x = M_x^T$, $\gamma_s = \gamma_{sx}$ and $h_0 = h_{0x}$, and taking M_x, f_y and γ_s into (2.24.3), there is

$$A_{sx2} = \frac{M_x}{f_y \gamma_s h_0} = \frac{2.16 \times 10^6}{360 \times 0.997 \times 165} = 36.5\,\text{mm}^2$$

Assuming that $M_y = M_y^T$, $\gamma_s = \gamma_{sy}$ and $h_0 = h_{0y}$, and taking M_y, f_y and γ_s into (2.24.3), there is

$$A_{sy2} = \frac{M_x}{f_y \gamma_s h_0} = \frac{0.945 \times 10^6}{360 \times 0.999 \times 155} = 17.0\,\text{mm}^2$$

In summary, the analysis results can be obtained under the action of temperature and load. That is

$$w = w_1 + w_2 = 0.00333 + 0.00064 = 0.004\,\text{m}$$
$$A_{sx} = A_{sx1} + A_{sx2} = 28.8 + 36.5 = 65.3\,\text{mm}^2$$
$$A_{sy} = A_{sy1} + A_{sy2} = 125.1 + 17.0 = 142.1\,\text{mm}^2$$

From the above results, the total deflection at the midspan point of the thin plate is 4 mm. The steel bar area per meter at the midspan point of the plate in the x direction is 65.3 mm^2 and the steel bar area per meter at the midspan point of the plate in the y direction is 142.1 mm^2.

3.4 Thermal Bending of Concrete Rectangular Thin Plate with Two Opposite Edges Clamped and One Edge Simply Supported and One Edge Free

3.4.1 Boundary Conditions

In Fig. 3.3, the boundary condition for the clamped edge is as follows:

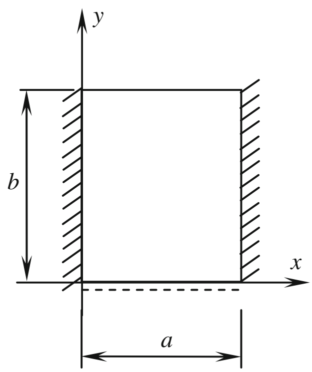


Fig. 3.3 Two opposite edges clamped, one edge simply supported, and one edge free

$$w\Big|_{\substack{x=0\\x=a}} = 0,\ \frac{\partial w}{\partial x}\Big|_{\substack{x=0\\x=a}} = 0 \tag{3.42}$$

$$w|_{y=0} = 0,\ \frac{\partial w}{\partial y}|_{y=0} = 0 \tag{3.43}$$

Giving that the deflection w is equal to zero on the whole boundary condition for the simply supported edge, according to (2.12), the above mentioned equations are as follows:

$$w|_{y=0} = 0,\ \frac{\partial^2 w}{\partial y^2}\Big|_{y=0} = -\frac{M^T}{D} \tag{3.44}$$

For the free edge, the bending moment M_y, torque M_{yz}, and transverse shear force F_{Qy} are equal to zero. Assuming that the deflection is expressed by the sine series on the boundary of $y = b$, M_y, M_{yz}, and F_{Qy} are as follows:

$$\begin{cases} M_y|_{y=b} = 0 \\ M_{yx}|_{y=b} = 0 \\ F_{Qy}|_{y=b} = 0 \end{cases} \tag{3.45}$$

$$w|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.46}$$

With the use of (2.12), the first formula of (3.44) turns into the following:

$$\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\Big|_{y=b} = -\frac{M^T}{D} \tag{3.47}$$

The distributed shear force on the free boundary is $\overline{F}_{Qy}$. Thus, the second and third formulas of (3.45) are merged into the following:

$$\overline{F}_{Qy}\Big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\Bigg|_{y=b} = 0 \tag{3.48}$$

3.4.2 Analytical Solution for the Thermal Elastic Problem

Searching for the deflection function, it is very difficult to be completely satisfied with the boundary conditions of (3.42), (3.43), (3.44), (3.46), (3.47), and (3.48). Thus, the superposition principle is adopted to solve this problem.

Given that $w = 0$ on the clamped boundary condition, according to (2.12) and (3.42), the following is obtained:

$$M_x\Bigg|_{\substack{x=0\\x=a}} = -M^T = M_T^-$$

Thus, the rectangular thin plate with two opposite clamped edges, one edge simply supported and one edge free under temperature variation that is perpendicular to the surface can be viewed as the superposition of two types of rectangular thin plate, namely, three edges simply supported and one edge free under temperature difference ΔT, and under the bending moment M_T^- on the two opposite edges. The other edge have no bending moment.

1. Analytic Solution of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Action of ΔT

According to (3.23), (3.24) and (3.25), analytic solution of rectangular thin plate with three edges simply supported and one edge free under the action of ΔT is

$$\begin{aligned} w_1 = &-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right] \\ &-\frac{M^T}{2D}(x-a)x + \frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)} \\ &\times\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\begin{array}{l}\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\} \end{aligned} \tag{3.49}$$

$$\begin{cases} M_x = \dfrac{4M^T}{\pi}(\mu-1)\sum\limits_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a} \\ \quad + \dfrac{2(3-2\mu)M^T}{\pi}\sum\limits_{m=1,3,\ldots}^{\infty}\left\{\dfrac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{pmatrix}\beta_m\coth\beta_m \\ +2\dfrac{1+\mu}{1-\mu}\end{pmatrix}\sinh\dfrac{m\pi y}{a} \atop -\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\right\} \\ M_y = \dfrac{4M^T}{\pi}(1-\mu)\sum\limits_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T+\dfrac{2(3-2\mu)M^T}{\pi} \\ \quad \times\sum\limits_{m=1,3,\ldots}^{\infty}\dfrac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a} \end{cases} \tag{3.50}$$

2. Solution of the Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free with the Bending Moment M_T^- on the Two Opposite Edges

Based on the boundary condition of (3.46), the rectangular thin plate with three edges simply supported and one edge free under the action of bending moment M_T^- on the two opposite edges can be viewed as the superposition of two types of rectangular thin plate. Namely, the rectangular thin plate with four edges simply supported under the bending moment M_T^- on the two opposite edges and that with three edges simply supported and one edge free under the deflection $w|_{y=b}$ on the free boundary.

According to (2.66) and (2.67), the expressions of deflection and bending moment of the rectangular thin plate with four edges simply supported under the bending moment M_T^- on two opposite edges are as follows:

$$w_2(x,y) = \frac{16M_T^-}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{3.51}$$

$$\begin{cases} M_{x2} = \dfrac{16M_T^-}{\pi^2 b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_{y2} = \dfrac{16M_T^-}{\pi^2 b^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \end{cases} \tag{3.52}$$

Substituting (3.51) into (3.48) obtains the following:

$$\overline{F}_{Qy}^{3}\Big|_{y=b} = -\frac{16M_T^-}{\pi a^2 b}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left\{i\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left[\frac{j^2}{b^2}+(2-\mu)\frac{i^2}{a^2}\right]\sin\frac{i\pi x}{a}\right\}$$

Letting $a_m = \bar{\bar{a}}_m$ on the free boundary by employing (3.48), the following is obtained:

$$\bar{F}^1_{Qy}\Big|_{y=b} + \bar{F}^3_{Qy}\Big|_{y=b} = 0$$

That is

$$\bar{\bar{a}}_m = -\frac{32 M_T^- a \sinh^2 \beta_m}{\pi^4 D b (1-\mu)^2} \frac{\sum\limits_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}$$

Substituting the above mentioned equation into (3.13) and (3.14) obtains the following:

$$w_3 = \frac{16 M^T a}{\pi^4 D b (1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left\{\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]\right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times \left[\left(\dfrac{2}{1-\mu}+\beta_m \coth \beta_m\right)\sinh \dfrac{m\pi y}{a} - \dfrac{m\pi y}{a}\cosh \dfrac{m\pi y}{a}\right] \sin \dfrac{m\pi x}{a} \end{array} \right\} \tag{3.53}$$

$$M_{x3} = \frac{16 M^T}{\pi^2 ab} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times \left[\begin{array}{r} \left(\beta_m \coth \beta_m + 2\dfrac{1+\mu}{1-\mu}\right) \times \sinh \dfrac{m\pi y}{a} - \\ \dfrac{m\pi y}{a}\cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \dfrac{m\pi x}{a} \end{array} \right\} \tag{3.54}$$

$$M_{y3} = \frac{16 M^T}{\pi^2 ab} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times \left(\dfrac{m\pi y}{a}\cosh \dfrac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \dfrac{m\pi y}{a}\right) \sin \dfrac{m\pi x}{a} \end{array} \right\} \tag{3.55}$$

Superimposing (3.49), (3.51) and (3.53) obtains the following:

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\
& +\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \\
& -\frac{16M^T}{\pi^4Da^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& +\frac{16M^Ta}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned} \tag{3.56}
$$

Superimposing (3.50), (3.52), and (3.54) obtains the following:

$$
\begin{aligned}
M_x = & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& +\frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right) \\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
& -\frac{16M^T}{\pi^2a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& +\frac{16M^T}{\pi^2ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned} \tag{3.57}
$$

Superimposing (3.50), (3.52), and (3.55) obtains the following:

$$
\begin{aligned}
M_y = {} & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{matrix}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}- \\ \beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{matrix}\right)\sin\frac{m\pi x}{a} \\
& + \frac{16M^T}{\pi^2 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{16M^T}{\pi^2 ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{matrix}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{matrix}\right\}
\end{aligned}
\tag{3.58}
$$

3.4.3 Results Analysis

To test the accuracy of (3.56), (3.57), and (3.58), the MATLAB software is used to program the formulas, and the results show that when $m = n = 17$ is taken, deflection w has converged to the exact solution. When $m = n = 259$ is taken for the bending moment M_x of unit width, the result has converged to the exact solution. When $m = n = 175$ is taken for the bending moment M_y of unit width, the result basically has converged to the exact solution, and the error is only 1/100,000 in comparison with the result when $m = n = 177$ is taken. For the convenience and engineering application, the calculated results are made into the form according to the length-width ratio of the plate (Table A.9).

The engineering application is the same with Sect. 2.3.4.

According to Table A.9, the relationship of l_x/l_y with k_x, k_y and f is shown in Fig. 3.4.

Figure 3.4 shows that both k_x and f decrease with l_x/l_y, but k_y increases with l_x/l_y, and f decreases slowly. When l_x/l_y changes from 0.5 to 2.0, k_x changes from greater than zero to less than zero, whereas k_y changes from less than zero to greater than zero.

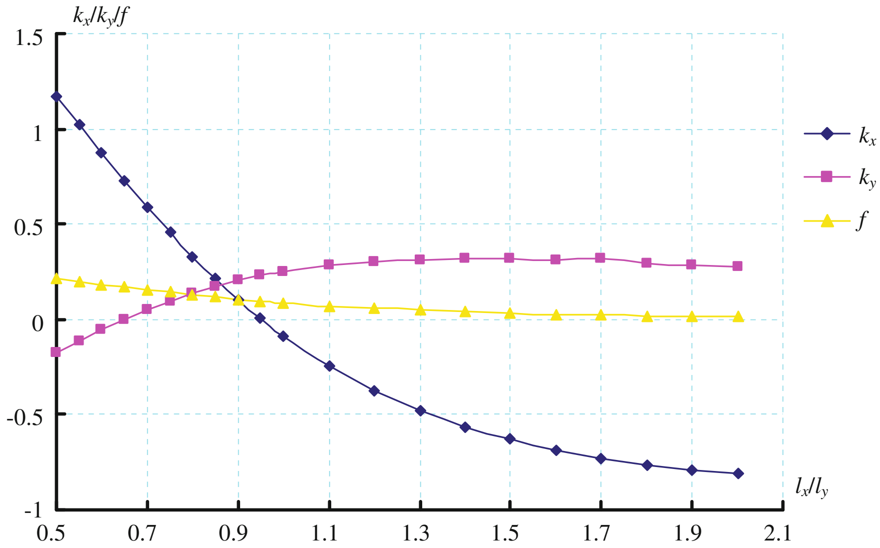


Fig. 3.4 Relationship of l_x/l_y with k_x, k_y and f

3.5 Thermal Bending of Concrete Rectangular Thin Plate with Two Adjacent Edges Clamped and One Edge Simply Supported and One Edge Free

3.5.1 Boundary Conditions

As is shown in Fig. 3.5, for clamped edge, the boundary conditions are:

$$w|_{x=0} = 0, \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \quad (3.59)$$

$$w|_{y=0} = 0, \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0 \quad (3.60)$$

For the simple supported edge, because the deflection w is equal to zero on the whole boundary condition, according to (2.12), there are

$$w|_{x=a} = 0, \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a} = -\frac{M^T}{D} \quad (3.61)$$

For free edge, the bending moment M_y, torque M_{yx} and transverse shear force F_{Qy} are all equal to zero. Assuming that the deflection is expressed by the sine series on the boundary of $y = b$, therefore, M_y, M_{yx} and F_{Qy} are

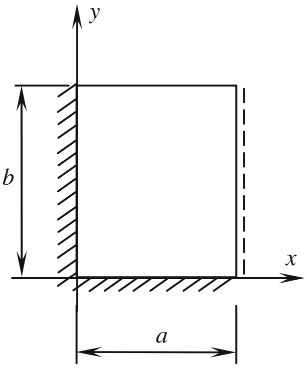


Fig. 3.5 Two adjacent edges clamped and one edge simply supported and one edge free

$$\begin{cases} M_y\big|_{y=b} = 0 \\ M_{yx}\big|_{y=b} = 0 \\ F_{Qy}\big|_{y=b} = 0 \end{cases} \tag{3.62}$$

$$w\big|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.63}$$

By (2.12), the first equation of (3.62) turns into

$$\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\bigg|_{y=b} = -\frac{M^T}{D} \tag{3.64}$$

The distributed shear force on the free boundary is $\overline{F}_{Qy}$, so the second and the third equation of (3.62) are merged into

$$\overline{F}_{Qy}\big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\bigg|_{y=b} = 0 \tag{3.65}$$

3.5.2 Analytical Solution for the Thermal Elastic Problem

As can be seen, it is very difficult to look for the deflection function which is completely satisfied with the boundary conditions (3.59), (3.60), (3.61), (3.63), (3.64), and (3.65), so the superposition principle is adopted to solve this problem.

Due to $w = 0$ on the clamped boundary condition, according to the (2.12), (3.59) and (3.60), there is

$$M_x\big|_{x=0} = -M^T = M_T^-,\ M_y\big|_{y=0} = -M^T = M_T^-$$

So the rectangular thin plate with two adjacent clamped edges and one simply supported edge and one free edge under temperature variation which is perpendicular to surface can be seen as the superposition of two kinds of rectangular thin plate, that are three simple supported edges and one free edge under temperature difference ΔT and three simple supported edges and one free edge under the bending moment M_T^- on the two adjacent edges and the other edges have not bending moment.

1. Analytic Solution of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Action of ΔT

According to (3.23), (3.24) and (3.25), analytic solution of Rectangular Thin Plate with three edges simply supported and one edge free under the action of ΔT

$$\begin{aligned} w_1 = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right] \\ & -\frac{M^T}{2D}(x-a)x + \frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)} \\ & \times\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\begin{array}{l}\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\} \end{aligned} \tag{3.66}$$

$$\left\{\begin{aligned} M_x = & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ & \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\begin{array}{l}\beta_m\coth\beta_m \\ +2\frac{1+\mu}{1-\mu}\end{array}\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\} \\ M_y = & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} + (\mu-1)M^T + \frac{2(3-2\mu)M^T}{\pi} \\ & \times\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \end{aligned}\right. \tag{3.67}$$

2. Analytical Solution of the Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free with the Bending Moment M_T^- on the Two Adjacent Edges

By boundary condition (3.64), the rectangular thin plate with three simply supported edges and one free edge under the action of bending moment M_T^- on the two adjacent edges can be seen as the superposition of two kinds of rectangular thin plate, which are the rectangular thin plate with four simple supported edges under the action of bending moment M_T^- on the two adjacent edges and that of three simply supported edges and one free edge with the deflection $w|_{y=b}$ on the free boundary.

According to (2.59) and (2.60), the expressions of deflection and bending moment of the rectangular thin plate with four edges simply supported under the bending moment M_T^- on two opposite edges are as follows:

$$w_3(x,y) = \frac{8M_T^-}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{3.68}$$

$$\begin{cases} M_{x3} = \dfrac{8M_{\mathrm{T}}^-}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{y3} = \dfrac{8M_{\mathrm{T}}^-}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \end{cases} \tag{3.69}$$

Substituting (3.68) into (3.65), there is

$$\overline{F}_{Qy}^3\Big|_{y=b} = -\frac{8M_T^-}{\pi b}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left\{\frac{1}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\left[\frac{j^2}{b^2}+(2-\mu)\frac{i^2}{a^2}\right]\sin\frac{i\pi x}{a}\right\}$$

Letting $a_m = \overline{\overline{a}}_m$, on the free boundary by (3.65), there is

$$\overline{F}_{Qy}^1\Big|_{y=b} + \overline{F}_{Qy}^3\Big|_{y=b} = 0$$

$$\overline{\overline{a}}_m = -\frac{16M_T^- a^3\sinh^2\beta_m}{\pi^4 Db(1-\mu)^2}\frac{\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}$$

Substituting the above equations into (3.13) and (3.14), there are

$$w_4 = -\frac{8M^T a^3}{\pi^4 Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left\{\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]\right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \tag{3.70}$$

$$M_{x4} = -\frac{8M^T a}{\pi^2 b} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times \left[\begin{array}{r} \left(\beta_m \coth \beta_m + 2\dfrac{1+\mu}{1-\mu}\right) \times \sinh \dfrac{m\pi y}{a} - \\ \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \dfrac{m\pi x}{a} \end{array} \right\} \quad (3.71)$$

$$M_{y4} = -\frac{8M^T a}{\pi^2 b} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times \left(\dfrac{m\pi y}{a}\cosh \dfrac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \dfrac{m\pi y}{a}\right) \sin \dfrac{m\pi x}{a} \end{array} \right\} \quad (3.72)$$

Superimposing (3.66), (3.68) and (3.70), there is

$$\begin{aligned} w = & -\frac{4a^2 M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh \alpha_m} \cosh\left(\frac{2\alpha_m y}{b} - \alpha_m\right) \sin \frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x \\ & + \frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \frac{\frac{\sinh \beta_m}{m^3 \cosh \alpha_m} \sinh\left(\frac{2\alpha_m y}{b} - \alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \times \left[\begin{array}{l} \left(\begin{array}{c} \dfrac{2}{1-\mu} \\ +\beta_m \coth \beta_m \end{array} \right) \sinh \dfrac{m\pi y}{a} \\ -\dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \right\} \\ & + \frac{8M^T}{\pi^4 D} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} - \frac{8M^T a^3}{\pi^4 Db(1-\mu)} \\ & \sum_{m=1,3,\ldots}^{\infty} \left\{ \frac{\sinh \beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \left\{ \frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right] \right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \times \left[\begin{array}{l} \left(\begin{array}{c} \dfrac{2}{1-\mu} \\ +\beta_m \coth \beta_m \end{array} \right) \sinh \dfrac{m\pi y}{a} \\ -\dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \right\} \end{aligned} \quad (3.73)$$

Superimposing (3.67), (3.69) and (3.71), there is

$$
\begin{aligned}
M_x = {} & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\begin{array}{l}\beta_m\coth\beta_m \\ +2\frac{1+\mu}{1-\mu}\end{array}\right)\times \\ \sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
& + \frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\times\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.74}
$$

Superimposing (3.67), (3.69) and (3.72), there is

$$
\begin{aligned}
M_y = {} & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{c}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}- \\ \beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{array}\right)\sin\frac{m\pi x}{a} \\
& + \frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times \\ \left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.75}
$$

3.5.3 Results Analysis

To test the accuracy of (3.73), (3.74) and (3.75), using MATLAB software to program and calculate, the results show that when taking $m = n = 17$, deflection w has converged to exact solution. For the bending moment M_x of unit width, when taking $m = n = 289$ the result has converged to the exact solution. For the bending moment M_y of unit width, when taking $m = n = 289$, the result has been basically converged to the exact solution. For the purposes of convenience and engineering application, according to the length-width ratio of the plate the calculated results are made into the form (Table A.10).

According to Table A.10, the relationship of l_x/l_y with k_x, k_y and f is shown in Fig. 3.6.

Figure 3.6 shows that both k_x and f decrease with l_x/l_y, but k_y increases with l_x/l_y, and f decreases slowly. When l_x/l_y changes from 0.5 to 2.0, k_x changes from greater than zero to less than zero, whereas k_y changes from less than zero to greater than zero. When l_x/l_y is almost equal to 2.0, f becomes less than zero.

The engineering application is the same with Sect. 2.3.4.

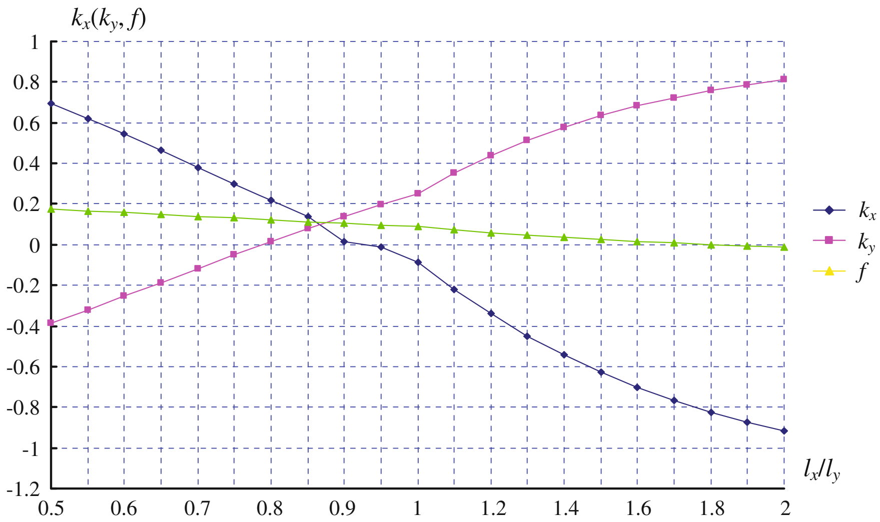


Fig. 3.6 Relationship of l_x/l_y *with* k_x, k_y and f

3.6 Thermal Bending of Concrete Rectangular Thin Plate with Two Opposite Edges Simply Supported and One Edge Clamped and One Edge Free

3.6.1 Boundary Conditions

Figure 3.7 shows that for clamped edges, the boundary conditions are given by

$$w|_{y=0} = 0, \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0 \tag{3.76}$$

For simply supported edges, given that the deflection w is equal to zero on the whole boundary condition and according to (2.12), the above equations become

$$w\Big|_{\substack{x=0\\x=a}} = 0, \left.\frac{\partial^2 w}{\partial x^2}\right|_{\substack{x=0\\x=a}} = -\frac{M^T}{D} \tag{3.77}$$

For free edges, the bending moment M_y, torque M_{yx}, and transverse shear force F_{Qy} are equal to zero. The deflection is assumed to be expressed by the sine series on the boundary of $y = b$. Therefore, M_y, M_{yx}, and F_{Qy} are

$$\begin{cases} M_y|_{y=b} = 0 \\ M_{yx}|_{y=b} = 0 \\ F_{Qy}|_{y=b} = 0 \end{cases} \tag{3.78}$$

$$w|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.79}$$

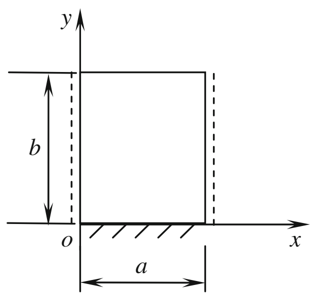


Fig. 3.7 Two opposite edges simply supported and one edge clamped and one edge free

Through (2.12), the first part of (3.78) becomes

$$\left.\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right|_{y=b} = -\frac{M^T}{D} \tag{3.80}$$

The distributed shear force on the free boundary is F_{Qy}. Thus, the second and third parts of (3.78) are merged into

$$\overline{F}_{Qy}\Big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\Bigg|_{y=b} = 0 \tag{3.81}$$

3.6.2 Analytical Solution for the Thermal Elastic Problem

The deflection function that completely satisfies the boundary condition (3.76), (3.77), (3.80), and (3.81) is difficult to be obtained. Thus, the superposition principle is adopted to solve this problem.

Given that $w = 0$ on the clamped boundary condition, according to (2.12) and (3.76), there is

$$M_y\Big|_{y=0} = -M^T = M_T^-$$

Thus, the rectangular thin plate with one edge clamped, two edges simply supported, and one edge free under temperature variation that is perpendicular to the surface can be considered as the superposition of two kinds of rectangular thin plate: another with three simply supported edges and one free edge under temperature difference ΔT and another with three simply supported edges and one free edge under the bending moment M_T^- on the $y = 0$ edge. The other edges do not have a bending moment.

1. Analytic Solution of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Action of ΔT

According to (3.23), (3.24) and (3.25), analytic solution of rectangular thin plate with three edges simply supported and one edge free under the action of ΔT

$$
\begin{aligned}
w_1 = & -\frac{4a^2 M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right] \\
& -\frac{M^T}{2D}(x-a)x + \frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)} \\
& \times \sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\begin{array}{l}\left(\begin{array}{l}\frac{2}{1-\mu}\\ \\ +\beta_m\coth\beta_m\end{array}\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\}
\end{aligned}
\tag{3.82}
$$

$$
\left\{
\begin{aligned}
M_x = & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\begin{array}{l}\beta_m\coth\beta_m\\ \\ +2\frac{1+\mu}{1-\mu}\end{array}\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\right\} \\
M_y = & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} + (\mu-1)M^T + \frac{2(3-2\mu)M^T}{\pi} \\
& \times\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}
\end{aligned}
\right.
\tag{3.83}
$$

2. Solution of the Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free with the Bending Moment M_T^- on the Edge $y = 0$

Using the boundary condition (3.79), the rectangular thin plate with three simply supported edges and one free edge under the action of bending moment M_T^- on the edge $y = 0$ can be considered as the superposition of two kinds of rectangular thin plate: one with four simply supported edges with the bending moment M_T^- on the edge $y = 0$ and another with three simply supported edges and one free edge with the deflection $w\big|_{y=b}$ on the free boundary.

According to (2.50) and (2.51), the expressions of deflection and bending moment of the rectangular thin plate with four edges simply supported under the bending moment M_T^- on the edge $y = 0$ are as follows:

$$
w(x, y) = \frac{8M_T^-}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}
\tag{3.84}
$$

$$\begin{cases} M_x = \dfrac{8M_T^-}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2} \sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_y = \dfrac{8M_T^-}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2} \sin\dfrac{i\pi x}{a}\sin\dfrac{i\pi y}{b} \\ M_{xy} = -(1-\mu)\dfrac{8M_T^-}{\pi^2 ab^3} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2} \cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b} \end{cases} \tag{3.85}$$

Substituting (3.84) with (3.81) yields

$$\overline{F}_{Qy}^3\Big|_{y=b} = -\frac{8M_T^-}{\pi b^3} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \left\{\frac{j^2}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left[\frac{j^2}{b^2}+(2-\mu)\frac{i^2}{a^2}\right] \sin\frac{i\pi x}{a}\right\}$$

Letting $a_m = \overline{\overline{a}}_m$ on the free boundary and using (3.81), we obtain

$$\overline{F}_{Qy}^1\Big|_{y=b} + \overline{F}_{Qy}^3\Big|_{y=b} = 0$$

That is,

$$\overline{\overline{a}}_m = -\frac{16M_T^- a^3 \sinh^2\beta_m}{\pi^4 D b^3 (1-\mu)^2} \frac{\sum\limits_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}$$

Substituting the above equation into (3.82) and (3.83) yields

$$w_4 = -\frac{8M^T a^3}{\pi^4 D b^3 (1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{c} \dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \left\{\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]\right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l} \left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a} \\ -\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a} \end{array}\right]\sin\dfrac{m\pi x}{a} \end{array} \right\} \tag{3.86}$$

$$M_{x4}=-\frac{8M^Ta}{\pi^2b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}\\ -\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \quad (3.87)$$

$$M_{y4}=-\frac{8M^Ta}{\pi^2b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a}\end{array}\right\} \quad (3.88)$$

Superimposing (3.82), (3.84), and (3.86) yields

$$\begin{aligned}w=&-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\\&+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}\\&+\frac{8M^T}{\pi^4Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\\&-\frac{8M^Ta^3}{\pi^4Db^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}\end{aligned} \quad (3.89)$$

Superimposing (3.83), (3.85), and (3.87) produces

$$
\begin{aligned}
M_x = {} & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
& + \frac{8M^T}{\pi^2 b^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.90}
$$

Superimposing (3.83), (3.85), and (3.88) produces

$$
\begin{aligned}
M_y = {} & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{l}\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\\ \beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\end{array}\right)\sin\frac{m\pi x}{a} \\
& + \frac{8M^T}{\pi^2 b^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T a}{\pi^2 b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.91}
$$

3.6.3 Results Analysis

MATLAB software was used to test the accuracy of (3.89), (3.90), and (3.91). The results show that when taking $m = n = 13$, the deflection w has converged to the exact solution. For the bending moment M_x of unit width, when taking $m = n = 175$, the result has converged to the exact solution. For the bending moment M_y of unit width, when taking $m = n = 259$, the result has converged to the exact solution, and

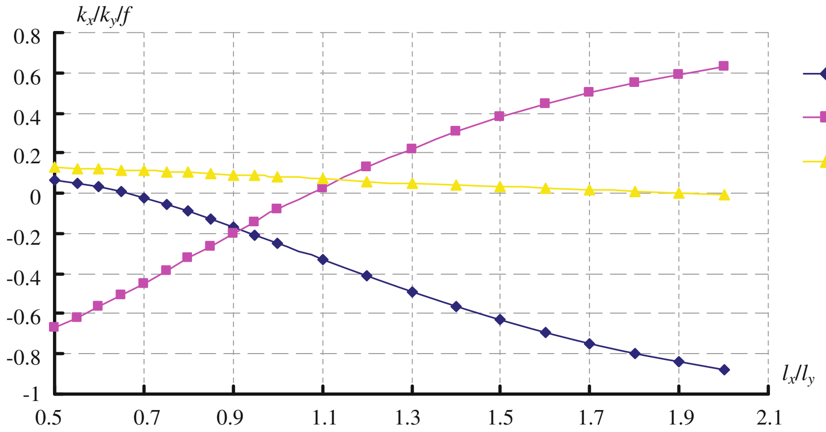


Fig. 3.8 Relationship of l_x/l_y with k_x, k_y and f

the error only is 1/100,000 in comparison with the result when taking $m = n = 261$. For the purpose of convenience and engineering application, the calculated results are given in the form according to the length-width ratio of the plate (Table A.11).

According to Table A.11, the relationship of l_x/l_y with k_x, k_y and f is shown in Fig. 3.8.

Figure 3.8 shows that both k_x and f decrease with l_x/l_y, but k_y increases with l_x/l_y, and f decreases slowly. When l_x/l_y changes from 0.5 to 2.0, k_x changes from greater than zero to less than zero, whereas k_y changes from less than zero to greater than zero. When l_x/l_y is almost equal to 2.0, f becomes less than zero.

The engineering application is the same with Sect. 2.3.4.

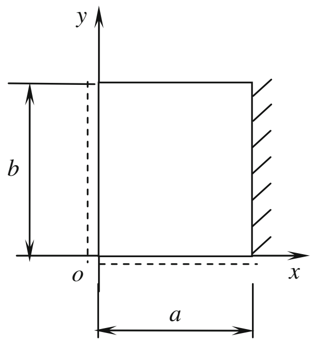
3.7 Thermal Bending of the Concrete Rectangular Thin Plate with Two Adjacent Edges Simply Supported and One Edge Clamped and One Edge Free

3.7.1 Boundary Conditions

Figure 3.9 shows that the boundary condition for the clamped edge is

$$w|_{x=a} = 0, \left.\frac{\partial w}{\partial y}\right|_{x=a} = -\frac{M^T}{D} \tag{3.92}$$

Fig. 3.9 Two adjacent edges simply supported and one edge clamped and one edge free



For the simply supported edge, given that deflection w is equal to zero on the whole boundary condition. According to (2.12), the above equations becomes

$$w\Big|_{\substack{x=0\\y=0}} = 0, \ \frac{\partial^2 w}{\partial x^2}\Big|_{\substack{x=0\\y=0}} = -\frac{M^T}{D} \tag{3.93}$$

For free edges, the bending moment M_y, torque M_{yx}, and transverse shear force F_{Qy} are equal to zero. The deflection is assumed to be expressed by the sine series on the boundary of $y = b$. Therefore, M_y, M_{yx}, and F_{Qy} are

$$\begin{cases} M_y\big|_{y=b} = 0 \\ M_{yx}\big|_{y=b} = 0 \\ F_{Qy}\big|_{y=b} = 0 \end{cases} \tag{3.94}$$

$$w\big|_{y=b} = \sum_{m=1,3,\ldots} a_m \sin\frac{m\pi x}{a} \tag{3.95}$$

Through (2.12), the first part of (3.94) becomes

$$\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\Big|_{y=b} = -\frac{M^T}{D} \tag{3.96}$$

The distributed shear force on the free boundary is F_{Oy}. Thus, the second and third parts of (3.94) are merged into

$$\overline{F}_{Qy}\big|_{y=b} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]\Bigg|_{y=b} = 0 \tag{3.97}$$

3.7.2 Analytical Solution for the Thermal Elastic Problem

The deflection function that completely satisfies the boundary condition (3.92), (3.93), (3.96), and (3.97) is difficult to be obtained. Thus, the superposition principle is adopted to solve this problem.

Given that $w = 0$ on the clamped boundary condition, according to the (2.12) and (3.92), there is

$$M_x|_{x=a} = -M^T = M_T^-$$

Thus, the rectangular thin plate with one edge clamped, two edges simply supported, and one edge free under temperature variation that is perpendicular to the surface can be considered as the superposition of two kinds of rectangular thin plate: another with three simply supported edges and one free edge under temperature difference ΔT and one with three simply supported edges and one free edge under the bending moment M_T^- on the $x = a$ edge. The other rectangular thin plate does not have a bending moment.

1. Analytic Solution of Rectangular Thin Plate with Three Edges Simply Supported and One Edge Free Under the Action of ΔT

According to (3.23), (3.24) and (3.25), analytic solution of rectangular thin plate with three edges simply supported and one edge free under the action of ΔT

$$w_1 = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right] - \frac{M^T}{2D}(x-a)x$$
$$+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\times\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a} - \frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \tag{3.98}$$

$$\begin{cases} M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \quad +\dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \\ M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T+\frac{2(3-2\mu)M^T}{\pi} \\ \quad \times\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \end{cases} \tag{3.99}$$

2. Solution for Rectangular Thin Plate with Three Simply Supported Edges and One Free Edge with Bending Moment M_T^- on the Edge $x = a$

Using the boundary condition (3.95), the rectangular thin plate with three simply supported edges and one free edge under the action of bending moment M_T^- on the edge $x = a$ can be considered as the superposition of two kinds of rectangular thin plate: one with four simply supported edges with the bending moment M_T^- on the edge $x = a$ and another with three simply supported edges and one free edge with the deflection $w\big|_{y=b}$ on the free boundary.

(1) Bending deformation energy of plate Fig. 2.1 shows that if shear force is ignored, the work of bending moment $M_x \mathrm{d}y$ is $-\frac{1}{2}M_x \frac{\partial^2 w}{\partial x^2}\mathrm{d}x\mathrm{d}y$. The work of bending moment $M_y \mathrm{d}y$ is $-\frac{1}{2}M_y \frac{\partial^2 w}{\partial y^2}\mathrm{d}x\mathrm{d}y$. The work of torque $M_{xy}\mathrm{d}y$ is $\frac{1}{2}M_{xy}\frac{\partial^2 w}{\partial y \partial x}\mathrm{d}x\mathrm{d}y$.

In addition, given that the work done by the torque does not affect the work done by the bending moment, the deformation energy of the isolation body is

$$\mathrm{d}V = -\frac{1}{2}\left[M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right]\mathrm{d}x\mathrm{d}y$$

Substituting (2.12) into the above equation and setting $M^T = 0$, it is obtained

$$\mathrm{d}V = \frac{1}{2}D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x \partial y}\right]\right\}$$

For the whole plate, the bending deformation energy is

$$V = \frac{1}{2}D\iint\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x \partial y}\right)\right]dxdy \tag{3.100}$$

(2) Navier solution under uniform bending moment on the $x = a$ edges. In (3.92), with the use of the Green formula, the second equation of the integrand is

$$\begin{aligned}\iint\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x \partial y}\right)\mathrm{d}x\mathrm{d}y &= -\iint\left[\frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)}{\partial x} - \frac{\partial\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right)}{\partial y}\right]\mathrm{d}x\mathrm{d}y \\ &= -\int\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\mathrm{d}x - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\mathrm{d}y\right)\end{aligned} \tag{3.101}$$

Given that (3.101) is integrated along the edge of the thin plate and owing to the rectangular thin plate with four simply supported edges, $\mathrm{d}x = 0$ and $\frac{\partial^2 w}{\partial y^2} = 0$ on the boundary of x = constant. Moreover, $\mathrm{d}y = 0$ and $\frac{\partial w}{\partial x} = 0$ on the boundary of y = constant. Thus, (3.100) can be simplified as

$$V = \frac{1}{2} D \iint \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 \mathrm{d}x\mathrm{d}y \tag{3.102}$$

According to the boundary conditions of the four simply supported edges plate and (2.13), the deflection function is assumed to be

$$w = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{3.103}$$

Substituting the above equation into (3.102), the deformation energy of the plate is

$$V = \frac{\pi^4 abD}{8} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} A_{ij}^2 \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 \tag{3.104}$$

The slopes of the points of the curved surface of plate along $x = a$ are

$$\left. \frac{\partial w}{\partial x} \right|_{x=a} = \pm \frac{\pi}{a} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} iA_{ij} \sin \frac{j\pi y}{b}$$

When A_{ij} increases $A_{ij} + \delta A_{ij}$, the increment of the slopes of each point on the curved surface of the plate along the $x = a$ is given by

$$\delta \left. \frac{\partial w}{\partial x} \right|_{x=a} = \pm \frac{\pi}{a} i \sin \frac{j\pi y}{b} \, \delta A_{ij}$$

The work of the bending moment along the plate boundary is

$$\int_0^b M_T^- \frac{\pi}{a} i \sin \frac{j\pi y}{b} \mathrm{d}x \, \delta A_{ij}$$

Because

$$\int_0^b M_T^- \sin \frac{j\pi y}{b} \mathrm{d}x = \frac{2bM_T^-}{j\pi}$$

The work done by the bending moment is $\frac{2ib}{ja} M_T^- \delta A_{ij}$
Using (3.104), the increment of the deformation energy is

$$\delta V = \frac{\pi^4 abD}{4} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^2 A_{ij} \delta A_{ij}$$

The principle of virtual displacement implies that

$$A_{ij} = \frac{8M_T^-}{\pi^4 Da^2} \frac{i}{j} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2}$$

Substituting the above equation into (3.101) yields

$$w_3(x, y) = \frac{8M_T^-}{\pi^4 Da^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{3.105}$$

Substituting the above equation into (2.12) and letting $M^T = 0$, the internal force calculation formula of rectangular thin plate with four edges simply supported with the bending moment M_T^- on the $x = a$ edge is

$$\begin{cases} M_{x3} = \dfrac{8M_T^-}{\pi^2 a^2} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j} \left(\frac{i^2}{a^2} + \mu \frac{j^2}{b^2} \right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \\ M_{y3} = \dfrac{8M_T^-}{\pi^2 a^2} \displaystyle\sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j} \left(\mu \frac{i^2}{a^2} + \frac{j^2}{b^2} \right) \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \end{cases} \tag{3.106}$$

Substituting (3.105) into (3.97) yields

$$\overline{F}_{Qy}^3 \Big|_{y=b} = -\frac{8M_T^-}{\pi a^2 b} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \left\{ i \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \left[\frac{j^2}{b^2} + (2 - \mu) \frac{i^2}{a^2} \right] \sin \frac{i\pi x}{a} \right\} \tag{3.107}$$

Letting $a_m = \overline{\overline{a}}_m$ and by using (3.97), on the free boundary, there is

$$\overline{F}_{Qy}^1 \Big|_{y=b} + \overline{F}_{Qy}^3 \Big|_{y=b} = 0 \tag{3.108}$$

That is,

$$\overline{\overline{a}}_m = -\frac{16M_T^- a\ \sinh^2\beta_m}{\pi^4 Db(1-\mu)^2}\frac{\sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \tag{3.109}$$

Substituting the above equation into (3.13) and (3.14), there is

$$w_4 = -\frac{8M^T a}{\pi^4 Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left\{\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]\right\}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \tag{3.110}$$

$$M_{x4} = -\frac{8M^T}{\pi^2 ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}\\ -\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \tag{3.111}$$

$$M_{y4} = -\frac{8M^T}{\pi^2 ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\right)\sin\dfrac{m\pi x}{a}\end{array}\right\} \tag{3.112}$$

Superimposing (3.98), (3.106), and (3.110) yields

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\
& +\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\} \\
& +\frac{8M^T}{\pi^4Da^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& -\frac{8M_Ta}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.113}
$$

Superimposing (3.100), (3.106), and (3.111), there is

$$
\begin{aligned}
M_x = & \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
& +\frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right) \\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
& +\frac{8M^T}{\pi^2a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& -\frac{8M^T}{\pi^2ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum\limits_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\tag{3.114}
$$

Superimposing (3.99), (3.106), and (3.112), there is

$$
\begin{aligned}
M_y = {} & \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
& + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{aligned}&\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\\ &-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{aligned}\right)\sin\frac{m\pi x}{a} \\
& + \frac{8M^T}{\pi^2 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^T}{\pi^2 ab}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{aligned}&\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ &\times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{aligned}\right\}
\end{aligned}
\tag{3.115}
$$

3.7.3 Results Analysis

MATLAB software was used to test the accuracy of (3.113), (3.114), and (3.115). The results show that when taking $m = n = 17$, deflection w has converged to the exact solution. For the bending moment M_x of unit width, when taking $m = n = 259$, the result converge has to the exact solution. For the bending moment M_y of unit width, when taking $m = n = 175$, the result has converged to the exact solution, and the error only is 1/100,000 in comparison with the result when taking

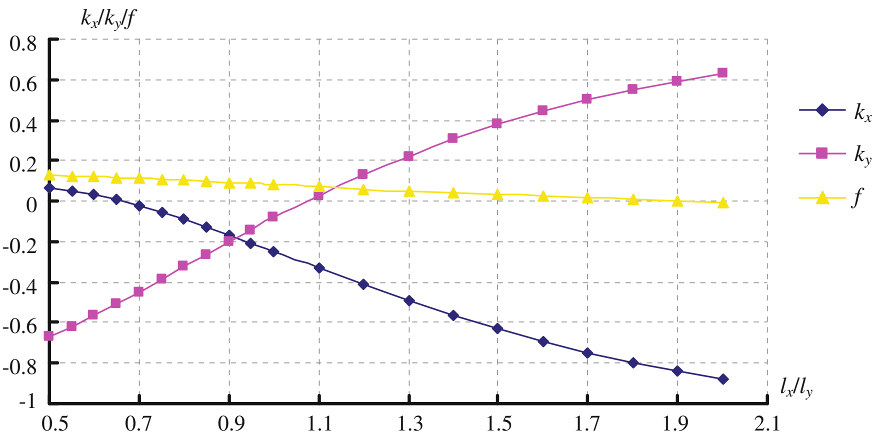


Fig. 3.10 The relationship of l_x/l_y with k_x, k_y and f

$m = n = 177$. For the convenience and engineering application, the calculated results are given in the form according to the length-width ratio of the plate (Table A.12).

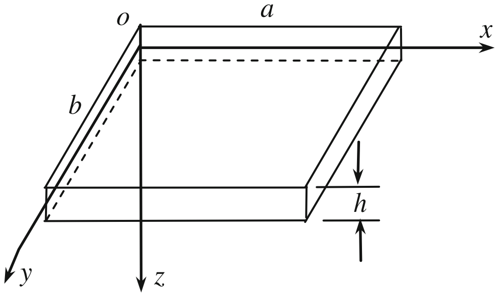
According to Table A.12, the relationship of l_x/l_y with k_x, k_y and f is shown in Fig. 3.10.

Figure 3.10 shows that both k_x and f decrease with l_x/l_y, but k_y increases with l_x/l_y. When l_x/l_y changes from 0.5 to 2.0, k_x changes from greater than zero to less than zero, k_y changes from less than zero to greater than zero, and f is always greater than zero.

The engineering application is the same with Sect. 2.3.4.

Chapter 4
Thermal Buckling of the Concrete Rectangular Thin Plate

Abstract Based on the small deflection theory of thin plate and the nonlinear constitutive equation of concrete, the closed form solutions of the critical buckling temperature variation about concrete rectangular thin plate with four edges simply supported under thermal loading condition are derived in this chapter.



4.1 Introduction

In this chapter, firstly aiming at an arbitrary rectangular thin plate, the equilibrium and stability equations of concrete rectangular plate subjected to thermal loading are derived. The close-form solution of the critical buckling temperature difference for a simply supported reinforced concrete rectangular plate under uniform temperature change is presented. The critical buckling temperature variations of concrete rectangular plates with simply supported boundary conditions in engineering generally used are calculated, and the influences of material parameters, geometric dimension (length-breadth ratio) and relative thickness on the critical buckling temperature variation are discussed. Then, according to the engineering application of the concrete rectangular thin plate structure, the equilibrium and stability equations of the concrete rectangular plate on elastic foundation under thermal loading are derived. The close-form solutions of the critical buckling temperature difference for a simply supported concrete rectangular plate on elastic foundation under

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temperature variation perpendicular to surface and uniform temperature change are presented. Through numerical examples, the influences of material parameters, length-breadth ratio, relative thickness and bending coefficient on the critical buckling temperature variation are discussed.

4.2 Equilibrium and Buckling Equations of Rectangular Thin Plate

4.2.1 Geometric Equation

Now considering a reinforced concrete rectangular thin plate with a size of $a \times b \times h$, the coordinates is taken as shown in Fig. 4.1.

Based on the rigidity plate and the small deflection theories [52], there is

$$u = -z\frac{\partial w}{\partial x}, v = -z\frac{\partial w}{\partial y}$$

where u, v and w represent the displacements along the x, y and z directions respectively.

Hence, the geometric equation of thin plate is

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{4.1}$$

where ε_x,ε_y and γ_{xy} are the components of strain.

4.2.2 Physical Equation

Based on the small elastic-plastic theory of Iliushin [54], the stress-strain relationship of the generally used materials can be expressed as

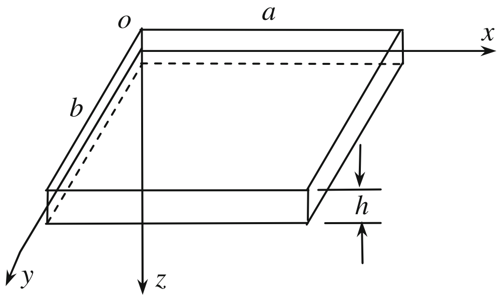


Fig. 4.1 Geometric graphic of rectangular thin plate

$$\sigma_i = \Psi(\varepsilon_i)$$

The above equation can be written in a polynomial form, that is

$$\sigma_i = E_1\varepsilon_i + E_2\varepsilon_i^2 + E_3\varepsilon_i^3 + \ldots$$

Namely

$$\sigma_i = E_1\left(\varepsilon_i + \frac{E_2}{E_1}\varepsilon_i^2 + \frac{E_3}{E_1}\varepsilon_i^3 + \ldots\right) \tag{4.2}$$

where E_1, E_2, E_3 ...are the material constant.

This paper is based on the concrete constitutive model proposed by Hognestad, as is shown in Fig. 4.2.

The rise segment of the model is quadratic parabola, the falling section is the oblique line, that is

$$\begin{cases} \sigma = f_c\left[2\frac{\varepsilon}{\varepsilon_0} - \left(\frac{\varepsilon}{\varepsilon_0}\right)^2\right] & \varepsilon \le \varepsilon_0 \\ \sigma = f_c\left(1 - 0.15\frac{\varepsilon - \varepsilon_0}{\varepsilon_{cu} - \varepsilon_0}\right) & \varepsilon_0 \le \varepsilon \le \varepsilon_{cu} \end{cases} \tag{4.3}$$

where f_c is the ultimate compressive strength of the prism body; ε_0 is the strain corresponding to the ultimate compressive strength of the prism body; ε_{cu} is the ultimate compressive strain.

Due to the cracks develop rapidly in the declining segment *AB* after reaching peak stress of concrete, the whole part of the internal structure is damaged increasingly. Therefore, the rise segment *OA* is only considered. If ordering:

$$E = E_1 = \frac{2f_c}{\varepsilon_0}, B = \frac{E_2}{E_1} = \frac{1}{2\varepsilon_0}$$

where *E* represents the initial elastic modulus of the material; *B* represents another new material constant.

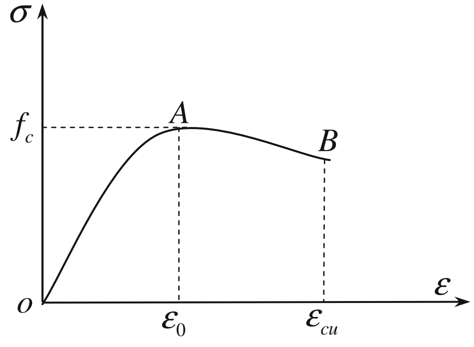


Fig. 4.2 The stress-strain relationship proposed by Hognestad

If combining (4.2) and (4.3), there is

$$\sigma_i = E\left(\varepsilon_i - B\varepsilon_i^2\right) \tag{4.4}$$

The nonlinear elastic constitutive equation of stress-strain under the condition of thin plate is

$$\begin{cases} \sigma_x = \frac{E}{1-\mu^2}\left[\left(\varepsilon_x - B\varepsilon_x^2\right) + \mu\left(\varepsilon_y - B\varepsilon_y^2\right) - (1+\mu)\alpha \mathrm{T}\right] \\ \sigma_y = \frac{E}{1-\mu^2}\left[\left(\varepsilon_y - B\varepsilon_y^2\right) + \mu\left(\varepsilon_x - B\varepsilon_x^2\right) - (1+\mu)\alpha \mathrm{T}\right] \\ \tau_{xy} = \frac{E}{2(1+\mu)}\left(\gamma_{xy} - B\gamma_{xy}^2\right) \end{cases} \tag{4.5}$$

where μ is the Poisson ratio; α is the linear expansion coefficient of the material; T is the temperature of any point in the thin plate, that is $T = T(x, y, z)$.

4.2.3 Equilibrium and Buckling Equations

Due to the thickness of the thin plate is very small compared to the other two dimensions, it can be assumed that there is only the longitudinal stress σ_x, σ_y and τ_{xy} which is parallel to the middle plane and invariable along the thickness [9]. So the internal force in the unit width of the plate can be obtained by integrating the stress along the direction of thickness

$$\begin{cases} N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} \mathrm{d}z \\ M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} z \mathrm{d}z \end{cases} \tag{4.6}$$

Substituting (4.1) and (4.5) into (4.6) yields

$$\begin{cases} N_x = -DB\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \mu\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] - \frac{\Phi}{1-\mu} \\ N_y = -DB\left[\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \mu\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] - \frac{\Phi}{1-\mu} \\ N_{xy} = -DB(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\ M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) - \frac{\varphi}{1-\mu} \\ M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) - \frac{\varphi}{1-\mu} \\ M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} \end{cases} \tag{4.7}$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$; $\Phi = E\alpha\int_{-h/2}^{h/2} T\mathrm{d}z$; $\varphi = E\alpha\int_{-h/2}^{h/2} Tz\mathrm{d}z$.

According to the assumption of small deflection bending problem of thin plate [9], [55–57], the equilibrium equation is

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + q = 0 \end{cases} \tag{4.8}$$

where, q is the distribution load of unit area. In (4.8), the first two equations are independent, therefore, substituting the fourth, fifth, sixth equations of (4.7) into the third equation of (4.8) yields

$$D\nabla^4 w + \frac{1}{1-\mu}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - q = 0 \tag{4.9}$$

where, $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

(4.9) is the stability equilibrium equation of concrete thin plate under the thermal load based on the small deflection theory.

The buckling equation of thin plate is derived by using the critical equilibrium method. w_0 and T_0 as the deflection and temperature of the critical state are set. Both the pre-buckling and post-buckling equilibrium equations are satisfied. In the above equation, a very small increment is given to w and T respectively, that is

$$w \to w_0 + \delta w, T \to T_0 + \delta T$$

By substituting $w = w_0 + \delta w$ and $T = T_0 + \delta T$ into (4.9) and subtracting the original equilibrium equation, then neglecting the high order, the buckling equation is obtained. If marking δw and $\delta \varphi$ as w* and T*, there is

$$D\nabla^4 w^* + \frac{1}{1-\mu}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) - N_{x0} \frac{\partial^2 w^*}{\partial x^2} - N_{y0} \frac{\partial^2 w^*}{\partial y^2} - 2N_{xy0} \frac{\partial^2 w^*}{\partial x \partial y} = 0 \tag{4.10}$$

where N_{x0}, N_{y0} and N_{xy0} are the pre-buckling internal force.

4.3 Thermal Buckling Temperature of Concrete Rectangular Thin Plate

4.3.1 Calculation Parameters

According to the existing literature, the relationship between the compressive strength of concrete when temperature is T and the compressive strength of concrete under normal temperature can be determined by the following equations [15]:

$$\begin{cases} f_c^T = f_c & T \leq 60^\circ\mathrm{C} \\ f_c^T = 0.88f_c - 0.94f_c & 60^\circ\mathrm{C} < T \leq 100^\circ\mathrm{C} \\ f_c^T = 0.95f_c - 1.08f_c & 100^\circ\mathrm{C} < T \leq 300^\circ\mathrm{C} \\ f_c^T = \left[1 + 18\left(\frac{T}{1000}\right)^{5.1}\right]^{-1} f_c & T > 300^\circ\mathrm{C} \end{cases} \tag{4.11}$$

The relationship between the peak strain ε_c^T and the peak train under normal temperature can be determined by the following equation:

$$\varepsilon_0^T = \left[1 + 5\left(\frac{T}{1000}\right)^{1.7}\right]\varepsilon_0 \tag{4.12}$$

Under the action of temperature, the linear expansion coefficient α^T can be determined by the following equation:

$$\alpha^T = 28\left(\frac{T}{1000}\right) \times 10^{-6} \tag{4.13}$$

Due to the high temperature, the cement slurry and aggregate in the confined concrete generate different expansion and contraction, and generate thermal stress and the cracks may appear. The experimental results show that the granite aggregate concrete generates thermal cracks at 550 °C, and the limestone aggregate concrete cracks at 700 °C. For the high strength concrete, it may burst and crack suddenly when $T > 400 \sim 500$ °C. Thus it is considered that the strength damage will occur when the temperature reaches these values, and the buckling calculation is not necessary.

4.3.2 Buckling Critical Temperature

Presuming the initial temperature of each point of the thin plate is the same, the boundary conditions are that the thin plate is clamped in the direction of the in-plane and simply supported in the bending direction. The critical temperature rising value ΔT_{cr} in buckling of the concrete thin plate is evaluated.

Given that the temperature varies uniformly in the plane direction, hence $N_{xy0} = 0$, according to the first two equations of (4.8), N_{x0} and N_{y0} also should be constant, thus applying it to the boundary, there is

$$\begin{cases} N_x|_{x=0,a} = -DB\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{\Phi}{1-\mu} \\ N_y\big|_{y=0,b} = -DB\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - \frac{\Phi}{1-\mu} \\ M_x|_{x=0,a} = -D\frac{\partial^2 w}{\partial x^2} - \frac{\varphi}{1-\mu} = 0 \\ M_y\big|_{y=0,b} = -D\frac{\partial^2 w}{\partial y^2} - \frac{\varphi}{1-\mu} = 0 \\ M_{xy} = 0 \end{cases} \tag{4.14}$$

Substituting the fourth and fifth equations into the first and the second equations produces

$$\begin{cases} N_x|_{x=0,a} = -\frac{B\varphi^2}{D(1-\mu)^2} - \frac{\Phi}{1-\mu} \\ N_y|_{y=0,b} = -\frac{B\varphi^2}{D(1-\mu)^2} - \frac{\Phi}{1-\mu} \end{cases} \tag{4.15}$$

Hence

$$N_{x0} = N_{y0} = -\frac{B\varphi^2}{D(1-\mu)^2} - \frac{\Phi}{1-\mu} \tag{4.16}$$

In addition, due to the uniform variation of the temperature in the plane direction, there is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.10) produces

$$D\nabla^4 w^* + \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) = 0 \tag{4.18}$$

When the temperature varies uniformly in the thin plate, integrating the φ and Φ in (4.7) yields

$$\varphi = 0, \Phi = E\alpha Th \tag{4.19}$$

Given that the edge constraint can increase the rigidity of the structure, for the sake of safety, only the condition that four edges are simply supported is discussed. The constraint equations of the rectangular thin plate with four edges simply supported are

$$\begin{cases} x = 0, \text{side a} : w^* = M_x^* = 0 \\ y = 0, \text{side b} : w^* = M_y^* = 0 \end{cases} \tag{4.20}$$

Assuming that the solution meeting the boundary conditions (4.20), the solution is

$$w^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{4.21}$$

where *m, n* represent the numbers of half wave along the directions of *x* and *y* when the thin plate is buckling; A_{mn} is an arbitrary constant.

Substituting (4.19) into (4.18), there is

$$D\nabla^4 w^* + \frac{Eh\alpha}{1-\mu} T * \left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) = 0 \tag{4.22}$$

Substituting (4.21) into (4.22), there is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 - \frac{Eh\alpha T^*}{1-\mu}\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0$$

In the above equations, A_{mn} can't be all zero, otherwise a trivial solution will be obtained, therefore the numerical value in bracket is required to be zero. That is

$$D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 = \frac{Eh\alpha T^*}{1-\mu}$$

Hence, the critical temperature expression can be written as

$$T^* = \frac{h^2\pi^2}{12(1+\mu)\alpha}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{4.23}$$

Obviously, to obtain the minimum value of T^*, it is necessary to the $m = n = 1$, thus the critical buckling temperature is turned into

$$\Delta T_{cr} = \frac{(h/a)^2\pi^2}{12(1+\mu)\alpha}\left[1 + (a/b)^2\right] \tag{4.24}$$

(4.24) is the buckling critical temperature change value of the reinforced concrete rectangular plate under the uniform temperature variation.

It can be seen that when the relative thickness of thin plate is constant, the critical temperature change decreases with the increase of the length-width ratio. When the length-width ratio is constant, the buckling critical temperature change increases monotonically with the increase of the relative thickness, but it is independent of the initial elastic modulus of the material.

4.3.3 Numerical Examples

If the initial working temperature or the lower temperature side of the thin plate is normal temperature. After calculation, the critical buckling temperature of the concrete rectangular thin plate with four edges simply supported is within 200 °C. For simplicity, the influence of temperature on the linear expansion coefficient is not considered.

According to (4.24) of the critical buckling temperature change, the buckling critical temperature of concrete rectangular thin plate with four simply supported edges is calculated, in which $\alpha = 1 \times 10^{-5}$/°C, $\mu = 1/6$.

In order to make the conclusion of this section can be directly applied to the calculation of structure engineering, take the length-width ratio $b/a = 1.0 \sim 3.0$, then the four cases are calculated when taking $h = a/30$, $h = a/35$, $h = a/40$ and $h = a/45$. Specific results are shown in Table 4.1.

As can be seen in Table 4.1, when taking the thickness of the plate $h = a/30$, the buckling critical temperature ΔT_{cr} varies in $156.6 \sim 87.1$ °C. When taking the thickness of the thin plate $h = a/35$, the critical buckling temperature ΔT_{cr} varies between 115.1 and 63.9 °C. When taking the thickness of the plate $h = a/40$, the buckling critical temperature ΔT_{cr} varies between 88.1 and 48.9 °C. When taking the thickness of the plate $h = a/45$, the buckling critical temperature ΔT_{cr} varies between 69.6 and 38.7 °C. Therefore, for the conventional reinforced concrete rectangular thin plate with four simply supported edges, the buckling critical temperature it can bear varies between 38.7 and 156.6 °C; The thinner the plate is, the smaller the critical temperature is, and the larger the length-width ratio is, the smaller the critical temperature is.

Table 4.1 Influences of b/a and h/a on the critical temperature difference ΔT_{cr}/°C

b/a	h/a	ΔT_{cr}	h/a	ΔT_{cr}	h/a	ΔT_{cr}	h/a	ΔT_{cr}
1.0	1/30	156.6	1/35	115.1	1/40	88.1	1/45	69.6
1.1	1/30	143.1	1/35	105.1	1/40	80.5	1/45	63.6
1.2	1/30	132.7	1/35	96.8	1/40	76.6	1/45	59.0
1.3	1/30	124.7	1/35	91.6	1/40	70.1	1/45	55.4
1.4	1/30	118.3	1/35	86.9	1/40	66.6	1/45	52.6
1.5	1/30	113.1	1/35	83.1	1/40	63.6	1/45	50.3
1.6	1/30	108.9	1/35	80.0	1/40	61.3	1/45	48.4
1.7	1/30	105.4	1/35	77.4	1/40	59.3	1/45	46.8
1.8	1/30	102.5	1/35	75.3	1/40	57.6	1/45	45.5
1.9	1/30	100.0	1/35	73.5	1/40	56.3	1/45	44.4
2.0	1/30	97.9	1/35	71.9	1/40	55.1	1/45	43.5
2.1	1/30	96.1	1/35	70.6	1/40	54.1	1/45	42.7
2.2	1/30	94.5	1/35	69.4	1/40	53.1	1/45	42.0
2.3	1/30	93.1	1/35	68.4	1/40	52.4	1/45	41.4
2.4	1/30	91.9	1/35	67.6	1/40	51.7	1/45	40.9
2.5	1/30	90.9	1/35	66.8	1/40	51.1	1/45	40.4
2.6	1/30	89.9	1/35	66.1	1/40	50.6	1/45	39.9
2.7	1/30	89.1	1/35	65.4	1/40	50.1	1/45	39.6
2.8	1/30	88.3	1/35	64.9	1/40	49.7	1/45	39.3
2.9	1/30	87.6	1/35	64.4	1/40	49.3	1/45	38.9
3.0	1/30	87.1	1/35	63.9	1/40	48.9	1/45	38.7

4.4 Thermal Buckling of Concrete Rectangular Thin Plate on the Elastic Foundation

4.4.1 Equilibrium and Buckling Equations

As is shown in Fig. 4.1, considering a rectangular thin plate on the elastic foundation, based on the classical small deflection theory of the thin plate, the equilibrium equation is

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + q - kw = 0 \end{cases} \tag{4.25}$$

where k is the foundation modulus of the elastic.

Similarly, the first two equations of (4.25) are independent, therefore, substituting the fourth, fifth, sixth equations of (4.7) into the third of Eq. (4.25) yields

$$D\nabla^4 w + \frac{1}{1-\mu}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + kw - q = 0 \tag{4.26}$$

(4.26) is the stability equilibrium equation of reinforced concrete thin plate on the elastic foundation under the thermal load and transverse load based on the small deflection theory.

4.4.2 Thermal Buckling of Thin Plate Under the Uniform Temperature Change

Similarly, with deriving (4.18), the same equation can be derived as

$$D\nabla^4 w^* + \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) + kw^* = 0 \tag{4.27}$$

When the temperature varies uniformly, substituting (4.19) into (4.27) yields

$$D\nabla^4 w^* + \frac{Eh\alpha}{1-\mu}T\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) + kw^* = 0 \tag{4.28}$$

Substituting (4.21) into (4.28) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[D\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \pi^4 - \frac{Eh\alpha T^*}{1-\mu}\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) + k\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

In the above equations, A_{mn} can't be all zero, otherwise a trivial solution will be obtained, therefore the numerical value in parentheses is required to be zero. That is

$$D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4+k-\frac{Eh\alpha T^*}{1-\mu}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2=0$$

Hence, the critical temperature expression can be written as

$$T^*=\frac{h^2\pi^2}{12(1+\mu)\alpha}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)+\frac{k(1-\mu)}{Eh\alpha\pi^2}\frac{a^2b^2}{m^2b^2+n^2a^2} \tag{4.29}$$

If ordering $\lambda=b/a$, $H = h/a$, there is

$$\Delta T=\frac{H^2\pi^2}{12(1+\mu)\alpha}\left(m^2+\frac{n^2}{\lambda^2}\right)+\frac{k(1-\mu)a}{EH\alpha\pi^2}\frac{\lambda^2}{m^2\lambda^2+n^2} \tag{4.30}$$

After calculation, when taking $m=n=1$, the minimum value was obtained. That is [49]

$$\Delta T_{cr}=\frac{H^2\pi^2}{12(1+\mu)\alpha}\left(1+\frac{1}{\lambda^2}\right)+\frac{k(1-\mu)a}{EH\alpha\pi^2}\frac{1}{\left(1+\frac{1}{\lambda^2}\right)} \tag{4.31}$$

(4.31) is the calculation formula of buckling critical temperature changed value of the reinforced concrete rectangular plate on the elastic foundation.

Given that the ground temperature varies within 3.2 m below the surface of the ground between −20 and 35 °C in the urban area [49], and after calculation, the critical buckling temperature is within 200 °C, thus there is no need to consider the influence of temperature on the material constant and linear expansion coefficient.

4.4.3 Numerical Examples

According to (4.31) of the critical buckling temperature variation, the buckling critical temperature of concrete rectangular thin plate with four simply supported edges is calculated. In which $\alpha=1\times10^{-5}/^\circ\mathrm{C}$, $\mu=1/6$, and the initial temperature is set as normal temperature. In order to make the conclusion of this section can be directly applied to the calculation of structure engineering, taking the length-width ratio b/a = 1.0 ~ 3.0, then $h=a/30$, $h=a/35$, $h=a/40$ and $h=a/45$, the concrete strength is 25 ~ 45 MPa.

For different initial elastic constants E, taking the length-width ratio $\lambda=1.0$, the short side a = 3.5 m, the bedding coefficient $k = 1\times10^6$ N/m^3, the relative thickness H = 1/35, the critical buckling temperature change is determined by (4.31). As is shown in Fig. 4.3.

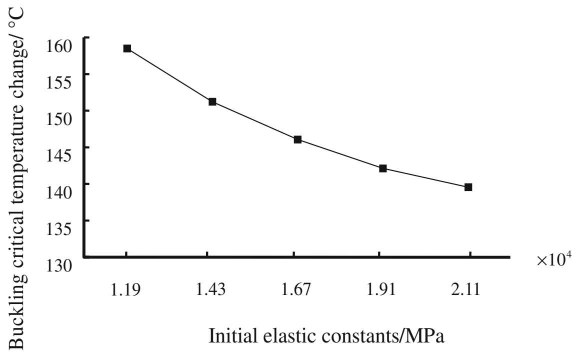


Fig. 4.3 The influence of material constant variation on the critical bucking temperature

It can be seen by Fig. 4.3 that the critical temperature change decreases with the increase of initial elastic constant E. Namely, the higher the concrete strength of thin plate is, the lower the critical temperature variation of the thin plate is. In addition, the effect of temperature on initial elastic constant is also considered when the temperature is higher.

For different the length-width ratio λ, taking short side a = 3.5 m, concrete strength is 40 MPa, bedding coefficient $k = 1 \times 10^6$ N/m^3, the relative thickness H=1/35, the critical buckling temperature change is determined by (4.31). As is shown in Fig. 4.4.

It can be seen Fig. 4.4 that the critical temperature change decreases with the increase of the length-width ratio λ. When $\lambda \geq 2.0$, the buckling critical temperature change value tends to be stable.

For different foundation modulus k, taking short edge a = 3.5 m, concrete strength is 40 MPa, $\lambda = 1.0$, H = 1/35, the critical buckling temperature change is determined by (4.31). As is shown in Fig. 4.5.

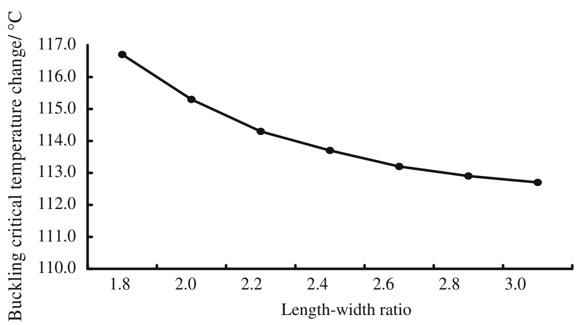


Fig. 4.4 The influence of length-width ratio on the critical bucking temperature

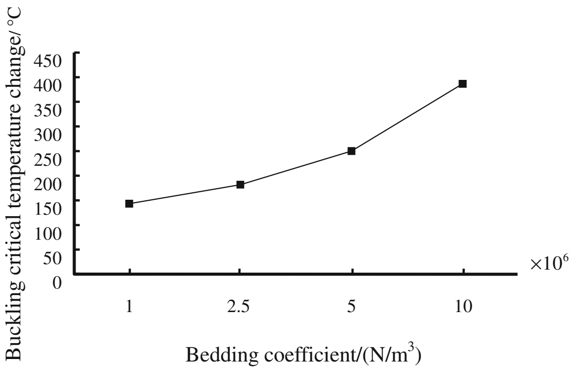


Fig. 4.5 The influence of the bedding coefficient on the critical buckling temperature variation

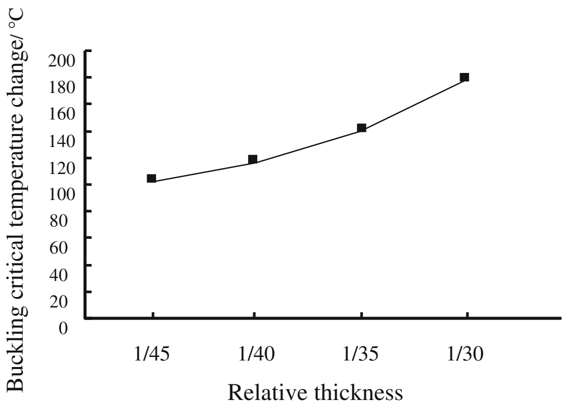


Fig. 4.6 The influence of the relative thickness on the critical buckling temperature variation

It can be seen in Fig. 4.5 that the critical buckling temperature increases with the increase of the bedding coefficient *k*. If the bedding coefficient $k \geq 1.0 \times 10^7$ N/m^3, the critical buckling temperature is close to 400 °C. Thus for conventional concrete strength is 25 ~ 45 MPa, it is only necessary to calculate the buckling critical temperature variation of soft soil, and clay and loam in medium dense soil. There is no need to calculate the buckling critical temperature for other soils. By this time, if the temperature variation is too large, the concrete plate will suddenly burst and crack, which should be paid enough attention to.

For different relative thickness *H,* taking the short side $a = 3.5$ m, $\lambda = 1.0$, the concrete strength is 40 MPa, bedding coefficient $k = 1 \times 10^6$ N/m^3, the critical buckling temperature variation is determined according to (4.31), as shown in Fig. 4.6.

As can be seen in Fig. 4.6, the critical buckling temperature increases with the increase of relative thickness. So for concrete rectangular thin plate with buckling temperature influence, the thickness of plate should be appropriately increased.

4.4.4 Thermal Buckling of Concrete Rectangular Thin Plate on Elastic Foundation in the Case of the Transverse Temperature

According to (4.10), there is

$$D\left(w^*_{,xxxx} + 2w^*_{,xxyy} + w^*_{,yyyy}\right) + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right)\left(w^*_{,xx} + w^*_{,yy}\right) + kw^* = 0 \tag{4.32}$$

The experimental results show that [15], temperature along the thickness direction is a nonlinear variation, but temperature is always transmitted through high temperature surface to low temperature. Hence, for the sake of simplicity, and considering the engineering practice, the temperature in plate is transmitted by the way of uniform change or linear change along the thick [50], that is

$$T(z) = T_u - \frac{T_u - T_d}{h} z, T_u \geq T_d$$

where T_{u} represents the top surface temperature of thin plate; T_{d} represents the below surface temperature of thin plate, that is ground temperature[17].

Substituting $T_{(z)}$ and (4.17) into φ and Φ produces

$$\varphi = \frac{E\alpha(T_u - T_d)h^2}{12}, \Phi = E\alpha T_u h \tag{4.33}$$

The constraint equations of the rectangular thin plate with four edges simply supported are

$$x = 0,\ a : w^* = M^*_x; y = 0,\ b : w^* = M^*_y \tag{4.34}$$

Assuming that the solution meeting the boundary conditions (4.34), there is

$$w^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{4.35}$$

where *m, n* represent the numbers of half wave along the directions of x and y when the thin plate is buckling. A_{mn} is an arbitrary constant.

Substituting (4.34), (4.35) into (4.32), there is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left\{D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4 - \left[\frac{BE\alpha^2h(1+\mu)(T_u-T_d)^2}{12(1-\mu)}+\frac{E\alpha T_u h}{1-\mu}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2+k\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0$$

In the above equations, A_{mn} can't be all zero, otherwise a trivial solution will be obtained, therefore the numerical value in parenthesis is required to be zero. That is

$$D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4-\left[\frac{BE\alpha^2h(1+\mu)(T_u-T_d)^2}{12(1-\mu)}+\frac{Eh\alpha T_u}{1-\mu}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2+k=0 \tag{4.36}$$

Obviously, to obtain the minimum value of T_u, it is necessary to require the m=n=1, thus the critical buckling temperature is turned into

$$\Delta T_{cr} = -\left[\frac{6}{B\alpha(1+\mu)}-T_d\right]+\sqrt{\left[\frac{6}{B\alpha(1+\mu)}-T_d\right]^2-T_d^2+\frac{1}{B\alpha^2(1+\mu)^2}\left[h^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\pi^2+12\frac{1-\mu^2}{Eh}\frac{k}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\pi^2}\right]} \tag{4.37}$$

In China city, ground temperature changes from - 20 °C to 35 °C within 3.2 meters under the ground surface [49], and through the calculation, the critical buckling temperature is within 200 °C, so the influence of temperature on material constant, the linear expansion coefficient and so on wasn't considered [15].

According to (4.37) of the critical buckling temperature change, the buckling critical temperature of concrete rectangular thin plate with four simply supported edges is calculated, in which $\alpha = 1\times10^{-5}$/°C, $\mu = 1/6$.

In order to make the conclusion of this section can be directly applied to the calculation of structure engineering, take the length-width ratio b/a=1.0–3.0, then the four cases are calculated when taking H=h/a=1/30, H=h/a=1/35, H=h/a=1/40, H=h/a=1/45. The concrete strength is 15∼45 MPa.

For different initial elastic constant E, taking the length-width ratio λ=1.0, the short side a= 3.5 m, the bedding coefficient $k = 1\times10^6$ N/m^3, the relative thickness H=1/35, T_d = 0°C, the critical buckling temperature change is determined by (4.37). As is shown in Fig. 4.7.

It can be seen by Fig. 4.7 that the critical temperature change decreases with the increase of initial elastic constant E but the change is very small.

For different the length-width ratio λ, taking short side a = 3.5 m, concrete is 40 MPa, T_d=0°C, bedding coefficient $k = 1\times10^6$ N/m^3, the relative thickness H=1/35, the critical buckling temperature change is determined by (4.37) . As is shown in Fig. 4.8.

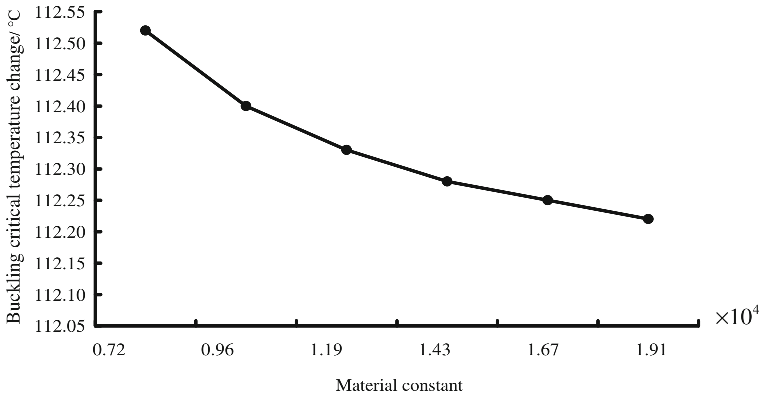


Fig. 4.7 Influences of material constant on the critical temperature difference

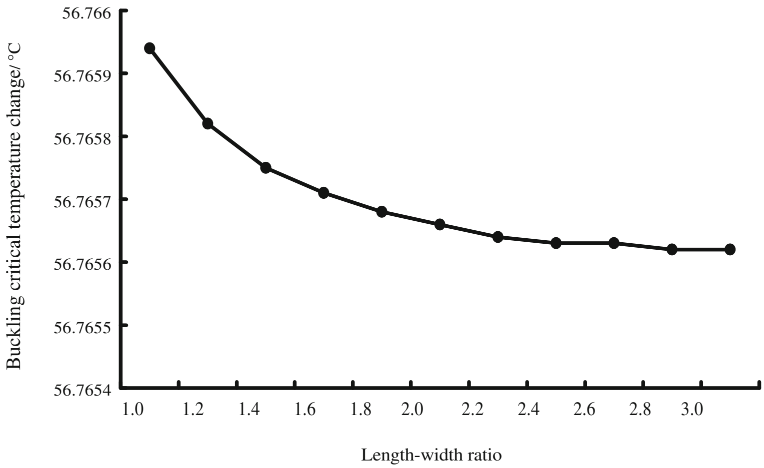


Fig. 4.8 Influences of length-breadth ratio on the variation of critical temperature

It can be seen Fig. 4.8 that the critical temperature change decreases with the increase of the length-width ratio, but change value is nearly equal. Therefore, for the sake of simplicity, length-width ratio can be taken as 1.

For different bedding coefficient k, it can directly be seen by (4.37) that the critical buckling temperature increases with the increases of bedding coefficient k, so concrete rectangular thin plate should be placed on good foundation if the buckling will appear because of temperature, in order to improve the critical buckling temperature values. For the sake of simplicity, the bed coefficient can be taken as is 1×10^{-3}N/mm^3 because the total change is not much.

For different relative thickness H, taking the short side a=3.5 m, λ=1.0, the concrete strength is 40 MPa, bedding coefficient $k = 1\times10^6$ N/m^3, the critical buckling temperature variation is determined according to (4.37), as shown in Fig. 4.9.

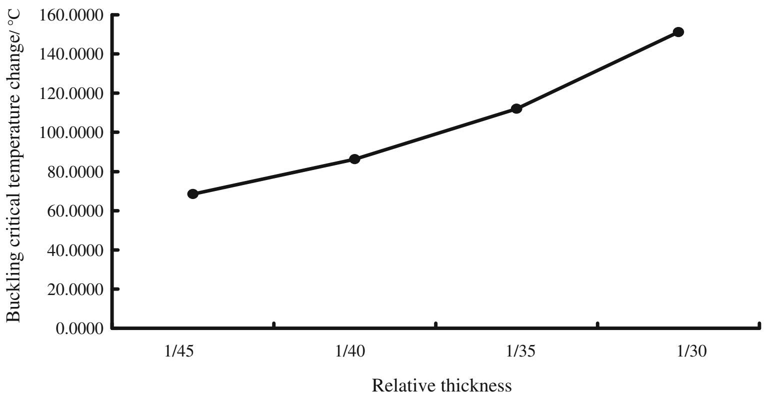


Fig. 4.9 Influences of relative thickness on the variation of critical temperature

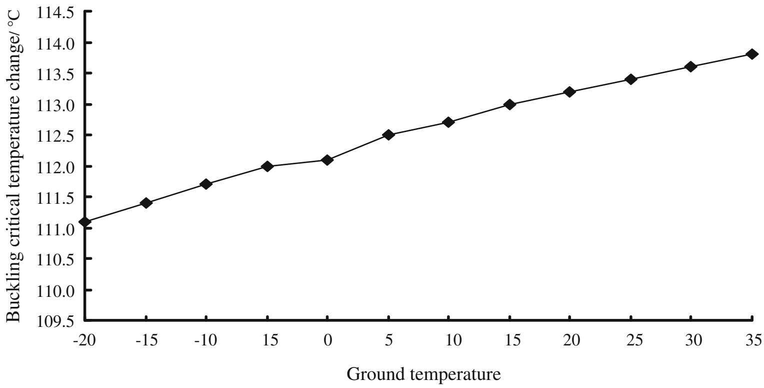


Fig. 4.10 Influences of ground temperature on the variation of critical temperature

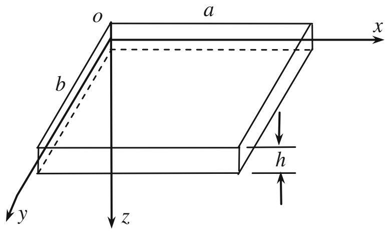
As can be seen in Fig. 4.9, the critical buckling temperature increases with the increase of relative thickness. So it is necessary to increase the thickness of concrete rectangular plate if the buckling will appear because of temperature.

For different ground temperature T_d, taking the short side a=3.5 m, λ=1.0, the concrete strength is 40 MPa, bedding coefficient $k = 1 \times 10^6$ N/m^3, T_d changes from -20°C to 35°C, the critical buckling temperature variation is determined according to (4.37), as shown in Fig. 4.10.

As can be seen in Fig. 4.10, the critical buckling temperature increases with the increase of ground temperature T_d, however, for engineering, its relative change value is not big. Therefore, for the sake of simplicity, ground temperature can be taken as 0 °C.

Chapter 5
Thermal Vibration of Concrete Rectangular Thin Plate

Abstract Based on the small deflection theory of thin plate, the calculation formula of natural frequency and the deflection function under forced vibration of rectangular thin plate with four edges simply supported under thermal loading condition are derived in this chapter.



5.1 Introduction

According to the research status, research reports about the vibration of concrete rectangular thin plate under the action of thermal load have yet been seen in the existing literature. In recent years, the nonlinear vibration of the thin plate with different geometric features has been extensively studied, and its content involves the influence of the geometric non-linearity, material non-linearity and anisotropy, shear deformation and moment of inertia, and deformation under static load. At present, due to the rapid development of science and technology and its wide use in engineering, a lot of research about vibration behavior of heating thin plate has been made. For example, Li analyzed vibration of heating ring plate, in these studies, the research of concrete material is less [46–48]. He et al. analyzed dynamic response of concrete plate under the action of the explosion load, but concrete is assumed to be the ideal rigid-plastic material. These studies laid a solid foundation for the vibration analysis of thin plate. But due to the particularity of concrete material, the current research results cannot be applied well. Therefore, in this book, based on the theory of small deflection, taking the quadratic double parameters model, the

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dynamic equation of thermal elastic problem of concrete rectangular thin plate is derived. Using the Galerkin method and Series method, the natural frequency and the deflection function of forced vibration about concrete rectangular thin plate under the thermal environment are deduced. For the purpose of the convenience of engineering design, the natural frequency under transverse temperature and uniform temperature changes and the deflection function under the action of uniformly distributed load about the concrete rectangular thin plate are given, and the influences of material elastic constants, length-width ratio, relative thickness and temperature on natural frequency and deflection function of concrete thin plate are discussed.

In this chapter, for any rectangular thin plates, the dynamic equation of the reinforced concrete rectangular thin plate is deduced first. And the nonlinear dynamic equation and analytical solution of concrete rectangular thin plate with four edges simply supported under the action of thermal load are given by using the Galerkin principle. Based on the concrete rectangular thin plate structure on elastic foundation, dynamic equation of concrete rectangular thin plate under the thermal environment on elastic foundation is deduced. And the formulas of natural frequency and deflection function of forced vibration about concrete rectangular thin plate with four edges simply supported on elastic foundation under the thermal environment are deduced using the Series method. For the convenience of engineering application, natural frequency and deflection function expression under uniformly distributed load about concrete rectangular thin plate with four edges simply supported under the action of transverse temperature and uniformly temperature change on elastic foundation are given.

5.2 Free Vibration of Rectangular Thin Plate Under Thermal Load

5.2.1 *Basic Equation About the Free Vibration of Rectangular Thin Plate*

As shown in Fig. 4.1, considering a rectangular thin plate, based on the classical small deflection theory of the thin plate, the equilibrium equation is

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \end{cases} \tag{5.1}$$

In (5.1), the first two equations are independent, therefore substituting the fourth, fifth, sixth equations of (4.7) into the third of Eq. (5.1), there is

$$D\nabla^4 w + \frac{1}{1-\mu}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{5.2}$$

(5.2) is a differential equation of free vibration based on the theory of small deflection of concrete thin plate under the action of thermal load.

Substituting (4.16) and (4.17) into (5.2) yields

$$D\nabla^4 w + \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{5.3}$$

For simplicity, considering the case of four edges simply supported, take the displacement mode shape as

$$w(x,y,t) = w^*(x,y)\sin(\omega t + \psi) \tag{5.4}$$

where, $w^*(x,y) = f\ \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$.

Substituting (5.4) into (5.3) yields

$$D\nabla^4 w + \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \omega^2 \rho h w = 0 \tag{5.5}$$

Based on Galerkin principle, there is

$$\iint_s \left[D\nabla^4 w + \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \omega^2 \rho h w\right]\delta w \mathrm{d}s = 0$$

Due to the arbitrary of δf, there is

$$D\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right) - \omega^2 \rho h = 0 \tag{5.6}$$

Through (5.6) yields

$$\omega^2 = \frac{1}{\rho h}\left\{D\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)\right\} \tag{5.7}$$

(5.7) is the basic frequency of the free vibration of a concrete rectangular thin plate with four edges simply supported under the thermal load.

Because the thermal load will make the mechanical properties of steel and concrete change greatly, therefore, in order to analyze the vibration rule of concrete thin plate under the action of temperature, temperature field distribution of thin plate must be determined in advance. The results of experiments show that, although temperature vary is nonlinear along the thickness, the temperature always transmits from higher temperature to the lower temperature [15]. Therefore, for the sake of simplicity and considering the actual project, temperature changes inside the plate are considered as uniformly or linear variation along the thickness [17], namely

$$T(z) = T_b - \frac{T_b - T_c}{h} z, T_b \geq T_c \tag{5.8}$$

where T_b represents the higher temperature of plate surface; T_c represents the lower temperature of plate surface.

Substituting (5.8) into expression of φ and Φ of (4.7), yields

$$\varphi = \frac{E\alpha(T_b - T_c)h^2}{12}, \Phi = E\alpha T_b h \tag{5.9}$$

Substituting (5.9) into (5.7), yields

$$\omega^2 = \frac{E\pi^2}{12\rho(1-\mu)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2\left\{\frac{\pi^2 h^2}{1+\mu}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{\alpha\left[(1+\mu)\alpha(T_b - T_c)^2 + 24\varepsilon_0 T_b\right]}{2\varepsilon_0}\right\} \tag{5.10}$$

When the temperature changes uniformly, substituting (4.19) into (5.7), yields

$$\omega^2 = \frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 - \frac{E\alpha T\pi^2}{\rho(1-\mu)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \tag{5.11}$$

In order to make (5.11) have a wide range of applicability in the engineering structure, letting $\lambda = b/a$ (length-width ratio), $H = h/a$ (relative thickness), then (5.11) becomes

$$\omega = \frac{\pi}{a}\sqrt{\frac{E}{\rho(1-\mu)}\left[\frac{H^2\pi^2}{12(1+\mu)}\left(1+\frac{1}{\lambda^2}\right) - \alpha T\right]\left(1+\frac{1}{\lambda^2}\right)} \tag{5.12}$$

Through (5.12), it can be seen that the natural frequency of thin plate increases with the increase of the initial elastic modulus E, and decreases with the increase of the length-width ratio λ, and the natural frequency of the square plate is the biggest, decreases with the increase of temperature T, and increases with the increase of the relative thickness.

Therefore, the influence of temperature on the natural frequency of thin plate should be fully estimated when calculating the structure, such as temperature change caused by sunshine (especially the temperature variation at day and night), temperature changes formed on the two plate surfaces.

In addition, by (4.24) can be known

$$\Delta T \leq \frac{H^2\pi^2}{12(1+\mu)\alpha}\left[1+\frac{1}{\lambda^2}\right] \tag{5.13}$$

Otherwise, the structure will be buckling failure.

5.2.2 Numerical Examples

According to (5.13), through calculating on the general concrete rectangular thin plates in engineering, the critical buckling temperature changes of concrete rectangular thin plate with the four edges simply supported are within 200 °C, so if the initial temperature is normal temperature or temperature inside the plate is less than 300 °C. For the sake of simplicity, the influence of the temperature on calculation of concrete parameters cannot be considered. As an example, only concrete rectangular thin plate with four edges simply supported is calculated when temperature is 60 °C, the calculated parameters are: a = 3.5 m, concrete strength is 30 MPa (f_c = 1.43×10^7 N/m^2), $\varepsilon_0 = 0.002$, H = 1/30, $\alpha = 1 \times 10^{-5}/°C$, $\mu = 1/6$, $\rho =$ 2500kg/m^3. Substituting the above parameters into (5.12) yields, ω = 103.40 rad/s.

5.3 Forced Vibration of Concrete Rectangular Thin Plate Under Thermal Load

5.3.1 Basic Equation of Forced Vibration of Rectangular Thin Plate

As shown in Fig. 4.1, considering a rectangular thin plate on the elastic foundation, based on the classical small deflection theory of the thin plate, the dynamic equilibrium equation is

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - \\ \rho h \frac{\partial^2 w}{\partial t^2} + F(x,y,t) = 0 \end{cases} \tag{5.14}$$

where, F (x, y, t) is vibration load force on the surface of the concrete plate.

In (5.14), the first two equations are independent, therefore substituting the fourth, fifth, sixth equations of (4.7) into the third of Eq. (5.14) yields

$$D\nabla^4 w - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} - F(x, y, t) = 0 \tag{5.15}$$

Substituting (4.16) and (4.17) into (5.15) yields

$$D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} - F(x, y, t) = 0 \tag{5.16}$$

(5.16) is the dynamic equation of the concrete plate under the thermal environment.

If $F(x, y, t)$ is 0, (5.16) is changed as

$$D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{5.17}$$

(5.17) is the equilibrium differential equation for the free vibration of concrete rectangular thin plate under thermal environment.

The situation of the four edges simply supported is considered, and the displacement pattern is taken as

$$w(x, y, t) = w^*(x, y) \sin(\omega t + \psi) \tag{5.18}$$

where, $w^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$

Submitting (5.18) into (5.17), there is

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left\{ D \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right] \times \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) - \omega^2 \rho h \right\} \times \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0 \tag{5.19}$$

Because the undetermined coefficients C_{mn} is not equal to zero, the quantity in the bracket must be zero, so the natural frequency of concrete rectangular thin plate under the thermal environment is

$$\omega_{mn} = \sqrt{ \frac{Eh^2\pi^4}{12\rho(1-\mu^2)} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \left[\frac{B\varphi^2}{D\rho h(1-\mu)^2} + \frac{\Phi}{\rho h(1-\mu)} \right] \times \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 } \tag{5.20}$$

Setting a simply supported rectangular plate subjected to a harmonic load, that is

$$F(x, y, t) = q(x, y)\sin(\theta t + \psi)$$

where, $q(x, y)$ is the load amplitude in unit area of the thin plate; θ is the frequency of the vibration load, ψ is the initial phase angle.

The equilibrium differential equation for the forced vibration of the rectangular thin plate under the thermal environment is

$$D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \rho h\frac{\partial^2 w}{\partial t^2} = q(x, y)\sin(\theta t + \psi) \tag{5.21}$$

Taking the displacement mode shape as

$$w(x, y, t) = w_0(x, y)\sin(\theta t + \psi) \tag{5.22}$$

where, $w_0(x, y) = \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$; m is the wave number of the half wave of Sine of the thin plate formed in the x direction when vibrating; n is the wave number of the half wave of Sine of the thin plate formed in the y direction when vibrating.

The load is expressed as a double trigonometric series.

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{mx}{a}\sin\frac{n\pi y}{b} \tag{5.23}$$

where

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y \tag{5.24}$$

Substituting (5.22), (5.23), (5.24) into (5.21) there is

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\times\right.\\&\left.\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) - \theta^2\rho h\right\}\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\end{aligned} \tag{5.25}$$

Thus

$$A_{mn} = \frac{q_{mn}}{\left\{D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu}\right]\times\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) - \theta^2\rho h\right\}}$$

The general formula of the deflection (amplitude) is

$$w_0(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left\{ \begin{array}{l} D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2}+\frac{\Phi}{1-\mu}\right]\times \\ \left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right) - \theta^2\rho h \end{array} \right\}} \tag{5.26}$$

Letting T_u represents the temperature value of the thin plate upward surface, and T_d indicates the temperature of the thin plate downward surface of the thin plate, then (5.9) becomes

$$\varphi = \frac{E\alpha(T_u - T_d)h^2}{12}, \Phi = E\alpha T_u h \tag{5.27}$$

Substituting (5.27) into (5.20), yields

$$\omega_{mn} = \sqrt{\begin{array}{l} \frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \left[\frac{BE\alpha^2(1+\mu)(T_u-T_d)^2}{12\rho(1-\mu)}+\frac{E\alpha T_u}{\rho(1-\mu)}\right]\times \\ \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 \end{array}} \tag{5.28}$$

Given that $T_u = T_d$, it becomes the case when the temperature changes.

(5.28) is vibration frequency calculation formula of concrete rectangular thin plate with four edges simply supported in the cases of transverse temperature change and uniform temperature change. For other non-uniform temperature field, as long as the temperature function $T = T(x, y, z)$ is known, based on expressions of Φ, φ and (5.20), vibration frequency formula under the action of any temperature $T\ (x,\ y,\ t)$ can be obtained.

Given that the pressure $q(x, y)$ on the thin plate is uniformly distributed, then by (5.24), there is

$$q_{mn} = \frac{4q}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y = \frac{16q}{mn\,\pi^2} \tag{5.29}$$

where, m and n are odd integer.

Substituting (5.27) and (5.29) into (5.26) yields

$$w_0(x,y) = \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty} \frac{16q\ \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left\{ \begin{array}{l} D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4 - \left[\frac{BE\alpha^2h(1+\mu)(T_u-T_d)^2}{12(1-\mu)} + \right. \\ \left. \frac{E\alpha T_u h}{1-\mu}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 - \theta^2\rho h \end{array} \right\}} \tag{5.30}$$

When $x = \frac{a}{2}$, $y = \frac{b}{2}$, the maximum deflection of the thin plate (amplitude) is

$$w_{\max} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} \frac{16q(-1)^{\frac{m+n}{2}-1}}{mn\left\{ D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \pi^4 - \left[\frac{BE\alpha^2 h(1+\mu)(T_u - T_d)^2}{12(1-\mu)} + \frac{E\alpha T_u h}{1-\mu}\right] \times \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 - \theta^2 \rho h \right\}} \tag{5.31}$$

5.3.2 Numerical Examples

As an example, only the cases of transverse temperature variation and the uniform temperature variation are discussed. By (5.28) there is

$$\omega_{mn} = \sqrt{\frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \left[\frac{BE\alpha^2(1+\mu)(T_u-T_d)^2}{12\rho(1-\mu)} + \frac{E\alpha T_u}{\rho(1-\mu)}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2} \tag{5.32}$$

(5.32) is the vibration natural frequency of concrete rectangular plate with four edges simply supported under transverse temperature change.

It can be seen that the greater the temperature difference is, the smaller the natural frequency is. Therefore, only the case of uniform temperature variation is considered, namely $T_u = T_d$, then (5.32) becomes

$$\omega_{mn} = \sqrt{\frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{E\alpha T\pi^2}{\rho(1-\mu)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \tag{5.33}$$

In order to make the conclusion apply in engineering structure calculation directly, letting a is short side, $\lambda = b/a$ (length-width ratio), $H = h/a$ (relative thickness), and then (5.33) becomes

$$\omega_{mn} = \sqrt{\frac{E\pi^4}{12\rho(1-\mu^2)}H^2\left(m^2+\frac{n^2}{\lambda^2}\right)^2 - \frac{E\alpha T\pi^2}{a^2\rho(1-\mu)}\left(m^2+\frac{n^2}{\lambda^2}\right)} \tag{5.34}$$

It can be seen that the vibration frequency of the thin plate decreases with the increase of the elastic constant E and the relative thickness H, decreases with the increase of T.

For the length and width ratio λ, for the ease of description, let the short side a = 3.5 m. In addition, let $\alpha = 1 \times 10^{-5}/°\text{C}$, $\mu = 1/6$, $\rho = 2500\text{kg/m}^3$, H = 1/30, T = 60 °C, concrete strength is 30 MPa, the first frequency of thin plate is shown as Fig. 5.1.

From Fig. 5.1, it can be seen that the natural frequency of thin plate decreases with the increase of length-width ratio, and the basic frequency tends to be stable when the ratio of length and width is greater than 2.

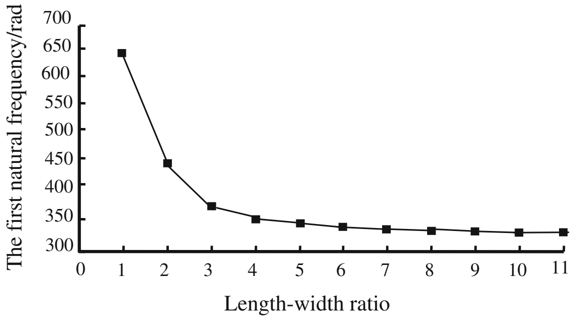

Fig. 5.1 The first frequency varies with length-width ratio

For the forced vibration under the action of uniformly distributed load, by (5.31) it can be seen that when the frequency of the load θ is given, the maximum deflection of thin plate can be determined. In addition, the bigger the temperature difference is, the greater the deflection is.

Through deriving, we know that, as for the forced vibration, when there is a big difference between the loading frequency θ and the natural frequency of the thin plate, arbitrary temperature of deflection (amplitude) can be calculated based on (5.26). As for common rectangular thin plate in engineering, to simplify the calculation, the transverse temperature change and uniform temperature change can be only considered, and the natural vibration frequency and forced vibration deflection (amplitude) can be calculated based on (5.32) and (5.31).

5.4 Thermal Vibration of Concrete Rectangular Thin Plate on Elastic Foundation

5.4.1 Dynamic Equation of Thin Plate

As shown in Fig. 4.1, considering a rectangular thin plate on the elastic foundation, based on the classical small deflection theory of the thin plate, the dynamic equilibrium equation is

$$\begin{cases} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - \\ \rho h \frac{\partial^2 w}{\partial t^2} - kw + F(x, y, t) = 0 \end{cases} \tag{5.35}$$

where, F (x, y, t) is the forced vibration load strength of the concrete thin plate surface.

In (5.35), the first two equations are independent, therefore substituting the fourth, fifth, sixth equations of (4.7) into the third of Eq. (5.35) yields

$$D\nabla^4 w - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} + kw - G_c \nabla^2 w - F(x, y, t) = 0 \tag{5.36}$$

Substituting (4.16) and (4.17) into (5.36) yields

$$D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} + kw - F(x, y, t) = 0 \tag{5.37}$$

(5.37) is the dynamic equation of the thin plate on the elastic foundation.

5.4.2 Vibration Problem of Concrete Rectangular Thin Plate on Elastic Foundation Under Thermal Environment

1. Free Vibration

If $F\ (x,\ y,\ t)$ is zero, then (5.37) becomes

$$D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} + kw = 0 \tag{5.38}$$

(5.38) is the equilibrium differential equation of the free vibration of the rectangular thin plate on the elastic foundation.

Considering the case of four edges simply supported, taking the displacement mode shape as

$$w(x, y, t) = w^*(x, y) \sin(\omega t + \psi) \tag{5.39}$$

where, $w^*(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$.

Substituting (5.39) into (5.38) yields

$$\begin{gathered} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left\{ D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2 - \left[\frac{B\varphi^2}{D(1-\mu)^2} + \frac{\Phi}{1-\mu} \right] \times \right. \\ \left. \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) - \omega^2 \rho h + k \right\} \times \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0 \end{gathered} \tag{5.40}$$

Since the undetermined coefficient C_{mn} is not equal to zero, the value of the bracket must be zero, so the natural vibration frequency of the reinforced concrete rectangular thin plate on the elastic foundation can be gotten.

$$\omega_{mn} = \sqrt{\frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \left[\frac{B\varphi^2}{D\rho h(1-\mu)^2}+\frac{\Phi}{\rho h(1-\mu)}\right]\times \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 + \frac{k}{\rho h}} \tag{5.41}$$

2. Forced Vibration

Considering a harmonic load on a rectangular thin plate with four edges simply supported, there is

$$F(x,y,t) = q(x,y)\sin(\theta t+\psi)$$

where, $q(x$, y) is the load amplitude on unit area of the thin plate; θ is the frequency of the vibration load; ψ is the initial phase angle.

The equilibrium differential equation of the forced vibration of the rectangular thin plate on the elastic foundation under the thermal environment is

$$\begin{aligned} &D\nabla^4 w + \left(\frac{B\varphi^2}{D(1-\mu)^2}+\frac{\Phi}{1-\mu}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) + \rho h\frac{\partial^2 w}{\partial t^2} + kw \\ &= q(x,y)\sin(\theta t+\psi) \end{aligned} \tag{5.42}$$

Take the displacement mode shape as

$$w(x,y,t) = w_0(x,y)\sin(\theta t+\psi) \tag{5.43}$$

where, $w_0(x,y) = \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$; m is the wave number of the half wave of Sine of the thin plate formed in the x direction when vibrating, n is the wave number of the half wave of Sine of the thin plate formed in the y direction when vibrating.

The load is expressed as a double trigonometric series $q(x,y)$

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5.44}$$

where

$$q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\mathrm{d}x\mathrm{d}y \tag{5.45}$$

Substituting (5.43), (5.44) and (5.45) into (5.42) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2-\left[\frac{B\varphi^2}{D(1-\mu)^2}+\frac{\Phi}{1-\mu}\right]\right.$$
$$\left.\times\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)+k-\theta^2\rho h\right\}\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5.46}$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

That is

$$A_{mn}=\frac{q_{mn}}{\left\{D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2-\left[\frac{B\varphi^2}{D(1-\mu)^2}+\frac{\Phi}{1-\mu}\right]\times\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)+k-\theta^2\rho h\right\}}$$

The general formula of the deflection (amplitude) is

$$w_0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left\{\begin{array}{l}D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2-\left[\frac{B\varphi^2}{D(1-\mu)^2}+\frac{\Phi}{1-\mu}\right]\times\\ \left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)+k-\theta^2\rho h\end{array}\right\}} \tag{5.47}$$

5.4.3 Forced Vibration of Concrete Rectangular Thin Plate on Elastic Foundation Under the Action of Geothermal

Let T_u represents the temperature value of the thin plate upward surface, and T_d indicates the temperature of the downward surface of the thin plate [50], then (5.9) becomes

$$\varphi=\frac{E\alpha(T_u-T_d)h^2}{12},\quad \Phi=E\alpha T_u h \tag{5.48}$$

Substituting (5.48) into (5.41) yields

$$\omega_{mn}=\sqrt{\begin{array}{l}\frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2-\left[\frac{BE\alpha^2(1+\mu)(T_u-T_d)^2}{12\rho(1-\mu)}+\right.\\ \left.\frac{E\alpha T_u}{\rho(1-\mu)}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2+\frac{k}{\rho h}\end{array}} \tag{5.49}$$

If $T_u=T_d$, it becomes the case when the temperature changes uniformly.

(5.49) is vibration frequency calculation formula of concrete rectangular thin plate with four edges simply supported in the cases of transverse temperature change and uniform temperature change on elastic foundation. For other non-uniform temperature field, as long as the temperature function $T = T(x, y, z)$ is known, based on expressions of Φ, φ and (5.20), vibration frequency formula under the action of arbitrariness temperature $T\ (x,\ y,\ t)$ can be obtained.

For the concrete thin plate, the pressure $q(x, y)$ is uniform load, through (5.24) there is

$$q_{mn} = \frac{4q}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y = \frac{16q}{mn\,\pi^2} \tag{5.50}$$

where m and n are odd.

Substituting (5.48) and (5.50) into (5.47) yields

$$w_0(x,y) = \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty}\frac{16q\ \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left\{\begin{array}{l} D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4 - \left[\frac{BE\alpha^2h(1+\mu)(T_u-T_d)^2}{12(1-\mu)} + \right. \\ \left.\frac{E\alpha T_u h}{1-\mu}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2 \\ +k-\theta^2\rho h \end{array}\right\}} \tag{5.51}$$

When $x = \frac{a}{2}$, $y = \frac{b}{2}$, the maximum deflection of the thin plate (amplitude) is

$$w_{\max} = \sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3\ldots}^{\infty}\frac{16q(-1)^{\frac{m+n}{2}-1}}{mn\left\{\begin{array}{l} D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2\pi^4 - \left[\frac{BE\alpha^2h(1+\mu)(T_u-T_d)^2}{12(1-\mu)}\right. \\ \left. +\frac{E\alpha T_u h}{1-\mu}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2+k-\theta^2\rho h \end{array}\right\}} \tag{5.52}$$

5.4.4 *Numerical Examples*

As an example, only the cases of transverse temperature variation and uniform temperature variation about the rectangular thin plate on the Winkler elastic foundation are discussed.

$$\omega_{mn} = \sqrt{\begin{array}{l} \frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \left[\frac{BE\alpha^2(1+\mu)(T_u-T_d)^2}{12\rho(1-\mu)}\right. \\ \left.+\frac{E\alpha T_u}{\rho(1-\mu)}\right]\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\pi^2+\frac{k}{\rho h} \end{array}} \tag{5.53}$$

(5.53) is the natural frequency of free vibration under transverse temperature variation on Winkler elastic foundation of concrete rectangular thin plate with four edges simply supported.

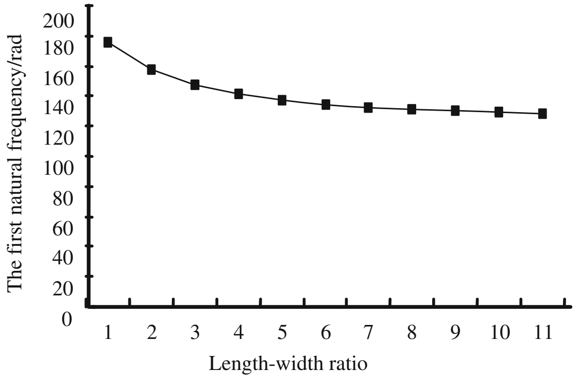

Fig. 5.2 The first frequency varies with length and width

It can be seen that the greater the k is, the greater the natural frequency is. That is, the harder the foundation is, the greater the natural frequency is. Therefore, as a numerical example, we only consider that k is constant; the greater the temperature difference is, the smaller the natural frequency is, thus, only we consider the condition of uniform temperature variation, namely $T_u = T_d$, then (5.53) becomes

$$\omega_{mn} = \sqrt{\frac{Eh^2\pi^4}{12\rho(1-\mu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{E\alpha T\pi^2}{\rho(1-\mu)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)+\frac{k}{\rho h}} \tag{5.54}$$

In order to make the conclusion apply to the engineering structure calculation directly, let a is short side, (length width ratio), $H = h/a$ (relative thickness), then (5.54) becomes

$$\omega_{mn} = \sqrt{\frac{E\pi^4}{12\rho(1-\mu^2)}H^2\left(m^2+\frac{n^2}{\lambda^2}\right)^2 - \frac{E\alpha T\pi^2}{a^2\rho(1-\mu)}\left(m^2+\frac{n^2}{\lambda^2}\right)+\frac{k}{\rho h}} \tag{5.55}$$

It can be seen that the natural frequency of thin plate increases with the increase of the elastic constant E and the relative thickness H, and decreases with the increase of temperature T.

As for the length width ratio λ, for the ease of description, let the short side a = 3.5 m, $k = 1\times10^6$ N/m^3 (wet soft clay). In addition, let $\alpha = 1 \times 10^{-5}$/°C, μ = 1/6, $\rho = 2500\text{kg/m}^3$, H = 1/30, T = 60 °C, concrete strength is 30 MPa, the first frequency of thin plate is shown as Fig. 5.2.

Through Fig. 5.2 it can be seen that the natural frequency of thin plate decreases with the increase of length-width ratio, and the basic frequency tends to be stable when the ratio of length to width is greater than 2.

For the forced vibration of uniform distribute load, through (5.52) we can see that the maximum deflection of the plate is determined when the load frequency θ is known, the maximum deflection of the plate can be determined. In addition, the bigger the temperature difference, the greater the deflection.

Appendix A
Thermal Bending Calculation Coefficient Tables

See Tables A.1, A.2, A.3, A.4, A.5, A.6, A.7, A.8, A.9, A.10, A.11 and A.12.

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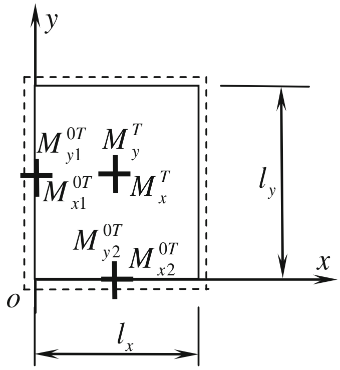
Table A.1 Thermal bending calculation coefficient of four edges simply supported under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

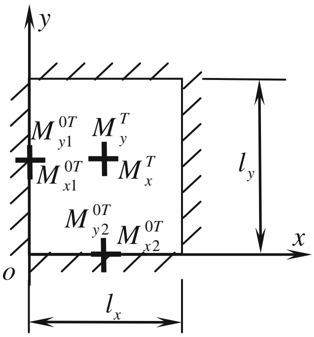
$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f \frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_x	k_y	f
0.50	0.0000	0.8333	0.8333	0.0000	0.0915	0.7419	0.1139
0.55	0.0000	0.8333	0.8333	0.0000	0.1215	0.7118	0.1102
0.60	0.0000	0.8333	0.8333	0.0000	0.1537	0.6796	0.1063
0.65	0.0000	0.8333	0.8333	0.0000	0.1874	0.6460	0.1022
0.70	0.0000	0.8333	0.8333	0.0000	0.2216	0.6117	0.0980
0.75	0.0000	0.8333	0.8333	0.0000	0.2561	0.5772	0.0937
0.80	0.0000	0.8333	0.8333	0.0000	0.2902	0.5431	0.0895
0.85	0.0000	0.8333	0.8333	0.0000	0.3235	0.5098	0.0854
0.90	0.0000	0.8333	0.8333	0.0000	0.3559	0.4775	0.0813
0.95	0.0000	0.8333	0.8333	0.0000	0.3870	0.4464	0.0774
1.00	0.0000	0.8333	0.8333	0.0000	0.4167	0.4167	0.0737
1.10	0.0000	0.8333	0.8333	0.0000	0.4717	0.3616	0.0666
1.20	0.0000	0.8333	0.8333	0.0000	0.5209	0.3125	0.0602
1.30	0.0000	0.8333	0.8333	0.0000	0.5640	0.2693	0.0545
1.40	0.0000	0.8333	0.8333	0.0000	0.6018	0.2315	0.0494
1.50	0.0000	0.8333	0.8333	0.0000	0.6346	0.1987	0.0448
1.60	0.0000	0.8333	0.8333	0.0000	0.6629	0.1704	0.0407
1.70	0.0000	0.8333	0.8333	0.0000	0.6873	0.1460	0.0371
1.80	0.0000	0.8333	0.8333	0.0000	0.7083	0.1250	0.0339
1.90	0.0000	0.8333	0.8333	0.0000	0.7264	0.1070	0.0310
2.00	0.0000	0.8333	0.8333	0.0000	0.7419	0.0915	0.0285

Table A.2 Thermal bending calculation coefficient of four edges clamped under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

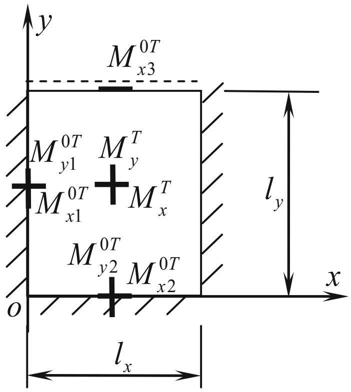
$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f\,\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_x	k_y	f
0.50	1.0000	0.8333	0.8333	1.0000	1.0000	1.0005	0.0000
0.55	1.0000	0.8333	0.8333	1.0000	1.0000	1.0006	0.0000
0.60	1.0000	0.8333	0.8333	1.0000	1.0000	1.0007	0.0000
0.65	1.0000	0.8333	0.8333	1.0000	1.0000	1.0008	0.0000
0.70	1.0000	0.8333	0.8333	1.0000	1.0000	1.0010	0.0000
0.75	1.0000	0.8333	0.8333	1.0000	1.0000	1.0010	0.0000
0.80	1.0000	0.8333	0.8333	1.0000	1.0000	1.0013	0.0000
0.85	0.0000	0.8333	0.8333	0.0000	1.0000	1.0014	0.0000
0.90	0.0000	0.8333	0.8333	0.0000	1.0000	1.0014	0.0000
0.95	0.0000	0.8333	0.8333	0.0000	1.0000	1.0016	0.0000
1.00	0.0000	0.8333	0.8333	0.0000	1.0000	1.0018	0.0000

Table A.3 Thermal bending calculation coefficient of three edges clamped and one edge simply supported under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

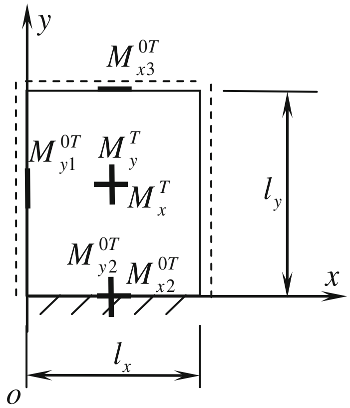
$$M_{x3}^{0T} = k_{x3} M^T$$

$$w(x, y) = f\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_{x3}	k_x	k_y	f
0.50	1.0000	0.8333	0.8333	1.0000	0.8333	0.4319	1.0000	0.0087
0.55	1.0000	0.8333	0.8333	1.0000	0.8333	0.4189	0.9866	0.0105
0.60	1.0000	0.8333	0.8333	1.0000	0.8333	0.3956	0.9809	0.0121
0.65	1.0000	0.8333	0.8333	1.0000	0.8333	0.3708	0.9984	0.0136
0.70	1.0000	0.8333	0.8333	1.0000	0.8333	0.3458	0.9877	0.0148
0.75	1.0000	0.8333	0.8333	1.0000	0.8333	0.3219	0.9750	0.0159
0.80	1.0000	0.8333	0.8333	1.0000	0.8333	0.3996	0.9609	0.0167
0.85	1.0000	0.8333	0.8333	1.0000	0.8333	0.2796	0.9456	0.0174
0.90	1.0000	0.8333	0.8333	1.0000	0.8333	0.2620	0.9294	0.0179
0.95	1.0000	0.8333	0.8333	1.0000	0.8333	0.2471	0.9125	0.0182
1.00	1.0000	0.8333	0.8333	1.0000	0.8333	0.2348	0.8951	0.0184
1.10	1.0000	0.8333	0.8333	1.0000	0.8333	0.2177	0.8603	0.0185
1.20	1.0000	0.8333	0.8333	1.0000	0.8333	0.2095	0.8258	0.0182
1.30	1.0000	0.8333	0.8333	1.0000	0.8333	0.2084	0.7930	0.0177
1.40	1.0000	0.8333	0.8333	1.0000	0.8333	0.2127	0.7621	0.0170
1.50	1.0000	0.8333	0.8333	1.0000	0.8333	0.2211	0.7335	0.0162
1.60	1.0000	0.8333	0.8333	1.0000	0.8333	0.2322	0.7075	0.0153
1.70	1.0000	0.8333	0.8333	1.0000	0.8333	0.2450	0.6839	0.0145
1.80	1.0000	0.8333	0.8333	1.0000	0.8333	0.2589	0.6626	0.0136
1.90	1.0000	0.8333	0.8333	1.0000	0.8333	0.2731	0.6435	0.0128
2.00	1.0000	0.8333	0.8333	1.0000	0.8333	0.2874	0.6264	0.0121

Table A.4 Thermal bending calculation coefficient of one edge clamped and three edges simply supported under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{y1}^{0T} = k_{y1} M^T$$

$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

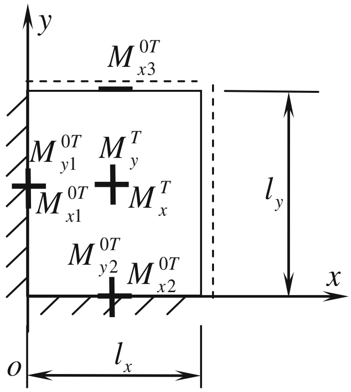
$$M_{x3}^{0T} = k_{x3} M^T$$

$$w(x, y) = f\,\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{x2}	k_{y2}	k_{x3}	k_x	k_y	f
0.50	0.8333	0.8333	1.0000	0.8333	0.1720	0.7253	0.1052
0.55	0.8333	0.8333	1.0000	0.8333	0.2196	0.6988	0.0997
0.60	0.8333	0.8333	1.0000	0.8333	0.2683	0.6277	0.0942
0.65	0.8333	0.8333	1.0000	0.8333	0.3169	0.6476	0.0886
0.70	0.8333	0.8333	1.0000	0.8333	0.3645	0.7357	0.0831
0.75	0.8333	0.8333	1.0000	0.8333	0.4105	0.6022	0.0778
0.80	0.8333	0.8333	1.0000	0.8333	0.4543	0.5823	0.0728
0.85	0.8333	0.8333	1.0000	0.8333	0.4955	0.5643	0.3072
0.90	0.8333	0.8333	1.0000	0.8333	0.5343	0.5482	0.3385
0.95	0.8333	0.8333	1.0000	0.8333	0.5702	0.5340	0.3687
1.00	0.8333	0.8333	1.0000	0.8333	0.6034	0.5216	0.3979
1.10	0.8333	0.8333	1.0000	0.8333	0.6621	0.5014	0.0481
1.20	0.8333	0.8333	1.0000	0.8333	0.7113	0.4867	0.0421
1.30	0.8333	0.8333	1.0000	0.8333	0.7518	0.4764	0.0368
1.40	0.8333	0.8333	1.0000	0.8333	0.7851	0.4695	0.0324
1.50	0.8333	0.8333	1.0000	0.8333	0.8124	0.4652	0.0286
1.60	0.8333	0.8333	1.0000	0.8333	0.8344	0.4630	0.0254
1.70	0.8333	0.8333	1.0000	0.8333	0.8523	0.4623	0.0226
1.80	0.8333	0.8333	1.0000	0.8333	0.8666	0.4625	0.0202
1.90	0.8333	0.8333	1.0000	0.8333	0.8783	0.4636	0.0182
2.00	0.8333	0.8333	1.0000	0.8333	0.8875	0.4651	0.0164

Table A.5 Thermal bending calculation coefficient of two adjacent edges clamped and two edges simply supported under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

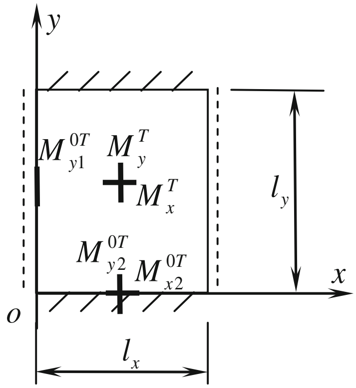
$$M_{x3}^{0T} = k_{x3} M^T$$

$$w(x, y) = f\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_{x3}	k_x	k_y	f
0.50	1.0000	0.8333	0.8333	1.0000	0.8333	0.5457	0.8710	0.0569
0.55	1.0000	0.8333	0.8333	1.0000	0.8333	0.5607	0.8560	0.0551
0.60	1.0000	0.8333	0.8333	1.0000	0.8333	0.5769	0.8398	0.0531
0.65	1.0000	0.8333	0.8333	1.0000	0.8333	0.5936	0.8231	0.0511
0.70	1.0000	0.8333	0.8333	1.0000	0.8333	0.6108	0.8059	0.0490
0.75	1.0000	0.8333	0.8333	1.0000	0.8333	0.6280	0.7887	0.0469
0.80	1.0000	0.8333	0.8333	1.0000	0.8333	0.6451	0.7716	0.0448
0.85	1.0000	0.8333	0.8333	1.0000	0.8333	0.6617	0.7550	0.0427
0.90	1.0000	0.8333	0.8333	1.0000	0.8333	0.6779	0.7388	0.0407
0.95	1.0000	0.8333	0.8333	1.0000	0.8333	0.6934	0.7233	0.0387
1.00	1.0000	0.8333	0.8333	1.0000	0.8333	0.7083	0.7084	0.0368
1.10	1.0000	0.8333	0.8333	1.0000	0.8333	0.7358	0.6809	0.0333
1.20	1.0000	0.8333	0.8333	1.0000	0.8333	0.7604	0.6563	0.0301
1.30	1.0000	0.8333	0.8333	1.0000	0.8333	0.7820	0.6347	0.0272
1.40	1.0000	0.8333	0.8333	1.0000	0.8333	0.8009	0.6158	0.0247
1.50	1.0000	0.8333	0.8333	1.0000	0.8333	0.8173	0.5994	0.0224
1.60	1.0000	0.8333	0.8333	1.0000	0.8333	0.8315	0.5852	0.0204
1.70	1.0000	0.8333	0.8333	1.0000	0.8333	0.8436	0.5731	0.0186
1.80	1.0000	0.8333	0.8333	1.0000	0.8333	0.8541	0.5626	0.0169
1.90	1.0000	0.8333	0.8333	1.0000	0.8333	0.8631	0.5536	0.0155
2.00	1.0000	0.8333	0.8333	1.0000	0.8333	0.8709	0.5458	0.0142

Table A.6 Thermal bending calculation coefficient of two opposite edges clamped and two edges simply supported under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{y1}^{0T} = k_{y1} M^T$$

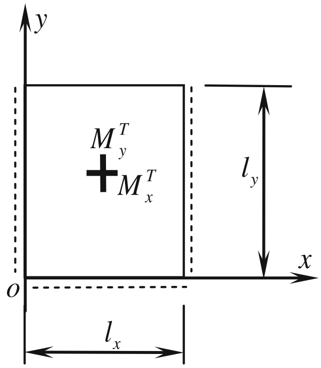
$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{y1}	k_{x2}	k_{y2}	k_x	k_y	f
0.50	0.8333	0.8333	1.0000	0.2528	0.7088	0.0965
0.55	0.8333	0.8333	1.0000	0.3176	0.6857	0.0893
0.60	0.8333	0.8333	1.0000	0.3828	0.6658	0.0821
0.65	0.8333	0.8333	1.0000	0.4464	0.6494	0.0750
0.70	0.8333	0.8333	1.0000	0.5073	0.6365	0.0683
0.75	0.8333	0.8333	1.0000	0.5648	0.6273	0.0620
0.80	0.8333	0.8333	1.0000	0.6183	0.6214	0.0561
0.85	0.8333	0.8333	1.0000	0.6675	0.6188	0.0506
0.90	0.8333	0.8333	1.0000	0.7127	0.6190	0.0456
0.95	0.8333	0.8333	1.0000	0.7535	0.6216	0.0410
1.00	0.8333	0.8333	1.0000	0.7902	0.6266	0.0368
1.10	0.8333	0.8333	1.0000	0.8525	0.6414	0.0297
1.20	0.8333	0.8333	1.0000	0.9017	0.6610	0.0239
1.30	0.8333	0.8333	1.0000	0.9395	0.6837	0.0192
1.40	0.8333	0.8333	1.0000	0.9684	0.7075	0.0155
1.50	0.8333	0.8333	1.0000	0.9901	0.7318	0.0125
1.60	0.8333	0.8333	1.0000	1.0000	0.7556	0.0101
1.70	0.8333	0.8333	1.0000	1.0000	0.7786	0.0081
1.80	0.8333	0.8333	1.0000	1.0000	0.8002	0.0066
1.90	0.8333	0.8333	1.0000	1.0000	0.8203	0.0053
2.00	0.8333	0.8333	1.0000	1.0000	1.0000	0.0044

Table A.7 Thermal bending calculation coefficient of three edges simply supported and one edge free under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

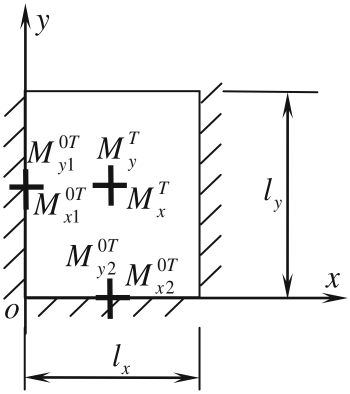
$$M_{y1}^{0T} = k_{y1} M^T, \quad M_{x2}^{0T} = k_{x2} M^T$$

$$w(x, y) = f\,\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{y1}	k_{x2}	k_x	k_y	f
0.50	0.8333	0.8333	0.0950	−0.7857	−0.0300
0.55	0.8333	0.8333	0.1230	−0.7955	−0.0330
0.60	0.8333	0.8333	0.1513	−0.7979	−0.0368
0.65	0.8333	0.8333	0.0973	−0.7962	−0.0412
0.70	0.8333	0.8333	0.1935	−0.7903	−0.0461
0.75	0.8333	0.8333	0.0979	−0.7804	−0.0514
0.80	0.8333	0.8333	0.0983	−0.7665	−0.0568
0.85	0.8333	0.8333	0.0975	−0.7487	−0.0623
0.90	0.8333	0.8333	0.0963	−0.7248	−0.0676
0.95	0.8333	0.8333	0.2020	−0.7037	−0.0728
1.00	0.8333	0.8333	0.1893	−0.6769	−0.0777
1.10	0.8333	0.8333	0.1543	−0.6173	−0.0864
1.20	0.8333	0.8333	0.1103	−0.5513	−0.0935
1.30	0.8333	0.8333	0.0623	−0.4815	−0.0988
1.40	0.8333	0.8333	0.0132	−0.4099	−0.1025
1.50	0.8333	0.8333	−0.0340	−0.3378	−0.1048
1.60	0.8333	0.8333	−0.0774	−0.2662	−0.1058
1.70	0.8333	0.8333	−0.1161	−0.1963	−0.1057
1.80	0.8333	0.8333	−0.1500	−0.1282	−0.1046
1.90	0.8333	0.8333	−0.1787	−0.0625	−0.1029
2.00	0.8333	0.8333	−0.1878	0.0006	−0.1006

Table A.8 Thermal bending calculation coefficient of three edges clamped and one edge free



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T,\ M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T,\ M_{y1}^{0T} = k_{y1} M^T$$

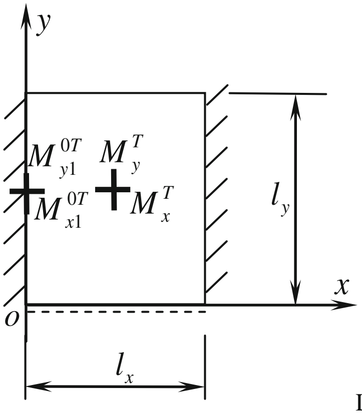
$$M_{x2}^{0T} = k_{x2} M^T,\ M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f\frac{l_x^2 M^T}{D}$$ (mid-span deflection)

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_x	k_y	f
0.50	1.0000	0.8333	0.8333	1.0000	0.2301	0.9699	0.0133
0.55	1.0000	0.8333	0.8333	1.0000	0.2350	0.9579	0.0191
0.60	1.0000	0.8333	0.8333	1.0000	0.2408	0.9398	0.0256
0.65	1.0000	0.8333	0.8333	1.0000	0.2437	0.9198	0.0326
0.70	1.0000	0.8333	0.8333	1.0000	0.2435	0.8985	0.0398
0.75	1.0000	0.8333	0.8333	1.0000	0.2403	0.8765	0.0470
0.80	1.0000	0.8333	0.8333	1.0000	0.2343	0.8542	0.0541
0.85	1.0000	0.8333	0.8333	1.0000	0.2260	0.8319	0.0608
0.90	1.0000	0.8333	0.8333	1.0000	0.2155	0.8099	0.0670
0.95	1.0000	0.8333	0.8333	1.0000	0.2031	0.7884	0.0726
1.00	1.0000	0.8333	0.8333	1.0000	0.1893	0.7676	0.0777
1.10	1.0000	0.8333	0.8333	1.0000	0.1584	0.7288	0.0858
1.20	1.0000	0.8333	0.8333	1.0000	0.1248	0.6935	0.0913
1.30	1.0000	0.8333	0.8333	1.0000	0.0905	0.6622	0.0943
1.40	1.0000	0.8333	0.8333	1.0000	0.0564	0.6345	0.0951
1.50	1.0000	0.8333	0.8333	1.0000	0.0234	0.6103	0.0941
1.60	1.0000	0.8333	0.8333	1.0000	−0.0077	0.5893	0.0915
1.70	1.0000	0.8333	0.8333	1.0000	−0.0368	0.5712	0.0878
1.80	1.0000	0.8333	0.8333	1.0000	−0.0638	0.5556	0.0832
1.90	1.0000	0.8333	0.8333	1.0000	−0.0885	0.5421	0.0779
2.00	1.0000	0.8333	0.8333	1.0000	−0.1113	0.5308	0.0722

Table A.9 Thermal bending calculation coefficient of two opposite edges clamped and one edge simply supported and one edge free under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

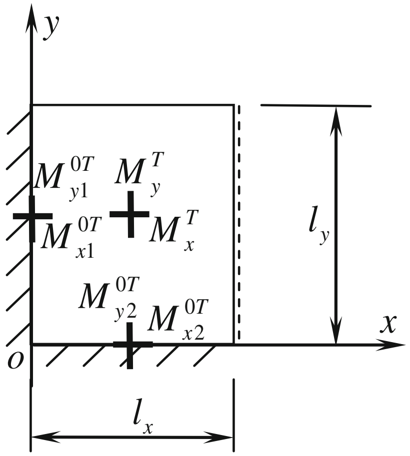
$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

$$w(x, y) = f\frac{l_x^2 M_T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_x	k_y	f
0.50	1.0000	0.8333	1.1670	−0.1714	0.2123
0.55	1.0000	0.8333	1.0194	−0.1120	0.1977
0.60	1.0000	0.8333	0.8720	−0.0543	0.1830
0.65	1.0000	0.8333	0.7274	0.0001	0.1686
0.70	1.0000	0.8333	0.5870	0.0513	0.1547
0.75	1.0000	0.8333	0.4551	0.0961	0.1415
0.80	1.0000	0.8333	0.3299	0.1368	0.1290
0.85	1.0000	0.8333	0.2131	0.1724	0.1174
0.90	1.0000	0.8333	0.1050	0.2031	0.1066
0.95	1.0000	0.8333	0.0055	0.2293	0.0966
1.00	1.0000	0.8333	−0.0853	0.2512	0.0875
1.10	1.0000	0.8333	−0.2433	0.2838	0.0716
1.20	1.0000	0.8333	−0.3732	0.3040	0.0586
1.30	1.0000	0.8333	−0.4781	0.3148	0.0480
1.40	1.0000	0.8333	−0.5629	0.3185	0.0393
1.50	1.0000	0.8333	−0.6310	0.3170	0.0323
1.60	1.0000	0.8333	−0.6853	0.3119	0.0266
1.70	1.0000	0.8333	−0.7288	0.3204	0.0219
1.80	1.0000	0.8333	−0.7635	0.2951	0.0181
1.90	1.0000	0.8333	−0.7911	0.2849	0.0150
2.00	1.0000	0.8333	−0.8129	0.2739	0.0125

Table A.10 Thermal bending calculation coefficient of the concrete rectangular thin plate with two adjacent edges clamped and one edge simply supported and one edge free under temperature disparity



$$\mu = \frac{1}{6},$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, M_{y1}^{0T} = k_{y1} M^T$$

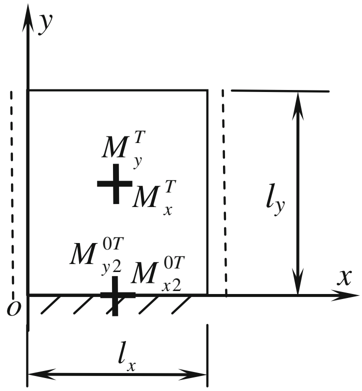
$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f \frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_{x2}	k_{y2}	k_x	k_y	f
0.50	1.0000	0.8333	0.8333	1.0000	0.6952	−0.3853	0.1754
0.55	1.0000	0.8333	0.8333	1.0000	0.6217	−0.3212	0.1672
0.60	1.0000	0.8333	0.8333	1.0000	0.5439	−0.2544	0.1585
0.65	1.0000	0.8333	0.8333	1.0000	0.4634	−0.1864	0.1494
0.70	1.0000	0.8333	0.8333	1.0000	0.3808	−0.1175	0.1403
0.75	1.0000	0.8333	0.8333	1.0000	0.2997	−0.0513	0.1310
0.80	1.0000	0.8333	0.8333	1.0000	0.2186	0.0140	0.1219
0.85	1.0000	0.8333	0.8333	1.0000	0.1390	0.0773	0.1129
0.90	1.0000	0.8333	0.8333	1.0000	0.0617	0.1379	0.1041
0.95	1.0000	0.8333	0.8333	1.0000	−0.0131	0.1956	0.0957
1.00	1.0000	0.8333	0.8333	1.0000	−0.0849	0.2503	0.0875
1.10	1.0000	0.8333	0.8333	1.0000	−0.2190	0.3500	0.0722
1.20	1.0000	0.8333	0.8333	1.0000	−0.3401	0.4375	0.0583
1.30	1.0000	0.8333	0.8333	1.0000	−0.4480	0.5132	0.0457
1.40	1.0000	0.8333	0.8333	1.0000	−0.5437	0.5786	0.0345
1.50	1.0000	0.8333	0.8333	1.0000	−0.6281	0.6345	0.0244
1.60	1.0000	0.8333	0.8333	1.0000	−0.7023	0.6823	0.0155
1.70	1.0000	0.8333	0.8333	1.0000	−0.7674	0.7231	0.0075
1.80	1.0000	0.8333	0.8333	1.0000	−0.8244	0.7579	0.0004
1.90	1.0000	0.8333	0.8333	1.0000	−0.8744	0.7876	−0.0059
2.00	1.0000	0.8333	0.8333	1.0000	−0.9182	0.8128	−0.0115

Table A.11 Thermal bending calculation coefficient of the concrete rectangular thin plate with two opposite edges simply supported and one edge clamped and one edge free under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

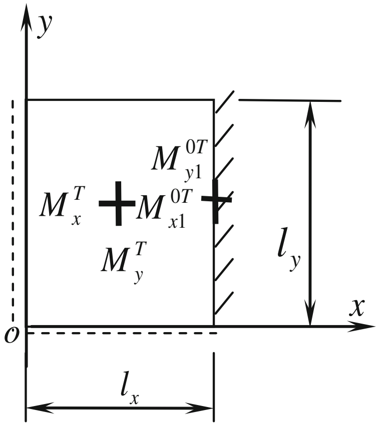
$$M_{x2}^{0T} = k_{x2} M^T, \quad M_{y2}^{0T} = k_{y2} M^T$$

$$w(x, y) = f \frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{y2}	k_{x2}	k_x	k_y	f
0.50	1.0000	0.8333	0.0656	−0.6708	0.1262
0.55	1.0000	0.8333	0.0508	−0.6212	0.1234
0.60	1.0000	0.8333	0.0307	−0.5672	0.1201
0.65	1.0000	0.8333	0.0056	−0.5097	0.1162
0.70	1.0000	0.8333	−0.0244	−0.4488	0.1119
0.75	1.0000	0.8333	−0.0563	−0.3881	0.1072
0.80	1.0000	0.8333	−0.0919	−0.3259	0.1022
0.85	1.0000	0.8333	−0.1297	−0.2637	0.0969
0.90	1.0000	0.8333	−0.1693	−0.2023	0.0916
0.95	1.0000	0.8333	−0.2098	−0.1420	0.0861
1.00	1.0000	0.8333	−0.2510	−0.0834	0.0806
1.10	1.0000	0.8333	−0.3336	0.0278	0.0697
1.20	1.0000	0.8333	−0.4143	0.1300	0.0591
1.30	1.0000	0.8333	−0.4913	0.2222	0.0490
1.40	1.0000	0.8333	−0.5635	0.3047	0.0395
1.50	1.0000	0.8333	−0.6302	0.3780	0.0307
1.60	1.0000	0.8333	−0.6913	0.4426	0.0226
1.70	1.0000	0.8333	−0.7469	0.4995	0.0151
1.80	1.0000	0.8333	−0.7971	0.5496	0.0083
1.90	1.0000	0.8333	−0.8424	0.5934	0.0021
2.00	1.0000	0.8333	−0.8829	0.6319	−0.0035

Table A.12 Thermal bending calculation coefficient of the concrete rectangular thin plate with two adjacent edges simply supported and one edge clamped and one edge free under temperature disparity



$$\mu = \frac{1}{6}$$

$$M_x^T = k_x M^T, \quad M_y^T = k_y M^T$$

$$M_{x1}^{0T} = k_{x1} M^T, \quad M_{y1}^{0T} = k_{y1} M^T$$

$$w(x, y) = f\,\frac{l_x^2 M^T}{D} \text{ (mid-span deflection)}$$

Lower temperature side is in tension

l_x/l_y	k_{x1}	k_{y1}	k_x	k_y	f
0.50	1.0000	0.8333	0.5378	−0.4566	0.1631
0.55	1.0000	0.8333	0.4490	−0.4199	0.1539
0.60	1.0000	0.8333	0.3591	−0.3670	0.1447
0.65	1.0000	0.8333	0.2700	−0.3230	0.1354
0.70	1.0000	0.8333	0.1822	−0.2799	0.1263
0.75	1.0000	0.8333	0.0995	−0.2408	0.1176
0.80	1.0000	0.8333	0.0199	−0.2035	0.1093
0.85	1.0000	0.8333	0.0552	−0.1691	0.1014
0.90	1.0000	0.8333	−0.1254	−0.1370	0.0939
0.95	1.0000	0.8333	−0.1907	−0.1091	0.0870
1.00	1.0000	0.8333	−0.2510	−0.0834	0.0806
1.10	1.0000	0.8333	−0.3575	−0.0398	0.0691
1.20	1.0000	0.8333	−0.4470	−0.0054	0.0594
1.30	1.0000	0.8333	−0.5211	0.0214	0.0512
1.40	1.0000	0.8333	−0.5824	0.0419	0.0443
1.50	1.0000	0.8333	−0.6328	0..0574	0.0385
1.60	1.0000	0.8333	−0.6741	0.0689	0.0336
1.70	1.0000	0.8333	−0.7081	0.0772	0.0295
1.80	1.0000	0.8333	−0.7359	0.0830	0.0260
1.90	1.0000	0.8333	−0.7587	0.0869	0.0230
2.00	1.0000	0.8333	−0.7774	0.0891	0.0205

Appendix B
Programs for the Rectangular Thin Plate with Four Edges Supported

Case 1: Four edges simply supported

(1) Deflection

$$w = -\frac{4a^2 M^T}{D\pi^3} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh\alpha_m} \cosh\frac{2\alpha_m y}{b} \sin\frac{m\pi x}{a} - \frac{M^T}{2D}(x-a)x$$

Let

$$a_1 = \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 \cosh\alpha_m} \cosh\frac{2\alpha_m y}{b} \sin\frac{m\pi x}{a}$$

So

$$w = \left[-\frac{4}{\pi^3} a_1 - \frac{1}{2a^2}(x-a)x\right] \frac{a^2 M^T}{D} = f\frac{a^2 M^T}{D}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$,

$a_1$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=1/2/L;
am=0.5*m*pi/L;
sum_x=1/(m^3*cosh(am))*cosh(m*pi*d-am)*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
```


X. Cheng, *Thermal Elastic Mechanics Problems of Concrete Rectangular Thin Plate*, Springer Tracts in Civil Engineering,
DOI 10.1007/978-981-10-4472-4

```
        sum_x1=sum_x;
        num=num+1;
        m=m+2;
        am=0.5*m*pi/L;
        sum_x2=1/(m^3*cosh(am))*cosh(m*pi*d-am)*sin(m*pi*c);
        sum_x=sum_x1+sum_x2;
    end
    sum_x
    num
```

(2) Bending moment

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\ \cosh\alpha_m} \cosh\frac{m\pi y}{a} \sin\frac{m\pi x}{a} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\ \cosh\alpha_m} \cosh\frac{m\pi y}{a} \sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)\sum\limits_{m=1,3,\ldots}^{\infty} \frac{1}{m\ \cosh\alpha_m} \sinh\frac{m\pi y}{a} \cos\frac{m\pi x}{a} \end{cases}$$

Let

$$b_1 = \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m\ \cosh\alpha_m} \cosh\frac{m\pi y}{a} \sin\frac{m\pi x}{a}$$

Hence

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)b_1 \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)b_1 + (\mu - 1)M^T \\ M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)b_1 \end{cases}$$

That is

$$\begin{cases} M_x^T = \left[\frac{4}{\pi}(\mu - 1)b_1\right]M^T = k_{x1}M^T = k_xM^T \\ M_y^T = \left[\frac{4}{\pi}(1 - \mu)b_1 + (\mu - 1)\right]M^T = k_{y1}M^T = k_yM^T \quad (\text{where } \mu = 1/6, \text{ the same below}) \\ M_{xy}^T = \left[\frac{4}{\pi}(1 - \mu)b_1\right]M^T = k_{xy1}M^T = k_{xy}M^T \end{cases}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$, there is,

b_1 is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=1/2/L;
am=0.5*m*pi/L;
sum_x=1/(m*cosh(am))*cosh(m*pi*d-am)*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;
    sum_x2=1/(m*cosh(am))*cosh(m*pi*d-am)*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
```

Case 2: Four edges clamped

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$-\frac{16M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

That is

$$w = \left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$
$$-\frac{16M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

In above equation, the first part can be obtained by Appendix B 2.1. For the second part, there is

$$w_2 = -\frac{16M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Let

$$f_2 = -\frac{16}{a^2\pi^4}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_2 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Hence

$$w_2 = \left(-\frac{16}{\pi^4 a^2}a_2\right)\frac{a^2M^T}{D} = f_2\frac{a^2M^T}{D}$$

$$w = (f_1 + f_2)\frac{a^2M^T}{D} = f\frac{a^2M^T}{D}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$, there is, a_2 is calculated as follows:

$$a_2 = a^2\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{mn}\left(m^2 + L^2n^2\right)^{-1}\sin m\pi c\,\sin n\pi d = a^2c_1$$

c_1 is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:69

    for n=1:2:69

        sum_x=1/m/n/(m^2+n^2*L^2)*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;

    end

end
sum_x1
```

(2) Bending moment

$$\begin{cases} M_x^T = \frac{4M^T}{\pi}(\mu - 1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad - \frac{16M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2} + \mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} + (\mu - 1)M^T \\ \qquad - \frac{16M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ \qquad + M_{xy}^T = \frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\cos\frac{m\pi x}{a} \\ \qquad + (1 - \mu)\frac{16M^T}{\pi^2 ab}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

That is

$$\begin{cases} M_x^T = \left\{\frac{4M^T}{\pi}(\mu - 1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a}\right\} \\ \qquad - \frac{16M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2} + \mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \left\{\frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\sin\frac{m\pi x}{a} + (\mu - 1)M^T\right\} \\ \qquad - \frac{16M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \left\{\frac{4M^T}{\pi}(1 - \mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a} - \alpha_m\right)\cos\frac{m\pi x}{a}\right\} \\ \qquad + (1 - \mu)\frac{16M^T}{\pi^2 ab}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$\begin{cases} M_{x2}^T = -\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{y2}^T = -\frac{16M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy2}^T = (1-\mu)\frac{16M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

$$\begin{cases} M_{x2}^T = \left[-\frac{16}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right]M^T \\ M_{y2}^T = \left[-\frac{16}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right]M^T \\ M_{xy2}^T = \left[(1-\mu)\frac{16}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}\right]M^T \end{cases}$$

Let

$$\begin{cases} \mathrm{b}_2 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ \mathrm{b}_3 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}, \\ \mathrm{b}_4 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

Hence

$$\begin{cases} M_{x2}^T = -\frac{16}{\pi^2}b_2 M^T = k_{x2}M^T \\ M_{y2}^T = -\frac{16}{\pi^2}b_3 M^T = k_{y2}M^T \\ M_{xy2}^T = (1-\mu)\frac{16}{\pi^2 ab}b_4 M^T = k_{xy2}M^T \end{cases}$$

$$\begin{cases} M_x^T = (k_{x1}+k_{x2})M^T = k_x M^T \\ M_y^T = \left(k_{y1}+k_{y2}\right)M^T = k_y M^T \\ M_{xy}^T = \left(k_{xy1}+k_{xy2}\right)M^T = k_{xy}M^T \end{cases}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_2 = \frac{1}{b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{m^2 + \mu n^2 L^2}{mnL^2} \sin m\pi c \, \sin n\pi d = \frac{1}{b^2} d_1 \\ b_3 = \frac{1}{b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{\mu m^2 + n^2 L^2}{mnL^2} \sin m\pi c \, \sin n\pi d = \frac{1}{b^2} d_2 \\ b_4 = ab \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{L^2}{m^2 + n^2 L^2} \cos m\pi c \, \cos n\pi d = abd_3 \end{cases}$$

$d_1$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:7999
    for n=1:2:7999
        sum_x=(m^2+u*n^2*L^2)/m/n/L^2*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_2$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:10999
    for n=1:2:10999
        sum_x=(u*m^2+n^2*L^2)/m/n/L^2*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
```

```
sum_x1
```

$d_3$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:69
    for n=1:2:69
        sum_x=L^2/(m^2+n^2*L^2)*cos(m*pi*c)*cos(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

Case 3: One edge simply supported and three edges clamped

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$-\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

That is

$$w = \left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$
$$-\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$w_3 = -\frac{8a^2M^T}{\pi^4 D}\frac{1}{a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Let

$$f_3 = -\frac{8}{\pi^4 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_3 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Hence

$$w_3 = \left[-\frac{8}{\pi^4 a^2}a_3\right]\frac{a^2 M^T}{D} = f_3\frac{a^2 M^T}{D}$$

$$w = (f_1 + f_3)\frac{a^2 M^T}{D} = f\frac{a^2 M^T}{D}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$, there is, a_2 is calculated as follows:

$$a_3 = a^2\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{2m^2+n^2L^2}{(m^2+n^2L^2)^2}\sin m\pi c\,\sin n\pi d = a^2 c_2$$

$c_2$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:39

    for n=1:2:39

        sum_x=1/m/n/(m^2+n^2*L^2)^2*(2*m^2+n^2*L^2)*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;

    end

end
sum_x1
```

(2) Bending moment

$$
\begin{cases}
M_x^T = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a} \\
\quad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_y^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T \\
\quad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_{xy}^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\cos\dfrac{m\pi x}{a} \\
\quad +\dfrac{8(1-\mu)M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}
\end{cases}
$$

That is

$$
\begin{cases}
M_x^T = \left\{\dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}\right\} \\
\quad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_y^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T\right\} \\
\quad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_{xy}^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\cos\dfrac{m\pi x}{a}\right\} \\
\quad +\dfrac{8(1-\mu)M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}
\end{cases}
$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$
\begin{cases}
M_{x3}^T = -\dfrac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_{y3}^T = -\dfrac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
M_{xy3}^T = \dfrac{8(1-\mu)M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}
\end{cases}
$$

$$
\begin{cases}
M_{x3}^T = \left[-\dfrac{8}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b}\right]M^T \\
M_{y3}^T = \left[-\dfrac{8}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b}\right]M^T \\
M_{xy3}^T = \left[\dfrac{8(1-\mu)}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}\right]M^T
\end{cases}
$$

Let

$$
\begin{cases}
b_5 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
b_6 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
b_7 = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{2i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}
\end{cases}
$$

Hence

$$
\begin{cases}
M_{x3}^T = -\dfrac{8}{\pi^2}b_5 M^T = k_{x3}M^T \\
M_{y3}^T = -\dfrac{8}{\pi^2}b_6 M^T = k_{y3}M^T \\
M_{xy3}^T = \dfrac{8(1-\mu)}{\pi^2 ab}b_7 M^T = k_{xy3}M^T
\end{cases}
$$

$$\begin{cases} M_x^T = (k_{x1} + k_{x3})M^T = k_x M^T \\ M_y^T = \left(k_{y1} + k_{y3}\right)M^T = k_y M^T \\ M_{xy}^T = \left(k_{xy1} + k_{xy3}\right)M^T = k_{xy} M^T \end{cases}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_5 = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{(m^2 + \mu n^2 L^2) \times (2m^2 + n^2 L^2)}{(m^2 + n^2 L^2)^2} \sin m\pi c \, \sin n\pi d = d_4 \\ b_6 = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{(\mu m^2 + n^2 L^2) \times (2m^2 + n^2 L^2)}{(m^2 + n^2 L^2)^2} \sin m\pi c \, \sin n\pi d = d_5 \\ b_7 = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{2m^2 + n^2 L^2}{(m^2 + n^2 L^2)^2} \cos m\pi c \, \cos n\pi d = a^2 d_6 \end{cases}$$

d_4 is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:7999

    for n=1:2:7999

        sum_x=(m^2+u*n^2*L^2)/(m^2+n^2*L^2)^2*(2*m^2+n^2*L^2)*sin
        (m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;

    end

end
sum_x1
```

d_5 is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:1999

    for n=1:2:1999
```

```
sum_x=(u*m^2+n^2*L^2)/m/n(m^2+n^2*L^2)^2*(2*m^2+n^2*L^2)*sin
(m*pi*c)*sin(n*pi*d);
    sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_6$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:39
    for n=1:2:39
sum_x=1/(m^2+n^2*L^2)^2*(2*m^2+n^2*L^2)*cos(m*pi*c)*cos(n*pi*d);
    sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

Case 4: Three edges simply supported and one edge clamped

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$-\frac{8M^T}{\pi^4 Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

That is

$$w = \left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$
$$-\frac{8M^T}{\pi^4 Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$w_4 = -\frac{8M^T}{\pi^4 Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Let

$$f_4 = -\frac{8}{\pi^4 a^2 b^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{n}{m}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$a_4 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{n}{m}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Hence

$$w_4 = -\frac{8}{\pi^4 a^2 b^2}a_4\frac{a^2M^T}{D} = f_4\frac{a^2M^T}{D}$$

$$w = (f_1 + f_4)\frac{a^2M^T}{D} = f\frac{a^2M^T}{D}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$, there is, a_4 is calculated as follows:

$$a_4 = a^4\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{n}{m}\frac{1}{(m^2+n^2L^2)^2}\sin m\pi c\ \sin n\pi d = a^4 c_3$$

$c_3$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:39
    for n=1:2:39
        sum_x=n/m/(m^2/L^2+n^2)^2*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
```

```
end
sum_x1
```

(2) Bending moment

$$\begin{cases} M_x^T = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a} \\ \qquad -\dfrac{8M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_y^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}+\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T \\ \qquad -\dfrac{8M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{i\pi y}{b} \\ M_{xy}^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}+\alpha_m\right)\cos\dfrac{m\pi x}{a} \\ \qquad -(\mu-1)\dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} j^2\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b} \end{cases}$$

That is

$$\begin{cases} M_x^T = \left\{\dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}\right\} \\ \qquad -\dfrac{8M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_y^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}+\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T\right\} \\ \qquad -\dfrac{8M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{j}{i}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\sin\dfrac{i\pi x}{a}\sin\dfrac{i\pi y}{b} \\ M_{xy}^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}+\alpha_m\right)\cos\dfrac{m\pi x}{a}\right\} \\ \qquad -(\mu-1)\dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty} j^2\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b} \end{cases}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$\begin{cases} M_{x4}^T = -\dfrac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{i^2}{a^2} + \dfrac{\mu j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ M_{y4}^T = -\dfrac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{i\pi y}{b} \\ M_{xy4}^T = -(\mu - 1) \dfrac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2 \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \cos \dfrac{i\pi x}{a} \cos \dfrac{j\pi y}{b} \end{cases}$$

$$\begin{cases} M_{x4}^T = \left[-\dfrac{8}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{i^2}{a^2} + \dfrac{\mu j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \right] M^T \\ M_{y4}^T = \left[-\dfrac{8}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{i\pi y}{b} \right] M^T \\ M_{xy4}^T = \left[-(\mu - 1) \dfrac{8M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2 \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \cos \dfrac{i\pi x}{a} \cos \dfrac{j\pi y}{b} \right] M^T \end{cases}$$

Let

$$\begin{cases} b_8 = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{i^2}{a^2} + \dfrac{\mu j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ b_9 = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2} \right) \sin \dfrac{i\pi x}{a} \sin \dfrac{i\pi y}{b} \\ b_{10} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2 \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-2} \cos \dfrac{i\pi x}{a} \cos \dfrac{j\pi y}{b} \end{cases}$$

Hence

$$\begin{cases} M_{x4}^T = -\dfrac{8}{\pi^2 b^2} b_8 M^T = k_{x4} M^T \\ M_{y4}^T = -\dfrac{8}{\pi^2 b^2} b_9 M^T = k_{y4} M^T \\ M_{xy4}^T = -(\mu - 1) \dfrac{8M^T}{\pi^2 ab} b_{10} M^T = k_{xy4} M^T \end{cases}$$

$$\begin{cases} M_x^T = (k_{x1} + k_{x4})M^T = k_x M^T \\ M_y^T = \left(k_{y1} + k_{y4}\right)M^T = k_y M^T \\ M_{xy}^T = \left(k_{xy1} + k_{xy4}\right)M^T = k_{xy} M^T \end{cases}$$

Taking $x = a/2$, $y = \mathrm{b}/2$, $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_8 = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{n}{m} \dfrac{m^2 + \mu n^2 L^2}{\left(m^2 + n^2 L^2\right)^2} \sin m\pi c \, \sin n\pi d = a^2 d_7 \\ b_9 = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{n}{m} \dfrac{\mu m^2 + n^2 L^2}{\left(m^2 + n^2 L^2\right)^2} \sin m\pi c \, \sin n\pi d = a^2 d_8 \\ b_{10} = a^4 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} n^2 \dfrac{1}{\left(m^2 + n^2 L^2\right)^2} \cos m\pi c \, \cos n\pi d = a^4 d_9 \end{cases}$$

$d_7$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:1999
    for n=1:2:1999
        sum_x=n/m(m^2+u*n^2*L^2)/(m^2+n^2*L^2)^2*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_8$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:7001
```

```
    for n=1:2:7001
        sum_x=n/m(u*m^2+n^2*L^2)/(m^2+n^2*L^2)^2*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_9$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:7999
    for n=1:2:7999
        sum_x=n^2/(m^2+n^2*L^2)^2*cos(m*pi*c)*cos(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

Case 5: Two adjacent edges simply supported and two edges clamped

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$-\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

That is

$$w = \left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$
$$-\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$w_5 = -\frac{8a^2M^T}{\pi^4 D}\frac{1}{a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Let

$$f_5 = -\frac{8}{\pi^4 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_5 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Hence

$$w_5 = \left(-\frac{8}{\pi^4 a^2}a_5\right)\frac{a^2M^T}{D} = f_5\frac{a^2M^T}{D}$$

$$w = (f_1 + f_5)\frac{a^2M^T}{D} = f\frac{a^2M^T}{D}$$

Taking $x = a/2$, y = $b/2$, $c = x/a$, $d = y/b$, $L = a/b$, so

$$a_5 = a^2\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{mn}\frac{1}{m^2+n^2L^2}\sin m\pi c\,\sin n\pi d = a^2 c_4$$

$c_4$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:69

    for n=1:2:69

        sum_x=1/m/n/(m^2+n^2*L^2)*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;

    end

end
sum_x1
```

(2) Bending moment

$$\begin{cases} M_x^T = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\ \qquad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-(\mu-1)M^T \\ \qquad +\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\ \qquad -(\mu-1)\dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

That is

$$\begin{cases} M_x^T = \left\{\dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}\right\} \\ \qquad -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_y^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-(\mu-1)M^T\right\} \\ \qquad +\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\ M_{xy}^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a}\right\} \\ \qquad -(\mu-1)\dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b} \end{cases}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$\begin{cases} M_{x5}^T = -\dfrac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_{y5}^T = \dfrac{8M^T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ M_{xy5}^T = -(\mu-1)\dfrac{8M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b} \end{cases}$$

$$\begin{cases} M_{x5}^T = \left[-\dfrac{8}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b}\right]M^T \\ M_{y5}^T = \left[\dfrac{8M_T}{\pi^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b}\right]M^T \\ M_{xy5}^T = \left[-(\mu-1)\dfrac{8M^T}{\pi^2 ab}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b}\right]M^T \end{cases}$$

Let

$$\begin{cases} b_{11} = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{i^2}{a^2}+\dfrac{\mu j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ b_{12} = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{1}{ij}\left(\dfrac{\mu i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ b_{13} = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-1}\cos\dfrac{i\pi x}{a}\cos\dfrac{j\pi y}{b} \end{cases}$$

Hence

$$\begin{cases} M_{x5}^T = -\dfrac{8}{\pi^2}b_{11}M^T = k_{x5}M^T \\ M_{y5}^T = \dfrac{8}{\pi^2}b_{12}M^T = k_{y5}M^T \\ M_{xy5}^T = -(\mu-1)\dfrac{8}{\pi^2 ab}b_{13}M^T = k_{xy5}M^T \end{cases}$$

$$\begin{cases} M_x^T = (k_{x1} + k_{x5})M^T = k_x M^T \\ M_y^T = \left(k_{y1} + k_{y5}\right)M^T = k_y M^T \\ M_{xy}^T = \left(k_{xy1} + k_{xy5}\right)M^T = k_{xy} M^T \end{cases}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{11} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{m^2 + \mu n^2 L^2}{m^2 + n^2 L^2} \sin m\pi c \, \sin n\pi d = d_{10} \\ b_{12} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{mn} \frac{\mu m^2 + n^2 L^2}{m^2 + n^2 L^2} \sin m\pi c \, \sin n\pi d = d_{11} \\ b_{13} = a^2 \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{1}{m^2 + n^2 L^2} \cos m\pi c \, \cos n\pi d = a^2 d_{12} \end{cases}$$

$d_{10}$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:5999
    for n=1:2:5999
        sum_x=(m^2+u*n^2*L^2)/m/n/(m^2+n^2*L^2)*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_{11}$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:5001
```

```
    for n=1:2:5001
        sum_x=(u*m^2+n^2*L^2)/m/n/(m^2+n^2*L^2)*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_{12}$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:69
    for n=1:2:69
        sum_x=1/(m^2+n^2*L^2)*cos(m*pi*c)*cos(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

Case 6: Two opposite edges simply supported and two edges clamped

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$-\frac{16M^T}{\pi^4b^2D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

That is

$$w = \left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$
$$-\frac{16M^T}{\pi^4b^2D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$w_6 = -\frac{16M^T}{\pi^4 b^2 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Let

$$f_6 = -\frac{16}{\pi^4 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_6 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Hence

$$w_6 = -\frac{16}{\pi^4 a^2}a_6\frac{a^2M^T}{D} = f_6\frac{a^2M^T}{D}$$

$$w = (f_1+f_6)\frac{a^2M^T}{D} = f\frac{a^2M^T}{D}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = 1/2/a/b$, $L = a/b$, so

$$a_6 = a^4\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{n}{m}\frac{1}{(m^2+n^2L^2)^2}\sin m\pi c\,\sin n\pi d = a^4c_5$$

$c_5$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:39

    for n=1:2:39

        sum_x=n/m/(m^2+n^2 L^2)^2*sin(m*pi*c)*sin(n*pi*d);
        sum_x1=sum_x1+sum_x;

    end

end
sum_x1
```

(2) Bending moment

$$
\begin{cases}
M_x^T = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
\qquad -\dfrac{16M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_y^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-(\mu-1)M^T \\
\qquad -\dfrac{16M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_{xy}^T = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a} \\
\qquad -(\mu-1)\dfrac{16M^T}{\pi\, ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}
\end{cases}
$$

That is

$$
\begin{cases}
M_x^T = \left\{\dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\ \cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}\right\} \\
\qquad -\dfrac{16M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_y^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}-(\mu-1)M^T\right\} \\
\qquad -\dfrac{16M^T}{\pi^2 b^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
M_{xy}^T = \left\{\dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\cos\frac{m\pi x}{a}\right\} \\
\qquad -(\mu-1)\dfrac{16M^T}{\pi^2 ab}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}j^2\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}
\end{cases}
$$

In above equation, the first part can be obtained by Case 1. For the second part, there is

$$\begin{cases} M_{x6}^T = -\frac{16M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{y6}^T = -\frac{16M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ M_{xy6}^T = -(\mu - 1)\frac{16M^T}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases}$$

$$\begin{cases} M_{x6}^T = \left[-\frac{16}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}\right] M^T \\ M_{y6}^T = \left[-\frac{16}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}\right] M^T \\ M_{xy6}^T = \left[-(\mu - 1)\frac{16}{\pi^2 ab} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b}\right] M^T \end{cases}$$

Let

$$\begin{cases} b_{14} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{i^2}{a^2} + \frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ b_{15} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{j}{i}\left(\frac{\mu i^2}{a^2} + \frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \\ b_{16} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} j^2\left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right)^{-1} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \end{cases}$$

Hence

$$\begin{cases} M_{x6}^T = -\frac{16}{\pi^2 b^2} b_{14} M^T = k_{x6} M^T \\ M_{y6}^T = -\frac{16}{\pi^2 b^2} b_{15} M^T = k_{y6} M^T \\ M_{xy6}^T = -(\mu - 1)\frac{16}{\pi^2 ab} b_{16} M^T = k_{xy6} M^T \end{cases}$$

$$\begin{cases} M_x^T = (k_{x1} + k_{x5}) M^T = k_x M^T \\ M_y^T = (k_{y1} + k_{y5}) M^T = k_y M^T \\ M_{xy}^T = (k_{xy1} + k_{xy5}) M^T = k_{xy} M^T \end{cases}$$

Taking $x = a/2$, $y = b/2$, $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{14} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{n}{m} \frac{m^2 + \mu n^2 L^2}{(m^2 + n^2 L^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{13} \\ b_{15} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \frac{n}{m} \frac{\mu m^2 + n^2 L^2}{(m^2 + n^2 L^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{14} \\ b_{16} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} n^2 \frac{1}{m^2 + n^2 L^2} \cos m\pi c \, \cos n\pi d = a^2 d_{15} \end{cases}$$

$d_{13}$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:5999
    for n=1:2:5999
        sum_x=n/m(m^2+u*n^2*L^2)/(m^2+n^2*L^2)^2*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

$d_{14}$is calculated as follows:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:9999
    for n=1:2:9999
        sum_x=n/m(u*m^2+n^2*L^2)/(m^2+n^2*L^2)^2*sin(m*pi*c)*sin
        (n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

d_{15} **is calculated as follows**:

```
syms a b am c d L
sum_x1=0;
u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
for m=1:2:69
    for n=1:2:69
        sum_x=n^2/(m^2+n^2*L^2)*cos(m*pi*c)*cos(n*pi*d);
        sum_x1=sum_x1+sum_x;
    end
end
sum_x1
```

Appendix C
Programs for the Rectangular Thin Plate with One edges Free

Case 1: Three edges simply supported and one edge free

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\left[\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}\right]-\frac{M^T}{2D}(x-a)x$$
$$+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}$$
$$\times\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\\ \left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

(where, $\mu = 1/6$, the same below)

In above equation, the first part can be obtained by Case 1. For the second part, Taking $x = a/2$, $y = b/2$, there is

$$\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\times\\ \left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$
$$=0$$

Therefore, for the rectangular thin plate with three edges simply supported and one edge free, the deflection solution in center point of the plate is the same with that with four edges simply supported.

X. Cheng, *Thermal Elastic Mechanics Problems of Concrete Rectangular Thin Plate*, Springer Tracts in Civil Engineering,
DOI 10.1007/978-981-10-4472-4

(2) Bending moment

$$
\begin{cases}
M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{pmatrix}\beta_m\coth\beta_m \\ +2\dfrac{1+\mu}{1-\mu}\end{pmatrix}\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\} \\
M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\times\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}
\end{cases}
$$

Taking $x = a/2$, $y = b/2$, so

$$
\begin{pmatrix}
\dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{pmatrix}\beta_m\coth\beta_m \\ +2\dfrac{1+\mu}{1-\mu}\end{pmatrix}\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\}=0 \\
\dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}=0
\end{pmatrix}
$$

Therefore, for the rectangular thin plate with three edges simply supported and one edge free, the bending moment solution in center point of the plate is the same with that with four edges simply supported.

Case 2 Three edges clamped and one edge free

(1) Deflection

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\
& + \frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
& - \frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& - \frac{8M^Ta^3}{\pi^4 Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}- \\ \qquad\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
$$

Taking $x = a/2$, $y = b/2$, so

$$\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\dfrac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

$$=0$$

Namely

$$w=-\frac{4a^2M^T}{D\,\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$

$$-\frac{8M^T}{\pi^4D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$-\frac{8M^Ta^3}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

That is

$$w=\left\{-\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x\right\}$$

$$-\frac{8M^T}{\pi^4D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$-\frac{8M^Ta^3}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, there is

$$w_7 = -\frac{8M^T}{\pi^4 D}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$-\frac{8M^T a^3}{\pi^4 Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

Let

$$f_7 = -\frac{8}{\pi^4 a^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$-\frac{8a}{\pi^4 b(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

$$a_7 = \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_8 = \sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

Hence

$$w_7 = \left(-\frac{8b^2}{\pi^4 a^2}a_7 - \frac{8a}{\pi^4 b(1-\mu)}a_8\right)\frac{a^2 M^T}{D} = f_7\frac{a^2 M^T}{D}$$

$$w = (f_1 + f_7)\frac{a^2 M^T}{D} = f\frac{a^2 M^T}{D}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$a_7 = b^2 \sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{L}{mn}\frac{2m^2 + n^2\times L^2}{m^2 + n^2 \times L^2}\sin m\pi c\,\sin n\pi d = b^2 c_6$$

$$a_8 = \sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty}\frac{2m^2+L^2\times k^2}{m^4}\frac{k^2\times L^2+(2-\mu)m^2}{(m^2+k^2\times L^2)^2}}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{l}\left(\dfrac{2}{L-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi d}{L}-\\ \qquad\dfrac{m\pi d}{L}\cosh\dfrac{m\pi d}{L}\end{array}\right]\sin m\pi c\end{array}\right\} = c_7$$

$c_6$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
am=0.5*m*pi/L;
sum_x=L*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;
    sum_x2=L*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);
    sum_x=sum_x1+sum_x2;
```

```
end
sum_x
num
```

c_7**is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1; u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;
sum_x=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2
+k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth
(bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

   sum_x1=sum_x;
   num=num+1;
   m=m+2;
   bm=m*pi/L;
   sum_x2=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2
   +k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth
   (bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
   sum_x=sum_x1+sum_x2;

end
sum_x
num
m
k
```

(2) Bending moment

$$
\begin{cases}
M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\begin{bmatrix}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right) \\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{bmatrix}\sin\frac{m\pi x}{a} \\
\quad - \dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
\quad - \dfrac{8M^T a}{\pi^2 b}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{matrix}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\begin{bmatrix}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\times \\ \sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{bmatrix}\sin\frac{m\pi x}{a}\end{matrix}\right\} \\
M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\begin{pmatrix}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \\ -\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{pmatrix}\sin\frac{m\pi x}{a} \\
\quad - \dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
\quad - \dfrac{8M^T a}{\pi^2 b}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{matrix}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{matrix}\right\}
\end{cases}
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero. so

$$
\begin{cases}
M_{x7} = -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
\qquad -\dfrac{8M^T a}{\pi^2 b}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times \\ \sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
M_{y7} = -\dfrac{8M^T}{\pi^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
\qquad -\dfrac{8M^T a}{\pi^2 b}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{cases}
$$

Let

$$
\begin{cases}
b_{17} = \displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\mu\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
b_{18} = \displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\mu\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{2i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
b_{19} = \displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l}\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a} \\ -\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
b_{20} = \displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left(\frac{2m^2}{a^2}+\frac{k^2}{b^2}\right)\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{cases}
$$

Hence

$$\begin{cases} M_{x7} = -\dfrac{8M^T}{\pi^2} \times b_{17} - \dfrac{8M^T a}{\pi^2 b} \times b_{19} = k_{x7}M^T \\ M_{y7} = -\dfrac{8M^T}{\pi^2} \times b_{18} - \dfrac{8M^T a}{\pi^2 b} \times b_{20} = k_{y7}M^T \end{cases}$$

$$\begin{cases} M_x^T = (k_{x1} + k_{x7})M^T = k_x M^T \\ M_y^T = \left(k_{y1} + k_{y7}\right)M^T = k_y M^T \end{cases}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{17} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{2m^2 + n^2 \times L^2}{mn} \dfrac{m^2 + \frac{1}{6} \times L^2 \times n^2}{\left(m^2 + L^2 \times n^2\right)^2} \sin m\pi c \, \sin n\pi d = d_{16} \\ b_{18} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{2m^2 + n^2 \times L^2}{mn} \dfrac{\frac{1}{6} \times m^2 + L^2 \times n^2}{\left(m^2 + L^2 \times n^2\right)^2} \sin m\pi c \, \sin n\pi d = d_{17} \\ b_{19} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \frac{2m^2 + L^2 \times k^2}{m^4} \frac{k^2 \times L^2 + (2-\mu)m^2}{\left(m^2 + k^2 \times L^2\right)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\frac{2}{L-\mu} + \beta_m \coth\beta_m\right) \sinh\frac{m\pi d}{L} - \\ \frac{m\pi d}{L} \cosh\frac{m\pi d}{L} \end{array} \right] \sin m\pi c \end{array} \right\} = d_{18} \\ b_{20} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \frac{2m^2 + L^2 \times k^2}{m^2} \frac{k^2 \times L^2 + (2-\mu)m^2}{\left(m^2 + k^2 \times L^2\right)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi d}{L} \cosh\frac{m\pi d}{L} - \beta_m \coth\beta_m \sinh\frac{m\pi d}{L}\right) \sin m\pi c \end{array} \right\} = d_{19} \end{cases}$$

$d_{16}$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=(m^2+u*n^2*L^2)*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/
(m^2+n^2*L^2)^2;
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
```

```
    m=m+2;
    am=0.5*m*pi/L;
sum_x2=(m^2+u*n^2*L^2)*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/
(m^2+n^2*L^2)^2;
    sum_x=sum_x1+sum_x2;
end
sum_x
num
```

$d_{17}$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=(u*m^2+n^2*L^2)*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/
(m^2+n^2*L^2)^2;
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;
sum_x2=(u*m^2+n^2*L^2)*(2*m^2+n^2*L^2)sin(m*pi*c)sin(n*pi*d)/m/n/
(m^2+n^2*L^2)^2;
    sum_x=sum_x1+sum_x2;
end
sum_x
num
```

$d_{18}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1; u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;
sum_x=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2
+k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth
(bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
```

```
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    bm=m*pi/L;
    sum_x2=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2
    +k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth
    (bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
k
```

d_{19}**is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1; u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;
sum_x=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2
+k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh
(m*pi*d/L))-bm*coth(bm)*sinh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    bm=m*pi/L;
    sum_x2=(2*m^2+L^2*k^2)*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2
    +k^2*L^2)^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh
    (m*pi*d/L))-bm*coth(bm)*sinh(m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
k
```

Case 3: Two opposite edges clamped and one edge simply supported and one edge free

(1) Deflection

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\
& +\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\dfrac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
& -\frac{16M^T}{\pi^4Da^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& +\frac{16M^Ta}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}- \\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$
\begin{aligned}
w_8 = & -\frac{16M^T}{\pi^4Da^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& +\frac{16M^Ta}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}- \\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
$$

Let

$$f_8 = -\frac{16}{\pi^4 a^4}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$+\frac{16}{\pi^4 ab(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\dfrac{m\pi x}{a}\end{array}\right\}$$

$$\left\{\begin{array}{l}a_9=\dfrac{1}{a^4}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b}\\ a_{10}=\dfrac{1}{ab}\sum\limits_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}\end{array}\right.$$

Hence

$$w_8 = -\frac{16a^2M^T}{\pi^4 D}\times a_9+\frac{16a^2M^T}{\pi^4 D(1-\mu)}\times a_{10}=f\frac{a^2M^T}{D}$$

$$w=(f_1+f_8)\frac{a^2M^T}{D}=f\frac{a^2M^T}{D}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$a_9 = a^4 \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{m}{n} \left(\frac{1}{m^2 + n^2 \times L^2} \right)^2 \sin m\pi c \, \sin n\pi d = a^4 c_8$$

$$a_{10} = ab \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \dfrac{L}{m^2} \dfrac{k^2 \times L^2 + (2-\mu)m^2}{(m^2 + k^2 \times L^2)^2}}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\dfrac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \dfrac{m\pi d}{L} - \\ \dfrac{m\pi d}{L} \cosh \dfrac{m\pi d}{L} \end{array} \right] \sin m\pi c \end{array} \right\} = abc_9$$

$c_8$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*m/(n*(m^2+n^2*L^2)^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*m/(n*(m^2+n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
n
```

c_9 is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)^2)/((3
+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh(m*pi*d/L)-
m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)^2)/
    ((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh
    (m*pi*d/L)-m*pi*d/L*cosh (m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
k
```

(2) Bending moment

$$
\begin{cases}
M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a} \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{\dfrac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{c}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\dfrac{m\pi x}{a} \\
\quad - \dfrac{16M^T}{\pi^2a^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
\quad - \dfrac{16M^T}{\pi^2 ab}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{c}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\times\\ \sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{\dfrac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{c}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\\ -\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{array}\right)\sin\dfrac{m\pi x}{a} \\
\quad + \dfrac{16M^T}{\pi^2a^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
\quad - \dfrac{16M^T}{\pi^2 ab}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{cases}
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$\begin{cases} M_{x8} = -\dfrac{16M^T}{\pi^2 a^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ \qquad -\dfrac{16M^T}{\pi^2 ab}\sum\limits_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\times \\ \sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\ M_{y8} = \dfrac{16M^T}{\pi^2 a^2}\sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ \qquad -\dfrac{16M^T}{\pi^2 ab}\sum\limits_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\} \end{cases}$$

Let

$$\begin{cases} b_{21} = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ b_{22} = \sum\limits_{i=1,3,\ldots}^{\infty}\sum\limits_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\ b_{23} = \sum\limits_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\dfrac{m^2}{a^2}+\dfrac{k^2}{b^2}\right)^{-2}\left[\dfrac{k^2}{b^2}+(2-\mu)\dfrac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\times \\ \sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\ b_{24} = \sum\limits_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\} \end{cases}$$

$$\begin{cases} M_{x8} = -\dfrac{16M^T}{\pi^2} \times b_{21} - \dfrac{16M^T}{\pi^2} \times b_{23} = k_{x8}M^T \\ M_{y8} = \dfrac{16M^T}{\pi^2} \times b_{22} - \dfrac{16M^T}{\pi^2} \times b_{24} = k_{y8}M^T \end{cases}$$

Taking $c=x/a$, $d=y/b$, $L= a/b$, there is

$$\begin{cases} b_{21} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{m}{n} \dfrac{m^2 + \frac{1}{6} \times L^2 \times n^2}{(m^2 + L^2 \times n^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{20} \\ b_{22} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{m}{n} \dfrac{\frac{1}{6} \times m^2 + L^2 \times n^2}{(m^2 + L^2 \times n^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{21} \\ b_{23} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} L \frac{k^2 \times L^2 + (2-\frac{1}{6})m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth \beta_m + 2\frac{1+\mu}{1-\mu}\right) \times \\ \sinh \frac{m\pi d}{L} - \frac{m\pi d}{L} \cosh \frac{m\pi d}{L} \end{array} \right] \sin \frac{m\pi c}{L} \end{array} \right\} = abd_{22} \\ b_{24} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} L \frac{k^2 \times L^2 + (2-\frac{1}{6})m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi d}{L} \cosh \frac{m\pi d}{L} - \beta_m \coth \beta_m \sinh \frac{m\pi d}{L}\right) \sin m\pi c \end{array} \right\} = abd_{23} \end{cases}$$

$d_{20}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*m*(m^2+1/6*n^2*L^2)/(n*(m^2+n^2*L^2)
^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*m*(m^2+1/6*n^2*L^2)/(n*(m^2
    +n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
```

```
end
sum_x
num
m
n
```

$d_{21}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*m*(1/6*m^2+n^2*L^2)/(n*(m^2+n^2*L^2)
^2);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*m*(1/6*m^2+n^2*L^2)/(n*(m^2
    +n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
n
```

$d_{22}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*sin(m*pi*c)*((2*(1+1/6)/(1-1/6)+bm*coth(bm))
*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L));
while abs(sum_x-sum_x1)>=1.0e-05
```

```
    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
    (1-1/6)*(sinh(2*bm)/2)+bm)*sin(m*pi*c)*((2*(1+1/6)/(1-1/6)+bm*coth
    (bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L));
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
k
```

$d_{23}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the waLue of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth(bm)*sinh
(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
    (1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth(bm)
    *sinh(m*pi*d/L))*sin(m*pi*c);
end
sum_x
num
m
k
```

Case 4: Two adjacent edges clamped and one edge simply supported and one edge free

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\dfrac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$
$$+\frac{8M^T}{\pi^4Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8M^Ta^3}{\pi^4Db^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$w_9 = \frac{8M^T}{\pi^4Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8M^Ta^3}{\pi^4Db^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\}$$

Let

$$f_9 = \frac{8}{\pi^4 a^2 b^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \times \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$

$$- \frac{8a}{\pi^4 b^3 (1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^4} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2} \right]}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\frac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \frac{m\pi y}{a} - \\ \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\}$$

$$a_{11} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$

$$a_{12} = \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2} \right]}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\frac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \frac{m\pi y}{a} - \\ \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\}$$

Hence

$$w_9 = \frac{8M^T}{\pi^4 D b^2} a_{11} - \frac{8M^T a^3}{\pi^4 D b^3 (1-\mu)} a_{12} = f_9 \frac{a^2 M^T}{D}$$

$$w = (f_1 + f_9) \frac{a^2 M^T}{D} = f \frac{a^2 M^T}{D}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$a_{11} = a^4 \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{n}{m} \left(\frac{1}{m^2 + n^2 \times L^2} \right)^2 \sin m\pi c \, \sin n\pi d = a^4 c_{10}$$

$$a_{12} = ab \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2 * L}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\left(\frac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \frac{m\pi d}{L} - \frac{m\pi d}{L} \cosh \frac{m\pi d}{L} \right] \sin m\pi c \end{array} \right\} = abc_{11}$$

$c_{10}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2= sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
n
```

$c_{11}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=k^2L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)^2)/((3
+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh(m*pi*d/L)-
m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=k^2L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)
    ^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh
    (m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
k
```

(2) Bending moment

$$
\left\{
\begin{aligned}
M_x ={}& \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
&+\frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\\ \times\ \sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
&+\frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
&-\frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\\ \times\ \sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
M_y ={}& \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
&+\frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{l}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\\ -\ \beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{array}\right)\sin\frac{m\pi x}{a} \\
&+\frac{8M^T}{\pi^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{1}{ij}\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-1}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
&-\frac{8M^T a}{\pi^2 b}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\right.
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x=a/2$, $y=b/2$, the second part is zero.

For the last two part, there is

$$\begin{cases} M_{x9} = \dfrac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{ij} \left(\dfrac{i^2}{a^2} + \dfrac{\mu j^2}{b^2} \right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-1} \sin\dfrac{i\pi x}{a} \sin\dfrac{j\pi y}{b} \\ \qquad - \dfrac{8M^T a}{\pi^2 b} \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-1} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth\beta_m + 2\frac{1+\mu}{1-\mu} \right) \\ \times \sinh\frac{m\pi y}{a} - \frac{m\pi y}{a} \cosh\frac{m\pi y}{a} \end{array} \right] \sin\frac{m\pi x}{a} \end{array} \right\} \\ M_{y9} = \dfrac{8M^T}{\pi^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{ij} \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2} \right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-1} \sin\dfrac{i\pi x}{a} \sin\dfrac{j\pi y}{b} \\ \qquad - \dfrac{8M^T a}{\pi^2 b} \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-1} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh\frac{m\pi y}{a} - \beta_m \coth\beta_m \sinh\frac{m\pi y}{a} \right) \sin\frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

Let

$$\begin{cases} b_{25} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{ij} \left(\dfrac{i^2}{a^2} + \mu\dfrac{j^2}{b^2} \right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-1} \sin\dfrac{i\pi x}{a} \sin\dfrac{j\pi y}{b} \\ b_{26} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{ij} \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2} \right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2} \right)^{-1} \sin\dfrac{i\pi x}{a} \sin\dfrac{j\pi y}{b} \\ b_{27} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-1} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth\beta_m + 2\dfrac{1+\mu}{1-\mu} \right) \\ \times \sinh\dfrac{m\pi y}{a} - \dfrac{m\pi y}{a} \cosh\dfrac{m\pi y}{a} \end{array} \right] \sin\frac{m\pi x}{a} \end{array} \right\} \\ b_{28} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-1} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh\frac{m\pi y}{a} - \beta_m \coth\beta_m \sinh\frac{m\pi y}{a} \right) \sin\frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

$$\begin{cases} M_{x9} = \dfrac{8M^T}{\pi^2} \times b_{25} - \dfrac{8M^T a}{\pi^2 b} \times b_{27} = k_{x9} M^T \\ M_{y9} = \dfrac{8M^T}{\pi^2} \times b_{26} - \dfrac{8M^T a}{\pi^2 b} \times b_{28} = k_{y9} M^T \end{cases}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{25} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{mn} \dfrac{m^2 + \frac{1}{6} \times L^2 \times n^2}{m^2 + L^2 \times n^2} \sin m\pi c \, \sin n\pi d = d_{24} \\ b_{26} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{1}{mn} \dfrac{\frac{1}{6} \times m^2 + L^2 \times n^2}{m^2 + L^2 \times n^2} \sin m\pi c \, \sin n\pi d = d_{25} \\ b_{27} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\dfrac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \dfrac{m\pi d}{L} \\ - \dfrac{m\pi d}{L} \cosh \dfrac{m\pi d}{L} \end{array} \right] \sin m\pi c \end{array} \right\} = d_{26} \\ b_{28} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi d}{L} \cosh \frac{m\pi d}{L} - \beta_m \coth \beta_m \sinh \frac{m\pi d}{L} \right) \sin m\pi c \end{array} \right\} = d_{27} \end{cases}$$

$d_{24}$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=(m^2+u*n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;

sum_x2=(m^2+u*n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);

    sum_x=sum_x1+sum_x2;
```

```
end
sum_x
num
```

$d_{25}$is calculated as follows:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=(u*m^2+n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;
sum_x2=(u*m^2+n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m/n/(m^2+n^2*L^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
```

$d_{26}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;
sum_x=sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2))/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(11/6)+bm*coth(bm))*sinh(m*pi*d/L)
m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    bm=m*pi/L;
```

```
sum_x2=sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2))/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(11/6)+bm*coth(bm))*sinh(m*pi*d/L)
m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);

    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
k
```

***d_{27}* is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the walue of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/m^2/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth(bm)*sinh
(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=     sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/m^2/(m^2+k^2*L^2)^2/((3
    +1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth
    (bm)*sinh(m*pi*d/L))*sin(m*pi*c);end
    sum_x

num
m
k
```

Case 5: Two opposite edges simply supported and one edge clamped and one edge free

(1) Deflection

$$w = -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x$$
$$+\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\frac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\}$$
$$+\frac{8M^T}{\pi^4Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8M^Ta^3}{\pi^4Db^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\}$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$w_{10} = \frac{8M^T}{\pi^4Db^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8M^Ta^3}{\pi^4Db^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^4}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m} \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\right\}$$

Let

$$f_{10} = \frac{8}{\pi^4 a^2 b^2} \sum_{i=1,3,}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \times \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$

$$- \frac{8a}{\pi^4 b^3 (1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^4} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\dfrac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \dfrac{m\pi y}{a} - \\ \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\}$$

$$a_{13} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{j}{i} \left(\frac{i^2}{a^2} + \frac{j^2}{b^2} \right)^{-2} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$$

$$a_{14} = \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^4} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2} \right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(\dfrac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \dfrac{m\pi y}{a} - \\ \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\}$$

Hence

$$w_{10} = \frac{8M^T}{\pi^4 D b^2} a_{13} - \frac{8M^T a^3}{\pi^4 D b^3 (1-\mu)} a_{14} = f_{10} \frac{a^2 M^T}{D}$$

$$w = (f_1 + f_{10}) \frac{a^2 M^T}{D} = f \frac{a^2 M^T}{D}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$a_{13} = a^4 \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{n}{m} \left(\frac{1}{m^2 + n^2 \times L^2} \right)^2 \sin m\pi c \, \sin n\pi d = a^4 c_{12}$$

$$a_{14} = ab\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty}\frac{k^2\times L}{m^4}\frac{k^2\times L^2+(2-\mu)m^2}{(m^2+k^2\times L^2)^2}}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi d}{L}-\\ \dfrac{m\pi d}{L}\cosh\dfrac{m\pi d}{L}\end{array}\right]\sin m\pi c\end{array}\right\} = abc_{13}$$

$c_{12}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
n
```

$c_{13}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2+k^2*L^2)^2)/
((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh
(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
```

```
        m=m+2;
        bm=m*pi/L;
        sum_x2= k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^4*(m^2+k^2*L^2)
        ^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh
        (m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
        sum_x=sum_x1+sum_x2;
    end
    sum_x
    num
    m
    k
```

(2) Bending moment

$$
\left\{
\begin{aligned}
M_x &= \frac{4M^T}{\pi}(\mu-1)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a} \\
&\quad + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{c}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\\ \times\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a} \\
&\quad + \frac{8M^T}{\pi^2 b^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\left(\frac{i^2}{a^2}+\frac{\mu j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
&\quad - \frac{8M^T a}{\pi^2 b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{c}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\\ \times\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
M_y &= \frac{4M^T}{\pi}(1-\mu)\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m\cosh\alpha_m}\cosh\left(\frac{m\pi y}{a}-\alpha_m\right)\sin\frac{m\pi x}{a}+(\mu-1)M^T \\
&\quad + \frac{2(3-2\mu)M^T}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{\frac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{c}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\\ -\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\end{array}\right)\sin\frac{m\pi x}{a} \\
&\quad + \frac{8M^T}{\pi^2 b^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{j}{i}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\times\left(\frac{\mu i^2}{a^2}+\frac{j^2}{b^2}\right)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
&\quad - \frac{8M^T a}{\pi^2 b^3}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{c}\frac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{k^2}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
\right.
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$\begin{cases} M_{x10} = \dfrac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \left(\dfrac{i^2}{a^2} + \dfrac{\mu j^2}{b^2}\right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ \quad - \dfrac{8M^T a}{\pi^2 b^3} \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{c} \left(\beta_m \coth \beta_m + 2\dfrac{1+\mu}{1-\mu}\right) \\ \times \sinh \dfrac{m\pi y}{a} - \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\} \\ M_{y10} = \dfrac{8M^T}{\pi^2 b^2} \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \times \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2}\right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ \quad - \dfrac{8M^T a}{\pi^2 b^3} \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh \frac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \frac{m\pi y}{a}\right) \sin \frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

Let

$$\begin{cases} b_{29} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \mu \dfrac{j^2}{b^2}\right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ b_{30} = \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{j}{i} \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \times \left(\dfrac{\mu i^2}{a^2} + \dfrac{j^2}{b^2}\right) \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ b_{31} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{r} \left(2\dfrac{1+\mu}{1-\mu} + \beta_m \coth \beta_m\right) \sinh \dfrac{m\pi y}{a} - \\ \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\} \\ b_{32} = \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2}\right]}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh \frac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \frac{m\pi y}{a}\right) \sin \frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

Hence

$$\begin{cases} M_{x10} = \dfrac{8M^T}{\pi^2 b^2} b_{29} - \dfrac{8M^T a}{\pi^2 b^3} b_{31} = k_{x10} M^T \\ M_{y10} = \dfrac{8M^T}{\pi^2 b^2} b_{30} - \dfrac{8M^T a}{\pi^2 b^3} b_{32} = k_{y10} M^T \end{cases}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{29} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{n}{m} \dfrac{m^2 + \frac{1}{6} \times L^2 \times n^2}{(m^2 + L^2 \times n^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{28} \\ b_{30} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{n}{m} \dfrac{\mu m^2 + n^2 L^2}{(m^2 + n^2 L^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{29} \\ b_{31} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2 * L}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(2 \dfrac{1+\mu}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \dfrac{m\pi d}{L} - \\ \dfrac{m\pi d}{L} \cosh \dfrac{m\pi d}{L} \end{array} \right] \sin m\pi c \end{array} \right\} = abd_{30} \\ b_{32} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{k^2 * L}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\dfrac{m\pi d}{L} \cosh \dfrac{m\pi d}{L} - \beta_m \coth \beta_m \sinh \dfrac{m\pi d}{L} \right) \sin m\pi c \end{array} \right\} = abd_{31} \end{cases}$$

$d_{28}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
```

```
sum_x=sin(m*pi*c)*sin(n*pi*d)*n*(m^2+1/6*n^2*L^2)/(m*(m^2+n^2*L^2)
^2);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*n*(m^2+1/6*n^2*L^2)/(m*(m^2
    +n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
n
```

$d_{29}a_{y3}$**is calculated as follows**:

```
syms a b am c d L
num=1;sum_x1=0;m=1;n=1;u=1/6;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=(u*m^2+n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m*n/(m^2+n^2*L^2)^2;
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    am=0.5*m*pi/L;
    sum_x2=(u*m^2+n^2*L^2)*sin(m*pi*c)sin(n*pi*d)/m*n/(m^2+n^2*L^2)^2;
    sum_x=sum_x1+sum_x2;
end
sum_x
num
```

d_{30}**is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
```

```
bm=m*pi/L;%bm=m*pi/L;
sum_x=k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)^2)/
((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2(1+1/6)/(1-1/6)+bm*coth(bm))*sinh
(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=  k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)
    ^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2(1+1/6)/(1-1/6)+bm*coth(bm))
    *sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;

end
sum_x
num
m
k
```

d_{31}**is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)^2)/
((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L))-bm*coth
(bm)*sinh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    k=k+2;
```

```
        m=m+2;
        bm=m*pi/L;
        sum_x2= k^2*L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2*(m^2+k^2*L^2)
        ^2)/((3+1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L))-
        bm*coth(bm)*sinh(m*pi*d/L))*sin(m*pi*c);
        sum_x=sum_x1+sum_x2;
    end
    sum_x
    num
    m
    k
```

Case 6: Two adjacent edges simply supported and one edge clamped and one edge free

(1) Deflection

$$
\begin{aligned}
w = & -\frac{4a^2M^T}{D\pi^3}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m^3\cosh\alpha_m}\cosh\left(\frac{2\alpha_m y}{b}-\alpha_m\right)\sin\frac{m\pi x}{a}-\frac{M^T}{2D}(x-a)x \\
& +\frac{2a^2(3-2\mu)M^T}{D\pi^3(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\dfrac{\sinh\beta_m}{m^3\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\frac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
& +\frac{8M^T}{\pi^4Da^2}\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \\
& -\frac{8M^Ta}{\pi^4Db(1-\mu)}\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\begin{array}{r}\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a}- \\ \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\}
\end{aligned}
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$w_{11} = \frac{8M^T}{\pi^4 Da^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8M^T a}{\pi^4 Db(1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a} - \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\dfrac{m\pi x}{a} \end{array} \right\}$$

Let

$$f_{11} = \frac{8}{\pi^4 a^4} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$
$$-\frac{8}{\pi^4 ab(1-\mu)} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh\beta_m \sum\limits_{k=1,3,\ldots}^{\infty} \frac{1}{m^2}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m} \\ \times\left[\left(\dfrac{2}{1-\mu}+\beta_m\coth\beta_m\right)\sinh\dfrac{m\pi y}{a} - \dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\right]\sin\frac{m\pi x}{a} \end{array} \right\}$$

$$a_{15} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{i}{j}\left(\frac{i^2}{a^2}+\frac{j^2}{b^2}\right)^{-2} \sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

$$a_{16} = \sum_{m=1,3,\ldots}^{\infty} \left\{ \frac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2} \right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu) \frac{m^2}{a^2} \right]}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \times \left[\left(\frac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \frac{m\pi y}{a} - \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a} \right\}$$

Hence

$$w_{11} = \frac{8a^2 M^T}{\pi^4 D} \times a_{15} - \frac{8a^2 M^T}{\pi^4 D(1-\mu)} \times a_{16} = f_{11} \frac{a^2 M^T}{D}$$

$$w = (f_1 + f_{11}) \frac{a^2 M^T}{D} = f \frac{a^2 M^T}{D}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$a_{15} = a^4 \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \frac{n}{m} \left(\frac{1}{m^2 + n^2 \times L^2} \right)^2 \sin m\pi c \, \sin n\pi d = a^4 c_{14}$$

$$a_{16} = ab \sum_{m=1,3,\ldots}^{\infty} \left\{ \frac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \frac{L}{m^2} \frac{k^2 \times L^2 + (2-\mu) m^2}{(m^2 + k^2 \times L^2)^2}}{\frac{3+\mu}{1-\mu} \frac{\sinh 2\beta_m}{2} + \beta_m} \times \left[\left(\frac{2}{1-\mu} + \beta_m \coth \beta_m \right) \sinh \frac{m\pi d}{L} - \frac{m\pi d}{L} \cosh \frac{m\pi d}{L} \right] \sin m\pi c \right\} = abc_{15}$$

c_{14} is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
```

```
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*n/(m*(m^2+n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
n
```

c_{15}**is calculated as follows**:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L/m^2*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh(m*pi*d/L)-
m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L/m^2*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3
    +1/6)/(1-1/6)*(sinh(2*bm)/2)+bm)*((2/(1-1/6)+bm*coth(bm))*sinh
    (m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L))*sin(m*pi*c);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
k
```

(2) Bending moment

$$
\begin{cases}
M_x = \dfrac{4M^T}{\pi}(\mu-1)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a} \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{\dfrac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\left[\begin{array}{l}\left(\beta_m\coth\beta_m+2\frac{1+\mu}{1-\mu}\right)\\ \times\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\end{array}\right]\sin\dfrac{m\pi x}{a} \\
\quad + \dfrac{8M^T}{\pi^2 a^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\dfrac{i^2}{a^2}+\mu\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
\quad - \dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left[\begin{array}{c}\left(\beta_m\coth\beta_m+2\dfrac{1+\mu}{1-\mu}\right)\\ \times\sinh\dfrac{m\pi y}{a}-\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\end{array}\right]\sin\frac{m\pi x}{a}\end{array}\right\} \\
M_y = \dfrac{4M^T}{\pi}(1-\mu)\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{1}{m\cosh\alpha_m}\cosh\left(\dfrac{m\pi y}{a}-\alpha_m\right)\sin\dfrac{m\pi x}{a}+(\mu-1)M^T \\
\quad + \dfrac{2(3-2\mu)M^T}{\pi}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\dfrac{\dfrac{\sinh\beta_m}{m\cosh\alpha_m}\sinh\left(\dfrac{2\alpha_m y}{b}-\alpha_m\right)}{\frac{3+\mu}{1-\mu}\frac{\sinh 2\beta_m}{2}+\beta_m}\left(\begin{array}{l}\dfrac{m\pi y}{a}\cosh\dfrac{m\pi y}{a}\\ -\beta_m\coth\beta_m\sinh\dfrac{m\pi y}{a}\end{array}\right)\sin\dfrac{m\pi x}{a} \\
\quad + \dfrac{8M^T}{\pi^2 a^2}\displaystyle\sum_{i=1,3,\ldots}^{\infty}\sum_{j=1,3,\ldots}^{\infty}\dfrac{i}{j}\left(\mu\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)\left(\dfrac{i^2}{a^2}+\dfrac{j^2}{b^2}\right)^{-2}\sin\dfrac{i\pi x}{a}\sin\dfrac{j\pi y}{b} \\
\quad - \dfrac{8M^T}{\pi^2 ab}\displaystyle\sum_{m=1,3,\ldots}^{\infty}\left\{\begin{array}{l}\dfrac{\sinh\beta_m\sum_{k=1,3,\ldots}^{\infty}\left(\frac{m^2}{a^2}+\frac{k^2}{b^2}\right)^{-2}\left[\frac{k^2}{b^2}+(2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu}\dfrac{\sinh 2\beta_m}{2}+\beta_m}\\ \times\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\beta_m\coth\beta_m\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}\end{array}\right\}
\end{cases}
$$

In above equation, the first part can be obtained by Appendix B Case 1. For the second part, Taking $x = a/2$, $y = b/2$, the second part is zero.

For the last two part, there is

$$\begin{cases} M_{x11} = \dfrac{8M^T}{\pi^2 a^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \dfrac{i}{j} \left(\dfrac{i^2}{a^2} + \mu \dfrac{j^2}{b^2}\right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ \qquad - \dfrac{8M^T}{\pi^2 ab} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth \beta_m + 2\dfrac{1+\mu}{1-\mu}\right) \\ \times \sinh \dfrac{m\pi y}{a} - \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\} \\ M_{y11} = \dfrac{8M^T}{\pi^2 a^2} \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \dfrac{i}{j} \left(\mu \dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ \qquad - \dfrac{8M^T}{\pi^2 ab} \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh \frac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \frac{m\pi y}{a}\right) \sin \frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

Let

$$\begin{cases} b_{33} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \dfrac{i}{j} \left(\dfrac{i^2}{a^2} + \mu \dfrac{j^2}{b^2}\right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ b_{34} = \sum_{i=1,3,\ldots}^{\infty} \sum_{j=1,3,\ldots}^{\infty} \dfrac{i}{j} \left(\mu \dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right) \left(\dfrac{i^2}{a^2} + \dfrac{j^2}{b^2}\right)^{-2} \sin \dfrac{i\pi x}{a} \sin \dfrac{j\pi y}{b} \\ b_{35} = \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth \beta_m + 2\dfrac{1+\mu}{1-\mu}\right) \\ \times \sinh \dfrac{m\pi y}{a} - \dfrac{m\pi y}{a} \cosh \dfrac{m\pi y}{a} \end{array} \right] \sin \frac{m\pi x}{a} \end{array} \right\} \\ b_{36} = \sum_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} \left(\frac{m^2}{a^2} + \frac{k^2}{b^2}\right)^{-2} \left[\frac{k^2}{b^2} + (2-\mu)\frac{m^2}{a^2}\right]}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi y}{a} \cosh \frac{m\pi y}{a} - \beta_m \coth \beta_m \sinh \frac{m\pi y}{a}\right) \sin \frac{m\pi x}{a} \end{array} \right\} \end{cases}$$

Hence

$$\begin{cases} M_{x11} = \dfrac{8M^T}{\pi^2} \times b_{33} - \dfrac{8M^T}{\pi^2} \times b_{35} = k_{x11} M^T \\ M_{y11} = \dfrac{8M^T}{\pi^2} \times b_{34} - \dfrac{8M^T}{\pi^2} \times b_{36} = k_{y11} M^T \end{cases}$$

Taking $c = x/a$, $d = y/b$, $L = a/b$, there is

$$\begin{cases} b_{33} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{m}{n} \dfrac{m^2 + \frac{1}{6} \times L^2 \times n^2}{(m^2 + L^2 \times n^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{32} \\ b_{34} = a^2 \sum\limits_{i=1,3,\ldots}^{\infty} \sum\limits_{j=1,3,\ldots}^{\infty} \dfrac{m}{n} \dfrac{\frac{1}{6} \times m^2 + L^2 \times n^2}{(m^2 + L^2 \times n^2)^2} \sin m\pi c \, \sin n\pi d = a^2 d_{33} \\ b_{35} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} L \frac{k^2 \times L^2 + \left(2 - \frac{1}{6}\right) m^2}{(m^2 + k^2 \times L^2)^2}}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left[\begin{array}{l} \left(\beta_m \coth \beta_m + 2 \dfrac{1+\mu}{1-\mu} \right) \\ \times \sinh \dfrac{m\pi d}{L} - \dfrac{m\pi d}{L} \cosh \dfrac{m\pi d}{L} \end{array} \right] \sin \frac{m\pi c}{L} \end{array} \right\} = abd_{34} \\ b_{36} = ab \sum\limits_{m=1,3,\ldots}^{\infty} \left\{ \begin{array}{l} \dfrac{\sinh \beta_m \sum_{k=1,3,\ldots}^{\infty} L \frac{k^2 \times L^2 + \left(2 - \frac{1}{6}\right) m^2}{(m^2 + k^2 \times L^2)^2}}{\dfrac{3+\mu}{1-\mu} \dfrac{\sinh 2\beta_m}{2} + \beta_m} \\ \times \left(\frac{m\pi d}{L} \cosh \frac{m\pi d}{L} - \beta_m \coth \beta_m \sinh \frac{m\pi d}{L} \right) \sin m\pi c \end{array} \right\} = abd_{35} \end{cases}$$

$d_{32}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*m*(m^2+1/6*n^2*L^2)/(n*(m^2+n^2*L^2)
^2);
while abs(sum_x-sum_x1)>=1.0e-05

    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
```

```
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*m*(m^2+1/6*n^2*L^2)/(n*(m^2
    +n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
n
```

$d_{33}$is calculated as follows:

```
syms a b c d L
num=1;sum_x1=0;m=1;n=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
sum_x=sin(m*pi*c)*sin(n*pi*d)*m*(1/6*m^2+n^2*L^2)/(n*(m^2+n^2*L^2)
^2);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    m=m+2;
    n=n+2;
    sum_x2=sin(m*pi*c)*sin(n*pi*d)*m*(1/6*m^2+n^2*L^2)/(n*(m^2
    +n^2*L^2)^2);
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
n
```

$d_{34}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the value of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*sin(m*pi*c)*((2*(1+1/6)/(1-1/6)+bm*coth(bm))
*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L));
while abs(sum_x-sum_x1)>=1.0e-05
```

```
    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
    (1-1/6)*(sinh(2*bm)/2)+bm)*sin(m*pi*c)*((2*(1+1/6)/(1-1/6)+bm*coth
    (bm))*sinh(m*pi*d/L)-m*pi*d/L*cosh(m*pi*d/L));
    sum_x=sum_x1+sum_x2;
end
sum_x
num
m
k
```

$d_{35}$is calculated as follows:

```
syms a b bm c d L
num=1;sum_x1=0;m=1;k=1;
L=input('enter the value of the ratio of a to b>');
c=input('enter the value of the ratio of x_axis coordinate to a>');
d=input('enter the walue of the ratio of y_axis coordinate to b>');
bm=m*pi/L;%bm=m*pi/L;
sum_x=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
(1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth(bm)*sinh
(m*pi*d/L))*sin(m*pi*c);
while abs(sum_x-sum_x1)>=1.0e-05
    sum_x1=sum_x;
    num=num+1;
    k=k+2;
    m=m+2;
    bm=m*pi/L;
    sum_x2=L*sinh(bm)*(k^2*L^2+(2-1/6)*m^2)/(m^2+k^2*L^2)^2/((3+1/6)/
    (1-1/6)*(sinh(2*bm)/2)+bm)*(m*pi*d/L*cosh(m*pi*d/L)-bm*coth(bm)
    *sinh(m*pi*d/L))*sin(m*pi*c);
end
sum_x
num
m
k
```

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