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Eli Gershon
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H_∞ Control and Estimation of State-multiplicative Linear Systems

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Eli Gershon · Uri Shaked · Isaac Yaesh

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With 24 Figures

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To Bina, Zippi and Marina

E.G U.S I.Y

Preface

This monograph considers the basic and fundamental control and estimation problems of noisy state-multiplicative uncertain continuous and discrete-time linear systems in an H_∞ setting. Applying various mathematical tools, these problems, and related ones, are solved in a comprehensive and detailed manner. In addition to the theoretical aspects of the above topics, the monograph contains various practical state of the art examples taken from the control engineering field.

Systems, where some of its uncertain parameters can be described by white noise processes, attracted the attention of scientists, particularly control theoreticians, in the early sixties, where most of the research was focused upon continuous-time problems. In fact, continuous-time state-multiplicative systems emerge quite naturally from basic principles in chemistry, biology, ecology and other fields even under somewhat restrictive assumptions. Furthermore nonlinear input-output maps can be approximated quite accurately by these systems. In principle, noisy state-multiplicative systems belong to a class of systems termed 'bilinear stochastic systems' which turns out to be a special case of the vast class of bilinear systems. The bilinearity of these systems implies that the system is linear in the product of the states and the stochastic multiplicative terms. Early research on the control of these systems concentrated mainly on stability issues and on basic control and estimation problems, analogous to their deterministic counterparts. In the eighties, along with the state-space formulation of the H_∞ control theory, a renewed interest in these systems yielded new results in both the analysis and synthesis aspects within the continuous-time domain. In the nineties, the theoretical treatment of state-multiplicative systems was extended to include discrete-time systems and many quite involved synthesis issues, including measurement-feedback, tracking control and mixed performance control and estimation solutions. By accumulating sufficient design experience with the above techniques, as demonstrated by the solution of some practical engineering problems, the applicability of the various topics tackled by the theoretical research has been facilitated.

In this monograph we address the major issues addressed by the research during the past two decades. Our aim is twofold: on the one hand we introduce and solve problems that were previously solved or partially solved. Here we apply methods which are taken from the mainstream of control theory, where we somewhat relax the mathematical burden typically encountered, by and large, in the field of stochastic processes. We thus simplify the introductory topics which are used at a later stage for the solution of much more considerably involved problems.

On the other hand we formulate and fully solve problems which were not previously tackled in this field. These problems, both from a theoretical and from a practical point of view, form the major part of this monograph.

In our treatment we apply new approaches, typically applied to deterministic systems in the past, to noisy state-multiplicative systems. Beside the theoretical treatment of the various problems contained in the book, we present six real practical engineering systems where extensive use is made of the theory we have developed. We especially emphasize this point to highlight the applicability of the theoretical results achieved in this field to practical problems, since this approach adds considerably to the value of the subject material under study.

The monograph is addressed to engineers engaged in control systems research and development, to graduate students specializing in control theory and to applied mathematicians interested in control problems.

The reader of this book is expected to have some previous acquaintance with control theory. The reader should also have taken, or be taking concurrently, introductory courses in state-space methods, emphasizing optimal control methods and theory. Some knowledge in stochastic processes would be an advantage, although the monograph is, in a sense, self contained and provides a basic background of the subject. The basic stochastic tools needed to master the subject material are given in the Appendix, where basic concepts are introduced and explained.

The book consists of four parts which include: introduction and literature survey, continuous-time stochastic systems, discrete-time systems and an especially extensive application section. The second part includes the following five chapters (Chapters 2-6): In Chapter 2 we present the formulation of the stochastic version of the bounded real lemma and the solutions to the state and the measurement-feedback control problems, utilizing a Luenberger-type observer. Chapter 3 contains a treatment of the stationary estimation problem where a general type of filter is considered. In Chapter 4 various preview-patterns are treated within the stochastic tracking problem. The contents of Chapter 5 deal with the stochastic static measurement control problem for nominal systems and for systems with polytopic uncertain parameters. Chapter 6 considers the stochastic passivity issue of state-multiplicative systems.

In Part 3 (discrete-time systems, Chapters 7-10) we present the discrete-time counterparts to Chapters 2-6. The last part (Part 4) introduces six practical examples of noisy state-multiplicative control and filtering problems, taken from various fields of control engineering, including tracking, guidance and navigation control. In addition to the above four parts, the book contains an extended Appendix consisting of three parts, the first of which contains a basic introduction to stochastic differential equations. The last two parts of the appendix consider theoretical and practical aspects of difference and differential linear matrix inequalities that are used in our solutions.

A few words about the numbering scheme used in the book are in order. Each chapter is divided into sections. Thus, Section 2.3 refers to the third section within the second Chapter. In each chapter, theorems, lemmas, corollaries, examples and figures are numbered consecutively within the chapter.

Acknowledgments

We owe a debt of gratitude to our colleagues from Romania: Professors V. Dragan, A. Stoica and T. Morozan, and the late Professor A. Halanay who have drawn our attention to the subject of multiplicative noise. We also acknowledge Dr. A. Ionita whose seminars inspired our interest in the subject.

We are grateful to Professor N. Berman of Ben Gurion University for sharing with us his experience in the field of stochastic processes and for his valuable comments that contributed to the rigor of the presentation of our results.

Finally, we thank C&M Maus Chair at Tel Aviv University, Holon Academic Institute of Technology and IMI Advanced Systems Division for supporting the research reported in this book.

Tel Aviv, Israel
March 2005

E. Gershon
U. Shaked
I. Yaesh

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Part III Discrete-time Systems

To Bina, Zippi and Marina

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Part III Discrete-time Systems

1 Introduction

This monograph considers linear systems that contain, in the general case, state, control and measurement multiplicative noise, in both the discrete-time and the continuous-time settings. In the discrete-time case, we consider the following system [43], [39]:

$$
\begin{aligned}
x_{k+1} &= (A_k + D_k v_k)x_k + B_{1,k} w_k + (B_{2,k} + G_k \eta_k) u_k, \quad x(0) = x_0 \\
y_k &= (C_k + F_k \zeta_k)x_k + D_{21,k} n_k, \\
z_k &= L_k x_k + D_{12,k} u_k, \qquad\qquad k = 0, 1, ..., N-1
\end{aligned} \tag{1.1}
$$

where $x_k \in R^n$ is the state vector, $w_k \in R^q$ is an exogenous disturbance, $u_k \in R^s$ is the control input signal, x_0 is an unknown initial state, $y_k \in R^r$ is the measured output, n_k is a measurement noise, $z_k \in R^m$ is the objective vector and where the sequences $\{v_k\}$,$\{\xi_k\}$ and $\{\eta_k\}$ are standard white noise scalar valued sequences with zero mean that satisfy:

$$
\begin{gathered}
E\{v_k v_j\} = \delta_{kj}, \quad E\{\eta_k \eta_j\} = \delta_{kj}, \quad E\{\eta_k v_j\} = \beta_k \delta_{kj}, \quad |\beta_k| < 1 \\
E\{\zeta_k \zeta_j\} = \delta_{kj}, \quad E\{\zeta_k \eta_j\} = \sigma_k \delta_{kj}, \quad E\{\zeta_k v_j\} = \alpha_k \delta_{kj}, \quad |\alpha_k| < 1, \quad |\sigma_k| < 1.
\end{gathered} \tag{1.2}
$$

The system (1.1) is a simplified version of a more involved system where: the control input feeds the measurements (i.e $D_{22,k} \neq 0$), the deterministic disturbance w_k appears also in z_k (i.e $D_{11,k} \neq 0$), and a finite sum of the multiplicative noise terms appears in the state equation instead of one multiplicative term. Notice that the system (1.1) is, in fact, the uncertain version of the linear system

$$
\begin{aligned}
x_{k+1} &= A_k x_k + B_{1,k} w_k + B_{2,k} u_k, \quad x(0) = x_0 \\
y_k &= C_k x_k + D_{21,k} n_k, \\
z_k &= L_k x_k + D_{12,k} u_k, \qquad\qquad k = 0, 1, ..., N-1
\end{aligned} \tag{1.3}
$$

where the system dynamic, control feed-through and measurement matrices, namely, A_k, $B_{2,k}$ and C_k are corrupted by the white noise signals $D_k v_k$, $G_k \eta_k$

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and $F_k\zeta_k$, respectively. In the present monograph, the exogenous disturbance w_k serves, in some cases, the purpose of analyzing both the H_∞ and H_2 norms of (1.1). Therefore, the disturbance feed-through matrix B_1 is intentionally not corrupted with noise in the above problem formulation in order to avoid terms that include products of noise signals.

In the continuous-time setting, we consider the following linear time invariant (LTI) Ito system with multiplicative noise:

$$\begin{aligned} dx(t) &= (Ax(t) + B_1 w(t))dt + B_2 u(t)dt + Gu(t)d\eta(t) + Dx(t)dv(t) \\ dy(t) &= (Cx(t) + D_{21}w(t))dt + Fx(t)d\zeta(t) \\ z &= Lx(t) + D_{12}u(t) \end{aligned} \tag{1.4}$$

where x, w, y, z are defined similarly to (1.1). The matrices $A,\ B_1,\ B_2,$ $C,\ L,\ D_{12},\ D_{21}$ and $D,\ F,\ G$ are constant matrices of the appropriate dimensions. Similarly to (1.1), the variables $v(t)$, $\eta(t)$ and $\zeta(t)$ are zero-mean real valued scalar Wiener processes that satisfy:

$$E\{dv(t)\} = 0,\ E\{d\zeta(t)\}=0,\ E\{d\eta(t)\}=0,\ E\{d\eta(t)^2\}=dt,$$

$$E\{dv(t)^2\}=dt,\ E\{d\zeta(t)^2\}=dt,\ E\{dv(t)d\zeta(t)\}=\alpha dt,\ |\alpha|< 1,$$

$$E\{d\eta(t)d\zeta(t)\}=\sigma dt,\ |\sigma|< 1,\ E\{dv(t)d\eta(t)\}=\beta dt,\ |\beta|< 1.$$

Namely, in the continuous-time case, adopting the formal notation of white noise signals being the derivative of corresponding Wiener processes, the system dynamic, control feed-through and measurement matrices A, B_2 and C are corrupted by the white noise signals $D\dot{v}$, $G\dot{\eta}$ and $F\dot{\zeta}$, respectively. Similarly to the discrete-time case, the system of (1.4) is a simplified version of a more involved system. We also consider the time varying version of (1.4).

The above systems belong to a class of systems called 'bilinear stochastic systems' (BLSS), which is a special case of the vast class of bilinear systems (BLS) [84]. The bilinearity of BLSS implies that the system is linear in the states and in the stochastic multiplicative terms. Deterministic bilinear systems (DBLS) include multiplicative terms where the states are multiplied by the control [13], meaning that while the system is linear in each variable, it is jointly non-linear. The latter systems evolved in the early 1960s out of a nuclear reactor research at Los Alamos where they arose naturally for neutron kinetics and heat transfer [13]. In fact, bilinear systems emerge, quite naturally, from basic principles in chemistry, biology, ecology and other fields [13]. Also, in somewhat restrictive cases, nonlinear input-output maps can be approximated quite accurately by BLS [84].

The diffusion processes found in the nuclear fission reaction can also be analyzed as BLSS. Other diffusion models, that can be described by the BLSS, have been developed for the migration of people and for biological cells. In a study of the immune system, it has been shown that BLSS may arise for cellular population and antibody concentration as a consequence of a stochastic coefficients for cell division, differentiation, and possibly antibody-antigen chemical affinity [13] In addition to the above, bilinear differential equations that are driven by standard Gaussian white-noise, have been used to analyze such diverse processes as sunspot activity and earthquakes (see [84] and the references therein). It is important to note that deterministic systems can be derived from BLSS models by considering the **mean** of the latter descriptions [84].

BLSS are also found in control and information systems. One important example is the human operator in a control task. From an intuitive point of view, the tracking error induced by the human operator in tracking an input signal depends, among other things, upon the magnitude of the input signal. When this input signal has a bandwidth significantly lower than the human operator induced error, the latter can be approximated to be a white noise signal. In such a case, a practical example for control-dependent noise (CDN) is obtained which has been investigated, both theoretically and experimentally, (see [79] and the references therein). Another example of CDN occurs in modelling thrust misalignment in a gas-jet thrusting system for the attitude control of a satellite, where the misalignment angle is being modelled as a stochastic process.[13]

Practical examples for state-dependent noise (SDN) are found in aerospace systems [96]. One example is the momentum exchange method for regulating the precession of a rotating spacecraft. Spacecrafts are often rotated about their symmetry axis in order to enhance their aerodynamic stability upon reentry, or to create an artificial sense of gravitational field to facilitate on board experiments in deep space [96]. Another aerospace related example is obtained when implementing gain scheduled controllers for large flight envelopes. In such cases the plant (i.e. the airframe model) is first linearized with respect to equilibrium points (referred to as 'trim' in the aerospace literature) where, naturally, the matrices A and B_2 strongly depend on the flight conditions of each envelope (operating) point, usually the corresponding dynamic pressure or the Mach number. A gain scheduled controller is then designed to stabilize the system.

Consider a state-feedback control

$$u = K(p_{meas})x$$

for the linear system

$$\dot{x} = Ax + B_2u$$

where

$$p_{meas} = p + \dot{v}$$

is the dynamic pressure corrupted by the broadband noise $\dot{v}$. Approximating to first order

$$K(p_{meas}) = K(p) + \frac{dK(p)}{dp}\dot{v}$$

one readily obtains a closed-loop system including the state-multiplicative noise, namely

$$dx(t) = (A + B_2K(p))xdt + D(p)xdv$$

where

$$D(p) = B_2\frac{dK(p)}{dp}.$$

One may be tempted to take the mean of the former and obtain a deterministic system that supposedly will possess the stability properties of the system with state-multiplicative noise, but such a simplifying approach may be erroneous.

The following simple example may serve the purpose of realizing the effect that multiplicative noise may have on stability. Consider

$$dx = -xdt - \delta xdv,$$

where $v(t)$ is a standard Wiener process (see Appendix A), and δ is a positive constant and where $E(x(0) = 0$ and $E(x(0)^2) = P_0$. Loosely speaking, the transfer function of this system is

$$G(s) = \frac{1}{s + 1 + \delta\dot{v}}.$$

Consider now $\phi(x) = x^2$. Then, by Ito lemma (see Appendix A), we have

$$d\phi = \phi_x dx + \delta^2x^2dt = 2x(-xdt + \delta xdv) + \delta^2x^2dt = (\delta^2 - 2)x^2dt + 2x^2\delta dv$$

Taking the expected value of both sides and defining:

$$P(t) = E(x^2(t)) = E(\phi(x)),$$

one readily sees that

$$\frac{dP}{dt} = (\delta^2 - 2)P.$$

Namely, the covariance $P(t)$ of $x(t)$ will tend to zero (meaning that the system is mean square stable as explained in the sequel) if and only if $\delta < \sqrt{2}$. Violating the latter condition will cause divergence of the system in spite of the fact that the expected value $dx/dt = -x$ of

$$dx = -xdt - \delta xdv$$

is stable. This simple example motivates the exact analysis of the effect of white multiplicative noise on system stability and performance. Few problems where such an analysis is required are listed next.

1.1 Stochastic Linear Quadratic Control and Estimation Problems

Many researchers have dealt in the past with various aspects of linear quadratic control and filtering problems in systems with multiplicative noise, mainly in the continuous-time case (for an overview of these works see [84]). Several important results in the area of stationary control with perfect measurements have appeared in [79], [108], [61], [105] and [106]. In [79], the problem of the output-feedback control with perfect measurement for stationary continuous-time system which contain both CDN and SDN, was solved using a special form of the minimum principle applied to matrix differential equations. Both the finite and the infinite time problems were solved in [79].

The linear quadratic control and H_2-type filtering problems for systems with multiplicative noise were solved in the discrete-time case by [90]. The continuous counterpart of the latter, in the stationary case, has been solved by [82]. In the case where the measurements are noisy, both the continuous time-varying estimation and quadratic control problems have been solved by [94]. The latter solution has been obtained in both the non-stationary and stationary cases, where in the former case a set of matrix nonlinear differential equations had to be solved. In the stationary case, non-linear algebraic equations had to be solved. It is important to note that the separation principle [57], [116] does not hold in the BLSS and, therefore, the filtering and the control problems have to be solved simultaneously [94]. In the absence of multiplicative noise the separation principle [57] is restored.

1.2 Stochastic H_∞ Control and Estimation Problems

Control and estimation of systems with state-multiplicative white noise has been a natural extension of the deterministic H_∞ control theory that was developed in the early 80s. The solutions, in the stochastic H_∞ context, of the latter problems were achieved by applying the stochastic version of the deterministic BRL [57], for both the continuous- and the discrete-time setups. Thus, for example, once the state-feedback solution for the stochastic H_∞ control problem was obtained, the corresponding measurement-feedback control problem has been solved, similarly to the deterministic case, by transforming the problem to one of filtering (with the aid of either a Luenberger observer [52] or a general-type filter [42]). The resulting filtering problem has then been solved by applying the stochastic BRL.

Although, in outline form, the stochastic problems are solved similarly to their deterministic counterpart, it is noted that the separation principle does not hold in the stochastic case [84], [13]. It is also noted that, once the covariance of the white noise process is set to zero (i.e, deleting the stochastic terms), the deterministic solutions should be fully recovered from their stochastic counterparts. In the sequel we bring a short survey of the various problems that were tackled in the stochastic H_∞ literature for the discrete- and the continuous-time setups.

1.2.1 Stochastic H_∞: The Discrete-time Case

The formulation of the stochastic BRL for the discrete-time case has been obtained by [87], [10], [26] and [43]. In [87], a version of the BRL for discrete-time varying systems with state-dependent noise is proved. For the time-invariant case, when the states and the control matrices are affected by noise, a corresponding result has been derived in [10]. A Linear Matrix Inequality (LMI) version for the following more involved time-varying system :

$$
\begin{aligned}
x_{k+1} &= (A_k^0 + \textstyle\sum_{i=1}^r A_k^i w_k^i)x_k + (B_{1,k}^0 + \textstyle\sum_{i=1}^r B_{1,k}^i w_k^i)w_k + B_{2,k}u_k,\\
y_k &= C_{2,k}x_k + D_{21,k}w_k,\\
z_k &= (C_{1,k}^0 + \textstyle\sum_{i=1}^r C_{1,k}^i w_k^i)x_k + (D_{11,k}^0 + \textstyle\sum_{i=1}^r D_{11,k}^i w_k^i)w_k + D_{12,k}u_k,
\end{aligned}
$$

was formulated by [87]. In the latter system, $\{w_k^i\}$ are standard non-correlated random scalar sequences with zero mean (note that w_k is the usual disturbance in the H_∞ sense). Based on a new version of the BRL for the above system, the problem of output-feedback control was solved in [26] for both the finite-horizon and the stationary cases. Necessary and sufficient conditions for the existence of a stabilizing deterministic controller were derived there which ensure an imposed level of attenuation. The latter conditions were derived in terms of a solution of two linear matrix inequalities that satisfy a complementary rank condition. It is noted that in the finite-horizon case the resulting necessary and sufficient conditions of [26] lead to an infinite number of linear inequalities that have to be solved. In the stationary case, only two LMIs have to be solved there. Also in [26], an explicit formulae for the resulting controller has been found under certain conditions. As a particular case, the above problem has been solved for discrete time-varying periodic systems where a periodic γ-attenuating controller can be computed as a function of a certain extended LMIs system. A major drawback of the work of [26] is that it does not allow for stochastic uncertainty in the measurement matrix- thus limiting its application to practical problems. In the H_∞ filtering, the work of [26] will be restricted to processes whose measurement matrix is perfectly known.

The stationary H_∞ and mixed H_2/H_∞ filtering problems were solved in [40], [46]. In these works the system matrices were allowed to reside in a polytopic setup.

1.2.2 Stochastic H_∞: The Continuous-time Case

In the continuous-time setting, the formulation of a stochastic BRL was obtained by [24], [86] for time-varying systems with only state-dependent uncertainty. A stationary stochastic BRL for LTI systems with state and input stochastic uncertainties has been derived by [59], where an LMI formulation has been obtained.

The problems of state and output feedback control have been solved for various versions of the system (1.4). The state-feedback control problem was solved by [85],[25],[56] and [103]. In [56] the following stationary LTI system is considered, which is similar to (1.4) but includes additional multiplicative white-noise terms:

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u u(t)dt + B_w w(t)dt + \textstyle\sum_{i=1}^{L}(A_i x(t) + B_{u,i}u(t) \\ &\quad + B_{w,i}w(t))dp_i(t), \\ dz(t) &= C_z x(t)dt + D_{zu}u(t)dt + D_{zw}w(t)dt + \textstyle\sum_{i=1}^{L}(C_{z,i}x(t) + D_{zu,i}u(t) \\ &\quad + D_{zw,i}w(t))dp_i(t). \end{aligned} \tag{1.5}$$

In this system, x, u, w and z are defined as in (1.4) and the process $p = (p_1, ..., p_L)$ is a vector of mutually independent standard zero mean Wiener processes. In [56], using the LMI optimization method, a static state-feedback law of the form $u = Kx$ was derived such that the closed-loop system is mean square stable and satisfies a given upper-bound on the output energy.

The state-feedback control problem has also been solved by [103], [104] for a simplified form of (1.5) (which only contain a state-multiplicative noise). A necessary and sufficient condition under which there exists a state-feedback stabilizing controller was derived. The latter condition is given in terms of a generalized game-type algebraic Riccati equation, arising from a stochastic linear quadratic game. Unlike the solution of [56], [103] considers stability of the exponentially mean square type.

We note that the solution of [103] applies the concept of entropy between two probability measures (see [30], for a review of this concept). Also in [110] the case of uncertain systems with state-multiplicative noise and time delays was treated. In [110], the system also contains time-varying parameters uncertainties, where a state-feedback controller was obtained.

The dynamic output-feedback control was solved by [59], [52](for systems with additional tracking signal) and [15] (which also includes the stochastic H_2 solution). The problem of dynamic output-feedback control for the system below has been solved by [59] in the infinite-horizon case.

$$\begin{aligned} dx(t) &= Ax(t)dt + A_0 x(t)dw_1(t) + B_0 w(t)dw_2(t) + B_1 w(t)dt + B_2 u(t)dt \\ y(t) &= C_2 x(t) + D_{21}w(t) \\ z(t) &= C_1 x(t) + D_{11}w(t) + D_{12}u(t). \end{aligned}$$

Based on a stationary stochastic BRL developed in [59], the solution was obtained in a form of two coupled non-linear matrix inequalities (instead of two uncoupled LMIs and a spectral radius condition in the deterministic case

[35]). A necessary and sufficient condition for the existence of a stabilizing controller was achieved there. In addition, a stability radii for the solution was obtained where a lower bound for these radii was determined. A major drawback of the work of [59] is that, beside the fact that the solution involves non-linearity, it does not allow for stochastic uncertainty in the measurement matrix. The stationary H_∞ filtering problems in the continuous-time case were solved in [45], [41], [2].

1.3 Some Concepts of Stochastic Stability

Stability is a quantitative property of the solutions to differential equations, which can be often studied without a direct recourse to solving the equations. Stability concepts are usually defined in terms of convergence relative to parameters such as the initial conditions, or the time parameter. The literature on the topic is abundant (see [69] and the references therein). Many concepts of stability have been studied and criteria have been derived mainly for the stability of deterministic systems.

The common modes of convergence which appear in the literature include: convergence in probability, convergence in the mean square and almost sure convergence. It is important to note that the stochastic stability in probability is too weak to be of practical significance [69] and examples can be constructed so that although $lim_{t\to\infty}Prob\{||x(t)|| > \epsilon\} = 0$, for $\epsilon > 0$, the samples $x(t)$ themselves do not converge to zero. Since also convergence in the mean square implies convergence in probability we do not use the latter definition of convergence [63]. In this section we review two types of mean square stability that are most applied in the literature of linear systems with stochastic state multiplicative noise [24],[56],[58],[59],[87] and that are, therefore, of major importance in our monograph.

1.3.1 Asymptotic Mean Square Stability

We consider the following discrete-time stochastic system [12]:

$$x_{k+1} = (A_0 + \sum_{i=1}^{L} A_i v_{i,k}) x_k \tag{1.6}$$

where $\{v_{i,k}\}$ are independent, identically distributed zero-mean random variables with the statistics of $E\{v_i v_i^T\} = \sigma_i^2$ and where the initial condition x_0 is independent of the above random sequence. Defining the state covariance matrix as $M_k = \underset{v_i}{E}\{x_k x_k^T\}$, it can be shown that M_k satisfies the the following linear recursion:

$$M_{k+1} = A_0 M_k A_0^T + \sum_{i=1}^{L} \sigma_i^2 A_i M_k A_i^T, \quad M_0 = E\{x_0 x_0^T\} \tag{1.7}$$

If the latter linear recursion is stable, meaning that $\lim_{k\to\infty} M_k = 0$ regardless of M_0, then the system is said to be **asymptotically mean square stable**.

Mean square stability can be verified directly by considering a stochastic Lyapunov function of the type $V(\xi) = \xi^T P \xi$, where $\underset{v_i}{E}\{V(x)\}$ decreases along the trajectories of (1.6) making the function $V(M) = Tr\{MP\}$ a linear Lyapunov function for the deterministic system (1.9) [12]. Similarly the system

$$dx = A_0 x dt + \sum_{i=1}^{L} A_i d\beta_i, \tag{1.8}$$

where $E\{d\beta^2\} = \sigma^2 dt$, is mean square stable whenever the following deterministic differential equation:

$$\dot{M} = A_0 M + M A_0^T + \sum_{i=1}^{L} \sigma_i^2 A_i M A_i^T, \quad M_0 = E\{x(0)x^T(0)\} \tag{1.9}$$

is stable.

1.3.2 Mean Square Exponential Stability

We describe this concept in the continuous-time setting. Consider the following linear system:

$$dx(t) = A(t)x(t)dt + \sum_{j=1}^{d} A_j(t)x(t)dw_j(t), \quad x(t_0) = x_0 \tag{1.10}$$

where $dw_j(t)$ are independent standard Wiener processes. The latter system is exponentially stable in the mean square if there exists ([24],[69]) $\beta \geq 1$ and $\alpha > 0$ such that the

$$E\{\|x(t)\|^2\} \leq \beta e^{-\alpha(t-t_0)}||x_0||^2$$

for all $t \geq t_0$ and for all $x_0 \in \mathcal{R}^n$.

Mean square stochastic stability can be verified also using quadratic stochastic Lyapunov functions where one has to require (see [58] and Appendix A) that $LV(x,t) < -\epsilon||x||^2$ for all $x \in \mathcal{R}^n$ and for all $t > t_0$ where $\epsilon > 0$.

Although in general mean square stability neither implies or is implied [63] by almost sure stability, in a certain class of stochastic systems to which (1.1) belongs mean square exponential stability implies that $x_k \to 0$ almost surely [63]. Since also when dealing with time-invariant systems mean square stability implies also exponential mean square stability, then most of the results in the present monograph deal with just mean square stability.

For additional discussion of the issue of stochastic stability the reader is referred to Appendix A.

1.4 Game-theory Approach to Stochastic H_∞ Control

The framework of dynamic (differential) game-theory stands out as the most natural approach among the different time-domain approaches to the solution of various problems in the deterministic H_∞ setting [3].

It has been shown that the original H_∞ optimal control problem (in its time domain equivalent) is in fact a zero-sum minmax optimization problem where the controller is the minimizing player that plays against nature (the disturbance) - which is the maximizing player [3]. In the 1990s, the game theory approach has been successfully applied to various H_∞ control and estimation problems. The problems of preview tracking control in both, the continuous and the discrete-time case were solved, for example, via a game theory approach in [17], [98]. The solution of [17] was also extended to the robust case (applying game-theory approach) in [18], where the system matrices reside in given uncertainty intervals.

Application of the game-theory approach to the field of state-multiplicative noise H_∞ control follows along lines similar to that of the deterministic case, where a saddle-point strategy is sought for both players of a cost function which both the minimizer (controller) and maximizer (disturbance) share. The stochastic state-feedback H_∞ control was solved, via the game theory approach, by [103] where a stochastic game is considered.

In this monograph we apply game-theoretic strategies in both, Chapters 4 and 9, where we solve the stochastic counterpart of the deterministic preview tracking control of [98] and [17].

1.5 The Use of $\sigma-$algebra for the Study of State-multiplicative Noisy Systems

The proper use of $\sigma-$algebra calculus is inherent in this monograph to the study of systems with state-multiplicative noise in the H_∞ context, given the nature of the multiplicative term (be it in the dynamics, the disturbance or the input matrices). This term is a white noise sequence (in the discrete-time case), or a Wiener process (in the continuous-time case). In the sequel we bring the description of a system with state-multiplicative noise and the signals involved in both the continuous- and the discrete-time setups.

1.5.1 Stochastic Setup: The Continuous-time Case

Given the following system with state-multiplicative noise:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)]dt + D(t)x(t)d\beta(t),\ x(0) = x_0, \\ y(t) &= C_2(t)x(t) + D_{21}(t)u(t), \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t), \end{aligned} \tag{1.11}$$

where, for simplicity, we consider a single, zero-mean, Wiener process, denoted by $\beta(t)$. We note that all the stochastic differential equations in this monograph are interpreted to be of the Ito type.

In order to elaborate on the nature of the signals involved in the system i.e: $x(t)$, $w(t)$, $u(t)$, $y(t)$ and $z(t)$ we introduce here the notations used for the continuous-time systems of Part 2.

In the continuous-time setup we provide all spaces $\mathcal{R}^k,\ k \geq 1$ with the usual inner product $< \cdot, \cdot >$, and with $|| \cdot ||$ we denote the standard Euclidean norm.

It is assumed here that the Wiener process β_t is defined on a probability space $(\Omega, \mathcal{F}, P)$. We then write $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ to denote the filtered probability space generated by the Wiener process β_t, that is, the increasing family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by β_t in the usual sense. The control signal u_t is taken to be a function of the observations, namely, $u_t = K(\mathcal{Y}_t)$, where $\mathcal{Y}_t = \{y(s) : s \leq t\}$. Thus, u_t is $\mathcal{F}_t$-measurable for all $t \in [0, T]$, and it is also non-anticipative process. Before characterizing the disturbance process w_t, we introduce the Hilbert space $\tilde{L}^2([0, T]; \mathcal{R}^k)$ which consists of, by definition, the the family of all non-anticipative stochastic processes, say f(t), defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and satisfy

$$||f||^2_{\tilde{L}^2} = E\{\int_0^T ||f(t)||^2 dt\} = \int_0^T E\{||f(t)||^2\}dt < \infty.$$

Thus, we assign for the system of (1.11) the above notations where $x(t) \in \tilde{L}^2([0, T]; \mathcal{R}^n$ is the state vector, $w(t) \in \tilde{L}^2([0, T]; \mathcal{R}^p)$ is an exogenous disturbance, $z(t) \in \tilde{L}^2([0, T]; \mathcal{R}^m$ is the objective vector and $u(t) \in \tilde{L}^2([0, T]; \mathcal{R}^l$ is the control input signal and $y(t) \in \tilde{L}^2([0, T]; \mathcal{R}^z$ is the measurement signal.

By writing $w(t) \in \tilde{L}^2([0, T]; \mathcal{R}^p)$ we thus mean that $w(t)$ is a finite (mean square) energy signal, measurable on $\mathcal{F}_t$, and consequently that $w(t)$ is, for example, a nonanticipative feedback strategy of the type $w(t) = f(x(\tau),\ \tau \leq t$.

1.5.2 Stochastic Setup: The Discrete-time Case

Similarly to the continuous-time case, we consider the following system with state-multiplicative noise:

$$\begin{aligned} x_{k+1} &= (A_k + D_k\beta_k)x_k + B_{2,k}u_k + B_{1,k}w_k, \qquad k = 0, 1, ..., N-1, \\ y_k &= C_{2,k}x_k + D_{21,k}u_k, \\ z_k &= C_{1,k}x_k + D_{12,k}u_k, \end{aligned} \tag{1.12}$$

where, for simplicity, we consider a single zero-mean white noise sequence, denoted by β_k. In order to describe the nature of the signals involved in the system, i.e: $x_k,\ w_k,\ u_k,\ y_k$ and z_k, we bring below the notations used for the discrete-time systems of Part 3.

Denoting by $\mathcal{N}$ the set of the positive integers in $[1\ N]$, we denote by $l^2(\Omega, \mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a $\sigma-$algebra of subsets of Ω called events, and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of $\sigma-$algebras which is generated by $\beta_j,\ 0 \le j \le k-1$. $\mathcal{F}_k \subset \mathcal{F}$. We also denote by $\tilde{l}^2(\mathcal{N};\mathcal{R}^n)$ the space of n-dimensional nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k \in L^2(\Omega,\mathcal{R}^n)$. On the latter space the following l^2-norm is defined:

$$||\{f_k\}||^2_{\tilde{l}_2} = E\{\textstyle\sum_0^{N-1} ||f_k||^2\} = \sum_0^{N-1} E\{||f_k||^2\} < \infty, \quad \{f_k\} \in \tilde{l}_2(\mathcal{N};\mathcal{R}^n) \tag{1.13}$$

where $||\cdot||$ is the standard Euclidean norm. Let $\phi : R^n \times R^m \to R$ be a continuous function and let $z \in R^n$ and W be an R^m valued random variable defined on $(\Omega,\mathcal{F},P)$. Define $E_W\{\phi(z,W)\} = \int_\Omega \phi(z,W(\omega))dP_W(\omega)$, where P_W is the measure induced by W (which is the restriction of P to the the $\sigma-$algebra generated by W). Note that it readily follows from this definition that $E_W\{\phi(z,W)\} = E\left[\phi(Z,W)/Z=z\right]$ where Z is any random variable defined on $(\Omega,\mathcal{F},P)$, which is independent on W.

Thus, we assign for the system (1.12) the above notations where $\{x_k\} \in \tilde{l}^2([0,N];\mathcal{R}^n)$ is the state vector, $\{w_k\} \in \tilde{l}^2([0,N-1];\mathcal{R}^p)$ is an exogenous disturbance, $\{z_k\} \in \tilde{l}^2([0,N];\mathcal{R}^m)$ is the objective vector and $\{u_k\} \in \tilde{l}^2([0,N-1];\mathcal{R}^l)$ is the control input vector and $\{y_k\} \in \tilde{l}^2([0,N];\mathcal{R}^z)$ is the measurement vector.

1.6 The LMI Optimization Method

The following is a short description of the LMI technique, which will be used extensively in the proposed research for stationary systems.

A LMI is any constraint of the form

$$A(x) \equiv A_0 + x_1A_1 + ... + x_NA_N < 0 \tag{1.14}$$

where $x = (x_1, ..., x_N)$ is a vector of unknown scalars and $A_0, ..., A_N$ are given symmetric matrices. The above LMI is a convex constraint on x since $A(y) < 0$ and $A(z) < 0$ imply that $A(\alpha y + (1-\alpha)z)) < 0$, for all $\alpha \in [0,\ 1]$. As a result, its solution set, if it exists, is called the feasible set and it is a convex subset of R^N. Within the feasible set one may choose to solve optimization problems which are still convex such as to minimize $c_0 + c_1x_1 + ... + c_nx_n$. It should be noted that cost functions for systems with state multiplicative noise often lead, as in the deterministic case, to quadratic inequalities in the search variables vector $x = (x_1, ..., x_N)$ of the following form :

$$Q(x) - S(x)R(x)^{-1}S^T(x) > 0, R(x) > 0$$

where $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$ and $S(x)$ depend affinely on x. In such cases, the application of the Schur complements (see e.g. [12]) shows that an equivalent LMI is :

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$$

where in many cases, repeated application of Schur's complements is needed to achieve an LMI which is affine not only in the search variables but also in some of the parameters (e.g. matrices A, B_2, etc.).

The LMI technique for solving convex optimization problems has led to a major development in the field of H_∞ control and estimation in the past few years [12], [35] motivated by the efficiency of the interior point algorithms to solve LMIs. In fact, all problems which are convex in the optimizing parameters in the field of system theory are amenable to this technique,and, therefore, since any matrix inequality of the form $L(X_1, X_2, ...) < 0$ being affine in matrix valued arguments $X_1, X_2, ...$ can be written at the form of (1.14). Consider, for example, the following Lyapunov inequality in matrices in $\mathcal{R}^{2\times 2}$:

$$A^T P + PA < 0.$$

Defining

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

The above inequality can be readily expressed in the form of (1.14). Note also that in the case where the dynamic matrix A is uncertain say, $A = \Sigma A_i f_i,\ f_i \in [0\ 1],\ \Sigma f_i = 1$, the resulting inequality should also be affine in A in order to be formulated as (1.14).

In a case where the system is linear and stationary, the LMI technique is easily applied to the solution of the H_∞ control problem, either with perfect state-measurement or with noisy one (and thereby also to the filtering problem). It is important to note that unlike the Riccati equation approach, which supplies the central solution, the LMI approach produces a solution which may be far from the central solution [57]. The inequality-based solution using the latter approach can be easily extended to other additional requirements such as the minimum upper-bound on variance, in the case of nominal H_∞ estimation problem (the "mixed H_2/H_∞" problem). It is also important to note that in the case where the system parameters are modelled in a polytopic framework, the LMI technique can be easily applied to allow solutions for both the control and the estimation problems (see, for example, [38] for the solution of the H_∞ filtering problem, in the presence of polytopic uncertainties) .

1.7 The DLMI Method

In this monograph, we apply the novel Difference LMI (DLMI) method which was originally developed for the solution of various control and estimation of finite horizon design problems for continuous and discrete Linear Time-Varying (LTV) systems.

To present the method, in the discrete-time case, we start by considering the solution P_k to the following problem:

$$\min Tr\{P_k\} \quad \text{subject to} \quad F(P_k, P_{k+1}) \le 0, P_N = 0$$

where $F(P_k, P_{k+1})$ is a Linear Matrix Inequality (LMI) which stems, say, from the formulation of the BRL. We note that this LMI contains elements which are linear in P_{k+1} and in P_k and that P_{k+1} is known, at each step k, since we assume a backward iterative solution starting from $P_N = 0$. We are then seeking P_k which will solve the problem. It turns out that the resulting P_k has the interesting property that it is as close to the Difference Riccati Equation (DRE) solution as the tolerance of the LMI solver routine. This feature, which is a result of the monotonicity properties of the DRE (see [7] pp. 277-278), allows one to mimic the DREs for LTV systems over a finite horizon and is thus useful as a consistency check. One is not restricted, however, to DRE's since convex constraints in the form of additional LMIs may be added which allow a solution to a wide range of multiobjective problems such as securing a given bound on the output peak[1].

In addition, the method allows one to solve problems over a finite horizon which have not been previously solved, such as the general type H_∞ filtering problem, whose solution was obtained by [37] for the LTI case over an infinite horizon only.

An attractive feature of LMIs in control and filtering has been their ability to cope with plant parameter uncertainties in stationary systems. In [12] a set of LMIs was introduced for the state-feedback problem, whose solution, if it exists, guarantees quadratic stability [93] and the required bound on the disturbance attenuation level of the closed-loop system, in spite of polytopic type uncertainties encountered in the plant. For the above set of LMIs, a single solution $(P,\ K)$ is sought where K is the required state-feedback gain and P is the kernel of a quadratic Lyapunov function common to all of the possible plants, occuring at different time instants k. While a single K is certainly justifiable (we seek a single state-feedback gain), the requirement for a common P, rather than P_k, is inherent in the optimization problem posed and it stems from the special structure of the LMI involved. With the new DLMI method, there is no longer a need to require the same P for all of the uncertain plants.

In Appendix B we bring the solution to the continuous-time BRL with the aid of the DLMI technique and we demonstrate the applicability and usefulness of the continuous DLMI in the solution of several control and estimation problems. Similarly, in Appendix C we bring the discrete-time DLMI counterpart technique which is somewhat simpler than the continuous-time method. We present there solutions to various important control problems [44] and we bring a numerical example that demonstrates the typical convergence of our algorithm.

1.8 Nomenclature

$(\cdot)^T$	matrix transposition.
$\mathcal{R}^n$	the n dimensional Euclidean space.
$\mathcal{R}^{n\times m}$	the set of all $n \times m$ real matrices.
$P>0, (P\geq 0)$	the matrix $P \in \mathcal{R}^{n\times n}$ is symmetric and positive definite (respectively, semi-definite).
$\|\|.\|\|_2^2$	the standard l_2-norm:$\|\|d\|\|_2^2 = (\Sigma_{k=0}^{N-1} d_k^T d_k)$.
$l_2[0 \quad N]$	the space of square summable functions over $[0 \quad N]$.
$\|\|f_k\|\|_R^2$	the product $f_k^T R f_k$.
$\\|f\\|$	the Euclidean norm of f.
$\underset{v}{E}\{\cdot\}$	the expectation with respect to v.
$[Q_k]_+$	the causal part of a sequence $\{Q_i,\ i = 1, 2, ..., N\}$.
$[Q_k]_-$	the anti causal part of a sequence $\{Q_i,\ i = 1, 2, ..., N\}$.
$Tr\{\cdot\}$	the trace of a matrix.
δ_{ij}	the Kronecker delta function.
$\delta(t)$	the Dirac delta function.
$\mathcal{N}$	the set of natural numbers.
Ω	the sample space.
$\mathcal{F}$	$\sigma-$algebra of subsets of Ω called events.
$\mathcal{P}$	the probability measure on $\mathcal{F}$.
$Pr(\cdot)$	probability of $(\cdot)$.
$l^2(\Omega, \mathcal{R}^n)$	the space of square-summable $\mathcal{R}^n-$ valued functions. on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.
$(\mathcal{F}_k)_{k\in\mathcal{N}}$	an increasing family of $\sigma-$algebras $\mathcal{F}_k \subset \mathcal{F}$.
$\tilde{l}^2([0, N]; \mathcal{R}^n)$	the space of nonanticipative stochastic processes: $\{f_k\} = \{f_k\}_{k\in[0,N]}$ in $\mathcal{R}^n$ with respect to $(\mathcal{F}_k)_{k\in[0,N)}$ satisfying $\|\|f_k\|\|_{\tilde{l}_2}^2 = E\{\sum_0^N \|\|f_k\|\|^2\} = \sum_0^N E\{\|\|f_k\|\|^2\} < \infty$, $f_k \in \tilde{l}_2([0, N); \mathcal{R}^n)$.
$\tilde{l}^2([0, \infty); \mathcal{R}^n)$	the above space for $N \to \infty$
$\tilde{L}^2([0, T); \mathcal{R}^k)$	the space of non anticipative stochastic processes: $f(\cdot) = (f(t))_{t\in[0,T]}$ in $\mathcal{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,T)}$ satisfying $\|\|f(\cdot)\|\|_{\tilde{L}_2}^2 = E\{\int_0^T \|\|f(t)\|\|^2 dt\} = \int_0^T E\{\|\|f(t)\|\|^2\}dt < \infty$.
$\tilde{L}^2([0, \infty); \mathcal{R}^n)$	the above space for $T \to \infty$.
$\begin{bmatrix} P & R \\ * & Q \end{bmatrix}$	$= \begin{bmatrix} P & * \\ R^T & Q \end{bmatrix}$, for symmetric P and Q, is the matrix $\begin{bmatrix} P & R \\ R^T & Q \end{bmatrix}$.
$diag\{A,\ B\}$	the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.
$col\{a, b\}$	the matrix (vector) $\begin{bmatrix} a \\ b \end{bmatrix}$.

1.9 Abbreviations

APN	Augmented Proportional navigation
BRL	Bonded Real Lemma
BLS	Bilinear System
BLSS	Bilinear Stochastic System
BRL	Bounded Real Lemma
CDN	Control Dependent Noise
DBLS	Deterministic Bilinear System
DLMI	Difference LMI (discrete-time)
	or
	Differential LMI (continuous-time)
DRE	Difference Riccati Equation
GBM	Geometrical Brownian Motion
LMI	Linear Matrix Inequality
LPD	Lyapunov Parameter Dependent
LTI	Linear Time Invariant
LTV	Linear Time Variant
MEL	Minimum Effort guidance Law
MSS	Mean Square Stable
NNC	Neural Network Controller
OPs	Operating Points
OUC	Open Unit Circle
RSL	Reduced Sensitivity guidance Law
SAC	Simplified Adaptive Control
SDN	State Dependent Noise
SISO	Single-Input-Single-Output
SNR	Signal to Noise Ratio
TRK	Stochastic Tracking Law

2

Continuous-time Systems: Control and Luenberger-type Filtering

2.1 Introduction

In this chapter we address the problems of continuous-time, state-multiplicative, H_∞ state-feedback control and estimation via the solution of the stochastic BRL which is formulated and proved at the beginning of the chapter. We then solve the dynamic output-feedback control problem by transforming the problem to an estimation one, to which we apply the result of the filtering solution. We derive solutions of both, the finite-horizon and the stationary cases for the above mentioned problems.

The general problems of the discrete-time and the continuous-time output-feedback control with state-multiplicative stochastic uncertainties have been treated by several groups. The discrete-time case has been solved by [26],[10] and [43]. The continuous-time counterpart case has been solved by [59], [52] (for systems with additional tracking signal) and [15]. The solution in [26] includes the finite and the infinite time horizon problems without transients. One drawback of [26] is the fact that in the infinite-time horizon case, an infinite number of LMI sets should be solved. Moreover, the fact that in [26] the measurement matrix has no uncertainty is a practical handicap: for example in cases where the measurements include state derivatives (e.g. altitude control of an aircraft or missile). The treatment of [59] includes a derivation of the stochastic BRL and it concerns only the stationary case where two coupled nonlinear inequalities were obtained.

In the present chapter we allow for a stochastic uncertainty in the measurement matrix and we address the problem via two new approaches: In the finite-horizon case we apply the DLMI method for the solution of the Riccati inequality obtained and in the stationary case we apply a special Lyapunov function which leads to an LMI based tractable solution. At the end of the chapter we bring an example which demonstrates the applicability of the theory developed.

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 21–44, 2005.

2.2 Problem Formulation

We treat the following problems:

i) H_∞ state-feedback control of systems with state-multiplicative noise:

We consider the following system:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)]dt + D(t)x(t)d\beta(t) + G(t)u(t)d\nu(t),\\ x(0) &= x_0,\\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t), \end{aligned} \tag{2.1}$$

where $x(t) \in R^n$ is the state vector, $w(t) \in \tilde{L}^2([0,T];\mathcal{R}^p)$ is an exogenous disturbance, $z(t) \in R^m$ is the objective vector and $u(t) \in R^l$ is the control input signal. It is assumed that $D_{12}(t)$ is of full rank for all $t \in [0,T]$.

The zero-mean real scalar Wiener processes of $\beta(t)$ and $\nu(t)$ satisfy:

$$E\{d\beta(t)^2\}=dt,\ E\{d\nu(t)^2\}=dt,\ E\{d\beta(t)d\nu(t)\}=\bar{\alpha}dt, |\bar{\alpha}|\leq 1,$$

Considering the following performance index for a prescribed $\gamma > 0$:

$$\begin{aligned} J_E &\stackrel{\triangle}{=} E\{\int_0^T ||z(t)||^2 dt - \gamma^2 \int_0^T ||w(t)||^2 dt\}\\ &+E\{x^T(T)P_T x(T)\} - \gamma^2 ||x_0||^2_{R^{-1}},\ R > 0,\ P_T \geq 0. \end{aligned} \tag{2.2}$$

Our objective is to find a state-feedback control law $u(t) = K(t)x(t)$ that achieves $J_E < 0$, for the worst-case of the process disturbance $w(t) \in \tilde{L}^2([0,T];\mathcal{R}^p)$ and the initial condition x_0 and for the prescribed scalar $\gamma > 0$.

ii) H_∞ filtering of systems with state-multiplicative noise:

We consider the following system:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t)]dt + D(t)x(t)d\beta(t),\ x(0) = x_0,\\ dy(t) &= [C_2(t)x(t) + D_{21}(t)w(t)]dt + F(t)x(t)d\zeta(t) + n(t)dt,\\ z(t) &= C_1(t)x(t), \end{aligned} \tag{2.3}$$

where $x(t),\ w(t),\ z(t)$ are defined above, $n(t) \in \tilde{L}^2([0,T];\mathcal{R}^z)$ is an additive measurement corruptive noise and $y(t) \in \mathcal{R}^z$. The zero-mean real scalar Wiener processes of $\beta(t)$ and $\zeta(t)$ satisfy:

$$E\{d\beta(t)^2\}=dt,\ E\{d\zeta(t)^2\}=dt, \quad E\{d\beta(t)d\zeta(t)\}= 0.$$

We consider the following filtered estimate system - the Luenberger type:

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + L(t)(dy(t) - C_2(t)d\hat{x}(t)), \quad \hat{x}_0 = 0. \tag{2.4}$$

Note that the initial value of the state vector x_0 of (2.3) is not known. It has a zero-mean value dictating, therefore, the zero initial value of the state estimator.

We denote

$$e(t) = x(t) - \hat{x}(t), \quad \text{and} \ \ \tilde{w}(t) = col\{w(t), n(t)\} \tag{2.5}$$

and we consider the following cost function:

$$J_F \stackrel{\triangle}{=} E\{\int_0^T ||z(t)||^2 dt - \gamma^2 [\int_0^T ||w(t)||^2 dt + \int_0^T ||n(t)||^2 dt]\}$$
$$-\gamma^2 ||x_0||^2_{R^{-1}}, \ R > 0. \tag{2.6}$$

Note that R represents the uncertainty in x_0. Namely, R of a norm tending to zero will force x_0 to zero.

Given $\gamma > 0$ and $R > 0$, we seek, in the filtering problem, an estimate $C_1(t)\hat{x}(t)$ of $C_1(t)x(t)$ over the finite time horizon $[0, T]$ such that J_F of (2.6) is negative for all nonzero $(\tilde{w}(t), x_0)$ where $\tilde{w}(t) \in \tilde{L}^2([0, T]; \mathcal{R}^{p+z})$ and $x_0 \in \mathcal{R}^n$. This estimate should be unbiased whenever $\tilde{w}(t)$ has zero-mean.

iii) H_∞ output-feedback control of systems with state-multiplicative noise - The finite-horizon case:

We consider the following system:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)]dt + D(t)x(t)d\beta(t) + G(t)u(t)d\nu(t),\\ x(0) &= x_0,\\ dy(t) &= [C_2(t)x(t) + D_{21}(t)w(t)]dt + F(t)x(t)d\zeta(t) + n(t)dt,\\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t). \end{aligned} \tag{2.7}$$

The zero-mean real scalar Wiener processes of $\beta(t)$, $\nu(t)$ and $\zeta(t)$ are defined in the above two problems where

$$E\{d\beta(t)d\zeta(t)\} = 0, \ E\{d\zeta(t)d\nu(t)\} = 0, \ E\{d\beta(t)d\nu(t)\} = 0.$$

We consider the following index of performance:

$$J_O \stackrel{\Delta}{=} E\{\int_0^T ||z(t)||^2 dt - \gamma^2 \int_0^T (||w(t)||^2 + ||n(t)||^2) dt\}$$
$$+Ex^T(T)P_T x(T) - \gamma^2 ||x_0||^2_{R^{-1}}, \ R > 0, \ P_T \geq 0, \tag{2.8}$$

We seek an output-feedback controller that achieves $J_O < 0$ for all nonzero $(w(t), x_0)$ where $w(t) \in \tilde{L}^2([0, T]; \mathcal{R}^p)$, $n(t) \in \tilde{L}^2([0, T]; \mathcal{R}^z)$ and $x_0 \in \mathcal{R}^n$. Similarly to the standard case [57], this problem involves an estimation of an appropriate combination of the states, and the application of the results of the state-feedback control problem with a proper modification.

iv) H_∞ output-feedback control of systems with state-multiplicative noise - The infinite-horizon case:

Given the following mean square stable system:

$$\begin{aligned} dx &= [Ax(t) + B_1 w(t) + B_2 u(t)]dt + Dx(t)d\beta(t) + Gu(t)d\nu(t),\ x(0) = 0,\\ dy(t) &= [C_2 x(t) + D_{21} w(t)]dt + Fx(t)d\zeta(t) + n(t)dt,\\ z(t) &= C_1 x(t) + D_{12} u(t), \end{aligned} \tag{2.9}$$

where the system matrices $A, B_1, B_2, D, G, C_2, D_{21}, F, C_1$ and D_{12} are all constant and $\beta(t),\ \nu(t),\ \zeta(t)$ are specified above. We seek an output-feedback controller that achieves, for a given $\gamma > 0$, $J_{SO} < 0$ where

$$J_{SO} \stackrel{\Delta}{=} E\{\int_0^\infty ||z(t)||^2 dt - \gamma^2 \int_0^\infty (||w(t)||^2 + ||n(t)||^2)dt\}$$
$$+E\{x^T(T)P_T x(T)\},\ P_T \geq 0, \tag{2.10}$$

for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$ and $n(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^z)$.

2.3 Bounded Real Lemma for Systems with State-multiplicative Noise

We address the problem of the stochastic state-feedback by deriving first a BRL for the following system:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t)]dt + D(t)x(t)d\beta(t),\ x(0) = x_0,\\ z(t) &= C_1(t)x(t), \end{aligned} \tag{2.11}$$

where x_0 is an unknown initial state and where $x(t) \in R^n$ is the state vector, $w(t) \in \tilde{L}^2([0,T];\mathcal{R}^p)$ is an exogenous disturbance and $z(t) \in R^m$ is the objective vector.

The zero-mean real scalar Wiener processes of $\beta(t)$ satisfies:

$$E\{d\beta(t)^2\} = dt.$$

Considering the cost function J_E of (2.2), our objective is to determine, for a given scalar $\gamma > 0$, whether J_E is negative for all nonzero $(w(t), x_0)$ where $x_0 \in \mathcal{R}^n$ and $w(t) \in \tilde{L}^2([0,N];\mathcal{R}^p)$.
We obtain the following result:

Theorem 2.1. *[26], [59], [86] Consider the system of (2.11). Given the scalar $\gamma > 0$, a necessary and sufficient condition for J_E of (2.2), to be negative for all nonzero $(w(t), x_0)$ where $w(t) \in \tilde{L}^2([0,T];\mathcal{R}^p)$ and $x_o \in \mathcal{R}^n$ is that there exists a solution $Q(t) > 0$ to the following Riccati-type equation:*

$$
\begin{aligned}
-\dot{Q}(t) &= Q(t)A(t) + A^T(t)Q(t) + \gamma^{-2}Q(t)B_1(t)B_1^T(t)Q(t) \\
&+C_1^T(t)C_1(t) + D^T(t)Q(t)D(t), \qquad Q(T) = P_T,
\end{aligned}
\tag{2.12}
$$

that satisfies $Q(0) < \gamma^2 R^{-1}$. When such $Q(t)$ exists, the corresponding optimal strategy of the disturbance $w(t)$ is:

$$
w^* = \gamma^2 B_1^T Q x. \tag{2.13}
$$

Proof: Sufficiency
Assume that there exists $Q(t)$ that satisfies (2.12) so that $Q(0) < \gamma^2 R^{-1}$. Consider:

$$
M(t) = E\{\int_0^T d(x^T Q(t)x)\} + x_0^T Q(0)x_0 - E\{x^T(T)P_T x(T)\}.
$$

Note that the fact that $Q(T) = P_T$ yields $M(t) = 0$. Using Ito lemma we have the following:

$$
d(x^T Q(t)x) = x^T \frac{\partial Q(t)}{\partial t} x dt + \frac{\partial}{\partial x}(x^T Q(t)x)dx_t + \frac{1}{2}Tr\{2Dxx^T D^T Q\}dt.
$$

Note that the last equation is the stochastic version of the chain rule for calculating differentials, where the last term is the sole contribution of the state-multiplicative term of $D(t)x(t)d\beta$ in (2.11).

We readily see that

$$
M(t) = x_0^T Q(0)x_0 - E\{x^T(T)P_T x(T)\} - \gamma^2 E\{\int_0^T ||w - \gamma^{-2}B_1^T Q(t)x||^2 dt\}
$$

$$
-E\{\int_0^T (||z||^2 - \gamma^2||w||^2)dt\},
$$

where we have used $Tr\{QDxx^T D^T\} = Tr\{x^T D^T QDx\}$ and the fact that mixed terms involving $x(t)$ and $\beta(t)$ drop under expectation (see Appendix A). Adding and subtracting $\gamma^2 x_0^T R^{-1}x_0$ to the above $M(t)$, we obtain that:

$$
M = -\gamma^2 E\{\int_0^T ||w - w^*||^2 dt\} - [E\{\int_0^T (||z||^2 - \gamma^2||w||^2)dt\} + E\{x^T(T)P_T x(T)\}
$$

$$
+\gamma^2 x_0^T R^{-1}x_0] + x_0^T[Q(0) - \gamma^2 R^{-1}]x_0 + Ex^T(T)[P_T - Q(T)]x(T) \tag{2.14}
$$

where $w^* = \gamma^{-2}B_1^T Qx$. Identifying the term in the brackets as J_E of (2.2) we see that:

$$
J_E = -\gamma^2 E\{\int_0^T ||w - w^*||^2 dt\} + x_0^T[Q(0) - \gamma^2 R^{-1}]x_0.
$$

Hence $J_E < 0$ if $Q(0) < \gamma^2 R^{-1}$.

This concludes the sufficiency part. Now, the condition of (2.12) with $Q(0) < \gamma^2 R^{-1}$ is also necessary and the reader is referred to [86] for the proof.

□

Remark 2.1. The differential equation (2.12) is solved by backward integration. An alternative result stems from (2.14) in the case where $Q(0)$ is taken to be $\gamma^2 R^{-1}$ and instead of requiring $Q(T) = P_T$ and $Q(0) < \gamma^2 R^{-1}$ we require that $P_T < Q(T)$.

2.3.1 The Stationary BRL

The derivation of the stationary stochastic BRL is obtained by two approaches. In the first approach we consider the following mean square stable system:

$$\begin{aligned} dx &= [Ax(t) + B_1 w(t)]dt + Dx(t)d\beta(t),\ x(0) = 0, \\ z(t) &= C_1 x(t), \end{aligned} \tag{2.15}$$

which is obtained from (2.11) for the case where the system matrices are constant and $T = \infty$. Considering the following index of performance:

$$J_{SE} \stackrel{\Delta}{=} E\{\int_0^\infty ||z(t)||^2 dt - \gamma^2 \int_0^\infty (||w(t)||^2)dt\} \tag{2.16}$$

we obtain the following result:

Theorem 2.2. *Consider the system of (2.15). Given the scalar $\gamma > 0$, a necessary and sufficient condition for J_{SE} of (2.16), to be negative for all nonzero $w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$ is that there exists a solution $\bar{Q} > 0$ to the following algebraic Riccati-type equation:*

$$\bar{Q}A + A^T\bar{Q} + \gamma^{-2}\bar{Q}B_1B_1^T\bar{Q} + C_1^TC_1 + D^T\bar{Q}D = 0. \tag{2.17}$$

When such $\bar{Q}$ exists, the corresponding optimal strategy of the disturbance $w(t)$ is:

$$w^* = \gamma^{-2}B_1^T\bar{Q}x. \tag{2.18}$$

Proof: The proof outline resembles that of the finite horizon case of Section 2.3. Thus, the optimal strategy w^* of (2.18) is obtained by completing to squares for $w(t)$, similarly to Section 2.3. Note that $J_{SE} < 0$ can also be achieved iff the following inequality holds:

$$\bar{Q}A + A^T\bar{Q} + \gamma^{-2}\bar{Q}B_1B_1^T\bar{Q} + C_1^TC_1 + D^T\bar{Q}D \le 0.$$

□

By using the nonstrict Schur's complements formula[12], the latter inequality can be readily transformed to the following LMI:

$$\Gamma_S \triangleq \begin{bmatrix} A^T\bar{Q} + \bar{Q}A & \bar{Q}B_1 & C_1^T & D^T\bar{Q} \\ * & -\gamma^2 I_p & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & -\bar{Q} \end{bmatrix} \leq 0. \tag{2.19}$$

The second approach for the solution of the stationary BRL is achieved by considering the finite horizon counterpart of this problem, assuming that the resulting system (2.15) is mean square stable. We consider the system of (2.11) for the case where the system matrices are constant and $T \to \infty$. Considering the results of [108] we obtain that the Riccati-type differential equation of (2.12) will converge to the algebraic equation of (2.17) where the pair $(A,\ C_1)$ is observable, $(A,\ B_1)$ is stabilizable and $P_T \geq 0$ (See Theorem 2.1, page 691 in [108]).

Remark 2.2. The optimal strategy of (2.18) is valid only if the resulting disturbance $w^*(t)$ is in $\tilde{L}^2([0,T];\mathcal{R}^p$. The question arises whether the resulting closed-loop system that is described by:

$$dx = [Ax(t) + \gamma^{-2}B_1B_1^T\bar{Q}]dt + Dx(t)d\beta(t),$$

is mean square stable. Namely, whether a solution of the algebraic equation (2.17) is strongly stabilizing [57]. A similar problem arises in the deterministic case ($D = 0$) where one has to show that the worst-case disturbance does not destabilize the system.

Two methods have been suggested, in the deterministic case, for proving that the corresponding algebraic Riccati equation (which is identical to (2.17) with $D = 0$) has a strong stabilizing solution $\bar{Q}$ that leads to a matrix $A + \gamma^{-2}B_1B_2^T\bar{Q}$ with eigenvalues that all reside in the open left half of the complex plane. The first method, which is the more direct one, is based on the properties of the corresponding Hamiltonian matrix [116]. Unfortunately, this method is not readily applicable to the stochastic case. The second method considers the algebraic Riccati equation as the limit of the differential Riccati equation in the case where the horizon T tends to infinity. It is shown in [57] that under the above mentioned conditions of stabilizability and observability there exists a solution to the algebraic Riccati equation which is strongly stabilizing. Since the same arguments of convergence of the solution of the differential Riccati equation (2.12) to a solution of (2.17) hold also in the stochastic case, strong stability is guaranteed if the latter solution is the one used in (2.18).

2.4 State-feedback Control of Systems with State-multiplicative Noise

We consider the following Riccati-type differential equation:

$$\begin{aligned}-\dot{Q}(t) &= Q(t)A(t) + A^T(t)Q(t) + \gamma^{-2}Q(t)B_1(t)B_1^T(t)Q(t)\\ &+C_1^T(t)C_1(t) - \bar{S}^T(t)\hat{R}^{-1}(t)\bar{S}(t) + D^T(t)Q(t)D(t),\\ Q(T) &= P_T,\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}\tilde{R} &= D_{12}^T D_{12}\\ \hat{R}(t) &= \tilde{R}(t) + G^T(t)Q(t)G(t),\\ \bar{S}(t) &= B_2^T(t)Q(t) + \bar{\alpha}G^T(t)Q(t)D(t) + D_{12}^T(t)C_1(t).\end{aligned} \tag{2.21}$$

The solution to the state-feedback control problem is obtained by the following:

Theorem 2.3. *[24], [59], [53] Consider the system of (2.1) with the feedback law $u(t) = K(t)x(t)$. Given $\gamma > 0$, a necessary and sufficient condition for J_E of (2.2) to be negative for all nonzero $(w(t), x_0)$ where $\tilde{w}(t) \in \tilde{L}^2([0,T];\mathcal{R}^p)$ and $x_o \in \mathcal{R}^n$ is that there exists a solution $Q(t) > 0$ to (2.20) that satisfies $Q_0 < \gamma^2 R^{-1}$.*

If there exists such $Q(t)$, then the state-feedback law is given by:

$$u(t) = K(t)x(t),$$

where (2.22)

$$K(t) = -\hat{R}^{-1}(t)[(B_2^T(t)Q(t) + D_{12}^T(t)C_1(t) + \bar{\alpha}G^T(t)Q(t)D(t))].$$

Remark 2.3. The general case, where a multiple set of correlated stochastic uncertainties appear in both the dynamics and the input matrices, can be readily solved by extending the results of Theorem 2.3. The proof outline of Theorem 2.3 in the later case is essentially unchanged. We bring below a simplified case which will be used in the infinite-horizon, stationary state-feedback control. We consider the following system:

$$\begin{aligned}dx &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)]dt + [\tilde{D}(t)d\nu(t)\\ &\quad +\hat{F}(t)d\zeta(t)]x(t) + G(t)u(t)d\nu(t), \quad x(0) = x_0,\\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t),\end{aligned} \tag{2.23}$$

where the variables $\beta(t)$ and $\nu(t)$ are zero-mean real scalar Wiener processes with the same statistic as above.

Note that the Wiener process of $\nu(t)$ appears in both, the dynamics and the input matrices. For system (2.23), we obtain the results of Theorem 2.3 where D^TQD in (2.20) is replaced by $\hat{F}^TQ\hat{F} + \tilde{D}^TQ\tilde{D} + 2\bar{\alpha}[\hat{F}^TQ\tilde{D} + \tilde{D}^TQ\hat{F}]$, and $\bar{\alpha}G^TQD$ is replaced by $\bar{\alpha}G^TQ\tilde{D} + G^TQ\hat{F}$ in $\bar{S}$ and $u(t)$ of (2.21) and (2.22), respectively.

2.4.1 The Infinite-horizon State-feedback Control

We treat the mean square stable system of (2.9) where we exclude the measurement equation and where T tends to infinity. Following [23] the solution $Q(t)$ of (2.20), if it exists for every $T > 0$, will tend to the mean square stabilizing solution of the following equation:

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + D^T\tilde{Q}D = 0, \qquad (2.24)$$

assuming that the pair $(\Pi C_1, A - B_2\tilde{R}^{-1}D_{12}^TC_1)$, where $\Pi = I - D_{12}\tilde{R}^{-1}D_{12}^T$, is detectable (see Theorem 5.8 in [23]). A strict inequality is achieved from (2.24) for $(w(t), x_o)$ that are not identically zero, iff the left side of (2.24) is strictly negative definite (for the equivalence between (2.24) and the corresponding inequality see [59]). The latter inequality can be expressed in a LMI form in the case where $\bar{\alpha} = 0$ and $D_{12}^TC_1 = 0$. We arrive at the following result:

Theorem 2.4. *[53] Consider the mean square stable system of (2.9) where the measurement equation is excluded. Given $\gamma > 0$, a necessary and sufficient condition for J_{SO} of (2.10), with $||n(t)||^2_{\tilde{l}_2} = 0$ to be negative for all nonzero $w(t)$ where $w(t) \in L^2([0,\infty);\mathcal{R}^p)$ is that there exists a positive-definite matrix $\tilde{P} \in \mathcal{R}^{n\times n}$ that satisfies the following LMI:*

$$\Gamma_1 \triangleq \begin{bmatrix} A\tilde{P} + \tilde{P}A^T - B_2\tilde{R}^{-1}B_2^T & B_1 & \tilde{P}C_1^T & B_2G^T & \tilde{P}D^T \\ * & -\gamma^2 I_p & 0 & 0 & 0 \\ * & * & -I_m & 0 & 0 \\ * & * & * & -(\tilde{P} + G\tilde{R}^{-1}G^T) & 0 \\ * & * & * & * & -\tilde{P} \end{bmatrix} < 0. \qquad (2.25)$$

If such a solution exists then the stationary state-feedback gain is obtained by:

$$K_s = -R^{-1}B_2^TP^{-1}.$$

Proof: The inequality that is obtained from (2.20) for $\bar{\alpha} = 0$ and $D_{12}^TC_1 = 0$ is:

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + D^T\tilde{Q}D < 0,$$

where

$$\tilde{S} = B_2^T\tilde{Q}.$$

Denoting $\tilde{P} = \tilde{Q}^{-1}$, we multiply the latter inequality by $\tilde{P}$ from both sides and obtain:

$$A\tilde{P} + \tilde{P}A^T + \gamma^{-2}B_1B_1^T + \tilde{P}C_1^TC_1\tilde{P} - B_2\bar{R}^{-1}B_2^T + \tilde{P}D^T\tilde{P}^{-1}D\tilde{P} < 0,$$

where

$$\bar{R} = \tilde{R} + G^T \tilde{P}^{-1} G.$$

Since

$$(\tilde{R} + G^T \tilde{P}^{-1} G)^{-1} = \tilde{R}^{-1/2}[I + \tilde{R}^{-1/2} G^T \tilde{P}^{-1} G \tilde{R}^{-1/2}]^{-1} \tilde{R}^{-1/2},$$

we obtain, using the matrix inversion lemma, the following equality:

$$[I + \tilde{R}^{-1/2} G^T \tilde{P}^{-1} G \tilde{R}^{-1/2}]^{-1} = I - \tilde{R}^{-1/2} G^T \tilde{P}^{-1} G \tilde{R}^{-1/2}$$
$$[I + \tilde{R}^{-1/2} G^T \tilde{P}^{-1} G \tilde{R}^{-1/2}]^{-1}.$$

Using the latter, together with the identity

$$\alpha[I + \ \beta\alpha]^{-1} = [I + \ \alpha\beta]^{-1}\alpha,$$

we readily obtain the following inequality:

$$A\tilde{P} + \tilde{P}A^T + \gamma^{-2} B_1 B_1^T + \tilde{P} C_1^T C_1 \tilde{P} - B_2 \tilde{R}^{-1} B_2^T + B_2 G^T [\tilde{P} + G \tilde{R}^{-1} G^T]^{-1}$$
$$G B_2^T + \tilde{P} D^T \tilde{P}^{-1} D \tilde{P} < 0.$$

By using Schur's complements formula, the latter inequality is equivalent to (2.25).

□

Remark 2.4. In the general case, where $D_{12}^T C_1 \neq 0$, a simple change of variables (see [57], page 195) can be readily used. Denoting:

$$\tilde{A} = A - B_2 \tilde{R}^{-1/2} D_{12}^T C_1,$$
$$\tilde{u} = u + \tilde{R}^{-1/2} D_{12}^T C_1 x,$$
$$\tilde{C}_1^T \tilde{C}_1 = C_1^T [I - D_{12} \tilde{R}^{-1} D_{12}^T] C_1,$$

we consider the following system:

$$dx = \tilde{A}xdt + B_1 wdt + B_2 \hat{u}dt + B_3 rdt + [Dd\beta - Gd\nu]x + G\hat{u}d\nu,$$

$$z = \begin{bmatrix} \tilde{C}_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} \tilde{u}.$$

Note that this system possesses multiple uncertainties which can be readily tackled using the arguments of Remark 2.3 with $\bar{\alpha} = 0$.

2.5 H_∞ filtering of Systems with State-multiplicative Noise

We consider the Luenberger-type state observer of (2.4) and $e(t)$ of (2.5). We obtain:

$$de(t) = [A - LC_2]e(t)dt + \hat{B}\tilde{w}(t)dt + [Dd\beta(t) - LFd\zeta(t)]x(t), \quad e(0) = x_0,$$

where

$$\hat{B} = \begin{bmatrix} B_1 - LD_{21} & -L \end{bmatrix},$$

and where $\tilde{w}(t)$ is defined in (2.5). Defining $\xi(t) = col\{x(t), e(t)\}$, we obtain the following.

$$\begin{aligned} &d\xi(t) = [\tilde{A}dt + \tilde{D}d\beta(t) + \tilde{F}d\zeta(t)]\xi(t) + \tilde{B}_1\tilde{w}(t)dt, \\ &\xi(0) = col\{x(0), x(0)\}, \\ &\tilde{z}(t) = \tilde{C}_1\xi(t), \end{aligned} \tag{2.26}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A - LC_2 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix},$$

$$\tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ B_1 - LD_{21} & -L \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 & 0 \\ -LF & 0 \end{bmatrix} \text{ and } \tilde{C}_1 = [0 \ \ C_1]. \tag{2.27}$$

Applying the stochastic BRL of Section 2.3 (also see [24], [59], [53]) to the system (2.26) with the matrices of (2.27), we obtain the following Riccati-type equation:

$$-\dot{\hat{P}} = \hat{P}\tilde{A} + \tilde{A}^T\hat{P} + \gamma^{-2}\hat{P}\tilde{B}_1\tilde{B}_1^T\hat{P} + \tilde{D}^T\hat{P}\tilde{D} + \tilde{F}^T\hat{P}\tilde{F} + \tilde{C}_1^T\tilde{C}_1,$$

$$\hat{P}(0) = \begin{bmatrix} I_n \\ I_n \end{bmatrix} \gamma^2 R^{-1} \begin{bmatrix} I_n & I_n \end{bmatrix}. \tag{2.28}$$

The solution of (2.28) involves the simultaneous solution of both $\hat{P}(t)$ and the filter gain L and can not be obtained readily due to mixed terms of the latter variables in (2.28). Considering, however, the monotonicity of $\hat{P}$ with respect to a free semi-positive definite term in (2.28) [59], the solution to the above Riccati-type equation can be obtained by solving the following DLMI [99],[44] :

$$\begin{bmatrix} \dot{\hat{P}} + \tilde{A}^T\hat{P} + \hat{P}\tilde{A} + \tilde{D}^T\hat{P}\tilde{D} & \hat{P}\tilde{B}_1 & \tilde{C}_1^T & \tilde{F}^T\hat{P} \\ * & -\gamma^2 I_{p+z} & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & -\hat{P} \end{bmatrix} \leq 0, \tag{2.29}$$

where $\hat{P} > 0$ and with $\hat{P}(0)$ of (2.28) and where we require that $Tr\{\hat{P}(\tau)\}$ be minimized at each time instant $\tau \in [0,\ T]$.
Recently, novel methods for solving DLMIs has been introduced in [99],[44]. Applying the method of [99], the above DLMI can be solved by discretizing the time interval $[0,\ T]$ into equally spaced time instances resulting in the following discretisized DLMI :

$$\begin{bmatrix} \Psi_{11} & \hat{P}_k\tilde{B}_{1,k} & \tilde{C}_{1,k}^T & \tilde{F}_k^T\hat{P}_k \\ * & -\gamma^2\tilde{\varepsilon}^{-1}I_p & 0 & 0 \\ * & * & -\tilde{\varepsilon}^{-1}I_q & 0 \\ * & * & * & -\tilde{\varepsilon}^{-1}\hat{P}_k \end{bmatrix} \leq 0, \qquad (2.30)$$

for $k = 0, 1, .., N-1$ and where:

$$\begin{aligned} \Psi_{11} &= \hat{P}_{k+1} - \hat{P}_k + \tilde{\varepsilon}(\tilde{A}_k^T\hat{P}_k + \hat{P}_k\tilde{A}_k) + \tilde{\varepsilon}\tilde{D}_k^T\hat{P}_k\tilde{D}_k, \\ \tilde{A}_k &= \tilde{A}(t_k), \\ \tilde{B}_{1,k} &= \tilde{B}_1(t_k), \\ \tilde{C}_{1,k} &= \tilde{C}_1(t_k), \\ \tilde{F}_k &= \tilde{F}(t_k), \\ \tilde{D}_k &= \tilde{D}(t_k), \ \text{ with} \\ &\{t_i,\ i = 0, ..N-1,\ t_N = T,\ t_0 = 0\} \text{ and} \\ t_{i+1} - t_i &\stackrel{\Delta}{=} \tilde{\varepsilon} = N^{-1}T,\ i = 0, ...N-1. \end{aligned} \qquad (2.31)$$

The discretized estimation problem thus becomes one of finding, at each $k \in [0,\ N-1]$, $\hat{P}_{k+1} > 0$ of minimum trace and L_k that satisfy (2.30).
The latter DLMI is initiated by the initial condition of (2.28) at the instant $k = 0$ and a solution for both, the filter gain L_k and $\hat{P}_{k+1}$ (i.e $\hat{P}_1$ and L_0) is sought, under the minimum trace requirement of $\hat{P}_{k+1}$. The latter procedure repeats itself by a forward iteration up to $k = N-1$, where N is chosen (and therefore $1/\tilde{\epsilon}$) to be large enough to allow for a smooth solution (see also [99]).
We summarize the above results by the following theorem:

Theorem 2.5. *Consider the system of (2.3) and J_F of (2.6). Given $\gamma > 0$ and $\tilde{\epsilon} > 0$, the stochastic state-multiplicative filtering problem achieves $J_F < 0$ if there exists $\hat{P}(t)$ that solves (2.30) $\forall t \in [0, T]$ starting from the initial condition of (2.28) for small enough $\tilde{\epsilon}$.*

2.5.1 The Stationary H_∞-filtering Problem

We consider the mean square stable system of (2.9) where $B_2 = 0$ and $D_{12} = 0$ and where, for simplicity, we take $G = 0$. We Introduce the following Lyapunov function:

$$V(t) = \xi^T(t)\tilde{P}\xi(t), \text{ with } \tilde{P} = \begin{bmatrix} P & \alpha\hat{P} \\ \alpha\hat{P} & \hat{P} \end{bmatrix}, \tag{2.32}$$

where $\xi(t) = [x^T(t) \quad e^T(t)]^T$, P and $\hat{P}$ are $n \times n$ matrices and α ia a tuning scalar. We obtain the following result:

Theorem 2.6. *Consider the system of (2.9) where $B_2 = 0$ and $D_{12} = 0$, $G = 0$ and where $\hat{x}(t)$ is defined in (2.4). Given the scalar $\gamma > 0$, the stationary state-multiplicative filtering problem possesses a solution if there exist $P = Q^T \in \mathcal{R}^{n\times n}$, $\hat{P} = \hat{P}^T \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times m}$ and a tuning scalar parameter α that satisfy the following LMI:*

$$\begin{bmatrix}
\Psi(1,1) & \Psi(1,2) & (P+\alpha\hat{P})B_1 & -\alpha Y & 0 & -\alpha F^T Y^T & -F^T Y^T & D^T(P+\alpha\hat{P}) & (1+\alpha)D^T\hat{P} \\
* & \Psi(2,2) & (\alpha+1)\hat{P}B_1 & -Y & C_1^T & 0 & 0 & 0 & 0 \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -I_p & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -P & -\alpha\hat{P} & 0 & 0 \\
* & * & * & * & * & * & -\hat{P} & 0 & 0 \\
* & * & * & * & * & * & * & -P & -\alpha\hat{P} \\
* & * & * & * & * & * & * & * & -\hat{P}
\end{bmatrix} < 0,$$

where

$$\Psi(1,1) \stackrel{\Delta}{=} A^T P + PA,$$

$$\Psi(1,2) \stackrel{\Delta}{=} \alpha[A^T\hat{P} + \hat{P}A] - \alpha Y C_2,$$

$$\Psi(2,2) \stackrel{\Delta}{=} A^T\hat{P} + \hat{P}A - YC_2 - C_2^T Y^T$$

Proof: Applying the results of continuous BRL of Section 2.3.1 for the stationary case (see also [86], [59]) to the latter system, the algebraic counterpart of (2.28) is obtained which similarly, to the finite horizon case, leads to the stationary version of (2.29). Thus we obtain:

$$\begin{bmatrix} \tilde{A}^T\tilde{P}+\tilde{P}\tilde{A} & \tilde{P}\tilde{B}_1 & \tilde{C}_1^T & \tilde{F}^T\tilde{P} & \tilde{D}^T\tilde{P} \\ * & -\gamma^2 I_{p+z} & 0 & 0 & 0 \\ * & * & -I_m & 0 & 0 \\ * & * & * & -\tilde{P} & 0 \\ * & * & * & * & -\tilde{P} \end{bmatrix} \leq 0. \tag{2.33}$$

Replacing for the structure of $\tilde{P}$ in (2.32) and the stationary version of the matrices of (2.27) and denoting $Y = \hat{P}L$, where L is the observer gain, and carrying out the various multiplications, the LMI of Theorem 2.6 is obtained. □

Remark 2.5. Note that contrary to the solution of the finite-horizon filtering of Section 2.5, where use has been made of the DLMI method, in the latter solution of the stationary case, an inherent overdesign is entailed in the solution, due to the special stricture of $\tilde{P}$ in (2.32). The advantage of the latter solution is however, in its simplicity and tractability, as will be seen also in Section 2.7.

2.6 Finite-horizon Stochastic Output-feedback Control

The solution of the output-feedback control problem is obtained below by transforming the problem to one of filtering, to which the result of the continuous-time stochastic state-multiplicative BRL of Section 2.3 is applied (see also [59], [24],[53]). In order to obtain the equivalent estimation problem, the optimal strategies of $w^*(t)$ and $u^*(t)$ of the state-feedback case are needed. One can not use the results of Section 2.3, for this transformation, since the solution there was obtained by applying the stochastic BRL to the closed-loop system of Section 2.4. The optimal strategies $w^*(t)$ and $u^*(t)$ are obtained below by completing to squares for both $w(t)$ and $u(t)$.

Consider the system of (2.1) and let $Q(t)$ be a solution of (2.20) that satisfies $Q(0) < \gamma^2 R^{-1}$. Considering (2.20) and applying Ito formula to $H(t, x(t)) = \langle x(t), Q(t)x(t)\rangle$, and taking expectation for every $T > 0$ we obtain the following, noting that $Q(T) = P_T$.

$$0 = E\{\int_0^T d\{x^T(t)Q(t)x(t)\} + x_0^T Q(0)x_0\} - E\{x^T(T)P_T x(T)\} =$$

$$E\{\int_0^T \langle x(t), \dot{Q}(t)x(t)\rangle dt\}$$

$$+2E\{\int_0^T \langle Q(t)x(t), A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)\rangle dt\}$$

$$+E\{\int_0^T Tr\{Q(t)[D(t)x(t)\ \ G(t)u(t)]$$

$$\bar{P}[D(t)x(t)\ \ G(t)u(t)]^T dt\}\} + x_0^T Q(0)x_0 - E\{x^T(T)P_T x(T)\},$$

where $\bar{P}dt \triangleq \begin{bmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & 1 \end{bmatrix} dt$ is the covariance matrix of the augmented Wiener process vector $col\{d\beta(t)\ \ d\nu(t)\}$. We also have the following:

$$Tr\{Q(t)[D(t)x(t)\ \ G(t)u(t)]\bar{P}[D(t)x(t)\ \ G(t)u(t)]^T\}$$

$$= Tr\{\begin{bmatrix} x^T(t)D^T(t) \\ u^T(t)G^T(t) \end{bmatrix} Q(t)[D(t)x(t)\ \ G(t)u(t)]\bar{P}\}$$

$$= Tr\{\begin{bmatrix} x^T(t)D^T(t)Q(t)D(t)x(t) & x^T(t)D^T(t)Q(t)G(t)u(t) \\ u^T(t)G^T(t)Q(t)D(t)x(t) & u^T(t)G^T(t)Q(t)G(t)u(t) \end{bmatrix} \begin{bmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & 1 \end{bmatrix}\}$$

$$= \langle \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} D^T(t)Q(t)D(t) & \bar{\alpha}D^T(t)Q(t)G(t) \\ \bar{\alpha}G^T(t)Q(t)D(t) & G^T(t)Q(t)G(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}\rangle$$

$$= x^T(t)D^T(t)Q(t)D(t)x(t) + 2\bar{\alpha}x^T(t)D^T(t)Q(t)G(t)u(t)$$

$$+u^T(t)G^T(t)Q(t)G(t)u(t)$$

Using the above derivation and adding the zero sum of

$$E\{\int_0^T ||w(t)||^2 dt - \int_0^T ||w(t)||^2 dt\}$$

$$+E\{\int_0^T ||z(t)||^2 dt - \int_0^T ||z(t)||^2 dt\}$$

we obtain after completing to squares for $w(t)$:

$$0 = E\{\int_0^T x^T(t)[\dot{Q}(t) + Q(t)A(t) + A^T(t)Q(t)$$

$$+D^T(t)Q(t)D(t) - \gamma^{-2}Q(t)B_1(t)B_1^T(t)Q(t)]x(t)dt\}$$

$$+2E\{\int_0^T x^T(t)Q(t)[B_2(t)u(t)]dt\}$$

$$+E\{\int_0^T \{||z||^2 + \gamma^2(||w||^2 + ||w - \gamma^{-2}B_1^T Qx||^2)\}dt\}$$

$$+E\{\int_0^T \{2\bar{\alpha} x^T(t) D^T(t) Q(t) G(t) u(t)$$
$$+u^T(t) G^T(t) Q(t) G(t) u(t)\} dt\} + x_0^T Q(0) x_0.$$
$$-E\{\int_0^T ||z(t)||^2 dt + \int_0^T [||C_1(t)x(t) + D_{12}(t)u(t)||^2] dt\} - E\{x^T(T) P_T x(T)\}$$

Completing to squares for $u(t)$, we obtain:

$$0 = E\{\int_0^T -||z(t)||^2 + \gamma^2(||w(t)||^2 - ||w(t) - \gamma^{-2} B_1^T(t) Q(t) x(t)||^2)\} dt$$
$$+E\{\int_0^T ||\hat{u}(t)||^2_{\hat{R}}\} dt + x_0^T Q(0) x_0 - E\{x^T(T) P_T x(T)\}$$

where

$$\hat{u}(t) = u(t) + \hat{R}^{-1}(t) \bar{S}(t) x(t), \tag{2.34}$$

where $\hat{R}(t)$, $\bar{S}(t)$ are defined in (2.21).

Adding $J_E(u, w, x_0)$ of 2.2 to the above zero quantity we obtain:

$$J_E = -\gamma^2 ||x_0||^2_{P_0^{-1}} + E\{\int_0^T [||u(t) + \hat{R}^{-1} \bar{S}(t) x(t)||^2_{\hat{R}}\} dt$$
$$-E\{\int_0^T ||w(t) - \gamma^{-2} B_1^T(t) Q(t) x(t)||^2\} dt$$

where

$$P_0 = [R^{-1} - \gamma^{-2} Q(0)]^{-1} \tag{2.35}$$

and with the optimal strategies being:

$$\begin{aligned} w^*(t) &= \gamma^2 B_1^T(t) Q(t) x(t), \\ u^* &= -\hat{R}^{-1}(t)[(B_2^T(t) Q(t) + D_{12}^T(t) C_1(t) + \bar{\alpha} G^T(t) Q(t) F(t))] x(t). \end{aligned} \tag{2.36}$$

Once the saddle point strategies are derived, we consider the system of (2.7) and take for simplicity $G(t) = 0$. We also suppress the time-dependence of the system matrices for clarity of representation. We assume that (2.20) has a solution $Q(t) > 0$ over $[0, T]$ that satisfies $Q(0) < \gamma^2 R^{-1}$. Using the optimal strategies of (2.36) for the above state-feedback case, the index of performance turns to be:

$$J_O(u, w, n, x_0) = -\gamma^2 ||x_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||w - w^*||^2 dt\}$$
$$+E\{\int_0^T ||[u - u^*]||^2_{\hat{R}} dt - \gamma^2 \int_0^T ||n(t)||^2 dt\}, \tag{2.37}$$

where $G \equiv 0$ is taken in both $\hat{R}$ and $\bar{S}$ of (2.21).

Following [21] we define:

$$\bar{w}(t) = w(t) - w^*(t), \tag{2.38}$$

where $w^*(t)$ is given in (2.36), we obtain:

$$J_O(u, w, n, x_0) = -\gamma^2 ||x_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||\bar{w}||^2 dt\}$$

$$+E\{\int_0^T ||\hat{R}^{1/2}[u + \hat{C}_1 x]||^2 dt\} - \gamma^2 E\{\int_0^T ||n(t)||^2 dt\},$$

where

$$\hat{C}_1 = \hat{R}^{-1}[B_2^T Q + D_{12}^T C_1]. \tag{2.39}$$

We seek a controller of the form

$$u(t) = -\hat{C}_1(t)\hat{x}(t).$$

Substituting (2.38) in (2.1) we re-formulate the state equation of the system and we obtain:

$$dx = [\bar{\bar{A}}(t)x(t) + B_1(t)\bar{w}(t) + B_2(t)u(t)]dt + D(t)x(t)d\beta(t), \tag{2.40}$$

where

$$\bar{\bar{A}}(t) = A + \gamma^{-2} B_1 B_1^T Q. \tag{2.41}$$

Considering the following Luenberger-type state observer:

$$\begin{aligned} &d\hat{x}(t) = \bar{\bar{A}}\hat{x}(t)dt + L[dy - \hat{C}_2\hat{x}(t)dt] + B_2(t)u(t)dt, \; \hat{x}(0) = 0, \\ &\hat{z}(t) = \hat{C}_1\hat{x}(t), \end{aligned} \tag{2.42}$$

where

$$\hat{C}_2 = C_2 + \gamma^{-2} D_{21} B_1^T Q,$$

and denoting $e(t) = x(t) - \hat{x}(t)$the following system for the error $e(t)$ is obtained.

$$de(t) = [\bar{\bar{A}} - L\hat{C}_2]e(t)dt + \hat{B}\hat{w}(t)dt + [Dd\beta(t) - LFd\zeta(t)]x(t),$$

where

$$\hat{w}(t) \stackrel{\triangle}{=} col\{\bar{w}(t), n(t)\}, \quad \text{and} \quad \hat{B} \stackrel{\triangle}{=} [\bar{B}_1 - LD_{21} \quad -L].$$

Defining $\xi(t) = col\{x(t), e(t)\}$, we obtain:

$$\begin{aligned} &d\xi(t) = [\tilde{A}dt + \tilde{D}d\beta(t) + \tilde{F}d\zeta(t)]\xi(t) + \tilde{B}_1\hat{w}(t)dt, \; \xi^T(0) = [x^T(0) \;\; x^T(0)]^T, \\ &\tilde{z}(t) = \tilde{C}_1\xi(t), \end{aligned} \tag{2.43}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{\bar{A}} - B_2\hat{C}_1 & B_2\hat{C}_1 \\ 0 & \bar{\bar{A}} - L\hat{C}_2 \end{bmatrix}, \ \tilde{D} = \begin{bmatrix} D\ 0 \\ D\ 0 \end{bmatrix},$$

$$\tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ B_1 - LD_{21} & -L \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 & 0 \\ -LF & 0 \end{bmatrix}, \ \tilde{C}_1 = [0 \ \ \hat{C}_1]. \tag{2.44}$$

Applying the stochastic BRL of Section 2.3 (see also [24], [53]) to the system of (2.43) with the matrices of (2.44), the following Riccati-type equation is obtained.

$$-\dot{\hat{P}} = \hat{P}\tilde{A} + \tilde{A}^T\hat{P} + \gamma^{-2}\hat{P}\tilde{B}_1\tilde{B}_1^T\hat{P} + \tilde{D}^T\hat{P}\tilde{D} + \tilde{F}^T\hat{P}\tilde{F} + \tilde{C}_1^T\tilde{C}_1,$$

$$\hat{P}(0) = \begin{bmatrix} I_n \\ I_n \end{bmatrix} (\gamma^2 R^{-1} - Q(0)) \begin{bmatrix} I_n \ I_n \end{bmatrix}, \tag{2.45}$$

where the latter initial condition follows from (2.35).

Similarly to the filtering problem of Section 2.5, the solution of (2.45) involves the simultaneous solution of both $\hat{P}(t)$ and the filter gain L and it can not be readily obtained due to mixed terms of the latter variables in (2.45). Considering, however, the monotonicity of $\hat{P}$ with respect to a free semi-positive definite term in (2.45) [59], the solution to the above Riccati-type equation can be obtained by solving the following Differential LMI (DLMI):

$$\begin{bmatrix} \dot{\hat{P}} + \tilde{A}^T\hat{P} + \hat{P}\tilde{A} + \tilde{D}^T\hat{P}\tilde{D} & \hat{P}\tilde{B}_1 & \tilde{C}_1^T & \tilde{F}^T\hat{P} \\ * & -\gamma^2 I_{p+z} & 0 & 0 \\ * & * & -I_m & 0 \\ * & * & * & -\hat{P} \end{bmatrix} \le 0, \tag{2.46}$$

for $\hat{P} > 0$ and with $\hat{P}(0)$ of (2.45) and where it is required that $Tr\{P(\tau)\}$ be minimized at each time instant $\tau \in [0, \ T]$.

Similarly to the filtering problem of Section 2.5, we apply the method of [99]. Thus the above DLMI can be solved by discretizing the time interval $[0, \ T]$ into equally spaced time instances resulting in the following discretisized DLMI:

$$\begin{bmatrix} \Psi_{11} & \hat{P}_k\tilde{B}_{1,k} & \tilde{C}_{1,k}^T & \tilde{F}_k^T\hat{P}_k \\ * & -\gamma^2\tilde{\varepsilon}^{-1}I_{p+z} & 0 & 0 \\ * & * & -\tilde{\varepsilon}^{-1}I_m & 0 \\ * & * & * & -\tilde{\varepsilon}^{-1}\hat{P}_k \end{bmatrix} \le 0, \tag{2.47}$$

for $k = 0, 1, .., N-1$ and where:

$$
\begin{aligned}
&\Psi_{11} = \hat{P}_{k+1} - \hat{P}_k + \tilde{\varepsilon}(\tilde{A}_k^T \hat{P}_k + \hat{P}_k \tilde{A}_k) + \tilde{\varepsilon}\tilde{D}_k^T \hat{P}_k \tilde{D}_k,\\
&\tilde{A}_k = \tilde{A}(t_k)\\
&\tilde{B}_{1,k} = \tilde{B}_1(t_k),\\
&\tilde{C}_{1,k} = \tilde{C}_1(t_k),\\
&\tilde{F}_k = \tilde{F}(t_k),\\
&\tilde{D}_k = \tilde{D}(t_k), \text{ with}\\
&\{t_i,\ i = 0, ..N-1,\ t_N = T,\ t_0 = 0\} \text{ and}\\
&t_{i+1} - t_i \stackrel{\Delta}{=} \tilde{\varepsilon} = N^{-1}T,\ i = 0, ...N-1.
\end{aligned}
\tag{2.48}
$$

The discretized estimation problem thus becomes one of finding, at each $k \in [0,\ N-1]$, $\hat{P}_{k+1} > 0$ of minimum trace that satisfies (2.47).

The latter DLMI is initiated with the initial condition of (2.45) at the instance $k = 0$ and a solution for both, the filter gain L_k and $\hat{P}_{k+1}$ (i.e $\hat{P}_1$ and L_0) is sought for, under the minimum trace requirement of $\hat{P}_{k+1}$. The latter procedure repeats itself by a forward iteration up to $k = N-1$, where N is chosen to be large enough to allow for a smooth solution (see also [99]). We summarize the above results by the following theorem:

Theorem 2.7. *Consider the system of (2.7) and J_O of (2.8). Given the positive scalar γ and $\tilde{\epsilon} > 0$, the output-feedback control problem possesses a solution if there exists $Q(t) > 0$, $\forall t \in [0, T]$ that solves (2.20) such that $Q(0) < \gamma^2 R^{-1}$, and $\hat{P}(t)$ that solves (2.47) $\forall t \in [0, T]$ starting from the initial condition of (2.45), where R is defined in (2.8). If a solution to (2.20) and (2.47) exists the following control law:*

$$
u_{of}(t) = -\hat{C}_1(t)\hat{x}(t) \tag{2.49}
$$

achieves the required H_∞ norm bound of γ for J_0 where $\hat{x}(t)$ is obtained by solving for (2.42).

Remark 2.6. We note that the solution of the latter DLMI proceeds the solution of the finite-horizon state-feedback problem of Section 2.4 that starts from $Q(T)$, in (2.20), for a given attenuation level of γ. Once a solution to the latter problem is achieved, the DLMI of (2.47) is solved for the same value of γ starting from the initial condition $\hat{P}_0$ of (2.45).

Remark 2.7. Similarly to the initial condition of (2.28), the initial condition $\hat{P}(0)$ of (2.45) corresponds to the case where a large weight is imposed on $\hat{x}(0)$ in order to force nature to select $e(0) = x(0)$ (i.e $\hat{x}(0) = 0$).

Remark 2.8. In the case where the augmented state-vector is chosen as $\xi(t) = col\{x(t), \hat{x}(t)\}$ the initial condition of $\hat{P}(0)$ of (2.45) would satisfy:

$$\hat{P}(0) = \begin{bmatrix} \gamma^2 R^{-1} - Q(0) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\gamma^2 R^{-1} - Q(0)$ is the initial weight. The latter$\hat{P}(0)$ can be readily transformed to account for the augmented state-vector of $\xi(t) = col\{x(t), e(t)\}$ by the pre- and post- multiplication of the above matrices, by Υ^T and Υ, respectively, where $\Upsilon \triangleq \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$. The resulting matrix will then be identical to the one of the initial condition of (2.45).

2.7 Stationary Stochastic Output-feedback Control

We consider the mean square stable system of (2.9) where, for simplicity, we take: $G = 0,\ E\{d\beta(t)\ d\zeta(t)\} = 0$ and $D_{12}^T C_1 = 0$. Similarly to the solution of the stationary filtering problem of Section 2.5.1, we introduce the following Lyapunov function:

$$V(t) = \xi^T(t)\tilde{Q}\xi(t), \text{ with } \tilde{Q} = \begin{bmatrix} Q & \alpha\hat{Q} \\ \alpha\hat{Q} & \hat{Q} \end{bmatrix}, \tag{2.50}$$

where $\xi(t)$ is the augmented state vector of the finite-horizon case, Q and $\hat{Q}$ are $n \times n$ matrices and α ia a tuning scalar. We obtain the following result:

Theorem 2.8. *Consider the system (2.9) and J_{SO} of (2.10) where the matrices $A, B_1, B_2, D, C_2, F, C_1$ and D_{12} are all constant, $G = 0$, $u(t) = K_s(t)\hat{x}(t)$ and where $\hat{x}(t)$ is defined in (2.42). Given a scalar $\gamma > 0$, there exists a controller that achieves $J_{SO} < 0$, $\forall(w(t), x(0)) \neq 0$ if there exist $Q = Q^T \in \mathcal{R}^{n\times n}$, $\hat{Q} = \hat{Q}^T \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times q}$ and a tuning scalar parameter α that satisfy the following LMI:*

$$
\begin{bmatrix}
\Upsilon(1,1) & \Upsilon(1,2) & (Q+\alpha\hat{Q})B_1-\alpha YD_{21} & -\alpha Y & 0 & -\alpha F^TY^T & -H^TY^T & F^T(Q+\alpha\hat{Q}) & D^T\hat{Q}(1+\alpha) \\
* & \Upsilon(2,2) & (\alpha+1)\hat{Q}B_1-YD_{21} & -Y & \hat{C}_1^T & 0 & 0 & 0 & 0 \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -I_p & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -Q & -\alpha\hat{Q} & 0 & 0 \\
* & * & * & * & * & * & -\hat{Q} & 0 & 0 \\
* & * & * & * & * & * & * & -Q & -\alpha\hat{Q} \\
* & * & * & * & * & * & * & * & -\hat{Q}
\end{bmatrix} < 0,
$$

where

$$
\begin{aligned}
\Upsilon(1,1) &\triangleq \bar{\bar{A}}^TQ + Q\bar{\bar{A}} - QB_2\hat{C}_1 - \hat{C}_1^TB_2^TQ, \\
\Upsilon(1,2) &\triangleq \alpha(\bar{\bar{A}}^T\hat{Q} + \hat{Q}\bar{\bar{A}}) + QB_2\hat{C}_1 - \alpha\hat{C}_1^TB_2^T\hat{Q} - \alpha\hat{C}_2, \\
\Upsilon(2,2) &\triangleq \bar{\bar{A}}^T\hat{Q} + \hat{Q}\bar{\bar{A}} + \alpha\hat{Q}B_2\hat{C}_1 + \alpha\hat{C}_1^TB_2^T\hat{Q} - Y\hat{C}_2^T - \hat{C}_2Y^T
\end{aligned}
$$

Proof: The proof outline for the above stationary case resembles the one for the finite-horizon problem. Considering the system (2.9) we first solve the stationary state-feedback problem to obtain the optimal stationary strategies of both $w_s^*(t)$ and $u_s^*(t)$ and the stationary controller gain K_s (see Section 2.4.1 for a derivation of the stationary Riccati-type inequality) . Similarly to the finite- horizon solution of the state-feedback control problem using completing to squares (See Section 2.6), the optimal strategies of the stationary case are given by :

$$u_s^*(t) \triangleq K_s x(t) \ \text{ where } \ K_s(t) = \hat{R}^{-1}[(B_2^T Q + D_{12}^T C_1)]$$

and

$$w^*(t) = \gamma^{-2} B_1^T Q x(t),$$

where $Q = \tilde{P}^{-1}$ is the solution of (2.25).

Using the latter optimal strategies the problem is transformed to an estimation one, thus arriving at the stationary counterpart of the augmented system for $\xi(t)$. Applying the stationary continuous BRL of Section 2.3.1 (see [86], [59]) to the latter system the algebraic counterpart of (2.45) is obtained which, similarly to the finite horizon case of Section 2.6, becomes the stationary version of (2.46). We thus obtain the following LMI in the decision variables $\tilde{Q}$ and L:

$$\begin{bmatrix} \tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} & \tilde{Q}\tilde{B}_1 & \tilde{C}_1^T & \tilde{F}^T\tilde{Q} & \tilde{D}^T\tilde{Q} \\ \tilde{B}_1^T\tilde{Q} & -\gamma^2 I_{p+z} & 0 & 0 & 0 \\ \tilde{C}_1 & 0 & -I_n & 0 & 0 \\ \tilde{Q}\tilde{F} & 0 & 0 & -\tilde{Q} & 0 \\ \tilde{Q}\tilde{D} & 0 & 0 & 0 & -\tilde{Q} \end{bmatrix} \leq 0. \qquad (2.51)$$

Substituting the structure of $\tilde{Q}$ and the stationary version of the matrices of 2.44 in (2.50), denoting $Y = \hat{Q}L$, where L is the observer gain, and carrying out the various multiplications, the LMI of Theorem 2.8 is obtained.

□

2.8 Example: Stationary Estimation and State-feedback Control

We consider the system of (2.9) where $B_2 = 0$ and $D_{12} = 0$ with the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix}, \ D = \begin{bmatrix} 0 & 0.3 \\ 0 & -0.12 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ C_1 = \begin{bmatrix} -0.5 & 1 \\ 0 & 0 \end{bmatrix},$$

where $C_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$ and where $G = 0, \ F = 0$. We apply the result of Theorem 2.6 and we obtain for a near minimum of $\gamma = 5.72$ and $\alpha = 0.1$ the following results:

$$P = \begin{bmatrix} 3.4333 & 0.5220 \\ 0.5220 & 2.9490 \end{bmatrix}, \ \hat{P} = \begin{bmatrix} 2.7807 & 0.2687 \\ 0.2687 & 1.7186 \end{bmatrix},$$

where the filter gain is $L^T = \begin{bmatrix}1.0292 & 1.5855\end{bmatrix}$. The resulting eigenvalues for the closed loop system are:

$$-0.2 \pm 0.9798i, \quad -0.3307 \pm 0.9937i.$$

We note that for the deterministic case, where $D = 0$, a near minimum of $\gamma = 4.96$ is achieved for $\alpha = 0.1$.

Assuming that there is an access to the states of the system, the stationary state-feedback control solution of this problem, where

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ .1 \end{bmatrix},$$

obtains a near minimum attenuation level of $\gamma = 1.39$, by applying the results of Theorem 2.4. The resulting state-feedback gain is $K = -R^{-1}B_2^T P^{-1} = [-32.1205 \quad -14.0154]$ where the eigenvalues of the closed-loop system are:-2.8683, -11.5471 and where

$$P = \begin{bmatrix} 0.2420 & -0.5546 \\ -0.5546 & 8.4061 \end{bmatrix},$$

is the matrix solution of (2.25). For the deterministic case, where $D = 0$, a near minimum attenuation level of $\gamma = 1.04$ is obtained.

2.9 Conclusions

In this chapter the solution of the stochastic state-multiplicative BRL is solved for both the finite- and infinite-horizon cases. Based on the BRl solution the problems of state-feedback control and filtering are solved where in the filtering case we make use of the simple and efficient DLMI technique. This technique was shown to produce, in the deterministic case, solutions that mimic the standard central solution of the Riccati equation associated with the solution of the deterministic BRL and the nominal state-feedback and filtering problems. The problem of H_∞-optimal output-feedback control of finite-horizon and stationary continuous-time linear systems with multiplicative stochastic uncertainties is solved. In both problems the solution is carried out along the standard approach where, using the optimal strategy for the state-feedback case, the problem is transformed to an estimation one, to which the stochastic BRL is applied. Unlike the previous works of [24], [59], [43], our solution is tractable and involves an LMI based recursion (DLMI) in the finite horizon case and a simple set of two LMIs for the stationary case. We note that in the latter case one has only to search for a single tuning parameter of α, a fact that renders our solution as a simple and easily implementable. We note that the stationary solution is based on a specific selection of a Lyapunov function which leads to a sufficient solution of the output-feedback problem. In the example the solution of both the stationary state-feedback control and the Luenberger observer based estimation is presented, where for the latter solution the efficiency of partitioning the matrix solution with a tuning parameter is demonstrated.

3

Continuous-time Systems: General Filtering

3.1 Introduction

In this chapter we bring the solution of the continuous-time stationary filtering using a general-type filter [45], [41]. Recently, the solution of the output-feedback problem for LTI continuous-time stochastic uncertain systems has been derived for the stationary case [59]. The solution in [59] is obtained by formulating a stochastic BRL and a general-type controller. It results in two coupled nonlinear matrix inequalities which reduce to the standard H_∞ output-feedback problem in the nominal case. The solution in [59] does not include uncertainty in the measurement matrix, which is most likely to be found in physical control systems.

In the present chapter, we treat the case where stochastic uncertainties appear in both the dynamic and the measurement matrices and correlations are allowed between these uncertain parameters [45], [41]. This case is often encountered in practice when one considers filtered estimation. Our solution is based on the stationary continuous-time BRL of Section 2.3.1 (for the derivation of the stationary BRL see also [24], [86] and [59]). In our solution we apply the techniques of [74] to the deterministic polytopic problem [36],[38]. Necessary and sufficient conditions are derived for the existence of a solution in terms of LMIs. The latter solution is extended to the case where the deterministic part of the system matrices lie in a convex bounded domain of polytopic-type. Our theory is also applicable to the case where the covariance matrices of the stochastic parameters are not perfectly known and lie in a polytopic domain. We also solve the mixed H_2/H_∞ problem where, of all the filters that solve the stochastic H_∞ filtering problem, the one that minimizes an upper-bound on the estimation error variance is found. The method developed is demonstrated by a practical example that is given in the Application part.

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 45–54, 2005.

3.2 Problem Formulation

We consider the following linear mean square stable system with state-dependent noise:

$$\begin{aligned} dx &= (Ax + B_1 w)dt + Dx d\beta, \\ dy &= (Cx + D_{21} w)dt + Fx d\zeta, \\ z &= Lx \end{aligned} \tag{3.1}$$

where $x \in \mathcal{R}^n$ is the system state vector , $x(0)$ is any norm bounded vector in $\mathcal{R}^n$, $y \in \mathcal{R}^r$ is the measurement, $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$ is the exogenous disturbance signal, $z \in \mathcal{R}^m$ is the state combination to be estimated and where A, B_1, C, D, F, D_{21} and L are constant matrices of the appropriate dimensions. The variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy:

$$E\{d\beta(t)\} = 0,\ E\{d\zeta(t)\}=0,\ E\{d\beta(t)^2\}=dt,\ E\{d\zeta(t)^2\}=dt,$$

$$E\{d\beta(t)d\zeta(t)\}=\alpha dt, |\alpha|< 1.$$

We consider the following filter for the estimation of $z(t)$:

$$d\hat{x} = A_f \hat{x} dt + B_f dy, \quad \hat{x}_0 = 0, \quad \hat{z} = C_f \hat{x}, \tag{3.2}$$

where $\hat{x} \in \mathcal{R}^n$ and $\hat{z} \in \mathcal{R}^m$. Denoting

$$\xi^T = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix} \quad \text{and} \quad \tilde{z} = z - \hat{z}, \tag{3.3}$$

we define, for a given scalar $\gamma > 0$, the following performance index

$$J_S \triangleq ||\tilde{z}(t)||^2_{\tilde{L}_2} - \gamma^2 ||w(t)||^2_{\tilde{L}_2} \tag{3.4}$$

The problems addressed in this chapter are:

i) Stochastic H_∞ filtering problem: Given $\gamma > 0$, find an asymptotically stable linear filter of the form (3.2) that leads to a mean square stable estimation error process $\tilde{z}$ such that J_S of (3.4) is negative for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$.

ii) Stochastic mixed H_2/H_∞ filtering problem: Of all the asymptotically error stabilizing filters that solve problem (i) find the one that minimizes an upper-bound on the estimation error variance: $\lim_{t\to\infty} E\left\{\tilde{z}^T \tilde{z}\right\}$, where $\dot{w}(t) = \dot{\eta}(t)$ is regarded as a standard white noise process, independent of $\beta(t)$ and $\zeta(t)$. Namely, in this case, (3.1) should be interpreted as:

$$dx = Axdt + Dxd\beta + B_1 d\eta$$

and

$$dy = Cxdt + Fxd\zeta + D_{21} d\eta,$$

where $E\{d\eta d\eta^T\} = I dt$.

3.3 The Stochastic BRL

We bring, for convenience, the lemma that is derived in Section 2.3.1 (see also [59]) for the following mean square stable system :

$$\begin{aligned} dx &= Axdt + (D_1 d\beta + D_2 d\zeta)x + B_1 w dt, \\ z &= Lx \end{aligned} \tag{3.5}$$

where the scalar Wiener processes $\beta(t)$ and $\zeta(t)$ and the disturbance $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$ are defined above. Using the cost function

$$\hat{J} = ||z(t)||^2_{\tilde{L}_2} - \gamma^2 ||w(t)||^2_{\tilde{L}_2}$$

and applying the arguments of [59] the following holds:

Lemma 3.1. *(Section 2.3.1, and [59]) For any $\gamma > 0$ the following statements are equivalent:*
i) *The system of (3.5) is mean square stable and $\hat{J}$ is negative for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$.*
ii) *There exists $Q > 0$ that satisfies the following inequality:*

$$QA + A^T Q + \gamma^{-2} QBB^T Q + L^T L + D_1^T Q D_1 + D_2^T Q D_2 + \alpha[D_1^T Q D_2 + D_2^T Q D_1] < 0.$$

3.4 The Stochastic Filter

Problem i is solved by applying Lemma 3.1 Considering the system of (3.1) and the definitions of (3.3) we obtain:

$$d\xi = \tilde{A}\xi dt + [\tilde{D}_1 d\beta + \tilde{D}_2 d\zeta]\xi + \tilde{B} w dt, \quad \tilde{z} = \tilde{C}\xi \tag{3.6}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{D}_2 = \begin{bmatrix} 0 & 0 \\ B_f F & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} L & -C_f \end{bmatrix}. \tag{3.7}$$

We arrive at the following result.

Theorem 3.1. *Consider the system of (3.1)-(3.4) and the filter of (3.2). Given $\gamma > 0$, the following holds:*

i) *A necessary and sufficient condition for (3.6) to be stable and J_S to be negative for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, is that there exist $R = R^T \in \mathcal{R}^{n\times n}$, $W = W^T \in \mathcal{R}^{n\times n}$, $Z \in \mathcal{R}^{n\times r}$, $S \in \mathcal{R}^{n\times n}$, and $T \in \mathcal{R}^{m\times n}$, such that*

$$\Sigma(R, W, Z, S, T) < 0 \tag{3.8}$$

where by $\Sigma(R, W, Z, S, T)$ *we define*

$$\Sigma = \begin{bmatrix} RA + A^T R & * & * & * & * & * & * \\ WA + ZC + S & S - S^T & * & * & * & * & * \\ L - T & T & -I_m & * & * & * & * \\ RD & 0 & 0 & -R & * & * & * \\ WD + \alpha ZF & 0 & 0 & 0 & -W & * & * \\ \tilde{\alpha} ZF & 0 & 0 & 0 & 0 & -W & * \\ B_1^T R & B_1^T W + D_{21}^T Z^T & 0 & 0 & 0 & 0 & -\gamma^2 I_q \end{bmatrix},$$

where $\tilde{\alpha} = (1 - \bar{\alpha}^2)^{0.5}$.

ii) *If (3.8) is satisfied, a mean square stabilizing filter in the form of (3.2) that achieves* $J_S < 0$ *is given by:*

$$A_f = -W^{-1} S, \quad B_f = -W^{-1} Z \quad \textit{and} \quad C_f = T. \tag{3.9}$$

Proof: The assertions that J_S is negative for all nonzero $w \in \tilde{L}^2([0, \infty); \mathcal{R}^q)$ and that the filter of (3.2) stabilizes the filtering error (namely, $\{\tilde{A}, \tilde{D}_1, \tilde{D}_2, \alpha\}$ defines a mean square stable evolution) are equivalent, by Lemma 3.1, to the solvability of the following Riccati inequality

$$\begin{aligned} &\tilde{A}^T Q + Q\tilde{A} + \gamma^{-2} Q\tilde{B}\tilde{B}^T Q + \tilde{C}^T \tilde{C} + \tilde{D}_1^T Q \tilde{D}_1 + \tilde{D}_2^T Q \tilde{D}_2 \\ &+ \bar{\alpha}[\tilde{D}_2^T Q \tilde{D}_1 + \tilde{D}_1^T Q \tilde{D}_2] < 0 \, , Q > 0. \end{aligned} \tag{3.10}$$

Since $\tilde{\alpha} \triangleq (1 - \bar{\alpha}^2)^{0.5}$, the inequality (3.10a) is equivalent to the following

$$\begin{aligned} \tilde{A}^T Q + Q\tilde{A} + \gamma^{-2} Q\tilde{B}\tilde{B}^T Q + \tilde{C}^T \tilde{C} + (\tilde{D}_1 + \bar{\alpha}\tilde{D}_2)^T Q (\tilde{D}_1 + \bar{\alpha}\tilde{D}_2) \\ + \tilde{\alpha}^2 \tilde{D}_2^T Q \tilde{D}_2 < 0, \end{aligned} \tag{3.11}$$

because

$$\begin{aligned} \tilde{D}_1^T Q \tilde{D}_1 + \tilde{D}_2^T Q \tilde{D}_2 + \bar{\alpha}\tilde{D}_2^T Q \tilde{D}_1 + \bar{\alpha}\tilde{D}_1^T Q \tilde{D}_2 = (\tilde{D}_1 + \bar{\alpha}\tilde{D}_2)^T Q \\ (\tilde{D}_1 + \bar{\alpha}\tilde{D}_2) + \tilde{\alpha}^2 \tilde{D}_2^T Q \tilde{D}_2. \end{aligned}$$

Applying Schur's complements, (3.11) can be readily rearranged into the following LMI:

$$\begin{bmatrix} \tilde{A}^T Q + Q\tilde{A} & \tilde{C}^T & (\tilde{D}_1 + \bar{\alpha}\tilde{D}_2)^T Q & \tilde{\alpha}\tilde{D}_2^T Q & Q\tilde{B} \\ * & -I & 0 & 0 & 0 \\ * & * & -Q & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\gamma^2 I_q \end{bmatrix} < 0, \tag{3.12}$$

and the negativity of J_S is thus guaranteed iff there exists $Q > 0$ that satisfies (3.12).

Following [38], we partition Q and Q^{-1} as follows

$$Q \triangleq \begin{bmatrix} X & M \\ M^T & U \end{bmatrix}, \quad Q^{-1} \triangleq \begin{bmatrix} Y & N \\ N^T & V \end{bmatrix}, \tag{3.13}$$

where we require that

$$X > Y^{-1}. \tag{3.14}$$

The latter inequality stems from the fact that for $Q > 0$ the following holds:

$$\begin{bmatrix} Q & I_{2n} \\ I_{2n} & Q^{-1} \end{bmatrix} \geq 0$$

which implies that

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0$$

Requiring, however, the filter of (3.2) to be of order n, a strict inequality is required in (3.14) (see[33], page 428). We also note that consequently

$$I_n - XY = MN^T$$

is of rank n.

Defining :

$$J \triangleq \begin{bmatrix} Y & I_n \\ N^T & 0 \end{bmatrix} \quad \text{and} \quad \tilde{J} \triangleq diag\left[J, I_m, J, J, I_q \right]$$

we pre- and post-multiply (3.12) by $\tilde{J}^T$ and $\tilde{J}$, respectively. Substituting for the matrices of (3.7) in (3.12) and carrying out the various multiplications in (3.12) we obtain:

$$\begin{bmatrix} AY + YA^T & A + YA^T X + YC^T Z^T + \hat{Z}^T & YL^T - \tilde{Z}^T & YD^T \\ * & XA + A^T X + ZC + C^T Z^T & L^T & D^T \\ * & * & -I & 0 \\ * & * & * & -Y \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$\left.\begin{array}{cccc} YD^TX+\bar{\alpha}YF^TZ^T & 0 & \tilde{\alpha}YF^TZ^T & B_1 \\ D^TX+\bar{\alpha}F^TZ^T & 0 & \tilde{\alpha}F^TZ^T & XB_1+ZD_{21} \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ -X & 0 & 0 & 0 \\ * & -Y & -I & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I_q \end{array}\right] < 0, \quad X > Y^{-1} > 0, \tag{3.15}$$

where

$$Z \stackrel{\triangle}{=} MB_f, \quad \tilde{Z} \stackrel{\triangle}{=} C_f N^T \quad \text{and} \quad \hat{Z} \stackrel{\triangle}{=} MA_f N^T. \tag{3.16}$$

Pre- and post-multiplying (3.15) by Υ and Υ^T, respectively, we obtain (3.8), where we define the following

$$\Upsilon \stackrel{\triangle}{=} diag\{\begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, I_m, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, I_q\},$$

$$S \stackrel{\triangle}{=} \hat{Z}R, \quad T \stackrel{\triangle}{=} \tilde{Z}R, \quad R \stackrel{\triangle}{=} Y^{-1}, \quad W = X - R. \tag{3.17}$$

ii) If a solution to (3.8) exists, we obtain from (3.16) that

$$A_f = M^{-1}\hat{Z}N^{-T}, \quad B_f = M^{-1}Z \quad \text{and} \quad C_f = \tilde{Z}N^{-T}. \tag{3.18}$$

Denoting the transfer function matrix of the filter of (3.2) by $H_{\hat{z}y}(s)$, we find from (3.18) that:

$$H_{\hat{z}y}(s) = \tilde{Z}N^{-T}(sI_n - M^{-1}\hat{Z}N^{-T})^{-1}M^{-1}Z$$

which leads to:

$$H_{\hat{z}y}(s) = \tilde{Z}(sMN^T - \hat{Z})^{-1}Z = \tilde{Z}[s(I_n - XY) - \hat{Z}]^{-1}Z.$$

The result of (3.9) follows using (3.17).

□

3.5 The Polytopic Case

Due to the affinity of (3.8) in A, B_1, C, D_{21}, D and F, the result of 3.1 can be extended to the case where these matrices lie in a convex bounded domain. In this case, we require that (3.8) holds for all the vertices of the uncertainty polytope for a single set of matrices (R, W, Z, S, T).

Assuming that A, B_1, C, D_{21}, D and F lie in the following uncertainty polytope:

$$\bar{\Omega} \stackrel{\triangle}{=} \{(A, B_1, C, D_{21}, D, F) | (A, B_1, C, D_{21}, D, F) = \sum_{i=1}^{l} \tau_i (A_i, B_{1i}, C_i, D_{21,i}, D_i, F_i) \; ; \tau_i \geq 0; \sum_{i=1}^{l} \tau_i = 1\}$$

and denoting the set of the l vertices of this polytope by $\bar{\Psi}$ we obtain the following result:

Corollary 3.1. *Consider the system of (3.1) and the filter of (3.2). The performance index of (3.4) is negative for a given $\gamma > 0$, for all nonzero $w \in \tilde{L}^2([0,\infty); \mathcal{R}^q)$ and for all $(A, B_1, C, D_{21}, D, F) \in \bar{\Omega}$, if (3.8) is satisfied at all the vertices in $\bar{\Psi}$ by a single set (R, W, Z, S, T). In the latter case the filter matrices are given by (3.9).*

We note here that while (3.8) requires (3.1) to be asymptotically stable for a single A, the requirement of the last corollary demands (3.1) to be quadratically stable [93] for all $A \in \bar{\Omega}$. Thus, while in 3.1 the conditions were necessary and sufficient, the results of the later corollary are only sufficient. It should be emphasized that the requirement for quadratic stability may sometimes be quite conservative.

3.6 Robust Mixed Stochastic H_2/H_∞ Filtering

The mixed stochastic H_2/H_∞ filter design is achieved by considering the set of filters that satisfy the H_∞ requirement and choosing the one that minimizes an upper-bound on the estimation error variance. The latter is described by the following H_2 objective function

$$J_2 = \lim_{t \to \infty} E\left\{\tilde{z}^T(t)\tilde{z}(t)\right\} = ||H_{\tilde{z}w}||_2^2,$$

where $||\cdot||_2$ stands for the standard H_2-norm and where $H_{\tilde{z}w}$ is the transference in the system of (3.6), from w to $\tilde{z}$. The signal $w(t)$ is considered, at the present context, to be a white noise.

Denoting $\bar{P} \stackrel{\triangle}{=} \lim_{t \to \infty} E\left\{\xi\xi^T\right\}$, we readily find that, in such a case, $||H_{\tilde{z}w}||_2^2 = Tr\{\tilde{C}\bar{P}\tilde{C}^T\}$ where $\bar{P}$ satisfies the following algebraic equation [105],[108]:

$$\tilde{A}\bar{P} + \bar{P}\tilde{A}^T + \tilde{D}_1\bar{P}\tilde{D}_1^T + \tilde{D}_2\bar{P}\tilde{D}_2^T + \bar{\alpha}(\tilde{D}_1\bar{P}\tilde{D}_2^T + \tilde{D}_2\bar{P}\tilde{D}_1^T) + \tilde{B}\tilde{B}^T = 0. \quad (3.19)$$

We consider next the following algebraic equation:

$$\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{D}_1^T\tilde{Q}\tilde{D}_1 + \tilde{D}_2^T\tilde{Q}\tilde{D}_2 + \bar{\alpha}(\tilde{D}_1^T\tilde{Q}\tilde{D}_2 + \tilde{D}_2^T\tilde{Q}\tilde{D}_1) + \tilde{C}^T\tilde{C} = 0. \quad (3.20)$$

We obtain the following:

Lemma 3.2. $Tr\{\tilde{C}\bar{P}\tilde{C}^T\} = Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}.$

Proof: (see also Proposition 1 in [86]) : We consider the following:

$$Tr\{[\tilde{A}\bar{P} + \bar{P}\tilde{A}^T + \tilde{D}_1\bar{P}\tilde{D}_1^T + \tilde{D}_2\bar{P}\tilde{D}_2^T + \bar{\alpha}\tilde{D}_1\bar{P}\tilde{D}_2^T]\tilde{Q}\} = Tr\{\bar{P}[\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{D}_1^T\tilde{Q}\tilde{D}_1 + \tilde{D}_2^T\tilde{Q}\tilde{D}_2 + \bar{\alpha}(\tilde{D}_1^T\tilde{Q}\tilde{D}_2 + \tilde{D}_2^T\tilde{Q}\tilde{D}_1)\}.$$

Using $Tr\{\bar{\alpha}\beta\} = Tr\{\beta\alpha\}$ the result of the lemma readily follows.

□

We note that $\tilde{Q}$ satisfies $\Gamma(\tilde{Q}) = 0$ where we define:

$$\Gamma(\tilde{Q}) = \tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{D}_1^T\tilde{Q}\tilde{D}_1 + \tilde{D}_2^T\tilde{Q}\tilde{D}_2 + \bar{\alpha}(\tilde{D}_1^T\tilde{Q}\tilde{D}_2 + \tilde{D}_2^T\tilde{Q}\tilde{D}_1) + \tilde{C}^T\tilde{C}. \tag{3.21}$$

We consider $\hat{\Gamma}(\Sigma, \hat{Q}) \stackrel{\triangle}{=} \Gamma(\hat{Q}) + \Sigma$ for some $0 \leq \Sigma \in \mathcal{R}^{n\times n}$ and relate solutions of the following equation (3.22) to those of $\Gamma(\tilde{Q}) = 0$:

$$\hat{\Gamma}(\hat{Q}, \Sigma) = 0. \tag{3.22}$$

It follows from the monotonicity, with respect to Σ, of the solutions to (3.22), say $\tilde{Q}$, that $\tilde{Q} - \hat{Q} < 0$ where $\tilde{Q}$ satisfies

$$\hat{\Gamma}(\tilde{Q}, 0) = \Gamma(\tilde{Q}) = 0.$$

Denoting the set

$$\tilde{\Omega} \stackrel{\triangle}{=} \{\hat{Q} | \Gamma(\hat{Q}) < 0 \quad ; \quad \hat{Q} > 0\},$$

it follows from the above monotonicity that

$$J_B = Tr\{\tilde{B}^T\hat{Q}\tilde{B}\} > Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}, \quad \forall \hat{Q} \in \tilde{\Omega}. \tag{3.23}$$

To solve the stochastic mixed H_2/H_∞ problem we seek to minimize an upper-bound on J_B over $\tilde{\Omega}$. Namely, assuming that there exists a solution to (3.11), we consider the following LMI:

$$\tilde{\Gamma}(\bar{Q}, \bar{H}) \stackrel{\triangle}{=} \begin{bmatrix} \bar{H} & -\tilde{B}^T\bar{Q} \\ -\bar{Q}\tilde{B} & \bar{Q} \end{bmatrix} > 0, \tag{3.24}$$

where we want to find $\bar{Q} \in \tilde{\Omega}$ and $\bar{H}$ that minimize $Tr\{\bar{H}\}$.

On the other hand (3.11) is equivalent to

$$\Gamma(Q) < -\gamma^{-2}Q\tilde{B}\tilde{B}^TQ,$$

where $\gamma^{-2}Q\tilde{B}\tilde{B}^TQ$ plays this role of $\Sigma \geq 0$. Restricting, therefore, $\bar{Q}$ of (3.24) to the set of the solutions to (3.12), we clearly have that $\Gamma(\bar{Q}) < 0$ and that $\bar{Q} \in \tilde{\Omega}$. We are thus looking for Q and $\bar{H}$ that satisfy (3.12) and $\tilde{\Gamma}(Q, \bar{H}) > 0$ so that $Tr(\bar{H})$ is minimized.

The minimization of $Tr\{\bar{H}\}$ can be put into LMI form that is affine in R, W and Z, by pre- and post-multiplying (3.24) by $diag\{I_q,\ J^T\}$ and $diag\{I_q,\ J\}$, respectively, substituting for $\tilde{B}$ (using (3.7) and (3.18)) and pre- and post-multiplying the result by $\bar{\Lambda}$ and $\bar{\Lambda}^T$, respectively, where: $\bar{\Lambda} \stackrel{\triangle}{=} diag\{I, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}\}$. We obtain the following result:

Theorem 3.2. *Consider the system of (3.6) and (3.4). Given $\gamma > 0$, a filter that yields $J_S < 0$ for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$ and that minimizes a bound on J_B of (3.23) is obtained if there exists a solution (R, S, Z, T, W) to (3.8). The minimizing filter is obtained by simultaneously solving (3.8) and*

$$\bar{\Upsilon} \stackrel{\triangle}{=} \begin{bmatrix} \bar{H} & B_1^T R & B_1^T W + D_{21}^T Z^T \\ * & R & 0 \\ * & * & W \end{bmatrix} > 0, \qquad (3.25)$$

and minimizing $Tr\{\bar{H}\}$. The filter matrices are given then by (3.9).

Remark 3.1. We note that the requirement for a simultaneous solution of (3.8) and (3.25) does not impose any special difficulty since both LMIs are affine with respect to the matrix variables. Their simultaneous solution can be obtained using any standard LMI solver.

Remark 3.2. The problem of finding the minimum of $Tr\{\bar{H}\}$ maybe ill-defined if strict inequalities are imposed on (3.8) and (3.25). In most cases, the infimum of the trace may be achieved on the closure of the feasibility region of (3.12). In our case, however, this infimum will be obtained for Q that is ϵ-close to the boundary of the feasibility region where ϵ is the tolerance of the LMI solver used.

3.7 Conclusions

In this chapter we solve the problem of stationary stochastic H_∞-filtering of continuous-time linear systems using LMI techniques. This problem has already been solved, indirectly, in [59], where a solution to the stochastic output-feedback problem was obtained. The solution of the filtering problem can be extracted from the latter in the case that the measurement matrix is not corrupted by stochastic uncertainty. The resulting filter will require the solution of two coupled nonlinear inequalities. Although our solution treats the case where the uncertainty appears in both, the dynamic and the measurement matrices, it can similarly be extended to the more general case where the stochastic uncertainty appears in all the system state-space representation.

The case that we have treated in this chapter is encountered with in many filtering problems.

Using the LMI approach and applying special transformations, the conditions for the existence of a solution to the problem have been obtained in terms of LMIs that are affine in the system and the filter parameters. This affinity also allows the consideration of deterministic uncertainty in the system, where the deterministic part of the system matrices lies in a given polytopic type domain and the bounded uncertainty in the covariance matrices of the stochastic parameters. An over-design that is inherent in our solution method stems from the quadratic stability nature of the solution. Under the requirement imposed by this type of stability, the conditions we obtained for the existence of a solution to the problem are both necessary and sufficient.

In Chapter 11 an altitude estimation example is given that utilizes the theory of the present chapter.

4

Continuous-time Systems: Tracking Control

4.1 Introduction

In this chapter we treat the problem of H_∞ tracking with stochastic multiplicative white noise. We extend the work of [98], which does not involve stochastic uncertainties, to the case where there are stochastic white noise parameter uncertainties in the matrices of the state-space model that describes the system. We treat the case where correlated parameter uncertainties appear in both the system dynamics and the input matrices for the state-feedback case, and in both, the input and the measurement matrices in the output-feedback case. An optimal finite-horizon state-feedback tracking strategy is derived which minimizes the expected value of the standard H_∞ performance index with respect to the unknown parameters and which applies game theoretic considerations. The solution of the latter problem and the stationary state-feedback case, appear in Section 4.3. In Section 4.4 we solve the output-feedback control problem where we allow for a state-multiplicative noise in the measurement matrix. We first introduce in Section 4.4.1 an auxiliary stochastic BRL for systems that contain, in addition to the standard stochastic continuous-time BRL [10], a reference signal in the system dynamics. The BRL is solved as a max-min problem and results in a modified Riccati equation.

The output-feedback tracking problem is solved in Section 4.4.2 via max-min strategy arguments, rather than the game approach that was applied in the state-feedback case of Section 4.3. Using the latter solution we re-formulate the problem to a filtering problem which we solve with the aid of the above auxiliary BRL. We then apply the theory to a simple state-feedback tracking example where we compare our solution with the solution achieved when the tracking signal is viewed as a disturbance signal and yet the state-multiplicative noise is taken into consideration in the design of the alternative controller.

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 55–73, 2005.

4.2 Problem Formulation

Given the following linear continuous time-varying system:

$$dx = [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) + B_3(t)r(t)]dt + F(t)x(t)d\beta(t)$$
$$+G(t)u(t)d\zeta(t), \quad x(0) = x_0, \tag{4.1}$$
$$z(t) = C_1(t)x(t) + D_{12}(t)u(t) + D_{13}(t)r(t)$$

where $x \in \mathcal{R}^n$ is the system state vector , $x(0)$ is any norm-bounded vector in $\mathcal{R}^n$, $w \in \tilde{L}^2([0,T);\mathcal{R}^p)$ is the exogenous disturbance signal, $z \in \mathcal{R}^q$ is the signal to be controlled and where $A(t),\ B_1(t),\ B_2(t),\ B_3(t),\ C_1(t),\ D_{12}(t),\ D_{13}(t),\ F(t)$ and $G(t)$ are real known, piecewise continuous time-varying matrices of the appropriate dimensions. The variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy:

$$E\{d\beta(t)\} = 0,\ E\{d\zeta(t)\} = 0,\ E\{d\beta(t)^2\} = dt,\ E\{d\zeta(t)^2\} = dt,$$
$$E\{d\beta(t)d\zeta(t)\} = \bar{\alpha}dt, |\bar{\alpha}| \leq 1.$$

We denote $\tilde{R}(t) \stackrel{\triangle}{=} D_{12}^T D_{12}$.
We consider the following problems:

i) State-feedback tracking: Our objective is to find a state-feedback control law $u(t)$ that minimizes, for the worst-case of the process disturbance $w(t)$ and the initial condition x_0, the mean energy of $z(t)$, with respect to the uncertain parameters, by using the available knowledge on the reference signal,. We, therefore, consider, for a given scalar $\gamma > 0$, the following performance index:

$$J_E \stackrel{\triangle}{=} E\{\int_0^T ||z(t)||^2 dt - \gamma^2 \int_0^T ||w(t)||^2 dt\} + E\{x^T(T)P_T x(T)\}$$
$$-\gamma^2 ||x_0||^2_{R^{-1}},\ R > 0,\ P_T \geq 0. \tag{4.2}$$

Similarly to [98] we consider three different tracking problems:
1) Stochastic H_∞-tracking with full preview of $r(t)$: The tracking signal is perfectly known over the interval $t \in [0, T]$.
2) Stochastic H_∞-tracking with zero preview of $r(t)$: The tracking signal is measured on line i.e at time t, $r(\tau)$ is known for $\tau \leq t$.
3) Stochastic H_∞ finite-fixed preview tracking of $r(t)$: The tracking signal $r(t)$ is previewed in a known fixed interval i.e $r(\tau)$ is known for $\tau \leq t+h$ where h is a known preview length.
In all three cases we seek a control law $u(t)$ of the form

$$u(t) = H_x(t)x(t) + H_r(t)r(t), \tag{4.3}$$

where $H_x(t)$ is a causal operator and where the causality of $H_r(t)$ depends on the information pattern of the reference signal.

For all of the above three tracking problems we consider a related linear quadratic game in which the controller plays against nature. We, thus, consider the following game:

Find $w^*(t) \in \tilde{L}^2([0,T);\mathcal{R}^p)$, $u^*(t) \in \tilde{L}^2([0,T);\mathcal{R}^l)$ and $x_0^* \in R^n$ that satisfy:

$$J_E(r,u^*,w,x_0) \leq J_E(r,u^*,w^*,x_0^*) \leq J_E(r,u,w^*,x_0^*), \tag{4.4}$$

where w^*, x_0^* and u^* are the saddle-point strategies and $r(t)$ is a fixed signal of finite energy.

ii) Output-feedback tracking: We consider the following system:

$$\begin{aligned}
&dx = [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) + B_3(t)r(t)]dt + F(t)x(t)d\beta(t),\\
&x(0) = x_0,\\
&dy(t) = [C_2(t)x(t) + D_{21}(t)w(t)]dt + H(t)x(t)d\zeta(t) + n(t)dt,\\
&z(t) = C_1(t)x(t) + D_{12}(t)u(t) + D_{13}(t)r(t)
\end{aligned} \tag{4.5}$$

where $y(t) \in \mathcal{R}^z$ and where we note that the measurement matrix is contaminated with a zero-mean real scalar white noise process $H(t)\dot{\zeta}(t)$, where $E\{d\zeta(t)^2\} = dt$, $E\{d\beta(t)d\zeta(t)\} = 0$. For simplicity, the stochastic uncertainty is removed from the input matrix. Similarly to the state-feedback case we seek a control law $u(t)$, based on the information on the reference signal $r(t)$, that minimizes the tracking error between the the system output and the tracking trajectory, for the worst case of the initial condition x_0, the process disturbances $w(t)$, and the measurement noise $n(t)$. We, therefore, consider the following performance index:

$$J_O(r,u,w,n,x_0) = J_E(r,u,w,x_0) - \gamma^2 E\{\int_0^T ||n(t)||^2 dt\}, \tag{4.6}$$

where J_E is given in (4.2). Similarly to the state-feedback case we solve the problem for the above three tracking patterns. We seek a controller $u(t) \in \tilde{L}^2([0,T);\mathcal{R}^l)$ of the form (4.3) where our design objective is to minimize

$$\max J_O(r,u,w,n,x_0)\ \forall w(t) \in \tilde{L}^2([0,T);\mathcal{R}^p),\ n(t) \in \tilde{L}^2([0,T);\mathcal{R}^z),\ x_o \in R^n.$$

For all the three tracking problems we derive a controller $u(t)$ which plays against its adversaries $w(t), n(t)$ and x_0.

4.3 The State-feedback Tracking

We consider the following Riccati-type differential equation:

$$
\begin{aligned}
&-\dot{Q} = QA + A^T Q + \gamma^{-2} Q B_1 B_1^T Q + C_1^T C_1 - \bar{S}^T \hat{R}^{-1} \bar{S} + F^T Q F, \\
&Q(T) = P_T
\end{aligned}
\tag{4.7}
$$

where

$$
\hat{R} = \tilde{R} + G^T Q G, \quad \bar{S} = B_2^T Q + \bar{\alpha} G^T Q F + D_{12}^T C_1. \tag{4.8}
$$

The solution to the state-feedback tracking problem is obtained by the following :

Theorem 4.1. *Consider the system of (4.1) and* J_E *of (4.2). Given* $\gamma > 0$*, the state-feedback tracking game possesses a saddle-point equilibrium solution iff there exists* $Q(t) > 0, \forall t \in [0, T]$ *that solves (4.7) such that* $Q(0) < \gamma^2 R^{-1}$*. When a solution exists, the saddle-point strategies are given by:*

$$
\begin{aligned}
x_0^* &= (\gamma^2 R^{-1} - Q_0)^{-1} \theta(0) \\
w^* &= \gamma^{-2} B_1^T (Qx + \theta) \\
u^* &= -\hat{R}^{-1} [(B_2^T Q + D_{12}^T C_1 + \bar{\alpha} G^T Q F) x + D_{12}^T D_{13} r + B_2^T \theta_c]
\end{aligned}
\tag{4.9}
$$

where w^*, x_0^* *and* u^* *are the maximizing and minimizing strategies of nature and the controller, respectively, and where*

$$
\dot{\theta}(t) = -\bar{A}^T \theta(t) + \bar{B}_r r(t), \quad t \in [0\ T],\ \theta(T) = 0, \tag{4.10}
$$

with

$$
\begin{aligned}
\bar{A} &= A - B_2 \hat{R}^{-1} (D_{12}^T C_1 + \bar{\alpha} G^T Q F) + (\gamma^{-2} B_1 B_1^T - B_2 \hat{R}^{-1} B_2^T) Q \\
\bar{B}_r &= \bar{S}^T \hat{R}^{-1} D_{12}^T D_{13} - (Q B_3 + C_1^T D_{13}),
\end{aligned}
\tag{4.11}
$$

and where $\theta_c \stackrel{\triangle}{=} [\theta(t)]_+$ *(i.e the causal part of* $\theta(\cdot)$*) satisfies:*

$$
\begin{aligned}
&\dot{\theta}_c(\tau) = -\bar{A}^T(\tau) \theta_c(\tau) + \bar{B}_r(\tau) r(\tau), \quad t \le \tau \le t_f, \\
&t_f = \begin{cases} t + h \ \textit{if} \ \ t + h < T \\ T \qquad \textit{if} \ \ t + h \ge T \end{cases} \\
&\theta_c(t_f) = 0.
\end{aligned}
\tag{4.12}
$$

The game value is then given by:

$$
J_E(r, u^*, w^*, x_0^*) = \bar{J}(r) + E\{\int_0^T ||B_2^T [\theta]_-||_{\hat{R}}^2 dt\} \tag{4.13}
$$

where $[\theta]_{-1} = \theta(t) - \theta_c(t),\ \forall t \in [0,\ \ T]$ is the anti causal part of $\theta(t)$ and where

$$
\bar{J}(r) = \int_0^T ||D_{13} r||^2 dt + \gamma^{-2} \int_0^T ||B_1^T \theta||^2 dt - \int_0^T ||\hat{R}^{-1/2} (B_2^T \theta + D_{12}^T D_{13} r)||^2 dt
$$

$$+2\int_0^T \theta^T B_3 r dt + \gamma^{-2}||\theta(0)||^2_{P_0} \tag{4.14}$$

with

$$P_0 = [R^{-1} - \gamma^{-2}Q(0)]^{-1}. \tag{4.15}$$

Proof: Sufficiency Let $Q(t)$ be a solution of (4.1) such that $Q(0) < \gamma^2 R^{-1}$. Considering (4.1) and applying Ito formula to $\varphi(t, x(t)) = \langle x(t), Q(t)x(t)\rangle$, and taking expectation for every $T > 0$ we obtain:

$$0 = E\{\int_0^T d\{x^T(t)Q(t)x(t)\} - x(T)^T P_T x(T)\} + x_0^T Q(0)x_0 = E\{\int_0^T \langle x(t), \dot{Q}(t)x(t)\rangle dt\}$$

$$+2E\{\int_0^T \langle Q(t)x(t), A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) + B_3(t)r(t)\rangle dt\}$$

$$+E\{\int_0^T Tr\{Q(t)[F(t)x(t) \;\; G(t)u(t)]\bar{P}[F(t)x(t) \;\; G(t)u(t)]^T\}dt\} + x_0^T Q(0)x_0$$

where $\bar{P} \triangleq \begin{bmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & 1 \end{bmatrix}$ is the covariance matrix of the augmented Wiener process vector $col\{\beta(t), \; \zeta(t)\}$. Now,

$$Tr\{Q(t)[F(t)x(t) \;\; G(t)u(t)]\bar{P}[F(t)x(t) \;\; G(t)u(t)]^T\}$$

$$= Tr\{\begin{bmatrix} x^T(t)F^T(t) \\ u^T(t)G^T(t) \end{bmatrix} Q(t)[F(t)x(t) \;\; G(t)u(t)]\bar{P}\}$$

$$= Tr\{\begin{bmatrix} x^T(t)F^T(t)Q(t)F(t)x(t) & x^T(t)F^T(t)Q(t)G(t)u(t) \\ u^T(t)G^T(t)Q(t)F(t)x(t) & u^T(t)G^T(t)Q(t)G(t)u(t) \end{bmatrix} \begin{bmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & 1 \end{bmatrix}\}$$

$$= \langle \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} F^T(t)Q(t)F(t) & \bar{\alpha}F^T(t)Q(t)G(t) \\ \bar{\alpha}G^T(t)Q(t)F(t) & G^T(t)Q(t)G(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \rangle$$

$$= x^T(t)F^T(t)Q(t)F(t)x(t) + 2\bar{\alpha}x^T(t)F^T(t)Q(t)G(t)u(t)$$

$$+u^T(t)G^T(t)Q(t)G(t)u(t)$$

Using the above derivation and adding the zero sum of

$$E\{\int_0^T ||w(t)||^2 dt\} - E\{\int_0^T ||w(t)||^2 dt\} + E\{\int_0^T ||z(t)||^2 dt\} - E\{\int_0^T ||z(t)||^2 dt\}$$

we obtain after completing to squares for $w(t)$:

$$0 = E\{\int_0^T [x^T(t)[\dot{Q}(t) + Q(t)A(t) + A^T(t)Q(t) + F^T(t)Q(t)F(t)$$

$$-\gamma^{-2}Q(t)B_1(t)B_1^T(t)Q(t)]x(t)dt\}+2E\{\int_0^T x^T(t)Q(t)[B_2(t)u(t)+B_3(t)r(t)]dt\}$$

$$+E\{\int_0^T ||z||^2+\gamma^2(||w||^2+||w-\gamma^{-2}B_1^TQx||^2)dt-x(T)^TP_Tx(T)\}$$

$$+E\{\int_0^T 2\bar{\alpha}x^T(t)F^T(t)Q(t)G(t)u(t)+u^T(t)G^T(t)Q(t)G(t)u(t)dt\}+x_0^TQ(0)x_0.$$

$$-E\{\int_0^T ||z(t)||^2dt+E\int_0^T [||C_1(t)x(t)+D_{12}(t)u(t)+D_{13}(t)r(t)||^2dt\}$$

Completing to squares for $u(t)$, we obtain:

$$0=E\{\int_0^T \{-||z(t)||^2+\gamma^2(||w(t)||^2-||w(t)-\gamma^{-2}B_1^T(t)Q(t)x(t)||^2)\}dt\}$$

$$+\{E\int_0^T ||\hat{u}(t)||^2_{\hat{R}}+||D_{13}(t)r(t)||^2dt-x(T)^TP_Tx(T)\}$$

$$+2E\{\int_0^T (x^T(t)[Q(t)B_3(t)+C_1^T(t)D_{13}(t)]+u^T(t)D_{12}^T(t)D_{13}(t)r(t))dt\}+x_0^TQ(0)x_0$$

where

$$\hat{u}(t)=u(t)+\hat{R}^{-1}\bar{S}(t)x(t). \tag{4.16}$$

Adding $J_E(r,u,w,x_0)$ of (4.2) to the above zero quantity we obtain:

$$J_E=-\gamma^2||x_0||^2_{P_0^{-1}}+E\{\int_0^T \{[||u(t)+\hat{R}^{-1}\bar{S}(t)x(t)||^2_{\hat{R}}+||D_{13}(t)r(t)||^2$$

$$+2x^T(t)Q(t)B_3(t)+C_1^T(t)D_{13}(t)]r(t)\}dt+E\{x^T(T)P_Tx(T)\}$$

$$+2E\{\int_0^T (\hat{u}(t)-\hat{R}^{-1}\bar{S}^T(t)x(t))^TD_{12}^T(t)D_{13}(t)r(t)dt\}$$

where

$$\hat{w}(t)=w(t)-\gamma^2B_1^T(t)Q(t)x(t). \tag{4.17}$$

Next we add the following identically zero-term to the above J_E, where we apply the Ito lemma:

$$0=2E\{\int_0^T d\theta^T(t)x(t)dt\}-2[\theta^T(0)x(0)]_+=2E\{\theta^T(T)x(T)\}-2\theta^T(0)x(0)$$

$$+2\theta^T(0)x(0)=2E\{\int_0^T \dot{\theta}^T(t)x(t)dt\}+2E\{\int_0^T \theta^T(t)\{\bar{A}(t)x(t)+B_1(t)\hat{w}(t)$$

$$+B_2(t)\hat{u}(t)+B_3(t)r(t)\}dt\}+2\theta^T(0)x(0)$$

where $\theta(t)$ and $\bar{A}(t)$ are defined in (4.10) and (4.11), respectively, and satisfy the following:

$$dx = \bar{A}(t)x(t) + B_1(t)\hat{w}(t) + B_2(t)\hat{u}(t) + B_3(t)r(t)]dt + F(t)x(t)d\beta(t)$$
$$+G(t)\hat{u}(t)d\zeta(t) - G(t)\hat{R}^{-1}\bar{S}xd\zeta(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)$$
$$+B_3(t)r(t)]dt + F(t)x(t)d\beta(t) + G(t)u(t)d\zeta(t).$$

By another completion to squares with respect to $\hat{u}(t)$ we obtain:

$$J_E = -\gamma^2||x_0 - \gamma^{-2}P_0\theta(0)||^2_{P_0^{-1}} + \gamma^{-2}||\theta(0)||^2_{P_0}$$
$$-\gamma^2 E\{\int_0^T ||[\hat{w} - \gamma^{-2}B_1^T\theta]||^2 dt\} + \gamma^{-2}E\{\int_0^T ||B_1^T\theta||^2 dt\}$$
$$+E\{\int_0^T ||\hat{R}^{1/2}[\hat{u} + \hat{R}^{-1}(B_2^T\theta + D_{12}^T D_{13}r)]||^2 dt\} - E\{\int_0^T ||\hat{R}_1^{-1/2}B_2^T\theta||^2 dt\}$$
$$+2E\{\int_T^0 [\dot{\theta} + \bar{A}^T\theta + [QB_3 + C_1^T D_{13} - \bar{S}^T\hat{R}^{-1}D_{12}^T D_{13}]r]^T x dt\} + E\{x^T(T)P_T x(T)\}$$
$$-E\{\int_0^T 2\theta^T(B_2\hat{R}^{-1}D_{12}^T D_{13} - B_3)r dt\} + E\{\int_0^T ||D_{13}r||^2 dt\}$$
$$-E\{\int_0^T ||\hat{R}^{-1/2}D_{12}^T D_{13}r||^2 dt\}.$$

Denoting next:

$$\hat{x}_0 = \gamma^{-2}P_0\theta(0) = [\gamma^2 R^{-1} - Q(0)]^{-1}\theta(0) \tag{4.18}$$

and considering (4.10), (4.16) and (4.17) and the above J_E can be rewritten as:

$$J_E(r, u, w, x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||w - \gamma^{-2}B_1^T(Qx + \theta)||^2 dt\}$$
$$+E\{\int_0^T ||\hat{R}^{1/2}[u + \hat{R}^{-1}\bar{S}^T x + \hat{R}^{-1}(B_2^T\theta + D_{12}^T D_{13}r)]||^2 dt\} + \bar{J}(r) + E\{x^T(T)P_T x(T)\} \tag{4.19}$$

where $\bar{J}(r)$ is given by (4.14) and is independent of u, w and x_0. Applying the fact that $P_0 > 0$ and that the admissible control signal $u(t)$ is based on the state measurements up to $\tau = t + h$, a saddle-point strategy for the tracking game is readily achieved. Considering (4.9), $[\theta]_-$ and the above J_E (see Chapter 9 about the minimization of quadratic functionals under causality constraints), we obtain:

$$J_E(r, u^*, w, x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||w - \gamma^{-2}B_1^T(Qx + \theta)||^2 dt\}$$

$$+E\{\int_0^T ||\hat{R}^{1/2}B_2^T[\theta]_-)||^2 dt\} + \bar{J}(r) \le J_E(r, u^*, w^*, x_0^*). \tag{4.20}$$

On the other hand, considering (4.9) and $[\theta]_-$ we obtain:

$$J_E(r, u, w^*, x_0^*) = E\{\int_0^T ||\hat{R}^{1/2}[u+\hat{R}^{-1}\bar{S}^T x+\hat{R}^{-1}(B_2^T\theta + D_{12}^T D_{13}r)]||^2 dt\}$$
$$+\bar{J}(r) \ge J_E(r, u^*, w^*, x_0^*). \tag{4.21}$$

Necessity The saddle-point strategies provide a saddle-point equilibrium also for the case of $r(t) \equiv 0$. In this case, one obtains the problem of stochastic state-multiplicative noisy state-feedback regulation game for which the condition of Theorem 4.1 is necessary [24], [86], [59].

If there exists a solution $Q(t) = Q^T(t), \forall t \in [0, T]$ that solves (4.7) with the condition of $Q(0) < \gamma^2 R^{-1}$, we obtain the following control strategies:

□

Corollary 4.1. Stochastic H_∞-Tracking with full preview: *In this case $\theta(t)$ is as in (4.10) and the control law is given by:*

$$u = K_x x + K_r r + K_\theta \theta,$$

where

$$K_x = -\hat{R}^{-1}(B_2^T Q + D_{12}^T C_1 + \bar{\alpha} G^T Q F),$$
$$K_r = -\hat{R}^{-1} D_{12}^T D_{13}, \tag{4.22}$$

and

$$K_\theta = -\hat{R}^{-1} B_2^T.$$

Furthermore, $J_E(r, u^*, w^*, x_0^*)$ of (4.4) coincides with $\bar{J}(r)$ of (4.14).

Corollary 4.2. Stochastic H_∞-Tracking with no preview: *In this case the control law is given by:*

$$u = K_x x + K_r r, \tag{4.23}$$

and the existence of (4.4) is guaranteed where

$$J_E(r, u^*, w^*, x_0^*) = E\{\int_0^T ||\hat{R}^{-1/2} B_2^T \theta||^2 dt\} + \bar{J}(r)$$

where $\theta(\cdot)$ and $\bar{J}(r)$ satisfy (4.10) and (4.14), respectively.

Corollary 4.3. Stochastic H_∞-Tracking with finite fixed-preview: *In this case the following control law is obtained:*

$$u = K_x x + K_r r + K_\theta \theta_c, \tag{4.24}$$

where K_x, K_r and K_θ are defined in (4.22) and θ_c is given by (4.12). The above controller achieves (4.4) with

$$J_E(r, u^*, w^*, x_0^*) = \bar{J}(r) + E\{\int_0^T ||\hat{R}^{-1/2} B_2^T [\theta]_-||^2 dt\}$$

and where $\bar{J}(r)$ is defined in (4.14). The latter is obtained since at time $\bar{t}$, $r(t)$ is known for $t \in [\bar{t} \ \ min(T, \bar{t} + h)]$.

Remark 4.1. The general case, where a multiple set of correlated stochastic uncertainties appear in both the dynamic and the input matrices, can be readily solved by extending the results of Theorem 4.1. The proof outline of Theorem 4.1 in the later case is essentially unchanged. We bring below a simplified case which will be used in the infinite-horizon, stationary state-feedback tracking. We consider the following system:

$$\begin{aligned} dx \ &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) + B_3(t)r(t)]dt + [\tilde{F}(t)d\beta(t) \\ &\quad + \hat{F}(t)d\zeta(t)]x(t) + G(t)u(t)d\zeta(t), \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) + D_{13}(t)r(t), \end{aligned} \tag{4.25}$$

where the variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes with the same statistic as above.

Note that the Wiener process of $\zeta(t)$ appears in both, the dynamics and the input matrices. In the above case, we obtain the results of Theorem 4.1 where $F^T Q F$ in (4.7) is replaced by $\hat{F}^T Q \hat{F} + \tilde{F}^T Q \tilde{F} + 2\bar{\alpha}[\hat{F}^T Q \tilde{F} + \tilde{F}^T Q \hat{F}]$, and $\bar{\alpha} G^T Q F$ is replaced by $\bar{\alpha} G^T Q \tilde{F} + G^T Q \hat{F}$ in $\bar{S}$, u^* and $\bar{A}$ of (4.8), (4.9) and (4.11), respectively. The results of Corollaries 4.1-4.3 are also retrieved when the latter changes are made in K_x of (4.22).

4.3.1 The Infinite-horizon Case

We treat the case where the matrices of the system in (4.1) are all time-invariant, T tends to infinity and the system is mean square stable. Since (4.7) is identical to the one encountered in the corresponding state-feedback regulator problem (where $r(t) \equiv 0$)[24], [86], the solution $Q(t)$ of (4.7), if it exists, will tend to the mean square stabilizing solution (see [23]) of the following equation:

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + F^T\tilde{Q}F = 0, \tag{4.26}$$

assuming that the pair $(\Pi C_1, A - B_2\tilde{R}^{-1}D_{12}^TC_1)$, $\Pi = I - D_{12}\tilde{R}^{-1}D_{12}^T$ is detectable (see Theorem 5.8 in [23]). A strict inequality is achieved from (4.26) for $(w(t), x_o)$ that are not identically zero, iff the left side of (4.26) is strictly negative definite (for the equivalence between (4.26) and the corresponding inequality see [59]). The latter inequality can be expressed in a LMI form in the case where $\bar{\alpha} = 0$ and $D_{12}^TC_1 = 0$. We arrive at the following result:

Theorem 4.2. *Consider the system of (4.1) and J_E of (4.2) with constant matrices, $D_{12}^T C_1 = 0$ and $\bar{\alpha} = 0$. Then, for a given $\gamma > 0$, there exists a strategy u^* that satisfies $\forall w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p),\ x_o \in R^n$,*

$$J_E(r,u^*,w,x_0) < \bar{J}(r) + E\{\int_0^\infty ||\hat{R}^{-1/2}B_2^T[\theta]_-||^2 dt\},$$

where

$$\bar{J}(r) = E\{\int_0^\infty ||D_{13}r||^2 dt\} + \gamma^{-2}E\{\int_0^\infty ||B_1^T\theta||^2 dt\} - E\{\int_0^\infty ||\hat{R}_1^{-1/2}(B_2^T\theta$$
$$+D_{12}^T D_{13}r)||^2 dt\} + 2E\{\int_0^\infty \theta^T B_3 r dt\} + \gamma^{-2}||\theta(0)||_{P_0}^2, \tag{4.27}$$

iff there exists a positive-definite matrix $\tilde{P} \in \mathcal{R}^{n\times n}$ that satisfies the following LMI:

$$\Gamma_1 \stackrel{\Delta}{=} \begin{bmatrix} A\tilde{P}+\tilde{P}A^T - B_2\tilde{R}^{-1}B_2^T & B_1 & \tilde{P}C_1^T & B_2G^T & \tilde{P}F^T \\ * & -\gamma^2 I_q & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -(\tilde{P}+G\tilde{R}^{-1}G^T) & 0 \\ * & * & * & * & -\tilde{P} \end{bmatrix} < 0. \tag{4.28}$$

Proof: The inequality that is obtained from (4.7) for $\bar{\alpha} = 0$ and $D_{12}^T C_1 = 0$ is

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + F^T\tilde{Q}F < 0,$$

where $\tilde{S} = B_2^T\tilde{Q}$.

Denoting $\tilde{P} = \tilde{Q}^{-1}$, we multiply the latter inequality by $\tilde{P}$ from both sides and obtain:

$$A\tilde{P} + \tilde{P}A^T + \gamma^{-2}B_1B_1^T + \tilde{P}C_1^TC_1\tilde{P} - B_2\bar{R}^{-1}B_2^T + \tilde{P}F^T\tilde{P}^{-1}F\tilde{P} < 0,$$

where $\bar{R} = \tilde{R} + G^T\tilde{P}^{-1}G$.
Since

$$(\tilde{R} + G^T\tilde{P}^{-1}G)^{-1} = \tilde{R}^{-1/2}[I + \tilde{R}^{-1/2}G^T\tilde{P}^{-1}G\tilde{R}^{-1/2}]^{-1}\tilde{R}^{-1/2},$$

we obtain, using the matrix inversion lemma, the following equality:

$$[I + \tilde{R}^{-1/2}G^T\tilde{P}^{-1}G\tilde{R}^{-1/2}]^{-1} = I - \tilde{R}^{-1/2}G^T\tilde{P}^{-1}G\tilde{R}^{-1/2}[I+$$
$$\tilde{R}^{-1/2}G^T\tilde{P}^{-1}G\tilde{R}^{-1/2}]^{-1}.$$

Using the latter, together with the identity $\alpha[I + \beta\alpha]^{-1} = [I + \alpha\beta]^{-1}\alpha$, we readily obtain the following inequality:

$$A\tilde{P} + \tilde{P}A^T + \gamma^{-2}B_1B_1^T + \tilde{P}C_1^TC_1\tilde{P} - B_2\tilde{R}^{-1}B_2^T + B_2G^T[\tilde{P} + G\tilde{R}^{-1}G^T]^{-1}$$
$$GB_2^T + \tilde{P}F^T\tilde{P}^{-1}F\tilde{P} < 0.$$

By using Schur's complements formula, the latter inequality is equivalent to (4.28)

□

Remark 4.2. In the general case, where $D_{12}^TC_1 \neq 0$, a simple change of variables (see [57], page 195) can be readily used. Denoting:

$$\tilde{A} = A - B_2\tilde{R}^{-1/2}D_{12}^TC_1,$$
$$\tilde{u} = u + \tilde{R}^{-1/2}D_{12}^TC_1x,$$

and

$$\tilde{C}_1^T\tilde{C}_1 = C_1^T[I - D_{12}\tilde{R}^{-1}D_{12}^T]C_1,$$

we consider the following mean square stable system:

$$dx = \tilde{A}xdt + B_1wdt + B_2\hat{u}dt + B_3rdt + [Fd\beta - Gd\zeta]x + G\hat{u}d\zeta,$$
$$z = \begin{bmatrix} \tilde{C}_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} \tilde{u}.$$

Note that this system possesses multiple uncertainties which can be readily tackled using the arguments of Remark 4.1 with $\bar{\alpha} = 0$.

4.4 The Output-feedback Tracking Control

The output-feedback problem is solved along the lines of the standard solution [57], where use is made of the state-feedback solution of Section 4.3, thus arriving at an estimation problem to which we apply an auxiliary BRL, which is partially derived from the state-feedback solution. We first bring the following BRL solution.

4.4.1 BRL for Systems with State-multiplicative Noise and Tracking Signal

We consider the system:

$$dx = [A(t)x(t) + B_1(t)w(t) + B_3(t)r(t)]dt + F(t)x(t)d\beta(t), \; x(0) = x_0,$$
$$z(t) = C_1(t)x(t) + D_{13}(t)r(t), \tag{4.29}$$

which is obtained from (4.1) by setting $B_2(t) \equiv 0$ and $D_{12}(t) \equiv 0$. We consider the following index of performance:

$$J_B(r,w,x_0) \stackrel{\triangle}{=} E\{\int_0^T ||z(t)||^2 dt\} - \gamma^2 E\{\int_0^T ||w(t)||^2 dt\} - \gamma^2 ||x_0||^2_{R^{-1}}, \; R > 0, \tag{4.30}$$

which is obtained from (4.2) by setting $P_T = 0$. We arrive at the following.

Theorem 4.3. *Consider the system of (4.29) and J_B of (4.30). Given $\gamma > 0$, J_B of (4.30) satisfies $J_B \leq \tilde{J}(r,\epsilon)$ $\forall w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, $x_o \in R^n$, where*

$$\tilde{J}(r,\epsilon) = E\{\int_0^T ||D_{13}r||^2 dt\} + \gamma^{-2} E\{\int_0^T ||B_1^T \tilde{\theta}||^2 dt\}$$

$$+2E\{\int_0^T \tilde{\theta}^T B_3 r dt + ||\tilde{\theta}(0)||^2_{\epsilon^{-1}}\}, \tag{4.31}$$

iff there exists $\tilde{Q}(t) > 0, \forall t \in [0,T]$ that solves the following Riccati-type equation:

$$-\dot{\tilde{Q}} = \tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 + F^T\tilde{Q}F, \quad \tilde{Q}(0) = \gamma^2 R^{-1} - \epsilon I, \tag{4.32}$$

for some $\epsilon > 0$, where

$$\dot{\tilde{\theta}}(t) = -\hat{A}^T\tilde{\theta}(t) + \hat{B}_r r(t), \quad t \in [0\ T],\ \tilde{\theta}(T) = 0, \tag{4.33}$$

and where

$$\hat{A} = A + \gamma^{-2}B_1B_1^T\tilde{Q}, \quad \hat{B}_r = -[\tilde{Q}B_3 + C_1^T D_{13}]. \tag{4.34}$$

Proof: The condition of the BRL does not involve saddle-point strategies since $u(t)$ is no longer an adversary. The sufficiency part of the proof can, however, be readily derived based on the first part of the sufficiency proof of Theorem 4.1, up to equation (4.19), where we set $B_2(t) \equiv 0$ and $D_{12}(t) \equiv 0$, and where we take $P_T = 0$. Analogously to (4.19) we obtain the following:

$$J_B(r,w,x_0) = -\gamma^2 ||x_0 - \hat{x}_0||^2_{\tilde{P}_0^{-1}} - \gamma^2 E\{\int_0^T ||w - \gamma^{-2}B_1^T(\tilde{Q}x + \tilde{\theta})||^2 dt\},$$

where we replace θ, Q by $\tilde{\theta}$ and $\tilde{Q}$, respectively and where

$$\begin{aligned} \tilde{P}_0 &= [R^{-1} - \gamma^{-2}\tilde{Q}(0)]^{-1}, \\ \tilde{x}_0 &= \gamma^{-2}\tilde{P}_0\tilde{\theta}(0) = [\gamma^2 R^{-1} - \tilde{Q}(0)]^{-1}\tilde{\theta}(0) \end{aligned} \tag{4.35}$$

The neccesity part follows from the fact that for $r(t) \equiv 0$, one gets $\tilde{J}(r,\epsilon) = 0$ (noting that in this case $\tilde{\theta} \equiv 0$ in (4.33) and therefore the last 3 terms in (4.31) are set to zero) and $J_B < 0$. Thus the existence of $\tilde{Q} > 0$ that solves (4.32) follows from the necessary condition in the stochastic BRL [24], [86], [59]. □

Remark 4.3. Note that the choice of $\epsilon > 0$ in $\tilde{Q}(0)$ of (4.32) reflects on both, the cost value (i.e $\tilde{J}(r,\epsilon)$) of (4.31) and the minimum achievable γ. Choosing $0 < \epsilon << 1$ causes the cost of $\tilde{J}(r,\epsilon)$ to increase while the solution of (4.32) is easier to achieve the result of which is a smaller γ. The choice of large ϵ, on the other hand causes the reverse effect, which leads to a larger γ.

4.4.2 The Output-feedback Control Solution

Due to the special structure of the stochastic uncertainty in the system of (4.5), the solution of the output-feedback control problem can not be obtained by applying saddle-point strategies but rather as a max-min problem. We consider the system of (4.5) and we assume that (4.7) has a solution $Q(t) > 0$ over $[0,T]$. Using the expression of (4.19) for $J_E(r,u,w,x)$ in the state-feedback case, the index of performance turns to be:

$$J_O(r,u,w,n,x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||w - \gamma^{-2}B_1^T(Qx+\theta)||^2 dt\}$$

$$+E\{\int_0^T ||[u+\hat{R}^{-1}\bar{S}^T x+\hat{R}^{-1}(B_2^T\theta+D_{12}^T D_{13}r)]||^2_{\hat{R}}dt\}+\bar{J}(r)-\gamma^2 E\{\int_0^T ||n(t)||^2 dt\}, \tag{4.36}$$

where $\bar{J}(r)$ is defined in (4.14) and where we take $G \equiv 0$ in both $\hat{R}$ and $\bar{S}$ of (4.8). We also note that in the full preview case $[\theta(t)]_+ = \theta(t)$.

We define:

$$\begin{aligned} \bar{w}(t) &= w(t) - w^*(t), \\ \bar{u}(t) &= u(t) + \hat{R}^{-1}[D_{12}^T D_{13}r + B_2^T\theta], \end{aligned} \tag{4.37}$$

where $w^*(t)$ is given in (4.9). We obtain:

$$J_O(r,u,w,n,x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2 E\{\int_0^T ||\bar{w}||^2 dt\}$$

$$+E\{\int_0^T ||\hat{R}^{1/2}[\bar{u} + \hat{C}_1 x]||^2 dt\} + \bar{J}(r) - \gamma^2 E\{\int_0^T ||n(t)||^2 dt\},$$

and

$$\hat{C}_1 = \hat{R}^{-1}[B_2^T Q + D_{12}^T C_1], \tag{4.38}$$

where P_0 is defined in (4.15).

We seek a controller of the form

$$\bar{u}(t) = -\hat{C}_1(t)\hat{x}(t).$$

We, therefore, re-formulate the state and measurement equations of (4.5) and we obtain:

$$\begin{aligned} dx &= [\bar{\bar{A}}(t)x(t) + B_1(t)\bar{w}(t) + B_2(t)\bar{u}(t) + \bar{r}(t)]dt + F(t)x(t)d\beta(t), \\ dy &= (C_2 + \gamma^{-2} D_{21} B_1^T Q)xdt + Hxd\zeta + ndt + \gamma^{-2} D_{21} B_1^T \theta, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} \bar{\bar{A}}(t) &= A + \gamma^{-2} B_1 B_1^T Q, \\ \bar{r}(t) &= [B_3 - B_2 \hat{R}^{-1} D_{12}^T D_{13}] r - [B_2 \hat{R}^{-1} B_2^T - \gamma^{-2} B_1 B_1^T]\theta. \end{aligned} \tag{4.40}$$

We consider the following Luenberger-type state observer:

$$\begin{aligned} d\hat{x}(t) &= \bar{\bar{A}}\hat{x}(t)dt + L[d\bar{y} - \hat{C}_2\hat{x}(t)dt] + g(t)dt, \quad \hat{x}(0) = 0, \\ \hat{z}(t) &= \hat{C}_1 \hat{x}(t), \\ \hat{C}_2 &= C_2 + \gamma^{-2} D_{21} B_1^T Q, \end{aligned} \tag{4.41}$$

where

$$\bar{y} = y - \gamma^{-2} D_{21} B_1^T \theta, \quad \text{and} \quad g(t) = B_2 \bar{u}(t) + \bar{r}(t). \tag{4.42}$$

We note that

$$d\bar{y} = \hat{C}_2 x(t)dt + Hx(t)d\zeta(t) + D_{21}\bar{w}dt + n(t)dt.$$

Denoting $e(t) = x(t) - \hat{x}(t)$ and using the latter we obtain:

$$de(t) = [\bar{\bar{A}} - L\hat{C}_2]e(t)dt + \hat{B}\hat{w}(t)dt + [Fd\beta(t) - LHd\zeta(t)]x(t),$$

where we define

$$\hat{w}(t) \stackrel{\triangle}{=} [\bar{w}^T(t) \quad n^T(t)]^T, \quad \text{and} \quad \hat{B} = [\bar{B}_1 - LD_{21} \quad -L].$$

Defining $\xi(t) = [x^T(t) \quad e^T(t)]^T$ and $\tilde{r}(t) = [r^T(t) \quad \theta^T(t)]^T$, we obtain:

$$\begin{aligned} d\xi(t) &= [\tilde{A}dt + \tilde{F}d\beta(t) + \tilde{H}d\zeta(t)]\xi(t) + \tilde{B}_1\hat{w}(t)dt + \tilde{B}_3\tilde{r}(t)dt, \\ \xi^T(0) &= [x^T(0) \;\; x^T(0)]^T, \\ \tilde{z}(t) &= \tilde{C}_1\xi(t), \end{aligned} \tag{4.43}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{\bar{A}} - B_2\hat{C}_1 & B_2\hat{C}_1 \\ 0 & \bar{\bar{A}} - L\hat{C}_2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ B_1 - LD_{21} & -L \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix},$$

$$\tilde{H} = \begin{bmatrix} 0 & 0 \\ -LH & 0 \end{bmatrix}, \quad \tilde{C}_1^T = \begin{bmatrix} 0 \\ \hat{C}_1^T \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} \tilde{B}_{3,11} & \tilde{B}_{3,12} \\ 0 & 0 \end{bmatrix}, \tag{4.44}$$

where $\tilde{B}_{3,11} \triangleq B_3 - B_2 \hat{R}^{-1} D_{12}^T D_{13}$ and $\tilde{B}_{3,12} \triangleq B_2 \hat{R}^{-1} B_2^T - \gamma^{-2} B_1 B_1^T$. Applying the results of Theorem 4.1 and Remark 4.1 to the system (4.43) with the matrices of (4.44a-e), we obtain the following Riccati-type equation:

$$-\dot{\hat{P}} = \hat{P}\tilde{A} + \tilde{A}^T \hat{P} + \gamma^{-2} \hat{P}\tilde{B}_1 \tilde{B}_1^T \hat{P} + \tilde{C}_1^T \tilde{C}_1 + \tilde{F}^T \hat{P}\tilde{F} + \tilde{H}^T \hat{P}\tilde{H},$$

where

$$\hat{P}(0) = \begin{bmatrix} \gamma^2 R^{-1} - Q(0) - \epsilon I + 0.5\gamma^2 \rho I & -0.5\gamma^2 \rho I \\ -0.5\gamma^2 \rho I & 0.5\gamma^2 \rho I \end{bmatrix}, \ \epsilon > 0, \ \rho >> 1. \quad (4.45)$$

The initial condition of (4.45) is derived from the fact that the initial condition of (4.43) corresponds to the case where a large weight of say, $\rho >> 1$, is imposed on $\hat{x}(0)$ to force nature to select $e(0) = x(0)$ (i.e $\hat{x}(0) = 0$) [44], [57]. In the case where the augmented state-vector is chosen as $\xi(t) = [x^T(t) \quad \hat{x}^T(t)]^T$, the initial condition of $\hat{P}_0$ of (4.45) would satisfy, following (4.32),

$$\hat{P}(0) = \begin{bmatrix} \gamma^2 R^{-1} - Q(0) & 0 \\ 0 & \gamma^2 \rho I \end{bmatrix} + \begin{bmatrix} -\epsilon I & 0 \\ 0 & -0.5\gamma^2 \rho I \end{bmatrix},$$

where $\gamma^2 R^{-1} - Q(0)$ is the initial weight and where the factor of 0.5 in $-0.5\gamma^2 \rho I$ is arbitrarily chosen such the $(2,2)$ block of $\hat{P}(0)$ is positive definite. The above $\hat{P}(0)$ can be readily transformed to account for the augmented state-vector of $\xi(t) = col\{x(t), \ e(t)\}$ by the pre- and post- multiplication of the above matrices, with Υ^T and Υ, respectively, where $\Upsilon \triangleq \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$, the result of which is the initial condition of (4.45).

The solution of the (4.45) involves the simultaneous solution of both $\hat{P}(t)$ and the filter gain L and can not be readily obtained due to mixed terms of the latter variables in (4.45). Considering, however, the monotonicity of $\hat{P}$ with respect to a free semi-positive definite term in (4.45) [59], the solution to the above Riccati-type equation can be obtained by solving the following Differential Linear Matrix Inequality (DLMI):

$$\Gamma(P) \triangleq \begin{bmatrix} \dot{\hat{P}} + \tilde{A}^T \hat{P} + \hat{P}\tilde{A} + \tilde{F}^T \hat{P}\tilde{F} & \hat{P}\tilde{B}_1 & \tilde{C}_1^T & \tilde{H}^T \hat{P} \\ * & -\gamma^2 I_{p+z} & 0 & 0 \\ * & * & -I_q & 0 \\ * & * & * & -\hat{P} \end{bmatrix} \leq 0, \quad \hat{P} > 0, \quad (4.46)$$

with $\hat{P}(0)$ of (4.45) and where we require that $Tr\{\hat{P}(\tau)\}$ be minimized at each time instant $\tau \in [0, \ T]$.

Recently, novel methods for solving both difference (for the discrete-time case) and DLMIs have been introduced in [99] and [44], respectively. Applying the method of [99], the above DLMI can be solved by discretizing the time interval $[0,\ T]$ into equally spaced time instances resulting in the following discretisized DLMI [99] :

$$\begin{bmatrix} \hat{P}_{k+1}-\hat{P}_k+\tilde{\varepsilon}(\tilde{A}_k^T\hat{P}_k+\hat{P}_k\tilde{A}_k)+\tilde{\varepsilon}\tilde{F}_k^T\hat{P}_k\tilde{F}_k & \hat{P}_k\tilde{B}_{1,k} & \tilde{C}_{1,k}^T & \tilde{H}_k^T\hat{P}_k \\ * & -\gamma^2\tilde{\varepsilon}^{-1}I_{p+z} & 0 & 0 \\ * & * & -\tilde{\varepsilon}^{-1}I_q & 0 \\ * & * & * & -\tilde{\varepsilon}^{-1}\hat{P}_k \end{bmatrix} \leq 0, \tag{4.47}$$

where $k = 0, 1, .., N-1$ and where $\tilde{A}_k = \tilde{A}(t_k),\ \tilde{B}_{1,k} = \tilde{B}_1(t_k),\ \tilde{C}_{1,k} = \tilde{C}_1(t_k),$ $\tilde{H}_k = \tilde{H}(t_k),$ and $\tilde{F}_k = \tilde{F}(t_k)$ with $\{t_i,\ i = 0, ..N-1,\ t_N = T,\ t_0 = 0\}$ and

$$t_{i+1} - t_i \stackrel{\triangle}{=} \tilde{\varepsilon} = N^{-1}T,\ i = 0, ...N-1. \tag{4.48}$$

The discretized estimation problem thus becomes one of finding, at each $k \in [0,\ N-1]$, $\hat{P}_{k+1} > 0$ of minimum trace that satisfies (4.47).

The latter DLMI is initiated with the initial condition of (4.45) at the instance $k = 0$ and a solution for both, the filter gain L_k and $\hat{P}_{k+1}$ (i.e $\hat{P}_1$ and L_0) is sought for, under the minimum trace requirement of $\hat{P}_{k+1}$. The latter procedure repeats itself by a forward iteration up to $k = N-1$, where N is chosen (and therefore $1/\tilde{\epsilon}$) to be large enough to allow for a smooth solution (see also [99]).

We summarize the above results, for the full preview case, by the following theorem:

Theorem 4.4. *Consider the system of (4.5) and J_O of (4.6). Given $\gamma > 0$ and $\tilde{\epsilon} > 0$, the output-feedback tracking control problem, where $r(t)$ is known a priori for all $t \leq T$ (the full preview case), possesses a solution iff there exist $Q(t) > 0,\ \forall t \in [0, T]$ that solves (4.7) such that $Q(0) < \gamma^2 R^{-1}$, and $\hat{P}(t)$ that solves (4.45) $\forall t \in [0, T]$ starting from the initial condition of (4.45), where R is defined in (4.2). If a solution to (4.7) and (4.45) exist we obtain the following control law:*

$$u_{of}(t) = -\hat{C}_1(t)\hat{x}(t) \tag{4.49}$$

where $\hat{x}(t)$ is obtained from (4.41).

In the case where $r(t)$ is measured on line, or with preview $h > 0$, we note that $w(t)$ which is not restricted by causality constraints, will be identical to the one in the case of the full preview. This stems from the fact that in (4.19) the optimal strategy of u^* leads to (4.20) where $[\theta]_-$ is not affected by w. We obtain the following:

Corollary 4.4. *H_∞ Output-feedback tracking with fixed-finite preview of $r(t)$: In this case:*

$$u_{of}(t) = -\hat{C}_1[\hat{x}]_+,$$

where

$$d[\hat{x}]_+ = [\bar{\bar{A}} + L\hat{C}_2]\hat{x}dt + Ld[\bar{y}]_+ + [g(t)]_+dt,$$
$$[g(t)]_+ = B_2\bar{u} + [B_3 - B_2\hat{R}^{-1}D_{12}^T D_{13}]r - [B_2\hat{R}^{-1}B_2^T - \gamma^{-2}B_1B_1^T][\theta]_+,$$
$$d[\bar{y}]_+ = dy - \gamma^{-1}D_{21}B_1^T[\theta]_+.$$

Corollary 4.5. H_∞ *Output-feedback tracking with no preview of* $r(t)$*: In this case* $[\theta(t)]_+ = 0$ *and*

$$u_{of}(t) = -\hat{C}_1[\hat{x}]_+,$$

where

$$d[\hat{x}]_+ = [\bar{\bar{A}} + L\hat{C}_2]\hat{x}dt + Ly + [g(t)]_+dt,$$
$$[g(t)]_+ = B_2\bar{u} + [B_3 - B_2\hat{R}^{-1}D_{12}^T D_{13}]r.$$

4.5 Example

We consider the system of (4.1) with the following objective function:

$$J = lim_{T\to\infty}E\{\int_0^T (||Cx - r||^2 + 0.01||u||^2 - \gamma^2||w||^2)d\tau\}$$

where there is an access to the states of the system, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \text{ and } \ C = \begin{bmatrix} -0.5 & 0.4 \end{bmatrix}.$$

and where $G = 0$. The case of $h = 0$ can be solved using the stochastic solution of [56] where r_k is considered as a disturbance. The disturbance vector w_k is augmented to include a finite-energy adversary r_k namely, $\tilde{w}_k \stackrel{\triangle}{=} col\{w_k,\ r_k\}$. Using the notation of the standard problem [57], we define

$$B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \ D_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } D_{12} = \begin{bmatrix} 0 \\ .1 \end{bmatrix}.$$

We obtain a minimum value of $\gamma = 2.07$ for the latter solution. Using the results of Theorem 4.2, we obtain $\gamma_{min} = 1.06$. We compared the two solutions for $\gamma = 2.1$ and we obtained for the standard solution of [56], the control law $u(t) = K_x x(t)$ where $K_x = [-500.65 \quad -53.01]$. For our solution, using Corollary 4.2, the resulting control law is: $u(t) = [-16.10 \quad -14.62]x(t), K_r = 0$. In Figure 4.1 the average tracking error $(Cx(t) - r(t))$, with respect to the statistics of the multiplicative noise, is depicted as a function of time for $r = sin(t)$. The improvement achieved by our new method is clearly visible.

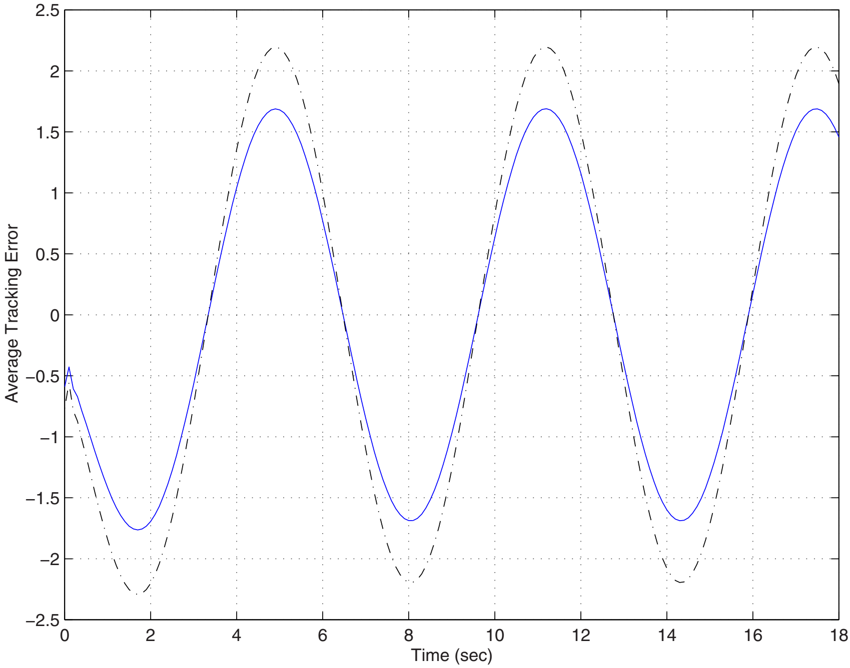

Fig. 4.1. Comparison between the tracking errors obtained in the standard solution (dashed lines) and by the new method (solid lines) for $r = sin(t)$, measured on- line.

4.6 Conclusions

In this chapter we solve the problem of tracking signals with preview in the presence of white-noise stochastic parameter uncertainties in the system state-space model. Applying the game theory approach, a saddle-point tracking strategy is obtained, for the state-feedback case, which is based on the perfect measurement of the system state and the previewed reference signal. The performance index that corresponds to the tracking game includes expectation over the statistics of the stochastic uncertainties in the system state-space model. The game value depends on the reference signal and is usually positive.

The state-feedback problem was first solved for the finite-horizon time varying case, where a nonzero correlation between the unknown parameters in the input and the dynamic matrices was allowed. The latter result has been extended to the time-invariant case with an infinite horizon. The problem of determining whether there exists a solution to the problem that guarantees a prechosen attenuation level became one of solving a single LMI, which can be obtained for the general case (i.e regardless of the possible orthogonality of $D_{12}^T C_1$). The general case requires, however, a simple re-formulation of the state equations which is readily implementable.

Similarly to the solution of the output-feedback control problem of Chapter 2, extension of the results of the state-feedback solution to the case where there is no access to the system states causes an error in the estimate of the system state, that depends not only on the past values of the error but also on the value of the state.

The solution of output-feedback tracking problem is, carried along the lines of the standard solution for both, the deterministic case [57] and the stochastic case [43], where we first apply the state-feedback solution to arrive to an estimation problem. The latter problem is solved by applying an auxiliary BRL which is solved as a max-min problem.

5

Continuous-time Systems: Static Output-feedback

5.1 Introduction

The static output-feedback problem has attracted the attention of many in the past [5], [111] and [100]. The main advantage of the static output-feedback is the simplicity of its implementation and the ability it provides for designing controllers of prescribed structure such as PI and PID. An algorithm has been presented recently by [62], which under some assumptions, is found to converge in stationary infinite horizon examples without uncertainty. A sufficient condition for the existence of a solution to a special case of the static output-feedback problem has been obtained in [20]. This condition is, in some cases, quite conservative.

A necessary and sufficient condition for the existence of a solution to the problem without uncertainty in terms of a matrix inequality readily follows from the standard Bounded Real Lemma [57]. It is, however, bilinear in the decision variable matrices and consequently standard convex optimization procedures could not be used in the past to solve the problem, even in the case where the system parameters were all known. Various methods have been proposed to deal with this difficulty [92] [72].

In the present chapter, we extend the work of [100] that considered the H_2 and the H_∞ static control problem in the absence of stochastic uncertainties, to the case where there exist stochastic white noise parameter uncertainties in the matrices of the state-space model that describes the system [51]. We apply a simple design method for deriving the static output-feedback gain that satisfies prescribed H_2 and H_∞performance criteria. Since a constant gain cannot be achieved in practice and all amplifiers have some finite bandwidth, we add, in series to the measured output of the system, a simple low-pass component with a very high bandwidth. A parameter dependent Lyapunov function is then assigned to the augmented system which is obtained by incorporating the states of the additional low-pass component into the state space description. This function does not require a specific structure for the matrix that corresponds to the states of the original system and a sufficient condi-

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 75–86, 2005.

tion is then obtained adopting a recent LPD (Linear Parameter Dependent) stabilization method that has been introduced in [91].

Static output-feedback is applied in many areas of control engineering including process and flight control. In the latter, designing flight control systems, engineers prefer the simple and physically sound controllers that are recommended as recipe structures [81], [8]. Only gains are included in these simple structures and the closed-loop poles are thus obtained by migration of the open-loop poles that have a clear physical meaning. In Chapter 11 we apply the theory of the present chapter to an altitude control example where the effect of the height on the Radar signal to noise ratio is modelled as a state-multiplicative noise.

5.2 Problem Formulation

We consider the following linear system

$$\begin{aligned} dx &= [Ax(t) + B_1(t)w(t) + B_2u(t)]dt + Dx(t)d\beta(t) + Gu(t)d\zeta(t),\ x(0) = x_0 \\ dy &= (C_2x + D_{21}w)dt + Fxd\nu \end{aligned} \tag{5.1}$$

with the objective vector

$$z(t) = C_1x(t) + D_{12}u(t) \tag{5.2}$$

where $x \in \mathcal{R}^n$ is the system state vector, $w \in \mathcal{R}^q$ is the exogenous disturbance signal, $u \in \mathcal{R}^\ell$ is the control input, $y \in \mathcal{R}^m$ is the measured output and $z \in \mathcal{R}^r \subset \mathcal{R}^n$ is the state combination (objective function signal) to be regulated and where the variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy:

$$E\{d\beta(t)\}=0,\ E\{d\zeta(t)\}=0,\ E\{d\beta(t)^2\}=dt,\ E\{d\zeta(t)^2\}=dt,\ E\{d\beta(t)d\zeta(t)\}=0,$$

$$E\{d\nu(t)\}=0,\ E\{d\nu(t)^2\}=dt,\ E\{d\nu(t)d\zeta(t)\}=0,\ E\{d\nu(t)d\beta(t)\}=\sigma dt,\ |\sigma|< 1.$$

Note that the dynamic and measurement Wiener-type processes are correlated. The latter correlation was also considered in the case of filtering of the above system in Chapter 3. The matrices in (5.1), (5.2) are constant matrices of appropriate dimensions.

We seek a constant controller of the following form:

$$u = Ky \tag{5.3}$$

that achieves a certain performance requirement. We treat the following two different performance criteria.

- **The stochastic H_2 control problem :** Assuming that w is a realization of a unit intensity white noise process and that either $D_{21} = 0$ or $D_{12} = 0$, the following performance index should be minimized:

$$J_2 \stackrel{\Delta}{=} \underset{w}{E}\{||\tilde{z}(t)||^2_{\tilde{L}_2}\}. \tag{5.4}$$

- **The stochastic H_∞control problem:** Assuming that the exogenous disturbance signal is energy bounded, a static control gain is sought which, for a prescribed scalar γ and for all nonzero $w \in \tilde{L}^2([0,\infty);\mathcal{R}^q)$, guarantees that $J_\infty < 0$ where

$$J_\infty \stackrel{\Delta}{=} ||z(t)||^2_{\tilde{L}_2} - \gamma^2||w(t)||^2_{\tilde{L}_2}. \tag{5.5}$$

Instead of considering the purely constant controller (5.3) we consider the following strictly proper controller

$$d\eta = -\rho\eta dt + \rho dy, \quad \eta(0) = 0, \quad u = K\eta \tag{5.6}$$

where $\eta \in \mathcal{R}^m$ and $1 << \rho$ is a scalar much larger than the open-loop and the desired closed-loop bandwidths. The latter controller is introduced in order to facilitate the convexity of the design method below. It represents, however, the actual situation where 'constant' gains are achieved in practice by amplifiers of finite bandwidths.

Augmenting the system (5.1) to include the states of (5.6) we define the augmented state vector $\xi = col\{x,\ \eta\}$ and obtain the following representation to the closed-loop.

$$\begin{aligned} d\xi &= [\tilde{A}\xi + \tilde{B}w]dt + \tilde{D}\xi d\beta + \tilde{G}\xi d\zeta + \tilde{F}\xi d\nu, \qquad \xi(0) = col\{x_0,\ 0\} \\ z &= \tilde{C}\xi, \end{aligned} \tag{5.7}$$

where:

$$\tilde{A} \stackrel{\Delta}{=} \begin{bmatrix} A & B_2K \\ \rho C_2 & -\rho I_m \end{bmatrix}, \ \tilde{D} \stackrel{\Delta}{=} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{G} \stackrel{\Delta}{=} \begin{bmatrix} 0 & G \\ 0 & 0 \end{bmatrix}, \ \tilde{F} \stackrel{\Delta}{=} \begin{bmatrix} 0 & 0 \\ \rho F & 0 \end{bmatrix},$$

$$\tilde{B} \stackrel{\Delta}{=} \begin{bmatrix} B_1 \\ \rho D_{21} \end{bmatrix} \quad \text{and} \quad \tilde{C} \stackrel{\Delta}{=} \begin{bmatrix} C_1 & D_{12}K \end{bmatrix}. \tag{5.8}$$

5.2.1 The Stochastic H_2 Control Problem

Applying to (5.7) the derivation of the stochastic H_2 control results [105], we obtain that $J_2 < \delta^2$ for a prescribed scalar δ if there exists a positive definite solution $\tilde{Q}$ of the structure (5.10) and $H \in \mathcal{R}^{q\times q}$ to the following LMIs that are derived in the proof for Theorem 5.1 below.

$$\begin{bmatrix} \tilde{Q}\tilde{A}^T + \tilde{A}\tilde{Q} & \tilde{Q}\tilde{C}^T & \tilde{Q}(\tilde{D}+\sigma\tilde{F})^T & \tilde{Q}\tilde{G}^T & \bar{\sigma}\tilde{Q}\tilde{F}^T \\ * & -I_r & 0 & 0 & 0 \\ * & * & -\tilde{Q} & 0 & 0 \\ * & * & * & -\tilde{Q} & 0 \\ * & * & * & * & -\tilde{Q} \end{bmatrix} < 0$$

$$\begin{bmatrix} H & \tilde{B}^T \\ \tilde{B} & \tilde{Q} \end{bmatrix} > 0, \quad \text{and} \quad Tr\{H\} < \delta^2. \tag{5.9}$$

where we consider the following structure for $\tilde{Q}$

$$\tilde{Q} = \begin{bmatrix} Q & C_2^T\hat{Q} \\ \hat{Q}C_2 & \alpha\hat{Q} \end{bmatrix}, \tag{5.10}$$

where $\bar{\sigma} = (1-\sigma^2)^{0.5}$ and where the parameter α is a positive scalar tuning parameter. We obtain the following result.

Theorem 5.1. *Consider the system of (5.1). The output-feedback control law (5.6) achieves a prescribed H_2-norm bound $0 < \delta$, for some $1 << \rho$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$, $Y \in \mathcal{R}^{\ell\times m}$ and $H \in \mathcal{R}^{q\times q}$ that, for some tuning scalar $0 < \alpha$, satisfy the following LMIs:*

$$\begin{bmatrix} \tilde{\Gamma}(1,1) & \tilde{\Gamma}(1,2) & \tilde{\Gamma}(1,3) & QD^T & \sigma\rho QF^T & C_2^T\hat{Q}G^T & 0 & 0 & QF^T\rho\bar{\sigma} \\ * & \tilde{\Gamma}(2,2) & \tilde{\Gamma}(2,3) & \hat{Q}C_2D^T & \sigma\hat{Q}C_2F^T & \alpha\hat{Q}G^T & 0 & 0 & \hat{Q}C_2F^T\rho\bar{\sigma} \\ * & * & -\gamma^2 I_r & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\alpha\hat{Q} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & -\alpha\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & * & -Q & -C_2^T\hat{Q} \\ * & * & * & * & * & * & * & * & -\alpha\hat{Q} \end{bmatrix} < 0,$$

$$\begin{bmatrix} H & B_1^T & \rho D_{21}^T \\ * & Q & C_2^T\hat{Q} \\ * & * & \alpha\hat{Q} \end{bmatrix} > 0, \quad \textit{and} \quad Tr\{H\} < \delta^2, \tag{5.11}$$

where

$$\tilde{\Gamma}(1,1) = AQ + QA^T + C_2^TY^TB_2^T + B_2YC_2,$$
$$\tilde{\Gamma}(1,2) = \alpha B_2Y + \rho QC_2^T + AC_2^T\hat{Q} - \rho C_2^T\hat{Q},$$
$$\tilde{\Gamma}(1,3) = QC_1^T + C_2^TY^TD_{12}^T,$$
$$\tilde{\Gamma}(2,2) = -2\rho\alpha\hat{Q} + \rho(\hat{Q}C_2C_2^T + C_2C_2^T\hat{Q}),$$
$$\tilde{\Gamma}(2,3) = \alpha Y^TD_{12}^T + \hat{Q}C_2C_1^T.$$

If a solution to the latter LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y\hat{Q}^{-1}. \tag{5.12}$$

Proof: Applying the result of Chapter 3 with a straightforward adaptation for the H_2 case, we obtain the following Riccati-type inequality:

$$\tilde{A}^T P + P\tilde{A} + \tilde{C}^T\tilde{C} + \tilde{D}^T P\tilde{D} + \tilde{F}^T P\tilde{F} + \sigma[\tilde{F}^T P\tilde{D} + \tilde{D}^T P\tilde{F}] + \tilde{G}^T P\tilde{G} < 0$$

$$P > 0. \tag{5.13}$$

Using the definition of $\bar{\sigma} = (1-\sigma^2)^{0.5}$, the inequality (5.13) is equivalent to the following

$$\tilde{A}^T P + P\tilde{A} + \tilde{C}^T\tilde{C} + (\tilde{D} + \sigma\tilde{F})^T P(\tilde{D} + \sigma\tilde{F}) + \bar{\sigma}^2\tilde{F}^T P\tilde{F} + \tilde{G}^T P\tilde{G} < 0, \tag{5.14}$$

since

$$\tilde{D}^T P\tilde{D} + \tilde{F}^T P\tilde{F} + \sigma\tilde{F}^T P\tilde{D} + \sigma\tilde{D}^T P\tilde{F} = (\tilde{D} + \sigma\tilde{F})^T P(\tilde{D} + \sigma\tilde{F}) + \bar{\sigma}^2\tilde{F}^T P\tilde{F}.$$

Multiplying (5.14) by P^{-1}, from the left and the right, we obtain, denoting $\tilde{Q} = P^{-1}$, the following inequality:

$$\tilde{Q}\tilde{A}^T + \tilde{A}\tilde{Q} + \tilde{Q}C^T\tilde{C}\tilde{Q} + \tilde{Q}(\tilde{D} + \sigma\tilde{F})^T\tilde{Q}^{-1}(\tilde{D} + \sigma\tilde{F})\tilde{Q} + \bar{\sigma}^2\tilde{Q}\tilde{F}^T\tilde{Q}^{-1}\tilde{F}\tilde{Q} + \tilde{Q}\tilde{G}^T\tilde{Q}^{-1}\tilde{G}\tilde{Q} < 0. \tag{5.15}$$

Applying Schur's complements, (5.15) can be readily rearranged into the LMI of (5.9). Substituting for $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ into (5.9) and assuming that either D_{12} or D_{21} is zero we obtain the LMIs of (5.11) where we define $Y = K\hat{Q}$.

□

Remark 5.1. In many practical cases amplifiers that produce constant gain have finite bandwidths and thus the result of Theorem 5.1 for $1 << \rho$, is directly applicable. It also stands to reason that since for $1 << \rho$ the controller (5.6) has the transference K, at the significant frequency range, the corresponding frequency-independent feedback controller of (5.3) will also achieve the prescribed H_2-norm. Note, that a good choice for the scalar ρ is one which is two order of magnitudes larger than the bandwidth of the system. It is also noted that a similar practice is adopted in gain-scheduling in the case where an uncertainty appears in the input or the output matrices, in the state space model of the system.

Remark 5.2. The structure of $\tilde{Q}$ of (5.10) contains in the (2,2) matrix block the term $\alpha\hat{Q}$. Originally, one could have chosen this block matrix to be an arbitrary positive square matrix. However, in order to be able to solve feedback problem, this matrix should be proportional to $\hat{Q}$. One could have chosen $\alpha = 1$ but then one puts an unnecessary restriction on this second diagonal block. Note also the over-design entailed in the structure of (5.10), an over-design that can not be avoided when solving this problem.

Remark 5.3. In the above, it was assumed that either D_{12} or D_{21} is zero. This requirement stems from the fact that if both these matrices are nonzero, then, by applying the control law (5.3), the disturbance w will have a direct access to the objective vector z, at the limit where ρ tends to infinity, which leads to an unbounded H_2-norm.

5.2.2 The Stochastic H_∞ Problem

The LMIs of Theorem 5.1 provide a sufficient condition for the existence of a static output-feedback gain that achieves a prescribed H_2-norm for the system (5.7). A similar result can be obtained if the H_∞-norm of the latter system is considered. Given a prescribed desired bound $0 < \gamma$ on the H_∞-norm of the system, the inequalities in (5.9) are replaced by the following Bounded Real Lemma (BRL) condition [59].

$$\begin{bmatrix} \tilde{A}^T\tilde{P}+\tilde{P}\tilde{A} & \tilde{P}\tilde{B} & \tilde{C}^T & (\tilde{D}+\sigma\tilde{F})^T\tilde{P} & \tilde{G}^T\tilde{P} & \bar{\sigma}\tilde{F}^T\tilde{P} \\ * & -I_q & 0 & 0 & 0 & 0 \\ * & * & -\gamma^2 I_r & 0 & 0 & 0 \\ * & * & * & -\tilde{P} & 0 & 0 \\ * & * & * & * & -\tilde{P} & 0 \\ * & * & * & * & * & -\tilde{P} \end{bmatrix} < 0, \quad \text{and} \quad \tilde{P} > 0. \tag{5.16}$$

Denoting $\tilde{Q} = \tilde{P}^{-1}$ and $Y = K\hat{Q}$, where it is assumed that $\tilde{Q}$ has the structure of (5.10) and multiplying (5.16) by $diag\{\tilde{Q},\ I_q,\ I_r,\ \tilde{Q},\ \tilde{Q},\ \tilde{Q}\}$, from both sides, we substitute for $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ into the latter LMI and obtain the following.

Theorem 5.2. *Consider the system of (5.1). The control law (5.6) achieves a prescribed H_∞-norm bound $0 < \gamma$, for some $1 << \rho$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalar $0 < \alpha$, satisfy the following LMIs:*

$$\begin{bmatrix} \tilde{\Gamma}(1,1) & \tilde{\Gamma}(1,2) & B_1 & \tilde{\Gamma}(1,4) & QD^T & \rho\sigma QF^T & C_2^T\hat{Q}G^T & 0 & 0 & QF^T\rho\bar{\sigma} \\ * & \tilde{\Gamma}(2,2) & \rho D_{21} & \tilde{\Gamma}(2,4) & \hat{Q}C_2D^T & \tilde{\Gamma}(2,6) & \alpha\hat{Q}G^T & 0 & 0 & \tilde{\Gamma}(2,10) \\ * & * & -I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I_r & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\alpha\hat{Q} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & * & -\alpha\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & * & * & -Q & -C_2^T\hat{Q} \\ * & * & * & * & * & * & * & * & * & -\alpha\hat{Q} \end{bmatrix} < 0$$

where

$$\begin{aligned}
\tilde{\Gamma}(1,4) &= QC_1^T + C_2^T Y^T D_{12}^T,\\
\tilde{\Gamma}(2,4) &= \alpha Y^T D_{12}^T + \hat{Q}C_2 C_1^T,\\
\tilde{\Gamma}(2,6) &= \rho\sigma\hat{Q}C_2 F^T,\\
\tilde{\Gamma}(2,10) &= \hat{Q}C_2 F^T \rho\bar{\sigma},
\end{aligned} \tag{5.17}$$

and

$$\begin{bmatrix} Q & C_2^T\hat{Q} \\ * & \alpha\hat{Q} \end{bmatrix} > 0.$$

If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by (5.12).

Proof: The proof follows the lines adopted in the proof of Theorem 5.1.

5.3 The Robust Stochastic H_2 Static Output-feedback controller

The system considered in Section 5.2 assumes that all the parameters of the system are known, including the matrices D, G and F. In the present section we consider the system (5.1) whose matrices are not exactly known. Denoting

$$\Omega = \begin{bmatrix} A & B_1 & B_2 & C_1 & D_{12} & D_{21} & D \end{bmatrix},$$

where either D_{12} or D_{21} is zero, we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1, ...N\}$, namely,

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \le f_j \le 1, \ \sum_{j=1}^{N} f_j = 1 \tag{5.18}$$

where the vertices of the polytope are described by

$$\Omega_j = \begin{bmatrix} A^{(j)} & B_1^{(j)} & B_2^{(j)} & C_1^{(j)} & D_{12}^{(j)} & D_{21}^{(j)} & D^{(j)} \end{bmatrix}, \quad j = 1, 2..., N.$$

For every point in Ω, say the one that is obtained by $\sum_{j=1}^{N} f_j \Omega_j$ for some $0 \le f_j \le 1$, $\sum_{j=1}^{N} f_j = 1$ we assign the following linear parameter varying Lyapunov function

$$V_L = \xi^T (\sum_{j=1}^{N} f_j \tilde{P}_j)\xi, \quad \tilde{P}_j \in \mathcal{R}^{(n+m)\times(n+m)} > 0 \tag{5.19}$$

For each vertex of Ω where, for simplicity we consider only the stochastic state-multiplicative noise of $\beta(t)$, the inequalities of (5.9) can be written as

$$\begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} \bar{A}^{(j)T} + \bar{A}^{(j)} \begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} < 0, \quad \bar{A}^{(j)} \triangleq \begin{bmatrix} \tilde{A}^{(j)} & 0 & 0 \\ \tilde{C}^{(j)} & -\frac{1}{2} I_r & 0 \\ \tilde{D}^{(j)} & 0 & -\frac{1}{2} I_{n+m} \end{bmatrix}$$

and

$$\begin{bmatrix} Z & \tilde{B}^{(j)T} \\ \tilde{B}^{(j)} & \tilde{Q}_j \end{bmatrix} > 0 \tag{5.20}$$

where $\tilde{Q}_j = \tilde{P}_j^{-1}$, $j = 1, 2, ..., N$.

The latter inequalities include products of Q_j by $A^{(j)}$ and $C^{(j)}$ and they cannot therefore be solved by standard convex optimization techniques. These inequalities are 'convexified' by the following.

Lemma 5.1. *The inequalities (5.20) in $\tilde{Q}_j$ and Z are satisfied if the following LMIs in in $\tilde{T}$, $\tilde{H} \in \mathcal{R}^{(2n+2m+r)\times(2n+2m+r)}$ and $0 < \tilde{Q}_j \in \mathcal{R}^{(n+m)\times(n+m)}$ possesses a solution.*

$$\begin{bmatrix} \tilde{T}^T \bar{A}^{(j)T} + \bar{A}^{(j)} \tilde{T} & -\begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} + \tilde{T}^T - \bar{A}^{(j)} \tilde{H} \\ * & -\tilde{H} - \tilde{H}^T \end{bmatrix} < 0$$

and

$$\begin{bmatrix} Z & \tilde{B}^{(j)T} \\ * & \tilde{Q}_j \end{bmatrix} > 0. \tag{5.21}$$

Proof : The equivalence between (5.21) and (5.20) is proved using the method of [91]. If (5.20) holds for a specific j then, choosing:

$$\tilde{T} = diag\{\tilde{Q}_j, \ I_r, \tilde{Q}_j\}$$

and taking $\tilde{H} = \beta I$ where $\beta > 0$ is arbitrarily small, (5.21) is satisfied. The latter implies that if there exists a quadratic stabilizing solution to (5.20) (namely a solution $\tilde{Q}$ is obtained independently of the vertices) then (5.21) will also have a solution for all the vertices. On the other hand, if (5.21) possesses a solution, multiplying the latter inequality from the left and the right by $\tilde{\Gamma}_j$ and $\tilde{\Gamma}_j^T$, respectively, where

$$\tilde{\Gamma}_j = \left[I_{2n+2m+r} \ -\bar{A}^{(j)} \right],$$

(5.20) is obtained.

□

In order to obtain from (5.21) LMIs in $\tilde{T}$, $\tilde{H}$, $\tilde{Q}_j$, $j = 1, 2, ..., N$, Z and K, for a given $0 < \delta$, $\tilde{G}$ and $\tilde{H}$ are sought that, for some positive tuning scalars β_i, $i = 1, 2, 3$ possess the following structure:

$$\tilde{T} = \begin{bmatrix} \begin{bmatrix} T_1 \\ T_2\,[C_2\ \beta_1 I_m] \end{bmatrix} & 0 \\ 0 & I_{r+m+n} \end{bmatrix},\ \tilde{H} = \begin{bmatrix} \begin{bmatrix} H_1 \\ T_2\,[\beta_2 C_2\ \beta_3 I_m] \end{bmatrix} & 0 \\ 0 & I_{r+m+n} \end{bmatrix}, \tag{5.22}$$

where $T_1,\ H_1 \in \mathcal{R}^{n\times(n+m)}$ and $T_2 \in \mathcal{R}^{m\times m}$ is a nonsingular matrix.

Substituting the latter and (5.8) in (5.21) and denoting $Y = KT_2$, the following result is obtained.

Theorem 5.3. *Consider the uncertain system of (5.1) where either D_{12} or D_{21} is zero. The control law (5.6) guarantees, for some $1 << \rho$, a prescribed H_2-norm bound $0 < \delta$ over the uncertainty polytope $\bar{\Omega}$ if there exist $T_1,\ H_1 \in \mathcal{R}^{n\times(n+m)}$, $T_2 \in \mathcal{R}^{m\times m}$, $Q_j \in \mathcal{R}^{(n+m)\times(n+m)}$ $j = 1, 2..., N$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalars $0 < \alpha,\ \beta_i,\ i = 1, 2, 3$, satisfy the following LMIs:*

$$\Gamma = \begin{bmatrix} \Gamma(1,1) & \Gamma(1,2) \\ * & -\tilde{H} - \tilde{H}^T \end{bmatrix} < 0,$$

$$\begin{bmatrix} Z & \begin{bmatrix} B_1^{(j)T} & \rho D_{21}^{(j)T} \end{bmatrix} \\ * & Q_j \end{bmatrix} > 0 \quad and \quad Tr\{Z\} < \delta^2, \quad j = 1, 2, ..., N \tag{5.23}$$

where

$$\Gamma(1,1) = \tilde{T}^T \hat{A}^{(j)T} + \hat{A}^{(j)} \tilde{T} + \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix} Y \begin{bmatrix} C_2^T \\ \beta_1 I_m \\ 0 \end{bmatrix}^T + \begin{bmatrix} C_2^T \\ \beta_1 I_m \\ 0 \end{bmatrix} Y^T \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix}^T,$$

$$\Gamma(1,2) = -\begin{bmatrix} Q_j 0 & 0 \\ 0\ \ I_r & 0 \\ 0\ \ \ 0 & Q_j \end{bmatrix} + \tilde{T}^T - \hat{A}^{(j)} \tilde{H} - \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix} Y \begin{bmatrix} \beta_2 C_2^T \\ \beta_3 I_m \\ 0 \end{bmatrix}^T,$$

and

$$\hat{A}^{(j)} \triangleq \begin{bmatrix} A^{(j)} & 0 & 0 \\ \rho C_2 & -\rho I_m & 0 \\ C_1^{(j)} & 0 & -\frac{1}{2} I_r \end{bmatrix}, \tag{5.24}$$

and where $\tilde{T}$ and $\tilde{H}$ posses the structure of (5.22).

If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y T_2^{-1}. \tag{5.25}$$

Remark 5.4. The solution provided by Theorem 5.3, being only sufficient, is inherently conservative. One could have chosen in $\tilde{T}$ and $\tilde{H}$ of (5.22), matrices of arbitrary structure. This however, would not allow a solution to be found by

LMIs. We thus constrain the structure of these matrices to be of the form given in (5.22) where we still use the degrees of freedom provided by β_1, β_2 and β_3 to tune the solution and to obtain a tighter upper-bound on the H_2-norm of the closed-loop system.

Remark 5.5. We note that $\tilde{H}$ in (5.22) must be invertible, being a diagonal element of the matrix of Γ of (5.23), once the feasibility of the latter matrix is obtained. Since the matrix $\tilde{H}$ of (5.22) has a lower block triangle structure, its diagonal block elements must be invertible and therefore $\beta_3\, T_2$ is nonsingular. Thus, the regularity of T_2 is ensured once the LMIs of (5.23) are found feasible, justifying therefore the controller gain of (5.25).

5.4 The Robust H_∞Control

For each point in Ω, say the one that is obtained by $\sum_{j=1}^N f_j \Omega_j$ for some $0 \le f_j \le 1$, $\sum_{j=1}^N f_j = 1$ we assign the parameter varying Lyapunov function of (5.19). For each vertex of Ω, say the j-th, the inequality of (5.16) can be written as:

$$\begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} \bar{A}^{(j)T} + \bar{A}^{(j)} \begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} + \gamma^{-2} \begin{bmatrix} \tilde{B}^{(j)} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{B}^{(j)T} & 0 & 0 \end{bmatrix} < 0 \quad (5.26)$$

where $0 < \tilde{Q}_j = \gamma^{-2} \tilde{P}_j^{-1}$. The latter inequality also stems directly from the Lyapunov function in (5.19), following the standard derivation of the BRL [57].

Inequality (5.26) has the form of a Lyapunov inequality and as such it is similar to (5.20). Following the same lines used to prove that the existence of a solution to (5.21) is a sufficient condition for (5.20) to hold, the following is obtained.

Lemma 5.2. *For a prescribed scalar γ the inequality (5.26) is satisfied for the j-th vertex of Ω by K and $0 < \tilde{Q}_j$ if the following LMI in $\tilde{T}$, $\tilde{H} \in \mathcal{R}^{(2n+2m+r)\times(2n+2m+r)}$ and $0 < \tilde{Q}_j \in \mathcal{R}^{(n+m)\times(n+m)}$ possesses a solution.*

$$\begin{bmatrix} \tilde{T}^T \bar{A}^{(j)T} + \bar{A}^{(j)} \tilde{T} & \begin{bmatrix} \tilde{B}^{(j)} \\ 0 \\ 0 \end{bmatrix} & -\begin{bmatrix} \tilde{Q}_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & \tilde{Q}_j \end{bmatrix} + \tilde{T}^T - \bar{A}^{(j)} \tilde{H} \\ * & -\gamma^2 I_q & 0 \\ * & * & -\tilde{H} - \tilde{H}^T \end{bmatrix} < 0 \qquad (5.27)$$

In order to obtain from the latter set of inequalities for $j = 1, ..., N$ for a given γ, $\tilde{T}$ and $\tilde{H}$ are sought that, for some positive tuning scalars β_i, $i = 1, 2, 3$ possess the structure of (5.22).

Substituting the latter and (5.8) in (5.27) and denoting $Y = KT_2$ the following result is obtained.

Theorem 5.4. *Consider the uncertain system of (5.1). The control law (5.6) guarantees for* $1 << \rho$ *a prescribed disturbance attenuation level* $0 < \gamma$ *over the uncertainty polytope* Ω *if there exist* $T_1,\ H_1 \in \mathcal{R}^{n\times(n+m)},\ T_2 \in \mathcal{R}^{m\times m}$, $Q_j \in \mathcal{R}^{(n+m)\times(n+m)},\ j = 1,2...,N$ *and* $Y \in \mathcal{R}^{\ell\times m}$ *that, for some scalars* $0 < \alpha,\ \beta_i,\ i = 1,2,3,$ *satisfy the following LMIs:*

$$\tilde{\Gamma} = \begin{bmatrix} \tilde{\Gamma}(1,1) & \begin{bmatrix} B_1^{(j)} \\ \rho D_{21}^{(j)} \\ 0 \\ 0 \end{bmatrix} & \tilde{\Gamma}(1,3) \\ * & -\gamma^2 I_q & 0 \\ * & * & -\tilde{H} - \tilde{H}^T \end{bmatrix} < 0$$

$$Q_j > 0, \quad j = 1,2,...,N. \tag{5.28}$$

where

$$\tilde{\Gamma}(1,2) = \tilde{T}^T \bar{A}^{(j)} + \bar{A}^{(j)T}\tilde{T} + \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix} Y \begin{bmatrix} C_2^T \\ \beta_1 I_m \\ 0 \end{bmatrix}^T + \begin{bmatrix} C_2^T \\ \beta_1 I_m \\ 0 \end{bmatrix} Y^T \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix}^T,$$

$$\tilde{\Gamma}(1,3) = -\begin{bmatrix} Q_j & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & Q_j \end{bmatrix} + \tilde{T}^T - \bar{A}^{(j)}\tilde{H} - \begin{bmatrix} B_2^{(j)} \\ 0 \\ D_{12}^{(j)} \\ 0 \end{bmatrix} Y \begin{bmatrix} \beta_2 C_2^T \\ \beta_3 I_m \\ 0 \end{bmatrix}^T,$$

where $\bar{A}^{(j)}$ *is defined in (5.20) and where* $\tilde{T}$ *and* $\tilde{H}$ *posses the structure of (5.22).*

If a solution to the latter set of LMIs exists, the gain matrix K *that stabilizes the system and achieves the required performance is given by*

$$K = Y T_2^{-1}.$$

5.5 Conclusions

A convex optimization method is presented which provides an efficient design of robust static output-feedback controllers. Linear systems with polytopic type uncertainties are considered and a sufficient condition is derived, based on a linear parameter varying Lyapunov function, for the existence of an almost constant output-feedback controller that stabilizes the system and achieves a prescribed bound on its performance over the entire uncertainty polytope.

Both H_2 and H_∞ performance criteria have been considered. For both criteria, conditions for quadratic stabilizing solution have been obtained. The conservatism entailed in these conditions is reduced either by using a recent method that enables the use of parameter varying Lyapunov based optimization, or by treating the vertices of the uncertainty polytope as distinct plants. The latter solution cannot guarantee the stability and the performance within the polytope whereas the former optimization method achieves the required bound over the entire polytope.

The proposed method can be used also in the mixed H_2/H_∞ case where a robust static output-feedback controller is sought that achieves, say, a prescribed attenuation level while minimizing a H_2-norm of the closed-loop. The standard way of achieving the mixed objective has been to apply the same single Lyapunov function for both the H_2 and the H_∞ criteria and for all the points in the uncertainty polytope. This practice is quite conservative and can be significantly reduced by using the LPD design of Lemma 5.1 and Section 5.4.

We note that in the Application chapter (Chapter 11) we bring an altitude control example which illustrates the design of static feedback controller in the presence of multiplicative noise. In this example it is shown how the multiplicative noise, resulting from the dependence of the noise level in the RADAR altitude measurement on the altitude itself, can be explicitly taken into account in the design procedure.

6

Stochastic Passivity

6.1 Introduction

Passivity of deterministic linear systems plays an important role in network theory and in adaptive control. The fact that a system is passive entails some stability properties that allow for robust control design of this system. It is the purpose of the present chapter to investigate the corresponding situation in linear systems with multiplicative noise. The concept of stochastic passivity is first considered and a positive-real like lemma for finite dimensional linear time invariant uncertain systems with state multiplicative noise is derived. The system uncertainties are assumed to be of the polytopic type.

These passivity conditions are applied to a certain class of direct adaptive controllers referred to as Simplified Adaptive Control (SAC) [64]. Adaptive controllers provide a possible alternative to fixed compensators when large parameter uncertainty is encountered in the model that describes the system. Often, the conditions for closed-loop stability when using adaptive controllers include a strict passivity requirement of the controlled plant. For example, when using SAC method [64], the passivity of the plant guarantees the robust stability of the closed-loop. The SAC applies a tracking error gain which is simply adapted by using proportional and integral versions of the squared tracking error. In fact, the relaxed condition of almost passivity, requiring the plant to be stabilizable and passive via static output-feedback gain, suffices in many cases. A similar situation is encountered when controlling uncertain plants with a class of neural network controllers (NNC) (see [76] and [77]). Also there, the plant is required to be almost passive to ensure closed loop stability. In some applications the system uncertainties can be fixed but unknown; other cases may involve stochastic uncertainties. The latter cases of stochastic uncertainties are described by state-multiplicative noise. In this case, the system matrices are corrupted by white noise while their deterministic components lie in a convex polytope.

In the present chapter, the concept of passivity is, therefore, generalized to the latter case of stochastic uncertainties. The new stochastic passivity

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condition is expressed in a form of LMIs to allow efficient solutions. It is shown that the stochastic passivity conditions ensure closed-loop stochastic stability when applying a class of SAC. A numerical example is given to demonstrate the use of the new theory in the control of a target tracking system with polytopic type uncertainties.

6.2 Problem Formulation

Consider the following time-invariant linear system with state-dependent noise:

$$\begin{aligned} dx_t &= (Ax_t + Bw_t)dt + Dx_t d\beta_t + Gw_t d\sigma_t, \\ dy_t &= (Cx_t + D_{21}w_t)dt \end{aligned} \tag{6.1}$$

defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the σ-algebra generated by the Wiener process:

$$W_t = col\{\beta_t,\ \sigma_t\} \tag{6.2}$$

where x_t is the $\mathcal{R}^n$- valued solution to (6.1); $x(0) = 0$, y_t is the $\mathcal{R}^q$-valued observation and $w \in \tilde{L}^2([0,\ \infty); \mathcal{R}^q)$. The stochastic processes β_t and σ_t are zero-mean scalar standard Wiener processes defined on the probability space $(\Omega, \mathcal{F}, P)$.

It is desired to verify whether

$$J = E\{\int_0^\infty y_\tau^T w_\tau d\tau\} \geq 0, \quad \forall\ w_t \in \tilde{L}^2([0,\ \infty)) \text{ over } \Omega, \tag{6.3}$$

for the case of polytopic uncertainties where the matrices $A,\ B,\ C,\ D_{21},\ D$ and G are unknown constant matrices that lie in the following uncertainty polytope:

$$\bar{\Omega} \stackrel{\Delta}{=} \sum_{i=1}^{N} \tau_i(A_i,\ B_i,\ C_i,\ D_{21,i},\ D_i,\ G_i);\ \tau_i \geq 0,\ \sum_{i=1}^{N} \tau_i = 1. \tag{6.4}$$

We denote by L the infinitesimal generator of the stochastic differential equation (6.1)(see Appendix A). Choosing the Lyapunov function and the supply rate to be, respectively,

$$V(x_t) = x_t^T P x_t, \quad S(x_t, w_t) = 2y_t^T w_t \tag{6.5}$$

where P is a positive-definite constant matrix in$\mathcal{R}^{n\times n}$, we find the following result:

Lemma 6.1.
i) *The system (6.1) is globally asymptotically stable in probability if for $w_t \equiv 0$ and for all $x \in \mathcal{R}^n$ the following holds over the polytope $\bar{\Omega}$.*

$$LV(x) < 0, \quad \forall x \in \mathcal{R}^n. \tag{6.6}$$

ii) *If the system (6.1) is stable in probability over Ω, then a sufficient condition for (6.3) to hold is:*

$$LV(x) \le S(x, w) \quad \forall x \in \mathcal{R}^n, \quad w \in \tilde{L}_2([0,\ \infty); \mathcal{R}^q) \quad over\ \Omega. \tag{6.7}$$

Proof: Part i is well known (see, e.g. [58] or [97]). To prove part ii we first realize that

$$dx_t = f(x, t)dt + g(x, t)d\gamma_t \tag{6.8}$$

where γ_t is a standard Wiener process and where $f(x, t) = Ax_t + Bw_t$, $g(x, t) = \begin{bmatrix} Dx_t \ Gw_t \end{bmatrix}$ and $d\gamma_t = col\{d\beta_t, d\sigma_t\}$. We then consider:

$$LV(x_t) = f^T \frac{\partial^T V(x_t)}{\partial x} + \frac{1}{2} Tr\{gg^T \frac{\partial^2}{\partial^2 x} V(x_t)\} =$$

$$V_x(x_t)(Ax_t + Bw_t) + \frac{1}{2}[x_t^T D^T V_{xx}(x_t) Dx_t + w_t^T G^T V_{xx}(x_t) Gw_t].$$

By Ito formula

$$V(x_t) = V(x_0) + \int_0^t LV(x_s)ds + \int_0^t V_x(x_s)Dx_s d\beta_s + \int_0^t V_x(x_s)Gw_s d\sigma_s \tag{6.9}$$

and since $x(0) = 0$ we find that

$$E\{\int_0^t LV(x_s)ds\} = E\{V(x_t)\} \ge 0, \quad \forall t \ge 0.$$

If (6.7) is satisfied, the results of (6.3) readily follows.

□

For the specific choice of (6.5) we obtain:

$$LV(x_t) = x_t^T P(Ax_t + Bw_t) + (Ax_t + Bw_t)^T Px_t + x_t^T D^T PDx_t + w_t^T G^T PGw_t. \tag{6.10}$$

The condition of (6.7) is thus

$$\begin{bmatrix} x_t^T \ w_t^T \end{bmatrix} \left[\begin{bmatrix} -PA - A^T P - D^T PD & -PB \\ -B^T P & -G^T PG \end{bmatrix} + \begin{bmatrix} 0 & C^T \\ C & D_{21} + D_{21}^T \end{bmatrix} \right] \begin{bmatrix} x_t \\ w_t \end{bmatrix} \ge 0 \tag{6.11}$$

The existence of $0 < P \in \mathcal{R}^{n \times n}$ that satisfies:

$$\begin{bmatrix} PA + A^T P + D^T PD & PB - C^T \\ B^T P - C & -D_{21} - D_{21}^T + G^T PG \end{bmatrix} < 0 \tag{6.12}$$

over Ω would thus ensure that (6.7) is satisfied. Since the requirement of (6.6) is also satisfied by the first block on the diagonal in (6.12), stability is also ensured. □

The latter inequality is not affine in the system parameters. Affinity is obtained by applying Schur's complements formula. The following is then achieved.

Theorem 6.1. *The system (6.1) is stable (in probability) over Ω and (6.3) is satisfied, over the uncertainty polytope $\bar{\Omega}$, if there exists $0 < P \in \mathcal{R}^{n\times n}$ that satisfies the following LMIs.*

$$\begin{bmatrix} PA_i + A_i^T P & PB_i - C_i^T & D_i^T P & 0 \\ * & -D_{21,i} - D_{21,i}^T & 0 & G_i^T P \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0, \quad i = 1, ..., N. \tag{6.13}$$

The latter result may turn out to be conservative since it applies the same decision variable P to all the vertices of $\bar{\Omega}$. Realizing that (6.13) can be written as

$$\bar{P}\bar{A}_i + \bar{A}_i^T \bar{P} < 0, \quad i = 1, ..., N$$

where

$$\bar{P} = diag\{P,\ I_q,\ P,\ P\} \quad \text{and} \quad \bar{A}_i = \begin{bmatrix} A_i & B_i & 0 & 0 \\ -C_i & -D_{21,i} & 0 & 0 \\ D_i & 0 & -\frac{1}{2}I_n & 0 \\ 0 & G_i & 0 & -\frac{1}{2}I_n \end{bmatrix}. \tag{6.14}$$

Applying then the result of [91], and defining $\bar{n} = 3n + q$, the following is obtained.

Corollary 6.1. *The system (6.1) is stable (in probability) over Ω and (6.3) is satisfied over the uncertainty polytope $\bar{\Omega}$, if there exist $0 < P_i \in \mathcal{R}^{n\times n}$, $\bar{G}$ and H in $\mathcal{R}^{\bar{n}\times\bar{n}}$ that satisfy the following LMIs.*

$$\begin{bmatrix} \bar{G}^T \bar{A}_i + \bar{A}_i^T \bar{G} & \bar{G}^T - diag\{P_i,\ I,\ P_i,\ P_i\} - \bar{A}_i^T H \\ * & -H - H^T \end{bmatrix} < 0, \quad i = 1, ..., N. \tag{6.15}$$

The above produced conditions for the stability in probability of the system. We next inquire what are the conditions for the exponential mean square stability of the system. The following result is standard:

Lemma 6.2. *([58])Assume there exists a positive function $V(x,t)$, which is twice differentiable in x and once in t, with $V(0,t) = 0$. Then, the system (6.1) is globally exponentially stable if for $w = 0$ there are positive numbers k_1, k_2, k_3 such that the following hold.*

$$\begin{aligned} k_1||x||^2 \le V(x,t) \le k_2||x||^2 \\ LV(x,t) \le -k_3||x||^2, \quad \forall t \ge 0 \quad \forall x \in \mathcal{R}^n \end{aligned} \tag{6.16}$$

In our case, as $V(x) = x^T Px$, the first condition is satisfied as long as $P > 0$. To satisfy the second condition we require $LV(x) \leq -\epsilon||x||^2$ for some $\epsilon > 0$ over Ω. Obviously, in terms of LMI, this sufficient condition would be: The system (6.1) (with $w = 0$) is exponentially stable in the mean square sense if there exist $P > 0$ and $0 < \epsilon$ such that

$$PA + A^T P + D^T PD + \epsilon I < 0, \quad \text{over} \ \ \Omega. \tag{6.17}$$

It is clear then that (6.17) is satisfied for small enough ϵ if there exists a solution $0 < P \in \mathcal{R}^{n\times n}$ to (6.12).

6.3 Application to Simplified Adaptive Control

Consider the following system

$$dx_t = (Ax_t + Bu_t)dt + Dx_t d\beta, \quad dy_t = Cx_t dt \tag{6.18}$$

where the matrices $A,\ B,\ C,\ D$ are again unknown constant matrices that lie in the following uncertainty polytope:

$$\hat{\Omega} \stackrel{\Delta}{=} \sum_{i=1}^{N} \tau_i (A_i,\ B_i,\ C_i,\ D_i);\ \tau_i \geq 0,\ \sum_{i=1}^{N} \tau_i = 1. \tag{6.19}$$

This system should be regulated using a direct adaptive controller [64] of the type:

$$u_t = -Ky_t \tag{6.20}$$

where

$$\dot{K} = y_t y_t^T. \tag{6.21}$$

In the context of deterministic systems, such a controller has been known to stabilize the plant and result in a finite gain matrix K if the plant is Almost Passive (AP). In our stochastic context we may conjecture that stochastic stability of this direct adaptive controller (which usually referred to as SAC) will be guaranteed by the stochastic version of the AP property. Namely, the existence of a constant output feedback matrix K_e which, using the control signal

$$u = u' - K_e x, \tag{6.22}$$

makes the transference relating u' and y stochastically passive. To see that this conjecture is true, we first substitute (6.20) and (6.21) in (6.1) and get the following closed-loop system:

$$dx_t = (Ax_t - BKCx_t)dt + Dx_t d\beta \tag{6.23}$$

Defining

$$\bar{A} = A - BK_e C \quad \text{and} \quad \bar{K} = K - K_e$$

the closed-loop system equations are given by :

$$dx_t = (\bar{A}x_t - B\bar{K}Cx_t)dt + Dx_t d\beta \tag{6.24}$$

and

$$\dot{\bar{K}} = Cxx^T C^T \tag{6.25}$$

In the sequel, we choose for simplicity to deal with SISO systems and define the augmented state vector

$$\bar{x} = \begin{bmatrix} x \\ \bar{K} \end{bmatrix} \triangleq \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

and choose the following Lyapunov function candidate (see [64]) :

$$V(\bar{x}_t) = \bar{x}_t^T \bar{P} \bar{x}_t$$

where

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}.$$

We note that $\bar{x}$ satisfies (6.8) with $\bar{x}$ replacing x and where:

$$f(\bar{x}) = \begin{bmatrix} \bar{A}\bar{x}_1 - B\bar{x}_2 C\bar{x}_1 \\ \bar{x}_1^T C^T C\bar{x}_1 \end{bmatrix}, \quad g(\bar{x}) = \begin{bmatrix} D\bar{x}_1 \\ 0 \end{bmatrix} \quad \text{and} \quad d\gamma_t = d\beta_t. \tag{6.26}$$

By Lemma 6.1, the closed-loop system (6.24) will be stable in probability if $LV < 0$. However,

$$LV(\bar{x}_t) = f^T \frac{\partial^T V(\bar{x}_t)}{\partial \bar{x}} + \frac{1}{2} Tr\{gg^T \frac{\partial^2 V(\bar{x}_t)}{\partial^2 \bar{x}}\}. \tag{6.27}$$

Substituting (6.26) into the last equation, we readily find that:

$$LV(\bar{x}_t) = \bar{x}_1{}^T[\bar{A}^T P + P\bar{A}]\bar{x}_1 - \zeta^T[B^T P - C]\bar{x}_1 - \bar{x}_1^T[PB - C^T]\zeta + \bar{x}_1{}^T D^T P D\bar{x}_1$$

where the last term is obtained from the second term in (6.27) and where $\zeta \triangleq \bar{x}_2 C\bar{x}_1$. Therefore, a sufficient condition for $LV < 0$ is

$$\bar{A}^T P + P\bar{A} + D^T P D < 0, \quad \text{where} \quad PB = C^T \tag{6.28}$$

A LMI version of (6.28) for the polytope $\hat{\Omega}$ is then the following.

$$\begin{bmatrix} P\bar{A}_i + \bar{A}_i^T P & PB_i - C_i^T & D_i^T P \\ B_i^T P - C_i & -\epsilon I & 0 \\ PD_i & 0 & -P \end{bmatrix} < 0, \quad i = 1, ..., N \tag{6.29}$$

for small enough $\epsilon > 0$, where $\bar{A}_i = A_i - B_i K_e C_i$. Namely, (6.18) can be stabilized and made stochastically passive using (6.22).

6.4 Numerical Example

In this section we bring a simple numerical example. Consider the system of (6.1) and (6.4) with two vertices differing only in A_i so that $A_i = -\alpha_i I_2, B = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, C = I_2, D_{21} = 10^{-3} I_2$ and $D = \frac{1}{2} I_2$ where $\alpha_1 = 1$ and $\alpha_2 = 10$. Solving (6.29) results in the following feasible positive definite solution $P = \begin{bmatrix} 0.9916 & -0.0485 \\ -0.0485 & 1.0066 \end{bmatrix}$ indicating that our system is stochastically passive throughout the convex polytope.

6.5 Conclusions

The concept of passivity has been generalized for a class of stochastic systems of practical significance. The passivity conditions in the form of LMIs can be efficiently solved using state of the art LMI solvers (e.g. [35]). It was shown that a relaxed version of the stochastic passivity conditions, namely the almost stochastic passivity, guarantees closed-loop stochastic stability. In Chapter 11 (the Application chapter) we bring a numerical example which is taken from the field of target tracking and which demonstrates the application of the theory.

7

Discrete-time Systems: Control and Luenberger-type Filtering

7.1 Introduction

In this chapter we bring the discrete-time counterpart of Chapter 2 where we solve the state-feedback control and filtering problems with the aid of the discrete-time stochastic BRL, which is fully derived in the chapter. We then, solve the finite-horizon and the stationary output-feedback control problems.

When dealing with linear time-varying systems with parameter uncertainties, both the stochastic state-multiplicative and the deterministic approaches are applied. In the deterministic H_∞ setting, the state-feedback control and the estimation problems have been treated by [102],[109], respectively. These problems have been solved by considering norm-bounded uncertainties that appear in the matrices of the state-space description. The main drawback of the method of [102],[109] is the significant over-design that is entailed in transforming the uncertainties into fictitious disturbances (in the discrete-time case this over-design is most accentuated when the system poles are close to the unit circle).

Using convex optimization over linear matrix inequalities, the stochastic state-multiplicative state-feedback H_∞ problem has been solved for the discrete-time setting in [12]. These solutions are not based on completing to squares, for both the exogenous and the control signals, and the expressions obtained there for the performance index can not, therefore, be readily extended to systems whose states are not directly accessible.

In [19], the discrete-time state-feedback H_∞-control problem has been solved for infinite-dimensional, stochastic, state-multiplicative systems using a probabilistic framework; it resulted in an algebraic Riccati-like operator equation. A necessary and sufficient condition for the existence of a state-feedback controller has been obtained there. The latter can not, however, be easily applied to the output-feedback case. A finite-dimensional approach to

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the problem has been offered in [24] where a bounded real lemma type result has been derived for the continuous-time case. Using the latter, the state-feedback problem has been solved in [85], [25].

Also recently, the corresponding discrete-time problem for output-feedback with stochastic uncertainties at the measured output has been considered in [26]. The solution there refers to both the finite and infinite time horizon problems without transients. One drawback of [26] is the fact that in the infinite-time horizon case, an infinite number sets of LMI have to be solved. Moreover, the fact that the measured output has no state-dependent uncertainties is practically a constraint. Specifically, in this chapter, we solve three problems that are defined in Section 7.2. We first solve in Sections 7.3 and 7.4 the time-varying BRL and the state-feedback problem for the discrete-time case, respectively, where we also allow for a nonzero correlation between the different stochastic variables of the system. The latter solution is obtained using the stochastic BRL which we introduce in order to derive both the necessary and sufficient conditions for the existence of a solution. We also address in Section 7.4.2 the convergence properties of our results in the stationary case.

Similarly to the state-feedback case, we solve the filtering problem, in Section 7.5, with possible uncertainty in the measurement matrix by applying the discrete-time BRL. In Section 7.7, we solve the output-feedback control problem by applying the state-feedback control solution and by transforming the resulting system to fit the filtering model. On the latter we apply the filtering solution of Section 7.5. The theory developed is demonstrated by three examples: a state-feedback control, filtering and measurement feedback examples. The first solution is compared to the solution obtained by applying the deterministic technique of [102], [109]. It is shown that in this problem our stochastic approach yields lower worst-case performance bound.

7.2 Problem Formulation

We treat the following three problems [43], [39]:

i) Stochastic state-feedback H_∞- control
Given the following linear discrete time-varying system:

$$x_{k+1} = (A_k + D_k v_k)x_k + (B_{2,k} + G_k r_k)u_k + B_{1,k}w_k, \qquad k = 0, 1, ..., N-1 \tag{7.1}$$

where x_k is the state vector, $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ is an exogenous disturbance, $\{u_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^l)$ is the control input signal and x_0 is an unknown initial state and where $\{v_k\}$ and $\{r_k\}$ are standard independent white noise scalar sequences with zero mean that satisfy:

$$E\{v_k v_j\} = \delta_{kj}, \quad E\{r_k r_j\} = \delta_{kj}, \quad E\{r_k v_j\} = \alpha_k \delta_{kj}, \quad |\alpha_k| < 1 \tag{7.2}$$

Denoting

$$z_k = L_k x_k + D_{12,k} u_k,$$

where $z_k \in \mathcal{R}^m$ and assuming, for simplicity, that:

$$[L_k^T \quad D_{12,k}^T] D_{12,k} = [0 \quad \tilde{R}_k], \quad \tilde{R}_k > 0 \tag{7.3}$$

we consider, for a given scalar $\gamma > 0$, the following performance index:

$$\begin{aligned} &J_1 \stackrel{\triangle}{=} \underset{r,v}{E}\{||z_k||_2^2 - \gamma^2 ||w_k||_2^2\} + \underset{r,v}{E}\{x_N^T \bar{Q}_N x_N\} - \gamma^2 x_0^T \bar{Q}_0 x_0, \\ &\bar{Q}_N \geq 0, \ \bar{Q}_0 > 0. \end{aligned} \tag{7.4}$$

The objective is to find a control law $\{u_k\}$ such that, J_1 is negative for all nonzero $(\{w_k\}, x_0)$ where $x_0 \in \mathcal{R}^n$ and $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and where w_k depends on $\mathcal{Y}_k \stackrel{\triangle}{=} \{x_0, ...x_k\}$ since it is $\mathcal{F}_{k-1}$ measurable.

ii) Stochastic H_∞-filtering

We consider the following system:

$$\begin{aligned} x_{k+1} &= (A_k + D_k v_k) x_k + B_{1,k} w_k, x_0 = x_0 \\ y_k &= (C_k + F_k \zeta_k) x_k + n_k, \\ z_k &= L_k x_k \end{aligned} \tag{7.5}$$

where $y_k \in \mathcal{R}^z$ is the measured output, $\{n_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^z)$ is a measurement noise, and where $\{v_k\}$ and $\{\zeta_k\}$ are standard independent white noise scalar sequences with zero mean that satisfy:

$$E\{v_k v_j\} = \delta_{kj}, \quad E\{\zeta_k \zeta_j\} = \delta_{kj}, \quad E\{\zeta_k v_j\} = \alpha_k \delta_{kj}, \quad |\alpha_k| < 1$$

We are looking for $L_k \hat{x}_k$, the filtered estimate of z_k, where:

$$\hat{x}_{k+1} = A_k \hat{x}_k + K_{o,k}(y_k - C_k \hat{x}_k) \quad , \hat{x}_0 = 0. \tag{7.6}$$

Note that similarly to the continuous-time counterpart problem, the initial value $\hat{x}_0 = 0$ of the above Luenberger-type estimator is chosen.

We define

$$e_k = x_k - \hat{x}_k, \quad \tilde{w}_k = col\{w_k, \ n_k\} \tag{7.7}$$

and we consider the following cost function:

$$J_2 \stackrel{\triangle}{=} \underset{v,\zeta}{E}\left\{||L_{k+1} e_{k+1}||_2^2 - \gamma^2 ||\tilde{w}_k||_2^2\right\} - \gamma^2 x_0^T P_0 x_0, \quad P_0 > 0. \tag{7.8}$$

Given the scalar $\gamma > 0$ and $P_0 > 0$, we look, in the filtering problem, for an estimate $L_k \hat{x}_k$ of $L_k x_k$ over the finite time horizon $[1, \quad N]$ such that J_2 of (7.8) is negative for all nonzero $(\{\tilde{w}_k\}, x_0)$ where $\{\tilde{w}_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^{p+z})$ and $x_0 \in \mathcal{R}^n$.

iii) Stochastic H_∞ output-feedback - The finite-horizon case
Given the following system:

$$\begin{aligned} x_{k+1} &= (A_k + D_k v_k)x_k + B_{1,k} w_k + (B_{2,k} + G_k r_k)u_k \\ y_k &= (C_k + F_k \zeta_k)x_k + n_k, \\ z_k &= L_k x_k + D_{12,k} u_k \end{aligned} \tag{7.9}$$

where v_k, r_k, w_k and n_k are defined in problems (i) and (ii) and ζ_k is similar to the former stochastic variables and satisfies the following:

$$E\{\zeta_k \zeta_j\} = \delta_{kj},\ E\{\zeta_k r_j\} = \sigma_k \delta_{kj},\ E\{\zeta_k v_j\} = \beta_k \delta_{kj},$$
$$E\{r_k v_j\} = \alpha_k \delta_{kj}, \quad |\alpha_k| < 1, |\beta_k| < 1, |\sigma_k| < 1$$

We seek an output-feedback controller that achieves, for a given scalar $\gamma > 0$,

$$\begin{aligned} &J_3 \stackrel{\triangle}{=} \underset{v,r,\zeta}{E} \left\{||z_k||_2^2 - \gamma^2 ||\tilde{w}_k||_2^2 + x_N^T \bar{Q}_N X_N\right\} - \gamma^2 x_0^T \bar{Q}_0 x_0 < 0, \\ &\bar{Q}_N \geq 0, \quad \bar{Q}_0 \geq 0 \end{aligned} \tag{7.10}$$

for all nonzero $(\{\tilde{w}_k\}, x_0)$ where $\{\tilde{w}_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^{p+z})$ and $x_0 \in \mathcal{R}^n$. Similar to the standard case [57], this problem involves the estimation of an appropriate combination of the states, for the worst case disturbance signal, and the application of the results of the state-feedback control with a proper modification.

iv) Stochastic H_∞ output-feedback - The Infinite-horizon case
Given the following system:

$$\begin{aligned} x_{k+1} &= (A + D v_k)x_k + B_1 w_k + (B_2 + G r_k)u_k, \quad x_0 = 0, \\ y_k &= (C + F \zeta_k)x_k + n_k, \\ z_k &= L x_k + D_{12} u_k \end{aligned} \tag{7.11}$$

where v_k, r_k, w_k and n_k are defined in the above problems and v_k, r_k and ζ_k are similar to the stochastic variables of problem iii).

We seek a stabilizing output-feedback controller that achieves, for a given scalar $\gamma > 0$,

$$J_4 \stackrel{\triangle}{=} \underset{v,r,\zeta}{E} \left\{||z_k||_2^2 - \gamma^2 ||\tilde{w}_k||_2^2 + x_N^T \bar{Q}_N X_N\right\} < 0, \quad \bar{Q}_N \geq 0, \tag{7.12}$$

for all nonzero $\{\tilde{w}_k\}$ where $\{\tilde{w}_k\} \in \tilde{l}^2([0, \infty); \mathcal{R}^{p+z})$. Similarly to the problem iii), this problem involves the stationary estimation of an appropriate combination of the states and the application of the results of the stationary state-feedback with a proper modification.

7.3 The Discrete-time Stochastic BRL

We address the problem of the stochastic state-feedback by deriving first a BRL for the following system:

$$x_{k+1} = (A_k + D_k v_k)x_k + (B_k + F_k r_k)w_k, \qquad k = 0, 1, ..., N-1 \tag{7.13}$$

where x_0 is an unknown initial state and where the scalar sequences $\{v_k\}$ and $\{r_k\}$ are defined in problem (i). The exogenous disturbance w_k is assumed to be of finite energy and may depend on current and past values of the state-vector namely, it is $\mathcal{F}_{k-1}$ measurable. We consider the cost function J_1 of (7.4) where

$$z_k = L_k x_k. \tag{7.14}$$

The objective is to determine, for a given scalar $\gamma > 0$, whether J_1 is negative for all nonzero $(\{w_k\}, x_0)$ where $x_0 \in \mathcal{R}^n$ and $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$.
We consider the following recursion:

$$Q_k = A_k^T Q_{k+1} A_k + (B_k^T Q_{k+1} A_k + \alpha_k F_k^T Q_{k+1} D_k)^T \Theta_k^{-1}$$
$$(B_k^T Q_{k+1} A_k + \alpha_k F_k^T Q_{k+1} D_k) + L_k^T L_k + D_k^T Q_{k+1} D_k, \qquad Q_N = \bar{Q}_N \tag{7.15}$$

where we define

$$\Theta_k \stackrel{\triangle}{=} \gamma^2 I - B_k^T Q_{k+1} B_k - F_k^T Q_{k+1} F_k.$$

The following result is obtained in [43]:

Theorem 7.1. *Consider the system (7.13),(7.14). Given the scalar $\gamma > 0$, a necessary and sufficient condition for J_1 of (7.4) to be negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ is that there exists a solution Q_k to (7.15) that satisfies $\Theta_k > 0, for \quad k = 1, 2, ..N-1$, and $Q_0 < \gamma^2 \bar{Q}_0$.*

Proof.
Sufficiency: We define:

$$\tilde{J}_k = x_{k+1}^T Q_{k+1} x_{k+1} - x_k^T Q_k x_k. \tag{7.16}$$

Substituting (7.13) in (7.16) we obtain:

$$\begin{aligned}\tilde{J}_k &= [x_k^T (A_k + D_k v_k)^T + w_k^T (B_k + F_k r_k)^T] Q_{k+1} [(A_k + D_k v_k)x_k + (B_k + F_k r_k)w_k] \\ &\quad - x_k^T Q_k x_k \\ &= x_k^T [A_k^T Q_{k+1} A_k + v_k D_k^T Q_{k+1} A_k + A_k^T Q_{k+1} D_k v_k + v_k^2 D_k^T Q_{k+1} D_k - Q_k + L_k^T L_k] x_k \\ &\quad + 2x_k^T A_k^T Q_{k+1} B_k w_k + 2x_k^T v_k D_k^T Q_{k+1} B_k w_k + 2r_k x_k^T A_k^T Q_{k+1} F_k w_k\end{aligned}$$

$+2w_k^T F_k^T Q_{k+1} D_k x_k r_k v_k - w_k^T \Theta_k w_k + 2w_k^T r_k F_k Q_{k+1} B_k w_k - z_k^T z_k + \gamma^2 w_k^T w_k$

where we added to $\tilde{J}_k$, the zero term $x_k^T L_k^T L_k x_k - z_k^T z_k + \gamma^2 w_k^T w_k - \gamma^2 w_k^T w_k$. Since w_k does not depend on v_k and r_k, taking the expectation with respect to these signals we obtain that:

$$\underset{v,r}{E}\left\{\tilde{J}_k\right\} = \underset{v,r}{E}\{x_k^T[A_k^T Q_{k+1} A_k + D_k^T Q_{k+1} D_k - Q_k + L_k^T L_k]x_k$$
$$+2x_k^T[D_k^T Q_{k+1} F_k \alpha_k + A_k^T Q_{k+1} B_k]w_k - w_k^T \Theta_k w_k - z_k^T z_k + \gamma^2 w_k^T w_k\} \tag{7.17}$$

Since $\Theta_k > 0$, completing to squares for w_k, we obtain using (7.15), that

$$\underset{v,r}{E}\left\{\tilde{J}_k\right\} = - \underset{v,r}{E}\{[w_k - w_k^*]^T \Theta_k [w_k - w_k^*] + z_k^T z_k - \gamma^2 w_k^T w_k\}$$

where

$$w_k^* \stackrel{\triangle}{=} \Theta_k^{-1}(B_k^T Q_{k+1} A_k + \alpha_k F_k^T Q_{k+1} D_K) x_k. \tag{7.18}$$

Taking the sum of the two sides of the latter, from 0 to $N-1$, we obtain using (7.16)

$$\underset{v,r}{E}\{\Sigma_{k=0}^{N-1} \tilde{J}_k\} = \underset{v,r}{E}\{x_N^T \bar{Q}_N x_N\} - x_0^T Q_0 x_0$$
$$= - \underset{v,r}{E}\{\Sigma_{k=0}^{N-1}[w_k - w_k^*]^T \Theta_k [w_k - w_k^*]\} - \underset{v,r}{E}\{\Sigma_{k=0}^{N-1}(z_k^T z_k - \gamma^2 w_k^T w_k)\}.$$

Hence J_1 of (7.4) is given by:

$$J_1 = - \underset{v,r}{E}\{\Sigma_{k=0}^{N-1}[w_k - w_k^*]^T \Theta_k [w_k - w_k^*]\} + x_0^T(Q_0 - \gamma^2 \bar{Q}_0)x_0. \tag{7.19}$$

Clearly J_1 is negative $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ that are not identically zero iff $Q_0 < \gamma^2 \bar{Q}_0$.

Necessity: We consider the case where J_1 is negative for all $(\{w_k\}, x_0) \neq 0$ and we solve (7.15) for $Q_i, \quad i = N-1, N-2, ...,$ starting with $Q_N = \bar{Q}_N$. Denoting the first instant for which Θ_i is not positive-definite by $i = k^*$, we distinguish between two cases. In the first, $\Theta_{k^*} \geq 0$ and it has a zero eigenvalue, namely there exists $d_{k^*} \neq 0$ that satisfies $\Theta_{k^*} d_{k^*} = 0$. In this case, we consider the following strategy of nature: $x_0 = 0, \quad w_i = 0, \quad i < k^*, \quad w_{k^*} = d_{k^*}$ and $w_i = w_i^*, \quad i > k^*$ where w_i^* is defined in (7.18). It follows from (7.17) that at the k^* instant the quadratic term in w_k is zero and from (7.13) that $x_{k^*} = 0$. Thus, the contribution of the terms in J_1 that corresponds to k^* is zero. It follows from (7.19) that the terms in J_1 that correspond to $i > k^*$ are also zero. Therefore, the above non-zero strategy yields a zero J_1, contradicting the above assumption. The second case is one where Θ_{k^*} is nonsingular but possesses a negative eigenvalue with a corresponding eigenvector d_{k*}. We consider then the following strategy: $w_i = w_i^*, i \neq k^* \quad and \quad w_{k^*} = w_{k^*}^* + d_{k^*}$ The resulting J_1 of (7.19) can be made arbitrarily large (positive) by choosing d_{k^*} to be of large norm.

□

Theorem 7.1 has been derived for the system (7.13) with a single stochastic parameter in A_k and B_k. The results of this theorem are easily extended to the following system:

$$
\begin{aligned}
x_{k+1} &= (A_k + \sum_{i=1}^{m_1} D_{i,k} v_{i,k}) x_k + (B_k + \sum_{i=1}^{m_2} G_{i,k} r_{i,k}) w_k, \\
z_k &= L_k x_k, \; k = 0, 1, ., N-1
\end{aligned}
\tag{7.20}
$$

where x_k, w_k and x_0 are defined in (7.1), and $\{v_{i,k}\}$ and $\{r_{i,k}\}$ are standard random scalar sequences with zero mean that satisfy:

$$
\begin{aligned}
E\{v_{i,k1} v_{j,k2}\} = \sigma_{ij,k} \delta_{k1k2}, \quad E\{r_{i,k1} r_{j,k2}\} = \beta_{ij,k} \delta_{k1k2}, \\
E\{r_{i,k1} v_{j,k2}\} = \alpha_{ij,k} \delta_{k1k2}
\end{aligned}
\tag{7.21}
$$

and where $\alpha_{ij,k}$, $\beta_{ij,k}$ and $\sigma_{ij,k}$ are predetermined scalars of absolute value less than 1. Denoting

$$
\begin{aligned}
\hat{G}_k &\stackrel{\triangle}{=} \sum_{i=1}^{m_2} \sum_{j=1}^{m_2} G_{i,k}^T Q_{k+1} G_{j,k} \beta_{ij,k}, \\
\hat{D}_k &\stackrel{\triangle}{=} \sum_{i=1}^{m_1} \sum_{j=1}^{m_1} D_{i,k}^T Q_{k+1} D_{j,k} \sigma_{ij,k}, \\
\hat{H}_k &\stackrel{\triangle}{=} \sum_{i=1}^{m_2} \sum_{j=1}^{m_1} G_{i,k}^T Q_{k+1} D_{j,k} \alpha_{ij,k},
\end{aligned}
\tag{7.22}
$$

we consider the following recursion:

$$
\begin{aligned}
&Q_k = A_k^T Q_{k+1} A_k + \hat{D}_k + L_k^T L_k + (B_k^T Q_{k+1} A_k + \hat{H}_k)^T \bar{\Theta}_k^{-1} (B_k^T Q_{k+1} A_k + \hat{H}_k), \\
&Q_N = \bar{Q}_N,
\end{aligned}
\tag{7.23}
$$

where

$$
\bar{\Theta}_k \stackrel{\triangle}{=} \gamma^2 I - B_k^T Q_{k+1} B_k - \hat{G}_k.
$$

We obtain the following result:

Corollary 7.1. *Consider the system (7.20) and a prescribed scalar $\gamma > 0$. A necessary and sufficient condition for J_1 of (7.4) to be negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ is that there exists a solution Q_k to (7.23) that satisfies $\bar{\Theta}_k > 0, \forall \quad k = 1, 2, ..N-1$, and $Q_0 < \gamma^2 \bar{Q}_0$.*

Remark 7.1. The above BRL can be used to solve the state-feedback H_∞ control problem by using $u_k = K_k x_k$ and looking for K_k that achieves the performance level of γ that involves both w_k and u_k. Another, more general, way is to derive an alternative expression for J_1 along the lines of the proof of Theorem 7.1. The advantage of the latter approach is its applicability to the case where the states are not all accessible. This approach will be used below in the solution of the output-feedback.

7.3.1 The Discrete-time Stochastic BRL: The Stationary Case

Similarly to the continuous-time counterpart of Section 2.3.1, the derivation of the discrete-time stationary stochastic BRL is obtained by two approaches. In the first approach we consider the following mean square stable system:

$$\begin{aligned} x_{k+1} &= (A + Dv_k)x_k + B_1 w_k, \ x_0 = 0, \\ z_k &= Lx_k \end{aligned} \tag{7.24}$$

which is obtained from (7.13) for the case where the system matrices are constant and $N = \infty$. Considering the following index of performance:

$$J_S \stackrel{\triangle}{=} \underset{v}{E}\{||z_k||_2^2 - \gamma^2 ||w_k||_2^2 + x_N^T \bar{Q}_N x_N\} < 0, \quad \bar{Q}_N \geq 0, \tag{7.25}$$

we obtain the following result:

Theorem 7.2. *Consider the system (7.24). Given the scalar $\gamma > 0$, a necessary and sufficient condition for J_S of (7.25), to be negative for all nonzero $\{w_k\} \in \tilde{l}^2([0, \infty); \mathcal{R}^p)$ is that there exists a solution $\bar{Q} > 0$ to the following algebraic Riccati-type difference equation:*

$$A^T \bar{Q} A + A^T \bar{Q} B_1 [\gamma^2 I - B_1^T \bar{Q} B_1]^{-1} B_1^T \bar{Q} A + L^T L + D^T \bar{Q} D = 0. \tag{7.26}$$

When such $\bar{Q}$ exists, the corresponding optimal strategy of the disturbance $\{w_k\}$ is:

$$w_k^* = \gamma^2 B_1^T \bar{Q} x. \tag{7.27}$$

Proof: The proof outline resembles that of the finite horizon case of Section 7.3. Thus, the optimal strategy w_k^* of (7.27) is obtained by completing to squares for $\{w_k\}$, similarly to Section 7.3. Note that $J_S < 0$ can also be achieved iff the following inequality holds:

$$A^T \bar{Q} A + A^T \bar{Q} B_1 [\gamma^2 I - B_1^T \bar{Q} B_1]^{-1} B_1 \bar{Q} A^T + L^T L + D^T \bar{Q} D \leq 0$$

By using Schur's complements formula, the latter inequality can be readily transformed to the following LMI:

$$\Gamma_S \stackrel{\triangle}{=} \begin{bmatrix} -\bar{Q} & 0 & A^T & D^T & L^T \\ * & -\gamma^2 I_p & B_1^T & 0 & 0 \\ * & * & -\bar{Q}^{-1} & 0 & 0 \\ * & * & * & -\bar{Q}^{-1} & 0 \\ * & * & * & * & -I_m \end{bmatrix} \leq 0. \tag{7.28}$$

The second approach for the solution of the stationary BRL is achieved by considering the finite horizon counterpart of this problem. We consider the system (7.13), (7.14) for the case where the system matrices are constant and $N \to \infty$. Considering the results of [87] we obtain that the Riccati-type difference equation of (7.15) will converge to the algebraic equation of (7.26) where the pair $(A,\ C_1)$ is observable, $(A,\ B_1)$ is stabilizable and $\bar{Q}_N \geq 0$.

□

7.4 Stochastic H_∞ State-feedback Control

We solve the problem of the state-feedback control by referring to (7.1). We apply the BRL of Theorem 7.1 replacing $A_k + D_k v_k$ of (7.13) by $A_k + D_k v_k + (B_{2,k} + G_k r_k)K_k$ and $B_k + F_k r_k$ by $B_{1,k}$. We also replace L_k of (7.14) by $L_k + D_{12,k} K_k$ and assume that (7.3) holds. We obtain the following system $k = 0, 1, ..., N-1$

$$
\begin{aligned}
x_{k+1} &= [A_k + D_k v_k + (B_{2,k} + G_k r_k)K_k]x_k + B_{1,k} w_k, \\
z_k &= (L_k + D_{12,k} K_k)x_k.
\end{aligned}
\tag{7.29}
$$

Applying Corollary 7.1, for $m_1 = 2,$ and $m_2 = 0$, we obtain the following Riccati equation:

$$
\begin{aligned}
Q_k &= (A_k + B_{2,k}K_k)^T Q_{k+1}(A_k + B_{2,k}K_k) + L_k^T L_k + K_k^T \tilde{R}_k K_k + D_k^T Q_{k+1} D_k \\
&+ K_k^T G_k^T Q_{k+1} G_k K_k + (A_k + B_{2,k}K_k)^T Q_{k+1} B_{1,k} R_k^{-1} B_{1,k}^T Q_{k+1}(A_k + B_{2,k}K_k) \\
&+ \alpha_k K_k^T G_k^T Q_{k+1} D_k + \alpha_k D_k^T Q_{k+1} G_k K_k \qquad , Q_N = \bar{Q}_N
\end{aligned}
\tag{7.30}
$$

where

$$
R_k \triangleq \gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}. \tag{7.31}
$$

Defining :

$$
\begin{aligned}
\bar{M}_k &\triangleq Q_{k+1}[I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}, \\
\Phi_k &\triangleq B_{2,k}^T \bar{M}_k B_{2,k} + G_k^T Q_{k+1} G_k + \tilde{R}_k
\end{aligned}
\tag{7.32}
$$

and completing to squares for K_k we obtain:

$$
Q_k = \bar{R}(Q_k) + (K_k + K_k^*)^T \Phi_k (K_k + K_k^*)
$$

where

$$
\begin{aligned}
&K_k^* \triangleq \Phi_k^{-1} \Delta_{1,k}, \\
&\Delta_{1,k} \triangleq B_{2,k}^T \bar{M}_k A_k + \alpha_k G_k^T Q_{k+1} D_k, \\
&\bar{R}(Q_k) \triangleq A_k^T \bar{M}_k A_k + D_k^T Q_{k+1} D_k + L_k^T L_k - \Delta_{1,k}^T \Phi_k^{-1} \Delta_{1,k}.
\end{aligned}
\tag{7.33}
$$

Clearly, by choosing $K_k = -K_k^*$ and using the monotonicity property of the Riccati equation [57] we arrive at the following:

Theorem 7.3. *Consider the system (7.29) and (7.4). Given a scalar $\gamma > 0$, a necessary and sufficient condition for J_1 of (7.4) to be negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ is that there exists a solution $Q_k > 0$ to the following Riccati equation*

$$Q_k = A_k^T \bar{M}_k A_k + D_k^T Q_{k+1} D_k + L_k^T L_k - \Delta_{1,k}^T \Phi_k^{-1} \Delta_{1,k} \quad , Q_N = \bar{Q}_N \quad (7.34)$$

that satisfies $R_k > 0$, $\forall k = 1, 2, ...N-1$ and $Q_0 < \gamma^2 \bar{Q}_0$, where $\bar{M}_k$ is defined in (7.32). If Q_k of (7.34) satisfies $R_k > 0$ for all $k = 1, 2, ..N-1$ and $Q_0 < \gamma^2 \bar{Q}_0$, then using the definition of (7.32) and (7.33) the state-feedback gain is given by:

$$K_k = -\Phi_k^{-1} \Delta_{1,k}. \quad (7.35)$$

7.4.1 The Multiple-noise Case

The result of Theorem 7.3 is extended to the case where the system is described by:

$$x_{k+1} = (A_k + \sum_{i=1}^{m_1} D_{i,k} v_{i,k}) x_k + (B_{2,k} + \sum_{i=1}^{m_2} G_{i,k} r_{i,k}) u_k + B_{1,k} w_k, \quad k = 0, 1, ..., N-1, \quad (7.36)$$

where x_k, w_k, u_k and x_0 are defined in (7.1) and $\{v_{i,k}\}$ and $\{r_{i,k}\}$ are standard random scalar sequences with zero mean that satisfy (7.21).

Using the notation of (7.22), we consider the following recursion

$$\begin{aligned} \hat{Q}_k &= A_k^T \hat{\hat{M}}_k A_k - (B_{2,k}^T \hat{\hat{M}}_k A_k + \hat{H}_k)^T (B_{2,k}^T \hat{\hat{M}}_k B_{2,k} + \hat{G}_k + \tilde{R}_k)^{-1} \\ &(B_{2,k}^T \hat{\hat{M}}_k A_k + \hat{H}_k) + \hat{D}_k + L_k^T L_k, \\ \hat{Q}_N &= \bar{Q}_N, \\ \hat{\hat{M}}_k &\stackrel{\Delta}{=} \hat{Q}_{k+1} [I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}. \end{aligned} \quad (7.37)$$

We obtain the following.

Corollary 7.2. *Consider the system (7.36). Given the scalar $\gamma > 0$, a necessary and sufficient condition for J_1 of (7.4) to be negative for all $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ that are not identically zero is that there exists a solution $\hat{Q}_k$ to (7.37) that satisfies $R_k > 0$, $\forall \ k = 1, 2, ..N-1$, and $\hat{Q}_0 < \gamma^2 \bar{Q}_0$.*

7.4.2 Infinite-horizon Stochastic H_∞ State-feedback control

We address the issue of closed-loop stability by considering the case where the system matrices in (7.1) and the correlation between them are all constants and N tends to infinity. The stability is considered in the mean square sense [105], [106] meaning that the states mean and the correlation matrix $P_k = \underset{v,r}{E}\{x_k x_k^T\}$ converge to zero as k tends to infinity, for all initial states x_0 and $\{w_k\} = 0$. In our case, when we consider (7.1), it is readily found that for $P_0 > 0$:

$$P_{k+1} = (A + B_2 K)P_k(A + B_2 K)^T + DP_k D^T + GKP_k K^T G^T \\ +\alpha DP_k K^T G^T + \alpha GKP_k D^T \tag{7.38}$$

where K is the state-feedback gain. If the latter linear recursion is stable, in the mean square stability sense [105], we obtain that x_k goes to the origin almost surely when k tends to infinity.

We derive next a sufficient condition for the mean square stability of the autonomous system (7.13) (with $B_k = F_k \equiv 0$), in the case where A and D are constant matrices and N tends to infinity. We then show that the closed-loop system that is obtained using the state-feedback of Theorem 7.3 satisfies this condition. We consider the following autonomous system

$$x_{k+1} = (A + \Sigma_{i=1}^{m_1} D_i v_{i,k})x_k \tag{7.39}$$

where $\{v_{ik}\}, i = 1, 2, ..m_1$ are random zero-mean sequences with $E\{v_{ik_1}, v_{ik_2}\} = \sigma_{ij}\delta_{k1k2}$. Defining

$$D^* = \sum_{i=1}^{m_1}\sum_{j=1}^{m_1} D_i^T Q D_j \sigma_{ij}$$

we obtain the following.

Lemma 7.1. *The system (7.39) is mean square stable if there exists a positive definite matrix $Q \in \mathcal{R}^{nxn}$ that satisfies $Tr\{\bar{\Gamma}P_i\} < 0, \quad \forall i$, where*

$$\bar{\Gamma} \triangleq A^T QA - Q + D^*.$$

Proof: We consider the following Lyapunov function:

$$V_i = \underset{v_{jk}, j=1,2,..L_1, k\leq i}{E}\{x_i^T Q x_i\} = Tr\{QP_i\}.$$

It follows from the positive definiteness of Q that V_i is nonnegative. We also find that

$$V_{i+1} - V_i = \underset{v_{jk},k\leq i}{E}\{x_i^T[(A^T + \sum_{i=1}^{m_1} D_i^T v_{ik})Q(A + \sum_{i=1}^{m_1} D_i v_{ik}) - Q]x_i\}$$

$$= \underset{v_{jk},k<i}{E}\{x_i^T \bar{\Gamma} x_i\} = Tr\{\bar{\Gamma} P_i\}.$$

The requirement of a negative $Tr\{\bar{\Gamma}P_i\}$ thus implies a negative V_{i+1}–V_i which leads to the convergence of $Tr\{QP_i\}$ to zero.

□

We apply the latter lemma to prove the following.

Theorem 7.4. *Given the system*

$$\begin{aligned} x_{k+1} &= (A + Dv_k)x_k + B_1 w_k + (B_2 + Gr_k)u_k, \\ z_k &= Lx_k + D_{12}u_k, \end{aligned} \tag{7.40}$$

where the matrices A, B_1, B_2, D, G, L and D_{12} are all constant and $\{v_k\}$ and $\{r_k\}$ are the standard random scalar sequences with zero-mean that satisfy (7.2). Assuming that (A, L) is observable and that $[L^T \;\; D_{12}^T]D_{12} = [0 \;\; \tilde{R}], \quad \tilde{R} > 0$, then the matrix K_k of Theorem 7.3 which solves the stochastic H_∞ state-feedback problem of Section 7.4, for a given $\gamma > 0$ and for N that tends to infinity, stabilizes the system in the mean square sense.

Proof: The closed-loop system (7.40) is described by:

$$\begin{aligned} x_{k+1} &= [(A + B_2K) + (Dv_k + GKr_k)]x_k + B_1 w_k, \\ z_k &= (L + D_{12}K)x_k \end{aligned} \tag{7.41}$$

It follows from Lemma 7.1 that a sufficient condition for the required stability is the existence of $Q > 0$ s.t. $Tr\{\Gamma_0 P_i\} < 0, \quad \forall i$, where P_i is the covariance of the state vector of the closed-loop system (7.41) (where $B_1 = 0$), and

$$\Gamma_0 \stackrel{\triangle}{=} (A + B_2K)^T Q(A + B_2K) - Q + D^T QD + K^T G^T QGK + \alpha K^T G^T QD$$
$$+\alpha D^T QGK = (A + B_2K)^T Q(A + B_2K) - Q + (1 - \alpha^2)D^T QD$$
$$+(K^T G^T + \alpha D^T)Q(GK + \alpha D).$$

On the other hand, we obtain from (7.30) that any state-feedback gain matrix K that solves the stationary state-feedback problem satisfies $Tr\{(\Gamma_0 + \Gamma_1)P_i\} = 0, \quad \forall i$ where

$$\Gamma_1 = (A + B_2K)^T QB_1 R^{-1} B_1^T Q(A + B_2K) + L^T L + K^T \tilde{R} K. \tag{7.42}$$

It is, thus, left to be shown that $Tr\{\Gamma_1 P_i\} > 0$, $\forall i$. Assuming that the latter trace is not positive for some value of i, we denote the first value of i for which the latter happens by i^*. We find from (7.42) that the only way for $Tr\{\Gamma_1 P_i\}$ to be nonpositive, in fact zero, is that

$$LP_{i^*}L^T = 0 \quad \text{and} \quad KP_{i^*}K^T = 0,$$

which also implies that P_{i^*} is singular. Since we started from $P_0 > 0$, and since the correlation between $\{v_k\}$ and $\{r_k\}$, $\alpha^2 < 1$, we obtain from (7.38) that

$$P_{i^*} - (A + B_2K)^{i^*} P_0 (A^T + K^T B_2^T)^{i^*} \geq 0.$$

The matrices $LP_{i^*}L^T$ and $L(A + B_2K)^{i^*} P_0(A^T + K^T B_2^T)^{i^*} L^T$ are therefore zero, and thus $L(A + B_2K)^{i^*}$ (and similarly $K(A + B_2K)^{i^*}$) are zero. Since

$$P_i \geq (A + B_2K)^{i-i^*} P_{i^*} (A^T + K^T B_2^T)^{i-i^*}, \quad i > i^* \tag{7.43}$$

it is obtained that

$$LP_i = 0 \quad \text{and} \quad KP_i = 0, \quad \forall i \geq i^*.$$

Denoting by d any eigenvector of P_{i^*} that corresponds to a nonzero eigenvalue of P_{i^*} we readily obtain that $Ld = 0$ and $Kd = 0$. We also find from (7.43) that both $L(A + B_2K)^j d$ and $K(A + B_2K)^{i-i^*} d$ are zero for $j > 0$. Since

$$(A + B_2K)^j d = A^{j-i}(A + B_2K)d = A^j d, \quad j > 0,$$

it follows that the vector d is perpendicular to L and LA^j and therefore to the observability matrix. This contradicts the assumption about the observability of the pair (A, L).

□

Remark 7.2. The solution of the stationary state-feedback control can be readily achieved in an LMI form by applying the stationary BRL of Section 7.3.1 to the closed-loop system (7.41). Thus one obtains the following stationary algebraic Riccati-type counterpart of 7.34:

$$-Q + A^T \bar{M} A + D^T Q D + L^T L - \Delta_1^T \Phi^{-1} \Delta_1 \leq 0 \quad , Q_N = \bar{Q}_N, \tag{7.44}$$

together with the following stationary optimal strategies:

$$\begin{aligned} w_{s,k}^* &= \hat{R}^{-1} B_1^T Q [A - B_2 \Phi^{-1} \Delta_1] x_k \\ u_{s,k}^* &= \Phi^{-1} \Delta_1 x_k, \end{aligned} \tag{7.45}$$

where

$$\begin{aligned} \hat{R} &= \gamma^2 I_p - B_1^T Q B_1, \\ \bar{M} &\stackrel{\triangle}{=} Q[I - \gamma^{-2} B_1 B_1^T Q]^{-1}, \\ \Delta_1 &\stackrel{\triangle}{=} B_2^T \bar{M} A + \alpha_k G^T Q D, \\ \Phi &\stackrel{\triangle}{=} B_2^T \bar{M} B_2 + G^T Q G + \tilde{R}, \\ \tilde{R} &= D_{12}^T D_{12} \end{aligned} \tag{7.46}$$

If Q of (7.44) satisfies $\hat{R} > 0$ then using the above definitions, the state-feedback gain is given by:

$$K_s = -\Phi^{-1}\Delta_1. \tag{7.47}$$

Defining $P = Q^{-1}$ and applying Schur's complements formula and carrying out various multiplications, the following inequalities are achieved:

$$\begin{bmatrix} -P & PA^T & 0 & PD^T & PC^T & 0 \\ * & -\Gamma(2,2) & B_2\tilde{R}^{-1} & 0 & 0 & B_1 \\ * & * & -P & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -I_l & 0 \\ * & * & * & * & * & -\gamma^2 I_p \end{bmatrix} \leq 0 \tag{7.48}$$

and

$$\begin{bmatrix} \gamma^2 I_p & B_1^T \\ * & P \end{bmatrix} > 0. \tag{7.49}$$

where $\Gamma(2,2) \stackrel{\Delta}{=} P + B_2\hat{R}^{-1}B_2^T$. We note that the optimal stationary strategies of (7.46) will be used for the solution of the stationary output-feedback of Section 7.8.

7.5 Stochastic State-multiplicative H_∞ Filtering

We investigate next the H_∞-filtering problem. Considering the system (7.5) and defining:

$$\xi_k = col\{x_k, e_k\},$$

we obtain the following system:

$$\begin{aligned} \xi_{k+1} &= \tilde{A}_k\xi_k + \tilde{D}_k\xi_k v_k + \tilde{F}_k\xi_k\zeta_k + \tilde{B}_k\tilde{w}_k,\ \xi_0 = \begin{bmatrix} x_0 \\ e_0 \end{bmatrix}, \\ z_k &= \tilde{C}_k\xi_k \end{aligned} \tag{7.50}$$

where

$$\tilde{A}_k = \begin{bmatrix} A_k & 0 \\ 0 & A_k - K_{ok}C_k \end{bmatrix}, \quad \tilde{B}_k = \begin{bmatrix} B_{1,k} & 0 \\ B_{1,k} & -K_{ok} \end{bmatrix}$$

$$\tilde{D}_k = \begin{bmatrix} D_k & 0 \\ D_k & 0 \end{bmatrix}, \quad \tilde{F}_k = \begin{bmatrix} 0 & 0 \\ -K_{ok}F_k & 0 \end{bmatrix}, \quad \tilde{C}_k = \begin{bmatrix} 0 & L_k \end{bmatrix}. \tag{7.51}$$

Similarly to the solution of the continuous-time counterpart problem, we seek for a gain observer K_{ok} that achieves $J_2 < 0$ where J_2 is defined in (7.8) and where we replace in the latter x_0 by ξ_0. We obtain the following theorem:

Theorem 7.5. *Consider the system (7.50) and (7.8). Given a scalar $\gamma > 0$, a necessary and sufficient condition for J_2 to be negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ is that there exists a solution (M_k, K_{ok}) to the following DLMI:*

$$\begin{bmatrix} -\hat{M}_k & \hat{M}_k\tilde{A}_k^T & 0 & \hat{M}_k\tilde{D}_k^T & \hat{M}_k\tilde{F}_k^T & \hat{M}_k\tilde{C}_{1,k}^T \\ * & -\hat{P}_{k+1} & \gamma^{-1}\tilde{B}_{1,k} & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -\hat{P}_{k+1} & 0 & 0 \\ * & * & * & * & -\hat{P}_{k+1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} \leq 0, \tag{7.52}$$

with a forward iteration, starting from the following initial condition:

$$\hat{P}_0 = \begin{bmatrix} I_n \\ I_n \end{bmatrix} \gamma^2 P_0 \begin{bmatrix} I_n & I_n \end{bmatrix} \quad \textit{where } \hat{M}_k = \hat{P}_k^{-1}, \quad k = 0, ..., N-1. \tag{7.53}$$

Proof: We apply the stochastic BRL of Section 7.3 to the system (7.50) and we obtain the following Riccati-type equation:

$$\begin{aligned} M_k = \tilde{A}_k^T M_{k+1}\tilde{A}_k + \tilde{A}_k^T M_{k+1}\tilde{B}_k\tilde{\Theta}_k^{-1}\tilde{B}_k^T M_{k+1}\tilde{A}_k + \tilde{C}_k^T\tilde{C}_k + \tilde{D}_k^T M_{k+1}\tilde{D}_k \\ +\tilde{F}_k^T M_{k+1}\tilde{F}_k + \alpha_k\tilde{F}_k^T M_{k+1}\tilde{D}_k + \alpha_k\tilde{D}_k^T M_{k+1}\tilde{F}_k, \\ M_0 = \begin{bmatrix} P_0 & P_0 \\ P_0 & P_0 \end{bmatrix}, \end{aligned} \tag{7.54}$$

where

$$\tilde{\Theta}_k \stackrel{\triangle}{=} \gamma^2 I - \tilde{B}_k^T M_{k+1}\tilde{B}_k > 0 \quad ,\forall \quad k = 1, 2, ..N-1. \tag{7.55}$$

In order to comply with the initial condition in (7.54), and with the choice of $\hat{x}_0 = 0$, we force nature to select the component e_0 in ξ_0 to be equal to x_0. □

The results of the last theorem can be extended to the case where the system is given by:

$$\begin{aligned} x_{k+1} &= (A_k + \sum_{i=1}^{m_1} D_{i,k} v_{i,k}) x_k + B_{1,k} w_k, \; x_0 = x_0, \\ y_k &= (C_k + \sum_{i=1}^{m_2} F_{i,k} \zeta_{i,k}) x_k + n_k, \\ z_k &= L_k x_k \end{aligned} \tag{7.56}$$

where x_k, w_k, n_k relate to the system (7.5) and $\{v_{i,k}\}$ and $\{\zeta_{i,k}\}$ are standard random scalar sequences with zero-mean that satisfy:

$$E\{v_{i,k1} v_{j,k2}\} = \sigma_{ij,k} \delta_{k1k2}, \quad E\{\zeta_{i,k1} \zeta_{j,k2}\} = \beta_{ij,k} \delta_{k1k2},$$

$$\text{and } E\{\zeta_{i,k1} v_{j,k2}\} = \alpha_{ij,k} \delta_{k1k2}$$

and where $\alpha_{ij,k}$, $\beta_{ij,k}$ and $\sigma_{ij,k}$ are predetermined scalar sequences. Denoting:

$$\begin{aligned} \tilde{F}_k &\stackrel{\triangle}{=} \sum_{i=1}^{m_2} \sum_{j=1}^{m_2} \bar{F}_{i,k}^T M_k \bar{F}_{j,k} \beta_{ij,k}, \\ \tilde{D}_k &\stackrel{\triangle}{=} \sum_{i=1}^{m_1} \sum_{j=1}^{m_1} \bar{D}_{i,k}^T M_k \bar{D}_{j,k} \sigma_{ij,k} \\ \tilde{S}_k &\stackrel{\triangle}{=} (\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \bar{D}_{i,k}^T M_k \bar{F}_{j,k} + \sum_{i=1}^{m_2} \sum_{j=1}^{m_1} \bar{F}_{i,k}^T M_k \bar{D}_{j,k}) \alpha_{ij,k} \end{aligned} \tag{7.57}$$

where

$$\bar{D}_{i,k} = \begin{bmatrix} D_{i,k} & 0 \\ D_{i,k} & 0 \end{bmatrix} \quad \text{and} \quad \bar{F}_{i,k} = \begin{bmatrix} 0 & 0 \\ -K_{ok} F_{i,k} & 0 \end{bmatrix},$$

we consider the following recursion:

$$\begin{aligned} M_k = \tilde{A}_k^T M_{k+1} \tilde{A}_k + \tilde{A}_k^T M_{k+1} \tilde{B}_k (\gamma^2 I - \tilde{B}_k^T M_{k+1} \tilde{B}_k)^{-1} \tilde{B}_k^T M_{k+1} \tilde{A}_k \\ + \tilde{C}_k^T \tilde{C}_k + \tilde{D}_k + \tilde{F}_k + \tilde{S}_k, \end{aligned} \tag{7.58}$$

where $\tilde{A}_k$, $\tilde{B}_k$, and $\tilde{C}_k$ are given in (7.51) and M_0 is given in (7.54). We obtain the following :

Corollary 7.3. *Consider the system (7.56) and (7.8). Given a scalar $\gamma > 0$, a sufficient condition for J_2 to be negative, for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$, is that there exists a solution (M_k, K_{ok}) to the Riccati equation of (7.58) that satisfies, $\tilde{\Theta}_k > 0 \quad , \forall \quad k = 1, 2, ..N-1$, where $\tilde{\Theta}_k$ is defined in Theorem 7.5.*

7.6 Infinite-horizon Stochastic Filtering

We consider the system (7.11) with $B_2 = G = 0,\ D_{12} = 0$ and where, for simplicity, we take zero correlation between ζ_k and v_k (i.e $\beta_k = 0$. Introducing the following Lyapunov function:

$$V_k = \xi_k^T \tilde{Q} \xi_k, \text{ with } \tilde{Q} = \begin{bmatrix} Q & \tilde{\alpha}\hat{Q} \\ \tilde{\alpha}\hat{Q} & \hat{Q} \end{bmatrix}, \tag{7.59}$$

where ξ_k is the augmented state vector of Section 7.5, Q and $\hat{Q}$ are $n \times n$ matrices and $\tilde{\alpha}$ is a tuning scalar. We obtain the following result:

Theorem 7.6. *Consider the system (7.11) and J_4 of (7.12) where the matrices A, B_1, D, C_2, F, C_1 and D_{12} are all constant, with $B_2 = G = 0,\ D_{12} = 0$, and where $\hat{x}_k$ is defined in (7.6). Given $\gamma > 0$, a necessary and sufficient condition for J_4 of (7.12) to be negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ is that there exist $Q = Q^T \in \mathcal{R}^{n\times n}$, $\hat{Q} = \hat{Q}^T \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times z}$ and a tuning scalar parameter $\tilde{\alpha}$ that satisfy the following LMIs:*

$$\begin{bmatrix} -Q & -\tilde{\alpha}\hat{Q} & 0 & 0 & \Upsilon(1,5) & \Upsilon(1,6) \\ * & -\hat{Q} & 0 & 0 & \Upsilon(2,5) & \tilde{\Upsilon}(2,6) \\ * & * & -\gamma^2 I_p & 0 & \Upsilon(3,5) & B_1^T\hat{Q}[1+\tilde{\alpha}] \\ * & * & * & -\gamma^2 I_z & -\tilde{\alpha}Y^T & -Y^T \\ * & * & * & * & -Q & -\tilde{\alpha}\hat{Q} \\ * & * & * & * & * & -\tilde{\alpha}\hat{Q} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{matrix}$$

$$
\left.\begin{matrix}
\Upsilon(1,7) & D^T\hat{Q}[1+\tilde{\alpha}] & \Upsilon(1,9) & -F^TY^T & 0 \\
0 & 0 & 0 & 0 & L^T \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-Q & -\tilde{\alpha}\hat{Q} & 0 & 0 & 0 \\
* & -\hat{Q} & 0 & 0 & 0 \\
* & * & -Q & -\tilde{\alpha}\hat{Q} & 0 \\
* & * & * & -\hat{Q} & 0 \\
* & * & * & 0 & -I_m
\end{matrix}\right] < 0,
$$

and

$$
\begin{bmatrix} \gamma^2 I_{p+z} & \tilde{B}^T \\ \tilde{B} & \tilde{Q} \end{bmatrix} > 0, \tag{7.60}
$$

where

$$
\begin{aligned}
\Upsilon(1,5) &\triangleq A^T[Q+\tilde{\alpha}\hat{Q}], \\
\Upsilon(1,6) &\triangleq A^T\hat{Q}[1+\tilde{\alpha}], \\
\Upsilon(1,7) &\triangleq D^T[Q+\tilde{\alpha}\hat{Q}], \\
\Upsilon(1,9) &\triangleq -F^TY^T, \\
\Upsilon(2,5) &\triangleq B_1^T[Q+\tilde{\alpha}\hat{Q}], \\
\Upsilon(2,6) &\triangleq B_1^T\hat{Q}[1+\tilde{\alpha}],
\end{aligned}
$$

Proof: The proof outline for the above stationary case resembles the one of the finite-horizon case. Considering the augmented state vector ξ_k of Section 7.5 and the stationary counterpart of both 7.5 and 7.51 where $N = \infty$, we obtain by applying the stationary stochastic BRL of Section 7.3.1, the stationary counterpart of 7.54. The latter can be readily expressed as the following inequality:

$$
\begin{aligned}
-\tilde{Q} + \tilde{A}^T\tilde{Q}\tilde{A} + \tilde{A}^T\tilde{Q}\tilde{B}[\gamma^2 I_{p+z} - \tilde{B}\tilde{Q}\tilde{B}]^{-1}\tilde{B}^T\tilde{Q}\tilde{A} + \tilde{C}^T\tilde{C} + \tilde{D}^T\tilde{Q}\tilde{D} \\
+\tilde{F}^T\tilde{Q}\tilde{F} \leq 0.
\end{aligned} \tag{7.61}
$$

Applying Schur's complements formula to (7.61) we obtain:

$$\Gamma_S \triangleq \begin{bmatrix} -\tilde{Q} & 0 & \tilde{A}^T & \tilde{D}^T & \tilde{F}^T & \tilde{C}^T \\ * & -\gamma^2 I_{p+z} & \tilde{B}^T & 0 & 0 & 0 \\ * & * & -\tilde{Q}^{-1} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}^{-1} & 0 & 0 \\ * & * & * & * & -\tilde{Q}^{-1} & 0 \\ * & * & * & * & * & -I_m \end{bmatrix} \leq 0. \qquad (7.62)$$

Multiplying the latter inequality by $diag\{I_{2n\times 2n},\ I_{p+z},\ \tilde{Q},\ \tilde{Q},\ \tilde{Q},\ I_m\}$ and carrying out the various multiplications and denoting $Y \triangleq \hat{Q}K_0$, where K_0 is the stationary observer gain, the LMIs of (7.60) are obtained.

□

7.7 Stochastic Output-feedback

The solution of the output-feedback problem is obtained below by transforming the problem to one of filtering which has been solved in the last section. We can not use the results of Section 7.4, for this transformation, since we need the optimal strategies of both $\{w_k\}$ and $\{u_k\}$. These are obtained below by completing to squares for both strategies.

Substituting from (7.1) in (7.16) we obtain:

$$\begin{aligned}\tilde{J}_k = & [x_k^T(A_k+D_k v_k)^T + u_k^T(B_{2,k}+G_k r_k)^T]Q_{k+1}[(A_k+D_k v_k)x_k+(B_{2,k}+G_k r_k)u_k] \\ & +2[x_k^T(A_k+D_k v_k)^T+u_k^T(B_{2,k}+G_k r_k)^T]Q_{k+1}B_{1,k}w_k+w_k^T B_{1,k}^T Q_{k+1}B_{1,k}w_k \\ & -x_k^T Q_k x_k - \gamma^2 w_k^T w_k + \gamma^2 w_k^T w_k + u_k^T \tilde{R}_k u_k + x_k^T C_k^T C_k x_k - z_k^T z_k \\ = & -w_k^T[\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}]w_k + 2[x_k^T(A_k + D_k v_k)^T + u_k^T(B_{2,k} + G_k r_k)^T] \\ & Q_{k+1}B_{1,k}w_k + u_k^T[\tilde{R}_k + (B_{2,k} + G_k r_k)^T Q_{k+1}(B_{2,k} + G_k r_k)]u_k \\ & +2x_k^T(A_k + D_k v_k)^T Q_{k+1}(B_{2,k} + G_k r_k)u_k \\ & +x_k^T[(A_k + D_k v_k)^T Q_{k+1}(A_k + D_k v_k) + C_k^T C_k - Q_k]x_k - z_k^T z_k + \gamma^2 w_k^T w_k.\end{aligned}$$

Taking the expectation with respect to r_k and v_k we obtain:

$$\begin{aligned}\underset{v,r}{E}\left\{\tilde{J}_k\right\} = & \underset{v,r}{E}\{-w_k^T[\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}]w_k + 2[x_k^T A_k^T Q_{k+1} B_{1,k} + u_k^T B_{2,k}^T Q_{k+1} B_{1,k}]w_k \\ & +u_k^T[\tilde{R}_k + B_{2,k}^T Q_{k+1} B_{2,k} + G_k^T Q_{k+1} G_k]u_k + 2x^T[A_k^T Q_{k+1} B_{2,k} + \alpha_k D_k^T Q_{k+1} G_k]u_k \\ & +x_k^T[A_k^T Q_{k+1} A_k + D_k^T Q_{k+1} D_k + C_k^T C_k - Q_k]x_k - z_k^T z_k + \gamma^2 w_k^T w_k\} + \underset{v,r}{E}\{\Psi_k\}\end{aligned}$$

where

$$\Psi_k = 2x_k^T D_k^T v_k Q_{k+1} B_{1,k} w_k + 2u_k^T G_k^T r_k Q_{k+1} B_{1,k} w_k + 2u_k^T G_k^T r_k Q_{k+1} B_{2,k} u_k$$

$$+2x_k^T D_k^T v_k Q_{k+1} B_{2,k} u_k + 2x_k^T A_k^T Q_{k+1} G_k r_k u_k + 2x_k^T D_k^T v_k Q_{k+1} A_k x_k$$

and where

$$\underset{v,r}{E}\{\Psi_k\} = 0, \qquad k = 0, 1, ..., N-1.$$

Completing to squares, first for w_k and then for u_k, we get:

$$\underset{v,r}{E}\left\{\tilde{J}_k\right\} = \underset{v,r}{E}\{-(w_k - w_k^*)^T R_k (w_k - w_k^*) + (u_k - u_k^*)^T \Phi_k (u_k - u_k^*)$$
$$-x_k^T [\Delta_{1,k} \Phi_k^{-1} \Delta_{1,k}^T - A_k^T Q_{k+1} B_{1,k} R_k^{-1} B_{1,k}^T Q_{k+1} A_k] x_k$$
$$+x_k^T [A_k^T Q_{k+1} A_k + D_k^T Q_{k+1} D_k + C_k^T C_k - Q_k] x_k$$
$$-z_k^T z_k + \gamma^2 w_k^T w_k\} + \underset{v,r}{E}\{\Psi_k\}$$

where R_k, Φ_k and $\Delta_{1,k}$ are defined in (7.31), (7.32), (7.33) and where we define:

$$u_k^* \triangleq K_k x_k,$$
$$w_k^* \triangleq K_{xk} x_k + K_{uk} u_k,$$
$$K_{xk} \triangleq R_k^{-1} B_{1,k}^T Q_{k+1} A_k,$$
$$K_{uk} \triangleq R_k^{-1} B_{1,k}^T Q_{k+1} B_{2,k}$$

where K_k is defined in (7.35).
Rearranging the last equation we obtain:

$$\underset{v,r}{E}\left\{\tilde{J}_k\right\} = \underset{v,r}{E}\{-(w_k - w_k^*)^T R_k (w_k - w_k^*) + (u_k - u_k^*)^T \Phi_k (u_k - u_k^*)$$
$$+x_k^T \bar{R}(Q_k) x_k - z_k^T z_k + \gamma^2 w_k^T w_k\} + \underset{v,r}{E}\{\Psi_k\} \tag{7.63}$$

where $\bar{R}(Q_k)$ is defined in (7.33) and Φ_k is defined in (7.32). Taking the sum of both sides of (7.16), from zero to $N-1$, we obtain using (7.63):

$$\underset{v,r}{E} \sum_{k=0}^{N-1} \left\{\tilde{J}_k\right\} = \underset{v,r}{E}\left\{x_N^T Q_N x_N\right\} - x_0^T Q_0 x_0$$

$$= \sum_{k=0}^{N-1} \underset{v,r}{E}\left\{-(w_k - w_k^*)^T R_k (w_k - w_k^*) + (u_k - u_k^*)^T \Phi_k (u_k - u_k^*)\right\}$$

$$+ \sum_{k=0}^{N-1} \underset{v,r}{E}\left\{x_k^T \bar{R}(Q_k) x_k\right\} + \sum_{k=0}^{N-1} \underset{v,r}{E}\left\{-z_k^T z_k + \gamma^2 w_k^T w_k\right\} + \underset{v,r}{E}\{\Psi_k\}$$

Hence

$$J_1 = \sum_{k=0}^{N-1} \underset{v,r}{E}\left\{-(w_k - w_k^*)^T R_k (w_k - w_k^*) + (u_k - u_k^*)^T \Phi_k (u_k - u_k^*)\right\}$$
$$+\sum_{k=0}^{N-1} \underset{v,r}{E}\left\{x_k^T \bar{R}(Q_k) x_k\right\} + x_0^T (Q_0 - \gamma^2 \bar{Q}_0) x_0 + \underset{v,r}{E}\{\Psi_k\}. \tag{7.64}$$

Clearly, the optimal strategy for u_k is given by $u_k = u_k^*$ where Q_k that is obtained by $\bar{R}(Q_k) = 0$ namely, Q_k satisfies:

$$Q_k = A_k^T Q_{k+1} A_k - \Delta_{1,k} \Phi_k^{-1} \Delta_{1,k}^T + D_k^T Q_{k+1} D_k + LT_k Lk$$
$$+A_k^T Q_{k+1} B_{1,k} R_k^{-1} B_{1,k}^T Q_{k+1} A_k, \quad Q_N = \bar{Q}_N, \tag{7.65}$$

$R_k > 0$ and $Q_0 < \gamma^2 \bar{Q}_0$, where R_k is defined in (7.31)

Remark 7.3. The recursion we obtained in (7.65) together with the feasibility condition of $\Phi_k > 0$ is identical to the necessary and sufficient condition we obtained in Theorem 7.3. We did not apply the proof of this section to derive the BRL in Section 7.3 since we could not prove the necessary condition of Theorem 7.3 using the arguments of the present proof.

We are now ready to use the results of the state-feedback to derive a solution to the output-feedback problem. Denoting $r_k \stackrel{\triangle}{=} w_k - w_k^*$ and using $u_k = K_k \hat{x}_k$, where $\hat{x}_k$ is yet to be found, we obtain from (7.9) that

$$x_{k+1} = (A_k + D_k v_k + B_{1,k} K_{xk}) x_k + B_{1,k} r_k + (B_{1,k} K_{uk} + B_{2,k} + G_k \nu_k) K_k \hat{x}_k,$$

$$x_0 = x_0, \quad y_k = (C_k + F_k \eta_k) x_k + n_k.$$

Substituting in (7.64) we look for $\hat{x}$ for which

$$J \stackrel{\triangle}{=} \underset{\nu,v,\eta}{E}\left\{\Sigma_{k=0}^{N-1} ||z_k||_{\Phi_k}^2 - ||r_k||_{R_k}^2 + \gamma^2 ||n_k||_2^2 + \Psi_k\right\} - x_0^T \tilde{S} x_0$$

is less then zero for all nonzero $(\{w_k\}, \{n_k\}, x_0)$, where

$$z_k = K_k (x_k - \hat{x}_k), \quad \text{and} \quad \tilde{S} = \gamma^2 \bar{Q}_0 - Q_0. \tag{7.66}$$

We consider the following state estimator

$$\hat{x}_{k+1} = (A_k + B_{1,k} K_{xk}) \hat{x}_k + (B_{1,k} K_{uk} + B_{2,k}) u_k + K_{0,k} (y_k - C_k \hat{x}_k). \tag{7.67}$$

Using the definition of (7.7) for e_k we obtain:

$$x_{k+1} = [A_k + B_{1,k} K_{xk} + B_{1,k} K_{uk} K_k + B_{2,k} K_k] x_k + D_k x_k v_k + G_k K_k x_k \nu_k$$
$$+B_{1,k} r_k - (B_{1,k} K_{uk} + B_{2,k}) K_k e_k - G_k K_k e_k \nu_k$$

and

$$\hat{x}_{k+1}=[A_k+B_{1,k}K_{xk}+(B_{2,k}+B_{1,k}K_{uk})K_k]\hat{x}_k+K_{0,k}[(C_k+F_k\eta_k)x_k+n_k-C_k\hat{x}_k].$$

Therefore,

$$\begin{aligned} e_{k+1} &= [A_k + D_k v_k + B_{1,k}K_{xk} + (B_{1,k}K_{uk} + B_{2,k} + G_k\nu_k)K_k]x_k \\ &-(B_{1,k}K_{uk} + B_{2,k} + G_k\nu_k)K_k e_k - (A_k + B_{1,k}K_{xk})\hat{x}_k - (B_{2,k} + B_{1,k}K_{uk})K_k\hat{x}_k \\ &-K_{0,k}[(C_k + F_k\eta_k)x_k + n_k - C_k\hat{x}_k] + B_{1,k}r_k = [A_k + B_{1,k}K_{xk} - K_{0,k}C_k]e_k \\ &+B_{1,k}r_k - K_{0,k}n_k + D_k x_k v_k + G_k K_k x_k \nu_k - K_{0,k}F_k x_k\eta_k - G_k K_k e_k\nu_k. \end{aligned}$$

Defining

$$\xi_k = col\{x_k,\ e_k\} \quad \text{and} \quad \tilde{w}_k = col\{r_k,\ n_k\}$$

we obtain the following system, which is equivalent to the one in (7.50):

$$\begin{aligned} \xi_{k+1} &= \tilde{A}_k\xi_k + \tilde{D}_{1,k}\xi_k v_k + \tilde{F}_k\xi_k\eta_k + \tilde{D}_{2,k}\xi_k\nu_k + \tilde{B}_k\tilde{w}_k,\ \xi_0 = \begin{bmatrix} x_0 \\ e_0 \end{bmatrix} \\ z_k &= \tilde{C}_{1,k}\xi_k \end{aligned} \tag{7.68}$$

where

$$\tilde{A}_k = \begin{bmatrix} A_k + (B_{1,k}K_{uk} + B_{2,k})K_k + B_{1,k}K_{xk} & -[B_{1,k}K_{uk} + B_{2,k}]K_k \\ 0 & A_k + B_{1,k}K_{xk} - K_{0,k}C_k \end{bmatrix}$$

$$\tilde{B}_k = \begin{bmatrix} B_{1,k} & 0 \\ B_{1,k} & -K_{0,k} \end{bmatrix}, \quad \tilde{D}_{1,k} = \begin{bmatrix} D_k & 0 \\ D_k & 0 \end{bmatrix}, \quad \tilde{F}_k = \begin{bmatrix} 0 & 0 \\ -K_{0,k}F_k & 0 \end{bmatrix}$$

$$\tilde{D}_{2,k} = \begin{bmatrix} G_k K_k & -G_k K_k \\ G_k K_k & -G_k K_k \end{bmatrix} \quad \text{and} \quad \tilde{C}_{1,k} = \begin{bmatrix} 0 & K_k \end{bmatrix}.$$

Using the above notation we arrive at the following theorem:

Theorem 7.7. *Consider the system (7.9) and J_3 of (7.10) where $u_k = K_k\hat{x}_k$, K_k is given in (7.35) and where $\hat{x}_k$ is defined above. Given a scalar $\gamma > 0$, there exists a controller that achieves $J_3 < 0$ if there exists a solution $(\hat{P}_k, K_{0,k})$ to the following difference linear matrix inequality (DLMI)[44]:*

$$\begin{bmatrix} -\hat{P}_k & \hat{P}_k\tilde{A}_k^T & 0 & \hat{P}_k\tilde{D}_{1,k}^T & \hat{P}_k\tilde{D}_{2,k}^T & \hat{P}_k\tilde{F}_k^T & \hat{P}_k\tilde{C}_{1,k}^T \\ * & -\hat{P}_{k+1} & \gamma^{-1}\tilde{B}_{1,k} & 0 & 0 & 0 & 0 \\ * & * & -I_{p+z} & 0 & 0 & 0 & 0 \\ * & * & * & -\hat{P}_{k+1} & 0 & 0 & 0 \\ * & * & * & * & -\hat{P}_{k+1} & 0 & 0 \\ * & * & * & * & * & -\hat{P}_{k+1} & 0 \\ * & * & * & * & * & * & -I_l \end{bmatrix} \le 0, \tag{7.69}$$

with a forward iteration, starting from the following initial condition:

$$\hat{P}_0 = \begin{bmatrix} I_n \\ I_n \end{bmatrix} (\gamma^2 \bar{Q}_0 - Q_0) \begin{bmatrix} I_n \ I_n \end{bmatrix}. \tag{7.70}$$

Proof: Applying the result of the discrete-time stochastic BRL of Section 7.3 (see also [43]) to the system (7.68), the following Riccati-type inequality is obtained:

$$\begin{array}{c} -\hat{Q}_k + \tilde{A}_k^T \hat{Q}_{k+1} \tilde{A}_k + \tilde{A}_k^T \hat{Q}_{k+1} \tilde{B}_k \Theta_k^{-1} \tilde{B}_k^T \hat{Q}_{k+1} \tilde{A}_k \\ + \tilde{D}_{1,k}^T \hat{Q}_{k+1} \tilde{D}_{1,k} + \tilde{D}_{2,k}^T \hat{Q}_{k+1} \tilde{D}_{2,k} + \tilde{F}_k^T \hat{Q}_{k+1} \tilde{F}_k + \tilde{C}_{1,k}^T \tilde{C}_{1,k} > 0, \\ \Theta_k = \gamma^2 I - \tilde{B}_k^T \hat{Q}_{k+1} \tilde{B}_k, \ \Theta_k > 0. \end{array} \tag{7.71}$$

By simple manipulations, including the matrix inversion Lemma, on the latter the following inequality is obtained:

$$\begin{array}{c} -\hat{Q}_k + \tilde{A}_k^T [\hat{Q}_{k+1}^{-1} - \gamma^{-2} \tilde{B}_k \tilde{B}_k^T]^{-1} \tilde{A}_k + \tilde{C}_{1,k}^T \tilde{C}_{1,k} \\ + \tilde{D}_{1,k}^T \hat{Q}_{k+1} \tilde{D}_{1,k} + \tilde{D}_{2,k}^T \hat{Q}_{k+1} \tilde{D}_{2,k} + \tilde{F}_k^T \hat{Q}_{k+1} \tilde{F}_k > 0. \end{array} \tag{7.72}$$

Denoting $\hat{P}_k = \hat{Q}_k^{-1}$ and using Schur's complements the result of (7.69) is obtained.

□

Remark 7.4. We note that the solution of the latter DLMI proceeds the solution of the finite-horizon state-feedback of Section 7.4 starting from Q_N in (7.34), for a given attenuation level of γ. Once a solution to the latter problem is achieved, the DLMI of (7.69) is solved for the same γ starting from the above initial condition.

Remark 7.5. We note that $\bar{P}_0 = E\{x_0 x_0^T\}$ where $\bar{P}_0 = \gamma^2 \bar{Q}_0 - Q_0)$. The latter suggests that the initial condition $\hat{P}_0$ of (7.70) is

$$\hat{P}_0 = E\{ \begin{bmatrix} x_0 \\ e_0 \end{bmatrix} \begin{bmatrix} x_0^T \ e_0^T \end{bmatrix} \} = \begin{bmatrix} \bar{P}_0 & \bar{P}_0 \\ \bar{P}_0 & \bar{P}_0 \end{bmatrix},$$

since $e_0 = x_0$, hence justifying the structure of (7.70).

7.8 Stationary Stochastic Output-feedback Control

We consider the mean square stable system (7.11), where for simplicity, we take $G = 0$. Introducing the following Lyapunov function:

$$V_k = \xi_k^T \tilde{Q} \xi_k, \text{ with } \tilde{Q} = \begin{bmatrix} Q & \tilde{\alpha} \hat{Q} \\ \tilde{\alpha} \hat{Q} & \hat{Q} \end{bmatrix}, \tag{7.73}$$

where ξ_k is the state vector of (7.68), Q and $\hat{Q}$ are $n \times n$ matrices and $\tilde{\alpha}$ is a tuning scalar, we obtain the following result:

Theorem 7.8. *Consider the mean square-stable system (7.11) and* J_4 *of (7.12) where the matrices* $A, B_1, B_2, D, C_2, F, C_1$ *and* D_{12} *are all constant,* $G = 0$*,* $u_k = K_s \hat{x}_k$ *and where* $\hat{x}_k$ *is defined in (7.67). Given* $\gamma > 0$*, there exists a controller that achieves* $J_4 < 0$ *if there exist* $Q = Q^T \in \mathcal{R}^{n \times n}$*,* $\hat{Q} = \hat{Q}^T \in \mathcal{R}^{n \times n}$*,* $Y \in \mathcal{R}^{n \times z}$ *and a tuning scalar parameter* $\tilde{\alpha}$ *that satisfy the following LMIs:*

$$\left[\begin{array}{cccccccccccc} -Q & \tilde{\alpha}\hat{Q} & \Upsilon(1,3) & \Upsilon(1,4) & 0 & 0 & -\tilde{\alpha}D^T Y & -D^T Y & F^T(Q+\tilde{\alpha}\hat{Q}) & F^T\hat{Q}(1+\tilde{\alpha}) & 0 \\ * & -\hat{Q} & \Upsilon(2,3) & \tilde{\Upsilon}(2,4) & 0 & 0 & 0 & 0 & 0 & 0 & \hat{C}_1^T \\ * & * & -Q & \tilde{\alpha}\hat{Q} & \Upsilon(3,5) & -\gamma^{-1}\tilde{\alpha}Y & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\hat{Q} & \Upsilon(4,5) & -\gamma^{-1}Y & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -I_p & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I_z & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q & \tilde{\alpha}\hat{Q} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\hat{Q} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Q & \tilde{\alpha}\hat{Q} & 0 \\ * & * & * & * & * & * & * & * & * & -\hat{Q} & 0 \\ * & * & * & * & * & * & * & * & * & 0 & -I_l \end{array}\right] < 0,$$

and

$$\begin{bmatrix} \gamma^2 I_{p+z} & \tilde{B}^T \\ \tilde{B} & \tilde{Q} \end{bmatrix} > 0. \tag{7.74}$$

where

$$\Upsilon(1,3) \stackrel{\Delta}{=} [K_{s,x}^T B_1^T + K_s^T(B_2^T + K_{s,u}^T B_1^T) + A^T]Q,$$
$$\Upsilon(1,4) \stackrel{\Delta}{=} \tilde{\alpha}\hat{Q}\Upsilon(1,3)Q^{-1},$$
$$\Upsilon(2,3) \stackrel{\Delta}{=} -\tilde{\alpha}C^T Y^T - K_s^T[B_2^T + K_{s,u}^T B_1^T]Q + \tilde{\alpha}[K_{s,x}^T B_1^T + A^T]\hat{Q},$$
$$\Upsilon(2,4) \stackrel{\Delta}{=} \tilde{\alpha}[K_{s,x}^T B_1^T + K_s^T(B_2^T + K_{s,u}^T B_1^T) + A^T]\hat{Q} + [K_{s,x}^T B_1^T + A^T]\hat{Q} - C^T Y^T,$$
$$\Upsilon(3,5) \stackrel{\Delta}{=} \gamma^{-1}[QB_1 + \tilde{\alpha}\hat{Q}B_1],$$
$$\Upsilon(4,5) \stackrel{\Delta}{=} \gamma^{-1}(\tilde{\alpha}+1)\hat{Q}B_1,$$
$$K_{s,x} = R^{-1}B_1^T P^{-1}A,$$
$$K_{s,u} = R^{-1}B_1^T P^{-1}B_2$$

and where P is the solution (7.48) of Section 7.4.2.

Proof: The proof outline for the above stationary case resembles the one of the finite-horizon case. Considering the system (7.11) we first solve the stationary state-feedback problem to obtain the optimal stationary strategies of both $w_{s,k}^*$ and $u_{s,k}^*$ and the stationary controller gain K_s. These are given in Section 7.4.2 in (7.45).

Using the optimal strategies we transform the problem to an estimation one, thus arriving to the stationary counterpart of the augmented system (7.68). Applying the stationary discrete BRL for the stationary case of Section 7.3.1 (see also [10]) to the latter system the algebraic counterpart of (7.71) is obtained [10] which, similarly to the finite horizon case, becomes the stationary version of (7.69). Multiplying the stationary version of (7.69) from the left and the right by $diag\{\hat{P}^{-1}, \hat{P}^{-1}, I_{p+z}, \hat{P}^{-1}, \hat{P}^{-1}, \hat{P}^{-1}, I_l\}$, denoting $\tilde{Q} = \hat{P}^{-1}$, $Y = \hat{Q}K_o$ where K_o is the observer gain and carrying out the various multiplications the LMI of Theorem 7.8 obtained.

7.9 Examples

In this section we solve two examples of stationary systems with stochastic parameter uncertainties. We first solve a stochastic state-feedback problem and we compare our results with those obtained using the method of [109] which is not specially developed for white noise uncertainty. In the second example we solve a stochastic H_∞-filtering problem, again using both methods.

7.9.1 Example 1: The State-feedback Case

We consider the system (7.1)

$$x_{k+1} = \begin{bmatrix} 0.1 & 0.6 + a_k \\ -1.0 & -0.5 \end{bmatrix} x_k + \begin{bmatrix} -0.225 \\ 0.45 \end{bmatrix} w_k + \begin{bmatrix} 0.04 + b_k \\ 0.05 \end{bmatrix} u_k,$$

$$z_k = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k$$

where $\{a_k\}$ and $\{b_k\}$ are zero-mean independent white noise Gaussian sequences with standard deviation of 0.63 and 0.04 respectively. Using the theory of Section 2.4 we have $D = [I \quad 0]^T [0 \quad 0.63]$ and $G = [0.04 \quad 0]^T$. Considering (7.34) for a large N and Q_N that tends to zero, we obtain that $\gamma = 1.74$ is close to the minimal possible level of attenuation and that a convergence of Q_k of (7.34) is obtained, up to a tolerance of 10^{-6}, after 80 steps. We notice that here (A, L) is observable. The resulting feedback gain is found by (7.35) to be $K = [9.73 \quad 1.45]$. In order to compare our results with those obtained by the method of [109] we choose the uncertainty interval in [109] to be one that provides an attenuation level similar to our result. For the latter comparison we choose, for example $a_k \in [-0.63 \quad 0.63]$ and $b_k \in [-0.01 \quad 0.01]$. We note that this choice allows for 32% of the values of a_k and 80% of b_k of our stochastic case to lie outside the above intervals. Using the method of [109] we search for one free parameter ϵ and we obtain a near minimum value of $\gamma = 1.67$ for $\epsilon = 4$. The resulting K is then $[11.12 \quad 0.152]$.

The two designs are compared first for the uncertainty intervals on which the design method of [109] was based. We find that while the latter design satisfies the attenuation level of 1.67, our design achieves, at one extreme point, an attenuation level of 2.29. On the other hand, our method allows for high probabilities for the parameters a_k and b_k to lie outside the latter intervals. Checking, therefore, the two designs for uncertain intervals that correspond to one standard deviation in our stochastic parameters, we achieve a worst attenuation level of 103 in the design of [109] compared with a level of 2.29 that is achieved by our method. In both methods the minimal achievable γ for the nominal system (with no uncertainties) is 0.6.

7.9.2 Example 2: The Output-feedback Case

We consider the mean square stable system (7.11) and the objective function of (7.12) with the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.6 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 \\ 0.08 & 0.16 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 \\ .1 \end{bmatrix}, C_1 = \begin{bmatrix} -0.5 & 0.4 \\ 0 & 0 \end{bmatrix}$$

and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ where $F = 0$.

We apply the result of Remark 7.2 and Theorem 7.8 where we solve first for (7.48) and then for the LMIs of (7.74) and obtain for a near minimum of $\gamma = 7.18$ and $\tilde{\alpha} = 0.15$ the following results:

$$Q = \begin{bmatrix} 0.5950 & -0.2184 \\ -0.2184 & 1.0805 \end{bmatrix}, \ \hat{Q} = \begin{bmatrix} 8.2663 & -3.1948 \\ -3.1948 & 2.7360 \end{bmatrix},$$

$K_s = \begin{bmatrix} 0.7790 & -0.9454 \end{bmatrix}$, $K_o^T = \begin{bmatrix} 1.2273 & 2.0104 \end{bmatrix}$. The resulting closed-loop transfer function, from w_k to z_k, is $G_z = (-0.9446z + 0.9646)(z^2 - 0.5350z - 0.3589)^{-1}$. We note that for the state-feedback solution of this problem (see Remark 7.2) (where we assume that there is an access to the states of the system) one obtains a near minimum attenuation level of $\gamma = 1.02$. We note also that for the deterministic counterpart of this example (where $D = 0$) we obtain for the output-feedback case, a near minimum γ of 4.88.

7.10 Conclusions

The problem of H_∞-optimal control and filtering of discrete time linear systems with multiplicative stochastic uncertainties has been solved. The main contribution of this chapter is the method it introduces to solve the problems in the time-varying finite-horizon case where the stochastic uncertainties appear in the input and the output matrices of the state-space description of the system, as well as in the dynamic matrix. The theory developed offers a necessary and sufficient condition for solving the state-feedback problem.

Comparing our stochastic approach to the deterministic approach of [102], [109], where the uncertainties are modelled as exogenous signals, it is shown in the state-feedback control of example 1 that the stochastic approach achieves an improved level of attenuation and estimation namely, lower values of γ, where we take the uncertainty interval in [102] to be within a one standard deviation of the distribution of our parameters. It should be pointed out that in the approach of [102] the prescribed bound on the index of performance is guaranteed for all the parameters in the uncertainty interval. In our approach, on the other hand, there is a finite nonzero probability of violating the index of performance. There is a trade-off here between a better attenuation and estimation level and certainty in achieving these levels. In many practical problems the designer may prefer achieving better performance on the average, knowing that in a small proportions of the cases, the design may not achieve its goals. Another approach for the analysis of this trade-off is presented in [112] and [11] where the uncertainty in the system parameters was assumed to be uniformly distributed over a prescribed convex region.

Another reason for the better performance of the stochastic approach is the inherent over design entailed in the method of [102] that is notably accentuated, in the discrete-time case, when the system poles are close to the unit circle. In the latter case, the solution is not readily found and one gets considerably higher values of the performance bound, especially when multiple uncertain parameters are involved.

Finally, we note that the solution of the filtering problem of Section 7.5 can be partially applied to the solution of the state-multiplicative H_∞ prediction problem. In the latter problem, for a given measurement interval say, $[0\ k]\ \ k = 1, ..., N$, a state prediction is sought for the time instant $k+h$, where h is the given prediction length. A special case of the latter problem is one where $h = 1$. This problem is called one step ahead prediction (see [9] for the deterministic case). The case where $h > 1$ but reasonably small enough can be possibly solved by augmenting the system states to include delayed states and by applying the BRL of Section 7.3. However, for large values of h this solution may prove to be inefficient and time-consuming. Therefore, one can easily find the prediction by applying the autonomous open loop dynamics of the filtered estimate and then obtain the corresponding $H_\infty-$norm of the prediction error by applying the stochastic BRL (one may expect to obtain larger attenuation levels as h is increased).

8

Discrete-time Systems: General Filtering

8.1 Introduction

In Chapter 7 we introduced the solution to the output-feedback control problem for the time-varying, discrete-time systems with state-multiplicative noise, for both the finite and the infinite time cases [43],[39]. This solution is based on solving the filtering part using a Luenberger-type observer and applying the stochastic BRL. In this chapter, similarly to Chapter 3, we apply a general type filter for the solution of both: the H_∞ and the mixed H_2/H_∞ stationary filtering cases. We also allow for a deterministic polytopic-type uncertainties in the system matrices.

LMIs for uncertainties that lie in a convex-bounded domain (polytopic type) have been studied by [36]- [38],[37]. In [38], applying the standard BRL [102] on the uncertain system, a Riccati inequality is obtained whose solution, over the whole uncertainty polytope, guarantees the existence of a single filter that achieves the prescribed estimation accuracy. This Riccati inequality is expressed in terms of an LMI that is affine in the uncertain parameters. A single solution to the latter, for all the vertices of the uncertainty polytope, produces the required result [35]. The mixed H_2/H_∞ problem has also been solved in [36].

In this chapter we treat the stationary case where the stochastic uncertainty appears in the dynamic, input and measurement matrices, and where we allow for correlations between the uncertain parameters[46], [40]. In our treatment, we considered the case that commonly appears in practice, namely the one where the stochastic uncertainties appear in A, B_1 and C. This treatment can also be readily extended to the more general case where the stochastic uncertainties appear in all the system matrices. The results one achieves in the latter case are, however, too cumbersome and the insight that could have been gained would be lost. Unlike the solution of the filtering problem of Chapter 7, we do not require a Luenberger-type structure for the filter[43]. This problem has been partially treated in [26], however, the solution there does not allow for uncertainty in the measurement matrix and for correlations between

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the parameters, and it does not treat the mixed H_2/H_∞ estimation problem. We use the techniques of [74], as applied in the solution of the deterministic polytopic problem [36]. Necessary and sufficient conditions are derived for the existence of a solution in terms of LMIs. Our solution is based on the stationary stochastic BRL of Chapter 7 and [10] and it is extended to cases where also the deterministic part of the system matrices is unknown and assumed to lie in a convex bounded domain of a polytopic-type. Our theory is also applicable to the case where the covariance matrices of the stochastic parameters are not perfectly known and lie in a polytope.

We also solve the mixed H_2/H_∞ problem where, of all the filters that solve the stochastic H_∞ filtering problem, the one that minimizes an upper-bound on the estimation error variance is found. The applicability of our method is demonstrated in a gain-scheduled estimation example which is brought is the Application part. In this example we treat a guidance motivated tracking problem and compare the results with those obtained by the Kalman-filter.

8.2 Problem Formulation

We consider the following mean square stable system:

$$\begin{aligned} x_{k+1} &= (A + D\beta_k)x_k + (B_1 + G\bar{\beta}_k)w_k, \quad x_0 = 0 \\ y_k &= (C + F\zeta_k)x_k + D_{21}w_k \\ z_k &= Lx_k \end{aligned} \tag{8.1}$$

where $x_k \in \mathcal{R}^n$ is the system states, $y_k \in \mathcal{R}^r$ is the measurement, $w_k \in \tilde{l}_2([0,\infty);\mathcal{R}^q)$ is the exogenous disturbance signal, $z_k \in \mathcal{R}^m$ is the state combination to be estimated and where A, B_1, C, D, D_{21}, F, G and L are constant matrices with the appropriate dimensions. The variables $\{\beta_k\}$, $\{\bar{\beta}_k\}$ and $\{\zeta_k\}$ are standard random scalar sequences with zero mean that satisfy:

$$E\{\beta_k\beta_j\} = \delta_{kj}, \quad E\{\bar{\beta}_k\bar{\beta}_j\} = \delta_{kj}, \quad E\{\zeta_k\zeta_j\} = \delta_{kj}, \quad E\{\zeta_k\beta_j\} = \alpha_k\delta_{kj},$$

$$|\alpha_k| < 1, \quad \forall k, j \quad \geq 0.$$

and where $\{\bar{\beta}_k\}$ is uncorrelated with $\{\beta_k\}$ and $\{\zeta_k\}$. The system model of (8.1) represents practical situations of systems with multiplicative noise. It could be readily generalized to cases with vector valued multiplicative noise at the expense of more complicated expressions.

We consider the following asymptotically stable filter for the estimation of z_k:

$$\begin{aligned} \hat{x}_{k+1} &= A_f\hat{x}_k + B_f y_k, \quad \hat{x}_0 = 0 \\ \hat{z}_k &= C_f\hat{x}_k. \end{aligned} \tag{8.2}$$

Denoting

$$e_k = x_k - \hat{x}_k, \quad \xi_k^T = col\{x_k,\ \hat{x}_k\} \quad \text{and} \quad \tilde{z}_k = z_k - \hat{z}_k, \tag{8.3}$$

we define, for a given scalar $\gamma > 0$, the following performance index

$$J_S \stackrel{\triangle}{=} ||\tilde{z}||_{\tilde{l}_2}^2 - \gamma^2 ||w_k||_{\tilde{l}_2}^2. \tag{8.4}$$

The problems addressed in this chapter are :

i) Stochastic H_∞ filtering problem : Given $\gamma > 0$, find an asymptotically stable linear filter of the form (8.2) that leads to an estimation error $\tilde{z}_k$ for which J_S of (8.4) is negative for all nonzero $w_k \in \tilde{l}_2([0,\infty);\mathcal{R}^q)$.

ii) Stochastic mixed H_2/H_∞ filtering problem : Of all the asymptotically stable filters that solve problem (i), find the one that minimizes an upper-bound on the estimation error variance:

$$\lim_{k\to\infty} \underset{w,\beta,\zeta,\bar{\beta}}{E} \left\{\tilde{z}_k^T \tilde{z}_k\right\},$$

where $\{w_k\}$ is assumed to be a zero-mean, standard, white-noise sequence.

8.3 A BRL for Systems with Stochastic Uncertainty

We first bring, for convenience, the lemma that is proven in Section 7.3.1 (which is similar to the one derived in [10]) for the following mean square stable system:

$$\begin{aligned} x_{k+1} &= (A + D_1\beta_k + D_2\zeta_k)x_k + (B + G\bar{\beta}_k)w_k, \\ z_k &= Lx_k \end{aligned} \tag{8.5}$$

where the scalar sequences $\{\bar{\beta}_k\}$, $\{\beta_k\}$, $\{\zeta_k\}$ and the exogenous disturbance $\{w_k\}$ are defined above. Considering the cost function

$$\hat{J} = ||z_k||_{\tilde{l}_2}^2 - \gamma^2 ||w_k||_{\tilde{l}_2}^2$$

and applying the arguments of [10] the following holds:

Lemma 8.1. *[43], [10]* *For any $\gamma > 0$ the following statements are equivalent:*
i) *The system of (8.5) is mean square stable and $\hat{J}$ is negative for all nonzero $w_k \in \tilde{l}_2([0,\infty);\mathcal{R}^q)$.*
ii) *There exists $Q > 0$ that satisfies the following inequality*

$$\begin{aligned} &-Q + A^TQA + A^TQB\Theta^{-1}B^TQA + L^TL + D_1^TQD_1 + D_2^TQD_2 \\ &+\alpha[D_1^TQD_2 + D_2^TQD_1] < 0 \end{aligned} \tag{8.6}$$

and also satisfies $\Theta > 0$, where $\Theta \stackrel{\triangle}{=} \gamma^2 I_q - B^TQB - G^TQG$.

8.4 Stochastic H_∞ Filtering

Problem (i) is solved by applying Lemma 8.1. Considering the system of (8.1) and the definitions of (8.3) we obtain:

$$
\begin{aligned}
\xi_{k+1} &= [\tilde{A} + \tilde{D}_1\beta_k + \tilde{D}_2\zeta_k]\xi_k + [\tilde{B} + \tilde{G}\bar{\beta}_k]w_k \\
\tilde{z}_k &= \tilde{C}\xi_k
\end{aligned}
\tag{8.7}
$$

where

$$
\begin{aligned}
&\tilde{A} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix}, \ \tilde{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}, \ \tilde{D}_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \\
&\tilde{D}_2 = \begin{bmatrix} 0 & 0 \\ B_f F & 0 \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} L & -C_f \end{bmatrix}.
\end{aligned}
\tag{8.8}
$$

We arrive at the following result:

Theorem 8.1. *Consider the system of (8.7). Given $\gamma > 0$, the following hold:*
i) A necessary and sufficient condition for J_S of (8.4) to be negative for all nonzero $w_k \in \tilde{l}_2([0,\infty);\mathcal{R}^q)$, is that there exist $R = R^T \in \mathcal{R}^{n\times n}$, $W = W^T \in \mathcal{R}^{n\times n}$, $Z \in \mathcal{R}^{n\times r}$, $S \in \mathcal{R}^{n\times n}$ and $T \in \mathcal{R}^{m\times n}$, such that

$$
\Sigma(R, W, Z, S, T, \gamma^2) < 0 \tag{8.9}
$$

where

$$
\Sigma(R, W, Z, S, T, \gamma^2) \triangleq \left[\begin{matrix}
-R & 0 & 0 & 0 \\
 & -W & 0 & 0 \\
 & * & -R & 0 \\
 & * & * & -W \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & *
\end{matrix}\right.
$$

$$\left.\begin{array}{ccccccc}
0 & 0 & 0 & -RA & 0 & -RB_1 & 0 \\
0 & 0 & 0 & -WA - ZC - S & S & -WB_1 - ZD_{21} & 0 \\
0 & 0 & 0 & -RD & 0 & 0 & 0 \\
0 & 0 & 0 & -WD - \alpha ZF & 0 & 0 & 0 \\
-W & 0 & 0 & -\bar{\alpha} ZF & 0 & 0 & 0 \\
* & -R & 0 & 0 & 0 & -RG & 0 \\
* & * & -W & 0 & 0 & -WG & 0 \\
* & * & * & -R & 0 & 0 & T^T - L^T \\
* & * & * & * & -W & 0 & -T^T \\
* & * & * & * & * & -\gamma^2 I_q & 0 \\
* & * & * & * & * & * & -I_m
\end{array}\right] \qquad (8.10)$$

and where

$$\Sigma(8,2) \stackrel{\Delta}{=} -A^T W - C^T Z^T - S^T \quad and \quad \Sigma(8,4) = -D^T W - \alpha F^T Z^T.$$

ii) If (8.10) is satisfied, a mean square stabilizing filter in the form of (8.2) that achieves the negative J_S is given by:

$$A_f = -W^{-1} S, \quad B_f = -W^{-1} Z \quad and \quad C_f = T. \qquad (8.11)$$

Proof: i) The assertion that J_S is negative for all nonzero $w_k \in \tilde{l}_2([0,\infty);\mathcal{R}^q)$ is equivalent, by Lemma 8.1, to the solvability of the following Riccati inequality

$$-Q + \tilde{A}^T Q \tilde{A} + \tilde{A}^T Q \tilde{B} \tilde{\Theta}^{-1} \tilde{B}^T Q \tilde{A} + \tilde{C}^T \tilde{C} + \tilde{D}_1^T Q \tilde{D}_1 + \tilde{D}_2^T Q \tilde{D}_2$$
$$+\alpha[\tilde{D}_2^T Q \tilde{D}_1 + \tilde{D}_1^T Q \tilde{D}_2] < 0,$$

and where

$$\tilde{\Theta} \stackrel{\triangle}{=} \gamma^2 I_q - \tilde{B}^T Q \tilde{B} - \tilde{G}^T Q \tilde{G} > 0. \qquad (8.12)$$

Lemma 8.1 can be applied to the system of (8.7) since it is composed of the mean square system of (8.1) and the asymptotically stable filter of (8.2). Denoting $\bar{\alpha} \stackrel{\triangle}{=} (1-\alpha^2)^{0.5}$, the inequality of (8.12) is equivalent to the following.

$$-Q + \tilde{A}^T Q \tilde{A} + \tilde{A}^T Q \tilde{B} \tilde{\Theta}^{-1} \tilde{B}^T Q \tilde{A} + \tilde{C}^T \tilde{C} + (\tilde{D}_1 + \alpha \tilde{D}_2)^T Q (\tilde{D}_1 + \alpha \tilde{D}_2)$$
$$+\bar{\alpha}^2 \tilde{D}_2^T Q \tilde{D}_2 < 0, \qquad (8.13)$$

since

$$\tilde{D}_1^T Q \tilde{D}_1 + \tilde{D}_2^T Q \tilde{D}_2 + \alpha \tilde{D}_2^T Q \tilde{D}_1 + \alpha \tilde{D}_1^T Q \tilde{D}_2$$
$$= (\tilde{D}_1 + \alpha \tilde{D}_2)^T Q (\tilde{D}_1 + \alpha \tilde{D}_2) + \bar{\alpha}^2 \tilde{D}_2^T Q \tilde{D}_2.$$

Applying Schur's complements, (8.13) can be readily rearranged into the following LMI:

$$\hat{\Gamma}(Q) \triangleq \begin{bmatrix} -Q^{-1} & 0 & 0 & 0 & \tilde{A} & \tilde{B} & 0 \\ * & -Q^{-1} & 0 & 0 & (\tilde{D}_1 + \alpha \tilde{D}_2) & 0 & 0 \\ * & * & -Q^{-1} & 0 & \bar{\alpha}\tilde{D}_2 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & \tilde{G} & 0 \\ * & * & * & * & -Q & 0 & \tilde{C}^T \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -I_m \end{bmatrix} < 0, \quad (8.14)$$

and the negativity of J_S is thus guaranteed iff there exists $Q > 0$ that satisfies the (8.14). Following [38], we partition Q and Q^{-1} as follows:

$$Q \triangleq \begin{bmatrix} X & M \\ M^T & U \end{bmatrix} \quad \text{and} \quad Q^{-1} \triangleq \begin{bmatrix} Y & N \\ N^T & V \end{bmatrix},$$

where we require that

$$X > Y^{-1}. \quad (8.15)$$

The latter inequality stems from the fact that $\begin{bmatrix} Q & I_{2n} \\ I_{2n} & Q^{-1} \end{bmatrix} \geq 0$, which leads to $X \geq Y^{-1}$. Requiring, however, the filter of (8.2) to be of order n , a strict inequality is required in (8.15) (see [35], page 428 for the discrete-time case). We also note that $I - XY = MN^T$ is of rank n.

Defining:

$$J \triangleq \begin{bmatrix} Y & I_n \\ N^T & 0 \end{bmatrix} \quad \text{and} \quad \tilde{J} \triangleq diag\left[QJ, QJ, QJ, QJ, J, I\ I\right],$$

we pre- and post-multiply (8.14) by $\tilde{J}^T$ and $\tilde{J}$, respectively. Substituting for the matrices of (8.8a-f) in (8.14) and carrying out the various multiplications in (8.14) we obtain:

$$
\left[\begin{array}{cccccccccccc}
-Y & -I_n & 0 & 0 & 0 & 0 & 0 & 0 & AY & A & B_1 & 0 \\
* & -X & 0 & 0 & 0 & 0 & 0 & 0 & XAY+ZCY+\hat{Z} & XA+ZC & XB_1+ZD_{21} & 0 \\
* & * & -Y & -I_n & 0 & 0 & 0 & 0 & DY & D & 0 & 0 \\
* & * & * & -X & 0 & 0 & 0 & 0 & XDY+\alpha ZFY & XD+\alpha ZF & 0 & 0 \\
* & * & * & * & -Y & -I_n & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -X & 0 & 0 & \bar{\alpha} ZFY & \bar{\alpha} ZF & 0 & 0 \\
* & * & * & * & * & * & -Y & -I_n & 0 & 0 & G & 0 \\
* & * & * & * & * & * & * & -X & 0 & 0 & XG & 0 \\
* & * & * & * & * & * & * & * & -Y & -I_n & 0 & YL^T-\hat{Z}^T \\
* & * & * & * & * & * & * & * & * & -X & 0 & L^T \\
* & * & * & * & * & * & * & * & * & * & -\gamma^2 I_q & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -I_m
\end{array}\right] < 0,
$$

$$
X > Y^{-1} > 0, \tag{8.16}
$$

where

$$
Z \triangleq MB_f, \quad \tilde{Z} \triangleq C_f N^T \quad \text{and} \quad \hat{Z} \triangleq MA_f N^T. \tag{8.17}
$$

Pre- and post-multiplying (8.16) by $\varUpsilon$ and $\varUpsilon^T$, respectively, where

$$
\varUpsilon \triangleq diag\{\begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}, I_q, I_m\},
$$

and where we denote $R \triangleq Y^{-1}$, we obtain, defining

$$S \stackrel{\triangle}{=} \hat{Z}R \quad \text{and} \quad T \stackrel{\triangle}{=} \tilde{Z}R \tag{8.18}$$

the requirement of (8.9), where we replace $X - R$ by W and multiply the resulting inequality by -1.
ii) If a solution to (8.9) exists it follows from (8.17) that

$$A_f = M^{-1}\hat{Z}N^{-T}, \quad B_f = M^{-1}Z \quad \text{and} \quad C_f = \tilde{Z}N^{-T}. \tag{8.19}$$

These results require the derivation of M, $\hat{Z}$, N and $\tilde{Z}$. We show next that a filter with the same transference as the one achieved by (8.2) and (8.19), can be obtained using (8.11). Denoting the transfer function matrix of the filter of (8.2) by $H_{\hat{z}y}$ we find that :

$$H_{\hat{z}y}(\rho) = \tilde{Z}N^{-T}(\rho I - M^{-1}\hat{Z}N^{-T})^{-1}M^{-1}Z,$$

where ρ is the Z-transform variable. The latter equation is similar to :

$$H_{\hat{z}y}(\rho) = \tilde{Z}(\rho MN^T - \hat{Z})^{-1}Z = \tilde{Z}[\rho(I - XY) - \hat{Z}]^{-1}Z,$$

and (8.11) follows using (8.18).

The fact that the filter of (8.2) stabilizes the filtering error (namely, $\{\tilde{A}, \tilde{D}_1, \tilde{D}_2, \alpha\}$ defines a mean square stable evolution) is readily implied by Lemma 8.1.

□

8.4.1 The Polytopic Case

Due to the affinity of Σ of (8.9) in A, B_1, C and D_{21}, the result of Theorem 8.1 can be easily extended to the case where these matrices lie in convex bounded domain. In this case, it is required that (8.9) holds for all the vertices of the uncertain polytope for a single quintuple (R, S, Z, T, W). We note that the system should be quadratically stable over the polytope [38].

Assuming that $A, B_1, C,$ and D_{21} lie in the following uncertainty polytope

$$\begin{aligned}\bar{\Omega} &\stackrel{\triangle}{=} \{(A, B_1, C, D_{21}) | (A, B_1, C, D_{21}) \\ &= \textstyle\sum_{i=1}^{l} \tau_i (A_i, B_{1i}, C_i, D_{21,i}); \tau_i \geq 0; \sum_{i=1}^{l} \tau_i = 1\}.\end{aligned} \tag{8.20}$$

and denoting the set of the l vertices of this polytope by $\bar{\Psi}$ we obtain the following result:

Corollary 8.1. *Consider the system of (8.1) and (8.2) and a given* $\gamma > 0$. *The performance index of (8.4) is negative for any nonzero* $w_k \in \tilde{l}_2([0, \infty); \mathcal{R}^q)$ *and for any* $(A, B_1, C, D_{21}) \in \bar{\Omega}$ *if (8.9) is satisfied for all the vertices in* $\bar{\Psi}$ *by a single* (R, Z, S, T, W). *In the latter case, the filter matrices are given by (8.11)*

Remark 8.1. In the above, the same matrix solution is considered (i.e Q in (8.14)). This is done in order to keep the convexity of the resulting LMI. However, taking the same matrix solution for all the vertices leads to an overdesign in the sense that in cases where the system of (8.1) is not quadratically stable, no solution can be reached. Also, in the case where a solution is obtained using the same matrix, a higher minimum γ is obtained. An improved result for the above polytopic filter may be achieved by using a less conservative stochastic version of the BRL of [89], where for each vertex a solution is obtained on a basis of a separate Lyapunov function, rather than applying the known conservative BRL where the solution is based on a single Lyapunov function [39].

8.5 Robust Mixed Stochastic H_2/H_∞ Filtering

The mixed stochastic H_2/H_∞ filter design is achieved by considering the filters that satisfy the H_∞ requirement and finding the one that minimizes an upper-bound on the estimation error variance when it is assumed that the sequence $\{w_k\}$ is zero mean, standard white noise that is uncorrelated with $\beta, \bar{\beta}$, and ζ. The latter is described by the following H_2 objective function :

$$J_2 = \lim_{k\to\infty} \underset{w,\beta,\bar{\beta},\zeta}{E} \left\{\tilde{z}_k^T \tilde{z}_k\right\}.$$

Denoting $\bar{P} \stackrel{\triangle}{=} \lim_{k\to\infty} \underset{w,\beta,\bar{\beta},\zeta}{E} \left\{\xi_k \xi_k^T\right\}$, we readily find that $J_2 = Tr\{\tilde{C}\bar{P}\tilde{C}^T\}$ where $\bar{P} = \lim_{k\to\infty} P_k$ and

$$-P_{k+1} + \tilde{A}P_k\tilde{A}^T + \tilde{D}_1P_k\tilde{D}_1^T + \tilde{D}_2P_k\tilde{D}_2^T + \alpha(\tilde{D}_1P_k\tilde{D}_2^T + \tilde{D}_2P_k\tilde{D}_1^T) + \tilde{B}\tilde{B}^T = 0. \tag{8.21}$$

The latter equation is readily achieved substituting for $\tilde{z}_{k+1}$ in J_2, using (8.7) and taking the expectation with respect to the stochastic parameters. We are interested in deriving the corresponding dual observability-type result [116], taking into account the stochastic nature of $\{\beta_k\}, \{\zeta_k\}, \{\bar{\beta}_k\}$. Considering the following recursion

$$\tilde{Q}_k = \tilde{A}^T\tilde{Q}_{k+1}\tilde{A} + \tilde{D}_1^T\tilde{Q}_{k+1}\tilde{D}_1 + \tilde{D}_2^T\tilde{Q}_{k+1}\tilde{D}_2 + \alpha(\tilde{D}_1^T\tilde{Q}_{k+1}\tilde{D}_2 + \tilde{D}_2^T\tilde{Q}_{k+1}\tilde{D}_1) + \tilde{C}^T\tilde{C},$$

we obtain :

$$Tr\{P_{k+1}\tilde{Q}_{k+1} - P_k\tilde{Q}_k\} = Tr\{[\tilde{A}P_k\tilde{A}^T + \tilde{D}_1P_k\tilde{D}_1^T + \tilde{D}_2P_k\tilde{D}_2^T + \alpha(\tilde{D}_1P_k\tilde{D}_2^T + \tilde{D}_2P_k\tilde{D}_1^T) + \tilde{B}\tilde{B}^T]\tilde{Q}_{k+1}\}$$

$$-Tr\{P_k[\tilde{A}^T\tilde{Q}_{k+1}\tilde{A}+\tilde{D}_1^T\tilde{Q}_{k+1}\tilde{D}_1+\tilde{D}_2^T\tilde{Q}_{k+1}\tilde{D}_2+\alpha(\tilde{D}_1^T\tilde{Q}_{k+1}\tilde{D}_2+\tilde{D}_2^T\tilde{Q}_{k+1}\tilde{D}_1)$$
$$+\tilde{C}^T\tilde{C}]\}.$$

Since

$$\lim_{k\to\infty} Tr\{P_{k+1}\tilde{Q}_{k+1} - P_k\tilde{Q}_k\} = 0$$

and $Tr\{\alpha\beta\} = Tr\{\beta\alpha\}$ it follows that $Tr\{\tilde{C}\bar{P}\tilde{C}^T\} = Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}$, where $\tilde{Q} = \lim_{k\to\infty}\tilde{Q}_k$.
Defining:

$$\Gamma(\tilde{Q})=-\tilde{Q}+\tilde{A}^T\tilde{Q}\tilde{A}+\tilde{D}_1^T\tilde{Q}\tilde{D}_1+\tilde{D}_2^T\tilde{Q}\tilde{D}_2+\alpha(\tilde{D}_1^T\tilde{Q}\tilde{D}_2+\tilde{D}_2^T\tilde{Q}\tilde{D}_1)+\tilde{C}^T\tilde{C}, \tag{8.22}$$

we denote the following set

$$\Phi \stackrel{\triangle}{=} \{\hat{Q}|\Gamma(\hat{Q}) \le 0 \quad ; \quad \hat{Q} > 0\}.$$

We also consider $\hat{\Gamma}(\hat{Q},\Sigma) \stackrel{\triangle}{=} \Gamma(\hat{Q}) + \Sigma$ for some $0 \le \Sigma \in \mathcal{R}^{n\times n}$. The monotonicity, with respect to Σ, of the solution $\hat{Q}$ to the equation

$$\hat{\Gamma}(\hat{Q},\Sigma) = 0, \tag{8.23}$$

implies, as in Lemma 3.2, that the solution $\tilde{Q}$ for (8.22) which is also the solution for (8.23) for $\Sigma = 0$, if it exists, is less than or equal, in the matrix inequality sense, to all other solutions of (8.23). We obtain

$$J_B = Tr\{\tilde{B}^T\hat{Q}\tilde{B}\} \ge Tr\{\tilde{B}^T\tilde{Q}\tilde{B}\}, \quad \forall\hat{Q}\in\Phi. \tag{8.24}$$

To solve the stochastic mixed H_2/H_∞ problem we assume that (8.13) has a solution. We define the set of all the solutions to (8.13) by Φ_Q, and we seek to minimize J_B over $\Phi\cap\Phi_Q$. Namely, we consider the following LMI:

$$\tilde{\Gamma}(\bar{Q},H) \stackrel{\triangle}{=} \begin{bmatrix} H & -\tilde{B}^T\bar{Q} \\ -\bar{Q}\tilde{B} & \bar{Q} \end{bmatrix} > 0, \quad \bar{Q}\in\Phi\cap\Phi_Q \tag{8.25}$$

and we want to find $\bar{Q}$ and H that minimize

$$J_\tau = Tr\{H\}. \tag{8.26}$$

It follows from (8.13) that $\Gamma(Q) < -\tilde{A}^TQ\tilde{B}\tilde{\Theta}^{-1}\tilde{B}^TQ\tilde{A}$ and thus $\Phi_Q \subset \Phi$. We are, therefore, looking for Q and H that satisfy (8.14) and $\tilde{\Gamma}(Q,H) > 0$ so that $Tr(H)$ is minimized.

Notice that the matrix built from the first and sixth column and row blocks in (8.14) resembles $\tilde{\Gamma}$. Hence Q of (8.13) satisfies also (8.25) for $H = \gamma^2 I_q$. This is in accordance with the well known fact that the solution to the H_∞ problem is an upper-bound to the solution of the corresponding H_2 problem [6]. We are clearly looking for a tighter bound on J_B.

The minimization of (8.26) can be put in an LMI form that is affine in B_f, by pre- and post-multiplying (8.25) by $diag\{I, J^T\}$ and $diag\{I, J\}$, respectively, substituting for $\tilde{B}$ (using (8.8) and (8.19)) and pre- and post-multiplying the result by $\bar{\Lambda}$ and $\bar{\Lambda}^T$, respectively, where :

$$\bar{\Lambda} \stackrel{\triangle}{=} diag\{I, \begin{bmatrix} R & 0 \\ -R & I_n \end{bmatrix}\}.$$

We obtain the following result :

Theorem 8.2. *Consider the system of (8.7) and (8.4) and $\gamma > 0$. A filter that yields $J_S < 0$ for all nonzero $w_k \in \tilde{l}_2([0, \infty); \mathcal{R}^q)$ and minimizes (8.24) is obtained if there exists a solution (R, S, Z, T, W, H) to (8.9). The H_2-norm minimizing filter is obtained by simultaneously solving for (8.9) and*

$$\tilde{\Upsilon} \stackrel{\triangle}{=} \begin{bmatrix} H & -B_1^T R & -B_1^T W - D_{21}^T Z^T \\ -RB_1 & R & 0 \\ -WB_1 - ZD_{21} & 0 & W \end{bmatrix} > 0$$

and minimizing (8.26). The filter matrices are then given by (8.11)

Remark 8.2. We note that the requirement for a simultaneous solution of (8.9) and $\tilde{\Upsilon}$ does not impose any special difficulty since both LMIs are affine with respect to the matrix variables. Their simultaneous solution can be easily obtained using any standard LMI solver.

8.6 Conclusions

In this chapter we solve the problem of stationary stochastic H_∞-filtering of discrete-time linear systems using LMI techniques [46], [40]. This problem was treated before in [43],[39] by restricting the filter to be of the Luenberger type. Using a Riccati recursion, the solution was obtained there only if in addition to the H_∞ requirement, an upper-bound on the covariance of the estimation error is minimized in each instant. The solution of the present chapter does not depend on the latter minimization and is not restricted to a specific structure of the filter.

Using the LMI approach, the conditions for the existence of a solution to the problem are obtained in term of LMIs that are affine in the system and the filter parameters. This affinity allows also the consideration of deterministic uncertainty in the system, when the deterministic part of the system matrices lie in a given polytopic type domain. Our solution entails an over-design that stems from the quadratic stability nature of the solution. Under the requirement for this type of stability, the conditions we obtained for the existence of a solution to the problem are both necessary and sufficient.

It may be argued that the solution of the estimation problem is a special case of the solution of the general output-feedback problem with state-multiplicative noise [10]. The latter, however, is confined to the case where the measurement is free of multiplicative noise . The theory we developed in this chapter allows for multiplicative uncertainties in the measurement matrix. As such it enables the design of gain scheduled filters for estimation problems with noisy measurement matrices. This possibility is illustrated via the guidance-motivated tracking example in which, unlike the standard Kalman filter, the multiplicative noise affects the filter gains. Moreover, these gains are scheduled according to the range measurements. Using the Kalman filter one can either ignore the multiplicative noise, or introduce an additional measurement corresponding to the range. In the latter case, a non-linear filtering scheme (e.g. extended kalman filter) can be applied. Our solution, which avoids this nonlinearity by gain scheduling, may be useful also in other similar applications where the nonlinear terms in the measurement can be used for gain-scheduling at the cost of introducing multiplicative noise.

9

Discrete-time Systems: Tracking Control

9.1 Introduction

A method for solving the deterministic discrete-time tracking problems with preview has been introduced in [17]. Similarly to the deterministic continuous-time tracking control of [98], this method processes the information that is gathered on the reference during the system operation and by applying the game-theory approach it derives the optimal tracking strategy.

An important extension of the above mentioned works on H_∞ tracking with preview is to allow for uncertainties in the plant parameters. Using the method of [17], the tracking problem was solved for discrete-time systems with norm-bounded uncertainties in [18]. The latter solution entails a significant overdesign, even in the case where the uncertain parameters in the plant state-space model are uniformly distributed in the assumed interval of uncertainty. It cannot, therefore, cope successfully with plant parameters that are non-uniformly distributed around given average values. One such example is encountered when the plant parameters are subject to white noise uncertainties resulting in state multiplicative noise.

In this chapter, we extend the work of [17] to systems with state-multiplicative noise [53], [42]. We treat the case where correlated parameter uncertainties appear in both the system dynamics and the input matrices. An optimal state-feedback tracking strategy is derived, in Section 9.3, which minimizes the expected value of the standard H_∞ performance index with respect to the unknown parameters, for three tracking patterns of the reference signal. In the finite horizon case, a game theory approach is applied where, given a specific preview pattern, the controller plays against nature which chooses the initial condition and the energy-bounded disturbance. In this case the optimal strategies of both nature and the designer are found by a achieving a saddle-point equilibrium. In the stationary case, however, a state-feedback control strategy is obtained in Section 9.3.2 for which the index of performance is less than or equal to a certain cost. In this case, the problem is solved by two

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easily implementable linear matrix inequalities. For both, the finite and the infinite horizon cases, necessary and sufficient conditions are obtained.

In Section 9.4 we solve the output-feedback control problem where we allow for a state-multiplicative noise in the measurement matrix. We first introduce an auxiliary stochastic BRL for systems that contain, in addition to the standard BRL, a reference signal in the system dynamics. The BRL is solved as a max-min problem and results in a modified Riccati equation.

The output-feedback tracking control problem is solved via a max-min strategy arguments in Section 9.4.2, rather than a game theory approach (applied to the solution of the finite-horizon state-feedback control). Using the solution of the finite-horizon state-feedback, we re-formulate the problem to a filtering problem which we solve with the above auxiliary BRL.

The theory developed in this chapter [53], [42] is applied to two examples. The first is a stochastic version of the example in [17] and in the second example, which is brought in Chapter 11 of the Application part, we synthesize an optimal guidance law for the case of noisy measurement in the time to go of a guidance system. It is shown, in the latter example, that the effect of the noise in the measurements can be modeled as a multiplicative white noise. The theory of the present chapter is, therefore, applied to derive a guidance law with a very low sensitivity to this noise.

9.2 Problem Formulation

Given the following linear discrete time-varying system:

$$\begin{aligned} x_{k+1} &= (A_k + F_k v_k)x_k + (B_{2,k} + G_k \eta_k)u_k + B_{1,k} w_k + B_{3,k} r_k \\ y_k &= (C_{2,k} + D_k \zeta_k)x_k + D_{21,k} n_k \end{aligned} \tag{9.1}$$

where $x_k \in R^n$ is the state vector, $w_k \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$ is an exogenous disturbance, $y_k \in \mathcal{R}^z$ is the measurement, $r_k \in R^r$ is a measured reference signal, $u_k \in \tilde{l}^2([0, N-1]; \mathcal{R}^l)$ is the control input signal and x_0 is an unknown initial state and where $\{v_k\}$ and $\{\eta_k\}$ are standard random scalar white noise sequences with zero mean that satisfy:

$$E\{v_k v_j\} = \delta_{kj}, \quad E\{\eta_k \eta_j\} = \delta_{kj}, \quad E\{v_k \eta_j\} = \alpha_k \delta_{kj}, \ |\alpha_k| \leq 1, \tag{9.2}$$

where $\{\zeta_k\}$ satisfies $E\{\zeta_k \zeta_j\} = \delta_{kj}$ and is uncorrelated with $\{v_k\}$ and $\{\eta_k\}$. We denote

$$z_k = C_k x_k + D_{2,k} u_k + D_{3,k} r_k, \quad z_k \in R^q, \ \ k \in [0, N]. \tag{9.3}$$

We assume, for simplicity, that:

$$[C_k^T \quad D_{3,k}^T \quad D_{2,k}^T] D_{2,k} = [0 \quad 0 \quad \tilde{R}_k], \quad \tilde{R}_k > 0.$$

Remark 9.1. We note that the formulation of (9.3) is most general. It reduces to an objective that includes the tracking error and a penalty on the control effort. For example, taking $C_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D_{3,k} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $D_{2,k} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ implies that we minimize the sum of squares of $e_k = y_k - r_k$ and u_k, where $y_k = [0\ 1]x_k$. We also note that the orthogonality between $[C_k^T \quad D_{3,k}^T]$ and $D_{2,k}$ leads to quadratic costs in both the error defined by $C_k x_k + D_{3,k} r_k$ and the control effort. The assumption on $\tilde{R}_k > 0$ is required to simplify the derivations contained in this chapter. This assumption can be readily relaxed [57].

Our objective is to find a control law $\{u_k\}$ that minimizes the mean energy of $\{z_k\}$ with respect to v and η, for any given energy of $\{w_k\}$, by using the available knowledge on the reference signal, for the worst-case of the process disturbance and measurement noise $\{w_k\}, \{n_k\}$ and the initial condition x_0. We, therefore, consider, for a given scalar $\gamma > 0$ the following two problems:

9.2.1 State-feedback Tracking

We consider the system of (9.1) and (9.3) where we define the following performance index:

$$J_E(r_k, u_k, w_k, x_0) \stackrel{\triangle}{=} \underset{v,\eta}{E} \{\|C_N x_N + D_{3,N} r_N\|^2 + ||z_k||_2^2 - \gamma^2 [||w_k||_2^2]\}$$
$$-\gamma^2 x_0^T R^{-1} x_0, \quad R^{-1} \geq 0. \tag{9.4}$$

Similarly to [17] we consider three different tracking problems differing on the information pattern over $\{r_k\}$:

1) Stochastic H_∞-Tracking with full preview of $\{r_k\}$: The tracking signal is perfectly known for the interval $k \in [0, N]$.

2) Stochastic H_∞-Tracking with no preview of $\{r_k\}$: The tracking signal measured at time k is known for $i \leq k$.

3) Stochastic H_∞-Tracking with fixed-finite preview of $\{r_k\}$: At time k, r_i is known for $i \leq min(N, k + h)$ where h is the preview length.
In all three cases we seek a control law $\{u_k\}$ of the form

$$u_k = H_x x_k + H_r r_k \tag{9.5}$$

where H_x is a causal operation on $\{x_k\}$, $i \leq k$ and where the causality of H_r depends on the information pattern of the reference signal. The design objective is to minimize

$$\max J_E(r_k, u_k, w_k, x_0)\ \forall \{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p), \{u_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^l),\ x_o \in R^n$$

where for all of the three tracking problems we consider a related linear quadratic game in which the controller plays against nature by choosing x_0

and $\{w_k\}$. We, thus, consider the following game:

Find $\{w_k^*\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$, $\{u_k^*\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^l)$, and $x_0^* \in R^n$ that satisfies, given $\{r_k\} \in l_2[0, N]$, the following inequalities:

$$J_E(r_k, u_k^*, w_k, x_0) \le J_E(r_k, u_k^*, w_k^*, x_0^*) \le J_E(r_k, u_k, w_k^*, x_0^*).$$

9.2.2 Output-feedback Control Tracking

We consider the system of (9.1) and (9.3) with the following index of performance:

$$\begin{aligned}\tilde{J}_E(r_k, u_k, w_k, n_k, x_0) &\stackrel{\triangle}{=} \underset{v,\zeta}{E}\{\|C_N x_N + D_{3,N} r_N\|^2\} \\ + \underset{v,\zeta}{E}\{||z_k||_2^2 - \gamma^2[||w_k||_2^2 + ||n_k||_2^2]\} &- \gamma^2 x_0^T R^{-1} x_0, \qquad R^{-1} \ge 0.\end{aligned} \tag{9.6}$$

Similarly to the state-feedback problem of Section 9.2.1, we consider the above three tracking problems differing on the information pattern over $\{r_k\}$. We seek a controller of the form (9.5) where our design objective is to minimize

$$\begin{aligned}&\max \tilde{J}_E(r_k, u_k, w_k, n_k, x_0) \ \forall \{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p), \\ &\{n_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^z), \ \{u_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^l), \ x_o \in R^n,\end{aligned}$$

where for all of the three tracking problems we derive a controller $\{u_k\}$ which plays against it's adversaries $\{w_k\}, \{n_k\}$ and x_0.

9.3 The State-feedback Control Tracking

We consider the following Riccati difference equation:

$$\begin{aligned}&Q_k = A_k^T M_{k+1} A_k + C_k^T C_k + F_k^T Q_{k+1} F_k - (F_k^T Q_{k+1} G_k \alpha_k + A_k^T M_{k+1} B_{2,k}) \Phi_k^{-1} \\ &(G_k^T Q_{k+1} F_k \alpha_k + B_{2,k}^T M_{k+1} A_k), \\ &Q(N) = C_N^T C_N\end{aligned} \tag{9.7}$$

where

$$M_{k+1} \stackrel{\triangle}{=} Q_{k+1}[I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}$$

and

$$\Phi_k = B_{2,k}^T M_{k+1} B_{2,k} + G_k^T Q_{k+1} G_k + \tilde{R}_k.$$

The solution of the state-feedback tracking problem is obtained by the following :

Theorem 9.1. *Consider the system of (9.1) and J_E of (9.4). Given $\gamma > 0$, the state-feedback tracking game possesses a saddle-point equilibrium solution iff there exists $Q_i > 0, \forall i \in [0, N]$ that solves (9.7) and satisfies*

$$R_{k+1} > 0, \quad k \in [0 \quad N-1], \quad \gamma^2 R^{-1} - Q_0 > 0, \tag{9.8}$$

where we define

$$R_{k+1} \stackrel{\triangle}{=} \gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}.$$

When a solution exists, the saddle-point strategies are given by:

$$\begin{aligned} x_0^* &= (\gamma^2 R^{-1} - Q_0)^{-1}\theta_0 \\ w_k^* &= R_{k+1}^{-1} B_{1,k}^T [\theta_{k+1} + Q_{k+1}(A_k x_k + B_{2,k} u_k + B_{3,k} r_k)] \\ u_k^* &= -\Phi_k^{-1}(B_{2,k}^T M_{k+1}[A_k x_k + B_{3,k} r_k + Q_{k+1}^{-1}\theta_{k+1}^c] + \alpha_k G_k^T Q_{k+1} F_k x_k) \end{aligned} \tag{9.9}$$

where

$$\theta_{k+1}^c = [\theta_{k+1}]_+ \tag{9.10}$$

and where θ_k satisfies

$$\theta_k = \bar{A}_k^T \theta_{k+1} + \bar{B}_k r_k, \quad \theta_N = C_N^T D_{3,N} r_N, \tag{9.11}$$

with

$$\begin{aligned} \bar{A}_k &= Q_{k+1}^{-1}(M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T)^{-1}(A_k - \alpha_k B_{2,k} T_{k+1}^{-1} G_k^T Q_{k+1} F_k) \\ \bar{B}_k &= \bar{A}_k^T Q_{k+1} B_{3,k} + C_k^T D_{3,k}. \end{aligned}$$

where

$$T_{k+1} \stackrel{\triangle}{=} \tilde{R}_k + G_k^T Q_{k+1} G_k.$$

The game value is then given by:

$$\begin{aligned} & J_E(r_k, u_k^*, w_k^*, x_0^*) = \|[Q_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T - \gamma^{-2} B_{1,k} B_{1,k}^T]^{-\frac{1}{2}} \\ & (Q_{k+1}^{-1}\theta_{k+1} + B_{3,k} r_k)\|_2^2 - \|Q_{k+1}^{-\frac{1}{2}}\theta_{k+1}\|_2^2 + \|D_{3,k} r_k\|_2^2 + \|D_{3,N} r_N\|^2 \\ & + \theta_0^T (\gamma^2 R^{-1} - Q_0)^{-1}\theta_0. \end{aligned} \tag{9.12}$$

Proof: In order to provide a full proof to Theorem 9.1, we need a preliminary result which deals with the causality issue that is essential for the structure of the control sequence:
We consider the following cost function:

$$J_o = \sum_{k=0}^{N-1} ||u_k - \mu_k||^2$$

where μ_k is the output of a linear time-varying system with input $w_k \in Y_k$ where $Y_k = \{y_j, j \leq k, \ \ r_j, j \leq min(k+h, N-1)\}$ and where we note that for

$h = 0$, $\{u_k\}$ is obtained using the zero preview on $\{r_k\}$. A finite h corresponds to finite preview and an infinite h (or h larger than $N-1$) corresponds to a full preview of $\{r_k\}$. We further note that in our case $y_k = x_k$ (full state-feedback control). In order to obtain $\{u_k\}$, using the above description, we consider the following:

$$\mu_k = \sum_{j=0}^{k} \alpha_j y_j + \sum_{j=0}^{N-1} \beta_j r_j$$

where $\alpha_j,\ \beta_j$ are matrices of the appropriate dimensions. Consider the following realization of $\{u_k\}$:

$$u_k = \sum_{j=0}^{k} \bar{\alpha}_j y_j + \sum_{j=0}^{k_h} \bar{\beta}_j r_j,$$

where $k_h \stackrel{\triangle}{=} min(k+h,\ N-1)$. We arrive at the following lemma:

Lemma A1: *Consider the above* $J_o,\ \mu_k$ *and* u_k*. The matrix parameters* $\bar{\alpha}_j,\ \bar{\beta}_j$ *that minimize* J_o *are given by*

$$\bar{\alpha}_j = \alpha_j,\ \ \bar{\beta}_j = \beta_j.$$

Proof: We define

$$[r_i]_+ = \begin{Bmatrix} r_i \ i \le k_h \\ 0 \ \ i > k_h \end{Bmatrix}, \quad [r_i]_- = \begin{Bmatrix} 0 \ \ i \le k_h \\ r_i \ i > k_h \end{Bmatrix}. \tag{9.13}$$

Subtracting u_k from μ_k we obtain:

$$\mu_k - u_k = \sum_{j=0}^{k} (\alpha_j - \bar{\alpha}_j) y_j + \sum_{j=0}^{k_h} (\beta_j - \bar{\beta}_j) r_j + \sum_{j=k_h+1}^{N-1} \beta_j r_j$$

By padding by zeros $\alpha_j,\ \bar{\alpha}_j$ for $k+1 \le j \le k_h$, we obtain:

$$\mu_k - u_k = \sum_{j=0}^{k_h} \{(\alpha_j - \bar{\alpha}_j) y_j + (\beta_j - \bar{\beta}_j) r_j\} + \sum_{j=k_h+1}^{N-1} \beta_j r_j.$$

We further define

$$\rho_j^T \stackrel{\triangle}{=} [y_j^T \ \ r_j^T]^T, \quad \epsilon_j \stackrel{\triangle}{=} [\alpha_j - \bar{\alpha}_j \ \ \beta_j - \bar{\beta}_j], \quad \nu_j \stackrel{\triangle}{=} [0 \ \ \beta_j]$$

and consider $[\rho_i]_+$ and $[\rho_i]_-$ similarly to (9.13). We have:

$$\mu_k - u_k = \sum_{j=0}^{k_h} \epsilon_j \rho_j + \sum_{j=k_h+1}^{N-1} \nu_j \rho_j = \sum_{j=0}^{N-1} \epsilon_j [\rho_j]_+ + \sum_{j=0}^{N-1} \nu_j [\rho_j]_-.$$

The result of the lemma readily follows by calculating $(\mu_k - u_k)^T(\mu_k - u_k)$ and using the orthogonality of $[\rho_k]_+$ and $[\rho_k]_-$. We note that this lemma implies that the minimization of $J_o = \sum_{k=0}^{N-1} ||u_k - \mu_k||^2$ by $\{u_k\}$ is accomplished by just $u_k = [\mu_k]_+$, namely the optimal control $\{u_k\}$ is just the causal part of $\{\mu_k\}$.

The remaining part of the proof follows the standard line of applying a Lyapunov-type quadratic function in order to comply with the index of performance. This is usually done by using two completing to squares operations however, since the reference signal of r_k is introduced in the dynamics of (9.1), we apply a third completion to squares operation with the aid of the signal θ_{k+1}. This latter signal finally affects the controller design through it's causal part $[\theta_{k+1}]_+$. Defining

$$J_k = ||C_k x_k + D_{3,k} r_k||^2 + ||D_{2,k} u_k||^2 - \gamma^2 ||w_k||^2,$$

we consider (9.4) and obtain:

$$J_E(r, u, w, x_0) = -\gamma^2 x_0^T R^{-1} x_0 + \underset{v,\eta}{E}\{\|C_N x_N + D_{3,N} r_N\|^2\} + \sum_{k=0}^{N-1} \underset{v,\eta}{E}\{J_k\}.$$

Denoting

$$\phi_k = x_{k+1}^T Q_{k+1} x_{k+1} - x_k^T Q_k x_k,$$

and substituting (9.1) in the latter, we find that

$$\begin{aligned}
\phi_k &= [x_k^T (A_k + F_k v_k)^T + u_k^T (B_{2,k} + G_k \eta_k)^T + r_k^T B_{3,k}^T] \\
&\quad Q_{k+1}[(A_k + F_k v_k) x_k + (B_{2,k} + G_k \eta_k) u_k + B_{3,k} r_k] \\
&\quad +2[x_k^T (A_k + F_k v_k)^T + u_k^T (B_{2,k} + G_k \eta_k)^T + r_k^T B_{3,k}] Q_{k+1} B_{1,k} w_k \\
&\quad +w_k^T B_{1,k}^T Q_{k+1} B_{1,k} w_k - \|x_k\|_{Q_k}^2 + \|C_k x_k + D_{3,k} r_k\|^2 \\
&\quad -\gamma^2 w_k^T w_k + \gamma^2 w_k^T w_k + \|u_k\|_{\tilde{R}_k}^2 - z_k^T z_k \\
&= -w_k^T [\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}] w_k + 2[x_k^T (A_k + F_k v_k)^T + u_k^T (B_{2,k} + G_k \eta_k)^T \\
&\quad +r_k^T B_{3,k}^T] Q_{k+1} B_{1,k} w_k + u_k^T [\tilde{R}_k + \|B_{2,k} + G_k \eta_k)\|_{Q_{k+1}}^2] u_k \\
&\quad +2x_k^T [(A_k + F_k v_k)^T + r_k^T B_{3,k}^T] Q_{k+1} (B_{2,k} + G_k \eta_k) u_k \\
&\quad +x_k^T [(A_k + F_k v_k)^T Q_{k+1} (A_k + F_k v_k) + C_k^T C_k - Q_k] x_k - z_k^T z_k + \gamma^2 w_k^T w_k \\
&\quad +r_k^T (D_{3,k}^T D_{3,k} + B_{3,k}^T Q_{k+1} B_{3,k}) r_k + 2 r_k^T (B_{3,k}^T Q_{k+1} (A_k + F_k v_k) + D_{3,k}^T C_k) x_k.
\end{aligned}$$

Taking the expectation with respect to η_k and v_k we obtain:

$$\underset{v,\eta}{E}\{\phi_k\} = \underset{v,\eta}{E}\{-w_k^T [\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}] w_k + 2[x_k^T A_k^T Q_{k+1} B_{1,k}$$

$$+(u_k^T B_{2,k}^T + r_k^T B_{3,k}^T)Q_{k+1}B_{1,k}]w_k + u_k^T[\tilde{R}_k + B_{2,k}^T Q_{k+1}B_{2,k} + G_k^T Q_{k+1}G_k]u_k$$
$$+2(x^T[A_k^T Q_{k+1}B_{2,k} + \alpha_k F_k^T Q_{k+1}G_k] + r_k^T B_{3,k}Q_{k+1}B_{2,k})u_k$$
$$+x_k^T[A_k^T Q_{k+1}A_k + F_k^T Q_{k+1}F_k + C_k^T C_k - Q_k]x_k$$
$$+r_k^T(D_{3,k}^T D_{3,k} + B_{3,k}^T Q_{k+1}B_{3,k})r_k + 2r_k^T(B_{3,k}^T Q_{k+1}A_k + D_{3,k}^T C_k)x_k\} - \underset{v,\eta}{E}\{J_k\}$$

Note that first order terms in both η_k or v_k have vanished as a result of the expectation operation. Completing to squares for w_k we get:

$$\underset{v,\eta}{E}\{\phi_k\} =$$
$$\underset{v,\eta}{E}\{-(w_k - \tilde{w}_k)^T R_{k+1}(w_k - \tilde{w}_k) + u_k^T[\tilde{R}_k + B_{2,k}^T M_{k+1}B_{2,k} + G_k^T Q_{k+1}G_k]u_k$$
$$+2(x^T[A_k^T M_{k+1}B_{2,k} + \alpha_k F_k^T Q_{k+1}G_k] + r_k^T B_{3,k}M_{k+1}B_{2,k})u_k$$
$$+x_k^T[A_k^T M_{k+1}A_k + F_k^T Q_{k+1}F_k + C_k^T C_k - Q_k]x_k$$
$$+r_k^T[D_{3,k}^T D_{3,k} + B_{3,k}^T M_{k+1}B_{3,k}]r_k + 2r_k^T(B_{3,k}^T M_{k+1}A_k + D_{3,k}^T C_k)x_k\} - \underset{v,\eta}{E}\{J_k\}.$$

Completing next to squares for u_k we obtain:

$$\underset{v,\eta}{E}\{\phi_k\} = \underset{v,\eta}{E}\{-(w_k - \tilde{w}_k)^T R_{k+1}(w_k - \tilde{w}_k) + (u_k - \tilde{u}_k)^T \Phi_k(u_k - \tilde{u}_k)\} - \underset{v,\eta}{E}\{J_k\}$$
$$+ \underset{v,\eta}{E}\{x_k^T[A_k^T M_{k+1}A_k - (A_k^T M_{k+1}B_{2,k} + \alpha_k F_k^T Q_{k+1}G_k)\Phi_k^{-1}(B_{2,k}^T M_{k+1}A_k$$
$$+\alpha G_k^T Q_{k+1}F_k) + F_k^T Q_{k+1}F_k + C_k^T C_k - Q_k]x_k$$
$$+r_k^T[D_{3,k}^T D_{3,k} + B_{3,k}^T(M_{k+1}^{-1} + B_{2,k}T_{k+1}^{-1}B_{2,k}^T)^{-1}B_{3,k}]r_k + 2r_k^T(B_{3,k}^T(M_{k+1}^{-1}$$
$$+B_{2,k}T_{k+1}^{-1}B_{2,k}^T)^{-1}A_k + D_{3,k}^T C_k - \alpha_k B_{3,k}^T M_{k+1}B_{2,k}\Phi_k G_k^T Q_{k+1}F_k)x_k\}$$

where:

$$\begin{aligned} \tilde{u}_k &= u_k^*, \\ \tilde{w}_k &= w_k^*, \\ \theta_{k+1} &= \theta_{k+1}^c = 0 \end{aligned} \tag{9.14}$$

and where w_k^*, u_k^* are given in (9.9).

Note that following the above two completion to square operations, the mixed terms in r_k and x_k still remain. Similarly to [17], in order to get rid of these latter terms in r_k and x_k we first define the following:

$$\hat{w}_k \stackrel{\triangle}{=} w_k - \tilde{w}_k, \quad \hat{u}_k \stackrel{\triangle}{=} u_k - \tilde{u}_k.$$

Considering the above definition of $\hat{w}_k$ and $\hat{u}_k$ we readily re formulate the state equation of (9.1) in the following way:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{2,k}u_k + B_{1,k}\hat{w}_k + B_{1,k}\tilde{w}_k + B_{3,k}r_k + F_k v_k x_k + G_k\eta_k u_k \\ &= A_k x_k + B_{1,k}\hat{w}_k + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}(A_k x_k + B_{2,k}u_k + B_{3,k}r_k) \\ &\quad + B_{2,k}u_k + B_{3,k}r_k + F_k v_k x_k + G_k\eta_k u_k \end{aligned}$$

$$= [I + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}]A_k x_k + [I + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}]B_{2,k}u_k$$
$$+[I + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}]B_{3,k}r_k + B_{1,k}\hat{w}_k + F_k v_k x_k + G_k \eta_k u_k.$$

Now, considering the definition of R_{k+1} in Theorem 9.1 and using the identity

$$\alpha[I - \beta\alpha]^{-1} = [I - \alpha\beta]^{-1}\alpha$$

we obtain the following.

$$I + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1} = I + \gamma^{-2}B_{1,k}[I - \gamma^{-2}B_{1,k}^T Q_{k+1}B_{1,k}]^{-1}B_{1,k}^T Q_{k+1}$$
$$= I + \gamma^{-2}B_{1,k}B_k^T Q_{k+1}[I - \gamma^{-2}B_{1,k}B_k^T Q_{k+1}]^{-1} = [I + \gamma^{-2}B_{1,k}B_k^T Q_{k+1}]^{-1}$$
$$= Q_{k+1}^{-1}M_{k+1},$$

where the definition of M_{k+1} follows (9.7). Considering the latter and $u_k = \tilde{u}_k + \hat{u}_k$, where $\tilde{u}_k$ is given in (9.14), we obtain the following.

$$x_{k+1} = Q_{k+1}^{-1}M_{k+1}A_k x_k + Q_{k+1}^{-1}M_{k+1}B_{2,k}\hat{u}_k + B_{1,k}\hat{w}_k + Q_{k+1}^{-1}M_{k+1}B_{3,k}r_k$$
$$-Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_k^{-1}[B_{2,k}^T M_{k+1}(A_k x_k + B_{3,k}r_k) + \alpha_k G_k^T Q_{k+1}F_k x_k]$$
$$+F_k v_k x_k + G_k \eta_k u_k = [Q_{k+1}^{-1}M_{k+1}(I - B_{2,k}\Phi_k^{-1}B_{2,k}^T M_{k+1})A_k$$
$$-Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_k^{-1}\alpha_k G_k^T Q_{k+1}F_k]x_k + F_k v_k x_k + G_k \eta_k u_k$$
$$+Q_{k+1}^{-1}M_{k+1}(I - B_{2,k}\Phi_k^{-1}B_{2,k}^T M_{k+1})r_k + Q_{k+1}^{-1}M_{k+1}B_{2,k}\hat{u}_k + B_{1,k}\hat{w}_k.$$

Noting that:

$$\begin{aligned}
&M_{k+1}B_{2,k}\Phi_k^{-1}B_{2,k}^T M_{k+1}\\
&= M_{k+1}B_{2,k}[I + T_{k+1}^{-1}B_{2,k}^T M_{k+1}B_{2,k}]^{-1}T_{k+1}^{-1}B_{2,k}^T M_{k+1}\\
&= [I + M_{k+1}B_{2,k}T_{k+1}^{-1}B_{2,k}^T]^{-1}M_{k+1}B_{2,k}T_{k+1}^{-1}B_{2,k}^T M_{k+1}\\
&= M_{k+1} - (M_{k+1}^{-1} + B_{2,k}T_{k+1}^{-1}B_{2,k}^T)^{-1},
\end{aligned}$$

where we use the identity

$$[I + L]^{-1}L = I - [I + L]^{-1}$$

and where Φ_k is given right after (9.7), we obtain:

$$\begin{aligned}
&-Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_k^{-1}\alpha_k G_k^T Q_{k+1}F_k\\
&= -Q_{k+1}^{-1}M_{k+1}B_{2,k}[I + T_{k+1}^{-1}B_{2,k}^T M_{k+1}B_{2,k}]^{-1}T_{k+1}^{-1}\alpha_k G_k^T Q_{k+1}F_k\\
&= -Q_{k+1}^{-1}M_{k+1}[I + B_{2,k}T_{k+1}^{-1}B_{2,k}^T M_{k+1}]^{-1}B_{2,k}T_{k+1}^{-1}\alpha_k G_k^T Q_{k+1}F_k\\
&= -Q_{k+1}^{-1}[M_{k+1}^{-1} + B_{2,k}T_{k+1}^{-1}B_{2,k}^T]^{-1}B_{2,k}T_{k+1}^{-1}\alpha_k G_k^T Q_{k+1}F_k.
\end{aligned}$$

Considering the above derivations, the new state equation of (9.1) in terms of $\hat{u}_k$ and $\hat{w}_k$ and the stochastic terms becomes:

$$x_{k+1} = \bar{A}_k x_k + Q_{k+1}^{-1} M_{k+1} B_{2,k} \hat{u}_k + B_{1,k} \hat{w}_k + Q_{k+1}^{-1} M_{k+1}$$

$$(I - B_{2,k} \Phi_k^{-1} B_{2,k}^T M_{k+1}) B_{3,k} r_k + F_k v_k x_k + G_k \eta_k u_k,$$

where

$$\bar{A}_k = Q_{k+1}^{-1} [M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T]^{-1} (A_k - B_{2,k} T_{k+1}^{-1} \alpha_k G_k^T Q_{k+1} F_k).$$

The next step which is taken in order to get rid of the mixed terms in r_k and x_k in $\underset{v,\eta}{E}\{\phi_k\}$ is the addition of the auxiliary term of $2(\theta_{k+1}^T x_{k+1} - \theta_k^T x_k)$ to ϕ_k before calculating $\underset{v,\eta}{E}\{\phi_k\}$.
In terms of the above new state equation we obtain:

$$\begin{aligned} &\underset{v,\eta}{E}\left\{\theta_{k+1}^T x_{k+1}\right\} \\ &= \underset{v,\eta}{E}\{\theta_{k+1}^T [\bar{A}_k x_k + Q_{k+1}^{-1} M_{k+1} B_{2,k} \hat{u}_k + B_{1,k} \hat{w}_k + Q_{k+1}^{-1} M_{k+1} \\ &(I - B_{2,k} \Phi_k^{-1} B_{2,k}^T M_{k+1}) B_{3,k} r_k + F_k v_k x_k + G_k \eta_k u_k].\} \end{aligned}$$

Defining :

$$\phi_k^* = \phi_k + 2(\theta_{k+1}^T x_{k+1} - \theta_k^T x_k),$$

we thus add the following quantity to ϕ_k

$$\begin{aligned} &\underset{v,\eta}{E}\left\{2(\theta_{k+1}^T x_{k+1} - \theta_k^T x_k)\right\} = \underset{v,\eta}{E}\left\{2(\theta_{k+1}^T \bar{A}_k - \theta_k^T) x_k\right\} \\ &+ \underset{v,\eta}{E}\{2\theta_{k+1}^T [Q_{k+1}^{-1} M_{k+1} B_{2,k} \hat{u}_k + B_{1,k} \hat{w}_k \\ &+ Q_{k+1}^{-1} M_{k+1} (I - B_{2,k} \Phi_k^{-1} B_{2,k}^T M_{k+1}) B_{3,k} r_k + F_k v_k x_k + G_k \eta_k u_k]\}. \end{aligned}$$

We obtain, after completing to squares for θ_k, the following expression.

$$\begin{aligned} \underset{v,\eta}{E}\{\phi_k^*\} &= -||\hat{w}_k - R_{k+1}^{-1} B_{1,k}^T \theta_{k+1}||_{R_{k+1}}^2 + ||\hat{u}_k + \Phi_k^{-1} B_{2,k}^T M_{k+1} Q_{k+1}^{-1} \theta_{k+1}||_{\Phi_k}^2 \\ &- \underset{v,\eta}{E}\{J_k\} + ||B_{1,k}^T \theta_{k+1}||_{R_{k+1}^{-1}}^2 - ||B_{2,k}^T M_{k+1} Q_{k+1}^{-1} \theta_{k+1}||_{\Phi_k^{-1}}^2 \\ &+ r_k^T [D_{3,k}^T D_{3,k} + B_{3,k}^T (M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T)^{-1} B_{3,k}] r_k + \\ &x_k^T [A_k^T M_{k+1} A_k - (A_k^T M_{k+1} B_{2,k} + \alpha_k F_k^T Q_{k+1} G_k) \Phi_k^{-1} (B_{2,k}^T M_{k+1} A_k + \alpha G_k^T Q_{k+1} F_k) \\ &+ F_k^T Q_{k+1} F_k + C_k^T C_k - Q_k] x_k \\ &+ 2\theta_{k+1}^T Q_{k+1}^{-1} (M_{k+1}^{-1} + B_{2,k} T_{k+1}^{-1} B_{2,k}^T)^{-1} B_{3,k} r_k. \end{aligned}$$

The performance index becomes:

$$J_E(r_k, u_k, w_k, x_0) = \sum_{k=0}^{N-1} -||\hat{w}_k - R_{k+1}^{-1} B_{1,k}^T \theta_{k+1}||_{R_{k+1}}^2 + \bar{J}(r)$$

$$+\sum_{k=0}^{N-1}\{||\hat{u}_k + \Phi_k^{-1}B_{2,k}^T M_{k+1}Q_{k+1}^{-1}\theta_{k+1}||_{\Phi_k}^2\} - \gamma^2||x_0 \\ -(\gamma^2 R^{-1} - Q_0)^{-1}\theta_0||_{R^{-1}-\gamma^{-2}Q_0}^2. \quad (9.15)$$

where

$$\bar{J}(r) \triangleq \sum_{k=0}^{N-1} \theta_{k+1}^T\{B_{1,k}R_{k+1}^{-1}B_{1,k}^T - Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_k^{-1}B_{2,k}^T M_{k+1}Q_{k+1}^{-1}\}\theta_{k+1}$$

$$+\sum_{k=0}^{N-1} r_k^T(D_{3,k}^T D_{3,k} + B_{3,k}^T(M_{k+1}^{-1}+B_{2,k}T_{k+1}^{-1}B_{2,k}^T)^{-1}B_{3,k})r_k$$

$$+2\sum_{k=0}^{N-1}\theta_{k+1}^T Q_{k+1}^{-1}(M_{k+1}^{-1} + B_{2,k}T_{k+1}^{-1}B_{2,k}^T)^{-1}B_{3,k}r_k + \theta_0^T(\gamma^2 R^{-1} - Q_0)^{-1}\theta_0$$

$$+||D_{3,N}r_N||^2 .$$

The result of (9.12) readily follows from (9.15) and Lemma A1, and the saddle-point strategies are obtained by taking:

$$\begin{aligned} \hat{w}^* &= R_{k+1}^{-1}B_{1,k}^T\theta_{k+1}, \\ \hat{u}^* &= -\Phi_k^{-1}B_{2,k}^T M_{k+1}Q_{k+1}^{-1}\theta_{k+1}^c, \\ x_0^* &= (\gamma^2 R^{-1} - Q_0)^{-1}\theta_0. \end{aligned} \quad (9.16)$$

□

Remark 9.2. For u_{N-1} to be defined, it is required that $C_N^T C_N$ is of full rank. This may not comply sometimes with actual requirements. In this case $C_N^T C_N$ may be replaced by $C_N^T C_N + \epsilon I$where $0 < \epsilon \ll 1$.

Remark 9.3. We note that θ_{k+1}^c is the causal part of θ_{k+1} which depends on the specific patterns of the three reference signals (1) - (3) of Section 9.2.1. Note also that only the saddle- point strategy of the controller u_k^* of (9.9) depends on the causality of the reference signal of the latter three problems.

9.3.1 Preview Control Tracking Patterns

Applying the result of Theorem 9.1 on the specific pattern of the reference signal we obtain:

Lemma 9.1. *H_∞-Tracking with full preview of $\{r_k\}$: In this case*

$$\theta_{k+1}^c = \hat{\Phi}_{k+1}\theta_N + \sum_{j=1}^{N-k-1}\Psi_{k+1,j}\bar{B}_{N-j}r_{N-j} \quad (9.17)$$

where

$$\begin{aligned} \hat{\Phi}_{k+1} &\triangleq \bar{A}_{k+1}^T\bar{A}_{k+2}^T...\bar{A}_{N-1}^T \\ \Psi_{k+1,j} &\triangleq \begin{Bmatrix} \bar{A}_{k+1}^T\bar{A}_{k+2}^T...\bar{A}_{N-j-1}^T & j < N-k-1 \\ I & j = N-k-1 \end{Bmatrix} \end{aligned} \quad (9.18)$$

The control law is given by (9.9) with θ_{k+1}^c given by (9.17).

Proof: Considering (9.11) and taking $k+1=N$ we obtain:

$$\theta_{N-1} = \bar{A}_{N-1}^T \theta_N + \bar{B}_{N-1} r_{N-1},$$

where θ_N is given in (9.11). Similarly we obtain for $N-2$

$$\theta_{N-2} = \bar{A}_{N-2}^T \theta_{N-1} + \bar{B}_{N-2} r_{N-2} = \bar{A}_{N-2}^T [\bar{A}_{N-1}^T \theta_N + \bar{B}_{N-1} r_{N-1}]$$
$$+\bar{B}_{N-2} r_{N-2} = \bar{A}_{N-2}^T \bar{A}_{N-1}^T \theta_N + \bar{A}_{N-2}^T \bar{B}_{N-1} r_{N-1} + \bar{B}_{N-2} r_{N-2}.$$

The above procedure is thus easily iterated to yield (9.18). Taking, for example $N=3$ one obtains from (9.11) the following equation for θ_1:

$$\theta_1 = \bar{A}_1^T \bar{A}_2^T \theta_3 + \bar{A}_1^T \bar{B}_2 r_2 + \bar{B}_1 r_1.$$

The same result is recovered by taking $k=0$ in (9.18) where $j=1,2$. We note that in this case of full preview of $\{r_k\}$, $\theta_{k+1}^c = [\theta_{k+1}]_+$.

□

Lemma 9.2. *H_∞-Tracking with zero preview of $\{r_k\}$: We obtain the following control law:*

$$u^* = -\Phi_k^{-1}[B_{2,k}^T M_{k+1}(A_k x_k + B_{3,k} r_k) + \alpha_k G_k^T Q_{k+1} F_k x_k].$$

Proof : In this case $\theta_{k+1}^c = 0$ since at time k, r_i is known only for $i \le k$. We obtain from (9.9) the above control law.

Lemma 9.3. *H_∞-Tracking with fixed-finite preview of $\{r_k\}$: We obtain the control law of (9.9) where θ_{k+1}^c satisfies :*

$$\theta_{k+1}^c = \begin{Bmatrix} \sum_{j=1}^{h} \bar{\Psi}_{k+1,j} \bar{B}_{k+h+1-j} r_{k+h+1-j} & k+h \le N-1 \\ \hat{\Phi}_{k+1}\theta_N + \sum_{j=1}^{h-1} \bar{\Psi}_{k+1,j} \bar{B}_{N-j} r_{N-j} & k+h = N \end{Bmatrix}.$$

where $\bar{\Psi}_{k+1,j}$ is obtained from (9.18) by replacing N by $k+h+1$.

The above results is obtained since r_i is known at time k for $i \le min(N, k+h)$. We note that the proof of Lemma 9.3 follows the same line as the proof of Lemma 9.1.

Remark 9.4. The case of multiple stochastic uncertainties in both the dynamic and the input matrices can be readily considered. For simplicity we bring here a case where two white noise uncertainties appear in the latter matrices (namely $F_k v_k$ and $G_k \eta_k$ in (9.1) are replaced by $F_{1,k} v_{1,k} + F_{2,k} v_{2,k}$ and $G_{1,k}\eta_{1,k} + G_{2,k}\eta_{2,k}$ respectively). We consider the special case where $\{v_{1,k}\}, \{v_{2,k}\}, \{\eta_{1,k}\}, \{\eta_{2,k}\}$ are uncorrelated standard random scalar white noise sequences with zero mean. For the latter case, we set $\alpha = 0$ in (9.7), (9.9) and in $\bar{A}$ and replace $F_k^T Q_{k+1} F_k$ by $F_{1,k}^T Q_{k+1} F_{1,k} + F_{2,k}^T Q_{k+1} F_{2,k}$ and $G_k^T Q_{k+1} G_k$ by $G_{1,k}^T Q_{k+1} G_{1,k} + G_{2,k}^T Q_{k+1} G_{2,k}$.

Remark 9.5. We note that since

$$\theta_k = \bar{A}_k^T \theta_{k+1} + \bar{B}_k r_k,\ u_k = g_{x,k} x_k + g_{\theta,k}\theta_k,$$

in the case where r_k is scalar and slowly varies with respect to the dynamics of θ_k, one may solve

$$\tilde{\theta}_k = \bar{A}_k^T \tilde{\theta}_{k+1} + \bar{B}_k$$

and approximate u_k by

$$u_k = g_{x,k} x_k + g_{\theta,k}\tilde{\theta}_k r_k.$$

In such cases the controller applies both causal state-feedback and disturbance/reference feedforward, whereas without this approximation the disturbance feedforward path is not causal. Of course, the error in such approximation is nulled when r_k is actually constant. This idea is widely applied in many guidance laws based on optimal control [115] where the target maneuvers are assumed to be constant for synthesis purposes, resulting in guidance laws which are applicable for slowly varying maneuvers.

9.3.2 State-feedback: The Infinite-horizon Case

We treat the case where the matrices of the system in (9.1) and (9.3) are all time-invariant and N tends to infinity. Since (9.7) and the condition of (9.8) are identical to those encountered in the corresponding state-multiplicative state-feedback regulator problem (where $r_k \equiv 0$) [87], the solution Q_k of (9.7) and (9.8), if it exists, will tend to the mean square stabilizing solution (see [12], page 134) of the following equation:

$$-\bar{Q}+A^T\bar{M}A+C^TC+F^T\bar{Q}F-(F^T\bar{Q}G\alpha+A^T\bar{M}B_2)\bar{\Phi}^{-1}(G^T\bar{Q}F\alpha+B_2^T\bar{M}A)=0 \tag{9.19}$$

where

$$\bar{M} \stackrel{\triangle}{=} \bar{Q}[I-\gamma^{-2}B_1B_1^T\bar{Q}]^{-1} \text{ and } \bar{\Phi} \stackrel{\triangle}{=} B_2^T\bar{M}B_2 + G^T\bar{Q}G + \tilde{R} \tag{9.20}$$

and where

$$\gamma^2 I_p - B_1^T \bar{Q} B_1 > 0. \tag{9.21}$$

The latter will guarantee that

$$J_E(r_k, u_k^*, w_k, x_0) \le \hat{J}(r),$$

where

$$\hat{J}(r) = \|[\bar{Q}^{-1} + B_2\bar{T}^{-1}B_2^T - \gamma^{-2}B_1B_1^T]^{-\frac{1}{2}}(\bar{Q}^{-1}\theta_{k+1} + B_3 r_k)\|_2^2$$
$$-\|\bar{Q}^{-\frac{1}{2}}\theta_{k+1}\|_2^2 + \|D_3 r_k\|_2^2 + \|D_3 r_N\|^2 + \theta_0^T(\gamma^2 R^{-1} - \bar{Q})^{-1}\theta_0.$$

In the latter, θ_k is given by the time-invariant version of (9.11), θ_k^c is defined in (9.10) and

$$\bar{T} \stackrel{\triangle}{=} \tilde{R} + G^T \bar{Q} G. \tag{9.22}$$

A strict inequality is achieved, for $(\{w_k\}, x_o)$ that is not identically zero, iff the left side of (9.19) is strictly less than zero. The latter inequality can be expressed in a LMI form in the case where $\alpha = 0$, as follows:

Theorem 9.2. *Consider the system of (9.1) and (9.3) with constant matrices and $\alpha = 0$. Then, for a given $\gamma > 0$,*

$$J_E(r_k, u_k^*, w_k, x_0) < \hat{J}(r).$$

for all $\{w_k\} \in \tilde{l}^2([0,\infty); \mathcal{R}^p)$ and $x_0 \in \mathcal{R}^n$ iff there exists a matrix $P = P^T \in \mathcal{R}^{n\times n}$ that satisfies the following LMIs:

$$\Gamma_1 \stackrel{\triangle}{=} \begin{bmatrix} -P & PA^T & 0 & PF^T & PC^T & 0 \\ * & -P - B_2\tilde{R}^{-1}B_2^T & B_2\tilde{R}^{-1}G^T & 0 & 0 & B_1 \\ * & * & -P - G\tilde{R}^{-1}G^T & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -I_q & 0 \\ * & * & * & * & * & -\gamma^2 I_p \end{bmatrix} < 0 \tag{9.23}$$

and

$$\begin{bmatrix} \gamma^2 I_p & B_1^T \\ * & P \end{bmatrix} > 0. \tag{9.24}$$

If such a solution exists then the optimal control strategy is given by:

$$u_k^* = -\bar{\Phi}^{-1} B_2^T \hat{M}(Ax_k + B_3 r_k + P\theta_{k+1}^c) \tag{9.25}$$

where

$$\hat{M} = (P - \gamma^{-2}B_1B_1^T)^{-1}, \text{ and } \theta_{k+1}^c = [\theta_{k+1}]_+$$

and θ_k satisfies

$$\theta_k = A^T[P - \gamma^{-2}B_1B_1^T + B_2(\tilde{R} + G^T P^{-1} G)^{-1} B_2^T]^{-1}(P\theta_{k+1} + B_3 r_k).$$

Proof: The inequality that is obtained from (9.19) for $\alpha = 0$ is

$$-\bar{Q} + A^T \bar{M}[I - B_2\bar{\Phi}^{-1}B_2^T\bar{M}]A + C^T C + F^T \bar{Q} F < 0$$

with the condition of (9.21). Since

$$\bar{M} - \bar{M}B_2\bar{\Phi}^{-1}B_2^T\bar{M} = (\bar{M}^{-1} + B_2 T^{-1} B_2^T)^{-1}$$

we obtain, using the definition of $\bar{M}$ in (9.20), the condition:

$$-\bar{Q} + A^T[\bar{Q}^{-1} - \gamma^{-2}B_1B_1^T + B_2\bar{T}^{-1}B_2^T]^{-1}A + C^TC + F^T\bar{Q}F < 0 \quad (9.26)$$

Applying the inversion lemma on $\bar{T}$ of (9.22) we obtain

$$\bar{T}^{-1} = \tilde{R}^{-1} - \tilde{R}^{-1}G^T(\bar{Q}^{-1} + G\tilde{R}^{-1}G^T)^{-1}G\tilde{R}^{-1}.$$

Substituting the latter in (9.26) and defining $P = \bar{Q}^{-1}$ we obtain:

$$P^{-1} - A^T[P - \gamma^{-2}B_1B_1^T + B_2\tilde{R}^{-1}B_2^T - B_2\tilde{R}^{-1}G^T(P + G\tilde{R}^{-1}G^T)^{-1}G\tilde{R}^{-1} B_2^T)]^{-1}A - C^TC - F^TP^{-1}F > 0.$$

Using Schur's complements the latter inequality is equivalent to

$$\begin{bmatrix} -P^{-1} & A^T & 0 & F^T & C^T & 0 \\ * & -P - B_2\tilde{R}^{-1}B_2^T & B_2\tilde{R}^{-1}G^T & 0 & 0 & B_1 \\ * & * & -P - G\tilde{R}^{-1}G^T & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -I_q & 0 \\ * & * & * & * & * & -\gamma^2 I_p \end{bmatrix} < 0 \qquad (9.27)$$

The LMI of (9.23) is then obtained by multiplying (9.27), from both sides, by $diag\{P,\ I_n,\ I_n,\ I_n,\ I_q,\ I_p\}$. Using Schur's complements the condition of (9.21) is readily given by (9.24).

□

Remark 9.6. The case of $D_3 = 0$ characterizes a situation where a part of the disturbance is measured with preview. Substituting for D_3 in the equations of Theorem 9.1 we obtain, by (9.12), the following game value.

$$\begin{aligned} &J_E(r_k, u_k^*, w_k^*, x_0^*) = \|[P + B_2T^{-1}B_2^T - \gamma^{-2}B_1B_1^T]^{-\frac{1}{2}}(P\theta_{k+1} + B_3r_k)\|_2^2 \\ &-\|P^{\frac{1}{2}}\theta_k\|_2^2 + \|P^{\frac{1}{2}}\theta_0\|^2 + \theta_0^T(\gamma^2R^{-1} - P^{-1})^{-1}\theta_0 \\ &= \|[P + B_2T^{-1}B_2^T - \gamma^{-2}B_1B_1^T]^{0.5}\eta_k\|_2^2 - \|P^{\frac{1}{2}}A^T\eta_k\|_2^2 + \|P^{\frac{1}{2}}\theta_0\|^2 \\ &+\theta_0^T(\gamma^2R^{-1} - P^{-1})^{-1}\theta_0 = \|\Theta\eta_k\|_2^2 + \|P^{\frac{1}{2}}\theta_0\|^2 + \theta_0^T(\gamma^2R^{-1} - P^{-1})^{-1}\theta_0, \end{aligned}$$

where

$$\eta_k \stackrel{\triangle}{=} [P + B_2T^{-1}B_2^T - \gamma^{-2}B_1B_1^T]^{-1}(P\theta_{k+1} + B_3r_k)$$

and where

$$\Theta \stackrel{\triangle}{=} P + B_2T^{-1}B_2^T - \gamma^{-2}B_1B_1^T - APA^T,$$

is the matrix block that is obtained by deleting the forth and fifth matrix rows and columns in Γ_1 of (9.23) and using the Schur's complements formula with respect to the (2,2) block in the resulting inequality. The fact that $\Gamma_1 > 0$ readily implies that the game value is positive. The latter also stems from the fact that in the case where $D_3 = 0$, r_k is a disturbance (measurable one) that does not appear in the performance index. If nature could choose also r_k, it would certainly win the game by choosing r_k of unbounded magnitude without being penalized. To include nature in the game and let it choose r_k one should add to the cost function of (9.4) a penalized term of $\{r_k\}$, namely $-\gamma^2||r_k||_2^2$. In such case, however, the maximizing future values of $r_k = r_k^*$ depend on x_k which in turn depends on r_k^* via u_k, resulting in an infinite dimensionality of the problem [65].

The optimal strategy for the control input u_k is obtained from the results of Theorem 9.2 according to the specific information we have on $\{r_k\}$. Assuming that the conditions of the latter theorem are satisfied we obtain the following.

Lemma 9.4. *Stationary H_∞-Tracking with zero preview of $\{r_k\}$: In this case $\theta^c_{k+1} = 0$ and*

$$u^* = -\bar{\Phi}^{-1} B_2^T \hat{M}(Ax_k + B_3 r_k).$$

Lemma 9.5. *Stationary H_∞ Fixed-finite Preview Tracking: The control law is given by (9.25) when θ^c_{k+1} is given by:*

$$\theta^c_{k+1} = \sum_{j=1}^{h} (\bar{A}^T)^{h-j} (\bar{A}^T P^{-1} B_3 + C^T D_3) r_{k+h+1-j}.$$

Proof: We note that in this case r_i is known at time k for $i \leq k+h$. The above expression is easily recovered from Lemma 9.3 by considering the case of $k+h < N-1$ (since N tends to infinity) and noting that $\bar{B}$ is time invariant. Replacing for $\bar{B}_k$ of (9.11) where we note that $P^{-1} = Q$ and noting that the summation over $\bar{\Psi}$ of Lemma 9.3 is done from $k+1$ to $k+1+h-j$ (since in this case N is replaced by $k+h+1$), we obtain the above expression for θ^c_{k+1}. Note also that in the stationary case $\bar{A}$ is time-invariant.

□

9.4 The Output-feedback Control Tracking

The solution of the deterministic counterpart of this problem (i.e with no white noise components) with different preview patterns was obtained, using a game theoretic approach, in [17]. Similarly to the standard dynamic output-feedback control problem solution [57], the solution in [17] involves two steps. The second step of this solution is a filtering problem of order n. A second Riccati equation is thus achieved by applying the bounded real lemma [57] to the dynamic equation of the estimation error. In our case we follow the same approach, however, it will be shown that the estimation error contains a state-multiplicative white noise component. The latter imposes augmentation of the system to order 2n. The augmented system contains also a tracking signal component and one needs, therefore, to apply a special BRL for state multiplicative system with tracking signal. We thus bring first the following auxiliary BRL:

9.4.1 BRL for Stochastic State-multiplicative Systems with Tracking Signal

We consider the following system:

$$\begin{aligned} x_{k+1} &= (A_k + F_k v_k)x_k + B_{1,k}w_k + B_{3,k}r_k \\ z_k &= C_k x_k + D_{3,k}r_k, \quad z_k \in R^q, \quad k \in [0, N] \end{aligned} \tag{9.28}$$

which is obtained from (9.1) and (9.3) by setting $B_{2,k} \equiv 0$ and $D_{2,k} \equiv 0$. We consider the following index of performance:

$$J_B(r_k, w_k, x_0) \stackrel{\triangle}{=} \underset{v}{E}\{\|C_N x_N + D_{3,N} r_N\|^2 + ||z_k||_2^2\} - \underset{v}{E}\{\gamma^2[||w_k||_2^2]\} - \gamma^2 x_0^T R^{-1} x_0, \quad R^{-1} \geq 0. \tag{9.29}$$

We arrive at the following theorem:

Theorem 9.3. *Consider the system of (9.28) and J_B of (9.29). Given $\gamma > 0$, J_B of (9.29) satisfies $J_B \leq \tilde{J}(r, \epsilon)$, $\forall \{w_k\} \in \tilde{l}^2([0, N-1]; \mathcal{R}^p)$, $x_o \in R^n$, where*

$$\tilde{J}(r, \epsilon) = \sum_{k=0}^{N-1} r_k^T (D_{3,k}^T D_{3,k}) r_k + ||D_{3,N} r_N||^2 + \sum_{k=0}^{N-1} \tilde{\theta}_{k+1}^T \{B_{1,k} R_{k+1}^{-1} B_{1,k}^T\} \tilde{\theta}_{k+1} + 2 \sum_{k=0}^{N-1} \tilde{\theta}_{k+1}^T \tilde{Q}_{k+1}^{-1} (M_{k+1}^{-1})^{-1} B_{3,k} r_k + \tilde{\theta}_0^T \epsilon^{-1} \tilde{\theta}_0,$$

if there exists $\tilde{Q}_k$ that solves the following Riccati-type equation

$$\tilde{Q}_k = A_k^T M_{k+1} A_k + C_k^T C_k + F_k^T \tilde{Q}_{k+1} F_k, \quad \tilde{Q}(0) = \gamma^2 R^{-1} - \epsilon I, \tag{9.30}$$

for some $\epsilon > 0$, where

$$\begin{aligned} \tilde{\theta}_k &= \hat{A}_k^T \tilde{\theta}_{k+1} + \hat{B}_k r_k, \quad \tilde{\theta}_N = C_N^T D_{3,N} r_N, \\ \hat{A}_k &= \tilde{Q}_{k+1}^{-1} M_{k+1} A_k, \quad \hat{B}_k = \hat{A}_k^T \tilde{Q}_{k+1} B_{3,k} + C_k^T D_{3,k}. \end{aligned} \tag{9.31}$$

Proof: Unlike the state-feedback tracking control, the solution of the BRL does not invoke saddle-point strategies (since the input signal u_k is no longer an adversary). The sufficiency part of the proof can, however, be readily derived based on the first part of the sufficiency proof of Theorem 9.1, up to equation (9.15), where we set $B_{2,k} \equiv 0$ and $D_{2,k} \equiv 0$, and where we take $P_N = 0$. Analogously to (9.15) we obtain the following:

$$J_B(r_k, u_k, w_k, x_0) = -\sum_{k=0}^{N-1} ||\hat{w}_k - R_{k+1}^{-1} B_{1,k}^T \tilde{\theta}_{k+1}||^2_{R_{k+1}}$$
$$-\gamma^2 ||x_0 - (\gamma^2 R^{-1} - \tilde{Q}_0)^{-1} \tilde{\theta}_0||^2_{R^{-1} - \gamma^{-2} \tilde{Q}_0}. \tag{9.32}$$

where we replace θ_{k+1}, Q_k by $\tilde{\theta}_{k+1}$ and $\tilde{Q}_k$, respectively, and where

$$\tilde{P}_0 = [R^{-1} - \gamma^{-2} \tilde{Q}_0]^{-1},$$
$$x_0 = \gamma^{-2} \tilde{P}_0 \tilde{\theta}_0 = [\gamma^2 R^{-1} - \tilde{Q}_0]^{-1} \tilde{\theta}_0 \tag{9.33}$$

The necessity follows from the fact that for $r_k \equiv 0$, one gets $\tilde{J}(r, \epsilon) = 0$ (noting that in this case $\tilde{\theta}_k \equiv 0$ in (9.31) and therefore the last 3 terms in the above $\tilde{J}(r, \epsilon)$ are set to zero) and $J_B < 0$. Thus the existence of $\tilde{Q} > 0$ that solves (9.30) is the necessary condition in the stochastic BRL [87], [10].

□

Remark 9.7. The choice of $\epsilon > 0$ in $\tilde{Q}(0)$ of (9.30) reflects on both, the cost value (i.e $\tilde{J}(r, \epsilon)$) of (9.30) and the minimum achievable γ. If one chooses $0 < \epsilon << 1$ then, the cost of $\tilde{J}(r, \epsilon)$ increases while the solution of (9.30) is easier to achieve, which results in a smaller γ. The choice of large ϵ, on the other hand, causes the reverse effect, which leads to a larger γ.

9.4.2 The Output-feedback Solution

We consider the following system:

$$x_{k+1} = (A_k + F_k v_k) x_k + B_{2,k} u_k + B_{1,k} w_k + B_{3,k} r_k$$
$$y_k = (C_{2,k} + D_k \zeta_k) x_k + D_{21,k} n_k, \tag{9.34}$$

where $y_k \in \mathcal{R}^z$ and where we note that the measurement matrix is contaminated with a white noise component of $D_k \zeta_k$ and where, for simplicity, the stochastic uncertainty is removed from the input matrix. Like in the state-feedback case we seek a control law $\{u_k\}$, based on the information of the reference signal $\{r_k\}$ that minimizes the tracking error between the the system output and the tracking trajectory, for the worst case of the initial condition x_0, the process disturbances $\{w_k\}$, and the measurement noise $\{n_k\}$. We, therefore, consider the following performance index:

$$\tilde{J}_E(r_k, u_k, w_k, n_k, x_0) \stackrel{\triangle}{=} \underset{v,\zeta}{E}\{\|C_N x_N + D_{3,N} r_N\|^2 + ||z_k||_2^2 - \gamma^2 [||w_k||_2^2 + ||n_k||_2^2]\}$$
$$-\gamma^2 x_0^T R^{-1} x_0 + \underset{v,\zeta}{E}\{x_N^T \bar{Q}_N x_N\}, \quad R^{-1} \geq 0, \ \bar{Q}_N \geq 0. \tag{9.35}$$

We assume that (9.7) has a solution $Q_{k+1} > 0$ over $[0, N-1]$ where (9.8) are satisfied. We introduce the following difference linear matrix inequality (DLMI) [99],[44]:

$$\hat{\Gamma} \triangleq \begin{bmatrix} -\hat{P}_k^{-1} & \tilde{A}_k^T & 0 & \tilde{D}_k^T & \tilde{F}_k^T & \tilde{C}_{1,k}^T \\ * & -\hat{P}_{k+1} & \gamma^{-1}\tilde{B}_{1,k} & 0 & 0 & 0 \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -\hat{P}_{k+1} & 0 & 0 \\ * & * & * & * & -\hat{P}_{k+1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} \leq 0, \quad \hat{P}_k = \hat{Q}_k^{-1}, \tag{9.36}$$

where $\tilde{A}_k,\ \tilde{B}_{1,k},\ \tilde{C}_k$ and $\tilde{D}_k,\ \tilde{F}_k$ are defined in (9.47) and $\hat{Q}_k,\ \hat{P}_k$ are given in (9.48) and (9.49), respectively.
The solution of the output-feedback problem is stated in the following theorem, for the a priori case, where u_k can use the information on $\{y_i,\ 0 \leq i < k\}$:

Theorem 9.4. *Consider the system of (9.34) and $\tilde{J}_E$ of (9.35). Given $\gamma > 0$, the output-feedback tracking control problem, where $\{r_k\}$ is known a priori for all $k \leq N$ (the full preview case) possesses a solution if there exists $\hat{P}_k \in \mathcal{R}^{2n\times 2n} > 0$, $A_{f,k} \in \mathcal{R}^{n\times n}$, $B_{f,k} \in \mathcal{R}^{n\times z}$, $C_{f,k} \in \mathcal{R}^{m\times l}$ $\forall k \in [0, N]$ that solves the DLMI of (9.36) with a forward iteration, starting from the initial condition:*

$$\hat{P}_0 = \hat{Q}_0^{-1} \triangleq \gamma^{-2} \begin{bmatrix} R & R \\ R & R+\epsilon I_n \end{bmatrix}, \quad 0 < \epsilon << 1. \tag{9.37}$$

where R is defined in (9.4).

Proof: Using the expression achieved for $J_E(r_k, u_k, w_k, x_0)$ in the state-feedback case of Section 9.3, the index of performance is now given by:

$$J_y(r_k, u_k, w_k, n_k, x_0) = J_E(r_k, u_k, w_k, x_0) - \gamma^2||n_k||_2^2 = -\gamma^2||\bar{w}_k||_2^2$$
$$-\gamma^2||x_0 - x_0^*||^2_{R^{-1}-\gamma^{-1}Q_0} + \sum_{k=0}^{N-1}[\{||\bar{u}_k + \hat{C}_{1,k}x_k||^2\}]_+ - \gamma^2||n_k||_2^2 + \bar{J}(r). \tag{9.38}$$

We note that in the full preview case $[\theta_{k+1}]_+ = \theta_{k+1}$.
Using the following definitions:

$$\bar{u}_k = \Phi_{k+1}^{1/2}u_k + \Phi_{k+1}^{-1/2}B_{2,k}^T M_{k+1}(B_{3,k}r_k + Q_{k+1}^{-1}\theta_{k+1})$$
$$\bar{w}_k = \gamma^{-1}R_{k+1}^{1/2}w_k - \gamma^{-1}R_{k+1}^{-1/2}B_{1,k}^T[Q_{k+1}(A_k x_k + B_{2,k}u_k + B_{3,k}r_k) + \theta_{k+1}], \tag{9.39}$$

we note that

$$\bar{w}_k = \gamma^{-1} R_{k+1}^{1/2}(w_k - w_k^*),$$

where w_k^* is defined in (9.9). We also note that

$$\bar{u}_k = \Phi_{k+1}^{1/2}(u_k - u_k^*),$$

where u_k^* is derived from (9.9) by excluding from u_k^* the terms that are not accessed by the controller (i.e the terms with x_k). Considering the above $\bar{w}_k$ and $\bar{u}_k$ we seek a controller of the form

$$\bar{u}_k = -\hat{C}_{1,k}\hat{x}_k.$$

We, therefore, re-formulate the state equation of (9.34) in terms of $^*_k$ and w_k^* rather than u_k and w_k.

Considering the above, we obtain the following state equation.

$$x_{k+1} = (\hat{A}_k + F_k v_k)x_k + \bar{B}_{1,k}\bar{w}_k + \bar{B}_{2,k}\bar{u}_k + \bar{B}_{3,k}r_k + \bar{B}_{4,k}\theta_{k+1}, \tag{9.40}$$

where

$$\begin{aligned}
\hat{A}_k &= Q_{k+1}^{-1}M_{k+1}A_k,\\
\bar{B}_{1,k} &= \gamma B_{1,k}R_{k+1}^{-1/2},\\
\bar{B}_{2,k} &= Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_{k+1}^{-1/2},\\
\bar{B}_{3,k} &= B_{3,k} + B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}B_{3,k} - \bar{B}_{2,k}\Phi_{k+1}^{-1/2}B_{2,k}^T M_{k+1}B_{3,k},\\
\bar{B}_{4,k} &= \bar{B}_{1,k}R_{k+1}^{-1}\bar{B}_{1,k}^T - \bar{B}_{2,k}\Phi_{k+1}^{-1/2}B_{2,k}^T M_{k+1}Q_{k+1}^{-1}.
\end{aligned} \tag{9.41}$$

Replacing for $\bar{w}_k$ and $\bar{u}_k$ we obtain:

$$\begin{aligned}
x_{k+1} = (Q_{k+1}^{-1}M_{k+1}A_k + F_k v_k)x_k + \bar{B}_{1,k}(\gamma^{-1}R_{k+1}^{1/2}(w_k - w_k^*))\\
+\bar{B}_{2,k}(\Phi_{k+1}^{1/2}(u_k - u_k^*)). + \bar{B}_{3,k}r_k + \bar{B}_{4,k}\theta_{k+1}
\end{aligned} \tag{9.42}$$

Replacing for w_k^* and u_k^* we obtain:

$$\begin{aligned}
&x_{k+1} = (Q_{k+1}^{-1}M_{k+1}A_k + F_k v_k)x_k + \gamma B_{1,k}R_{k+1}^{-1/2}\gamma^{-1}R_{k+1}^{1/2}(w_k - R_{k+1}^{-1}B_{1,k}^T[\theta_{k+1}\\
&+Q_{k+1}(A_k x_k + B_{2,k}u_k + B_{3,k}r_k)]) + Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_{k+1}^{-1/2}(\Phi_{k+1}^{1/2}(u_k\\
&+\Phi_{k+1}^{-1}B_{2,k}^T M_{k+1}(B_{3,k}r_k + Q_{k+1}^{-1}\theta_{k+1}))) + \bar{B}_{3,k}r_k + \bar{B}_{4,k}\theta_{k+1},
\end{aligned} \tag{9.43}$$

or obtain:

$$\begin{aligned}
&x_{k+1} = [(I - \gamma^{-2}B_{1,k}B_{1,k}^T Q_{k+1}^{-1})^{-1}A_k - B_{1,k}R_{k+1}^{-1}B_{1,k}^T Q_{k+1}A_k]x_k + F_k x_k v_k\\
&+\gamma B_{1,k}R_{k+1}^{-1/2}\gamma^{-1}R_{k+1}^{1/2}w_k - (\gamma B_{1,k}R_{k+1}^{-1/2}\gamma^{-1}R_{k+1}^{1/2}R_{k+1}^{-1}B_{1,k}^T[\theta_{k+1}\\
&+Q_{k+1}(A_k x_k + B_{2,k}u_k + B_{3,k}r_k)]) + Q_{k+1}^{-1}M_{k+1}B_{2,k}\Phi_{k+1}^{-1/2}\\
&(\Phi_{k+1}^{1/2}(u_k + \Phi_{k+1}^{-1}B_{2,k}^T M_{k+1}(B_{3,k}r_k + Q_{k+1}^{-1}\theta_{k+1}))) + \bar{B}_{3,k}r_k + \bar{B}_{4,k}\theta_{k+1}
\end{aligned} \tag{9.44}$$

where we note that:

$$\begin{aligned}
\hat{A}_k &= A_k - B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} A_k \\
&= (I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}^{-1})^{-1} A_k - B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} A_k, \\
B_{1,k} &= \gamma^{-1} \bar{B}_{1,k} R_{k+1}^{1/2}, \\
B_{2,k} &= -B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} B_{2,k} + \bar{B}_{2,k} \Phi_{k+1}^{1/2} \\
B_{3,k} &= -B_{1,k} R_{k+1}^{-1} B_{1,k}^T Q_{k+1} B_{3,k} + \bar{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} + \bar{B}_{3,k} \\
\bar{B}_{2,k} \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} Q_{k+1}^{-1} &= Q_{k+1}^{-1} M_{k+1} B_{2,k} \Phi_{k+1}^{-1} B_{2,k}^T M_{k+1} Q_{k+1}^{-1}.
\end{aligned}$$

We consider next the following *a priori*-type state observer:

$$\begin{aligned}
\hat{x}_{k+1} &= A_{f,k} \hat{x}_k + B_{f,k} y_k + d_k, \quad \hat{x}_0 = 0, \\
\hat{z}_k &= C_{f,k} \hat{x}_k,
\end{aligned} \tag{9.45}$$

where

$$d_k = \bar{B}_{2,k} \bar{u}_k + \bar{B}_{3,k} r_k + \bar{B}_{4,k} \theta_{k+1}.$$

Denoting $e_k = x_k - \hat{x}_k$ and using the latter we obtain:

$$e_{k+1} = (\hat{A}_k - B_{f,k} C_{2,k} - A_{f,k}) x_k + A_{f,k} e_k + F_k x_k v_k - B_{f,k} D_k x_k \zeta_k + \hat{B}_{1,k} \hat{w}_k,$$

where we define

$$\hat{w}_k \stackrel{\triangle}{=} col\{\bar{w}_k,\ n_k\}, \quad \hat{B}_{1,k} = [\bar{B}_{1,k} \quad -B_{f,k} D_{21,k}].$$

Defining also

$$\xi_k \stackrel{\triangle}{=} col\{x_k,\ e_k\}, \ \ \bar{r}_k \stackrel{\triangle}{=} col\{r_k,\ \theta_{k+1}\},$$

we obtain

$$\begin{aligned}
\xi_{k+1} &= (\tilde{A}_k + \tilde{F}_k v_k + \tilde{D}_k \zeta_k) \xi_k + \tilde{B}_{1,k} \hat{w}_k + \hat{B}_{3,k} \bar{r}_k, \ \xi_0 = col\{x_0,\ e_0\}, \\
\tilde{z}_k &= \tilde{C}_{1,k} \xi_k,
\end{aligned} \tag{9.46}$$

where

$$\tilde{A}_k = \begin{bmatrix} \hat{A}_k - \bar{B}_{2,k} C_{f,k} & \bar{B}_{2,k} C_{f,k} \\ \hat{A}_k - B_{f,k} C_{2,k} - A_{f,k} & A_{f,k} \end{bmatrix}, \quad \tilde{B}_{1,k} = \begin{bmatrix} \bar{B}_{1,k} & 0 \\ \bar{B}_{1,k} & -B_{f,k} D_{21,k} \end{bmatrix},$$

$$\begin{aligned}
\hat{B}_{3,k} &= \begin{bmatrix} \bar{B}_{3,k} & \bar{B}_{4,k} \\ 0 & 0 \end{bmatrix}, \tilde{F}_k = \begin{bmatrix} F_k & 0 \\ F_k & 0 \end{bmatrix}, \quad \tilde{D}_k = \begin{bmatrix} 0 & 0 \\ -B_{f,k} D_k & 0 \end{bmatrix}, \\
\hat{C}_{1,k} &= \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} A_k, \quad \tilde{C}_{1,k} = [\hat{C}_{1,k} - C_{f,k} \ \ C_{f,k}].
\end{aligned} \tag{9.47}$$

Applying the results of Theorem 9.3 to the system of (9.46) we obtain the following Riccati-type equation:

$$\hat{Q}_k=\tilde{A}_k^T[\hat{Q}_{k+1}^{-1}-\gamma^{-2}\tilde{B}_{1,k}\tilde{B}_{1,k}^T]^{-1}\tilde{A}_k+\tilde{C}_k^T\tilde{C}_k+\tilde{F}_k^T\hat{Q}_{k+1}\tilde{F}_k+\tilde{D}_k^T\hat{Q}_{k+1}\tilde{D}_k, \tag{9.48}$$

where $\hat{Q}_0$ is given in (9.37).
Denoting

$$\hat{P}_k=\hat{Q}_k^{-1} \tag{9.49}$$

and using Schur complement we obtain the DLMI of (9.36). The latter DLMI is initiated with the initial condition of (9.37) which corresponds to the case where a weighting $\gamma^2\epsilon^{-1}I_n$ is applied to $\hat{x}_0$ in order to force nature to select $\hat{x}_0=0$ in the corresponding differential game [99], [57].

□

In the case where $\{r_k\}$ is measured on line, or with a preview $h>0$, we note that nature's strategy, which is not restricted to causality constrains, will be the same as in the case of full preview of $\{r_k\}$, meaning that, in contrast to $\bar{u}_k$ which depends on the preview length h, $\bar{w}_k$ of (9.39) remains unchanged. We obtain the following.

Lemma 9.6. *H_∞ Output-feedback Tracking with full preview of $\{r_k\}$: In this case*

$$\bar{u}_k=\Phi_{k+1}^{1/2}u_k+\Phi_{k+1}^{-1/2}B_{2,k}^TM_{k+1}(B_{3,k}r_k+Q_{k+1}^{-1}\theta_{k+1}).$$

Solving (9.11) we obtain:

$$\theta_{k+1}=\hat{\Phi}_{k+1}\theta_N+\sum_{j=1}^{N-k-1}\Psi_{k+1,j}\bar{B}_{N-j}r_{N-j}$$

where

$$\begin{aligned}\hat{\Phi}_{k+1}&\triangleq\bar{A}_{k+1}^T\bar{A}_{k+2}^T...\bar{A}_{N-1}^T\\ \Psi_{k+1,j}&\triangleq\left\{\begin{matrix}\bar{A}_{k+1}^T\bar{A}_{k+2}^T...\bar{A}_{N-j-1}^T & j<N-k-1\\ I & j=N-k-1\end{matrix}\right\}\end{aligned} \tag{9.50}$$

Proof: We note that in this case $[\theta_{k+1}]_+=\theta_{k+1}$. Considering (9.11) and taking $k+1=N$ we obtain:

$$\theta_{N-1}=\bar{A}_{N-1}^T\theta_N+\bar{B}_{N-1}r_{N-1},$$

where θ_N is given in (9.11). Similarly we obtain for $N-2$

$$\begin{aligned}\theta_{N-2}&=\bar{A}_{N-2}^T\theta_{N-1}+\bar{B}_{N-2}r_{N-2}=\bar{A}_{N-2}^T[\bar{A}_{N-1}^T\theta_N\\ &+\bar{B}_{N-1}r_{N-1}]+\bar{B}_{N-2}r_{N-2}=\bar{B}_{N-2}r_{N-2}+\bar{A}_{N-2}^T\bar{A}_{N-1}^T\theta_N+\bar{A}_{N-2}^T\bar{B}_{N-1}r_{N-1}.\end{aligned}$$

The above procedure is thus easily iterated to yield (9.50). Taking, for example $N=3$ one obtains from (9.11) the following equation for θ_1 :

$$\theta_1 = \bar{A}_1^T \bar{A}_2^T \theta_3 + \bar{A}_1^T \bar{B}_2 r_2 + \bar{B}_1 r_1.$$

The same result is recovered by taking $k = 0$ in (9.50) where $j = 1, 2$.

□

Lemma 9.7. H_∞ *Output-feedback Tracking with no preview of* $\{r_k\}$*: In this case*

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} B_{3,k} r_k.$$

The proof follows from the fact that in this case $[\theta_{k+1}]_+ = 0$ since at time k, r_i is known only for $i \leq k$.

Lemma 9.8. H_∞ *Output-feedback tracking with fixed-finite preview of* $\{r_k\}$*: In this case:*

$$\bar{u}_k = \Phi_{k+1}^{1/2} u_k + \Phi_{k+1}^{-1/2} B_{2,k}^T M_{k+1} (B_{3,k} r_k + Q_{k+1}^{-1} [\theta_{k+1}]_+)$$

and

$$d_k = \bar{B}_{2,k} \bar{u} + \bar{B}_{3,k} r_k + \bar{B}_{4,k} [\theta_{k+1}]_+.$$

where $[\theta_{k+1}]_+$ *satisfies :*

$$[\theta_{k+1}]_+ = \left\{ \begin{array}{ll} \sum_{j=1}^{h} \bar{\Psi}_{k+1,j} \bar{B}_{k+h+1-j} r_{k+h+1-j} & k+h \leq N-1 \\ \hat{\Phi}_{k+1} \theta_N + \sum_{j=1}^{h-1} \bar{\Psi}_{k+1,j} \bar{B}_{N-j} r_{N-j} & k+h = N \end{array} \right\}.$$

and where $\bar{\Psi}_{k+1,j}$ *is obtained from (9.50) by replacing* N *by* $k+h+1$.

9.5 Example: A Stochastic Finite Preview Tracking

We consider the system of (9.1) with the following objective function:

$$J = \Sigma_0^\infty E_{\bar{R}_k} \{ \|Cx_k - r_k\|^2 + 0.01 \|u_k\|^2 \} - \gamma^2 \Sigma_{k=0}^\infty \|w_k\|^2$$

where there is an access to the states of the system, where

$$A = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.6 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 \\ 0.12 & 0.24 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} -0.5 \, 0.4 \end{bmatrix}.$$

and where $G = 0$. The case of $h = 0$ can be solved using the standard H_∞ model where r_k is considered as a disturbance [21]. The disturbance vector w_k becomes the augmented disturbance vector $\tilde{w}_k \stackrel{\triangle}{=} col\{w_k, \; r_k\}$. Using the notation of the standard problem, we define:

$$B_1 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \; D_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } D_{12} = \begin{bmatrix} 0 \\ .1 \end{bmatrix}.$$

Applying the theory of [12] we get that the minimum possible value of γ is near $\gamma_s = 2.031$. The obtained control law for this γ is $u_k = K_s x_k$, $K_s = \begin{bmatrix} 0.472 & -2.359 \end{bmatrix}$ and the resulting closed-loop transfer function, from r_k to Cx_k, is $G_s = -(z + 1.021)(z^2 + 0.759z + 0.327)^{-1}$.

We apply the results obtained in Section 9.3.2 with preview of $h = 0$. We find that for $\gamma = 1.016$, a value very close to the lowest achievable value of γ, the solution of (9.23) is $P = \begin{bmatrix} 6.1846 & 3.1776 \\ 3.1776 & 4.2601 \end{bmatrix}$, and the resulting closed-loop transfer function, from r_k to Cx_k, is $G_s = -(0.3645z)(z^2 - 0.794z)^{-1}$. We note that one should not draw a conclusion from the fact that the minimal γ achieved by the two methods is significantly different. Unlike our solution, in [21] the energy of r_k is also taken into account in the performance index. The comparison between the two methods should, therefore, be based on time simulation. The resulting control law of (9.25) becomes:

$$\tilde{u}_k = [0.7944 \quad -1.0][Ax_k + B_3 r_k].$$

We compare the results of [12] with our result in Figure 9.1 where the average tracking error $(Cx_k - r_k)$, with respect to the statistics of the multiplicative noise, is depicted as a function of time for $r_k = sin(k)$. The improvement achieved by our new method, in this frequency, is clearly visible.

The latter controller is extended to the case of $h > 0$ using the result of Theorem 9.2. The only addition to the control law of $\tilde{u}_k$ is the term $K_\theta \theta^c_{k+1}$ where $K_\theta = -\Phi^{-1} B_2^T \hat{M} P = [1.7354 \quad -1.7358]$. The results that are obtained for the case of $h > 0$ are described in Figure 9.2, where the average power of the tracking error, with respect to the statistics of the multiplicative noise, is depicted as a function of the preview length.

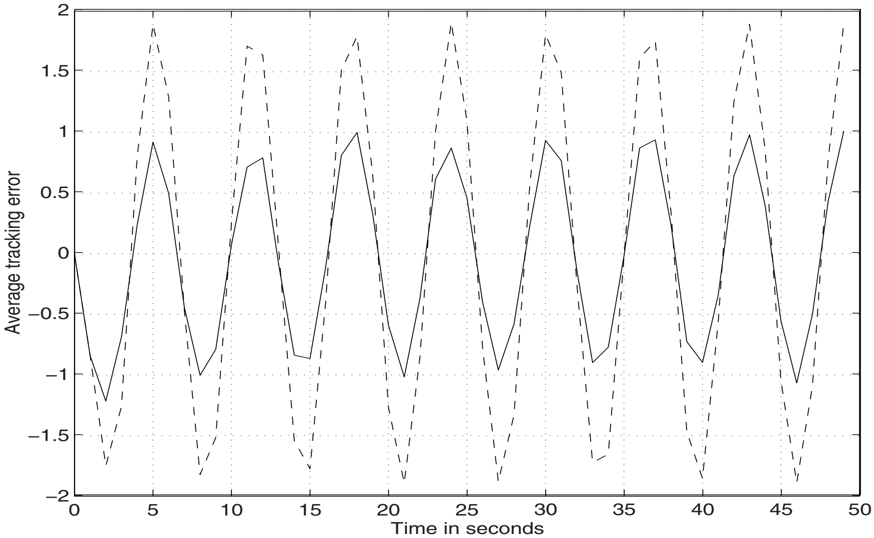

Fig. 9.1. Comparison between the tracking errors obtained in the standard solution (dashed lines) and the new method (solid lines) for $r_k = sin(k)$, measured on line

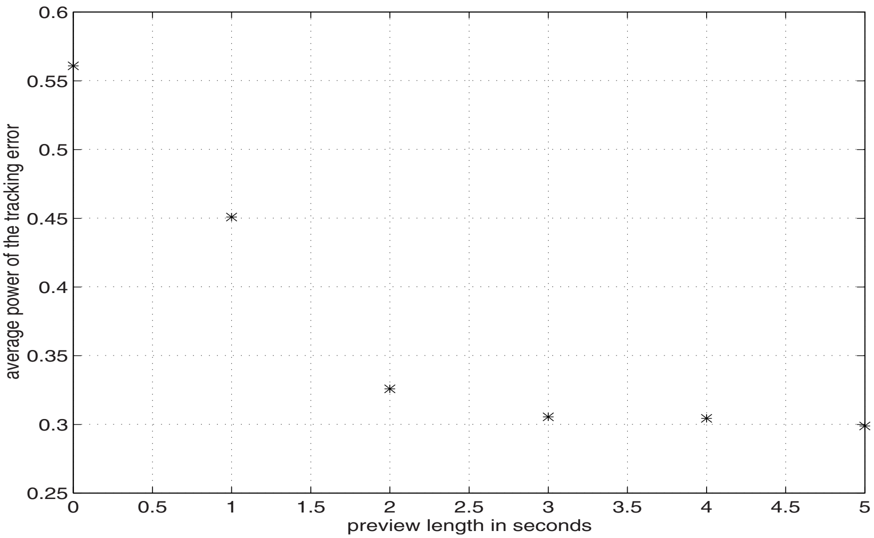

Fig. 9.2. The power of the tracking error as a function of the preview length

9.6 Conclusions

In this chapter we solve the problem of both, the state-feedback and output-feedback control tracking with preview in the presence of white noise parameter uncertainties in the system state-space model. Applying the game theory approach, a saddle-point tracking strategy is obtained for the finite-horizon state-feedback case which is based on the measurement of the system state and the previewed reference signal. The performance index that corresponds to the tracking game includes averaging over the statistics of the white noise parameters in the system state-space model. The game value depends on the reference signal sequence and is usually greater than zero.

For the finite-horizon time varying state-feedback case, a nonzero correlation between the unknown parameters in the input and the dynamic matrices was allowed. The latter result has been extended to the time invariant case with an infinite horizon. The problem of determining whether there exists a solution that guarantees a pre chosen attenuation level became one of solving a set of two LMI's, in the case where the latter correlation is zero. The fact that we could not formulate the solution to the problem in a LMI form, in the case where this correlation in not zero, may indicate that the problem is not convex.

The problem is probably nonconvex in the case where this correlation is nonzero.

Extension of the results of the state-feedback solution to the case where there is no access to the system states and the controller has to rely on noisy measurements of the output is not immediate. Unlike the treatment in the deterministic uncertain case where the uncertainties are norm-bounded, the stochastic uncertainties of the plant in our case lead to an error in the estimate of the system state that depends not only on the past values of the error but also on the value of the state multiplied by the stochastic uncertainty. To solve the output-feedback tracking control we needed, therefore, to formulate and solve an auxiliary BRL which also contains an additional tracking signal in the system dynamics. Thus, using an *a priori*-type state-observer the latter problem can be formulated as an estimation problem, to which we apply the auxiliary BRL. Applying the latter we arrive at a linear matrix inequality over a finite-horizon (i.e DLMI) which can be solved with the aid of the DLMI technique [99], [44] presented in the introduction and in Appendices B and C. This technique was accurately shown to mimic the standard Riccati recursions of various H_∞ control and estimation problems. It was also shown to converge to the stationary solution of the latter problems in the case where the system is LTI and the horizon length tends to infinity [99], [44]. Alternatively the latter DLMI can be also solved using homotopy algorithm.

10

Discrete-time Systems: Static Output-feedback

10.1 Introduction

In this chapter we bring the solution of the discrete-time counterpart of Chapter 5 where we solve both the H_2 and the H_∞ static output-feedback control problems. In both cases we treat systems where that contain stochastic white-noise parameter uncertainties in the matrices of the state-space model, including the measurement matrix, that describes the system.

Similarly to Chapter 5 we apply the simple design method of [100] for deriving the static output-feedback gain that satisfies prescribed H_2 and H_∞performance criteria. A parameter dependent Lyapunov (LPD) function is applied and a sufficient condition is obtained by adopting a stochastic counterpart of a recent LPD stabilization method that has been introduced in [36]. The theory we develop is demonstrated by an example of a two- output one-input system where we design a static controller for a nominal and uncertain system.

10.2 Problem Formulation

We consider the following linear system:

$$\begin{aligned} x_{k+1} &= (A + D\nu_k)x_k + B_1 w_k + (B_2 + G\zeta_k)u_k, \quad x_0 = 0 \\ y_k &= C_2 x_k + D_{21} n_k \end{aligned} \tag{10.1}$$

with the objective vector

$$z_k = C_1 x_k + D_{12} u_k, \tag{10.2}$$

where $\{x_k\} \in \mathcal{R}^n$ is the system state vector, $\{w_k\} \in \tilde{l}^2([0,\infty);\mathcal{R}^q)$ is the exogenous disturbance signal, $\{n_k\} \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$ is the the measurement noise sequence, $\{u_k\} \in \tilde{l}^2([0,\infty);\mathcal{R}^\ell)$ is the control input, $\{y_k\} \in \mathcal{R}^m$ is the

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 163–172, 2005.

measured output and $\{z_k\} \in \mathcal{R}^r \subset \mathcal{R}^n$ is the state combination (objective function signal) to be regulated. The variables $\{\zeta_k\}$ and $\{\nu_k\}$ are zero-mean real scalar white-noise sequences that satisfy:

$$E\{\nu_k \nu_j\} = \delta_{kj}, \quad E\{\zeta_k \zeta_j\} = \delta_{kj} \quad E\{\zeta_k \nu_j\} = 0, \;\; \forall k, j \quad \geq 0.$$

The matrices in (10.1), (10.2) are assumed to be constant matrices of appropriate dimensions.

We seek a constant output-feedback controller

$$u_k = K y_k, \tag{10.3}$$

that achieves a certain performance requirement. We treat the following two different performance criteria.

- **The stochastic H_2 control problem:** Assuming that $\{w_k\}$, $\{n_k\}$ are realizations of a unit variance, stationary, white noise sequences that are uncorrelated with $\{\nu_k\}$, $\{\zeta_k\}$, rather than adversaries in l_2, the following performance index should be minimized:

$$J_2 \stackrel{\Delta}{=} \mathop{E}_{w,n}\{||z_k||^2_{\tilde{l}_2}\}. \tag{10.4}$$

- **The stochastic H_∞control problem:** Assuming that the exogenous disturbance signal is in $\tilde{l}_2$, a static control gain is sought which, for a prescribed scalar $\gamma > 0$ and for all nonzero $\{w_k\} \in \tilde{l}^2([0,\infty);\mathcal{R}^q)$ and $\{n_k\} \in \tilde{l}^2([0,\infty);\mathcal{R}^p)$, guarantees that $J_\infty < 0$ where

$$J_\infty \stackrel{\Delta}{=} ||z_k||^2_{\tilde{l}_2} - \gamma^2[||w_k||^2_{\tilde{l}_2} + ||n_{k+1}||^2_{\tilde{l}_2}]. \tag{10.5}$$

Augmenting system (10.1) and (10.2) to include the measured output y_k we define the augmented state vector $\xi_k = col\{x_k,\; y_k\}$ and obtain the following representation to the closed-loop system.

$$\begin{aligned} \xi_{k+1} &= \tilde{A}\xi_k + \tilde{B}\tilde{w}_k + \tilde{D}\xi_k\nu_k + \tilde{G}\xi_k\zeta_k, \qquad \xi_0 = 0 \\ z_k &= \tilde{C}\xi_k \end{aligned} \tag{10.6}$$

where:

$$\tilde{w}_k \stackrel{\Delta}{=} \begin{bmatrix} w_k \\ n_{k+1} \end{bmatrix}, \; \tilde{A} \stackrel{\Delta}{=} \begin{bmatrix} A & B_2K \\ C_2A & C_2B_2K \end{bmatrix}, \; \tilde{D} \stackrel{\Delta}{=} \begin{bmatrix} D & 0 \\ C_2D & 0 \end{bmatrix},$$

$$\tilde{G} \stackrel{\Delta}{=} \begin{bmatrix} 0 & GK \\ 0 & C_2GK \end{bmatrix}, \; \tilde{B} \stackrel{\Delta}{=} \begin{bmatrix} B_1 & 0 \\ C_2B_1 & D_{21} \end{bmatrix}, \; \tilde{C} \stackrel{\Delta}{=} \begin{bmatrix} C_1 & D_{12}K \end{bmatrix}. \tag{10.7}$$

We consider the following Lyapunov function

$$V_L = \xi^T \tilde{P} \xi \quad \text{with} \quad \tilde{P} = \begin{bmatrix} P & -\alpha^{-1}PC_2^T \\ -\alpha^{-1}C_2P & \hat{P} \end{bmatrix}, \; \tilde{P} > 0, \tag{10.8}$$

where $P \in \mathcal{R}^{n\times n}$ and $\hat{P} \in \mathcal{R}^{m\times m}$. The parameter α is a positive scalar tuning parameter.

10.2.1 The Stochastic H_2 Control Problem

Applying (10.8) to the derivation of the stochastic H_2 control results [105] it is obtained that $J_2 < \delta^2$ for a prescribed δ if there exist a positive definite solution $\tilde{Q} = \tilde{P}^{-1}$, where $\tilde{P}$ is of the structure (10.8), and $H \in \mathcal{R}^{(q+p)\times(q+p)}$ that solve the following Linear Matrix Inequalities (LMIs) :

$$\begin{bmatrix} -\tilde{Q} & \tilde{A}\tilde{Q} & 0 & 0 & 0 \\ * & -\tilde{Q} & \tilde{Q}\tilde{C}^T & \tilde{Q}\tilde{D}^T & \tilde{Q}\tilde{G}^T \\ * & * & -I_r & 0 & 0 \\ * & * & * & -\tilde{Q} & 0 \\ * & * & * & * & -\tilde{Q} \end{bmatrix} < 0, \begin{bmatrix} H & \tilde{B}^T \\ * & \tilde{Q} \end{bmatrix} > 0, \quad \text{and} \quad Tr\{H\} < \delta^2. \tag{10.9}$$

Applying [100] it is found that $\tilde{Q}$ possesses the following structure:

$$\tilde{Q} = \begin{bmatrix} Q & C_2^T\hat{Q} \\ \hat{Q}C_2 & \alpha\hat{Q} \end{bmatrix}, \tag{10.10}$$

where $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$.

Substituting for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, $\tilde{G}$ into the latter LMIs we obtain the following:

Theorem 10.1. *Consider the system (10.1),(10.2). The output-feedback control law (10.3) achieves a prescribed H_2-norm bound $0 < \delta$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$, $Y \in \mathcal{R}^{\ell\times m}$ and $H \in \mathcal{R}^{(q+p)\times(q+p)}$ that, for some tuning scalar $0 < \alpha$, satisfy the following LMIs:*

$$\tilde{\Gamma} \stackrel{\Delta}{=} \begin{bmatrix} -Q & -C_2^T\hat{Q} & \tilde{\Gamma}(1,3) & \tilde{\Gamma}(1,4) & 0 & 0 & 0 & 0 & 0 \\ * & -\alpha\hat{Q} & \tilde{\Gamma}(2{,}3) & \tilde{\Gamma}(2{,}4) & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & -C_2^T\hat{Q} & \tilde{\Gamma}(3{,}5) & QD^T & \tilde{\Gamma}(3{,}7) & C_2^TY^TG^T & \tilde{\Gamma}(3{,}9) \\ * & * & * & -\alpha\hat{Q} & \tilde{\Gamma}(4{,}5) & \hat{Q}C_2D^T & \tilde{\Gamma}(4{,}7) & \alpha Y^TG^T & \tilde{\Gamma}(4{,}9) \\ * & * & * & * & -I_r & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q & -C_2^T\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & -\alpha\hat{Q} & 0 & 0 \\ * & * & * & * & * & * & * & -Q & -C_2^T\hat{Q} \\ * & * & * & * & * & * & * & * & -\alpha\hat{Q} \end{bmatrix} < 0, \tag{10.11}$$

$$\begin{bmatrix} H_{11} & H_{12} & B_1^T & B_1^T C_2^T \\ * & H_{22} & 0 & D_{21}^T \\ * & * & Q & C_2^T \hat{Q} \\ * & * & * & \alpha \hat{Q} \end{bmatrix} > 0, \quad Tr\{H\} < \delta^2, \tag{10.11}$$

where

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

$$\begin{aligned}
\tilde{\Gamma}(1,3) &= AQ + B_2 Y C_2, \\
\tilde{\Gamma}(1,4) &= \alpha B_2 Y + A C_2^T \hat{Q}, \\
\tilde{\Gamma}(2,3) &= [C_2 AQ + C_2 B_2 Y C_2 - \hat{Q} C_2] + \hat{Q} C_2, \\
\tilde{\Gamma}(2,4) &= [C_2 A C_2^T \hat{Q} + \alpha C_2 B_2 Y - \alpha \hat{Q}] + \alpha \hat{Q}, \\
\tilde{\Gamma}(3,5) &= Q C_1^T + C_2^T Y^T D_{12}^T, \\
\tilde{\Gamma}(3,7) &= Q D^T C_2^T, \\
\tilde{\Gamma}(3,9) &= C_2^T Y^T G^T C_2^T, \\
\tilde{\Gamma}(4,5) &= \alpha Y^T D_{12}^T + \hat{Q} C_2 C_1^T, \\
\tilde{\Gamma}(4,7) &= \hat{Q} C_2 D^T C_2^T, \\
\tilde{\Gamma}(4,9) &= \alpha Y^T G^T C_2^T.
\end{aligned} \tag{10.12}$$

If a solution to the latter LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y \hat{Q}^{-1}. \tag{10.13}$$

10.2.2 The Stochastic H_∞ Problem

The LMIs of Theorem 10.1 provide a sufficient condition for the existence of a static output-feedback gain that achieves a prescribed H_2-norm for the system (10.6). A similar result can be obtained if the H_∞-norm of the latter system is considered. Given a prescribed desired bound $0 < \gamma$ on the H_∞-norm of the system, the inequalities in (10.9) are replaced by the following Bounded Real Lemma (BRL) condition (see 7.3.1).

$$\begin{bmatrix} -\tilde{Q} & \tilde{A}\tilde{Q} & \tilde{B} & 0 & 0 & 0 \\ * & -\tilde{Q} & 0 & \tilde{Q}\tilde{C}^T & \tilde{Q}\tilde{D}^T & \tilde{Q}\tilde{G}^T \\ * & * & -\gamma^2 I_{q+p} & 0 & 0 & 0 \\ * & * & * & -I_r & 0 & 0 \\ * & * & * & * & -\tilde{Q} & 0 \\ * & * & * & * & * & -\tilde{Q} \end{bmatrix} < 0. \tag{10.14}$$

Using the definition of (10.10), and substituting for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, $\tilde{G}$ in the latter LMI we obtain the following.

Theorem 10.2. *Consider the system of (10.1), (10.2) . The control law (10.3) achieves a prescribed H_∞-norm bound $0 < \gamma$, if there exist $Q \in \mathcal{R}^{n\times n}$, $\hat{Q} \in \mathcal{R}^{m\times m}$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some scalar $0 < \alpha$, satisfy the following LMI:*

$$\begin{bmatrix} \tilde{\Gamma} & \begin{bmatrix} B_1 & 0 \\ C_2B_1 & D_{21} \\ 0 & 0 \end{bmatrix} \\ * & -\gamma^2 I_{q+p} \end{bmatrix} < 0. \tag{10.15}$$

where $\tilde{\Gamma}$ is defined in (10.11).
If a solution to the latter LMI exists, the gain matrix K that stabilizes the system and achieves the required performance is given by (10.13).

10.3 The Robust Stochastic H_2 Static Output-feedback Controller

The system considered in Section 10.2 assumes that all the parameters of the system are known, including the matrices D and G. In the present section we consider the system (10.1),(10.2) whose matrices are not exactly known. Denoting

$$\Omega = \begin{bmatrix} A \ B_1 \ B_2 \ C_1 \ D_{12} \ D_{21} \ D \ G \end{bmatrix},$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1, ...N\}$, namely,

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \le f_j \le 1, \sum_{j=1}^{N} f_j = 1 \tag{10.16}$$

where the vertices of the polytope are described by

$$\Omega_j = \begin{bmatrix} A^{(j)} \ B_1^{(j)} \ B_2^{(j)} \ C_1^{(j)} \ D_{12}^{(j)} \ D_{21}^{(j)} \ D^{(j)} \ G^{(j)} \end{bmatrix}, \quad j = 1, 2..., N.$$

The solution of the robust problem in this section is based on the derivation of a specially devised BRL for polytopic-type uncertainties [36] with a simple straightforward adaptation to the stochastic case. Considering (10.9), multiplying it by $diag\{I_{n+m},\ \tilde{Q}^{-1},\ I_r,\ I_{n+m},\ I_{n+m}\}$, from the left and the right and using the method of [36] we obtain that a sufficient condition for achieving the H_2-norm bound of δ for the system at the i-th vertex of Ω is that there exists a solution $\tilde{Q}_i,\ Z,\ H$ to the following LMIs:

$$\begin{bmatrix} -\tilde{Q}_i & \tilde{A}_i Z & 0 & 0 & 0 \\ * & \tilde{Q}_i - Z - Z^T & Z^T \tilde{C}_i^T & Z^T \tilde{D}_i^T & Z^T \tilde{G}_i^T \\ * & * & -I_r & 0 & 0 \\ * & * & * & -\tilde{Q}_i & 0 \\ * & * & * & * & -\tilde{Q}_i \end{bmatrix} < 0,$$

$$\begin{bmatrix} H & \tilde{B}_i^T \\ * & \tilde{Q}_i \end{bmatrix} > 0, \text{ and } \quad Tr\{H\} < \delta^2, \tag{10.17}$$

for $i = 1, 2, ...N$, where $H \in \mathcal{R}^{(q+p)\times(q+p)}$. Denoting

$$Z \stackrel{\Delta}{=} \begin{bmatrix} Z_1 \\ Z_2[C_2 \ \ \beta I_m] \end{bmatrix} \tag{10.18}$$

where $\beta > 0$ is a tuning scalar parameter, we arrive at the following result:

Theorem 10.3. *Consider the uncertain system of (10.1), (10.2). The control law (10.3) guarantees a prescribed H_2-norm bound $0 < \delta$ over the entire uncertainty polytope Ω if there exist $Z_1 \in \mathcal{R}^{n\times(n+m)}$, $Z_2 \in \mathcal{R}^{m\times m}$, $\tilde{Q}_i \in \mathcal{R}^{(n+m)\times(n+m)}$, $i = 1, 2..., N$ and $Y \in \mathcal{R}^{\ell\times m}$ that, for some positive scalar β, satisfy the following LMIs:*

$$\Upsilon^i \stackrel{\Delta}{=} \begin{bmatrix} -\tilde{Q}_{i11} & -\tilde{Q}_{i12} & \Upsilon(1,3) & \Upsilon(1,4) & 0 & 0 & 0 & 0 & 0 \\ * & -\tilde{Q}_{i22} & \Upsilon(2,3) & \Upsilon(2,4) & 0 & 0 & 0 & 0 & 0 \\ * & * & \Upsilon(3,3) & \Upsilon(3,4) & \Upsilon(3,5) & Z_{11}^T D_i^T & Z_{11}^T D_i^T C_2^T & C_2^T Y^T G_i^T & \Upsilon(3,9) \\ * & * & * & \Upsilon(4,4) & \Upsilon(4,5) & Z_{12}^T D_i^T & Z_{12}^T D_i^T C_2^T & \beta Y^T G_i^T & \Upsilon(4,9) \\ * & * & * & * & -I_r & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\tilde{Q}_{i11} & -\tilde{Q}_{i12} & 0 & 0 \\ * & * & * & * & * & * & -\tilde{Q}_{i22} & 0 & 0 \\ * & * & * & * & * & * & * & -\tilde{Q}_{i11} & -\tilde{Q}_{i12} \\ * & * & * & * & * & * & * & * & -\tilde{Q}_{i22} \end{bmatrix} < 0,$$

$$\begin{bmatrix} H_{11} & H_{12} & B_{1i}^T & B_{1i}^T C_2^T \\ * & H_{22} & 0 & D_{21i}^T \\ * & * & \tilde{Q}_{i11} & \tilde{Q}_{i12} \\ * & * & * & \tilde{Q}_{i22} \end{bmatrix} > 0, \ i = 1, 2, ...N, \quad \textit{and} \quad Tr\{H\} < \delta^2, \tag{10.19}$$

where

$$\tilde{Q}_{i11} = \bar{\Upsilon}_1 \tilde{Q}_i \bar{\Upsilon}_1^T, \ \tilde{Q}_{i12} = \bar{\Upsilon}_1 \tilde{Q}_i \bar{\Upsilon}_2^T, \ \tilde{Q}_{i22} = \bar{\Upsilon}_2 \tilde{Q}_i \bar{\Upsilon}_2^T, \ Z_{11} \stackrel{\Delta}{=} Z_1 \bar{\Upsilon}_1^T, \ Z_{12} \stackrel{\Delta}{=} Z_1 \bar{\Upsilon}_2^T,$$

$$\bar{\Upsilon}_1 \triangleq [I_n \ \ 0],\ \bar{\Upsilon}_2 \triangleq [0 \ \ I_m], \tag{10.20}$$

$$\begin{aligned}
&\Upsilon(1,3) = A_i Z_{11} + B_{2i} Y C_2,\ \Upsilon(1,4) = A_i Z_{12} + \beta B_{2i} Y,\\
&\Upsilon(2,3) = [C_2 A_i Z_{11} + C_2 B_{2i} Y C_2 - Z_2 C_2] + Z_2 C_2,\\
&\Upsilon(2,4) = [\beta C_2 B_{2i} Y + C_2 A_i Z_{12} - \beta Z_2] + \beta Z_2,\\
&\Upsilon(3,3) = \tilde{Q}_{i11} - Z_{11} - Z_{11}^T,\ \Upsilon(3,4) = \tilde{Q}_{i12} - Z_{12} - C_2^T Z_2^T,\\
&\Upsilon(3,5) = Z_{11}^T C_{1i}^T + C_2^T Y^T D_{12i}^T,\ \Upsilon(3,9) = C_2^T Y^T G_i^T C_2^T,\\
&\Upsilon(4,4) = Q_{i22} - \beta[Z_2 + Z_2^T],\\
&\Upsilon(4,5) = Z_{12}^T C_{1i}^T + \beta Y^T D_{12i}^T,\ \Upsilon(4,9) = \beta Y^T G_i^T C_2^T.
\end{aligned} \tag{10.21}$$

If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by

$$K = Y Z_2^{-T}. \tag{10.22}$$

Remark 10.1. We note that the existence of Z_2^{-T} in (10.22) is guaranteed since $\Upsilon(4,4) = Q_{i22} - \beta[Z_2 + Z_2^T]$ in (10.19) must be negative definite in order to satisfy the inequality of (10.19), and $Q_{i22} > 0,\ \beta > 0$.

10.4 The Robust Stochastic H_∞ Static Output-feedback Controller

Similarly to the previous section, at each point in Ω, say the one that is obtained by $\sum_{j=1}^{N} f_j \Omega_j$ for some $0 \le f_j \le 1$, $\sum_{j=1}^{N} f_j = 1$ we assign a special matrix solution $\tilde{Q}$. For each vertex of Ω, say the i-th, the inequality of (10.14) can be written, following a modified result of [36] as the following set of LMIs:

$$\begin{bmatrix}
-\tilde{Q}_i & \tilde{A}_i Z & \tilde{B}_i & 0 & 0 & 0\\
* & \tilde{Q}_i - Z - Z^T & 0 & Z^T \tilde{C}_i^T & Z^T \tilde{D}_i^T & Z^T \tilde{G}_i^T\\
* & * & -\gamma^2 I_{q+p} & 0 & 0 & 0\\
* & * & * & -I_r & 0 & 0\\
* & * & * & * & -\tilde{Q}_i & 0\\
* & * & * & * & * & -\tilde{Q}_i
\end{bmatrix} < 0 \tag{10.23}$$

where Z is defined in (10.18). We therefore arrive at the following result:

Theorem 10.4. *Consider the uncertain system (10.1), (10.2). The control law (10.3) guarantees a prescribed H_∞-norm $\gamma > 0$ over the entire uncertainty polytope Ω, if there exist $Z_1 \in \mathcal{R}^{n \times (n+m)}$, $Z_2 \in \mathcal{R}^{m \times m}$, $\tilde{Q}_i \in \mathcal{R}^{(n+m)\times(n+m)}$, $i = 1, 2..., N$ and $Y \in \mathcal{R}^{\ell \times m}$ that, for some positive scalar β, satisfy the following LMI:*

$$\begin{bmatrix} \Upsilon^i & \begin{bmatrix} B_{1i} & 0 \\ C_2 B_{1i} & D_{21i} \\ 0 & 0 \end{bmatrix} \\ * & -\gamma^2 I_{q+p} \end{bmatrix} < 0, \quad i = 1, 2, ...N \tag{10.24}$$

where Υ^i is defined in (10.19).

If a solution to the latter set of LMIs exists, the gain matrix K that stabilizes the system and achieves the required performance is given by (10.22).

10.5 Example

To demonstrate the solvability of the various LMIs in this chapter we bring a 3rd-order, two-output, one-input example where we seek static-output feedback controllers in 3 cases: the stochastic H_2 control, the stochastic H_∞ control and the robust stochastic H_∞ control problems. We consider the system of (10.1), (10.2) where:

$$A = \begin{bmatrix} 0.9813 & 0.3420 & 1.3986 \\ 0.0052 & 0.9840 & -0.1656 \\ 0 & 0 & 0.5488 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.0198 & 0.0034 & 0.0156 \\ 0.0001 & 0.0198 & -0.0018 \\ 0 & 0 & 0.0150 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -1.47 \\ -0.0604 \\ 0.4512 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; D_{21} = 0, \; C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \; G = 0. \tag{10.25}$$

We obtain the following results:

- **The stochastic H_2 controller**: Applying the result of Theorem 10.1 and solving (10.11) a minimum H_2-norm bound of $\delta = 0.0449$ is obtained for $\alpha = 2.4$. The corresponding static output-feedback controller of (10.13) is:

$$K = \begin{bmatrix} 0.3469 & 0.6216 \end{bmatrix}.$$

  The corresponding closed-loop poles are: 0.0695, $0.9486+0.1059i$, $0.9486-0.1059i$. These poles reside in the Open Unit Circle (OUC).
- **The stochastic H_∞ controller**: Using Theorem 10.2 and solving (10.15) a minimum value of $\gamma = 0.8916$ is obtained for $\alpha = 2.4$. The corresponding static output-feedback controller of (10.13) is:

$$K = \begin{bmatrix} 0.3567 & 1.2622 \end{bmatrix},$$

and the corresponding closed-loop poles are 0.0961, $0.9087+0.0882i$, and $0.9087-0.0882i$. Also these poles are all in the OUC. For the nominal case, where $D=0$ (i.e with no state-multiplicative noise), we obtain for the above value of α an attenuation level of $\gamma=0.6572$.

- **The robust stochastic H_∞ controller:** We consider the system of (10.1), (10.2) where the system matrices A, B_1 and B_2 of (10.1) reside in a polytope of 4 vertices. The system matrices include the matrices of (10.25a-i) and the following 3 sets of A^j, B_1^j and B_2^j for $j=2,3,4$:

$$A^2=\begin{bmatrix}0.9872 & 0.3575 & 1.2273\\ 0.0016 & 0.9872 & -0.1603\\ 0 & 0 & 0.5488\end{bmatrix},\ B_1^2=\begin{bmatrix}0.0199 & 0.0036 & 0.0137\\ 0.0000 & 0.0199 & -0.0018\\ 0 & 0 & 0.0150\end{bmatrix},$$

$$A^3=\begin{bmatrix}0.9687 & 0.9840 & 3.6304\\ 0.0043 & 0.9742 & -0.4647\\ 0 & 0 & 0.5488\end{bmatrix},\ B_1^3=\begin{bmatrix}0.0197 & 0.0099 & 0.0412\\ 0.0000 & 0.0197 & -0.0052\\ 0 & 0 & 0.0150\end{bmatrix},$$

$$A^4=\begin{bmatrix}0.9857 & 0.5881 & 2.5226\\ -0.0135 & 0.9717 & -0.4702\\ 0 & 0 & 0.5488\end{bmatrix},\ B_1^4=\begin{bmatrix}0.0199 & 0.0059 & 0.0284\\ -0.0001 & 0.0197 & -0.0051\\ 0 & 0 & 0.0150\end{bmatrix},$$

$$B_2^2=\begin{bmatrix}-4.9990\\ -0.0576\\ 0.4512\end{bmatrix},\ B_2^3=\begin{bmatrix}-0.4376\\ -0.1589\\ 0.4512\end{bmatrix},\ B_2^4=\begin{bmatrix}-1.4700\\ -0.0604\\ 0.4512\end{bmatrix}.$$

Applying the result of Theorem 10.4 and solving (10.24) a minimum value of $\gamma=5.39$ is obtained for $\beta=1.02$. The corresponding static output-feedback controller of (10.22) is:

$$K=\begin{bmatrix}0.2007 & 1.9022\end{bmatrix}.$$

10.6 Conclusions

In this chapter we present a convex optimization method that provides an efficient design of robust static output-feedback controllers for linear systems with state multiplicative noise. Our treatment is similar to the one of Chapter 5 with the clear advantage that in the present chapter the results are obtained without adding the low-pass component that was required in Chapter 5 in order to achieve convexity. We also consider linear systems with polytopic type uncertainties in which case and based on a linear parameter dependent Lyapunov function, sufficient conditions are derived for the existence of a constant output-feedback gain that stabilizes the system and achieves a prescribed bound on its performance over the entire uncertainty polytope.

Both stochastic H_2 and H_∞performance criteria have been considered. For both, conditions for quadratic stabilizing solution have been obtained. The conservatism entailed in these conditions are reduced either by using a recent method that enables the use of parameter dependent Lyapunov based optimization (adopted for the stochastic case), or by treating the vertices of the uncertainty polytope as distinct plants. The latter solution cannot guarantee the stability and performance within the polytope whereas the former optimization method achieves the required bound over the entire polytope.

To the best of our knowledge, no other solution has been published in the literature that concerns the discrete-time static output-feedback for state-multiplicative noise system. Due to the latter fact we can not compare our results, in the example, to any existing solution. However, it is shown, in the deterministic counterpart of this work (for the continuous-time case [100]), that the results obtained in [100] are less conservative than any other result obtained by other existing method.

In the example we demonstrate the tractable solvability of the various LMIs obtained in our study. We also note that the proposed method can be used also to solve the mixed stochastic H_2/H_∞ problem where a robust static output-feedback controller is sought that achieves, say, a prescribed attenuation level while minimizing a bound on H_2-norm of the closed-loop.

11

Systems with State-multiplicative Noise: Applications

11.1 Altitude Estimation

In order to demonstrate the application of the theory developed in Chapter 3, we consider the problem of altitude estimation with measurements from a RADAR altimeter and a baro altimeter. The barometric altitude measurement is based on a static pressure measurement. As a result of various sources of error, (e.g. initial reference error, static pressure measurement bias, or temperature measurement errors) the baro altimeter is corrupted with a bias error (see page 32 [75]) up to 1000 ft together with a small white noise component. Denoting the true altitude above ground by h, we have the following approximate model for the altitude hold loop which is commanded by the altitude command $w(t)$:

$$dh = -1/\bar{\tau}(h - w)dt, \quad h_{baro} = h + b + \zeta_1 \tag{11.1}$$

where $\bar{\tau}$ is the time constant of the command response, b represents the baro altitude measurement bias and ζ_1 is a standard zero-mean white noise with intensity R_1, that is: $E\{\zeta_1(t)\zeta_1(\tau)\} = R_1\delta(t - \tau)$.

The RADAR altimeter measures the height above ground level without bias, however, its output is corrupted by a broad band measurement noise, the intensity of which increases with height due to a lower SNR (signal to noise ratio) effect at higher altitude (see [73], page 196 for various models depending on the SNR levels).

A reasonable model for this effect is:

$$\begin{aligned} h_{radar} &= h(1 + \eta) + \zeta_2, \\ E\{\zeta_2(t)\zeta_2(\tau)\} &= R_2\delta(t - \tau), \\ E\{\eta(t)\eta(\tau)\} &= R_3\delta(t - \tau). \end{aligned} \tag{11.2}$$

In this model, η represents the altitude-dependent broad-band measurement noise and ζ_2 represents the altitude-independent white noise. The overall model is completed by:

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 175–204, 2005.

$$\dot{b} = \sqrt{2}\bar{w}$$

where $\bar{w}$ is a standard white noise signal which interferes with the measurement of the bias. The later signal is not correlated with the other white noise signals in the system.

To obtain a state-space description we denote:

$$x \stackrel{\triangle}{=} col\{h,\ b\}, \quad y \stackrel{\triangle}{=} col\{[h_{baro},\ h_{radar}\}$$

where x is the state vector and y is the measurement vector. Using the measurements $Y(t) \stackrel{\triangle}{=} \{y(\tau), \quad \tau < t\}$, we want to estimate $z(t) = [h(t) \quad b(t)]^T$.

Combining the dynamical descriptions in (11.1), (11.2) and that of $\dot{b}$, we obtain:

$$dx = (Ax + B_1\tilde{w})dt, \quad dy = (Cx + D_{21}\tilde{w})dt + Fxd\eta, \quad z = Lx$$

where

$$\tilde{w} = \begin{bmatrix} w \\ \bar{w} \\ \zeta_1 \\ \zeta_2 \end{bmatrix} \quad A = \begin{bmatrix} -1/\bar{\tau} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1/\bar{\tau} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 \\ \sqrt{R_3} & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 & \sqrt{R_1} & 0 \\ 0 & 0 & 0 & \sqrt{R_2} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In our problem, the altitude hold-loop dynamics is characterized by a time-constant of $\bar{\tau} = 30sec$. We consider a step maneuver in w at a time $5.6sec$ from an initial condition of $w = h(0) = 500ft$ to the value of $w = 5000ft$. We also take an initial bias of $b = 0ft$ and we solved the case where $R_1 = R_2 = 0.4$ and $R_3 = 0.0016$.

We consider 3 filters:

1) **Kalman filter**: This filter ignores the multiplicative noise $\eta(t)$. For the design of the filter we assume that w is an unbiased white-noise signal with: $E\{w(t)w(\tau)\} = Q\delta(t-\tau)$ where Q should be determined to achieve the best result for the filter (a tuned Q).

2) H_∞ **filter**: The multiplicative noise $\eta(t)$ is neglected and a standard H_∞ design is used. In order to compare the results with those obtained by the Kalman filter we multiply B by $\sqrt{Q}$. The measurement equation is changed to

$$y = Cx + \sqrt{R}\tilde{\zeta}$$

where $\tilde{\zeta} = [\zeta_1 \quad \zeta_2]^T$ and where $R = diag\{R_1, R_2\}$.

3) **Stochastic** H_∞ **filter**: The term $F\eta$ is taken into consideration using the results of 3.1 In our case (A, B) is not detectable. We therefore add a

very small perturbation term to the second diagonal element in A (we used $-2 \cdot 10^{-6}$).

Since the best Q-tuned Kalman filter is obtained for $Q = 1000$, we designed the three filters for this value of Q. We obtain the following:

- The Kalman filter: The following transfer function matrix for the tuned Kalman filter relates the two measurements of h_{radar} and h_{baro} to the estimate of the height h:

$$G_{Kalman} = \frac{[0.64s \;\; 1.50s + 3.65]}{s^2 + 4.24s + 3.72}.$$

  The behaviour of the resulting L_2 estimate is depicted in Fig. 11.1b, where the true trajectory of h is described in Fig. 11.1a. In these figures, and the figures below, we depict the time behaviour for 50 seconds into the future, in order to accentuate the difference between the various filter designs. The Kalman estimate is poor with a mean square error of $34890 ft^2$.
- The nominal H_∞ design: We designed this filter with $\gamma = 1.12$. We choose this value for γ in spite of the fact that the minimum possible value is 0.632. This is because we want to compare this design with the one achieved by the stochastic H_∞ filter which achieves a minimum value of $\gamma_0 = 1.10$. For the value of $\gamma = 1.12$, we obtain the following transfer function matrix which predicts h from h_{radar} and h_{baro}:

$$G_{nom} = \frac{[0.99s \;\; 2.61s + 9.10]}{(s^2 + 6.46s + 9.07)}.$$

  The time behaviour of the resulting estimate is given in Fig. 11.1c. It is seen in this case that the estimate is even worse than the Kalman filter given above. The corresponding mean square estimation error is $50277 ft^2$.
- The stochastic H_∞ filter : This filter is calculated for the same value of $\gamma = 1.12$, where the minimal achievable value in this case is $\gamma = 1.10$. The corresponding transfer function matrix is given by

$$G_{sto} = \frac{[1.52s \;\; 1.03s + 4.12]}{(s^2 + 5.06s + 4.12)}.$$

  The time behavior of the estimate that is achieved by this filter is given in Fig. 11.1d. This is, by far, the best of all the three filters considered. Its estimate is closest to the actual trajectory of h and its mean squares error is only $19106 ft^2$.

11.2 Altitude Control

In this section an altitude control example is presented to illustrate the design of static-feedback controller in presence of multiplicative noise, as considered

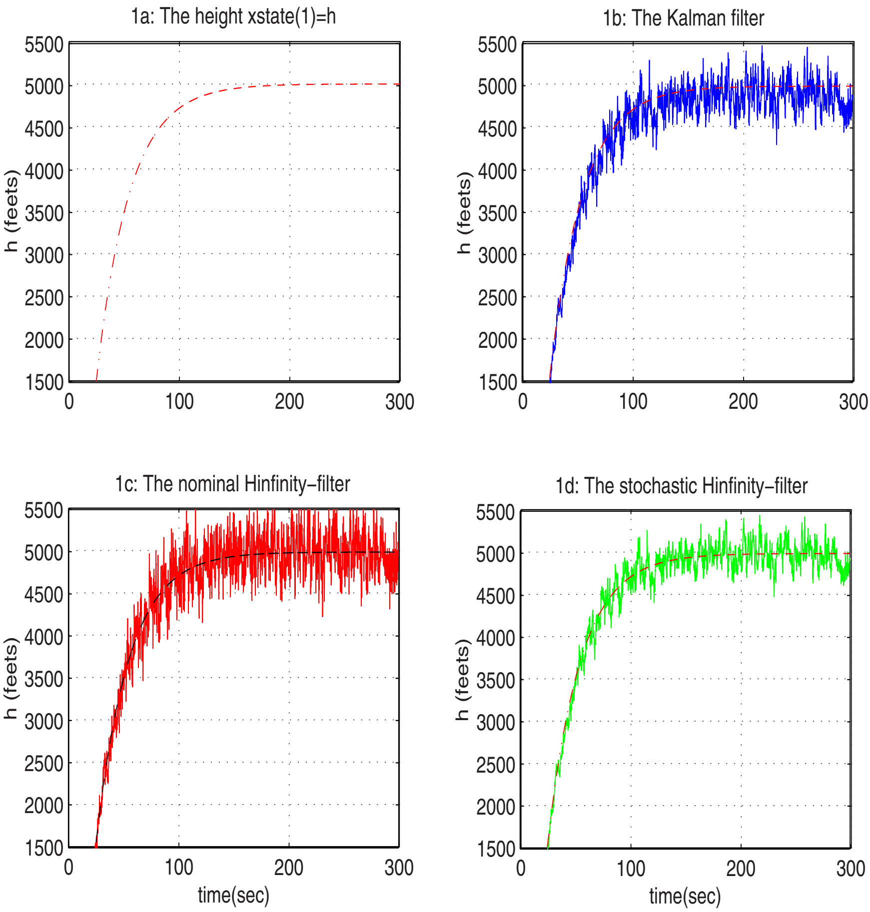

Fig. 11.1. Comparison between the stochastic H_∞, the H_∞, and the Kalman filters:

in Chapter 5. Our example is adapted from (McLean 1990,[80]) where a non-causal control system was designed, based on complete *a priori* information about the altitude level. A causal version of this altitude control problem is considered in the sequel where the effect of the height on the RADAR altimeter Signal to Noise Ratio (SNR) is considered.

We consider the system of (5.1) where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -.088 & .0345 & 0 & 0 & 1 & -.0032 & 0 & 0 \\ 0 & 0 & \Theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\tau_d & 0 & 0 & 0 & 0 & -1/\tau_d \end{bmatrix} \tag{11.3}$$

and

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\Theta & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.09 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{11.4}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 & 0 & 0 & -0.15 \end{bmatrix}, \tag{11.5}$$

where τ_d is a time constant of an approximate differentiator, $\Theta = 0.025[rad/sec]$ corresponds to the required closed-loop time-constant and where, rather than zero, $\epsilon_2 = -0.01$ is chosen, in order to obtain finite gain at low frequencies of the thrust control loop. The state vector, measurements vectors and the exogenous signals, are given by the following table:

Variable	Physical Variable
x_1	Vertical acceleration
x_2	Height rate
x_3	Height
x_4	Thrust Command
x_5	Thrust
x_6	Airspeed
x_7	Height error integral
x_8	Filtered Height
y_1	RADAR altimeter height meausurement
y_2	Integrator Output
y_3	Vertical acceleration meausurement
y_4	Height rate of RADAR altimeter output
u_1	Vertical acceleration command
u_2	Thrust rate command
w	Height command
ν	Height measurement multiplicative noise

The fact that the height measurement errors become larger with the height (Levanon 1988, [73]) due to SNR decrease is reflected by $F_{1,3}$. Note that the estimate of the height rate is obtained by feeding the RADAR altimeter output to the approximate differentiator $s/(1+s\tau_d) = [(1-1/(1+s\tau_d)]/\tau_d$. Since we have taken $\tau_d = 1sec$, the nonzero terms in $F_{4,3}$ and $F_{4,8}$ reflect the multiplicative noise which is produced by the approximate differentiation of the RADAR altimeter output.

The control problem definition is then completed by (5.2) where

$$C_1 = \begin{bmatrix} 0\,0\,0\,0\,0 & 0 & 1\,0 \\ 0\,0\,0\,0\,0 & 2.23 & 0\,0 \\ 0\,0\,0\,0\,0 & 0 & 0\,0 \\ 0\,0\,0\,0\,0 & 0 & 0\,0 \end{bmatrix} \quad \text{and} \quad D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{3} & 0 \\ 0 & \sqrt{0.3} \end{bmatrix}. \tag{11.6}$$

Notice that C_1 reflects the goals of controlling the closed-loop bandwidth to be Θ (namely, a rise-time of about 40 seconds) and minimizing the airspeed changes.

Notice also that the above selection of z and the fact that the exogenous disturbance signal is the height command, imply that

$$||\Theta T_{e,w}/s||_\infty < \gamma,$$

where $T_{e,w}$ is the transference between the height command w and the height tracking error $e = x_3 - w$. Therefore, the time constant τ of the transference that describes the attenuation of the height command effect is approximately given by $\tau = \gamma\Theta^{-1}$. The exact time constant is affected by the control weighting D_{12} which, in turn, affects γ and τ.

We next design a static output-feedback type controller using the result of Theorem 5.2. To this end we first relocate the double integrator poles, which correspond to the transfer function between acceleration and position, to a stable complex pair. This is achieved by the static output-feedback controller

$$u = \tilde{u} + K_{aux}y$$

where

$$K_{aux} = \begin{bmatrix} -0.02 & 0 & 0 & -0.2 \\ 0.002 & 0 & 0 & -0.02 \end{bmatrix}. \tag{11.7}$$

This auxiliary control signal u, which is needed in order to avoid numerical problems with the LMI solver, includes, however, the multiplicative noise in y. We, therefore, have both nonzero F and D with a unit correlation coefficient ($\sigma = 1$) between $\beta(t)$ and $\nu(t)$ in (5.1), namely

$$D = B_2 K_{aux} F = 10^{-4} \begin{bmatrix} 0 & 0 & -330 & 0 & 0 & 0 & 0 & 300 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11.8}$$

The design was performed using the result of Theorem 5.2 and using the $MATLAB^{TM}$ LMI Toolbox [35] to solve the relevant matrix inequalities. We obtained $\gamma = 0.6625$ (actual closed-loop norm is 0.6495) and

$$K = \begin{bmatrix} -0.00124 & -0.06018 & -0.05012 & 0.00759 \\ 0.000321 & 0.00650 & 1.32 \times 10^{-5} & 0.00134 \end{bmatrix}. \tag{11.9}$$

The simulations results are depicted in Fig. 11.2-11.3. In Fig. 11.2 the altitude as a function of time is depicted where the smooth line in Fig. 11.2 show the altitude command. The acceleration command a_{z_c} (acceleration, z-axis, command) and thrust-rate command (usually denoted as δ_{TH_c} (thrust, command) are respectively depicted in Figs. 11.3-11.4. The effect of the multiplicative noise in these measurements is clearly observed as the noise magnitude increases with the altitude. Notice that the time-response of Fig. 11.2 can be made faster by applying an anti-causal feed-forward in the style suggested in (McLean 1990, [80]) but the latter strategy is out of the scope of the this example.

11.3 Guidance-motivated Tracking Filter

We illustrate the use of the theory of Chapter 8 in a guidance motivated tracking problem. In this problem, a scheduled estimation is obtained in spite of a significant noise intensity that is encountered in the measurement of the scheduling parameter.

Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \omega$$

where x_1 is the relative separation between an interceptor and an evader, normal to a collision course, x_2 is its derivative, with respect to time, and ω represents the relative interceptor-evader maneuvers. The state x_2 has to be estimated via the following measurements :

$$y = x_1/R + \nu \qquad R_m = R(1+\zeta) \tag{11.10}$$

where ν and ζ are additive and multiplicative white noise zero-mean signals in the bearing measurement and the measurement R_m of the range R, respectively. These noise signals stem from the characteristics of the measuring devices. Substituting (11.10) in (11.10) we have

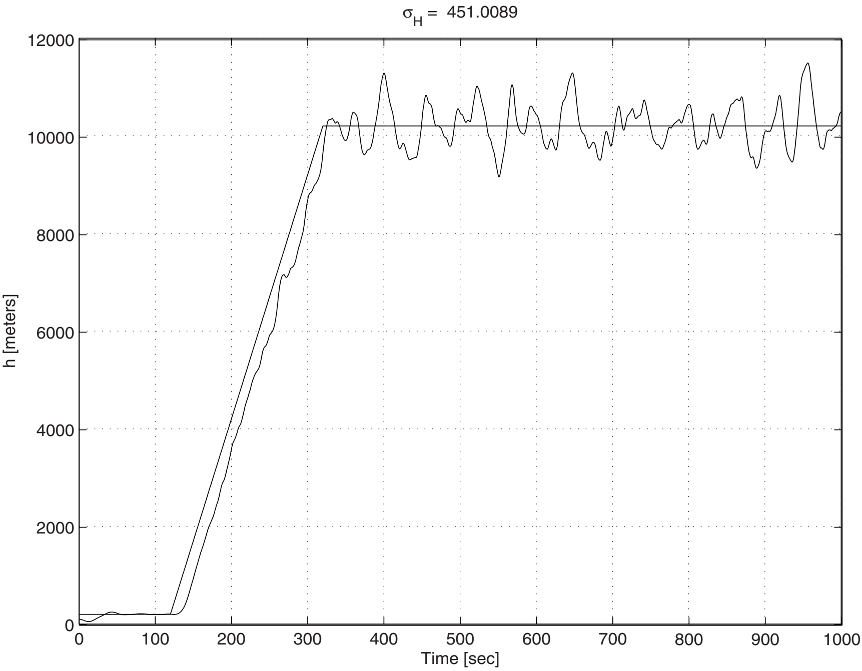

Fig. 11.2. - Height and Height Command (smooth line)

$$y = x_1(1+\zeta)/R_m + \nu.$$

Given the variances of ν and ζ, it is desired to obtain an estimate that is scheduled by the measurement of R_m and achieves a given H_∞ estimation level. We solve the problem in discrete-time where we sample the continuous system with a sampling period T. The resulting discrete-time system is the one described in (8.1) with $D = 0$, $G = 0$,

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T^2/2 & 0 \\ T & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & \bar{\rho} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The range measurement of R_k satisfies the discrete-time version of (11.10) and is given by

$$R_{m,k} = R_k(1+\zeta_k), \tag{11.11}$$

where $\{\zeta_k\}$ is a zero-mean white-noise sequence which is a reasonable modeling of the sampled version of ζ. The corresponding time-varying matrix C_k in (8.1) is $C_k = \begin{bmatrix} \frac{1}{Rm_k} & 0 \end{bmatrix}$, and the corresponding time-varying version of F in (8.1) becomes $F_k = \begin{bmatrix} \frac{\sigma}{Rm_k} & 0 \end{bmatrix}$, where σ is the standard deviation of $\{\zeta_k\}$. In our

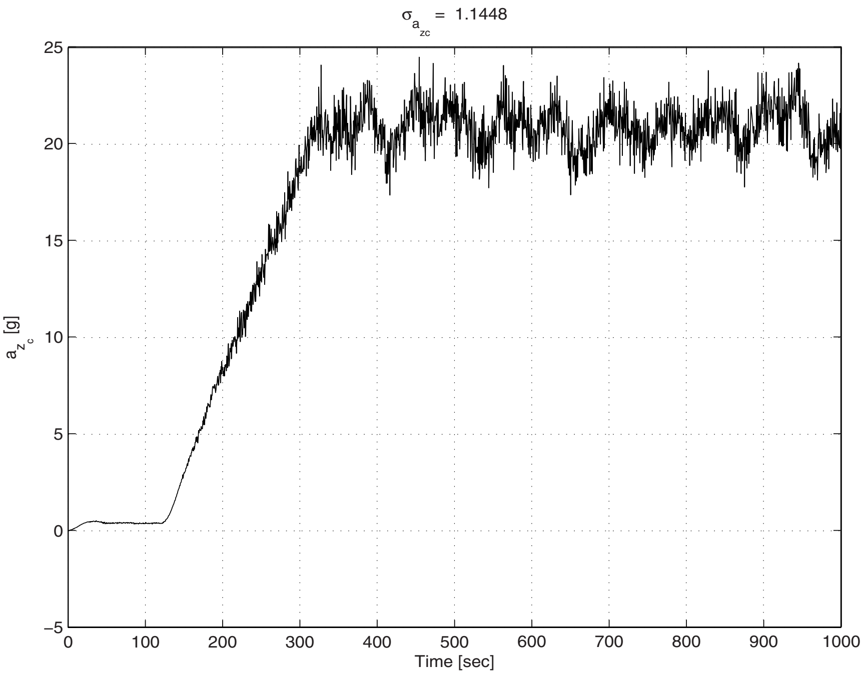

Fig. 11.3. - Acceleration Command a_{z_c}

example, we take $T = 0.025$sec., $\bar{\rho} = 0.001$, and $\sigma = 0.3$. The matrix R_k in (11.11) is taken as

$$R_k = V_c(50 - k/40), \quad k \in [0, N], \text{ where } V_c = 300\text{m/sec}$$

and where $N = 1880$ is taken to match a time range of $[0, 47]$ sec. Since C_k varies significantly during the system operation in $k \in [0 \quad N]$, it cannot be possibly represented by an average value. We therefore take C_k to be affinely dependent on $R_{m,k}^{-1}$ of (11.11) and consider it to be uncertain, varying in the interval described by the two vertices $[g_1 \quad 0]$ and $[g_2 \quad 0]$ where

$$g_1 \stackrel{\triangle}{=} 1/R_1 = 1/15,000, \quad \text{and} \quad g_2 \stackrel{\triangle}{=} 1/R_N = 1/900.$$

The matrix F_k that corresponds to $k = 0$ and $k = N$ is similarly considered as an uncertain matrix lying between $\left[\,0.3g_1 \ 0\,\right]$ and $\left[\,0.3g_2 \ 0\,\right]$. The attempt to design a robust filter over the whole range of $R_{m,k}$ by using the results of [38], without taking into account the state multiplicative noise, would clearly cause a significant over-design since it leads to a single H_∞ filter that satisfies the required estimation level over the whole interval of uncertainty. Instead,

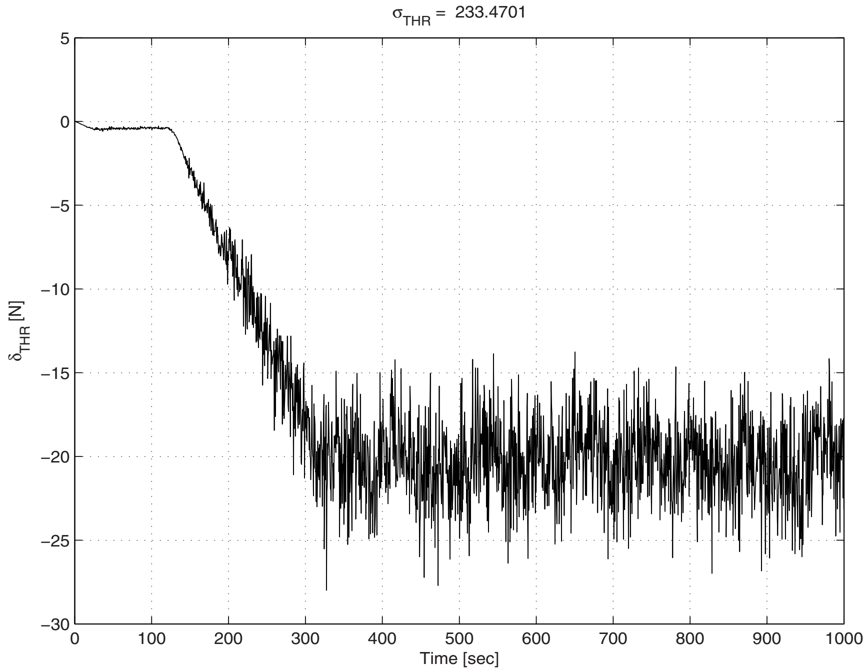

Fig. 11.4. - Thrust Rate Command δ_{TH_c}

since we measure $R_{m,k}$, we may use this noisy measurement of R_k to schedule the filter at time k. This scheduling is based on the fact that the LMI of (8.9) is affine in the products ZC and ZF. We thus define $\Psi_C \stackrel{\triangle}{=} Z_k C_k$ and $\Psi_F \stackrel{\triangle}{=} Z_k F_k$, bearing in mind that $\Psi_F = \sigma \Psi_C$. Keeping Ψ_C constant for all k we want to solve for (W, S, Z_k, T, R) in (8.9) for k in $[1,\ N]$. This is done by first solving the LMI for $k = 1$, namely for C_k and F_k that correspond to R_1, we obtain the matrix Z_1. The resulting Ψ_C and Ψ_F are easily calculated. Since the last two matrices are fixed for all $k\ \in\ [1\ N]$, we obtain Z_N from either $\Psi_C = Z_N C_N$ or $\Psi_F = Z_N F_N$.

For any $k\ \in\ [1\ N]$, $R_{m,k}$ can be expressed as a convex combination of R_1 and R_N, say $R_{mk} = \alpha_k R_1 + (1-\alpha_k) R_N, \quad \alpha \in [0, 1]$, The matrix Z_k that satisfies $Z_k C_k = \Psi_C$ and $Z_k F_k = \Psi_F$ is obtained by $Z_k = \alpha_k Z_1 + (1-\alpha_k) Z_N$. The corresponding B_{fi} in (8.2) is then given, at any instant $i \in [1, N]$, by $B_{fi} = \alpha_i B_{f1} + (1-\alpha_i) B_{fN}$, where B_{f1} and B_{fN} are obtained from Z_1 and Z_N by (8.11). The matrices A_f and C_f in (8.2) are constant. They are obtained by (8.11).

We solved the problem for $\gamma = 30$ where we also added the requirement for a minimum upper-bound on the H_2-norm of the estimation error. We obtained $Z_1^T = \begin{bmatrix} -0.1153 & -20.088 \end{bmatrix}$, $B_{f1}^T = \begin{bmatrix} 1.895 & 0.088 \end{bmatrix}$, $Z_N^T = \begin{bmatrix} -1.9231 & -333.8954 \end{bmatrix}$, and $B_{fN}^T = \begin{bmatrix} 31.653 & 1.474 \end{bmatrix}$. The pair (A_f, C_f) is

$$A_f = \begin{bmatrix} 0.9978 & 0.025 \\ -0.0001 & 0.9999 \end{bmatrix}, \quad C_f = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The resulting scheduled estimate is given by $\hat{z}_k = C_f \hat{x}_k$ where

$$\hat{x}_{k+1} = A_f \hat{x}_k + [\alpha_k B_{f1} + (1 - \alpha_k) B_{fN}] y_k, \quad \hat{x}_0 = 0.$$

Fig 11.5 describes the ensemble average of the estimation error and the plus and minus standard deviation (sd) plots for ten randomly selected white noise sequences $\{\zeta_k\}$. We compare the latter results to those obtained by applying the standard time-varying aposteriori Kalman filter. It is seen that although our filter is based on *a priori* measurements, we achieve better results than those obtained by the aposteriori Kalman filter. It is also seen that our method, where we use $1/R_{m,k}$ to schedule the gain instead of computing the gain on-line (as with the Kalman filter), is more robust at small ranges, where the effect of the additive noise in the range measurement on the bearing is more significant.

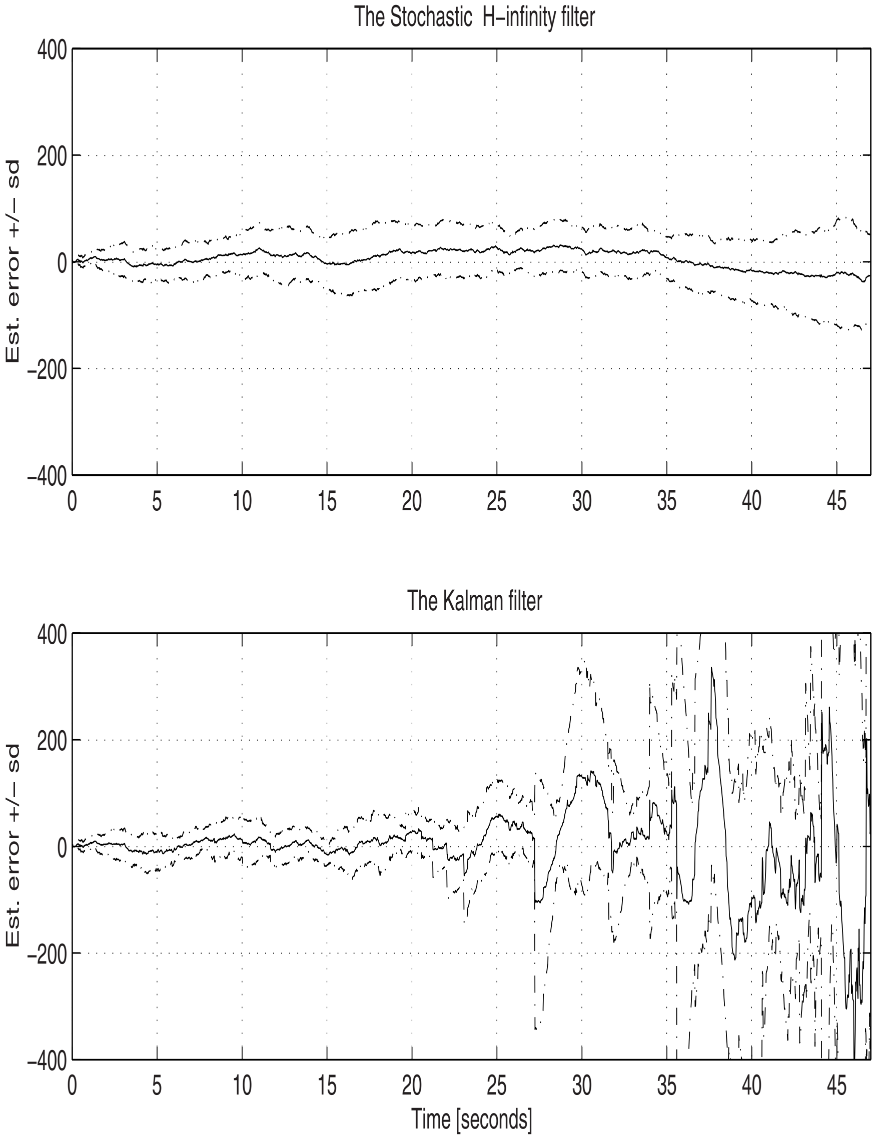

Fig. 11.5. Comparison between the H_∞ and the Kalman filters. In both parts: the solid line describes error mean and the dashed lines describe the +/- sd as indicated.

11.4 Terrain Following

In this section we bring a second example to illustrate the design of static output-feedback controllers in presence of multiplicative noise, as considered in Chapter 5 and Section 11.2. We present a terrain following example which is adapted from [80] where a non-causal control system is designed, based on complete *a priori* information about the terrain level. A causal version of this terrain following problem is considered in the sequel where the effect of the height on the RADAR altimeter Signal to Noise Ratio (SNR) is modeled as a state-multiplicative noise.

We consider the system of (5.1) where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -.088 & .0345 & 0 & 0 & 1 & -.0032 & 0 \\ 0 & 0 & \bar{\Theta} & 0 & 0 & 0 & \epsilon_2 \end{bmatrix} \tag{11.12}$$

and

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\Theta} \end{bmatrix}^T, \; B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.09 & 0 & 0 & 0 \end{bmatrix}^T, \tag{11.13}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 & 0.04 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11.14}$$

where $\bar{\Theta} = 1/20[rad/sec]$ and $\epsilon_2 = -10^{-5}$. Note that in comparison with Section 11.2, here the height rate is assumed to be directly measured. Also the multiplicative noise level reflected in F is somewhat smaller in order to allow comparison to other methods. The state vector, measurements vector, and the exogenous signals and control inputs, are given in the table of the next page.

The fact that errors in height measurement are larger with the height [73] due to a decrease in SNR is reflected by the none zero entry of $F_{1,3}$.

The control problem definition is then completed by (5.2) where

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2.23 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{3} & 0 \\ 0 & \sqrt{0.3} \end{bmatrix} \tag{11.15}$$

State Variable	Physical Variable
x_1	Vertical acceleration
x_2	Height rate
x_3	Height
x_4	Thrust Command
x_5	Thrust
x_6	Airspeed
x_7	Height error integral
Measured Variable	Physical Variable
y_1	RADAR altimeter height meausurement
y_2	RADAR altimeter height rate meausurement
y_3	Integrator Output
Disturbance Variable	Physical Variable
w	Height command
ν	Height measurement multiplicative noise
Control Variable	Physical Variable
u_1	Acceleration command
u_2	Thrust rate command
Minimized Variable	Physical Variable
z_1	Integral of tracking error
z_2	Weighted true air speed
z_3	Weighted acceleration command
z_4	Weighted thrust rate command

Notice that C_1 reflects the goals of controlling the closed-loop bandwidth to be $\bar{\Theta}$ (namely a rise-time of about 20 seconds) and minimizing the airspeed changes. Notice, also, that the above selection of z_1 and the fact that the exogenous disturbance signal is the height command, imply (as in Section 11.2 that

$$||T_{z_1,w}||_\infty < \gamma$$

in the sense of (5.5) where

$$T_{z_1,w} = \bar{\Theta}\frac{1}{s}(x_3 - w)$$

is the transference between the height command w and the weighted integral height tracking error, and where s is the differentiation operator. Therefore, the time constant τ of the transference that describes the attenuation of the height command effect is approximately given by $\tau = \gamma\bar{\Theta}^{-1}$. The exact time constant is affected by the control weighting D_{12} which, in turn, affects γ and τ. We note that the approach of modeling the reference signal (the height command in our case) in tracking problems as an exogenous finite energy signal is rather standard in the H_∞ literature (see [32], pp. 18-19). The latter approach is useful when no future information is available on the reference signal. Otherwise, an anti-causal feed-forward can be used (e.g. [98]).

We compare a couple of controller designs, both of the static output-feedback type characterized in Theorem 5.2. The first one (Design 1), ignores F of (11.14) (namely assumes $F = 0$) for design purposes but the effect of the nonzero F is tested by simulations. The second one (Design 2), takes F of (11.14) into account, both in design and simulations. Both designs were performed using the result of Theorem 5.2 and using the MATLAB's LMI Toolbox [34] to solve the relevant matrix inequalities.

For Design 1, a minimum attenuation level of $\gamma = 1.0944$ is obtained, using $\rho = 1000$ and $\alpha = 7442.7$, (the actual closed-loop norm is 0.985) and the corresponding K is :

$$K = \begin{bmatrix} -0.0280 & -0.2164 & -0.0286 \\ 0.0001 & 0.0000 & 0.0001 \end{bmatrix}. \tag{11.16}$$

For Design 2 we obtained, using again $\rho = 1000$ and $\alpha = 7442.7$, that $\gamma = 2.7138$ (the actual closed-loop norm is 1.55) and

$$K = \begin{bmatrix} -0.0170 & -0.1593 & -0.0111 \\ 0.0000 & -0.0001 & 0.0000 \end{bmatrix}. \tag{11.17}$$

Notice that the gains in Design 2 on the measured altitude are of smaller magnitude. A simulation was performed using the static gains obtained above, without the dynamic augmentation including $\rho/(s+\rho)$. Namely, the dynamic augmentation is used for design purposes only. The simulations results that compare the performance of these two designs are depicted in Fig. 11.6- 11.9. In Fig. 11.6 the altitude as a function of time is depicted, where the dash-dotted lines correspond to Design 1 and the solid lines to Design 2. The dotted lines in Fig. 11.6 show the altitude command. The magnitude of the steady-state (i.e. $t > 400$ sec.) oscillations estimated by their standard deviation (that are the result of the state-multiplicative noise corrupting the altitude measurements) is somewhat smaller by about 8 percent at the cost ,however, of a somewhat more sluggish response. The merit of Design 2 is better seen in Figs. 11.7- 11.9 where the true-air-speed, acceleration command and thrust-rate command are respectively depicted with dash-dotted lines corresponding to Design 1 and solid lines corresponding to Design 2. The improvement in these variables, in terms of standard deviation of the steady-state oscillations, as a result of taking F of (11.14) into account is 30, 40 and 40 percent respectively. Namely, the oscillations in the height rate and control variables is significantly reduced at the cost of somewhat slower time-response. Notice that this time-response can be made faster by applying an anti-causal feed-forward in the style suggested in [80].

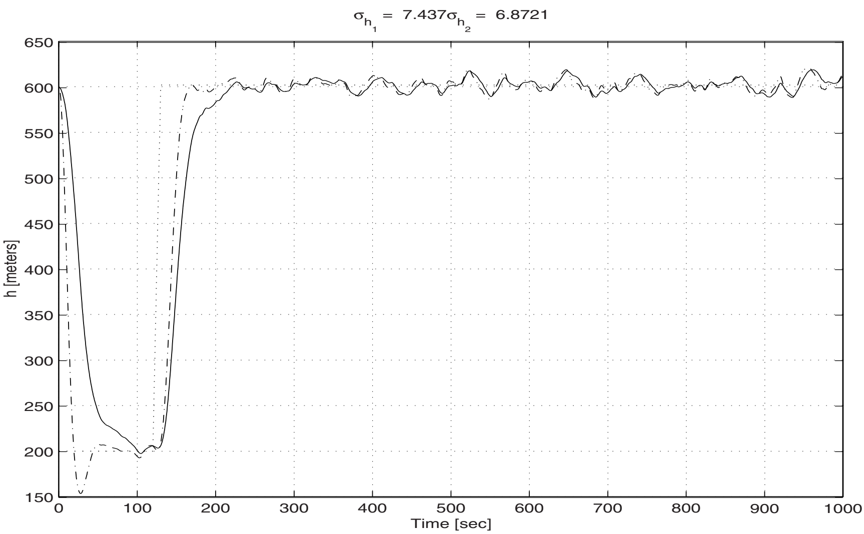

Fig. 11.6. Height and Height Command (dash-dotted lines Design 1, solid lines Design 2, dotted lines altitude command)

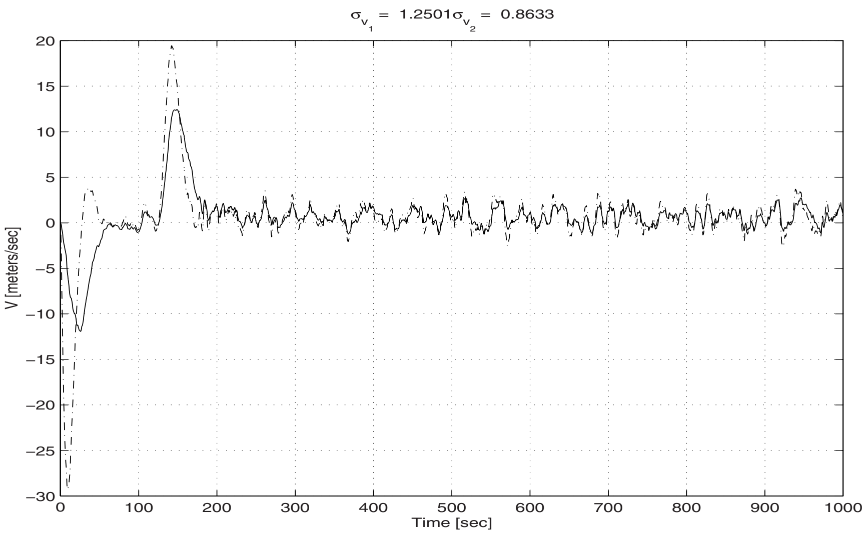

Fig. 11.7. Height Rate (dash-dotted lines Design 1, solid lines Design 2)

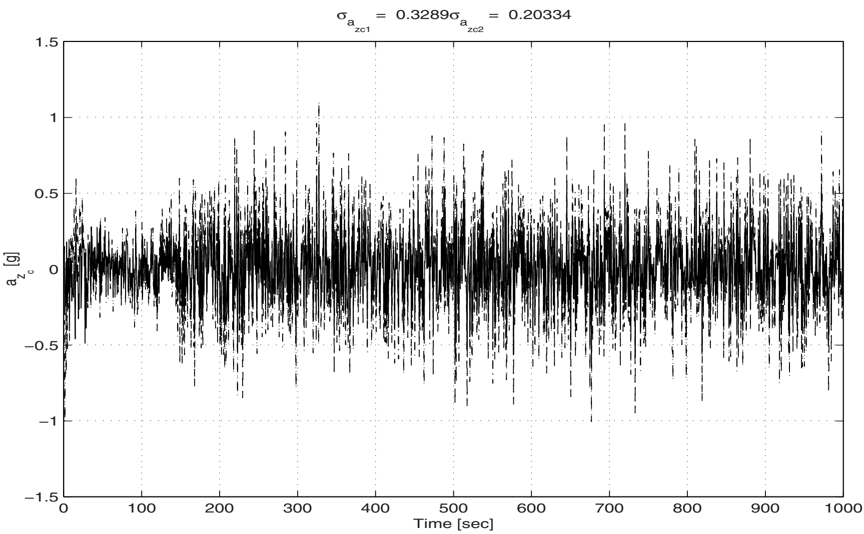

Fig. 11.8. Acceleration Command (dash-dotted lines Design 1, solid lines Design 2)

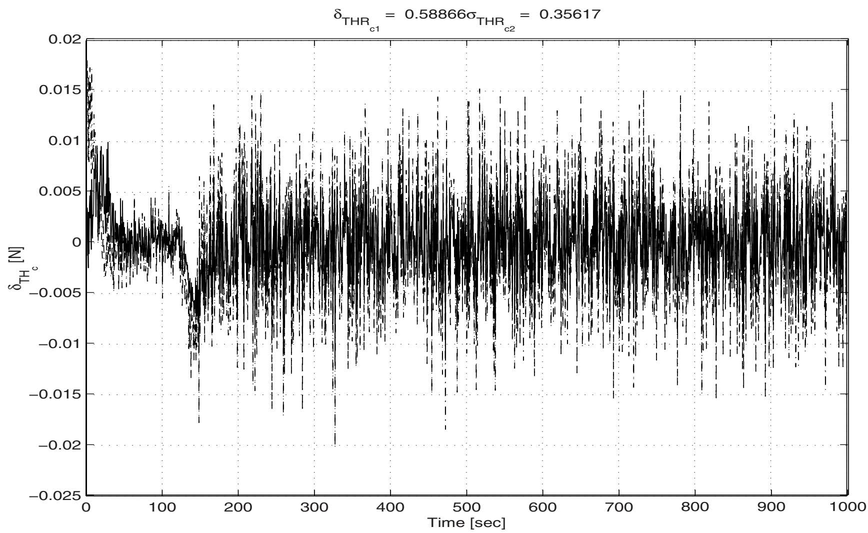

Fig. 11.9. Thrust Rate Command (dash-dotted lines Design 1, solid lines Design 2)

11.5 Stochastic Passivity: Adaptive Motion Control

This example applies the theory of Chapter 6 to adaptive motion control. We consider the following system

$$\begin{aligned}
\tfrac{dx_1}{dt} &= x_2,\\
\tfrac{dx_2}{dt} &= \alpha(-x_2+u),\\
\tfrac{dx_3}{dt} &= \alpha(x_1-x_3)+0.5x_1\dot{\beta}+\dot{v}_1,\\
\tfrac{dx_4}{dt} &= 5(x_2-x_4)+\dot{v}_2,\\
z &= x_3+x_4
\end{aligned}$$

where the range x_1 and the range rate x_2 of a velocity controlled air vehicle, are measured by their low-pass filtered versions x_3 and x_4 respectively. The low-pass filtered version x_3 of the target range is assumed to be corrupted with $\dot{\beta}x_1$, namely the measurement noise increases with the measured range, where β is a standard Wiener process. Both x_3 and x_4 are also driven by white noise processes $\dot{v}_1$ and $\dot{v}_2$ of intensity 10^{-4} (namely $E\{dv_i^2\} = 10^{-4}dt, i = 1,2$) mutually independent and also independent of β. The bandwidth $\alpha \in [5,10]$ of the velocity control loop is assumed to vary due to changes in the flight conditions. It is desired to achieve a regulation of $z(t)$ using the measurements x_3 and x_4 in the presence of variations of the bandwidth α and the various source of measurement noise, including also the state multiplicative noise. A simplified adaptive control is suggested for this control task. Note that while the transference relating u and x_1 is not passive, the one relating u and z is passive (see [113] for a similar idea where the actual controlled variable is chosen as close as possible to the desired control variable under a passivity constraint). We suggest the following stochastic controller :

$$u(t) = -Ky(t)$$

where

$$y(t) = z(t),$$

$z = Cx$ and $x = col\{x_1,\ x_2\ x_3\ x_4\}$, and where $C = \begin{bmatrix} 0\ 0\ 1\ 1 \end{bmatrix}$ and where K obeys the simplified adaptation law of (6.21).

By the results of Section 6.3, the closed-loop stability is ensured if $\dot{x} = Ax - BK_e y + Bu, y = Cx$ is passive where K_e may possibly depend on α. This passivity condition can be verified by $K_e \in [1000, 500]$ corresponding to $\alpha \in [5,10]$ by solving (6.29) which results in the following positive definite solution :

$$P = \begin{bmatrix} 28.1441 & 0.0061 & -10.1908 & -28.1443 \\ 0.0061 & 0.0020 & 0.0039 & 0.0007 \\ -10.1908 & 0.0039 & 21.0051 & 20.8977 \\ -28.1443 & 0.0007 & 20.8977 & 38.8519 \end{bmatrix}$$

A simulation of the above system was run where α linearly varied between 5 and 10 and where the initial range was 500 meters . The simulation results are depicted in Fig. 11.10-11.13 whereas the variation of α is seen in Fig. 11.14: The position x_1 and velocity x_2 are depicted in Fig. 11.10 and Fig. 11.11, respectively, whereas the control signal is depicted in Fig. 11.12. The adaptive gain K is depicted in Fig. 11.13. The results in Fig. 11.10-11.11 show a satisfactory regulation quality of x_1 in spite of the multiplicative noise with a rather smooth behavior of the adaptive gain (see Fig. 11.13). Note also the gradual decrease in the noise component observed in the control signal u in Fig. 11.12 as the range decreases and consequently the effect of multiplicative noise diminishes.

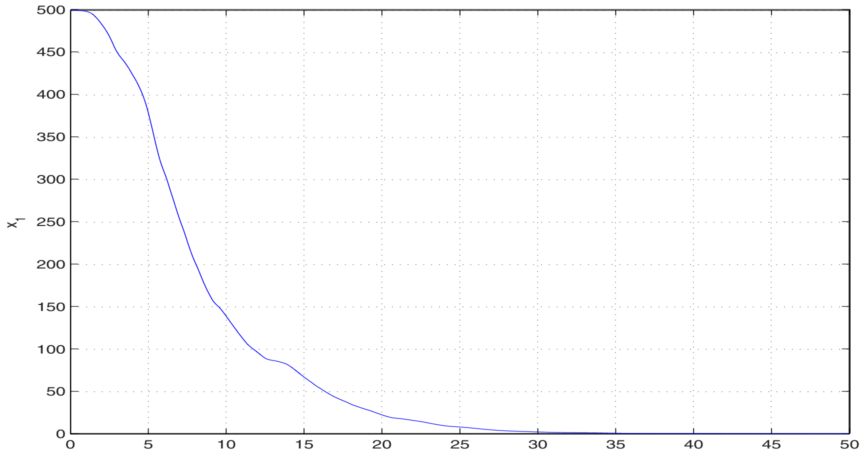

Fig. 11.10. Position

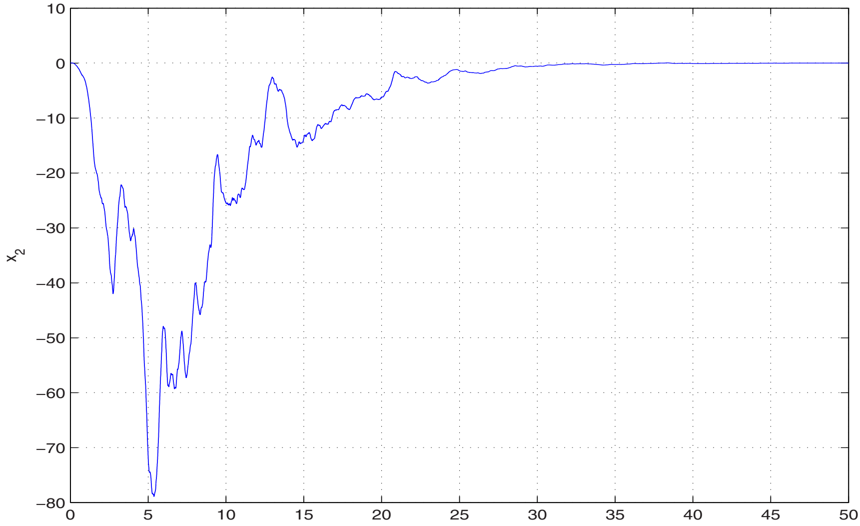

Fig. 11.11. Velocity

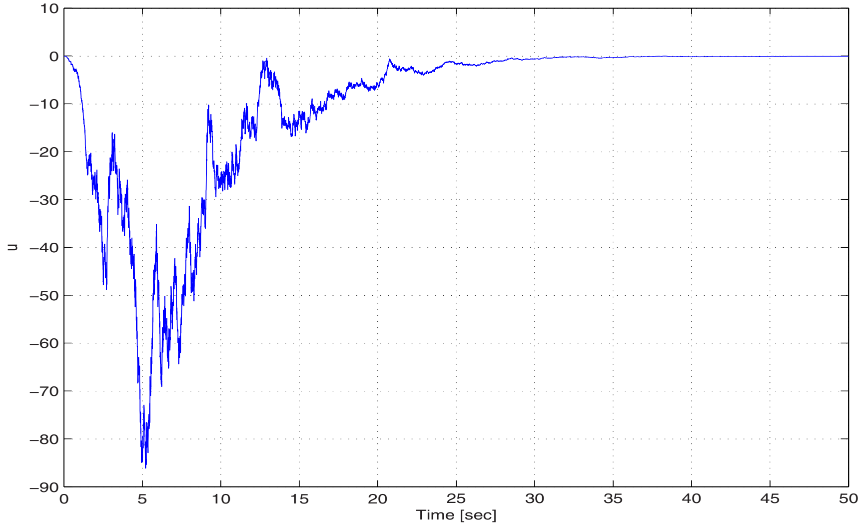

Fig. 11.12. Adaptive Gain

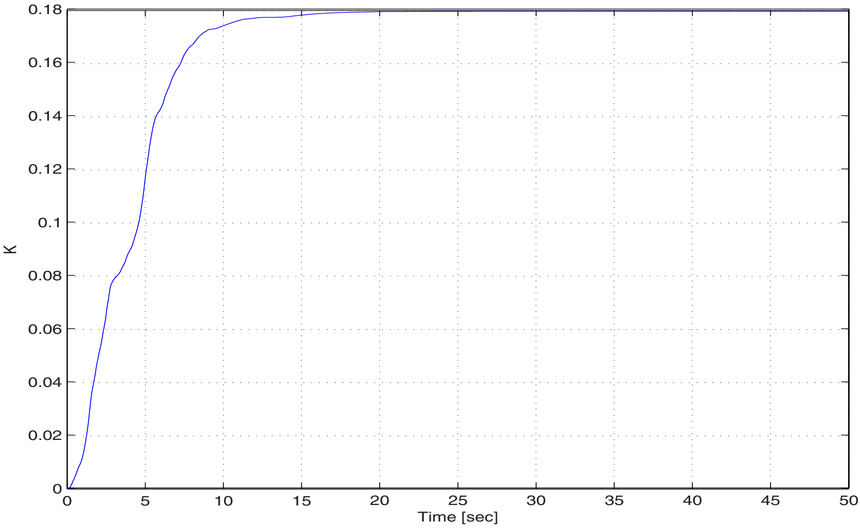

Fig. 11.13. Velocity Command

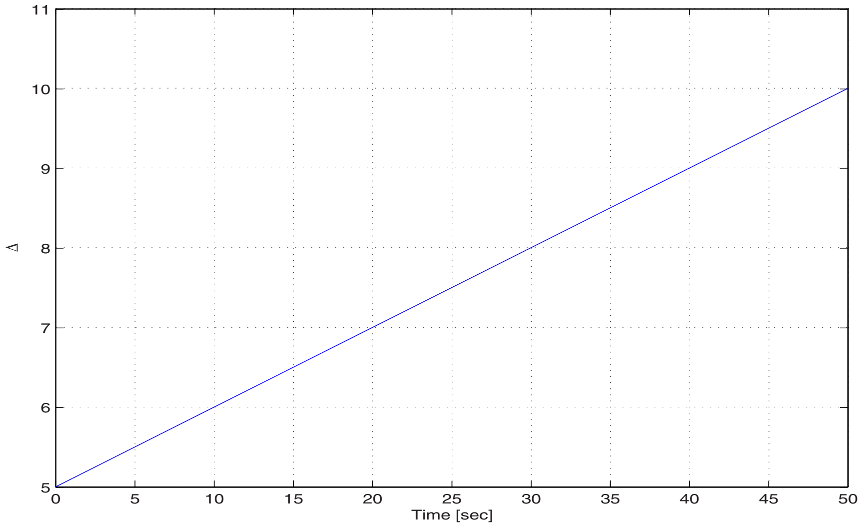

Fig. 11.14. Velocity Loop Bandwidth

11.6 Finite-horizon Disturbance Attenuation: Guidance

This example utilizes the theory of Chapter 9 and is taken from the field of guidance. It is well known ,[115], that modern guidance laws strongly depend on the time-to-go. When the time-to-go is not exactly known, a severe performance degradation can occur. In [4], the effect of bias errors was shown to be significantly reduced, by an optimal rendezvous guidance, where besides the miss-distance (i.e. the relative pursuer-evader position at intercept) the relative velocity and pursuer acceleration at intercept were minimized as well. The rational behind the fact that rendezvous guidance is less sensitive to time-to-go errors stems from the fact that in the last moments before the rendezvous instant, the pursuer is already heading the evader and timing errors can not significantly affect the miss distance. For the same reasons, rendezvous guidance should cope well also with noise effects in the time-to-go measurements, although it is not specifically tuned to deal with such noise . In this example we offer a systematic development of an optimal guidance law which explicitly considers the time-to-go measurements noise effect and compare the performance of the resulting guidance law (which we denote in the sequel by TRK) to the Minimum Effort guidance Law (MEL) of [115] and the Reduced Sensitivity guidance Law (RSL) of [4]. We also refer to the Augmented Proportional navigation (APN) guidance law (see e.g. [115]) which is very popular in missile guidance. Our motivation for analyzing the effect of time-to-go noise is the guidance of missiles equipped with a RADAR seeker. Few applications regarding such missiles are mentioned in [115] from which we adopt some of the data we use for realistic evaluation of the effects of time-to-go measurements noise on various guidance laws.

We consider the following system :

$$\dot{x} = A_c x + B_{2_c} u + B_{3_c} r$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/T \end{bmatrix}, \quad B_{2_c} = \begin{bmatrix} 0 \\ 0 \\ 1/T \end{bmatrix}, \quad B_{3_c} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The first two components of the state vector x are the pursuer-evader relative position and velocity, respectively, whereas the third component is the actual pursuer's acceleration. The control u represents the pursuer's acceleration command and the disturbance r is the evader's maneuver. Due to preference of discrete-time realization, we take a discrete-time model of the above system, where we assume for simplicity that both r and u are the outputs of a zero order hold device with a sampling time of h. The equivalent discrete-time system becomes then [55] the one of (9.1) where :

$$A_k = \begin{bmatrix} 0 & h & -T^2 + Th + T^2 e^{-h/T} \\ 0 & 1 & T(1 - e^{-h/T}) \\ 0 & 0 & e^{-h/T} \end{bmatrix}, \quad B_{2_k} = \begin{bmatrix} h^2/2 - Th + T^2(1 - e^{-h/T}) \\ h - T(1 - e^{-h/T}) \\ 1 - e^{-h/T} \end{bmatrix}$$

$$B_{1_k} = 0 \quad \text{and} \quad B_{3_k} = \begin{bmatrix} h^2/2 \\ h \\ 0 \end{bmatrix}.$$

For design purposes we assume, following [115] and [4], that the evader's maneuver r is constant throughout the interception conflict. Since we also assume that r is measured online by the pursuer, the apparently anti-causal feed-forward part of the control θ_k of (9.11) can be interpreted as a time-varying gain (obtained by setting $r = 1$), multiplying the measured constant disturbance. Although it may seem at first glance that the assumption of constant r is limiting, the resulting guidance law is quite powerful dealing also with slowly time-varying maneuvers [115]. Indeed, we show in the sequel that the guidance law that we obtain is very effective in the presence of a weave maneuver which is typical to tactical ballistic missiles ([115] , page 433).

We assume that the true time-to-go $t_{go_{true_k}} = t_f - kh$ is noisily measured. Namely,

$$t_{go_{meas_k}} = t_{go_{true_k}} + v_k$$

where adopting the RADAR noise model of [115], page 379, we assume that v_k is a standard $N(0, 0.0225)$ white noise sequence (derived from a variance of about 500ft^2 and a closing velocity of $V_c = 300$m/sec) with a sample time of $h = 0.05$sec, and where t_f is the interception conflict duration and t the time from conflict start.

We take the MEL law of [115] as a starting point and try to improve its performance in the presence of the noise v_k in the time-to-go measurements. To this end, we consider the cost function of (9.4) where we take there

$$C_i = 0, \ \forall i < N, \ \ C_N = \begin{bmatrix} 10^4 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \quad \text{and} \quad D_{12_k} = 1.$$

Besides the $\epsilon = 1$ terms which make our problem well posed, taking $D_{3,N} = 0$ and γ which tends to infinity, the guidance law which results from Theorem 9.1 nearly recovers the MEL gains. The gains obtained do not exactly match the MEL gains only due the finite sampling time h and the ϵ terms. We denote the resulting gains matrix by $K_{MEL}(t_{go})$ and compute the first two terms in the Taylor series of these gains. Namely,

$$K_{MEL}(t_{go}) = K_0(t_{go}) + K_1(t_{go})v + O(v^2).$$

If we take the discrete-time version of this Taylor development and substitute in (9.1) we readily obtain a first order approximation to the time-to-go noise effect by setting:

$$F_k = B_{2_c} K_1(k) h \quad \text{and} \quad H_k = D_{12_k} K_1(k)$$

We denote the gains obtained then from Theorem 9.1 by $K_{TRK(t_{go})}$. The RSL gains of [4] obtained by taking

$$C_i = 0,\ \forall i < N,\ \ C_N = \begin{bmatrix} 10^4 & 0 & 0 \\ 0 & 10^3 & 0 \\ 0 & 0 & 10^3 \end{bmatrix} \quad \text{and} \quad \gamma = 12,$$

are denoted by $K_{RSL(t_{go})}$, whereas the APN gains are taken from [115], page 145, and are computed using the noise corrupted time-to-go.

The gains of the MEL, RSL and APN divided by those of the TRK are depicted in Fig. 11.15-11.18. We see already there that the MEL relative velocity gain is lower than the one of the TRK, indicating that the former is less suitable to be a rendezvous type law and is likely to be more sensitive to time-to-go noise. On the other hand, the relative RSL velocity gain is higher (especially at small time-to-go) than the corresponding TRK gain, indicating that it is a rendezvous type law. Apparently, as it seems from the simulation results in the sequel, the RSL gains are too high and lead,therefore, to higher amplification of the time-to-go noise than those of the TRK (at small time-to-go the RSL gain is also higher than those of the MEL) . The relative TRK velocity gain which was especially tailored, by using our multiplicative noise theory, to deal with the time to go noise is just as high as needed. We note that the APN evader acceleration gain is too small to achieve a small miss distance for short conflict durations. We also note that the APN law is the only one of the four guidance laws considered here that is derived by ignoring the pursuer's time-constant. Both its relative insensitivity to time-to-go noise (since it is not "aware" of the need of compensating for this time constant) and its poor performance for maneuvering target in short duration (in terms of pursuer time-constants) are the results of ignoring the pursuer's time constant.

We next simulate the four guidance laws to illustrate the performance under time-to-go noise effects. We use an integration step of $\Delta t = 0.0005$ and take an initial heading error of 10deg. (corresponding to $x_2(0) = 10\pi V_c/180$ where $V_c = 300$m/sec is the closing velocity and the evader maneuvers with 6g$cos(3t)$ (namely $r = 6 * 9.81cos(3t)$ m/sec^2. The latter maneuver model is taken from [115], page 435, and is typical for tactical ballistic missiles weaving into resonance. The resulting performance of the MEL, TRK , RSL and APN laws is depicted in Fig. 11.19 a-d, Fig. 11.20 a-d, Fig. 11.21 a-d and Fig. 11.22 a-d respectively (a-relative separation, b-relative velocity, c-commanded and actual acceleration and d-control effort). Obviously both our new TRK law and the RSL efficiently attenuate the noise effect while MEL fails to do so. The TRK which is especially tailored to deal with the noise performs better.

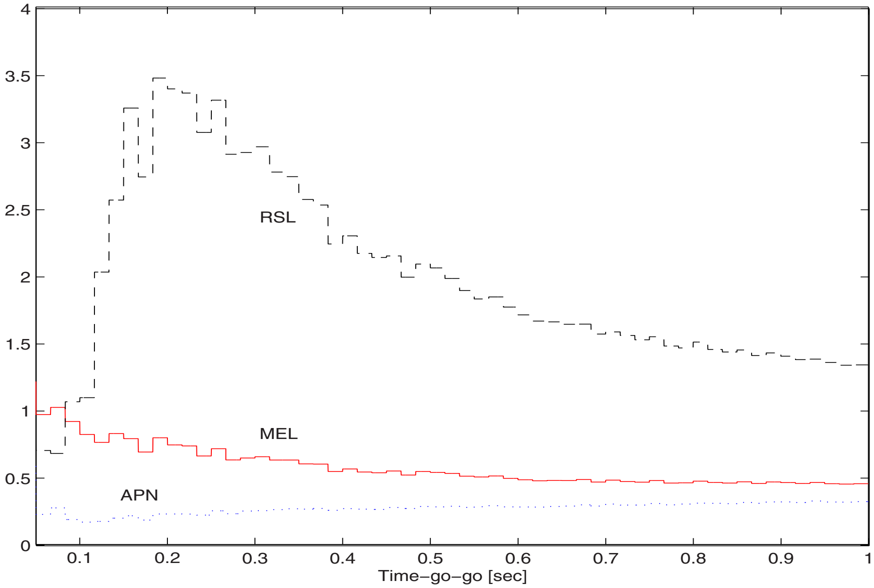

Fig. 11.15. Guidance Position Gains of MEL, APN and RSL as a function of the normalized time to go in seconds

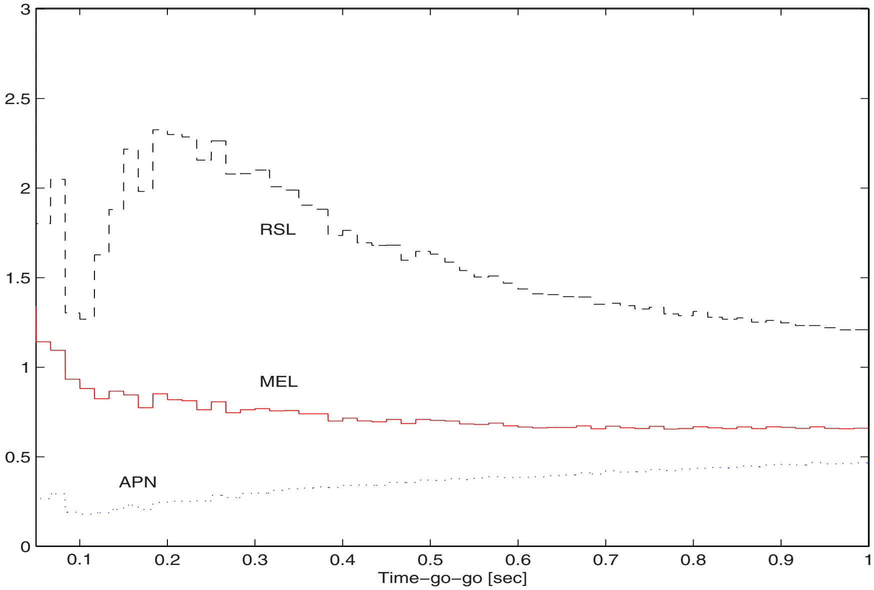

Fig. 11.16. Guidance Velocity Gains of MEL, APN and RSL as a function of the normalized time to go in seconds

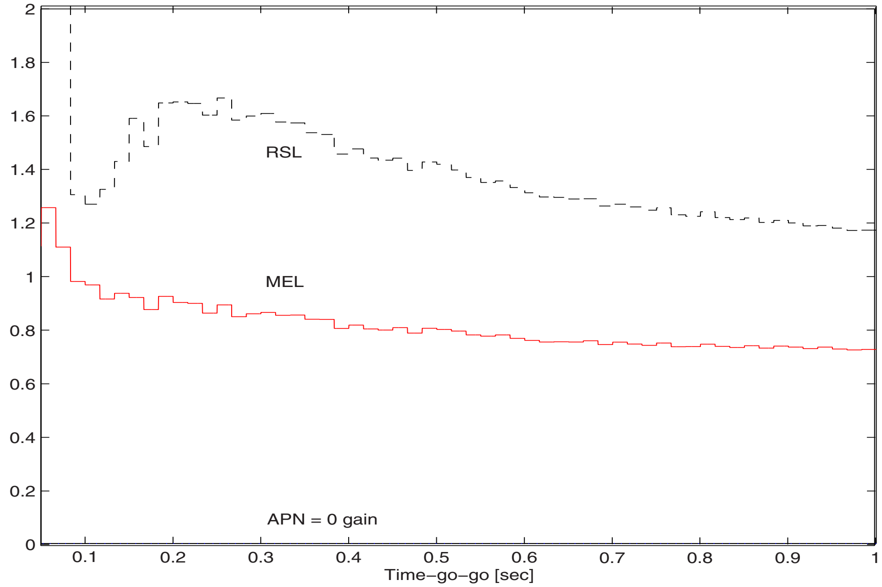

Fig. 11.17. Guidance Pursuer Acceleration Gains of MEL, APN and RSL as a function of the normalized time to go in seconds

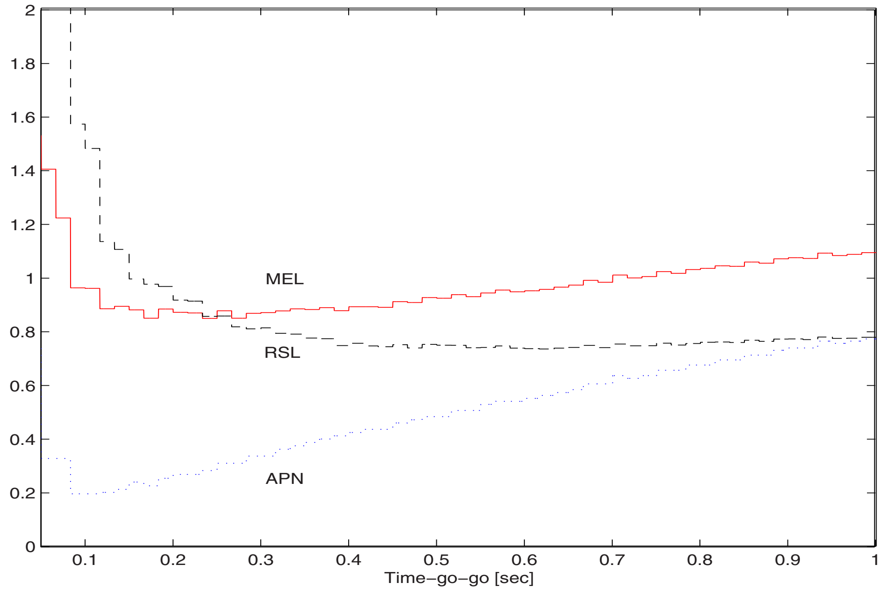

Fig. 11.18. Guidance Evader Gains of MEL, RSL and APN divided by those of TRK as a function of the normalized time to go

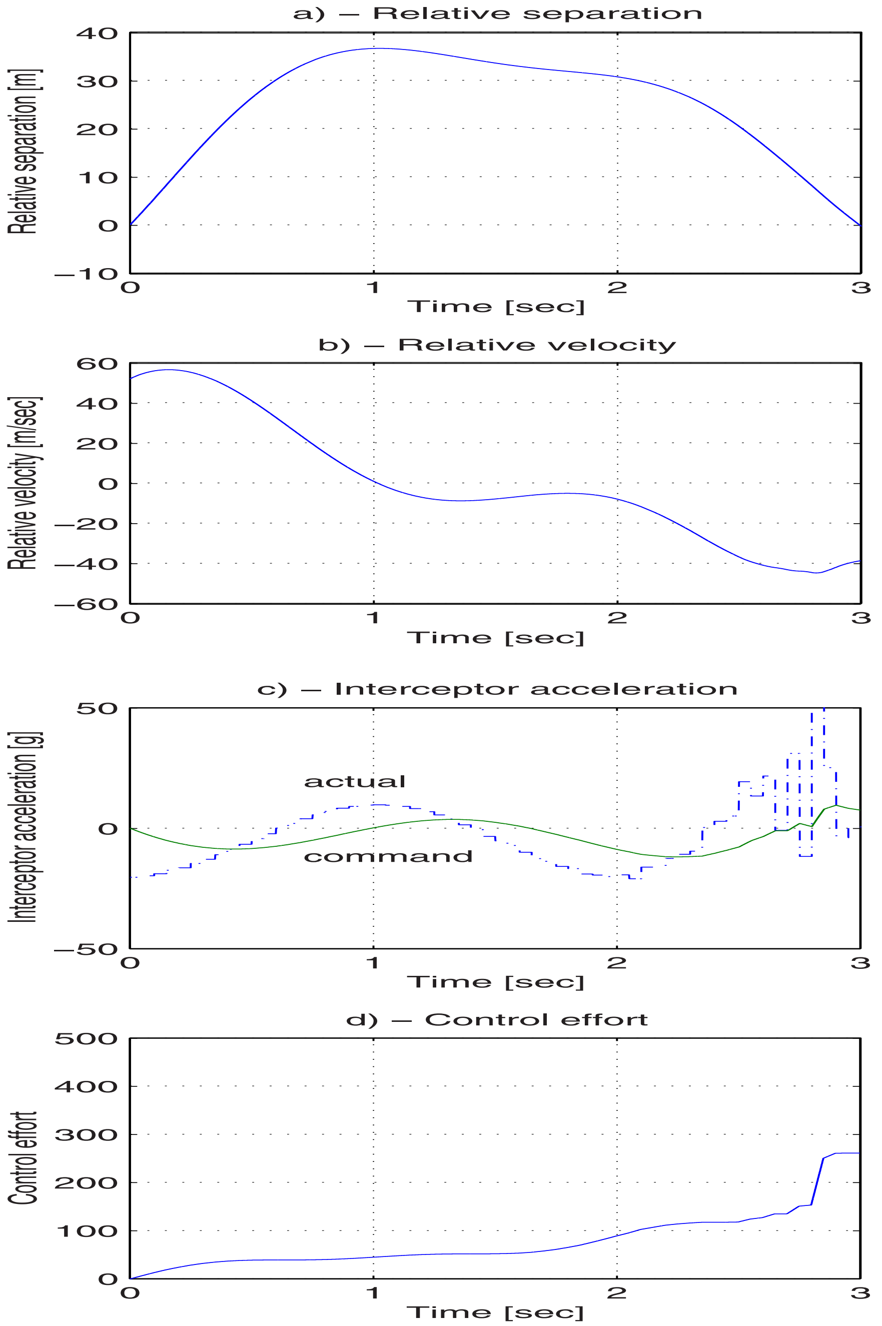

Fig. 11.19. Simulations results for MEL: a)-relative separation, b)-relative velocity, c)-commanded and actual acceleration and d)-control effort. The miss-distance is -0.299 m, the interception angle is -7.302 degrees and $t_f^{-1} g^{-2} \int_0^{t_f} u^2 d\tau = 261.786$.

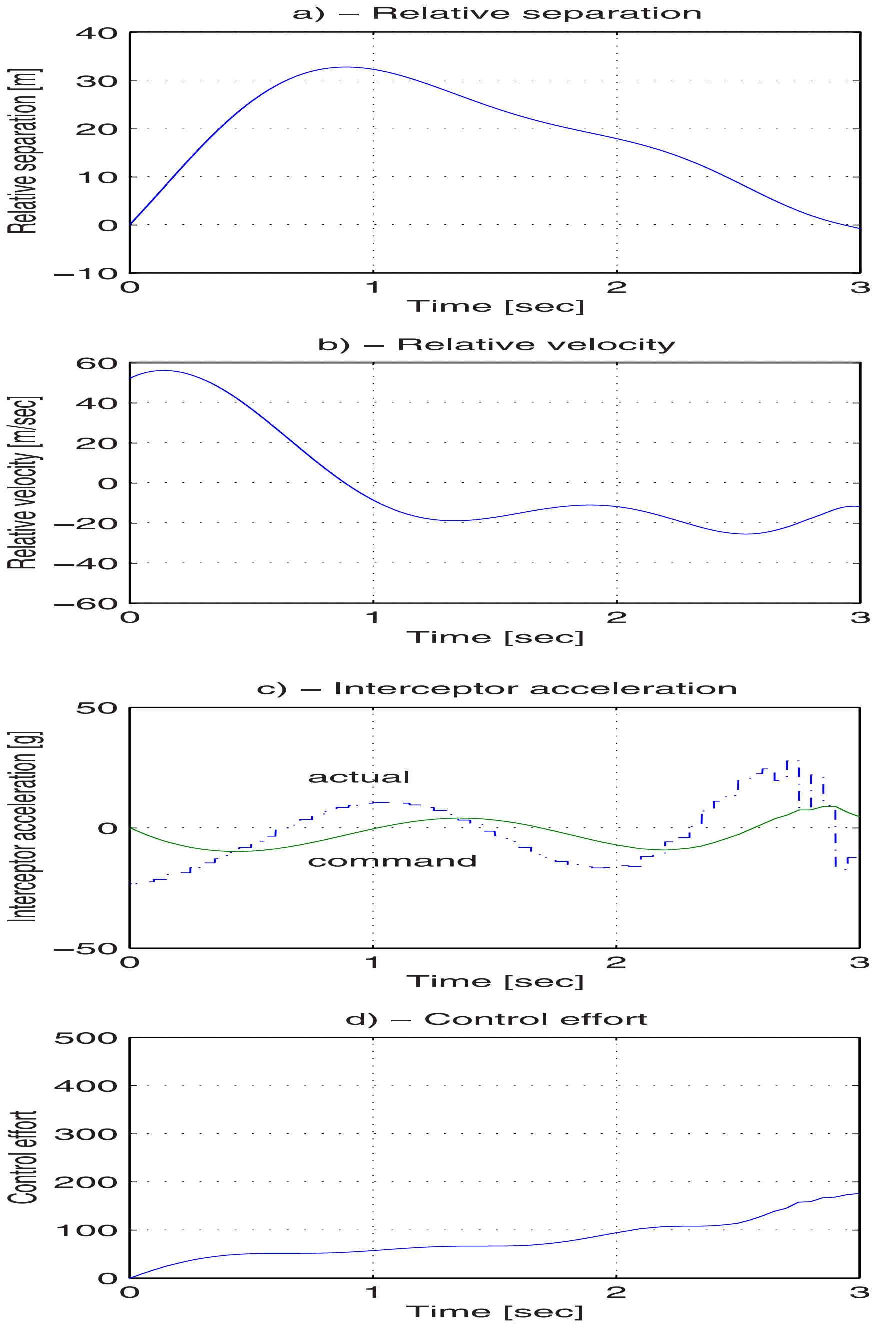

Fig. 11.20. Simulations results for TRK: a)-relative separation, b)-relative velocity, c)-commanded and actual acceleration and d)-control effort. The miss-distance is -0.851 m, the interception angle is -2.185 degrees and $t_f^{-1} g^{-2} \int_0^{t_f} u^2 d\tau = 176.530$.

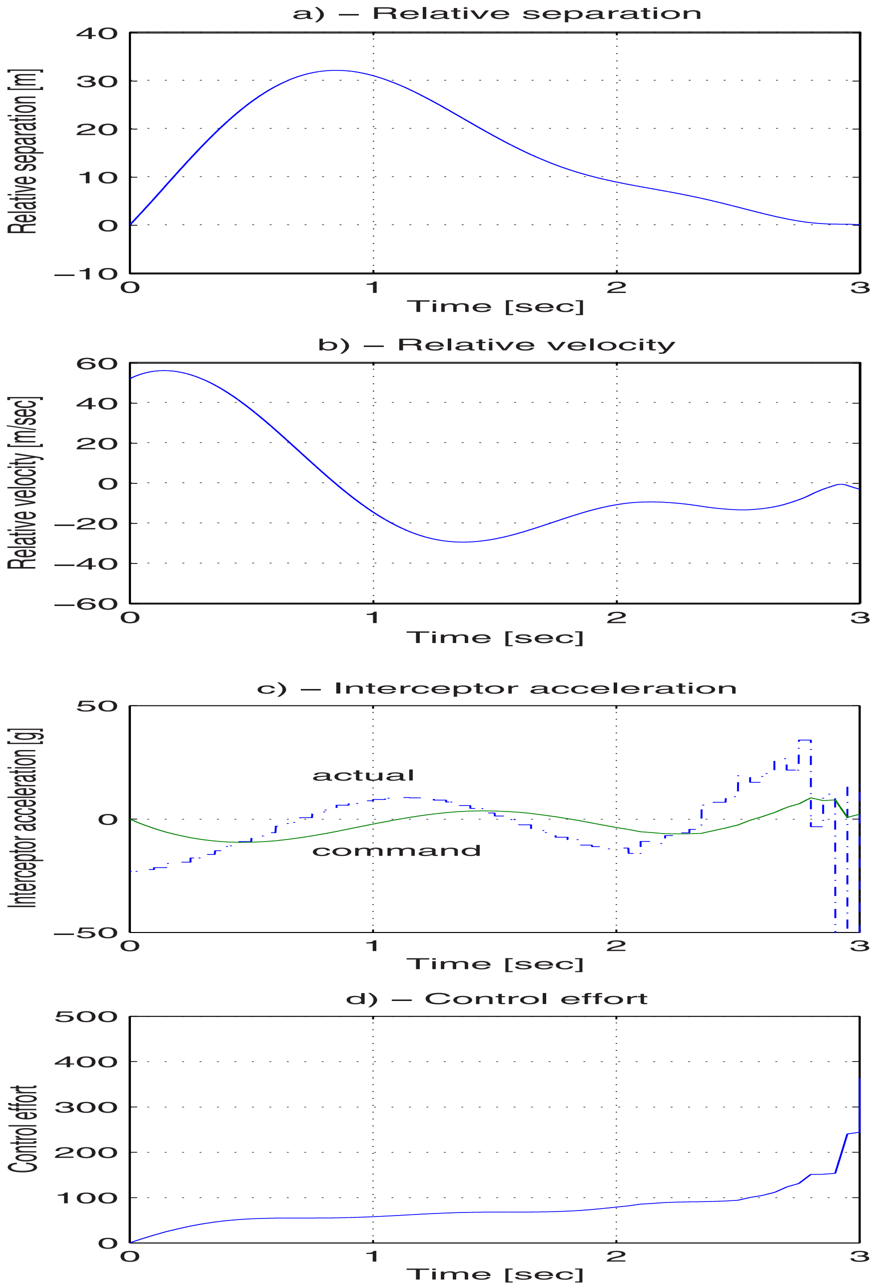

Fig. 11.21. Simulations results for RSL: a)-relative separation, b)-relative velocity, c)-commanded and actual acceleration and d)-control effort. The miss-distance is 0.031 m, the interception angle is -0.514 degrees and $t_f^{-1} g^{-2} \int_0^{t_f} u^2 d\tau = 363.516$.

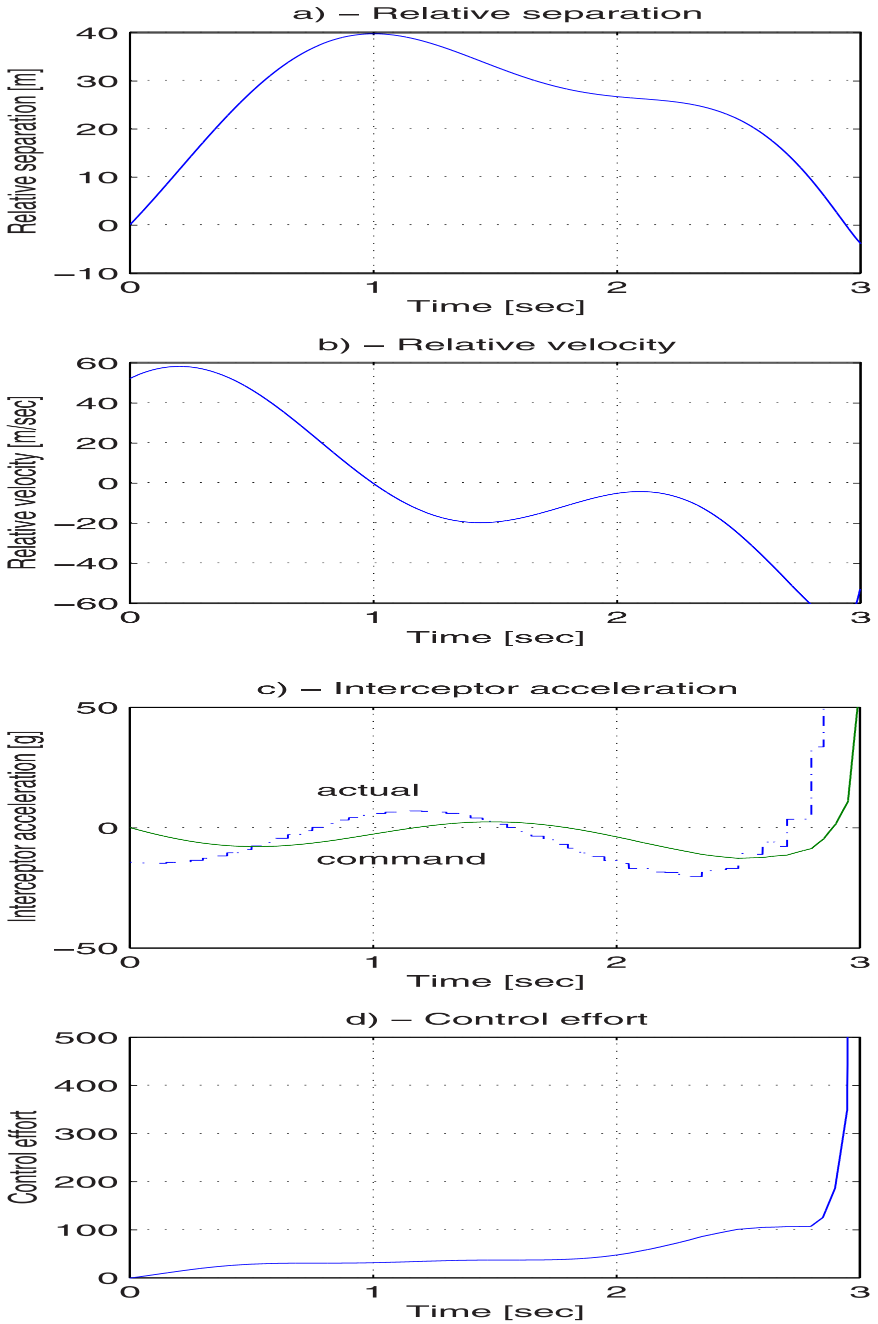

Fig. 11.22. Simulations results for APN: a)-relative separation, b)-relative velocity, c)-commanded and actual acceleration and d)-control effort. The miss-distance is about 3 m.

A

Appendix: Introduction to Stochastic Differential Equations

A.1 Introduction

In this appendix we bring some basic results concerning stochastic differential equations of the Ito type which systems with state-multiplicative noise constitute a special case of. Stochastic differential equations received a comprehensive treatment in [63], mainly aimed at providing a rigorous framework for optimal state estimation of nonlinear stochastic processes. In the present Appendix, we provide only the main facts that are required to assimilate the main concepts and results which are useful in deriving optimal estimators and controllers for linear systems with state-multiplicative white noise. While the expert reader may skip this appendix, graduate students or practicing engineers may find it to be a useful summary of basic facts and concepts, before they read the text of [63]. Comprehensive treatment of stochastic differential equations in a form accessible to graduate students and practicing engineers is given also in [97] where the close connections between Ito type stochastic differential equations and statistical physics are explored and where a few additional topics such as stochastic stability are covered. Also in [97] many communications oriented examples of nonlinear estimation theory can be found.

A.2 Stochastic Processes

Stochastic processes are a family of random variables parameterized by time $t \in \mathcal{T}$. Namely, at each instant t, $x(t)$ is a random variable. When t is continuous (namely $\mathcal{T} = \mathcal{R}$), we say that $x(t)$ is a continuous-time stochastic process, and if t is discrete (namely $\mathcal{T} = \{1, 2, ...\}$), we say that $x(t)$ is a discrete-time variable. For any finite set of $\{t_1, t_2, ...t_n\} \in \mathcal{T}$ we can define the joint distribution $F(x(t_1), x(t_2), ..., x(t_n))$ and the corresponding joint density $p(x(t_1), x(t_2), ..., x(t_n))$.

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 205–216, 2005.

The first and the second order distribution functions, $p(x(t))$ and $p(x(t), x(\tau))$, respectively, play an important role in our discussion. Also the mean $m_x(t) \triangleq E\{x(t)\}$ and the autocorrelation $\gamma_x(t,\tau) \triangleq E\{x(t)x(\tau)\}$ are useful characteristics of the stochastic process $x(t)$. When $x(t)$ is vector valued, the autocorrelation is generalized to be $\Gamma_x(t,\tau) = E\{x(t)x(\tau)^T\}$. The covariance matrix of a vector valued stochastic process $x(t)$ is a measure of its perturbations with respect to its mean value and is defined by $P_x(t) \triangleq E\{(x(t) - m_x(t))(x(t) - m_x(t))^T\}$.

A process $x(t)$ is said to be stationary if

$$p(x(t_1), x(t_2), ..., x(t_n)) = p(x(t_1 + \tau), x(t_2 + \tau), ..., x(t_n + \tau))$$

for all n and τ. If the latter is true only for $n = 1$, then the process $x(t)$ is said to be stationary of order 1 and then $p(x(t))$ does not depend on t. Consequently, the mean $m_x(t)$ is constant and $p(x(t), x(\tau))$ depends only on $t - \tau$. Also in such a case, the autocorrelation function of two time instants depends only on the time difference, namely $\gamma_x(t, t - \tau) = \gamma_x(\tau)$.

An important class of stochastic processes is one of Markov processes. A stochastic process $x(t)$ is called a Markov process if for any finite set of time instants $t_1 < t_2 < ... < t_{n-1} < t_n$ and for any real λ it satisfies

$$Pr\{x(t_n) < \lambda | x(t_1), x(t_2), ..., x(t_{n-1}), x(t_n)\} = Pr\{x(t_n) | x(t_{n-1}\}.$$

Stochastic processes convergence properties of a process $x(t)$ to a limit x can be analyzed using different definitions. The common definitions are almost sure or with probability 1 convergence (namely $x(t) \rightarrow x$ almost surely, meaning that this is satisfied except for an event with a zero probability), convergence in probability (namely for all $\epsilon > 0$, the probability of $|x(t) - x| \geq \epsilon$ goes to zero), and mean square convergence, where given that $E\{x(t)^2\}$ and $E\{x^2\}$ are both finite, $E\{(x(t) - x)^2\} \rightarrow 0$. In general, almost sure convergence neither implies nor it is implied by mean square convergence, but both imply convergence in probability. In the present book we adopt the notion of mean square convergence and the corresponding measure of stability, namely mean square stability.

A.3 Mean Square Calculus

Dealing with continuous-time stochastic processes in terms of differentiation, integration, etc. is similar to the analysis of deterministic functions, but it requires some extra care in evaluation of limits. One of the most useful approaches to calculus of stochastic processes is the so called mean square calculus where mean square convergence is used when evaluating limits.

The full scope of mean square calculus is covered in [63] and [97] but we bring here only a few results that are useful to our discussion.

The notions of mean square continuity and differentiability are key issues in our discussion. A process $x(t)$ is said to be mean square continuous if $lim_{h \to 0} x(t+h) = x(t)$. It is easy to see that if $\gamma_x(t, \tau)$ is continuous at (t, t) then also $x(t)$ is mean square continuous. Since the converse is also true, then mean square continuity of $x(t)$ is equivalent to continuity of $\gamma(t, \tau)$ in (t, t). Defining mean square derivative by the mean square limit as $h \to 0$ of $(x(t+h) - x(t))/h$, then it is similarly obtained that $x(t)$ is mean square differentiable (i.e. its derivative exists in the mean square sense) if and only if $\gamma_x(t, \tau)$ is differentiable at (t, t). A stochastic process is said to be mean square integrable, whenever $\sum_{i=0}^{n-1} x(\tau_i)(t_{i+1} - ti)$ is mean square convergent where $a = t_0 < t_1 < ... < t_n = b$, where $\tau_i \in [t_i, t_{i+1}]$ and where $|t_{i+1} - ti| \to 0$. In such a case, the resulting limit is denoted by $\int_a^b x(t)dt$. It is important to know that $x(t)$ is mean square integrable on $[a, b]$ if and only if $\gamma_x(t, \tau)$ is integrable on $[a, b] \times [a, b]$. The fundamental theorem of mean square calculus then states that if $\dot{x}(t)$ is mean square integrable on $[a, b]$, then for any $t \in [a, b]$, we have

$$x(t) - x(a) = \int_a^t \dot{x}(\tau)d\tau.$$

The reader is referred to [63] for a more comprehensive coverage of mean square calculus.

A.4 Wiener Process

A process $\beta(t)$ is said to be a Wiener Process (also referred to as Wiener-Levy process or Brownian motion) if it has the initial value of $\beta(0) = 0$ with probability 1, has stationary independent increments and is normally distributed with zero mean for all $t \geq 0$. The Wiener process has then the following properties : $\beta(t) - \beta(\tau)$ is normally distributed with zero mean and variance $\sigma^2(t-\tau)$ for $t > \tau$ where σ^2 is an empirical positive constant. Consider now for $t > \tau$ the autocorrelation

$$\begin{aligned}\gamma_\beta(t, \tau) = E\{\beta_t \beta_\tau)\} &= E\{(\beta(t) - \beta(\tau) + \beta(\tau))\beta(\tau)\} \\ &= E\{(\beta(t) - \beta(\tau))\beta(\tau)\} + E\{\beta^2(\tau)\}.\end{aligned}$$

Since the first term is zero, due to the independent increments property of the Wiener process, it is readily obtained that $\gamma_\beta(t, \tau) = \sigma^2 \tau$. Since we have assumed that $t > \tau$ we have in fact that $\gamma_\beta(t, \tau) = \sigma^2 min(t, \tau)$. Since the latter is obviously continuous at (t, t), it follows that $\beta(t)$ is mean square continuous. However, a direct calculation (see [63]) of the second order derivative of $\gamma_\beta(t, \tau)$, with respect to t and τ at (t, t), shows that

$$\frac{min(t+h, t+h') - min(t, t)}{hh'} = 1/max(h, h')$$

which is clearly unbounded as h and h' tend to zero. Therefore, $\gamma_\beta(t,\tau)$ is not differentiable at any (t,t) and consequently $\beta(t)$ is not mean square differentiable anywhere. It is, therefore, concluded that the Wiener process is continuous but not differentiable in the mean square sense. In fact, it can be shown that the latter conclusion holds also in the sense of almost sure convergence.

A.5 White Noise

We begin this section by considering discrete-time white noise type stochastic processes. A discrete-time process is said to be white if it is a Markov process and if all $x(k)$ are mutually independent. Such a process is said to be a white Gaussian noise if, additionally, its samples are normally distributed. The mutual independence property leads, in the vector valued case, to $E\{x(n)x^T(m)\} = Q_n\delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta function (1 for equal arguments and zero otherwise) and where $Q_n \geq 0$. The discrete-time white noise is a useful approximation of measurement noise in many practical cases. Its continuous-time analog also appears to be useful. Consider a stationary process $x(t)$ whose samples are mutually independent, taken at large enough intervals. Namely,

$$\gamma(\tau) = E\{x(t+\tau)x(t)\} = \sigma^2\frac{\rho}{2}e^{-\rho|\tau}$$

where $\rho >> 1$. As ρ tends to infinity $\gamma(\tau)$ rapidly decays as a function of τ, and therefore the samples of $x(t)$ become virtually independent and the process becomes white. Noting that for ρ that tends to infinity, $\frac{\rho}{2}e^{-\rho|\tau} \to \delta(\tau)$ where δ is the Dirac delta function [63], a vector valued white process $x(t)$ is formally considered to have the autocorrelation of $\gamma(\tau) = Q(t)\delta(\tau)$ where $Q(t) \geq 0$. Namely, $E\{x(t)x(\tau)\} = Q(t)\delta(t-\tau)$. Defining the spectral density of $x(t)$ by the Fourier transform of its autocorrelation, namely by

$$f(\omega) = \int_{-\infty}^{\infty} e^{-i\tau\omega}\sigma^2\frac{\rho}{2}e^{-\rho|\tau|}d\tau = \frac{\sigma^2}{1+\omega^2/\rho^2}$$

we see that this spectral density is constant and has the value of σ^2 up to about the frequency ρ where it starts dropping to zero. Namely, the spectrum of $x(t)$ is nearly flat independently of the frequency, which is the source of the name "white" noise, in analogy to white light including all frequencies or wavelengths. We note that for finite $\rho >> 1$, $x(t)$ is said to be a wide-band noise (where 1 may represent the measured process bandwidth and ρ the measurement noise bandwidth). In such a case, modelling $x(t)$ as a white noise is a reasonable approximation. We note, however, that constant $f(\omega)$ or white spectrum corresponds to infinite energy by Parseval's theorem . Alternatively, looking at the autocorrelation at $\tau = 0$, we see that

$$\gamma(0) = E\{x^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)d\omega \rightarrow \infty.$$

Therefore, white noise is not physically realizable but is an approximation to wide band noise. To allow mathematical manipulations of white noise, we relate it to Wiener processes which are well defined. To this end we recall that the autocorrelation of a Wiener process $\beta(t)$ is given by

$$\gamma(t,\tau) = E\{x(t)x(\tau)\} = \sigma^2 min(t,\tau).$$

Since expectation and derivatives can be interchanged, namely

$$E\{\frac{d\beta(t)}{dt}\frac{d\beta(\tau)}{d\tau}\} = \frac{d^2}{dtd\tau}E\{\beta_t\beta_\tau\},$$

it follows that the autocorrelation of $\dot{\beta(t)}$ is given by $\sigma^2\frac{d}{d\tau}[\frac{d}{dt}min(t,\tau)]$. However, $min(t,\tau)$ is τ for $\tau < t$ and t otherwise; therefore, its partial derivative with respect to t is a step function of τ rising from 0 to 1 at $\tau = t$. Consequently, the partial derivative of this step function is just $\sigma^2\delta(t-\tau)$. The autocorrelation of $\dot{\beta}(t)$ is thus $\sigma^2\delta(t-\tau)$, just as the autocorrelation of white noise, and we may, therefore, formally conclude that white noise is the derivative, with respect to time, of a Wiener process.

A.6 Stochastic Differential Equations

Many stochastic processes are formally described (see [63] and [97]) by the Langevin's equation:

$$\frac{dx}{dt} = f(x(t),t) + g(x(t),t)\dot{\beta}(t)$$

where $\dot{\beta}(t)$ is a white noise process. For example, in this monograph, the so called state-multiplicative process is obtained when $f(x(t),t) = Ax$ and $g(x(t),t) = Dx$ leading to

$$\frac{dx}{dt} = Ax + Dx\dot{\beta}(t).$$

When we write the latter in terms of differentials rather than in terms of derivatives, we obtain the following equation

$$dx = Axdt + Dxd\beta$$

where the physically unrealizable $\dot{\beta}(t)$ no longer appears but instead the differential $d\beta$ of $\beta(t)$ drives the equation. Note that

$$d\beta(t) \triangleq \beta(t) - \beta(t-dt)$$

is normally distributed with zero mean and $\sigma^2(t-(t-dt)) = \sigma^2 dt$. When $\sigma^2 = 1$ we say that the Wiener process $\beta(t)$ and the corresponding white noise process $\dot{\beta}(t)$ are standard. Back to Langevin's equation, we may realize that it can also be written in terms of differentials as

$$dx(t) = f(x(t),t)dt + g(x(t),t)d\beta(t).$$

This equation is, in fact, interpreted by

$$x(t) - x(t_0) = \int_{t_0}^{t} f(x(\tau,\tau)d\tau + \int_{t_0}^{t} g(x(\tau,\tau)d\beta(\tau),$$

where the first term is a Lebesgue-Stieltges integral and the second term is an Ito integral with respect to the Wiener process $\beta(t)$. Namely, this integral is defined via approximation by the sum:

$$\sum_{i=0}^{n-1} g_{t_i}[\beta(t_{i+1}) - \beta(t_i)], \quad \text{where} \;\; a = t_0 < t_1 < ... < t_n = b,$$

where g_{t_i} is a random variable at the fixed time t_i which is $\mathcal{F}_{t_i}$ measurable (see Section 1.4.1). It is assumed that g_{t_i} is independent on future increments $\beta(t_k)-\beta(t_l)$ for $t_i \leq t_l \leq t_k \leq b$ of $\beta(t)$. The stochastic integral is then defined by choosing a series $g^n$of piecewise step functions which converge to g, in the sense that the mean square of the integral of $g^n - g$ tends to zero as n tends to infinity. Whenever g_t is mean square integrable and is independent of future increments of $\beta(t)$, the stochastic Ito sense integral exists. Furthermore, it satisfies two useful identities:

$$E\{\int g_t d\beta(t)\} = 0, \;\; \text{and} \;\; E\{\int g_t d\beta(t) \int f_t d\beta(t)\} = \sigma^2 \int E\{g_t f_t\} dt.$$

In fact, we have defined in the above the first order stochastic integral. The second order stochastic integral in the Ito sense, $\int_0^t g_t d\beta^2(t)$, is similarly defined by taking the limit of n to infinity in $\sum_{i=0}^{n-1} g_{t_i}[\beta(t_{i+1}) - \beta(t_i)]$. It can be shown [63] that the latter converges, in mean square, to just $\int_{t_0}^t \sigma^2 g_t dt$.

We consider next

$$dx(t) = x(t)d\beta(t) + \frac{1}{2}x(t)d\beta^2(t)$$

where $x(0) = 1$ almost surely. Integrating the latter yields

$$x(t) - 1 = \int_0^t x(t)d\beta(t) + \frac{1}{2}\int_0^t x(t)d\beta^2(t) = \int_0^t x(t)d\beta(t) + \frac{\sigma^2}{2}\int_0^t x(t)dt.$$

The latter is simply the integral form in the Ito sense of:

$$dx(t) = x(t)d\beta(t) + \frac{\sigma^2}{2}x(t)dt,$$

meaning that in stochastic differential equations, $d\beta^2(t)$ can be replaced in the mean square sense by $\sigma^2 dt$.

A.7 Ito Lemma

Ito lemma is a key lemma which is widely used in the present monograph to evaluate differentials of nonlinear scalar valued functions $\varphi(x(t))$ of solutions $x(t)$ of Ito type stochastic differential equations. Consider a scalar process $x(t)$ which satisfies

$$\frac{dx}{dt} = f(x(t), t) + g(x(t), t)\dot{\beta}(t).$$

Then, using Taylor expansion, we have

$$d\varphi = \varphi_t dt + \varphi_x dx + \frac{1}{2}\varphi_{xx} dx^2 + \frac{1}{3}\varphi_{xxx} dx^3 + ...$$

Discarding terms of the order $o(dt)$, recalling that $d\beta^2(t)$ is of the order of dt, and substituting for dx in the above Taylor expansion, it is found that [63]:

$$d\varphi = \varphi_t dt + \varphi_x dx + \frac{1}{2}\varphi_{xx} g^2 d\beta^2(t).$$

Substituting $\sigma^2 dt$ for $d\beta^2(t)$ we obtain

$$d\varphi = \varphi_t dt + \varphi_x dx + \frac{\sigma^2}{2}\varphi_{xx} g^2 dt.$$

For vector valued $x(t)$, where $Qdt = E\{d\beta d\beta^T\}$, the latter result reads:

$$d\varphi = \varphi_t dt + \varphi_x dx + \frac{1}{2} Tr\{gQg^T \varphi_{xx}\} dt$$

where φ_{xx} is the Hessian of φ with respect to x.

A.8 Application of Ito Lemma

Ito lemma is useful in evaluating the covariance of state multiplicative processes considered in the present monograph. Consider

$$dx = Axdt + Dxd\xi + Bd\beta$$

where ξ is a scalar valued standard Wiener process and where β is also a scalar Wiener process independent of ξ so that $E\{d\beta d\beta^T\} = Qdt$. We define $w = col\{\beta,\ \xi\}$. The intensity of w is given by $\tilde{Q} \triangleq \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}$. We also define $\tilde{G} = \begin{bmatrix} B & Dx \end{bmatrix}$

Defining $\varphi(x(t)) = x_i(t)x_j(t)$, where x_i is the i-th component of x, we get (see [63]), using Ito lemma, that

$$d\varphi = x_i(t)dx_j(t) + x_j(t)dx_i(t) + \frac{1}{2} Tr\{\tilde{G}\tilde{Q}\tilde{G}^T \Sigma\},$$

where the only non zero entries in Σ are at locations i, j and j, i. Consequently, we have that

$$d(xx^T) = (xdx^T + dxx^T + \tilde{G}\tilde{Q}\tilde{G}^T)dt.$$

Taking the expectation of both sides in the latter, and defining: $P(t) = E\{x(t)x^T(t)\}$, the following result is obtained.

$$\dot{P} = AP + PA^T + BQB^T + DPD^T.$$

A.9 Simulation of Stochastic Differential Equations

Consider again

$$dx = Axdt + Dxd\xi + Bd\beta.$$

Consider now the discrete-time stochastic process

$$\frac{x(k+1) - x(k)}{h} = Ax(k) + Dx(K)\bar{\xi}(k)/\sqrt{h} + D\bar{\beta}(k)/\sqrt{h},$$

where $\bar{\xi}(k)$ is a normally distributed random sequence of zero mean and unit variance and $\bar{\beta}(k)$ is a random vector sequence of zero mean and covariance Q. Notice that $x(k)$, in fact, satisfies

$$x(k+1) = Fx(k) + Gw(k),$$

where $F = I + Ah$ and $G = \sqrt{h}\left[B\ Dx(k)\right]$ and where $w(k) = col\{\bar{\beta}(K),\ \bar{\xi}(K)\}$. Defining $S(k) = E\{x(k)x^T(k)\}$ we see, by the independence of $x(k)$ on $w(k)$, that

$$P(k+1) = FP(k)F^T + E\{GQG^T\} = FP(k)F^T + hBQB^T + hDP(k)D^T.$$

Substituting for F, neglecting terms of the order of h^2 and dividing both sides of the resulting equation by h we get

$$\frac{P(k+1) - P(k)}{h} = AP(k) + P(k)A^T + BQB^T + DP(k)D^T,$$

meaning that the discrete-time process $x(k)$ correctly represents the continuous-time process $x(t)$, in the sense that the first two moments are identical in the limit where the integration step h tends to zero. Furthermore, we note that the increments $\xi((k+1)h) - \xi(kh)$ are represented in the discrete-time model by $\bar{\xi}(k)\sqrt{h}$, which is zero mean and of variance h thus approximating the property of Wiener process increments $E\{d\xi^2(t)\} = dt$.

A.10 The Kalman Filter for Systems with State-multiplicative Noise

The present monograph deals with H_∞ or mixed H_2/H_∞ estimators and controllers for systems with state-multiplicative noise. It is appropriate, for the sake of completeness, to derive also the pure H_2-optimal estimator for this case which is a variant of the celebrated Kalman filter.

We consider the system

$$dx = Axdt + Dxd\xi + Bd\beta,$$

where $E\{d\xi^2(t)\} = dt$, $E\{d\beta d\beta^T\} = Qdt$, and $E\{x_0 x_0^T\} = P_0$, measured by

$$dy = Cxdt + d\eta,$$

where $E\{d\eta d\eta^T\} = Rdt$ and where the Wiener processes ξ, β, η are assumed to be mutually independent. Considering the observer

$$d\hat{x} = A\hat{x} + K(dy - C\hat{x}dt)$$

and defining the estimation error $e = x - \hat{x}$, we readily obtain that the augmented state vector $\mu = col\{e,\ x\}$, satisfies the following equation:

$$d\mu = F\mu dt + Gdw + \Delta\mu d\xi,$$

where

$$F = \begin{bmatrix} A-KC & 0 \\ 0 & A \end{bmatrix},\ G = \begin{bmatrix} B & -K \\ B & 0 \end{bmatrix},\ \text{and } \Delta = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}$$

where $w = col\{\beta,\ \eta\}$ and where

$$E\{dwdw^T\} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} dt \stackrel{\Delta}{=} \tilde{Q}.$$

The covariance S of the augmented state vector obviously satisfies then

$$\dot{S} = FS + SF^T + \Delta S\Delta^T + G\tilde{Q}G^T, \quad S(0) = \begin{bmatrix} P_0 & P_0 \\ P_0 & P_0 \end{bmatrix}.$$

Partitioning the covariance matrix S in conformity with the dimensions of e and x so that

$$S \stackrel{\Delta}{=} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix},$$

it is readily obtained that

$$\dot{S_{11}} = (A-KC)S_{11} + S_{11}(A-KC)^T + BQB^T + KRK^T + DS_{11}D^T$$

$$\dot{S_{12}} = (A-KC)S_{12} + S_{12}A^T + BQB^T + KRK^T + DS_{11}D^T$$

and

$$\dot{S_{22}} = AS_{22} + S_{22}A^T + BQB^T + DS_{11}D^T.$$

Fortunately, the equation for S_{11} can be solved independently of the other two. Denoting $P \triangleq S_{11}$ and completing to squares we obtain that

$$\dot{P} = AP + PA^T + BQB^T - PC^TR^{-1}CP + DPD^T$$
$$+(K - PC^TR^{-1})R(K - PC^TR^{-1})^T, \quad P(0) = P_0.$$

In order to minimize P, the last term should be nulled and therefore the Kalman filter for our case turns out to be:

$$d\hat{x} = A\hat{x}dt + PC^TR^{-1}(dy - C\hat{x}dt)$$

where

$$\dot{P} = AP + PA^T + BQB^T - PC^TR^{-1}CP + DPD^T, \quad P(0) = P_0.$$

Note that the last term is the only contribution of the state multiplicative noise. In fact, the Kalman filter for linear systems with state multiplicative noise, is one of the very few examples for which finite dimensional minimum mean square optimal estimators are obtained for nonlinear systems.

A.11 Stochastic Stability

It is appropriate to conclude the present appendix with few words about stochastic stability. Consider the following stochastic system

$$dx = f(x(t), t)dt + g(x(t), t)d\beta(t) \tag{A.1}$$

for $x \in \mathcal{R}^n$ and the corresponding Ito formula:

$$d\varphi = \varphi_t dt + \varphi_x dx + \frac{1}{2}Tr\{gQg^T\varphi_{xx}\}$$

$$= \varphi_t dt + \varphi_x(f + gd\beta) + \frac{1}{2}Tr\{gQg^T\varphi_{xx}\}$$

$$= \varphi_t dt + \varphi_x f + \frac{1}{2}Tr\{gQg^T\varphi_{xx}\} + \varphi_x gd\beta = L\varphi + \varphi_x gd\beta$$

where

$$L \triangleq \frac{\partial}{\partial t} + f^T\frac{\partial^T}{\partial x} + \frac{1}{2}\Sigma_{i,j=1}^n(gg^T)_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}$$

is referred to as the infinitesimal generator of the system (A.1). Since

$$d\varphi = L\varphi + Md\beta$$

where $M \stackrel{\Delta}{=} \varphi_x g$, then if $\varphi(x) = V(x) > 0$ for all x and $LV \leq 0$, then V is said to be Lyapunov function.

From the above we have:

$$V(x(t)) = V(x_0) + \int_0^t LV(x(t))dt + \int_0^t M d\beta \leq V(x_0) + \int_0^t M d\beta. \quad \text{(A.2)}$$

Using results about Martingales, which are beyond the scope of the present monograph, it is shown in [97] that (A.2) implies that all trajectories with x_0 sufficiently close to the origin remain, at all times, in the origin's neighborhood, except for a set of trajectories with arbitrarily small probability. The latter result, which corresponds to almost sure stability, can be understood also intuitively: if $V(x)$ is reduced so that $x(t)$ gets closer to the origin, then its gradient V_x becomes small and consequently $M = V_x g$ tends to zero. Therefore, $V(x)$ is reduced in a manner which is similar to the deterministic case. In fact, when $g(x,t) \equiv 0$ the above discussion reduces to the stability analysis of the deterministic system

$$\dot{x} = f(x(t), t).$$

In this case, $LV \leq 0$ reduces to $\frac{dV}{dt} \leq 0$. Furthermore, if $g(x,t) = x$, then the term

$$M = V_x g = V_x x$$

tends to zero if $x(t) \to 0$, thus allowing the effect of $\beta(t)$ to vanish, ensuring the convergence of $x(t)$ to zero. If in addition, $lim_{t\to\infty} x(t) = 0$ with probability 1, then the origin is stochastically asymptotically stable. Furthermore, if the above results hold for all x_0 rather than those that are in a small neighborhood, then the origin is globally asymptotically stable. It turns out that when $LV(x) < 0$ holds for all nonzero x then the origin is asymptotically stable in probability. If also $LV(x) < -kV(x)$ for all nonzero x, for some $k > 0$, then also mean square global stability is obtained.

Consider the class of linear systems with state-multiplicative noise that is treated in the present monograph where

$$f(x) = Ax, \text{ and } g(x) = Dx,$$

and where all eigenvalues of A are in the open left half of the complex plane. Choosing

$$V(x) = x^T P x,$$

with $P > 0$ and requiring that $LV < -kV$, for $k > 0$, (which generally means small enough $||D||$), imply square stability, in addition to the global asymptotic stability in probability.

It may be interesting to see a case [78] where the different concepts of stability may lead to different conclusions.

Consider the so-called Geometrical Brownian Motion (GBM) (see also [97]) which evolves according the following stochastic differential equation:

$$dx(t) = ax(t)dt + bx(t)d\beta(t), \quad x(0) = x_0 \tag{A.3}$$

where $\beta(t)$ is again a standard Wiener process (or Brownian Motion), namely $E\{d\beta^2(t)\} = dt$ and where $a - b^2/2 < 0$ but $a > 0$ so that the deterministic case with $b = 0$ would be unstable.

Define $dz(t) = bd\beta(t), z(0) = 0$ (meaning that $z(t) = b\beta(t)$) and consider $\varphi(z) = e^{(a-b^2/2)t+z(t)}$. Applying Ito lemma to evaluate $d\varphi$ we obtain, since z satisfies (A.1) with $f(z) = 0$, and $g(z) = b$, that

$$d\varphi = \varphi_t dt + \varphi_z dz + \tfrac{1}{2}\varphi_{zz}dt = [(a - b^2/2)\varphi + \varphi dz + \tfrac{b^2}{2}\varphi]dt = a\varphi dt + b\varphi dz$$
$$= a\varphi dt + b\varphi d\beta.$$

Since φ satisfies (A.3) we arrive at the conclusion (see [97]) that the solution (A.3) is

$$x(t) = e^{(a-b^2/2)t+b\beta(t)} = e^{[(a-b^2/2)+b\beta(t)/t]t}$$

Since $\beta(t)/t$ tends to zero as t tends to infinity ($\beta(t)$ is of the order of $\sqrt{t}$), $x(t)$ tends to zero as t tends to infinity (see [97]). Also, the corresponding Lyapunov exponential is negative [78]:

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} [log(x(t))]/t = a - b^2/2 < 0.$$

Consider now $V(x) = px^2$ where $p > 0$. Then,

$$LV = (2a + b^2)px^2 > 0$$

and also $P(t) = E\{x^2(t)\}$ is shown to satisfy

$$dP/dt = (2a + b^2)P$$

which means that $P(t) = x_0^2 e^{(2a+b^2)t}$. The latter tends to infinity as t tends to infinity. Namely, (A.3) can not be shown to be stochastically stable in probability, using a quadratic Lyapunov function, but it is clearly unstable in the mean square sense. Considering a non quadratic Lyapunov function $V(x) = p|x|^\alpha$ where $p > 0$, then

$$LV = [a + \frac{1}{2}b^2(\alpha - 1)]\alpha p|x|^\alpha,$$

which, by choosing $0 < \alpha < 1 - \frac{2a}{b^2}$, leads to $LV < 0$. Therefore, the system (A.3) is stochastically stable, in probability, but not mean square stable.

If both $a < 0$ and $a - b^2/2 < 0$ are satisfied, then for $b^2 < -2a$ (in this case b plays the role of D in $dx = Axdt + Dxd\beta$), the above discussion shows that (A.3) is mean square stable.

B

Appendix: The Continuous DLMI Method

B.1 Introduction

In the past, LMIs were used mainly to solve stationary problems. The algebraic nature of the Riccati equations to which these problems correspond, enables the construction of equivalent LMIs by applying the Schur complements formula [60].

Unfortunately, in cases where the systems involved are time-varying or when the time-horizon is finite, differential or difference LMIs appear. The degree of freedom that is entailed in solving these inequalities at each instant of time should be exploited to derive the best solution that will enable the optimal solution at future instances of time. In the discrete-time case, recursive sets of LMIs are obtained, and the question which was raised in [44] was how to find a solution for a given set of difference LMIs at, say, the k-th instant which enables the best solution to these LMIs at instances $i > k$. The method developed in [44] was used to solve robust control and filtering problems and the relationship between the proposed solutions and the corresponding 'central' solutions was discussed.

In this appendix we adopt a similar approach for continuous-time systems by discretizing the time scale and developing LMIs that resemble the ones obtained in [44] for the discrete-time case. The results we obtain here enable the solution of the state and output-feedback control problems for time-varying systems over a finite horizon. They also provide an efficient means for solving the control problem for multiple operating points and, by an appropriate gridding of the uncertainty intervals, for the robust control of systems with polytopic uncertainties.

B.2 Solution of the BRL via Discretization

Given the following system $\mathcal{S}(A,\ B,\ C)$:

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 217–228, 2005.

$$\dot{x} = Ax + Bw, \quad z = Cx, \quad x_0 = 0, \tag{B.1}$$

where $x \in \mathcal{R}^n$ is the system state vector, $w \in \mathcal{L}_2^q[t,\ T]$ is the exogenous disturbance signal and $z \in \mathcal{R}^m \subset \mathcal{R}^n$ is the state combination (objective function signal) to be attenuated. The matrices $A,\ B$ and C are time-varying matrices of appropriate dimensions. For a prescribed scalar $\gamma > 0$ and for a given $P_T \geq 0$ we define the performance index:

$$J(w,t,T) = x^T(T)P_T x(T) + \int_t^T (z^T z - \gamma^2 w^T w)d\tau \tag{B.2}$$

It is well known that a necessary and sufficient condition for $J(w,t,T) < 0$ for all $w \in \mathcal{L}_2^q[t,\ T]$ is the existence of a solution P, on $[t,\ T]$, to the following Differential Riccati Equation (DRE):

$$-\dot{P} = PA + A^T P + \gamma^{-2} PBB^T P + C^T C, \quad P(T) = P_T. \tag{B.3}$$

It follows from the monotonicity of P with respect to an additional positive semidefinite term on the right hand side of (B.3), that the solution to the above DRE can be obtained by solving the following Differential Linear Matrix Inequality (DLMI):

$$\Gamma(P) \stackrel{\Delta}{=} \begin{bmatrix} \dot{P} + A^T P + PA & PB & C^T \\ * & -\gamma^2 I_q & 0 \\ * & * & -I_m \end{bmatrix} \leq 0, \quad P \geq 0, \quad P(T) = P_T \tag{B.4}$$

where we require that $Tr\{P(\tau)\}$ be minimized at each time instant $\tau \in [t,\ \ T]$. The above DLMI can be solved by discretizing the time interval $[t,\ T]$ into equally spaced time instances $\{t_i,\ i = 1,..N,\ t_N = T,\ t_1 = t\}$, where:

$$t_i - t_{i-1} \stackrel{\Delta}{=} \varepsilon = N^{-1}(T - t),\ i = 1, ...N. \tag{B.5}$$

The discretized BRL problem thus becomes one of finding, at each $k \in [1,\ N]$, $P_{k-1} > 0$ of minimum trace that satisfies

$$\begin{bmatrix} -P_{k-1} + P_k + \varepsilon(A_k^T P_k + P_k A_k) & P_k B_k & C_k^T \\ * & -\frac{\gamma^2}{\varepsilon} I_q & 0 \\ * & * & -\varepsilon^{-1} I_m \end{bmatrix} \leq 0, \quad P_N = P_T \tag{B.6}$$

where the index k implies that the matrix concerned is evaluated at $t = t_k$.

For relatively large N, $P(t_i)$ may be a good approximation to $P(\tau)$ on $[t_i - \varepsilon/2,\ t_i + \varepsilon/2]$. The problem with the above solution procedure arises, however, when the matrices $A,\ B$ and C depend on some other matrix variable, for example the gain matrix K in the state-feedback problem. When applying the above method to the latter problem, at instant k, the matrix P_k is given and the matrices P_{k-1} and K_k are sought. The fact that the latter matrix variables correspond to different time instances may cause instability in the iterative process for small feasible values of γ, especially when $T - t$ tends to infinity. One may overcome this difficulty by replacing $\dot{P}(t_k)$ by $\varepsilon^{-1}(P_{k+1} - P_k)$. The DLMI of (B.6) will then become:

$$\begin{bmatrix} P_{k+1} - P_k + \varepsilon(A_k^T P_k + P_k A_k) & P_k B_k & C_k^T \\ * & -\frac{\gamma^2}{\varepsilon} I_q & 0 \\ * & * & -\varepsilon^{-1} I_m \end{bmatrix} \le 0, \quad P_N = P_T \tag{B.7}$$

where the $P_k \ge 0$ of minimum trace is sought that satisfies the inequality.

Another, more serious, disadvantage of the above method is evident in the case where A, B and C are time-invariant uncertain matrices lying within a polytope Ω. Solving (B.4) for this case and for $T - t$ which tends to infinity, we seek a $P > 0$ of minimum trace that satisfies

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ * & -\gamma^2 I_q & 0 \\ * & * & -I_m \end{bmatrix} \le 0 \tag{B.8}$$

When the matrices in Ω are affine functions of a common matrix variable (for example, in the state-feedback case a single constant gain matrix K should be found for all of the systems in Ω), a single matrix P is sought which solves (B.8) for all of the matrices belonging to the set Ω. This procedure guarantees the quadratic stability [93] of the resulting closed-loop systems, but the overall design is quite conservative.

It would, therefore, be of interest to find another discrete representation for the DLMI of (B.4) that will overcome the above mentioned difficulties. Defining

$$\tilde{Q} \stackrel{\Delta}{=} P^{-1} \tag{B.9}$$

we find that (B.3) is equivalent to:

$$\dot{\tilde{Q}} = \tilde{Q}A^T + A\tilde{Q} + \gamma^{-2}BB^T + \tilde{Q}C^T C\tilde{Q}, \quad \tilde{Q}(T) = P_T^{-1}. \tag{B.10}$$

Further defining $\epsilon\bar{Q} \stackrel{\Delta}{=} \tilde{Q}$, where $0 < \epsilon << 1$ is the scalar frame time (integration step) defined in (B.5), we obtain:

$$\epsilon\dot{\bar{Q}} = \epsilon(\bar{Q}A^T + A\bar{Q}) + \gamma^{-2}BB^T + \epsilon^2\bar{Q}C^T C\bar{Q}, \quad \bar{Q}(T) = \epsilon^{-1}P_T^{-1}.$$

The monotonicity of the solution $\bar{Q}$, with respect to an additional positive semidefinite term on the right hand side of the above equation, implies that the same solution can be obtained by solving

$$-\epsilon\dot{\bar{Q}} + \epsilon(\bar{Q}A^T + A\bar{Q}) + \gamma^{-2}BB^T + \epsilon^2\bar{Q}C^T C\bar{Q} \le 0, \quad \bar{Q}(T) = \epsilon^{-1}P_T^{-1} \tag{B.11}$$

and choosing the minimal solution at each $t \in [0 \quad T]$.

Applying the previously described time discretization scheme and replacing $\dot{\bar{Q}}(t_k)$ by $\epsilon^{-1}(\bar{Q}_{k+1}-\bar{Q}_k)$ it readily follows from (B.11) that the discretized BRL problem is one of finding $\{Q_k\}$ that will satisfy

$$-\overline{Q}_{k+1}+\overline{Q}_k+\epsilon(A_k\overline{Q}_k+\overline{Q}_kA_k^T)+\gamma^{-2}B_kB_k^T+\epsilon^2\overline{Q}_kC_k^TC_k\overline{Q}_k \leq 0 \quad \bar{Q}_N=\epsilon^{-1}P_T \tag{B.12}$$

with $\bar{Q}_k > 0$ of minimum trace for all $k = N-1, ...0$. The latter leads to the following:

Lemma B.1. *Given $\gamma > 0$, P_T and $0 < \epsilon << 1$. The discretized BRL problem has a solution if for all $k = N-1, ...1$ there exists a solution Q_k of minimum trace to the following DLMIs*

$$\begin{bmatrix} -Q_k & I_n+\epsilon A_k^T & 0 & C_k^T \\ * & -Q_{k+1}^{-1} & B_k & 0 \\ * & * & -\gamma^2 I_q & 0 \\ * & * & * & -\epsilon^{-2}I_m \end{bmatrix} \leq 0, \quad Q_k > 0, \quad Q_N = \epsilon P_T \tag{B.13}$$

Proof: We have that

$$\overline{Q}_k+\epsilon(A_k\overline{Q}_k+\overline{Q}_kA_k^T) = (I+\epsilon A_k)\overline{Q}_k(I+\epsilon A_k^T)-\epsilon^2A_k\overline{Q}_kA_k^T$$

so that inequality (B.12) becomes:

$$-\overline{Q}_{k+1}+(I+\epsilon A_k)\overline{Q}_k(I+\epsilon A_k^T)-\epsilon^2A_k\overline{Q}_kA_k^T+\gamma^{-2}B_kB_k^T+\epsilon^2\overline{Q}_kC_k^TC_k\overline{Q}_k \leq 0 \tag{B.14}$$

We also have that

$$\begin{aligned}(I-\epsilon^2C_k\overline{Q}_kC_k^T)^{-1} &= I+\epsilon^2C_k\overline{Q}_kC_k^T+\epsilon^4C_k\overline{Q}_kC_k^TC_k\overline{Q}_kC_k^T+... \\ &= I+\epsilon^2C_k\overline{Q}_kC_k^T+O(\epsilon^4)\end{aligned}$$

hence

$$\epsilon^2\overline{Q}_kC_k^TC_k\overline{Q}_k = \epsilon^2\overline{Q}_kC_k^T(I-\epsilon^2C_k\overline{Q}_kC_k^T)^{-1}C_k\overline{Q}_k+O(\epsilon^4)$$

Therefore:

$$\begin{aligned}&(I+\epsilon A_k)\overline{Q}_k(I+\epsilon A_k^T)+\epsilon^2\overline{Q}_kC_k^TC_k\overline{Q}_k \\ &= (I+\epsilon A_k)\left[\overline{Q}_k+\epsilon^2\overline{Q}_kC_k^T(I-\epsilon^2C_k\overline{Q}_kC_k^T)^{-1}C_k\overline{Q}_k\right](I+\epsilon A_k^T)+O(\epsilon^3).\end{aligned}$$

This in turn implies that inequality (B.14) becomes:

$$\begin{aligned}&-\overline{Q}_{k+1}+(I+\epsilon A_k)\left[\overline{Q}_k+\epsilon^2\overline{Q}_kC_k^T(I-\epsilon^2C_k\overline{Q}_kC_k^T)^{-1}C_k\overline{Q}_k\right](I+\epsilon A_k^T) \\ &-\epsilon^2A_k\overline{Q}_kA_k^T+\gamma^{-2}B_kB_k^T+O(\epsilon^3) \leq 0, \qquad \bar{Q}_N=\epsilon^{-1}P_T^{-1}.\end{aligned} \tag{B.15}$$

Defining $\Sigma_k \triangleq \epsilon^2A_k\overline{Q}_kA_k^T+O(\epsilon^3)$, (B.15) is then:

$$-\overline{Q}_{k+1} + (I + \epsilon A_k) \left[\overline{Q}_k + \epsilon^2 \overline{Q}_k C_k^T (I - \epsilon^2 C_k \overline{Q}_k C_k^T)^{-1} C_k \overline{Q}_k\right] (I + \epsilon A_k^T)$$

$$-\Sigma_k + \gamma^{-2} B_k B_k^T \leq 0, \qquad \bar{Q}_N = \epsilon^{-1} P_T^{-1}. \qquad \text{(B.16)}$$

The latter can be put in the following form:

$$\begin{bmatrix} -\bar{Q}_k^{-1} & I_n + \epsilon A_k^T & 0 & C_k^T \\ * & -\bar{Q}_{k+1} - \Sigma_k & B_k & 0 \\ * & * & -\gamma^2 I_q & 0 \\ * & * & * & -\epsilon^{-2} I_m \end{bmatrix} \leq 0, \quad \bar{Q}_N = \epsilon^{-1} P_T^{-1} \qquad \text{(B.17)}$$

The term $\Sigma_k(\bar{Q}_k)$ is not linear in $\bar{Q}_k$. However, since $\Sigma_k(\bar{Q}_k) \geq 0$ for a small enough ϵ, we solve (B.17) by replacing Σ_k with a zero matrix. For a small enough ϵ, the resulting solution sequence $\{\bar{Q}_k\}$ will also solve (B.17) with a non-zero Σ_k. Denoting $Q_k = \bar{Q}_k^{-1}$ (which means that $Q_k = \epsilon P_k$) and replacing $\bar{Q}_k$ by Q_k^{-1}, (B.17) readily leads to (B.13).

The first inequality in (B.17) does not suffer from the discrepancy between the time indices in the state space matrices and $\bar{Q}_k$. It will be shown below that it also overcomes the difficulties entailed in the design when subjected to parameter uncertainty.

B.3 State-feedback Control

We now apply the preceding DLMIs to the finite-horizon, time-varying, state-feedback design problem. Given the system $\mathcal{S}(A,\ B_1,\ B_2,\ C_1,\ D_{12})$:

$$\dot{x} = Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{12} u, \quad x_0 = 0, \qquad \text{(B.18)}$$

where x, w and z are defined in Section B.2, $u \in \mathcal{R}^\ell$ is the control input and $A,\ B_1,\ B_2,\ C_1$ and D_{12} are time-varying matrices of appropriate dimensions. For a prescribed scalar $\gamma > 0$ and for a given $P_T > 0$ we consider the performance index of (B.2). We look for the state-feedback gain matrix $K(t)$ which, by means of the control strategy of $u = Kx$, achieves $J(w, t, T) \leq 0$ for all $w \in \mathcal{L}_2^q[t,\ T]$.

Replacing A_k and C_k of (B.13) by $A_k + B_{2,k} K_k$ and $C_{1,k} + D_{12,k} K_k$, respectively, the discretized state-feedback problem becomes one of finding sequences $\{Q_k\}$ and $\{K_k\}$ such that, for $k = N-1, N-2, ..., 0$, they simultaneously satisfy the following LMI:

$$\begin{bmatrix} -Q_k & I_n + \epsilon(A_k^T + K_k^T B_{2,k}^T) & 0 & C_{1,k}^T + K_k^T D_{12,k}^T \\ * & -Q_{k+1}^{-1} & B_{1,k} & 0 \\ * & * & -\gamma^2 I_q & 0 \\ * & * & * & -\epsilon^{-2} I_m \end{bmatrix} \leq 0, \quad Q_N = \epsilon P_T \qquad \text{(B.19)}$$

and Q_k possesses a minimum trace.

The applicability of the above method is demonstrated by the following example:

Example 1:

We consider the system of (B.18) with:

$$A_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_{1,k} = B_{2,k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{1,k} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } D_{12,k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \; T = 20\text{secs}$$

with the cost function of (B.2) and with $P_T = 10^{-10} I_2$, which implies that we are not interested in the final value of $x(20)$. The minimum achievable value of γ, for the corresponding infinite horizon case ($T \rightarrow \infty$) is 1, and we thus look for a solution in the vicinity of the minimum value of γ, say $\gamma = 1.01$. The time interval of interest (in seconds) is $[0 \quad 20]$. We thus choose $\epsilon = .005$ seconds and sequentially solve (B.19) for $k = 3999, ..., 0$. The following results were obtained:

$$Q_0 = \begin{bmatrix} 3.7823 & 7.1366 \\ 7.1366 & 26.9322 \end{bmatrix} \epsilon, \quad K_0 = \begin{bmatrix} -7.1177 & -26.8965 \end{bmatrix}.$$

The eigenvalues of the resulting closed-loop matrix $A_0 + B_{2,0} K_0$ are: -0.2673 and -26.6292. The maximum singular value of $Q_1 - Q_0$ is $1.28 \times 10^{-6} \epsilon$, which indicates the convergence of the sequence $\{Q_i, i = 4000, 3999, ...\}$ to the nearly stationary value of Q_0.
In comparison, the 'central solution' for the stationary case, which is derived by solving

$$A^T P + PA + C_1^T C_1 + P(\gamma^{-2} B_1 B_1^T - B_2 R^{-1} B_2^T) P = 0,$$

$$R = D_{12}^T D_{12}, \quad \text{and} \quad K = R^{-1} B_2^T P$$

for the same value of $\gamma = 1.01$ is:

$$P = \begin{bmatrix} 3.7746 & 7.1240 \\ 7.1240 & 26.8906 \end{bmatrix}, \text{ and } K = \begin{bmatrix} -7.1240 & -26.8906 \end{bmatrix}.$$

with closed-loop poles at -0.2676 and -26.6230 !

B.3.1 The Uncertain Case:

We assume in the following that $A, \; B_1, \; B_2, \; C_1$ and D_{21} are appropriately dimensioned matrices which reside in the following uncertainty polytope

$$\Omega \stackrel{\Delta}{=} \{(A,\ B_1,\ B_2,\ C_1,\ D_{21})\ |(A,\ B_1,\ B_2,\ C_1,\ D_{21})$$

$$= \sum_{i=1}^{N_v} \tau_i (A_i,\ B_{1,i},\ B_{2,i},\ C_{1,i},\ D_{21,i});\ \tau_i \geq 0,\ \sum_{i=1}^{N_v} \tau_i = 1\}, \tag{B.20}$$

and we look for a single state-feedback gain matrix $K(t)$ that will yield a negative performance index in (B.2) for $x(0) = 0$ and for all nonzero $w \in \mathcal{L}_2^q[t,\ T]$. The preceding state-feedback problem can be solved by applying the above discretization and by replacing $A_k,\ B_k$ and C_k of (B.17) by $A_{i,k} + B_{2,i,k}K_k,\ B_{1,i}$ and $C_{1,i,k} + D_{12,i,k}K_k$, respectively, where $i = 1, ...N_v$ denote the vertices of Ω. Multiplying the first row block in (B.17), from the left, by Q_k and the first column block ,from the right, by the same matrix and taking $\Sigma_k = 0$ we have:

$$\begin{bmatrix} -\bar{Q}_k & \bar{Q}_k + \epsilon(\bar{Q}_k A_{i,k}^T + \bar{Q}_k K_k^T {B_{2,i,k}}^T) & 0 & \bar{Q}_k C_{1,i,k}^T + \bar{Q}_k K_k^T D_{12,i,k}^T \\ * & -\bar{Q}_{k+1} & B_{1,i,k} & 0 \\ * & * & -\gamma^2 I_q & 0 \\ * & * & * & -\epsilon^{-2} I_m \end{bmatrix} \leq 0,$$

$$\bar{Q}_N = \epsilon^{-1} P_T^{-1} \tag{B.21}$$

Letting $\bar{Q}_k K_k^T \stackrel{\Delta}{=} Y_k^T$,(B.21) becomes:

$$\begin{bmatrix} -\bar{Q}_k & \bar{Q}_k + \epsilon(\bar{Q}_k A_{i,k}^T + Y_k^T {B_{2,i,k}}^T) & 0 & \bar{Q}_k C_{1,i,k}^T + Y_k^T D_{12,i,k}^T \\ * & -\bar{Q}_{k+1} & B_{1,i,k} & 0 \\ * & * & -\gamma^2 I_q & 0 \\ * & * & * & -\epsilon^{-2} I_m \end{bmatrix} \leq 0,$$

$$\bar{Q}_N = \epsilon^{-1} P_T^{-1} \tag{B.22}$$

and we look for the sequences of $\{\bar{Q}_k\}$ with maximum trace and $\{Y_k\}$ which, for $k = N–1,\ N–2....0$, satisfy the above LMIs for $i = 1, ...N_v$. If a solution to the latter set of LMIs exists, the solution for K_k is derived from:

$$K_k = Y_k \bar{Q}_k^{-1}. \tag{B.23}$$

Example 2:

We solved the above set of LMIs (B.22) for the case where $g \in [g_1 \quad g_2] = [-.8 \quad .8]$, and the system matrices of (B.18) are respectively:

$$A_{i,k} = \begin{bmatrix} 0 & 1 \\ 2 & g_i \end{bmatrix},\ B_{1,i,k} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ B_{2,i,k} = \begin{bmatrix} -.5 \\ 1 \end{bmatrix},\ C_{1,,i,k} = \begin{bmatrix} -.4 & 1 \\ 0 & 0 \end{bmatrix},$$

and

$$D_{12,i,k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Once again, we consider the cost function (B.2) with $T = 5$ secs. and $P_T = 10^{-15} I_2$. The uncertainty polytope is described by the $N_v = 2$ vertices, g_1 and g_2. We solved (B.22), with $\epsilon = .005$ secs. and a tolerance of 10^{-6}, for a nearly minimum value of $\gamma = 84$ and for $\gamma = 86$. For both values of γ we obtained solutions. The resulting gain matrices $K(t) = [K_1(t) \quad K_2(t)]$ are depicted in Fig. 1. We note here that the robust stationary state-feedback solution ($T \to \infty$), which can also be obtained using the method of Boyd *et.al.* [12], admits a solution only for $\gamma > 106.6$. All of these results entail an overdesign since they are based on the same sequence $\{Q_k\}$ for all the vertices of Ω.

The LMIs of (B.19) imply another important application of the DLMI method to robust design. Assuming that the system of (B.18) belongs to a set containing a finite number of systems, that is:

$$(A,\ B_1,\ B_2,\ C_1\ D_{21})\ \subset \Gamma$$

where

$$\Gamma \stackrel{\Delta}{=} \{(A_i,\ B_{1,i},\ B_{2,i},\ C_{1,i},\ D_{21,i}) \quad i = 1,\ 2, ...N_s\}$$

we derive the following set of coupled LMIs from (B.19):

$$\begin{bmatrix} -Q_{i,k} & I_n + \epsilon(A_{i,k}^T + K_k^T B_{2,i,k}^T) & 0 & C_{1,i,k}^T + K_k^T D_{12,i,k}^T \\ I_n + \epsilon(A_{i,k} + B_{2,i,k} K_k) & -Q_{i,k+1}^{-1} & B_{1,i,k} & 0 \\ 0 & B_{1,i,k}^T & -\gamma^2 I_q & 0 \\ C_{1,i,k} + D_{12,i,k} K_k & 0 & 0 & -\epsilon^{-2} I_m \end{bmatrix} \leq 0,$$

$$i{=}1,..,N_s, \qquad Q_{i,N} = \epsilon P_T \tag{B.24}$$

For each $k < N$ we seek $\{Q_{i,k},\ i = 1, ..., N_s\}$ of minimum trace and K_k that satisfy (B.24), based on the preceding values of $\{Q_{i,k+1},\ i = 1, ..., N_s\}$. A solution to the latter set of LMIs, may significantly reduce the overdesign of the previous method which applied a single Q_k to all of the plants. We demonstrate the use of (B.24) in the following example.

Example 3:

The system of Example 2, where g is either g_1 or g_2, may represent the two operating points of a practical process. With the time interval of $[0 \quad 5]$ seconds, $\epsilon = .005$ seconds and emphasizing the trace of $Q_{1,k}$, we obtained a solution for the close to minimum value of $\gamma = 51$. The resulting state-feedback gain vector $K(t) = [K_1(t) \quad K_2(t)]$ is depicted in Fig. 2. Extending our horizon time to $T = 20$ seconds, resulted in a nearly minimum value of $\gamma = 88$ and with

a converging gain vector $K(0)$ consisting of $K(0) = -[162.5901 \quad 86.9370]$. This $K(0)$ is, in fact, the stationary feedback gain vector which guarantees the H_∞norm for the two operating points in the infinite horizon case. Indeed, a maximum value of 83.91 has been actually obtained (for the case with g_1).
Remark: The above method is aimed at solving the problem for a distinct set of plants. Since the design procedure does not preserve convexity, the resulting state-feedback controller may not satisfy the requirement of achieving a disturbance attenuation level of γ for all the convex combinations of the state space matrices of these distinct plants. If such a requirement is made, one may superimpose a grid onto the uncertainty polytope and the above method can then be applied to all of the plants that correspond to the points on the grid. It can be shown that, by choosing a dense enough grid, the specifications can be satisfied for all the points in the uncertainty polytope [14]. Indeed, the results for the stationary K, show that the maximum achievable H_∞norm, for all $g \in [g_1 \quad g_2]$ is less than 83.91, which demonstrates that in our case, a grid of two points was sufficient to guarantee robust performance over the whole uncertainty polytope.

B.4 Output-feedback

The previously described method of solving the continuous-time BRL equation via recursive LMIs can also be applied to the output-feedback control problem. We consider the following linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \qquad x_0 = 0 \\ y &= C_2 x + D_{21} w \end{aligned} \tag{B.25}$$

with the objective vector

$$z = C_1 x + D_{12} u + D_{11} w \tag{B.26}$$

where x, w, u and z are defined in the previous section and $y \in \mathcal{R}^m$ is the measured output.

We seek a controller

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x + D_c y \end{aligned} \tag{B.27}$$

such that for prescribed $\gamma > 0$ and $\bar{P}_T > 0$ the following holds:

$$J \stackrel{\Delta}{=} x(T)^T \bar{P}_T x(T) + \int_0^T \{z^T z - \gamma^2 w^T w\} dt < 0 \quad \forall w \in \mathcal{L}_2^q[t,\ T]. \tag{B.28}$$

Defining the augmented state vector ξ as:

$$\xi = \begin{bmatrix} x \\ x_c \end{bmatrix}$$

we obtain the following:

$$\begin{aligned}\dot{\xi} &= \tilde{A}\xi + \tilde{B}w, \quad \xi_0 = 0 \\ z &= \tilde{C}\xi + \tilde{D}w\end{aligned} \tag{B.29}$$

where:

$$\begin{aligned}\tilde{A} &\triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \Theta \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \\ \tilde{B} &\triangleq \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \Theta \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \\ \tilde{C} &\triangleq \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \Theta \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \\ \tilde{D} &\triangleq D_{11} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \Theta \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}\end{aligned}$$

and where

$$\Theta \triangleq \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}.$$

Considering the system of (B.29) we apply the discretization as in Section B.2. It follows from the standard BRL and a derivation similar to that of (B.13), that the objective of (B.28) is equivalent to the existence of $\{Q_k\}$, $Q_k = Q_k^T$ and $\{\Theta_k\}$ that, for $0 < \delta,\ \epsilon << 1$ satisfy:

$$\begin{bmatrix} -Q_k & I_{2n} + \epsilon \tilde{A}_k^T & 0 & \epsilon \tilde{C}_k^T \\ I_{2n} + \epsilon \tilde{A}_k & -Q_{k+1}^{-1} & \tilde{B}_k & 0 \\ 0 & \tilde{B}_k^T & -\gamma^2 I_q & \tilde{D}_k^T \\ \epsilon \tilde{C}_k & 0 & \tilde{D}_k & -I_m \end{bmatrix} \leq 0, \quad Q_N = \epsilon diag\{P_T,\ \delta I_n\}, \tag{B.30}$$

with minimum $Tr\{\mathcal{T} Q_k \mathcal{T}^T\}$ for each $k = N-1,\ N-2, ..., 0$, where $N = \epsilon^{-1}T$, and $\mathcal{T} = \begin{bmatrix} I & I \\ 0 & -I \end{bmatrix}$.

In the above DLMI, ϵ is the integration time step, the index k implies the value of the matrix at time t_k (as in Section B.2) and δ is a small positive scalar which implies that we hardly weight the final value of x_c. The matrix $\mathcal{T}^T$ transforms ξ into $col\{x,\ x - x_c\}$ so that by minimizing the latter trace we are, in fact, applying the BRL to the augmented system consisting of x and the 'error' vector $x - x_c$. Recall that the DLMI stems from a differential Riccati inequality which was derived by differentiation of a Lyapunov candidate function (LCF - see [54]) of the form $V = \xi^T Q_k \xi$. Hence, if we use the transformation $\chi = \mathcal{T}^T \xi$, our LCF becomes:

$$V = \chi^T \mathcal{T} Q_k \mathcal{T}^T \chi.$$

While we have not changed the underlying LCF, we can force a coordinate change by asking for the minimization of $\mathcal{T}Q_k\mathcal{T}^T$, without changing the stucture or entries of the DLMI in (B.30).

Using the the definitions of $\tilde{A}_K$, $\tilde{B}_k$, $\tilde{C}_k$, and $\tilde{D}_k$ and substituting them into (B.30), the problem becomes one of searching for $\{Q_k\}$ and $\{\Theta_k\}$ which minimize $Tr\{\mathcal{T}Q_k\mathcal{T}^T\}$ for each $k < N$ and satisfy:

$$\Phi_k(Q_{k+1}) - \mathcal{I}Q_k\mathcal{I}^T + \mathcal{P}_k\Theta_k\mathcal{D}_k^T + \mathcal{D}_k\Theta_k^T\mathcal{P}_k^T < 0 \qquad \text{(B.31)}$$

where

$$\mathcal{I} \triangleq \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{P}_k \triangleq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_{2,k} \\ I_n & 0 \\ 0 & 0 \\ 0 & D_{12,k} \end{bmatrix}, \quad \mathcal{D}_k^T \triangleq \begin{bmatrix} 0 & \epsilon I_n & 0 & 0 & 0 & 0 \\ \epsilon C_{2,k} & 0 & 0 & 0 & D_{21,k} & 0 \end{bmatrix},$$

and

$$Q_N = \epsilon diag\{P_T, \delta I_n\} \text{ and } \Phi_k(Q_{k+1}) \triangleq \begin{bmatrix} 0 & \begin{bmatrix} I+\epsilon A_k^T & 0 \\ 0 & I_n \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \epsilon C_{1,k}^T \\ 0 \end{bmatrix} \\ * & -Q_{k+1}^{-1} & \begin{bmatrix} B_{1,k} \\ 0 \end{bmatrix} & 0 \\ * & * & -\gamma^2 I_q & D_{11,k}^T \\ * & * & * & -I_m \end{bmatrix} \qquad \text{(B.32)}$$

At each step, say the k-th, $\Phi_k(Q_{k+1})$ in (B.31) is known and we then look for $\mathcal{T}Q_k\mathcal{T}^T$ of minimum trace and Θ_k that satisfy (B.31).

The latter set of LMIs can readily be applied to the situation where (in similarity to Example 3), the plant belongs to a finite set of possible plants. The LMI of (B.31) should then be solved for each plant in the set with different $Q_{i,k}$ but with a Θ_k that is common to all of the resulting LMIs. We demonstrate the applicability of the method in the following example.

Example 4:

We consider the system of (B.25) with the state space matrices, the time interval ([0 5] secs.) and the performance index as in Example 2 and with

$$B_{1,k} = \begin{bmatrix} -.5 & 0 \\ 1 & 0 \end{bmatrix}, \; C_{2,k} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_{21,k} = \begin{bmatrix} 0 & -.1 \end{bmatrix}.$$

We solved (B.31) and (B.32) for the $N_s = 2$ plants characterized by $g_1 = -0.8$ and $g_2 = 0.8$. Applying the discretization period of $\epsilon = 0.04$ seconds and putting an emphasis on the trace that corresponds to $g = -.8$, a near minimum value of $\gamma = 80$ was obtained, for which the time varying sequence $\{\Theta_k,\ k = 125, ...0\}$ was derived. The time evolution of the eigenvalues of the controller dynamical matrix A_c are depicted in Fig. 3. It is seen in the figure that at the end of the time interval, at $t = 0$, the eigenvalues of A_c tend to reside on the imaginary axis.

Unfortunately, no solution could be found for the infinite horizon case, even for large γ.

B.5 Conclusions

A method for solving the continuous-time BRL has been proposed which enables solutions to various time-varying finite-horizon H_∞control problems with uncertainty and with multiple objectives. This method also resolves the difficulty that was encountered in the stationary case with uncertainties in the state space matrices, where the matrix variable in the corresponding LMI multiplies the system state-space matrices. The recursive nature of the proposed algorithm leads to inequalities that are affine in all that matrix variables. This property enables the solution of the robust output-feedback control problem in cases where the uncertain system to be controlled is included in a finite set of known systems. The control of systems with polytopic type uncertainty can also be solved by the new method by imposing a grid on the uncertainty polytope and solving the problem for the grid point systems.

The fact that at each of the discrete instances the resulting DLMI is affine in the controller parameters at this instant also enables a solution that guarantees the stability of the controller in the stationary control problem. This stability, which is quite difficult to guarantee by the standard H_∞solution method, is achieved by directly solving the corresponding Lyapunov inequality for the dynamic matrix of the controller.

C

Appendix: The Discrete DLMI Method

C.1 A BRL for Discrete-time LTV Systems

We consider the following system:

$$x_{k+1} = A_k x_k + B_k w_k, \qquad k = 0, 1, ..., N-1, \quad z_k = L_k x_k, \tag{C.1}$$

where $x_k \in \mathcal{R}^n$ is the state vector, $w_k \in \mathcal{R}^q$ is the exogenous disturbance which is assumed to be of finite energy, $z_k \in \mathcal{R}^m$ is the objective vector and x_0 is an unknown initial state. We consider, for a given scalar $\gamma > 0$, the following performance index:

$$J_B \stackrel{\triangle}{=} ||z_k||_2^2 - \gamma^2 ||w_k||_2^2 + x_N^T \bar{Q}_N x_N - \gamma^2 x_0^T \bar{Q}_0 x_0, \quad \bar{Q}_N > 0, \quad \bar{Q}_0 > 0. \tag{C.2}$$

The objective is to determine, for a given $\gamma > 0$, whether J_B is nonpositive for all possible $\{x_0, \ \{w_k\}\} \in \mathcal{R}^n \times l_2[0, \ N-1]$.
We obtain the following:
Theorem C.1.1: *Consider the system of (C.1) and a given $\gamma > 0$. The following statements are equivalent:*

1. *J_B of (C.2) is negative for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in l_2[0 \quad N-1]$ and $x_0 \in \mathcal{R}^n$.*
2. *There exists a solution Q_k to*
$$\Gamma_k(0) = 0, \quad s.t \quad \Theta_k > 0, \ \ k = 0, 1, ..., N-1, \qquad Q_N = \bar{Q}_N, \ \ Q_0 \leq \gamma^2 \bar{Q}_0 \tag{C.3}$$
where
$$\Gamma_k(\Sigma_k) \stackrel{\triangle}{=} -Q_k + A_k^T Q_{k+1} A_k + (B_k^T Q_{k+1} A_k)^T \Theta_k^{-1} (B_k^T Q_{k+1} A_k) + L_k^T L_k + \Sigma_k$$
and
$$\Theta_k \stackrel{\triangle}{=} \gamma^2 I_q - B_k^T Q_{k+1} B_k. \tag{C.4}$$
3. *At each $i > k$, minimizing the trace of the solution Q_i to the following DLMIs:*

E. Gershon et al.: H_∞ Contr. and Est. of State-multipl. Lin. Sys., LNCIS 318, pp. 229–237, 2005.

$$\begin{bmatrix} -Q_{i+1}^{-1} & A_i & B_i & 0 \\ A_i^T & -Q_i & 0 & L_i^T \\ B_i^T & 0 & -\gamma^2 I_q & 0 \\ 0 & L_i & 0 & -I_m \end{bmatrix} \leq 0, \quad Q_N = \bar{Q}_N, \tag{C.5}$$

$$\begin{bmatrix} \gamma^2 I_q & B_i^T Q_{i+1} \\ B_i Q_{i+1} & Q_{i+1} \end{bmatrix} > 0$$

results in a solution to (C.5) also for $i = k$, *for all* $k = N-1,\ N-2, ...0$, *with* $Q_0 \leq \gamma^2 \bar{Q}_0$.

Proof: The equivalence between assertions 1 and 2 is standard [57]. The proof that assertion 1 is equivalent to assertion 2 is based on the fact that, using Schur's complements, the DLMI of (C.5) is equivalent to the following DRE:

$$\Gamma_k(\Sigma_k) = 0 \tag{C.6}$$

for some $0 \leq \Sigma_k \in \mathcal{R}^{n \times n}$. The monotonicity property of the latter DRE (see for example [101], pp. 38-39) implies that its solution for $\Sigma_k \equiv 0$ (the so called 'central solution'), if it exists, is less than or equal to, in the matrix inequality sense, all other solutions of (C.6) that are obtained for $0 \leq \Sigma_k$, $\Sigma_k \neq 0$. Denoting the set of all the solutions to (C.5) for i=k by $\bar{\Omega}_k$, we find that the central solution $\{Q_k\} \in \bar{\Omega}_k$. Since Q_k is less than or equal to all of the other elements in the sequences of $\bar{\Omega}_k$, it attains the minimum trace over $\bar{\Omega}_k$.

Remark C.1.1: Using Schur's complements, it is easily verified that the feasibility condition of (C.3) is satisfied by all the solutions to (C.5). The results of Theorem C.1.1 thus suggest an alternative solution procedure to the BRL problem. We begin at the stage $i = N - 1$ by substituting the given terminal condition (i.e $Q_{k+1} = \bar{Q}_N$) into (C.5). At each instant, say the k-th, we solve for Q_k, using the previously established solution to Q_{k+1} that attained the minimum trace at the instant $i = k + 1$.

Remark C.1.2: Consider the case where the system of (C.1) is LTI, asymptotically stable, reachable and observable. Due to the equivalence between the solutions of the above DLMI (with the minimization of $Tr\{Q_k\}$) and the recursion of (C.3), the DLMIs solution to Q_k will converge, in the limit where $N - k$ tends to infinity, to the central stationary solution Q_s, if the latter exists and if $\bar{Q}_N$ is less than the antistabilizing solution of the corresponding algebraic Riccati equation [88].

Remark C.1.3: The DLMI of (C.5), with the requirement of minimum trace for Q_k is a convex problem and can be readily solved with existing programs (for example the MATLAB LMI toolbox [34]). In some filtering problems it will be important to minimize the trace of the inverse of the solution to the corresponding DLMI. This minimization can be achieved by imposing the additional requirement that for each k, the trace of the matrix $0 \leq H_k \in \mathcal{R}^{n\times n}$ that satisfies: $\begin{bmatrix} H_k & I_n \\ I_n & Q_k \end{bmatrix} \geq 0, \quad k = 0, 1, ..., N-1$, is minimized.

C.2 Application of the DLMI Technique

C.2.1 H_∞ State-feedback Control

The state-feedback control problem is solved using the above BRL [57]. We consider the following system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{2,k} u_k + B_{1,k} w_k, \quad x_0 = x_0, \quad k = 0, 1, ..., N-1 \\ z_k &= L_k x_k + D_{12,k} u_k, \quad k = 1, 2, ..., N \end{aligned} \tag{C.7}$$

where x_k, w_k, z_k and x_0 are defined as in the system of (C.1) and where $u_k \in \mathcal{R}^s$ is the control input signal. We seek a state-feedback law $u_k = K_k x_k$ that achieves $J_B \leq 0$ of (C.2) for all $\{w_k\} \in l_2[0 \quad N-1]$ and $x_0 \in \mathcal{R}^n$. The central solution to the above problem is well known [57]. It can also be achieved by substituting u_k into (C.7) and applying the DLMI of Theorem C.1.1. The following lemma solves the H_∞ state-feedback problem:

Lemma C.2.1: *Consider the system of (C.7) and $u_k = K_k x_k$ with the performance index of (C.2). For a prescribed $\gamma > 0$ there exists a control law of $u_k = K_k x_k$ that guarantees a non positive J_B for all $\{w_k\} \in l_2[0 \quad N-1]$ and $x_0 \in \mathcal{R}^n$ if there exist $Q_k \in \mathcal{R}^{n\times n}$ and $K_k \in \mathcal{R}^{s\times n}$ that satisfy the following DLMIs:*

$$\begin{aligned} &\begin{bmatrix} -Q_{k+1}^{-1} & \tilde{A}_k & \tilde{B}_k & 0 \\ \tilde{A}_k^T & -Q_k & 0 & \tilde{L}_k^T \\ \tilde{B}_k^T & 0 & -\gamma^2 I_q & 0 \\ 0 & \tilde{L}_k & 0 & -I_m \end{bmatrix} \leq 0, \; k = N-1, ..., 0, \; Q_N = \bar{Q}_N, \\ &\begin{bmatrix} \gamma^2 I_q & \tilde{B}_k^T Q_{k+1} \\ \tilde{B}_k Q_{k+1} & Q_{k+1} \end{bmatrix} > 0 \end{aligned} \tag{C.8}$$

where

$$\tilde{A}_k \stackrel{\triangle}{=} A_k + B_{2,k} K_k, \quad \tilde{B}_k = B_{1,k}, \quad \textit{and} \quad \tilde{L}_k = L_k + D_{12,k} K_k,$$

with Q_k of minimum trace that also satisfies $Q_0 \leq \gamma^2 \bar{Q}_0$.

Unlike the solution to the BRL which sought a single matrix Q_k for (C.5), in Lemma C.2.1 we seek, at each instant k, two matrix variables namely: Q_k and the state-feedback gain K_k.

Remark C.2.1: In a way dual to the state-feedback control, one can apply (C.8) to design both *a priori* and *a posteriori* Luenberger-type filters. The DLMI of (C.8) should then be solved iteratively where a minimum for $Tr(Q_{k+1}^{-1})$ is sought, beginning with an initial condition that equals to the weight on the initial state in the performance index.

C.2.2 Robust H_∞ State-feedback Control of Uncertain Systems

A clear advantage of the new DLMI method emerges when dealing with uncertain systems. We consider the case where, due to uncertainty in the plant's parameters, the system is described by a finite set of plants. It is desired to obtain a single controller (in the present application a state-feedback one) for all of the plants in the set that will meet some pre-specified stability and disturbance attenuation requirements. Such requirements arise in many practical situations where a single controller is sought that will satisfy prescribed performance criteria for various operating points. For example, in [71], a single controller is sought that will satisfy some strict requirements on the behavior of a flexible arm for three different loads.
The above problem can be solved, in the stationary case, using the method of [12] for state-feedback design in the presence of polytopic-type uncertainty. Embedding the set of plants into a convex polytope may provide a solution to the problem, if a single matrix solution exists for a set of LMIs that are solved at the vertices of the polytope. This method is quite conservative since it requires the quadratic stabilizability of the set of systems involved. The solution guarantees, however, the required performance for all of the possible plants in the polytope.
The advantages of the DLMI approach to the design of the required robust state-feedback control are two-fold. First of all, it can provide a solution to the time-varying case over a finite horizon and secondly it is not restricted to quadratically stabilizing solutions.
Representing the finite set of plants involved by $\Phi \stackrel{\Delta}{=} \{\mathcal{S}_i,\ i = 1, ...n_p\}$, where the system $\mathcal{S}_i$ is described by:

$$\begin{aligned} x_{k+1} &= A_{k,i}x_k + B_{2,k,i}u_k + B_{1,k,i}w_k, \quad k = 0, 1, ..., N-1, \quad x_0 = x_{0,i} \\ z_k &= L_{k,i}x_k + D_{12,k,i}u_k, \quad k = 1, 2, ..., N. \end{aligned} \tag{C.9}$$

we look for a single state-feedback law of $u_k = K_k x_k, \quad k = 0, ...N-1$ which ensures that all of the n_p systems in Φ, achieve (C.2), for all $\{w_k\} \in l_2[0 \quad N\text{–}1]$ and $x_{0,i} \in \mathcal{R}^n$. The following corollary provides the solution to the robust H_∞ state-feedback problem:

Corollary C.2.1 : *Consider the system of (C.9) with the performance index of (C.2). For a prescribed* $\gamma > 0$ *there exists a control law of* $u_k = K_k x_k$ *that guarantees a non positive* J_B*, over* Φ*, for all* $\{w_k\} \in l_2[0 \quad N-1]$ *and* $x_0 \in \mathcal{R}^n$ *if there exist* $Q_{k,i} \in \mathcal{R}^{n\times n}$ *and* $K_k \in \mathcal{R}^{s\times n}$ *that satisfy the following DLMIs:*

$$\begin{bmatrix} -Q_{k+1,i}^{-1} & \tilde{A}_{k,i} & \tilde{B}_{k,i} & 0 \\ \tilde{A}_{k,i}^T & -Q_{k,i} & 0 & \tilde{L}_{k,i}^T \\ \tilde{B}_{k,i}^T & 0 & -\gamma^2 I_q & 0 \\ 0 & \tilde{L}_{k,i} & 0 & -I_m \end{bmatrix} \leq 0, \; Q_{N,i} = \bar{Q}_N,$$

$$\begin{bmatrix} \gamma^2 I_q & \tilde{B}_{k-1,i}^T Q_{k,i} \\ \tilde{B}_{k-1,i} Q_{k,i} & Q_{k,i} \end{bmatrix} > 0, \tag{C.10}$$

for $i = 1, 2, ..., n_p,\; k = N-1, ..., 0,$ *with* $Q_{k,i}$ *of minimum trace that also satisfy* $Q_{i,0} \leq \gamma^2 \bar{Q}_0$*, where*

$$\tilde{A}_{k,i} \stackrel{\Delta}{=} A_{k,i} + B_{2,k,i}K_k, \quad \tilde{B}_{k,i} = B_{1,k,i}, \;\; \textit{and} \;\; \tilde{L}_{k,i} = L_{k,i} + D_{12,k,i}K_k.$$

C.2.3 The Static Output-feedback Control Problem

The DLMI method allows us to solve the static output-feedback control problem in much the same way as the state-feedback control problem of Section C.2.1. We consider the following system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_{2,k} u_k + B_{1,k} w_k, \qquad && k = 0, 1, ..., N-1 \\ y_k &= C_k x_k, \quad z_k = L_k x_k + D_{12,k} u_k, && k = 1, 2, ..., N \end{aligned} \tag{C.11}$$

where $x_k,\ w_k,\ z_k$ and x_0 are defined as in the system (C.1) and where $u_k \in \mathcal{R}^s$ is the control input signal. We seek a static output-feedback law $u_k = K_k y_k$ that achieves $J_B \leq 0$ of (C.2) for all $\{w_k\} \in l_2[0 \quad N-1]$ and $x_0 \in \mathcal{R}^n$.
The ‘central-type’ solution to the above problem can be achieved by substituting $u_k = K_k y_k$ into (C.11) and applying the DLMI of Theorem C.1.1. The H_∞ static output-feedback problem is solved with the aid of the next lemma:

Lemma C.2.2: *Consider the system of (C.11) with the performance index of (C.2). For a prescribed* $\gamma > 0$ *there exists a control law of* $u_k = K_k y_k$ *that guarantees a non positive* J_B *for all* $\{w_k\} \in l_2[0 \quad N-1]$ *and* $x_0 \in \mathcal{R}^n$ *if there exist* $Q_k \in \mathcal{R}^{n\times n}$ *and* $K_k \in \mathcal{R}^{s\times r}$ *that satisfy the DLMI of (C.8) where*

$$\tilde{A}_k \stackrel{\triangle}{=} A_k + B_{2,k}K_kC_k, \quad \tilde{B}_k = B_{1,k}, \;\; \textit{and} \;\; \tilde{L}_k = L_k + D_{12,k}K_kC_k,$$

with Q_k *of minimum trace that also satisfies* $Q_0 \leq \gamma^2 \bar{Q}_0$.

As with the robust state-feedback control of Section C.2.1, the advantage of the DLMI approach in the static output-feedback case lies in its ability to provide a finite horizon solution and, in particular, the fact that it is not restricted to quadratically stabilizing solutions (see [38] and the references therein). Representing the finite set of plants involved by $\tilde{\Phi} \stackrel{\triangle}{=} \{\mathcal{S}_i,\ i = 1, ...n_p\}$, where the system $\mathcal{S}_i$ is described by:

$$\begin{aligned} x_{k+1} &= A_{k,i}x_k + B_{2,k,i}u_k + B_{1,k,i}w_k, \qquad k = 0, 1, ..., N-1, \quad x_0 = x_{0,i} \\ y_k &= C_{k,i}x_k, \quad z_k = L_{k,i}x_k + D_{12,k,i}u_k, \quad k = 1, 2, ..., N \end{aligned} \tag{C.12}$$

and where, similar to Section C.2.1 we seek a single static-feedback law of the form $u_k = K_k y_k$ which ensures that all of the n_p systems in $\tilde{\Phi}$, achieve $J_B \leq 0$, for all $\{w_k\} \in l_2[0 \quad N-1]$ and $x_{0,i} \in \mathcal{R}^n$. The solution to the robust H_∞ static output feedback problem is obtained from the following corollary:

Corollary C.2.2: *The robust static output-feedback problem of (C.12), $u_k = K_k y_k$ and (C.2) possesses a solution over $\tilde{\Phi}$ for a prescribed $\gamma > 0$ if for every $k = N-1, N-2, ..., 0$ and for $i = 1, 2, ..., n_p$ there exist $Q_k \in \mathcal{R}^{n\times n}$ and $K_k \in \mathcal{R}^{s\times r}$ that satisfy the following DLMI*

$$\begin{bmatrix} -Q_{k+1,i}^{-1} & \tilde{A}_{k,i} & \tilde{B}_{k,i} & 0 \\ \tilde{A}_{k,i}^T & -Q_{k,i} & 0 & \tilde{L}_{k,i}^T \\ \tilde{B}_{k,i}^T & 0 & -\gamma^2 I_q & 0 \\ 0 & \tilde{L}_{k,i} & 0 & -I_m \end{bmatrix} \leq 0, \ Q_{N,i} = \bar{Q}_N,$$
$$\begin{bmatrix} \gamma^2 I_q & \tilde{B}_{k,i}^T Q_{k+1,i} \\ \tilde{B}_{k,i} Q_{k+1,i} & Q_{k+1,i} \end{bmatrix} > 0 \tag{C.13}$$

with $Q_{k,i}$ of minimum trace that also satisfy $Q_{0,i} \leq \gamma^2 \bar{Q}_0$, where

$$\tilde{A}_{k,i} \stackrel{\triangle}{=} A_{k,i} + B_{2,k,i}K_k C_{k,i}, \quad \tilde{B}_{k,i} = B_{1,k,i}, \ \ and \ \tilde{L}_{k,i} = L_{k,i} + D_{12,k,i}K_k C_{k,i}.$$

Remark C.2.2: The above procedure can be readily performed such that a single matrix solution Q_k is found for all the n_p DLMIs at the kth iteration step, $k = N-1, N-2, ..., 0$. The latter corresponds to the selection of a single Lyapunov function $\sum_{k=0}^{N-1} x_{k+1}^T Q_{k+1} x_{k+1} - x_k^T Q_k x_k$ for all the plants in $\tilde{\Phi}$.

C.3 Example: Robust H_∞ State-feedback Control

To demonstrate the applicability of the DLMI technique for the discrete-time setup we bring a practical problem of terrain following control system which is encountered in aerospace engineering. The problem is taken from [80] and a full description of the physics involved appears in Section 11.11, pages 404-408 there.

We consider the following continuous-time uncertain system:

$$\dot{x} = Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{12} u,$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -10^{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -10^{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10^{-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ A(6,1) & .0345 & 0 & 0 & 1 & A(6,6) & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & -10^{-6} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.02 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0\,0\,0\,0\,0 & 0 & 1 \\ 0\,0\,0\,0\,0 & \sqrt{5} & 0 \\ 0\,0\,0\,0\,0 & 0 & 0 \\ 0\,0\,0\,0\,0 & 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3162 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.09 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and where the uncertain parameters reside in the following intervals:

$$A(6,1) = [-0.0704 \quad -0.1056], \quad A(6,6) = [-0.00256 \quad -0.00384].$$

Note that the original problem was of order 6 and we have added a seventh state (an integrator) in order to provide a zero tracking error for the tracking altitude command. The discrete-time robust state-feedback version of the above continuous-time problem was solved, by discretization with a ZOH (zero-order hold) at a sampling rate of $10H_z$. The control design consisted of the following 3 distinct plants:

$$(A(6,1),\ A(6,6)) = \begin{array}{c}(-0.0704 \ -0.00256),\ (-0.1056 \ -0.00384), \\ (-0.088 \ -0.0032)\end{array}.$$

The state-feedback controller for the above system, in the stationary case, was readily found by applying the DLMIs procedure of (C.10) of Corollary C.2.1. We note that at each instant k, starting from $k = N = 400$ where $Q_{N,i} = \bar{Q}_N = 10^{-8} I_7,\ i = 1, 2, 3$, we substituted for Q_k and obtained K_{k-1}

and Q_{k-1}. For a near minimum value of $\gamma = 0.080$ we obtained the following robust state-feedback controller K where K is:

$$\begin{bmatrix} -6.4646 & -10.9603 & -5.8185 & -0.1620 & -0.3736 & -0.5934 & -72.8956 \\ -13.5683 & -39.5879 & -18.3110 & -45.9064 & -97.8663 & -180.6088 & -228.7737 \end{bmatrix}.$$

The top part of Figure C.1 below depicts the traces of the three matrix solutions $\{Q_{k,i}\}$ that are obtained for the three OPs, along the interval of the recursive solution of the DLMI of Corollary C.1.1. The bottom part of Figure C.1 describes the trace of KK^T, where K is the controller gain, along the solution trajectory. Both parts of the figure show the convergence properties of the DLMI method.

C.4 Conclusions

In this appendix a DLMI method for discrete-time linear systems is presented which successfully mimics the Riccati recursion in the solution of the standard time-varying H_∞control and filtering problems over a finite horizon. Due to their structure, the DLMIs that are solved at each instant are linear in the matrices Q_k or Q_{k+1}, depending on the nature of the iterative solution and in the matrices that constitute the required controllers or filters. This linearity enables a simple and easily implementable treatment of plant uncertainty in both the finite horizon time-varying and the stationary cases.

The problems of robust output-feedback control, either static or dynamic, and the H_∞ estimation that uses a general filter, is solved here for the first time. The existing H_∞ stationary solution for uncertain systems (i.e convex (polytopic) method [12]), [38], [37] are not applicable to many practical control problems (i.e tracking control), which are of the finite-horizon type. It has been shown in [44] and partially in the present appendix that the DLMI method easily tackles these latter types of problems. In the stationary case, the DLMI method may provide solutions that are significantly better than those obtained by the polytopic type design of [12] and [38] (see [44]).

We note that while the convex methods [12], [38], [37], guarantee a solution for all of the operating points included within the convex domain, the non-convex DLMI approach guarantees a solution for only the operating points taken into account by the design procedure. In some control systems, most notably in aerospace, a design for distinct plants is of major importance.

Dealing with robust problems, such as the finite-horizon robust state-feedback, the DLMI method easily allows the implementation of both the quadratic and the non quadratic design procedures. An additional important advantage of the DLMI method is that, besides satisfying a required H_∞ performance, it may deal successfully with additional design requirements, such as the minimization of bounds on the peak-to-peak gain and with the control and estimation problems of systems with state-multiplicative noise.

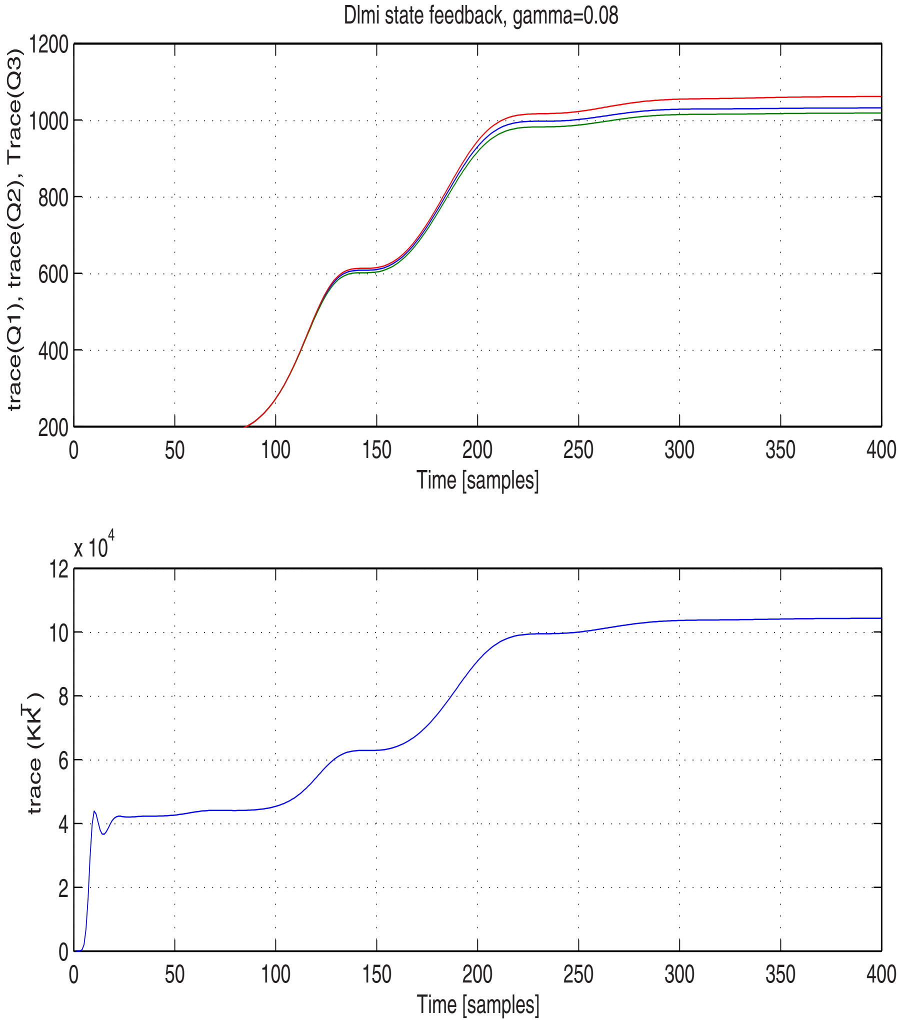

Fig. C.1. The results of the H_∞ robust state-feedback solution. Shown are the traces of the 3 matrix solutions for the 3 distinct OPs and the trace of K*K'.

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