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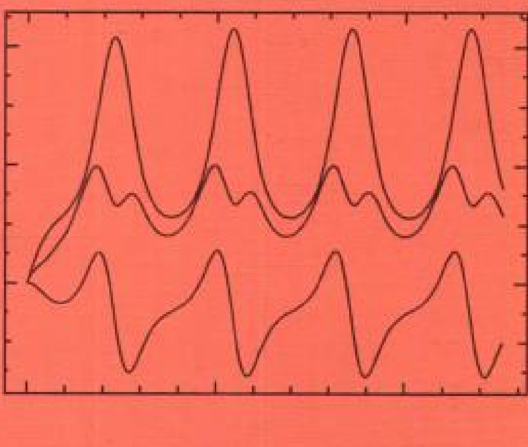
Akira Ichikawa and Hitoshi Katayama

Linear Time Varying Systems and Sampled-data Systems



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Linear Time Varying Systems and Sampled-data Systems

With 40 Figures

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Preface

The system theory for linear time-invariant systems is now mature and offers a wide range of system theoretic concepts, design methods and solutions to optimal or suboptimal control problems including the design of state feedback controllers and observers, optimal quadratic regulators, Kalman filters, coprime factorization and Youla-parametrization of stabilizing controllers, H_2 control, differential games, H_∞ control and robust control. One of the most important recent developments is, without doubt, H_∞ control. Since the beginning of the linear systems theory many researchers have made constant efforts to extend the theory to time-varying systems and sampled-data systems as well as to infinite dimensional systems. Although there are many excellent books on the systems theory of linear time-invariant systems, there are not many books covering recent developments for time-varying systems. In this monograph we consider linear optimal regulators, H_2 control, differential games, H_∞ control and filtering, and develop the theory for time-varying systems and jump systems. Jump systems arise when impulse controls are involved. As is well-known sampled-data systems can be written as jump systems with constant coefficients which are regarded as periodic systems with period equal to the sampling period. One of our main motivations for writing this monograph is to develop the H_2 and H_∞ theory of sampled-data systems from the jump system point of view. The jump system is a natural state-space representation of sampled-data systems and original signals and parameters are maintained in the new system. The H_2 and H_∞ problems for jump systems can be treated in a unified manner as for time-invariant systems. Moreover, they can be directly extended to more general cases of delayed observation, first-order hold and infinite dimensional systems. Jump systems are also useful to design stabilizing controllers for certain nonlinear systems. Since jump systems with constant coefficients are periodic systems and hence time-varying systems, it is useful to develop the system theory for time-varying systems. Extension of the system theory to time-varying systems seems routine, but there are some inherent features of time-varying systems. For example, frequency domain arguments cannot be extended and the state-space approach is needed. Some arguments for time-invariant systems may not have easy extensions to time-varying systems. The H_∞ theory based on X and Y Riccati equations is such an example as we see in Chapter 2. Hence the systems theory for time-varying systems itself is important and interesting and gives some new points of view or new insights into the system theory of time-invariant systems.

In Chapter 2 we consider continuous-time systems and consider stability, quadratic control, differential games, H_∞ control, H_∞ filtering and H_2 control. In H_∞ control and filtering we allow for initial uncertainty in the system and develop the general theory of this case. We give examples and computer simulations for most of main results. Chapter 3 is concerned with discrete-time systems and discusses the same topics as in Chapter 2. Chapter 4 introduces the jump system which contains both continuous- and discrete-time features and discusses the same problems as in earlier chapters. Chapter 5 covers a special case of jump systems which arises from the sampled-data systems with zero-order hold and applies the main results of Chapter 4 to them. Finally in Chapter 6 we discuss further developments in the theory of jump systems. We first give an extension to infinite dimensions and as an example we consider H_2 and H_∞ control for sampled-data systems with first-order hold. We also introduce sampled-data fuzzy systems which can express certain nonlinear sampled-data systems and show how to design stabilizing output feedback controllers using jump systems.

Chapter 2 is an introduction to time-varying continuous-time systems while Chapter 3 is an introduction to discrete-time systems and either of them can be read independently of the rest of the monograph. To read Chapter 4 the materials in Chapters 2 and 3 will be very helpful. To read Section 6.1 elements of functional analysis are necessary.

Contents

1. Introduction

The linear system theory offers basic concepts, design methods and optimization problems. The underlying systems it deals with are usually time-invariant continuous-time systems or discrete-time systems. There are many excellent books [1, 19, 21, 46, 55, 66, 80, 93] for time-invariant systems which cover stability theory, quadratic control, H_2 control and H_∞ control. The sampled-data system with sampler and zero-order hold is a linear system but involves both continuous-time and discrete-time signals. The system theory for it, which covers topics above, is usually developed via the system theory for discrete-time systems after transforming the original problems to those for discrete-time systems.

The purpose of this monograph is two fold. We introduce the linear system theory for time-varying systems which covers H_2 and H_∞ control. There are some inherent features of time-varying systems, and not all arguments used for time-invariant systems are easily extended to them. Thus we regard the extension as important and hope that it gives some new insight into the linear systems theory. Secondly we develop the H_2 and H_∞ control theory for sampled-data systems from the point of view of jump systems. The jump system, which contains jumps in the state variable, is a natural state-space representation of sampled-data systems and has an advantage that the continuous-time nature and discrete-time signals of the original system are maintained in the new system. Hence the system theory for jump systems can be viewed as an extension of the theory of continuous-time or discrete-time systems. In H_∞ control the initial conditions are usually taken to be zero, but in this monograph initial uncertainty is incorporated and a general theory is developed.

As the jump system may not be familiar to the reader, we shall introduce below all the systems which appear in this monograph. We also introduce the jump system which is obtained from a sampled-data system.

1.1 Continuous-time Systems and Discrete-time Systems

In Chapter 2 we consider continuous-time systems of the form

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + D_{21}(t)w \end{aligned} \tag{1.1}$$

where x is the state of the system, w is a disturbance, u is a control input, z is a controlled output and y is the output to be used for control.

In Chapter 3 we consider discrete-time systems of the form

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k). \end{aligned} \tag{1.2}$$

We consider stability, quadratic control, disturbance attenuation problems, differential games, H_∞ control and H_2 control. When we introduce new problems, we sometimes start with the results for time-invariant systems. Moreover, all the results in the time-invariant case are given as corollaries. However, we give proofs only for time-varying systems. For proofs typical to time-invariant systems we refer the reader to other books in the reference.

1.2 Jump Systems

A general form of jump systems is given by

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_dx(k\tau) + B_{1d}w_d(k) + B_{2d}u_d(k), \\ z_c &= C_1x + D_{12}u, \\ z_d(k) &= C_{1d}x(k\tau) + D_{12d}u_d(k), \\ y_c &= C_2x + D_{21}w, \\ y_d(k) &= C_{2d}x(k\tau) + D_{21d}w_d(k) \end{aligned} \tag{1.3}$$

where the continuous part satisfies (1.1) while the jump part satisfies (1.2). We assume that all matrices in the system are constant. Then it is a τ-periodic system and a special case of time-varying systems. We can easily see the following :

(a) If $B_1 = 0$, $B_2 = 0$, $A_d = I$ and $B_{1d} = 0$, then it is a system with impulse control.
(b) If $A_d = I$, $B_{1d} = 0$ and $B_{2d} = 0$, it is a continuous-time system.
(c) If $A = 0$ $B_1 = 0$ and $B_2 = 0$, then it can be regarded as a discrete-time system.

Hence the jump system is a natural extension of continuous-time and discrete-time systems [37, 51, 65, 67] and we can expect some potential applications in the areas of mechanical systems [83], chemical processes [56] and economic systems [6] where impulsive inputs naturally appear. The system (1.3) is often too general and for H_2 or H_∞ control we shall restrict ourselves to the system of the form

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w, \;\; k\tau < t < (k+1)\tau, \\
x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \\
z_c &= C_1 x, \\
z_d(k) &= D_{12} u(k), \\
y(k) &= C_2 x(k\tau) + D_{21} w_d(k).
\end{aligned} \tag{1.4}
$$

This system still keeps the essential features of jump systems and covers sampled-data systems. In Chapter 4 we consider stability and control problems for (1.4) as in Chapters 2 and 3.

1.3 Sampled-data Systems

In Chapter 5 we consider the sampled-data system [8, 16]

$$
\begin{aligned}
\dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\
z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\
y(k) &= C_2 x(k\tau) + D_{21} w_d(k)
\end{aligned} \tag{1.5}
$$

where $\tau > 0$ is a sampling period and $\tilde{u}$ is the control input realized through the zero-order hold [18, 85]

$$\tilde{u}(t) = u(k), \;\; k\tau < t \leq (k+1)\tau.$$

We introduce the following state space representation of the control $\tilde{u}(t)$:

$$\dot{\bar{x}} = 0, \;\; \bar{x}(k\tau^+) = u(k), \;\; k\tau < t \leq (k+1)\tau.$$

Then clearly $\tilde{u}(t) = \bar{x}(t)$. Let $x_e(t) = [x' \;\; \bar{x}']'(t)$ be the new state variable. Then the system (1.5) is equivalent to the following jump system

$$
\begin{aligned}
\dot{x}_e(t) &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} x_e(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t), \;\; k\tau < t < (k+1)\tau, \\
x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \;\; i = 0, 1, 2, ..., \\
z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} [C_1 \;\; 0]\, x_e(t) \\ \sqrt{\tau} D_{12} u(k) \end{bmatrix}, \\
y(k) &= [C_2 \;\; 0]\, x_e(k\tau) + D_{21} w_d(k)
\end{aligned}
$$

where $z_d = \sqrt{\tau} D_{12} u(k)$ is introduced when we consider

$$\int_0^\infty | D_{12}\tilde{u}(t) |^2 = \sum_{k=0}^\infty \int_0^\tau | D_{12}u(k) |^2 = \sum_{k=0}^\infty | \sqrt{\tau} D_{12}u(k) |^2 .$$

The jump system is a natural state-space representation of sampled-data systems and original signals and parameters are maintained in the new system. The H_2 and H_∞ problems for jump systems can be treated in a unified manner as for time-invariant systems.

1.4 Infinite Dimensional Systems and Sampled-data Fuzzy Systems

The infinite dimensional system in Chapter 6 is written as (1.4) but the state, inputs and outputs lie in Hilbert spaces. We assume that A is the infinitesimal generator of a strongly continuous semigroup in a Hilbert space and other operators are bounded. Thus the system covers partial differential equations and delay differential equations. It will be shown that the sampled-data system with first-order hold can be expressed in this form and H_2 and H_∞ control problems are considered.

The sampled-data fuzzy system in Section 6.2 is given by

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^r \lambda_i(z(t))\{A_i x(t) + B_i \tilde{u}(t)\}, \\ y(k) &= \sum_{i=1}^r \lambda_i(z(k\tau)) C_i x(k\tau) \end{aligned}$$

where $\lambda_i(z) \geq 0$, $\sum_{i=1}^r \lambda_i(z) = 1$ and z is a premise variable in the IF-THEN rules. It can represents certain nonlinear sampled-data systems. Using local jump systems we give a design method of stabilizing output feedback controllers.

1.5 Notation

$$\begin{aligned} | x | &= \sqrt{x'x}, \; x \in \mathbf{R}^n. \\ | M | &: \text{ norm of } M \in \mathbf{R}^{n\times m} \text{ induced by the Euclidean norm.} \\ < f, g > &= \int_{t_0}^T f'(t)g(t)dt, \; f, \; g \in L^2(t_0, T; \mathbf{R}^n) \\ & \quad \text{where } T \text{ can be finite or } T = \infty. \\ \| f \|_2 &= \sqrt{< f, f >}. \end{aligned}$$

$$
\begin{aligned}
< f, g > &= \sum_{k=k_0}^{N} f'(k) g(k), \;\; f, \;\; g \in l^2(k_0, N; \mathbf{R}^n) \\
&\quad \text{where } N \text{ can be finite or } N = \infty. \\
\| f \|_2 &= \sqrt{< f, f >}. \\
x(s^+) &= \lim_{t \downarrow s} x(t). \\
x(s^-) &= \lim_{t \uparrow s} x(t).
\end{aligned}
$$

2. Continuous-time Systems

In this chapter we are concerned with time-varying continuous-time systems and consider stability, quadratic control, disturbance attenuation problems, differential games, H_∞ control, H_∞ filtering and H_2 control. In H_∞ control and filtering we allow for initial uncertainty in the systems and develop the general theory.

2.1 Stability

2.1.1 Lyapunov Equations

Consider

$$\dot{x} = A(t)x, \quad x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n\times n}$ is a piecewise continuous matrix with

$$| A(t) | \leq a, \ ^\forall t \geq t_0 \text{ for some } a > 0.$$

Let $S(t, r)$ be the state transition matrix of the system (2.1). Then

$$\frac{d}{dt}S(t,s) = A(t)S(t,s), \quad S(s,s) = I.$$

If

$$A(t)A(s) = A(s)A(t) \ ^\forall t, \ s$$

then

$$S(t,s) = e^{\int_s^t A(r)dr}.$$

If A is θ-periodic, i.e., $A(t+\theta) = A(t)$, then

$$S(t+\theta, s+\theta) = S(t,s).$$

If $A(t) = A$, then $S(t,s) = e^{A(t-s)}$.

Definition 2.1 *The system (2.1) (or simply A) is said to be exponentially stable on* $[t_0, \infty)$ *if*

$$| S(t,s) | \leq Me^{-\alpha(t-s)} \textit{ for any } t_0 \leq s \leq t < \infty$$

for some positive constants M and α independent of t_0 and t. (The system (2.1) is also called internally stable.)

If $A(t) = A$, then A is stable if and only if every eigenvalue of A has a negative real part. The following result is also well-known.

Proposition 2.1 *The following statements are equivalent.*
(a) A is exponentially stable.
(b) There exists a positive definite matrix X satisfying

$$A'X + XA + I = 0. \tag{2.2}$$

(c) There exists a positive definite matrix Y satisfying

$$AY + YA' + I = 0.$$

The equation (2.2) is called the Lyapunov equation. We generalize this result to the time-varying system. We need the following lemma.

Lemma 2.1 *(a) $| S(t,s) | \leq e^{a(t-s)}$, $t_0 \leq s \leq t$.*
(b) For a given $\epsilon \in (0,1)$, there exists a $\delta > 0$ such that

$$S'(t,s)S(t,s) \geq (1-\epsilon)I \textit{ for any } 0 \leq t-s \leq \delta.$$

Proof. Since

$$S(t,s) = I + \int_s^t A(r)S(r,s)dr$$

we have

$$| S(t,s) | \leq 1 + \int_s^t a \mid S(r,s) \mid dr.$$

Hence by Gronwall's inequality, we obtain $| S(t,s) | \leq e^{a(t-s)}$. We also have

$$\begin{aligned} \left| \int_s^t A(r)S(r,s)dr \right| &\leq \int_s^t a \mid S(r,s) \mid dr \\ &\leq \int_s^t a e^{a(r-s)} dr = e^{a(t-s)} - 1. \end{aligned}$$

Now

$$\begin{aligned} x'S'(t,s)S(t,s)x &= x'(I + \int_s^t A(r)S(r,s)dr)'(I + \int_s^t A(r)S(r,s)dr)x \\ &\geq \mid x \mid^2 + 2x' \int_s^t A(r)S(r,s)drx \\ &\geq \mid x \mid^2 - 2(e^{a(t-s)} - 1) \mid x \mid^2 \\ &\geq (1-\epsilon) \mid x \mid^2 \end{aligned}$$

for any $0 < t - s \leq \delta$ where $\delta = \frac{1}{a}\log(1 + \frac{\epsilon}{2})$ so that $2(e^{a(t-s)} - 1) \leq \epsilon$, $0 \leq t - s \leq \delta$. ∎

Proposition 2.2 *The following statements are equivalent.*
(a) The system (2.1) is exponentially stable.
(b) There exists a symmetric matrix $X(t)$ such that

$$\begin{array}{ll} (i) & c_1 I \le X(t) \le c_2 I, \ \forall t \ge t_0 \text{ for some } c_i > 0,\ i = 1, 2. \\ (ii) & -\dot{X} = A'(t)X + XA(t) + I. \end{array}$$

(c) $\int_s^\infty | S(t,s)x |^2 \, dt \le c \, | x |^2, \ \forall x, \ \forall s \ge t_0$ *and for some* $c > 0$.
If A is θ-periodic, then X is also θ-periodic.

Proof. Suppose (a) holds. Then (c) also holds and

$$X(t) = \int_t^\infty S'(r,t)S(r,t)dr$$

is well-defined and bounded, i.e., $X(t) \le c_2 I$. Let $\epsilon \in (0,1)$ be given and choose $\delta > 0$ as Lemma 2.1 such that

$$S'(t,s)S(t,s) \ge (1-\epsilon)I \text{ for } 0 \le t - s \le \delta.$$

Then

$$X(t) \ge \int_t^{t+\delta} S'(r,t)S(r,t)dr \ge (1-\epsilon)\delta I.$$

Hence (i) of (b) has been shown. (ii) of (b) follows from differentiating $X(t)$.

Now we assume (b). Then for $x(t) = S(t,s)x_0$

$$\frac{d}{dt}(x'(t)X(t)x(t)) = - | x(t) |^2 \le -\frac{1}{c_2} x'(t)X(t)x(t)$$

which implies

$$x'(t)X(t)x(t) \le e^{-\frac{1}{c_2}(t-s)} x'(s)X(s)x(s).$$

Using the property (i) we have

$$c_1 | x(t) |^2 \le c_2 e^{-\frac{1}{c_2}(t-s)} | x_0 |^2 .$$

Hence

$$| S(t,s) | \le \sqrt{\frac{c_2}{c_1}} e^{-\frac{1}{2c_2}(t-s)}$$

and (a) follows.

Finally let $A(t)$ be θ-periodic. Then

$$\begin{aligned} X(t) &= \int_t^\infty S'(r,t)S(r,t)dr \\ &= \int_t^\infty S'(r+\theta, t+\theta)S(r+\theta, t+\theta)dr \\ &= \int_{t+\theta}^\infty S'(s, t+\theta)S(s, t+\theta)ds \\ &= X(t+\theta). \end{aligned}$$

∎

Definition 2.2 *The equation (ii) of (b) is called the Lyapunov equation for the system (2.1).*

If A is exponentially stable, we can show that any solution of the Lyapunov equation coincides with $X(t)$ given in the proof of Proposition 2.2. Hence the Lyapunov equation has a unique solution. See also Theorem 2.4.

Example 2.1 Consider the periodic system with period 2π:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - 0.5\cos t & -1 - \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.3}$$

which is exponentially stable. In fact there exists a 2π-periodic nonnegative solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ of the condition (b) in Proposition 2.2 (Figures 2.1 and 2.2).

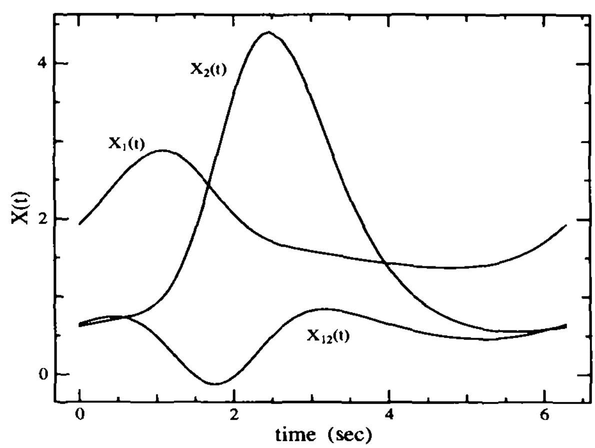

Figure 2.1: The periodic solution $X(t)$ of the Lyapunov equation

Consider the adjoint equation of (2.1)

$$-\dot{\xi} = A(t)\xi, \quad \xi(T) = \xi_1. \tag{2.4}$$

Let $\xi(t; T, \xi_1)$ be the solution of (2.4).

Definition 2.3 *The system (2.4) is said to be exponentially stable if*

$$| \xi(t; T, \xi_1) | \leq M e^{-\alpha(T-t)} | \xi_1 | \quad \textit{for any } t_0 \leq t \leq T < \infty$$

for some $M > 0$ and $\alpha > 0$ independent of t, T and ξ_1.

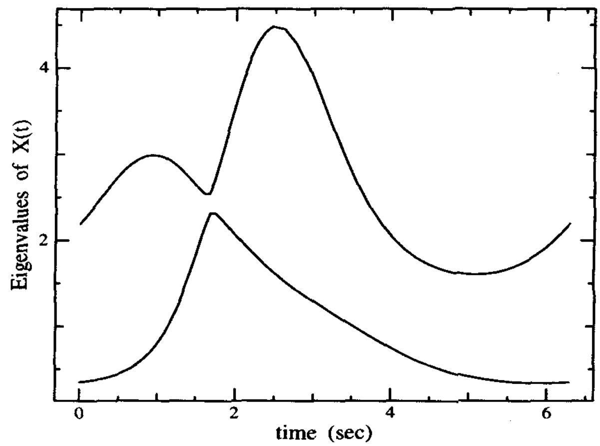

Figure 2.2: Eigenvalues of the periodic solution $X(t)$

Since $\xi(t;T,\xi_1) = S'(T,t)\xi_1$, the system (2.4) is exponentially stable if and only if the system (2.1) is exponentially stable.

We have a dual result to Proposition 2.2.

Proposition 2.3 *The following statements are equivalent.*
(a) The system (2.4) (and hence (2.1)) is exponentially stable.
(b) There exists a symmetric matrix $Y(t)$ and a $\delta > 0$ such that

(i) $0 < Y(t)$, $\forall t \geq t_0$ *and* $c_1 I \leq Y(t)$, $\forall t \geq t_0 + \delta$ *for some* $c_1 > 0$.
(ii) $Y(t) \leq c_2 I$, $t_0 \leq \forall t < \infty$ *for some* $c_2 > 0$.
(iii) $\dot{Y} = A(t)Y + YA'(t) + I$, $Y(t_0) = 0$.

(c) $\int_s^T | S'(T,t)\xi |^2 dt \leq c | \xi |^2$, $\forall s, T$ *with* $t_0 \leq s \leq T < \infty$ *and for some* $c > 0$.

Proof. Suppose (a) holds. Then (c) is true and

$$Y(t) = \int_{t_0}^{t} S(t,s)S'(t,s)ds$$

is well-defined, positive for $t > t_0$ and bounded. Hence (ii) of (b) holds. To show (i), let $\epsilon \in (0,1)$ and choose $\delta > 0$ such that

$$S(t,s)S'(t,s) \geq (1-\epsilon)I \text{ for } 0 \leq t-s \leq \delta.$$

Now let $t \geq \delta$, then

$$Y(t) \geq \int_{t-\delta}^{t} S(t,s)S'(t,s)ds \geq (1-\epsilon)\delta I.$$

Hence (i) follows. The equation (iii) follows from differentiating $Y(t)$.

Now we suppose (b) holds. Then

$$\frac{d}{dt}(\xi'(t)Y(t)\xi(t)) = \mid \xi(t) \mid^2 \geq \frac{1}{c_2}\xi'(t)Y(t)\xi(t)$$

from which follows

$$\xi'(s)Y(s)\xi(s) \leq e^{-\frac{1}{c_2}(T-s)}\xi'(T)Y(T)\xi(T).$$

Hence for $t_0 + \delta \leq s \leq T < \infty$

$$c_1 \mid \xi(s) \mid^2 \leq c_2 e^{-\frac{1}{c_2}(T-s)} \mid \xi_1 \mid^2$$

which yields

$$\mid S'(T,s) \mid \leq \sqrt{\frac{c_2}{c_1}} e^{-\frac{1}{2c_2}(T-s)}.$$

For $t_0 \leq s \leq t_0 + \delta \leq T < \infty$

$$\begin{aligned}
\mid S'(T,s) \mid &= \mid (S(T,t_0+\delta)S(t_0+\delta,s))' \mid \\
&\leq \mid S'(t_0+\delta,s) \mid\mid S'(T,t_0+\delta) \mid \\
&\leq \sqrt{\frac{c_2}{c_1}} c_0 e^{-\frac{1}{2c_2}(T-t_0-\delta)}
\end{aligned}$$

since $\mid S'(t_0+\delta,s) \mid \leq c_0$ for $t_0 \leq s \leq t \leq t_0 + \delta$ for some $c_0 > 0$. Hence

$$\mid S'(T,s) \mid \leq \sqrt{\frac{c_2}{c_1}} c_0 e^{\frac{1}{2c_2}\delta} e^{-\frac{1}{2c_2}(T-s)}.$$

For $t_0 \leq s \leq t \leq t_0 + \delta$

$$\mid S'(t,s) \mid \leq c_0 \leq c_0 e^{\frac{1}{2c_2}\delta} e^{-\frac{1}{2c_2}(t-s)}.$$

Choosing

$$M = \max(\sqrt{\frac{c_2}{c_1}},\ \sqrt{\frac{c_2}{c_1}} c_0 e^{\frac{1}{2c_2}\delta},\ c_0 e^{\frac{1}{2c_2}\delta})$$

we obtain

$$\mid S'(t,s) \mid \leq M e^{-\frac{1}{2c_2}(T-s)} \text{ for any } t_0 \leq s \leq T < \infty.$$

Hence (a) holds. ∎

Definition 2.4 *The equation (iii) of (b) is called the Lyapunov equation of the backward system (2.4) (or simply the backward Lyapunov equation).*

Corollary 2.1 *Let $A(t)$ be θ-periodic. Then the system (2.4) is exponentially stable if and only if there exists a θ-periodic solution of the backward Lyapunov equation with $c_1 I \leq Y(t) \leq c_2 I$, $\forall t$ for some c_1, $c_2 > 0$.*
Moreover, the θ-periodic solution is unique if A is exponentially stable.

Proof. We shall show that $Y(t+n\theta)$ is increasing in n and hence converges to $Y_\theta(t)$ which is θ-periodic. In fact

$$\begin{aligned}
Y(t+n\theta) &= \int_{t_0}^{t+n\theta} S(t+n\theta, s)S'(t+n\theta, s)ds \\
&= \int_{t_0}^{t+n\theta} S(t+(n+1)\theta, s+\theta)S'(t+(n+1)\theta, s+\theta)ds \\
&= \int_{t_0+\theta}^{t+(n+1)\theta} S(t+(n+1)\theta, \sigma)S'(t+(n+1)\theta, \sigma)d\sigma \\
&\le \int_{t_0}^{t+(n+1)\theta} S(t+(n+1)\theta, \sigma)S'(t+(n+1)\theta, \sigma)d\sigma \\
&= Y(t+(n+1)\theta).
\end{aligned}$$

Let $Y_\theta(t)$ be the limit of $Y(t+n\theta)$ as $n \to \infty$.

$$\begin{aligned}
Y_\theta(t+\theta) &= \lim_{n\to\infty} Y(t+\theta+n\theta) \\
&= \lim_{n\to\infty} Y(t+(n+1)\theta) = Y_\theta(t).
\end{aligned}$$

For the proof of uniqueness, see the proof of Theorem 2.4. ∎

Example 2.2 Consider the system (2.3) in Example 2.1, which is exponentially stable. There exists a bounded nonnegative solution $Y = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}$ satisfying the condition (b) in Proposition 2.3 which converges to a 2π-periodic solution (Figure 2.3).

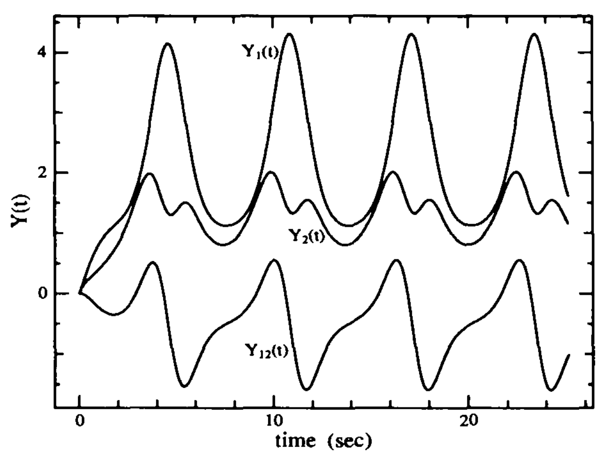

Figure 2.3: The bounded solution $Y(t)$

Consider

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \\ y &= C(t)x \end{aligned} \tag{2.5}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$, $y \in \mathbf{R}^{p_2}$ and A, B, C are bounded piecewise continuous matrices of appropriate dimensions. Then the solution $x(t)$ with $x(t_0) = x_0$ is given by

$$x(t) = S(t, t_0)x_0 + \int_{t_0}^{t} S(t, r)B(r)u(r)\ dr$$

and

$$y(t) = C(t)S(t, t_0)x_0 + C(t)\int_{t_0}^{t} S(t, r)B(r)u(r)\ dr.$$

Definition 2.5 *The system (2.5) is said to be input-output stable (or simply IO-stable) on* $[t_0, \infty)$ *if for* $x(s) = 0$, $s \geq t_0$ *and any* $u \in L^2(s, \infty; \mathbf{R}^{m_2})$

$$y \in L^2(s, \infty; \mathbf{R}^{p_2}) \ \text{and} \ \| y \|_2 \leq c \| u \|_2$$

for some $c > 0$ *independent of* s.

Definition 2.6 *(a) The pair* (A, B) *is said to be stabilizable on* $[t_0, \infty)$ *if there exists a bounded piecewise continuous matrix* K *such that* $A + BK$ *is exponentially stable on* $[t_0, \infty)$.
(b) The pair (C, A) *is detectable on* $[t_0, \infty)$ *if there exists a bounded piecewise continuous matrix* J *such that* $A + JC$ *is exponentially stable on* $[t_0, \infty)$.
(c) If (a) and (b) hold, the system (2.5) (or (A, B, C)*) is said to be stabilizable and detectable.*

Note that (A, I, I) is stabilizable and detectable.

Proposition 2.4 *Suppose* (A, B, C) *is stabilizable and detectable on* $[t_0, \infty)$. *Then the system (2.5) is exponentially stable if and only if it is IO-stable.*

Proof. It is enough to show sufficiency. First we shall show

$$C(t)S(t, s)x_0 \in L^2(s, \infty; \mathbf{R}^{p_2}).$$

Since (A, B) is stabilizable, there exists a bounded piecewise continuous matrix K such that the system

$$\dot{x} = (A + BK)(t)x, \ x(s) = x_0 \tag{2.6}$$

is exponentially stable. Hence $x \in L^2(s, \infty; \mathbf{R}^n)$. Then

$$\begin{aligned} \dot{x} &= A(t)x + B(t)K(t)x, \ x(s) = x_0, \\ x(t) &= S(t, s)x_0 + \int_{s}^{t} S(t, r)B(r)K(r)x(r)dr \end{aligned}$$

and

$$C(t)x(t) = C(t)S(t,s)x_0 + C(t)\int_s^t S(t,r)B(r)K(r)x(r)dr.$$

Since (2.5) is IO-stable

$$C(t)\int_s^t S(t,r)B(r)K(r)x(r)dr \in L^2(s,\infty;\mathbf{R}^{p_2})$$

and hence $C(t)S(t,s)x_0 \in L^2(s,\infty;\mathbf{R}^{p_2})$ and $\| C(t)S(t,s)x_0 \|_2 \leq c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. Since the system

$$x(t) = A(t)x, \quad x(s) = x_0$$

is equivalent to

$$x(t) = (A + LC)(t)x - L(t)C(t)x, \quad x(s) = x_0$$

where $L(t)$ is a bounded piecewise continuous matrix such that $A + LC$ is exponentially stable. Then we have

$$x(t) = S_L(t,s)x_0 + \int_s^t S_L(t,r)L(r)C(r)x(r)dr$$

where $S_L(t,r)$ is the state transition matrix of $A + LC$. Since

$$C(t)x(t) = C(t)S(t,s)x_0,$$

$x \in L^2(s,\infty;\mathbf{R}^n)$ and $\| x \|_2 \leq c \mid x_0 \mid$ which implies (2.5) is exponentially stable. ∎

Proposition 2.5 *(a) Suppose that (C, A) is detectable. The system (2.5) is exponentially stable if and only if there exists a bounded nonnegative solution to*

$$-\dot{X} = A'(t)X + XA(t) + C'(t)C(t). \tag{2.7}$$

(b) Suppose that (A, B) is stabilizable. Then the system (2.5) is exponentially stable if and only if there exists a bounded nonnegative solution to

$$\dot{Y} = A(t)Y + YA'(t) + B(t)B'(t). \tag{2.8}$$

Proof. We shall show (a) only. If A is exponentially stable,

$$X(t) = \int_t^\infty S'(r,t)C'(t)C(t)S(r,t)dr$$

is a bounded nonnegative solution of (2.7). Conversely, let $X(t)$ be a nonnegative solution of (2.7) and $x(t) = S(t,s)x_0$. Then differentiating $x'Xx$ we easily obtain

$$x'(T)X(T)x(T) + \int_s^T \mid C(t)x(t) \mid^2 dt = x_0'X(s)x_0.$$

Hence $C(t)S(t,s)x_0 \in L^2(s,\infty;\mathbf{R}^{p_2})$ with $\| CS(t,s)x_0 \|_2 \leq c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. As in the last part of the proof of Proposition 2.4, we can show $x \in L^2(s,\infty;\mathbf{R}^n)$ with $\| x \|_2 \leq c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. $Y(t)$ given by

$$Y(t) = \int_{t_0}^{t} S(t,r)B(r)B'(r)S'(t,r)dr$$

is a bounded nonnegative solution of (2.8). ∎

The equation (2.7) is reduced to

$$A'X + XA + C'C = 0, \tag{2.9}$$

if the system is time-invariant and its solution is called the observability gramian. The equation (2.8) is reduced to

$$AY + YA' + BB' = 0 \tag{2.10}$$

when the system is time-invariant and Y is called the controllability gramian.

Remark 2.1 Proposition 2.1 (b) is a special case of Proposition 2.5 (a) since (I, A) is detectable.

2.1.2 Performance Measures of Stable Systems

Consider the system $\mathbf{G}$:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w, \\ z &= C(t)x \end{aligned} \tag{2.11}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $z \in \mathbf{R}^{p_1}$, A, B, C are bounded piecewise continuous matrices of appropriate dimensions and A is exponentially stable. First we assume that the system is time-invariant and recall the following definitions.

Definition 2.7 *The H_2-norm of the system* $\mathbf{G}$, *denoted by* $\| G \|_2$ *is*

$$\begin{aligned} \| G \|_2 &= \left(\sum_{i=1}^{m_1} \int_0^\infty \mid Ce^{At}Be_i \mid^2 \, dt \right)^{\frac{1}{2}} \\ &= \left(tr. \int_0^\infty B'e^{A't}C'Ce^{At}B \, dt \right)^{\frac{1}{2}} \end{aligned}$$

where (e_i) are unit vectors in $\mathbf{R}^{m_1}$.

$\| G \|_2$ can be regarded as the total energy of impulse responses. Let $G(s)$ be the transfer function of the system so that $G(s) = C(sI - A)^{-1}B$. Then via Fourier transform we have

$$\| G \|_2 = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} tr.G^*(j\omega)G(j\omega)\, d\omega \right]^{\frac{1}{2}} \tag{2.12}$$

where $G^*(\cdot)$ is the Hermitian transpose of $G(\cdot)$. We also have the following.

Lemma 2.2

$$\| G \|_2^2 = tr.B'XB = tr.CYC'$$

where X , Y are observability- and controllability gramians respectively of the system given by (2.9) and (2.10).

Definition 2.8 *The H_∞-norm of the system* **G**, *denoted by $\| G \|_\infty$ is given by*

$$\| G \|_\infty = \sup_{0 \neq w \in L^2} \frac{\| z \|_2}{\| w \|_2}.$$

$\| G \|_\infty$ is the supremum of the ratio of the energies of the output and input. As is known

$$\| G \|_\infty = \sup_\omega \sigma[G(j\omega)] \tag{2.13}$$

where $\sigma(M)$ is the maximum singular value of the matrix M. The H_2- and H_∞-norms of transfer functions $G(s)$ are denoted by (2.12) and (2.13).

The following result is known as the Bounded Real Lemma.

Lemma 2.3 *The following statements are equivalent.*
(a) $\| G \|_\infty < \gamma$.
(b) There exists a nonnegative solution X to

$$A'X + XA + C'C + \frac{1}{\gamma^2} XBB'X = 0$$

such that $A + \frac{1}{\gamma^2}BB'X$ is exponentially stable.
(c) There exists a nonnegative solution Y to

$$AY + YA' + BB' + \frac{1}{\gamma^2} YC'CY = 0$$

such that $A + \frac{1}{\gamma^2}YC'C$ is exponentially stable.

Now we generalize Definitions 2.7 and 2.8 to time-varying systems.

Definition 2.9 *The H_2-norm of the system* **G** *on* $[t_0, \infty)$ *is defined by*

$$\begin{aligned}\| G \|_{2,t_0} &= \left[\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{m_1} \int_s^\infty \mid C(t)S(t,s)B(s)e_i \mid^2 dtds\right]^{\frac{1}{2}} \\ &= \left[\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.B'(s)\right. \\ &\qquad \left. \times \int_s^\infty S'(t,s)C'(t)C(t)S(t,s)dtB(s)ds\right]^{\frac{1}{2}}.\end{aligned}$$

For θ-periodic systems

$$\| G \|_{2,\theta} = \left[\frac{1}{\theta} \int_{t_0}^{t_0+\theta} tr.B'(s) \int_s^\infty S'(t,s)C'(t)C(t)S(t,s)dtB(s)\ ds\right]^{\frac{1}{2}}.$$

Note that two norms are equal for periodic systems.

Remark 2.2 Note that

$$\| G \|_{2,t_0}^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.B'(s) \int_s^{t_0+T} S'(t,s)C'(t)C(t)S(t,s)dtB(s)ds.$$

and

$$\begin{aligned}\| G \|_{2,t_0}^2 &= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr. \int_s^{t_0+T} C(t)S(t,s)B(s)B'(s)S'(t,s)C'(t)dtds \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.C(t) \int_{t_0}^{t} S(t,s)B(s)B'(s)S'(t,s)dsC'(t)dt\end{aligned}$$

where we have used the property of the trace and Fubini's theorem. From the last equality $\| G \|_{2,t_0}$ can be also interpreted as the H_2-norm of the backward system $\mathbf{G}^*$

$$\begin{aligned} -\dot{\tilde{x}} &= A'(t)\tilde{x} + C'(t)\tilde{w}, \\ \tilde{z}(t) &= B'(t)\tilde{x}. \end{aligned} \tag{2.14}$$

Let $\tilde{z}(t;s,i)$ be the impulse response of (2.14) with $u(t) = \delta(t-s)e_i$ where (e_i) are unit vectors in $\mathbf{R}^{p_1}$. Then

$$\tilde{z}(t;s,i) = \begin{cases} B'(t)S'(s,t)C'(s)e_i, & t \le s, \\ 0, & t > s. \end{cases}$$

Definition 2.10 *The H_2-norm of the backward system* $\mathbf{G}^*$ *is defined by*

$$\| G^* \|_{2,t_0}^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{i=1}^{m_1} \int_{t_0}^\infty \mid \tilde{z}(t;s,i) \mid^2 dtds.$$

Then clearly

$$\| G^* \|_{2,t_0}^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{s} \sum_{i=1}^{p_1} | B'(t)S'(s,t)C'(s)e_i |^2 \, dsdt$$

and $\| G \|_{2,t_0} = \| G^* \|_{2,t_0}$.

Lemma 2.4

$$\begin{aligned} \| G \|_{2,t_0}^2 &= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.B'(s)X(s)B(s) \; ds \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.C(s)Y(s)C'(s) \; ds \end{aligned}$$

where X and Y are the observability and controllability gramians of the system $\mathbf{G}$ *given by (2.7) and (2.8) with $Y(t_0) = 0$, respectively. Moreover, for θ-periodic systems X is θ-periodic and*

$$\begin{aligned} \| G \|_{2,\theta}^2 &= \frac{1}{\theta} \int_{t_0}^{t_0+\theta} tr.B'(s)X(s)B(s) \; ds \\ &= \frac{1}{\theta} \int_{t_0}^{t_0+\theta} tr.C(s)Y_\theta(s)C'(s) \; ds \end{aligned}$$

where Y_θ is the θ-periodic solution of (2.8).

Definition 2.11 *The H_∞-norm of the system* $\mathbf{G}$ *is that of the map $w \to z$: $L^2(t_0, \infty; \mathbf{R}^{m_1}) \to L^2(t_0, \infty; \mathbf{R}^{p_1})$.*

To generalize the bounded real lemma we need to consider a quadratic optimization problem. But we first introduce the standard quadratic control problems.

2.1.3 Quadratic Control

Consider

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$ and A, B are bounded piecewise continuous matrices of compatible dimension. For this system we introduce the functional

$$J_T(u; t_0, x_0) = \int_{t_0}^{T} [| C(t)x(t) |^2 + | u(t) |^2] dt + | Fx(T) |^2$$

which is minimized where $F \in \mathbf{R}^{q\times n}$ and $C \in \mathbf{R}^{p_2\times n}$ is bounded piecewise continuous.

We need the following Riccati equation

$$\begin{aligned} -\dot{X} &= A'(t)X + XA(t) + C'(t)C(t) - XB(t)B'(t)X, && (2.15) \\ X(T) &= F'F. && (2.16) \end{aligned}$$

Theorem 2.1 *There exists a unique nonnegative solution $X = X_T(t)$ to the Riccati eqaution (2.15) and (2.16). Moreover, the state feedback law*

$$\bar{u}(\cdot) = -B'(\cdot)X(\cdot)x(\cdot)$$

is optimal and

$$J_T(\bar{u}; t_0, x_0) = x_0' X(t_0) x_0.$$

We omit the proof of this theorem. Instead we shall give a proof for a similar problem (2.41). See Lemma 2.8.

Now consider the infinite horizon problem

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \ x(s) = x_0, \quad s \geq t_0, \\ J(u; s, x_0) &= \int_s^\infty [\| C(t)x(t) |^2 + | u(t) |^2] dt \end{aligned}$$

where $u \in L^2(s, \infty; \mathbf{R}^{m_2})$ is admissible if its response $x \in L^2(s, \infty; \mathbf{R}^n)$ and $\lim_{t\to\infty} x(t) = 0$.

RC: We assume that for each (s, x_0) there exists a control $u(\cdot; x_0)$ such that $J(u(\cdot, x_0); s, x_0) \leq c(x_0)$ for some constant c independent of s.

If (A, B) is stabilizable, then **RC** holds.

Lemma 2.5 *Assume* **RC** *holds. Then there exists a bounded nonnegative solution to the Riccati equation (2.15).*

Proof. By Theorem 2.1 there exists a nonnegative solution to (2.15) on $[t_0, T]$ with $X(T) = 0$. Then for any $s \geq t_0$ $X_T(s) \leq X_{\bar{T}}(s)$ if $s \leq T \leq \bar{T}$. In fact let

$$\bar{u}_T = -B' X_T x$$

then

$$\begin{aligned} x_0' X_T(s) x_0 &= J_T(\bar{u}_T; s, x_0) \\ &\leq J_T(\tilde{u}_{\bar{T}}; s, x_0) \\ &\leq J_{\bar{T}}(\bar{u}_{\bar{T}}; s, x_0) = x_0' X_{\bar{T}}(s) x_0 \end{aligned}$$

where we set $F = 0$ in J_T and $\tilde{u}_{\bar{T}}$ in J_T is the restriction of the feedback law $\bar{u}_{\bar{T}}(\cdot)$ to $[s, T]$. We note that

$$\begin{aligned} x_0' X_T(s) x_0 &= J_T(\bar{u}_T; s, x_0) \\ &\leq J_T(u(\cdot; x_0); s, x_0) \\ &\leq J(u(\cdot; x_0); s, x_0) < \infty. \end{aligned}$$

Hence $x_0'X_T(s)x_0$ is monotone increasing and uniformly bounded in s and T. Since x_0 is arbitrary, there exists a nonnegative bounded matrix X such that

$$X_T(s) \to X(s) \text{ for any } s.$$

Then X satisfies the Riccati equation (2.15). ∎

Lemma 2.6 *Suppose (C, A) is detectable. Then $A - BB'X$ is exponentially stable.*

Proof. The Riccati equation (2.15) can be written as

$$-\dot{X} = (A - BB'X)'X + X(A - BB'X) + [C' \quad XB] \begin{bmatrix} C \\ B'X \end{bmatrix}.$$

Hence, if x is the solution of the state feedback system

$$\dot{x} = (A - BB'X)(t)x, \; x(s) = x_0$$

then

$$\begin{bmatrix} C \\ B'X \end{bmatrix} x \in L^2(s, \infty; \mathbf{R}^{p_2+m_2})$$

with

$$\| \begin{bmatrix} C \\ B'X \end{bmatrix} x \|_2 \leq c \mid x_0 \mid \text{ for some } c > 0.$$

Since (C, A) is detectable, it is easy to see that $(\begin{bmatrix} C \\ B'X \end{bmatrix}, A - BB'X)$ is also detectable. Hence by Proposition 2.5, $A - BB'X$ is exponentially stable. ∎

We say that X is a stabilizing solution of the Riccati equation (2.15) if $A - BB'X$ is exponentially stable.

Theorem 2.2 *Suppose (C, A) is detectable and* **RC** *holds. Then there exists a nonnegative stabilizing solution of the Riccati equation (2.15). Moreover the feedback law*

$$\bar{u}(\cdot) = -B'(\cdot)X(\cdot)x(\cdot)$$

is optimal and

$$J(\bar{u}; s, x_0) = x_0'X(s)x_0. \tag{2.17}$$

If A, B and C are θ-periodic, then X is also θ-periodic.

Proof. The first part follows from Lemmas 2.5 and 2.6. Differentiating $x'Xx$ we obtain

$$x'(T)X(T)x(T) + J_T(u; s, x_0) = x_0'X(s)x_0 + \int_s^T \mid u + B'Xx \mid^2 dt$$

where u is an admissible control and x is its response. Since $x'(T)X(T)x(T) \to 0$ as $T \to \infty$, we obtain

$$J(u; s, x_0) = x_0' X(s) x_0 + \int_s^\infty | u + B'Xx |^2 \, dt.$$

Hence the optimality of $\bar{u}$ and (2.17) follow immediately.

By Lemma 2.5 the bounded stabilizing solution X of (2.15) is constructed as $\lim_{n\to\infty} X_T(t)$ where $X_T(t)$ is the solution of (2.15) with $X_T(T) = 0$. If A, B, C are θ-periodic, $X_T(t+\theta) = X_{T-\theta}(t)$. Hence

$$X(t+\theta) = \lim_{n\to\infty} X_T(t+\theta) = \lim_{n\to\infty} X_{T-\theta}(t) = X(t). \quad \blacksquare$$

Corollary 2.2 *(A, B) is stabilizable if and only if there exists a control $u(\cdot; s, x_0)$ for each s and x_0 such that*

$$\| x \|_2^2 + \| u \|_2^2 \le c(x_0)$$

for some constant $c(x_0)$.

Proof. We only need to show sufficiency. Consider the regulator problem with $C = I$. By Theorem 2.2 $A - BB'X$ is exponentially stable where X is the bounded nonnegative solution of the Riccati equation (2.15) with $C = I$. $\blacksquare$

Example 2.3 Consider the periodic system with period 3:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1 + 0.3\cos\frac{2\pi}{3}t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{2.18}$$

which is called the Mathieu's equation if $u(t) = 0$ [26]. This system is unstable, but by the feedback $u = fx$ with

$$f(t) = [\, -0.3\cos\tfrac{2\pi}{3}t - 0.5\cos t \quad -1 - \cos t \,]$$

it is stabilized (see Example 2.1). For

$$c = [\, 1 \quad 0 \,],$$

the system (2.18) is detectable and there exists a 3-periodic nonnegative stabilizing solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ of the Riccati equation (2.15) (Figure 2.4). The optimal response of this system with initial condition $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is given in Figure 2.5.

Consider the backward system

$$-\dot{\xi} = A'(t)\xi + C'(t)v, \ \xi(T) = \xi_1.$$

As in Theorem 2.1 we consider

$$\dot{Y} = A(t)Y + YA'(t) + B(t)B'(t) - YC'(t)C(t)Y, \tag{2.19}$$

$$Y(t_0) = HH'. \tag{2.20}$$

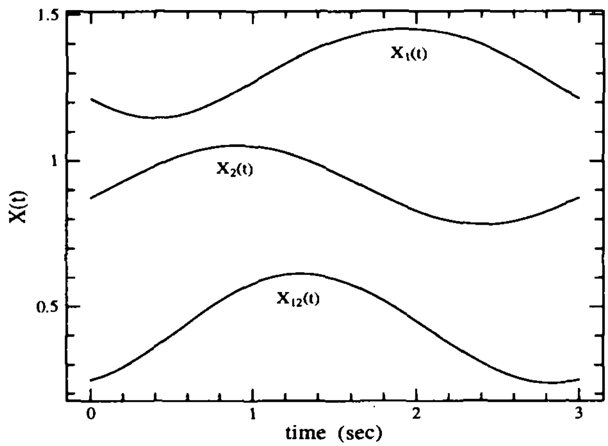

Figure 2.4: The periodic solution $X(t)$ of the Riccati equation

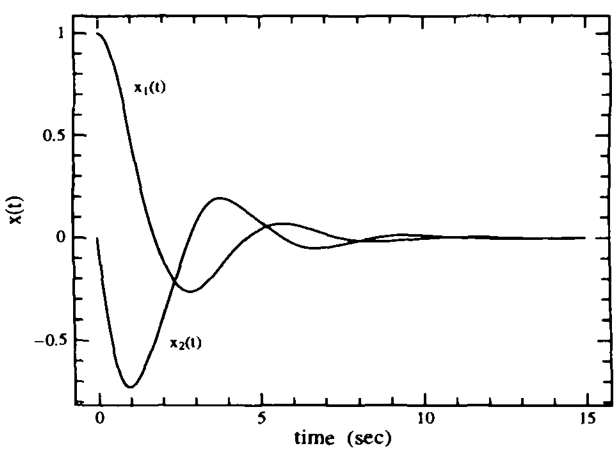

Figure 2.5: Simulation result

Theorem 2.3 *(a) There exists a nonnegative solution of the Riccati equation (2.19) and (2.20) on any* $[t_0, T]$.
(b) Let $H = 0$ *and suppose there exists a control* $v(\cdot; T, \xi_1)$ *such that*

$$\| B'\xi \|^2_{L^2(t_0,T;\mathbf{R}^{m_2})} + \| v \|^2_{L^2(t_0,T;\mathbf{R}^{p_2})} \leq c(\xi_1)$$

for some constant $c(\xi_1)$. *Then the solution of the Riccati equation (2.19) with* $Y(t_0) = 0$ *is bounded. If, further,* (A, B) *is stabilizable, then* $A - YC'C$ *is exponentially stable.*
(c) (C, A) *is detectable if and only if there exists a control* $v(\cdot; T, \xi_1)$ *such that*

$$\| \xi \|^2_{L^2(t_0,T;\mathbf{R}^n)} + \| v \|^2_{L^2(t_0,T;\mathbf{R}^{p_2})} \leq c(\xi_1)$$

for some constant $c(\xi_1)$.

Corollary 2.3 *Let A, B and C be θ-periodic. Let Y be a bounded nonnegative solution of (2.19) with* $Y(t_0) = 0$ *such that* $A - YC'C$ *is exponentially stable. Then* $\lim_{n\to\infty} Y(t + n\theta)$ *exists (denoted by* Y_θ*) and* Y_θ *is a θ-periodic nonnegative solution of (2.19) such that* $A - Y_\theta C'C$ *is exponentially stable.*

Proof. It is enough to show that $Y(t + n\theta)$ is monotone increasing in n. Let $Y(t; Y(t_0))$ be the solution of (2.19) with initial condition $Y(t_0) \geq 0$. Then $Y(t) = Y(t; 0)$. Since A, B and C are θ-periodic, we have $Y(t) = Y(t - n\theta; Y(n\theta))$ for $n\theta \leq t < (n + 1)\theta$. Hence

$$Y(t + 2\theta) = Y(t + \theta; \bar{Y}(\theta)) \geq Y(t + \theta; 0) = Y(t + \theta).$$

Similarly, we have

$$Y(t + (n + 1)\theta) \geq Y(t + n\theta)$$

and $Y(t + n\theta)$ is monotone increasing in n. Since Y is bounded, there exists a limit $Y_\theta(t)$ of $Y(t + n\theta)$ as $n \to \infty$. Note that

$$Y_\theta(t) = \lim_{n\to\infty} Y(t + n\theta) = \lim_{n\to\infty} Y(t + \theta + (n - 1)\theta) = Y_\theta(t + \theta).$$

Hence $Y_\theta(t)$ is θ-periodic. Let s, $t \in [0, \theta)$. Integrating (2.19) from $n\theta + s$ to $n\theta + t$ and passing to the limit $n \to \infty$ and then differentiating again, we can easily show that $Y_\theta(t)$ satisfies (2.19).

Next we shall show that $A - Y_\theta C'C$ is exponentially stable. Let $t_0 < T < \infty$ be arbitrary but fixed. Let x_θ be solution of

$$\dot{x} = (A - Y_\theta C'C)(t)x, \ x(t_0) = x_0. \tag{2.21}$$

Consider

$$\dot{x} = (A - YC'C)(t)x, \ x(t_0) = x_0.$$

and denote by $x_n(t)$ the solution at $t + n\theta$. Then

$$\begin{aligned} \dot{x}_n(t) &= \dot{x}(t + n\theta) \\ &= [A(t + n\theta) - Y(t + n\theta)C'(t + n\theta)C(t + n\theta)]x(t + n\theta) \\ &= [A(t) - Y(t + n\theta)C'(t)C(t)]x_n(t) \end{aligned}$$

and we have

$$\lim_{n\to\infty} x_n(t) = x_\theta(t),\ t \in [t_0, T].$$

Since $A - YC'C$ is stabilizing

$$\int_{t_0}^{T} | x_n(t) |^2 \, dt \le c \mid x_0 \mid^2 \text{ for any } n$$

where $c > 0$ is a constant independent of T. Hence by Fatou's lemma we obtain

$$\int_{t_0}^{T} | x_\theta(t) |^2 \, dt = \liminf_{n\to\infty} \int_{t_0}^{T} | x_n(t) |^2 \, dt \le c \mid x_0 \mid^2 .$$

Since T is arbitrary, the system (2.21) is exponentially stable. Suppose that $Y(t; H'H)$ is a bounded nonnegative stabilizing solution of (2.19) such that $A - YC'C$ is exponentially stable. Then by Theorem 2.4 below, $\lim_{n\to\infty} Y(t + n\theta; H'H) = Y_\theta(t)$. ∎

We generalize the notion of the stabilizing solution of the Riccati equations, which will be useful in later sections. Consider the Riccati equations on $[t_0, \infty)$

$$-\dot{X} = A'(t)X + XA(t) + P(t) + XR(t)X, \tag{2.22}$$
$$\dot{Y} = A(t)Y + YA'(t) + Q(t) + YS(t)Y \tag{2.23}$$

where P, Q, R and S are bounded piecewise continuous symmetric matrices.

Definition 2.12 *Let X (Y) be the solution of (2.22) (respectively, (2.23)).*
(a) X (Y) is bounded if $\mid X(t) \mid \le cI$ ($\mid Y(t) \mid \le cI$) for some $c \ge 0$.
(b) A bounded symmetric solution X of (2.22) is called stabilizing if $A + RX$ is exponentially stable.
(c) A bounded symmetric solution Y of (2.23) is called stabilizing if $A + YS$ is exponentially stable.

These definitions are consistent with those of Theorems 2.2 and 2.3.

Theorem 2.4 *(a) A bounded stabilizing solution (2.22), if one exists, is unique.*
(b) Let Y and $\bar{Y}$ be two stabilizing solutions of (2.23). Then

$$Y(t) - \bar{Y}(t) \to 0 \text{ as } t \to \infty.$$

Proof. (a) Let X and $\bar{X}$ be two stabilizing solutions of (2.22). Then

$$-\frac{d}{dt}(X - \bar{X}) = (A + RX)'(t)(X - \bar{X}) + (X - \bar{X})(A + R\bar{X})(t).$$

Hence

$$(X - \bar{X})(t) = S'_X(T,t)[X(T) - \bar{X}(T)]S_{\bar{X}}(T,t)$$

where S_X is the state transition matrix of $A + RX$. Hence

$$| X(t) - \bar{X}(t) |= M_1 e^{-\alpha_1(T-t)} c M_2 e^{-\alpha_2(T-t)}$$

for some positive constant M_1, M_2, c, α_1 and α_2. Letting $T \to \infty$ we obtain $X(t) - \bar{X}(t) = 0$ for any $t_0 \leq t < \infty$.
(b) Since

$$\frac{d}{dt}(Y - \bar{Y}) = (A + YS)(t)(Y - \bar{Y}) + (Y - \bar{Y})(A + \bar{Y}S)'(t),$$

we have

$$Y(t) - \bar{Y}(t) = S_Y(t, t_0)[Y(t_0) - \bar{Y}(t_0)]S'_{\bar{Y}}(t, t_0)$$

where S_Y and $S_{\bar{Y}}$ are the state transition matrices of $A - YS$ and $A - \bar{Y}S$, respectively. Hence $Y(t) - \bar{Y}(t) \to 0$ as $t \to \infty$. ∎

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \quad x(t_0) = x_0, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + D_{21}(t)w \end{aligned}$$

and the controller $u = Ky$ of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \quad \hat{x}(t_0) = 0, \\ u &= \hat{C}(t)\hat{x} + \hat{D}(t)y. \end{aligned} \tag{2.24}$$

Then the closed-loop system **G** and $u = Ky$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}(t)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 + B_2\hat{D}D_{21} \\ \hat{B}D_{21} \end{bmatrix}(t)w, \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t_0) &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ z &= [\,C_1 + D_{12}\hat{D}C_2 \quad D_{12}\hat{C}\,](t)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + (D_{12}\hat{D}D_{21})(t)w. \end{aligned} \tag{2.25}$$

Definition 2.13 *Consider the system* **G** *on* $[t_0, \infty)$. *A controller* $u = Ky$ *of the form (2.24) is said to be IO-stabilizing if the closed-loop system (2.25) is IO-stable. If, further, the closed-loop system is exponentially stable (or*

$$\begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}$$

is exponentially stable) then the controller is said to be (internally) stabilizing.

Proposition 2.6 *Consider the system* **G** *and the controller* $u = Ky$ *of the form (2.24). If the controller* $u = Ky$ *is internally stabilizing, then* (A, B_2, C_2) *and* $(\hat{A}, \hat{B}, \hat{C})$ *are stabilizable and detectable.*

Proof. Let $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t)$ be the solution of

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}(t)\begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(s) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \tag{2.26}$$

Then by assumption $x,\ \hat{x} \in L^2$. Rewriting (2.26) as

$$\begin{aligned} \dot{x} &= A(t)x + B_2(t)(\hat{D}C_2x + B_2\hat{C}\hat{x}),\ x(s) = x_0, \\ \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)(C_2x),\ \hat{x}(s) = 0 \end{aligned}$$

and applying Corollary 2.2, we conclude that (A, B_2) and $(\hat{A}, \hat{B})$ are stabilizable. The detectability of (C_2, A) and $(\hat{C}, \hat{A})$ also follow from the adjoint system of (2.26) and Theorem 2.3. ∎

2.1.4 Disturbance Attenuation Problems

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w, \\ z &= C(t)x, &\quad (2.27) \\ z_1 &= Fx(T) &\quad (2.28) \end{aligned}$$

with initial condition

$$x(t_0) = Hh \tag{2.29}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $z \in \mathbf{R}^{p_1}$, $z_1 \in \mathbf{R}^q$, $h \in \mathbf{R}^{n_1}$, $H \in \mathbf{R}^{n \times n_1}$, $F \in \mathbf{R}^{q \times n}$ and other matrices are bounded piecewise continuous of appropriate dimensions. For each input $(h, w) \in \mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1})$ we have the output $(z_1, z) \in \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{p_1})$. Thus we can define the input-output operator G_{Tt_0} of the system (2.27)-(2.29) by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G_{Tt_0}\begin{pmatrix} h \\ w \end{pmatrix} = \begin{pmatrix} G_{1Tt_0}\begin{pmatrix} h \\ w \end{pmatrix} \\ G_{2Tt_0}\begin{pmatrix} h \\ w \end{pmatrix} \end{pmatrix} \tag{2.30}$$

where

$$\begin{aligned} G_{1Tt_0}\begin{pmatrix} h \\ w \end{pmatrix} &= FS(T, t_0)Hh + F\int_{t_0}^{T} S(T, r)B(r)w(r)dr, \\ G_{2Tt_0}\begin{pmatrix} h \\ w \end{pmatrix} &= C(t)S(t, t_0)Hh + C(t)\int_{t_0}^{t} S(t, r)B(r)w(r)dr. \end{aligned}$$

Let $\mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0,T;\mathbf{R}^{m_1});\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}))$ be the space of bounded linear operators mapping $\mathbf{R}^{n_1} \times L^2(t_0,T;\mathbf{R}^{m_1})$ into $\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1})$. Then $G_{Tt_0} \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0,T;\mathbf{R}^{m_1});\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}))$. We regard (h,w) as the disturbance and for a given $\gamma > 0$ we wish to find necessary and sufficient conditions for $\| G_{Tt_0} \| < \gamma$, i.e.,

$$| z_1 |^2 + \| z \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2) \text{ for some } 0 < d < \gamma. \tag{2.31}$$

In this case the system $\mathbf{G}$ is said to fulfil the γ-disturbance attenuation.

The adjoint $G^*_{Tt_0}$ of G_{Tt_0} is given by

$$G^*_{Tt_0}\begin{pmatrix} f \\ v \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \zeta \end{pmatrix}, \quad (f,v) \in \mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}) \tag{2.32}$$

where

$$\begin{aligned} -\dot{\xi} &= A'(t)\xi + C'(t)v, \\ \zeta &= B'(t)\xi. \\ \xi(T) &= F'f, \\ \zeta_0 &= H'\xi(t_0), \end{aligned} \tag{2.33}$$

Since $\| G^*_{Tt_0} \| = \| G_{Tt_0} \|$ by Theorem A.2, (2.31) is equivalent to

$$| \zeta_0 |^2 + \| \zeta \|_2^2 \leq d^2(| f |^2 + \| v \|_2^2). \tag{2.34}$$

To give necessary and sufficient conditions for $\| G_{Tt_0} \| < \gamma$, we need the Riccati equations

$$-\dot{X} = A'(t)X + XA(t) + C'(t)C(t) + \frac{1}{\gamma^2}XB(t)B'(t)X, \tag{2.35}$$

$$X(T) = F'F, \tag{2.36}$$

$$H'X(t_0)H \leq d^2 I \text{ for some } 0 < d < \gamma \tag{2.37}$$

and

$$\dot{Y} = A(t)Y + YA'(t) + B(t)B'(t) + \frac{1}{\gamma^2}YC'(t)C(t)Y, \tag{2.38}$$

$$Y(t_0) = HH', \tag{2.39}$$

$$FY(T)F' \leq d^2 I \text{ for some } 0 < d < \gamma. \tag{2.40}$$

To give the solution of this problem, we introduce the following functional

$$J(w;t_0,x_0) = \int_{t_0}^{T} [| z(t) |^2 - \gamma^2 | w(t) |^2]dt + | Fx(T) |^2 \tag{2.41}$$

subject to

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w, \quad x(t_0) = x_0, \\ z &= C(t)x \end{aligned}$$

and consider the maximization of $J(w; t_0, x_0)$ over all $w \in L^2(t_0, T; \mathbf{R}^{m_1})$. Let

$$\begin{aligned} \bar{G}_{Tt_0} w &= G_{Tt_0} \begin{pmatrix} 0 \\ w \end{pmatrix}, \\ \bar{G}_{iTt_0} w &= G_{iTt_0} \begin{pmatrix} 0 \\ w \end{pmatrix}, \quad i = 1, 2. \end{aligned}$$

Lemma 2.7 $\| \bar{G}_{2Ls} \| \leq \| \bar{G}_{2Tt_0} \|$, $\| \bar{G}_{Ts} \| \leq \| \bar{G}_{Tt_0} \|$ *for any* $t_0 \leq s \leq L \leq T$.

Proof. We shall show only the first inequality. Let $\tilde{w}$ be the extension of $w \in L^2(s, L; \mathbf{R}^{m_1})$ to $[t_0, T]$ by zero, i.e,

$$\tilde{w}(t) = \begin{cases} 0, & t_0 \leq t < s, \\ w(t), & s \leq t \leq L, \\ 0, & L < t \leq T. \end{cases}$$

Then we have

$$\begin{aligned} \| \bar{G}_{2Ls} w \|_2^2 &= \int_s^L | C(t) \int_s^t S(t,r)B(r)w(r)dr |^2 \, dt \\ &= \int_{t_0}^L | C(t) \int_{t_0}^t S(t,r)B(r)\tilde{w}(r)dr |^2 \, dt \\ &\leq \int_{t_0}^T | C(t) \int_{t_0}^t S(t,r)B(r)\tilde{w}(r)dr |^2 \, dt \\ &= \| \bar{G}_{2Tt_0} \tilde{w} \|_2^2 \\ &\leq \| \bar{G}_{2Tt_0} \|^2 \| \tilde{w} \|_2^2 = \| \bar{G}_{2Tt_0} \|^2 \| w \|_2^2 . \end{aligned}$$

∎

Consider the optimal control problem (2.27)-(2.29) and (2.41) with t_0, T replaced by arbitrary s, L, $t_0 \leq s \leq L \leq T$.

Lemma 2.8 *Assume* $\| \bar{G}_{Tt_0} \| < \gamma$. *Then the following statements are true.*
(a) There exists a unique optimal maximizing element $w_{Ts} \in L^2(s, T; \mathbf{R}^{m_1})$ *of* $J(w; s, x_0)$. *Moreover*

$$\| w_{Ts} \|_2 \leq \delta \mid x_0 \mid, \quad J(w_{Ts}; s, x_0) \leq \delta \mid x_0 \mid^2 \tag{2.42}$$

for some $\delta = \delta(\gamma) > 0$ *independent of* s *and* x_0.
(b) There exists a unique nonnegative solution to (2.35) and (2.36). The optimal control for (2.41) is given by the feedback law

$$w_{Tt_0}(\cdot) = \frac{1}{\gamma^2} B'(\cdot) X(\cdot) x(\cdot)$$

and

$$J(w_{Tt_0}, t_0, x_0) = x_0' X(t_0) x_0. \tag{2.43}$$

Proof. (a) By Lemma 2.7, $\| \bar{G}_{Ts} \| < \gamma$ for any $t_0 \leq s \leq T$. Hence $\gamma^2 I - \bar{G}_{Ts}^* \bar{G}_{Ts} > aI$ for some $a > 0$ and the quadratic functional $J(w; s, x_0)$ is strictly concave and $J(w; s, x_0) \to -\infty$ as $\| w \|_2 \to \infty$. Then there exists a unique optimal w_{Ts} for $J_T(w; s, x_0)$ which is given by

$$(\gamma^2 I - \bar{G}_{Ts}^* \bar{G}_{Ts}) w = \bar{G}_{Ts}^* z_0, \quad z_0(t) = \begin{pmatrix} C(t)S(t,s)x_0 \\ FS(T,s)x_0 \end{pmatrix}.$$

Hence

$$w_{Ts} = (\gamma^2 I - \bar{G}_{Ts}^* \bar{G}_{Ts})^{-1} \bar{G}_{Ts}^* z_0.$$

Since $\| \bar{G}_{Ts} \| \leq d < \gamma$, we have

$$\| (\gamma^2 I - \bar{G}_{Ts}^* \bar{G}_{Ts})^{-1} \| \leq \frac{1}{\gamma^2 - d^2}.$$

Hence

$$\| w_{Ts} \|_2 \leq \frac{d}{\gamma^2 - d^2} \mid x_0 \mid$$

from which the assertion follows.
(b) Suppose that there exists a symmetric solution to (2.35) and (2.36) on some interval $[t_1, T]$. Then for any s, $t_1 \leq s \leq T$

$$J(w_{Ts}; s, x_0) = x_0' X(s) x_0 - \gamma^2 \int_s^T \mid v(t) \mid^2 dt$$

where

$$v(t) = w(t) - \frac{1}{\gamma^2} B' X(t) x(t)$$

and x is the response of (2.27) to $w \in L^2(s, T; \mathbf{R}^{m_1})$ with $x(s) = x_0$. This follows as in the standard quadratic problem. Hence

$$w_{Ts}(t) = \frac{1}{\gamma^2} B' X(t) x(t)$$

and

$$J(w_{Ts}; s, x_0) = x_0' X(s) x_0.$$

In view of (2.42) the norm of X is bounded, i.e., $\mid X(s) \mid \leq \delta$ for some $\delta > 0$. Since the Riccati equation (2.35) is locally Lipschitz and its solution is a priori bounded, there exists a global solution on $[t_0, T]$. The uniqueness and nonnegativity of X follows from (2.43). ∎

We are now ready to give the solution to our original problem.

Theorem 2.5 *The following statements are equivalent.*
(a) $\| G_{Tt_0} \| < \gamma$.
(b) There exists a nonnegative solution to (2.35)-(2.37).
(c) There exists a nonnegative solution to (2.38)-(2.40).

Proof. Suppose (a) holds. Then $\| \bar{G}_{Tt_0} \| < \gamma$ and (b) except (2.37) follows from Lemma 2.8. Moreover for (2.27) and (2.35) the following equality holds:

$$| z_1 |^2 + \| z \|_2^2 = \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh - \gamma^2 \| w - \frac{1}{\gamma^2}B'Xx \|_2^2 . \quad (2.44)$$

Setting $w = \frac{1}{\gamma^2}B'Xx$ and using (2.31) we obtain

$$d^2(| h |^2 + \| w \|_2^2) \geq \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh.$$

Hence $d^2 | h |^2 \geq h'H'X(t_0)Hh$ which implies (2.37).

Conversely suppose (b) holds. Then by (2.44)

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &\leq \gamma^2 \| w \|_2^2 + d^2 | h |^2 - \gamma^2 \| r \|_2^2 \\ &\leq \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2) \end{aligned}$$

where $r = w - \frac{1}{\gamma^2}B'Xx$. Since there exists $a > 0$ such that

$$| h |^2 + \| w \|_2^2 \leq a(| h |^2 + \| r \|_2^2),$$

we have

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &\leq \gamma^2(| h |^2 + \| w \|_2^2) - \frac{\gamma^2 - d^2}{a}(| h |^2 + \| w \|_2^2) \\ &= (\gamma^2 - \frac{\gamma^2 - d^2}{a})(| h |^2 + \| w \|_2^2). \end{aligned}$$

Hence $\| G_{Tt_0} \| < \gamma$. The equivalence of (a) and (c) also follows since (c) is the dual of (b) concerning the adjoint system (2.33) of G_{Tt_0}. ∎

If we assume that initial conditions are known, we can set $h = 0$.

Corollary 2.4 *The following statements are equivalent.*
(a) $\| \bar{G}_{Tt_0} \| < \gamma$.
(b) There exists a nonnegative solution to (2.35) and (2.36).
(c) There exists a nonnegative solution to (2.38) and (2.40) with $Y(t_0) = 0$.

Now we consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w, \\ z &= C(t)x, \\ x(t_0) &= Hh \end{aligned}$$

on $[t_0, \infty)$ and assume that this system is exponentially stable. Then we define the input-output operator $G \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1}); L^2(t_0, \infty; \mathbf{R}^{p_1}))$ by

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

In this case we wish to find the condition for $\| G \| < \gamma$. We replace (2.30) and (2.41) by

$$\begin{aligned} G\begin{pmatrix} h \\ w \end{pmatrix} &= C(t)S(t,t_0)Hh + C(t)\int_{t_0}^{t} S(t,r)B(r)w(r)dr, \\ J(w;t_0,x_0) &= \int_{t_0}^{\infty} [\mid z(t) \mid^2 - \gamma^2 \mid w(t) \mid^2]dt. \end{aligned}$$

We also need the functional (2.41) with $F = 0$, i.e.,

$$J_T(w;t_0,x_0) = \int_{t_0}^{T} [\mid z(t) \mid^2 - \gamma^2 \mid w(t) \mid^2]dt.$$

Let $\bar{G}w = G\begin{pmatrix} 0 \\ w \end{pmatrix}$. Proceeding as in the finite horizon case we have the following.

Lemma 2.9 *$\| \bar{G}_{2Tt_0} \| \leq \| \bar{G} \|$ for any $t_0 \leq T < \infty$.*

Lemma 2.10 *Assume $\| G \| < \gamma$. Then the following statements are true.*
(a) There exists a unique control w_{Tt_0} (w_{t_0}) maximizing $J_T(w;t_0,x_0)$ ($J(w;t_0,x_0)$, respectively). Moreover

$$\begin{array}{ll} \| w_{Tt_0} \|_2 \leq \delta \mid x_0 \mid, & \| w_{t_0} \|_2 \leq \delta \mid x_0 \mid, \\ J_T(w_{Tt_0};t_0,x_0) \leq \delta \mid x_0 \mid^2, & J(w_{t_0};t_0,x_0) \leq \delta \mid x_0 \mid^2 \end{array} \tag{2.45}$$

for some $\delta = \delta(\gamma) > 0$ independent of T and x_0.
(b) There exists a unique bounded nonnegative stabilizing solution to (2.35). Moreover if the conditions above are satisfied, the optimal control w_{t_0} of $J(w;t_0,x_0)$ exists and is given by the feedback law

$$w_{t_0}(\cdot) = \frac{1}{\gamma^2}B'(\cdot)X(\cdot)x(\cdot)$$

and $J(w_{t_0};t_0,x_0) = x_0'X(t_0)x_0$.

Proof. (a) Since $\| \bar{G} \| \leq \| G \| < \gamma$, by Lemma 2.9 we have $\| \bar{G}_{2Tt_0} \| < \gamma$. Hence from Lemma 2.8, we have

$$w_{Tt_0} = (\gamma^2 I - \bar{G}^*_{2Tt_0}\bar{G}_{2Tt_0})^{-1}\bar{G}^*_{2Tt_0}z_0$$

and

$$w_{t_0} = (\gamma^2 I - \bar{G}^* \bar{G})^{-1} \bar{G}^* z_0.$$

Since $\bar{G}_{2Tt_0}$, $(\gamma^2 I - \bar{G}^*_{2Tt_0} \bar{G}_{2Tt_0})^{-1}$ are uniformly bounded in T, we have the assertion.

(b) Since $\| \bar{G} \| < \gamma$ implies $\| \bar{G}_{2Tt_0} \| < \gamma$, we have a nonnegative solution $X_T(t)$ to (2.35) with $X_T(T) = 0$. Moreover for each t, $X_T(t)$ is monotone increasing in T. In fact let $L < T$ and define a control on $[t_0, T]$ by

$$\tilde{w}_{Tt_0}(t) = \begin{cases} \frac{1}{\gamma^2} B' X(t) x(t), & t \in [t_0, L], \\ 0, & t \in (L, T]. \end{cases}$$

Then

$$\begin{aligned} x_0' X_L(t_0) x_0 = J_L(w_{Lt_0}; t_0, x_0) &\leq J_T(\tilde{w}_{Tt_0}; t_0, x_0) \\ &\leq J_T(w_{Tt_0}; t_0, x_0) = x_0' X_T(t_0) x_0. \end{aligned}$$

The mononicity of $X_T(t)$ also follows from $J_T(w; t, x_0)$. Note that $X_T(t)$ is bounded uniformly in T. This follows from (2.45) and

$$J_T(w_{Tt_0}; t_0, x_0) = x_0' X_T(t_0) x_0.$$

Hence $X_T(t)$ converges to a limit $X(t)$ as $T \to \infty$ and it satisfies (2.35). Now it remains to show that $A + \frac{1}{\gamma^2} BB'X$ is exponentially stable. Let $\tilde{w}_{Tt_0} \in L^2(t_0, \infty; \mathbf{R}^{m_1})$ be given by

$$\tilde{w}_{Tt_0}(t) = \begin{cases} \frac{1}{\gamma^2} B' X_T(t) x_T(t), & t \in [t_0, T], \\ 0, & t \in (T, \infty). \end{cases}$$

Then by Lemma 2.10, $\{\tilde{w}_{Tt_0}\}$ is bounded in $L^2(t_0, \infty; \mathbf{R}^{m_1})$. Hence there exists a subsequence again denoted by $\{\tilde{w}_{Tt_0}\}$ which is weakly convergent to $\tilde{w} \in L^2(t_0, \infty; \mathbf{R}^{m_1})$ with $\| \tilde{w} \|_2 \leq c \mid x_0 \mid$, $c > 0$ (see Theorem A.5). Let $\tilde{x}$ be the response to $\tilde{w}$, i.e, the solution of

$$\dot{\tilde{x}} = A(t)\tilde{x} + B(t)\tilde{w}, \quad \tilde{x}(t_0) = x_0.$$

Then for each t, $x_T(t) \to \tilde{x}(t)$ as $T \to \infty$. On the other hand $x_T(t) \to \bar{x}(t)$ in any interval, where $\bar{x}$ is the solution of equation

$$\dot{\bar{x}} = (A + \frac{1}{\gamma^2} BB'X)(t)\bar{x}(t), \quad \bar{x}(t_0) = x_0.$$

Hence we can identify $\tilde{x} = \bar{x}$. Since A is exponentially stable and $\| \tilde{w} \|_2 \leq c \mid x_0 \mid^2$, we conclude $\| \tilde{x} \|_2 = \| \bar{x} \|_2 \leq c_1 \mid x_0 \mid$ for some $c_1 > 0$. The same conclusion holds when we replace t_0 by $s \geq t_0$. Hence by Proposition 2.2 $A + \frac{1}{\gamma^2} BB'X$ is exponentially stable. ∎

Theorem 2.6 *Assume that the system* $\mathbf{G}$ *is exponentially stable on* $[t_0, \infty)$. *Then the following statements are equivalent:*
(a) $\| G \| < \gamma$.
(b) There exists a bounded nonnegative stabilizing solution of (2.35) on $[t_0, \infty)$ *satisfying (2.37).*
(c) There exists a bounded nonnegative stabilizing solution of (2.38) and (2.39) on $[t_0, \infty)$.
The solutions in (b) and (c) are unique.

Assume that the initial condition is known so that $h = 0$.

Corollary 2.5 *The following statements are equivalent.*
(a) $\| \bar{G} \| < \gamma$.
(b) There exists a nonnegative solution to (2.35).
(c) There exists a nonnegative solution to (2.38) with $Y(t_0) = 0$.

Proof of Theorem 2.6. Suppose (a) holds. Then the existence of a stabilizing solution follows from Lemma 2.10. The condition (2.37) follows as in Theorem 2.5. Hence (a) implies (b). The converse is also similar to Theorem 2.5. We only need to show

$$| h |^2 + \| w \|_2^2 \leq a(| h |^2 + \| r \|_2^2) \text{ for some } a > 0.$$

But this follows from

$$\begin{aligned} \dot{x} &= (A + \frac{1}{\gamma^2} BB'X)(t)x + B(t)r, \\ w &= \frac{1}{\gamma^2} B'(t)X(t)x + r \end{aligned}$$

since $A + \frac{1}{\gamma^2} BB'X$ is exponentially stable.

(c) is the dual of (b) and (a) implies the existence of a bounded nonnegative solution of (2.38) with property (2.39). In fact we consider the adjoint system

$$\begin{aligned} -\dot{\xi} &= A'(t)\xi + C'(t)v, \quad \xi(T) = \xi_1, \\ \zeta &= B'(t)\xi \end{aligned}$$

and

$$J(v; T, \xi_1) = \int_{t_0}^{T} [| \zeta(t) |^2 - \gamma^2 | v(t) |^2] dt$$

and proceed as in Lemma 2.10. To show the exponential stability of $A + \frac{1}{\gamma^2} YC'C$, let $v_T(t) = \frac{1}{\gamma^2} CY(t)\xi(t)$ be the maximizing element of $J(v; T, \xi_1)$, then

$$\| v_T \|_{L^2(t_0,T;\mathbf{R}^{p_1})} \leq c_0 \, | \xi_1 | \quad \text{for some } c_0 > 0.$$

We extend v_T to $[t_0, \infty)$ by zero which we denote by $\tilde{v}_T \in L^2(t_0, \infty; \mathbf{R}^{p_1})$. Then there exists a subsequence again denoted by $\tilde{v}_T$ convergent weakly to

$$\tilde{v} \in L^2(t_0, \infty; \mathbf{R}^{p_1}) \text{ with } \| \tilde{v} \|_{L^2(t_0,\infty;\mathbf{R}^{p_1})} \leq c_0 \mid \xi_1 \mid.$$

Now let $t_0 < L < \infty$ be a fixed but arbitrary number and consider

$$\begin{aligned} -\dot{\xi}_T &= A'(t)\xi_T + C'(t)\tilde{v}_T, \quad \xi_T(L) = \xi_1, \\ -\dot{\tilde{\xi}} &= A'(t)\tilde{\xi} + C'(t)\tilde{v}, \quad \tilde{\xi}_T(L) = \xi_1 \end{aligned}$$

and

$$-\dot{\xi} = A'(t)\xi + \frac{1}{\gamma^2}C'(t)C(t)Y(t)\tilde{\xi}, \quad \xi(L) = \xi_1. \tag{2.46}$$

Then as in Lemma 2.10, we can show $\xi_T(t) \to \tilde{\xi}_T(t)$ as $T \to \infty$ for any $t \in [t_0, L]$ and $\tilde{\xi}(t) = \xi(t)$, $t \in [t_0, L]$. Since $\| \tilde{v} \|_{L^2(t_0,\infty;\mathbf{R}^{p_1})} \leq c_0 \mid \xi_1 \mid$,

$$\int_{t_0}^{L} \mid \tilde{\xi}(t) \mid^2 dt \leq c \mid \xi_1 \mid^2 \quad \text{for some } c > 0,$$

which implies

$$\int_{t_0}^{L} \mid \xi(t) \mid^2 dt \leq c \mid \xi_1 \mid^2 \quad \text{for any } t_0 \leq L < \infty.$$

Hence by Proposition 2.3, the system (2.46) is exponentially stable and so is $A + \frac{1}{\gamma^2}YC'C$.

The converse of (c) follows if we consider the adjoint of the system **G** and proceed as the converse of (b). ∎

Corollary 2.6 *Let the system* **G** *be θ-periodic, i.e.,* $A(t+\theta) = A(t)$, $B(t+\theta) = B(t)$ *and* $C(t+\theta) = C(t)$. *Then*
(a) The stabilizing solution of (b) in Theorem 2.6 is θ-periodic.
(b) There exists a θ-periodic nonnegative stabilizing solution $Y_\theta(t)$ *to (2.38) and* $Y(t) - Y_\theta(t) \to 0$ *as* $t \to \infty$ *where* Y *is a bounded nonnegative stabilizing solution of (2.38) and (2.39).*

Proof. Proofs of (a) and (b) are similar to those of Theorem 2.2 and Corollary 2.3, respectively. ∎

If the system **G** is time-invariant, then we need the algebraic Riccati equations:

$$A'X + XA + C'C + \frac{1}{\gamma^2}XBB'X = 0, \tag{2.47}$$

$$H'XH \leq d^2 I \text{ for some } 0 < d < \gamma, \tag{2.48}$$

$$AY + YA' + BB' + \frac{1}{\gamma^2}YC'CY = 0. \tag{2.49}$$

We define the stabilizing solutions of (2.47) and (2.49) as above. We can set $t_0 = 0$.

Corollary 2.7 *Let the system* **G** *be time-invariant. Suppose A is exponentially stable. Then the following statements are equivalent:*
(a) $\| G \| < \gamma$.
(b) There exists a nonnegative stabilizing solution of (2.47) satisfying (2.48).
(c) There exists a bounded nonnegative stabilizing solution of (2.38) with $Y(0) = H'H$. *Moreover, there exists a unique nonnegative stabilizing solution of (2.49) and* $Y(t) \to Y_\infty$ *as* $t \to \infty$ *where* Y_∞ *is the nonnegative stabilizing solution of (2.49).*

Proof. The last property follows from Theorem 2.4. ∎

Corollary 2.8 *Let the system* **G** *be time-invariant. Suppose A is exponentially stable. Then the following statements are equivalent:*
(a) $\| \bar{G} \| < \gamma$.
(b) There exists a nonnegative stabilizing solution of (2.47).
(c) There exists a nonnegative stabilizing solution to (2.49).

Example 2.4 Consider the periodic system with period 2π:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 - 0.5\cos t & -1 - \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \ x(0) = Hh, \\ z &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

which is exponentially stable (see Example 2.1). For this system we consider the following two cases

$$\text{(a) } H = 0, \ \text{(b)} H = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where 2 in the case (b) can be regarded as weight on the initial uncertainty. First we consider the case (a). For $\gamma \geq 2.3515$, there exists a 2π-periodic nonnegative stabilizing solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix} (t)$ of the Riccati equation (2.35) (Figure 2.6) and there exists a bounded nonnegative stabilizing solution $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix} (t)$ of the Riccati equation (2.38) with $Y(0) = 0$ which converges to a $2\pi-$periodic solution (Figure 2.7). Next we consider the case (b). Then for all $\gamma \geq 2.7751$, there exist a $2\pi-$periodic nonnegative stabilizing solution $X(t)$ of (2.35) and a bounded nonnegative stabilizing solution $Y(t)$ of (2.37) and (2.38) which converges to a 2π-periodic solution (Figure 2.8).

2.2 H_∞ Control and Differential Games

In this section we consider the differential games related to the H_∞ control problems. We consider finite and infinite horizon problems.

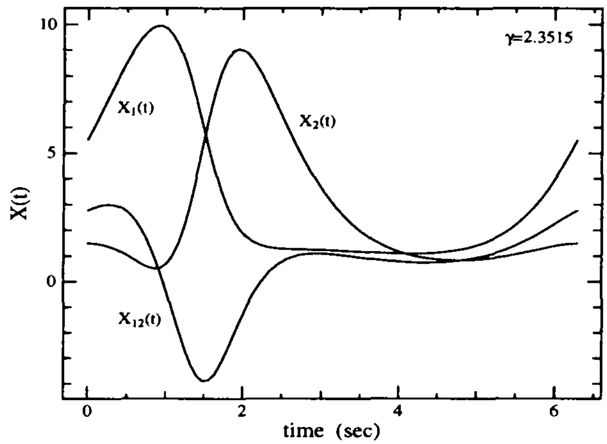

Figure 2.6: The periodic solution $X(t)$

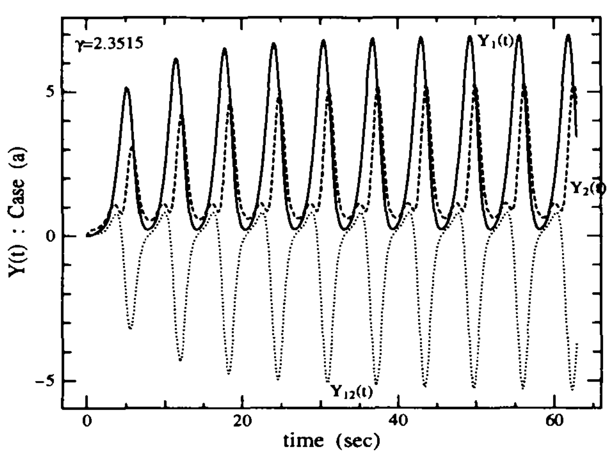

Figure 2.7: The bounded nonnegative stabilizing solution $Y(t)$: Case (a)

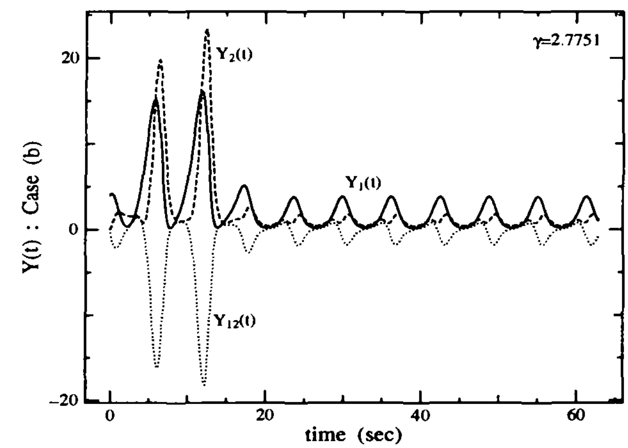

Figure 2.8: The bounded nonnegative stabilizing solution $Y(t)$: Case (b)

2.2.1 Finite Horizon Problems

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \ x(t_0) = x_0, \\ z &= C_1(t)x + D_{12}(t)u, \qquad (2.50) \\ y &= C_2(t)x + D_{21}(t)w, \\ z_1 &= Fx(T) \qquad (2.51) \end{aligned}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement, $F \in \mathbf{R}^{q \times n}$ and A, B_1, etc are bounded and piecewise continuous matrices of appropriate dimensions. For this system we assume

$$\mathbf{C1}: D_{12}'(t) \left[C_1(t) \quad D_{12}(t) \right] = \left[0 \quad I \right] \text{ for any } t.$$

The standard H_∞-control is to find necessary and sufficient conditions for the existence of a controller of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \ \ \hat{x}(t_0) = 0, \\ u &= \hat{C}(t)\hat{x} + \hat{D}(t)y \qquad (2.52) \end{aligned}$$

such that $\| \bar{G} \| < \gamma$, i.e.,

$$\| z \|_2^2 + | z_1 |^2 \leq d^2 \| w \|_2^2 \text{ for some } 0 < d < \gamma$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are bounded piecewise continuous matrices and $\bar{G}$ is the input-output operator: $w \to \begin{pmatrix} z_1 \\ z \end{pmatrix}$. In this case the controller (2.52) is called γ-suboptimal.

Now we assume that a γ-suboptimal controller K of the form (2.52) exists and study its consequence to the following quadratic game:

$$J(u,w;s,x_0) = \int_s^T [| z(t) |^2 - \gamma^2 | w(t) |^2]dt + | Fx(T) |^2 \tag{2.53}$$

where u is the minimizer and w is the maximizer. The response to (2.50)-(2.52) can be written

$$\begin{aligned} x_K(t) &= (\Phi_{1K}x_0)(t) + (\Phi_{2K}w)(t), \\ z_K(t) &= (\Psi_{1K}x_0)(t) + (\Psi_{2K}w)(t), \\ u_K(t) &= (\Pi_{1K}x_0)(t) + (\Pi_{2K}w)(t), \\ z_{1K} &= F\Phi_{1K}(T)x_0 + F\Phi_{2K}(T)w \end{aligned} \tag{2.54}$$

where

$$\begin{aligned} \Phi_{1K},\ \Psi_{1K},\ \Pi_{1K} &\in \mathcal{L}(\mathbf{R}^n, L^2(s,T;\mathbf{E})) \\ \Phi_{2K},\ \Psi_{2K},\ \Pi_{2K} &\in \mathcal{L}(L^2(s,T;\mathbf{R}^{m_1}), L^2(s,T;\mathbf{E})) \end{aligned}$$

with $\mathbf{E} = \mathbf{R}^n,\ \mathbf{R}^{p_1},\ \mathbf{R}^{m_2}$, respectively and

$$\Phi_{1K}(T)x_0 = (\Phi_{1K}x_0)(T),\ \ \Phi_{2K}(T)w = (\Phi_{2K}w)(T).$$

Moreover Φ_{2K}, Ψ_{2K} and Π_{2K} are causal and $\| \bar{G} \| < \gamma$ is equivalent to

$$\| \bar{\Psi}_K \| = \left\| \begin{pmatrix} F\Phi_{2K}(T) \\ \Psi_{2K} \end{pmatrix} \right\| \leq d \text{ for some } 0 < d < \gamma \tag{2.55}$$

which implies

$$\| \Psi_{2K}w \|_2^2 + | F\Phi_{2K}(T)w |^2 \leq d^2 \| w \|_2^2 .$$

Now consider the functional (2.53). Since

$$| z(t) |^2 = | C_1(t)x(t) |^2 + | u(t) |^2$$

by **C1**, $J(u,w;s,x_0)$ is strictly convex in u. Hence by Theorem A.4 for any x_0 and $w \in L^2(s,T;\mathbf{R}^{m_1})$ there exists a unique $u_s = u_s(w,x_0) \in L^2(s,T;\mathbf{R}^{m_2})$ such that

$$\min_u J(u,w;s,x_0) = J(u_s,w;s,x_0).$$

The response of the system $\mathbf{G}$ to u_s can be written

$$\begin{aligned} x_s(t) &= (\Phi_{1s}x_0)(t) + (\Phi_{2s}w)(t), \\ z_s(t) &= (\Psi_{1s}x_0)(t) + (\Psi_{2s}w)(t), \\ u_s(t) &= (\Pi_{1s}x_0)(t) + (\Pi_{2s}w)(t), \\ z_{1s} &= F\Phi_{1s}(T)x_0 + F\Phi_{2s}(T)w. \end{aligned} \tag{2.56}$$

Since

$$J(u_s, w; s, x_0) \leq J(u_K, w; s, x_0), \tag{2.57}$$

we have

$$\| \bar{\Psi}_s \| = \| \begin{pmatrix} F\Phi_{2s}(T) \\ \Psi_{2s} \end{pmatrix} \| \leq d. \tag{2.58}$$

Now

$$\begin{aligned} J(u_s, w; s, x_0) &= \| z_s \|_2^2 - \gamma^2 \| w \|_2^2 + | z_{1s} |^2 \\ &= \| \begin{pmatrix} F\Phi_{1s}(T) \\ \Psi_{1s} \end{pmatrix} x_0 + \bar{\Psi}_s w \|_2^2 - \gamma^2 \| w \|_2^2 \end{aligned}$$

where

$$\| \begin{pmatrix} F\Phi_{1s}(T) \\ \Psi_{1s} \end{pmatrix} x_0 + \bar{\Psi}_s w \|^2 = | F\Phi_{1s}(T)x_0 + F\Phi_{2s}(T)w |^2 + \| \Psi_{1s}x_0 + \Psi_{2s}w \|_2^2 .$$

By (2.58) $\gamma^2 I - \bar{\Psi}_s^* \bar{\Psi}_s$ is bounded both from below and above. So its inverse exists (Theorem A.3) and is uniformly bounded in s. Hence there exists a unique maximizing element of $J(u_s, w; s, x_0)$ given by

$$w_s = (\gamma^2 I - \bar{\Psi}_s^* \bar{\Psi}_s)^{-1} \bar{\Psi}_s^* \begin{pmatrix} F\Phi_{1s}(T) \\ \Psi_{1s} \end{pmatrix} x_0. \tag{2.59}$$

Next we shall show that $w_s = w_s(x_0)$ and $u_s(w_s, x_0)$ are uniformly bounded in s. Setting $w = 0$ in (2.57) we have

$$\| u_s(0, x_0) \|_2^2 \leq J(u_s(0, x_0), 0; s, x_0) \leq J(u_K, 0; s, x_0) = \| z_K \|_2^2 + | z_{1K} |^2$$

or

$$\| \Pi_{1s} x_0 \|_2^2 \leq \| \Psi_{1s} x_0 \|_2^2 + | F\Phi_{1s}(T) x_0 |^2 \leq \| \Psi_{1K} x_0 \|_2^2 + | F\Phi_{1K}(T) x_0 |^2 .$$

Hence Π_{1s} and Ψ_{1s} are uniformly bounded. From (2.58) and (2.59) we have

$$\| w_s \|_2 \leq a_1 | x_0 | \tag{2.60}$$

for some $a_1 > 0$ independent of s and w_s is uniformly bounded. Setting $x_0 = 0$ in (2.57) we also have

$$\begin{aligned} \| u_s(w, 0) \|_2^2 - \gamma^2 \| w \|_2^2 + | z_{1s} |^2 &\leq J(u_s(w, 0), w; s, 0) \\ &\leq J(u_K, w; s, 0) \\ &\leq \| z_K \|_2^2 - \gamma^2 \| w \|_2^2 + | z_{1K} |^2 \end{aligned}$$

and

$$\begin{aligned} \| \Pi_{2s} w \|_2^2 + | F\Phi_{2s}(T) |^2 &\leq \| \Psi_{2s} w \|_2^2 + | F\Phi_{2s}(T) |^2 \\ &\leq \| \Psi_{2K} w \|_2^2 + | F\Phi_{2s}(T) |^2 \\ &\leq d^2 \| w \|_2^2 . \end{aligned}$$

This implies that Π_{2s} is uniformly bounded. Now (2.56) and (2.60) yield

$$\| u_s(w_s, x_0) \|_2 \leq a_2 \mid x_0 \mid \tag{2.61}$$

for some $a_2 > 0$ independent of s. Thus we have shown the following.

Lemma 2.11 *(a) $\Pi_{1s}, \Psi_{1s}, \Pi_{2s}$ and Ψ_{2s} are uniformly bounded.*
(b) $w_s(x_0)$ and $u_s(w_s, x_0)$ are uniformly bounded and

$$\max_w \min_u J(u, w; s, x_0) = J(u_s, w_s; s, x_0) \leq a \mid x_0 \mid^2$$

for some $a > 0$ independent of s.

Now consider the Riccati equation

$$\begin{aligned} -\dot{X} &= A'(t)X + XA(t) + C_1'(t)C_1(t) \\ &\qquad + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')(t)X, \qquad (2.62) \\ X(T) &= F'F. \qquad (2.63) \end{aligned}$$

If there exists a symmetric solution to (2.62) on $[t_1, T]$, $t_1 \geq t_0$, then for any $t_1 \leq s \leq T$

$$\begin{aligned} J(u, w; s, x_0) &= x_0'X(s)x_0 - \gamma^2 \int_s^T \mid w(t) - \frac{1}{\gamma^2}B_1'X(t)x(t) \mid^2 dt \\ &\qquad + \int_s^T \mid u(t) + B_2'X(t)x(t) \mid^2 dt \qquad (2.64) \end{aligned}$$

where x is the solution of (2.50) with $t_0 = s$. Define feedback laws

$$\bar{w}(\cdot) = \frac{1}{\gamma^2}B_1'(\cdot)X(\cdot)x(\cdot), \; \bar{u}(\cdot) = -B_2'(\cdot)X(\cdot)x(\cdot) \tag{2.65}$$

and let x^* be the solution of (2.50) with $t_0 = s$ corresponding to (2.65). Set

$$w^*(t) = \frac{1}{\gamma^2}B_1'(t)X(t)x^*(t), \; u^*(t) = -B_2'(t)X(t)x^*(t). \tag{2.66}$$

We shall show the value of the game exists, i.e.,

$$\sup_w \inf_u J(u, w; s, x_0) = \inf_u \sup_w J(u, w; s, x_0).$$

Lemma 2.12 *Suppose that there exists a solution X of (2.62) and (2.63) on $[t_1, T]$. Then it is nonnegative. Moreover*

$$\begin{aligned} J(\bar{u}_s, w; s, x_0) &\leq J(\bar{u}, \bar{w}; s, x_0) = x_0'X(s)x_0 \leq J(u, \bar{w}; s, x_0), \quad (2.67) \\ J(u^*, w^*; s, x_0) &= x_0'X(s)x_0 \leq J(u, w^*; s, x_0) \qquad (2.68) \end{aligned}$$

for any $(w,u) \in L^2(s,T;\mathbf{R}^{m_1}) \times L^2(s,T;\mathbf{R}^{m_2})$. *The max-min of* J *is attained by the pair* $(\bar{u}, w^*)$ *and*

$$\begin{aligned}\max_w \min_u J(u,w;s,x_0) &= J(\bar{u},w^*;s,x_0) \\ &= J(\bar{u},\bar{w};s,x_0) \qquad (2.69)\\ &= J(u^*,w^*;s,x_0) \\ &= x_0' X(s) x_0 = \inf_u \sup_w J(u,w;s,x_0).\end{aligned}$$

Proof. We note that (2.67) follows from (2.64). Setting $w = 0$ in (2.67), we have

$$0 \le J(\bar{u}_s, 0; s, x_0) \le J(\bar{u}_s, \bar{w}_s; s, x_0) = x_0' X(s) x_0.$$

Hence $X(s)$ is nonnegative. From (2.64) we have

$$J(\bar{u}, w; s, x_0) \le J(\bar{u}, \bar{w}; s, x_0) = x_0' X(s) x_0$$

and hence

$$\min_u J(u,w;s,x_0) \le J(\bar{u},w;s,x_0) \le x_0' X(s) x_0$$

for any $w \in L^2(s,T;\mathbf{R}^{m_1})$. This implies

$$\sup_w \min_u J(u,w;s,x_0) \le x_0' X(s) x_0.$$

Now we shall show

$$\min_u J(u,w^*;s,x_0) = J(u^*,w^*;s,x_0) = x_0' X(s) x_0. \qquad (2.70)$$

For this purpose, we consider $e = x - x^*$, where x is the solution of

$$\dot{x} = Ax + \frac{1}{\gamma^2} B_1 B_1' X(t) x + B_2 u, \ \ x(s) = x_0.$$

Then

$$\dot{e} = Ae + B_2(u - u^*), \ \ e(s) = 0$$

and

$$J(u, w_s^*; s, x_0) = \| C_1(e + x^*) \|_2^2 + \| u \|_2^2 - \gamma^2 \| w^* \|_2^2 + | F(e + x^*)(T) |^2 .$$

Define

$$\begin{aligned}(\mathbf{H}u)(t) &= \int_s^t S(t,r) B_2(r) u(r)\, dr, \\ \mathbf{H}_s u &= \int_s^T S(T,r) B_2(r) u(r)\, dr\end{aligned}$$

where $S(t,r)$ is the state transition matrix of A. Then

$$\begin{aligned} e &= \mathbf{H}(u-u^*), \\ e(T) &= \mathbf{H}_s(u-u^*). \end{aligned}$$

Since $J(u,w_s^*;s,x_0)$ is strictly convex in u, there exists a unique minimizing element u given by the solution of

$$u+\mathbf{H}^*C_1'C_1\mathbf{H}(u-u^*)+\mathbf{H}^*C_1'C_1x^*+\mathbf{H}_s^*F'F\mathbf{H}_s(u-u^*)+\mathbf{H}_s^*F'Fx^*(T)=0.$$

We shall show that $u=u^*$ is the solution. Note that for $h\in L^2(s,T;\mathbf{R}^n)$ and $\tilde{h}\in\mathbf{R}^n$

$$\begin{aligned} (\mathbf{H}^*h)(t) &= B_2'(t)\int_s^T S'(t,r)h(r)\,dr, \\ (\mathbf{H}_s^*\tilde{h})(t) &= B_2'(t)S'(T,t)\tilde{h}. \end{aligned}$$

It is enough to show that $u^*(t)=-B_2'(t)X(t)x^*(t)$ coincides with

$$-\mathbf{H}^*C_1'C_1x^*-\mathbf{H}_s^*F'Fx^*(T)$$

which is equal to

$$-B_2'(t)\int_s^T S'(t,r)C_1'(r)C_1(r)X^*(r)\,dr-B_2'(t)S'(T,t)F'FX^*(T).$$

But differentiating

$$g(t)=X(t)x^*(t)-\int_s^T S'(t,r)C_1'C_1(r)x^*(r)\,dr-S'(T,t)F'Fx^*(T)$$

we obtain

$$\dot{g}=-A'(t)g,\quad g(T)=0$$

and hence $g=0$. This yields (2.70) and hence (2.68). It remains to show the last equality in (2.69). From (2.64)

$$x_0'X(s)x_0\le J(u,\bar{w};s,x_0)\le\sup_w J(u,w;s,x_0)$$

for any u and hence

$$x_0'X(s)x_0\le\inf_u\sup_w J(u,w;s,x_0).$$

But

$$\max_w J(\bar{u},w;s,x_0)=x_0'X(s)x_0$$

and $x_0'X(s)x_0=J(\bar{u},w^*;s,x_0)=\inf_u\sup_w J(u,w;s,x_0)$. ∎

Lemma 2.13 *There exists a unique nonnegative solution of the Riccati equation (2.62) and (2.63) on* $[t_0, T]$.

Proof. Since the Riccati equation is locally Lipschitz, there exists a local solution. But by Lemma 2.11 and (2.69) the solution is a priori bounded. Hence there exists a global solution on $[t_0, T]$. ∎

Summing up we have the following.

Theorem 2.7 *Assume* **C1** *and the controller (2.52) is* γ*-suboptimal for the system* **G**. *Then there exists a unique nonnegative solution* X *on* $[s, T]$ *to the Riccati equation (2.62) and (2.63). Moreover*

$$\begin{aligned}\max_w \min_u J(u, w; s, x_0) &= J(\bar{u}, w^*; s, x_0)\\ &= J(\bar{u}, \bar{w}; s, x_0)\\ &= J(u^*, w^*; s, x_0)\\ &= x_0' X(s) x_0 = \inf_u \sup_w J(u, w; s, x_0).\end{aligned}$$

Consider the backward system

$$\begin{aligned}-\dot{\tilde{x}} &= A'(t)\tilde{x} + C_1'(t)\tilde{w} + C_2'(t)\tilde{u},\\ \tilde{z} &= B_1'(t)\tilde{x} + D_{21}'(t)\tilde{u},\\ \tilde{y} &= B_2'(t)\tilde{x} + D_{12}'(t)\tilde{w},\\ \tilde{z}_1 &= H'\tilde{x}(t_0)\end{aligned} \tag{2.71}$$

which is the adjoint of the system **G** with $x(t_0) = Hh$, $h \in \mathbf{R}^{n_1}$. For the system (2.71), we introduce the controller $u = Ky$ of the form

$$\begin{aligned}-\dot{\hat{x}} &= \hat{A}'(t)\hat{x} + \hat{C}'(t)\tilde{y},\\ \tilde{u} &= \hat{B}'(t)\hat{x} + \hat{D}'(t)\tilde{y}\end{aligned} \tag{2.72}$$

which satisfies

$$\| \tilde{z} \|_2^2 + | \tilde{z}_1 |^2 \leq d^2 \| \tilde{w} \|_2^2 \text{ for some } 0 < d < \gamma.$$

Now we introduce the functional

$$\tilde{J}(\tilde{u}, \tilde{w}; T, \tilde{x}(T)) = \int_s^T [| \tilde{z}(t) |^2 - \gamma^2 | \tilde{w}(t) |^2] dt + | H'\tilde{x}(s) |^2$$

subject to (2.71) and we consider the following Riccati equation

$$\begin{aligned}\dot{Y} &= A(t)Y + YA'(t) + B_1(t)B_1'(t)\\ &\qquad + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)(t)Y,\end{aligned} \tag{2.73}$$

$$Y(t_0) = H'H. \tag{2.74}$$

Then as in Lemmas 2.11-2.13 we have the following result.

Corollary 2.9 *Assume* **C2** *and that a γ-suboptimal controller (2.72) exists for the system (2.71). Then there exists a nonnegative solution Y of (2.73) and (2.74) and*

$$\max_{\tilde{w}} \min_{\tilde{u}} \tilde{J}(\tilde{u}, \tilde{w}; T, \tilde{x}(T)) = \tilde{x}'(T)Y(T)\tilde{x}(T).$$

2.2.2 The Infinite Horizon Problem

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \ x(t_0) = x_0, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + D_{21}(t)w \end{aligned}$$

with the assumption **C1**. We further assume that (A, B_2, C_1) is stabilizable and detectable. As in the finite horizon problem we assume the existence of a controller K of the form (2.52) with property

$$\| z \|_2 \leq d \| w \|_2 \ \text{ for some } 0 < d < \gamma \tag{2.75}$$

and study its consequence to the quadratic game defined by the functional

$$J(u, w; t_0, x_0) = \int_{t_0}^{\infty} [| z(t) |^2 - \gamma^2 | w(t) |^2] dt. \tag{2.76}$$

Such a controller is called IO-stabilizing with disturbance attenuation γ (IO-γ-suboptimal) and is called γ-suboptimal if it is internally stabilizing. We also consider the finite horizon problem associated with

$$J_T(u, w; t_0, x_0) = \int_{t_0}^{T} [| z(t) |^2 - \gamma^2 | w(t) |^2] dt. \tag{2.77}$$

Note that if a controller K of the form (2.52) is IO-γ-suboptimal, it is also γ-suboptimal on any $[t_0, T]$. Since (A, B_2) is stabilizable, Ψ_{1s} in (2.56) is uniformly bounded. Then by Lemmas 2.11, 2.12 and Theorem 2.7 we have the following.

Lemma 2.14 *There exists a unique nonnegative solution X_T of the Riccati equation (2.62) with $X_T(T) = 0$ on any interval $[t_0, T]$ such that*

$$| X_T(t) | \leq c \ \textit{ independent of } t_0 \leq t \leq T < \infty.$$

Lemma 2.15 *For each $t \geq t_0$, $X_T(t)$ is monotone increasing in T.*

Proof. Let $L \leq T$ and we shall show $X_L(t_0) \leq X_T(t_0)$. This follows from

$$\begin{aligned} x_0' X_L(t_0) x_0 = J_L(\bar{u}_L, \bar{w}_L; t_0, x_0) &\leq J_L(\bar{u}_T, w_L; t_0, x_0) \\ &\leq J_T(\bar{u}_T, \hat{w}_T; t_0, x_0) \\ &\leq J_T(\bar{u}_T, \bar{w}_T; t_0, x_0) = x_0' X_T(t_0) x_0 \end{aligned}$$

where $\tilde{u}_T$ is the restriction of $\bar{u}_T$ on $[t_0, L]$ and $\hat{w}_T$ is the extension of $\bar{w}_L$ to $[t_0, T]$ by zero. The proof of a general t is similar. ∎

By Lemmas 2.14 and 2.15 $X_T(t)$ is bounded and monotone increasing in T. Hence it converges to some $X \geq 0$ which is a solution of the Riccati equation

$$-\dot{X} = A'(t)X + A(t)X + C_1'(t)C_1(t) + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')(t)X. \quad (2.78)$$

Now we show that $A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X$ is exponentially stable. Let x_T^* be the solution of

$$\dot{x} = [A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X_T](t)x, \ x(t_0) = x_0. \quad (2.79)$$

Then for any fixed interval $[t_0, L]$ the solution x_T^* converges to the solution $\bar{x}$ of

$$\dot{\bar{x}} = [A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X](t)\bar{x}, \ \bar{x}(t_0) = x_0.$$

We can rewrite (2.79) as

$$\dot{x} = (A - JC_1)x + JC_1x_T^* + B_1w_T^* + B_2u_T^*, \ x(t_0) = x_0 \quad (2.80)$$

where $J \in \mathbf{R}^{n \times p_1}$ is chosen such that $A - JC_1$ is exponentially stable. The solution of (2.80) coincides with $x_T^*(t)$ on $[t_0, T]$. We extend it to $[t_0, \infty)$ by the homogenous equation of (2.80). By Lemma 2.12 $\| C_1x_T^* \|_2$, $\| w_T^* \|_2$, $\| u_T^* \|_2 \leq a \mid x_0 \mid$ for some $a > 0$ and $C_1x_T^*$, w_T^* and u_T^* converge weakly to $\tilde{h}$, $\tilde{w}$ and $\tilde{u}$ in $L^2(t_0, \infty; \mathbf{E})$, $\mathbf{E} = \mathbf{R}^{p_1}$, $\mathbf{R}^{m_1}$ and $\mathbf{R}^{m_2}$ respectively, along a subsequence $T \to \infty$. Let $\tilde{x}$ be the solution of

$$\dot{x} = (A - JC_1)x + J\tilde{h} + B_1\tilde{w} + B_2\tilde{u}, \ x(t_0) = x_0.$$

Since the restriction of $C_1x_T^*$ etc on any interval $[t_0, L]$ converge weakly to those of $\tilde{h}$, etc, we can identify $\tilde{x}$ and $\bar{x}$ on $[t_0, L]$. Since $A - JC_1$ is exponentially stable, $\tilde{x} \in L^2(t_0, \infty; \mathbf{R}^n)$. Hence $\bar{x} \in L^2(t_0, \infty; \mathbf{R}^n)$ for each x_0 and $\| \bar{x} \|_2 \leq c \mid x_0 \mid$ for some $c > 0$ independent of x_0. Hence by Propostion 2.2 $A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X$ is exponentially stable.

Define feedback laws

$$\bar{w}(\cdot) = \frac{1}{\gamma^2}B_1'(\cdot)X(\cdot)x(\cdot), \ \bar{u}(\cdot) = -B_2'(\cdot)X(\cdot)x(\cdot). \quad (2.81)$$

Let x^* be the solution of (2.50) corresponding to (2.81) and let

$$w^*(t) = \frac{1}{\gamma^2}B_1'(t)X(t)x^*(t), \ u^*(t) = -B_2'(t)X(t)x^*(t). \quad (2.82)$$

First we show that the feedback law $\bar{u}$ is stabilizing.

Lemma 2.16 *Suppose that X is a nonnegative solution of the Riccati equation (2.78) such that $A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X$ is exponentially stable. Then $A - B_2B_2'X$ is exponentially stable.*

Proof. Since $A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X$ is exponentially stable,

$$\left(\begin{bmatrix} C_1 \\ \frac{1}{\gamma}B_1'X \\ B_2'X \end{bmatrix}, A - B_2B_2'X \right)$$

is detectable. Rewrite now the Riccati equation (2.78) in the form

$$-\dot{X} = (A - B_2B_2'X)'X + X(A - B_2B_2'X) + \begin{bmatrix} C_1 \\ \frac{1}{\gamma}B_1X \\ B_2'X \end{bmatrix}' \begin{bmatrix} C_1 \\ \frac{1}{\gamma}B_1X \\ B_2'X \end{bmatrix}.$$

Hence by Proposition 2.5 $A - B_2B_2'X$ is exponentially stable. Note that the detectability of (C_1, A) is not necessary. ∎

Let **SF** be the set of stabilizing feedback laws. As Lemma 2.12 we shall show

$$\begin{aligned} \sup_w \inf_{u\in\mathbf{SF}} J(u, w; t_0, x_0) &= J(\bar{u}, w^*; t_0, x_0) \\ &= J(\bar{u}, \bar{w}; t_0, x_0) \\ &= J(u^*, w^*; t_0, x_0) \\ &= x_0'X(t_0)x_0 \\ &= \inf_{u\in\mathbf{SF}} \sup_w J(u, w; t_0, x_0). \end{aligned} \tag{2.83}$$

Note that

$$\inf_{u\in\mathbf{SF}} \sup_w J(u, w; t_0, x_0) \le \sup_w J(\bar{u}, w; t_0, x_0) = J(\bar{u}, w^*; t_0, x_0) = x_0'X(t_0)x_0.$$

It suffices to show

$$x_0'X(t_0)x_0 \le J(\bar{u}, w^*; t_0, x_0) = \inf_{u\in\mathbf{SF}} J(u, w^*; t_0, x_0). \tag{2.84}$$

In fact this implies

$$x_0'X(t_0)x_0 = \inf_{u\in\mathbf{SF}} J(u, w^*; t_0, x_0) \le \sup_w \inf_{u\in\mathbf{SF}} J(u, w; t_0, x_0)$$

and (2.83) follows. To show (2.84), we proceed as in the proof of Lemma 2.12. Consider

$$\begin{aligned} \dot{x} &= Ax + B_1w^* + B_2u \\ &= (A - B_2B_2'X)x + B_1w^* + B_2v \end{aligned}$$

where $v = u + B_2'Xx$. Then $e = x - x^*$ satisfies

$$\dot{e} = (A - B_2B_2'X)e + B_2v$$

and $J(u, w^*; t_0, x_0)$ can be rewritten as

$$\begin{aligned}\tilde{J}(v, w^*; t_0, x_0) &= \| C_1x \|_2^2 + \| v - B_2'Xx \|_2^2 - \gamma^2 \| w^* \|_2^2 \\ &= \| C_1(\mathbf{H}v + x^*) \|_2^2 + \| v - B_2'X(\mathbf{H}v + x^*) \|_2^2 \\ &\qquad - \gamma^2 \| w^* \|_2^2\end{aligned}$$

where

$$(\mathbf{H}v)(t) = \int_{t_0}^{t} S_X(t, r)B_2(r)v(r)\, dr$$

and $S_X(t, r)$ is the state transition matrix of $A - B_2B_2'X$. The unique minimum of $\tilde{J}$ exists and is given by the solution of

$$\begin{aligned}&\mathbf{H}^*C_1'C_1\mathbf{H}v + \mathbf{H}^*C_1'C_1x^* \\ &\quad +(I - B_2'X\mathbf{H} - \mathbf{H}^*XB_2 + \mathbf{H}^*XB_2B_2'\mathbf{H})v - (I - \mathbf{H}^*XB_2)B_2'Xx^* = 0.\end{aligned}$$

We shall show that $v = 0$ is the solution. This follows if

$$B_2'Xx^* = \mathbf{H}^*(C_1'C_1 + XB_2B_2'X)x^*$$

which is true if

$$g(t) = Xx^*(t) - \int_t^\infty S_X'(t, r)(C_1'C_1 + XB_2B_2'X)(r)x^*(r)\, dr$$

is identically zero. Differentiating g we obtain

$$\dot{g} = -(A - B_2B_2'X)'g, \;\; g(\infty) = 0.$$

Hence $g(t) = 0$ and $v = 0$ minimizes $\tilde{J}$ which implies $\bar{u} = -B_2'Xx$ minimizes $J(u, w^*; t_0, x_0)$. Thus the value of the game $J(u, w; t_0, x_0)$ over $\mathbf{SF} \times L^2(t_0, \infty; \mathbf{R}^{m_1})$ exists.

Summing up we have the following.

Theorem 2.8 *Assume* **C1** *and* (A, B_2, C_1) *is stabilizable and detectable. Suppose an IO-stabilizing controller with property (2.75) exists. Then there exists a bounded nonnegative stabilizing solution to the Riccati equation (2.78). Moreover* $\bar{u} \in \mathbf{SF}$ *and*

$$\begin{aligned}\sup_w \inf_{u \in \mathbf{SF}} J(u, w; t_0, x_0) &= J(\bar{u}, w^*; t_0, x_0) \\ &= J(\bar{u}, \bar{w}; t_0, x_0) \\ &= J(u^*, w^*; t_0, x_0) \\ &= x_0'X(t_0)x_0 = \inf_{u \in \mathbf{SF}} \sup_w J(u, w; t_0, x_0).\end{aligned}$$

If **G** *is* θ*-periodic, then* X *is also* θ*-periodic.*

Corollary 2.10 *Suppose that the conditions of Theorem 2.8 hold. Then there exists a stabilizing state feedback law such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution to the Riccati equation (2.78).*

Proof. $\bar{u} = -B_2' X x$ is such a law. ∎

Corollary 2.11 *Consider the system (2.71) and assume* **C2** *and* (A, B_1, C_2) *is stabilizable and detectable. Suppose an IO-stabilizing controller of the form (2.72) with property*

$$\| \tilde{z} \|_2^2 + | \tilde{z}_1 |^2 \leq d^2 \| \tilde{w} \|_2^2 \ \textit{for some}\ 0 < d < \gamma$$

exists. Then there exists a bounded nonnegative stabilizing solution to the Riccati equation (2.73) and (2.74). Moreover, if (2.71) is θ*-periodic, the* $\lim_{n\to\infty} Y(t+n\theta)$ *exists (denoted by* $Y_\theta(t)$*) and* Y_θ *is a* θ*-periodic nonnegative stabilizng solution of (2.73).*

2.3 H_∞ Control

In this section we consider H_∞-control problems with initial uncertainty as in Khargonekar et al. [49], but we assume that initial conditions lie in some subspace. We shall introduce a general framework for H_∞-control and define our main problems. Then we consider two special problems called the full information- and the disturbance feedforward problems, which lead us eventually to the solutions of our main problems.

2.3.1 Main Results

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \qquad (2.85) \\ y &= C_2(t)x + D_{21}(t)w, \\ z_1 &= Fx(T), \qquad (2.86) \\ x(t_0) &= Hh \qquad (2.87) \end{aligned}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement, $h \in \mathbf{R}^{n_1}$, $F \in \mathbf{R}^{q\times n}$, $H \in \mathbf{R}^{n\times n_1}$ and A, B_1, etc are bounded and piecewise continuous matrices of appropriate dimensions. For the system **G** we assume

$$\mathbf{C1}: \quad D_{12}'(t)\,[\,C_1(t) \quad D_{12}(t)\,] = [\,0 \quad I\,] \ \text{ for any } t,$$
$$\mathbf{C2}: \quad D_{21}(t)\,[\,B_1'(t) \quad D_{21}'(t)\,] = [\,0 \quad I\,] \ \text{ for any } t.$$

Consider the controller $u = Ky$ of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \quad (2.88) \\ u &= \hat{C}(t)\hat{x} + \hat{D}(t)y, \\ \hat{x}(t_0) &= 0. \quad (2.89) \end{aligned}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are bounded piecewise continuous matrices of appropriate dimensions. Let $\gamma > 0$ be given. Then the H_∞-control problem on $[t_0, T]$ with initial uncertainty is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., a controller such that

$$\| z \|_2^2 + | z_1 |^2 \leq d^2(| h |^2 + \| w \|_2^2) \text{ for some } 0 < d < \gamma.$$

Without loss of generality we assume that H and F have full column rank and full row rank, respectively.

To give the solution of this problem, we introduce the following Riccati equations

$$\begin{aligned} -\dot{X} &= A'(t)X + XA(t) + C_1'(t)C_1(t) \\ &\quad + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')(t)X, \quad (2.90) \\ X(T) &= F'F, \quad (2.91) \\ H'X(t_0)H &\leq d^2 I \text{ for some } 0 < d < \gamma \quad (2.92) \end{aligned}$$

and

$$\begin{aligned} \dot{Y} &= A(t)Y + YA'(t) + B_1(t)B_1'(t) \\ &\quad + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)(t)Y, \quad (2.93) \\ Y(t_0) &= HH'. \quad (2.94) \end{aligned}$$

We also need the following Riccati equation depending on X:

$$\begin{aligned} \dot{Z} &= (A + \frac{1}{\gamma^2}B_1B_1'X)(t)Z + Z(A + \frac{1}{\gamma^2}B_1B_1'X)'(t) + B_1(t)B_1'(t) \\ &\quad + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)(t)Z, \quad (2.95) \\ Z(t_0) &= H(I - \frac{1}{\gamma^2}H'X(t_0)H)^{-1}H'. \quad (2.96) \end{aligned}$$

Lemma 2.17 *(a) Suppose X, Y and Z are solutions of (2.90), (2.93) and (2.95), respectively. If $Z(s) - Y(s) - \frac{1}{\gamma^2}Z(s)X(s)Y(s) = 0$ for some $s \geq t_0$, then $Z(t) - Y(t) - \frac{1}{\gamma^2}Z(t)X(t)Y(t) = 0$ for all $t \geq s$.*
(b) If (2.92), (2.94) and (2.96) hold, then

$$Z(t_0) - Y(t_0) - \frac{1}{\gamma^2}Z(t_0)X(t_0)Y(t_0) = 0.$$

Proof. (a) Let $Q = Z - Y - \frac{1}{\gamma^2}ZXY$, then by direct calculation

$$\dot{Q} = [A+\frac{1}{\gamma^2}B_1B_1'X+Z(\frac{1}{\gamma^2}XB_2B_2'X-C_2'C_2)]Q+Q[A+Y(\frac{1}{\gamma^2}C_1'C_1-C_2'C_2)]'.$$

Hence

$$Q(t) = S_Z(t,s)Q(s)S_Y'(t,s)$$

where S_Z and S_Y are state transition matrices of

$$A + \frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)$$

and $A + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)$, respectively. Hence if $Q(s)=0$, then $Q(t) = 0$ for all $t \geq t_0$.
(b) By (2.92), $Z(t_0)$ is well-defined. Moreover

$$\begin{aligned} Q(t_0) &= Z(t_0)(I - \frac{1}{\gamma^2}XY)(t_0) - Y(t_0) \\ &= H(I - \frac{1}{\gamma^2}H'X(t_0)H)^{-1}H'(I - \frac{1}{\gamma^2}X(t_0)HH') - HH' \\ &= HH'(I - \frac{1}{\gamma^2}X(t_0)HH')^{-1}(I - \frac{1}{\gamma^2}X(t_0)HH') - HH' \\ &= 0. \end{aligned}$$

∎

Lemma 2.18 *Let X, Y and Z be matrices of the same order with property*

$$Z - Y - \frac{1}{\gamma^2}ZXY = 0.$$

Then
(a) $I + \frac{1}{\gamma^2}XZ$, $I - \frac{1}{\gamma^2}XY$ are nonsingular and

$$Z = Y(I - \frac{1}{\gamma^2}XY)^{-1}, \quad Y = Z(I + \frac{1}{\gamma^2}XZ)^{-1}.$$

(b) λ is an eigenvalue of XZ if and only if $\mu = \frac{\gamma^2\lambda}{\gamma^2+\lambda}$ is an eigenvalue of XY.
(c) If X and Z are nonnegative, then every eigenvalue of XZ is nonnegative and

$$\rho(XY) = \max_{\lambda\in\lambda(XZ)} \frac{\gamma^2\lambda}{\gamma^2+\lambda} < \gamma^2$$

where $\lambda(A)$ denotes the set of eigenvalues of A and $\rho(\cdot)$ denotes the spectral radius of a matrix.

Proof. (a) Since

$$(I + \frac{1}{\gamma^2}XZ)(I - \frac{1}{\gamma^2}YX) = I + \frac{1}{\gamma^2}X(Z - Y - \frac{1}{\gamma^2}ZXY) = I,$$

$I+\frac{1}{\gamma^2}XZ$ and $I-\frac{1}{\gamma^2}XY$ and hence $I+\frac{1}{\gamma^2}ZX$ and $I-\frac{1}{\gamma^2}YX$ are nonsingular and

$$\begin{aligned} Y &= Z(I+\frac{1}{\gamma^2}XZ)^{-1} = (I+\frac{1}{\gamma^2}ZX)^{-1}Z, \\ Z &= Y(I-\frac{1}{\gamma^2}XY)^{-1} = (I-\frac{1}{\gamma^2}YX)^{-1}Y. \end{aligned}$$

(b) Since $XY = XZ(I+\frac{1}{\gamma^2}XZ)^{-1}$, the assertion readily follows.
(c) The first part is well-known and by (b) XY has only nonnegative eigenvalues. Thus the second part also follows from (b). ∎

Lemma 2.19 *(a) Let X, Y and Z be the solutions of (2.90), (2.93) and (2.95), respectively. Suppose $I-\frac{1}{\gamma^2}XY$ is nonsingular. If x satisfies*

$$-\dot{x} = [A+Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)]'(t)x \tag{2.97}$$

then $\tilde{x} = (I-\frac{1}{\gamma^2}XY)x$ satisfies

$$-\dot{\tilde{x}} = [A+\frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)]'(t)\tilde{x}. \tag{2.98}$$

(b) Let X, Y and Z be bounded on $[t_0,\infty)$ and suppose $I-\frac{1}{\gamma^2}XY$ is nonsingular and its inverse is uniformly bounded in t. Then Y is a stabilizing solution of (2.93) if and only if Z is a stabilizing solution of (2.95).

Proof. Differentiating $\tilde{x}$ we obtain

$$\begin{aligned} -\dot{\tilde{x}} &= [A+\frac{1}{\gamma^2}B_1B_1'X + Y(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)Y]'x \\ &\qquad -\frac{1}{\gamma^2}(A+\frac{1}{\gamma^2}B_1B_1'X)'XYx \\ &= (A+\frac{1}{\gamma^2}B_1B_1'X)'(I-\frac{1}{\gamma^2}XY)x \\ &\qquad +(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)Y(I-\frac{1}{\gamma^2}XY)^{-1}(I-\frac{1}{\gamma^2}XY)x \\ &= [A+\frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)]'\tilde{x}. \end{aligned}$$

(b) If $(I-\frac{1}{\gamma^2}XY)^{-1}$ is uniformly bounded, then (2.97) and (2.98) are equivalent. ∎

The following are our main results.

Theorem 2.9 *Assume* **C1** *and* **C2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following hold:
(i) There exists a nonnegative solution X to (2.90)-(2.92).
(ii) There exists a nonnegative solution Z to (2.95) and (2.96).
(b) In this case the set of all γ-suboptimal controllers is given by

$$
\begin{aligned}
\dot{\hat{x}} &= [A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X - ZC_2'C_2](t)\hat{x} \\
&\qquad + Z(t)C_2'(t)y + [I + \frac{1}{\gamma^2}ZX](t)B_2(t)\hat{v}, \\
u &= -B_2'(t)X(t)\hat{x} + \hat{v}, \qquad (2.99)\\
\hat{r} &= -C_2(t)\hat{x} + y, \\
\hat{v} &= Q\hat{r}, \ \| Q \| < \gamma, \\
\hat{x}(t_0) &= 0.
\end{aligned}
$$

Theorem 2.10 *Assume* **C1** *and* **C2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following hold:
(i) There exists a nonnegative solution X to (2.90)-(2.92).
(ii) There exists a nonnegative solution Y to (2.93) and (2.94).
(iii) $\rho(X(t)Y(t)) \leq d^2$ for any $t \in [t_0, T]$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (2.99) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$.

Remark 2.3 The controller (2.99) with $Q = 0$ is called central.

Next we consider the system **G**:

$$
\begin{aligned}
\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\
z &= C_1(t)x + D_{12}(t)u, \\
y &= C_2(t)x + D_{21}(t)w, \\
x(t_0) &= Hh
\end{aligned}
$$

on $[t_0, \infty)$ and the controller $u = Ky$ of the form (2.88) and (2.89). Here we assume **C1**, **C2** and

$$
\begin{aligned}
&\mathbf{C3}: \quad (A, B_1, C_1) \text{ is stabilizable and detectable,} \\
&\mathbf{C4}: \quad (A, B_2, C_2) \text{ is stabilizable and detectable.}
\end{aligned}
$$

Then the H_∞-control problem is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., an internally stabilizing controller such that

$$\| z \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2) \text{ for some } 0 < d < \gamma.$$

The solution of this problem is given by the following.

Theorem 2.11 *Assume* **C1-C4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if the following hold:
(i) There exists a bounded nonnegative stabilizing solution X to (2.90) and (2.92).
(ii) There exists a bounded nonnegative stabilizing solution Z to (2.95) and (2.96).
(b) In this case the set of all γ-suboptimal controllers is given by (2.99) with Q internally stable.

Theorem 2.12 *Assume* **C1-C4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if the following hold:
(i) There exists a bounded nonnegative stabilizing solution X to (2.90) and (2.92).
(ii) There exists a bounded nonnegative stabilizing solution Y to (2.93) and (2.94).
(iii) $\rho(X(t)Y(t)) \leq d^2$ for any $t \in [t_0, \infty)$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (2.99) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ and Q internally stable.

Now we assume that the system **G** is θ-periodic and the conditions **C1-C4** hold. Then by Theorem 2.8 and Corollary 2.11 the solution X in Theorems 2.11 and 2.12 is θ-periodic and there exist θ-periodic nonnegative stabilizing solutions Y_θ and Z_θ such that

$$\lim_{n\to\infty} Y(t+n\theta) = Y_\theta(t), \quad \lim_{n\to\infty} Z(t+n\theta) = Z_\theta(t).$$

If we further assume $h = 0$, then we have the following corollaries.

Corollary 2.12 *(a) There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a θ-periodic nonnegative stabilizing solution to (2.90) and (2.92).
(ii) There exists a θ-periodic nonnegative stabilizing solution to (2.95).
(b) In this case the controllers is given by (2.99) with Q internally stable is γ-suboptimal. If Q is θ-periodic, the controller (2.99) is θ-periodic and γ-suboptimal.

Corollary 2.13 *(a) There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a θ-periodic nonnegative stabilizing solution to (2.90) and (2.92).
(ii) There exists a θ-periodic nonnegative stabilizing solution to (2.93).
(iii) $\rho(X(t)Y(t)) \leq d^2$ for any $t \in [t_0, t_0 + \theta)$ and for some $0 < d < \gamma$.
(b) In this case the controllers given by (2.99) with $Z = (I - \frac{1}{\gamma^2}YX)^{-1}Y$

and internally stable Q are γ-suboptimal. If further Q is θ-periodic, they are θ-periodic.

Let the system **G** be time-invariant and assume the conditions **C1-C4** hold. Then we need the algebraic Riccati equations

$$A'X + XA + C_1'C_1 + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X = 0, \quad (2.100)$$

$$AY + YA' + B_1B_1' + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)Y = 0, \quad (2.101)$$

$$(A + \frac{1}{\gamma^2}B_1B_1'X)Z + Z(A + \frac{1}{\gamma^2}B_1B_1'X)' + B_1B_1' + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)Z = 0. \quad (2.102)$$

We define the stabilizing solutions of (2.100), (2.101) and (2.102) as in Definition 2.12. Without loss of generality we can set $t_0 = 0$. Then we have the following corollaries.

Corollary 2.14 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (2.100) with $H'XH \leq d^2I$ for some $0 < d < \gamma$.
(ii) There exists a nonnegative stabilizing solution of (2.95) with $Z(0) = H(I - \frac{1}{\gamma^2}H'XH)^{-1}H'$.
Moreover, there exists a nonnegative stabilizing solution Z_∞ of (2.102) and $\lim_{t\to\infty} Z(t) = Z_\infty$.

Corollary 2.15 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of the algebraic Riccati equation of (2.100) with $H'XH \leq d^2I$ for some $0 < d < \gamma$.
(ii) There exists a nonnegative stabilizing solution of (2.93) and (2.94).
Moreover, there exists a nonnegative stabilizing solution Y_∞ of (2.101) and $\lim_{t\to\infty} Y(t) = Y_\infty$.
(iii) $\rho(X_\infty Y(t)) \leq d^2$ for any $t \in [t_0, \infty)$ and for some $0 < d < \gamma$.

If we further assume that there is no initial uncertainty, i.e., $h = 0$, we obtain the following.

Corollary 2.16 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (2.100).
(ii) There exists a nonnegative stabilizing solution Z_∞ of (2.102).

Corollary 2.17 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (2.100).
(ii) There exists a nonnegative stabilizing solution Y_∞ of (2.101).
(iii) $\rho(X_\infty Y_\infty) \le d^2$ for some $0 < d < \gamma$.

Example 2.5 Consider the H_∞-control problem for the following system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -(1+0.3\cos\frac{2\pi}{3}t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ z(t) &= \begin{bmatrix} x_1(t) \\ u(t) \end{bmatrix}, \; x(0) = Hh, \\ y(t) &= x_1(t) + w_2(t). \end{aligned}$$

This system comes from the Mathieu's equation and is unstable. Obviously this system satisfies the assumptions **C1-C4**. We consider the two cases

$$\text{(a) } H = 0, \quad \text{(b) } H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In each case there exist a periodic nonnegative stabilizing solution $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}$ of (2.90) with period 3 and a nonnegative stabilizing solution $Y = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}$ of (2.93) and (2.94) which satisfy $\rho(X(t)Y(t)) \le d^2$, $0 < d < \gamma$ for all $\gamma \ge 2.01$. Moreover both $Y(t)$'s converge to the same 3-periodic solution solution. Figures 2.9, 2.10 and 2.11 show $X(t)$, $Y(t)$ and the eigenvalues of $X(t)Y(t)$, respectively in the case (a). Figure 2.12 shows the simulation results of the closed-loop system with the central controllers where $\gamma = 2.01$, the initial conditions are $x_1(0) = 1$, $x_2(0) = 0$ and the disturbances are $w_1(t) = 10e^{-10t}\sin 10t$ and $w_2(t) = 0$. The controller of the case (b) gives a better response.

2.3.2 Full Information Problem

Consider the system $\mathbf{G}_{FI}$:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned} \tag{2.103}$$

with

$$\begin{aligned} x(t_0) &= Hh, \\ z_1 &= Fx(T) \end{aligned}$$

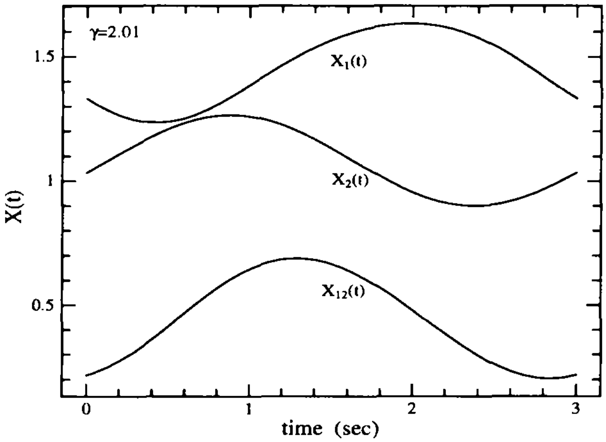

Figure 2.9: The periodic nonnegative stabilizing solution $X(t)$

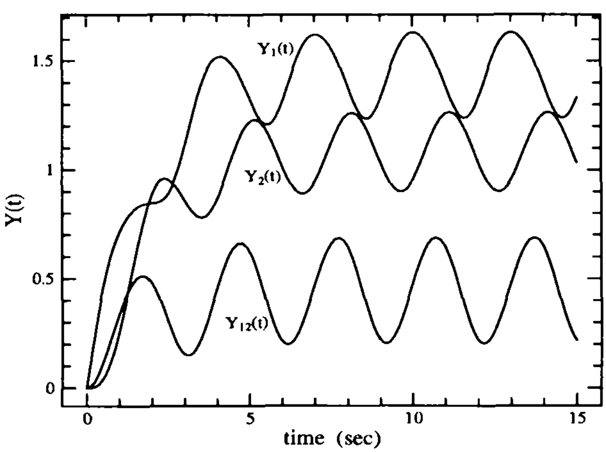

Figure 2.10: The bounded nonnegative stabilizing solution $Y(t)$

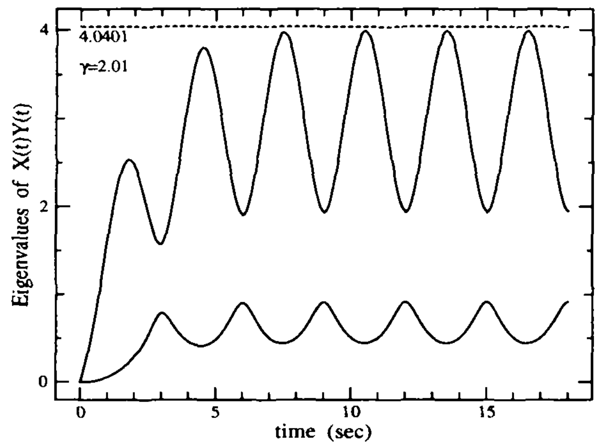

Figure 2.11: Eigenvalues of $X(t)Y(t)$

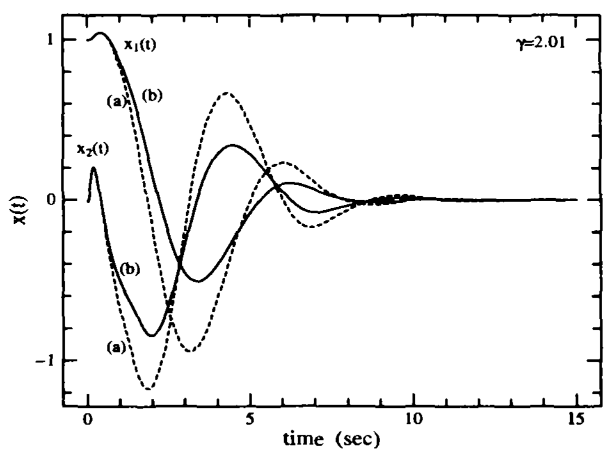

Figure 2.12: Simulation results

where we have replaced the observation in (2.85) by $y = [x', w']'$. We assume **C1**, i.e,

$$D_{12}'(t)\,[\,C_1(t) \quad D_{12}(t)\,] = [\,0 \quad I\,] \quad \text{for any } t.$$

The H_∞-control problem for the system $\mathbf{G}_{FI}$ is called the full information (FI) problem and the solution to this problem is needed to solve the H_∞-problem for the system $\mathbf{G}$. Since the state x is now available, we may allow for nonzero initial condition for the controller

$$\hat{x}(t_0) = \hat{H}h \text{ for some } \hat{H}. \tag{2.104}$$

In this case the controller (2.88) and (2.104) is written as $u = K\begin{pmatrix} h \\ y \end{pmatrix}$. First we consider the finite horizon problem. For each controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ define the input-output operator G by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

Let X be the solution of (2.90)-(2.92). Define the set of controllers $Q\begin{pmatrix} h \\ r \end{pmatrix} \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}); \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{m_2}))$ of the form (2.88) and (2.104):

$$\begin{aligned} Q_\gamma \quad = \quad & \{Q : \| Q\begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 + h'H'X(t_0)Hh \le d^2(|\, h\,|^2 + \|\, r\,\|_2^2) \\ & \qquad\qquad\qquad \text{for some } 0 < d < \gamma\}. \end{aligned} \tag{2.105}$$

We can now generalize Theorem 3.1 of Khargonekar et al. [49].

Theorem 2.13 *Assume* **C1**.
(a) There exists a controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ *of the form (2.88) and (2.104) such that* $\| G \| < \gamma$ *if and only if there exists a nonnegative solution of (2.90)-(2.92).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$u(t) = -B_2'(t)X(t)x(t) + \left[Q\begin{pmatrix} h \\ -\frac{1}{\gamma^2}B_1'Xx + w \end{pmatrix}\right](t), \; Q \in Q_\gamma. \tag{2.106}$$

Proof. Suppose $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is γ-suboptimal. Then setting $h = 0$ and applying Theorem 2.7 we obtain an $X \ge 0$ satisfying (2.90) and (2.91). Moreover for the system $\mathbf{G}_{FI}$ the following holds:

$$\begin{aligned} |\, z_1\,|^2 + \|\, z\,\|_2^2 \quad = \quad & \gamma^2 \|\, w\,\|_2^2 + h'H'X(t_0)Hh \\ & + \|\, u + B_2'Xx\,\|_2^2 - \gamma^2 \|\, w - \frac{1}{\gamma^2}B_1'Xx\,\|_2^2 . \end{aligned}$$

Setting $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ and $w = \frac{1}{\gamma^2}B_1'Xx$ we obtain

$$\begin{aligned} d^2(|\, h\,|^2 + \|\, w\,\|_2^2) &\geq |\, z_1\,|^2 + \|\, z\,\|_2^2 \\ &\geq \gamma^2 \|\, w\,\|_2^2 + h'H'X(t_0)Hh. \end{aligned}$$

Hence

$$d^2 \,|\, h\,|^2 \geq h'H'X(t_0)Hh$$

which yields (2.92).
Sufficiency of (a) and the characterization in (b) follow from Lemmas 2.20 and 2.21 below.

To complete the proof we consider

$$\begin{aligned} \dot{x} &= (A - B_2B_2'X)x + B_1w + B_2v, \\ z &= (C_1 - D_{12}B_2'X)x + D_{12}v, \\ r &= -\frac{1}{\gamma^2}B_1'Xx + w, \\ z_1 &= Fx(T), \\ x(t_0) &= Hh \end{aligned} \tag{2.107}$$

and

$$\begin{aligned} \dot{\bar{x}} &= (A + \frac{1}{\gamma^2}B_1B_1'X)\bar{x} + B_1r + B_2u, \\ v &= B_2'X\bar{x} + u, \\ y &= \begin{bmatrix} \bar{x} \\ \frac{1}{\gamma^2}B_1'X\bar{x} + r \end{bmatrix}, \\ \bar{x}(t_0) &= Hh. \end{aligned} \tag{2.108}$$

Lemma 2.20 *Let X be the solution of (2.90)-(2.92).*
(a) For (2.107) the following holds:

$$|\, z_1\,|^2 + \|\, z\,\|_2^2 = \gamma^2 \|\, w\,\|_2^2 + h'H'X(t_0)Hh + \|\, v\,\|_2^2 - \gamma^2 \|\, r\,\|_2^2 . \tag{2.109}$$

(b) The system $\mathbf{G}_{FI}$ with controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to the interconnection of (2.107) and the feedback system (2.108) with $u = K\begin{pmatrix} h \\ y \end{pmatrix}$.

Proof. (a) follows from direct calculation. To show (b) we set $e = x - \bar{x}$. Then

$$\dot{e} = [A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X]e, \;\; e(t_0) = 0.$$

Moreover (2.107) and (2.108) with $u = Ky$ is written as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u - B_2 B_2' X e, \\ z &= C_1 x + D_{12} u - D_{12} B_2' X e, \\ y &= C_2 x + D_{21} w - C_2 e, \\ u &= Ky \end{aligned}$$

and hence (b) follows. ∎

Now introduce a feedback

$$v = Q \begin{pmatrix} h \\ r \end{pmatrix} \tag{2.110}$$

to (2.107), where Q is of the form (2.88) and (2.104).

Lemma 2.21 *Consider the closed-loop system (2.107) and (2.110). Let*

$$G \begin{pmatrix} h \\ w \end{pmatrix} = \begin{pmatrix} z_1 \\ z \end{pmatrix}$$

be the input-output operator. Then $\| G \| < \gamma$ *if and only if* $Q \in \mathcal{Q}_\gamma$.

Proof. For each $r_0 \in L^2(t_0, T; \mathbf{R}^{m_1})$ there exists a $w \in L^2(t_0, T; \mathbf{R}^{m_1})$ such that the internal signal r in (2.107) and (2.110) coincides with r_0 and

$$c_1(| h |^2 + \| r_0 \|_2^2) \leq | h |^2 + \| w \|_2^2 \leq c_2(| h |^2 + \| r_0 \|_2^2) \tag{2.111}$$

for some $c_i > 0$, $i = 1, 2$. It suffices to take w_0 given by

$$\begin{aligned} \dot{x} &= (A - B_2 B_2' X)x + B_1(r_0 + \frac{1}{\gamma^2} B_1' X x) + B_2 v_0, \\ w_0 &= r_0 + \frac{1}{\gamma^2} B_1' X x, \\ x(t_0) &= Hh \end{aligned}$$

where $v_0 = Q r_0$. Now suppose $\| G \| < \gamma$ for (2.107) and (2.110). Then for some $0 < d < \gamma$

$$\begin{aligned} d^2(| h |^2 + \| w \|_2^2) &\geq | z_1 |^2 + \| z \|_2^2 \\ &= \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh + \left\| Q \begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \| r \|_2^2 \end{aligned}$$

by (2.109). Hence

$$\begin{aligned} & \left\| Q \begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 + h'H'X(t_0)Hh \\ \leq\ & \gamma^2(| h |^2 + \| r \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \\ \leq\ & [\gamma^2 - c_1(\gamma^2 - d^2)](| h |^2 + \| r \|_2^2) \qquad \text{by (2.111)} \end{aligned}$$

which implies $Q \in Q_\gamma$.

Conversely, let $Q \in Q_\gamma$. Then

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &= \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh + \| Q\begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 - \gamma^2 \| r \|_2^2 \\ &\le \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2) \\ &\le (\gamma^2 - \frac{\gamma^2 - d^2}{c_2})(| h |^2 + \| w \|_2^2). \end{aligned}$$

Hence $\| G \| < \gamma$. ∎

Remark 2.4 If $\| G \| < \gamma$, then

$$\begin{aligned} \| Q\begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 &\le \gamma^2(| h |^2 - \frac{1}{\gamma^2} h'H'X(t_0)Hh + \| r \|_2^2) \\ &\qquad -(\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \\ &= \gamma^2(| \tilde{h} |^2 + \| r \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \end{aligned}$$

where $\tilde{h} = \left(I - \frac{1}{\gamma^2} H'X(t_0)H\right)^{\frac{1}{2}} h$. Using

$$| h |^2 + \| w \|_2^2 \le c_2'(| \tilde{h} |^2 + \| r \|_2^2)$$

we can show $Q \in Q_\gamma'$ where

$$Q_\gamma' = \{Q : \| Q\begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 \le d^2(| \tilde{h} |^2 + \| r \|_2^2) \text{ for some } 0 < d < \gamma\}.$$

To conclude the proof of Theorem 2.13, we note that u given by (2.106) is γ-suboptimal by Lemma 2.21. Now let $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ be an arbitrary γ-suboptimal controller. Let Q be the input-output operator of the closed-loop system (2.108) with $u = K\begin{pmatrix} h \\ y \end{pmatrix}$. Then by Lemma 2.21, $Q \in Q_\gamma$. Hence $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to

$$\begin{aligned} u &= -B_2'Xx + Q\begin{pmatrix} h \\ r \end{pmatrix} \\ &= -B_2'Xx + Q\begin{pmatrix} h \\ -\frac{1}{\gamma^2}B_1'Xx + w \end{pmatrix}, \end{aligned}$$

which implies (b) of Theorem 2.13. ∎

Next we consider the system $\mathbf{G}_{FI}$ on the infinite horizon $[t_0, \infty)$. In this case we assume

C5 : (A, B_2, C_1) is stabilizable and detectable.

For each IO-stabilizing controller ($z \in L^2$ for each h and $w \in L^2$) we can define the input-output operator as follows:

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

The notion of IO-stabilzing controller is needed when we consider the filtering problem, for which internal stability is not in general expected.

Theorem 2.14 *Assume* **C1** *and* **C5**.
(a) There exists an IO-stabilizing controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ *on* $[t_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* X *of (2.90) and (2.92).*
(b) In this case the set of all such controllers is given by

$$u(t) = -B_2'(t)X(t)x(t) + \left[Q\begin{pmatrix} h \\ -\frac{1}{\gamma^2}B_1'Xx + w \end{pmatrix}\right](t), \; Q \in Q_\gamma \tag{2.112}$$

where $Q_\gamma \subset \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1}); L^2(t_0, \infty; \mathbf{R}^{m_2}))$ *is defined as in (2.105). In particular the set of all internally stabilizing controllers with* $\| G \| < \gamma$ *is given by (2.112) with internally stable* Q.

Proof of Theorem 2.14. (i) Necessity of (a): Suppose there exists an IO-stabilizing controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ such that $\| G \| < \gamma$. Consider the system $\mathbf{G}_{FI}$ with $h = 0$. Then for each $w \in L^2(t_0, \infty; \mathbf{R}^{m_1})$ there exists a control $u \in L^2(t_0, \infty; \mathbf{R}^{m_2})$ such that $\| z \|_2 \leq d \| w \|_2$ for some $0 < d < \gamma$. Then by Theorem 2.8 this assures, under the assumption **C5**, the existence of stabilizing solution of (2.90). To show (2.92) consider the restriction of $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ on $[t_0, T]$. Then we obtain the solution X_T of (2.90) satisfying (2.92) and $X(T) = 0$. Since $X_T(t)$ converges to $X(t)$ on $[t_0, \infty)$ we conclude $H'X(t_0)H \leq d^2 I$.

Sufficiency of (a) and the characterization of controllers will be shown below. Consider systems (2.107) and (2.108) on $[t_0, \infty)$. Note that $A - B_2B_2'X$ is exponentially stable by Lemma 2.16 and hence we have as in Lemma 2.20

$$\| z \|_2^2 = \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh + \| v \|_2^2 - \gamma^2 \| r \|_2^2 . \tag{2.113}$$

The system $\mathbf{G}_{FI}$ with controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to the interconnection of (2.107) and the feedback system (2.108) with $u = K\begin{pmatrix} h \\ y \end{pmatrix}$.

First we assume $h = 0$ and consider (2.107) with feedback

$$v = Qr \tag{2.114}$$

where Q is of the form (2.88) and (2.89).

Lemma 2.22 *Consider the closed-loop system (2.107) and (2.114) and let $Gw = z$ be the input-output operator. Suppose Q is IO-stabilizing. Then*
(a) x, r, v are square integrable and

$$\| x \|_2, \ \| r \|_2, \ \| v \|_2 \leq a \| w \|_2 \ \textit{for some } a > 0.$$

(b) If $\| G \| < \gamma$, then the map: $w \to r$ is onto and Q is IO-stable.
(c) $\| G \| < \gamma$ if and only if Q is IO-stable with $\| Q \| < \gamma$.
(d) If, further, Q is internally stabilizing then Q is internally stable.

Proof. (a) Since $z \in L^2(t_0, \infty; \mathbf{R}^{p_1})$, **C1** implies $C_1 x$ and $v - B_2' X x$ are L^2 and

$$\| C_1 x \|_2, \ \| v - B_2' X x \|_2 \leq a \| w \|_2 \ \text{ for some } a > 0.$$

Now we write (2.107) as

$$\begin{aligned} \dot{x} &= (A - JC_1)x + JC_1 x + B_1 w + B_2(v - B_2' X x), \\ x(t_0) &= 0 \end{aligned}$$

where J is a piecewise continuous bounded matrix such that $A - JC_1$ is exponentially stable. Hence x is L^2 and $\| x \|_2 \leq a \| w \|_2$ for some $a > 0$. The rest is an immediate consequence of this.
(b) We write (2.107) as

$$\begin{pmatrix} z \\ r \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix}.$$

Then P_{ij} are exponentially stable. Moreover P_{21}^{-1} is realized by

$$\begin{aligned} \dot{x} &= [A + (\frac{1}{\gamma^2} B_1 B_1' - B_2 B_2')X]x + B_1 r, \\ w &= \frac{1}{\gamma^2} B_1' X x + r \end{aligned}$$

which is exponentially stable. For the closed-loop system r and v are the solutions of

$$\begin{aligned} r &= P_{21} w + P_{22} v, \\ v &= Qr. \end{aligned}$$

By (2.113)

$$\| P_{22} \| \leq \frac{1}{\gamma} \ \text{and} \ \| v \|_2 \leq \gamma \| r \|_2 . \tag{2.115}$$

Now let $r_0 \in L^2(t_0, \infty; \mathbf{R}^{m_1})$ be arbitrary and define

$$s = (I - P_{22} Q) r_0.$$

Then s is locally square integrable. Now let s_T be the truncation of s at T so that $s_T \in L^2(t_0,\infty;\mathbf{R}^{m_1})$. Now set $w_T = P_{21}^{-1}s_T \in L^2(t_0,\infty;\mathbf{R}^{m_1})$ and let r_T be the internal signal of the closed-loop system corresponding to w_T. Then $r_T = s_T + P_{22}Qr_T$ and by (2.115)

$$\| r_T - s_T \|_2 = \| P_{22}Qr_T \|_2 \leq \| r_T \|_2,$$

which implies

$$\| r_T - s_T \|_{L^2(t_0,T;\mathbf{R}^{m_1})} \leq \| r_T \|_{L^2(t_0,T;\mathbf{R}^{m_1})} .$$

Since $r_T = r_0$ on $[t_0,T]$ we conclude

$$\| r_0 - s_T \|_{L^2(t_0,T;\mathbf{R}^{m_1})} = \| P_{22}Qr_0 \|_{L^2(t_0,T;\mathbf{R}^{m_1})} \leq \| r_0 \|_{L^2(t_0,T;\mathbf{R}^{m_1})} \leq \| r_0 \|_2 .$$

Since T is arbitrary, $P_{22}Qr_0$ is L^2. Now set $w_0 = P_{21}^{-1}(I - P_{22}Q)r_0$. Then r_0 is the responce to the input w_0 and the map: $w \to r$ is onto. Since $\| Qr \|_2 \leq \gamma \| r \|_2$, for any r, Q is IO-stable.
(c) Now let r be the response to w. Then from (b) we have

$$c_1 \| r \|_2 \leq \| w \|_2 \leq c_2 \| r \|_2 \quad \text{for some } c_i > 0,\ i = 1,2. \tag{2.116}$$

Now assume Q is IO-stabilizing and $\| G \| < \gamma$. Then for some $d < \gamma$

$$d^2 \| w \|_2^2 \geq \| z \|_2^2 = \gamma^2 \| w \|_2^2 + \| v \|_2^2 - \gamma^2 \| r \|_2^2 .$$

Hence

$$\begin{aligned} \| v \|_2^2 &\leq \gamma^2 \| r \|_2^2 - (\gamma^2 - d^2) \| w \|_2^2 \\ &\leq [\gamma^2 - c_1(\gamma^2 - d^2)] \| r \|_2^2, \end{aligned}$$

which implies $\| Q \| < \gamma$.
The converse follows from (2.113) and (2.116) in a similar manner.
(d) If Q is internally stabilizing, then by Proposition 2.6, Q is stabilizable and detectable. But Q is IO-stable by (b). This together with Proposition 2.4 implies that Q is exponentially stable. ∎

Lemma 2.23 *Consider the closed-loop system (2.107) and (2.110). Let*

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}$$

be the input-output operator.
(a) Q is IO-stabilizing and $\| G \| < \gamma$ if and only if Q is IO-stable and $Q \in Q_\gamma$.
(b) Q is internally stabilizing and $\| G \| < \gamma$ if and only if Q is internally stable and $Q \in Q_\gamma$.

Proof. (a) Suppose Q is IO-stabilizing and $\| G \| < \gamma$. We write

$$Q\begin{pmatrix} h \\ r \end{pmatrix} = Q_0 h + Q_1 r.$$

Setting $h = 0$, Q_1 is IO-stabilizing. Hence by Lemma 2.22, Q_1 is IO-stable and $\| Q_1 \| < \gamma$. Recall that r and v are written

$$\begin{aligned} r &= P_{20}h + P_{21}w + P_{22}v, \\ v &= Q_0 h + Q_1 r \end{aligned} \tag{2.117}$$

where P_{20} is exponentially stable. Since v and $Q_1 r$ are L^2, $Q_0 h$ is also L^2 for any h. Hence Q_0 is bounded and Q is IO-stable. Since $\| G \| < \gamma$, for some $d < \gamma$ we have

$$\begin{aligned} & d^2(| h |^2 + \| w \|_2^2) \\ & \quad \geq \| z \|_2^2 = \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \| r \|_2^2. \end{aligned}$$

Hence

$$\begin{aligned} & \gamma^2(| h |^2 + \| r \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \\ & \qquad \geq h'H'X(t_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2. \end{aligned}$$

Since $| h |^2 + \| r \|_2^2 \leq a(| h |^2 + \| w \|_2^2)$ for some $a > 0$, we conclude that

$$(\gamma^2 - \frac{\gamma^2 - d^2}{a})(| h |^2 + \| r \|_2^2) \geq h'H'X(t_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2.$$

Thus $Q \in \mathcal{Q}_\gamma$.

Conversely let Q be IO-stable and $Q \in \mathcal{Q}_\gamma$. Then for each $(h, w) \in \mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1})$ there exists a unique $(v, r) \in L^2(t_0, \infty; \mathbf{R}^{m_2}) \times L^2(t_0, \infty; \mathbf{R}^{m_1})$ satisfying (2.117) such that

$$\| r \|_2^2, \ \| v \|_2^2 \leq a(| h |^2 + \| w \|_2^2).$$

The pair coincides with the signal r, v of the closed-loop system. Hence x and z are in L^2 and by virtue of (2.113)

$$\begin{aligned} \| z \|_2^2 &= \gamma^2 \| w \|_2^2 + h'H'X(t_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \| r \|_2^2 \\ &\leq \gamma^2 \| w \|_2^2 - d^2(| h |^2 + \| r \|_2^2) - \gamma^2 \| r \|_2^2 \ \text{ for some } 0 < d < \gamma \\ &= \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2). \end{aligned} \tag{2.118}$$

Now for each $(h, r_0) \in \mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1})$ consider

$$\dot{x} = [A + (\frac{1}{\gamma^2}B_1 B_1' - B_2 B_2')X]x + B_1 r_0 + B_2 v,$$

$$\begin{aligned} w_0 &= \frac{1}{\gamma^2} B_1' X x + r_0, \\ x(t_0) &= Hh, \\ v &= Q \begin{pmatrix} h \\ r_0 \end{pmatrix}. \end{aligned} \tag{2.119}$$

Then $w_0 \in L^2(t_0, \infty; \mathbf{R}^{m_1})$ and

$$| h |^2 + \| w_0 \|_2^2 \leq \frac{1}{a}(| h |^2 + \| r_0 \|_2^2) \text{ for some } a > 0.$$

Since (r, w) of the closed-loop system (2.107) and (2.110) is one of (r_0, w_0) above we conclude

$$\| z \|_2^2 \leq [\gamma^2 - a(\gamma^2 - d^2)](| h |^2 + \| w \|_2^2).$$

Hence $\| G \| < \gamma$. ∎

Now the proof of sufficiency of (a) and (b) in Theorem 2.14 follows from Lemma 2.23 as in the case of Theorem 2.13. ∎

2.3.3 Disturbance Feedforward Problem

We consider the H_∞-problem for the special system $\mathbf{G}_{DF}$:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + w \end{aligned} \tag{2.120}$$

with

$$\begin{aligned} z_1 &= Fx(T), \\ x(t_0) &= 0. \end{aligned}$$

This problem needed later to solve the general H_∞-problem is called the disturbance feedforward (DF) problem. As we see below it can be reduced to the FI problem. In fact consider the observer

$$\dot{\hat{x}} = A\hat{x} + B_1(y - C_2\hat{x}) + B_2 u, \ \hat{x}(t_0) = 0.$$

Then $e = x - \hat{x}$ satisfies

$$\dot{e} = (A - B_1 C_2)e, \ e(t_0) = 0$$

and hence $\hat{x} = x$. Moreover w is observable since

$$w = y - C_2 x = y - C_2 \hat{x}.$$

Thus we can use the controllers of the FI problem with $h = 0$:

$$u = -B_2' X x + Q(-\frac{1}{\gamma^2} B_1' X x + w), \ \| Q \| < \gamma.$$

Theorem 2.15 *For each controller define* $Gw = \begin{pmatrix} z_1 \\ z \end{pmatrix}$ *and assume* **C1**.
(a) There exists a γ*-suboptimal controller on* $[t_0, T]$ *if and only if there exists a nonnegative solution* X *satisfying (2.90) and (2.91).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$\begin{aligned}
\dot{\hat{x}} &= (A - B_1C_2 - B_2B_2'X)(t)\hat{x} + B_1(t)y + B_2(t)v, \ \hat{x}(t_0) = 0, \\
u &= -B_2'(t)X(t)\hat{x} + v, \qquad (2.121)\\
r &= -(C_2 + \frac{1}{\gamma^2}B_1'X)(t)\hat{x} + y, \\
v &= Qr, \ \| Q \| < \gamma
\end{aligned}$$

where Q *is a controller of the form (2.88) and (2.89).*

Proof. The necessity of (a) follows from Theorem 2.13. The sufficiency and (b) follow from Theorem 2.13 and the observation

$$\begin{aligned}
u &= -B_2'Xx + Qr = -B_2'X\hat{x} + Qr, \\
r &= -\frac{1}{\gamma^2}B_1'Xx + w = -(C_2 + \frac{1}{\gamma^2}B_1'X)\hat{x} + y.
\end{aligned}$$

∎

We now consider the infinite horizon problem. We assume **C5** and

C6 : $A - B_1C_2$ is exponentially stable.

Theorem 2.16 *Assume* **C1**, **C5** *and* **C6**.
(a) There exists a γ*-suboptimal* $u = Ky$ *on* $[t_0, \infty)$ *if and only if there exists a bounded nonnegative stabilizing solution* X *for (2.90).*
(b) In this case the set of all γ*-suboptimal controllers is given by (2.121) with* Q *internally stable.*

Consider the H_∞-problem for the system $\mathbf{G}_{OE}$:

$$\begin{aligned}
\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\
z &= C_1(t)x + u, \qquad (2.122)\\
y &= C_2(t)x + D_{21}(t)w, \\
x(t_0) &= Hh
\end{aligned}$$

which is called the output estimation (OE) problem. The adjoint of (2.122) is the backward version of the DF problem. Hence we have the following.

Theorem 2.17 *For each controller define* $G\begin{pmatrix} h \\ w \end{pmatrix} = z$ *and assume* **C2**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *on* $[t_0, T]$ *if and only if there exists a nonnegative matrix* Y *satisfying (2.93) and (2.94), i.e.,*

$$\begin{aligned}
\dot{Y} &= A(t)Y + YA'(t) + B_1(t)B_1'(t) + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)(t)Y, \\
Y(t_0) &= HH'.
\end{aligned}$$

(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned}
\dot{\hat{x}} &= (A - B_2C_1 - YC_2'C_2)(t)\hat{x} + Y(t)C_2'(t)y + (B_2 + \frac{1}{\gamma^2}YC_1')(t)v, \\
\hat{x}(t_0) &= 0, \\
u &= -C_1(t)\hat{x} + v, \\
r &= -C_2(t)\hat{x} + y, \\
v &= Qr, \ \| Q \| < \gamma.
\end{aligned} \tag{2.123}$$

Theorem 2.18 *Suppose* **C2**, *(A, B_1, C_2) is stabilizable and detectable and that $A - B_2C_1$ is exponentially stable.*
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if there exists a bounded nonnegative stabilizing solution Y for (2.93) and (2.94).
(b) In this case the set of all γ-suboptimal controllers is given by (2.123) with Q internally stable.

To give the proofs of Theorems 2.17 and 2.18 we consider the FI- and DF problems for the backward systems in the next subsection.

2.3.4 Backward Systems

Consider the backward system $\mathbf{G}_{FI}$:

$$\begin{aligned}
-\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\
z &= C_1(t)x + D_{12}(t)u, \\
y &= \begin{bmatrix} x \\ w \end{bmatrix}, \\
z_1 &= Fx(t_0)
\end{aligned} \tag{2.124}$$

with $x(T) = 0$ and a controller $u = Ky$ of the form

$$\begin{aligned}
-\dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \ \hat{x}(T) = 0, \\
u &= \hat{C}(t)\hat{x} + \hat{D}(t)y
\end{aligned} \tag{2.125}$$

where all matrices are piecewise continuous and uniformly bounded. The H_∞-control problem for the system $\mathbf{G}_{FI}$ is the FI-problem for the backward system and the solution to this problem is need to the H_∞ filtering problem. For the system $\mathbf{G}_{FI}$ we assume **C1**. To give the solution of this problem, we need the following Riccati equation

$$\begin{aligned}
\dot{P} &= A'(t)P + PA(t) + C_1'(t)C_1(t) \\
&\qquad + P(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')(t)P, \qquad (2.126) \\
P(t_0) &= F'F. \qquad (2.127)
\end{aligned}$$

Then we have the following.

Theorem 2.19 *Assume* **C1**.
(a) There exists a γ-suboptimal controller $u = Ky$, $t \in [t_0, T]$ of the form (2.125) if and only if there exists a nonnegative solution $P(t)$, $t \in [t_0, T]$ of (2.126) and (2.127).
(b) In this case the set of all γ-suboptimal controllers is given by

$$u(t) = -B_2'(t)P(t)x(t) + \left[Q\left(-\frac{1}{\gamma^2}B_1'Px + w\right)\right](t), \ \| Q \| < \gamma. \quad (2.128)$$

Proof. **Necessity of (a).** Suppose that the controller (2.125) is γ-suboptimal. Then by Corollary 2.9 we obtain a nonnegative solution $P(t)$, $t \in [t_0, T]$ to (2.126) and (2.127).

Sufficiency of (a) and (b). Now we assume the existence of a nonnegative solution $P(t)$, $t \in [t_0, T]$ to (2.126) and (2.127). Then as in the previous subsections, we consider the following systems

$$\begin{aligned} -\dot{x} &= (A - B_2B_2'P)x + B_1w + B_2v, \\ z &= (C_1 - D_{12}B_2'P)x + D_{12}v, \\ r &= -\frac{1}{\gamma^2}B_1'Px + w, \\ z_1 &= Fx(t_0), \\ x(T) &= 0 \end{aligned} \quad (2.129)$$

and

$$\begin{aligned} -\dot{\bar{x}} &= (A + \frac{1}{\gamma^2}B_1B_1'P)\bar{x} + B_1r + B_2u, \\ v &= B_2'P\bar{x} + u, \\ y &= \begin{bmatrix} \bar{x} \\ \frac{1}{\gamma^2}B_1'P\bar{x} + r \end{bmatrix}, \\ \bar{x}(T) &= 0. \end{aligned} \quad (2.130)$$

Then as in the proof of Theorem 2.13, we can show the sufficiency of (a) and the characterization in (b) using the following lemmas. ∎

Lemma 2.24 *Let P be the solution of (2.126) and (2.127).*
(a) For (2.129) the following holds

$$| z_1 |^2 + \| z \|_2^2 = \gamma^2 \| w \|_2^2 + \| v \|_2^2 - \gamma^2 \| r \|_2^2 . \quad (2.131)$$

(b) The system $\mathbf{G}_{FI}$ with a controller $u = Ky$ is equivalent to the interconnection of (2.129) and the feedback system (2.130) with a controller $u = Ky$.

Lemma 2.25 *Consider the closed-loop system (2.129) and $v = Qr$ of the form (2.125). Let G be the input-output operator given by $\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw$. Then $\| G \| < \gamma$ if and only if $\| Q \| < \gamma$.*

Next we consider the system $\mathbf{G}_{FI}$ on the infinite horizon $[t_0, \infty)$. In this case we assume **C5**. For each IO-stabilizing controller we can define the input-output operator as follows:

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw \quad \text{on } [t_0, \infty).$$

Theorem 2.20 *Assume* **C1** *and* **C5**.
(a) There exists an IO-stabilizing controller $u = Ky$ on $[t_0, \infty)$ such that $\| G \| < \gamma$ if and only if there exists a bounded nonnegative stabilizing solution $P(t)$, $t \in [t_0, \infty)$ of (2.126) and (2.127).
(b) In this case the set of all such controllers is given by

$$u(t) = -B_2'(t)P(t)x(t) + \left[Q\left(-\frac{1}{\gamma^2}B_1'Px + w\right)\right](t), \; \| Q \| < \gamma. \tag{2.132}$$

In particular the set of all internally stabilizing controllers with $\| G \| < \gamma$ is given by (2.132) with internally stable Q.

Proof. **Necessity of (a).** Suppose that there exists an IO-stabilizing controller $u = Ky$ such that $\| G \| < \gamma$. Then by Corollary 2.11 there exists a bounded nonnegative stabilizing solution $P(t)$, $t \in [t_0, \infty)$ to (2.126) and (2.127).
Sufficiency of (a) and (b). Consider the system (2.129) and (2.130) on $[t_0, \infty)$. Note that $A - B_2B_2'P$ is exponentially stable by Proposition 2.5 and hence we have as in Lemma 2.24

$$| z_1 |^2 + \| z \|_2^2 = \gamma^2 \| w \|_2^2 + \| v \|_2^2 - \gamma^2 \| r \|_2^2 . \tag{2.133}$$

The system $\mathbf{G}_{FI}$ with a controller $u = Ky$ is equivalent to the interconnection of (2.129) and the feedback system (2.130) with $u = Ky$.

As in the proof of Theorem 2.14, we can complete the proof using the following lemma. ∎

Lemma 2.26 *Let G be the input-output operator of the closed-loop system (2.129) and $v = Qr$ of the form (2.125). Then $\| G \| < \gamma$ if and only if Q is internally stable and $\| Q \| < \gamma$.*

Proof. We only need to show necessity. We identify $L^2(t_0, T; \cdot)$ as the subspace $L^2(t_0, \infty; \cdot)$ with support on $[t_0, T]$. Let $w \in L^2(t_0, T; \mathbf{R}^{m_1})$. Then the corresponding r and v have the same support. As in Lemma 2.25

$$\| v \|_2 = \| Qr \|_2 \leq \gamma \| r \|_2 .$$

For each $r \in L^2(t_0,T;\mathbf{R}^{m_1})$ there exists a $w \in L^2(t_0,T;\mathbf{R}^{m_1})$ such that r is the response to w. Hence $\| Qr \|_2 \leq \gamma \| r \|_2$ for any $r \in L^2(t_0,T;\mathbf{R}^{m_1})$.

Consider the adjoint of the closed-loop system (2.129) and $v = Qr$. Then it is given by the closed-loop system

$$\begin{aligned}
\dot{\xi} &= (A - B_2B_2'P)'\xi + (C_1 - D_{12}B_2'P)'\nu - \frac{1}{\gamma^2}PB_1\mu, \\
\tau &= B_1'\xi + \mu, \\
\eta &= B_2'\xi + D_{12}'\nu, \\
\xi(t_0) &= F'f,\ f \in \mathbf{R}^q
\end{aligned} \tag{2.134}$$

and $\mu = Q^*\eta$ of the form

$$\begin{aligned}
\dot{\hat{\xi}} &= \hat{A}'\hat{\xi} + \hat{C}'\eta, \\
\mu &= \hat{B}'\hat{\xi} + \hat{D}'\eta.
\end{aligned} \tag{2.135}$$

Since the closed-loop system (2.134) and (2.135) is internally stable, (2.135) is stabilizable and detectable. Moreover

$$\| Q^*\eta \|_{L^2(t_0,T;\mathbf{R}^{m_1})} \leq \gamma \| \eta \|_{L^2(t_0,T;\mathbf{R}^{m_2})} \quad \text{for any } T.$$

This implies $Q^*\eta \in L^2(t_0,T;\mathbf{R}^{m_1})$ for any $\eta \in L^2(t_0,T;\mathbf{R}^{m_2})$ and $\| Q^*\eta \|_2 \leq \gamma \| \eta \|_2$ for any $\eta \in L^2(t_0,T;\mathbf{R}^{m_2})$. Since $L^2(t_0,T;\cdot)$, $t_0 \leq T < \infty$ is dense in $L^2(t_0,\infty;\cdot)$, $Q^*\eta \in L^2(t_0,\infty;\mathbf{R}^{m_1})$ for any $\eta \in L^2(t_0,\infty;\mathbf{R}^{m_2})$ and $\| Q^* \| \leq \gamma$. Hence Q^* is IO-stable. Since (2.135) is stabilizable and detectable, Q^* is internally stable by Proposition 2.4 and so is Q. $\| Q \| < \gamma$ follows as in Lemma 2.22. ∎

Corollary 2.18 *Assume that the system* $\mathbf{G}_{FI}$ *is* θ*-periodic and that the conditions* **C1** *and* **C5** *hold. Let* $F = 0$*. Then*
(a) There exists an IO-stabilizing controller $u = Ky$ *on* $[t_0,\infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a* θ*-periodic nonnegative stabilizing solution* P *of (2.126).*
(b) In this case the controller (2.132) is IO-stabilizing such that $\| G \| < \gamma$*. If* Q *is* θ*-periodic, such a controller is also* θ*-periodic.*
In particular, if Q *is internally stable, the controller (2.132) is internally stabilizing.*

Proof. Necessity of (a) follows from Corollary 2.11. Now we assume the existence of a θ-periodic nonnegative stabilizing solution P of (2.126) and consider the systems (2.129) and (2.130) on $[t_0,\infty)$. Then similarly to the proof of Lemma 2.24 we obtain

$$\| z \|_2^2 \leq x_0'P(t_0)x_0 + \| z \|_2^2 = \gamma^2 \| w \|_2^2 + \| v \|_2^2 - \gamma^2 \| r \|_2^2 \,.$$

Hence the sufficiency of (a) and (b) follow from Lemma 2.26. ∎

We now consider the H_∞-control problem for the system $\mathbf{G}_{DF}$:

$$\begin{aligned} -\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + w, \\ z_1 &= Fx(t_0) \end{aligned} \tag{2.136}$$

with $x(T) = 0$ and a controller $u = Ky$ of the form (2.125). This problem is the DF-problem for the backward system. Since it can be reduced to the FI-problem for the backward system, we have the following result.

Theorem 2.21 *For each controller define* $Gw = \begin{pmatrix} z_1 \\ z \end{pmatrix}$ *and assume* **C1**.
(a) There exists a controller $u = Ky$ *on* $[t_0, T]$ *such that* $\| G \| < \gamma$ *if and only if there exists a nonnegative solution* $P(t)$, $t \in [t_0, T]$ *to (2.126) and (2.127).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$\begin{aligned} -\dot{\hat{x}} &= (A - B_1C_2 - B_2B_2'P)(t)\hat{x} + B_1(t)y + B_2(t)v, \\ u &= -B_2'(t)P(t)\hat{x} + v, \\ r &= -(C_2 + \frac{1}{\gamma^2}B_1'P)(t)\hat{x} + y, \\ v &= Qr, \ \| Q \| < \gamma \end{aligned} \tag{2.137}$$

where Q *is a controller of the form (2.125).*

We consider the infinite horizon problem. We further assume **C5** and **C6**.

Theorem 2.22 *Assume* **C1**, **C5** *and* **C6**.
(a) There exists an internally stabilizing controller $u = Ky$ *on* $[t_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* $P(t)$, $t \in [t_0, \infty)$ *to (2.126) and (2.127).*
(b) In this case the set of all γ*-suboptimal controllers is given by (2.137) with* Q *internally stable.*

2.3.5 Proofs of Main Results

Proof of Theorem 2.9: Necessity of (a). Suppose that there exists a controller $u = Ky$ on $[t_0, T]$ such that $\| G \| < \gamma$. Then by Theorem 2.13 (i) holds. Now consider (2.107)

$$\begin{aligned} \dot{x} &= (A - B_2B_2'X)x + B_1w + B_2v, \\ z &= (C_1 - D_{12}B_2'X)x + D_{12}v, \\ r &= -\frac{1}{\gamma^2}B_1'Xx + w, \\ z_1 &= Fx(T), \\ x(t_0) &= Hh \end{aligned}$$

and

$$\begin{aligned}
\dot{\bar{x}} &= (A + \frac{1}{\gamma^2} B_1 B_1' X)\bar{x} + B_1 r + B_2 u, \\
v &= B_2' X \bar{x} + u, \\
y &= C_2 \bar{x} + D_{21} r, \\
\bar{x}(t_0) &= Hh
\end{aligned} \tag{2.138}$$

with a controller

$$u = Ky. \tag{2.139}$$

Then $e = x - \bar{x}$ satisfies

$$\dot{e} = [A + (\frac{1}{\gamma^2} B_1 B_1' - B_2 B_2')X]e, \; e(t_0) = 0$$

and hence the system **G** with $u = Ky$ is equivalent to the interconnection of (2.107) and (2.138) with $u = Ky$. Let $\tilde{Q}$ be the input-output operator of the closed-loop system (2.138) and (2.139) so that $v = \tilde{Q}\begin{pmatrix} h \\ r \end{pmatrix}$. Then by Remark 2.4, $\tilde{Q} \in Q_\gamma'$ and hence $u = Ky$ is γ-suboptimal for the H_∞-problem defined by (2.138) with H and h replaced by $\tilde{H} = H(I - \frac{1}{\gamma^2} H' X(t_0) H)^{-\frac{1}{2}}$ and $\tilde{h} = (I - \frac{1}{\gamma^2} H' X(t_0) H)^{\frac{1}{2}} h$, respectively. Since this is an OE problem, the condition (ii) holds by Theorem 2.17.

Sufficiency of (a) and (b). Consider the system (2.138) and (2.99). Then by Theorem 2.17, the set of the controllers $u = Ky$ given by (2.99) satisfies $\tilde{Q} \in Q_\gamma'$ where $\tilde{Q}$ is the input-output operator of the closed-loop system (2.138) and (2.99). By Lemma 2.21 it is enough to show $\tilde{Q} \in Q_\gamma$ to complete the proof. To do so, let

$$e = \bar{x} - \hat{x}.$$

Then

$$\begin{aligned}
\dot{e} &= (A + \frac{1}{\gamma^2} B_1 B_1' X - Z C_2' C_2) e + (B_1 - Z C_2' D_{21}) r - \frac{1}{\gamma^2} Z X B_2 \mu, \\
v &= B_2' X e + \mu, \\
\eta &= C_2 e + D_{21} r, \\
\mu &= Q\eta, \\
v_1 &= 0, \\
e(t_0) &= \tilde{H}\tilde{h}
\end{aligned}$$

and its adjoint is given by

$$-\dot{\tilde{e}} = (A + \frac{1}{\gamma^2} B_1 B_1' X - Z C_2' C_2)' \tilde{e} + X B_2 \tilde{r} + C_2' \tilde{\mu},$$

$$\begin{aligned}
\tilde{v} &= (B_1 - ZC_2'D_{21})'\tilde{e} + D_{21}'\tilde{\mu}, \\
\tilde{\eta} &= -\frac{1}{\gamma^2}B_2'XZ\tilde{e} + \tilde{r}, \\
\tilde{\mu} &= Q^*\tilde{\eta}, \\
\tilde{v}_1 &= \tilde{H}'\tilde{e}(t_0), \\
\tilde{e}(T) &= 0
\end{aligned}$$

where Q^* is the adjoint operator of Q. Then it is enough to show

$$\| v \|_2^2 \le d^2(\| r \|_2^2 + | \tilde{h} |^2) \text{ for some } 0 < d < \gamma$$

which is equivalent to

$$\| \tilde{v} \|_2^2 + | \tilde{v}_1 |^2 \le d^2 \| \tilde{r} \|_2^2 .$$

By direct calculation, we obtain

$$\begin{aligned}
\frac{d}{dt}[\tilde{e}'Z\tilde{e}] &= -| \mu |^2 + | (B_1 - ZC_2D_{21})'\tilde{e} + D_{21}'\mu |^2 \\
&\quad -\gamma^2 | \tilde{r} |^2 + \gamma^2 | \tilde{r} + \frac{1}{\gamma^2}B_2'XZ\tilde{e} |^2 \\
&= -| \mu |^2 + | \tilde{v} |^2 - \gamma^2 | \tilde{r} |^2 + \gamma^2 | \eta |^2 .
\end{aligned}$$

Integrating this from t_0 to T we obtain

$$\begin{aligned}
-\tilde{e}(t_0)Z(t_0)\tilde{e}(t_0) &= -\| \mu \|_2^2 + \| \tilde{v} \|_2^2 - \gamma^2 \| \tilde{r} \|_2^2 + \gamma^2 \| \eta \|_2^2 \\
\| \tilde{v} \|_2^2 + | \tilde{v}_1 |^2 &= \| \mu \|_2^2 + \gamma^2 \| \tilde{r} \|_2^2 - \gamma^2 \| \eta \|_2^2 . \qquad (2.140)
\end{aligned}$$

Recall that

$$\| \mu \|_2 \le \| Q^* \| \| \eta \|_2 \le \sqrt{\gamma^2 - \epsilon} \| \eta \|_2 \qquad (2.141)$$

for some $\epsilon > 0$. Since

$$\tilde{r} = \eta + \frac{1}{\gamma^2}B_2'XZ\tilde{e}$$

the map: $\eta \longrightarrow \tilde{r}$ is given by

$$\begin{aligned}
-\dot{\xi} &= [A + \frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)]'\xi + XB_2\eta + C_2'\tilde{\mu}, \\
\tilde{r} &= \frac{1}{\gamma^2}B_2'XZ\xi + \eta, \\
\tilde{\mu} &= Q^*\tilde{\eta}.
\end{aligned}$$

Hence the map: $\eta \longrightarrow \tilde{r}$ is bounded and $\| \tilde{r} \|_2 \le \delta \| \eta \|_2$ for some $\delta > 0$. Combining this with (2.140) and (2.141) we obtain

$$\| \tilde{v} \|_2^2 + | \tilde{v}_1 |^2 \le \left(\gamma^2 - \frac{\epsilon}{\delta^2}\right) \| \tilde{r} \|_2^2$$

and we have the assertion. ∎

Proof of Theorem 2.10: (a) Suppose a γ-suboptimal controller exists. Then by Theorem 2.7 and Corollary 2.9, there exist nonnegative solutions X, Y and Z of (2.90)-(2.92), (2.93), (2.94) and (2.95), (2.96). By Lemmas 2.17 and 2.18, $I-\frac{1}{\gamma^2}XY$ is nonsingular and the eigenvalues of XY have the form $\frac{\gamma^2\lambda}{\gamma^2+\lambda}$, $\lambda \in \lambda(XZ)$. Since X and Z are nonnegative and uniformly bounded in T, $\lambda \in \lambda(XZ)$ are nonnegative and uniformly bounded. Hence $\rho(X(t)Y(t)) \le d^2$ for some $0 < d < \gamma$. Hence the condition (iii) follows. To show sufficiency of (a) we note $I-\frac{1}{\gamma^2}X(t)Y(t)$ is nonsingular and $[I-\frac{1}{\gamma^2}X(t)Y(t)]^{-1}$ is uniformly bounded in $t \in [t_0, T]$. Define $Z(t) = Y(t)[I - \frac{1}{\gamma^2}X(t)Y(t)]^{-1}$. Then

$$Z(I - \frac{1}{\gamma^2}XY) - Y = 0$$

and

$$\begin{aligned}\dot{Z}(I - \frac{1}{\gamma^2}XY) &= (A + \frac{1}{\gamma^2}B_1B_1'X)Y + Z(A + \frac{1}{\gamma^2}B_1B_1'X)'(I - \frac{1}{\gamma^2}XY) \\ &\quad + B_1B_1'(I - \frac{1}{\gamma^2}XY) + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)Y.\end{aligned}$$

Hence Z satisfies the Riccati equation (2.95) and (2.96). The rest follows from Theorem 2.9. ∎

Proof of Theorem 2.11: Since (A, B_1) is stabilizable and $A + (\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X$ is exponentially stable, we can easily show that (2.138) satisfies the assumptions of Theorem 2.18 except for the detectability of $(C_2, A + \frac{1}{\gamma^2}B_1B_1'X)$. Since $A + \frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)$ is exponentially stable,

$$\left(A + \frac{1}{\gamma^2}B_1B_1'X - ZC_2'C_2,\ \frac{1}{\gamma}ZXB_2\right)$$

is stabilizable and so is

$$\left(A + \frac{1}{\gamma^2}B_1B_1'X - ZC_2'C_2, \begin{bmatrix} \frac{1}{\gamma}ZXB_2 & ZC_2' & B_1 \end{bmatrix}\right).$$

Since we can rewrite the Riccati equation (2.95) in the form

$$\begin{aligned}\dot{Z} &= (A + \frac{1}{\gamma^2}B_1B_1'X - ZC_2'C_2)Z + Z(A + \frac{1}{\gamma^2}B_1B_1'X - ZC_2'C_2)' \\ &\quad + \begin{bmatrix} \frac{1}{\gamma}ZXB_2 & ZC_2' & B_1 \end{bmatrix}\begin{bmatrix} \frac{1}{\gamma}ZXB_2 & ZC_2' & B_1 \end{bmatrix}',\end{aligned}$$

$A + \frac{1}{\gamma^2}B_1B_1'X - ZC_2'C_2$ is exponentially stable by Proposition 2.5 and hence $(C_2, A + \frac{1}{\gamma^2}B_1B_1'X)$ is detectable. Thus the system (2.138) satisfies the assumptions of Theorem 2.18 and we can complete the proof as for Theorem 2.9. ∎

Proof of Theorem 2.12: The proof is similar to that of Theorem 2.10. We only need to show $Z = Y(I - \frac{1}{\gamma^2}XY)^{-1}$ is a stabilizing solution of (2.95). But this follows from Lemma 2.19 and the stabilizing property of Y. ∎

Proof of Corollary 2.12: Necessity of (a) follows from Theorem 2.8 and Corollary 2.11. To complete the proof it is enough to show that the controller (2.99) is γ-suboptimal for the system (2.138). But we can show it as in the proofs of Theorems 2.9 and 2.11. ∎

2.4 H_∞ Filtering

In this section we consider the H_∞ filtering problem with initial uncertainty for time-varying systems.

Consider the system $\mathbf{G}_F$:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w, \\ z &= L(t)x, && (2.142) \\ y &= C(t)x + D(t)w, \\ x(t_0) &= Hh, && (2.143) \\ z_1 &= Fx(T) && (2.144) \end{aligned}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the state to be estimated, $y \in \mathbf{R}^{p_2}$ is the measurement, $h \in \mathbf{R}^{n_1}$, $F \in \mathbf{R}^{q\times n}$, $H \in \mathbf{R}^{n\times n_1}$ and A, B, etc are bounded piecewise continuous matrices of appropriate dimensions. For this system we assume

$$\mathbf{CF1}: \quad [B(t) \quad D(t)]\, D'(t) = [0 \quad I] \text{ for any } t.$$

We wish to estimate z_1 and z by the causal filter of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \ \hat{x}(t_0) = 0, \\ \hat{z} &= \hat{C}(t)\hat{x} + \hat{D}(t)y, && (2.145) \\ \hat{z}_1 &= \hat{F}\hat{x}(T) \end{aligned}$$

and to achieve the following:

$$| z_1 - \hat{z}_1 |^2 + \| z - \hat{z} \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2), \text{ for some } 0 < d < \gamma \quad (2.146)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are bounded piecewise continuous matrices and $\hat{F}$ is a constant matrix of appropriate dimension. Such a filter is called γ-suboptimal. We can write (2.142)-(2.145) as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ \hat{B}C & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ \hat{B}D \end{bmatrix} w,$$

$$\begin{aligned}
\begin{bmatrix} x(t_0) \\ \hat{x}(t_0) \end{bmatrix} &= \begin{bmatrix} Hh \\ 0 \end{bmatrix}, \\
e_1 &= z_1 - \hat{z}_1 = [\, F \quad -\hat{F} \,] \begin{bmatrix} x(T) \\ \hat{x}(T) \end{bmatrix}, \\
e &= z - \hat{z} = [\, L - \hat{D}C \quad -\hat{C} \,] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} - \hat{D}Dw.
\end{aligned}$$

Define the operator $G \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}); \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{p_1}))$ by

$$\begin{pmatrix} e_1 \\ e \end{pmatrix} = G \begin{pmatrix} h \\ w \end{pmatrix}. \tag{2.147}$$

Then (2.146) is equivalent to $\| G \| \leq d$. The adjoint G^* is given by

$$\begin{pmatrix} \zeta_0 \\ \zeta \end{pmatrix} = G^* \begin{pmatrix} f \\ v \end{pmatrix}$$

where

$$\begin{aligned}
-\begin{bmatrix} \dot{\xi} \\ \dot{\hat{\xi}} \end{bmatrix} &= \begin{bmatrix} A' & C'\hat{B}' \\ 0 & \hat{A}' \end{bmatrix} \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} + \begin{bmatrix} L' - C'\hat{D}' \\ -\hat{C}' \end{bmatrix} v, \\
\zeta &= [\, B' \quad D'\hat{B}' \,] \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} - D'\hat{D}'v, \\
\zeta_0 &= [\, H' \quad 0 \,] \begin{bmatrix} \xi(0) \\ \hat{\xi}(0) \end{bmatrix}, \\
\begin{bmatrix} \xi(T) \\ \hat{\xi}(T) \end{bmatrix} &= \begin{bmatrix} F' \\ -\hat{F}' \end{bmatrix} f.
\end{aligned} \tag{2.148}$$

This may be regarded as a closed-loop system

$$\begin{aligned}
-\dot{\xi} &= A'\xi + L'v + C'\mu, \\
\zeta &= B'\xi + D'\mu, \\
\eta &= \begin{bmatrix} \xi \\ v \end{bmatrix}, \\
\zeta_0 &= H'\xi(t_0), \\
\xi(T) &= F'f
\end{aligned} \tag{2.149}$$

with controller $\mu = K^* \begin{pmatrix} f \\ \eta \end{pmatrix}$

$$\begin{aligned}
-\dot{\hat{\xi}} &= \hat{A}'\hat{\xi} - [\, 0 \quad \hat{C}' \,]\eta, \\
\mu &= \hat{B}'\hat{\xi} - [\, 0 \quad \hat{D}' \,]\eta, \\
\hat{\xi}(T) &= -\hat{F}'f.
\end{aligned} \tag{2.150}$$

The system (2.149) is of the FI type and (2.146) is equivalent to

$$|\zeta_0|^2 + \|\zeta\|_2^2 \leq d^2(|f|^2 + \|v\|_2^2). \tag{2.151}$$

The Riccati equation corresponding to this is

$$\begin{aligned} \dot{Y} &= A(t)Y + YA'(t) + B(t)B'(t) \\ &\qquad + Y(\frac{1}{\gamma^2}L'L - C'C)(t)Y, \end{aligned} \tag{2.152}$$

$$Y(t_0) = HH', \tag{2.153}$$

$$FY(T)F' \leq d^2 I \text{ for some } 0 < d < \gamma. \tag{2.154}$$

As Q_γ in Section 2.3.2 we define the set of controllers of backward type:

$$\begin{aligned} Q_\gamma^* = \{ & Q^* \in \mathcal{L}(\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_2}); L^2(t_0,T;\mathbf{R}^{p_1})) : \\ & f'FY(T)F'f + \| Q^* \begin{pmatrix} f \\ \rho \end{pmatrix} \|_2^2 \leq d^2(|f|^2 + \|\rho\|_2^2) \\ & \text{for some } 0 < d < \gamma \}. \end{aligned} \tag{2.155}$$

Let $\tilde{Q}_\gamma$ be the set of adjoint systems of $Q^* \in Q_\gamma^*$. Modifying Theorems 2.13 and 2.19 we have the following.

Theorem 2.23 *(a) There exists a γ-suboptimal filter if and only if there exists a nonnegative solution Y to the Riccati equation (2.152)-(2.154).*
(b) In this case the set of filters with property (2.146) is given by

$$\begin{aligned} \dot{\hat{x}} &= (A - YC'C)(t)\hat{x} + Y(t)C'(t)y + \frac{1}{\gamma^2}Y(t)L'(t)v, \ \hat{x}(t_0) = 0, \\ \hat{z} &= L(t)\hat{x} - v, \\ r &= -C(t)\hat{x} + y, \\ v &= Q_1 r, \end{aligned} \tag{2.156}$$

$$\hat{z}_1 = F\hat{x}(T) - Q_0 r, \quad Q = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} \in \tilde{Q}_\gamma. \tag{2.157}$$

Proof. (a) follows from a modification of Theorem 2.19. To show (b) recall that the set of all controllers $\mu = K^* \begin{pmatrix} f \\ \eta \end{pmatrix}$ with $\| G^* \| < \gamma$ is given by

$$\mu = -CY\xi + Q^* \begin{pmatrix} f \\ -\frac{1}{\gamma^2}LY\xi + v \end{pmatrix}, \quad Q^* \in Q_\gamma^*. \tag{2.158}$$

Then the closed-loop system (2.149) with (2.158) is written as

$$\begin{aligned} -\dot{\xi} &= (A' - C'CY)\xi + [0 \quad L']\eta + C'\bar{\mu}, \\ \rho &= -\frac{1}{\gamma^2}LY\xi + [0 \quad I]\eta, \\ \bar{\mu} &= Q^* \begin{pmatrix} f \\ \rho \end{pmatrix}, \\ \xi(T) &= F'f. \end{aligned} \tag{2.159}$$

In view of this we can show that the controller (2.158) is equivalent to

$$\begin{aligned}
-\dot{\hat{\xi}} &= (A' - C'CY)\hat{\xi} + [0 \quad L']\eta + C'\bar{\mu}, \\
\mu &= -CY\hat{\xi} + \bar{\mu}, \qquad (2.160)\\
\rho &= -\frac{1}{\gamma^2}LY\hat{\xi} + [0 \quad I]\eta, \\
\bar{\mu} &= Q^*\begin{pmatrix} f \\ \rho \end{pmatrix}, \\
\hat{\xi}(T) &= F'f.
\end{aligned}$$

In fact for (2.149) and (2.160) $e = \xi - \hat{\xi}$ satisfies

$$-\dot{e} = A'e, \; e(T) = 0$$

and ξ satisfies (2.159). Now consider the adjoint of (2.149) and (2.160):

$$\begin{aligned}
\dot{x} &= Ax + Bw + [I \quad 0]u, \\
\tilde{z} &= Lx + [0 \quad I]u, \qquad (2.161)\\
y &= Cx + Dw, \\
x(t_0) &= Hh, \\
\tilde{z}_1 &= Fx(T) + u_1, \qquad (2.162)\\
\dot{\hat{x}} &= (A - YC'C)\hat{x} + YC'y + \frac{1}{\gamma^2}YL'v, \; \hat{x}(t_0) = 0, \\
u &= -\begin{bmatrix} 0 \\ L \end{bmatrix}\hat{x} + \begin{bmatrix} 0 \\ I \end{bmatrix}v, \qquad (2.163)\\
r &= -C\hat{x} + y, \\
v &= Q_1 r, \; \| Q_1 \| < \gamma, \\
u_1 &= -F\hat{x}(T) + Q_0 r, \; Q = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} \in \tilde{Q}_\gamma. \qquad (2.164)
\end{aligned}$$

Then $\| G^* \| < \gamma$ is equivalent to

$$| \tilde{z}_1 |^2 + \| \tilde{z} \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2) \text{ for some } d < \gamma. \qquad (2.165)$$

Note that (2.161) except $\tilde{z}$, $\tilde{z}_1$ coincide with (2.142) and (2.143). Thus (2.163)-(2.165) can be easily interpreted as the filtering result in (b). ∎

Remark 2.5 The filter (2.156) with $Q = 0$ is called central.

Consider the system $\mathbf{G}_F$:

$$\begin{aligned}
\dot{x} &= A(t)x + B(t)w, \\
z &= L(t)x, \\
y &= C(t)x + D(t)w, \\
x(t_0) &= Hh
\end{aligned}$$

on $[t_0, \infty)$ Then the H_∞-filtering problem is to find a γ-suboptimal filter, i.e., a filter of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \ \hat{x}(t_0) = 0, \\ \hat{z} &= \hat{C}(t)\hat{x} + \hat{D}(t)y \end{aligned} \tag{2.166}$$

such that $z - \hat{z} \in L^2(t_0, \infty; \mathbf{R}^{p_1})$ and

$$\| z - \hat{z} \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2), \text{ for some } 0 < d < \gamma. \tag{2.167}$$

We further assume

CF2 : (A, B, C) is stabilizable and detectable.

Again considering the FI problem for (2.149) on $[t_0, \infty)$ and modifying Theorem 2.20 we have the following result.

Theorem 2.24 *Assume* **CF1** *and* **CF2**. *Then*
(a) There exists a γ-suboptimal filter if and only if there exists a nonnegative bounded stabilizing solution to the Riccati equation (2.152) and (2.153).
(b) In this case the set of all γ-suboptimal filters is given by (2.156), where Q_1 is an IO-stable system with $\| Q_1 \| < \gamma$. Moreover, the set of all internally stable filters is given by (2.156) restricting Q_1 to be internally stable.

We may incorpolate the estimate of z_1 on the infinite horizon.

Corollary 2.19 *There exists a filter of the form (2.145) such that*

$$\sup_{T \geq T_0} [\| z_1 - \hat{z}_1 |^2 + \| z - \hat{z} \|_2^2] \leq d^2(| h |^2 + \| w \|_2^2), \text{ for some } d < \gamma$$

if and only if there exists a bounded nonnegative stabilizing solution of (2.152) and (2.153) with

$$FY(T)F' \leq d^2 I, \ T \geq T_0 \text{ for some } 0 < d < \gamma.$$

Modifying Corollary 2.18 we have also the following result.

Corollary 2.20 *Let* $\mathbf{G}_F$ *be θ-periodic and assume that* **CF1** *and* **CF2**. *Assume further that the initial conditions are known, i.e., $h = 0$. Then*
(a) There exists a filter of the form (2.166) with property (2.167) if and only if there exists a θ-periodic nonnegative stabilizing solution to the Riccati equation (2.152).
(b) In this case the filters given by (2.156) is γ-suboptimal where Q_1 is an IO-stable system with $\| Q_1 \| < \gamma$. If Q_1 is θ-periodic, the filter is θ-periodic and γ-suboptimal. Moreover, the filters given by (2.156) is internally stabilizing if Q_1 is internally stable.

Corollary 2.21 *Let the system* $\mathbf{G}_F$ *be time-invariant. Then* $Y(t)$ *in (a) converges as* $t \to \infty$ *to the stabilizing solution* Y_∞ *of the algebraic Riccati equation*

$$AY + YA' + BB' + Y(\frac{1}{\gamma^2}L'L - C'C)Y = 0.$$

Moreover the filter (2.156) with Y_∞ *gives the set of all* γ*-suboptimal filters when* $h = 0$.

Remark 2.6 The filtering results with known initial conditions i.e., $x(t_0) = 0$ in Nagpal et al. [58] follow from Theorems 4.1 and 4.2 setting $F = 0$ and $H = 0$.

Example 2.6 Consider the H$_\infty$-filtering problem for the following periodic system with period 3:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -(1 + 0.3\cos\frac{2\pi}{3}t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w, \; x(0) = Hh, \\
z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t)
\end{aligned}$$

which is unstable. We give its solutions both for

$$\text{(a) } H = 0, \;\; \text{(b) } H = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In the case (a) there exists a bounded nonnegative stabilizing solution $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}(t)$ of the Riccati equation (2.152) and (2.153) for all $\gamma \geq 1.26$ and in the case (b) there exists a bounded nonnegative stabilizing solution for all $\gamma \geq 1.3475$. Figures 2.13 and 2.14 show the solution $Y(t)$ of the case (a) and (b), respectively with $\gamma = 2$ and Figure 2.15 shows the asymptotic convergence of the outputs of central filters of (a) and (b) to the estimate z where $\gamma = 2$, the initial conditions are $x_1(0) = 1$, $x_2(0) = 0$ and the disturbances $w_1(t) = 10e^{-10t}\sin 10t$, $w_2(t) = 0$. The central filter in the case (b) gives a better estimate, since initial uncertainty is incorporated.

2.5 H$_2$ Control

In this section we consider the H$_2$ control problem. The theory of H$_2$ control for a time-invariant system is now well-known [14, 21, 93]. Here we extend the H$_2$ theory to time-varying systems.

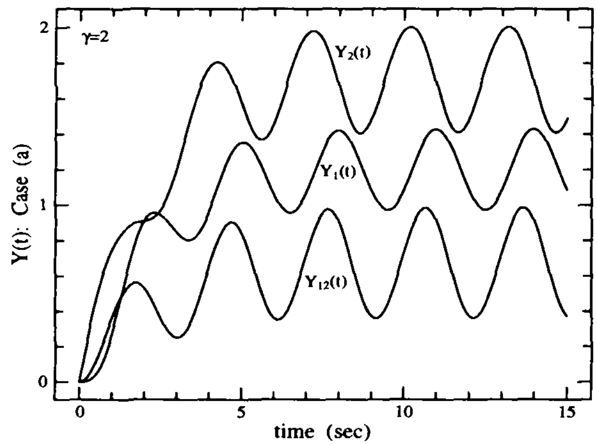

Figure 2.13: The bounded nonnegative stabilizing solution $Y(t)$ of the case (a)

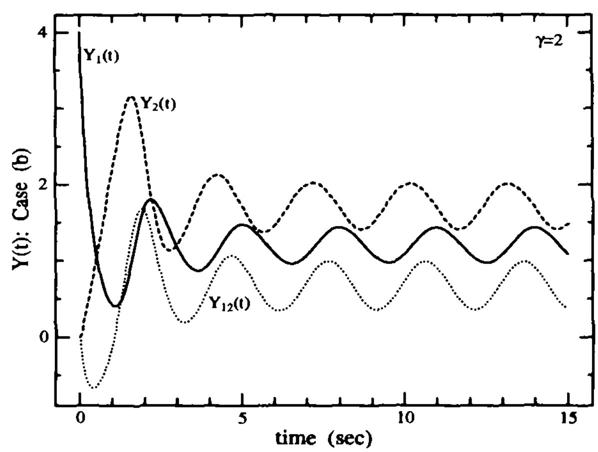

Figure 2.14: The bounded nonnegative stabilizing solution $Y(t)$ of the case (b)

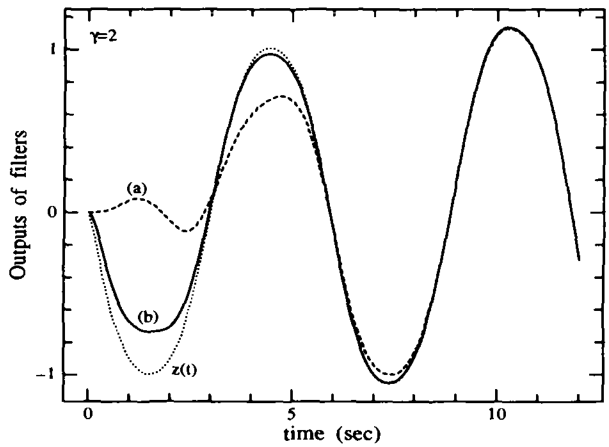

Figure 2.15: The outputs of the central fiters

2.5.1 Main Results

Consider the system **G**:

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + D_{21}(t)w \end{aligned} \tag{2.168}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement and A, B_1, etc are bounded and piecewise continuous of appropriate dimensions. For this system we assume **C1-C4**, i.e.,

$$\begin{aligned} &\mathbf{C1}: && D_{12}'(t)\,[\,C_1(t) \quad D_{12}(t)\,] = [\,0 \quad I\,] \ \text{ for any } t, \\ &\mathbf{C2}: && D_{21}(t)\,[\,B_1'(t) \quad D_{21}'(t)\,] = [\,0 \quad I\,] \ \text{ for any } t, \\ &\mathbf{C3}: && (A, B_1, C_1) \text{ is stabilizable and detectable,} \\ &\mathbf{C4}: && (A, B_2, C_2) \text{ is stabilizable and detectable.} \end{aligned}$$

Consider a controller $u = Ky$ of the form:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \\ u &= \hat{C}(t)\hat{x} \end{aligned} \tag{2.169}$$

for some bounded and piecewise continuous matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$.

To formulate the H_2-control problem for the system **G**, we introduce the following set of controllers

$\mathbf{K} = \{K : K$ is of the form (2.169) and internally stabilizes the system $\mathbf{G}\}$.

Then the H$_2$-norm, $\| G \|_2$ of the closed-loop system **G** and a controller $u = Ky$ is well-defined and our H$_2$-problem is to find a controller $K \in \mathbf{K}$ which minimizes $\| G \|_2$. To give the solution of this problem we introduce the following Riccati equations:

$$-\dot{X} = A'(t)X + XA(t) + C_1'(t)C_1(t) - XB_2(t)B_2'(t)X \tag{2.170}$$

and

$$\begin{aligned} \dot{Y} &= A(t)Y + YA'(t) + B_1(t)B_1'(t) - YC_2'(t)C_2(t)Y, & (2.171)\\ Y(t_0) &= 0. & (2.172) \end{aligned}$$

By Theorems 2.2 and 2.3, we have the following result.

Lemma 2.27 *Assume* **C1-C4**. *Then*
(a) There exists a bounded nonnegative stabilizing solution $X(t)$, $t \in [t_0, \infty)$ *to (2.170).*
(b) There exists a bounded nonnegative stabilizing solution $Y(t)$, $t \in [t_0, \infty)$ *to (2.171) and (2.172).*

Remark 2.7 By Lemma 2.5, a bounded nonnegative solution $X(t)$, $t \in [t_0, \infty)$ of (2.170) is obtained as the limit of $X_T(t)$, $t \in [t_0, T]$ where X_T is a nonnegative solution of (2.170) with $X_T(T) = 0$.

Consider the stabilizing controller based on the feedback gain

$$\hat{F}(t) = -B_2'(t)X(t)$$

and the observer gain $\hat{H}(t) = -Y(t)C_2'(t)$:

$$\begin{aligned} \dot{\hat{x}} &= A(t)\hat{x} + B_2(t)u(t) + \hat{H}(t)[C_2(t)\hat{x} - y],\\ u &= \hat{F}(t)\hat{x} \end{aligned}$$

or

$$\begin{aligned} \dot{\hat{x}} &= (A + B_2\hat{F} + \hat{H}C_2)(t)\hat{x} - \hat{H}(t)y, & (2.173)\\ u &= \hat{F}(t)\hat{x}. \end{aligned}$$

Theorem 2.25 *Assume* **C1-C4** *and consider the H_2-problem for the system* **G**. *Then the controller (2.173) is optimal and*

$$\begin{aligned} \min_{K \in \mathbf{K}} \| G \|_2^2 &= \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \{tr.B_1'(s)X(s)B_1(s)\\ &\qquad + tr.\hat{F}(s)Y(s)\hat{F}'(s)\} ds. \quad (2.174) \end{aligned}$$

Corollary 2.22 *Let* **G** *be* θ*-periodic. Then* $X(t)$ *is* θ*-periodic and there exists a* θ*-periodic nonnegative stabilizing solution* $Y_\theta(t)$ *of (2.171). Moreover, the controller (2.173) with* Y *replaced by* Y_θ *is optimal and*

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \frac{1}{\theta}\int_{t_0}^{t_0+\theta} tr.[B_1'(s)X(s)B_1(s) + \hat{F}(s)Y(s)\hat{F}'(s)]ds.$$

Let **G** be time-invariant. Then there exist nonnegative stabilizing solutions X and Y, respectively of algebraic Riccati equations

$$A'X + XA + C_1'C_1 - XB_2B_2'X = 0$$

and

$$AY + YA' + B_1B_1' - YC_2'C_2Y = 0.$$

Corollary 2.23 *Let* **G** *be time-invariant. Then the controller (2.173) with* $(X(t), Y(t))$ *replaced by* (X, Y) *is optimal and*

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = tr.[B_1'XB_1 + \hat{F}Y\hat{F}'].$$

2.5.2 Proofs of Main Results

To prove Theorem 2.25 we need some preliminary results. Consider the system **G** and the controller $u = Ky$ of the form (2.169). Let X be the solution of (2.170). We introduce

$$v = u - \hat{F}x$$

and the system $\tilde{\mathbf{G}}$:

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u, \\ v &= -\hat{F}x + u, \\ y &= C_2x + D_{21}w. \end{aligned} \tag{2.175}$$

Then z can be written using v as follows:

$$\begin{aligned} \dot{x} &= (A + B_2\hat{F})x + B_1w + B_2v, \\ z &= (C_1 + D_{12}\hat{F})x + D_{12}v. \end{aligned}$$

This system is exponentially stable and

$$z = G_cw + Uv$$

where G_c and U are given by

$$\begin{aligned} \dot{\xi} &= (A + B_2\hat{F})\xi + B_1w, \\ \zeta &= (C_1 + D_{12}\hat{F})\xi \end{aligned}$$

and

$$\begin{aligned}\dot{x} &= (A + B_2\hat{F})x + B_2 v,\\ z &= (C_1 + D_{12}\hat{F})x + D_{12}v\end{aligned}$$

respectively. Then we can easily see the following.

Lemma 2.28 *(a) The system* $\mathbf{G}$ *is equivalent to the interconnection of the system* $\bar{\mathbf{G}}$ *and* (G_c, U).
(b) K *stabilizes the system* $\mathbf{G}$ *if and only if it stabilizes the system* $\bar{\mathbf{G}}$.

Next we shall show the properties of G_c and U.

Lemma 2.29 *(a)* $\| Uv \|_2 = \| v \|_2$ *for any* $v \in L^2(s, \infty; \mathbf{R}^{m_2})$.
(b) $< G_c\delta(\cdot - s)w_0, Uv >= 0$ *for any* $w_0 \in \mathbf{R}^{m_1}$ *and* $v \in L^2(s, \infty; \mathbf{R}^{m_2})$.

Proof. (a) We can rewrite the Riccati equation (2.170) as

$$-\dot{X} = (A + B_2\hat{F})'X + X(A + B_2\hat{F}) + (C_1 + D_{12}\hat{F})'(C_1 + D_{12}\hat{F}).$$

By direct calculation, we have

$$\frac{d}{dt}[x'(t)X(t)x(t)] = - \mid z(t) \mid^2 + \mid v(t) \mid^2$$

and integrating it from s to T we have

$$x'(T)X(T)x(T) - x'(s)X(s)x(s) = \int_s^T (\mid v(t) \mid^2 - \mid z(t) \mid^2)dt.$$

Since $x(s) = 0$ and $\lim_{T\to\infty} x(T) = 0$ we have the assertion.
(b) Consider G_c with $w(t) = \delta(t - s)w_0$. Then $\xi(t) = S_F(t, s)B_1(s)w_0$ where $S_F(\cdot, \cdot)$ is the state transition matrix of $A + B_2\hat{F}$. As in (a) we have

$$\frac{d}{dt}[\xi'(t)X(t)x(t)] = -\zeta'(t)z(t)$$

and integrating it from s to T, we obtain

$$-\xi'(T)X(T)x(T) + \xi'(s)X(s)x(s) = \int_s^T \zeta'(t)z(t)dt.$$

Since $\lim_{T\to\infty} \xi(T) = \lim_{T\to\infty} x(T) = 0$ and $x(s) = 0$, we have shown (b). ∎

Now we return to the H2-control problem for the system $\mathbf{G}$. Suppose K stabilizes the system $\mathbf{G}$ and hence the system $\bar{\mathbf{G}}$. Let $\bar{G}$ be the input-output operator of the closed-loop system $\bar{\mathbf{G}}$ with $u = Ky$, i.e.,

$$v = \bar{G}w.$$

Then by Lemma 2.29

$$\begin{aligned} \| G \|_2^2 &= \| G_c + U\bar{G} \|_2^2 \\ &= \| G_c \|_2^2 + \| U\bar{G} \|_2^2 \\ &= \| G_c \|_2^2 + \| \bar{G} \|_2^2 \end{aligned} \tag{2.176}$$

and

$$\min_{K \in \mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \min_{K \in \mathbf{K}} \| \bar{G} \|_2^2 .$$

Thus our original H_2-problem has been reduced to the one for the system $\bar{\mathbf{G}}$. By Remark 2.2, $\min_{K \in \mathbf{K}} \| \bar{G} \|_2^2$ is equivalent to the H_2-problem for the backward system

$$\begin{aligned} -\dot{\tilde{x}} &= A'(t)\tilde{x} - \hat{F}'(t)\tilde{w} + C_2'(t)\tilde{u}, \\ \tilde{v} &= B_1'(t)\tilde{x} + D_{21}'(t)\tilde{u}, \\ \tilde{y} &= B_2'(t)\tilde{x} + \tilde{w} \end{aligned} \tag{2.177}$$

with an internally stabilizing controller of the form

$$\begin{aligned} -\dot{\hat{x}} &= \hat{A}'(t)\hat{x} + \hat{C}'(t)\tilde{y}, \\ \tilde{u} &= \hat{B}'(t)\hat{x}. \end{aligned}$$

The H_2-problem for the system (2.177) is the DF problem. Its solution will be given below.

Backward Systems

We take a general backward system and consider special H_2 problems. First consider the system with full information (denoted by $\mathbf{G}_{FI}$):

$$\begin{aligned} -\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= \begin{bmatrix} x \\ w \end{bmatrix}. \end{aligned} \tag{2.178}$$

We take a controller $u = Ky$ of the form

$$\begin{aligned} -\dot{\hat{x}} &= \hat{A}(t)\hat{x} + \hat{B}(t)y, \\ u &= \hat{C}(t)\hat{x} \end{aligned} \tag{2.179}$$

where all matrices are uniformly bounded and of compatible dimensions. Let G_{FI} be the input-output operator of the closed-loop system $\mathbf{G}_{FI}$ with $u = Ky$. To formulate the H_2-problem for the system $\mathbf{G}_{FI}$ we introduce the following set of controllers:

$$\begin{aligned} \mathbf{K} = \{K : K &\text{ is of the form (2.179)} \\ &\text{and internally stabilizes the system } \mathbf{G}_{FI}\}. \end{aligned}$$

Then the H_2-problem for the system $\mathbf{G}_{FI}$ (FI-problem) is to find a controller $K \in \mathbf{K}$ which minimizes $\| G_{FI} \|_2$.

For the system $\mathbf{G}_{FI}$, we assume **C1** and **C5**, i.e.,

C5 (A, B_2, C_1) is stabilizable and detectable.

Then as in Lemma 2.27, we have the following.

Lemma 2.30 *Assume* **C1** *and* **C5**. *Then there exists a unique bounded non-negative stabilizing solution* $P(t)$, $t \in [t_0, \infty)$ *to the Riccati equation*

$$\begin{aligned} \dot{P} &= A'(t)P + PA(t) + C_1'(t)C_1(t) - PB_2(t)B_2'(t)P, \quad (2.180) \\ P(t_0) &= 0. \end{aligned}$$

As in the previous subsection, we introduce

$$v = u - F_P x, \; F_P(t) = -B_2'(t)P(t)$$

and the system $\bar{\mathbf{G}}^b$:

$$\begin{aligned} -\dot{\bar{x}} &= A\bar{x} + B_1 w + B_2 u, \\ v &= -F_P \bar{x} + u, \\ y &= \begin{bmatrix} x \\ w \end{bmatrix}. \end{aligned} \tag{2.181}$$

Then z can be written using v as follows:

$$\begin{aligned} -\dot{x} &= (A + B_2 F_P)x + B_1 w + B_2 v, \\ z &= (C_1 + D_{12} F_P)x + D_{12} v. \end{aligned}$$

Hence

$$z = G_c^b w + U^b v$$

where G_c^b and U^b are given by

$$\begin{aligned} -\dot{\xi} &= (A + B_2 F_P)\xi + B_1 w, \\ \zeta &= (C_1 + D_{12} F_P)\xi \end{aligned}$$

and

$$\begin{aligned} -\dot{x} &= (A + B_2 F_P)x + B_2 v, \\ z &= (C_1 + D_{12} F_P)x + D_{12} v, \end{aligned}$$

respectively. Then we have the following.

(a) The system $\mathbf{G}_{FI}$ is equivalent to the interconnection of the system $\bar{\mathbf{G}}^b$ and (G_c^b, U^b).
(b) K stabilizes the system $\mathbf{G}_{FI}$ if and only if it stabilizes $\bar{\mathbf{G}}^b$.

Next we need the following lemma.

Lemma 2.31 *(a)* $\| U^b v \|_2 = \| v \|_2$ *for any* $v \in L^2(t_0, \infty; \mathbf{R}^{m_2})$.
(b) $< G_c^b \delta(\cdot - s) w_0, U^b v >= 0$ *for any* $w_0 \in \mathbf{R}^{m_1}$ *and* $v \in L^2(t_0, \infty; \mathbf{R}^{m_2})$ *with support in* $[t_0, s]$.

Proof. (a) We can rewrite the Riccati equation (2.180) as

$$\begin{aligned} \dot{P} &= (A + B_2 F_P)' P + P(A + B_2 F_P) + (C_1 + D_{12} F_P)'(C_1 + D_{12} F_P), \\ P(t_0) &= 0. \end{aligned}$$

Then by direct calculation

$$\frac{d}{dt}[x'(t)P(t)x(t)] = - \mid z(t) \mid^2 + \mid v(t) \mid^2$$

and integrating it from t_0 to s, we obtain

$$x'(s)P(s)x(s) - x'(t_0)P(t_0)x(t_0) = \int_{t_0}^{s} [\mid v(t) \mid^2 - \mid z(t) \mid^2] dt.$$

Since $x(s) = 0$ and $P(t_0) = 0$, we have the assertion.
(b) Consider the system G_c^b with $w(t) = \delta(t - s) w_0$, $t_0 \leq s < \infty$. Then $\xi(t) = S_F'(s,t) B_1(s) w_0$ and

$$\frac{d}{dt}[\xi'(t)P(t)x(t)] = \zeta'(t)z(t), \ t < s$$

where $S_F(\cdot, \cdot)$ is the state transition matrix of $(A + B_2 F_P)'$. Moreover $x(t) = 0$, $t > s$ and $x(s^+) = 0$ where x is the state of the system U^b. Integrating $\frac{d}{dt}[\xi'(t)P(t)x(t)]$ from t_0 to t, we have

$$\int_{t_0}^{t} \zeta'(r)z(r)dr = \xi'(t)P(t)x(t) - \xi'(t_0)P(t_0)x(t_0) = \xi'(t)P(t)x(t).$$

Letting $t \uparrow s$, $\int_{t_0}^{s} \zeta'(r)z(r)dr = 0$. Since $\xi(t) = 0$ and $z(t) = 0$, $t > s$, $\int_{t_0}^{\infty} \zeta'(r)z(r)dr = 0$. ∎

Let $u = Ky$ be an internally stabilizing controller and $\bar{G}^b$ the input-output operator of the closed-loop system $\bar{\mathbf{G}}^b$ with $u = Ky$ given by

$$v = \bar{G}^b w.$$

Then $v(t) = \bar{G}^b \delta(\cdot - s) w_0$ has support in $[t_0, s]$ and by Lemma 2.31, we have

$$\begin{aligned} \| G_{FI} \|_2^2 &= \| G_c^b + U^b \bar{G}^b \|_2^2 \\ &= \| G_c^b \|_2^2 + \| U^b \bar{G}^b \|_2^2 \\ &= \| G_c^b \|_2^2 + \| \bar{G}^b \|_2^2 . \end{aligned}$$

Hence we have

$$\min_{K\in\mathbf{K}} \| G_{FI} \|_2^2 = \| G_c^b \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G}^b \|_2^2 .$$

Thus the H_2-problem of the system $\mathbf{G}_{FI}$ is reduced to the one for the system $\bar{\mathbf{G}}^b$. Since $u = F_P(t)x$ is stabilizing, $u = [\,F_P(t) \quad 0\,]\,y$ internally stabilizes the system $\bar{\mathbf{G}}^b$ and this yields $v = 0$ or $\bar{G}^b = 0$. Hence $u = [\,F_P(t) \quad 0\,]\,y$ is the optimal controller for the system $\mathbf{G}_{FI}$ and

$$\min_{K\in\mathbf{K}} \| G_{FI} \|_2^2 = \| G_c^b \|_2^2 .$$

The controllability gramian for the backward system associated with G_c^b is a unique nonnegative solution and is given by

$$\dot{L}_o = (A + B_2F_P)'L_o + L_o(A + B_2F_P) + (C_1 + D_{12}F_P)'(C_1 + D_{12}F_P)$$

which implies $L_o = P$. Hence by Lemma 2.4

$$\| G_c^b \|_2^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.B_1'(s)P(s)B_1(s)ds.$$

Summarizing the above we have the following.

Theorem 2.26 *Assume* **C1**, **C5** *and consider the H_2-problem for the system* $\mathbf{G}_{FI}$. *Then*
(a) $\min_{K\in\mathbf{K}} \| G_{FI} \|_2^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} tr.B_1'(s)P(s)B_1(s)ds.$
(b) $K = [\,F_P(t) \quad 0\,]$ *is optimal.*

Next we consider the H_2-problem for the system (denoted by $\mathbf{G}_{DF}$):

$$\begin{aligned} -\dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + w \end{aligned} \tag{2.182}$$

and we take a controller $u = K_{DF}y$ of the form (2.179). Here we assume **C1**, **C5** and **C6**, i.e.,

$$\mathbf{C6}: \ A - B_1C_2 \text{ is exponentially stable.}$$

As we see below, this problem is equivalent to the FI-problem.

Proposition 2.7 *A controller K_{DF} internally stabilizes $\mathbf{G}_{DF}$ if and only if $K = K_{DF}[C_2 \ D_{21}]$ internally stabilizes $\mathbf{G}_{FI}$. In this case $G_{DF} = G_{FI}$ where G_{DF} is the input-output operator of the closed-loop system $\mathbf{G}_{DF}$ with $u = K_{DF}y$ defined by $z = G_{DF}w$.*

Proof. The proof follows from $u = K_{DF}y = K_{DF}\begin{bmatrix} C_2 & I \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}$. ▮

Consider the controller K_{DF}:

$$\begin{aligned} -\dot{\hat{x}} &= A(t)\hat{x} + B_1(t)[y - C_2(t)\hat{x}] + B_2(t)u_{FI}, \\ u &= u_{FI}, \\ u_{FI} &= Ky_{FI}, \\ y_{FI} &= \begin{bmatrix} \hat{x} \\ y - C_2(t)\hat{x} \end{bmatrix}. \end{aligned} \tag{2.183}$$

Proposition 2.8 *The controller* K *internally stabilizes the system* $\mathbf{G}_{FI}$ *if and only if* K_{DF} *given by (2.183) internally stabilizes* $\mathbf{G}_{DF}$*. In this case* $G_{FI} = G_{DF}$.

Proof. Let $e = x - \hat{x}$ where x and $\hat{x}$ are the states of the system $\mathbf{G}_{DF}$ and (2.183), respectively. Then e satisfies

$$-\dot{e} = (A - B_1C_2)e$$

which is exponentially stable. Moreover

$$\begin{aligned} -\dot{\hat{x}} &= A\hat{x} + B_1\hat{w} + B_2u, \\ u &= u_{FI} = K\begin{bmatrix} x \\ w \end{bmatrix} = K\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \end{aligned}$$

where $\hat{w} = w + C_2e$. Hence

$$\begin{aligned} -\dot{\hat{x}} &= A\hat{x} + B_1\hat{w} + B_2u, \\ u &= K\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}. \end{aligned} \tag{2.184}$$

Now suppose K stabilizes $\mathbf{G}_{FI}$. Then $\hat{x} \in L^2$, but $e \in L^2$ and hence $x \in L^2$. Thus K_{DF} stabilizes $\mathbf{G}_{DF}$. Conversely suppose K_{DF} stabilizes G_{DF}. Then (2.184) is exponentially stable. Finally z is given

$$z = C_1x + D_{12}u = C_1(\hat{x} + e) + D_{12}u_{FI}$$

subject to (2.184). Hence $G_{FI} = G_{DF}$. ▮

Now it is easy to obtain the solution of DF-problem. Since $K = [F_P(t)\ 0]$ is optimal for the system $\mathbf{G}_{FI}$, the optimal controller for $\mathbf{G}_{DF}$ is given by

$$u = \begin{bmatrix} F_P(t) & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ y - C_2(t)\hat{x} \end{bmatrix}$$

and (2.183) in this case

$$\begin{aligned} -\dot{\hat{x}} &= A(t)\hat{x} + B_1(t)[y - C_2(t)\hat{x}] + B_2(t)\begin{bmatrix} F_P(t) & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ y - C_2(t)\hat{x} \end{bmatrix} \\ &= (A - B_1C_2 + B_2F_P)(t)\hat{x} + B_1(t)y, \qquad (2.185) \\ u &= F_P(t)\hat{x}. \end{aligned}$$

Theorem 2.27 *Assume* **C1**, **C5** *and* **C6** *and consider the* H_2*-problem for the system* $\mathbf{G}_{DF}$. *Then*
(a) $\min_{K\in\mathbf{K}} \| G_{DF} \|_2^2 = \| G_c^b \|_2^2 = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} tr.B_1'(s)P(s)B_1(s)ds.$
(b) The controller (2.185) is optimal.

Proof of Theorem 2.25

Now we return to the H2-problem for the system **G**. By (2.176) we have

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2$$

and the original H2-problem was reduced to the H2-problem for the backward system (2.177), which is a DF-problem. Since the conditions **C1**, **C5** and **C6** are satisfied for (2.177), we can apply Theorem 2.27 to obtain

$$\min_{K\in\mathbf{K}} \| \bar{G} \|_2^2 = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} tr.\hat{F}(s)Y(s)\hat{F}'(s)ds$$

and the optimal controller is given by

$$\begin{aligned} -\dot{\tilde{x}} &= (A' + \hat{F}'B_2' + C_2'C_2Y)\tilde{x} - \hat{F}'\tilde{y}, \\ \tilde{u} &= C_2Y\tilde{x}. \end{aligned}$$

Hence the forward controller (2.173) is optimal for the system $\bar{\mathbf{G}}$ and hence for the system **G**. We also have

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} tr.\hat{F}(s)Y(s)\hat{F}'(s)ds.$$

Now we express $\| G_c \|_2^2$ using the observability gramian of G_c which is a unique nonnegative solution of

$$-\dot{L}_o = (A + B_2\hat{F})'L_o + L_o(A + B_2\hat{F}) + (C_1 + D_{12}\hat{F})'(C_1 + D_{12}\hat{F}).$$

But X satisfies the equation above and hence $L_o = X$. Then by Lemma 2.4, we have (2.174) and the proof of Theorem 2.25 is complete.

2.6 Notes and References

The stability results in Section 2.1 are obtained using the basic ideas of [11, 27, 61] which deal with infinite dimensional systems. More results on stability can be found in [84]. The H_2 and H_∞ norms are defined as in [21]. The formulation of the quadratic control follows [12, 13]. The disturbance attenuation problem is discussed in [28, 30, 62]. On finite horizons we allow for initial uncertainty and an output of the terminal state and obtained symmetric results in X and Y. We modified [30], whose extension to infinite dimensions is found in [31].

The results on differential games in Section 2.2 are obtained following [28, 29]. Standard results of differential games are found in [5]. The relation between H_∞ control amd differential games is discussed by many authors [4, 52, 63, 78, 79].

The H_∞ control theory in Section 2.3 is based on [30, 38]. Initial uncertainty is considered in the problem formulation and the output of the terminal state is included in the finite horizon problem. We have given the necessary and sufficient conditions for the existence of γ-suboptimal controllers and the characterization of all γ-suboptimal controllers. The necessary and sufficient conditions in terms of the solutions of two independent Riccati equations and a coupling condition were not available for some time and were established in [38]. The H_∞ theory for time-invariant systems is complete and found in the original papers [14, 20, 69] or in the books [21, 66, 93]. The state space theory of H_∞ control was extended to time-varying systems [49, 52, 62, 70]. The finite horizon problem is considered in [52] via game theoretic approach and necessary and sufficient conditions and the characterization of all suboptimal controllers are given (see also [21]). The infinite horizon problem is considered by Ravi et al [62]. A more general setting involving initial uncertainty is given in [49]. These papers give necessary and sufficient conditions using two coupled Riccati equations and a γ-suboptimal controller. The H_∞ theory for infinite dimensional time-varying systems is given in [31, 70].

The H_∞ filtering theory in Section 2.4 is based on [30]. The H_∞-filtering problem was first considered by Nagpal and Khargonekar [58]. They gave necessary and sufficient conditions for finite and infinite horizon problems and a suboptimal filter. Limebeer and Shaked [53] give a stochastic interpretation of H_∞-filtering. For time-invariant systems with zero initial condition they considered the infinite horizon problem and gave the set of all stable suboptimal filters. The same characterization is also given by Takaba and Katayama [71] via model matching.

The H_2 control theory for time-invariant systems is well-known and can be found in [14, 21, 93]. No extension to time-varying systems seems to be available. We have taken the approach in [14].

3. Discrete-time Systems

In this chapter, we take time-varying discrete-time systems and consider stability, quadratic games, H_∞ control, H_∞ filtering and H_2 control.

3.1 Stability

3.1.1 Lyapunov Equations

Consider

$$x(k+1) = A(k)x(k), \quad x(k_0) = x_0 \tag{3.1}$$

where $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n\times n}$ is a bounded matrix of k, i.e.,

$$| A(k) | \leq a, \ {}^\forall k \geq k_0 \text{ for some } a > 0.$$

Let $S(k,j)$ is the state transition matrix of A. Then

$$S(k,j) = \begin{cases} A(k-1)A(k-2)\cdots A(j), & k > j, \\ I, & k = j \end{cases}$$

and $x(k)$, ${}^\forall k \geq k_0$ is given by

$$x(k) = S(k, k_0)x_0.$$

If A is θ-periodic, i.e., $A(k+\theta) = A(k)$, then

$$S(k+\theta, j+\theta) = S(k,j).$$

If $A(k) = A$, then $S(k,j) = A^{k-j}$.

Definition 3.1 *The system (3.1) (or simply A) is said to be exponentially stable on* $[k_0, \infty)$ *if*

$$| S(k,j) | \leq M\alpha^{k-j}, \quad \textit{for any } k_0 \leq j \leq k < \infty$$

for some constants $M > 0$ *and* $0 < \alpha < 1$ *independent of* j *and* k*. (The system (3.1) is also called internally stable.)*

If $A(k) = A$, then A is stable if and only if the magnitude of every eigenvalue of A is less than 1. The following result is well-known.

Proposition 3.1 *The following statements are equivalent.*
(a) A is exponentially stable.
(b) There exists a positive definite matrix X satisfying

$$X = A'XA + I. \tag{3.2}$$

(c) There exists a positive definite matrix Y satisfying

$$Y = AYA' + I.$$

The equation (3.2) is called the Lyapunov equation. We now generalize this result to the time-varying system.

Proposition 3.2 *The following statements are equivalent.*
(a) The system (3.1) is exponentially stable.
(b) There exists a symmetric matrix $X(k)$ such that

$$\begin{array}{ll} (i) & c_1 I \leq X(k) \leq c_2 I, \ {}^\forall k \geq k_0 \text{ for some } c_i \geq 1,\ i = 1, 2, \\ (ii) & X(k) = A'(k)X(k+1)A(k) + I. \end{array}$$

(c) $\sum_{j=s}^{\infty} \mid S(j,s)x \mid^2 \leq c \mid x \mid^2$, ${}^\forall x$, ${}^\forall s \geq k_0$ and for some $c \geq 1$.
If A is θ-periodic, then X is also θ-periodic.

Proof. Suppose (a) holds. Then (c) also holds and

$$X(k) = \sum_{j=k}^{\infty} S'(j,k)S(j,k)$$

is well-defined and uniformly bounded, i.e., $X(k) \leq cI$ for some $c \geq 1$. Since

$$X(k) \geq S'(k,k)S(k,k) = I,$$

(i) of (b) has been shown. Since

$$\begin{aligned} X(k) &= S'(k,k)S(k,k) + \sum_{j=k+1}^{\infty} (S(j,k+1)A(k))'S(j,k+1)A(k) \\ &= I + A'(k)[\sum_{j=k+1}^{\infty} S'(j,k+1)S(j,k+1)]A(k) \end{aligned}$$

we have (ii) of (b).

Now assume (b). Then

$$\begin{aligned} & x'(k+1)X(k+1)x(k+1) - x'(k)X(k)x(k) \\ & \qquad = - \mid x(k) \mid^2 \leq -\frac{1}{c_2} x'(k)X(k)x(k) \end{aligned}$$

and

$$x'(k+1)X(k+1)x(k+1) \leq (1 - \frac{1}{c_2})x'(k)X(k)x(k)$$

which implies

$$x'(k)X(k)x(k) \leq (1 - \frac{1}{c_2})^{k-s}x'(s)X(s)x(s).$$

Using the property (i), we have

$$c_1 \mid x(k) \mid^2 \leq c_2(1 - \frac{1}{c_2})^{k-s} \mid x(s) \mid^2 .$$

Hence

$$\mid S(k,s) \mid \leq \sqrt{\frac{c_2}{c_1}}[(1 - \frac{1}{c_2})^{\frac{1}{2}}]^{k-s}$$

for any $k_0 \leq s \leq k < \infty$. Since $(1 - \frac{1}{c_2})^{\frac{1}{2}} < 1$, (a) holds.

Finally let $A(k)$ be θ-periodic. Then

$$\begin{aligned} X(k) &= \sum_{j=k}^{\infty} S'(j,k)S(j,k) \\ &= \sum_{j=k}^{\infty} S'(j+\theta, k+\theta)S(j+\theta, k+\theta) \\ &= \sum_{j=k+\theta}^{\infty} S'(j, k+\theta)S(j, k+\theta) \\ &= X(k+\theta). \end{aligned}$$

∎

Definition 3.2 *The equation (ii) of (b) is called the Lyapunov equation of the system (3.1).*

If A is exponentially stable, we can show that any solution of the Lyapunov equation coincides with $X(k)$ given in the proof of Proposition 3.2. Hence the Lyapunov equation has a unique solution. See Theorem 3.4 for the proof in a more general case.

Consider the adjoint equation of (3.1)

$$\xi(k) = A'(k)\xi(k+1), \quad \xi(N) = \xi_1. \tag{3.3}$$

Let $\xi(k; N, \xi_1)$ be the solution of (3.3).

Definition 3.3 *The system (3.3) is said to be exponentially stable if*

$$\mid \xi(k; N, \xi_1) \mid \leq M\alpha^{N-k} \mid \xi_1 \mid, \text{ for any } k_0 \leq k \leq N < \infty$$

for some constants $M > 0$ and $0 < \alpha < 1$ independent of k and N.

Since $\xi(k; N, \xi_1) = S'(N,k)\xi_1$, the system (3.3) is exponentially stable if and only if the system (3.1) is exponentially stable.

We have a dual result to Proposition 3.2.

Proposition 3.3 *The following statements are equivalent.*
(a) The system (3.3) is exponentially stable.
(b) There exists a symmetric matrix $Y(k)$ such that

$$\begin{array}{ll} (i) & c_1 I \le Y(k) \le c_2 I, \ \forall k \ge k_0 \text{ for some } c_i \ge 1,\ i = 1,2, \\ (ii) & Y(k+1) = A(k)Y(k)A'(k) + I, \ \ Y(k_0) = I. \end{array}$$

(c) $\sum_{j=s}^{N} | S'(N,j)\xi |^2 \le c | \xi |^2$, $\forall k_0 \le s \le N < \infty$ *and for some* $c \ge 1$.

Proof. Suppose (a) holds. Then (c) is true and

$$Y(k) = \sum_{j=k_0}^{k} S(k,j)S'(k,j)$$

is well-defined and uniformly bounded, i.e., $Y(k) \le cI$, $\forall k \ge k_0$ for some $c \ge 1$. We also have

$$Y(k) \ge S(k,k)S'(k,k) = I.$$

Hence (i) of (b) holds. Since $S'(k_0,k_0)S(k_0,k_0) = I$ and

$$\begin{aligned} Y(k+1) &= \sum_{j=k_0}^{k+1} S(k+1,j)S'(k+1,j) \\ &= S(k+1,k+1)S'(k+1,k+1) + \sum_{j=k_0}^{k} A(k)S(k,j)S'(k,j)A'(k) \\ &= I + A(k)Y(k)A'(k) \end{aligned}$$

we have (ii) of (b).

Now assume (b). Then

$$\begin{aligned} &\xi'(k)Y(k)\xi(k) - \xi'(k-1)Y(k-1)\xi(k-1) \\ &\qquad = | \xi(k) |^2 \ge \frac{1}{c_2}\xi'(k)Y(k)\xi(k) \end{aligned}$$

and

$$\xi'(k-1)Y(k-1)\xi(k-1) \le (1 - \frac{1}{c_2})\xi'(k)Y(k)\xi(k),$$

from which we have

$$\xi'(k)Y(k)\xi(k) \le (1 - \frac{1}{c_2})^{N-k}\xi'(N)Y(N)\xi(N).$$

Hence

$$| \xi(k) |^2 \le \frac{c_2}{c_1}(1 - \frac{1}{c_2})^{N-k} | \xi_1 |^2$$

and (a) holds. ▮

Definition 3.4 *The equation (ii) of (b) is called the Lyapunov equation of the backward system (3.3) (or simply the backward Lyapunov equation).*

Corollary 3.1 *Let $A(k)$ be θ-periodic. The system (3.3) is exponentially stable if and only if there exists a θ-periodic solution of the backward Lyapunov equation with $c_1 I \le Y(k) \le c_2 I$, $^\forall k \ge k_0$ for some c_1, $c_2 \ge 1$.*
Moreover, the θ-periodic solution is unique if A is exponentially stable.

Proof. We shall show that $Y(k+n\theta)$ is increasing in n and hence converges to $Y_\theta(k)$ which is θ-periodic. In fact

$$\begin{aligned}
Y(k+n\theta) &= \sum_{j=k_0}^{k+n\theta} S(k+n\theta, j)S'(k+n\theta, j) \\
&= \sum_{j=k_0}^{k+n\theta} S(k+(n+1)\theta, j+\theta)S'(k+(n+1)\theta, j+\theta) \\
&= \sum_{s=k_0+\theta}^{k+(n+1)\theta} S(k+(n+1)\theta, s)S'(k+(n+1)\theta, s) \\
&\le \sum_{s=k_0}^{k+(n+1)\theta} S(k+(n+1)\theta, s)S'(k+(n+1)\theta, s) \\
&= Y(k+(n+1)\theta).
\end{aligned}$$

Let $Y_\theta(k)$ be the limit of $Y(k+n\theta)$ as $n \to \infty$. Then

$$\begin{aligned}
Y_\theta(k+\theta) &= \lim_{n\to\infty} Y(k+n\theta+\theta) \\
&= \lim_{n\to\infty} Y(k+(n+1)\theta) = Y_\theta(k).
\end{aligned}$$

For the proof of uniqueness, see the proof of Theorem 3.4. ▮

Consider

$$\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k), \\
y(k) &= C(k)x(k) + D(k)u(k)
\end{aligned} \tag{3.4}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$, $y \in \mathbf{R}^{p_2}$ and A, B, C, D are bounded matrices of appropriate dimensions. Then $x(k)$ with $x(k_0) = x_0$ is given by

$$x(k) = S(k, k_0)x_0 + \sum_{j=k_0}^{k-1} S(k, j+1)B(j)u(j)$$

and

$$y(k) = C(k)S(k,k_0)x_0 + C(k)\sum_{j=k_0}^{k-1} S(k,j+1)B(j)u(j) + D(k)u(k).$$

Definition 3.5 *The system (3.4) is said to be input-output stable (or simply IO-stable) on* $[k_0, \infty)$ *if for* $x(s) = 0$, $s \geq k_0$ *and any* $u \in l^2(s, \infty; \mathbf{R}^{m_2})$

$$y \in l^2(s, \infty; \mathbf{R}^{p_2}) \text{ and } \| y \|_2 \leq c \| u \|_2$$

for some $c > 0$ *independent of* s.

Definition 3.6 *(a) The pair* (A, B) *is said to be stabilizable on* $[k_0, \infty)$ *if there exists a bounded matrix* $K(\cdot)$ *such that* $A + BK$ *is exponentially stable on* $[k_0, \infty)$.
(b) The pair (C, A) *is detectable on* $[k_0, \infty)$ *if there exists a bounded matrix* $J(\cdot)$ *such that* $A + JC$ *is exponentially stable on* $[k_0, \infty)$.
(c) If (a) and (b) hold, the system (3.4) or (A, B, C) *is said to be stabilizable and detectable.*

Proposition 3.4 *Suppose that* (A, B, C) *is stabilizable and detectable on* $[k_0, \infty)$. *Then the system (3.4) is exponentially stable if and only if it is IO-stable.*

Proof. It is enough to show sufficiency. Without loss of generality, let $D = 0$. First we shall show $C(k)S(k,s)x_0 \in l^2(s, \infty; \mathbf{R}^{p_2})$ for any $x_0 \in \mathbf{R}^n$. Since (A, B) is stabilizable, there exists a bounded matrix $K(\cdot)$ such that the system

$$x(k+1) = (A + BK)(k)x(k), \; x(s) = x_0 \tag{3.5}$$

is exponentially stable and hence $x \in l^2(s, \infty; \mathbf{R}^n)$. Now

$$x(k+1) = A(k)x(k) + B(k)K(k)x(k), \; x(s) = x_0$$

and we have

$$\begin{aligned} x(k) &= S(k,s)x_0 + \sum_{j=s}^{k-1} S(k,j+1)B(j)K(j)x(j), \\ C(k)x(k) &= C(k)S(k,s)x_0 + C(k)\sum_{j=s}^{k-1} S(k,j+1)B(j)K(j)x(j). \end{aligned}$$

Since (3.4) is IO-stable

$$C(k)\sum_{j=s}^{k-1} S(k,j+1)B(j)K(j)x(j) \in l^2(s, \infty; \mathbf{R}^{p_2})$$

and hence $C(k)S(k,s)x_0 \in l^2(s,\infty;\mathbf{R}^{p_2})$ and $\| C(k)S(k,s)x_0 \|_2 \le c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. Since the system

$$x(k+1) = A(k)x(k), \quad x(s) = x_0$$

is equivalent to

$$x(k+1) = (A + LC)(k)x(k) - L(k)C(k)x(k), \quad x(k_0) = x_0$$

where $L(\cdot)$ is a bounded matrix such that $A + LC$ is exponentially stable. Then we have

$$x(k) = S_L(k,s)x_0 + \sum_{j=s}^{k-1} S_L(k, j+1)L(j)C(j)x(j)$$

where $S_L(k,j)$ is the state transition matrix of $A + LC$. Since

$$C(k)x(k) = C(k)S(k,s)x_0,$$

$x \in l^2(s,\infty;\mathbf{R}^n)$ and $\| x \|_2 \le c \mid x_0 \mid$, which implies (3.4) is exponentially stable. ∎

Proposition 3.5 *(a) Suppose that (C,A) is detectable. Then the system (3.4) is exponentially stable if and only if there exists a bounded nonnegative solution to*

$$X(k) = A'(k)X(k+1)A(k) + C'(k)C(k). \tag{3.6}$$

(b) Suppose (A,B) is stabilizable. The system (3.4) is exponentially stable if and only if there exists a bounded nonnegative solution to

$$Y(k+1) = A(k)Y(k)A'(k) + B(k)B'(k), \quad Y(k_0) = 0. \tag{3.7}$$

Proof. We shall show (a) only. If A is exponentially stable,

$$X(k) = \sum_{s=k}^{\infty} S'(s,k)C'(s)C(s)S(s,k)$$

is a bounded nonnegative solution of (3.6). Conversely, let $X(k)$ be a bounded nonnegative solution of (3.6) and $x(k) = S(k,s)x_0$. Then

$$x'(k+1)X(k+1)x(k+1) - x'(k)X(k)x(k) = - \mid C(k)x(k) \mid^2$$

and

$$x'(N+1)X(N+1)x(N+1) + \sum_{k=s}^{N} \mid C(k)x(k) \mid^2 = x_0'X(s)x_0.$$

Hence $C(k)S(k,s)x_0 \in l^2(s,\infty;\mathbf{R}^{p_2})$ with $\| C(k)S(k,s)x_0 \|_2 \le c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. As in the last part of the proof of Proposition 3.4, we can show $x \in l^2(s,\infty;\mathbf{R}^n)$ with $\| x \|_2 \le c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. $Y(k)$ given by

$$Y(k) = \sum_{s=k_0}^{k-1} S(k,s+1)B(s)B'(s)S'(k,s+1)$$

is a bounded nonnegative solution of (3.7). ∎

If the system is time-invariant, the equation (3.6) is reduced to

$$X = A'XA + C'C \tag{3.8}$$

and its solution is called the observability gramian. The equation (3.7) is reduced to

$$Y = AYA' + BB' \tag{3.9}$$

and Y is called the controllability gramian.

Remark 3.1 Proposition 3.2 is a special case of Proposition 3.5 (a) since (I,A) is detectable.

3.1.2 Performance Measures of Stable Systems

Consider the system $\mathbf{G}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \\ z(k) &= C(k)x(k) + D(k)w(k) \end{aligned} \tag{3.10}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $z \in \mathbf{R}^{p_1}$, A, B, C, D are bounded matrices of appropriate dimensions and A is exponentially stable. First we assume that the system is time-invariant and recall the following definitions.

Definition 3.7 *The H_2-norm of the system* $\mathbf{G}$, *denoted by* $\| G \|_2$ *is*

$$\begin{aligned} \| G \|_2 &= \left(\sum_{i=1}^{m_1} [\sum_{k=1}^{\infty} \mid CA^{k-1}Be_i \mid^2 + \mid De_i \mid^2] \right)^{\frac{1}{2}} \\ &= \left(tr.[\sum_{k=1}^{\infty} B'(A')^{k-1}C'CA^{k-1}B + D'D] \right)^{\frac{1}{2}} \end{aligned}$$

where (e_i) *are unit vectors in* $\mathbf{R}^{m_1}$.

$\| G \|_2$ can be regarded as the total energy of impulse responses. Let $G(z)$ be the transfer function of the system so that $G(z) = C(zI - A)^{-1}B + D$. Then via Fourier transform we have

$$\| G \|_2 = \left[\frac{1}{2\pi} \int_0^{2\pi} tr.[G^*(e^{j\theta})G(e^{j\theta})]d\theta \right]^{\frac{1}{2}} \tag{3.11}$$

where $G^*(\cdot)$ is the Hermitian transpose of $G(\cdot)$. We also have the following.

Lemma 3.1

$$\| G \|_2^2 = tr.[B'XB + D'D] = tr.[CYC' + DD']$$

where X , Y are observability- and controllability gramians respectively of the system given by (3.8) and (3.9).

Definition 3.8 *The H_∞-norm of the system* **G**, *denoted by $\| G \|_\infty$ is given by*

$$\| G \|_\infty = \sup_{0 \neq w \in l^2} \frac{\| z \|_2}{\| w \|_2}.$$

$\| G \|_\infty$ is the supremum of the ratio of the energies of the output and input. As is known

$$\| G \|_\infty = \sup_\theta \sigma[G(e^{j\theta})] \tag{3.12}$$

where $\sigma(M)$ is the maximum singular value of the matrix M. The H_2- and H_∞-norms of transfer functions $G(z)$ are defined by (3.11) and (3.12).

The following result is known as the Bounded Real Lemma.

Lemma 3.2 *The following statements are equivalent.*
(a) $\| G \|_\infty < \gamma$.
(b) There exists a nonnegative solution X to

$$\begin{aligned} T_1 &> 0, \\ X &= A'XA + C'C + R_1'T_1^{-1}R_1 \end{aligned}$$

such that $A + BT_1^{-1}R_1$ is exponentially stable where $T_1 = \gamma^2 I - D'D - B'XB$ and $R_1 = B'XA + D'C$.
(c) There exists a nonnegative solution Y to

$$\begin{aligned} T_{1Y} &> 0, \\ Y &= AYA' + BB' + R_{1Y}'T_{1Y}^{-1}R_{1Y} \end{aligned}$$

such that $A + R_{1Y}'T_{1Y}^{-1}C$ is exponentially stable where $T_{1Y} = \gamma^2 I - DD' - CYC'$ and $R_{1Y} = CYA' + DB'$.

Now we generalize Definitions 3.7 and 3.8 to time-varying systems.

Definition 3.9 *The H_2-norm of the system* **G** *on* $[s,\infty)$ *is defined by*

$$\begin{aligned}\| G \|_{2,k_0}^2 &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{m_1}[\sum_{s=k_0}^{k_0+N-1}\sum_{k=s+1}^{\infty}\{| C(k)S(k,s+1)B(s)e_i |^2 \\ &\qquad + | D(s)e_i |^2\}] \\ &= \lim_{N\to\infty}\frac{1}{N}tr.[\sum_{s=k_0}^{k_0+N-1}\{B'(s)\sum_{k=s+1}^{\infty}S'(k,s+1)(C'C)(k) \\ &\qquad \times S(k,s+1)B(s) + (D'D)(s)\}].\end{aligned}$$

For θ-periodic systems

$$\begin{aligned}\| G \|_{2,\theta}^2 &= \frac{1}{\theta}\sum_{s=k_0}^{k_0+\theta-1} tr.[B'(s)\sum_{k=s+1}^{\infty}S'(k,s+1)(C'C)(k) \\ &\qquad \times S(k,s+1)B(s) + (D'D)(s)].\end{aligned}$$

Note that two norms above coincide for θ-periodic systems.

Remark 3.2 Note that

$$\begin{aligned}\| G \|_{2,k_0}^2 &= \lim_{N\to\infty}\frac{1}{N}tr.[\sum_{s=k_0}^{k_0+N-1}\{B'(s)\sum_{k=s+1}^{k_0+N}S'(k,s+1)(C'C)(k) \\ &\qquad \times S(k,s+1)B(s) + (D'D)(s)\}.\end{aligned}$$

and

$$\begin{aligned}\| G \|_{2,k_0}^2 &= \lim_{N\to\infty}\frac{1}{N}tr.[\sum_{s=k_0}^{k_0+N-1}\{\sum_{k=s+1}^{k_0+N}C(k)S(k,s+1)(BB')(s) \\ &\qquad \times S'(k,s+1)C'(k) + (DD')(s)\}] \\ &= \lim_{N\to\infty}\frac{1}{N}tr.[\sum_{k=k_0+1}^{k_0+N}\{C(k)\sum_{s=k_0}^{k-1}S(k,s+1)(BB')(s) \\ &\qquad \times S'(k,s+1)C'(k) + (DD')(k)\}]\end{aligned}$$

where we have used the property of the trace and

$$\sum_{s=k_0}^{k_0+N-1}\sum_{k=s+1}^{k_0+N} = \sum_{k=k_0+1}^{k_0+N}\sum_{s=k_0}^{k-1}.$$

From the last equality, $\| G \|_{2,k_0}$ is equal to the H_2-norm of the backward system $\mathbf{G}^*$

$$\begin{aligned} \tilde{x}(k) &= A'(k)\tilde{x}(k+1) + C'(k)\tilde{w}(k), \\ \tilde{z}(k) &= B'(k)\tilde{x}(k+1) + D'(k)\tilde{w}(k). \end{aligned} \tag{3.13}$$

Let $\tilde{z}(k; s, i)$ be the impulse response of (3.13) with $\tilde{w}(s) = e_i$ where (e_i) are unit vectors in $\mathbf{R}^{p_1}$. Then

$$\tilde{z}(k; s, i) = \begin{cases} D'(s)e_i, & k = s, \\ B'(k)S'(s, k+1)C'(s)e_i, & k < s, \\ 0, & k > s. \end{cases}$$

Definition 3.10 *The H_2-norm of the backward system $\mathbf{G}^*$ is defined by*

$$\| G^* \|_2^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{p_1} \{ \sum_{k=k_0}^{k_0+N-1} \sum_{s=k_0}^{\infty} | \tilde{z}(k; s, i) |^2 \}.$$

Then clearly

$$\begin{aligned} \| G^* \|_2^2 &= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{p_1} \sum_{k=k_0+1}^{k_0+N} [\sum_{s=k_0}^{k-1} | B'(s)S'(s, k+1)C'(k)e_i |^2 \\ &\qquad + | D'(k)e_i |^2] \end{aligned}$$

and $\| G^* \|_{2,k_0} = \| G \|_{2,k_0}$.

Lemma 3.3

$$\begin{aligned} \| G \|_{2,k_0}^2 &= \lim_{N \to \infty} \frac{1}{N} tr.[\sum_{s=k_0}^{k_0+N-1} B'(s)X(s+1)B(s) + (D'D)(s)] \\ &= \lim_{N \to \infty} \frac{1}{N} tr.[\sum_{s=k_0+1}^{k_0+N} (CYC' + DD')(s)] \end{aligned}$$

where X and Y are the observability and controllability gramians of the system (3.10) given by (3.6) and (3.7), respectively. Moreover, for θ-periodic systems X is θ-periodic and

$$\begin{aligned} \| G \|_{2,\theta}^2 &= \frac{1}{\theta} \sum_{s=k_0}^{k_0+\theta-1} tr.[B'(s)X(s+1)B(s) + (D'D)(s)] \\ &= \frac{1}{\theta} \sum_{s=k_0}^{k_0+\theta-1} tr.(CY_\theta C' + DD')(s) \end{aligned}$$

where Y_θ is the θ-periodic solution of (3.7).

Definition 3.11 *The H_∞-norm of the system* $\mathbf{G}$ *is that of the map* $w \to z$: $l^2(k_0, \infty; \mathbf{R}^{m_1}) \to l^2(k_0, \infty; \mathbf{R}^{p_1})$.

To generalize the bounded real lemma we need to consider a quadratic optimization problem. But we first introduce the standard quadratic control problems.

3.1.3 Quadratic Control

Consider the system

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad x(k_0) = x_0$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$ and A, B are bounded matrices of compatible dimension. For this system we introduce the functional

$$J_N(u; k_0, x_0) = \sum_{k=k_0}^{N} [| C(k)x(k) |^2 + | u(k) |^2] + | Fx(N+1) |^2$$

which is minimized where $F \in \mathbf{R}^{q \times n}$ and $C \in \mathbf{R}^{p_2 \times n}$ are unifomly bounded.

We need the following Riccati equation

$$\begin{aligned} X(k) &= A'(k)X(k+1)A(k) + C'(k)C(k) \\ &\quad -(R_2 T_2^{-1} R_2)(k), \qquad (3.14) \\ X(N+1) &= F'F \qquad (3.15) \end{aligned}$$

where $T_2(k) = I + B'(k)X(k+1)B(k)$ and $R_2(k) = B'(k)X(k+1)A(k)$.

Theorem 3.1 *There exists a unique nonnegative solution* $X = X_N(k)$ *to the Riccati eqaution (3.14) and (3.15). Moreover, the state feedback law*

$$\bar{u}(\cdot) = -(T_2^{-1} R_2)(\cdot)x(\cdot)$$

is optimal and

$$J_N(\bar{u}; k_0, x_0) = x_0' X(k_0) x_0.$$

We omit the proof of this theorem. Instead we shall give a proof for a similar problem (3.39). See Lemma 3.8.

Now consider the infinite horizon problem

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(s) = x_0, \quad s \geq k_0, \\ J(u; s, x_0) &= \sum_{k=s}^{\infty} [| C(k)x(k) |^2 + | u(k) |^2] \end{aligned}$$

where $u \in l^2(s,\infty;\mathbf{R}^{m_2})$ is admissible if its response $x \in l^2(s,\infty;\mathbf{R}^n)$ and $\lim_{k\to\infty} x(k) = 0$. As in the continuous-time case we assume the following condition.

RD: We assume that for each (s, x_0) there exists a control $u(\cdot; x_0)$ such that $J(u(\cdot, x_0); s, x_0) \leq c(x_0)$ for some constant c independent of s.

If (A, B) is stabilizable, then **RD** holds.

Lemma 3.4 *Assume that* **RD** *holds. Then there exists a bounded nonnegative solution to the Riccati equation (3.14).*

Proof. By Theorem 3.1 there exists a nonnegative solution to (3.14) on $[k_0, N+1]$ with $X(N+1) = 0$. Then for any $s \geq k_0$, $X_N(s) \leq X_{\bar{N}}(s)$ if $s \leq N \leq \bar{N}$. In fact let

$$\bar{u}_N(\cdot) = -(T_2^{-1}R_2)(\cdot)x(\cdot)$$

then

$$\begin{aligned} x_0'X_N(s)x_0 &= J_N(\bar{u}_N; s, x_0) \\ &\leq J_N(\tilde{u}_{\bar{N}}; s, x_0) \\ &\leq J_{\bar{N}}(\bar{u}_{\bar{N}}; s, x_0) = x_0'X_{\bar{N}}(s)x_0 \end{aligned}$$

where we set $F = 0$ in J_N and $\tilde{u}_{\bar{N}}$ in J_N is the restriction of the feedback law $\bar{u}_{\bar{N}}(\cdot)$ to $[s, N]$. We note that

$$\begin{aligned} x_0'X_N(s)x_0 &= J_N(\bar{u}_N; s, x_0) \\ &\leq J_N(u(\cdot; x_0); s, x_0) \\ &\leq J(u(\cdot; x_0); s, x_0) < \infty. \end{aligned}$$

Hence $x_0'X_N(s)x_0$ is monotone increasing and uniformly bounded in s and N. Since x_0 is arbitrary, there exists a bounded nonnegative matrix X such that

$$X_N(s) \to X(s) \text{ for any } s.$$

Then X satisfies the Riccati equation (3.14). ∎

Lemma 3.5 *Suppose (C, A) is detectable. Then $A - BT_2^{-1}R_2$ is exponentially stable.*

Proof. The Riccati equation (3.14) can be written as

$$X(k) = (A - BT_2^{-1}R_2)'X(k+1)(A - BT_2^{-1}R_2) + \begin{bmatrix} C \\ T_2^{-1}R_2 \end{bmatrix}' \begin{bmatrix} C \\ T_2^{-1}R_2 \end{bmatrix}.$$

Hence, if x is the solution of the state feedback system

$$x(k+1) = (A - BT_2^{-1}R_2)(k)x(k), \quad x(s) = x_0$$

then as in the proof of Lemma 2.6 we can show

$$\begin{bmatrix} C \\ T_2^{-1}R_2 \end{bmatrix} x \in l^2(s,\infty;\mathbf{R}^{p_2+m_2})$$

and that

$$\| \begin{bmatrix} C \\ T_2^{-1}R_2 \end{bmatrix} x \|_2 \leq c \mid x_0 \mid \text{ for some } c > 0.$$

Since (C, A) is detectable, it is easy to see that $(\begin{bmatrix} C \\ T_2^{-1}R_2 \end{bmatrix}, A - BT_2^{-1}R_2)$ is also detectable. Hence by Proposition 3.5, $A - BT_2^{-1}R_2$ is exponentially stable. ∎

We say that X is a stabilizing solution of the Riccati equation (3.14) if $A - BT_2^{-1}R_2$ is exponentially stable.

Theorem 3.2 *Suppose (C, A) is detectable and that* **RD** *holds. Then there exists a bounded nonnegative stabilizing solution of (3.14). Moreover the feedback law*

$$\bar{u}(\cdot) = -(T_2^{-1}R_2)(\cdot)x(\cdot)$$

is optimal and

$$J(\bar{u}; s, x_0) = x_0'X(s)x_0. \tag{3.16}$$

If A, B and C are θ-periodic, then X is also θ-periodic.

Proof. The first part follows from Lemmas 3.4 and 3.5. Since

$$\begin{aligned} & x'(k+1)X(k+1)x(k+1) - x'(k)X(k)x(k) \\ = \quad & -(\mid C(k)x(k) \mid^2 + \mid u(k) \mid^2) + \mid T_2^{\frac{1}{2}}(k)[u(k) + (T_2^{-1}R_2)(k)x(k)] \mid^2, \end{aligned}$$

we obtain

$$\begin{aligned} & x'(N+1)X(N+1)x(N+1) + J_N(u; s, x_0) \\ & \quad = x_0'X(s)x_0 + \sum_{k=s}^{N} \mid T_2^{\frac{1}{2}}(k)[u(k) + (T_2^{-1}R_2)(k)x(k)] \mid^2 \end{aligned}$$

where u is an admissible control and x is its response. Since

$$x'(N+1)X(N+1)x(N+1) \to 0 \text{ as } N \to \infty,$$

we obtain

$$J(u; s, x_0) = x_0'X(s)x_0 + \sum_{k=s}^{\infty} \mid T_2^{\frac{1}{2}}(k)[u(k) + (T_2^{-1}R_2)(k)x(k)] \mid^2 .$$

Hence the optimality of $\bar{u}$ and (3.16) follow immediately.

By Lemma 3.4, the bounded nonnegative stabilizing solution X of (3.14) is constructed as $\lim_{N\to\infty} X_N(k)$ where $X_N(k)$ is the solution of (3.14) with $X_N(N+1) = 0$. If A, B and C are θ-periodic, $X_N(k+\theta) = X_{N-\theta}(k)$. Hence

$$X(k+\theta) = \lim_{N\to\infty} X_N(k+\theta) = \lim_{N\to\infty} X_{N-\theta}(k) = X(k). \quad \blacksquare$$

Corollary 3.2 *(A, B) is stabilizable if and only if there exists a control $u(\cdot; s, x_0)$ for each s and x_0 such that*

$$\| x \|_2^2 + \| u \|_2^2 \leq c(x_0)$$

for some constant $c(x_0)$.

Proof. We only need to show sufficiency. Consider the regulator problem with $C = I$. By Theorem 3.2 $A - BT_2^{-1}R_2$ is exponentially stable where X is the bounded nonnegative solution of the Riccati equation (3.14) with $C = I$. $\blacksquare$

Consider the backward system

$$\xi(k) = A'(k)\xi(k+1) + C'(k)v(k), \;\; \xi(N+1) = \xi_1$$

and the functional

$$J(u; N+1, \xi_1) = \sum_{k=k_0}^{N} [\| B'(k)\xi(k+1) |^2 + | v(k) |^2] + | H'\xi(k_0) |^2$$

which is minimized. As in Theorem 3.1 we consider

$$\begin{aligned} Y(k+1) &= A(k)Y(k)A'(k) + B(k)B'(k) - (R_{2Y}'T_{2Y}^{-1}R_{2Y})(k), &(3.17)\\ Y(k_0) &= HH' &(3.18) \end{aligned}$$

where $T_{2Y}(k) = I + C(k)Y(k)C'(k)$ and $R_{2Y}(k) = C(k)Y(k)A'(k)$. Then similarly to Theorem 3.2 and Corollary 3.2 we have the following result.

Theorem 3.3 *(a) There exists a nonnegative solution of the Riccati equation (3.17) and (3.18) on any $[k_0, N+1]$.*
(b) Let $H = 0$ and suppose there exists a control $v(\cdot; N+1, \xi_1)$ such that

$$\| B'\xi \|_{l^2(k_0,N+1;\mathbf{R}^{m_2})}^2 + \| v \|_{l^2(k_0,N;\mathbf{R}^{p_2})}^2 \leq c(\xi_1)$$

for some constant $c(\xi_1)$. Then the solution of the Riccati equation (3.17) with $Y(k_0) = 0$ is bounded. If, further, (A, B) is stabilizable, then $A - R_{2Y}'T_{2Y}^{-1}C$ is exponentially stable.
(c) (C, A) is detectable if and only if there exists a control $v(\cdot; N+1, \xi_1)$ such that

$$\| \xi \|_{l^2(k_0,N+1;\mathbf{R}^n)}^2 + \| v \|_{l^2(k_0,N;\mathbf{R}^{p_2})}^2 \leq c(\xi_1)$$

for some constant $c(\xi_1)$.

We say that a bounded nonnegative solution Y of the Riccati equation (3.17) is stabilizing if $A - R'_{2Y}T_{2Y}^{-1}C$ is exponentially stable.

Corollary 3.3 *Let A, B and C be θ-periodic. Let Y be a bounded nonnegative stabilizing solution Y of (3.17) with $Y(k_0) = 0$. Then $\lim_{n\to\infty} Y(k+n\theta)$ exists (denoted by Y_θ) and Y_θ is a θ-periodic nonnegative stabilizing solution of (3.17).*

Proof. It is enough to show that $Y(k+n\theta)$ is monotone increasing in n. Let $Y(k;Y(k_0))$ be the solution of (3.17) with initial condition $Y(k_0) \geq 0$. Then $Y(k) = Y(k;0)$. Since A, B and C are θ-periodic, we have $Y(k) = Y(k-n\theta, Y(n\theta))$ for $n\theta \leq k < (n+1)\theta$. Hence

$$Y(k+2\theta) = Y(k+\theta, \bar{Y}(\theta)) \geq Y(k+\theta, 0) = Y(k+\theta).$$

Similarly, we have

$$Y(k+(n+1)\theta) \geq Y(k+n\theta)$$

and $Y(k+n\theta)$ is monotone increasing in n. Since Y is bounded, there exists a limit $Y_\theta(k)$ of $Y(k+n\theta)$ as $n \to \infty$. Note that

$$Y_\theta(k) = \lim_{n\to\infty} Y(k+n\theta) = \lim_{n\to\infty} Y(k+\theta+(n-1)\theta) = Y_\theta(k+\theta).$$

Hence $Y_\theta(k)$ is θ-periodic. Since

$$Y(k+1+n\theta) = A(k)Y(k+n\theta)A'(k) + B(k)B'(k) + (R'_{2Y}T_{2Y}^{-1}R_{2Y})(k+n\theta),$$

taking the limit $n \to \infty$ on the both side, $Y_\theta(k)$ satisfies (3.17).

Next we shall show the stabilizing property of Y_θ. Let $k_0 < N < \infty$ be arbitrary but fixed. Let x_θ be solution of

$$x(k+1) = (A - R_{2Y_\theta}T_{2Y_\theta}^{-1}C)(k)x(k), \; x(k_0) = x_0. \tag{3.19}$$

Consider

$$x(k+1) = (A - R_{2Y}T_{2Y}^{-1}C)(k)x(k), \; x(k_0) = x_0.$$

and denote by $x_n(k)$ the solution at $k+n\theta$. Then

$$\begin{aligned} x_n(k+1) &= x(k+n\theta+1) \\ &= (A - R_{2Y}T_{2Y}^{-1}C)(k+n\theta)x(k+n\theta) \\ &= [A(k) - (R_{2Y}T_{2Y}^{-1})(k+n\theta)C(k)]x_n(k) \end{aligned}$$

and we have

$$\lim_{n\to\infty} x_n(k) = x_\theta(k), \; k \in [k_0, N].$$

Since Y is stabilizing

$$\sum_{k=k_0}^{N} | x_n(k) |^2 \, dt \leq c \,| x_0 |^2 \text{ for any } n$$

where $c > 0$ is a constant independent of N. Hence we obtain

$$\sum_{k=k_0}^{N} \mid x_\theta(k) \mid^2 dt = \lim_{n\to\infty} \sum_{k=k_0}^{N} \mid x_n(k) \mid^2 dt \leq c \mid x_0 \mid^2 .$$

Since N is arbitrary, the system (3.19) is exponentially stable.

Suppose $Y(k; H'H)$ is a bounded nonnegative stabilizing solution of (3.17). Then $\lim_{n\to\infty} Y(k+n\theta; H'H) = Y_\theta(k)$ by Theorem 3.4 below. ∎

As in the continuous-time case we have the following property for the stabilizing solutions.

Theorem 3.4 *(a) A bounded stabilizing solution of (3.14), if one exists, is unique.*
(b) Let Y and $\bar{Y}$ be two stabilizing solutions of (3.17). Then

$$Y(k) - \bar{Y}(k) \to 0 \text{ as } k \to \infty.$$

Proof. (a) Let X and $\bar{X}$ be two stabilizing solutions of (3.14). Then by direct calculation we have

$$(A - BT_2^{-1}R_2)'(k)(X - \bar{X})(k+1)(A - B\bar{T}_2^{-1}\bar{R}_2)(k) = X(k) - \bar{X}(k)$$

where $\bar{T}_2(k) = I + B'(k)\bar{X}(k+1)B(k)$ and $\bar{R}_2(k) = B'(k)\bar{X}(k+1)A(k)$. Hence

$$X(k) - \bar{X}(k) = S'_X(N,k)(X - \bar{X})(N)S_{\bar{X}}(N,k)$$

where S_X and $S_{\bar{X}}$ are the state transition matrices of $A - BT_2^{-1}R_2$ and $A - B\bar{T}_2^{-1}\bar{R}_2$, respectively. Hence

$$\mid X(k) - \bar{X}(k) \mid \leq M_1\alpha_1^{N-k}cM_2\alpha_2^{N-k}$$

for some constants $M_i > 0$, $0 < \alpha_i < 1$, $i = 1,2$ and $c > 0$. Letting $N \to \infty$ we obtain $X(k) - \bar{X}(k) = 0$, $^\forall k \geq k_0$.
(b) Since

$$Y(k+1) - \bar{Y}(k+1) = (A - R'_{2Y}T_{2Y}^{-1}C)(k)(Y - \bar{Y})(k)(A - \bar{R}'_{2Y}\bar{T}_{2Y}^{-1}C)'(k)$$

we have

$$Y(k) - \bar{Y}(k) = S_Y(k,k_0)(Y - \bar{Y})(k_0)S'_{\bar{Y}}(k,k_0)$$

where $\bar{T}_{2Y}(k) = I + C(k)\bar{Y}(k)C'(k)$, $\bar{R}_{2Y}(k) = C(k)\bar{Y}(k)A'(k)$ and S_Y and $S_{\bar{Y}}$ are the state transition matrices of $A - R'_{2Y}T_{2Y}^{-1}C$ and $A - \bar{R}'_{2Y}\bar{T}_{2Y}^{-1}C$, respectively. Hence $Y(k) - \bar{Y}(k) \to 0$ as $k \to \infty$, since $A - R'_{2Y}T_{2Y}^{-1}C$ and $A - \bar{R}'_{2Y}\bar{T}_{2Y}^{-1}C$ are exponentially stable. ∎

Consider the system $\mathbf{G}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \ x(k_0) = x_0, \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned}$$

and the controller $u = Ky$ of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \ \hat{x}(k_0) = 0, \\ u(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k). \end{aligned} \tag{3.20}$$

Then the closed-loop system $\mathbf{G}$ and $u = Ky$ is given by

$$\begin{aligned} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k+1) &= \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k) + \begin{bmatrix} B_1 + B_2\hat{D}D_{21} \\ \hat{B}D_{21} \end{bmatrix} w(k), \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k_0) &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ z(k) &= \begin{bmatrix} C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k) + D_{12}\hat{D}D_{21}w(k). \end{aligned} \tag{3.21}$$

Definition 3.12 *Consider the system* $\mathbf{G}$ *on* $[k_0, \infty)$. *A controller* $u = Ky$ *of the form (3.20) is said to be IO-stabilizing if the closed-loop system (3.21) is IO-stable. If, further, the closed-loop system is exponentially stable (or*

$$\begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}$$

is exponentially stable) then the controller is said to be internally stabilizing.

Proposition 3.6 *Consider the system* $\mathbf{G}$ *and the controller* $u = Ky$ *of the form (3.20). If the controller is internally stabilizing, then* (A, B_2, C_2) *and* $(\hat{A}, \hat{B}, \hat{C})$ *are stabilizable and detectable.*

Proof. Let $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k)$ be the solution of

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k+1) = \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k), \quad \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k_0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \tag{3.22}$$

Then by assumption $x, \hat{x} \in l^2$. Rewriting (3.22) as

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_2(k)(\hat{D}C_2x + B_2\hat{C}\hat{x})(k), \ x(k_0) = x_0, \\ \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)(C_2x)(k), \ \hat{x}(k_0) = 0 \end{aligned}$$

and applying Corollary 3.2 we conclude that (A, B_2) and $(\hat{A}, \hat{B})$ are stabilizable. The detectability of (C_2, A) and $(\hat{C}, \hat{A})$ follows from the adjoint system of (3.22) and Theorem 3.3. ∎

3.1.4 Disturbance Attenuation Problems

Consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \\ z(k) &= C(k)x(k) + D(k)w(k), \\ z_1 &= Fx(N+1) \end{aligned} \tag{3.23, 3.24}$$

with initial condition

$$x(k_0) = Hh \tag{3.25}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $z \in \mathbf{R}^{p_1}$, $z_1 \in \mathbf{R}^q$, $h \in \mathbf{R}^{n_1}$, $H \in \mathbf{R}^{n\times n_1}$, $F \in \mathbf{R}^{q\times n}$ and other matrices are uniformly bounded of compatilbe dimensions. For each input $(h, w) \in \mathbf{R}^{n_1} \times l^2(k_0, N; \mathbf{R}^{m_1})$ we have the output $(z_1, z) \in \mathbf{R}^q \times l^2(k_0, N; \mathbf{R}^{p_1})$. Thus we can define the input-output operator G_{Nk_0} of the system (3.23)-(3.25)

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G_{Nk_0}\begin{pmatrix} h \\ w \end{pmatrix} = \begin{pmatrix} G_{1Nk_0}\begin{pmatrix} h \\ w \end{pmatrix} \\ G_{2Nk_0}\begin{pmatrix} h \\ w \end{pmatrix} \end{pmatrix} \tag{3.26}$$

where

$$\begin{aligned} G_{1Nk_0}\begin{pmatrix} h \\ w \end{pmatrix} &= FS(N+1, k_0)Hh + F\sum_{j=k_0}^{N} S(N+1, j+1)B(j)w(j), \\ G_{2Nk_0}\begin{pmatrix} h \\ w \end{pmatrix}(k) &= C(k)S(k, k_0)Hh + C(k)\sum_{j=k_0}^{k-1} S(k, j+1)B(j)w(j) \\ &\quad + D(k)w(k). \end{aligned}$$

Then $G_{Nk_0} \in \mathcal{L}(\mathbf{R}^{n_1} \times l^2(k_0, N; \mathbf{R}^{m_1}); \mathbf{R}^q \times l^2(k_0, N; \mathbf{R}^{p_1}))$. We regard (h, w) as the disturbance and for a given $\gamma > 0$ we wish to find necessary and sufficient conditions for $\| G_{Nk_0} \| < \gamma$, i.e.,

$$| z_1 |^2 + \| z \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2), \text{ for some } 0 < d < \gamma. \tag{3.27}$$

If (3.27) holds, the system **G** is said to fulfil the γ-disturbance attenuation.

The adjoint $G^*_{Nk_0}$ of G_{Nk_0} is given by

$$G^*_{Nk_0}\begin{pmatrix} f \\ v \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \zeta \end{pmatrix}, \quad (f, v) \in \mathbf{R}^q \times l^2(k_0, N; \mathbf{R}^{p_1}) \tag{3.28}$$

where

$$\begin{aligned} \xi(k) &= A'(k)\xi(k+1) + C'(k)v(k), \\ \zeta(k) &= B'(k)\xi(k+1) + D'(k)v(k), \\ \xi(N+1) &= F'f, \\ \zeta_0 &= H'\xi(k_0). \end{aligned} \tag{3.29}$$

Since $\| G^*_{Nk_0} \| = \| G_{Nk_0} \|$ by Theorem A.2, (3.27) is equivalent to

$$| \zeta_0 |^2 + \| \zeta \|_2^2 \leq d^2(| f |^2 + \| v \|_2^2). \tag{3.30}$$

To give necessary and sufficient conditions for $\| G_{Nk_0} \| < \gamma$, we need the Riccati equations

$$\begin{aligned} T_1(k) &> aI \text{ for some } a > 0, && (3.31)\\ X(k) &= A'X(k+1)A + C'C + (R_1'T_1^{-1}R_1)(k), && (3.32)\\ X(N+1) &= F'F, && (3.33)\\ H'X(k_0)H &\leq d^2 I \text{ for some } 0 < d < \gamma && (3.34) \end{aligned}$$

and

$$\begin{aligned} T_{1Y}(k) &> aI \text{ for some } a > 0, && (3.35)\\ Y(k+1) &= AY(k)A' + BB' + (R_{1Y}'T_{1Y}^{-1}R_{1Y})(k), && (3.36)\\ Y(k_0) &= HH', && (3.37)\\ FY(N+1)F' &\leq d^2 I \text{ for some } 0 < d < \gamma && (3.38) \end{aligned}$$

where

$$\begin{aligned} T_1(k) &= \gamma^2 I - D'D - B'X(k+1)B, \quad && R_1(k) = D'C + B'X(k+1)A,\\ T_{1Y}(k) &= \gamma^2 I - DD' - CY(k)C', && R_{1Y}(k) = DB' + CY(k)A' \end{aligned}$$

and for simplicity we have omitted k in all matrices of (3.23).

To give the solution of this problem, we introduce the following functional

$$J(w; k_0, x_0) = \sum_{k=k_0}^{N} [| z(k) |^2 - \gamma^2 | w(k) |^2] + | Fx(N+1) |^2 \tag{3.39}$$

subject to

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \ x(k_0) = x_0,\\ z(k) &= C(k)x(k) + D(k)w(k) \end{aligned}$$

and consider the maximization of $J(w; k_0, x_0)$ over all $w \in l^2(k_0, N; \mathbf{R}^{m_1})$. Let

$$\begin{aligned} \bar{G}_{Nk_0} w &= G_{Nk_0} \begin{pmatrix} 0 \\ w \end{pmatrix},\\ \bar{G}_{iNk_0} w &= G_{iNk_0} \begin{pmatrix} 0 \\ w \end{pmatrix}, \ i = 1, 2. \end{aligned}$$

Lemma 3.6

$\| \bar{G}_{2Ls} \| \leq \| \bar{G}_{2Nk_0} \|$, $\| \bar{G}_{Ns} \| \leq \| \bar{G}_{Nk_0} \|$, *for any* $0 \leq k_0 \leq s \leq L \leq N$.

Proof. We shall show only the first inequality. Let $\tilde{w}$ be the extension of $w \in l^2(s, L; \mathbf{R}^{m_1})$ to $[k_0, N]$ by zero, i.e,

$$\tilde{w}(k) = \begin{cases} k = k_0, .., s-1, & \tilde{w}(k) = 0, \\ k = s, .., L, & \tilde{w}(k) = w(k), \\ k = L+1, .., N, & \tilde{w}(k) = 0. \end{cases}$$

Then we have

$$\begin{aligned} \| \bar{G}_{2Ls} w \|_2^2 &= \sum_{k=s}^{L} | C(k) \sum_{j=i}^{k-1} S(k, j+1) B(j) w(j) + D(k) w(k) |^2 \\ &\leq \sum_{k=k_0}^{N} | C(k) \sum_{j=i}^{k-1} S(k, j+1) B(j) \tilde{w}(j) + D(k) \tilde{w}(k) |^2 \\ &= \| \bar{G}_{2Nk_0} \tilde{w} \|_2^2 \\ &\leq \| \bar{G}_{2Nk_0} \|^2 \| \tilde{w} \|_2^2 = \| \bar{G}_{2Nk_0} \|^2 \| w \|_2^2 . \end{aligned}$$

∎

Consider the maximization problem (3.23)-(3.25) and (3.39) with k_0, N replaced by arbitrary s, L, $k_0 \leq s \leq L \leq N$.

Lemma 3.7 *Assume* $\| \bar{G}_{Nk_0} \| < \gamma$. *Then for any* $k_0 \leq s \leq N$, $J(w; s, x_0)$ *is strictly concave in* w *and there exists a unique optimal maximizing element* $w_{Ns} \in l^2(s, N; \mathbf{R}^{m_1})$. *Moreover*

$$\| w_{Ns} \|_2 \leq \delta \mid x_0 \mid, \quad J(w_{Ns}; s, x_0) \leq \delta \mid x_0 \mid^2$$

for some $\delta = \delta(\gamma) > 0$ *independent of* s *and* x_0.

Proof. By Lemma 3.6, $\| \bar{G}_{Ns} \| < \gamma$ for any $k_0 \leq s \leq N$. Hence $\gamma^2 I - \bar{G}^*_{Ns} \bar{G}_{Ns} > aI$ for some $a > 0$ and the quadratic functional $J(w; s, x_0)$ is strictly concave and $J(w; s, x_0) \to -\infty$ as $\| w \|_2 \to \infty$. Then there exists a unique optimal w_{Ns} for $J(w; s, x_0)$ which is given by

$$(\gamma^2 I - \bar{G}^*_{Ns} \bar{G}_{Ns}) w = \bar{G}^*_{Ns} z_0, \quad z_0(k) = \begin{pmatrix} FS(N+1, s) x_0 \\ C(k) S(k, s) x_0 \end{pmatrix}.$$

Hence

$$w_{Ns} = (\gamma^2 I - \bar{G}^*_{Ns} \bar{G}_{Ns})^{-1} \bar{G}^*_{Ns} z_0.$$

Thus we have

$$\| w_{Ns} \|_2 \leq \delta \mid x_0 \mid$$

for some δ independent of s and x_0. ∎

Lemma 3.8 *Suppose* $\| \bar{G}_{Nk_0} \| < \gamma$. *Then there exists a nonnegative solution* $X(k)$, $k = k_0, .., N+1$ *to (3.31)-(3.33). The optimal control for (3.39) is given by the feedback law*

$$w_{Nk_0}(\cdot) = (T_1^{-1} R_1)(\cdot) x(\cdot)$$

and

$$J(w_{Nk_0}; k_0, x_0) = x_0' X(k_0) x_0.$$

Proof. First we set $k_0 = N$ and $x(N) = 0$. Then $\| \bar{G}_{NN} \| < \gamma$ is equivalent to

$$\begin{aligned} d^2 \mid w(N) \mid^2 &\geq \mid z_1 \mid^2 + \mid z(N) \mid^2 \\ &= \mid FB(N)w(N) \mid^2 + \mid D(N)w(N) \mid^2 \end{aligned}$$

which implies that

$$T_1(N) \geq (\gamma^2 - d^2)I$$

and we can define $X(N)$ by (3.32). Now we assume that (3.31) and (3.32) are true for $k = N, ..., j+1, j \geq k_0$. Then $X(k)$, $k = N+1, .., j+1$ is well-defined. Furthermore we obtain

$$\begin{aligned} \sum_{k=j+1}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2] + \mid z_1 \mid^2 &= x'(j+1)X(j+1)x(j+1) \\ &- \sum_{k=j+1}^{N} \mid [T_1^{\frac{1}{2}}(w - T_1^{-1}R_1 x)](k) \mid^2 . \end{aligned}$$

Now we consider

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \ x(j) = 0, \ k \geq j, \\ z(k) &= C(k)x(k) + D(k)w(k). \end{aligned}$$

Then $\| \bar{G}_{Nj} \| < \gamma$ implies

$$\sum_{k=j}^{N} d^2 \mid w(k) \mid^2 \geq \sum_{k=j}^{N} \mid z(k) \mid^2 + \mid z_1 \mid^2$$

and

$$d^2 \mid w(j) \mid^2 \geq \mid z(j) \mid^2 + \sum_{k=j+1}^{N} [\mid z(k) \mid^2 - d^2 \mid w(k) \mid^2] + \mid z_1 \mid^2$$

for any $w \in l^2(j, N; \mathbf{R}^{m_1})$. Hence we have

$$\begin{aligned} d^2 \mid w(j) \mid^2 &\geq \mid z(j) \mid^2 + \max_{w} \{ \sum_{k=j+1}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(j) \mid^2] + \mid z_1 \mid^2 \} \\ &= \mid z(j) \mid^2 + x'(j+1)X(j+1)x(j+1) \\ &= \mid D(j)w(j) \mid^2 + w'(j)B'(j)X(j+1)B(j)w(j) \end{aligned}$$

which implies

$$T_1(j) \geq (\gamma^2 - d^2)I$$

and we can define $X(j)$. Since

$$J(w; s, x(s)) = x'(s)X(s)x(s) - \sum_{k=s}^{N} \mid [T_1^{\frac{1}{2}}(w - T_1^{-1}R_1 x)](k) \mid^2,$$

we have

$$x'(s)X(s)x(s) = \max_w J(w; s, x(s)) = J(w_{Ns}; s, x(s))$$

where $w_{Ns}(k) = (T_1^{-1}R_1)(k)x(k)$. ∎

We are now ready to give the solution of our original problem.

Theorem 3.5 *The following statements are equivalent.*
(a) $\| G_{Nk_0} \| < \gamma$.
(b) There exists a nonnegative solution to (3.31)-(3.34).
(c) There exists a nonnegative solution to (3.35)-(3.38).

Proof. Suppose (a) holds. Then $\| \bar{G}_{Nk_0} \| < \gamma$ and (b) except (3.34) follows from Lemma 3.8. Moreover for (3.23) and (3.32) the following equality holds:

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &= \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh \\ &\quad - \| T_1^{-\frac{1}{2}}(w - T_1^{-1}R_1 x) \|_2^2 . \end{aligned} \quad (3.40)$$

Setting $w = T_1^{-1}R_1 x$ and using (3.27) we obtain

$$d^2(| h |^2 + \| w \|_2^2) \geq \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh.$$

Hence $d^2 | h |^2 \geq h'H'X(k_0)Hh$ which implies (3.34).
Conversely suppose (b) holds. Then by (3.40)

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &\leq \gamma^2 \| w \|_2^2 + d^2 | h |^2 - \gamma^2 \| r \|_2^2 \\ &\leq \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2) \end{aligned}$$

where $r = T_1^{\frac{1}{2}}(w - T_1^{-1}R_1 x)$. Since there exists $a > 0$ such that

$$| h |^2 + \| w \|_2^2 \leq a(| h |^2 + \| r \|_2^2),$$

we have

$$\begin{aligned} | z_1 |^2 + \| z \|_2^2 &\leq \gamma^2(| h |^2 + \| w \|_2^2) - \frac{\gamma^2 - d^2}{a}(| h |^2 + \| w \|_2^2) \\ &= (\gamma^2 - \frac{\gamma^2 - d^2}{a})(| h |^2 + \| w \|_2^2). \end{aligned}$$

Hence $\| G_{Nk_0} \| < \gamma$. The equivalence of (a) and (c) also follows since (c) is the dual of (b) concerning the adjoint (3.29) of G_{Nk_0}. ∎

If the initial condition is known, we can set $h = 0$.

Corollary 3.4 *The following statements are equivalent.*
(a) $\| \bar{G}_{Nk_0} \| < \gamma$.
(b) There exists a nonnegative solution to (3.31)-(3.33).
(c) There exists a nonnegative solution to (3.35), (3.36) and (3.38) with $Y(k_0) = 0$.

We now consider the system $\mathbf{G}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \\ z(k) &= C(k)x(k) + D(k)w(k), \\ x(k_0) &= Hh \end{aligned}$$

on $[k_0, \infty)$ and assume that this system is exponentially stable. Then we can define the input-output operator $G \in \mathcal{L}(\mathbf{R}^{n_1} \times l^2(k_0, \infty; \mathbf{R}^{m_1}); l^2(k_0, \infty; \mathbf{R}^{p_1}))$ by

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

Again we wish to find the condition for $\| G \| < \gamma$. We replace (3.26) and (3.39) by

$$\begin{aligned} G\begin{pmatrix} h \\ w \end{pmatrix} &= C(k)S(k,k_0)Hh \\ &\quad + C(k)\sum_{k=k_0}^{k-1} S(k,j+1)B(j)w(j) + D(k)w(k), \\ J(w;k_0,x_0) &= \sum_{k=k_0}^{\infty} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2]. \end{aligned}$$

We also need the functional (3.39) with $F = 0$, i.e.,

$$J_N(w;k_0,x_0) = \sum_{k=k_0}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2].$$

Let $\bar{G}w = G\begin{pmatrix} 0 \\ w \end{pmatrix}$. Proceeding as in the finite horizon case we have the following.

Lemma 3.9 $\| \bar{G}_{2Nk_0} \| \leq \| \bar{G} \|$ *for any* $k_0 \leq N < \infty$.

Lemma 3.10 *Assume* $\| G \| < \gamma$. *Then* $J_N(w;k_0,x_0)$ $(J(w;k_0,x_0))$ *is strictly concave and there exists a unique control* w_{Nk_0} (w_{k_0}) *maximizing* $J_N(w;k_0,x_0)$ $(J(w;k_0,x_0)$*, respectively). Moreover*

$$\begin{array}{ll} \| w_{Nk_0} \|_2 \leq \delta \mid x_0 \mid, & \| w_{k_0} \|_2 \leq \delta \mid x_0 \mid, \\ J_N(w_{Nk_0};k_0,x_0) \leq \delta \mid x_0 \mid^2, & J(w_{k_0};k_0,x_0) \leq \delta \mid x_0 \mid^2 \end{array}$$

for some $\delta = \delta(\gamma)$ *independent of* N *and* x_0.

Proof. Since $\| \bar{G} \| \leq \| G \| < \gamma$ and Lemma 3.9, we have $\| \bar{G}_{2Nk_0} \| < \gamma$. Hence from Lemma 3.7, we have

$$w_{Nk_0} = (\gamma^2 I - \bar{G}^*_{2Nk_0}\bar{G}_{2Nk_0})^{-1}\bar{G}^*_{2Nk_0}z_0, \quad z_0 = C(k)S(k,k_0)x_0,$$

and

$$w_{k_0} = (\gamma^2 I - \bar{G}^*\bar{G})^{-1}\bar{G}^* z_0$$

where $\bar{G}^*$ is the adjoint of $\bar{G}$. Since $\bar{G}_{2Nk_0}$ and $(\gamma^2 I - \bar{G}^*_{2Nk_0}\bar{G}_{2Nk_0})^{-1}$ are uniformly bounded in N, we have the assertion. ∎

Definition 3.13 *(a) A bounded nonnegative solution X of (3.32) is called the stabilizing solution if $A + BT_1^{-1}R_1$ is exponentially stable.*
(b) A bounded nonnegative solution Y of (3.36) is called the stabilizing solution if $A + R'_{1Y}T_{1Y}^{-1}C$ is exponentially stable.

As in Theorem 3.4, we have the following property for the stabilizing solutions of the Riccati equations (3.32) and (3.36).

Lemma 3.11 *(a) A bounded stabilizing solution of (3.32), if one exists, is unique.*
(b) Let Y and $\bar{Y}$ be two stabilizing solutions of (3.36). Then $Y(k) - \bar{Y}(k) \to 0$ as $k \to \infty$.

Lemma 3.12 *Suppose $\| G \| < \gamma$. Then there exists a bounded nonnegative stabilizing solution to (3.31) and (3.32). Moreover if the conditions above are satisfied, a unique maximizing element w_{k_0} of $J(w; k_0, x_0)$ exists and is given by the feedback law*

$$w_{k_0}(\cdot) = (T_1^{-1}R_1)(\cdot)x(\cdot)$$

and $J(w_{k_0}; k_0, x_0) = x_0' X(k_0)x_0$.

Proof. Since $\| G \| < \gamma$ implies $\| \bar{G} \| < \gamma$ and $\| \bar{G}_{Nk_0} \| < \gamma$, we have a nonnegative solution $X_N(k)$ to (3.31) and (3.32) with $X_N(N+1) = 0$. Moreover for each k, $X_N(k)$ is monotone increasing in N. In fact let $L < N$ and define a control on $[k_0, N]$ by

$$\tilde{w}_{Nk_0}(k) = \begin{cases} (T_{1L}^{-1}R_{1L})(k)x_L(k), & k \in [k_0, L], \\ 0, & k \in [L+1, N] \end{cases}$$

where $R_1 = R_{1L}$ to denote the denpendency on X_L and x_L is the response to the feedback pair $w_{Lk_0} = T_{1L}^{-1}R_{1L}x_L$ in (3.23). Then

$$\begin{aligned} x_0'X_L(k_0)x_0 = J_L(w_{Lk_0}; k_0, x_0) &\le J_N(\tilde{w}_{Nk_0}; k_0, x_0) \\ &\le J_N(w_{Nk_0}; k_0, x_0) = x_0'X_N(k_0)x_0. \end{aligned}$$

The monotonicity of $X_N(k)$ also follows from $J_N(w; k, x_0)$. Note that $X_N(k_0)$ is bounded uniformly in N. This follows from Lemma 3.10 and

$$J_N(w_{Nk_0}; k_0, x_0) = x_0'X_N(k_0)x_0.$$

Hence $X_N(k_0)$ converges to a limit$X(k_0)$. Changing the initial time, $X_N(k)$, $k \ge k_0$ converges to a limit $X(k)$. As we have seen in the proof of Lemma 3.8,

$T_1(k) \geq (\gamma^2 - d^2)I$ in (3.31) independently of N and hence $T_1(k) \geq (\gamma^2 - d^2)I$ for the limit $X(k)$. So X satisfies (3.31) and (3.32). Now it remains to show that $A + BT_1^{-1}R_1$ is exponentially stable. Let x_N be the response to w_{Nk_0} and let $\tilde{w}_{Nk_0} \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ given by

$$\tilde{w}_{Nk_0}(k) = \begin{cases} (T_{1N}^{-1}R_{1N})(k)x_N(k), & k \in [k_0, N], \\ 0, & k \in [N+1, \infty). \end{cases}$$

Then

$$0 \leq x_0' X_N(k_0) x_0 \leq J(\tilde{w}_{Nk_0}; k_0, x_0) \leq J(w_{k_0}; k_0, x_0)$$

and $\{\tilde{w}_{Nk_0}\}$ is bounded in $l^2(k_0, \infty; \mathbf{R}^{m_1})$. Hence there exists a subsequence again denoted by $\{\tilde{w}_{Nk_0}\}$ which is weakly convergent to $\tilde{w} \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ with $\| \tilde{w} \|_2 \leq c \mid x_0 \mid$, $c > 0$ (see Theorem A.5). Let $\tilde{x}$ be the response to $\tilde{w}$, i.e, the solution of

$$\tilde{x}(k+1) = A(k)\tilde{x}(k) + B(k)\tilde{w}(k), \quad \tilde{x}(k_0) = x_0.$$

Since the restriction of $\tilde{w}_{Nk_0}$ on any subinterval converges weakly to that of $\tilde{w}$, $x_N(k) \to \tilde{x}(k)$ in $\mathbf{R}^n$ for each k as $N \to \infty$. On the other hand $x_N(k) \to \bar{x}(k)$ in any finite interval, where $\bar{x}$ is the solution of

$$\bar{x}(k+1) = (A + BT_1^{-1}R_1)(k)\bar{x}(k), \quad \bar{x}(k_0) = x_0.$$

Hence we can identify $\tilde{x} = \bar{x}$. Since A is exponentially stable and $\tilde{w} \in l^2(k_0, \infty; \mathbf{R}^{m_1})$, we conclude $\tilde{x} \in l^2(k_0, \infty; \mathbf{R}^n)$ and $\bar{x} \in l^2(k_0, \infty; \mathbf{R}^n)$. This is true for any x_0, which via Proposition 3.2 implies that $A + BT_1^{-1}R_1$ is exponentially stable. ∎

Theorem 3.6 *Assume that the system* $\mathbf{G}$ *is exponentially stable on* $[k_0, \infty)$. *Then the following statements are equivalent:*
(a) $\| G \| < \gamma$.
(b) There exists a bounded nonnegative stabilizing solution of (3.31) and (3.32) on $[k_0, \infty)$ *satisfying (3.34).*
(c) There exists a bounded nonnegative stabilizing solution of (3.35)-(3.37) on $[k_0, \infty)$.

Proof. Suppose (a) holds. Then the existence of a stabilizing solution follows from Lemma 3.12. The condition (3.34) follows as in Theorem 3.5. Hence (a) implies (b). The converse is also similar to Theorem 3.5. We only need to show

$$| h |^2 + \| w \|_2^2 \leq a(| h |^2 + \| r \|_2^2) \text{ for some } a > 0.$$

But this follows from

$$\begin{aligned} x(k+1) &= (A + BT_1^{-1}R_1)(k)x(k) + (BT_1^{-\frac{1}{2}})(k)r(k), \\ w(k) &= (T_1^{-1}R_1)(k)x(k) + T_1^{-\frac{1}{2}}(k)r(k) \end{aligned}$$

since $A + BT_1^{-1}R_1$ is exponentially stable.

(c) is the dual of (b) and (a) implies that there is a bounded nonnegative solution of (3.36) with properties (3.35) and (3.37). In fact we consider the adjoint system

$$\begin{aligned} \xi(k) &= A'(k)\xi(k+1) + C'(k)v(k), \quad \xi(N+1) = \xi_1, \\ \zeta(k) &= B'(k)\xi(k+1) + D'(k)v(k) \end{aligned}$$

and

$$J(v; N+1, \xi_1) = \sum_{k=k_0}^{N} [| \zeta(k) |^2 - \gamma^2 \mid v(k) |^2] + | H'\xi(k_0) |^2$$

and proceed as in Lemma 3.12. To show the exponential stability of $A + R'_{1Y}T_{1Y}^{-1}C$, let $v_N(k) = T_{1Y}^{-1}R_{1Y}\xi(k)$ be the maximizing element of $J(v; N + 1, \xi_1)$, then

$$\| v_N \|_{l^2(k_0,N;\mathbf{R}^{p_1})} \leq c_0 \mid \xi_1 \mid \quad \text{for some } c_0 > 0.$$

We extend v_N to $[k_0, \infty)$ by zero which we denote by $\tilde{v}_N \in l^2(k_0, \infty; \mathbf{R}^{p_1})$. Then there exists a subsequence again denoted by $\tilde{v}_N$ convergent weakly to $\tilde{v}_N \in l^2(k_0, \infty; \mathbf{R}^{p_1})$. Now let $k_0 < L < \infty$ be a fixed but arbitrary number and consider

$$\begin{aligned} \xi_N(k) &= A'(k)\xi_N(k+1) + C'(k)\tilde{v}_N(k), \quad \xi_N(L+1) = \xi_1, \\ \tilde{\xi}(k) &= A'(k)\tilde{\xi}(k+1) + C'(k)\tilde{v}(k), \quad \tilde{\xi}(L+1) = \xi_1 \end{aligned}$$

and

$$\xi(k) = A'(k)\xi(k+1) + C'(k)(T_{1Y}^{-1}R_{1Y})(k)\tilde{\xi}(k), \quad \xi(L+1) = \xi_1. \tag{3.41}$$

Then as in Lemma 3.12, we can show $\xi_N(k) \to \tilde{\xi}(k)$ for any $k \in [k_0, L+1]$ and $\tilde{\xi}(k) = \xi(k)$, $k \in [k_0, L+1]$. Since $\| \tilde{v} \|_{l^2(k_0,\infty;\mathbf{R}^{p_1})} \leq c_0 \mid \xi_1 \mid$,

$$\sum_{k=k_0}^{L+1} | \tilde{\xi}(k) |^2 \leq c \mid \xi_1 |^2 \quad \text{for some } c > 0,$$

which implies

$$\sum_{k=k_0}^{L+1} | \xi(k) |^2 \leq c \mid \xi_1 |^2 \quad \text{for any } k_0 < L < \infty.$$

Hence by Proposition 3.3, the system (3.41) is exponenetially stable and so is $A + R'_{1Y}T_{1Y}^{-1}C$. Thus (a) implies (c).

The converse follws concerning the adjoint of the system **G** and proceed as the converse of (b). ∎

Corollary 3.5 *Let the system* **G** *be θ-periodic, i.e., $A(k+\theta) = A(k)$, $B(k+\theta) = B(k)$, $C(k+\theta) = C(k)$ and $D(k+\theta) = D(k)$. Then*
(a) The stabilizing solution of (b) in Theorem 3.6 is θ-periodic.
(b) There exists a θ-periodic nonnegative stabilizing solution $Y_\theta(k)$ to (3.35) and (3.36) such that $Y(k) - Y_\theta(k) \to 0$ as $k \to \infty$ where Y is a bounded nonnegative stabilizing solution of (3.35)-(3.37).

Proof. Proofs of (a) and (b) are similar to those of Theorem 3.2 and Corollary 3.3, respectively. ∎

If the system **G** is time-invariant, then we need the algebraic Riccati equations:

$$
\begin{aligned}
T_1 &> 0, &(3.42)\\
X &= A'XA + C'C + R_1'T_1^{-1}R_1, &(3.43)\\
H'XH &\le d^2 I \text{ for some } 0 < d < \gamma, &(3.44)\\
T_{1Y} &> 0, &(3.45)\\
Y &= AYA' + BB' + R_{1Y}'T_{1Y}^{-1}R_{1Y}. &(3.46)
\end{aligned}
$$

We define the stabilizing solutions of (3.43) and (3.46) as above. We can set $k_0 = 0$.

Corollary 3.6 *Let the system* **G** *be time-invariant. Suppose A is exponentially stable. Then the following statements are equivalent.*
(a) $\| G \| < \gamma$.
(b) There exists a nonnegative stabilizing solution X_∞ of (3.42)-(3.44).
(c) There exists a bounded nonnegative stabilizing solution Y of (3.45) and (3.46) with $Y(0) = 0$. Moreover, there exists a unique nonnegative stabilizing solution Y_∞ of (3.45) and (3.46) and $Y(k) \to Y_\infty$ as $k \to \infty$.

Proof. The last property follows from Lemma 3.11. ∎

Corollary 3.7 *Let the system* **G** *be time-invariant. Suppose A is exponentially stable. Then the following statements are equivalent.*
(a) $\| \bar{G} \| < \gamma$.
(b) There exitsts a nonnegative stabilizing solution X_∞ of (3.42) and (3.43).
(c) There exists a bounded nonnegative stabilizing solution Y_∞ of (3.45) and (3.46).

3.2 H_∞ Control and Quadratic Games

As in Section 2.2 we consider the quadratic games related to the H_∞ control problems.

3.2.1 Finite Horizon Problems

Consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \; x(k_0) = x_0, \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned} \tag{3.47}$$

with

$$z_1 = Fx(N+1) \tag{3.48}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement, $F \in \mathbf{R}^{q \times n}$ and A, B_1, etc are bounded matrices of appropriate dimensions. For this system we assume

D1': $D_{12}'(k)\,[C_1(k) \quad D_{11}(k) \quad D_{12}(k)] = [0 \quad 0 \quad I]$ for any k.

The standard H_∞-control is to find necessary and sufficient conditions for the existence of a controller of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \quad \hat{x}(k_0) = 0, \\ u(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k) \end{aligned} \tag{3.49}$$

such that $\| \bar{G} \| < \gamma$, i.e.,

$$\| z \|_2^2 + | z_1 |^2 \le d^2 \| w \|_2^2 \quad \text{for some } 0 < d < \gamma$$

where $\bar{G}$ is the input-output operator:$w \to \begin{pmatrix} z_1 \\ z \end{pmatrix}$. In this case the controller (3.49) is called γ-suboptimal.

Now we assume that a γ-suboptimal controller exists and study its consequence to the following quadratic game:

$$J(u, w; s, x_0) = \sum_{k=s}^{N} [| z(k) |^2 - \gamma^2 | w(k) |^2] + | Fx(N+1) |^2 \tag{3.50}$$

where u is the minimizer and w is the maximizer. The response to (3.47) and (3.49) can be written

$$\begin{aligned} x_K(k) &= (\Phi_{1K}x_0)(k) + (\Phi_{2K}w)(k), \\ z_K(k) &= (\Psi_{1K}x_0)(k) + (\Psi_{2K}w)(k), \\ u_K(k) &= (\Pi_{1K}x_0)(k) + (\Pi_{2K}w)(k), \\ z_{1K} &= F\Phi_{1K}(N+1)x_0 + F\Phi_{2K}(N+1)w \end{aligned} \tag{3.51}$$

where

$$\begin{array}{rcl}\Phi_{1K} & \in & \mathcal{L}(\mathbf{R}^n, l^2(s, N+1; \mathbf{R}^n)),\\ \Phi_{2K} & \in & \mathcal{L}(l^2(s, N; \mathbf{R}^{m_1}), l^2(s, N+1; \mathbf{R}^n)),\\ \Psi_{1K},\ \Pi_{1K} & \in & \mathcal{L}(\mathbf{R}^n, l^2(s, N; \mathbf{E})),\\ \Psi_{2K},\ \Pi_{2K} & \in & \mathcal{L}(l^2(s, N; \mathbf{R}^{m_1}), l^2(s, N; \mathbf{E}))\end{array}$$

with $\mathbf{E} = \mathbf{R}^{p_1}$, $\mathbf{R}^{m_2}$, respectively and $\Phi_{1K}(N+1)x_0 = (\Phi_{1K}x_0)(N+1)$, $\Phi_{2K}(N+1)w = (\Phi_{2K}w)(N+1)$. Moreover Φ_{2K}, Ψ_{2K} and Π_{2K} are causal and $\| \bar{G} \| < \gamma$ is equivalent to

$$\| \bar{\Psi}_K \| = \| \begin{pmatrix} F\Phi_{2K}(N+1) \\ \Psi_{2K} \end{pmatrix} \| \leq d \ \text{ for some } 0 < d < \gamma \tag{3.52}$$

which implies

$$\| \Psi_{2K}w \|_2^2 + | F\Phi_{2K}(N+1)w |^2 \leq d^2 \| w \|_2^2 .$$

Now consider the functional (3.50). Since

$$\| z \|_2 = \| C_1 x + D_{11} w \|_2 + \| u \|_2$$

by **D1'**, $J(u, w; s, x_0)$ is strictly convex in u. Hence by Theorem A.4 for any x_0 and $w \in l^2(s, N; \mathbf{R}^{m_1})$ there exists a unique $u_s = u_s(w, x_0) \in l^2(s, N; \mathbf{R}^{m_2})$ such that

$$\min_u J(u, w; s, x_0) = J(u_s, w; s, x_0).$$

The response of (3.47) and (3.48) to u_s can be written

$$\begin{array}{rcl} x_s(k) & = & (\Phi_{1s}x_0)(k) + (\Phi_{2s}w)(k),\\ z_s(k) & = & (\Psi_{1s}x_0)(k) + (\Psi_{2s}w)(k),\\ u_s(k) & = & (\Pi_{1s}x_0)(k) + (\Pi_{2s}w)(k),\\ z_{1s} & = & F\Phi_{1s}(N+1)x_0 + F\Phi_{2s}(N+1)w. \end{array} \tag{3.53}$$

Since

$$J(u_s, w; s, x_0) \leq J(u_K, w; s, x_0) \tag{3.54}$$

we have

$$\| \bar{\Psi}_s \| = \| \begin{pmatrix} F\Phi_{2s}(N+1) \\ \Psi_{2s} \end{pmatrix} \| \leq d. \tag{3.55}$$

Now

$$\begin{array}{rcl} J(u_s, w; s, x_0) & = & \| z_s \|_2^2 - \gamma^2 \| w \|_2^2 + | Fx(N+1) |^2 \\ & = & \| \begin{pmatrix} F\Phi_{1s}(N+1) \\ \Psi_{1s} \end{pmatrix} x_0 + \bar{\Psi}_s w \|^2 - \gamma^2 \| w \|_2^2 \end{array}$$

where

$$\| \begin{pmatrix} F\Phi_{1s}(N+1) \\ \Psi_{1s} \end{pmatrix} x_0 + \bar{\Psi}_s w \|^2 = | F\Phi_{1s}(N+1)x_0 + F\Phi_{2s}(N+1)w |^2 + \| \Psi_{1s}x_0 + \Psi_{2s}w \|_2^2 .$$

By (3.55) $\gamma^2 I - \bar{\Psi}_{2s}^* \bar{\Psi}_{2s}$ is bounded both from below and above. So its inverse exists (Theorem A.3) and is uniformly bounded in s. Hence there exists a unique maximizing element of $J(u_s, w; s, x_0)$ given by

$$w_s = (\gamma^2 I - \bar{\Psi}_s^* \bar{\Psi}_s)^{-1} \bar{\Psi}_s^* \begin{pmatrix} F\Phi_{1s}(N+1) \\ \Psi_{1s} \end{pmatrix} x_0. \tag{3.56}$$

Next we shall show that $w_s = w_s(x_0)$ and $u_s(w_s, x_0)$ are uniformly bounded in s. Setting $w = 0$ in (3.54) we have

$$\| u_s(0, x_0) \|_2^2 \leq J(u_s(0, x_0), 0; s, x_0) \leq J(u_K, 0, s, x_0) = \| z_K \|_2^2 + | z_{1K} |^2$$

or

$$\begin{aligned} \| \Pi_{1s} x_0 \|_2^2 &\leq \| \Psi_{1s} x_0 \|_2^2 + | F\Phi_{1s}(N+1)x_0 |^2 \\ &\leq \| \Psi_{1K} x_0 \|_2^2 + | F\Phi_{1K}(N+1)x_0 |^2 . \end{aligned}$$

Hence Π_{1s} and Ψ_{1s} are uniformly bounded. By (3.56) we have

$$\| w_s \|_2 \leq a_1 | x_0 | \tag{3.57}$$

for some $a_1 > 0$ independent of s and w_s is uniformly bounded. Setting $x_0 = 0$ in (3.54) we also have

$$\begin{aligned} \| u_s(w, 0) \|_2^2 - \gamma^2 \| w \|_2^2 &\leq J(u_s(w, 0), w; s, 0) \\ &\leq J(u_K, w; s, 0) \\ &\leq \| z_K \|_2^2 - \gamma^2 \| w \|_2^2 + | z_{1K} |^2 \end{aligned}$$

and

$$\| \Pi_{2s} w \|_2^2 \leq \| \Psi_{2s} w \|_2^2 \leq \| \Psi_{2K} w \|_2^2 \leq d^2 \| w \|_2^2 .$$

This implies that Π_{2s} is uniformly bounded. Now (3.53) and (3.57) yields

$$\| u_s(w_s, x_0) \|_2 \leq a_2 | x_0 | \tag{3.58}$$

for some $a_2 > 0$ independent of N. Thus we have shown:

Lemma 3.13 *(a) Π_{1s}, Ψ_{1s}, Π_{2s} and Ψ_{2s} are uniformly bounded.*
(b) $w_s(x_0)$ and $u_s(w_s, x_0)$ are uniformly bounded and

$$\max_w \min_u J(u, w; s, x_0) = J(u_s, w_s; s, x_0) \leq a | x_0 |^2$$

for some $a > 0$ independent of s.

Now we consider the Riccati equation

$$\begin{aligned} V(k) &> aI \text{ for some } a > 0, && (3.59)\\ X(k) &= A'X(k+1)A + C_1'C_1 \\ &\qquad -(R_2'T_2^{-1}R_2)(k) + (F_1'VF_1)(k), && (3.60)\\ X(N+1) &= F'F && (3.61) \end{aligned}$$

where

$$\begin{aligned} &T_1(k) = \gamma^2 I - D_{11}'D_{11} - B_1'X(k+1)B_1, && T_2(k) = I + B_2'X(k+1)B_2,\\ &R_1(k) = B_1'X(k+1)A + D_{11}'C_1, && R_2(k) = B_2'X(k+1)A,\\ &S(k) = B_2'X(k+1)B_1, && V(k) = (T_1 + S'T_2^{-1}S)(k),\\ &F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k), && F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k) \end{aligned}$$

and for simplicity we have omitted k in all system matrices of (3.47).

First we assume that there exists a sequence of symmetric matrices $X(k)$, $k \in [s, N+1]$ satisfying (3.59)-(3.61) and examine the properties of $X(k)$. By direct calculation, we obtain

$$\begin{aligned} J(u,w;s,x_0) &= x_0'X(s)x_0 + \sum_{k=s}^{N} \mid [T_2^{\frac{1}{2}}\{u + T_2^{-1}(Sw + R_2x)\}](k) \mid^2 \\ &\qquad -\gamma^2 \sum_{k=s}^{N} \mid \frac{1}{\gamma}[V^{\frac{1}{2}}(w - F_1x)](k) \mid^2 \qquad (3.62) \end{aligned}$$

where x is the response of the system (3.47) to the pair $(u,w) \in l^2(s,N;\mathbf{R}^{m_2}) \times l^2(s,N;\mathbf{R}^{m_1})$. Define feedback laws

$$\bar{w}(\cdot) = F_1(\cdot)x(\cdot), \ \bar{u}(\cdot) = -[T_2^{-1}(Sw + R_2x)](\cdot) \qquad (3.63)$$

and let x^* be the solution of (3.47) with $k_0 = s$ corresponding to (3.63). Set

$$w^*(k) = F_1(k)x^*(k), \ u^*(k) = F_2(k)x^*(k). \qquad (3.64)$$

We shall show that the value of the game exists i.e,

$$\sup_w \inf_u J(u,w;s,x_0) = \inf_u \sup_w J(u,w;s,x_0).$$

Lemma 3.14 *Suppose that there exists a sequence $X(k)$, $k \in [s, N+1]$ satisfying (3.59)-(3.61). Then X is nonnegative. Moreover*

$$\begin{aligned} J(\bar{u},w;s,x_0) &\le J(\bar{u},\bar{w};s,x_0) \\ &= x_0'X(s)x_0 \le J(u,\bar{w};s,x_0), && (3.65)\\ J(u^*,w^*;s,x_0) &= x_0'X(s)x_0 \le J(u,w^*;s,x_0) && (3.66) \end{aligned}$$

for any $(w,u) \in l^2(s,N;\mathbf{R}^{m_1}) \times l^2(s,N;\mathbf{R}^{m_2})$. *The max-min of* $J(u,w;s,x_0)$ *is attained by the pair* $(\bar{u}, w^*)$ *and*

$$\begin{aligned} \max_w \min_u J(u,w;s,x_0) &= J(\bar{u},w^*;s,x_0) \\ &= J(\bar{u},\bar{w};s,x_0) \\ &= J(u^*,w^*;s,x_0) \\ &= x_0'X(s)x_0 = \inf_u \sup_w J(u,w;s,x_0). \end{aligned} \tag{3.67}$$

Proof. We note that (3.65) follows from (3.62). Setting $w = 0$ in (3.65), we have

$$0 \le J(\bar{u},0;s,x_0) \le J(\bar{u},\bar{w};s,x_0) = x_0'X(s)x_0.$$

Hence $X(s)$ is nonnegative. From (3.62) we have

$$J(\bar{u},w;s,x_0) \le J(\bar{u},\bar{w};s,x_0) = x_0'X(s)x_0$$

and hence

$$\min_u J(u,w;s,x_0) \le J(\bar{u},w;s,x_0) \le x_0'X(s)x_0$$

for any $w \in l^2(s,N;\mathbf{R}^{m_1})$. This implies

$$\sup_w \min_u J(u,w;s,x_0) \le x_0'X(s)x_0.$$

Now we shall show

$$\min_u J(u,w^*;s,x_0) = J(u^*,w^*;s,x_0) = x_0'X(s)x_0. \tag{3.68}$$

For this purpose, we consider $e = x - x^*$, where x is given by

$$x(k+1) = Ax(k) + B_1F_1x^*(k) + B_2u(k), \; x(s) = x_0.$$

Then

$$e(k+1) = Ae(k) + B_2[u(k) - u^*(k)], \; e(s) = 0$$

and

$$\begin{aligned} J(u,w^*;s,x_0) \;\; = \;\; & \| C_1(e+x^*) + D_{11}w^* \|_2^2 + \| u \|_2^2 - \gamma^2 \| w^* \|_2^2 \\ & + | F(e+x^*)(N+1) |^2 . \end{aligned}$$

Define

$$\begin{aligned} (\mathbf{H}u)(k) &= \sum_{j=s}^{k-1} S(k,j+1)B_2(j)u(j), \\ \mathbf{H}_s u &= \sum_{j=s}^{N} S(N+1,j+1)B_2(j)u(j) \end{aligned}$$

where $S(k,j)$ is the state transition matrix of A. Then

$$\begin{aligned} e(k) &= [\mathbf{H}(u-u^*)](k), \\ e(N+1) &= \mathbf{H}_s(u-u^*). \end{aligned}$$

Since $J(u, w^*; s, x_0)$ is strictly convex in u, there exists a unique minimizing element u. It is given by the solution of

$$\begin{aligned} u + \mathbf{H}^* C_1' C_1 \mathbf{H}(u-u^*) + \mathbf{H}^* C_1'(C_1 x^* + D_{11} w^*) \\ + \mathbf{H}_s^* F'F \mathbf{H}_s(u-u^*) + \mathbf{H}_s^* F'F x^*(N+1) = 0. \end{aligned} \tag{3.69}$$

We shall show that $u = u^*$ is the solution. Note that for $h \in l^2(s, N; \mathbf{R}^n)$ and $\tilde{h} \in \mathbf{R}^n$

$$\begin{aligned} (\mathbf{H}^* h)(k) &= B_2'(k) \sum_{j=k+1}^{N} S'(j, k+1) h(j), \\ (\mathbf{H}_s^* \tilde{h})(k) &= B_2'(k) S'(N+1, k+1) \tilde{h}. \end{aligned}$$

It is enough to show that $u^*(k) = F_2(k) x^*(k)$ coincides with

$$-\mathbf{H}^* C_1' C_1 x^* - \mathbf{H}^* C_1' D_{11} w^* - \mathbf{H}_s^* F'F x^*(N+1)$$

which is equal to

$$\begin{aligned} -B_2'(k) \sum_{j=k+1}^{N} S'(j, k+1)[C_1' C_1 x^* + C_1' D_{11} w^*](j) \\ -B_2'(k) S'(N+1, k+1) F'F x^*(N+1). \end{aligned}$$

Since

$$\begin{aligned} & S'(l, k+1)[C_1' C_1 x^* + C_1' D_{11} w^*](l) + S'(l+1, k+1) X(l+1) x^*(l+1) \\ = \; & S'(l, k+1)[C_1' C_1 + C_1' D_{11} F_1](l) x^*(l) \\ & \quad + S'(l, k+1) A'(l) X(l+1)(A + B_1 F_1 + B_2 F_2)(l) x^*(l) \\ = \; & S'(l, k+1)[C_1' C_1 + C_1' D_{11} F_1 + A' X(l+1)(A + B_1 F_1 + B_2 F_2)] x^*(l) \end{aligned}$$

and

$$\begin{aligned} & C_1' C_1 + C_1' D_{11} F_1 + A' X(l+1)(A + B_1 F_1 + B_2 F_2) \\ = \; & C_1' C_1 + A' X(l+1) A + R_1' F_1 + R_2' F_2 \\ = \; & C_1' C_1 + A' X(l+1) A - R_2' T_2^{-1} R_2 + F_1' V F_1 \\ = \; & X(l), \end{aligned}$$

we have

$$\begin{aligned} & S'(l, k+1)[C_1' C_1 x^* + C_1' D_{11} w^*](l) + S'(l+1, k+1) X(l+1) x^*(l+1) \\ = \; & S'(l, k+1) X(l) x^*(l) \end{aligned}$$

and

$$-B_2'(k)[\sum_{j=k+1}^{N} S'(j,k+1)[C_1'C_1x^* + C_1'D_{11}w^*](j)$$

$$+S'(N+1,k+1)F'Fx^*(N+1)] = -B_2'(k)X(k+1)x^*(k+1).$$

On the other hand we have

$$\begin{aligned} & F_2(k)x^*(k) \\ = \quad & -[I + B_2'(k)X(k+1)B_2(k)]^{-1}B_2'X(k+1)(A+B_1F_1)(k)x^*(k) \\ = \quad & -B_2'(k)X(k+1)[I+(B_2B_2')(k)X(k+1)]^{-1}(A+B_1F_1)(k)x^*(k) \\ = \quad & -B_2'(k)X(k+1)x^*(k+1) \end{aligned}$$

and hence $u = u^*$ is the solution of (3.69).

It remains to show the last equality in (3.67). From (3.62)

$$x_0'X(s)x_0 \leq J(u,\bar{w};s,x_0) \leq \sup_w J(u,w;s,x_0)$$

for any u and hence

$$x_0'X(s)x_0 \leq \inf_u \sup_w J(u,w;s,x_0).$$

But

$$\max_w J(\bar{u},w;s,x_0) = x_0'X(s)x_0$$

and $x_0'X(s)x_0 = J(\bar{u},w^*;s,x_0) = \inf_u \sup_w J(u,w;s,x_0)$. ∎

Next we shall show the existence of a solution to the Riccati equation (3.59)-(3.61). Recall that we are assuming the existence of a γ-suboptimal controller.

Lemma 3.15 *There exists a nonnegative solution $X(k)$, $k \in [s, N+1]$ to (3.59)-(3.61) and*

$$\max_w \min_u J(u,w;s,x_0) = x_0'X(s)x_0.$$

Furthermore the controller

$$u(\cdot) = -(T_2^{-1}R_2)(\cdot)x(\cdot) - (T_2^{-1}S)(\cdot)w(\cdot)$$

satisfies $\| \bar{G} \| < \gamma$.

Proof. We prove it by induction. Consider the functional

$$J(u,w;N,x_0) = \mid z(N) \mid^2 - \gamma \mid w(N) \mid^2 + \mid Fx(N+1) \mid^2$$

subject to

$$\begin{aligned} x(N+1) &= A(N)x(N) + B_1(N)w(N) + B_2(N)u(N), \;\; x(N) = x_0, \\ z(N) &= C_1(N)x(N) + D_{11}(N)w(N) + D_{12}(N)u(N). \end{aligned}$$

Then $J(u,w;N,x_0)$ is rewritten as

$$\begin{aligned} & J(u,w;N,x_0) \\ = \; & x_0'[(C_1'C_1)(N) + A'(N)X(N+1)A(N) - (R_2T_2^{-1}R_2)(N)]x_0 \\ & + \mid [T_2^{\frac{1}{2}}\{u + T_2^{-1}(Sw + R_2x)\}](N) \mid^2 \\ & - w'(N)V(N)w(N) + 2w'(N)(R_1 - S'T_2^{-1}R_2)(N)x_0 \end{aligned}$$

with $X(N+1) = F'F$. Since there exists a γ-suboptimal controller on $[s,N]$, it is also γ-suboptimal on $[N,N]$ and by Lemma 3.13 we obtain

$$\max_w \min_u J(u,w;N,x_0) \le c \mid x_0 \mid^2 \;\; \text{for some } c > 0.$$

Hence $V(N) > aI$ for some $a > 0$ and we can define $X(N)$ by (3.60).

We assume the existence of a solution $X(k)$, $k \in [j+1,N]$ to (3.59)-(3.61). Consider the functional

$$J(u,w;j,x_0) = \sum_{k=j}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2] + \mid Fx(N+1) \mid^2$$

subject to

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \;\; x(j) = x_0, \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k). \end{aligned}$$

Then by (3.62) and the above argument we can rewrite $J(u,w;j,x_0)$ as

$$\begin{aligned} J(u,w;j,x_0) &= \mid z(j) \mid^2 - \gamma^2 \mid w(j) \mid^2 \\ & \quad + \sum_{k=j+1}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2] + \mid Fx(N+1) \mid^2 \\ &= \mid z(j) \mid^2 - \gamma^2 \mid w(j) \mid^2 + x'(j+1)X(j+1)x(j+1) \\ & \quad + \sum_{k=j+1}^{N} \mid [T_2^{\frac{1}{2}}\{u + T_2^{-1}(Sw + R_2x)\}](k) \mid^2 \\ & \quad - \gamma^2 \sum_{k=j+1}^{N} \mid \frac{1}{\gamma}[V^{\frac{1}{2}}(w - F_1x)](k) \mid^2 \end{aligned}$$

$$
\begin{aligned}
= \;& x_0'[(C_1'C_1)(j) + A'(j)X(j+1)A(j) - (R_2T_2^{-1}R_2)(j)]x_0 \\
&+ \sum_{k=j+1}^{N} \mid [T_2^{\frac{1}{2}}\{u + T_2^{-1}(Sw + R_2x)\}](k) \mid^2 \\
&-w'(j)V(j)w(j) + 2w'(j)(R_1 - S'T_2^{-1}R_2)(j)x_0 \\
&-\gamma^2 \sum_{k=j+1}^{N} \mid \frac{1}{\gamma}[V^{\frac{1}{2}}(w - F_1x)](k) \mid^2 .
\end{aligned}
$$

By Lemma 3.13 and the above argument

$$\max_w \min_u J(u, w; j, x_0) \leq c \mid x_0 \mid^2 \text{ for some } c > 0.$$

and $V(j) > aI$. We can define $X(j)$ by (3.60) and the rest follows from Lemma 3.14. ∎

Summing up we have the following.

Theorem 3.7 *Assume* **D1'**. *Suppose the controller (3.49) is γ-suboptimal for the system* **G**. *Then there exists a nonnegative solution $X(k)$, $k \in [s, N+1]$ to the Riccati equation (3.59)-(3.61). Moreover*

$$
\begin{aligned}
\max_w \min_u J(u, w; s, x_0) &= J(\bar{u}, \bar{w}; s, x_0) \\
&= J(u^*, w^*; s, x_0) \\
&= x_0'X(s)x_0 = \inf_u \sup_w J(u, w; s, x_0).
\end{aligned}
$$

Consider the backward system

$$
\begin{aligned}
\tilde{x}(k) &= A'(k)\tilde{x}(k+1) + C_1'(k)\tilde{w}(k) + C_2'(k)\tilde{u}(k), \\
\tilde{z}(k) &= B_1'(k)\tilde{x}(k+1) + D_{11}'(k)\tilde{w}(k) + D_{21}'(k)\tilde{u}(k), \\
\tilde{y}(k) &= B_2'(k)\tilde{x}(k+1) + D_{12}'(k)\tilde{w}(k), \\
\tilde{z}_1 &= H'\tilde{x}(k_0)
\end{aligned}
\tag{3.70}
$$

with

$$\tilde{x}(N+1) = F'f, \; f \in \mathbf{R}^{p_1}$$

which is the adjoint system of **G** with $x(k_0) = Hh$. For the system (3.70) we introduce the controller of the form

$$
\begin{aligned}
\hat{x}(k) &= \hat{A}'(k)\hat{x}(k+1) + \hat{C}'(k)\tilde{y}(k), \\
\tilde{u}(k) &= \hat{B}'(k)\hat{x}(k+1) + \hat{D}'(k)\tilde{y}(k)
\end{aligned}
\tag{3.71}
$$

which satisfies

$$\| \tilde{z} \|_2^2 + \mid \tilde{z}_1 \mid^2 \leq d^2 \| \tilde{w} \|_2^2 \text{ for some } 0 < d < \gamma.$$

Now we introduce the following functional

$$\tilde{J}(\tilde{u},\tilde{w};N+1,\tilde{x}(N+1)) = \sum_{k=s}^{N}[\mid \tilde{z}(k)\mid^2 - \gamma^2 \mid \tilde{w}(k)\mid^2] + \mid H'\tilde{x}(s)\mid^2$$

subject to (3.70) and we consider the following Riccati equation

$$\begin{aligned} V_Y(k) &> aI \text{ for some } a>0, & (3.72)\\ Y(k+1) &= AY(k)A' + B_1B_1 \\ &\qquad -(R'_{2Y}T_{2Y}^{-1}R_{2Y})(k) + (F'_{1Y}V_YF_{1Y})(k), & (3.73)\\ Y(k_0) &= HH' & (3.74)\end{aligned}$$

where

$$\begin{aligned} &T_{1Y}(k) = \gamma^2 I - D_{11}D'_{11} - C_1Y(k)C'_1, && T_{2Y}(k) = I + C_2Y(k)C'_2,\\ &R_{1Y}(k) = C_1Y(k)A' + B'_1D_{11}, && R_{2Y}(k) = C_2Y(k)A',\\ &S_Y(k) = C_2Y(k)C'_1, && V_Y(k) = (T_{1Y} + S'_YT_{2Y}^{-1}S_Y)(k),\\ &F_{1Y}(k) = [V_Y^{-1}(R_{1Y} - S'_YT_{2Y}^{-1}R_{2Y})](k), &&\\ &F_{2Y}(k) = -[T_{2Y}^{-1}(R_{2Y} + S_YF_{1Y})](k) && \end{aligned}$$

and for simplicity we have omitted k in all system matrices of (3.70). Then as in Lemmas 3.13-3.15, considering the max-min problem for $\tilde{J}(\tilde{u},\tilde{w};N+1,\tilde{x}(N+1))$ and hence we have the following result.

Corollary 3.8 *Assume the condition*

D2': $D_{21}(k)\,[B'_1(k) \quad D'_{11}(k) \quad D'_{21}(k)] = [0 \quad 0 \quad I]$ *for any* k.

Suppose the controller (3.71) is γ-suboptimal for the system (3.70). Then
(a) There exists a nonnegative solution $Y(k)$, $k \in [s, N+1]$ to the Riccati equation (3.72)-(3.74) and

$$\max_{\tilde{w}}\min_{\tilde{u}} \tilde{J}(\tilde{u},\tilde{w};N+1,\tilde{x}(N+1)) = \tilde{x}'(N+1)Y(N+1)\tilde{x}(N+1).$$

(b) For $\epsilon > 0$ small, there exists a nonnegative solution $Y_{\gamma-\epsilon}(k)$, $k \in [s, N+1]$ of (3.72)-(3.74) with γ replaced by $\gamma - \epsilon$.

3.2.2 The Infinite Horizon Problem

Consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k),\\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k),\\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned}$$

with $x(k_0) = x_0$ and the assumption **D1'**. We further assume that (A, B_2, C_1) is stabilizable and detectable. As in the finite horizon case, we assume the existence of a controller K of the form (3.49) with property

$$\| z \|_2 \leq d \| w \|_2 \quad \text{for some } 0 < d < \gamma \tag{3.75}$$

and study its consequence to the quadratic game defined by the functional

$$J(u, w; k_0, x_0) = \sum_{k=k_0}^{\infty} [| z(k) |^2 - \gamma^2 | w(k) |^2]. \tag{3.76}$$

Such a controller is called IO-stabilizing with γ-disturbance attenuation (IO-γ-suboptimal) and is called γ-suboptimal if it is internally stabilizing. We also consider the finite horizon problem associated with

$$J_N(u, w; k_0, x_0) = \sum_{k=k_0}^{N} [| z(k) |^2 - \gamma^2 | w(k) |^2]. \tag{3.77}$$

Note that if a controller K of the form (3.49) is IO-γ-suboptimal, it is also γ-suboptimal on any $[k_0, N]$. Since (A, B_2) is stabilizable, Ψ_{1s} in (3.53) is uniformly bounded. Then by Lemmas 3.13, 3.14 and Theorem 3.7 we have the following.

Lemma 3.16 *There exists a unique nonnegative solution $X_N(k)$, $k \in [k_0, N+1]$ of the Riccati equation (3.59) and (3.60) with $X_N(N+1) = 0$ such that*

$$| X_N(k) | \leq c \quad \text{independent of } k_0 \leq k \leq N + 1 < \infty.$$

Lemma 3.17 *For each $k \geq k_0$, $X_N(k)$ of Lemma 3.15 is monotone increasing in N.*

Proof. Let $L \leq N$ and we shall show $X_L(k_0) \leq X_N(k_0)$. This follows from

$$\begin{aligned} x_0' X_L(k_0) x_0 &= J_L(\bar{u}_L, \bar{w}_L; k_0, x_0) \\ &\leq J_L(\tilde{u}_N, w_L; k_0, x_0) \\ &\leq J_N(\bar{u}_N, \hat{w}_N; k_0, x_0) \\ &\leq J_N(\bar{u}_N, \bar{w}_N; k_0, x_0) = x_0' X_N(k_0) x_0 \end{aligned}$$

where $\tilde{u}_N$ is the restriction of $\bar{u}_N$ on $[k_0, L]$ and $\hat{w}_N$ is the extension of $\bar{w}_L$ to $[k_0, N]$ by zero. The proof of a general k is similar. ∎

Next we shall show that $V^{-1}(k)$ is uniformly bounded. To do this we first introduce the following result.

Lemma 3.18 *Let* $0 \leq P \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ *and define*

$$\mathbf{W}[P] = P[I - B(I + B'PB)^{-1}B'P]. \tag{3.78}$$

Then if $P_1 \geq P_2 \geq 0$, $\mathbf{W}[P_1] \geq \mathbf{W}[P_2]$.

Proof. Since $I + BB'P$ is invertible, we can easily show

$$\mathbf{W}[P] = P(I + BB'P)^{-1}.$$

Moreover

$$\mathbf{W}[P] = (P^{-1} + BB')^{-1}$$

for $P > 0$ and the assertion follows in this case. Let $\epsilon > 0$ be arbitrary but fixed. Define

$$P_{i\epsilon} = P_i + \epsilon I, \quad i = 1,\ 2.$$

Then $P_{1\epsilon} \geq P_{2\epsilon} > 0$ and hence $\mathbf{W}[P_{1\epsilon}] \geq \mathbf{W}[P_{2\epsilon}]$. Now letting $\epsilon \to 0$ we have $\mathbf{W}[P_1] \geq \mathbf{W}[P_2]$. ∎

Lemma 3.19 $V^{-1}(k)$, $k \in [k_0, N+1]$ *is uniformly bounded.*

Proof. Let K be a γ-suboptimal controller, i.e., an internally stabilizing controller such that $\| \bar{G} \| < \gamma$. Then $\| \bar{G} \| < \gamma - \epsilon$ for some $\epsilon > 0$. Thus we have two sequences $X_N(k) \geq 0$ and $X_N^\epsilon(k) \geq 0$ where X_N^ϵ is defined as X_N with γ replaced by $\gamma - \epsilon$. Since

$$\begin{aligned} x'X_N(k_0)x &= \max_w \min_u \sum_{k=k_0}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2] \\ &\leq \max_w \min_u \sum_{k=k_0}^{N} [\mid z(k) \mid^2 - (\gamma - \epsilon)^2 \mid w(k) \mid^2] \\ &= x'X_N^\epsilon(k_0)x \end{aligned}$$

we have $X_N(k_0) \leq X_N^\epsilon(k_0)$. By Lemma 3.18 we also have

$$\mathbf{W}[X_N] \leq \mathbf{W}[X_N^\epsilon]$$

with $\mathbf{W}[X]$ is defined by (3.78) with B, P replaced by B_2, $X(k+1)$, respectively. We write $V[X]$ to show the dependence of V on X. Since $V[X_N^\epsilon] > 0$, we have

$$\begin{aligned} 0 < V[X_N^\epsilon] &= (\gamma - \epsilon)^2 I - D_{11}'D_{11} - B_1'\mathbf{W}[X_N^\epsilon]B_1 \\ &\leq (\gamma - \epsilon)^2 I - D_{11}'D_{11} - B_1'\mathbf{W}[X_N]B_1 \end{aligned}$$

which implies

$$V[X_N] = \gamma^2 I - D_{11}'D_{11} - B_1'\mathbf{W}[X_N]B_1 > \delta I$$

where $\delta = 2\gamma\epsilon - \epsilon^2$ and independent of N. Consequently $V^{-1}[X_N](k)$ is uniformly boumded. ∎

In view of Lemmas 3.16 and 3.17, $X_N(k)$ is uniformly bounded and monotone increasing in N. So $X_N(k)$ converges to a limit $X(k)$. Since $V^{-1}[X_N](k)$ is uniformly bounded, X satisfies (3.59) and (3.60). Next we shall show that $A + B_1F_1 + B_2F_2$ is exponentially stable.

Lemma 3.20 *$A + B_1F_1 + B_2F_2$ is exponentially stable.*

Proof. For this purpose we consider

$$\hat{x}_N(k+1) = (A + B_1F_1[X_N] + B_2F_2[X_N])(k)\hat{x}_N(k), \ \hat{x}_N(k_0) = x_0. \quad (3.79)$$

Then for any interval $[k_0, N]$ the solution $\hat{x}_N$ converges to the solution $\bar{x}$ of

$$\bar{x}(k+1) = (A + B_1F_1[X] + B_2F_2[X])(k)\bar{x}(k), \ \bar{x}(k_0) = x_0.$$

We can rewrite (3.79) as

$$\begin{aligned} x(k+1) &= Ax(k) + B_1\hat{w}_N(k) + B_2\hat{u}_N(k) & (3.80)\\ &= (A - JC_1)x(k) + JC_1\hat{x}_N(k) + B_1\hat{w}_N(k) + B_2\hat{u}_N(k), \\ x(k_0) &= x_0 \end{aligned}$$

where $J \in \mathbf{R}^{n\times p_1}$ is chosen such that $A - JC_1$ is exponentially stable. The solution of (3.80) coincides with $\hat{x}_N$ on $[k_0, N]$. We extend it to $[k_0, \infty)$ by the homogenous equation of (3.80). By Lemma 3.14 $\| C_1\hat{x}_N \|_2$, $\| \hat{w}_N \|_2$, $\| \hat{u}_N \|_2 \le a \mid x_0 \mid$ for some $a > 0$ and $C_1\hat{x}_N$, $\hat{w}_N$ and $\hat{u}_N$ converges weakly to $\tilde{h}$, $\tilde{w}$, $\tilde{u}$ in $l^2(k_0, \infty; \mathbf{E})$, $\mathbf{E} = \mathbf{R}^{p_1}$, $\mathbf{R}^{m_1}$ and $\mathbf{R}^{m_2}$, respectively along a subsequence $N \to \infty$. Let $\tilde{x}$ be the solution of

$$\begin{aligned} \tilde{x}(k+1) &= (A - JC_1)\tilde{x}(k) + J\tilde{h}(k) + B_1\tilde{w}(k) + B_2\tilde{u}(k), \\ \tilde{x}(k_0) &= x_0 \end{aligned}$$

then $\hat{x}_N$ converges to $\tilde{x}$ and we can identify $\tilde{x}$ and $\bar{x}$. Since $A - JC_1$ is exponentially stable, $\tilde{x} \in l^2(k_0, \infty; \mathbf{R}^n)$. Hence $\bar{x} \in l^2(k_0, \infty; \mathbf{R}^n)$ for each x_0 and $\| \bar{x} \|_2 \le c \mid x_0 \mid$ for some $a > 0$ independent of x_0. Hence by Proposition 3.2, $A + B_1F_1 + B_2F_2$ is exponentially stable. ∎

Define feedback laws

$$\bar{w}(\cdot) = F_1(\cdot)x(\cdot), \ \bar{u}(\cdot) = -(T_2^{-1}R_2)(\cdot)x(\cdot) - (T_2^{-1}S)(\cdot)w(\cdot). \quad (3.81)$$

Let x^* be the solution of (3.47) corresponding to (3.81) and let

$$w^*(k) = F_1(k)x^*(k), \ u^*(k) = F_2(k)x^*(k). \quad (3.82)$$

First we show that the feedback law $\bar{u}$ is stabilizing.

Lemma 3.21 *Suppose there exists a bounded nonnegative solution $X(k)$, $k \in [k_0, \infty)$ of (3.59) and (3.60) such that $A+B_1F_1+B_2F_2$ is exponentially stable. Then $A - B_2T_2^{-1}R_2$ is exponentially stable.*

Proof. Since

$$A + B_1F_1 + B_2F_2 = (A - B_2T_2^{-1}R_2) + (B_1 - B_2T_2^{-1}S)F_1$$

is exponentially stable, $(F_1, A - B_2T_2^{-1}R_2)$ is detectable and so is

$$\left(\begin{bmatrix} C_1 \\ T_2^{-1}R_2 \\ V^{\frac{1}{2}}F_1 \end{bmatrix}, A - B_2T_2^{-1}R_2 \right).$$

Rewrite now the Riccati equation (3.60) in the form

$$\begin{aligned} X(k) &= (A - B_2T_2^{-1}R_2)'X(k+1)(A - B_2T_2^{-1}R_2) \\ &\quad + \begin{bmatrix} C_1 \\ T_2^{-1}R_2 \\ V^{\frac{1}{2}}F_1 \end{bmatrix}' \begin{bmatrix} C_1 \\ T_2^{-1}R_2 \\ V^{\frac{1}{2}}F_1 \end{bmatrix}. \end{aligned}$$

Hence by Proposition 3.5, $A - B_2T_2^{-1}R_2$ is exponentially stable. ∎

Let **FI** be the set of stabilizing feedback laws of the form $u(\cdot) = K_2(\cdot)x(\cdot) + K_1(\cdot)w(\cdot)$. As Lemma 3.14 we shall show

$$\begin{aligned} \sup_w \inf_{u \in \mathbf{FI}} J(u, w; k_0, x_0) &= J(\bar{u}, w^*; k_0, x_0) \\ &= J(\bar{u}, \bar{w}; k_0, x_0) \\ &= J(u^*, w^*; k_0, x_0) \\ &= x_0'X(k_0)x_0 \\ &= \inf_{u \in \mathbf{FI}} \sup_w J(u, w; k_0, x_0). \end{aligned} \quad (3.83)$$

Note that

$$\inf_{u \in \mathbf{FI}} \sup_w J(u, w; k_0, x_0) \le \sup_w J(\bar{u}, w; k_0, x_0) = J(\bar{u}, w^*; k_0, x_0) = x_0'X(k_0)x_0.$$

It suffices to show

$$x_0'X(k_0)x_0 \le J(\bar{u}, w^*; k_0, x_0) = \inf_{u \in \mathbf{FI}} J(u, w^*; k_0, x_0). \quad (3.84)$$

In fact this implies

$$x_0'X(k_0)x_0 = \inf_{u \in \mathbf{FI}} J(u, w^*; k_0, x_0) \le \sup_w \inf_{u \in \mathbf{FI}} J(u, w; k_0, x_0)$$

and (3.83) follows. To show (3.84), we proceed as in the proof of Lemma 3.14. Consider

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w^*(k) + B_2 u(k) \\ &= (A - B_2 T_2^{-1} R_2) x(k) + (B_1 - B_2 T_2^{-1} S) w^*(k) + B_2 v(k) \end{aligned}$$

with $x(k_0) = x_0$ where $v(k) = u(k) + (T_2^{-1} R_2) x(k) + (T_2^{-1} S) w^*(k)$. Then $e = x - x^*$ satisfies

$$e(k+1) = (A - B_2 T_2^{-1} R_2) e(k) + B_2 v(k), \quad e(k_0) = 0$$

and $J(u, w^*; k_0, x_0)$ can be written as

$$\begin{aligned} \tilde{J}(v, w^*; k_0, x_0) &= \| C_1(e + x^*) + D_{11} w^* \|_2^2 - \gamma^2 \| w^* \|_2^2 \\ &\quad + \| v - T_2^{-1} R_2 x - T_2^{-1} S w^* \|_2^2 \\ &= \| C_1(\mathbf{H}v + x^*) + D_{11} w^* \|_2^2 - \gamma^2 \| w^* \|_2^2 \\ &\quad + \| v - T_2^{-1} R_2 (\mathbf{H}v + x^*) - T_2^{-1} S w^* \|_2^2 \end{aligned}$$

where

$$(\mathbf{H}v)(k) = \sum_{j=k_0}^{k-1} S_X(k, j+1) B_2(j) v(j)$$

where $S_X(k, j)$ is the state transition matrix of $A - B_2 T_2^{-1} R_2$. The unique minimizing element of $\tilde{J}$ given by the solution of

$$\begin{aligned} &\mathbf{H}^* C_1' C_1 \mathbf{H} v + \mathbf{H}^* C_1' (C_1 x^* + D_{11} w^*) + (I - T_2^{-1} R_2 \mathbf{H})^* (I - T_2^{-1} R_2 \mathbf{H}) v \\ &\quad - (I - T_2^{-1} R_2 \mathbf{H})^* T_2^{-1} (R_2 x^* + S w^*) = 0. \end{aligned}$$

We shall show that $v = 0$ is the solution. This follows if

$$\mathbf{H}^* C_1' (C_1 x^* + D_{11} w^*) - (I - T_2^{-1} R_2 \mathbf{H})^* T_2^{-1} (R_2 x^* + S w^*) = 0.$$

Since $A - B_2 T_2^{-1} R_2$ is exponentially stable, we have for $h \in l^2(k_0, \infty; \mathbf{R}^n)$

$$(\mathbf{H}^* h)(k) = B_2'(k) \sum_{j=k+1}^{\infty} S_X'(j, k+1) h(j).$$

Then as in the proof of Lemma 3.14,

$$\begin{aligned} &\mathbf{H}^* C_1' (C_1 x^* + D_{11} w^*) - (I - T_2^{-1} R_2 \mathbf{H})^* T_2^{-1} (R_2 x^* + S w^*) \\ &= F_2 x^*(k) + \mathbf{H}^* [C_1' C_1 + C_1' D_{11} - R_2' T_2^{-1} F_2] x^* \\ &= B_2'(k) [-X(k+1) x^*(k+1) \\ &\qquad + \sum_{j=k+1}^{\infty} S_X(j, k+1) [C_1' C_1 + C_1' D_{11} - R_2' T_2^{-1} F_2] x^*(j)]. \end{aligned}$$

Since

$$\begin{aligned}&-X(k)+[C_1'C_1+C_1'D_{11}-R_2'T_2^{-1}F_2](k)\\&\qquad=-(A-B_2T_2^{-1}R_2)'(k)X(k+1)(A+B_1F_1+B_2F_2)(k),\end{aligned}$$

we have

$$\begin{aligned}&\mathbf{H}^*C_1'(C_1x^*+D_{11}w^*)-(I-T_2^{-1}R_2\mathbf{H})^*T_2^{-1}(R_2x^*+Sw^*)\\=\;&B_2'(k)[-(A-B_2T_2^{-1}R_2)'(k+1)X(k+2)\\&\qquad\times(A+B_1F_1+B_2F_2)(k+1)x^*(k+2)\\&\qquad+\sum_{j=k+2}^{\infty}S_X(j,k+1)[C_1'C_1+C_1'D_{11}-R_2'T_2^{-1}F_2]x^*(j)].\end{aligned}$$

Repeating this argument, we have

$$\begin{aligned}&\mathbf{H}^*C_1'(C_1x^*+D_{11}w^*)-(I-T_2^{-1}R_2\mathbf{H})^*T_2^{-1}(R_2x^*+Sw^*)\\=\;&B_2'(k)[-S_X(N,k+1)X(N)x^*(N+1)\\&\qquad+\sum_{j=N}^{\infty}S_X(j,k+1)[C_1'C_1+C_1'D_{11}-R_2'T_2^{-1}F_2]x^*(j)]\\=\;&0\text{ as }N\to\infty.\end{aligned}$$

Hence $v=0$ minimizes $\tilde{J}$ which implies $\bar{u}$ minimizes $J(u,w^*;k_0,x_0)$. Thus the value of the game $J(u,w;k_0,x_0)$ over $\mathbf{FI}\times l^2(k_0,\infty;\mathbf{R}^{m_1})$ exists.

Summing up we have the following.

Theorem 3.8 *Assume* **D1'** *and* (A,B_2,C_1) *is stabilizable and detectable. Suppose an IO-stabilizing controller with property (3.75) exists. Then there exists a unique bounded nonnegative solution ($A+B_1F_1+B_2F_2$ is exponentially stable) to the Riccati equation (3.59) and (3.60). Moreover* $\bar{u}\in\mathbf{FI}$ *and*

$$\begin{aligned}\sup_w\inf_{u\in\mathbf{FI}}J(u,w;k_0,x_0)&=J(\bar{u},w^*;k_0,x_0)\\&=J(\bar{u},\bar{w};k_0,x_0)=J(u^*,w^*;k_0,x_0)\\&=x_0'X(k_0)x_0=\inf_{u\in\mathbf{FI}}\sup_w J(u,w;k_0,x_0).\end{aligned}$$

If the system $\mathbf{G}$ *is* θ*-periodic, then* X *is also* θ*-periodic.*

Corollary 3.9 *Consider the system (3.70) and assume* **D2'** *and* (A,B_1,C_2) *is stabilizable and detectable. Suppose an IO-stabilizing controller of the form (3.71) with property*

$$|\tilde{z}_1|^2+\|\tilde{z}\|_2^2\leq d^2\|\tilde{w}\|_2^2\;\;\textit{for some}\;0<d<\gamma$$

exists. Then
(a) There exists a bounded nonnegative solution ($A + F'_{1Y}C_1 + F'_{2Y}C_2$ is exponentially stable) to the Riccati equation (3.72)-(3.74).
(b) For $\epsilon > 0$ small, there exists a bounded nonnegative stabilizing solution $Y_{\gamma-\epsilon}$ of (3.72)-(3.74) with γ replaced by $\gamma - \epsilon$.
If the system (3.70) is θ-periodic, then there exists a θ-periodic nonnegative stabilizing solution Y_θ of (3.72) and (3.73) such that $Y(k) - Y_\theta(k) \to 0$ as $k \to \infty$.

3.3 H_∞ Control

In this section we consider H_∞-control problems with initial uncertainty as in Section 2.3. First we shall introduce the general framework for H_∞-control and define our main problems. Then we consider two special problems called the full information- and the disturbance feedforward problems.

3.3.1 Main Results

Consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned} \tag{3.85}$$

with

$$z_1 = Fx(N+1), \tag{3.86}$$

$$x(k_0) = Hh \tag{3.87}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement, $h \in \mathbf{R}^{n_1}$, $F \in \mathbf{R}^{q\times n}$, $H \in \mathbf{R}^{n\times n_1}$ and A, B_1, etc are bounded matrices of appropriate dimensions. For this system we assume

$$\begin{aligned} &\mathbf{D1}: \quad D'_{12}(k)\,[\,C_1(k) \quad D_{12}(k)\,] = [\,0 \quad I\,] \text{ for any } k, \\ &\mathbf{D2}: \quad D_{21}(k)\,[\,B'_1(k) \quad D'_{21}(k)\,] = [\,0 \quad I\,] \text{ for any } k. \end{aligned}$$

Consider a controller $u = Ky$ of the form

$$\hat{x}(k+1) = \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \tag{3.88}$$

$$u(k) = \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k),$$

$$\hat{x}(k_0) = 0 \tag{3.89}$$

for some bounded matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$. Let $\gamma > 0$ be given. Then the H_∞-control on $[k_0, N]$ with initial uncertainty is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., a controller such that

$$\| z \|_2^2 + | z_1 |^2 \leq d^2(| h |^2 + \| w \|_2^2) \text{ for some } 0 < d < \gamma.$$

Without loss of generality we assume that H and F have full column rank and full row rank, respectively.

To give the solution of this problem, we introduce the following Riccati equations

$$\begin{aligned}
V(k) &> aI \text{ for some } a > 0, && (3.90)\\
X(k) &= A'X(k+1)A + C_1'C_1 \\
&\quad -(R_2'T_2^{-1}R_2)(k) + (F_1'VF_1)(k), && (3.91)\\
X(N+1) &= F'F, && (3.92)\\
H'X(k_0)H &\leq d^2 I \text{ for some } 0 < d < \gamma && (3.93)
\end{aligned}$$

and

$$\begin{aligned}
V_Y(k) &> aI \text{ for some } a > 0, && (3.94)\\
Y(k+1) &= AY(k)A' + B_1B_1' \\
&\quad -(R_{2Y}'T_{2Y}^{-1}R_{2Y})(k) + (F_{1Y}'V_YF_{1Y})(k), && (3.95)\\
Y(k_0) &= HH' && (3.96)
\end{aligned}$$

where

$T_1(k) = \gamma^2 I - B_1'X(k+1)B_1,$
$T_2(k) = I + B_2'X(k+1)B_2,$
$R_1(k) = B_1'X(k+1)A,$
$R_2(k) = B_2'X(k+1)A,$
$S(k) = B_2'X(k+1)B_1,$
$V(k) = (T_1 + S'T_2^{-1}S)(k),$
$F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k),$
$F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k),$
$T_{1Y}(k) = \gamma^2 I - C_1Y(k)C_1',$
$T_{2Y}(k) = I + C_2Y(k)C_2',$
$R_{1Y}(k) = C_1Y(k)A',$
$R_{2Y}(k) = C_2Y(k)A',$
$S_Y(k) = C_2Y(k)C_1',$
$V_Y(k) = (T_{1Y} + S_Y'T_{2Y}^{-1}S_Y)(k),$
$F_{1Y}(k) = [V_Y^{-1}(R_{1Y} - S_Y'T_{2Y}^{-1}R_{2Y})](k),$
$F_{2Y}(k) = -[T_{2Y}^{-1}(R_{2Y} + S_YF_{1Y})](k)$

and we have omitted k in all system matrices of (3.85). We also need the following Riccati equation depending on X:

$$\begin{aligned}
V_Z(k) &> aI \text{ for some } a > 0, && (3.97)\\
Z(k+1) &= A_XZ(k)A_X' + B_{1X}B_{1X}' \\
&\quad -(R_{2Z}'T_{2Z}^{-1}R_{2Z})(k) + (F_{1Z}'V_ZF_{1Z})(k), && (3.98)\\
Z(k_0) &= H(I - \frac{1}{\gamma^2}H'X(k_0)H)^{-1}H' && (3.99)
\end{aligned}$$

where

$$A_X(k) = (A + B_1F_1)(k), \qquad B_{1X}(k) = \gamma(B_1V^{-\frac{1}{2}})(k),$$
$$C_{1X}(k) = [T_2^{-\frac{1}{2}}(R_2 + SF_1)](k), \qquad D_{11X}(k) = \gamma(T_2^{-\frac{1}{2}}SV^{-\frac{1}{2}})(k),$$
$$D_{12X}(k) = T_2^{\frac{1}{2}}(k), \qquad D_{21X}(k) = \gamma(D_{21}V^{-\frac{1}{2}})(k),$$
$$T_{1Z}(k) = \gamma^2 I - D_{11X}D'_{11X} - C_{1X}Z(k)C'_{1X}, \qquad T_{2Z}(k) = I + C_2Z(k)C'_2,$$
$$R_{1Z}(k) = C_{1X}Z(k)A'_X + D_{11X}B'_{1X}, \qquad R_{2Z}(k) = C_2Z(k)A'_X,$$
$$S_Z(k) = C_2Z(k)C'_{1X},$$
$$V_Z(k) = (T_{1Z} + S'_ZT_{2Z}^{-1}S_Z)(k),$$
$$F_{1Z}(k) = [V_Z^{-1}(R_{1Z} - S'_ZT_{2Z}^{-1}R_{2Z})](k),$$
$$F_{2Z}(k) = -[T_{2Z}^{-1}(R_{2Z} + S_ZF_{1Z})](k).$$

We can rewrite (3.91), (3.95) and (3.98) as

$$\begin{aligned} X(k) &= A'X(k+1)A + C_1'C_1 - \left(\begin{bmatrix} R_2 \\ R_1 \end{bmatrix}' \begin{bmatrix} T_2 & S \\ S' & -T_1 \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ R_1 \end{bmatrix} \right)(k), \\ Y(k+1) &= AY(k)A' + B_1B_1' - \left(\begin{bmatrix} R_{2Y} \\ R_{1Y} \end{bmatrix}' \begin{bmatrix} T_{2Y} & S_Y \\ S_Y' & -T_{1Y} \end{bmatrix}^{-1} \begin{bmatrix} R_{2Y} \\ R_{1Y} \end{bmatrix} \right)(k) \end{aligned}$$

and

$$Z(k+1) = A_XZ(k)A'_X + B_{1X}B'_{1X} - \left(\begin{bmatrix} R_{2Z} \\ R_{1Z} \end{bmatrix}' \begin{bmatrix} T_{2Z} & S_Z \\ S_Z' & -T_{1Z} \end{bmatrix}^{-1} \begin{bmatrix} R_{2Z} \\ R_{1Z} \end{bmatrix} \right)(k)$$

respectively. Here note that the existence and uniform boundedness of

$$\begin{bmatrix} T_2 & S \\ S' & -T_1 \end{bmatrix}^{-1}, \begin{bmatrix} T_{2Y} & S_Y \\ S_Y' & -T_{1Y} \end{bmatrix}^{-1} \text{and} \begin{bmatrix} T_{2Z} & S_Z \\ S_Z' & -T_{1Z} \end{bmatrix}^{-1}$$

are guaranteed by (3.90), (3.94) and (3.97), respectively. We can also rewrite

$$\begin{aligned} A_{Xcl}(k) &= (A + B_1F_1 + B_2F_2)(k), \\ A_{Ycl}(k) &= (A + F'_{1Y}C_1 + F'_{2Y}C_2)(k) \end{aligned}$$

and

$$A_{Zcl}(k) = (A_X + F'_{1Z}C_{1X} + F'_{2Z}C_2)(k)$$

as

$$\begin{aligned} A_{Xcl}(k) &= \left(A - [B_2 \quad B_1] \begin{bmatrix} T_2 & S \\ S' & -T_1 \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ R_1 \end{bmatrix} \right)(k), \\ A_{Ycl}(k) &= \left(A - \begin{bmatrix} R_{2Y} \\ R_{1Y} \end{bmatrix}' \begin{bmatrix} T_{2Y} & S_Y \\ S_Y' & -T_{1Y} \end{bmatrix}^{-1} \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \right)(k) \end{aligned}$$

and

$$A_{Zcl}(k) = \left(A_X - \begin{bmatrix} R_{2Z} \\ R_{1Z} \end{bmatrix}' \begin{bmatrix} T_{2Z} & S_Z \\ S_Z' & -T_{1Z} \end{bmatrix}^{-1} \begin{bmatrix} C_2 \\ C_{1X} \end{bmatrix} \right)(k)$$

respectively. As in the continuous-time case we have the following results.

Lemma 3.22 *(a) Suppose X, Y and Z are solutions of (3.91), (3.95) and (3.98), respectively. If $Z(s) - Y(s) - \frac{1}{\gamma^2}Z(s)X(s)Y(s) = 0$ for some $s \geq k_0$, then $Z(k) - Y(k) - \frac{1}{\gamma^2}Z(k)X(k)Y(k) = 0$, for all $k \geq s$.*
(b) If (3.93), (3.96) and (3.99) hold, then

$$Z(k_0) - Y(k_0) - \frac{1}{\gamma^2}Z(k_0)X(k_0)Y(k_0) = 0.$$

Lemma 3.23 *(a) Let X, Y and Z be solutions of (3.91), (3.95) and (3.98), respectively. Suppose x satisfies*

$$x(k) = A'_{Ycl}(k)x(k+1) \tag{3.100}$$

then $\tilde{x}(k) = (I - \frac{1}{\gamma^2}XY)(k)x(k)$ satisfies

$$\tilde{x}(k) = A'_{Zcl}(k)\tilde{x}(k+1). \tag{3.101}$$

(b) Let X, Y and Z be bounded solutions on $[k_0, \infty)$ of (3.91), (3.95) and (3.98), respectively. Assume that $I - \frac{1}{\gamma^2}XY$ has a bounded inverse on $[k_0, \infty)$. Then A_{Zcl} is exponentially stable if and only if so is A_{Ycl}.

We give the proofs of Lemmas 3.22 and 3.23 in section 3.3.5. The following are our main results.

Theorem 3.9 *Assume* **D1** *and* **D2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[k_0, N]$ if and only if the following hold:
(i) There exists a nonnegative solution $X(k)$, $k \in [k_0, N+1]$ to (3.90)-(3.93).
(ii) There exists a nonnegative solution $Z(k)$, $k \in [k_0, N+1]$ to (3.97)-(3.99).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}_1(k)y(k) + \hat{B}_2(k)\hat{v}(k), \\ u(k) &= \hat{C}_1(k)\hat{x}(k) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)\hat{v}(k), \\ \hat{r}(k) &= \hat{C}_2(k)\hat{x}(k) + \hat{D}_{21}(k)y(k), \\ \hat{v} &= Q\hat{r}, \quad \| Q \| < \gamma, \\ \hat{x}(k_0) &= 0 \end{aligned} \tag{3.102}$$

where $\hat{A}(k) = [A_{Xcl} - (R'_{2Z} - B_2T_2^{-\frac{1}{2}}S_Z)T_{2Z}^{-1}C_2](k)$ and

$$\begin{aligned} &\hat{B}_1(k) = [(R'_{2Z} - B_2T_{2X}^{-\frac{1}{2}}S_Z)T_{2Z}^{-1}](k), \quad &&\hat{B}_2(k) = \tfrac{1}{\gamma}(F'_{1Z} + B_2T_2^{-\frac{1}{2}}V_Z^{\frac{1}{2}})(k), \\ &\hat{C}_1(k) = -(F_2 + T_2^{-\frac{1}{2}}S'_ZT_{2Z}^{-1}C_2)(k), &&\hat{C}_2(k) = -(T_{2Z}^{-\frac{1}{2}}C_2)(k), \\ &\hat{D}_{11}(k) = -(T_2^{-\frac{1}{2}}S'_ZT_{2Z}^{-1})(k), &&\hat{D}_{12}(k) = \tfrac{1}{\gamma}(T_2^{-\frac{1}{2}}V_Z^{\frac{1}{2}})(k), \\ &\hat{D}_{21}(k) = T_{2Z}^{-\frac{1}{2}}(k). \end{aligned}$$

Theorem 3.10 *Assume* **D1** *and* **D2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[k_0, N]$ if and only if the following hold:
(i) There exists a nonnegative solution $X(k)$, $k \in [k_0, N+1]$ to (3.90)-(3.93).
(ii) There exists a nonnegative solution $Y(k)$, $k \in [k_0, N+1]$ to (3.94)-(3.96).
(iii) $\rho(X(k)Y(k)) \leq d^2$ for any $k \in [k_0, N+1]$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (3.102) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$.

We now consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k), \\ x(k_0) &= Hh \end{aligned}$$

on $[k_0, \infty)$ and the controller $u = Ky$ of the form (3.88) and (3.89). Here we assume **D1**, **D2** and

D3 : (A, B_1, C_1) is stabilizable and detectable,

D4 : (A, B_2, C_2) is stabilizable and detectable.

Then the H_∞-control is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., an internally stabilizing controller such that

$$\| z \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2) \quad \text{for some } 0 < d < \gamma.$$

To give the solution of this problem, we need the following definition.

Definition 3.14 *(a) The solution X of (3.91) is called a stabilizing solution if A_{Xcl} is exponentially stable.*
(b) The solution Y of (3.95) is called a stabilizing solution if A_{Ycl} is exponentially stable.
(a) The solution Z of (3.98) is called a stabilizing solution if A_{Zcl} is exponentially stable.

As in Theorem 3.4, we have the following properties for stabilizing solutions.

Lemma 3.24 *(a) A bounded stabilizing solution of (3.91), if one exists, is unique.*
(b) Let Y and $\bar{Y}$ be two stabilizing solutions of (3.95). Then $Y(k) - \bar{Y}(k) \to 0$ as $k \to \infty$.
(c) Let Z and $\bar{Z}$ be two stabilizing solutions of (3.98). Then $Z(k) - \bar{Z}(k) \to 0$ as $k \to \infty$.

Theorem 3.11 *Assume* **D1-D4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[k_0, \infty)$ if and only if the following hold:
(i) There exists a bounded nonnegative stabilizing solution $X(k)$, $k \in [k_0, \infty)$ to (3.90), (3.91) and (3.93).
(ii) There exists a bounded nonnegative stabilizing solution $Z(k)$, $k \in [k_0, \infty)$ to (3.97)-(3.99).
(b) In this case the set of all γ-suboptimal controllers is given by (3.102) with Q internally stable.

Theorem 3.12 *Assume* **D1-D4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[k_0, \infty)$ if and only if the following hold:
(i) There exists a bounded nonnegative stabilizing solution $X(k)$, $k \in [k_0, \infty)$ to (3.90), (3.91) and (3.93).
(ii) There exists a nonnegative solution $Y(k)$, $k \in [k_0, \infty)$ to (3.94)-(3.96).
(iii) $\rho(X(k)Y(k)) \le d^2$ for any $k \in [k_0, \infty)$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (3.102) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ and Q internally stable.

Now we assume that the system **G** is θ-periodic and the conditions **D1-D4** hold. Then by Theorem 3.8 and Corollary 3.9 the solution X in Theorems 3.11 and 3.12 is θ-periodic and there exist θ-periodic nonnegative stabilizing solutions Y_θ and Z_θ such that

$$\lim_{n\to\infty} Y(k + n\theta) = Y_\theta(k), \quad \lim_{n\to\infty} Z(k + n\theta) = Z_\theta(k).$$

If we further assume $h = 0$, then we have the following corollaries.

Corollary 3.10 *(a) There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a θ-periodic nonnegative stabilizing solution to (3.90) and (3.91).
(ii) There exists a θ-periodic nonnegative stabilizing solution to (3.97) and (3.98).
(b) In this case the controllers given by (3.102) with internally stable Q are γ-suboptimal. If Q is θ-periodic, the controllers (3.102) are θ-periodic.

Corollary 3.11 *(a) There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a θ-periodic nonnegative stabilizing solution to (3.90) and (3.91).
(ii) There exists a θ-periodic nonnegative stabilizing solution to (3.94) and (3.95).
(iii) $\rho(X(k)Y(k)) \le d^2$ for any $t \in [k_0, k_0 + \theta)$ and for some $0 < d < \gamma$.
(b) In this case the controllers given by (3.102) with $Z = (I - \frac{1}{\gamma^2}YX)^{-1}Y$ and internally stable Q are γ-suboptimal. If Q is θ-periodic, they are θ-periodic.

Let the system **G** be time-invariant and assume the conditions **D1-D4**. Then we need the algebraic Riccati equations.

$$V > 0, \tag{3.103}$$
$$X = A'XA + C_1'C_1 - R_2'T_2^{-1}R_2 + F_1'VF_1, \tag{3.104}$$
$$V_Y > 0, \tag{3.105}$$
$$Y = AYA' + B_1B_1' - R_{2Y}'T_{2Y}^{-1}R_{2Y} + F_{1Y}'V_YF_{1Y}, \tag{3.106}$$
$$V_Z > 0, \tag{3.107}$$
$$Z = A_XZA_X' + B_{1X}B_{1X}' - R_{2Z}'T_{2Z}^{-1}R_{2Z} + F_{1Z}'V_ZF_{1Z}. \tag{3.108}$$

We define the stabilizing soutions of (3.104), (3.106) and (3.108) as in Definition 3.14. Without loss of generality, we can set $k_0 = 0$. Then we have the following corollaries.

Corollary 3.12 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (3.103) and (3.104) with $H'XH \leq d^2I$ for some $0 < d < \gamma$.
(ii) There exists a bounded nonnegative stabilizing solution of (3.97) and (3.98) with $Z(0) = H(I - \frac{1}{\gamma^2}H'XH)^{-1}H'$.
Moreover, there exists a nonnegative stabilizing solution Z_∞ of (3.107) and (3.108) such that $\lim_{k\to\infty} Z(k) = Z_\infty$.

Corollary 3.13 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (3.103) and (3.104) with $H'XH \leq d^2I$ for some $0 < d < \gamma$.
(ii) There exists a bounded nonnegative stabilizing solution of (3.94) and (3.95).
Moreover, there exists a nonnegative stabilizing solution Y_∞ of (3.105) and (3.106) such that $\lim_{t\to\infty} Y(t) = Y_\infty$.
(iii) $\rho(X_\infty Y(k)) \leq d^2$ for any $k \in [k_0, \infty)$ and for some $0 < d < \gamma$.

We further assume that there is no initial uncertainty, i.e., $h = 0$, we obtain the following.

Corollary 3.14 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (3.103) and (3.104).
(ii) There exists a nonnegative stabilizing solution Z_∞ of (3.107) and (3.108).

Corollary 3.15 *There exists a γ-suboptimal controller if and only if the following hold:*
(i) There exists a nonnegative stabilizing solution X_∞ of (3.103) and (3.104).
(ii) There exists a nonnegative stabilizing solution Y_∞ of (3.105) and (3.106).
(iii) $\rho(X_\infty Y_\infty) \leq d^2$ for some $0 < d < \gamma$.

3.3.2 Full Information Problem

Consider the system $\mathbf{G}_{FI}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= \begin{bmatrix} x \\ w \end{bmatrix}(k) \end{aligned} \tag{3.109}$$

with (3.87) and (3.86)

$$\begin{aligned} x(k_0) &= Hh, \\ z_1 &= Fx(N+1) \end{aligned}$$

where we assume **D1'**. The solution to this problem is needed to solve the H_∞-problem for the system **G**. Moreover, the filtering problem in Section 3.4 turns out to be the dual of this problem. Since the state x is now available, we may allow for nonzero initial condition for the controller

$$\hat{x}(k_0) = \hat{H}h \text{ for some } \hat{H}. \tag{3.110}$$

In this case the controller (3.88) and (3.110) is written as $u = K\begin{pmatrix} h \\ y \end{pmatrix}$. First we consider the finite horizon problem. For each controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ define the input-output operator G by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

Define the set of controllers

$$Q\begin{pmatrix} h \\ r \end{pmatrix} \in \mathcal{L}(\mathbf{R}^{n_1} \times l^2(k_0, N; \mathbf{R}^{m_1}); l^2(k_0, N; \mathbf{R}^{m_2}))$$

of the form (3.88) and (3.110):

$$\begin{aligned} Q_\gamma = \{Q : \| Q\begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 + h'H'X(k_0)Hh \le d^2(| h |^2 + \| r \|_2^2) \\ \text{for some } 0 < d < \gamma\}. \end{aligned} \tag{3.111}$$

Then we have the following.

Theorem 3.13 *Assume* **D1'**.
(a) There exists a controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ *of the form (3.88) and (3.110) such that* $\| G \| < \gamma$ *if and only if there exists a nonnegative solution* $X(k)$,

$k \in [k_0, N+1]$ to (3.90)-(3.93).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} u(k) &= -(T_2^{-1}R_2)(k)x(k) - (T_2^{-1}S)(k)w(k) \\ &\quad + T_2^{-\frac{1}{2}}(k)\left[Q\begin{pmatrix} h \\ \frac{1}{\gamma}V^{\frac{1}{2}}[w - F_1 x]\end{pmatrix}\right](k), \ Q \in \mathcal{Q}_\gamma. \end{aligned} \tag{3.112}$$

Proof. Suppose $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is γ-suboptimal. Then setting $h = 0$ and applying Theorem 3.7 we obtain an $X(k) \geq 0$, $k \in [k_0, N+1]$ satisfying (3.90)-(3.92). Moreover for (3.109) the following holds:

$$\begin{aligned} |\, z_1 \,|^2 + \|\, z \,\|_2^2 &= \gamma^2 \,\|\, w \,\|_2^2 + h'H'X(k_0)Hh - \gamma^2 \,\|\, \frac{1}{\gamma}V^{\frac{1}{2}}[w - F_1 x] \,\|_2^2 \\ &\quad + \|\, T_2^{\frac{1}{2}}[u + T_2^{-1}(Sw + R_2 x)] \,\|_2^2 . \end{aligned}$$

Setting $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ and $w = F_1 x$ we obtain

$$\begin{aligned} d^2(|\, h \,|^2 + \|\, w \,\|_2^2) &\geq |\, z_1 \,|^2 + \|\, z \,\|_2^2 \\ &\geq \gamma^2 \,\|\, w \,\|_2^2 + h'H'X(k_0)Hh. \end{aligned}$$

Hence

$$d^2 \,|\, h \,|^2 \geq h'H'X(k_0)Hh$$

which yields (3.93).

Sufficiency of (a) and the characterization in (b) follow from Lemmas 3.25 and 3.26 below.

To complete the proof we consider

$$\begin{aligned} x(k+1) &= A_I x(k) + B_{1I} w(k) + B_{2I} v(k), \\ z(k) &= C_{1I} x(k) + D_{11I} w(k) + D_{12I} v(k), \\ r(k) &= C_{2I} x(k) + D_{21I} w(k), \\ x(k_0) &= Hh \end{aligned} \tag{3.113}$$

with $z_1 = Fx(N+1)$ and

$$\begin{aligned} \bar{x}(k+1) &= A_X \bar{x}(k) + B_{1X} r(k) + B_2 u(k), \\ v(k) &= C_{1X} \bar{x}(k) + D_{11X} r(k) + D_{12X} u(k), \\ y(k) &= \begin{bmatrix} \bar{x} \\ F_1 \bar{x} + \gamma V^{-\frac{1}{2}} r \end{bmatrix}(k), \\ \bar{x}(k_0) &= Hh \end{aligned} \tag{3.114}$$

where

$$\begin{aligned}
&A_I(k) = (A - B_2T_2^{-1}R_2)(k), \\
&B_{1I}(k) = (B_1 - B_2T_2^{-1}S)(k), \qquad B_{2I}(k) = (B_2T_2^{-\frac{1}{2}})(k), \\
&C_{1I}(k) = (C_1 - D_{12}T_2^{-1}R_2)(k), \qquad C_{2I}(k) = -\tfrac{1}{\gamma}(V^{\frac{1}{2}}F_1)(k), \\
&D_{11I}(k) = (D_{11} - D_{12}T_2^{-1}S)(k), \quad D_{12I}(k) = (D_{12}T_2^{-\frac{1}{2}})(k), \\
&D_{21I}(k) = \tfrac{1}{\gamma}V^{\frac{1}{2}}(k).
\end{aligned}$$

Lemma 3.25 *Let X be the solution of (3.90)-(3.93).*
(a) For (3.113) the following holds:

$$|\, z_1 \,|^2 + \|\, z \,\|_2^2 = \gamma^2 \,\|\, w \,\|_2^2 + h'H'X(k_0)Hh + \|\, v \,\|_2^2 - \gamma^2 \,\|\, r \,\|_2^2 \,. \tag{3.115}$$

(b) The system $\mathbf{G}_{FI}$ with controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to the interconnection of (3.113) and the feedback system (3.114) with $u = K\begin{pmatrix} h \\ y \end{pmatrix}$.

Proof. (a) follows from direct calculation. Noting that $e = x - \bar{x}$ satisfies

$$e(k+1) = A_{X_{cl}}e(k), \;\; e(k_0) = 0$$

we can show (b) as in Lemma 2.20. ∎

Now introduce a feedback

$$v = Q\begin{pmatrix} h \\ r \end{pmatrix} \tag{3.116}$$

to (3.113), where Q is of the form (3.88) and (3.110).

Lemma 3.26 *Consider the closed-loop system (3.113) and (3.116). Let*

$$G\begin{pmatrix} h \\ w \end{pmatrix} = \begin{pmatrix} z_1 \\ z \end{pmatrix}$$

be the input-output operator. Then $\|\, G \,\| < \gamma$ if and only if $Q \in \mathcal{Q}_\gamma$.

Proof. For each $r_0 \in l^2(k_0, N; \mathbf{R}^{m_1})$ there exists a $w \in l^2(k_0, N; \mathbf{R}^{m_1})$ such that the internal signal r in (3.113) and (3.116) coincides with r_0 and

$$c_1(|\, h \,|^2 + \|\, r_0 \,\|_2^2) \le |\, h \,|^2 + \|\, w \,\|_2^2 \le c_2(|\, h \,|^2 + \|\, r_0 \,\|_2^2) \tag{3.117}$$

for some $c_i > 0$, $i = 1, 2$. Now suppose $\|\, G \,\| < \gamma$ for (3.113) and (3.116). Then for some $0 < d < \gamma$

$$\begin{aligned}
& d^2(|\, h \,|^2 + \|\, w \,\|_2^2) \\
\ge \;& |\, z_1 \,|^2 + \|\, z \,\|_2^2 \\
= \;& \gamma^2 \,\|\, w \,\|_2^2 + h'H'X(k_0)Hh + \left\|\, Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \,\|\, r \,\|_2^2 \text{ by (3.115).}
\end{aligned}$$

Hence

$$
\begin{aligned}
&\| Q \begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 + h'H'X(k_0)Hh \\
\leq\ & \gamma^2(| h |^2 + \| r \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \\
\leq\ & [\gamma^2 - c_1(\gamma^2 - d^2)](| h |^2 + \| r \|_2^2) \qquad \text{by (3.117)}
\end{aligned}
$$

which implies $Q \in Q_\gamma$.

Conversely, let $Q \in Q_\gamma$. Then

$$
\begin{aligned}
| z_1 |^2 + \| z \|_2^2 &= \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh + \| Q \begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 - \gamma^2 \| r \|_2^2 \\
&\leq \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2) \\
&\leq (\gamma^2 - \frac{\gamma^2 - d^2}{c_2})(| h |^2 + \| w \|_2^2).
\end{aligned}
$$

Hence $\| G \| < \gamma$. ∎

Remark 3.3 If $\| G \| < \gamma$, then as in Remark 2.4 $Q \in Q'_\gamma$ where

$$
\begin{aligned}
Q'_\gamma &= \{Q : \| Q \begin{pmatrix} h \\ r \end{pmatrix} \|_2^2 \leq d^2(| \tilde{h} |^2 + \| r \|_2^2) \text{ for some } 0 < d < \gamma\}, \\
\tilde{h} &= \left(I - \frac{1}{\gamma^2} H'X(k_0)H\right)^{\frac{1}{2}} h.
\end{aligned}
$$

To conclude the proof of Theorem 3.13, we note that u given by (3.112) is γ-suboptimal by Lemma 3.26. Now let $u = K \begin{pmatrix} h \\ y \end{pmatrix}$ be an arbitrary γ-suboptimal controller. Let Q be the input-output operator of the closed-loop system (3.114) with $u = K \begin{pmatrix} h \\ y \end{pmatrix}$. Then by Lemma 3.26, $Q \in Q_\gamma$. Hence $u = K \begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to

$$
\begin{aligned}
u(k) &= T_2^{-\frac{1}{2}}[v - T_2^{-\frac{1}{2}}(R_2 + SF_1)\bar{x} - \gamma T_2^{-\frac{1}{2}} S V^{\frac{1}{2}} r](k) \\
&= -T_2^{-1} R_2 x(k) - T_2^{-1} S w(k) + T_2^{-\frac{1}{2}} Q \begin{pmatrix} h \\ \frac{1}{\gamma} V^{\frac{1}{2}}[w - F_1 x] \end{pmatrix}
\end{aligned}
$$

which implies (b) of Theorem 3.13. ∎

Next we consider the system $\mathbf{G}_{FI}$ on the infinite horizon $[k_0, \infty)$. In this case we assume

D5 : (A, B_2, C_1) is stabilizable and detectable.

For each IO-stabilizing controller we can define the input-output operator as follows:

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}.$$

The notion of IO-stabilizing controller is needed when we consider the filtering problem, for which internal stability is not in general expected.

Theorem 3.14 *Assume* **D1'** *and* **D5.**
(a) There exists an IO-stabilizing controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ *on* $[k_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* $X(k)$, $k \in [k_0, \infty)$ *to (3.90), (3.91) and (3.93).*
(b) In this case the set of all such controllers is given by

$$\begin{aligned} u(k) &= -(T_2^{-1}R_2)(k)x(k) - (T_2^{-1}S)(k)w(k) \\ &\quad + T_2^{-\frac{1}{2}}(k)\left[Q\begin{pmatrix} h \\ \frac{1}{\gamma}V^{\frac{1}{2}}[w - F_1x] \end{pmatrix}\right](k), \quad Q \in Q_\gamma \end{aligned} \tag{3.118}$$

where $Q_\gamma \subset \mathcal{L}(\mathbf{R}^{n_1} \times l^2(k_0, \infty; \mathbf{R}^{m_1}); l^2(k_0, \infty; \mathbf{R}^{m_2}))$ *is defined as in (3.111). In particular the set of all internally stabilizing controllers with* $\| G \| < \gamma$ *is given by (3.118) with internally stable* Q.

Proof. **(i) Necessity of (a).** Suppose there exists an IO-stabilizing controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ such that $\| G \| < \gamma$. Consider the system $\mathbf{G}_{FI}$ with $h = 0$. Then for each $w \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ there exists a control $u \in l^2(k_0, \infty; \mathbf{R}^{m_2})$ such that $\| z \|_2 \leq d \| w \|_2$ for some $0 < d < \gamma$. Then by Theorem 3.8, there exists a bounded nonnegative stabilizing solution of (3.90) and (3.91) under the assumptions **D1'** and **D5**. To show (3.93) consider the restriction of $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ on $[k_0, N]$. Then we obtain the solution X_N of (3.90) and (3.91) satisfying (3.93) and $X_N(N+1) = 0$. Since $X_N(k)$ converges to $X(k)$ on $[k_0, \infty)$ we conclude $H'X(k_0)H \leq d^2 I$.

Sufficiency of (a) and the characterization of all γ-suboptimal controllers will be shown below. Consider systems (3.113) and (3.114) on $[k_0, \infty)$. Note that $A - B_2T_2^{-1}R_2$ is exponentially stable by Lemma 3.21 and hence we have as in Lemma 3.25

$$\| z \|_2^2 = \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh + \| v \|_2^2 - \gamma^2 \| r \|_2^2 . \tag{3.119}$$

The system $\mathbf{G}_{FI}$ with a controller $u = K\begin{pmatrix} h \\ y \end{pmatrix}$ is equivalent to the interconnection of (3.113) with the feedback system (3.114) with $u = K\begin{pmatrix} h \\ y \end{pmatrix}$.

First we assume $h = 0$ and consider (3.113) with feedback

$$v = Qr \tag{3.120}$$

where Q is of the form (3.88) and (3.89).

Lemma 3.27 *Consider the closed-loop system (3.113) and (3.120) and let $Gw = z$ be the input-output operator. Suppose Q is IO-stabilizing. Then*
(a) x, r, v are square summable and

$$\| x \|_2, \ \| r \|_2, \ \| v \|_2 \leq a \| w \|_2 \ \textit{for some } a > 0.$$

(b) If $\| G \| < \gamma$, then the map: $w \to r$ is onto and Q is IO-stable.
(c) $\| G \| < \gamma$ if and only if Q is IO-stable with $\| Q \| < \gamma$.
(d) If, further, Q is internally stabilizing then Q is internally stable.

Proof. (a) Since $z \in l^2(k_0, \infty; \mathbf{R}^{p_1})$, **D1'** implies $C_1 x$ and $T_2^{\frac{1}{2}} v - T_2^{-1} R_2 x$ are l^2 and

$$\| C_1 x \|_2, \ \| T_2^{\frac{1}{2}} v - T_2^{-1} R_2 x \|_2 \leq a \| w \|_2 \ \text{for some } a > 0.$$

Now we write (3.113) as

$$\begin{aligned} x(k+1) &= (A - JC_1)x(k) + JC_1 x(k) + (B_1 - B_2 T_2^{-1} S) w(k) \\ &\qquad + B_2 [T_2^{\frac{1}{2}} v(k) - T_2^{-1} R_2 x(k)], \\ x(k_0) &= 0 \end{aligned}$$

where J is a bounded matrix such that $A - JC_1$ is exponentially stable. Hence x is l^2 and $\| x \|_2 \leq a \| w \|_2$ for some $a > 0$. The rest is an immediate consequence of this.
(b) We write (3.113) as

$$\begin{pmatrix} z \\ r \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix}.$$

Then P_{ij} are exponentially stable. Moreover P_{21}^{-1} is realized by

$$\begin{aligned} x(k+1) &= A_{Xcl} x(k) + \gamma (B_1 - B_2 T_2^{-1} S) V^{-\frac{1}{2}} r(k), \\ w(k) &= F_1 x(k) + \gamma V^{-\frac{1}{2}} r(k) \end{aligned}$$

which is exponentially stable. For the closed-loop system r and v are the solutions of

$$\begin{aligned} r &= P_{21} w + P_{22} v, \\ v &= Qr. \end{aligned}$$

By (3.119)

$$\| P_{22} \| \leq \frac{1}{\gamma} \ \text{and} \ \| v \|_2 \leq \gamma \| r \|_2 . \tag{3.121}$$

Now let $r_0 \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ be arbitrary and define

$$s = (I - P_{22}Q)r_0.$$

Then s is locally square summable. Now let s_N be the truncation of s at N so that $s_N \in l^2(k_0, \infty; \mathbf{R}^{m_1})$. Now set $w_N = P_{21}^{-1}s_N \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ and let r_N be the internal signal of the closed-loop system corresponding to w_N. Then $r_N = s_N + P_{22}Qr_N$ and by (3.121)

$$\| r_N - s_N \|_2 = \| P_{22}Qr_N \|_2 \leq \| r_N \|_2,$$

which implies

$$\| r_N - s_N \|_{l^2(k_0,N;\mathbf{R}^{m_1})} \leq \| r_N \|_{l^2(k_0,N;\mathbf{R}^{m_1})} .$$

Since $r_N = r_0$ on $[k_0, N]$ we conclude

$$\| r_0 - s_N \|_{l^2(k_0,N;\mathbf{R}^{m_1})} = \| P_{22}Qr_0 \|_{l^2(k_0,N;\mathbf{R}^{m_1})} \leq \| r_0 \|_{l^2(k_0,N;\mathbf{R}^{m_1})} \leq \| r_0 \|_2 .$$

Since N is arbitrary, $P_{22}Qr_0$ is l^2. Now set $w_0 = P_{21}^{-1}(I - P_{22}Q)r_0$. Then r_0 is the response to the input w_0 and the map: $w \to r$ is onto. Since $\| Qr \|_2 \leq \gamma \| r \|_2$, for any r, Q is IO-stable.
(c) Now let r be the response to w. Then from (b) we have

$$c_1 \| r \|_2 \leq \| w \|_2 \leq c_2 \| r \|_2 \quad \text{for some } c_i > 0,\ i = 1, 2. \tag{3.122}$$

Now assume Q is IO-stabilizing and $\| G \| < \gamma$. Then for some $0 < d < \gamma$

$$d^2 \| w \|_2^2 \geq \| z \|_2^2 = \gamma^2 \| w \|_2^2 + \| v \|_2^2 - \gamma^2 \| r \|_2^2 .$$

Hence

$$\begin{aligned} \| v \|_2^2 &\leq \gamma^2 \| r \|_2^2 - (\gamma^2 - d^2) \| w \|_2^2 \\ &\leq [\gamma^2 - c_1(\gamma^2 - d^2)] \| r \|_2^2, \end{aligned}$$

which implies $\| Q \| < \gamma$. The converse follows from (3.119) and (3.122) in a similar manner.
(d) If Q is internally stabilizing, then by Proposition 3.6 Q is stabilizable and detectable. But Q is IO-stable by (b). This together with Proposition 3.4 implies that Q is exponentially stable. ∎

Lemma 3.28 *Consider the closed-loop system (3.113) and (3.116). Let*

$$z = G\begin{pmatrix} h \\ w \end{pmatrix}$$

be the input-output operator.
(a) Q is IO-stabilizing and $\| G \| < \gamma$ if and only if Q is IO-stable and $Q \in Q_\gamma$.
(b) Q is internally stabilizing and $\| G \| < \gamma$ if and only if Q is internally stable and $Q \in Q_\gamma$.

Proof. (a) Suppose Q is IO-stabilizing and $\| G \| < \gamma$. We write

$$Q\begin{pmatrix} h \\ r \end{pmatrix} = Q_0 h + Q_1 r.$$

Setting $h = 0$, Q_1 is IO-stabilizing. Hence by Lemma 3.27, Q_1 is IO-stable and $\| Q_1 \| < \gamma$. Recall that r and v are written

$$\begin{aligned} r &= P_{20}h + P_{21}w + P_{22}v, \\ v &= Q_0 h + Q_1 r, \end{aligned} \tag{3.123}$$

where P_{20} is exponentially stable. Since v and $Q_1 r$ are l^2, $Q_0 h$ is also l^2 for any h. Hence Q_0 is bounded and Q is IO-stable. Since $\| G \| < \gamma$, for some $0 < d < \gamma$ we have

$$\begin{aligned} & d^2(| h |^2 + \| w \|_2^2) \\ & \qquad \geq \| z \|_2^2 = \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \| r \|_2^2. \end{aligned}$$

Hence

$$\begin{aligned} & \gamma^2(| h |^2 + \| r \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| w \|_2^2) \\ & \qquad \qquad \geq h'H'X(k_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2. \end{aligned}$$

Since $| h |^2 + \| r \|_2^2 \leq a(| h |^2 + \| w \|_2^2)$ for some $a > 0$, we conclude that

$$(\gamma^2 - \frac{\gamma^2 - d^2}{a})(| h |^2 + \| r \|_2^2) \geq h'H'X(k_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2.$$

Thus $Q \in \mathcal{Q}_\gamma$.

Conversely let Q be IO-stable and $Q \in \mathcal{Q}_\gamma$. Then for each $(h, w) \in \mathbf{R}^{n_1} \times l^2(k_0, \infty; \mathbf{R}^{m_1})$ there exists a unique $(v, r) \in l^2(k_0, \infty; \mathbf{R}^{m_2}) \times l^2(k_0, \infty; \mathbf{R}^{m_1})$ satisfying (3.123) such that

$$\| r \|_2^2, \ \| v \|_2^2 \leq a(| h |^2 + \| w \|_2^2).$$

The pair coincides with the signal r, v of the closed-loop system. Hence x and z are in l^2 and by virtue of (3.119)

$$\begin{aligned} \| z \|_2^2 &= \gamma^2 \| w \|_2^2 + h'H'X(k_0)Hh + \left\| Q\begin{pmatrix} h \\ r \end{pmatrix} \right\|_2^2 - \gamma^2 \| r \|_2^2 \\ &\leq \gamma^2 \| w \|_2^2 - d^2(| h |^2 + \| r \|_2^2) - \gamma^2 \| r \|_2^2 \ \text{ for some } 0 < d < \gamma \\ &= \gamma^2(| h |^2 + \| w \|_2^2) - (\gamma^2 - d^2)(| h |^2 + \| r \|_2^2). \end{aligned} \tag{3.124}$$

Now for each $(h, r_0) \in \mathbf{R}^{n_1} \times l^2(k_0, \infty; \mathbf{R}^{m_1})$ consider

$$\begin{aligned} x(k+1) &= A_{Xcl}x(k) + \gamma(B_1 - B_2T_2^{-1}S)V^{-\frac{1}{2}}r_0(k) + B_2T_2^{-\frac{1}{2}}v(k), \\ w_0(k) &= F_1 x(k) + \gamma V^{-\frac{1}{2}} r_0(k), \\ x(k_0) &= Hh, \\ v &= Q\begin{pmatrix} h \\ r_0 \end{pmatrix}. \end{aligned} \tag{3.125}$$

Then $w_0 \in l^2(k_0, \infty; \mathbf{R}^{m_1})$ and

$$| h |^2 + \| w_0 \|_2^2 \leq \frac{1}{a}(| h |^2 + \| r_0 \|_2^2) \text{ for some } a > 0.$$

Since (r, w) of the closed-loop system (3.113) and (3.116) is one of (r_0, w_0) above we conclude

$$\| z \|_2^2 \leq [\gamma^2 - a(\gamma^2 - d^2)](| h |^2 + \| w \|_2^2).$$

Hence $\| G \| < \gamma$. ∎

Now the proof of sufficiency of (a) and (b) in Theorem 3.14 follows from Lemma 3.28 as in the case of Theorem 3.13. ∎

Remark 3.4 It follows from Theorems 3.13 and 3.14 that the controllers u with $Q = 0$ in (3.112) and (3.118) are γ-suboptimal.

3.3.3 Disturbance Feedforward Problem

We consider the H_∞-problem for the special system $\mathbf{G}_{DF}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned} \tag{3.126}$$

with

$$\begin{aligned} z_1 &= Fx(N+1), \\ x(k_0) &= 0 \end{aligned}$$

where D_{21} is a nonsingular and its inverse is bounded. The H_∞ control problem for this system is called the disturbance feedforward (DF) problem and as in the continuous-time case it can be reduced to the FI problem. In fact consider the observer

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + [B_1 D_{21}^{-1}(y - C_2\hat{x})](k) + B_2(k)u(k), \\ \hat{x}(k_0) &= 0. \end{aligned}$$

Then $e = x - \hat{x}$ satisfies

$$e(k+1) = (A - B_1 D_{21}^{-1} C_2)(k)e(k), \;\; e(k_0) = 0$$

and hence $\hat{x} = x$. Moreover w is observable since

$$w(k) = D_{21}^{-1}(k)[y(k) - C_2(k)x(k)] = D_{21}^{-1}(k)[y(k) - C_2(k)\hat{x}(k)].$$

Thus we can use the controllers of the FI problem with $h = 0$:

$$\begin{aligned} u(k) = & -(T_2^{-1}R_2)(k)x(k) - (T_2^{-1}S)(k)w(k) \\ & + T_2^{-\frac{1}{2}}(k)[Q(\frac{1}{\gamma}V^{\frac{1}{2}}[w - F_1 x])](k), \;\; \| Q \| < \gamma. \end{aligned}$$

Theorem 3.15 *For each controller define* $\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw$ *and assume* **D1'**.
(a) There exists a controller $u = Ky$ *on* $[k_0, N]$ *such that* $\| G \| < \gamma$ *if and only if there exists a nonnegative solution* $X(k)$, $k \in [k_0, N+1]$ *of (3.90)-(3.92).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}_1(k)y(k) + \hat{B}_2(k)v(k), \quad \hat{x}(k_0) = 0, \\ u(k) &= \hat{C}_1(k)\hat{x}(k) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\ r(k) &= \hat{C}_2(k)\hat{x} + \hat{D}_{21}y(k), \\ v &= Qr, \ \| Q \| < \gamma \end{aligned} \tag{3.127}$$

where $\hat{A}(k) = [A - B_1 D_{21}^{-1} C_2 - B_2 T_2^{-1}(R_2 - S D_{21}^{-1} C_2)](k)$ *and*

$$\begin{array}{ll} \hat{B}_1(k) = [(B_1 - B_2 T_2^{-1} S) D_{21}^{-1}](k), & \hat{B}_2(k) = (B_2 T_2^{-\frac{1}{2}})(k), \\ \hat{C}_1(k) = -[T_2^{-1}(R_2 - S D_{21}^{-1} C_2)](k), & \hat{C}_2(k) = -\frac{1}{\gamma}[V^{\frac{1}{2}}(F_1 + D_{21}^{-1} C_2)](k), \\ \hat{D}_{11}(k) = -(T_2^{-1} S D_{21}^{-1})(k), & \hat{D}_{12}(k) = T_2^{-\frac{1}{2}}(k), \\ \hat{D}_{21}(k) = \frac{1}{\gamma}(V^{\frac{1}{2}} D_{21}^{-1})(k) & \end{array}$$

and Q *is a controller of the form (3.88) and (3.89).*

Proof. The necessity of (a) follows from Theorem 3.13. The sufficiency and (b) follow from Theorem 3.13 and the observation

$$\begin{aligned} u(k) &= -T_2^{-1} R_2 x(k) - T_2^{-1} S w(k) + T_2^{-\frac{1}{2}} Qr \\ &= -T_2^{-1}(R_2 - S D_{21}^{-1} C_2)\hat{x}(k) - T_2^{-1} S D_{21}^{-1} y(k) + T_2^{-\frac{1}{2}} Qr, \\ r(k) &= \frac{1}{\gamma} V^{\frac{1}{2}} [w(k) - F_1 x(k)] \\ &= \frac{1}{\gamma} V^{\frac{1}{2}} [-(F_1 + D_{21}^{-1} C_2)\hat{x}(k) + D_{21}^{-1} y(k)]. \end{aligned}$$

∎

We now consider the infinite horizon problem. We assume **D5** and

D6 : $A - B_1 D_{21}^{-1} C_2$ is exponentially stable.

Theorem 3.16 *Assume* **D1'**, **D5** *and* **D6**.
(a) There exists an internally stabilizing controller $u = Ky$ *on* $[k_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* X *for (3.90) and (3.91).*
(b) In this case the set of all γ*-suboptimal controllers is given by (3.127) with* Q *internally stable.*

Consider the H_∞-problem for the system $\mathbf{G}_{OE}$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k), \\ x(k_0) &= Hh \end{aligned} \tag{3.128}$$

where D_{12} is invertible and has bounded inverse. This problem is called the output estimation (OE) problem. The adjoint of (3.128) is the backward version of the DF problem. Hence modifying Theorems 3.15 and 3.16 we have the following.

Theorem 3.17 *For each controller define* $z = G\begin{pmatrix} h \\ w \end{pmatrix}$ *and assume* **D2'**.
(a) There exists a controller $u = Ky$ *on* $[k_0, N]$ *such that* $\| G \| < \gamma$ *if and only if there exists a nonnegative solution* $Y(k)$, $k \in [k_0, N+1]$ *of (3.94)-(3.96).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}_1(k)y(k) + \hat{B}_2(k)v(k), \quad \hat{x}(k_0) = 0, \\ u(k) &= \hat{C}_1(k)\hat{x}(k) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\ r(k) &= \hat{C}_2(k)\hat{x} + \hat{D}_{21}y(k), \\ v &= Qr, \ \| Q \| < \gamma \end{aligned} \tag{3.129}$$

where $\hat{A}(k) = [A - B_2D_{12}^{-1}C_1 - (R'_{2Y} - B_2D_{12}^{-1}S'_Y)T_{2Y}^{-1}C_2](k)$ *and*

$$\begin{aligned} &\hat{B}_1(k) = [(R'_{2Y} - B_2D_{12}^{-1}S'_Y)T_{2Y}^{-1}](k), && \hat{B}_2(k) = \tfrac{1}{\gamma}[(F'_{1Y} + B_2D_{12}^{-1})V_Y^{\frac{1}{2}}](k), \\ &\hat{C}_1(k) = -[D_{12}^{-1}(C_1 - S'_YT_{2Y}^{-1}C_2)](k), && \hat{C}_2(k) = -(T_{2Y}^{-\frac{1}{2}}C_2)(k), \\ &\hat{D}_{11}(k) = -(D_{12}^{-1}S'_YT_{2Y}^{-1})(k), && \hat{D}_{12}(k) = \tfrac{1}{\gamma}(D_{12}^{-1}V_Y^{\frac{1}{2}})(k), \\ &\hat{D}_{21}(k) = T_{2Y}^{-\frac{1}{2}}(k) \end{aligned}$$

and Q *is a controller of the form (3.88) and (3.89).*

Theorem 3.18 *Suppose* **D2'**, (A, B_1, C_2) *is stabilizable and detectable and that* $A - B_2D_{12}^{-1}C_1$ *is exponentially stable.*
(a) Then there exists an internally stabilizing controller $u = Ky$ *on* $[k_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* Y *for (3.94)-(3.96).*
(b) In this case the set of all γ*-suboptimal controllers is given by (3.129) with* Q *internally stable.*

To give the proofs of Theorems 3.17 and 3.18, we consider the FI- and DF problems for the backward systems below.

3.3.4 Backward Systems

Consider the backward system $\mathbf{G}_{FI}$:

$$\begin{aligned} x(k) &= A(k)x(k+1) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k+1) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= \begin{bmatrix} x(k+1) \\ w(k) \end{bmatrix}, \\ z_1 &= Fx(k_0) \end{aligned} \tag{3.130}$$

with $x(N+1) = 0$ and a controller $u = Ky$ of the form

$$\begin{aligned} \hat{x}(k) &= \hat{A}(k)\hat{x}(k+1) + \hat{B}(k)y(k), \ \hat{x}(N+1) = 0, \\ u(k) &= \hat{C}(k)\hat{x}(k+1) + \hat{D}(k)y(k) \end{aligned} \tag{3.131}$$

where all matrices are uniformly bounded. The H_∞-control problem for the system $\mathbf{G}_{FI}$ is the FI-problem and the solution to this problem is needed to the H_∞ filtering problem. We assume **D1'**. To give the solution of this problem, we need the following Riccati equation

$$V(k) > aI \text{ for some } a > 0, \tag{3.132}$$

$$P(k+1) = A'P(k)A + C_1'C_1 - (R_2'T_2^{-1}R_2)(k) + (F_1'VF_1)(k), \tag{3.133}$$

$$P(k_0) = F'F \tag{3.134}$$

where

$$\begin{aligned} &T_1(k) = \gamma^2 I - D_{11}'D_{11} - B_1'P(k)B_1, && T_2(k) = I + B_2'P(k)B_2, \\ &R_1(k) = B_1'P(k)A + D_{11}'C_1, && R_2(k) = B_2'P(k)A, \\ &S(k) = B_2'P(k)B_1, && V(k) = (T_1 + S'T_2^{-1}S)(k), \\ &F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k), && F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k) \end{aligned}$$

and we have omitted k in all system matrices of (3.130). Then we have the following result.

Theorem 3.19 *Assume* **D1'**.
(a) There exists a γ-suboptimal of the form (3.131) if and only if there exists a nonnegative solution $P(k)$, $k \in [k_0, N+1]$ of (3.132)-(3.134).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} u(k) &= -(T_2^{-1}R_2)(k)x(k+1) - (T_2^{-1}S)(k)w(k) \\ &\quad + T_2^{-\frac{1}{2}}(k)\left[Q\left(\frac{1}{\gamma}V^{\frac{1}{2}}[w - F_1x]\right)\right](k), \ \parallel Q \parallel < \gamma. \end{aligned} \tag{3.135}$$

Proof. Necessity of (a) follows from Corollary 3.8. Similar to the proof of Theorems 2.19 and 3.13, we can show the sufficiency of (a) and (b). ∎

Next we consider the system $\mathbf{G}_{FI}$ on the infinite horizon $[k_0, \infty)$. In this case we assume **D5**. For each IO-stabilizing controller we can define the input-output operator as follows:

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw \ \text{ on } [k_0, \infty).$$

Theorem 3.20 *Assume* **D1'** *and* **D5**.
(a) There exists an IO-stabilizing controller $u = Ky$ on $[k_0, \infty)$ such that

$\| G \| < \gamma$ if and only if there exists a bounded nonnegative stabilizing solution $P(k)$, $k \in [k_0, \infty)$ of (3.132)-(3.134).
(b) In this case the set of all such controllers is given by (3.135). In particular the set of all internally stabilizing controllers with $\| G \| < \gamma$ is given by (3.135) with internally stable Q.

Proof. Necessity of (a) follows from Corollary 3.9. Proof of sufficiency of (a) and (b) is similar to the proof of Theorems 2.20, 3.14 and 3.19. ∎

Corollary 3.16 *Let the system $\mathbf{G}_{FI}$ be θ-periodic and $F = 0$. Assume* **D1'** *and* **D5**. *Then*
(a) There exists an IO-stabilizing controller $u = Ky$ on $[k_0, \infty)$ such that $\| G \| < \gamma$ if and only if there exists a θ-periodic nonnegative stabilizing solution P of (3.132) and (3.133).
(b) In this case the controllers (3.135) are IO-stabilizing such that $\| G \| < \gamma$. If Q is θ-periodic, the controllers (3.135) are also θ-periodic.
In particular, if Q is internally stable, the controllers (3.135) are internally stabilizing.

We now consider the H_∞-control problem for the system $\mathbf{G}_{DF}$:

$$\begin{aligned} x(k) &= A(k)x(k+1) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k+1) + D_{11}(k)w(k) + D_{12}(k)u(k), \qquad (3.136) \\ y(k) &= C_2(k)x(k+1) + D_{21}(k)w(k), \\ z_1 &= Fx(k_0), \\ x(N+1) &= 0 \end{aligned}$$

and a controller $u = Ky$ of the form (3.131) where D_{21} is a nonsingular and its inverse is bounded. This problem is the DF-problem for the backward system. Since it can be reduced to the FI-problem for the backward system, we have the following result.

Theorem 3.21 *For each controller define $\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw$ and assume* **D1'**.
(a) There exists a controller $u = Ky$ on $[k_0, N]$ such that $\| G \| < \gamma$ if and only if there exists a nonnegative solution $P(k)$, $k \in [k_0, N+1]$ of (3.132)-(3.134).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} \hat{x}(k) &= \hat{A}(k)\hat{x}(k+1) + \hat{B}_1(k)y(k) + \hat{B}_2(k)v(k), \ \hat{x}(N+1) = 0, \\ u(k) &= \hat{C}_1(k)\hat{x}(k+1) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \qquad (3.137) \\ r(k) &= \hat{C}_2(k)\hat{x}(k+1) + \hat{D}_{21}(k)y(k), \\ v &= Qr, \ \| Q \| < \gamma \end{aligned}$$

where $\hat{A}(k) = [A - B_1 D_{21}^{-1} C_2 - B_2 T_2^{-1}(R_2 - S D_{21}^{-1} C_2)](k)$ *and*

$$\begin{array}{ll} \hat{B}_1(k) = [(B_1 - B_2 T_2^{-1} S) D_{21}^{-1}](k), & \hat{B}_2(k) = (B_2 T_2^{-\frac{1}{2}})(k), \\ \hat{C}_1(k) = -[T_2^{-1}(R_2 - S D_{21}^{-1} C_2)](k), & \hat{C}_2(k) = -\frac{1}{\gamma}[V^{\frac{1}{2}}(F_1 + D_{21}^{-1} C_2)](k), \\ \hat{D}_{11}(k) = -(T_2^{-1} S D_{21}^{-1})(k), & \hat{D}_{12}(k) = T_2^{-\frac{1}{2}}(k), \\ \hat{D}_{21}(k) = \frac{1}{\gamma}(V^{\frac{1}{2}} D_{21}^{-1})(k) & \end{array}$$

and Q *is a controller of the form (3.131).*

We consider the infinite horizon problem. We further assume **D5** and **D6**.

Theorem 3.22 *Assume* **D1'**, **D5** *and* **D6**.
(a) There exists an internally stabilizing controller $u = Ky$ *on* $[k_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution* $P(k)$, $k \in [k_0, \infty)$ *of (3.132)-(3.134).*
(b) In this case the set of all γ*-suboptimal controllers is given by (3.137) with* Q *internally stable.*

3.3.5 Proofs of Main Results

We now give the proofs of our main results using Theorems 3.17 and 3.18. We first prove Lemmas 3.22 and 3.23. To do so, we first rewrite the Riccati equations in compact forms. Using the equalities (provided all inverses exist)

$$\begin{aligned} E(I + LE)^{-1} &= (I + EL)^{-1} E, \quad E \in \mathbf{R}^{n \times m}, \ L \in \mathbf{R}^{m \times n}, \\ I - (I + G)^{-1} &= G(I + G)^{-1} = (I + G)^{-1} G, \quad G \in \mathbf{R}^{n \times n} \end{aligned}$$

we have

$$\begin{aligned} A_X(k) &= (A + B_1 F)(k) \\ &= [I + B_2 B_2' X(k+1)][I + (B_2 B_2' - \frac{1}{\gamma^2} B_1 B_1') X(k+1)]^{-1} A. \end{aligned}$$

Let $M(k) = I + B_2 B_2' X(k+1)$ and $N(k) = [M(k) - \frac{1}{\gamma^2} B_1 B_1' X(k+1)]^{-1}$. Then we can rewrite (3.91) as

$$\begin{aligned} X(k) &= C_1' C_1 + A' X(k+1) N(k) A \\ &= C_1' C_1 + A' X(k+1) M^{-1}(k) A_X. \end{aligned} \tag{3.138}$$

Similarly we can rewrite (3.95) and (3.98) as follows

$$Y(k+1) = B_1 B_1' + A Y(k) N_Y(k) A', \tag{3.139}$$

$$Z(k+1) = [I - \frac{1}{\gamma^2} \Phi(k) X(k+1) B_2 T_2^{-1}(k) B_2' X(k+1)]^{-1} \times \Phi(k) \tag{3.140}$$

where

$$\begin{aligned} N_Y(k) &= [I + C_2'C_2Y(k) - \frac{1}{\gamma^2}C_1'C_1Y(k)]^{-1}, \\ \Phi(k) &= (MN)(k)B_1B_1 + \Psi(k), \\ \Psi(k) &= A_X Z(k)[I + C_2'C_2Z(k)]^{-1}A_X'. \end{aligned}$$

We also have

$$\begin{aligned} A_{Ycl}(k) &= AN_Y'(k), \\ A_{Zcl}(k) &= [I - \frac{1}{\gamma^2}\Phi(k)X(k+1)B_2T_2^{-1}(k)B_2'X(k+1)]^{-1} \\ &\quad \times A_X[I + Z(k)C_2'C_2]^{-1}. \end{aligned} \tag{3.141}$$

By (3.140), we have

$$\begin{aligned} \Phi(k) &= Z(k+1)[I + \frac{1}{\gamma^2}X(k+1)B_2T_2^{-1}(k)B_2'X(k+1)Z(k+1)]^{-1} \\ &= [I + \frac{1}{\gamma^2}Z(k+1)X(k+1)B_2T_2^{-1}(k)B_2'X(k+1)]^{-1}Z(k+1) \end{aligned}$$

and hence we can rewrite (3.141) as

$$\begin{aligned} A_{Zcl}(k) &= [I + \frac{1}{\gamma^2}Z(k+1)X(k+1)B_2T_2^{-1}(k)B_2'X(k+1)] \\ &\quad \times A_X[I + Z(k)C_2'C_2]^{-1}. \end{aligned} \tag{3.142}$$

Proof of Lemma 3.22. We shall prove the equality by induction. Set

$$Q(k) = Z(k) - Y(k) - \frac{1}{\gamma^2}ZXY(k).$$

Let $k = k_0$. Then $Q(k) = 0$ since $Z(k_0) = 0 = Y(k_0)$. Now we assume $Q(k) = 0$. Then by Lemma 2.18,

$$\begin{aligned} Z(k) &= Y(k)(I - \frac{1}{\gamma^2}XY)^{-1}(k), \\ Y(k) &= Z(k)(I - \frac{1}{\gamma^2}XZ)^{-1}(k). \end{aligned}$$

Since $Q(k+1) = -Y(k+1) + [Z(I - \frac{1}{\gamma^2}XY)](k+1)$, it is enough to show

$$Y(k+1) = [Z(I - \frac{1}{\gamma^2}XY)](k+1). \tag{3.143}$$

Now

$$\begin{aligned} Y(k+1) &= B_1B_1' + AYN_YA' \\ &= B_1B_1' + AZ[I + C_2C_2'Z + \frac{1}{\gamma^2}(X - C_1C_1')Z]^{-1}A' \\ &= B_1B_1' + AZ[I + C_2C_2'Z + \frac{1}{\gamma^2}A'X(k+1)M^{-1}A_XZ]^{-1}A' \end{aligned}$$

where $Y(k) = Z(I + \frac{1}{\gamma^2}XZ)^{-1}(k)$ is substituted in the second equality and the Riccati equation (3.138) is used in the third equality. Since

$$\begin{aligned}
& AZ[I + C_2C_2'Z + \frac{1}{\gamma^2}A'\hat{X}M^{-1}A_XZ]^{-1}A' \\
= & (MN)^{-1}A_XZ(I + C_2C_2'Z)^{-1} \\
& \qquad \times[I + \frac{1}{\gamma^2}A'\hat{X}M^{-1}A_XZ(I + C_2C_2'Z)^{-1}]^{-1}A' \\
= & (MN)^{-1}A_XZ(I + C_2C_2'Z)^{-1}A' \\
& \qquad \times[I + \frac{1}{\gamma^2}\hat{X}M^{-1}A_XZ(I + C_2C_2'Z)^{-1}A']^{-1} \\
= & (MN)^{-1}A_XZ(I + C_2C_2'Z)^{-1}A_X' \\
& \qquad \times[(MN)' + \frac{1}{\gamma^2}\hat{X}M^{-1}A_XZ(I + C_2C_2'Z)^{-1}A_X']^{-1},
\end{aligned}$$

we have

$$\begin{aligned}
Y(k+1) &= (MN)^{-1}[MNB_1B_1' + \Psi(N'M' + \frac{1}{\gamma^2}\hat{X}M^{-1}\Psi)^{-1}] \\
&= (MN)^{-1}[MNB_1B_1' \\
& \qquad +\{I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}\}^{-1}\Psi(N'M')^{-1}] \\
&= (MN)^{-1}[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]^{-1} \\
& \qquad \times[MNB_1B_1' + \Psi(N'M')^{-1}(I + \frac{1}{\gamma^2}\hat{X}NB_1B_1')] \\
&= (MN)^{-1}[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]^{-1}\Phi \qquad (3.144)
\end{aligned}$$

where we set $\hat{X} = X(k+1)$ and we have used $N'M' = I + \frac{1}{\gamma^2}\hat{X}NB_1B_1'$ in the last equality. On the other hand

$$\begin{aligned}
& Z(k+1)[I - \frac{1}{\gamma^2}\hat{X}Y(k+1)] \\
= & [I - \frac{1}{\gamma^2}\Phi\hat{X}B_2T_2^{-1}B_2'\hat{X}]^{-1}\Phi \\
& \qquad \times\{I - \frac{1}{\gamma^2}\hat{X}(MN)^{-1}[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]^{-1}\Phi\} \\
= & [I - \frac{1}{\gamma^2}\Phi\hat{X}B_2T_2^{-1}B_2'\hat{X}]^{-1} \\
& \qquad \times\{I - \frac{1}{\gamma^2}\Phi\hat{X}(MN)^{-1}[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]^{-1}\}\Phi \\
= & [I - \frac{1}{\gamma^2}\Phi\hat{X}B_2T_2^{-1}B_2'\hat{X}]^{-1}
\end{aligned}$$

$$\times\{[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]MN - \frac{1}{\gamma^2}\Phi\hat{X}\}Y(k+1)$$

where we have used (3.140) and (3.144) in the second equality and (3.144) again in the last equality. By direct calculation we have

$$[I + \frac{1}{\gamma^2}\Psi(N'M')^{-1}\hat{X}M^{-1}]MN - \frac{1}{\gamma^2}\Phi\hat{X} = I - \frac{1}{\gamma^2}\Phi\hat{X}B_2T_2^{-1}B_2'\hat{X}$$

which implies (3.143). ∎

Proof of Lemma 3.23. We shall show (a) only. Using (3.142) and $Z(k) = Y(I - \frac{1}{\gamma^2}XY)^{-1}(k)$ we have

$$\begin{aligned} A'_{Zcl}(k) &= (I - \frac{1}{\gamma^2}XY)(I - \frac{1}{\gamma^2}XY + C_2'C_2Y)^{-1}A'_X \\ &\quad \times(I - \frac{1}{\gamma^2}\hat{X}\hat{Y} + \hat{X}B_2T_2^{-1}B_2'\hat{X}\hat{Y})(I - \frac{1}{\gamma^2}\hat{X}\hat{Y})^{-1} \end{aligned}$$

where $\hat{Y} = Y(k+1)$. Note

$$\begin{aligned} &(I - \frac{1}{\gamma^2}XY + C_2'C_2Y)^{-1}A'_X(I - \frac{1}{\gamma^2}\hat{X}\hat{Y} + \hat{X}B_2T_2^{-1}B_2'\hat{X}\hat{Y}) \\ = \; &(I + C_2'C_2Y - \frac{1}{\gamma^2}C_1C_1Y - \frac{1}{\gamma^2}A'\hat{X}NAY)^{-1}A'_X \\ &\quad \times(I - \frac{1}{\gamma^2}\hat{X}\hat{Y} + \hat{X}B_2T_2^{-1}B_2'\hat{X}\hat{Y}) \\ = \; &N_Y(I - \frac{1}{\gamma^2}A'\hat{X}NAYN_Y)^{-1}A'N'M'[I - \frac{1}{\gamma^2}(I + \hat{X}B_2B_2')^{-1}\hat{X}\hat{Y}] \\ = \; &N_YA'(I - \frac{1}{\gamma^2}\hat{X}NAYN_YA')^{-1}N'[M' - \frac{1}{\gamma^2}\hat{X}\hat{Y}] \end{aligned}$$

where we have used (3.138) in the second equality. By direct calculation, we have

$$\begin{aligned} (I - \frac{1}{\gamma^2}\hat{X}NAYN_YA')^{-1}N' &= [(N')^{-1} - \frac{1}{\gamma^2}(N')^{-1}\hat{X}NAYN_YA']^{-1} \\ &= [M' - \frac{1}{\gamma^2}\hat{X}B_1B_1' - \frac{1}{\gamma^2}\hat{X}AYN_YA']^{-1} \\ &= [M' - \frac{1}{\gamma^2}\hat{X}(B_1B_1' + AYN_YA')]^{-1} \\ &= (M' - \frac{1}{\gamma^2}\hat{X}\hat{Y})^{-1} \end{aligned}$$

where we have used $\hat{X}N = N'\hat{X}$ in the third equality and (3.139) in the last equality. Hence

$$A'_{Zcl}(k) = (I - \frac{1}{\gamma^2}XY)N_YA'(I - \frac{1}{\gamma^2}\hat{X}\hat{Y})^{-1}$$

$$= (I - \frac{1}{\gamma^2}XY)(k)A'_{Ycl}(k)(I - \frac{1}{\gamma^2}XY)^{-1}(k+1)$$

and we have shown the assertion. ∎

We now give the proofs of Theorems 3.9-3.12.

Proof of Theorem 3.9: Necessity of (a). Suppose that there exists a γ-suboptimal controller $u = Ky$ on $[k_0, N+1]$ for the system **G**. Then by Theorem 3.13 (i) holds. Now consider (3.113)

$$\begin{aligned} x(k+1) &= A_I x(k) + B_{1I} w(k) + B_{2I} v(k), \\ z(k) &= C_{1I} x(k) + D_{11I} w(k) + D_{12I} v(k), \\ r(k) &= C_{2I} x(k) + D_{21I} w(k), \\ x(k_0) &= Hh, \\ z_1 &= Fx(N+1) \end{aligned} \tag{3.145}$$

and

$$\begin{aligned} \bar{x}(k+1) &= A_X \bar{x}(k) + B_{1X} r(k) + B_2 u(k), \\ v(k) &= C_{1X} \bar{x}(k) + D_{11X} r(k) + D_{12X} u(k), \\ y(k) &= C_2 \bar{x}(k) + D_{21X} r(k), \\ \bar{x}(k_0) &= Hh \end{aligned} \tag{3.146}$$

with a controller

$$u = Ky. \tag{3.147}$$

Note that

$$\begin{aligned} D_{21X} B'_{1X} &= \gamma^2 D_{21} V^{-1} B'_1 \\ &= \gamma^2 D_{21} [\gamma^2 I - B'_1 X B_1 + B'_1 X B_2 T_2^{-1} B'_2 X B_1]^{-1} B'_1 \\ &= \gamma^2 D_{21} B'_1 [\gamma^2 I - X B_1 B'_1 + X B_2 T_2^{-1} B'_2 X B_1 B'_1]^{-1} \\ &= 0 \end{aligned}$$

and similarly $D_{21X} D'_{21X} = I$ and $D_{21X} D'_{11X} = 0$. Hence the condition **D2'** for the system (3.146) is satisfied. Then $e = x - \bar{x}$ satisfies

$$e(k+1) = (A + B_1 F_1 + B_2 F_2) e(k), \; e(k_0) = 0$$

and hence similar to Lemma 3.25, the system **G** with $u = Ky$ is equivalent to the interconnection of (3.145) and (3.146) with $u = Ky$. Let $\tilde{Q}$ be the input-output operator of the closed-loop system (3.146) and (3.147) so that $v = \tilde{Q}\begin{pmatrix} h \\ r \end{pmatrix}$. Then by Remark 3.3, $\tilde{Q} \in Q'_\gamma$. Hence $u = Ky$ is γ-suboptimal for the H$_\infty$-problem defined by (3.146) with H and h replaced by $\tilde{H} = H(I -$

$\frac{1}{\gamma^2}H'X(k_0)H)^{-\frac{1}{2}}$ and $\tilde{h} = (I - \frac{1}{\gamma^2}H'X(k_0)H)^{\frac{1}{2}}h$, respectively. Since $D_{12X} = T_2^{\frac{1}{2}}$, this is an OE problem and hence the condition (ii) holds.

Sufficiency of (a) and (b). Consider the systems (3.145) and (3.146). Then by Theorem 3.17, the set of the controllers $u = Ky$ given by (3.102) satisfies $\tilde{Q} \in Q'_\gamma$ where $\tilde{Q}$ is the input-output operator of the closed-loop system (3.146) and (3.102). Considering $e = \bar{x} - \hat{x}$ and the adjoint system as in the proof of Theorem 2.9, we can directly show that the controller (3.102) is γ-suboptimal, i.e., $\tilde{Q} \in Q_\gamma$. Hence sufficiency of (a) and (b) hold. ∎

Proof of Theorem 3.10: Necessity of (a). Suppose a γ-suboptimal controller exists. Then by Theorem 3.7 and Corollary 3.8, there exist nonnegative solutions X, Y and Z of (3.90)-(3.93), (3.94)-(3.96) and (3.97)-(3.99), respectively. By Lemmas 2.18 and 3.22, $I - \frac{1}{\gamma^2}XY$ is nonsingular and the set of eigenvalues of XY has the form

$$\frac{\gamma^2\lambda}{\gamma^2+\lambda},\ \lambda \in \lambda(XZ).$$

Since X and Z are nonnegative and uniformly bounded in N, $\lambda \in \lambda(XZ)$ are nonnegative and uniformly bounded. Hence $\rho(X(k)Y(k)) \le d^2$ for some $0 < d < \gamma$. Hence the condition (iii) holds.

Sufficiency of (a) and (b). Note that $I - \frac{1}{\gamma^2}X(k)Y(k)$ is nonsingular and $[I - \frac{1}{\gamma^2}X(k)Y(k)]^{-1}$ is uniformly bounded in $k \in [k_0, N+1]$. Define

$$Z(k) = Y(k)[I - \frac{1}{\gamma^2}X(k)Y(k)]^{-1},\ k \in [k_0, N+1].$$

Then $Z(k_0) = H(I - \frac{1}{\gamma^2}H'X(k_0)H)^{-1}H'$ and similar to the proof of Lemma 3.22, we can show

$$[Y(I-\frac{1}{\gamma^2}XY)^{-1}](k+1) = [I-\frac{1}{\gamma^2}\Phi(k)X(k+1)B_2T_2^{-1}(k)B_2'X(k+1)]^{-1}\Phi(k).$$

Hence $Z(k) = Y(k)[I - \frac{1}{\gamma^2}X(k)Y(k)]^{-1}$ satisfies the Riccati equation (3.97)-(3.99). The rest follows from Theorem 3.9. ∎

Proof of Theorem 3.11: Since (A, B_1) is stabilizable and $A+B_1F_1+B_2F_2$ is exponentially stable, we can easily show that (3.146) satisfies the assumptions of Theorem 3.18 except for the detectability of (C_2, A_X). Since

$$A_X + F'_{1Z}C_{1X} + F'_{2Z}C_2 = A_X - R'_{2Z}T_{2Z}^{-1}C_2 + F'_{1Z}(C_{1X} - S_ZT_{2Z}^{-1}C_2)$$

is exponentially stable, $(A_X - R'_{2Z}T_{2Z}^{-1}C_2, F'_{1Z})$ is stabilizable and so is

$$\left(A_X - R'_{2Z}T_{2Z}^{-1}C_2,\ [\,F'_{1X}V^{\frac{1}{2}} \quad B_{1X} \quad R'_{2Z}T_{2Z}^{-1}\,]\right).$$

Since we can rewrite the Riccati equation (3.98) in the form

$$
\begin{aligned}
Z(k+1) &= (A_X - R'_{2Z}T_{2Z}^{-1}C_2)Z(k)(A_X - R'_{2Z}T_{2Z}^{-1}C_2)' \\
&\quad + [\,F'_{1X}V^{\frac{1}{2}} \quad B_{1X} \quad R'_{2Z}T_{2Z}^{-1}\,][\,F'_{1X}V^{\frac{1}{2}} \quad B_{1X} \quad R'_{2Z}T_{2Z}^{-1}\,]',
\end{aligned}
$$

$A_X - R'_{2Z}T_{2Z}^{-1}C_2$ is exponentially stable by Proposition 3.5 and hence (C_2, A_X) is stabilizable. Thus the system (3.146) satisfies the assumptions of Theorem 3.18 and we can proceed as in Theorem 3.9. ∎

Proof of Theorem 3.12: The proof is similar to that of Theorem 3.10. We only need to show that $Z = Y(I - \frac{1}{\gamma^2}XY)^{-1}$ is a stabilizing solution of (3.97)-(3.99). But this follows from Lemma 3.23 and the stabilizing property of Y. ∎

Proof of Corollary 3.10: Necessity of (a) follows from Theorem 3.8 and Corollary 3.9. Proof of the sufficiency of (a) and (b) is similar to that of Corollary 2.12. ∎

3.4 H_∞ Filtering

As in Section 2.4 we consider the H_∞ filtering problem with initial uncertainty. We consider the problem both on finite and infinite horizons.

Consider the system $\mathbf{G}_F$:

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)w(k), \\
z(k) &= L(k)x(k), \qquad (3.148)\\
y(k) &= C(k)x(k) + D(k)w(k), \\
x(k_0) &= Hh, \qquad (3.149)\\
z_1 &= Fx(N+1) \qquad (3.150)
\end{aligned}
$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the outputs to be estimated, $y \in \mathbf{R}^{p_2}$ is the measurement, $h \in \mathbf{R}^{n_1}$, $H \in \mathbf{R}^{n \times n_1}$ and A, B, C, D and L are bounded matrices of appropriate dimensions. Here we assume

$$
\mathbf{DF1}: \; [\,B(k) \quad D(k)\,]\,D'(k) = [\,0 \quad I\,] \text{ for any } k.
$$

We wish to estimate z_1 and z by the causal filter of the form

$$
\begin{aligned}
\hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \;\; \hat{x}(k_0) = 0, \\
\hat{z}(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k), \qquad (3.151)\\
\hat{z}_1 &= \hat{F}\hat{x}(N+1)
\end{aligned}
$$

and to achieve the following:

$$|z_1 - \hat{z}_1|^2 + \| z - \hat{z} \|_2^2 \leq d^2(|h|^2 + \| w \|_2^2) \text{ for some } 0 < d < \gamma \qquad (3.152)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are bounded matrices of appropriate dimensions and $\hat{F}$ is a constant matrix. Such a filter is called γ-suboptimal. We can write (3.148)-(3.151) as

$$\begin{aligned}
\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k+1) &= \begin{bmatrix} A & 0 \\ \hat{B}C & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k) + \begin{bmatrix} B \\ \hat{B}D \end{bmatrix} w(k), \\
\begin{bmatrix} x(k_0) \\ \hat{x}(k_0) \end{bmatrix} &= \begin{bmatrix} Hh \\ 0 \end{bmatrix}, \\
e_1 &= z_1 - \hat{z}_1 = [F \quad -\hat{F}]\begin{bmatrix} x(N+1) \\ \hat{x}(N+1) \end{bmatrix}, \\
e(k) &= z(k) - \hat{z}(k) = [L - \hat{D}C \quad -\hat{C}]\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k) - \hat{D}Dw(k).
\end{aligned}$$

Define the operator $G \in \mathcal{L}(\mathbf{R}^{n_1} \times l^2(k_0, N; \mathbf{R}^{m_1}); \mathbf{R}^q \times l^2(k_0, N; \mathbf{R}^{p_1}))$ by

$$\begin{pmatrix} e_1 \\ e \end{pmatrix} = G\begin{pmatrix} h \\ w \end{pmatrix}. \qquad (3.153)$$

Then (3.152) is equivalent to $\| G \| \leq d$. The adjoint G^* is given by

$$\begin{pmatrix} \zeta_0 \\ \zeta \end{pmatrix} = G^*\begin{pmatrix} f \\ v \end{pmatrix}$$

where

$$\begin{aligned}
\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}(k) &= \begin{bmatrix} A' & C'\hat{B}' \\ 0 & \hat{A}' \end{bmatrix}\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}(k+1) + \begin{bmatrix} L' - C'\hat{D}' \\ -\hat{C}' \end{bmatrix} v(k), \\
\zeta(k) &= [B' \quad D'\hat{B}']\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}(k+1) - D'\hat{D}'v(k), \\
\zeta_0 &= [H' \quad 0]\begin{bmatrix} \xi(k_0) \\ \hat{\xi}(k_0) \end{bmatrix}, \qquad (3.154) \\
\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix}(N+1) &= \begin{bmatrix} F' \\ -\hat{F}' \end{bmatrix} f.
\end{aligned}$$

This may be regarded as a closed-loop system

$$\begin{aligned}
\xi(k) &= A'(k)\xi(k+1) + L'(k)v(k) + C'(k)\mu(k), \\
\zeta(k) &= B'(k)\xi(k+1) + D'(k)\mu(k), \qquad (3.155) \\
\eta(k) &= \begin{bmatrix} \xi(k+1) \\ v(k) \end{bmatrix}, \\
\zeta_0 &= H'\xi(k_0), \\
\xi(N+1) &= F'f
\end{aligned}$$

with controller $\mu = K^* \begin{pmatrix} f \\ \eta \end{pmatrix}$ of the form

$$\begin{aligned} \hat{\xi}(k) &= \hat{A}'(k)\hat{\xi}(k+1) - [0 \quad \hat{C}'](k)\eta(k), \\ \mu(k) &= \hat{B}'(k)\hat{\xi}(k+1) - [0 \quad \hat{D}'](k)\eta(k), \\ \hat{\xi}(N+1) &= -F'f. \end{aligned} \tag{3.156}$$

The system (3.155) is of the full information type and (3.152) is equivalent to

$$|\zeta_0|^2 + \|\zeta\|_2^2 \leq d^2(|f|^2 + \|v\|_2^2). \tag{3.157}$$

The Riccati equation corresponding to this is

$$V_Y(k) > aI \text{ for some } a > 0, \tag{3.158}$$

$$Y(k+1) = AY(k)A' + BB' - (R'_{2Y}T_{2Y}^{-1}R_{2Y})(k) + (F'_{1Y}V_Y F_{1Y})(k), \tag{3.159}$$

$$Y(k_0) = HH', \tag{3.160}$$

$$FY(N+1)F' \leq d^2 I \text{ for some } 0 < d < \gamma \tag{3.161}$$

where

$T_{1Y}(k) = \gamma^2 I - LY(k)L'$, $\quad T_{2Y}(k) = I + CY(k)C'$,
$R_{1Y}(k) = LY(k)A'$, $\quad R_{2Y}(k) = CY(k)A'$,
$S_Y(k) = CY(k)L'$, $\quad V_Y(k) = (T_{1Y} + S'_Y T_{2Y}^{-1} S_Y)(k)$,
$F_{1Y}(k) = [V_Y^{-1}(R_{1Y} - S'_Y T_{2Y}^{-1} R_{2Y})](k)$,
$F_{2Y}(k) = -[T_{2Y}^{-1}(R_{2Y} + S_Y F_{1Y})](k)$.

As Q_γ in Section 3.3.2 we define the set of controllers of backward type:

$$\begin{aligned} Q^*_\gamma = \{ & Q^* \in \mathcal{L}(\mathbf{R}^q \times l^2(k_0, N; \mathbf{R}^{p_1}); l^2(k_0, N; \mathbf{R}^{p_2})) : \\ & f'FY(N+1)F'f + \left\| Q^* \begin{pmatrix} f \\ \rho \end{pmatrix} \right\|_2^2 \leq d^2(|f|^2 + \|\rho\|_2^2) \\ & \text{for some } 0 < d < \gamma \}. \end{aligned} \tag{3.162}$$

Let $\tilde{Q}_\gamma$ be the set of adjoint system of $Q^* \in Q^*_\gamma$. Modifying Theorems 3.13 and 3.19 we have the following.

Theorem 3.23 *(a) There exists a γ-suboptimal filter if and only if there exists a nonnegative solution Y to the Riccati equation (3.158)-(3.161).*
(b) In this case the set of filters with property (3.152) is given by

$$\begin{aligned} \hat{x}(k+1) &= (A - R'_{2Y}T_{2Y}^{-1}C)(k)\hat{x}(k) + (R'_{2Y}T_{2Y}^{-1})(k)y(k) \\ &\quad + \frac{1}{\gamma}(F'_{1Y}V_Y^{\frac{1}{2}})(k)v(k), \\ \hat{x}(k_0) &= 0, \end{aligned}$$

$$\begin{aligned}
\hat{z}(k) &= (L - S_Y' T_{2Y}^{-1} C)(k)\hat{x}(k) + (S_Y' T_{2Y}^{-1})(k) y(k) \\
&\qquad -\frac{1}{\gamma} V_Y^{\frac{1}{2}}(k) v(k), \qquad (3.163) \\
r(k) &= -(T_{2Y}^{\frac{1}{2}} C)(k)\hat{x}(k) + T_{2Y}^{-\frac{1}{2}}(k) y(k), \\
v &= Q_1 r, \\
\hat{z}_1 &= F\hat{x}(N+1) - Q_0 r, \quad Q = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} \in \tilde{Q}_\gamma. \qquad (3.164)
\end{aligned}$$

Proof. (a) follows from a modification of Theorem 3.19. To show (b) recall that the set of all controllers $\mu = K^* \begin{pmatrix} f \\ \eta \end{pmatrix}$ with $\| G^* \| < \gamma$ is given by

$$\begin{aligned}
\mu(k) &= -T_{2Y}^{-1} R_{2Y} \xi(k+1) - T_{2Y}^{-1} S_Y v(k) \\
&\quad + T_{2Y}^{-\frac{1}{2}} Q^* \begin{pmatrix} f \\ \frac{1}{\gamma} V_Y^{\frac{1}{2}} [v - F_{1Y}\xi] \end{pmatrix}, \quad Q^* \in Q_\gamma^*. \qquad (3.165)
\end{aligned}$$

Then the closed-loop system (3.155) with (3.165) is written as

$$\begin{aligned}
\xi(k) &= (A' - C' T_{2Y}^{-1} R_{2Y}) \xi(k+1) + [\,0 \quad L' - C' T_{2Y}^{-1} S_Y\,] \eta(k) \\
&\quad + C' T_{2Y}^{-\frac{1}{2}} \bar{\mu}(k), \\
\rho(k) &= -\frac{1}{\gamma} V_Y^{\frac{1}{2}} F_{1Y} \xi(k+1) + \left[0 \quad \tfrac{1}{\gamma} V_Y^{\frac{1}{2}} \right] \eta(k), \qquad (3.166) \\
\bar{\mu} &= Q^* \begin{pmatrix} f \\ \rho \end{pmatrix}, \\
\xi(N+1) &= F' f.
\end{aligned}$$

In view of this we can show that the controller (3.165) is equivalent to

$$\begin{aligned}
\hat{\xi}(k) &= (A' - C' T_{2Y}^{-1} R_{2Y}) \hat{\xi}(k+1) + [\,0 \quad L' - C' T_{2Y}^{-1} S_Y\,] \eta(k) \\
&\quad + C' T_{2Y}^{-\frac{1}{2}} \bar{\mu}(k), \\
\mu(k) &= -T_{2Y}^{-1} R_{2Y} \hat{\xi}(k+1) - T_{2Y}^{-1} S_Y v(k) + T_{2Y}^{-\frac{1}{2}} \bar{\mu}(k), \qquad (3.167) \\
\rho(k) &= -\frac{1}{\gamma} V_Y^{\frac{1}{2}} F_{1Y} \hat{\xi}(k+1) + \left[0 \quad \tfrac{1}{\gamma} V_Y^{\frac{1}{2}} \right] \eta(k), \\
\bar{\mu} &= Q^* \begin{pmatrix} f \\ \rho \end{pmatrix}, \\
\hat{\xi}(N+1) &= F' f.
\end{aligned}$$

In fact for (3.155) and (3.167) $e = \xi - \hat{\xi}$ satisfies

$$e(k) = A' e(k+1), \; e(N+1) = 0$$

and ξ satisfies (3.166). Now consider the adjoint of (3.155) and (3.167):

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) + [I \quad 0]\,u(k), \\ \tilde{z}(k) &= Lx(k) + [0 \quad I]\,u(k), && (3.168) \\ y(k) &= Cx(k) + Dw(k), \\ x(k_0) &= Hh, \\ \tilde{z}_1 &= Fx(N+1) + u_1, && (3.169) \\ \hat{x}(k+1) &= (A - R'_{2Y}T_{2Y}^{-1}C)\hat{x}(k) - R'_{2Y}T_{2Y}^{-1}y(k) - \frac{1}{\gamma}F'_{1Y}V_Y^{\frac{1}{2}}v(k), \\ \hat{x}(k_0) &= 0, \\ u(k) &= \begin{bmatrix} 0 \\ (L - S'_Y T_{2Y}^{-1}C)\hat{x}(k) - S'_Y T_{2Y}^{-1}y(k) + \frac{1}{\gamma}V_Y^{\frac{1}{2}}v(k) \end{bmatrix}, && (3.170) \\ r(k) &= T_{2Y}^{-1}C\hat{x}(k) + T_{2Y}^{-\frac{1}{2}}y(k), \\ v &= Q_1 r, \ \| Q_1 \| < \gamma, \\ u_1 &= -F\hat{x}(N+1) + Q_0 r, \ Q = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} \in \tilde{Q}_\gamma. && (3.171) \end{aligned}$$

Then $\| G^* \| < \gamma$ is equivalent to

$$| \tilde{z}_1 |^2 + \| \tilde{z} \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2) \ \text{ for } d < \gamma. \tag{3.172}$$

Note that (3.168) except $\tilde{z}$, $\tilde{z}_1$ coincides with (3.148) and (3.149). Thus (3.170)-(3.172) can be easily interpreted as the filtering result in (b). ∎

Consider the system $\mathbf{G}_F$:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)w(k), \\ z(k) &= L(k)x(k), \\ y(k) &= C(k)x(k) + D(k)w(k), \\ x(k_0) &= Hh \end{aligned}$$

on $[t_0, \infty)$. Then the H_∞-filtering problem is to find a γ-suboptimal filter, i.e., a filter of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \ \hat{x}(k_0) = 0, && (3.173) \\ \hat{z}(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k) \end{aligned}$$

such that $z - \hat{z} \in l^2(k_0, \infty; \mathbf{R}^{p_1})$ and

$$\| z - \hat{z} \|_2^2 \leq d^2(| h |^2 + \| w \|_2^2), \ \text{for some } 0 < d < \gamma. \tag{3.174}$$

In this case we further assume

DF2 : (A, B, C) is stabilizable and detectable.

Again considering the FI problem for (3.155) on $[k_0, \infty)$ and modifying Theorem 3.20 we have the following.

Theorem 3.24 *Assume* **DF1** *and* **DF2**.
(a) Then there exists a γ-suboptimal filter if and only if there exists a nonnegative bounded stabilizing solution to the Riccati equation (3.158)-(3.160).
(b) In this case the set of all γ-suboptimal filters is given by (3.163), where Q_1 is an IO-stable system with $\| Q_1 \| < \gamma$. Moreover, the set of all internally stable filters is given by (3.163) restricting Q_1 to be internally stable.

We may incorporate the estimate of z_1 on the infinite horizon.

Corollary 3.17 *There exists a filter of the form (3.151) such that*

$$\sup_{N \geq N_0} [\mid z_1 - \hat{z}_1 \mid^2 + \| z - \hat{z} \|_2^2] \leq d^2(\mid h \mid^2 + \| w \|_2^2) \text{ for some } d < \gamma$$

if and only if there exists a bounded nonnegative stabilizing solution of (3.158)-(3.160) with

$$FY(N+1)F' \leq d^2 I, \ N \geq N_0 \text{ for some } 0 < d < \gamma.$$

Modifying Corollary 3.16 we have also the following result.

Corollary 3.18 *Let* $\mathbf{G}_F$ *be θ-periodic and assume* **DF1** *and* **DF2**. *Assume further that the initial condition is known, i.e., $h = 0$. Then*
(a) There exists a filter of the form (3.173) with property (3.174) if and only if there exists a θ-periodic nonnegative stabilizing solution of (3.158) and (3.159).
(b) In this case the filters given by (3.163) are γ-suboptimal where Q_1 is an IO-stable system with $\| Q_1 \| < \gamma$. If Q_1 is θ-periodic, the filter is θ-periodic and γ-suboptimal. Moreover, the filters are given by (3.163) is internally stabilizing if Q_1 is internally stable.

Corollary 3.19 *Let the system* $\mathbf{G}_F$ *be time-invariant. Then $Y(k)$ in (a) converges as $t \to \infty$ to the stabilizing solution Y_∞ of the algebraic Riccati equation*

$$\begin{aligned} V_Y &> 0, \\ Y &= AYA' + BB' - R_{2Y}' T_{2Y}^{-1} R_{2Y} + F_{1Y} V_Y F_{1Y}. \end{aligned}$$

Moreover the filter (3.163) with Y_∞ gives the set of all γ-suboptimal filters when $h = 0$.

3.5 H_2 Control

In this section we consider the H_2 control problem. The H_2 theory for time-invariant systems is now well-known [21, 93]. Here we give an extension to time-varying systems.

3.5.1 Main Results

Consider the system **G**:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned} \tag{3.175}$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the measurement and A, B_1, etc are bounded matrices of appropriate dimensions. For this system we assume **D1-D4**, i.e.,

$$\begin{aligned} &\mathbf{D1}: \ D'_{12}(k)\,[\,C_1(k) \quad D_{12}(k)\,] = [\,0 \quad I\,] \ \text{ for any } k, \\ &\mathbf{D2}: \ D_{21}(k)\,[\,B'_1(k) \quad D'_{21}(k)\,] = [\,0 \quad I\,] \ \text{ for any } k, \\ &\mathbf{D3}: \ (A, B_1, C_1) \text{ is stabilizable and detectable,} \\ &\mathbf{D4}: \ (A, B_2, C_2) \text{ is stabilizable and detectable.} \end{aligned}$$

Consider a controller $u = Ky$ of the form:

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k) \end{aligned} \tag{3.176}$$

for some bounded matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$

To formulate the H2-control problem for the system **G**, we introduce the following set of controllers

$$\mathbf{K} = \{K : K \text{ is of the form (3.176) and internally stabilizes the system } \mathbf{G}\}.$$

Then the H2-norm, $\| G \|_2$, of the closed-loop system **G** and a controller $u = Ky$ is well-defined and our H2-problem is to find a controller $K \in \mathbf{K}$ which minimizes $\| G \|_2$. To give the solution of this problem we introduce the following Riccati equations:

$$X(k) = A'(k)X(k+1)A(k) + C'_1(k)C_1(k) - (R'_2T_2^{-1}R_2)(k) \tag{3.177}$$

and

$$Y(k+1) = A(k)Y(k)A'(k) + B_1(k)B'_1(k) - (R'_{2Y}T_{2Y}^{-1}R_{2Y})(k), \tag{3.178}$$

$$Y(k_0) = 0 \tag{3.179}$$

where

$$\begin{aligned} &T_2(k) = I + B'_2(k)X(k+1)B_2(k), \quad R_2(k) = B'_2(k)X(k+1)A(k), \\ &T_{2Y}(k) = I + C_2(k)Y(k)C'_2(k), \quad R_{2Y}(k) = C_2(k)Y(k)A'(k). \end{aligned}$$

Definition 3.15 *(a) The solution X of (3.177) is called a stabilizing solution if* $A + B_2\hat{F}$, $\hat{F}(\cdot) = -(T_2^{-1}R_2)(\cdot)$ *is exponentially stable.*
(b) The solution Y of (3.178) is called a stabilizing solution if $A + \hat{H}C_2$, $\hat{H}(\cdot) = -(R'_{2Y}T_{2Y}^{-1})(\cdot)$ *is exponentially stable.*

By Theorems 3.2 and 3.3, we have the following result.

Lemma 3.29 *Assume* **D1**-**D4**. *Then*
(a) There exists a bounded nonnegative stabilizing solution $X(k)$, $k \in [k_0, \infty)$ *to (3.177).*
(b) There exists a bounded nonnegative stabilizing solution $Y(k)$, $k \in [k_0, \infty)$ *to (3.178) and (3.179).*

Consider the stabilizing controller

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k) \end{aligned} \tag{3.180}$$

where

$$\begin{aligned} \hat{A}(k) &= (A + B_2\hat{F} + \hat{H}C_2 - B_2LC_2)(k), \\ \hat{B}(k) &= -(\hat{H} - B_2L)(k), \\ \hat{C}(k) &= (\hat{F} - LC_2)(k), \\ \hat{D}(k) &= L(k) \end{aligned}$$

and $L(k) = (\hat{F}YC_2'T_{2Y}^{-1})(k)$.

Theorem 3.25 *Assume* **D1**-**D4** *and consider the* H_2*-problem for the system* **G**. *Then the controller (3.180) is optimal and*

$$\begin{aligned} \min_{K \in \mathbf{K}} \| G \|_2^2 &= \lim_{N \to \infty} \frac{1}{N}\{ \sum_{s=k_0}^{k_0+N-1} tr.[\bar{B}_1'(s)X(s+1)\bar{B}_1(s) + (S'T_2^{-2}S)(s)] \\ &\quad + \sum_{s=k_0+1}^{k_0+N} tr.[T_2(\hat{C}Y\hat{C}' + \bar{D}_{21}\bar{D}_{21}')](s)\} \end{aligned} \tag{3.181}$$

where $\bar{B}_1(k) = (B_1 - B_2T_2^{-1}S)(k)$, $\bar{D}_{21}(k) = (T_2^{-1}S + LD_{21})(k)$ *and* $S(k) = B_2'(k)X(k+1)B_1(k)$.

Corollary 3.20 *Let* **G** *be θ-periodic. Then* $X(k)$ *is θ-periodic and there exists a θ-periodic nonnegative stabilizing solution* $Y_\theta(k)$ *of (3.178). Moreover, the controller (3.180) with Y replaced by* Y_θ *is optimal and*

$$\begin{aligned} \min_{K \in \mathbf{K}} \| G \|_2^2 &= \frac{1}{\theta}\{ \sum_{s=k_0}^{k_0+\theta-1} tr.[\bar{B}_1'(s)X(s+1)\bar{B}_1(s) + (S'T_2^{-2}S)(s)] \\ &\quad + \sum_{s=k_0+1}^{k_0+\theta} tr.[T_2(\hat{C}Y_\theta\hat{C}' + \bar{D}_{21}\bar{D}_{21}')](s)\}. \end{aligned}$$

Let **G** be time-invariant. Then there exist nonnegative stabilizing solutions X and Y, respectively of algebraic Riccati equations

$$A'XA + C_1'C_1 - R_2'T_2^{-1}R_2 = 0$$

and

$$AYA' + B_1B_1' - R_{2Y}'T_{2Y}^{-1}R_{2Y} = 0.$$

Corollary 3.21 *Let* **G** *be time-invariant. Then the controller (3.180) with* $(X(k),\ Y(k))$ *replaced by* (X, Y) *is optimal and*

$$\min_{K \in \mathbf{K}} \| G \|_2^2 = tr.[\bar{B}_1'X\bar{B}_1 + S'T_2^{-2}S + T_2(\hat{C}Y\hat{C}' + \bar{D}_{21}\bar{D}_{21}')].$$

3.5.2 Proofs of Main Results

To prove Theorem 3.25 we need some preliminary results. Consider the system **G** and the controller $u = Ky$ of the form (3.176). Let X be the solution of (3.177). We introduce

$$v(k) = T_2^{\frac{1}{2}}[u(k) + T_2^{-1}Sw(k) - \hat{F}x(k)]$$

and the following system $\bar{\mathbf{G}}$:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1w(k) + B_2u(k), \\ v(k) &= T_2^{-\frac{1}{2}}R_2x(k) + T_2^{-\frac{1}{2}}Sw(k) + T_2^{\frac{1}{2}}u(k), \\ y(k) &= C_2x(k) + D_{21}w(k). \end{aligned} \tag{3.182}$$

Then z can be written using v as follows:

$$\begin{aligned} x(k+1) &= (A + B_2\hat{F})x(k) + (B_1 - B_2T_2^{-1}S)w(k) + B_2T_2^{-\frac{1}{2}}v(k), \\ z(k) &= (C_1 + D_{12}\hat{F})x(k) - D_{12}T_2^{-1}Sw(k) + D_{12}T_2^{-\frac{1}{2}}v(k). \end{aligned}$$

Note that the above system is exponentially stable. Hence

$$z = G_cw + Uv$$

where G_c and U are given by

$$\begin{aligned} \xi(k+1) &= (A + B_2\hat{F})\xi(k) + (B_1 - B_2T_2^{-1}S)w(k), \\ \zeta(k) &= (C_1 + D_{12}\hat{F})\xi(k) - D_{12}T_2^{-1}Sw(k) \end{aligned}$$

and

$$\begin{aligned} x(k+1) &= (A + B_2\hat{F})x(k) + B_2T_2^{-\frac{1}{2}}v(k), \\ z(k) &= (C_1 + D_{12}\hat{F})x(k) + D_{12}T_2^{-\frac{1}{2}}v(k), \end{aligned}$$

respectively. Then we can easily see the following.

Lemma 3.30 *(a) The system* $\mathbf{G}$ *is equivalent to the interconnection of the system* $\bar{\mathbf{G}}$ *and* (G_c, U).
(b) K *stabilizes the system* $\mathbf{G}$ *if and only if it stabilizes the system* $\bar{\mathbf{G}}$.

Next we shall show the properties of G_c and U.

Lemma 3.31 *(a)* $\| Uv \|_2 = \| v \|_2$ *for any* $v \in l^2(s,\infty;\mathbf{R}^{m_2})$.
(b) $< G_c\delta_{\cdot s}w_0, Uv >= 0$ *for any* $w_0 \in \mathbf{R}^{m_1}$ *and* $v \in l^2(s,\infty;\mathbf{R}^{m_2})$ *where*

$$\delta_{ks}w_0 = \begin{cases} w_0, & k = s, \\ 0, & otherwise. \end{cases}$$

Proof. (a) We can rewrite the Riccati equation (3.177) as

$$X(k) = (A + B_2\hat{F})'X(k+1)(A + B_2\hat{F}) + (C_1 + D_{12}\hat{F})'(C_1 + D_{12}\hat{F}).$$

By direct calculation, we have

$$x'(k+1)X(k+1)x(k+1) - x'(k)X(k)x(k) = - | z(k) |^2 + | v(k) |^2$$

where we have used the following equality

$$B_2'(k)X(k+1)(A + B_2\hat{F})(k) = -\hat{F}(k).$$

Hence we have

$$x'(N+1)X(N+1)x(N+1) - x'(s)X(s)x(s) = \sum_{k=s}^{N}[| v(k) |^2 - | z(k) |^2].$$

Since $x(s) = 0$ and $\lim_{N\to\infty} x(N+1) = 0$ we have the assertion.
(b) Consider G_c with $w(s) = w_0$ and $w(k) = 0$, $s+1 \le k$. Then

$$\xi(k) = \begin{cases} 0, & k = s, \\ S_F(k, s+1)(B_1 - B_2T^{-1}S)(s)w_0 & k \ge s+1 \end{cases}$$

where $S_F(\cdot,\cdot)$ is the state transition matrix of $A + B_2\hat{F}$. Using the equality

$$(B_1 - B_2T_2^{-1}S)'(k)X(k+1)B_2(k) = S'(k)T_2^{-1}(k)$$

we have

$$\begin{aligned} & \xi'(s+1)X(s+1)x(s+1) - \xi'(s)X(s)x(s) \\ = \; & w_0'(B_1 - B_2T_2^{-1}S)(s)X(s+1)B_2(s)T_2^{-\frac{1}{2}}(k)v(s) \\ = \; & [(D_{12}T_2^{-1}S)(s)w_0]'[(D_{12}T_2^{-\frac{1}{2}})(s)v(s)] \\ = \; & -\zeta'(s)z(s) \end{aligned}$$

and as in (a) we have

$$\xi'(k+1)X(k+1)x(k+1) - \xi'(k)X(k)x(k) = -\zeta'(k)z(k),\ k \geq s+1.$$

Hence

$$\xi'(N+1)X(N+1)x(N+1) - \xi'(s)X(s)x(s) = \sum_{k=s}^{N} \zeta'(k)z(k)$$

and we have the assertion since $\lim_{N\to\infty} \xi(N+1) = \lim_{N\to\infty} x(N+1) = 0$ and $x(s) = 0$. ∎

Now we return to the H_2-control problem for the system **G**. Suppose K stabilizes the system **G** and hence the system $\bar{\mathbf{G}}$. Let $\bar{G}$ be the input-output operator of the closed-loop system $\bar{\mathbf{G}}$ with $u = Ky$, i.e.,

$$v = \bar{G}w.$$

By Lemma 3.31

$$\begin{aligned} \| G \|_2^2 &= \| G_c + U\bar{G} \|_2^2 \\ &= \| G_c \|_2^2 + \| U\bar{G} \|_2^2 \\ &= \| G_c \|_2^2 + \| \bar{G} \|_2^2 \end{aligned} \tag{3.183}$$

and

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2 .$$

Thus our original H_2-problem has been reduced to the one for the system $\bar{\mathbf{G}}$. By Remark 3.2, $\min_{K\in\mathbf{K}} \| \bar{G} \|_2^2$ is equivalent to the H_2-problem for the backward system

$$\begin{aligned} \tilde{x}(k) &= A'(k)\tilde{x}(k+1) + (R_2'T_2^{-\frac{1}{2}})(k)\tilde{w}(k) + C_2'(k)\tilde{u}(k), \\ \tilde{v}(k) &= B_1'(k)\tilde{x}(k+1) + (S'T_2^{-\frac{1}{2}})(k)\tilde{w}(k) + D_{21}'(k)\tilde{u}(k), \\ \tilde{y}(k) &= B_2'(k)\tilde{x}(k+1) + T_2^{\frac{1}{2}}(k)\tilde{w}(k) \end{aligned} \tag{3.184}$$

with an internally stabilizing controller of the form

$$\begin{aligned} \hat{x}(k) &= \hat{A}'(k)\hat{x}(k+1) + \hat{C}'(k)\tilde{y}(k), \\ \tilde{u}(k) &= \hat{B}'(k)\hat{x}(k+1) + \hat{D}'(k)\tilde{y}(k). \end{aligned}$$

The H_2-problem for the system (3.184) is the DF problem. Its solution will be given below.

Backward Systems

We take a general backward system and consider special H_2 problems. First we consider the system with full information (denoted by $\mathbf{G}_{FI}$:)

$$\begin{aligned} x(k) &= A(k)x(k+1) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k+1) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= \begin{bmatrix} x(k+1) \\ w(k) \end{bmatrix}. \end{aligned} \tag{3.185}$$

We take a controller $u = Ky$ of the form

$$\begin{aligned} \hat{x}(k) &= \hat{A}(k)\hat{x}(k+1) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k+1) + \hat{D}(k)y(k) \end{aligned} \tag{3.186}$$

where all matrices are uniformly bounded and of compatible dimensions. Let G_{FI} be the input-output operator of the closed-loop system $\mathbf{G}_{FI}$ with $u = Ky$. To formulate the H_2-problem for the system $\mathbf{G}_{FI}$ we introduce the following set of controllers:

$$\begin{aligned} \mathbf{K} \;=\; \{K : K &\text{ is of the form (3.186) and} \\ &\text{internally stabilizes the system } \mathbf{G}_{FI}\}. \end{aligned}$$

Then the H_2-problem for the system $\mathbf{G}_{FI}$ (FI-problem) is to find a controller $K \in \mathbf{K}$ which minimizes $\| G_{FI} \|_2$.

For the system $\mathbf{G}_{FI}$, we assume **D1'** and **D5**, i.e.,

$$\mathbf{D5} : (A, B_2, C_1) \text{ is stabilizable and detectable.}$$

Then as in Lemma 3.29, we have the following.

Lemma 3.32 *Assume* **D1'** *and* **D5**. *Then there exists a unique bounded nonnegative stabilizing solution $P(k)$, $k \in [k_0, \infty)$ to the Riccati equation*

$$\begin{aligned} P(k+1) &= A'(k)P(k)A(k) + C_1'(k)C_1(k) - (R_P'T_P^{-1}R_P)(k), \\ P(k_0) &= 0 \end{aligned} \tag{3.187}$$

where $T_P(k) = I + B_2'(k)P(k)B_2(k)$ and $R_P(k) = B_2'(k)P(k)A(k)$.

As in the previous subsection, we introduce

$$v(k) = T_P^{\frac{1}{2}}(k)[u(k) + (T_P^{-1}S_P)(k)w(k) - F_P(k)x(k+1)]$$

and the system $\bar{\mathbf{G}}^b$:

$$\begin{aligned} x(k) &= A(k)\bar{x}(k+1) + B_1(k)w(k) + B_2(k)u(k), \\ v(k) &= (T_P^{-\frac{1}{2}}R_P)(k)x(k+1) + (T_P^{-\frac{1}{2}}S_P)(k)w(k) + T_P^{\frac{1}{2}}(k)u(k), \\ y(k) &= \begin{bmatrix} x(k+1) \\ w(k) \end{bmatrix} \end{aligned} \tag{3.188}$$

where $F_P(k) = -(T_P^{-1}R_P)(k)$ and $S_P(k) = B_2'(k)P(k)B_1(k)$. Then z can be written using v as follows:

$$\begin{aligned} x(k) &= (A + B_2F_P)x(k+1) + (B_1 - B_2T_P^{-1}S_P)w(k) + B_2T_P^{-\frac{1}{2}}v(k), \\ z(k) &= (C_1 + D_{12}F_P)x(k+1) + (D_{11} - D_{12}T_P^{-1}S_P)w(k) + D_{12}T_P^{-\frac{1}{2}}v(k). \end{aligned}$$

Hence

$$z = G_c^b w + U^b v$$

where G_c^b and U^b are given by

$$\begin{aligned} \xi(k) &= (A + B_2F_P)\xi(k+1) + (B_1 - B_2T_P^{-1}S_P)w(k), \\ \zeta(k) &= (C_1 + D_{12}F_P)\xi(k+1) + (D_{11} - D_{12}T_P^{-1}S_P)w(k) \end{aligned}$$

and

$$\begin{aligned} x(k) &= (A + B_2F_P)x(k+1) + B_2T_P^{-\frac{1}{2}}v(k), \\ z(k) &= (C_1 + D_{12}F_P)x(k+1) + D_{12}T_P^{-\frac{1}{2}}v(k), \end{aligned}$$

respectively. Then we have the following.

(a) The system $\mathbf{G}_{FI}$ is equivalent to the interconnection of the system $\bar{\mathbf{G}}^b$ and (G_c^b, U^b).
(b) K stabilizes the system $\mathbf{G}_{FI}$ if and only if it stabilizes $\bar{\mathbf{G}}^b$.

Next we need the following lemma.

Lemma 3.33 *(a)* $\| U^b v \|_2 = \| v \|_2$ *for any* $v \in l^2(k_0, \infty; \mathbf{R}^{m_2})$.
(b) $< G_c^b \delta_{\cdot s} w_0, U^b v >= 0$ *for any* $w_0 \in \mathbf{R}^{m_1}$ *and* $v \in l^2(k_0, \infty; \mathbf{R}^{m_2})$ *with support in* $[k_0, s]$.

Proof. (a) We can rewrite the Riccati equation (3.187) as

$$\begin{aligned} P(k+1) &= (A + B_2F_P)'P(k)(A + B_2F_P) \\ &\qquad + (C_1 + D_{12}F_P)'(C_1 + D_{12}F_P), \\ P(k_0) &= 0. \end{aligned}$$

The by direct calculation

$$x'(k+1)P(k+1)x(k+1) - x'(k)P(k)x(k) = | z(k) |^2 - | v(k) |^2$$

and hence

$$\begin{aligned} \sum_{k=k_0}^{s} [| z(k) |^2 - | v(k) |^2] &= x'(s+1)P(s+1)x(s+1) - x'(k_0)P(k_0)x(k_0) \\ &= x'(s+1)P(s+1)x(s+1). \end{aligned}$$

Since $x(s+1) = 0$ and $x(k) = 0$, $^{\forall}k > s$, we obtain (a).
(b) Consider the system G_c^b with $w(s) = w_0$ and $w(k) = 0, k \neq s, k_0 \leq s < \infty$. Then as in the proof of Lemma 3.31 we obtain

$$\begin{aligned}\sum_{k=k_0}^{s} \zeta'(k)z(k) &= \xi'(s+1)P(s+1)x(s+1) - \xi'(k_0)P(k_0)x(k_0) \\ &= \xi'(s+1)P(s+1)x(s+1).\end{aligned}$$

Since $\xi(s+1) = 0$, $\sum_{k=k_0}^{s} \xi'(k)z(k) = 0$. Since $\xi(k) = 0$ and $z(k) = 0$ $k > s$, we have $\sum_{k=k_0}^{\infty} \xi'(k)z(k) = 0$. ∎

Let $u = Ky$ be an internally stabilizing controller and $\bar{G}^b$ the input-output operator of the closed-loop system $\bar{\mathbf{G}}_{FI}$ with $u = Ky$ given by

$$v = \bar{G}^b w.$$

Then $v(k) = \bar{G}^b \delta_{\cdot s} w_0$ has support in $[k_0, s]$. and by Lemma 3.33

$$\begin{aligned}\| G_{FI} \|_2^2 &= \| G_c^b + U^b \bar{G}^b \|_2^2 \\ &= \| G_c^b \|_2^2 + \| U^b \bar{G}^b \|_2^2 \\ &= \| G_c^b \|_2^2 + \| \bar{G}^b \|_2^2 .\end{aligned}$$

Hence we have

$$\min_{K \in \mathbf{K}} \| G_{FI} \|_2^2 = \| G_c^b \|_2^2 + \min_{K \in \mathbf{K}} \| \bar{G}^b \|_2^2 .$$

Thus the H_2-problem of the system $\mathbf{G}_{FI}$ is reduced to the one for the system $\bar{\mathbf{G}}^b$. Since $u(k) = F_P(k)x(k)$ is stabilizing,

$$u(k) = [\, F_P \quad -T_P^{-1} S_P \,]\, (k) y(k)$$

internally stabilizes the system $\bar{\mathbf{G}}^b$ and this yields $v = 0$ or $\bar{G}^b = 0$. Hence this controller is optimal for the system $\mathbf{G}_{FI}$ and

$$\min_{K \in \mathbf{K}} \| G_{FI} \|_2^2 = \| G_c^b \|_2^2 .$$

The controllability gramian for the backward system associated with G_c^b is a unique nonnegative solution is given by

$$L_o(k+1) = (A + B_2 F_P)' L_o(k)(A + B_2 F_P) + (C_1 + D_{12} F_P)'(C_1 + D_{12} F_P)$$

which implies $L_o = P$. Hence by Lemma 3.3

$$\| G_c^b \|_2^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{s=k_0+1}^{k_0+N} tr.[\bar{B}_1' P \bar{B}_1 + \bar{D}_{11}' \bar{D}_{11}](s)$$

where $\bar{B}_1(k) = (B_1 - B_2 T_P^{-1} S_P)(k)$ and $\bar{D}_{11}(k) = (D_{11} - D_{12} T_p^{-1} S_p)(k)$.

Summarizing the above we have the following.

Theorem 3.26 *Assume* **D1'**, **D5** *and consider the H_2-problem for the system* $\mathbf{G}_{FI}$. *Then*

(a) $\min_{K \in \mathbf{K}} \| G \|_2^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{s=k_0+1}^{k_0+N} tr.[\bar{B}_1' P \bar{B}_1 + \bar{D}_{11}' \bar{D}_{11}](s).$

(b) $K = [\, F_P \quad -T_P^{-1} S_P \,](k)$ *is optimal.*

Next we consider the H_2-problem for the system $\mathbf{G}_{DF}$:

$$\begin{aligned} x(k) &= A(k)x(k+1) + B_1(k)w(k) + B_2(k)u(k), \\ z(k) &= C_1(k)x(k+1) + D_{11}(k)w(k) + D_{12}(k)u(k), \\ y(k) &= C_2(k)x(k+1) + D_{21}(k)w(k) \end{aligned} \tag{3.189}$$

where D_{21}^{-1} exists and is uniformly bounded and we take a controller $u = K_{DF} y$ of the form (3.186). Here we assume **D1'**, **D5** and **D6**, i.e.,

$$\mathbf{D6}: \; A - B_1 D_{21}^{-1} C_2 \text{ is exponentially stable.}$$

As we see below, this problem is equivalent to the FI-problem.

Proposition 3.7 *A controller K_{DF} internally stabilizes $\mathbf{G}_{DF}$ if and only if $K = K_{DF}[C_2 \;\; D_{21}]$ internally stabilizes $\mathbf{G}_{FI}$. In this case $G_{DF} = G_{FI}$ where G_{DF} is the input-output operator of the closed-loop system $\mathbf{G}_{DF}$ with $u = K_{DF} y$ defined by $z = G_{DF} w$.*

Proof. The proof follows from $u = K_{DF} y = K_{DF} \, [\, C_2 \quad D_{21} \,] \begin{bmatrix} x \\ w \end{bmatrix}$. ∎

Consider the controller K_{DF}:

$$\begin{aligned} \hat{x}(k) &= A(k)\hat{x}(k+1) + (B_1 D_{21}^{-1})(k)[y(k) - C_2(k)\hat{x}(k+1)] \\ &\qquad + B_2(k) u_{FI}(k), \\ u(k) &= u_{FI}(k), \\ u_{FI} &= K y_{FI}, \\ y_{FI}(k) &= \begin{bmatrix} \hat{x}(k+1) \\ D_{21}^{-1}(k)[y(k) - C_2(k)\hat{x}(k+1)] \end{bmatrix}. \end{aligned} \tag{3.190}$$

Proposition 3.8 *The controller K internally stabilizes the system $\mathbf{G}_{FI}$ if and only if K_{DF} given by (3.190) internally stabilizes $\mathbf{G}_{DF}$. In this case $G_{FI} = G_{DF}$.*

Proof. Let $e = x - \hat{x}$ where x and $\hat{x}$ are the states of the system $\mathbf{G}_{DF}$ and (3.190), respectively. Then e satisfies

$$e(k) = (A - B_1 D_{21}^{-1} C_2) e(k+1)$$

which is exponentially stable. Moreover

$$\begin{aligned} \hat{x}(k) &= A\hat{x}(k+1) + B_1\hat{w}(k) + B_2u(k), \\ u(k) &= u_{FI}(k) = K\begin{bmatrix} x(k+1) \\ w(k) \end{bmatrix} = K\begin{bmatrix} \hat{x}(k+1) \\ \hat{w}(k) \end{bmatrix} \end{aligned}$$

where $\hat{w}(k) = w(k) + D_{21}^{-1}C_2e(k+1)$. Hence

$$\begin{aligned} \hat{x}(k) &= A\hat{x}(k+1) + B_1\hat{w}(k) + B_2u(k), \\ u(k) &= K\begin{bmatrix} \hat{x}(k+1) \\ \hat{w}(k) \end{bmatrix}. \end{aligned} \tag{3.191}$$

Now suppose K stabilizes $\mathbf{G}_{FI}$. Then $\hat{x} \in l^2$, but $e \in l^2$ and hence $x \in l^2$. Thus K_{DF} stabilizes $\mathbf{G}_{DF}$. Conversely suppose K_{DF} stabilizes the system $\mathbf{G}_{DF}$. then (3.191) is exponentially stable. Finally z is given

$$\begin{aligned} z(k) &= C_1x(k+1) + D_{11}w(k) + D_{12}u(k) \\ &= C_1(\hat{x}+e)(k+1) + D_{11}[\hat{w}(k) - D_{21}^{-1}C_2e(k+1)] + D_{12}u_{FI}(k) \end{aligned}$$

subject to (3.191). Hence $G_{FI} = G_{DF}$. ∎

Now it is easy to obtain the solution of DF-problem. Since

$$K = [\,F_P \quad -T_P^{-1}S_P\,]\,(k)$$

is optimal for the system $\mathbf{G}_{FI}$, the optimal controller for $\mathbf{G}_{DF}$ is given by

$$u(k) = [\,F_P \quad -T_P^{-1}S_P\,]\,(k)\begin{bmatrix} \hat{x}(k+1) \\ D_{21}^{-1}(k)[y(k) - C_2(k)\hat{x}(k+1)] \end{bmatrix}$$

and (3.190) in this case

$$\begin{aligned} \hat{x}(k) &= \hat{A}(k)\hat{x}(k+1) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k+1) + \hat{D}(k)y(k) \end{aligned} \tag{3.192}$$

where

$$\begin{aligned} \hat{A}(k) &= [A - (B_1 - B_2T_P^{-1}S_P)D_{21}^{-1}C_2 + B_2F_P](k), \\ \hat{B}(k) &= [(B_1 - B_2T_P^{-1}S_P)D_{21}^{-1}](k), \\ \hat{C}(k) &= (F_P + T_P^{-1}S_PD_{21}^{-1}C_2)(k), \\ \hat{D}(k) &= -(T_P^{-1}S_PD_{21}^{-1})(k) \end{aligned}$$

Theorem 3.27 *Assume* **D1'**, **D5** *and* **D6** *and consider the H_2-problem for the system $\mathbf{G}_{DF}$. Then*
(a) $\min_{K\in\mathbf{K}} \| G_{DF} \|_2^2 = \| G_c^b \|_2^2$.
(b) The controller (3.192) is optimal.

Proof of Theorem 3.25

Now we return to the H2-problem for the system **G**. By (3.183) we have

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2$$

and the original H2-problem was reduced to the H2-problem for the backward system (3.184), which is a DF-problem. Since the conditions **D1'**, **D5** and **D6** are satisfied for (3.184), we can apply Theorem 3.27 to obtain

$$\begin{aligned}
&\min_{K\in\mathbf{K}} \| \bar{G} \|_2^2 \\
= &\lim_{N\to\infty} \frac{1}{N} \sum_{s=k_0+1}^{k_0+N} tr.[T_2^{\frac{1}{2}} \hat{F}(I - C_2' T_{2Y}^{-1} C_2 Y)' Y (I - C_2' T_{2Y}^{-1} C_2 Y) \hat{F}' T_2^{\frac{1}{2}} \\
&\qquad + T_2^{\frac{1}{2}} (T_2^{-1} S + L D_{21})(T_2^{-1} S + L D_{21})' T_2^{\frac{1}{2}}](s) \\
= &\lim_{N\to\infty} \frac{1}{N} \sum_{s=k_0+1}^{k_0+N} tr.[T_2(\hat{C}' Y \hat{C}' + \bar{D}_{21} \bar{D}_{21}')](s)
\end{aligned}$$

and the optimal controller is given by

$$\begin{aligned}
\tilde{x}(k) &= [A' + (\hat{F}' - C_2' L') B_2' + C_2' \hat{H}](k) \tilde{x}(k+1) \\
&\qquad - (\hat{F}' - C_2' L')(k) \tilde{y}(k), \\
\tilde{u}(k) &= (\hat{H}' - L' B_2')(k) \tilde{x}(k+1) + L'(k) \tilde{y}(k).
\end{aligned}$$

Hence the forward controller (3.180) is optimal for the system $\bar{\mathbf{G}}$ and hence for the system **G**. We also have

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \lim_{N\to\infty} \frac{1}{N} \sum_{s=k_0+1}^{k_0+N} tr.[T_2(\hat{C} Y \hat{C}' + \bar{D}_{21} \bar{D}_{21}')](s).$$

Now we express $\| G_c \|_2^2$ using the observability gramian of G_c which is a unique nonnegative solution of

$$L_o(k) = (A + B_2 \hat{F})' L_o(k+1)(A + B_2 \hat{F}) + (C_1 + D_{12} \hat{F})'(C_1 + D_{12} \hat{F}).$$

But X satisfies the equation above and hence $L_o = X$. Then by Lemma 3.3, we have

$$\| G_c \|_2^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{s=k_0}^{k_0+N-1} tr.[\bar{B}_1'(s) X(s+1) \bar{B}_1(s) + (S' T_2^{-2} S)(s)].$$

and we obtain (3.181) and the proof of Theorem 3.25 is complete.

3.6 Notes and References

This chapter contains a discrete version of the results in Chapter 2.

The stability results in Section 3.1 are obtained as in Section 2.1 and we can find similar results in [23]. The H_2 and H_∞ norms are defined as in [21]. The formulation of the quadratic control follows [1, 21]. The results of the disturbance attenuation problems with initial uncertainty are obtained as in Section 2.1.4 and part of the results are found in [41, 42].

The results on quadratic games in Section 3.2 are obtained following the theory in Section 2.2 and part of the results are also found in [42, 44].

The H_∞ control theory in Section 3.3 is based on [44, 38]. As in Section 2.3 initial uncertainty is considered in the problem formulation and the output of the terminal state is included in the finite horizon problem. We have given the necessary and sufficient conditions for the existence of all γ-suboptimal controllers. The necessary and sufficient conditions in terms of the solutions of two independent Riccati equations and a coupling conditions were not available for some time and were established in [38]. The H_∞ theory for time-invariant systems is complete and found in the original papers [39, 40] or in the books [21, 66]. The state space theory of H_∞ control was extended to time-varying systems [15, 21, 44]. The finite horizon problem is considered in [21].

The H_∞ filtering theory is found in [21], but in Section 3.4 we have introduced initial uncertainty in the problem formulation and included the output of the terminal state to be estimated for the finite horizon problem. The H_∞ filtering problem has been considered in [21, 86, 87]. Green and Limebeer [21] gave necessary and sufficient conditions for the existence of γ-suboptimal filters and its characterization for finite horizon case. For a time-invariant system they considered the infinite horizon problem.

The H_2 control theory for time-invariant systems is complete and can be found in [21, 93]. As in the continuous-time systems, we extended the H_2 theory to time-varying systems.

4. Jump Systems

In this chapter we consider jump systems which are the mixture of continuous- and discrete-time systems. We consider the same stability and control problems as in earlier chapters.

4.1 Stability

4.1.1 Lyapunov Equations

Consider

$$\begin{aligned} \dot{x} &= Ax, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau), \\ x(t_0) &= x_0, \ 0 \le t_0 < \tau \end{aligned} \tag{4.1}$$

where $x \in \mathbf{R}^n$ and A, A_d are $n \times n$ constant matrices. Let $S(t, s)$ be the state transition matrix of the system (4.1) (or simply (A, A_d)). Then

$$\begin{aligned} \frac{d}{dt} S(t,s) &= AS(t,s), \quad k\tau < t < (k+1)\tau, \\ S(k\tau^+, s) &= A_d S(k\tau, s), \\ S(s,s) &= I. \end{aligned}$$

Let $0 < t_0 < \tau$. Then the solution $x(t)$, $t \ge t_0$ of (4.1) is continuous except at $t = k\tau$, $k = 1, 2, ..$ where the state jumps according to the second equation and is defined by

$$\begin{aligned} x(t) &= S(t, t_0)x_0 \\ &= \begin{cases} e^{A(t-t_0)}x_0, & t_0 \le t \le \tau, \\ e^{A(t-\tau)}A_d e^{A(\tau - t_0)}x_0, & \tau < t \le 2\tau, \\ \cdots, & \cdots \end{cases} \end{aligned}$$

Here $x(t)$ is left-continuous at $t = k\tau$. The following properties of $S(t, s)$ will be used later

$$S(k\tau^+, k\tau) = A_d,$$

$$\begin{aligned} S(t, k\tau^-) &= S(t, k\tau^+)A_d, \ t > k\tau, \\ S(k\tau, k\tau^-) &= I. \end{aligned}$$

Definition 4.1 *The system (4.1) (or simply (A, A_d)) is said to be exponentially stable on $[t_0, \infty)$ if*

$$| S(t, s) | \leq M e^{-\alpha(t-s)} \text{ for any } t_0 \leq s \leq t < \infty$$

for some positive constants M and α independent of s and t. (The system (4.1) is also called internally stable).

Since $x(k\tau)$ satisfies the discrete-time system

$$x(k+1) = e^{A\tau} A_d x(k), \quad x(0) = e^{A(\tau - t_0)} x_0,$$

(4.1) is exponentially stable if and only if the magnitude of every eigenvalue of $e^{A\tau} A_d$ is less than 1 and by Proposition 3.1, we have the following result.

Proposition 4.1 *The following statements are equivalent.*
(a) The system (4.1) is exponentially stable.
(b) There exists a positive definite matrix X satisfying

$$X = (e^{A\tau} A_d)' X e^{A\tau} A_d + I.$$

(c) There exists a positive definite matrix Y satisfying

$$Y = e^{A\tau} A_d Y (e^{A\tau} A_d)' + I.$$

We also give the stability result using the Lyapunov equation of the jump system (4.1).

Proposition 4.2 *The following statements are equivalent.*
(a) The system (4.1) is exponentially stable.
(b) There exists a τ-periodic symmetric matrix $X(t)$ such that

$$\begin{aligned} &(i) \quad c_1 I \leq X(t) \leq c_2 I, \ \forall t \geq t_0 \text{ for some } c_i > 0, \ i = 1, 2. \\ &(ii) \quad -\dot{X} = A'X + XA + I, \ k\tau < t < (k+1)\tau, \\ &\qquad X(k\tau^-) = A_d' X(k\tau) A_d + I. \end{aligned}$$

(c) $\int_s^\infty | S(t, s)x |^2 \, dt \leq c \, | x |^2$, $\forall x$, $\forall s \geq t_0$ for some $c > 0$.

Proof. Suppose (a) holds. Then (c) also holds and

$$X(t) = \int_t^\infty S'(r, t) S(r, t) dr + \sum_{k\tau > t} S'(k\tau, t) S(k\tau, t)$$

is well-defined and bounded, i.e., $X(t) \leq c_2 I$ and τ-periodic. Since the first term is greater than $a_1 I$ in $(0, \tau - \delta)$ for some $a_1 > 0$ and the second term is greater than $a_2 I$ for some $a_2 > 0$ in $(\tau - \delta, \tau)$, we obtain $X(t) \geq c_1 I$ and hence (i) of (b) has been shown. Differentiating $X(t)$ on $k\tau < t < (k+1)\tau$, we obtain (ii) of (b). The τ-periodicity of $X(t)$ follows as in the proof of Proposition 2.2.

Now we assume (b). Then we have for $k\tau < t < (k+1)\tau$

$$\frac{d}{dt}[x'(t)X(t)x(t)] = -\mid x(t) \mid^2 \leq -\frac{1}{c_2}x'(t)X(t)x(t).$$

At $t = k\tau$, $x'(t)X(t)x(t)$ has left and right limits and

$$x'(k\tau^+)X(k\tau)x(k\tau^+) - x'(k\tau)X(k\tau^-)x(k\tau) = -\mid x(k\tau) \mid^2 \leq 0.$$

Hence

$$x'(t)X(t)x(t) \leq e^{-\frac{1}{c_2}(t-s)}x'(s)X(s)x(s)$$

where $t_0 \leq s \leq t$. Using the property (i) we have

$$c_1 \mid x(t) \mid^2 \leq c_2 e^{-\frac{1}{c_2}(t-s)} \mid x_0 \mid^2 .$$

Hence

$$\mid S(t,s) \mid \leq \sqrt{\frac{c_2}{c_1}} e^{-\frac{1}{2c_2}(t-s)}$$

and (a) follows. ∎

Definition 4.2 *The equation (ii) of (b) is called the Lyapunov equation of the system (4.1).*

Example 4.1 Consider the jump system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x, \ k < t < k+1, \\ x(k^+) &= \begin{bmatrix} 1 & 0 \\ -0.6 & -0.5 \end{bmatrix} x(k). \end{aligned} \tag{4.2}$$

This is exponentially stable. In fact there exists a periodic nonnegative solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ of the condition (b) in Proposition 4.2 (Figures 4.1 and 4.2).

Consider the adjoint equation of (4.1)

$$\begin{aligned} -\dot{\xi} &= A'\xi, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= A'_d \xi(k\tau), \\ \xi(T) &= \xi_1 \end{aligned} \tag{4.3}$$

where $N\tau \leq T < (N+1)\tau$. Let $\xi(t; T, \xi_1)$ be the solution of (4.3).

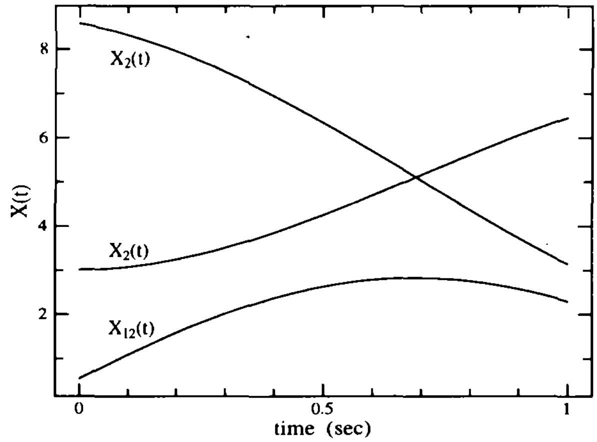

Figure 4.1: The periodic nonnegative solution $X(t)$

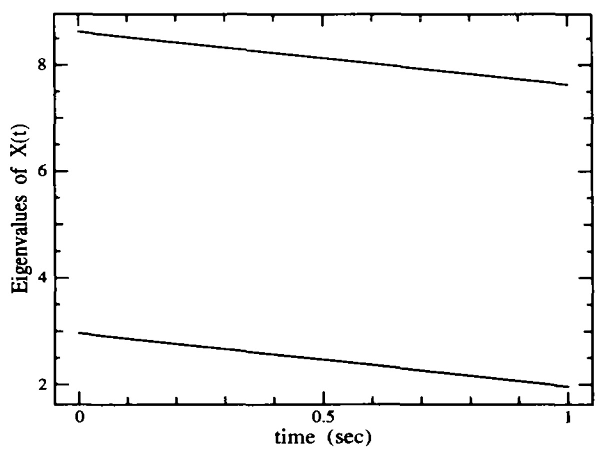

Figure 4.2: Eigenvalues of $X(t)$

Definition 4.3 *The system (4.3) is said to be exponentially stable if*

$$| \xi(t; T, \xi_1) | \leq M e^{-\alpha(T-t)} | \xi_1 | \quad \text{for any } t \leq T < \infty$$

for some positive constants M and α independent of t, T and ξ_1.

We have a dual result to Proposition 4.2.

Proposition 4.3 *The following statements are equivalent.*
(a) The system (4.3) (and hence (4.1)) is exponentially stable.
(b) There exists a symmetric matrix $Y(t)$ and a $0 < \delta < \tau - t_0$ such that

(i) $0 < Y(t)$, $^\forall t > t_0$ *and* $c_1 I \leq Y(t)$, $^\forall t \geq t_0 + \delta$ *for some* $c_1 > 0$.
(ii) $Y(t) \leq c_2 I$, $t_0 \leq {}^\forall t < \infty$ *for some* $c_2 > 0$.
(iii) $\dot{Y} = AY + YA' + I$, $k\tau < t < (k+1)\tau$,
$Y(k\tau^+) = A_d Y(k\tau) A_d' + I$,
$Y(t_0) = 0$.

(c) $\int_s^T | S'(T,t)\xi |^2 \, dt \leq c | \xi |^2$, $^\forall s, T$ *with* $t_0 \leq s \leq T < \infty$ *and for some* $c > 0$.

Proof. Suppose (a) holds. Then (c) is true and

$$Y(t) = \int_{t_0}^{t} S(t,s) S'(t,s) ds + \sum_{k\tau < t} S(t, k\tau^+) S'(t, k\tau^+)$$

is well-defined, positive for $t > t_0$ and bounded. Hence (ii) of (b) holds. Combining the arguments of the proof of Propositions 2.3 and 4.2, we obtain $Y(t) \geq c_1 I$, $^\forall t \geq t_0 + \delta$ for some $c_1 > 0$ and hence (ii) follows. Differentiating $Y(t)$ on $k\tau < t < (k+1)\tau$, we obtain (iii) of (b).

Now we suppose (b) holds. Then for $k\tau < t < (k+1)\tau$

$$\frac{d}{dt}[\xi'(t) Y(t) \xi(t)] = | \xi(t) |^2 \geq \frac{1}{c_2} \xi'(t) Y(t) \xi(t)$$

and at $t = k\tau$

$$\xi'(k\tau) Y(k\tau^+) \xi(k\tau) - \xi'(k\tau^-) Y(k\tau) \xi(k\tau^-) = | \xi(k\tau) |^2 \geq 0.$$

Hence

$$\xi'(s) Y(s) \xi(s) \leq e^{-\frac{1}{c_2}(T-s)} \xi'(T) Y(T) \xi(T).$$

Hence for $t_0 + \delta \leq s \leq \tau \leq T < \infty$

$$c_1 | \xi(s) |^2 \leq c_2 e^{-\frac{1}{c_2}(T-s)} | \xi_1 |^2$$

which yields

$$| S'(T,s) | \leq \sqrt{\frac{c_2}{c_1}} e^{-\frac{1}{2c_2}(T-s)}.$$

Similar to the proof of Proposition 2.3

$$| S'(T,s) | \leq \sqrt{\frac{c_2}{c_1}} c_0 e^{\frac{1}{2c_2}\delta} e^{-\frac{1}{2c_2}(T-s)} \text{ for } t_0 \leq s \leq t_0 + \delta \leq \tau \leq T < \infty$$

and

$$| S'(T,s) | \leq c_0 \leq c_0 e^{\frac{1}{2c_2}\delta} e^{-\frac{1}{2c_2}(T-s)} \text{ for } t_0 \leq s \leq t_0 + \delta.$$

Choosing

$$M = \max(\sqrt{\frac{c_2}{c_1}}, \sqrt{\frac{c_2}{c_1}} c_0 e^{\frac{1}{2c_2}\delta}, c_0 e^{\frac{1}{2c_2}\delta})$$

we obtain

$$| S'(t,s) | \leq M e^{-\frac{1}{2c_2}(T-s)} \text{ for any } t_0 \leq s \leq T < \infty.$$

Hence (a) holds. ∎

Definition 4.4 *The equation (iii) of (b) is called the Lyapunov equation of the backward system (4.3) (or simply the backward Lyapunov equation).*

Corollary 4.1 *The system (4.1) is exponentially stable if and only if there exists a τ-periodic solution of the Lyapunov equation with $c_1 I \leq Y(t) \leq c_2 I$ for some c_1, $c_2 > 0$.*

Proof. Similar to the proof of Corollary 2.1. ∎

Example 4.2 Consider the system (4.2) in Example 4.1 which is exponentially stable. In fact there exists a bounded nonnegative solution $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}(t)$ of the condition (b) in Proposition 4.3 which converges to a periodic solution with period 1 (Figure 4.3).

Consider the jump system

$$\begin{aligned} \dot{x} &= Ax + Bu, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d u_d(k), \\ y &= Cx, \\ y_d(k) &= C_d x(k\tau) + D_d u_d(k) \end{aligned} \tag{4.4}$$

with initial condition

$$x(t_0) = x_0$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$, $u_d \in \mathbf{R}^{m_{2d}}$, $y \in \mathbf{R}^{p_2}$, $y_d \in \mathbf{R}^{p_{2d}}$ and all matrices are of compatible dimensions. Since the system (4.4) is τ-periodic, without loss of generality we can set $0 < t_0 \leq \tau$. If $A_d = I$, $B_d = 0$, $C_d = 0$ and

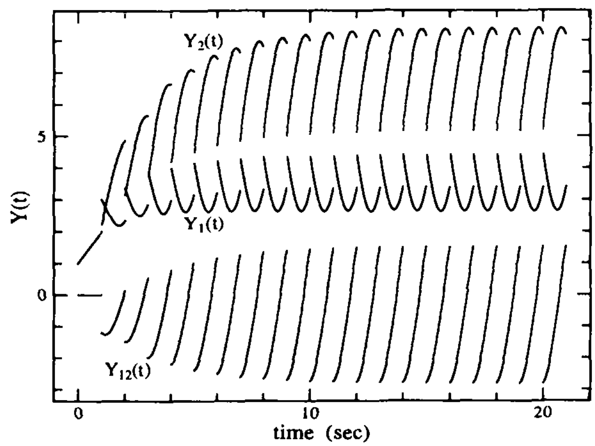

Figure 4.3: The bounded nonnegative solution $Y(t)$

$D_d = 0$, the system (4.4) is a usual continuous-time system. On the other hand if $B = 0$ and $C = 0$, then (4.4) is equivalent to a discrete-time system

$$\begin{aligned} x(k+1) &= \tilde{A}_d x(k) + \tilde{B}_d w_d(k), \quad x(0) = e^{A(\tau - t_0)} x_0, \\ z_d(k) &= C_d x(k\tau) + D_d w_d(k) \end{aligned}$$

where $\tilde{A}_d = e^{A\tau} A_d$ and $\tilde{B}_d = e^{A\tau} B_d$. Hence the jump system is a natural extension of the continuous- and discrete-time systems.

The solution $x(t)$ with $x(s) = x_0$, $0 < s \le \tau$ of (4.4) is defined in a piecewise manner as follows

$$x(t) = S(t, k\tau) x(k\tau^+) + \int_{k\tau}^{t} S(t,r) B u(r) dr, \quad k\tau < t \le (k+1)\tau.$$

We can express $x(t)$ in terms of $S(t,s)$ as

$$\begin{aligned} x(t) &= S(t, t_0) x_0 + \int_{t_0}^{t} S(t,r) B u(r) dr + \sum_{j=1}^{k} S(t, j\tau^+) B_d u_d(j), \\ & \qquad k\tau < t \le (k+1)\tau. \end{aligned}$$

Definition 4.5 *The system (4.4) is said to be input-output stable (or simply IO-stable) if for* $x(s) = 0$, $0 < s \le \tau$ *and any* $(u, u_d) \in \times L^2(s, \infty; \mathbf{R}^{m_2}) \times l^2(1, \infty; \mathbf{R}^{m_{2d}})$,

$$(y, y_d) \in L^2(s, \infty; \mathbf{R}^{p_2}) \times l^2(1, \infty; \mathbf{R}^{p_{2d}})$$

and

$$\| y \|^2_{L^2(t_0,\infty;\mathbf{R}^{p_2})} + \| y_d \|^2_{l^2(1,\infty;\mathbf{R}^{p_{2d}})} \leq c(\| u \|^2_{L^2(s,\infty;\mathbf{R}^{m_2})} + \| u_d \|^2_{l^2(1,\infty;\mathbf{R}^{m_{2d}})})$$

for some c independent of s.

Definition 4.6 *(a) The system (4.4) (or $([A, A_d], [B, B_d])$) is said to be stabilizable if there exist matrices K and K_d such that $(A + BK, A_d + B_dK_d)$ is exponentially stable.*
(b) The system (4.4) (or $([C, C_d], [A, A_d])$) is detectable if there exist matrices J and J_d such that $(A + JC, A_d + J_dC_d)$ is exponentially stable.
(c) If (a) and (b) hold, the system (4.4) or $([A, A_d], [B, B_d], [C, C_d])$ is said to be stabilizable and detectable.

Proposition 4.4 *Suppose that the system (4.4) is stabilizable and detectable. Then it is exponentially stable if and only if it is IO-stable.*

Proof. It is enough to show sufficiency. Without loss of generality we assume $D_d = 0$. First we shall show $CS(t, s)x_0 \in L^2(s, \infty; \mathbf{R}^{p_2})$ and $C_dS(k\tau, s)x \in l^2(1, \infty; \mathbf{R}^{p_{2d}})$. Since (4.4) is stabilizable, there exist matrices K and K_d such that the system

$$\begin{aligned} \dot{x} &= (A + BK)x, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d + B_dK_d)x(k\tau), \\ x(s) &= x_0, \ 0 < s \leq \tau \end{aligned}$$

is exponentially stable. Hence $x \in L^2(s, \infty; \mathbf{R}^n)$. Then

$$\begin{aligned} \dot{x} &= Ax + BKx, \ x(s) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_dx(k\tau) + B_dK_dx(k\tau) \end{aligned}$$

and

$$\begin{aligned} Cx(t) &= CS(t, s)x_0 + C\int_s^t S(t, r)BKx(r)dr \\ &\quad + C\sum_{j=1}^{k} S(t, j\tau^+)B_dK_dx(j\tau), \ k\tau < t < (k+1)\tau, \\ C_dx(k\tau) &= C_dS(k\tau, s)x_0 + C_d\int_s^{k\tau} S(k\tau, r)BKx(r)dr \\ &\quad + C_d\sum_{j=1}^{k-1} S(k\tau, j\tau^+)B_dK_dx(j\tau). \end{aligned}$$

Since (4.4) is IO-stable,

$$C[\int_s^t S(t, r)BKx(r)dr + \sum_{j=1}^{k} S(t, j\tau^+)B_dK_dx(j\tau)] \in L^2(s, \infty; \mathbf{R}^{p_2})$$

and

$$C_d[\int_s^{k\tau} S(t,r)BKx(r)dr + \sum_{j=1}^{k} S(k\tau, jh^+)B_dK_dx(j\tau)] \in l^2(1,\infty;\mathbf{R}^{p_{2d}}).$$

Hence

$$CS(t,s)x_0 \in L^2(s,\infty;\mathbf{R}^{p_2}), \quad C_dS(k\tau,s)x_0 \in l^2(1,\infty;\mathbf{R}^{p_{2d}})$$

and

$$\| CS(t,s)x_0 \|_{L^2}, \ \| C_dS(k\tau,s)x_0 \|_{l^2} \le c \mid x_0 \mid$$

for some $c > 0$ independent of s and x_0. Since the system

$$\begin{aligned} \dot{x} &= Ax, \ x(s) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_dx(k\tau) \end{aligned}$$

is equivalent to

$$\begin{aligned} \dot{x} &= (A+LC)x - LCx, \ x(s) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d + L_dC_d)x(k\tau) - L_dC_dx(k\tau) \end{aligned}$$

where L and L_d are chosen such that $(A + LC, A_d + L_dC_d)$ is exponentially stable. Then we have

$$\begin{aligned} x(t) &= S_L(t,s)x_0 - \int_s^t S_L(t,r)LCx(r)dr \\ &\quad - \sum_{j=1}^{k} S_L(t, j\tau^+)L_dC_dx(j\tau), \ k\tau < t < (k+1)\tau \end{aligned}$$

where $S_L(t,s)$ is the state transition matrix of $(A + LC, A_d + L_dC_d)$. Since

$$\begin{aligned} Cx(t) &= CS(t,s)x_0 \in L^2(s,\infty;\mathbf{R}^{p_2}), \\ C_dx(k\tau) &= C_dS(k\tau,s)x_0 \in l^2(1,\infty;\mathbf{R}^{p_{2d}}), \end{aligned}$$

$x \in L^2(s,\infty;\mathbf{R}^n)$ and $\| x \|_{L^2} \le c \mid x_0 \mid$ which implies (4.4) is exponentially stable. ∎

Proposition 4.5 *(a) Suppose that $([C, C_d], [A, A_d])$ is detectable. Then the system (4.4) is exponentially stable if and only if there exists a τ-periodic nonnegative solution to*

$$\begin{aligned} -\dot{X} &= A'X + XA + C'C, \ \ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A_d'X(k\tau)A_d + C_d'C_d. \end{aligned}$$

(b) Suppose that $([A, A_d], [B, B_d])$ is stabilizable. Then the system (4.4) is exponentially stable if and only if there exists a τ-periodic nonnegative solution to

$$\begin{aligned} \dot{Y} &= AY + YA' + BB', \ \ k\tau < t < (k+1)\tau, \\ Y(k\tau^+) &= A_dY(k\tau)A_d' + B_dB_d'. \end{aligned}$$

Proof. We shall show (a) only. Let $x(t) = S(t,s)x_0$. Then for $k\tau < t < (k+1)\tau$ we have

$$\frac{d}{dt}[x'(t)X(t)x(t)] = - \mid Cx(t) \mid^2$$

and at $t = k\tau$

$$x'(k\tau^+)X(k\tau)x(k\tau^+) - x'(k\tau)X(k\tau^-)x(k\tau) = - \mid C_d x(k\tau) \mid^2 .$$

Integrating $\frac{d}{dt}[x'(t)X(t)x(t)]$ from s to T, we obtain

$$x'(T)X(T)x(T) + \int_s^T \mid Cx(t) \mid^2 dt + \sum_{k=1}^{N} \mid C_d x(k\tau) \mid^2 = x_0' X(s) x_0$$

where $0 < s \le \tau \le N\tau \le T < (N+1)\tau < \infty$. Hence $CS(t,s)x_0 \in L^2(s,\infty;\mathbf{R}^{p_2})$ and $C_d S(k\tau,s)x_0 \in L^2(1,\infty;\mathbf{R}^{p_{2d}})$ with $\| CS(t,s)x_0 \|_{L^2}$, $\| C_d S(k\tau,s)x_0 \|_{l^2} \le c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. As in the last part of the proof of Proposition 4.4, we can show $x \in L^2(s,\infty;\mathbf{R}^n)$ with $\| x \|_{L^2} \le c \mid x_0 \mid$ for some $c > 0$ independent of s and x_0. The τ-periodicity of X follows as in the proof of Proposition 4.2. ∎

4.1.2 Performance Measures of Stable Systems

Consider the system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + Bw, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k), \\ z_c &= Cx, \\ z_d(k) &= C_d x(k\tau) + D_d w_d(k) \end{aligned} \tag{4.5}$$

with initial condition

$$x(0) = 0$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $w_d \in \mathbf{R}^{m_{1d}}$, $z_c \in \mathbf{R}^{p_1}$, $z_d \in \mathbf{R}^{p_{1d}}$ and all matrices are of compatible dimensions. Here we assume that (A, A_d) is exponentially stable. Let T_{zw} and T_{zw_d} be the operators from w and w_d to $z = \begin{bmatrix} z_c \\ z_d \end{bmatrix}$, given by

$$z = \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = T_{zw} w = \begin{bmatrix} C \int_0^t S(t,r)Bw(r)dr \\ C_d \int_0^{k\tau} S(k\tau,r)Bw(r)dr \end{bmatrix}$$

and

$$z = \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = T_{zw_d} w_d = \begin{bmatrix} C \sum_{j=0}^{k} S(t, j\tau^+) B_d w_d(j) \\ C_d \sum_{j=0}^{k-1} S(k\tau, j\tau^+) B_d w_d(j) + D_d w_d(k) \end{bmatrix}$$

respectively, where $k\tau < t < (k+1)\tau$. Let (e_i) and (f_j) be the unit vectors of $\mathbf{R}^{m_1}$ and $\mathbf{R}^{m_{1d}}$, respectively. As in continuous- and discrete time systems, we consider the impulse

$$w(t) = \delta(t-s)e_i, \quad 0 \le s < \tau$$

and

$$w_d(0) = f_j \text{ and } w_d(k) = 0 \ \ ^\forall k \ge 1.$$

Then

$$T_{zw}\delta(\cdot - s)e_i = \begin{bmatrix} CS(t,s)Be_i \\ C_dS(k\tau,s)Be_i \end{bmatrix}$$

and

$$T_{zw_d}\delta_{\cdot 0}f_j = \begin{cases} \begin{bmatrix} CS(t,0^+)B_df_j \\ C_dS(k\tau,0^+)B_df_j \end{bmatrix}, & k \ge 1, \\ \begin{bmatrix} 0 \\ D_df_j \end{bmatrix}, & k = 0 \end{cases}$$

where $k\tau < t < (k+1)\tau$ and δ_{ks} is the Kronecker delta. Now we define the H$_2$ norm of the system $\mathbf{G}_j$ as follows:

Definition 4.7 *The H_2-norm of the system $\mathbf{G}_j$, denoted by $\| G \|_2$ is*

$$\| G \|_2^2 = \sum_{i=1}^{m_1} \frac{1}{\tau} \int_0^\tau \| T_{zw}\delta(\cdot - s)e_i \|_{L^2\times l^2}^2 \, ds + \sum_{j=1}^{m_{1d}} \| T_{zw_d}\delta_{\cdot 0}f_j \|_{L^2\times l^2}^2 \, .$$

where (e_i) and (f_j) are unit vectors in $\mathbf{R}^{m_1}$ and $\mathbf{R}^{m_{1d}}$, respectively and

$$\| \begin{pmatrix} \xi_c \\ \xi_d \end{pmatrix} \|_{L^2\times l^2} = \sqrt{\| \xi_c \|_{L^2}^2 + \| \xi_d \|_{l^2}^2}.$$

If $A_d = I$, $B_d = 0$, $C_d = 0$ and $D_d = 0$, then $\| G \|_2$ is the H$_2$-norm of continuous-time systems, while the case $A = 0$, $B = 0$ and $C = 0$ yields the H$_2$-norm of discrete-time systems.

Remark 4.1 Using the state transition matrix of (A, A_d), we can express $\| G \|_2$ as

$$\begin{aligned} \| G \|_2^2 \;=\; & \frac{1}{\tau}\int_0^\tau tr.\{B'[\int_s^\infty S'(t,s)C'CS(t,s)dt \\ & \qquad + \sum_{k=1}^\infty S'(k\tau,s)C_d'C_dS(k\tau,s)]B\}ds \\ & + tr.\{B_d'[\int_0^\infty S'(t,0^+)C'CS(t,0^+)dt \end{aligned}$$

$$+\sum_{k=1}^{\infty} S'(k\tau,0^+)C_d'C_dS(k\tau,0^+)]B_d + D_d'D_d\}$$

$$= \frac{1}{\tau}\int_0^\tau tr.\{C\int_s^\infty S(t,s)BB'S'(t,s)dtC'$$
$$+C_d\sum_{k=1}^{\infty} S(k\tau,s)BB'S'(k\tau,s)C_d'\}ds$$
$$+tr.\{C\int_0^\infty S(t,0^+)B_dB_d'S'(t,0^+)dtC'$$
$$+C_d\sum_{k=1}^{\infty} S(k\tau,0^+)B_dB_d'S'(k\tau,0^+)C_d' + D_dD_d'\}.$$

Since (A, A_d) is exponentially stable, by Proposition 4.5 there exist a unique τ-periodic nonnegative solution X called the observability gramian such that

$$\begin{aligned} -\dot{X} &= A'X + XA + C'C,\ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A_d'X(k\tau)A_d + C_d'C_d \end{aligned} \tag{4.6}$$

and a unique τ-periodic nonnegative solution Y called the controllability gramian:

$$\begin{aligned} \dot{Y} &= AY + YA' + \frac{1}{\tau}BB',\ k\tau < t < (k+1)\tau, \\ Y(k\tau^+) &= A_dY(k\tau)A_d' + B_dB_d'. \end{aligned} \tag{4.7}$$

We can express $\| G \|_2$ in terms of X or Y. The following lemma is useful.

Lemma 4.1 *Let N be a positive integer. Then*

$$(a) \quad \int_0^\tau\int_s^{N\tau} S(t,s)dtds = \int_0^\tau\int_0^{N\tau-\xi} S(N\tau-\xi,s)dsd\xi,$$

$$(b) \quad \int_0^{N\tau} S(t,0)dt = \sum_{j=0}^{N-1}\int_{(N-1-j)\tau}^{(N-j)\tau} S(N\tau-\xi,j\tau)d\xi.$$

Proof. (a)

$$\begin{aligned} \int_\tau^{N\tau} S(t,s)dt &= \sum_{j=0}^{N-2}\int_{(N-1-j)\tau}^{(N-j)\tau} S(t,s)dt,\quad 0\le s\le\tau \\ &= \sum_{j=0}^{N-2}\int_0^\tau S((N-j)\tau-\xi,s)d\xi, \\ &= \sum_{j=0}^{N-2}\int_0^\tau S(N\tau-\xi,j\tau+s)d\xi \end{aligned}$$

where the last equality follows from the τ-periodicity of $S(\cdot,\cdot)$. We also have

$$\int_s^\tau S(t,s)dt = \int_0^{\tau-s} S(N\tau-\xi,(N-1)\tau+s)d\xi$$

and

$$\begin{aligned}\int_0^\tau\int_s^\tau S(t,s)dtds &= \int_0^\tau\int_0^{\tau-\xi} S(N\tau-\xi,(N-1)\tau+s)d\xi ds\\ &= \int_0^\tau\int_0^{\tau-\xi} S(N\tau-\xi,(N-1)\tau+s)dsd\xi\\ &= \int_0^\tau\int_{(N-1)\tau}^{N\tau-\xi} S(N\tau-\xi,s)dsd\xi\end{aligned}$$

where we have used the Fubini's theorem in the second equality. Hence

$$\int_0^\tau\int_s^{N\tau} S(t,s)dtds = \int_0^\tau\int_0^{N\tau-\xi} S(N\tau-\xi,s)dsd\xi.$$

(b) Similar. ∎

Theorem 4.1

$$\begin{aligned}\| G\|_2^2 &= \frac{1}{\tau}\int_0^\tau tr.B'X(s)B\ ds + tr.[B_d'X(0)B_d + D_d'D_d]\\ &= \int_0^\tau tr.CY(s)C'\ ds + tr.[C_dY(0)C_d' + D_dD_d']. \qquad (4.8)\end{aligned}$$

Proof. Consider the system $\mathbf{G}_j$ with the initial condition $x(s)$ and $w=0$, $w_d=0$. Then for $k\tau < t \le (k+1)\tau$

$$\frac{d}{dt}[x'(t)X(t)x(t)] = -\mid Cx(t)\mid^2 = -\mid z_c(t)\mid^2$$

and at $t=k\tau$

$$\begin{aligned}&x'(k\tau^+)X(k\tau)x(k\tau^+) - x'(k\tau)X(k\tau^-)x(k\tau)\\ &\qquad = -\mid C_dx(k\tau)\mid^2 = -\mid z_d(k)\mid^2.\end{aligned}$$

Integrating the above derivative from s to ∞, $0<s<\tau$ we have

$$x'(s)X(s)x(s) = \int_s^\infty \mid z_c(t)\mid^2 ds + \sum_{k=1}^\infty \mid z_d(k)\mid^2 .$$

Let $x(s^+) = Be_i$ which corresponds to the case when $w_d = 0$ and $w(t) = \delta(t-s)e_i$. Then

$$e_i'B'X(s)Be_i = \int_s^\infty \mid z_c[e_i](t)\mid^2 ds + \sum_{k=1}^\infty \mid z_d[e_i](k)\mid^2$$

and

$$\frac{1}{\tau}\int_0^\tau tr.B'X(s)B\,ds = \sum_{i=1}^{m_1}\frac{1}{\tau}\int_0^\tau\{\int_s^\infty |z_c[e_i](t)|^2\,dt + \sum_{k=1}^{\infty}|z_d[e_i](k)|^2\}ds.$$

Similarly let $x(0^+) = B_d f_j$ which corresponds to the case when $w(t) = 0$ and $w_d(0) = f_j$ with $w_d(k) = 0$, $^\forall k > 0$ in (4.5). Then we have

$$tr.[B_d'X(0)B_d + D_d'D_d] = \sum_{j=1}^{m_{1d}}\{\int_0^\infty |z_c[f_j](t)|^2\,dt + \sum_{k=0}^{\infty}|z_d[f_j](k)|^2\}.$$

Hence we have obtained the first equality in (4.8).

To show the second equality of (4.8), first assume $C = 0$ and consider

$$\begin{aligned}\| G_N \|^2 &= \frac{1}{\tau}\int_0^\tau tr.C_d\sum_{k=1}^{N}S(k\tau,s)BB'S'(k\tau,s)C_d'\,ds \\ &\quad + tr.[C_d\sum_{k=1}^{N}S(k\tau,0^+)B_dB_d'S'(k\tau,0^+)C_d' + D_dD_d'].\end{aligned}$$

By Remark 4.1

$$\| G \|_2^2 = \lim_{N\to\infty}\| G_N \|^2 .$$

Using the τ-periodicity of $S(t,s)$, i.e.,

$$S(t+\tau, s+\tau) = S(t,s) \text{ for any } t \geq s,$$

$\| G_N \|^2$ is rewritten as

$$\begin{aligned}\| G_N \|^2 &= \frac{1}{\tau}\int_0^\tau tr.[C_d\sum_{k=1}^{N}S(N\tau,(N-k)\tau+s)BB' \\ &\qquad\times S'(N\tau,(N-k)\tau+s)C_d']ds \\ &\quad + tr.[C_d\sum_{k=1}^{N}S(N\tau,(N-k)\tau^+)B_dB_d' \\ &\qquad\times S'(N\tau,(N-k)\tau^+)C_d' + D_dD_d'].\end{aligned}$$

This is equivalent to

$$\| G_N \|^2 = \sum_{j=1}^{m_d}\{\int_0^{N\tau}|\tilde{z}_{jc}(t)|^2\,dt + \sum_{k=0}^{N}|\tilde{z}_{jd}(k)|^2\}$$

where $\tilde{z}_{jc}$ and $\tilde{z}_{jd}$ are the outputs of the system

$$\begin{aligned} -\dot{\tilde{x}} &= A'\tilde{x}, \ k\tau < t < (k+1)\tau, \\ \tilde{x}(k\tau^-) &= A_d'\tilde{x}(k\tau) + C_d'\tilde{w}_d(k), \\ \tilde{z}_{jc} &= \frac{1}{\sqrt{\tau}}B'\tilde{x}, \\ \tilde{z}_{jd}(k) &= B_d'\tilde{x}(k\tau) + D_d'\tilde{w}_d(k) \end{aligned} \tag{4.9}$$

with the terminal condition

$$\tilde{x}(T) = 0, \ N\tau \leq T < (N+1)\tau$$

and

$$\tilde{w}_d(N) = f_j \text{ and } \tilde{w}_d(k) = 0, \ 1 \leq k \leq N-1.$$

As in the first part, we have

$$\| G_N \|^2 = tr.[C_d\tilde{Y}(N\tau)C_d' + D_dD_d']$$

where $\tilde{Y}$ is a unique nonnegative solution of (4.7) with $\tilde{Y}(0) = 0$. Since

$$\lim_{n\to\infty} \tilde{Y}(t+n\tau) = Y(t),$$

we conclude

$$\| G \|_2^2 = \lim_{N\to\infty} tr.[C_d\tilde{Y}(N\tau)C_d' + D_dD_d'] = tr.[C_dY(0)C_d' + D_dD_d']. \tag{4.10}$$

Now we assume $C_d = 0$ and set

$$\begin{aligned} \| G \|_{2N}^2 &= \frac{1}{\tau}\int_0^\tau tr.B'\int_s^{N\tau} S'(t,s)C'CS(t,s)dtBds \\ &\quad + tr.B_d'\int_0^{N\tau} S'(t,0^+)C'CS(t,0^+)dtB_d. \end{aligned}$$

Using Lemma 4.1, we can show

$$\| G \|_{2N}^2 = \int_0^\tau tr.C\tilde{Y}(N\tau - t)C' \ dt.$$

Hence

$$\| G \|^2 = \lim_{N\to\infty} \| G \|_{2N}^2 = \int_0^\tau tr.CY(\xi)C' \ d\xi. \tag{4.11}$$

Combining (4.10) and (4.11), we obtain the second equality in (4.8). ∎

Remark 4.2 As in Chapters 2 and 3 we can define the H_2-norm of the system (4.9) (denoted by $\mathbf{G}^b$) by

$$\begin{aligned}\| G^b \|_2^2 &= \lim_{N\to\infty} \sum_{j=1}^{m_{1d}} \| G^b \delta_{\cdot N} e_i \|_{L^2\times l^2}^2 \\ &= \lim_{N\to\infty} \sum_{j=1}^{m_{1d}} \{ \int_0^{N\tau} \mid \tilde{z}_{jc}(t) \mid^2 dt + \sum_{k=0}^{N} \mid \tilde{z}_{jd}(k) \mid^2 \}.\end{aligned}$$

Definition 4.8 *The H_∞-norm of the system $\mathbf{G}_j$, denoted by $\| G \|_\infty$ is given by*

$$\| G \|_\infty = \sup_{0\neq(w,w_d)\in L^2\times l^2} \frac{\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2\times l^2}}{\| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2\times l^2}}.$$

As in the previous chapters we extend the Bounded Real Lemma to jump systems. To do so, we need to consider a quadratic optimization problem. First we consider the quadratic control problems for jump systems.

4.1.3 Quadratic Control

Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d u_d(k), \\ x(t_0) &= x_0, \ 0 < t_0 \leq \tau\end{aligned}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^{m_2}$, $u_d \in \mathbf{R}^{m_{2d}}$ and all matrices are of compatible dimensions. For this system we introduce the functional to be minimized

$$\begin{aligned}J_T(u, u_d; t_0, x_0) &= \int_{t_0}^{T} [\| Cx(t) \mid^2 + \mid u(t) \mid^2] dt \\ &\quad + \sum_{k=1}^{N} [\| C_d x(k) \mid^2 + \mid u_d(k) \mid^2] + \mid Fx(T) \mid^2\end{aligned}$$

where $t_0 \leq N\tau \leq T < (N+1)\tau$, $C \in \mathbf{R}^{p_2\times n}$, $C_d \in \mathbf{R}^{p_{2d}\times n}$ and $F \in \mathbf{R}^{q\times n}$.

We need the following Riccati equation with jumps

$$\begin{aligned}-\dot{X} &= A'X + XA + C'C - XBB'X, \qquad (4.12) \\ &\qquad k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A_d' X(k\tau) A_d + C_d' C_d - (R_2' T_2^{-1} R_2)(k), \qquad (4.13) \\ X(T) &= F'F \qquad (4.14)\end{aligned}$$

where $T_2(k) = I + B_d' X(k\tau) B_d$ and $R_2(k) = B_d' X(k\tau) A_d$. Considering Theorems 2.1 and 3.1, we obtain the following.

Theorem 4.2 *There exists a unique nonnegative solution $X = X_T(t)$ to the Riccati eqaution (4.12)-(4.14). Moreover, the state feedback law*

$$\begin{aligned} \bar{u}(t) &= -B' X(t) x(t), \ k\tau < t < (k+1)\tau, \\ \bar{u}_d(k) &= -(T_2^{-1} R_2)(k) x(k\tau), \ k = 1, .., N \end{aligned}$$

is optimal and

$$J_T(\bar{u}, \bar{u}_d; t_0, x_0) = \begin{cases} x_0' X(t_0) x_0, & \text{if } t_0 \neq \tau, \\ x_0' X(\tau^-) x_0, & \text{if } t_0 = \tau. \end{cases}$$

We omit the proof of this theorem. Instead we shall give a proof for a similar problem (4.39). See Lemma 4.6.

Now consider the infinite horizon problem

$$\begin{aligned} \dot{x} &= Ax + Bu, \ x(t_0) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d u_d(k), \\ J(u, u_d; t_0, x_0) &= \int_{t_0}^{\infty} [| Cx(t) |^2 + | u(t) |^2] dt \\ &\quad + \sum_{k=1}^{\infty} [| C_d x(k) |^2 + | u_d(k) |^2] \end{aligned}$$

where $(u, u_d) \in L^2(t_0, \infty; \mathbf{R}^{m_2}) \times l^2(1, \infty; \mathbf{R}^{m_{2d}})$ is admissible if its response $x \in L^2(t_0, \infty; \mathbf{R}^n)$ and $\lim_{t\to\infty} x(t) = 0$.

RJ: We assume that for each x_0 there exists a control $(u(\cdot; x_0), u_d(\cdot; x_0))$ such that $J(u(\cdot, x_0), u_d(\cdot, x_0); t_0, x_0) \leq c(x_0)$ for some constant $c(x_0)$.

If $([A, A_d], [B, B_d])$ is stabilizable, then **RJ** holds.

Lemma 4.2 *Assume* **RJ** *holds. Then there exists a τ-periodic nonnegative solution to the Riccati equation (4.12) and (4.13).*

Proof. By Theorem 4.2 there exists a nonnegative solution to (4.12) and (4.13) on $[t_0, T]$ with $X(T) = 0$. Then $X_T(t_0) \leq X_{\bar{T}}(t_0)$ if $t_0 \leq T \leq \bar{T}$. In fact let

$$\begin{aligned} \bar{u}_T(t) &= -B' X_T(t) x(t), \ k\tau < t < (k+1)\tau, \\ \bar{u}_{dT}(k) &= -(T_2^{-1} R_2)(k) x(k\tau) \end{aligned}$$

then

$$\begin{aligned} x_0' X_T(t_0) x_0 &= J_T(\bar{u}_T, \bar{u}_{dT}; t_0, x_0) \\ &\leq J_T(\tilde{u}_{\bar{T}}, \tilde{u}_{d\bar{T}}; t_0, x_0) \\ &\leq J_{\bar{T}}(\bar{u}_{\bar{T}}, \bar{u}_{d\bar{T}}; t_0, x_0) = x_0' X_{\bar{T}}(t_0) x_0 \end{aligned}$$

where we set $F = 0$ in J_T and $\tilde{u}_T$, $\tilde{u}_{dT}$ in J_T is the restriction of the feedback law $\tilde{u}_T$, $\tilde{u}_{dT}$ to $[s, T]$. We note that

$$\begin{aligned} x_0' X_T(t_0) x_0 &= J_T(\bar{u}_T, \bar{u}_{dT}; t_0, x_0) \\ &\leq J_T(u(\cdot; x_0), u_d(\cdot; x_0); t_0, x_0) \\ &\leq J(u(\cdot; x_0), u_d(\cdot; x_0); t_0, x_0) < \infty. \end{aligned}$$

Hence $x_0' X_T(t_0) x_0$ is monotone increasing and uniformly bounded in T. Since x_0 is arbitrary, there exists a bounded nonnegative matrix X such that

$$X_T(t_0) \to X(t_0).$$

Changing the initial time, we also have

$$X_T(t) \to X(t) \text{ for any } t.$$

Then X satisfies the Riccati equation (4.12) and (4.13).

Finally we shall show that X is τ-periodic. Since all system matrices in (4.5) are constant, we have

$$X_{T+\tau}(t + \tau) = X_T(t) \text{ for any } t \geq 0.$$

Letting $T \to \infty$, we have $X(t + \tau) = X(t)$ and hence X is τ-periodic. ∎

Since X is τ-periodic, $R_2(k)$ and $T_2(k)$ are constant matrices and we write $R_2 = R_2(k)$ and $T_2 = T_2(k)$.

Lemma 4.3 *Suppose that* $([C, C_d], [A, A_d])$ *is detectable. Then*

$$(A - BB'X, A - B_d T_2^{-1} R_2)$$

is exponentially stable.

Proof. The Riccati equation (4.12) and (4.13) can be written as

$$\begin{aligned} -\dot{X} &= (A - BB'X)'X + X(A - BB'X) \\ &\qquad + \begin{bmatrix} C \\ B'X \end{bmatrix}' \begin{bmatrix} C \\ B'X \end{bmatrix}, \quad k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= (A_d - B_d T_2^{-1} R_2)' X(k\tau)(A_d - B_d T_2^{-1} R_2) \\ &\qquad + \begin{bmatrix} C_d \\ T_2^{-1} R_2 \end{bmatrix}' \begin{bmatrix} C_d \\ T_2^{-1} R_2 \end{bmatrix}. \end{aligned}$$

Hence, if x is the solution of the state feedback system

$$\begin{aligned} \dot{x} &= (A - BB'X)x, \ x(t_0) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d - B_d T_2^{-1} R_2) x(k\tau) \end{aligned}$$

then

$$\begin{bmatrix} C \\ B'X \end{bmatrix} x \in L^2(t_0, \infty; \mathbf{R}^{p_2+m_2})$$

and

$$\begin{bmatrix} C_d \\ T_2^{-1}R_2 \end{bmatrix} x(\cdot\tau) \in l^2(1, \infty; \mathbf{R}^{p_{2d}+m_{2d}})$$

with

$$\| \begin{bmatrix} C \\ B'X \end{bmatrix} x \|_{L^2}, \ \| \begin{bmatrix} C_d \\ T_2^{-1}R_2 \end{bmatrix} x(\cdot\tau) \|_{l^2} \leq c \mid x_0 \mid \ \text{ for some } c > 0.$$

Since $([C, C_d], [A, A_d])$ is detectable, it is easy to see that

$$\left(\left[\begin{bmatrix} C \\ B'X \end{bmatrix}, \begin{bmatrix} C_d \\ T_2^{-1}R_2 \end{bmatrix} \right], \ [A - BB'X, \ A_d - B_d T_2^{-1} R_2] \right)$$

is also detectable. Hence by Proposition 4.5, $(A - BB'X, A_d - B_d T_2^{-1} R_2)$ is exponentially stable. ∎

We say that X is a stabilizing solution of the Riccati equation (4.12) and (4.13) if $(A - BB'X, A_d - B_d T_2^{-1} R_2)$ is exponentially stable.

Theorem 4.3 *Suppose $([C, C_d], [A, A_d])$ is detectable and* **RJ** *holds. Then there exists a τ-periodic nonnegative stabilizing solution of the Riccati equation (4.12) and (4.13). Moreover the feedback law*

$$\begin{aligned} \bar{u}(t) &= -B'X(t)x(t), \ k\tau < t < (k+1)\tau, \\ \bar{u}_d(k) &= -T_2^{-1} R_2 x(k\tau) \end{aligned}$$

is optimal and

$$J(\bar{u}, \bar{u}_d; t_0, x_0) = \begin{cases} x_0' X(t_0) x_0, & \text{if } t_0 \neq \tau, \\ x_0' X(\tau^-) x_0, & \text{if } t_0 = \tau. \end{cases} \tag{4.15}$$

Proof. The first part follows from Lemmas 4.2 and 4.3. Differentiating $x'Xx$ for $k\tau < t < (k+1)\tau$ we have

$$\frac{d}{dt}[x'(t)X(t)x(t)] = -[\mid Cx(t) \mid^2 + \mid u(t) \mid^2] + \mid u(t) + B'X(t)x(t) \mid^2$$

and at $t = k\tau$

$$\begin{aligned} & x'(k\tau^+)X(k\tau)x(k\tau^+) - x'(k\tau)X(k\tau^-)x(k\tau) \\ = \ & -[\mid C_d x(k\tau) \mid^2 + \mid u_d(k) \mid^2] + \mid T_2^{\frac{1}{2}} [u(k) + T_2^{-1} R_2 x(k\tau)] \mid^2 . \end{aligned}$$

Hence

$$\begin{aligned} & x'(T)X(T)x(T) + J_T(u, u_d; t_0, x_0) \\ = \; & x_0'X(t_0)x_0 + \int_{t_0}^{T} \mid u(t) + B'X(t)x(t) \mid^2 \; dt \\ & + \sum_{k=1}^{N} \mid T_2^{\frac{1}{2}}[u(k) + T_2^{-1}R_2x(k\tau)] \mid^2 \end{aligned}$$

where (u, u_d) is an admissible control and x is its response. Since

$$x'(T)X(T)x(T) \to 0 \text{ as } T \to \infty,$$

we obtain

$$\begin{aligned} J(u, u_d; t_0, x_0) \; = \; & x_0'X(t_0)x_0 + \int_{t_0}^{\infty} \mid u + B'Xx \mid^2 dt \\ & + \sum_{k=1}^{\infty} \mid T_2^{\frac{1}{2}}[u(k) + T_2^{-1}R_2x(k\tau)] \mid^2 . \end{aligned}$$

Hence the optimality of $\bar{u}$ and (4.15) follow immediately. ∎

Corollary 4.2 *($[A, A_d], [B, B_d]$) is stabilizable if and only if there exists a control $(u(\cdot; x_0), u_d(\cdot; x_0))$ for each x_0 such that*

$$\begin{aligned} & \| x \|^2_{L^2(t_0,\infty;\mathbf{R}^n)} + \| x(\cdot\tau) \|^2_{l^2(1,\infty;\mathbf{R}^n)} \\ & \quad + \| u \|^2_{L^2(t_0,\infty;\mathbf{R}^{m_2})} + \| u_d \|^2_{l^2(1,\infty;\mathbf{R}^{m_{2d}})} \leq c(x_0) \end{aligned}$$

for some constant $c(x_0)$.

Proof. We only need to show sufficiency. Consider the regulator problem with $C = I$ and $C_d = I$. Then by Theorem 4.3 $(A - BB'X, A_d - B_dT_2^{-1}R_2)$ is exponentially stable where X is the bounded nonnegative solution of the Riccati equation (4.12) and (4.13) with $C = I$ and $C_d = I$. ∎

Example 4.3 Consider the system with impulse control

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; k < t < k+1, \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(k^+) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k). \end{aligned}$$

This system is unstable, but by setting

$$u(k) = -0.6x_1(k) - 0.5x_2(k)$$

we can easily show that it is stabilized (see Example 4.1). For this system we set $c = [1 \quad 0]$. Then the system is detectable and there exists a periodic nonnegative solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ of the Riccati equation (4.12) and (4.13) with period 1 (Figure 4.4). The simulation result of the system with $x_1(0) = 1$ and $x_2(0) = 0$ under the optimal input is given in Figure 4.5.

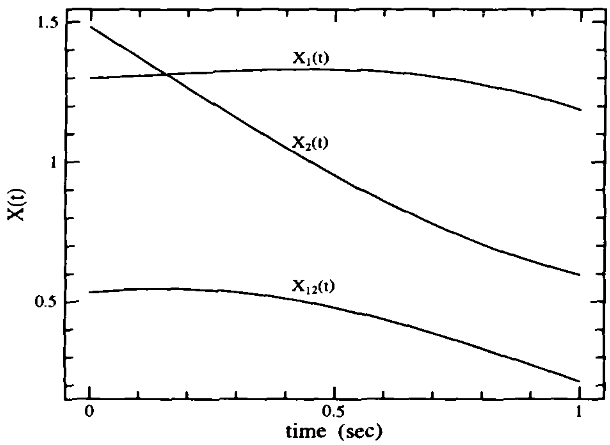

Figure 4.4: The periodic nonnegative solution $X(t)$

Consider the backward system

$$\begin{aligned} -\dot{\xi} &= A'\xi + C'v, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= A_d'\xi(k\tau) + C_d'v_d(k), \\ \xi(T) &= \xi_1, \ t_0 \le N\tau \le T < (N+1)\tau. \end{aligned} \tag{4.16}$$

Then as in Theorem 4.2 we consider

$$\dot{Y} = AY + YA' + BB' - YC'CY, \ k\tau < t < (k+1)\tau, \tag{4.17}$$

$$Y(k\tau^+) = A_dY(k\tau)A_d' + B_dB_d' - (R_{2Y}T_{2Y}^{-1}R_{2Y})(k), \tag{4.18}$$

$$Y(t_0) = HH' \tag{4.19}$$

where $T_{2Y}(k) = I + C_dY(k\tau)C_d'$ and $R_{2Y}(k) = C_dY(k\tau)A_d'$.

Theorem 4.4 *(a) There exists a nonnegative solution of the Riccati equation (4.17)-(4.18) on any $[t_0, T]$.*
(b) Let $H = 0$ and suppose there exists a control $(v(\cdot;T,\xi_1), v_d(\cdot;T,\xi_1))$ such that

$$\| B'\xi \|^2_{L^2(t_0,T;\mathbf{R}^{m_2})} + \| B_d'\xi \|^2_{l^2(1,N;\mathbf{R}^{m_{2d}})} + \| v \|^2_{L^2(t_0,T;\mathbf{R}^{p_2})} + \| v_d \|^2_{l^2(1,N;\mathbf{R}^{p_{2d}})} \le c(\xi_1)$$

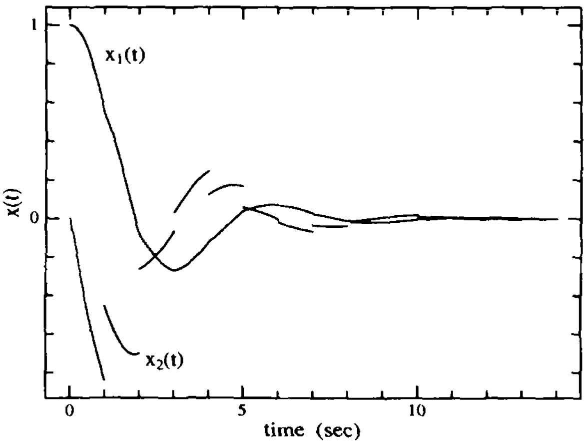

Figure 4.5: Simulation result

for some constant $c(\xi_1)$. *Then the solution of the Riccati equation (4.17) and (4.18) with* $Y(0) = 0$ *is bounded. If, further,* $([A, A_d], [B, B_d])$ *is stabilizable, then* $(A - YC'C, A_d - R'_{2Y}T_{2Y}^{-1}C_d)$ *is exponentially stable.*
(c) $([C, C_d], [A, A_d])$ *is detectable if and only if there exists a control* $(v(\cdot; T, \xi_1), v_d(\cdot; T, \xi_1))$ *such that*

$$\| \xi \|^2_{L^2(t_0,T;\mathbf{R}^n)} + \| \xi(\cdot\tau) \|^2_{l^2(1,N;\mathbf{R}^n)} + \| v \|^2_{L^2(t_0,T;\mathbf{R}^{p_2})} + \| v_d \|^2_{l^2(1,N;\mathbf{R}^{p_{2d}})} \leq c(\xi_1)$$

for some constant $c(\xi_1)$.

We say that Y is a stabilizing solution if $(A - YC'C, A_d - R'_{2Y}T_{2Y}^{-1}C_d)$ is exponentially stable. Since the system (4.16) is τ-periodic, we obtain the following result. The proof is similar to that of Corollary 2.3.

Corollary 4.3 *Suppose that there exists a bounded nonnegative stabilizing solution* Y *of (4.17)-(4.19). Then the* $\lim_{n\to\infty} Y(t + n\tau)$ *exists (denoted by* $Y_\tau(t)$*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.17) and (4.18).*

Similarly to Theorems 2.4 and 3.4, we have the following result.

Theorem 4.5 *(a) A* τ*-periodic nonnegative stabilizing solution of (4.12) and (4.13), if one exists, is unique.*
(b) Let Y *and* $\bar{Y}$ *be two stabilizing solutions of (4.17) and (4.18). Then*

$$Y(t) - \bar{Y}(t) \to 0 \text{ as } t \to \infty.$$

Proof. (a) Let X and $\bar{X}$ be two stabilizing solutions of (4.12) and (4.13). As in the proofs of Theorems 2.4 and 3.4, we have

$$\begin{aligned} -\frac{d}{dt}(X-\bar{X}) &= (A-BB'X)'(X-\bar{X})+(X-\bar{X})(A-BB'\bar{X}), \\ & \qquad k\tau < t < (k+1)\tau, \\ (X-\bar{X})(k\tau^-) &= (A_d-B_dT_2^{-1}R_2)'(X-\bar{X})(k\tau)(A_d-B_d\bar{T}_2^{-1}\bar{R}_2) \end{aligned}$$

where $\bar{T}_2 = I + B_d'\bar{X}(k\tau)B_d$ and $\bar{R}_2 = B_d'\bar{X}(k\tau)A_d$. Hence

$$X(t) - \bar{X}(t) = S_X'(T,t)(X-\bar{X})(T)S_{\bar{X}}(T,t)$$

where S_X and $S_{\bar{X}}$ are the state transition matrices of $(A - BB'X, A_d - B_dT_2^{-1}R_2)$ and $(A - BB'\bar{X}, A_d - B_d\bar{T}_2^{-1}\bar{R}_2)$, respectively. Thus

$$\mid X(t) - \bar{X}(t) \mid \le M_1 e^{-\alpha_1(T-t)} c M_2 e^{-\alpha_2(T-t)}$$

for some positive constants M_i, α_i, $i = 1,2$ and c. Letting $T \to \infty$ we obtain $X(t) - \bar{X}(t) = 0$ for any $t \ge t_0$.
(b) Combine the proofs of Theorems 2.4 and 3.4. ∎

Consider the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u, \; k\tau < t < (k+1)\tau, \\ x(t_0) &= x_0, \; 0 < t_0 \le \tau, \\ x(k\tau^+) &= A_dx(k\tau) + B_{1d}w_d(k) + B_{2d}u_d(k), \\ z = \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} &= \begin{bmatrix} C_1x + D_{12}u \\ C_{1d}x(k\tau) + D_{12d}u_d(k) \end{bmatrix}, \\ y = \begin{bmatrix} y_c \\ y_d(k) \end{bmatrix} &= \begin{bmatrix} C_2x + D_{21}w \\ C_{2d}x(k\tau) + D_{21d}w_d(k) \end{bmatrix} \end{aligned}$$

and the controller $u = Ky$ of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}y, \; \hat{x}(t_0) = 0, \; k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d\hat{x}(k\tau) + \hat{B}_dy_d(k), \\ \begin{bmatrix} u \\ u_d(k) \end{bmatrix} &= \begin{bmatrix} \hat{C}\hat{x} + \hat{D}y \\ \hat{C}_d\hat{x}(k\tau) + \hat{D}_dy_d(k) \end{bmatrix}. \end{aligned} \tag{4.20}$$

Then the closed-loop system $\mathbf{G}_j$ with $u = Ky$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= A_e \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + B_ew, \; k\tau < t < (k+1)\tau, \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t_0) &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau^+) &= A_{ed}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau) + B_{ed}w_d(k), \end{aligned} \tag{4.21}$$

$$z = \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_e \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_e w \\ C_{ed} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} (k\tau) + D_{ed} w_d(k) \end{bmatrix}$$

where

$$\begin{aligned} A_e &= \begin{bmatrix} A + B_2\hat{D}C_2 & B_2\hat{C} \\ \hat{B}C_2 & \hat{A} \end{bmatrix}, & A_{ed} &= \begin{bmatrix} A_d + B_{2d}\hat{D}_d C_{2d} & B_{2d}\hat{C}_d \\ \hat{B}_d C_{2d} & \hat{A}_d \end{bmatrix}, \\ B_e &= \begin{bmatrix} B_1 + B_2\hat{D}D_{21} \\ \hat{B}D_{21} \end{bmatrix}, & B_{ed} &= \begin{bmatrix} B_{1d} + B_{2d}\hat{D}_d D_{21d} \\ \hat{B}_d D_{21d} \end{bmatrix}, \\ C_e &= \begin{bmatrix} C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} \end{bmatrix}, & C_{ed} &= \begin{bmatrix} C_{1d} + D_{12d}\hat{D}_d C_{2d} & D_{12d}\hat{C}_d \end{bmatrix}, \\ D_e &= D_{12}\hat{D}D_{21}, & D_{ed} &= D_{12d}\hat{D}_d D_{21d}. \end{aligned}$$

Definition 4.9 *Consider the system $\mathbf{G}_j$ on $[t_0, \infty)$. A controller $u = Ky$ of the form (4.20) is said to be IO-stabilizing if the closed-loop system (4.21) is IO-stable. If, further, the closed-loop system is exponentially stable (or (A_e, A_{ed}) is exponentially stable) then the controller is said to be (internally) stabilizing.*

Proposition 4.6 *If there exists an internally stabilizing controller $u = Ky$ of the form (4.20), then $([A, A_d], [B_2, B_{2d}], [C_2, C_{2d}])$, $([\hat{A}, \hat{A}_d], [\hat{B}, \hat{B}_d], [\hat{C}, \hat{C}_d])$ are stabilizable and detectable.*

Proof. Let $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t)$ be the solution of

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= A_e \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t_0) = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix}, \quad k\tau < t < (k+1)\tau, \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau^+) &= A_{ed} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau). \end{aligned} \tag{4.22}$$

Then by assumption x, $\hat{x} \in L^2$. Rewriting (4.22) as

$$\begin{aligned} \dot{x} &= Ax + B_2[\hat{D}C_2 x + B_2\hat{C}\hat{x}], \; x(t_0) = x_0, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_{2d}[\hat{D}_d C_{2d} x(k\tau) + B_{2d}\hat{C}_d \hat{x}(k\tau)] \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}C_2 x, \; \hat{x}(t_0) = \hat{x}_0, \; k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d \hat{x}(k\tau) + \hat{B}_d C_{2d} x(k\tau) \end{aligned}$$

and applying Corollary 4.2, we conclude that $([A, A_d], [B_2, B_{2d}])$ and $([\hat{A}, \hat{A}_d], [\hat{B}, \hat{B}_d])$ are stabilizable. The detectability of $([C_2, C_{2d}], [A, A_d])$, $([\hat{C}, \hat{C}_d], [\hat{A}, \hat{A}_d])$ also follows from the adjoint of the system $\mathbf{G}_j$ and Theorem 4.4. ∎

4.1.4 Disturbance Attenuation Problems

The Finite Horizon Problem

Consider the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + Bw, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k), \qquad (4.23) \\ z_c &= Cx, \\ z_d(k) &= C_d x(k\tau) + D_d w_d(k), \\ z_1 &= Fx(T), \ t_0 \le N\tau \le T < (N+1)\tau \qquad (4.24) \end{aligned}$$

with initial condition

$$x(t_0) = Hh, \ 0 < t_0 \le \tau \qquad (4.25)$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $w_d \in \mathbf{R}^{m_{1d}}$, $z_c \in \mathbf{R}^{p_1}$, $z_d \in \mathbf{R}^{p_{1d}}$, $z_1 \in \mathbf{R}^q$, $h \in \mathbf{R}^{n_1}$ and all matrices are of compatible dimensions. For each input $(h, w, w_d) \in \mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}) \times l^2(1, N; \mathbf{R}^{m_{1d}})$ we have the output $(z_1, z_c, z_d) \in \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{p_1}) \times l^2(1, N; \mathbf{R}^{p_{1d}})$. Thus we can define the input-output operator G_{Tt_0} of the system $\mathbf{G}_j$ by

$$\begin{pmatrix} z_1 \\ z_c \\ z_d \end{pmatrix} = G_{Tt_0} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} = \begin{pmatrix} G_{1Tt_0} \\ G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}$$

where

$$\begin{aligned} G_{1Tt_0} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} &= FS(T, t_0)Hh + F \int_{t_0}^{T} S(T, r)Bw(r)dr \\ &\quad + F \sum_{j=k_0}^{N} S(N+1, j\tau^+) B_d w_d(j), \\ G_{2Tt_0} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} &= CS(t, t_0)Hh + C \int_{t_0}^{t} S(t, r)Bw(r)dr \\ &\quad + C \sum_{j=k_0}^{k} S(t, j\tau^+) B_d w_d(j) \end{aligned}$$

and

$$\begin{aligned} G_{3Tt_0} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} &= C_d S(k\tau, t_0)Hh + C_d \int_{t_0}^{k\tau} S(k\tau, r)Bw(r)dr \\ &\quad + C_d \sum_{j=k_0}^{k-1} S(k\tau, j\tau^+) B_d w_d(j) + D_d w_d(k). \end{aligned}$$

Then $G_{Tt_0} \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0,T;\mathbf{R}^{m_1}) \times l^2(1,N;\mathbf{R}^{m_{1d}}); \mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}) \times l^2(1,N;\mathbf{R}^{p_{1d}}))$. We regard (h, w, w_d) as the disturbance and we wish to find necessary and sufficient conditions for $\| G_{Tt_0} \| < \gamma$, i.e.,

$$| z_1 |^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \leq d^2 (| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2}) \tag{4.26}$$

for some $0 < d < \gamma$. In this case the system (4.23) and (4.24) is said to fulfil the γ-disturbance attenuation.

The adjoint $G^*_{Tt_0}$ of G_{Tt_0} is given by

$$G^*_{Tt_0} \begin{pmatrix} f \\ v \\ v_d \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \zeta_c \\ \zeta_d \end{pmatrix}$$

where $(f, v, v_d) \in \mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}) \times l^2(1,N;\mathbf{R}^{p_{1d}})$ and

$$\begin{aligned} -\dot{\xi} &= A'\xi + C'v, \quad k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= A'_d\xi(k\tau) + C'_d v_d(k), \\ \zeta_c &= B'\xi, \\ \zeta_d(k) &= B'_d\xi(k\tau) + D'_d v_d(k), \\ \xi(T) &= F'f, \\ \zeta_0 &= H'\xi(t_0). \end{aligned} \tag{4.27}$$

Since $\| G^*_{Tt_0} \| = \| G_{Tt_0} \|$ (Theorem A.2), (4.26) is equivalent to

$$| \zeta_0 |^2 + \left\| \begin{pmatrix} \zeta_c \\ \zeta_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \leq d^2 (| f |^2 + \left\| \begin{pmatrix} v \\ v_d \end{pmatrix} \right\|^2_{L^2 \times l^2}). \tag{4.28}$$

To give necessary and sufficient conditions for $\| G_{Tt_0} \| < \gamma$, we need the Riccati equations with jumps. For definiteness we assume $0 < t_0 < \tau$.

$$-\dot{X} = A'X + XA + C'C + \frac{1}{\gamma^2} XBB'X, \qquad k\tau < t < (k+1)\tau, \tag{4.29}$$

$$T_1(k) > aI \text{ for some } a > 0, \tag{4.30}$$

$$X(k\tau^-) = A'_d X(k\tau) A_d + C'_d C_d + (R'_1 T_1^{-1} R_1)(k), \tag{4.31}$$

$$X(T) = F'F, \tag{4.32}$$

$$H'X(t_0)H \leq d^2 I \text{ for some } 0 < d < \gamma \tag{4.33}$$

and

$$\dot{Y} = AY + YA + BB' + \frac{1}{\gamma^2} YC'CY, \tag{4.34}$$

$$
\begin{aligned}
& && k\tau < t < (k+1)\tau, \\
T_{1Y}(k) &> aI \text{ for some } a > 0, && (4.35)\\
Y(k\tau^+) &= A_d Y(k\tau) A_d' + B_d B_d' + (R_{1Y}' T_{1Y}^{-1} R_{1Y})(k), && (4.36)\\
Y(t_0) &= HH', && (4.37)\\
FY(T)F' &\le d^2 I \text{ for some } 0 < d < \gamma && (4.38)
\end{aligned}
$$

where

$$
\begin{aligned}
T_1(k) &= \gamma^2 I - D_d' D_d - B_d' X(k\tau) B_d, & R_1(k) &= D_d' C_d + B_d' X(k\tau) A_d,\\
T_{1Y}(k) &= \gamma^2 I - D_d D_d' - C_d Y(k\tau) C_d', & R_{1Y}(k) &= D_d B_d' + C_d Y(k\tau) A_d'.
\end{aligned}
$$

If we wish to take $t_0 = \tau$, the condition (4.33) becomes

$$H' X(\tau^-) H \le d^2 I.$$

To give the solution of this problem, we introduce the following functional

$$
\begin{aligned}
J(w, w_d; t_0, x_0) &= \int_{t_0}^{T} [\mid z(t) \mid^2 - \gamma^2 \mid w(t) \mid^2] dt \\
&\quad + \sum_{k=1}^{N} [\mid z(k) \mid^2 - \gamma^2 \mid w(k) \mid^2] + \mid Fx(T) \mid^2 \quad (4.39)
\end{aligned}
$$

subject to

$$
\begin{aligned}
\dot{x} &= Ax + Bw, \ k\tau < t < (k+1)\tau,\\
x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k),\\
z_c &= Cx,\\
z_d(k) &= C_d x(k\tau) + D_d w_d(k)
\end{aligned}
$$

with initial condition $x(t_0) = x_0$ and consider the maximization $J(w, w_d; t_0, x_0)$ over all $(w, w_d) \in L^2(t_0, T; \mathbf{R}^{m_1}) \times l^2(1, N; \mathbf{R}^{m_{1d}})$. Let

$$
\begin{aligned}
\bar{G}_{Tt_0} \begin{pmatrix} w \\ w_d \end{pmatrix} &= G_{Tt_0} \begin{pmatrix} 0 \\ w \\ w_d \end{pmatrix},\\
\bar{G}_{iTt_0} \begin{pmatrix} w \\ w_d \end{pmatrix} &= G_{iTt_0} \begin{pmatrix} w \\ w_d \end{pmatrix}, \ i = 1, 2, 3.
\end{aligned}
$$

Lemma 4.4

$$
\begin{aligned}
\left\| \begin{pmatrix} \bar{G}_{2Ls} \\ \bar{G}_{3Ls} \end{pmatrix} \right\| &\le \left\| \begin{pmatrix} \bar{G}_{2Tt_0} \\ \bar{G}_{3Tt_0} \end{pmatrix} \right\|,\\
\| \bar{G}_{Ts} \| &\le \| \bar{G}_{Tt_0} \| \quad \textit{for any } 0 \le t_0 \le s \le L \le N.
\end{aligned}
$$

Proof. We shall show only the first inequality. Let $(\tilde{w}, \tilde{w}_d)$ be the extension of $(w, w_d) \in L^2(s, L; \mathbf{R}^{m_1}) \times l^2(k_s, k_L; \mathbf{R}^{m_{1d}})$ to $[t_0, T]$ by zero, i.e,

$$\tilde{w}(t) = \begin{cases} 0, & t_0 \le t < s, \\ w(t), & s \le t \le L, \\ 0, & L < t \le T, \end{cases} \quad \tilde{w}_d(k) = \begin{cases} 0, & k_0 \le k < k_s, \\ w_d(k), & k_s \le k \le k_L, \\ 0, & k_L < k \le N \end{cases}$$

where $(k_s - 1)\tau < s \le k_s\tau$ and $k_L\tau \le L < (k_L + 1)\tau$. Then by the proof of Lemmas 2.7 and 3.6 we have

$$\begin{aligned} \left\| \begin{pmatrix} \bar{G}_{2Ls} \\ \bar{G}_{3Ls} \end{pmatrix} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} &= \left\| \bar{G}_{2Ls} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2} + \left\| \bar{G}_{3Ls} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{l^2} \\ &\le \left\| \bar{G}_{2Tt_0} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2} + \left\| \bar{G}_{3Tt_0} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{l^2} \\ &= \left\| \begin{pmatrix} \bar{G}_{2Tt_0} \\ \bar{G}_{3Tt_0} \end{pmatrix} \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2}. \end{aligned}$$

∎

Consider the optimal control problem for the system $\mathbf{G}_j$ with t_0, T replaced by arbitrary s, L, $t_0 \le s \le L \le T$, $(k_s - 1)\tau < s \le k_s\tau$, $k_L\tau \le L < (k_L + 1)\tau$.

Lemma 4.5 *Assume* $\| \bar{G}_{Tt_0} \| < \gamma$. *Then for any* $t_0 \le s \le T$, $J(w, w_d; s, x_0)$ *is strictly concave in* (w, w_d) *and there exists a unique optimal maximizing element* $(w_{Ts}, w_{dTs}) \in L^2(s, T; \mathbf{R}^{m_1}) \times l^2(k_s, N; \mathbf{R}^{m_{1d}})$. *Moreover*

$$\begin{aligned} \left\| \begin{pmatrix} w_{Ts} \\ w_{dTs} \end{pmatrix} \right\|^2_{L^2 \times l^2} &\le \delta \mid x_0 \mid^2, \\ J(w_T, w_{dN}; s, x_0) &\le \delta \mid x_0 \mid^2 \end{aligned}$$

for some $\delta = \delta(\gamma) > 0$ *independent of* s *and* x_0.

Proof. By Lemma 4.4 $\| G_{Ts} \| < \gamma$ for any $t_0 \le s \le T$ and hence $\gamma^2 I - G^*_{Ts} G_{Ts} > aI$ for some $a > 0$ and the quadratic functional $J(w, w_d; s, x_0)$ is strictly concave and $J(w, w_d; s, x_0) \to -\infty$ as $\left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2 \times l^2} \to \infty$. Then by Theorem A.4 there exists a unique optimal (w_{Ts}, w_{dTs}) for $J(w, w_d; s, x_0)$ which is given by

$$\begin{aligned} (\gamma^2 I - G^*_{Ts} G_{Ts}) \begin{pmatrix} w \\ w_d \end{pmatrix} &= G^*_{Ts} z_0, \\ z_0 &= \begin{pmatrix} CS(t, s)x_0 \\ C_d S(k\tau, s)x_0 \\ FS(T, s)x_0 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{pmatrix} w_{Ts} \\ w_{dTs} \end{pmatrix} = (\gamma^2 I - G^*_{Ts} G_{Ts})^{-1} G^*_{Ts} z_0.$$

Thus we have

$$\| \begin{pmatrix} w_{Ts} \\ w_{dTs} \end{pmatrix} \|_{L^2 \times l^2} \leq \delta \mid x_0 \mid$$

for some δ independent of s and x_0. ∎

Lemma 4.6 *Suppose $\| \bar{G}_{Tt_0} \| < \gamma$. Then there exists a nonnegative solution to (4.29)-(4.32). The optimal control for (4.39) is given by the feedback law*

$$\begin{aligned} w_{Tt_0}(t) &= \frac{1}{\gamma^2} B' X(t) x(t), \quad k\tau < t < (k+1)\tau, \\ w_{dTt_0}(k) &= (T_1^{-1} R_1)(k) x(k\tau), \quad k = k_0, k_0+1, .., N \end{aligned}$$

and

$$J(w_{Tt_0}, w_{dTt_0}, t_0, x_0) = \begin{cases} x_0' X(t_0) x_0, & \text{if } t_0 \neq \tau, \\ x_0' X(\tau^-) x_0, & \text{if } t_0 = \tau. \end{cases}$$

Proof. A brief outline of the proof is as follows. We first establish the existence of $X(t)$ of (4.29) and (4.32) on the interval $[N\tau, T]$, i.e., the last interval. Then using the jump equation, we show the existence of $X(N\tau^-)$ defined by (4.30) and (4.31). Next we show the existence of $X(t)$ on the interval $[(N-1)\tau, N\tau)$. The existence of $X(t)$, $t \in [t_0, T]$ will be established by repeating these arguments.

Step 1. Consider the functional

$$\begin{aligned} J(w, w_d; s, x_0) &= J(w; s, x_0) \\ &= \int_s^T [\mid z_c(t) \mid^2 - \gamma^2 \mid w(t) \mid^2] dt + \mid Fx(T) \mid^2 \end{aligned}$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bw, \ x(s) = x_0, \\ z_c &= Cx \end{aligned}$$

where $N\tau < s < T$. Since $\| \bar{G}_{Tt_0} \| < \gamma$, by Lemma 4.4 $\| \bar{G}_{Ts} \| < \gamma$. Hence by Lemma 2.8, there exists a unique nonnegative solution $X(t)$, $t \in [s, T]$ to (4.29) and (4.32). We write this solution X_T to show the dependence of T. We also have

$$\begin{aligned} \max_{(w, w_d)} J(w, w_d; s, x_0) &= \max_w J(w; s, x_0) \\ &= J(w_{Ts}; s, x_0) \\ &= x_0' X_T(s) x_0. \end{aligned}$$

Step 2. We introduce the functional

$$\begin{aligned} J(w, w_d; N\tau, x_0) &= \int_{N\tau}^T [\mid z_c(t) \mid^2 - \gamma^2 \mid w(t) \mid^2] dt \\ &\quad + \mid z_d(N) \mid^2 - \gamma^2 \mid w_d(N) \mid^2 + \mid Fx(T) \mid^2 \end{aligned}$$

subject to (4.23) and (4.24) with $x(N\tau) = x_0$ and consider the maximization $J(w, w_d; N\tau, x_0)$ over all $w \in L^2(N\tau, T; \mathbf{R}^{m_1})$ and $w_d(N)$. Since $\| \tilde{G}_{TN\tau} \| < \gamma$, we have

$$d^2 \left(\int_{N\tau}^{T} | w(t) |^2 dt + | w_d(N) |^2 \right) \geq \int_{N\tau}^{T} | z_c(t) |^2 dt + | z_d(N) |^2 + | Fx(T) |^2$$

for any $w \in L^2(N\tau, T; \mathbf{R}^{m_1})$ and $w_d(N)$ and for some $0 < d < \gamma$. Hence we have

$$\begin{aligned} d^2 | w_d(N) |^2 &\geq | z_d(N) |^2 \\ &\quad + \max_{w} \left[\int_{N\tau}^{T} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt + | Fx(T) |^2 \right] \\ &= | z_d(N) |^2 + x'(N\tau^+) X_T(N\tau) x(N\tau^+). \end{aligned}$$

Using the jump system (4.23) with $x(N\tau) = x_0 = 0$, we have

$$d^2 | w_d(N) |^2 \geq | D_d w_d(N) |^2 + w_d'(N) B_d' X_T(N\tau) B_d w_d(N).$$

Hence

$$T_1(N) = T_1[X_T](N) \geq (\gamma^2 - d^2) I$$

and we can define $X_T(N\tau^-)$ by (4.31). Since

$$\begin{aligned} J(w, w_d; N\tau, x(N\tau)) &= x'(N\tau) X_T(N\tau^-) x(N\tau) \\ &\quad - \int_{N\tau}^{T} | w(t) - \frac{1}{\gamma^2} B' X_T(t) x(t) |^2 dt \\ &\quad - | T_1^{\frac{1}{2}}(N) [w_d(N) - (T_1^{-1} R_1)(N) x(N\tau)] |^2, \end{aligned}$$

we have

$$\begin{aligned} x'(N\tau) X_T(N\tau^-) x(N\tau) &= \max_{(w, w_d)} J(w, w_d; N\tau, x(N\tau)) \\ &= J(w_{TN\tau}, w_{dTN\tau}; N\tau, x(N\tau)) \end{aligned}$$

where

$$\begin{aligned} w_{TN\tau}(t) &= \frac{1}{\gamma^2} B' X_T(t) x(t), \; N\tau < t < (N+1)\tau, \\ w_{dTN\tau}(N) &= (T_1^{-1} R_1)(N) x(N\tau). \end{aligned}$$

Step 3. Now we assume that $X_T(t), t \in (N\tau, T]$ is well-defined and introduce the functional

$$\begin{aligned} J(w, w_d; s, x_0) &= \int_{s}^{T} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt \\ &\quad + | z_d(N) |^2 - \gamma^2 | w_d(N) |^2 + | Fx(T) |^2 \end{aligned}$$

subject to (4.23) with $x(s) = x_0$, $(N-1)\tau < s \leq N\tau$. Then

$$\begin{aligned} J(w, w_d; s, x_0) &= \int_s^{N\tau} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt + x'(N\tau) X_T(N\tau^-) x(N\tau) \\ &\quad - \int_{N\tau}^{T} | w(t) - \frac{1}{\gamma^2} B' X_T(t) x(t) |^2 \, dt \\ &\quad - | T_1^{\frac{1}{2}}(N)[w_d(N) - (T_1^{-1} R_1)(N) x(N\tau)] |^2 \end{aligned}$$

and hence

$$\begin{aligned} \max_{(w,w_d)} J(w, w_d; s, x_0) &= \max_{w} [\int_s^{N\tau} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt \\ &\quad + x'(N\tau) X_T(N\tau^-) x(N\tau)]. \end{aligned}$$

As in the proof of **Step 1**, we can show the existence of a unique nonnegative solution $X(t)$, $t \in [s, N\tau)$ of (4.29) with $X(N\tau) = X_T(N\tau^-)$.

Continuing these arguments we can show the existence of a unique nonnegative solution to (4.29)-(4.32). Since

$$\begin{aligned} J(w, w_d; t_0, x_0) &= x_0' X_T(t_0) x_0 \\ &\quad - \int_{t_0}^{T} | w(t) - \frac{1}{\gamma^2} B' X_T(t) x(t) |^2 \, dt \\ &\quad - \sum_{k=1}^{N} | T_1^{\frac{1}{2}}(k)[w_d(k) - (T_1^{-1} R_1)(k) x(k\tau)] |^2, \end{aligned}$$

we have

$$\begin{aligned} x_0' X_T(t_0) x_0 &= \max_{(w,w_d)} J(w, w_d; t_0, x_0) \\ &= J(w_{Tt_0}, w_{dTt_0}; t_0, x_0) \end{aligned}$$

where

$$\begin{aligned} w_{Tt_0}(t) &= \frac{1}{\gamma^2} B' X_T(t) x(t), \; k\tau < t < (k+1)\tau, \\ w_{dTt_0}(k) &= (T_1^{-1} R_1)(k) x(k\tau) \end{aligned}$$

and the proof is complete. ∎

We are now ready to give the solution of our original problem.

Theorem 4.6 *The following statements are equivalent.*
(a) $\| G_{Tt_0} \| < \gamma$.
(b) There exists a nonnegative solution $X(t)$, $t \in [t_0, T]$ *to (4.29)-(4.33).*
(c) There exists a nonnegative solution $Y(t)$, $t \in [t_0, T]$ *to (4.34)-(4.38).*

Proof. Suppose (a) holds. Then (b) except (4.33) follows from Lemma 4.6. Moreover for the system $\mathbf{G}_j$ the following equality holds:

$$\begin{aligned} \mid z_1 \mid^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2\times l^2} &= \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2} \\ &+h'H'X(t_0)Hh - \left\| \begin{pmatrix} w - \frac{1}{\gamma^2}B'Xx \\ T_1^{-\frac{1}{2}}(w_d - T_1^{-1}R_1x) \end{pmatrix} \right\|^2_{L^2\times l^2}. \end{aligned} \tag{4.40}$$

Setting $w(t) = \frac{1}{\gamma^2}B'X(t)x(t)$, $w_d(k) = (T_1^{-1}R_1)(k)x(k\tau)$ and using (4.26) we obtain

$$d^2(\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2}) \geq \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2} + h'H'X(t_0)Hh.$$

Hence $d^2 \mid h \mid^2 \geq h'H'X(k_0)Hh$ which implies (4.33).

Conversely suppose (b) holds. Then by (4.40)

$$\begin{aligned} \mid z_1 \mid^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2\times l^2} &\leq \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2} \\ &\quad + d^2 \mid h \mid^2 - \gamma^2 \left\| \begin{pmatrix} r \\ r_d \end{pmatrix} \right\|^2_{L^2\times l^2} \\ &= \gamma^2(\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2}) \\ &\quad -(\gamma^2 - d^2)(\mid h \mid^2 + \left\| \begin{pmatrix} r \\ r_d \end{pmatrix} \right\|^2_{L^2\times l^2}) \end{aligned}$$

where

$$\begin{aligned} r(t) &= \frac{1}{\gamma^2}B'X(t)x(t), \ k\tau < t < (k+1)\tau, \\ r_d(k) &= (T_1^{-1}R_1)(k)x(k\tau). \end{aligned}$$

Since there exists $a > 0$ such that

$$\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2} \geq a(\mid h \mid^2 + \left\| \begin{pmatrix} r \\ r_d \end{pmatrix} \right\|^2_{L^2\times l^2}),$$

we have

$$\begin{aligned} \mid z_1 \mid^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2\times l^2} &\leq \gamma^2(\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2}) \\ &\quad - \frac{\gamma^2 - d^2}{a}(\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2}) \\ &= \left(\gamma^2 - \frac{\gamma^2 - d^2}{a}\right)(\mid h \mid^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2\times l^2}). \end{aligned}$$

Hence $\| G_{Tt_0} \| < \gamma$. The equivalence of (a) and (c) also follows since (c) is the dual of (b) concerning the adjoint (4.27) of G_{Tt_0}. ∎

If initial conditions are known, we can set $h = 0$.

Corollary 4.4 *The following statements are equivalent.*
(a) $\| \bar{G}_{Tt_0} \| < \gamma$.
(b) There exists a nonnegative solution $X(t)$, $t \in [t_0, T]$ *to (4.29)-(4.32).*
(c) There exists a nonnegative solution $Y(t)$, $t \in [t_0, T]$ *to (4.34)-(4.36) and (4.38) with* $Y(t_0) = 0$.

The Infinite Horizon Problem

We now consider the system $\mathbf{G}_j$

$$\begin{aligned} \dot{x} &= Ax + Bw, \ \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k), \\ z_c &= Cx, \\ z_d(k) &= C_d x(k\tau) + D_d w_d(k), \\ x(t_0) &= Hh, \ \ 0 < t_0 \le \tau \end{aligned}$$

on $[t_0, \infty)$ and we assume that (A, A_d) is exponentially stable on $[t_0, \infty)$. Then we can define the input-output operator

$$G \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}}); L^2(t_0, \infty; \mathbf{R}^{p_1}) \times l^2(1, \infty; \mathbf{R}^{p_{1d}}))$$

by

$$\begin{pmatrix} z_c \\ z_d \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} = \begin{pmatrix} G_2 \\ G_3 \end{pmatrix} \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}$$

where

$$G_2 \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} = CS(t, t_0)Hh + C \int_{t_0}^{t} S(t, r)Bw(r)dr + C \sum_{j=1}^{k} S(t, j\tau^+)B_d w_d(j)$$

and

$$\begin{aligned} G_3 \begin{pmatrix} h \\ w \\ w_d \end{pmatrix} &= C_d S(k\tau, t_0)Hh + C_d \int_{t_0}^{k\tau} S(k\tau, r)Bw(r)dr \\ &\quad + C_d \sum_{j=1}^{k-1} S(k\tau, j\tau^+)B_d w_d(j) + D_d w_d(k). \end{aligned}$$

In this case we wish to find the condition $\| G \| < \gamma$. We replace (4.39) by

$$J(w, w_d; t_0, x_0) = \int_{t_0}^{\infty} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt + \sum_{k=1}^{\infty} [| z_d(k) |^2 - \gamma^2 | w_d(k) |^2].$$

We also need the functional (4.39) with $F = 0$, i.e.,

$$J_T(w, w_d; t_0, x_0) = \int_{t_0}^{T} [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt + \sum_{k=1}^{N} [| z_d(k) |^2 - \gamma^2 | w_d(k) |^2].$$

Let

$$\bar{G}\begin{pmatrix} w \\ w_d \end{pmatrix} = G\begin{pmatrix} 0 \\ w \\ w_d \end{pmatrix},$$
$$\bar{G}_i\begin{pmatrix} w \\ w_d \end{pmatrix} = G_i\begin{pmatrix} 0 \\ w \\ w_d \end{pmatrix}, \quad i = 2, 3.$$

Proceeding as in the finite horizon case we have the following

Lemma 4.7 $\left\| \begin{pmatrix} \bar{G}_{2Tt_0} \\ \bar{G}_{3Tt_0} \end{pmatrix} \right\| \leq \| \bar{G} \|$ *for any* $t_0 \leq T < \infty$.

Lemma 4.8 *Assume* $\| G \| < \gamma$. *Then* $J_T(w, w_d; t_0, x_0)$ $(J(w, w_d; t_0, x_0))$ *is strictly concave and there exists a unique optimal element* (w_{Tt_0}, w_{dTt_0}) $((w_{t_0}, w_{dt_0}))$ *maximizing* $J_T(w, w_d; t_0, x_0)$ $(J(w, w_d; t_0, x_0)$, *respectively). Moreover*

$$\left\| \begin{pmatrix} w_{Tt_0} \\ w_{dTt_0} \end{pmatrix} \right\|_{L^2 \times l^2} \leq \delta | x_0 |, \qquad \left\| \begin{pmatrix} w_{t_0} \\ w_{dt_0} \end{pmatrix} \right\|_{L^2 \times l^2} \leq \delta | x_0 |,$$

$$J_T(w_{Tt_0}, w_{dTt_0}; t_0, x_0) \leq \delta | x_0 |^2, \quad J(w_{t_0}, w_{dt_0}; t_0, x_0) \leq \delta | x_0 |^2$$

for some $\delta = \delta(\gamma) > 0$ *independent of* T *and* x_0.

Proof. Since $\| \bar{G} \| \leq \| G \| < \gamma$, $\left\| \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix} \right\| < \gamma$ by Lemma 4.7. Hence by Lemma 4.5 we have

$$\begin{pmatrix} w_{Tt_0} \\ w_{dTt_0} \end{pmatrix} = \left[\gamma^2 I - \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix}^* \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix} \right]^{-1} \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix}^* z_0,$$
$$z_0 = \begin{pmatrix} CS(t, t_0)x_0 \\ C_d S(k\tau, t_0)x_0 \end{pmatrix}$$

and

$$\begin{pmatrix} w_{t_0} \\ w_{dt_0} \end{pmatrix} = (\gamma^2 I - \bar{G}^* \bar{G})^{-1} \bar{G}^* z_0.$$

Since $\begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix}$ and $(\gamma^2 I - \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix}^* \begin{pmatrix} G_{2Tt_0} \\ G_{3Tt_0} \end{pmatrix})^{-1}$ are uniformly bounded in T, we have the assertion. ∎

Definition 4.10 *(a) A bounded nonnegative solution X of (4.29)-(4.31) is called the stabilizing solution if $(A + \frac{1}{\gamma^2} BB'X, A + BT_1^{-1}R_1)$ is exponentially stable.*
(b) A bounded nonnegative solution Y of (4.34)-(4.36) is called the stabilizing solution if $(A + \frac{1}{\gamma^2} YC'C, A + R'_{1Y} T_{1Y}^{-1} C)$ is exponentially stable.

Similarly to Theorem 4.5, we have the following property for the stabilizing solutions.

Lemma 4.9 *(a) A bounded nonnegative stabilizing solution of (4.29)-(4.31), if exists, is unique.*
(b) Let Y and $\bar{Y}$ be two stabilizing solutions of (4.34)-(4.36). Then $Y(t) - \bar{Y}(t) \to 0$ as $t \to \infty$.

Lemma 4.10 *Suppose $\| G \| < \gamma$. Then there exists a τ-periodic nonnegative stabilizing solution to (4.29)-(4.31). Moreover if the conditons above are satisfied, a unique maximizing element (w_{t_0}, w_{dt_0}) of $J(w, w_d; x_0)$ exists and is given by the feedback law*

$$\begin{aligned} w_{t_0}(t) &= \frac{1}{\gamma^2} B'X(t)x(t), \quad k\tau < t < (k+1)\tau, \\ w_{dt_0}(k) &= T_1^{-1} R_1 x(k\tau), \quad k = 1, 2, .. \end{aligned}$$

and $J(w_{t_0}, w_{dt_0}, x_0) = x_0' X(t_0) x_0$ where $T_1 = T_1(k)$ and $R_1 = R_1(k)$, for any $k \geq 1$.

Proof. Since $\| G \| < \gamma$ implies $\| \bar{G} \| < \gamma$ and $\| \begin{pmatrix} \bar{G}_{2Tt_0} \\ \bar{G}_{3Tt_0} \end{pmatrix} \| < \gamma$, we have a nonnegative solution $X_T(t)$ to (4.29)-(4.31) with $X_T(T) = 0$. Moreover for each t, $X_T(t)$ is monotone increasing in T. In fact let $L < T$ and define a control on $[t_0, T]$ by

$$\begin{aligned} \tilde{w}_{Tt_0}(t) &= \begin{cases} \frac{1}{\gamma^2} B'X_L(t)x_L(t), & t \in [t_0, L], \\ 0, & t \in (L, T], \end{cases} \\ \tilde{w}_{dTt_0}(k) &= \begin{cases} (T_{1L}^{-1} R_{1L})(k) x_L(k\tau), & k \in [1, k_L], \\ 0, & k \in [k_L + 1, N] \end{cases} \end{aligned}$$

where $T_{1L}(\cdot)$, $R_{1L}(\cdot)$ are defined by $T_1(\cdot)$, $R_1(\cdot)$ respectively with $X(k\tau)$ replaced $X_L(k\tau)$, $k_L\tau \leq L < (k_L + 1)\tau$ and x_L is the response to the feedback pair $(w_{Lt_0}, w_{dLt_0}) = (\frac{1}{\gamma^2} B'X_L x_L, (T_{1L}^{-1} R_{1L})(\cdot) x_L(\cdot\tau))$ in the system $\mathbf{G}_j$.

Then

$$\begin{aligned} x_0' X_L(t_0) x_0 &= J_L(w_{Lt_0}, w_{dLt_0}; t_0, x_0) \\ &\le J_T(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0}; t_0, x_0) \\ &\le J_T(w_{Tt_0}, w_{dTt_0}; t_0, x_0) = x_0' X_T(t_0) x_0. \end{aligned}$$

The mononicity of $X_T(t)$ also follows from $J_T(w, w_d; t, x_0)$. Note that $X_T(t)$ is bounded uniformly in T. This follows from Lemma 4.8 and

$$J_T(w_{Tt}, w_{dTt}; t, x_0) = x_0' X_T(t) x_0.$$

Hence $X_T(t)$ converges to a limit $X(t)$ as $T \to \infty$. As we have seen in the proof of Lemma 4.6, $X_T(k\tau)$ satisfies $T_1(k) \ge (\gamma^2 - d^2)I$ independent of T and hence $X(k\tau)$ satisfies $T_1(k) \ge (\gamma^2 - d^2)I$. So $X(k\tau^-)$ is defined by (4.30) and (4.31). A τ-periodicity of $X(t)$ follows from the proof of Lemma 4.2.

Now it remains to show that $(A + \frac{1}{\gamma^2} BB'X, A_d + B_d T_1^{-1} R_1)$ is exponentially stable. Let x_T be the response (w_{Tt_0}, w_{dTt_0}) and let $(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0}) \in L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}})$ be given by

$$\begin{aligned} \tilde{w}_{Tt_0}(t) &= \begin{cases} \frac{1}{\gamma^2} B' X_T(t) x_T(t), & t \in [t_0, T], \\ 0, & t \in (T, \infty), \end{cases} \\ \tilde{w}_{dTt_0}(k) &= \begin{cases} (T_1^{-1} R_1)(k) x_T(k\tau), & k \in [1, N], \\ 0, & k \in [N+1, \infty). \end{cases} \end{aligned}$$

Then

$$0 \le x_0' X_T(t_0) x_0 \le J(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0}; t_0, x_0) \le J(w_{t_0}, w_{dt_0}; t_0, x_0)$$

and $\{(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0})\}$ is bounded in $L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}})$. Hence there exists a subsequence again denoted by $\{(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0})\}$ which is weakly convergent to $(\tilde{w}, \tilde{w}_d) \in L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}})$ (Theorem A.5). Let $\tilde{x}$ be the response to $(\tilde{w}, \tilde{w}_d)$, i.e, the solution of

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{w}, \ \tilde{x}(t_0) = x_0, \ k\tau < t < (k+1)\tau, \\ \tilde{x}(k\tau^+) &= A_d \tilde{x}(k\tau) + B_d \tilde{w}_d(k). \end{aligned}$$

Since the restriction of $(\tilde{w}_{Tt_0}, \tilde{w}_{dTt_0})$ on any subinterval converges weakly to that of $(\tilde{w}, \tilde{w}_d)$, for any t, $x_T(t) \to \tilde{x}(t)$ in $\mathbf{R}^n$ as $T \to \infty$. On the other hand $x_T(t) \to \bar{x}(t)$ in any interval, where $\bar{x}$ is the solution of

$$\begin{aligned} \dot{\bar{x}} &= (A + \frac{1}{\gamma^2} BB' X(t))\bar{x}, \ \bar{x}(t_0) = x_0, \ k\tau < t < (k+1)\tau, \\ \bar{x}(k\tau^+) &= (A_d + B_d T_1^{-1} R_1) x(k\tau). \end{aligned}$$

Hence we can identify $\tilde{x} = \bar{x}$. Since (A, A_d) is exponentially stable and $(\tilde{w}, \tilde{w}_d) \in L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}})$, we conclude

$$\tilde{x} \in L^2(t_0, \infty; \mathbf{R}^n) \text{ and } \bar{x} \in L^2(t_0, \infty; \mathbf{R}^n).$$

This is true for any x_0 which implies that $(A + \frac{1}{\gamma^2}BB'X, A_d + B_dT_1^{-1}R_1)$ is exponentially stable. ∎

Theorem 4.7 *Assume that (A, A_d) is exponentially stable on $[t_0, \infty)$. Then the following statements are equivalent.*
(a) $\| G \| < \gamma$.
(b) There exists a τ-periodic nonnegative stabilizing solution $X(t)$, $t \in [t_0, \infty)$ to (4.29)-(4.31) satisfying (4.33).
(c) There exists a bounded nonnegative stabilizing solution $Y(t)$, $t \in [t_0, \infty)$ to (4.34)-(4.37).
Moreover the $\lim_{n\to\infty} Y(t + n\tau)$ exists (denoted by $Y_\tau(t)$) and Y_τ is a τ-periodic nonnegative stabilizing solution of (4.34)-(4.36).

Proof. Suppose (a) holds. Then the existence of a τ-periodic nonnegative stabilizing solution follows from Lemma 4.10. The condition (4.33) follows as in Theorem 4.6. Hence (a) implies (b). The converse is also similar to Theorem 4.6. We only need to show

$$| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|^2_{L^2 \times l^2} \le a(| h |^2 + \| \begin{pmatrix} r \\ r_d \end{pmatrix} \|^2_{L^2 \times l^2}) \text{ for some } a > 0.$$

But this follows from

$$\begin{aligned} \dot{x} &= [A + \frac{1}{\gamma^2}BB'X(t)]x + Br, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d + B_dT_1^{-1}R_1)x(k\tau) + B_dT_1^{-\frac{1}{2}}r_d(k), \\ w &= \frac{1}{\gamma^2}B'X(t)x + r, \\ w_d(k) &= T_1^{-1}R_1x(k\tau) + T_1^{-\frac{1}{2}}r_d(k) \end{aligned}$$

since $(A + \frac{1}{\gamma^2}BB'X, A_d + B_dT_1^{-1}R_1)$ is exponentially stable.

(c) is the dual of (b) and (a) implies the solution of a bounded nonnegative solution of (4.34)-(4.37). In fact we consider the adjoint system

$$\begin{aligned} -\dot{\xi} &= A'\xi + C'v, \; k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= A_d'\xi(k\tau) + C_d'v_d(k), \\ \zeta_c &= B'\xi, \\ \zeta_d(k) &= B_d'\xi(k\tau) + D_d'v_d(k), \\ \xi(T) &= \xi_1 \end{aligned}$$

and

$$J(v, v_d; T, \xi_1) = \int_{t_0}^{T} [| \zeta_c(t) |^2 - \gamma^2 | v(t) |^2]dt + \sum_{k=1}^{N}[| \zeta_d(k) |^2 - \gamma^2 | v_d(k) |^2]$$

and proceed as in Lemma 4.10.

To show the exponential stability of $(A + \frac{1}{\gamma^2}YC'C, A_d + R'_{1Y}T_{1Y}^{-1}C_d)$, let

$$\begin{aligned} v_T(t) &= \frac{1}{\gamma^2}CY(t)\xi(t), \; k\tau < t < (k+1)\tau, \\ v_{dT}(k) &= (T_{1Y}^{-1}R_{1Y})(k)\xi(k\tau) \end{aligned}$$

be the maximizing element of $J_T(v, v_d; \xi_1)$, then

$$\| \begin{pmatrix} v_T \\ v_{dT} \end{pmatrix} \|_{L^2(t_0,T;\mathbf{R}^{p_1})\times l^2(1,N;\mathbf{R}^{p_{1d}})} \le c_0 \mid \xi_1 \mid \quad \text{for some } c_0 > 0.$$

We extend (v_T, v_{dT}) to $[t_0, \infty)$ by zero which we denote by

$$(\tilde{v}_T, \tilde{v}_{dT}) \in L^2(t_0, \infty; \mathbf{R}^{p_1}) \times l^2(1, \infty; \mathbf{R}^{p_{1d}}).$$

Then there exists a subsequence again denoted by $(\tilde{v}_T, \tilde{v}_{dT})$ convergent weakly to

$$(\tilde{v}, \tilde{v}_d) \in L^2(t_0, \infty; \mathbf{R}^{p_1}) \times l^2(1, \infty; \mathbf{R}^{p_{1d}})$$

with $\| \begin{pmatrix} \tilde{v} \\ \tilde{v}_d \end{pmatrix} \|_{L^2(t_0,\infty;\mathbf{R}^{p_1})\times l^2(1,\infty;\mathbf{R}^{p_{1d}})} \le c_0 \mid \xi_1 \mid$. Now let $t_0 < L < \infty$ be a fixed but arbitrary number and consider

$$\begin{aligned} -\dot{\xi}_T &= A'\xi_T + C'\tilde{v}_T, \; \xi_T(L) = \xi_1, \; k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= A'_d\xi_T(k\tau) + C'_d v_{dT}(k), \\ -\dot{\tilde{\xi}} &= A'\tilde{\xi} + C'\tilde{v}, \; \tilde{\xi}_T(L) = \xi_1 \; k\tau < t < (k+1)\tau, \\ \tilde{\xi}(k\tau^-) &= A'_d\tilde{\xi}(k\tau) + C'_d\tilde{v}_d(k) \end{aligned}$$

and

$$\begin{aligned} -\dot{\xi} &= A'\xi + \frac{1}{\gamma^2}C'CY(t)\tilde{\xi}, \; \xi(L) = \xi_1, \; k\tau < t < (k+1)\tau, \quad (4.41) \\ \xi(k\tau^-) &= A'_d\xi(k\tau) + C'_d(T_{1Y}^{-1}R_{1Y})(k)\tilde{\xi}(k\tau). \end{aligned}$$

Then as in Lemma 4.10, we can show $\xi_T(t) \to \tilde{\xi}_T(t)$ as $T \to \infty$ for any $t \in [t_0, L]$ and $\tilde{\xi}(t) = \xi(t)$, $t \in [t_0, L]$. Since $\| \begin{pmatrix} \tilde{v} \\ \tilde{v}_d \end{pmatrix} \|_{L^2(t_0,\infty;\mathbf{R}^{m})\times l^2(1,\infty;\mathbf{R}^{m_d})} \le c_0 \mid \xi_1 \mid$,

$$\int_{t_0}^{L} \mid \tilde{\xi}(t) \mid^2 dt \le c \mid \xi_1 \mid^2 \quad \text{for some } c > 0,$$

which implies

$$\int_{t_0}^{L} \mid \xi(t) \mid^2 dt \le c \mid \xi_1 \mid^2 \quad \text{for any } t_0 \le L < \infty.$$

Hence by Proposition 4.3, the system (4.41) is exponenetially stable and so is $(A+\frac{1}{\gamma^2}YC'C, A_d+R'_{1Y}T_{1Y}^{-1}C_d)$. Hence we have shown that (a) implies the existence of a bounded nonnegative stabilizing solution $Y(t)$ of (4.34)-(4.37). The rest of the proof is similar to that of Corollary 4.3.

The converse follows if we consider the adjoint of (4.23) and proceed as in the first part. ∎

Now we assume that $h = 0$. Then we have the following.

Corollary 4.5 *The following statements are equivalent.*
(a) $\| \bar{G} \| < \gamma$.
(b) There exists a τ*-periodic nonnegative stabilizing solution to (4.29)-(4.31).*
(c) There exists a bounded nonnegative stabilizing solution $Y(t)$, $t \in [t_0, \infty)$ *to (4.34)-(4.36) with* $Y(t_0) = 0$.
Moreover the $\lim_{n\to\infty} Y(t + n\tau)$ *exists (denoted by* $Y_\tau(t)$*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.34)-(4.36).*

Corollary 4.6 *The following statements are equivalent.*
(a) $\| \bar{G} \| < \gamma$.
(b) There exists a τ*-periodic nonnegative stabilizing solution to (4.29)-(4.31).*
(c) There exists a τ*-periodic nonnegative stabilizing solution to (4.34)-(4.36).*

Example 4.4 Consider the following jump system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \; k < t < k+1, \\ x(k^+) &= \begin{bmatrix} 1 & 0 \\ -0.6 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_d(k), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \\ z_d(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k). \end{aligned}$$

For this system we consider the disturbance attenuation problems to the following two cases

$$\text{(a) } H = 0, \; \text{(b) } H = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For all $\gamma \geq 8.3$, there exists a periodic nonnegative stabilizing solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ of the Riccati equation (4.29)-(4.31) to both cases (Figure 4.6) and there exist bounded nonnegative solutions $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}(t)$ of the Riccati equation (4.34)-(4.37) which converge to a periodic solution with period 1 (Figure 4.7).

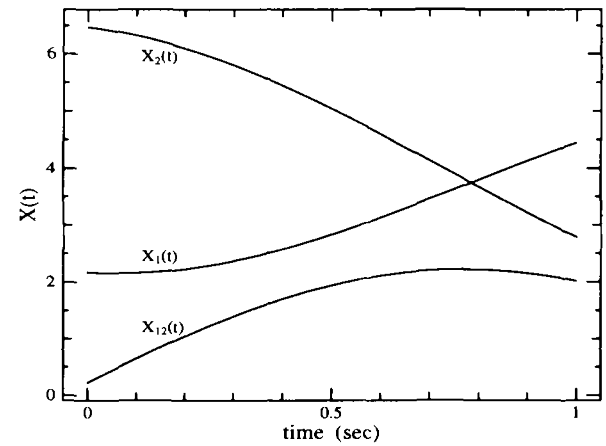

Figure 4.6: The periodic nonnegative stabilizing solution $X(t)$

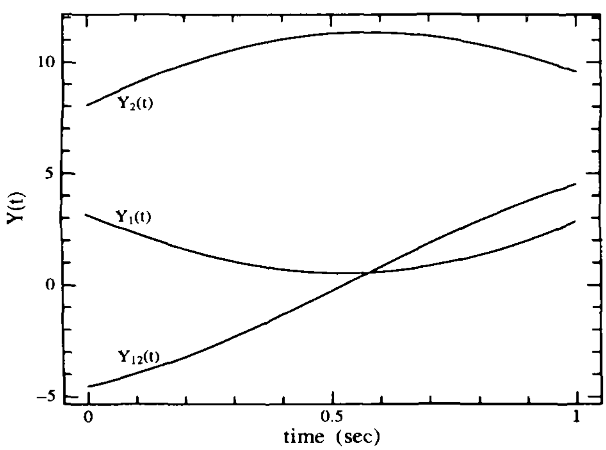

Figure 4.7: The periodic nonnegative stabilizing solution $Y(t)$

4.2 H_∞ Control

In this section we consider the H_∞ control problem for jump systems with initial uncertainty. The main results will be applied to sampled-data systems in Chapter 5.

4.2.1 Main Results

Consider the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T), \ t_0 \le N\tau \le T < (N+1)\tau \end{aligned} \tag{4.42, 4.43}$$

with initial condition

$$x(t_0) = Hh, \ 0 < t_0 \le \tau \tag{4.44}$$

where $x \in \mathbf{R}^n$ is the state, $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_1, z_c, z_d) \in \mathbf{R}^q \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_{1d}}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the sampled observation, $h \in \mathbf{R}^{n_1}$ and all matrices are of compatible dimensions. For the jump system $\mathbf{G}_j$ we assume

$$\begin{aligned} \mathbf{J1}: &\ D_{12}' D_{12} = I, \\ \mathbf{J2}: &\ D_{21} D_{21}' = I. \end{aligned}$$

We consider feedback controllers $u = Ky$ of the form:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k\tau) + \hat{D}(k)y(k) \end{aligned} \tag{4.45}$$

where all matrices are uniformly bounded.

Remark 4.3 As we have mentioned in Section 4.1.1, the feedback controller of the form (4.45) is equivalent to the following discrete-time controller

$$\begin{aligned} \hat{x}(k+1) &= S((k+1)\tau, k\tau^+)\hat{A}_d(k)\hat{x}(k) + S((k+1)\tau, k\tau^+)\hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}(k)y(k) \end{aligned}$$

where $S(\cdot,\cdot)$ is the state transition matrix of $\hat{A}$. If all matrices in (4.45) are constant, the above discrete-time controller becomes the following time-invariant controller:

$$\begin{aligned} \hat{x}(k+1) &= e^{\hat{A}\tau}\hat{A}_d\hat{x}(k) + e^{\hat{A}\tau}\hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k). \end{aligned}$$

On the other hand any discrete-time feedback controller is rewritten as the jump system of the form (4.45).

Consider the system $\mathbf{G}_j$ and the controller $u = Ky$ on $[t_0, T]$. Define the input-output operator of the closed-loop system by

$$\begin{pmatrix} z_1 \\ z_c \\ z_d \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.$$

Then

$$\begin{aligned} G \in \; & \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}) \times l^2(1, N; \mathbf{R}^{m_{1d}}); \\ & \qquad \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{p_1}) \times l^2(1, N; \mathbf{R}^{p_{1d}})). \end{aligned}$$

The H_∞-control problem with initial uncertainty is to find necessary and sufficient conditions for the existence of a controller $u = Ky$ of the form (4.45) such that $\| G \| < \gamma$, i.e.,

$$| z_1 |^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \leq d^2 \left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \right) \text{ for some } 0 < d < \gamma.$$

Such a controller is called γ-suboptimal. Without loss of generality we assume that H and F have full column rank and full row rank, respectively.

To give the solution of this problem, we introduce the following Riccati equations with jumps. For definiteness we assume $0 < t_0 < \tau$.

$$\begin{aligned} -\dot{X} &= A'X + XA + C_1'C_1 + \frac{1}{\gamma^2} X B_1 B_1' X, \\ & \qquad\qquad k\tau < t < (k+1)\tau, && (4.46) \\ X(k\tau^-) &= A_d' X(k\tau) A_d - (R_2' T_2^{-1} R_2)(k), && (4.47) \\ X(T) &= F'F, && (4.48) \\ H'X(t_0)H &\leq d^2 I \text{ for some } 0 < d < \gamma && (4.49) \end{aligned}$$

and

$$\begin{aligned} \dot{Y} &= AY + YA' + B_1 B_1' + \frac{1}{\gamma^2} Y C_1' C_1 Y, \\ & \qquad\qquad k\tau < t < (k+1)\tau, && (4.50) \\ Y(k\tau^+) &= A_d Y(k\tau) A_d' - (R_{2Y}' T_{2Y}^{-1} R_{2Y})(k), && (4.51) \\ Y(t_0) &= H'H && (4.52) \end{aligned}$$

where

$$\begin{aligned} T_2(k) &= I + B_2' X(k\tau) B_2, \quad & R_2(k) &= B_2' X(k\tau) A_d, \\ T_{2Y}(k) &= I + C_2 Y(k\tau) C_2', \quad & R_{2Y}(k) &= C_2 Y(k\tau) A_d'. \end{aligned}$$

If we wish to take $t_0 = \tau$, the condition (4.49) becomes

$$H'X(\tau^-)H \leq d^2 I.$$

We also need the following Riccati equation depending on X:

$$\begin{aligned}\dot{Z} &= (A+\frac{1}{\gamma^2}B_1B_1'X)Z + Z(A+\frac{1}{\gamma^2}B_1B_1'X)' + B_1B_1', \\ &\qquad k\tau < t < (k+1)\tau, & (4.53)\\ V_Z(k) &> aI \text{ for some } a>0, & (4.54)\\ Z(k\tau^+) &= A_dZ(k\tau)A_d' - (R_{2Z}'T_{2Z}^{-1}R_{2Z})(k) + (F_{1Z}'V_ZF_{1Z})(k), & (4.55)\\ Z(t_0) &= H(I-\frac{1}{\gamma^2}H'X(t_0)H)^{-1}H' & (4.56)\end{aligned}$$

where

$$\begin{aligned}&T_{1Z}(k) = \gamma^2 I - T_2^{-\frac{1}{2}}R_2Z(k\tau)R_2'T_2^{-\frac{1}{2}}, && T_{2Z}(k) = I + C_2Z(k\tau)C_2',\\ &R_{1Z}(k) = T_2^{-\frac{1}{2}}R_2Z(k\tau)A_d', && R_{2Z}(k) = C_2Z(k\tau)A_d',\\ &S_Z(k) = C_2Z(k\tau)R_2'T_2^{-\frac{1}{2}}, && V_Z(k) = [T_{1Z}+S_Z'T_{2Z}^{-1}S_Z](k),\\ &F_{1Z}(k) = [V_Z^{-1}(R_{1Z}-S_Z'T_{2Z}^{-1}R_{2Z})](k), &&\\ &F_{2Z}(k) = -[T_{2Z}^{-1}(R_{2Z}+S_ZF_{1Z})](k). &&\end{aligned}$$

As in the continuous- and discrete-time H_∞-control problems we have the following relationship between X, Y and Z. Proofs of lemmas below will be given in Section 4.2.4.

Lemma 4.11 *(a) Suppose X, Y and Z are solutions of (4.46), (4.47), (4.50), (4.51) and (4.53)-(4.55), respectively. If $Z(s) - Y(s) - \frac{1}{\gamma^2}Z(s)X(s)Y(s) = 0$ for some $s \geq t_0$, then $Z(t) - Y(t) - \frac{1}{\gamma^2}Z(t)X(t)Y(t) = 0$ for all $t \geq s$.*
(b) If (4.52) and (4.56) hold, then $Z(t_0) - Y(t_0) - \frac{1}{\gamma^2}Z(t_0)X(t_0)Y(t_0) = 0$.

Lemma 4.12 *(a) Let X, Y and Z be the solutions of (4.46), (4.47), (4.50), (4.51) and (4.53)-(4.55), respectively. Suppose $I - \frac{1}{\gamma^2}XY$ is nonsingular. If x satisfies*

$$\begin{aligned}-\dot{x} &= (A+\frac{1}{\gamma^2}YC_1'C_1)'(t)x, \quad k\tau < t < (k+1)\tau,\\ x(k\tau^-) &= (A_d - R_{2Y}'T_{2Y}^{-1}C_2)'(k)x(k\tau),\end{aligned}$$

then $\tilde{x} = (I - \frac{1}{\gamma^2}XY)x$ satisfies

$$\begin{aligned}-\dot{\tilde{x}} &= (A+\frac{1}{\gamma^2}B_1B_1'X)'(t)\tilde{x}, \quad k\tau < t < (k+1)\tau,\\ \tilde{x}(k\tau^-) &= (A_d + F_{1Z}'T_2^{-\frac{1}{2}}R_2 + F_{2Z}'C_2)'(k)\tilde{x}(k\tau).\end{aligned}$$

(b) Let X, Y and Z be bounded on $[t_0,\infty)$ and suppose $I-\frac{1}{\gamma^2}XY$ is nonsingular and its inverse is uniformly bounded in t. $(A+\frac{1}{\gamma^2}YC_1'C_1, A_d - R_{2Y}'T_{2Y}^{-1}C_2)$ is exponentially stable if and only if $(A+\frac{1}{\gamma^2}B_1B_1'X, A_d+F_{1Z}'T_2^{-\frac{1}{2}}R_2+F_{2Z}'C_2)$ is exponentially stable.

The following are our main results.

Theorem 4.8 *Assume* **J1** *and* **J2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following conditions hold:
(i) There exists a nonnegative solution X to (4.46)-(4.49).
(ii) For the solution X in (i), there exists a nonnegative solution Z to (4.53)-(4.56).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} \dot{\hat{x}} &= [A + \frac{1}{\gamma^2} B_1 B_1' X(t)]\hat{x}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}_1(k)y(k) + \hat{B}_2(k)\hat{v}(k), \\ u(k) &= \hat{C}(k)\hat{x}(k\tau) + \hat{D}_1(k)y(k) + \hat{D}_2(k)\hat{v}(k), \\ \hat{r}(k) &= T_{2Z}^{-\frac{1}{2}}(k)[-C_2\hat{x}(k\tau) + y(k)], \\ \hat{v} &= Q\hat{r}, \ Q \in Q_\gamma \end{aligned} \tag{4.57}$$

where

$$\begin{aligned} &\hat{A}_d(k) = [(A_d - B_2 T_2^{-1} R_2)\Psi](k), \\ &\hat{B}_1(k) = (A_d - B_2 T_2^{-1} R_2)(k) Z(k\tau) C_2' T_{2Z}^{-1}(k), \\ &\hat{B}_2(k) = \tfrac{1}{\gamma}([F_{1Z}' + B_2 T_2^{-\frac{1}{2}}] V_Z^{\frac{1}{2}})(k), \\ &\hat{C}(k) = -T_2^{-1} R_2 \Psi(k), \\ &\hat{D}_1(k) = -(T_2^{-1} R_2)(k) Z(k\tau) C_2' T_{2Z}^{-1}(k), \\ &\hat{D}_2(k) = \tfrac{1}{\gamma}(T_2^{-\frac{1}{2}} V_Z^{\frac{1}{2}})(k), \\ &\Psi(k) = I - Z(k\tau) C_2' T_{2Z}^{-1}(k) C_2 \end{aligned}$$

and

$$\begin{aligned} Q_\gamma = \{ & Q \in \mathcal{L}(l^2(1, N; \mathbf{R}^{p_2}); l^2(1, N; \mathbf{R}^{m_2})) : \\ & Q \text{ is of the form (4.45) and } \| Q \| < \gamma \}. \end{aligned}$$

Theorem 4.9 *Assume* **J1** *and* **J2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following conditions hold:
(i) There exists a nonnegative solution X to (4.46)-(4.49).
(ii) There exists a nonnegative solution Y to (4.50)-(4.52).
(iii) $\rho(X(t)Y(t)) \le d^2 I$ for any $t \in [t_0, T]$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (4.57) with Z replaced by $(I - \frac{1}{\gamma^2} YX)^{-1} Y$.

Next we consider the system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \end{aligned}$$

$$
\begin{aligned}
z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\
y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\
x(t_0) &= Hh,\ 0 < t_0 \le \tau
\end{aligned}
$$

on $[t_0, \infty)$ and the controller $u = Ky$ of the form (4.45). We assume **J1**, **J2** and

$$
\begin{aligned}
&\mathbf{J3}: \quad ([A, A_d], [B_1, 0], [C_1, 0]) \text{ is stabilizable and detectable,} \\
&\mathbf{J4}: \quad ([A, A_d], [0, B_2], [0, C_2]) \text{ is stabilizable and detectable.}
\end{aligned}
$$

Remark 4.4 (a) As we see in Remark 4.3 the condition **J4** is equivalent to the stabilizability and detectability of $(e^{A\tau} A_d, e^{A\tau} B_2, C_2)$.
(b) Since the condition **J3** is equivalent to the stabilizability and detectability of the system

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w,\ k\tau < t < (k+1)\tau, \\
x(k\tau^+) &= A_d x(k\tau), \\
z &= C_1 x,
\end{aligned}
$$

J3 is equivalent to the existence of matrices $K \in \mathbf{R}^{m_1 \times n}$ and $J \in \mathbf{R}^{n \times p_1}$ such that $e^{(A+B_1K)\tau} A_d$ and $e^{(A+JC_1)\tau} A_d$ are exponentially stable.

If the controller is IO-stabilizing (or internally stabilizing), we define the input-output map of the closed-loop system

$$
\begin{pmatrix} z_c \\ z_d \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.
$$

Then

$$
\begin{aligned}
G \in \mathcal{L}(&\mathbf{R}^{n_1} \times L^2(t_0, \infty; \mathbf{R}^{m_1}) \times l^2(1, \infty; \mathbf{R}^{m_{1d}}); \\
&L^2(t_0, \infty; \mathbf{R}^{p_1}) \times l^2(1, \infty; \mathbf{R}^{p_{1d}})).
\end{aligned}
$$

The H_∞-control problem with initial uncertainty on $[t_0, \infty)$ is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., an internally stabilizing controller $u = Ky$ of the form (4.45) such that $\| G \| < \gamma$, i.e.,

$$
\left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \le d^2 \left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \right) \text{ for some } 0 < d < \gamma. \quad (4.58)
$$

To give the solution of this problem, we need the following definition.

Definition 4.11 *(a) The solution X of (4.46) and (4.47) is called stabilizing if $(A + \frac{1}{\gamma^2} B_1 B_1' X, A_d - B_2 T_2^{-1} R_2)$ is exponentially stable.*

(b) The solution Y of (4.50) and (4.51) is called stabilizing if $(A+\frac{1}{\gamma^2}YC_1'C_1, A_d - R_{2Y}'T_{2Y}^{-1}C_2])$ is exponentially stable.
(c) The solution Z of (4.53)-(4.55) is called stabilizing if $(A+\frac{1}{\gamma^2}B_1B_1'X, A_d + F_{1Z}'T_2^{-\frac{1}{2}}R_2 + F_{2Z}'C_2)$ is exponentially stable.

Theorem 4.10 *Assume* **J1-J4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing solution X to (4.46), (4.47) and (4.49).
(ii) For the solution X in (i), there exists a bounded nonnegative stabilizing solution Z to (4.53)-(4.56).
(b) In this case the set of all γ-suboptimal controllers is given by (4.57) with Q internally stable.
Moreover the $\lim_{n\to\infty} Z(t + n\tau)$ exists (denoted by $Z_\tau(t)$) and Z_τ is a τ-periodic nonnegative stabilizing solution of (4.53)-(4.55).

Theorem 4.11 *Assume* **J1-J4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing solution X to (4.46), (4.47) and (4.49).
(ii) There exists a bounded nonnegative stabilizing solution Y of (4.50)-(4.52).
(iii) $\rho(X(t)Y(t)) \le d^2$, for any $t \in [t_0, \infty)$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (4.57) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ and Q internally stable.
Moreover the $\lim_{n\to\infty} Y(t + n\tau)$ exists (denoted by $Y_\tau(t)$) and Y_τ is a τ-periodic nonnegative stabilizing solution of (4.50) and (4.51).

If $h = 0$ we can construct τ-periodic γ-suboptimal controllers. Proofs of the following corollaries are similar to that of Corollary 2.12.

Corollary 4.7 *Consider the system $\mathbf{G}_j$ with $h = 0$ and assume* **J1-J4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing solution X to (4.46), (4.47) and (4.49).
(ii) For the solution X in (i), there exists a τ-periodic nonnegative stabilizing solution Z to (4.53)-(4.55).
(b) In this case the controllers given by (4.57) are γ-suboptimal. If Q is τ-periodic, then the controller (4.57) is τ-periodic.

Corollary 4.8 *Consider the system $\mathbf{G}_j$ with $h = 0$ and assume* **J1-J4**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, \infty)$ if and only if

the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing solution X to (4.46), (4.47) and (4.49).
(ii) There exists a τ-periodic nonnegative stabilizing solution Y of (4.50) and (4.51).
(iii) $\rho(X(t)Y(t)) \leq d^2$, for any $t \in [t_0, t_0 + \tau)$ and for some $0 < d < \gamma$.
(b) In this case the controllers given by (4.57) with Z replaced by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ and Q internally stable are γ-suboptimal. If Q is τ-periodic, then the controllers are τ-periodic and γ-suboptimal.

We consider an example and apply Theorem 4.11.

Example 4.5 Consider the H_∞-control problem for the following system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \ k < t < k+1, \ x(0) = Hh, \\
x(k^+) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \\
z(t) &= \begin{bmatrix} [1 \quad 0]\, x \\ u(k) \end{bmatrix}, \\
y(k) &= [1 \quad 0]\, x(k) + w_d(k)
\end{aligned}$$

which satisfies the assumptions **J1-J4**. For this system we consider the following two cases

$$\text{(a) } H = 0, \ \text{(b) } H = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then in both cases, the conditions (i)-(iii) of Theorem 4.11 are satisfied for all $\gamma \geq 1.85$. Figure 4.8 shows the periodic nonnegative stabilizing solution $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12} & X_2 \end{bmatrix}(t)$ with $\gamma = 1.85$ and Figures 4.9 and 4.10 shows the nonnegative stabilizing solution $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}(t)$ with $\gamma = 1.85$ of the cases (a) and (b), respectively which converges to the same periodic solution. Figure 4.11 shows the eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ of $X(t)Y(t)$ in the case (b) with $\gamma = 1.85$ and $\lambda_1(t) \leq 3.3931 < 1.85^2$. Figures 4.12 and 4.13 show the simulation results of the closed-loop systems with central controllers of the case (a) and (b), respectively where $\gamma = 1.85$, the initial conditions $x_1(0) = 1$, $x_2(0) = 0$ and the disturbances $w(t) = e^{-10t}\sin 10t$, $w_d(k) = 0$. The controller of the case (b) gives a better response.

4.2.2 H_∞ Riccati Equations

Before proving our main results, we first consider the relationship between H_∞-problems and quadratic games as in continuous- and discrete-time cases.

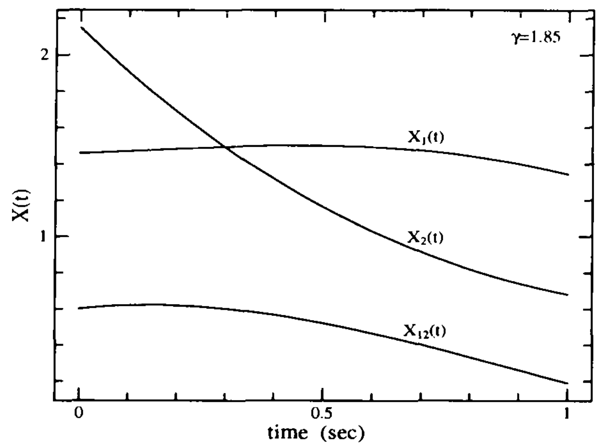

Figure 4.8: The periodic nonnegative stabilizing solution $X(t)$

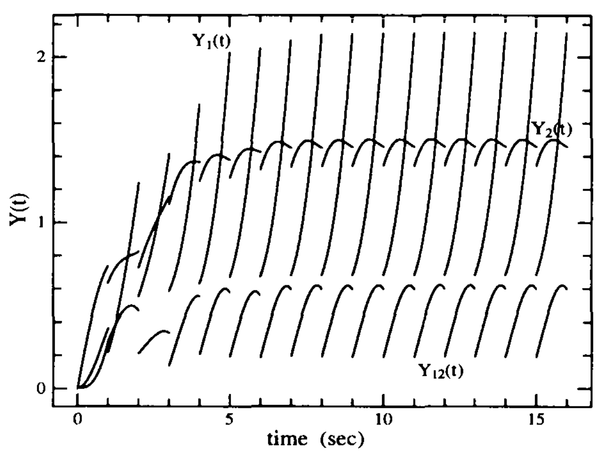

Figure 4.9: The bounded nonnegative stabilizing solution $Y(t)$ of the case (a)

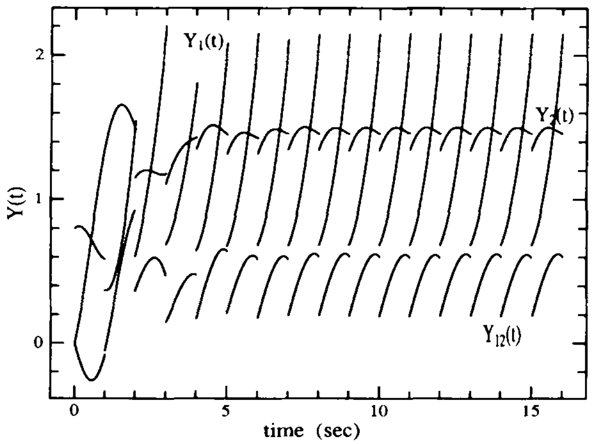

Figure 4.10: The bounded nonnegative stabilizing solution $Y(t)$ of the case (b)

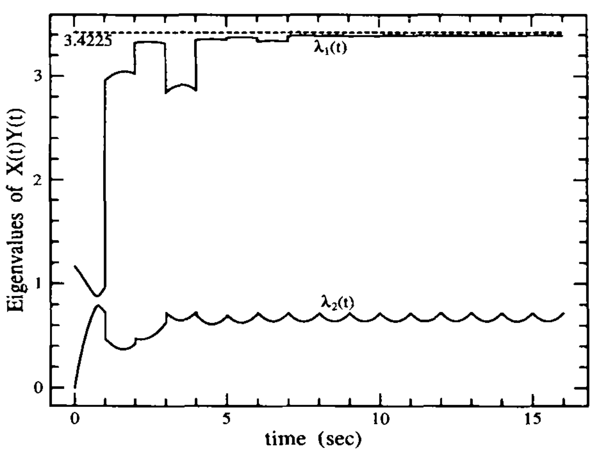

Figure 4.11: Eigenvalues of $X(t)Y(t)$ of the case (b)

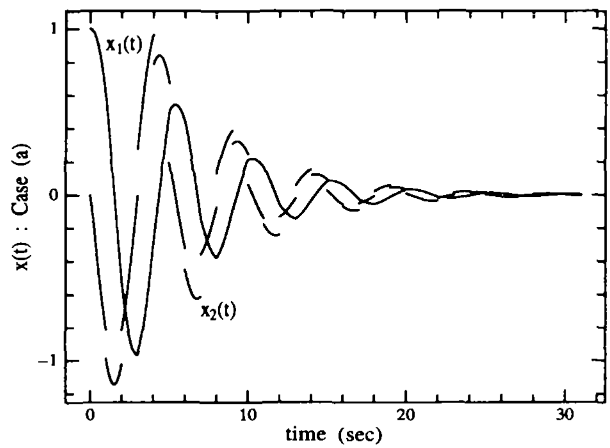

Figure 4.12: Simulation result: Case (a)

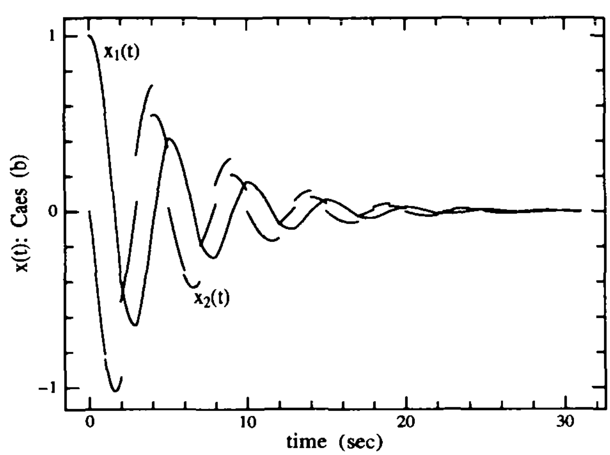

Figure 4.13: Simulation result: Case (b)

Then we shall show that the existence of solutions to the Riccati equations with jumps (4.46)-(4.48) and (4.50)-(4.52) under the existence of a γ-suboptimal controller.

The Finite Horizon Problem

Consider the system which is slightly generalized than the system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_{1d} w_d(k) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ C_{1d} x(k\tau) + D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{4.59}$$

with

$$z_1 = Fx(T) \tag{4.60}$$

and the controller $u = Ky$ of the form (4.45). For the system (4.59) we assume

$$\textbf{J1'}: D_{12}' C_{1d} = 0 \text{ and } D_{12}' D_{12} = I.$$

The H_∞-control problem with no initial uncertainty on $[t_0, T]$ is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller such that $\| \bar{G} \| < \gamma$, i.e.,

$$\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2 \times l^2}^2 \leq d^2 \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2 \ \text{ for some } 0 < d < \gamma$$

where $\bar{G}$ is the input-output operator defined by

$$\begin{pmatrix} z_c \\ z_d \end{pmatrix} = G \begin{pmatrix} w \\ w_d \end{pmatrix}.$$

Now we assume the existence of a γ-suboptimal controller and study its consequence to the following quadratic game

$$\begin{aligned} J(u, (w, w_d); s, x_0) = &\int_s^T [| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt \\ &+ \sum_{k=k_s}^{N} [| z_d(k) |^2 - \gamma^2 | w_d(k) |^2] + | Fx(T) |^2 \end{aligned} \tag{4.61}$$

where $s \geq t_0$, u is the minimizer and (w, w_d) is the maximizer. We assume $(k_s - 1)\tau < s \leq k_s \tau \leq N\tau \leq T < (N+1)\tau$. Then the response to (4.59) can be written

$$\begin{aligned} x_K(t) &= (\Phi_{1K} x_0)(t) + (\Phi_{2K} w)(t) + (\Phi_{3K} w_d)(t), \\ z_{cK}(t) &= (\Psi_{11K} x_0)(t) + (\Psi_{12K} w)(t) + (\Psi_{13K} w_d)(t), \\ z_{dK}(k) &= (\Psi_{21K} x_0)(k) + (\Psi_{22K} w)(k) + (\Psi_{23K} w_d)(k), \\ u_K(k) &= (\Pi_{1K} x_0)(k) + (\Pi_{2K} w)(k) + (\Pi_{3K} w_d)(k), \\ z_{1K} &= F\Phi_{1K}(T) x_0 + F\Phi_{2K}(T) w + F\Phi_{3K}(T) w_d \end{aligned} \tag{4.62}$$

where

$$\begin{aligned}&\Phi_{1K},\ \Psi_{11K},\ \Psi_{21K}\ \Pi_{1K} \in \mathcal{L}(\mathbf{R}^n, \mathbf{E}),\\ &\Phi_{2K},\ \Psi_{12K},\ \Psi_{22K}\ \Pi_{2K} \in \mathcal{L}(L^2(s,T;\mathbf{R}^{m_1}), \mathbf{E}),\\ &\Phi_{3K},\ \Psi_{13K},\ \Psi_{23K}\ \Pi_{3K} \in \mathcal{L}(l^2(k_s,N;\mathbf{R}^{m_{1d}}), \mathbf{E})\end{aligned}$$

$\mathbf{E} = L^2(s,T;\mathbf{R}^n)$, $L^2(s,T;\mathbf{R}^{p_1})$, $l^2(k_s,N;\mathbf{R}^{p_{1d}})$ and $l^2(k_s,N;\mathbf{R}^{m_2})$, respectively and

$$\begin{aligned}\Phi_{1K}(T)x_0 &= (\Phi_{1K}x_0)(T),\\ \Phi_{2K}(T)w &= (\Phi_{2K}w)(T),\\ \Phi_{3K}(T)w_d &= (\Phi_{3K}w_d)(T).\end{aligned}$$

Moreover Φ_{jK}, Π_{jK} and Ψ_{ijK}, $i = 1, 2$, $j = 2, 3$ are causal and $\| \tilde{G} \| < \gamma$ is equivalent to

$$\| \bar{\Psi}_K \| = \left\| \begin{pmatrix} F\Phi_{2K}(T) & F\Phi_{3K}(T) \\ \Psi_{12K} & \Psi_{13K} \\ \Psi_{22K} & \Psi_{23K} \end{pmatrix} \right\| \leq d \text{ for some } 0 < d < \gamma \qquad (4.63)$$

which implies

$$\begin{aligned}&| F\Phi_{2K}(T)w + F\Phi_{3K}(T)w_d |^2\\ &\quad + \left\| \begin{pmatrix} \Psi_{12K}w + \Psi_{13K}w_d \\ \Psi_{22K}w + \Psi_{23K}w_d \end{pmatrix} \right\|_{L^2\times l^2} \leq d^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}.\end{aligned}$$

Now consider the functional (4.61). Since

$$\| z_d \|_{l^2}^2 = \| C_d x \|_{l^2}^2 + \| u \|_{l^2}^2,$$

$J(u,(w,w_d);s,x_0)$ is strictly convex in u. Hence by Theorem A.4 for any x_0 and $(w,w_d) \in L^2(s,T;\mathbf{R}^{m_1}) \times l^2(k_s,N;\mathbf{R}^{m_{1d}})$ there exists a unique $u_s = u_s((w,w_d),x_0) \in l^2(k_s,N;\mathbf{R}^{m_2})$ such that

$$\min_u J(u,(w,w_d);s,x_0) = J(u_s,(w,w_d);s,x_0).$$

The response of (4.59) to u_s is written

$$\begin{aligned}x_s(t) &= (\Phi_{1s}x_0)(t) + (\Phi_{2s}w)(t) + (\Phi_{3s}w_d)(t),\\ z_{cs}(t) &= (\Psi_{11s}x_0)(t) + (\Psi_{12s}w)(t) + (\Psi_{13s}w_d)(t),\\ z_{ds}(k) &= (\Psi_{21s}x_0)(k) + (\Psi_{22s}w)(k) + (\Psi_{23s}w_d)(k),\\ u_s(k) &= (\Pi_{1s}x_0)(k) + (\Pi_{2s}w)(k) + (\Pi_{3s}w_d)(k),\\ z_{1K} &= F\Phi_{1K}(T)x_0 + F\Phi_{2K}(T)w + F\Phi_{3K}(T)w_d.\end{aligned} \qquad (4.64)$$

Since

$$J(u_s,(w,w_d);s,x_0) \leq J(u_K,(w,w_d);s,x_0), \qquad (4.65)$$

we have

$$\| \bar{\Psi}_s \| = \left\| \begin{pmatrix} F\Phi_{2s}(T) & F\Phi_{3s}(T) \\ \Psi_{12s} & \Psi_{13s} \\ \Psi_{22s} & \Psi_{23s} \end{pmatrix} \right\| \leq d \text{ for some } 0 < d < \gamma. \tag{4.66}$$

Now

$$\begin{aligned} & J(u_s, (w, w_d); s, x_0) \\ = & \left\| \begin{pmatrix} z_{cs} \\ z_{ds} \end{pmatrix} \right\|^2_{L^2 \times l^2} - \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} + | z_{1s} |^2 \\ = & \left\| \begin{pmatrix} F\Phi_{1s}(T) \\ \Psi_{11s} \\ \Psi_{21s} \end{pmatrix} x_0 + \bar{\Psi}_s \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2 - \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2}. \end{aligned}$$

By (4.66), $\gamma^2 I - \bar{\Psi}_s^* \bar{\Psi}_s$ is bounded both from below and above. So its inverse exists (Theorem A.3) and is uniformly bounded in s. Hence there exists a unique maximizing element of $J(u, (w, w_d); s, x_0)$ given by

$$\begin{pmatrix} w_s \\ w_{ds} \end{pmatrix} = (\gamma^2 I - \bar{\Psi}_s^* \bar{\Psi}_s)^{-1} \bar{\Psi}_s^* \begin{pmatrix} F\Phi_{1s}(T) \\ \Psi_{11s} \\ \Psi_{21s} \end{pmatrix} x_0. \tag{4.67}$$

Next we shall show that $\begin{pmatrix} w_s \\ w_{ds} \end{pmatrix} = \begin{pmatrix} w_s \\ w_{ds} \end{pmatrix}(x_0)$ and $u_s((w_s, w_{ds}), x_0)$ are uniformly bounded in s. Setting $w = 0$ and $w_d = 0$ in (4.65) we have

$$\begin{aligned} \| u_s((0,0); x_0) \|^2_{l^2} & \leq J(u_s((0,0); x_0), (0,0); s, x_0) \\ & \leq J(u_K, (0,0); s, x_0) = \left\| \begin{pmatrix} z_{cK} \\ z_{dK} \end{pmatrix} \right\|^2_{L^2} + | z_{1K} |^2 \end{aligned}$$

or

$$\begin{aligned} \| \Pi_{1s} x_0 \|^2_{l^2} & \leq \left\| \begin{pmatrix} \Psi_{11s} x_0 \\ \Psi_{21s} x_0 \end{pmatrix} \right\|^2_{L^2 \times l^2} + | F\Phi_{1s}(T) x_0 |^2 \\ & \leq \left\| \begin{pmatrix} \Psi_{11K} x_0 \\ \Psi_{21K} x_0 \end{pmatrix} \right\|^2_{L^2 \times l^2} + | F\Phi_{1K}(T) x_0 |^2 . \end{aligned}$$

Hence Π_{1s}, Ψ_{11s} and Ψ_{21s} are uniformly bounded. From (4.66) and (4.67), we have

$$\left\| \begin{pmatrix} w_s \\ w_{ds} \end{pmatrix} \right\|_{L^2 \times l^2} \leq a | x_0 | \tag{4.68}$$

for some $a > 0$ independent of s. Setting $x_0 = 0$ in (4.65), we also have

$$\begin{aligned} & \| u_s((w, w_d); 0) \|^2_{l^2} - \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \\ \leq & J(u_s((w, w_d); 0), (w, w_d); s, 0) \\ \leq & J(u_K, (w, w_d); s, 0) \\ = & \left\| \begin{pmatrix} z_{cK} \\ z_{dK} \end{pmatrix} \right\|^2_{L^2 \times l^2} - \gamma^2 \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \end{aligned}$$

and

$$
\begin{aligned}
&\| \Pi_{2s}w + \Pi_{3s}w_d \|_{l^2} + | F\Psi_{2s}(T)w + F\Psi_{3s}(T)w_d |^2 \\
\leq\ & \| \begin{pmatrix} F\Psi_{2s}(T) & F\Psi_{3s}(T) \\ \Psi_{12s} & \Psi_{13s} \\ \Psi_{22s} & \Psi_{23s} \end{pmatrix} \begin{pmatrix} w \\ w_d \end{pmatrix} \|^2 \\
\leq\ & \| \begin{pmatrix} F\Psi_{2K}(T) & F\Psi_{3K}(T) \\ \Psi_{12K} & \Psi_{13K} \\ \Psi_{22K} & \Psi_{23K} \end{pmatrix} \begin{pmatrix} w \\ w_d \end{pmatrix} \|^2 \\
\leq\ & d^2 \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|^2_{L^2\times l^2}
\end{aligned}
$$

for some $0 < d < \gamma$. Hence Π_{js}, Ψ_{ijs}, $i = 1,2$, $j = 2,3$ are uniformly bounded. Now (4.64) and (4.68) yield

$$\| u_s((w,w_d),x_0) \|_{l^2} \leq a \mid x_0 \mid$$

for some $a > 0$ independent of s. Thus we have shown the following.

Lemma 4.13 *(a)* Π_{js}, Ψ_{ijs}, $i = 1,2$, $j = 1,2,3$ *are uniformly bounded.*
(b) $w_s(x_0)$, $w_{ds}(x_0)$ *and* $u_s((w_s,w_{ds}),x_0)$ *are uniformly bounded and*

$$\max_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) = J(u_s,(w_s,w_{ds});s,x_0) \leq a \mid x_0 \mid^2$$

for some $a > 0$ *independent of* s.

We consider

$$
\begin{aligned}
-\dot{X} &= A'X + XA + C_1'C_1 + \frac{1}{\gamma^2}XB_1B_1'X, &(4.69)\\
&\qquad k\tau < t < (k+1)\tau, \\
V(k) &> aI \text{ for some } a > 0, &(4.70)\\
X(k\tau^-) &= A_d'X(k\tau)A_d + C_{1d}'C_{1d} \\
&\qquad -(R_2'T_2^{-1}R_2)(k) + (F_1'VF_1)(k), &(4.71)\\
X(T) &= F'F &(4.72)
\end{aligned}
$$

where

$$
\begin{aligned}
&T_1(k) = \gamma^2 I - B_{1d}'X(k\tau)B_{1d}, && T_2(k) = I + B_2'X(k\tau)B_2,\\
&R_1(k) = B_{1d}'X(k\tau)A_d, && R_2(k) = B_2'X(k\tau)A_d,\\
&S(k) = B_2'X(k\tau)B_{1d}, && V(k) = [T_1 + S'T_2^{-1}S](k),\\
&F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k), && F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k).
\end{aligned}
$$

First we assume that there exists a nonnegative solution X to (4.69)-(4.72) and examine the properties of X. By direct calculation, we obtain

$$
\begin{aligned}
J(u,(w,w_d);s,x_0) &= x_0'X(s)x_0 + \| T_2^{\frac{1}{2}}[u + T_2^{-1}(Sw_d + R_2x)] \|^2_{l^2} \\
&\quad -\gamma^2 \| \begin{pmatrix} w - \frac{1}{\gamma^2}B_1'Xx \\ \frac{1}{\gamma}V^{\frac{1}{2}}(w_d - F_1x) \end{pmatrix} \|^2_{L^2\times l^2} \qquad (4.73)
\end{aligned}
$$

where x is the response of the system (4.59) to the pair $(u,(w,w_d)) \in l^2(k_s,N;\mathbf{R}^{m_2}) \times L^2(s,T;\mathbf{R}^{m_1}) \times l^2(k_s,N;\mathbf{R}^{m_{1d}})$. Define feedback laws

$$\begin{aligned} \bar{w}(t) &= \frac{1}{\gamma^2}B_1'X(t)x(t), \\ \bar{w}_d(k) &= F_1(k)x(k\tau), \\ \bar{u}(k) &= -(T_2^{-1}R_2)(k)x(k\tau) - (T_2^{-1}S)(k)w_d(k) \end{aligned} \tag{4.74}$$

and let x^* be the solution of (4.59) corresponding to (4.74). Set

$$\begin{aligned} w^*(t) &= \frac{1}{\gamma^2}B_1'X(t)x^*(t), \\ w_d^*(k) &= F_1(k)x^*(k\tau), \\ u^*(k) &= F_2(k)x^*(k\tau). \end{aligned} \tag{4.75}$$

We shall show that the value of the game exists, i.e.,

$$\sup_{(w,w_d)} \inf_u J(u,(w,w_d);s,x_0) = \inf_u \sup_{(w,w_d)} J(u,(w,w_d);s,x_0).$$

Lemma 4.14 *Suppose that there exists an X satisfying (4.69)-(4.72). Then X is nonnegative. Moreover*

$$\begin{aligned} J(\bar{u},(w,w_d);s,x_0) &\le J(\bar{u},(\bar{w},\bar{w}_d);s,x_0) \\ &= x_0'X(s)x_0 \le J(u,(\bar{w},\bar{w}_d);s,x_0), \qquad (4.76) \\ J(u^*,(w^*,w_d^*);s,x_0) &= x_0'X(s)x_0 \le J(u,(w^*,w_d^*);s,x_0) \qquad (4.77) \end{aligned}$$

for any $(u,(w,w_d)) \in l^2(k_s,N;\mathbf{R}^{m_2}) \times L^2(s,T;\mathbf{R}^{m_1}) \times l^2(k_s,N;\mathbf{R}^{m_{1d}})$. The max-min of $J(u,(w,w_d);s,x_0)$ is attained by the pair $(\bar{u}_s,(w_s^*,w_{ds}^*))$ and

$$\begin{aligned} \max_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) &= J(\bar{u},(w^*,w_d^*);s,x_0) \\ &= J(\bar{u},(\bar{w},\bar{w}_d);s,x_0) \qquad (4.78) \\ &= J(u^*,(w^*,w_d^*);s,x_0) \\ &= x_0'X(s)x_0 \\ &= \inf_u \sup_{(w,w_d)} J(u,(w,w_d);s,x_0). \end{aligned}$$

Proof. We note that (4.76) follows from (4.73). Setting $w=0$, $w_d=0$ in (4.76), we have

$$0 \le J(\bar{u},(0,0);s,x_0) \le J(\bar{u},(\bar{w},\bar{w}_d);s,x_0) = x_0'X(s)x_0.$$

Hence $X(s)$ is nonnegative. Changing the initial time, we also have $X(t) \ge 0$. From (4.76), we have

$$J(\bar{u},(w,w_d);s,x_0) \le J(\bar{u},(\bar{w},\bar{w}_d);s,x_0) = x_0'X(s)x_0$$

and hence

$$\min_u J(u,(w,w_d);s,x_0) \le J(\bar{u},(w,w_d);s,x_0) \le x_0' X(s) x_0$$

for any $(w,w_d) \in L^2(s,T;\mathbf{R}^{m_1}) \times l^2(k_s,N;\mathbf{R}^{m_{1d}})$. This implies

$$\sup_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) \le x_0' X(s) x_0.$$

Next we shall show

$$\min_u J(u,(w^*,w_d^*);s,x_0) = J(u^*,(w^*,w_d^*);s,x_0) = x_0' X(s) x_0. \qquad (4.79)$$

For this purpose we consider $e = x - x^*$ where x is given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w^*, \ x(s) = x_0, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_{1d} w_d^*(k) + B_2 u(k). \end{aligned}$$

Then

$$\begin{aligned} \dot{e} &= Ae, \ e(s) = 0, \ k\tau < t < (k+1)\tau, \\ e(k\tau^+) &= A_d e(k\tau) + B_2[u(k) - u^*(k)] \end{aligned}$$

and

$$\begin{aligned} J(u,(w^*,w_d^*);s,x_0) &= \| \begin{pmatrix} C_1(e+x^*) \\ C_{1d}(e+x^*) \end{pmatrix} \|_{L^2\times l^2}^2 + \| u \|_{l^2}^2 \\ &\quad -\gamma^2 \| \begin{pmatrix} w^* \\ w_d^* \end{pmatrix} \|_{L^2\times l^2}^2 + | F(e+x^*)(T) |^2 . \end{aligned}$$

Define

$$\begin{aligned} (\mathbf{H}u)(t) &= \sum_{j=k_s}^{k} S(t, j\tau^+) B_2 u(j) \\ &= e^{A(t-k\tau)} \sum_{j=k_s}^{k} (A_d e^{A\tau})^{k-j} B_2 u(j), \ k\tau < t \le (k+1)\tau, \\ (\mathbf{H}_d u)(k\tau) &= \sum_{j=k_s}^{k-1} S(k\tau, j\tau^+) B_2 u(j) \\ &= e^{A\tau} \sum_{j=k_s}^{k-1} (A_d e^{A\tau})^{k-1-j} B_2 u(j), \\ \mathbf{H}_s u &= \sum_{j=k_s}^{N} S(T, j\tau^+) B_2 u(j) \\ &= e^{A(T-N\tau)} \sum_{j=k_s}^{N} (A_d e^{A\tau})^{k-j} B_2 u(j) \end{aligned}$$

where $S(t,r)$ is the state transition matrix of (A, A_d). Then

$$\begin{aligned} e(t) &= [\mathbf{H}(u-u^*)](t), \\ e(k\tau) &= [\mathbf{H}_d(u-u^*)](k\tau) \end{aligned}$$

and

$$e(T) = \mathbf{H}_s(u-u^*).$$

Since $J(u,(w^*,w_d^*);s,x_0)$ is strictly convex in u, there exists a unique minimizing element u. It is given by the solution of

$$\begin{aligned} u + \mathbf{H}^*C_1'C_1[\mathbf{H}(u-u^*)+x^*] + \mathbf{H}_d^*C_{1d}'C_{1d}[\mathbf{H}_d(u-u^*)+x^*] \\ +\mathbf{H}_s^*F'F[\mathbf{H}_s(u-u^*)+x^*] = 0. \end{aligned}$$

Next we shall show that $u = u^*$ is the solution. Note that for $(\tilde{h}, h, h_d) \in \mathbf{R}^{n_1} \times L^2(s,T;\mathbf{R}^n) \times l^2(k_s,N;\mathbf{R}^n)$

$$\begin{aligned} (\mathbf{H}^*h)(k) &= B_2' \int_{k\tau}^{T} S'(t,k\tau^+)h(t)dt, \\ (\mathbf{H}_d^*h_d)(k) &= B_2' \sum_{j=k+1}^{N} S'(j\tau,k\tau^+)h_d(j\tau) \end{aligned}$$

and

$$(\mathbf{H}_s^*\tilde{h})(k) = B_2'S'(T,k\tau^+)\tilde{h}.$$

As in the proof of Lemma 3.14 we can show

$$u^*(k) = F_2(k)x^*(k) = -B_2'X(k\tau)x^*(k\tau^+).$$

It is enough to show

$$\begin{aligned} X(k\tau)x^*(k\tau^+) - \int_{k\tau}^{T} S'(t,k\tau^+)C_1'C_1x^*(t)dt \\ - \sum_{j=k+1}^{N} S'(j\tau,k\tau^+)C_{1d}'C_{1d}x^*(j\tau) - S'(T,k\tau^+)F'Fx^*(T) = 0. \end{aligned}$$

For this purpose we define for $t \neq k\tau$

$$\begin{aligned} g(t) = X(t)x^*(t) - \int_{t}^{T} S'(s,t)C_1'C_1x^*(s)ds \\ - \sum_{T>j\tau>t} S'(j\tau,t)C_{1d}'C_{1d}x^*(j\tau) - S'(T,t)F'Fx^*(T). \end{aligned}$$

Then g is continuous except at $t = k\tau$ and has right and left limits at $t = k\tau$. We need to show $g(k\tau^+) = 0$, but we shall show $g(t) = 0$. First note that

$$g(T) = X(T)x^*(T) - F'Fx * (T) = 0.$$

For $t \neq k\tau$

$$\begin{aligned}
\dot{g}(t) &= \dot{X}(t)x^*(t) + X(t)\dot{x}^*(t) + C_1'C_1x^*(t) \\
&\quad + A'[\int_t^T S'(s,t)C_1'C_1x^*(s)ds \\
&\quad + \sum_{T>j\tau>t} S'(j\tau,t)C_{1d}'C_{1d}x^*(j\tau) + S'(T,t)F'Fx^*(T)] \\
&= -A'[X(t)x^*(t) - \int_t^T S'(s,t)C_1'C_1x^*(s)ds \\
&\quad - \sum_{T>j\tau>t} S'(j\tau,t)C_{1d}'C_{1d}x^*(j\tau) - S'(T,t)F'Fx^*(T)] \\
&= -A'g(t)
\end{aligned}$$

and

$$\begin{aligned}
g(k\tau^-) &= \lim_{t\uparrow k\tau}[X(t)x^*(t) - \int_t^T S'(s,t)C_1'C_1x^*(s)ds \\
&\quad - \sum_{T>j\tau>t} S'(j\tau,t)C_{1d}'C_{1d}x^*(j\tau) - S'(T,t)F'Fx^*(T)] \\
&= X(k\tau^-)x^*(k\tau) - \int_{k\tau}^T S'(s,k\tau^-)C_1'C_1x^*(s)ds \\
&\quad - \sum_{j=k}^N S'(j\tau,k\tau^-)C_{1d}'C_{1d}x^*(j\tau) - S'(T,k\tau^-)F'Fx^*(T) \\
&= [A_d'X(k\tau)A_d - (R_2'T_2^{-1}R_2)(k) \\
&\qquad + (F_1'VF_1)(k) + C_{1d}'C_{1d}]X(k\tau^-)x^*(k\tau) \\
&\quad - A_d'[\int_{k\tau}^T S'(s,k\tau^+)C_1'C_1x^*(s)ds \\
&\quad + \sum_{j=k+1}^N S'(j\tau,k\tau^+)C_{1d}'C_{1d}x^*(j\tau) + S'(T,k\tau^+)F'Fx^*(T)] \\
&\quad - S'(k\tau,k\tau^-)C_{1d}'C_{1d}x^*(k\tau) \\
&= A_d'[X(k\tau)x^*(k\tau^+) - \int_{k\tau}^T S'(s,k\tau^+)C_1'C_1x^*(s)ds \\
&\quad - \sum_{j=k+1}^N S'(j\tau,k\tau^+)C_{1d}'C_{1d}x^*(j\tau) - S'(T,k\tau^+)F'Fx^*(T)] \\
&= A_d'g(k\tau^+).
\end{aligned}$$

Hence $g(t) = 0$ and $g(k\tau^+) = g(k\tau^-) = 0$. This yields (4.79) and hence (4.77).

It remains to show the last equality in (4.78). From (4.76) we have

$$x_0' X(s) x_0 \le J(u,(\bar{w},\bar{w}_d);s,x_0) \le \sup_{(w,w_d)} J(u,(w,w_d);s,x_0) \text{ for any } u$$

which implies

$$x_0' X(s) x_0 \le \inf_u \sup_{(w,w_d)} J(u,(w,w_d);s,x_0).$$

But

$$\max_{(w,w_d)} J(\bar{u},(w,w_d);s,x_0) = x_0' X(s) x_0$$

and

$$x_0' X(s) x_0 = J(\bar{u},(w^*,w_d^*);s,x_0) = \inf_u \sup_{(w,w_d)} J(u,(w,w_d);s,x_0). \qquad \blacksquare$$

Next we shall show the existence of a solution to the Riccati equation (4.69)-(4.72) under the assumption that a γ-suboptimal controller exists.

Lemma 4.15 *There exists a nonnegative solution X to (4.69)-(4.72) and*

$$\max_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) = x_0' X(s) x_0.$$

Furthermore for the controller

$$u(k) = -(T_2^{-1}R_2)(k)x(k\tau) - (T_2^{-1}S)(k)w_d(k)$$

$\| \bar{G} \| < \gamma$ *holds.*

Proof. As in the proof of Lemma 2.13 we first establish the existence of X on the interval $[N\tau, T]$, i.e., the last subinterval. Then using the max-min game theory to the functional (4.61) and the jump equation in (4.59), we show the existence of $X(N\tau^-)$ satisfying (4.70) and (4.71). Next we show the existence of X on the interval $[(N-1)\tau, N\tau)$. The existence of $X(t)$ for all $t \in [s,T]$ will be established by repeating these arguments.

Step 1: Consider the functional

$$\begin{aligned} J(u,(w,w_d);\bar{s},x_0) &= J(w;\bar{s},x_0) \\ &= \int_{\bar{s}}^{T} [\| z_c(t) |^2 - \gamma^2 | w(t) |^2] dt + | Fx(T) |^2 \end{aligned}$$

subject to

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ x(\bar{s}) = x_0, \\ z_c &= Cx \end{aligned}$$

where $N\tau < \bar{s} \leq T$. Since $u = Ky$ is γ-suboptimal on $[s, T]$, it is also γ-suboptimal on $[\bar{s}, T]$ and by Lemma 2.13 there exists a nonnegative solution $X(t)$, $t \in [\bar{s}, T]$ to (4.69) and (4.72). We write this solution as X_T to show the dependence on T. We also have

$$\begin{aligned} \max_{(w,w_d)} \min_u J(u,(w,w_d);\bar{s},x_0) &= \max_w J(w;\bar{s},x_0) \\ &= x_0 X_T(\bar{s}) x_0. \end{aligned}$$

Step 2: We introduce the functional

$$\begin{aligned} J(u,(w,w_d);N\tau,x_0) &= \int_{N\tau}^T [\mid z_c(t) \mid^2 - \gamma^2 \mid w(t) \mid^2] dt \\ &\quad + \mid z_d(N) \mid^2 - \gamma^2 \mid w_d(N) \mid^2 + \mid Fx(T) \mid^2 \end{aligned}$$

for the system (4.59) on $[N\tau, T]$ with $x(N\tau) = x_0$. Then by (4.73) and the same arguments in the proof of Lemma 3.15, we have

$$\begin{aligned} & J(u,(w,w_d);N\tau,x_0) \\ = \; & x'(N\tau^+) X_T(N\tau) x(N\tau^+) - \gamma^2 \int_{N\tau}^T \mid w(t) - \frac{1}{\gamma^2} B_1' X_T(t) x(t) \mid^2 dt \\ & + \mid z_d(N) \mid^2 - \gamma^2 \mid w_d(N) \mid^2 \\ = \; & x_0'[(C_{1d}'C_{1d})(N) + A_d X_T(N\tau) A_d - (R_2 T_2^{-1} R_2)(N)] x_0 \\ & + \mid T_2^{\frac{1}{2}}[u + T_2^{-1}(S w_d + R_2 x)](N) \mid^2 - w_d'(N) V(N) w_d(N) \\ & + 2 w_d(N)(R_1 - S' T_2^{-1} R_2)(N) x_0 \\ & - \gamma^2 \int_{N\tau}^T \mid w(t) - \frac{1}{\gamma^2} B_1' X_T(t) x(t) \mid^2 dt. \end{aligned}$$

By Lemma 4.13

$$\max_{(w,w_d)} \min_u J(u,(w,w_d);N\tau,x_0) \leq a \mid x_0 \mid^2 \text{ for some } a > 0$$

and we obtain $V(N) > aI$ for some $a > 0$. Hence we can define $X(N\tau^-)$ by (4.71) and

$$\max_{\substack{(w(t),w_d(N)) \\ N\tau < t \leq T}} \min_{u(N)} J(u,(w,w_d);N\tau,x_0) = x_0' X(N\tau^-) x_0.$$

Step 3: Now we assume that $X_T(t)$, $t \in (N\tau, T]$ is well-defined and introduce the functional

$$\begin{aligned} J(u,(w,w_d);\bar{s},x_0) &= \int_{\bar{s}}^T [\mid z_c(t) \mid^2 - \gamma^2 \mid w(t) \mid^2] dt \\ &\quad + \mid z_d(N) \mid^2 - \gamma^2 \mid w_d(N) \mid^2 + \mid Fx(T) \mid^2 \end{aligned}$$

subject to the system (4.59) with $x(\bar{s}) = x_0$ on $[\bar{s}, T]$, $(N-1)\tau < \bar{s} \leq N\tau$. Then

$$\begin{aligned} & J(u,(w,w_d);\bar{s},x_0) \\ = & \int_{\bar{s}}^{N\tau} [| z_c(t) |^2 - \gamma^2 | w(t) |^2]dt + x'(N\tau)X_T(N\tau^-)x(N\tau) \\ & -\gamma^2 \int_{N\tau}^{T} | w(t) - \frac{1}{\gamma^2}B_1'X_T(t)x(t) |^2 \, dt \\ & + | T_2^{\frac{1}{2}}[u + T_2^{-1}(Sw_d + R_2x)](N) |^2 \\ & -\gamma^2 | \frac{1}{\gamma}[V^{\frac{1}{2}}(w_d - F_1x)](N) |^2 \end{aligned}$$

and

$$\begin{aligned} \max_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) &= \max_{\substack{w(t) \\ \bar{s}\leq t\leq N\tau}} [\int_{\bar{s}}^{N\tau} [| z_c(t) |^2 - \gamma^2 | w |^2]dt \\ & \qquad + x'(N\tau)X(N\tau^-)x(N\tau)]. \end{aligned}$$

As in the proof of **Step 1** we can show that there exists a nonnegative solution $X(t)$, $t \in [s, N\tau]$ to

$$\begin{aligned} -\dot{X} &= A'X + XA + C'C + \frac{1}{\gamma^2}XB_1B_1'X, \\ X(N\tau) &= X_T(N\tau^-). \end{aligned}$$

Continuing in this way we can show the existence of a nonnegative solution to (4.69)-(4.72). The rest of the proof is similar to that of Lemma 4.14. ∎

Summing up we have the following result.

Theorem 4.12 *Assume* **J1'**. *Suppose that there exists a γ-suboptimal controller $u = Ky$ on $[s,T]$ for the system (4.59). Then there exists a nonnegative solution $X(t)$, $t \in [s,T]$ to the Riccati equation with jumps (4.69)-(4.72). Moreover*

$$\begin{aligned} \max_{(w,w_d)} \min_u J(u,(w,w_d);s,x_0) &= J(\bar{u},(w^*,w_d^*);s,x_0) \\ &= J(\bar{u},(\bar{w},\bar{w}_d);s,x_0) \\ &= J(u^*,(w^*,w_d^*);s,x_0) \\ &= x_0'X(s)x_0 = \inf_u \sup_{(w,w_d)} J(u,w;s,x_0). \end{aligned}$$

Consider the backward system

$$
\begin{aligned}
-\dot{\tilde{x}} &= A'\tilde{x} + C_1'\tilde{w}, \; k\tau < t < (k+1)\tau, \\
\tilde{x}(k\tau^-) &= A_d'\tilde{x}(k\tau) + C_{1d}'\tilde{w}_d(k) + C_2'\tilde{u}(k), \\
\tilde{z} &= \begin{bmatrix} \tilde{z}_c \\ \tilde{z}_d(k) \end{bmatrix} = \begin{bmatrix} B_1'\tilde{x} \\ B_{1d}'\tilde{x}(k\tau) + D_{21}'\tilde{u}(k) \end{bmatrix}, \\
\tilde{y}(k) &= B_2'\tilde{x}(k\tau) + D_{12}'\tilde{w}_d(k), \\
\tilde{z}_1 &= H'\tilde{x}(s)
\end{aligned} \tag{4.80}
$$

which is the adjoint system of (4.59) with $x(s) = Hh$. For this system, we introduce the controller $\tilde{u} = K\tilde{y}$ of the form

$$
\begin{aligned}
-\dot{\hat{x}} &= \hat{A}'(t)\hat{x}, \; k\tau < t < (k+1)\tau, \\
\hat{x}(k\tau^+) &= \hat{A}_d(k)'\hat{x}(k\tau) + \hat{C}'(k)\tilde{y}(k), \\
\tilde{u}(k) &= \hat{B}(k)\hat{x}(k\tau) + \hat{D}'(k)\tilde{y}(k)
\end{aligned} \tag{4.81}
$$

which is also the adjoint system of (4.45).

Corollary 4.9 *Assume*

$$\mathbf{J2'}\text{: } D_{21}B_{1d}' = 0, \; D_{21}D_{21}' = I.$$

Suppose that there exists a γ-suboptimal controller $\tilde{u} = K\tilde{y}$ on $[s,T]$ for the system (4.80). Then there exists a nonnegative solution $Y(t)$, $t \in [s,T]$ to the Riccati equation with jumps

$$
\dot{Y} = AY + YA + B_1B_1' + \frac{1}{\gamma^2}YC_1'C_1Y, \quad k\tau < t < (k+1)\tau, \tag{4.82}
$$

$$
V_Y(k) > aI \text{ for some } a > 0, \tag{4.83}
$$

$$
Y(k\tau^+) = A_dY(k\tau)A_d' + B_{1d}B_{1d}' - (R_{2Y}'T_{2Y}^{-1}R_{2Y})(k) + (F_{1Y}'V_YF_{1Y})(k), \tag{4.84}
$$

$$
Y(s) = HH' \tag{4.85}
$$

where

$T_{1Y}(k) = \gamma^2 I - C_{1d}Y(k\tau)C_{1d}'$,
$R_{1Y}(k) = C_{1d}Y(k\tau)A_d'$,
$S_Y(k) = C_2X(k\tau)C_{1d}'$,
$F_{1Y}(k) = [V_Y^{-1}(R_{1Y} - S_Y'T_{2Y}^{-1}R_{2Y})](k)$,
$F_{2Y}(k) = -[T_{2Y}^{-1}(R_{2Y} + S_YF_{1Y})](k)$.

$T_{2Y}(k) = I + C_2Y(k\tau)C_2'$,
$R_{2Y}(k) = C_2Y(k\tau)A_d'$,
$V_Y(k) = [T_{1Y} + S_Y'T_{2Y}^{-1}S_Y](k)$,

The Infinite Horizon Problem

Consider the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_{1d} w_d(k) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ C_{1d} x(k\tau) + D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{4.86}$$

with $x(s) = x_0$, $(k_s - 1)\tau < s < k_s \tau$. We assume **J1'** and that

$$([A, A_d], [0, B_2], [C_1, C_{1d}]) \text{ is stabilizable and detectable.}$$

As in the finite horizon problem we assume the existence of a controller $u = Ky$ of the form (4.45) with property

$$\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|^2_{L^2 \times l^2} \le d^2 \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|^2_{L^2 \times l^2} \ \text{ for some } 0 < d < \gamma \tag{4.87}$$

and study its consequence to the quadratic game defined by the functional

$$\begin{aligned} J(u, (w, w_d); s, x_0) &= \int_s^\infty [| \, z_c(t) \, |^2 - \gamma^2 \, | \, w(t) \, |^2] dt \\ &\quad + \sum_{k=k_s}^{\infty} [| \, z_d(k) \, |^2 - \gamma^2 \, | \, w_d(k) \, |^2]. \end{aligned}$$

Note that such a controller is IO-stabilizing with disturbance attenuation γ (IO-γ-suboptimal) and we call it γ-suboptimal if it is internally stabilizing. We also consider the finite horizon problem associated with

$$\begin{aligned} J_T(u, (w, w_d); s, x_0) &= \int_s^T [| \, z_c(t) \, |^2 - \gamma^2 \, | \, w(t) \, |^2] dt \\ &\quad + \sum_{k=k_s}^{N} [| \, z_d(k) \, |^2 - \gamma^2 \, | \, w_d(k) \, |^2]. \end{aligned}$$

Note that if a controller $u = Ky$ of the form (4.45) is IO-γ-suboptimal, it is also γ-suboptimal on any $[s, T]$. Since $([A, A_d], [0, B_2])$ is stabilizable, Ψ_{11s} and Ψ_{21s} in (4.64) are uniformly bounded. Then by Lemma 4.13 and Theorem 4.12 we have the following:

Lemma 4.16 *There exists a unique nonnegative solution X_T of the Riccati equation with jumps (4.69)-(4.71) with $X_T(T) = 0$ on any interval $[s, T]$ such that*

$$| \, X_T(t) \, | \le c \text{ independent of } s \le t \le T < \infty.$$

Lemma 4.17 *For each $t \geq s$, $X_T(t)$ is monotone increasing in T.*

Proof. Let $L \leq T$ and we shall show $X_L(s) \leq X_T(s)$. This follows from

$$\begin{aligned} x_0' X_L(s) x_0 &= J_L(\bar{u}_L, (\bar{w}_L, \bar{w}_{dL}); s, x_0) \\ &\leq J_L(\tilde{u}_T, (\bar{w}_L, \bar{w}_{dL}); s, x_0) \\ &\leq J_T(\bar{u}_T, (\hat{w}_T, \hat{w}_{dT}); s, x_0) \\ &\leq J_T(\bar{u}_T, (\bar{w}_T, \bar{w}_{dT}); s, x_0) = x_0' X_T(s) x_0 \end{aligned}$$

where $\tilde{u}_T$ is the restriction of $\bar{u}_T$ on $[s, L]$ and $(\hat{w}_T, \hat{w}_{dT})$ is the extension of $(\bar{w}_L, \bar{w}_{dL})$ to $[s, N]$ by zero. Changing the initial time, we also show $X_L(t) \leq X_T(t)$. ∎

Lemma 4.18 *There exists a τ-periodic nonnegative solution $X(t)$, $t \in [s, \infty)$ to (4.69)-(4.71).*

Proof. Let X_T be a nonnegative solution to (4.69)-(4.71) with $X_T(T) = 0$. In view of Lemmas 4.16 and 4.17, X_T is uniformly bounded and monotone increasing in T and hence $X_T(t)$ converges to a limit $X(t)$. Then as in the proof of Lemma 3.19, we can show $V(k) > aI$ for some $a > 0$ and hence $X(t)$ is a nonnegative solution to (4.69)-(4.71). The τ-periodicity of $X(t)$ follows from the proof of Lemma 4.2. ∎

Next we shall show the stabilizing property of the solution.

Lemma 4.19 $(A + \frac{1}{\gamma^2} B_1 B_1' X, A_d + B_{1d} F_1 + B_2 F_2)$ *is exponentially stable.*

Proof. Let x_T^* be the solution of

$$\begin{aligned} \dot{x} &= (A + \frac{1}{\gamma^2} B_1 B_1' X_T) x, \; x(s) = x_0, \; k\tau < t < k+1)\tau, \\ x(k\tau^+) &= (A_d + B_{1d} F_{1T} + B_2 F_{2T}) x(k\tau) \end{aligned} \quad (4.88)$$

where F_{1T} and F_{2T} show the dependency on T of F_1 and F_2, respectively. Then for any interval $[s, L]$, the solution x_T^* converges to the solution $\bar{x}$ of

$$\begin{aligned} \dot{\bar{x}} &= (A + \frac{1}{\gamma^2} B_1 B_1' X) \bar{x}, \; \bar{x}(s) = x_0, \; k\tau < t < (k+1)\tau, \\ \bar{x}(k\tau^+) &= (A_d + B_{1d} F_1 + B_2 F_2) \bar{x}(k\tau). \end{aligned}$$

We can rewrite (4.88) as

$$\begin{aligned} \dot{x} &= (A - JC_1) x + JC_1 x_T^* + B_1 w_T^*, \; x(s) = x_0, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d - J_d C_{1d}) x(k\tau) + J_d C_{1d} x_T^*(k\tau) \\ &\quad + B_{1d} w_{dT}^*(k) + B_2 u_T^*(k) \end{aligned} \quad (4.89)$$

where J, J_d are chosen such that $(A + JC_1, A_d + J_dC_{1d})$ is exponentially stable. The solution of (4.89) coincides with x_T^* on $[s,T]$. We extend it to $[s,\infty)$ by the homogenous equation of (4.89). By Lemma 4.14

$$\| \begin{pmatrix} C_1 x_T^* \\ C_{1d} x_T^* \end{pmatrix} \|_{L^2\times l^2}, \ \| \begin{pmatrix} w_T^* \\ w_{dt}^* \end{pmatrix} \|_{L^2\times l^2}, \ \| u \|_{l^2} \leq a \mid x_0 \mid \ \text{ for some } a > 0$$

and $(C_1 x_T^*, C_{1d} x_T^*)$, (w_T^*, w_{dT}^*), u_T^* converges weakly to

$$\begin{aligned}
(\tilde{h}, \tilde{h}_d) &\in L^2(s,\infty; \mathbf{R}^{p_1}) \times l^2(k_s,\infty; \mathbf{R}^{p_{1d}}), \\
(\tilde{w}, \tilde{w}_d) &\in L^2(s,\infty; \mathbf{R}^{m_1}) \times l^2(k_s,\infty; \mathbf{R}^{m_{1d}}), \\
\tilde{u} &\in l^2(k_s,\infty; \mathbf{R}^{m_2})
\end{aligned}$$

respectively along a subsequence $T \to \infty$. Let $\tilde{x}$ be the solution of

$$\begin{aligned}
\dot{x} &= (A - JC_1)x + J\tilde{h} + B_1\tilde{w}, \ x(s) = x_0, \ k\tau < t < (k+1)\tau, \\
x(k\tau^+) &= (A_d - J_dC_{1d})x(k\tau) + J_d\tilde{h}_d(k) + B_{1d}\tilde{w}_d(k) + B_2\tilde{u}(k).
\end{aligned}$$

Since the restriction of $C_1 x_T^*$, etc on any interval $[s,L]$ converge weakly to those of $\tilde{h}$, etc, we can identify $\tilde{x}$ and $\bar{x}$ on $[s,L]$. Since $(A - JC_1, A_d - J_dC_{1d})$ is exponentially stable, $\tilde{x} \in L^2(s,\infty;\mathbf{R}^n)$. Hence $\bar{x} \in L^2(s,\infty;\mathbf{R}^n)$ for each x_0 and $\| \bar{x} \|_{L^2} \leq a \mid x \mid$ for some $a > 0$ independent of x_0. Hence by Proposition 4.2, $(A + \frac{1}{\gamma^2}B_1B_1'X, A_d + B_{1d}F_1 + B_2F_2)$ is exponentially stable. ∎

Define feedback laws

$$\begin{aligned}
\bar{w}(t) &= \frac{1}{\gamma^2}B_1'X(t)x(t), \\
\bar{w}_d(k) &= F_1 x(k\tau), \\
\bar{u}(k) &= -T_2^{-1}R_2 x(k\tau) - T_2^{-1}Sw_d(k).
\end{aligned} \tag{4.90}$$

Let x^* be the solution of (4.86) corresponding to (4.90) and let

$$\begin{aligned}
w^*(t) &= \frac{1}{\gamma^2}B_1'X(t)x^*(t), \\
w_d^*(k) &= F_1 x^*(k\tau), \\
u^*(k) &= F_2 x^*(k\tau).
\end{aligned} \tag{4.91}$$

First we show that the feedback law $\bar{u}$ is stabilizing.

Lemma 4.20 *Suppose X is a τ-periodic nonnegative solution to (4.69)-(4.71) such that $(A + \frac{1}{\gamma^2}B_1B_1'X, A_d + B_{1d}F_1 + B_2F_2)$ is exponentially stable. Then $(A, A_d - B_2T_2^{-1}R_2)$ is exponentially stable.*

Proof. Since $(A + \frac{1}{\gamma^2}B_1B_1'X, A_d + B_{1d}F_1 + B_2F_2)$ is exponentially stable and

$$\begin{aligned} A + \frac{1}{\gamma^2}B_1B_1'X &= A + \frac{1}{\gamma}B_1(\frac{1}{\gamma}B_1'X), \\ A_d + B_{1d}F_1 + B_2F_2 &= (A_d - B_2T_2^{-1}R_2) + (B_{1d} - B_2T_2^{-1}S)F_1, \end{aligned}$$

$([A, A_d - B_2T_2^{-1}R_2], [\frac{1}{\gamma}B_1'X, F_1])$ is detectable and so is

$$\left([A, A_d - B_2T_2^{-1}R_2], \left[\begin{bmatrix} \frac{1}{\gamma}B_1'X_h \\ C_1 \end{bmatrix}, \begin{bmatrix} V^{\frac{1}{2}}F_1 \\ T_2^{-1}R_2 \\ C_{1d} \end{bmatrix} \right] \right).$$

Since

$$\begin{aligned} -\dot{X} &= A'X + XA + \begin{bmatrix} \frac{1}{\gamma}B_1'X \\ C_1 \end{bmatrix}' \begin{bmatrix} \frac{1}{\gamma}B_1'X \\ C_1 \end{bmatrix}, \; k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= (A_d - B_2T_2^{-1}R_2)'X(k\tau)(A_d - B_2T_2^{-1}R_2) \\ &\quad + \begin{bmatrix} V^{\frac{1}{2}}F_1 \\ T_2^{-1}R_2 \\ C_{1d} \end{bmatrix}' \begin{bmatrix} V^{\frac{1}{2}}F_1 \\ T_2^{-1}R_2 \\ C_{1d} \end{bmatrix}, \end{aligned}$$

$(A, A_d - B_2T_2^{-1}R_2)$ is exponentially stable by Proposition 4.5. ∎

Let **FI** be the set of stabilizing feedback laws of the form $u = K_1x + K_2w_d$. As Lemma 4.14, we shall show

$$\begin{aligned} \sup_{(w,w_d)} \inf_{u \in \mathbf{FI}} J(u, (w, w_d); s, x_0) &= J(\bar{u}, (w^*, w_d^*); s, x_0) \\ &= J(\bar{u}, (\bar{w}, \bar{w}_d); s, x_0) \qquad (4.92) \\ &= J(u^*, (w^*, w_d^*); s, x_0) \\ &= x_0'X(s)x_0 \\ &= \inf_{u \in \mathbf{FI}} \sup_{(w,w_d)} J(u, (w, w_d); s, x_0). \end{aligned}$$

Note that

$$\begin{aligned} \inf_{u \in \mathbf{FI}} \sup_{(w,w_d)} J(u, (w, w_d); s, x_0) &\le \sup_{(w,w_d)} J(\bar{u}, (w, w_d); s, x_0) \\ &= J(\bar{u}, (w^*, w_d^*); s, x_0) = x_0'X(s)x_0. \end{aligned}$$

It suffices to show

$$x_0'X(s)x_0 \le J(\bar{u}, (w^*, w_d^*); s, x_0) = \inf_{u \in \mathbf{FI}} J(u, (w^*, w_d^*); s, x_0). \qquad (4.93)$$

In fact this implies

$$x_0'X(s)x_0 = \inf_{u \in \mathbf{FI}} J(u, (w^*, w_d^*); s, x_0) \le \sup_{(w,w_d)} \inf_{u \in \mathbf{FI}} J(u, (w, w_d); s, x_0)$$

and (4.92) follows. To show (4.93), we proceed as in the proof of Lemma 4.14. Consider

$$\begin{aligned} \dot{x} &= Ax + B_1 w^*, \; x(s) = x_0, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_{1d} w_d^*(k) + B_2 u(k). \end{aligned}$$

Then $e = x - x^*$ satisfies

$$\begin{aligned} \dot{e} &= Ae, \; e(s) = 0, \; k\tau < t < (k+1)\tau, \\ e(k\tau^+) &= (A_d - B_2 T_2^{-1} R_2) e(k\tau) + B_2 v(k) \end{aligned}$$

where $v(k) = u(k) + T_2^{-1} R_2 x(k\tau) + T_2^{-1} S w_d(k)$ and $J(u, (w^*, w_d^*); x_0)$ can be rewritten as

$$\begin{aligned} \tilde{J}(v, (w^*, w_d^*); s, x_0) &= \left\| \begin{pmatrix} C_1(e + x^*) \\ C_{1d}(e + x^*) \end{pmatrix} \right\|_{L^2 \times l^2}^2 - \gamma^2 \left\| \begin{pmatrix} w^* \\ w_d^* \end{pmatrix} \right\|_{L^2 \times l^2}^2 \\ &\quad + \| v - T_2^{-1} R_2 x - T_2^{-1} S w_d^* \|_{l^2}^2 \\ &= \left\| \begin{pmatrix} C_1(\mathbf{H}v + x^*) \\ C_{1d}(\mathbf{H}_d v + x^*) \end{pmatrix} \right\|_{L^2 \times l^2}^2 - \gamma^2 \left\| \begin{pmatrix} w^* \\ w_d^* \end{pmatrix} \right\|_{L^2 \times l^2}^2 \\ &\quad + \| v - T_2^{-1} R_2 (\mathbf{H}_d v + x^*) - T_2^{-1} S w_d^* \|_{l^2}^2 \end{aligned}$$

where

$$\begin{aligned} (\mathbf{H}v)(t) &= \sum_{j=k_s}^{k} S_F(t, j\tau^+) B_2 v(j), \; k\tau < t < (k+1)\tau, \\ &= e^{A(t-k\tau)} \sum_{j=k_s}^{k} [(A_d - B_2 T_2^{-1} R_2) e^{Ah}]^{k-j} B_2 v(j), \\ (\mathbf{H}_d v)(k\tau) &= \sum_{j=k_s}^{k-1} S_F(k\tau, j\tau^+) B_2 v(j) \\ &= e^{A\tau} \sum_{j=k_s}^{k-1} [(A_d - B_2 T_2^{-1} R_2) e^{A\tau}]^{k-1-j} B_2 v(j) \end{aligned}$$

and $S_F(\cdot, \cdot)$ is the state transition matrix associated with $(A, A_d - B_2 T_2^{-1} R_2)$. The unique minimizing element v of $\tilde{J}$ given by the solution of

$$\begin{aligned} &\mathbf{H}^* C_1' C_1 (\mathbf{H}v + x^*) + \mathbf{H}_d^* C_{1d}' C_{1d} (\mathbf{H}_d v + x^*) \\ &\quad + (I - T_2^{-1} R_2 \mathbf{H}_d)^* [v - T_2^{-1} R_2 (\mathbf{H}_d v + x^*) - T_2^{-1} S w_d^*] = 0. \end{aligned}$$

We shall show that $v = 0$ is the solution. This follows if

$$\mathbf{H}^* C_1' C_1 x^* + \mathbf{H}_d^* C_{1d}' C_{1d} x^* - (I - T_2^{-1} R_2 \mathbf{H}_d)^* (T_2^{-1} R_2 x^* - T_2^{-1} S w_d^*) = 0.$$

Since $(A, A_d - B_2T_2^{-1}R_2)$ is exponentially stable, we have for $\tilde{h} \in L^2(s,\infty;\mathbf{R}^n)$ and $\tilde{h}_d \in l^2(k_s,\infty;\mathbf{R}^n)$

$$\begin{aligned}(\mathbf{H}^*\tilde{h})(k) &= B_2' \int_{k\tau}^{\infty} S_F'(t,k\tau^+)\tilde{h}(t)dt,\\ (\mathbf{H}_d^*\tilde{h}_d)(k) &= B_2' \sum_{j=k+1}^{\infty} S_F'(jh,k\tau^+)\tilde{h}_d(j).\end{aligned}$$

Then as the proofs of Lemmas 2.12, 3.14 and 4.14 we have

$$\mathbf{H}^*C_1'C_1x^* + \mathbf{H}_d^*C_{1d}'C_{1d}x^* - T_2^{-1}R_2x^* - T_2^{-1}Sw_d^* = 0.$$

Hence we have $u = \bar{u}$. Thus the value of the game $J(u,(w,w_d);s,x_0)$ over $\mathbf{FI} \times L^2(s,\infty;\mathbf{R}^{m_1}) \times l^2(k_s,\infty;\mathbf{R}^{m_{1d}})$ exists.

Summing up we have the following.

Theorem 4.13 *Assume* **J1'** *and* $([A,A_d],[0,B_2],[C_1,C_{1d}])$ *is stabilizable and detectable. Suppose an IO-stabilizing controller with property (4.87) exists. Then there exists a* τ*-periodic nonnegative solution to (4.69)-(4.71) such that* $(A + \frac{1}{\gamma^2}B_1B_1'X, A_d + B_{1d}F_1 + B_2F_2)$ *is exponentially stable. Moreover* $\bar{u} \in \mathbf{FI}$ *and*

$$\begin{aligned}\sup_{(w,w_d)} \inf_{u\in\mathbf{FI}} J(u,(w,w_d);s,x_0) &= J(\bar{u},(w^*,w_d^*);x_0)\\ &= J(\bar{u},(\bar{w},\bar{w}_d);s,x_0)\\ &= J(u^*,(w^*,w_d^*);s,x_0)\\ &= x_0'X(s)x_0\\ &= \inf_{u\in\mathbf{FI}} \sup_{(w,w_d)} J(u,(w,w_d);s,x_0).\end{aligned}$$

Corollary 4.10 *Consider the system (4.80). Assume* **J2'** *and*

$$([A,A_d],[B_1,B_{1d}],[0,C_2]) \text{ is stabilizable and detectable.}$$

Suppose an IO-stabilizing controller of the form (4.81) with property

$$|\,\tilde{z}_1\,|^2 + \left\| \begin{pmatrix} \tilde{z}_c \\ \tilde{z}_d \end{pmatrix} \right\|_{L^2\times l^2}^2 \leq d^2 \left\| \begin{pmatrix} \tilde{w} \\ \tilde{w}_d \end{pmatrix} \right\|_{L^2\times l^2}^2 \quad \text{for some } 0 < d < \gamma$$

exists. Then there exists a bounded nonnegative stabilizing solution ($(A + \frac{1}{\gamma^2}YC_1'C_1, A_d + F_{1Y}'C_{1d} + F_{2Y}'C_2)$ *is exponentially stable) to (4.82)-(4.85). Moreover, the* $\lim_{n\to\infty} Y(t+n\tau)$ *exists (denoted by* $Y_\tau(t)$*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.82)-(4.84).*

4.2.3 Backward Systems

To prove our main results, we use the FI- and DF problems for backward systems.

Full Information Problem

Consider first the FI problem given by the backward system $\mathbf{G}_{FIj}$

$$\begin{aligned} -\dot{x} &= Ax, \ k\tau < t < (k+1)\tau, \\ x(k\tau^-) &= A_d x(k\tau) + B_1 w(k) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= \begin{bmatrix} x(k\tau) \\ w(k) \end{bmatrix}, \\ z_1 &= Fx(t_0), \ 0 < t_0 \leq \tau \end{aligned} \tag{4.94}$$

with $x(T) = 0$, $N\tau \leq T < (N+1)\tau$ and a controller $u = Ky$ of the form

$$\begin{aligned} -\dot{p} &= \hat{A}(t)p, \ k\tau < t < (k+1)\tau, \\ p(k\tau^-) &= \hat{A}_d(k)p(k\tau) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)p(k\tau) + \hat{D}(k)y(k) \end{aligned} \tag{4.95}$$

where all matrices are uniformly bounded and we assume **J1**. The solution of this problem is needed to solve the H_∞-control problems for the system $\mathbf{G}_j$. Moreover, the filtering problem turns out to be the dual of this problem.

First we consider the finite horizon problem. For each controller, define the input-output operator G by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw.$$

To give the solution of this FI-problem, we need the following Riccati equation with jumps:

$$\dot{P} = A'P + PA + C_1'C_1, \ k\tau < t < (k+1)\tau, \tag{4.96}$$

$$V(k) > aI, \text{ for some } a > 0, \tag{4.97}$$

$$P(k\tau^+) = A_d'P(k\tau)A_d - (R_2'T_2^{-1}R_2)(k) + (F_1'VF_1)(k), \tag{4.98}$$

$$P(t_0) = F'F \tag{4.99}$$

where

$$\begin{aligned} &T_1(k) = \gamma^2 I - B_1'P(k\tau)B_1, && T_2(k) = I + B_2'P(k\tau)B_2, \\ &R_1(k) = B_1'P(k\tau)A_d, && R_2(k) = B_2'P(k\tau)A_d, \\ &S(k) = B_2'P(k\tau)B_1, && V(k) = [T_1 + S'T_2^{-1}S](k), \\ &F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k), && F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k). \end{aligned}$$

Let P be the solution of (4.96)-(4.99). Define the set of controllers $v = Qr$ of the form (4.95)

$$Q_\gamma = \{Q : Q \in \mathcal{L}(l^2(1,N;\mathbf{R}^{m_1}); l^2(1,N;\mathbf{R}^{m_2})) : \| Q \| < \gamma\}.$$

Then we have the following.

Theorem 4.14 *Assume* **J1**.
(a) There exists a γ-suboptimal controller $u = Ky$ of the form (4.95) if and only if there exists a nonnegative solution $P(t)$, $t \in [t_0, T]$ to (4.96)-(4.99).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} u(k) &= -(T_2^{-1}R_2)(k)x(k\tau) - (T_2^{-1}S)(k)w(k) \\ &\quad + T_2^{-\frac{1}{2}}(k)[Q(\frac{1}{\gamma}V^{\frac{1}{2}}(w - F_1 x))](k), \; Q \in Q_\gamma. \end{aligned} \tag{4.100}$$

Proof. Suppose that $u = Ky$ is γ-suboptimal. Then by Corollary 4.9, we obtain a nonnegative solution of (4.96)-(4.99).

To show the sufficiency of (a) and the characterization of (b), we need two lemmas below. As in the continuous-time and discrete-time FI problems, we consider

$$\begin{aligned} -\dot{x} &= Ax, \; k\tau < t < (k+1)\tau, \\ x(k\tau^-) &= A_{dI}(k)x(k\tau) + B_{1I}(k)w(k) + B_{2I}(k)v(k), \\ z &= \begin{bmatrix} C_1 x \\ C_{1d}(k)x(k\tau) + D_{11I}(k)w(k) + D_{12I}(k)v(k) \end{bmatrix}, \\ r(k) &= \frac{1}{\gamma}V^{\frac{1}{2}}(k)[w(k) - F_1(k)x(k\tau)], \\ z_1 &= Fx(t_0) \end{aligned} \tag{4.101}$$

and

$$\begin{aligned} -\dot{\bar{x}} &= A\bar{x}, \; k\tau < t < (k+1)\tau, \\ \bar{x}(k\tau^-) &= A_{dX}(k)\bar{x}(k\tau) + B_{1X}(k)r(k) + B_2 u(k), \\ v(k) &= C_{1X}(k)\bar{x}(k\tau) + D_{11X}(k)r(k) + T_2^{\frac{1}{2}}(k)u(k), \\ y(k) &= \begin{bmatrix} \bar{x}(k\tau) \\ F_1(k)\bar{x}(k\tau) + \gamma V^{-\frac{1}{2}}(k)r(k) \end{bmatrix} \end{aligned} \tag{4.102}$$

where

$$\begin{aligned} &A_{dI}(k) = (A_d - B_2 T_2^{-1}R_2)(k), && B_{1I}(k) = (B_1 - B_2 T_2^{-1}S)(k), \\ &B_{2I}(k) = B_2 T_2^{-\frac{1}{2}}(k), && C_{1dI}(k) = -D_{12}(T_2^{-1}R_2)(k), \\ &D_{11I}(k) = -D_{12}(T_2^{-1}S)(k)w(k), && D_{12I}(k) = D_{12}T_2^{-\frac{1}{2}}(k), \\ &A_{dX}(k) = (A_d + B_1 F_1)(k), && B_{1X}(k) = \gamma B_1 V^{-\frac{1}{2}}(k), \\ &C_{1X}(k) = [T_2^{-\frac{1}{2}}(R_2 + SF_1)](k), && D_{11X}(k) = \gamma(T_2^{-\frac{1}{2}}SV^{-\frac{1}{2}})(k). \end{aligned}$$

Then we have the following.

Lemma 4.21 *Let P be the solution of (4.96)-(4.99).*
(a) For the system (4.101), the following holds:

$$| z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|^2_{L^2 \times l^2} = \gamma^2 \| w_d \|^2_{l^2} + \| v \|^2_{l^2} - \gamma^2 \| r \|^2_{l^2} .$$

(b) The system $\mathbf{G}_{FIj}$ with a controller $u = Ky$ is equivalent to the interconnection of (4.101) and the feedback system (4.102) with $u = Ky$.

Proof. By direct calculation, we have

$$\begin{aligned} & x'(T)P(T)x(T) - x'(t_0)P(t_0)x(t_0) \\ & \quad = \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|^2_{L^2 \times l^2} - \gamma^2 \| w \|^2_{l^2} - \| v \|^2_{l^2} + \gamma^2 \| r \|^2_{l^2} . \end{aligned}$$

Since $x(T) = 0$ and $P(t_0) = F'F$, we obtain (a). The rest of the proof is similar to the proof of Lemma 2.24. ∎

Now introduce the feedback $v = Qr$ to (4.101) where Q is of the form (4.95).

Lemma 4.22 *Let G be the input-output operator of the closed-loop system (4.101) and $v = Qr$. Then $\| G \| < \gamma$ if and only if $\| Q \| < \gamma$.*

Proof. Similar to the proof of Lemma 2.25. ∎

We are now ready to complete the proof of Theorem 4.14. We note that $u(k)$ given by (4.100) is γ-suboptimal by Lemma 4.22. Now let $u = Ky$ be an arbitrary γ-suboptimal controller. Let Q be the input-output operator of the closed-loop system (4.102) with $u = Ky$. Then Q is of the form (4.95) and by Lemma 4.22, $Q \in Q_\gamma$. Hence $u = Ky$ is equivalent to

$$\begin{aligned} u(k) &= -[T_2^{-1}(R_2 + SF_1)]\bar{x}(k) + \gamma T_2^{-1} S V^{-\frac{1}{2}} r(k) + T_2^{-\frac{1}{2}} v(k) \\ &= -T_2^{-1} R_2 \bar{x}(k) + T_2^{-1} S w(k) + T_2^{-\frac{1}{2}} Q[\frac{1}{\gamma} V^{\frac{1}{2}} (w_d - F_1 x)] \end{aligned}$$

which implies (b) and the sufficiency of (a). ∎

Next we consider the infinite horizon case. Consider the system $\mathbf{G}_{FIj}$ on $[t_0, \infty)$ with the assumption **J1**. We further assume

J5 : $([A, A_d], [0, B_2], [C_1, 0])$ is stabilizable and detectable.

For each IO-stabilizing controller, we can define the input-output operator by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = Gw.$$

Theorem 4.15 *Assume* **J1** *and* **J5**.
(a) There exists an IO-stabilizing controller $u = Ky$ *of the form (4.95) on* $[t_0, \infty)$ *such that* $\| G \| < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution to (4.96)-(4.99).*
(b) In this case the set of all γ*-suboptimal controllers is given by*

$$\begin{aligned} u(k) &= -(T_2^{-1}R_2)(k)x(k\tau) - (T_2^{-1}S)(k)w(k) \\ &\quad + T_2^{-\frac{1}{2}}(k)[Q(\frac{1}{\gamma}V^{\frac{1}{2}}(w - F_1 x))](k), \ Q \in Q_\gamma \end{aligned} \tag{4.103}$$

where

$$Q_\gamma = \{Q : Q \in \mathcal{L}(l^2(1,\infty;\mathbf{R}^{m_1}); l^2(1,\infty;\mathbf{R}^{m_2})) :\| Q \| < \gamma\}.$$

In particular, the set of all γ*-suboptimal controller is given by (4.103) with* Q *internally stable.*
Moreover, the $\lim_{n\to\infty} P(t + n\tau)$ *exists (denoted by* $P_\tau(t)$*) and* P_τ *is a* τ*-periodic nonnegative stabilizng solution of (4.96)-(4.98).*

Proof. **(i) Necessity of (a)** Suppose that there exists an IO-stabilizing controller $u = Ky$ such that $\| G \| < \gamma$. Then under the assumptions **J1** and **J5**, we obtain a nonnegative solution of (4.96)-(4.99) by Corollary 4.10.

To show the sufficiency of (a) and the characterization of (b), we need two lemmas below. Consider the systems (4.101) and (4.102) on $[t_0, \infty)$. Note that $(A, A_d - B_2T_2^{-1}R_2)$ is exponentially stable by Lemma 4.20. Then as in Lemma 4.21, we have the following results.

Lemma 4.23 *Let* P *be the solution of (4.96)-(4.99).*
(a) For the system (4.101), the following holds:

$$| z_1 |^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2\times l^2} = \gamma^2 \| w \|^2_{l^2} + \| v \|^2_{l^2} - \gamma^2 \| r \|^2_{l^2} .$$

(b) The system $\mathbf{G}_{FIj}$ *with a controller* $u = Ky$ *is equivalent to the interconnection of (4.101) and the feedback system (4.102) with* $u = Ky$.

Proof. As in the proof of Lemma 4.21, we have

$$\begin{aligned} & x'(T)P(T)x(T) - x'(t_0)P(t_0)x(t_0) \\ = & \int_{t_0}^{T} | z_c(t) |^2 \, dt + \sum_{k=1}^{N} [| z_d(k) |^2 - \gamma^2 | w(k) |^2 - | v(k) |^2 + \gamma^2 | r(k) |^2]. \end{aligned}$$

Since $x(T) = 0$, we let T tend to ∞ to obtain (a). The rest of the proof is similar to the proof of Lemma 2.24. ∎

Now introduce the feedback $v = Qr$ to (4.101) where Q is of the form (4.95).

Lemma 4.24 *Let G be the input-output operator of the closed-loop system (4.101) and $v = Qr$. Then $\| G \| < \gamma$ if and only if Q is internally stable and $\| Q \| < \gamma$.*

Proof. Similarly to the proof of Lemma 2.26. ∎

We are now ready to complete the proof of Theorem 4.15. We note that $u(k)$ given by (4.100) is γ-suboptimal by Lemma 4.24. Now let $u = Ky$ be an arbitrary γ-suboptimal controller. Let Q be the input-output operator of the closed-loop system (4.102) with $u = Ky$. Then Q is of the form (4.95) and by Lemma 4.24, $Q \in \mathcal{Q}_\gamma$. Hence $u = Ky$ is equivalent to

$$\begin{aligned} u(k) &= -T_2^{-1}(R_2 + SF_1)\bar{x}(k) + \gamma T_2^{-1}SV^{-\frac{1}{2}}r(k) + T_2^{-\frac{1}{2}}v(k) \\ &= -T_2^{-1}R_2\bar{x}(k) + T_2^{-1}Sw(k) + T_2^{-\frac{1}{2}}Q(\frac{1}{\gamma}V^{\frac{1}{2}}(w - F_1x)) \end{aligned}$$

which implies (b) and the sufficiency of (a). ∎

Corollary 4.11 *Consider the system* $\mathbf{G}_{FIj}$ *with* $F = 0$ *and assume* **J1** *and* **J5**.
(a) There exists an IO-stabilizing controller $u = Ky$ of the form (4.95) on $[t_0, \infty)$ such that $\| G \| < \gamma$ if and only if there exists a τ-periodic nonnegative stabilizing solution to (4.96)-(4.98).
(b) In this case the controllers (4.103) is γ-suboptimal. If Q is τ-periodic, then the controllers (4.103) are also τ-periodic.

Proof. Similar to the proof of Corollary 2.18. ∎

Disturbance Feedforward Problem

We consider the H_∞-problem for the special system $\mathbf{G}_{DFj}$

$$\begin{aligned} -\dot{x} &= Ax, \ k\tau < t < (k+1)\tau, \\ x(k\tau^-) &= A_d x(k\tau) + B_1 w(k) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12}u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21}w(k), \\ z_1 &= Fx(t_0) \end{aligned} \tag{4.104}$$

with $x(T) = 0$ where D_{21} is nonsingular. Here we assume **J1**. As in the continuous-time (or discrete-time) case, it can be reduced to the FI-problem. In fact consider the observer

$$\begin{aligned} -\dot{\hat{x}} &= A\hat{x}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^-) &= A_d x(k\tau) + B_1 D_{21}^{-1}[y(k) - C_2\hat{x}(k\tau)] + B_2 u(k), \\ \hat{x}(T) &= 0. \end{aligned}$$

Then $e = x - \hat{x}$ satisfies

$$\begin{aligned} -\dot{e} &= Ae, \ k\tau < t < (k+1)\tau, \\ e(k\tau^-) &= (A_d - B_1 D_{21}^{-1} C_2) e(k\tau), \\ e(T) &= 0 \end{aligned}$$

and hence $\hat{x} = x$. Moreover w is observable since

$$w(k) = D_{21}^{-1}[y(k) - C_2 x(k\tau)] = D_{21}^{-1}[y(k) - C_2 \hat{x}(k\tau)].$$

Thus we can use the controller (4.100) of the FI problem.

Theorem 4.16 *Assume* **J1**.
(a) There exists a γ-suboptimal controller $u = Ky$ of the form (4.95) if and only if there exists a nonnegative solution P of (4.96)-(4.99).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} -\dot{\hat{x}} &= A\hat{x}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^-) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}_1(k)y(k) + \hat{B}_2(k)v(k), \\ u(k) &= \hat{C}_1(k)\hat{x}(k\tau) - (T_2^{-1}SD_{21}^{-1})(k)y(k) + T_2^{-\frac{1}{2}}(k)v(k), \\ r(k) &= \hat{C}_2(k)\hat{x}(k\tau) + \frac{1}{\gamma}V^{\frac{1}{2}}(k)D_{21}^{-1}y(k), \\ v &= Qr, \ Q \in \mathcal{Q}_\gamma \end{aligned} \tag{4.105}$$

where $\hat{A}_d(k) = [A_d - B_1 D_{21}^{-1} C_2 - B_2 T_2^{-1}(R_2 - SD_{21}^{-1}C_2)](k)$ *and*

$$\begin{aligned} \hat{B}_1(k) &= (B_1 - B_2 T_2^{-1} S)(k) D_{21}^{-1}, & \hat{B}_2(k) &= B_2 T_2^{-\frac{1}{2}}(k), \\ \hat{C}_1(k) &= -[T_2^{-1}(R_2 - SD_{21}^{-1}C_2)](k), & \hat{C}_2(k) &= -\tfrac{1}{\gamma}[V^{\frac{1}{2}}(D_{21}^{-1}C_2 + F_1)](k). \end{aligned}$$

Proof. The necessity of (a) follows from Theorem 4.14. The sufficiency and (b) follow from Theorem 4.14 and the observation

$$\begin{aligned} u(k) &= -T_2^{-1}R_2 x(k\tau) - T_2^{-1}Sw(k) + T_2^{-\frac{1}{2}}Qr \\ &= -T_2^{-1}R_2\hat{x}(k\tau) - T_2^{-1}S[D_{21}^{-1}(y(k) - C_2\hat{x}(k\tau)] + T_2^{-\frac{1}{2}}Qr \\ &= -T_2^{-1}(R_2 - SD_{21}^{-1}C_2)\hat{x}(k\tau) - T_2^{-1}SD_{21}^{-1}y(k) + T_2^{-\frac{1}{2}}Qr, \\ r(k) &= \frac{1}{\gamma}V^{\frac{1}{2}}[w(k) - F_1 x(k\tau)] \\ &= \frac{1}{\gamma}V^{\frac{1}{2}}\{D_{21}^{-1}[y(k) - C_2\hat{x}(k\tau)] - F_1 x(k\tau)\} \\ &= -\frac{1}{\gamma}V^{\frac{1}{2}}(D_{21}^{-1}C_2 + F_1)\hat{x}(k\tau) + \frac{1}{\gamma}V^{\frac{1}{2}}D_{21}^{-1}y(k). \end{aligned}$$

∎

We consider the infinite horizon problem. We assume **J1**, **J5** and

$$\mathbf{J6}: \ (A, A_d - B_1 D_{21}^{-1} C_2) \text{ is exponentially stable.}$$

Theorem 4.17 *Assume* **J1**, **J5** *and* **J6**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (4.95) if and only if there exists a bounded nonnegative stabilizing solution to (4.96)-(4.99).*
(b) In this case the set of all γ*-suboptimal controllers is given by (4.105) with* Q *internally stable.*

4.2.4 Proofs of Main Results.

We now give the proofs of our main results using Theorems 4.16 and 4.17. We first prove Lemmas 4.11 and 4.12. As in the discrete-time H_∞-control problem, we can rewrite (4.47), (4.51) and (4.55) as

$$\begin{aligned} X(k\tau^-) &= A_d'X(k\tau)N(k)A_d, \quad N(k) = [I + B_2B_2'X(k\tau)]^{-1}, \quad (4.106)\\ Y(k\tau^+) &= A_dY(k\tau)N_Y(k)A_d', \quad N_Y(k) = [I + C_2'C_2Y(k\tau)]^{-1} \end{aligned}$$

and

$$Z(k\tau^+) = [I - \frac{1}{\gamma^2}\Phi(k)X(k\tau)B_2T_2^{-1}(k)B_2'X(k\tau)]^{-1}\Phi(k) \qquad (4.107)$$

respectively, where $\Phi(k) = A_dZ(k\tau)[I + C_2'C_2Z(k\tau)]^{-1}A_d'$. By (4.107) we have

$$\begin{aligned} \Phi(k) &= Z(k\tau^+)[I + \frac{1}{\gamma^2}X(k\tau)B_2T_2^{-1}(k)B_2'X(k\tau)Z(k\tau^+)]^{-1}\\ &= [I + \frac{1}{\gamma^2}Z(k\tau^+)X(k\tau)B_2T_2^{-1}(k)B_2'X(k\tau)]^{-1}Z(k\tau^+). \quad (4.108) \end{aligned}$$

We also have

$$\begin{aligned} (A_d - B_2T_2^{-1}R_2)(k) &= N(k)A_d,\\ (A_d - R_{2Y}'T_{2Y}^{-1}C_2)(k) &= A_dN_Y'(k) \qquad (4.109) \end{aligned}$$

and

$$\begin{aligned} &A_d + (F_{1Z}'T_2^{-\frac{1}{2}}R_2)(k) + F_{1Z}(k)C_2\\ = \ &[I - \frac{1}{\gamma^2}\Phi(k)X(k\tau)B_2T_2^{-1}(k)B_2'X(k\tau)]^{-1}\\ &\times A_d[I + Z(k\tau)C_2'C_2]^{-1}. \qquad (4.110) \end{aligned}$$

Using (4.108), we can rewrite (4.110) as

$$\begin{aligned} &A_d + (F_{1Z}'T_2^{-\frac{1}{2}}R_2)(k) + F_{1Z}(k)C_2\\ = \ &[I + \frac{1}{\gamma^2}Z(k\tau^+)X(k\tau)B_2T_2^{-1}(k)B_2'X(k\tau)]\\ &\times A_d[I + Z(k\tau)C_2'C_2]^{-1}. \qquad (4.111) \end{aligned}$$

Proof of Lemma 4.11: Let $Q(t) = Z(t) - Y(t) - \frac{1}{\gamma^2}Z(t)X(t)Y(t)$ and $(k_s - 1)\tau < s \le k_s\tau$. Then as in the proof of Lemma 2.17 for $s \le t \le k_s\tau$ and $k\tau < t \le (k+1)\tau$, $k \ge k_s$ we have

$$\dot{Q}(t) = [A + \frac{1}{\gamma^2}B_1B_1'X(t)]Q(t) + Q(t)[A + \frac{1}{\gamma^2}Y(t)C_1'C_1]'.$$

Since $Q(s) = 0$,

$$Q(t) = 0,\ s \le t \le k_s\tau.$$

Moreover, if $Q(k\tau^+) = 0$,

$$Q(t) = 0,\ k\tau < t \le (k+1)\tau.$$

To complete the proof, it is enough to show $Q(k\tau^+) = 0$ when $Q(k\tau) = 0$. Since

$$\begin{aligned} Q(k\tau^+) &= Y(k\tau^+) - Z(k\tau^+)[I - \frac{1}{\gamma^2}X(k\tau^+)Y(k\tau^+)] \\ &= Y(k\tau^+) - Z(k\tau^+)[I - \frac{1}{\gamma^2}X(k\tau)Y(k\tau^+)], \end{aligned}$$

it suffices to show

$$Y(k\tau^+) = Z(k\tau^+)[I - \frac{1}{\gamma^2}X(k\tau)Y(k\tau^+)]. \tag{4.112}$$

Since $Q(k\tau) = 0$, by Lemma 2.18 we have

$$\begin{aligned} Y(k\tau) &= Y(k\tau^-) \\ &= Z(k\tau^-)[I + \frac{1}{\gamma^2}X(k\tau^-)Z(k\tau^-)] \\ &= Z(k\tau)[I + \frac{1}{\gamma^2}X(k\tau^-)Z(k\tau)] \end{aligned}$$

and using the argument in the proof of Lemma 3.22

$$\begin{aligned} Y(k\tau^+) &= A_dY(k\tau)[I + C_2'C_2Y(k\tau)]^{-1}A_d' \\ &= [I + \frac{1}{\gamma^2}\Phi(k)X(k\tau)[I + B_2B_2'X(k\tau)]^{-1}]^{-1}\Phi(k). \quad (4.113) \end{aligned}$$

By (4.106), (4.113) and the argument in the proof of Lemma 3.22, we obtain (4.112). ∎

Proof of Lemma 4.12: As in the proof of Lemma 2.19, $\tilde{x} = (I - \frac{1}{\gamma^2}XY)x$ satisfies

$$\dot{\tilde{x}} = -[A + \frac{1}{\gamma^2}B_1B_1'X(t)]'\tilde{x}$$

for $s \le t < k_s\tau$ and $k\tau < t < (k+1)\tau$, $k \ge k_s$. Using a similar calculation in the proof of Lemma 3.23 with (4.106), (4.112) and

$$Z(k\tau) = Y(k\tau)[I - \frac{1}{\gamma^2}X(k\tau^-)Y(k\tau)]^{-1},$$

we obtain

$$\begin{aligned} &(A_d + F_{1Z}'T_2^{-\frac{1}{2}}R_2 + F_{2Z}'C_2)'(k) \\ = \ & [I - \frac{1}{\gamma^2}X(k\tau^-)Y(k\tau)](A_d - R_{2Y}'T_{2Y}^{-1}C_2)'(k)[I - \frac{1}{\gamma^2}X(k\tau)Y(k\tau^+)]^{-1} \end{aligned}$$

and hence we have the assertion. ∎

Proof of Theorem 4.8: Necessity of (a). Suppose that there exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ for the system $\mathbf{G}_j$ given by (4.42)-(4.44). Then by Theorem 4.12, there exists a nonnegative solution $X(t)$, $t \in [t_0, T]$ to (4.46)-(4.48). Moreover for the system $\mathbf{G}_j$ the following holds:

$$\begin{aligned} | z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2\times l^2} &= h'H'X(t_0)Hh + \| T_2^{\frac{1}{2}}(u + T_2^{-1}R_2x) \|_{l^2}^2 \\ &+\gamma^2 \| w \|_{L^2} - \gamma^2 \| w - \frac{1}{\gamma^2}B_1'Xx \|_{L^2}^2 . \end{aligned}$$

Setting $u = Ky$ and $w = \frac{1}{\gamma^2}B_1'Xx$, we obtain

$$\begin{aligned} d^2(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2\times l^2}) &\ge | z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2\times l^2} \\ &\ge h'H'X(t_0)Hh + \gamma^2 \| w \|_{L^2} \end{aligned}$$

and

$$d^2(| h |^2 + \| w_d \|_{l^2}) \ge h'H'X(t_0)Hh$$

for any w_d. Hence

$$d^2 \ | h |^2 \ge h'H'X(t_0)Hh$$

which implies (4.49) and (i) holds.

Now consider the systems

$$\begin{aligned} \dot{x} &= Ax + B_1w, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d - B_2T_2^{-1}R_2)(k)x(k\tau) + B_2T_2^{-\frac{1}{2}}(k)v(k), \\ \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} &= \begin{bmatrix} C_1x \\ -D_{12}(T_2^{-1}R_2)(k)x(k\tau) + D_{12}T_2^{-\frac{1}{2}}(k)v(k) \end{bmatrix}, \\ r &= -\frac{1}{\gamma^2}B_1'X(t)x + w, \\ z_1 &= Fx(T), \\ x(t_0) &= Hh \end{aligned} \tag{4.114}$$

and

$$\begin{aligned}\dot{\bar{x}} &= [A+\frac{1}{\gamma^2}B_1B_1'X(t)]\bar{x}+B_1r,\ k\tau<t<(k+1)\tau,\\ \bar{x}(k\tau^+) &= A_d\bar{x}(k\tau)+B_2u(k),\\ v(k) &= (T_2^{-\frac{1}{2}}R_2)(k)\bar{x}(k\tau)+T_2^{\frac{1}{2}}(k)u(k),\\ y(k) &= C_2\bar{x}(k\tau)+D_{21}w_d(k)\end{aligned} \tag{4.115}$$

with a controller

$$u = Ky. \tag{4.116}$$

Then we have the following.

Lemma 4.25 *Let X be the solution of (4.46)-(4.49).*
(a) For the system (4.114), the following holds:

$$\begin{aligned}| z_1 |^2 + \left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2\times l^2} &= \gamma^2 \| w \|^2_{L^2} + h'H'X(t_0)Hh\\ &\quad + \| v \|^2_{l^2} - \gamma^2 \| r \|^2_{L^2}.\end{aligned} \tag{4.117}$$

(b) The system $\mathbf{G}_j$ with a controller $u = Ky$ is equivalent to the interconnection of (4.114) and the feedback system (4.115) with $u = Ky$.

Proof. By direct calculation, we can show (a). The proof of (b) is similar to the proof of Lemma 4.23. ∎

Now introduce the feedback

$$v = Q\begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \tag{4.118}$$

of the form

$$\begin{aligned}\dot{p} &= \hat{A}(t)p+\hat{B}(t)r,\ k\tau<t<(k+1)\tau,\\ p(k\tau^+) &= \hat{A}_d(k)p(k\tau)+\hat{B}_d(k)w_d(k),\\ r &= \hat{C}_d(k)p(k\tau)+\hat{D}(k)w_d(k).\end{aligned} \tag{4.119}$$

Lemma 4.26 *Let*

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G\begin{pmatrix} h \\ w \\ w_d \end{pmatrix} \tag{4.120}$$

be the input-output operator of the closed-loop system (4.114) and (4.118). Then $\| G \| < \gamma$ if and only if $Q \in Q_\gamma$ where

$$Q_\gamma = \{Q : Q \in \mathcal{L}(\mathbf{R}^{n_1}\times L^2(t_0,T;\mathbf{R}^{m_1})\times l^2(1,N;\mathbf{R}^{m_{1d}});l^2(1,N;\mathbf{R}^{m_2}));$$

Q is of the form (4.119) with

$$\| Q \begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \|_{l^2}^2 + h'H'X(t_0)Hh \leq d^2(| h |^2 + \| \begin{pmatrix} r \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2)$$

for some $0 < d < \gamma\}$.

Proof. For each $r_0 \in L^2(t_0, T; \mathbf{R}^{m_1})$, there exists a $w \in L^2(t_0, T; \mathbf{R}^{m_1})$ such that the internal signal r in (4.114) and (4.118) coincides with r_0 and

$$\begin{aligned} c_1(| h |^2 + \| \begin{pmatrix} r_0 \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) &\leq | h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2 \\ &\leq c_2(| h |^2 + \| \begin{pmatrix} r_0 \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) \end{aligned}$$

for some $c_i > 0$, $i = 1, 2$. It suffices to take w_0 given by

$$\begin{aligned} \dot{x} &= Ax + B_1[r_0 + \frac{1}{\gamma^2}B_1'Xx], \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d - B_2T_2^{-1}R_2)x(k\tau) + B_2T_2^{-\frac{1}{2}}v_0(k), \\ w_0 &= r_0 + \frac{1}{\gamma^2}B_1'Xx, \\ x(t_0) &= Hh \end{aligned}$$

where $v_0 = Qr_0$. Now suppose $\| G \| < \gamma$ for (4.114) and (4.118). Then for some $0 < d < \gamma$

$$\begin{aligned} d^2(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) &\geq | z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2\times l^2}^2 \\ &\geq \gamma^2 \| w \|_{L^2}^2 + h'H'X(t_0)Hh \\ &\qquad + \| v \|_{l^2}^2 - \gamma^2 \| r \|_{L^2}^2 \end{aligned}$$

by (4.117). Hence

$$\begin{aligned} &\| Q \begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \|_{l^2}^2 + h'H'X(t_0)Hh \\ \leq\ & \gamma^2(| h |^2 + \| \begin{pmatrix} r \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) - (\gamma^2 - d^2)(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) \\ \leq\ & [\gamma^2 - c_1(\gamma^2 - d^2)](| h |^2 + \| \begin{pmatrix} r \\ w_d \end{pmatrix} \|_{L^2\times l^2}^2) \end{aligned}$$

which implies $Q \in \mathcal{Q}_\gamma$. Conversely, let $Q \in \mathcal{Q}_\gamma$. Then by (4.117) we have

$$| z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2\times l^2}^2$$

$$
\begin{aligned}
&= \gamma^2 \| w \|_{L^2}^2 + h'H'X(t_0)Hh + \left\| Q \begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \right\|_{l^2}^2 - \gamma^2 \| r \|_{L^2}^2 \\
&\le \gamma^2 \| w \|_{L^2}^2 + d^2\left(| h |^2 + \left\| \begin{pmatrix} r \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) - \gamma^2 \| r \|_{L^2}^2 \\
&\le \gamma^2\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) - (\gamma^2 - d^2)\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) \\
&\le \left(\gamma^2 - \frac{\gamma^2 - d^2}{c_2}\right)\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right).
\end{aligned}
$$

Hence $\| G \| < \gamma$. ∎

Remark 4.5 If $\| G \| < \gamma$, then

$$
\begin{aligned}
\left\| Q \begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \right\|_{l^2}^2 &= \gamma^2\left(| h |^2 + \frac{1}{\gamma^2} h'H'X(t_0)Hh + \left\| \begin{pmatrix} r \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) \\
&\quad - (\gamma^2 - d^2)\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) \\
&= \gamma^2\left(| \tilde{h} |^2 + \| r \|_{L^2}^2 + \| w_d \|_{l^2}^2\right) \\
&\quad - (\gamma^2 - d^2)\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right)
\end{aligned}
$$

where

$$
\tilde{h} = \left(I - \frac{1}{\gamma^2} H'X(t_0)H\right)^{\frac{1}{2}} h.
$$

Using

$$
| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2 \le c_2'\left(| \tilde{h} |^2 + \left\| \begin{pmatrix} r \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right),
$$

we can show $Q \in Q'_\gamma$ where

$$
\begin{aligned}
Q'_\gamma = \Big\{ &Q : Q \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}) \times l^2(1, N; \mathbf{R}^{m_{1d}}); l^2(1, N; \mathbf{R}^{m_2})); \\
&Q \text{ is of the form (4.119) with} \\
&\left\| Q \begin{pmatrix} h \\ r \\ w_d \end{pmatrix} \right\|_{l^2}^2 + h'H'X(t_0)Hh \le d^2\left(| \tilde{h} |^2 + \left\| \begin{pmatrix} r \\ w_d \end{pmatrix} \right\|_{L^2\times l^2}^2\right) \\
&\text{for some } 0 < d < \gamma \Big\}.
\end{aligned}
$$

Let $\tilde{Q}$ be the input-output operator of the closed-loop system (4.115) and $u = Ky$ so that

$$
v = \tilde{Q} \begin{pmatrix} h \\ r \\ w_d \end{pmatrix}.
$$

Then $\tilde{Q}$ has the form (4.119) and by Lemma 4.26 we have $\tilde{Q} \in Q'_\gamma$. Hence $u = Ky$ is γ-suboptimal for the H_∞-problem defined by

$$\begin{aligned}
\dot{\bar{x}} &= [A + \frac{1}{\gamma^2} B_1 B_1' X(t)]\bar{x} + B_1 r, \quad k\tau < t < (k+1)\tau, \\
\bar{x}(k\tau^+) &= A_d \bar{x}(k\tau) + B_2 u(k), \\
v(k) &= (T_2^{-\frac{1}{2}} R_2)(k)\bar{x}(k\tau) + T_2^{\frac{1}{2}}(k) u(k), \\
y(k) &= C_2 \bar{x}(k\tau) + D_{21} w_d(k), \\
\bar{x}(t_0) &= H(I - \frac{1}{\gamma^2} H' X(t_0) H)^{-\frac{1}{2}} \tilde{h}
\end{aligned} \tag{4.121}$$

with $\tilde{h} = (I - \frac{1}{\gamma^2} H' X(t_0) H)^{\frac{1}{2}} h$. The adjoint of (4.121) is given by

$$\begin{aligned}
-\dot{\tilde{x}} &= [A + \frac{1}{\gamma^2} B_1 B_1' X(t)]' \tilde{x}, \quad k\tau < t < (k+1)\tau, \\
\tilde{x}(k\tau^-) &= A_d' \tilde{x}(k\tau) + (R_2' T_2^{-\frac{1}{2}})(k) \tilde{w}_d(k) + C_2' \tilde{u}(k), \\
\tilde{z} &= \begin{bmatrix} \tilde{z}_c \\ \tilde{z}_d(k) \end{bmatrix} = \begin{bmatrix} B_1' \tilde{x} \\ D_{21}' \tilde{u}(k) \end{bmatrix}, \\
\tilde{y}(k) &= B_2' \tilde{x}(k\tau) + T_2^{\frac{1}{2}}(k) \tilde{w}_d(k), \\
\tilde{z}_1 &= (I - \frac{1}{\gamma^2} H' X(t_0) H)^{-\frac{1}{2}} H \tilde{x}(t_0).
\end{aligned} \tag{4.122}$$

Since $T_2^{\frac{1}{2}}$ is nonsingular and its inverse is uniformly bounded, the H_∞-control problem for this system is the DF-problem for the backward type and hence by Theorem 4.16, there exists a nonnegative stabilizing solution $Z(t)$ $t \in [t_0, T]$ to (4.53)-(4.56).

Sufficiency of (a) and the characterization in (b) of Theorem 4.8. Consider the systems (4.114) and (4.115). Then by Theorem 4.16, the set of the controllers given by (4.57) satisfies $\tilde{Q} \in Q'_\gamma$ where $\tilde{Q}$ is the input-output operator of the closed-loop system (4.115) and (4.116). Similarly to the proof of Theorem 2.9, we consider $e = \bar{x} - \hat{x}$ and the adjoint system. Then we can directly show that the controller (4.57) is γ-suboptimal, i.e., $Q \in Q_\gamma$. Hence sufficiency of (a) and (b) hold. ∎

Proof of Theorem 4.9. Necessity of (a): Suppose a γ-suboptimal controller exists. Then by Theorem 4.12 and Corollary 4.9, there exist nonnegative solutions X, Y and Z of (4.46)-(4.49), (4.50)-(4.52) and (4.53)-(4.56), respectively. By Lemmas 2.18 and 4.11, $I - \frac{1}{\gamma^2} X(t) Y(t)$, $t \in [t_0, T]$ is nonsingular and the eigenvalues of XY have the form

$$\frac{\gamma^2 \lambda}{\gamma^2 + \lambda}, \quad \lambda \in \lambda(XZ).$$

Since X and Z are nonnegative and uniformly bounded in T, $\lambda \in \lambda(XZ)$ are nonnegative and uniformly bounded. Hence $\rho(X(t)Y(t)) \le d^2$ for some $0 < d < \gamma$ and the condition (iii) holds.

Sufficiency of (a) and the characterization in (b). Note that $I - \frac{1}{\gamma^2}X(t)Y(t)$ is nonsingular and $[I - \frac{1}{\gamma^2}X(t)Y(t)]^{-1}$ is uniformly bounded in $t \in [t_0, T]$. Define

$$Z(t) = Y(t)[I - \frac{1}{\gamma^2}X(t)Y(t)]^{-1},\ t \in [t_0, T].$$

Then $Z(t_0) = H[I - \frac{1}{\gamma^2}H'X(t_0)H]^{-1}H'$. Similarly to the proof of Lemmas 2.17, 3.22 and 4.11 we can show $Z(t) = Y(t)[I - \frac{1}{\gamma^2}X(t)Y(t)]^{-1}$ satisfies the Riccati equation (4.53)-(4.56). The rest of the proof follows from the proof of Theorem 4.8. ∎

Proof of Theorem 4.10. Note that if X and Z are the nonnegative stabilizing solutions of (4.46), (4.47), (4.49) and (4.53)-(4.56), respectively. As in the proofs of Theorems 2.11 and 3.11, we can show that the assumptions of Theorem 4.17 are satisfied for the system (4.122). Then the proof is similar to the proof of Theorem 4.8. ∎

Proof of Theorem 4.11. The proof is similar to that of Theorem 4.9. We only need to show $Z = Y(I - \frac{1}{\gamma^2}XY)^{-1}$ is a bounded nonnegative stabilizing solution of (4.53)-(4.56). But this follows from Lemma 4.12 and the stabilizing property of Y. ∎

4.2.5 The General Case

Consider the general jump system $\mathbf{G}_j$

$$\begin{aligned}
\dot{x} &= Ax + B_1 w_c + B_2 u_c,\ k\tau < t < (k+1)\tau, \\
x(k\tau^+) &= A_d x(k\tau) + B_{1d} w_d(k) + B_{2d} u_d(k), \\
z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x + D_{12} u_c \\ C_{1d} x(k\tau) + D_{12d} u_d(k) \end{bmatrix}, \\
y &= \begin{bmatrix} y_c \\ y_d(k) \end{bmatrix} = \begin{bmatrix} C_2 x + D_{21} w_c \\ C_{2d} x(k\tau) + D_{21d} w_d(k) \end{bmatrix}, \\
z_1 &= Fx(T),\ t_0 \le N\tau \le T < (N+1)\tau, \\
x(t_0) &= Hh,\ 0 < t_0 \le \tau
\end{aligned} \tag{4.123}$$

where $x \in \mathbf{R}^n$ is the state, $(w_c, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbance, $(u_c, u_d) \in \mathbf{R}^{m_2} \times \mathbf{R}^{m_{2d}}$ is the control input, $(z_1, z_c, z_d) \in \mathbf{R}^q \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_{1d}}$ is the controlled output, $(y_c, y_d) \in \mathbf{R}^{p_2} \times \mathbf{R}^{p_{2d}}$ is the sampled observation,

$h \in \mathbf{R}^{n_1}$ and all matrices are of compatible dimensions. For the jump system (4.123), we assume

$$
\begin{array}{ll}
\mathbf{JG1}: & D_{12}'[C_1 \quad D_{12}] = [0 \quad I],\ D_{12d}'[C_{1d} \quad D_{12d}] = [0 \quad I], \\
\mathbf{JG2}: & D_{21}[B_1' \quad D_{21}'] = [0 \quad I],\ D_{21d}[B_{1d}' \quad D_{21d}'] = [0 \quad I], \\
\mathbf{JG3}: & ([A, A_d], [B_1, B_{1d}], [C_1, C_{1d}]) \text{ is stabilizable and detectable}, \\
\mathbf{JG4}: & ([A, A_d], [B_2, B_{2d}], [C_2, C_{2d}]) \text{ is stabilizable and detectable}.
\end{array}
$$

We assume that any feedback controller $u = Ky$ of the form:

$$
\begin{aligned}
\dot{p} &= \hat{A}(t)p + \hat{B}(t)y_c,\ k\tau < t < (k+1)\tau, \\
p(k\tau^+) &= \hat{A}_d(k)p(k\tau) + \hat{B}_d(k)y_d(k), \\
u_c &= \hat{C}(t)p + \hat{D}(t)y_c, \\
u_d(k) &= \hat{C}_d(k)p(k\tau) + \hat{D}_d(k)y_d(k)
\end{aligned}
\tag{4.124}
$$

where all matrices are compatible dimensions and uniformly bounded. Consider the system $\mathbf{G}_j$ and the controller $u = Ky$ of the form (4.124) on $[t_0, T]$. Define the input-output operator of the closed-loop system by

$$
\begin{pmatrix} z_1 \\ z_c \\ z_d \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.
$$

Then

$$
\begin{aligned}
G \ \in \ & \mathcal{L}(\mathbf{R}^{n_1} \times L^2(t_0, T; \mathbf{R}^{m_1}) \times l^2(1, N; \mathbf{R}^{m_{1d}}); \\
& \qquad \mathbf{R}^q \times L^2(t_0, T; \mathbf{R}^{p_1}) \times l^2(1, N; \mathbf{R}^{p_{1d}})).
\end{aligned}
$$

To give the solution of the H_∞-control problem for the system $\mathbf{G}_j$, we introduce the following Riccati equations with jumps. For definiteness we assume $0 < t_0 < \tau$.

$$
\begin{aligned}
-\dot{X} &= A'X + XA + X(\frac{1}{\gamma^2}B_1B_1' - B_2B_2')X + C_1'C_1, \\
& \qquad\qquad k\tau < t < (k+1)\tau, && (4.125) \\
V(k) &> aI \text{ for some } a > 0, && (4.126) \\
X(k\tau^-) &= A_d'X(k\tau)A_d - (R_2'T_2^{-1}R_2)(k) \\
& \qquad + (F_1'VF_1)(k) + C_{1d}'C_{1d}, && (4.127) \\
X(T) &= F'F, && (4.128) \\
H'X(t_0)H &\le d^2I \text{ for some } 0 < d < \gamma && (4.129)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{Y} &= AY + YA' + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)Y + B_1B_1', \\
& \qquad\qquad k\tau < t < (k+1)\tau, && (4.130)
\end{aligned}
$$

$$\begin{aligned}
V_Y(k) &> aI \text{ for some } a>0, &(4.131)\\
Y(k\tau^+) &= A_dY(k\tau)A_d' - (R_{2Y}'T_{2Y}^{-1}R_{2Y})(k)\\
&\qquad +(F_{1Y}'V_YF_{1Y})(k) + B_{1d}B_{1d}', &(4.132)\\
Y(t_0) &= H'H &(4.133)
\end{aligned}$$

where

$T_1(k) = \gamma^2 I - B_{1d}'X(k\tau)B_{1d}$, $T_2(k) = I + B_{2d}'X(k\tau)B_{2d}$,
$R_1(k) = B_{1d}'X(k\tau)A_d$, $R_2(k) = B_{2d}'X(k\tau)A_d$,
$S(k) = B_{2d}'X(k\tau)B_{1d}$, $V(k) = (T_1 + S'T_2^{-1}S)(k)$,
$F_1(k) = [V^{-1}(R_1 - S'T_2^{-1}R_2)](k)$, $F_2(k) = -[T_2^{-1}(R_2 + SF_1)](k)$,
$T_{1Y}(k) = \gamma^2 I - C_{1d}Y(k\tau)C_{1d}'$, $T_{2Y}(k) = I + C_{2d}Y(k\tau)C_{2d}'$,
$R_{1Y}(k) = C_{1d}Y(k\tau)A_d'$, $R_{2Y}(k) = C_{2d}Y(k\tau)A_d'$,
$S_Y(k) = C_{2d}Y(k\tau)C_{1d}'$, $V_Y(k) = (T_{1Y} + S_Y'T_{2Y}^{-1}S_Y)(k)$,
$F_{1Y}(k) = [V_Y^{-1}(R_{1Y} - S_Y'T_{2Y}^{-1}R_{2Y})](k)$,
$F_{2Y}(k) = -[T_{2Y}^{-1}(R_{2Y} + S_YF_{1Y})](k)$.

If we wish to take $t_0 = \tau$, the condition (4.129) becomes

$$H'X(\tau^-)H \le d^2 I.$$

We also need the following Riccati equation depending on X:

$$\begin{aligned}
\dot Z &= (A + \frac{1}{\gamma^2}B_1B_1'X)Z + Z(A + \frac{1}{\gamma^2}B_1B_1'X)' + B_1B_1'\\
&\quad +Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2)Z,\ k\tau < t < (k+1)\tau, &(4.134)\\
V_Z(k) &> aI \text{ for some } a>0, &(4.135)\\
Z(k\tau^+) &= A_dZ(k\tau)A_d' - (R_{2Z}'T_{2Z}^{-1}R_{2Z})(k)\\
&\qquad +(F_{1Z}'V_ZF_{1Z})(k) + B_{1d}B_{1d}', &(4.136)\\
Z(t_0) &= H(I - \frac{1}{\gamma^2}H'X(t_0)H)^{-1}H' &(4.137)
\end{aligned}$$

where

$A_X(k) = (A + B_{1d}F_1)(k)$, $B_{1X}(k) = \gamma(B_{1d}V^{-\frac{1}{2}})(k)$,
$C_{1X}(k) = [T_2^{-\frac{1}{2}}(R_2 + SF_1)](k)$,
$D_{11X}(k) = \gamma(T_2^{-\frac{1}{2}}SV^{-\frac{1}{2}})(k)$,
$D_{12X}(k) = T_2^{\frac{1}{2}}(k)$, $D_{21X}(k) = \gamma(D_{21d}V^{-\frac{1}{2}})(k)$,
$T_{1Z}(k) = \gamma^2 I - D_{11X}D_{11X}' - C_{1X}Z(k\tau)C_{1X}'$, $T_{2Z}(k) = I + C_{2d}Z(k\tau)C_{2d}'$,
$R_{1Z}(k) = C_{1X}Z(k\tau)A_X' + D_{11X}B_{1X}'$, $R_{2Z}(k) = C_{2d}Z(k\tau)A_X'$,
$S_Z(k) = C_{2d}Z(k\tau)C_{1X}'$,
$V_Z(k) = (T_{1Z} + S_Z'T_{2Z}^{-1}S_Z)(k)$,
$F_{1Z}(k) = [V_Z^{-1}(R_{1Z} - S_Z'T_{2Z}^{-1}R_{2Z})](k)$,
$F_{2Z}(k) = -[T_{2Z}^{-1}(R_{2Z} + S_ZF_{1Z})](k)$.

Now generalize the results in the Section 4.2.1.

Theorem 4.18 *Assume* **JG1** *and* **JG2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following conditions hold:
(i) There exists a nonnegative solution $X(t)$, $t \in [t_0, T]$ to (4.125)-(4.129).
(ii) For the solution X in (i), there exists a nonnegative solution $Z(t)$, $t \in [t_0, T]$ to (4.134)-(4.137).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned} \dot{p} &= [A + \frac{1}{\gamma^2} B_1 B_1' X(t)]p, \ k\tau < t < (k+1)\tau, \\ p(k\tau^+) &= \hat{A}_d(k)p(k\tau) + \hat{B}_1(k)y(k) + \hat{B}_2(k)s(k), \\ u(k) &= \hat{C}(k)p(k\tau) + \hat{D}_1(k)y(k) + \hat{D}_2(k)s(k), \\ g(k) &= T_2^{-\frac{1}{2}}(k)[-C_2 p(k\tau) + y(k)], \\ s &= Qg, \ Q \in Q_\gamma \end{aligned} \tag{4.138}$$

where

$$\begin{aligned} &\hat{A}_d(k) = [(A_d - B_2 T_2^{-1} R_2)\Psi](k), \\ &\hat{B}_1(k) = (A_d - B_2 T_2^{-1} R_2)(k) Z(k\tau) C_2' T_{2Z}^{-1}(k), \\ &\hat{B}_2(k) = \tfrac{1}{\gamma}([F_{1Z}' + B_2 T_2^{-\frac{1}{2}}] V_Z^{\frac{1}{2}})(k), \\ &\hat{C}(k) = -T_2^{-1} R_2 \Psi(k), \\ &\hat{D}_1(k) = -(T_2^{-1} R_2)(k) Z(k\tau) C_2' T_{2Z}^{-1}(k), \\ &\hat{D}_2(k) = \tfrac{1}{\gamma}(T_2^{-\frac{1}{2}} V_Z^{\frac{1}{2}})(k), \\ &\Psi(k) = I - Z(k\tau) C_2' T_{2Z}^{-1}(k) C_2 \end{aligned}$$

and

$$\begin{aligned} Q_\gamma = \{ & Q \in \mathcal{L}(L^2(t_0, T; \mathbf{R}^{p_2}) \times l^2(1, N; \mathbf{R}^{p_{2d}}); \\ & L^2(t_0, T; \mathbf{R}^{m_2}) \times l^2(1, N; \mathbf{R}^{m_{2d}})) : \\ & Q \text{ is of the form (4.124) and } \| Q \| < \gamma \}. \end{aligned}$$

Theorem 4.19 *Assume* **JG1** *and* **JG2**.
(a) There exists a γ-suboptimal controller $u = Ky$ on $[t_0, T]$ if and only if the following conditions hold:
(i) There exists a nonnegative solution X to (4.125)-(4.129).
(ii) There exists a nonnegative solution Y to (4.130)-(4.133).
(iii) $\rho(X(t)Y(t)) \le d^2$ for any $t \in [t_0, T]$ and for some $0 < d < \gamma$.
(b) In this case the set of all γ-suboptimal controllers is given by (4.138) with Z replaced by $(I - \frac{1}{\gamma^2} YX)^{-1} Y$.

Next we consider the H_∞-control problem on the infinite horizon $[t_0, \infty)$. Then we need the following definition.

Definition 4.12 *(a) The solution X of (4.125)-(4.127) is called the stabilizing solution if*

$$\left(A + (\frac{1}{\gamma^2} B_1 B_1' - B_2 B_2') X, A_d - B_{1d} F_1 + B_{2d} F_2 \right)$$

is exponentially stable.
(b) The solution Y *of (4.130)-(4.132) is called the stabilizing solution if*

$$\left(A + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2), A_d + F_{1Y}'C_{1d} + F_{2Y}'C_{2d}\right)$$

is exponentially stable.
(c) The solution Z *of (4.134)-(4.136) is called the stabilizing solution if*

$$\left(A + \frac{1}{\gamma^2}B_1B_1'X + Z(\frac{1}{\gamma^2}XB_2B_2'X - C_2'C_2), A_d + F_{1Z}'C_{1X} + F_{2Z}'C_{2d}\right)$$

is exponentially stable.

Theorem 4.20 *Assume* **JG1-JG4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *on* $[t_0, \infty)$ *if and only if the following conditions hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* X *to (4.125)-(4.127) and (4.129).*
(ii) For the solution X *in (i), there exists a bounded nonnegative stabilizing solution* Z *to (4.134)-(4.137).*
(b) In this case the set of all γ*-suboptimal controllers is given by (4.138) with* Q *internally stable.*
Moreover the $\lim_{n\to\infty} Z(t + n\tau)$ *exists (denoted by* $Z_\tau(t)$*) and* Z_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.134)-(4.136).*

Theorem 4.21 *Assume* **JG1-JG4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *on* $[t_0, \infty)$ *if and only if the following conditions hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* X *to (4.125)-(4.127) and (4.129).*
(ii) There exists a bounded nonnegative stabilizing solution Y *of (4.130)-(4.133).*
(iii) $\rho(X(t)Y(t)) \le d^2$*, for any* $t \in [t_0, \infty)$ *and for some* $0 < d < \gamma$*.*
(b) In this case the set of all γ*-suboptimal controllers is given by (4.138) with* Z *replaced by* $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ *and* Q *internally stable.*
Moreover the $\lim_{n\to\infty} Y(t + n\tau)$ *exists (denoted by* $Y_\tau(t)$*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.130)-(4.132).*

4.3 H_∞ Filtering

The filtering problem is to find an estimate of the state based on the given observation. The H_∞ filtering theory is well-known for continuous- and discrete-time systems as we see in Chapter 2 and 3. Sun et al [67] considered the H_∞ filtering for a time-invariant continuous system with sampled observation. In this section we consider the H_∞-filtering problems for jump systems.

Consider the jump system $\mathbf{G}_{Fj}$:

$$\begin{aligned} \dot{x} &= Ax + Bw, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k), \qquad (4.139) \\ z &= Lx, \\ y(k) &= Cx(k\tau) + Dw_d(k), \\ z_1 &= Fx(T), \ 0 \le N\tau \le T < (N+1)\tau \qquad (4.140) \end{aligned}$$

with initial condition

$$x(t_0) = Hh, \ 0 < t_0 \le \tau \qquad (4.141)$$

where $x \in \mathbf{R}^n$ is the state, $(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1}$ is the state to be estimated, $y \in \mathbf{R}^{p_2}$ is the sampled measurement, $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbances, $h \in \mathbf{R}^{n_1}$ and all matrices are of compatible dimensions. For this system we assume

$$\mathbf{JF1}: \ D[B_d' \quad D'] = [0 \quad I].$$

Then the filtering problem on $[t_0, T]$ is to is to find necessary and sufficient conditions for the existence of a causal filter based on y of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}(t)\hat{x}, \ \hat{x}(t_0) = 0, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}(k)y(k), \qquad (4.142) \\ \hat{z} &= \hat{C}(t)\hat{x}, \\ \hat{z}_1 &= \hat{F}\hat{x}(T) \end{aligned}$$

such that

$$| z_1 - \hat{z}_1 |^2 + \| z - \hat{z} \|_{L^2}^2 \le d^2(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2) \qquad (4.143)$$

for some $0 < d < \gamma$ where all matrices in (4.142) are uniformly bounded in t and k and $(\hat{z}_1, \hat{z})$ is the estimate of (z_1, z) given by the filter. We give necessary and sufficient conditions for the existence of a filter with property (4.143) (γ-suboptimal filter) and characterize all γ-suboptimal filters.

We can rewrite (4.139)-(4.141) and (4.142) as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A}(t) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w, \ k\tau < t < (k+1)\tau, \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau^+) &= \begin{bmatrix} A_d & 0 \\ \hat{B}(k)C & \hat{A}_d(k) \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(k\tau) + \begin{bmatrix} B_d \\ \hat{B}(k)D \end{bmatrix} w_d(k), \\ e(t) &= z(t) - \hat{z}(t) = [L \quad -\hat{C}(t)] \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t), \\ e_1 &= z_1 - \hat{z}_1 = [F \quad -\hat{F}] \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(T), \\ \begin{bmatrix} x \\ \hat{x} \end{bmatrix}(t_0) &= \begin{bmatrix} Hh \\ 0 \end{bmatrix}. \end{aligned}$$

Define the operator

$$G \in \mathcal{L}(L^2(t_0,T;\mathbf{R}^{m_1}) \times l^2(1,N;\mathbf{R}^{m_{1d}});\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}))$$

by

$$\begin{pmatrix} e_1 \\ e \end{pmatrix} = G \begin{pmatrix} w \\ w_d \end{pmatrix}. \tag{4.144}$$

Then (4.143) is equivalent to $\| G \| \leq d$. The adjoint G^* is given by

$$\begin{pmatrix} \zeta_0 \\ \zeta_c \\ \zeta_d \end{pmatrix} = G^* \begin{pmatrix} f \\ \nu \end{pmatrix}$$

where

$$\begin{aligned}
-\begin{bmatrix} \dot{\xi} \\ \dot{\hat{\xi}} \end{bmatrix} &= \begin{bmatrix} A' & 0 \\ 0 & \hat{A}'(t) \end{bmatrix} \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} + \begin{bmatrix} L' \\ -\hat{C}'(t) \end{bmatrix} \nu, \ k\tau < t < (k+1)\tau, \\
\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} (k\tau^-) &= \begin{bmatrix} A'_d & C'\hat{B}'(k) \\ 0 & \hat{A}'_d(k) \end{bmatrix} \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} (k\tau), \\
\begin{bmatrix} \zeta_c \\ \zeta_d \end{bmatrix} &= \begin{bmatrix} B'\xi \\ B'_d\xi(k\tau) + D'\hat{B}'(k)\hat{\xi}(k\tau) \end{bmatrix}, \\
\zeta_0 &= [H' \quad 0] \begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} (t_0), \\
\begin{bmatrix} \xi \\ \hat{\xi} \end{bmatrix} (T) &= \begin{bmatrix} F' \\ -\hat{F}' \end{bmatrix} f.
\end{aligned}$$

This may be regarded as a closed-loop system

$$\begin{aligned}
-\dot{\xi} &= A'\xi + L'\nu, \ k\tau < t < (k+1)\tau, \\
\xi(k\tau^-) &= A'_d\xi(k\tau) + C'\mu(k), \\
\zeta_c &= B'\xi, \\
\zeta_d(k) &= B'_d\xi(k\tau) + D'\mu(k), \\
\eta &= \begin{bmatrix} \xi \\ \nu \end{bmatrix}, \\
\zeta_0 &= H'\xi(t_0), \\
\xi(T) &= F'f
\end{aligned} \tag{4.145}$$

with controller $\mu = K^*\eta$

$$\begin{aligned}
-\dot{\hat{\xi}} &= \hat{A}'(t)\hat{\xi} + [0 \quad -\hat{C}'(t)]\,\nu, \ k\tau < t < (k+1)\tau, \\
\hat{\xi}(k\tau^-) &= \hat{A}'_d(k)\hat{\xi}(k\tau), \\
\mu(k) &= \hat{B}'(k)\hat{\xi}(k\tau), \\
\hat{\xi}(T) &= -\hat{F}f.
\end{aligned}$$

The system (4.145) is of full information type and (4.143) is equivalent to

$$|\zeta_0|^2 + \left\| \begin{pmatrix} \zeta_c \\ \zeta_d \end{pmatrix} \right\|^2_{L^2\times l^2} \le d^2 \|\nu\|^2_{L^2} \text{ for some } 0 < d < \gamma.$$

The Riccati equation with jumps corresponding to this is

$$\begin{aligned} \dot{Y} &= AY + YA' + BB' + \frac{1}{\gamma^2} YL'LY, \quad (4.146) \\ & \qquad k\tau < t < (k+1)\tau, \\ Y(k\tau^+) &= A_d Y(k\tau) A_d' + B_d B_d' - (R_{2Y}' T_{2Y}^{-1} R_{2Y})(k), \quad (4.147) \\ Y(t_0) &= HH', \quad (4.148) \\ FY(T)F' &\le d^2 I \text{ for some } 0 < d < \gamma \quad (4.149) \end{aligned}$$

where $R_{2Y}(k) = CY(k\tau)A_d'$ and $T_{2Y}(k) = I + CY(k\tau)C'$. As Q_γ in Chapter 2, we define the set of controllers of backward type:

$$\begin{aligned} Q_\gamma^* = \{ & Q^* \in \mathcal{L}(\mathbf{R}^q \times L^2(t_0,T;\mathbf{R}^{p_1}); l^2(1,N;\mathbf{R}^{p_2})) : \\ & f'FY(T)F'f + \left\| Q^* \begin{pmatrix} f \\ \rho \end{pmatrix} \right\|^2_{l^2} \le d^2(|f|^2 + \|\rho\|^2_{L^2}) \\ & \text{for some } 0 < d < \gamma \}. \end{aligned}$$

Let $\tilde{Q}_\gamma$ be the set of adjoint systems of $Q^* \in Q_\gamma^*$. Modifying Theorem 4.14 we have the following.

Theorem 4.22 *Assume* **JF1**.
(a) There exists a γ-suboptimal filter of the form (4.142) if and only if there exists a nonnegative solution $Y(t)$, $t \in [t_0,T]$ to (4.146)-(4.148).
(b) In this case the set of γ-suboptimal filters is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \frac{1}{\gamma^2} YL'v, \ \hat{x}(0) = 0, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= [A_d - (R_{2Y}' T_{2Y}^{-1})(k) C]\hat{x}(k\tau) + (R_{2Y}' T_{2Y}^{-1})(k) y(k), \quad (4.150) \\ \hat{z} &= L\hat{x} - v, \\ r(k) &= T_{2Y}^{-\frac{1}{2}}(k)[-C\hat{x}(k\tau) + y(k)], \\ v &= Q_1 r, \\ \hat{z}_1 &= F\hat{x}(T) - Q_0 r, \ Q = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} \in \tilde{Q}_\gamma. \end{aligned}$$

Proof. The existence of (4.146)-(4.148) follows from Theorem 4.14. The condition (4.149) can be obtained similar to the proofs of Theorems 2.13 and 3.13.

To show (b) recall that the set of all controllers $\mu = K^*\eta$ with $\|G^*\| < \gamma$ is given by

$$\mu(k) = -(T_{2Y}^{-1} R_{2Y})(k)\xi(k\tau) + T_{2Y}^{-\frac{1}{2}}(k)[Q^* \begin{pmatrix} f \\ -\frac{1}{\gamma^2} LY\xi + \nu \end{pmatrix}](k) \quad (4.151)$$

where $Q \in Q_\gamma$. Then the closed-loop system (4.145) with (4.151) is equivalent to

$$\begin{aligned} -\dot{\xi} &= A'\xi + [0 \quad L']\,\eta, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^-) &= [A_d' - C'(T_{2Y}^{-1}R_{2Y})(k)]\xi(k\tau) + C'T_{2Y}^{-\frac{1}{2}}(k)s(k), \quad (4.152) \\ \zeta &= B'\xi, \\ \zeta_d(k) &= [B_d' - D'(T_{2Y}^{-1}R_{2Y})(k)]\xi(k) + D'T_{2Y}^{-\frac{1}{2}}(k)s(k), \\ \rho &= -\frac{1}{\gamma^2}LY\xi + [0 \quad I]\,\eta, \\ s &= Q^*\begin{pmatrix} f \\ \rho \end{pmatrix}, \\ \xi(T) &= F'f. \end{aligned}$$

In view of this we can show that the controller (4.151) is equivalent to

$$\begin{aligned} -\dot{\hat{\xi}} &= A'\hat{\xi} + [0 \quad L']\,\eta, \ k\tau < t < (k+1)\tau, \\ \hat{\xi}(k\tau^-) &= [A_d' - C'(T_{2Y}^{-1}R_{2Y})(k)]\hat{\xi}(k\tau) + C'T_{2Y}^{-\frac{1}{2}}(k)s(k), \quad (4.153) \\ \mu(k) &= -(T_{2Y}^{-1}R_{2Y})(k)\hat{\xi}(k\tau) + T_{2Y}^{-\frac{1}{2}}(k)s(k), \\ \rho &= -\frac{1}{\gamma^2}LY\hat{\xi} + [0 \quad I]\,\eta, \\ s &= Q^*\rho, \\ \hat{\xi}(T) &= F'f. \end{aligned}$$

In fact for (4.145) and (4.153), $e = \xi - \hat{\xi}$ satisfies

$$\begin{aligned} -\dot{e} &= A'e, \ k\tau < t < (k+1)\tau, \\ e(k\tau^-) &= A_d'e(k\tau), \\ e(T) &= 0 \end{aligned}$$

and ξ satisfies (4.152). Now consider the adjoint of (4.145) and (4.153) which is given by the closed-loop system:

$$\begin{aligned} \dot{x} &= Ax + Bw + [I \quad 0]\,u, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_dx(k\tau) + B_dw_d(k), \quad (4.154) \\ \tilde{z} &= Lx + [0 \quad I]\,u, \\ y(k) &= Cx(k\tau) + Dw_d(k), \\ x(t_0) &= Hh, \\ \tilde{z}_1 &= Fx(T) + u \end{aligned}$$

and

$$\dot{\hat{x}} = A\hat{x} + \frac{1}{\gamma^2}YL'v, \ k\tau < t < (k+1)\tau,$$

$$
\begin{aligned}
\hat{x}(k\tau^+) &= [A_d - (R'_{2Y}T_{2Y}^{-1})(k)C]\hat{x}(k\tau) + (R_{2Y}T_{2Y}^{-1})(k)y(k), \quad (4.155)\\
u &= -\begin{bmatrix} 0 \\ L \end{bmatrix}\hat{x} + \begin{bmatrix} 0 \\ I \end{bmatrix} v,\\
r(k) &= T_{2Y}^{-\frac{1}{2}}(k)[-C\hat{x}(k\tau) + y(k)],\\
v &= Q_1 r, \ \| Q_1 \| < \gamma,\\
u_1 &= -Fx(T) + Q_0 r,\\
\hat{x}(t_0) &= 0
\end{aligned}
$$

where

$$
Q = \begin{pmatrix} Q_1 \\ Q_0 \end{pmatrix} \in \tilde{Q}_\gamma. \tag{4.156}
$$

Then $\| G^* \| < \gamma$ is equivalent to

$$
| \tilde{z}_1 |^2 + \| \tilde{z} \|_{L^2}^2 \leq d^2 \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2 \ \text{ for some } 0 < d < \gamma. \tag{4.157}
$$

Note that the (4.154) except for $\tilde{z}$ coincides with the system $\mathbf{G}_{Fj}$. Thus (4.155) and (4.156) can be easily interpreted as the filtering result in (b). ∎

Next we consider the filtering problem on the infinite horizon $[t_0, \infty)$. Again consider the system $\mathbf{G}_{Fj}$:

$$
\begin{aligned}
\dot{x} &= Ax + Bw, \ k\tau < t < (k+1)\tau,\\
x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k),\\
z &= Lx,\\
y(k) &= Cx(k\tau) + Dw_d(k),\\
x(t_0) &= Hh, \ 0 < t_0 \leq \tau.
\end{aligned} \tag{4.158}
$$

Then the H_∞-filtering problem on $[t_0, \infty)$ is to find a γ-suboptimal filter, i.e., a filter of the form

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{A}(t)\hat{x}, \ \hat{x}(t_0) = 0, \ k\tau < t < (k+1)\tau,\\
\hat{x}(k\tau^+) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}(k)y(k),\\
\hat{z} &= \hat{C}(k)\hat{x}
\end{aligned} \tag{4.159}
$$

such that

$$
\| z - \hat{z} \|_{L^2}^2 \leq d^2(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2) \tag{4.160}
$$

for some $0 < d < \gamma$. For the system (4.158) we assume **JF1** and

JF2 : $([A, A_d], [B, B_d], [C, 0])$ is stabilizable and detectable.

Then modifying Theorem 4.15 we have the following.

Theorem 4.23 *Assume* **JF1** *and* **JF2**.
(a) There exists a γ*-suboptimal filter if and only if there exists a bounded nonnegative stabilizing solution* $Y(t)$, $t \in [t_0, \infty)$ $((A + \frac{1}{\gamma^2} Y L'L, A_d - R_Y' T_Y^{-1} C)$ *is exponentially stable) to (4.146)-(4.148).*
(b) In this case the set of all γ*-suboptimal filters is given by (4.150) where* $Q_\gamma \subset \mathcal{L}(l^2(1,\infty; \mathbf{R}^{p_2}); L^2(t_0,\infty;\mathbf{R}^{p_1}))$ *is defined as in Theorem 4.22. Moreover, the set of internally stable filters is given by (4.150) if we restrict* Q *to be internally stable.*
Moreover, the $\lim_{n\to\infty} Y(t + n\tau)$ *exists (denoted by* $Y_\tau(t)$*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution of (4.146) and (4.147).*

Proof. Using Y and repeating the same procedure as in the proof of Theorem 4.22 for the system (4.145), we can show (b). ∎

Corollary 4.12 *There exists a filter of the form (4.142) such that*

$$\sup_{T \geq T_0} [\mid z_1 - \hat{z}_1 \mid^2 + \parallel z - \hat{z} \parallel_{L^2}^2] \leq d^2(\mid h \mid^2 + \parallel \begin{pmatrix} w \\ w_d \end{pmatrix} \parallel_{L^2 \times l^2}^2)$$

for some $0 < d < \gamma$ *if and only if there exists a bounded nonnegative stabilizing solution of (4.146)-(4.148) with*

$$FY(T)F' \leq d^2 I, \ T \geq T_0 \ \textit{for some} \ 0 < d < \gamma.$$

If $h = 0$, we can construct the τ-periodic γ-suboptimal filters.

Corollary 4.13 *Consider the system* $\mathbf{G}_F$ *with* $h = 0$ *and assume* **JF1** *and* **JF2**.
(a) There exists a γ*-suboptimal filter if and only if there exists a* τ*-periodic nonnegative stabilizing solution* $Y(t)$, $t \in [t_0, \infty)$ *to (4.146) and (4.147).*
(b) In this case the filters (4.150) is γ*-suboptimal where*

$$Q_\gamma \subset \mathcal{L}(l^2(1,\infty; \mathbf{R}^{p_2}); L^2(t_0,\infty;\mathbf{R}^{p_1}))$$

is defined as in Theorem 4.22. If Q *is* τ*-periodic, then the filter (4.150) is* τ*-periodic and* γ*-suboptimal*

Example 4.6 Consider the H_∞-filtering problem for the following system with a sampled observation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \ x(0) = Hh, \\ z(t) &= [0 \quad 1] x, \\ y(k) &= [1 \quad 0] x(k) + w_d(k) \end{aligned}$$

which satisfies the assumptions **JF1** and **JF2**. We consider the H_∞-filtering problems for the following two cases

$$(a)\ H = 0,\ (b)\ H = 2\begin{bmatrix}1\\0\end{bmatrix}.$$

In the case (a) there exists a bounded nonnegative stabilizing solution $Y(t) = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12} & Y_2 \end{bmatrix}(t)$ of (4.146)-(4.148) for any $\gamma \geq 1.317$ which converges to a periodic solution. In the case (b) there exists a bounded nonnegative stabilizing solution $Y(t)$ for any $\gamma \geq 1.318$ which also converges to a periodic solution. Figures 4.14 and 4.15 show the bounded nonnegative solutions with $\gamma = 2$ of the cases (a) and (b), respectively. Figure 4.16 gives the asymptotic convergence of the outputs of central filters in the cases (a) and (b) to z where $\gamma = 2$, the initial conditions $x_1(0) = 1$, $x_2(0) = 0$ and the disturbances are $w(t) = e^{-10t}\sin 10t$, $w_d = 0$. The central filter in the case (b) incorporating initial uncertainty gives a better estimate.

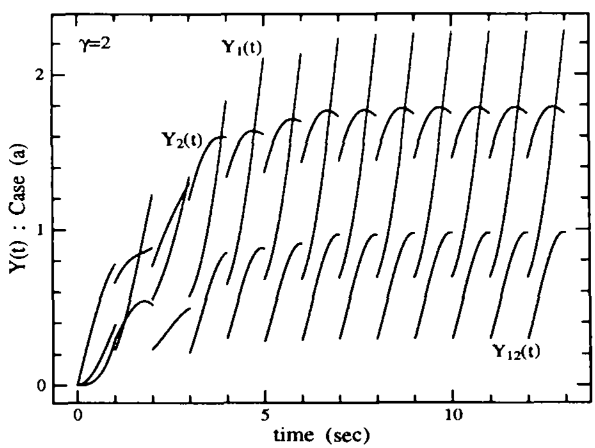

Figure 4.14: The bounded nonnegative stabilizing soultion $Y(t)$ of the case (a)

4.4 H_2 Control

In this section we consider the H_2 control problem for jump systems which covers the sampled-data H_2 control problem in Chapter 5.

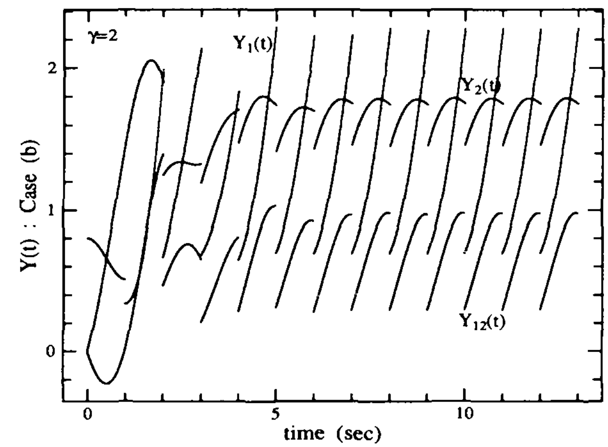

Figure 4.15: The bounded nonnegative stabilizing soultion $Y(t)$ of the case (b)

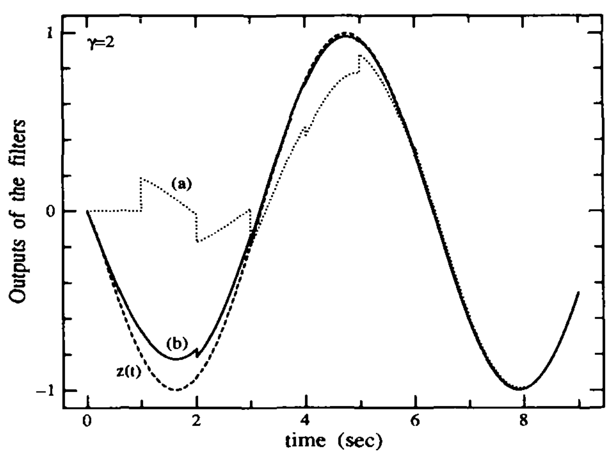

Figure 4.16: The outputs of the filters with $\gamma = 2$

4.4.1 Main Results

Consider the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{4.161}$$

with initial condition

$$x(0) = 0$$

where $x \in \mathbf{R}^n$ is the state, $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbance, $u \in \mathbf{R}^{m_2}$ is the control input, $(z_c, z_d) \in \mathbf{R}^{p_1} \times \mathbf{R}^{p_{1d}}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the sampled observation and all matrices are of compatible dimensions. For this system we assume

J1 : $D_{12}' D_{12} = I$,
J2 : $D_{21} D_{21}' = I$,
J3 : $([A, A_d], [B_1, 0], [C_1, 0])$ is stabilizable and detectable,
J4 : $([A, A_d], [0, B_2], [0, C_2])$ is stabilizable and detectable.

Consider a controller $u = Ky$ of the form:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x}, \; k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}_d \hat{x}(k\tau) + \hat{B} y(k), \\ u(k) &= \hat{C}\hat{x}(k\tau) + \hat{D} y(k). \end{aligned} \tag{4.162}$$

To formulate the H_2-problem for the system $\mathbf{G}_j$, we introduce the following set of controllers

$\mathbf{K} = \{K : K$ is of the form (4.162) and internally stabilizes the system $\mathbf{G}_j\}$.

Then the H_2-norm, $\| G \|_2$, of the closed-loop system $\mathbf{G}_j$ and a controller $u = Ky$ is well-defined and our H_2-problem is to find a controller $K \in \mathbf{K}$ which minimizes $\| G \|_2$. To give the solution of this problem we introduce the following Riccati equations with jumps:

$$-\dot{X} = A'X + XA + C_1'C_1, \; k\tau < t < (k+1)\tau, \tag{4.163}$$
$$X(k\tau^-) = A_d' X(k\tau) A_d - (R_2' T_2^{-1} R_2)(k) \tag{4.164}$$

and

$$\dot{Y} = AY + YA' + \frac{1}{\tau} B_1 B_1', \; k\tau < t < (k+1)\tau, \tag{4.165}$$
$$Y(k\tau^+) = A_d Y(k\tau) A_d' - (R_{2Y}' T_{2Y}^{-1} R_{2Y})(k) \tag{4.166}$$

where $R_2(k) = B_2' X(k\tau) A_d$, $T_2(k) = I + B_2' X(k\tau) B_2$, $R_{2Y}(k) = C_2 Y(k\tau) A_d'$ and $T_{2Y}(k) = I + C_2 Y(k\tau) C_2'$.

Definition 4.13 *(a) The solution X of (4.163) and (4.164) is called a stabilizing solution if $(A, A_d + B_2\hat{F})$, $\hat{F} = -T_2^{-1}(0)R_2(0)$ is exponentially stable.*
(b) The solution Y of (4.165) and (4.166) is called a stabilizing solution if $(A, A_d + \hat{H}C_2)$, $\hat{H} = -R'_{2Y}(0)T_{2Y}^{-1}(0)$ is exponentially stable.

By Theorems 4.3 and 4.4, we have the following result.

Lemma 4.27 *Assume* **J1-J4**.
(a) There exists a unique τ-periodic nonnegative stabilizing solution X to (4.163) and (4.164).
(b) There exists a unique τ-periodic nonnegative stabilizing solution Y to (4.165) and (4.166).

Remark 4.6 The τ-periodic solution Y of (4.165) and (4.166) is obtained as

$$\lim_{n \to \infty} \hat{Y}(t + n\tau)$$

where $\hat{Y}$ is a solution of (4.165) and (4.166) with $\hat{Y}(0) = 0$.

Consider the stabilizing controller based on the feedback gain $\hat{F}$ and the observer gain $\hat{H}$:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= (A_d + B_2\hat{F} + \hat{H}C_2 - B_2LC_2)\hat{x}(k\tau) \\ &\quad -(\hat{H} - B_2L)y(k), \qquad (4.167) \\ u(k) &= (\hat{F} - LC_2)\hat{x}(k\tau) + Ly(k) \end{aligned}$$

where $L = \hat{F}Y(0)C_2'T_{2Y}^{-1}(0)$

Theorem 4.24 *Assume* **J1-J4** *and consider the H_2-problem for the system* $\mathbf{G}_j$*. Then the controller (4.167) is optimal and*

$$\begin{aligned} \min_{K \in \mathbf{K}} \| G \|_2^2 &= \frac{1}{\tau}\int_0^\tau tr.B_1'X(s)B_1 \ ds \\ &\quad + tr.T_2\hat{F}(I + Y(0)C_2'C_2)^{-1}Y(0)\hat{F}' \quad (4.168) \end{aligned}$$

where $T_2 = T_2(0) = T_2(k\tau)$.

4.4.2 Proofs of Main Results

To prove Theorem 4.24 we need some preliminary results. Consider the system (4.161) and the controller $u = Ky$ of the form (4.162). Let X be the solution of (4.163) and (4.164). We introduce

$$v(k) = T_2^{\frac{1}{2}}[u(k) - \hat{F}x(k\tau)]$$

and the system $\bar{\mathbf{G}}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \;\; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \\ v(k) &= -T_2^{\frac{1}{2}} \hat{F} x(k\tau) + T_2^{\frac{1}{2}} u(k), \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k). \end{aligned} \tag{4.169}$$

Then $z = \begin{bmatrix} z_c \\ z_d \end{bmatrix}$ can be written using v as follows:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \;\; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d + B_2\hat{F})x(k\tau) + B_2 T_2^{-\frac{1}{2}} v(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12}\hat{F}x(k\tau) + D_{12}T_2^{-\frac{1}{2}} v(k) \end{bmatrix}. \end{aligned}$$

This system is exponentially stable and

$$z = G_{cj} w + U_j v$$

where G_{cj} and U_j are given by

$$\begin{aligned} \dot{\xi} &= A\xi + B_1 w, \;\; k\tau < t < (k+1)\tau, \\ \xi(k\tau^+) &= (A_d + B_2\hat{F})\xi(k\tau), \\ z &= \begin{bmatrix} \zeta_c \\ \zeta_d(k) \end{bmatrix} = \begin{bmatrix} C_1 \xi \\ D_{12}\hat{F}\xi(k\tau) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= Ax, \;\; k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= (A_d + B_2\hat{F})x(k\tau) + B_2 T_2^{-\frac{1}{2}} v(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12}\hat{F}x(k\tau) + D_{12}T_2^{-\frac{1}{2}} v(k) \end{bmatrix} \end{aligned}$$

respectively. Then we can easily see:

Lemma 4.28 *(a) The system $\mathbf{G}_j$ is equivalent to the interconnection of the system $\bar{\mathbf{G}}_j$ and (G_{cj}, U_j).*
(b) K stabilizes the system $\mathbf{G}_j$ if and only if it stabilizes the system $\bar{\mathbf{G}}_j$.

Next we shall show the properties of G_{cj} and U_j.

Lemma 4.29 *(a) $\| U_j v \|_{L^2 \times l^2} = \| v \|_{l^2}$ for any $v \in l^2(0,\infty;\mathbf{R}^{m_2})$.*
(b) $< G_{cj}\delta(\cdot - s)w_0, U_j v >_{L^2 \times l^2} = 0$ for any $w_0 \in \mathbf{R}^{m_1}$, $v \in l^2(0,\infty;\mathbf{R}^{m_2})$ with $v(0) = 0$ and $0 < s < \tau$.

Proof. (a) We can rewrite the Riccati equation (4.163) and (4.164) as

$$\begin{aligned} -\dot{X} &= A'X + XA + C_1'C_1, \ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= (A_d + B_2\hat{F})'X(k\tau)(A_d + B_2\hat{F}) + \hat{F}'\hat{F}. \end{aligned}$$

By direct calculation we have for $k\tau < t < (k+1)\tau$

$$\frac{d}{dt}[x'(t)X(t)x(t)] = -\mid z_c(t) \mid^2$$

and at $t = k\tau$

$$x'(k\tau^+)X(k\tau)x(k\tau^+) - x'(k\tau)X(k\tau^-)x(k\tau) = \mid v(k) \mid^2 - \mid z_d(k) \mid^2,$$

where we have used

$$B_2'X(0)(A_d + B_2\hat{F}) = -\hat{F}.$$

Upon integration from 0 to ∞ we have

$$\begin{aligned} -\int_0^\infty \mid z_c(t) \mid^2 dt + \sum_{k=1}^{\infty}[\mid v(k) \mid^2 - \mid z_d(k) \mid^2] &= \int_0^\infty \frac{d}{dt}[x'(t)X(t)x(t)]dt \\ &= -x'(0^+)X(0)x(0^+) \\ &= \mid v(0) \mid^2 - \mid z_d(0) \mid^2 \end{aligned}$$

and we conclude

$$\| v \|_{l^2} = \| z_c \|_{L^2} + \| z_d \|_{l^2} = \| U_j v \|_{L^2 \times l^2} .$$

(b) Consider G_{cj} with $w(t) = \delta(t-s)w_0$, $0 < s < \tau$. Then

$$\xi(t) = \begin{cases} \hat{S}(t,s)B_1 w_0, & s < t, \\ 0, & s \geq t \end{cases}$$

where $\hat{S}(\cdot,\cdot)$ is the state transition matrix of $(A, A_d + B_2\hat{F})$. As in the proof of (a) we have

$$\begin{aligned} -\int_s^\infty \zeta_c'(t)z_c(t)dt - \sum_{k=1}^{\infty}\zeta_d'(k)z_d(k) &= \int_s^\infty \frac{d}{dt}[\xi'(t)X(t)x(t)]dt \\ &= -\xi'(s^+)X(s)x(s^+) \end{aligned}$$

where x is the state of the system U_j. Since $\xi(t) = 0$, $t \leq s$ and $z_d(0) = 0$, we have

$$0 = \int_0^\infty \zeta_c'(t)z_c(t)dt + \sum_{k=0}^{\infty}\zeta_d'(k)z_d(k) = < G_{cj}\delta(\cdot - \tau)w_0, Uv >_{L^2 \times l^2} . \quad \blacksquare$$

Now we return to the H2-control problem for the system $\mathbf{G}_j$. Suppose K stabilizes the system $\mathbf{G}_j$ and hence $\bar{\mathbf{G}}_j$. Let $\bar{G}$ be the input-output operator of the closed-loop system $\bar{\mathbf{G}}_j$ with $u = Ky$, i.e.,

$$v = \bar{G}\begin{pmatrix} w \\ w_d \end{pmatrix}.$$

Note that

$$\begin{aligned} G\begin{bmatrix} \delta(\cdot - s)w_0 \\ 0 \end{bmatrix} &= G_{cj}\delta(\cdot - s)w_0 + U_j\bar{G}\begin{bmatrix} \delta(\cdot - s)w_0 \\ 0 \end{bmatrix}, \\ \bar{G}\begin{bmatrix} \delta(\cdot - s)w_0 \\ 0 \end{bmatrix}(k) &= 0, \text{ for } k = 0 \end{aligned}$$

and by Lemma 4.29

$$\begin{aligned} &\sum_{i=1}^{m_1} \| G_{cj}\delta(\cdot - s)e_i + U_j\bar{G}\begin{bmatrix} \delta(\cdot - s)e_i \\ 0 \end{bmatrix} \|^2_{L^2\times l^2} \\ = &\sum_{i=1}^{m_1} \| G_{cj}\delta(\cdot - s)e_i \|^2_{L^2\times l^2} + \sum_{i=1}^{m_1} \| U_j\bar{G}\begin{bmatrix} \delta(\cdot - s)e_i \\ 0 \end{bmatrix} \|^2_{L^2\times l^2}. \end{aligned}$$

Then

$$\begin{aligned} \| G \|_2^2 &= \| G_{cj} + U_j\bar{G} \|_2^2 \\ &= \| G_{cj} \|_2^2 + \| U_j\bar{G} \|_2^2 \\ &= \| G_{cj} \|_2^2 + \| \bar{G} \|_2^2 \end{aligned}$$

and

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_{cj} \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2. \tag{4.170}$$

Thus our original H2-problem has been reduced to the one for the system $\bar{\mathbf{G}}_j$. By Theorem 4.1 and the arguments of its proof, $\min_{K\in\mathbf{K}} \| \bar{G} \|_2^2$ is equivalent to the H2-problem for the backward system

$$\begin{aligned} -\dot{\tilde{x}} &= A'\tilde{x}, \ k\tau < t < (k+1)\tau, \\ \tilde{x}(k\tau^-) &= A_d'\tilde{x}(k\tau) - \hat{F}'T_2^{\frac{1}{2}}\tilde{w}_d(k) + C_2'\tilde{u}(k), \\ \tilde{v} &= \begin{bmatrix} \tilde{v}_c \\ \tilde{v}_d(k) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\tau}}B_1'\tilde{x} \\ D_{21}'\tilde{u}(k) \end{bmatrix}, \\ \tilde{y}(k) &= B_2'x(k\tau) + T_2^{\frac{1}{2}}\tilde{w}_d(k) \end{aligned} \tag{4.171}$$

with an internally stabilizing controller $\tilde{u} = K^b\tilde{y}$ of the form

$$\begin{aligned} -\dot{\tilde{p}} &= \hat{A}'\tilde{p}, \ k\tau < t < (k+1)\tau, \\ \tilde{p}(k\tau^-) &= \hat{A}_d'\tilde{p}(k\tau) + \hat{C}'\tilde{y}(k), \\ \tilde{u}(k) &= \hat{B}'\tilde{p}(k\tau) + \hat{D}'\tilde{y}(k). \end{aligned}$$

The H2-problem for the system (4.171) is the DF problem and its solution will be given in the next subsection.

Backward Systems

First we consider the FI problem for the backward system $\mathbf{G}_{FIj}$

$$\begin{aligned}
-\dot{x} &= Ax, \ k\tau < t < (k+1)\tau, \\
x(k\tau^-) &= A_d x(k\tau) + B_1 w_d(k) + B_2 u(k), \\
z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\
y(k) &= \begin{bmatrix} x(k\tau) \\ w_d(k) \end{bmatrix}.
\end{aligned} \tag{4.172}$$

We take a controller $u = Ky$ of the form

$$\begin{aligned}
-\dot{p} &= \hat{A}p, \ k\tau < t < (k+1)\tau, \\
p(k\tau^-) &= \hat{A}_d p(k\tau) + \hat{B}y(k), \\
u(k) &= \hat{C}p(k\tau) + \hat{D}y(k).
\end{aligned} \tag{4.173}$$

Let G_{FI} be the input-output operator of the closed-loop system $\mathbf{G}_{FIj}$ with $u = Ky$. To formulate the H_2-problem for the system $\mathbf{G}_{FIj}$ we introduce the following set of controllers:

$$\begin{aligned}
\mathbf{K} = \{K : K & \text{ is of the form (4.173) and} \\
& \text{internally stabilizes the system } \mathbf{G}_{FIj}\}.
\end{aligned}$$

Then the closed-loop system $\mathbf{G}_{FIj}$ with a controller $u = Ky$ of the form (4.173) is given by

$$\begin{aligned}
-\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \quad k\tau < t < (k+1)\tau, \\
\begin{bmatrix} x \\ p \end{bmatrix}(k\tau^-) &= \begin{bmatrix} A_d & B_2\hat{C} \\ \hat{B}_1 & \hat{A}_d \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}(k\tau) + \begin{bmatrix} B_1 \\ \hat{B}_2 \end{bmatrix} w_d(k), \\
z_c &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \\
z_d(k) &= \begin{bmatrix} 0 & D_{12}\hat{C} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}(k\tau)
\end{aligned}$$

which is the backward form of (4.5) with $B = 0$ and $D_d = 0$ and by Remark 4.2 we can define the H_2-norm $\| G_{FI} \|_2$ as

$$\| G_{FI} \|_2^2 = \lim_{N \to \infty} \sum_{i=1}^{m_{1d}} \| G_{FI} \delta_{\cdot N} e_i \|_{L^2 \times l^2}^2 .$$

Hence the H_2-problem for the system $\mathbf{G}_{FIj}$ is to find a controller $K \in \mathbf{K}$ which minimizes $\| G_{FI} \|_2$.

For the system $\mathbf{G}_{FIj}$ we assume **J1** and **J5**, i.e.,

J5 $([A, A_d], [0, B_2], [C_1, 0])$ is stabilizable and detectable.

Then as in Lemma 4.27, we have the following.

Lemma 4.30 *Assume* **J1** *and* **J5**. *Then there exists a unique τ-periodic nonnegative stabilizing solution ($(A, A_d+B_2F_P)$, $F_P = -T_P^{-1}R_P$ is exponentially stable) to the Riccati equation with jumps:*

$$\begin{aligned} \dot{P} &= A'P + PA + C_1'C_1, \;\; k\tau < t < (k+1)\tau, && (4.174)\\ P(k\tau^+) &= A_d'P(k\tau)A_d - (R_P'T_P^{-1}R_P)(k) && (4.175)\end{aligned}$$

where $T_P = I + B_2'P(0)B_2$ and $R_P = B_2'P(0)A_d$.

As in the previous subsection, we introduce

$$v(k) = T_P^{\frac{1}{2}}[u(k) + T_P^{-1}S_P w_d(k) - F_P x(k\tau)]$$

and the system $\bar{\mathbf{G}}^b$:

$$\begin{aligned} -\dot{\bar{x}} &= A\bar{x}, \;\; k\tau < t < (k+1)\tau,\\ \bar{x}(k\tau^-) &= A_d\bar{x}(k\tau) + B_1w_d(k) + B_2u(k),\\ v(k) &= -T_P^{-\frac{1}{2}}R_P\bar{x}(k\tau) + T_P^{-\frac{1}{2}}S_Pw_d(k) + T_P^{\frac{1}{2}}u(k), && (4.176)\\ y(k) &= \begin{bmatrix}\bar{x}(k\tau)\\ w_d(k)\end{bmatrix}\end{aligned}$$

where $S_p = B_2'P(0)B_1$. Then $z = \begin{bmatrix} z_c \\ z_d \end{bmatrix}$ can be written using v as follows:

$$\begin{aligned} -\dot{x} &= Ax, \;\; k\tau < t < (k+1)\tau,\\ x(k\tau^-) &= (A_d + B_2F_P)x(k\tau) + (B_1 - B_2T_P^{-1}S_P)w_d(k) + B_2T_P^{-\frac{1}{2}}v(k),\\ z &= \begin{bmatrix} z_c \\ z_d(k)\end{bmatrix} = \begin{bmatrix} C_1x \\ D_{12}F_Px(k\tau) - D_{12}T_P^{-1}S_Pw_d(k) + D_{12}T_P^{-\frac{1}{2}}v(k)\end{bmatrix}.\end{aligned}$$

Hence

$$z = G_{cj}^b w_d + U_j^b v$$

where G_{cj}^b and U_j^b are given by

$$\begin{aligned} -\dot{\xi} &= A\xi, \;\; k\tau < t < (k+1)\tau,\\ \xi(k\tau^-) &= (A_d + B_2F_P)\xi(k\tau) + (B_1 - B_2T_P^{-1}S_P)w_d(k),\\ \zeta &= \begin{bmatrix} \zeta_c \\ \zeta_d(k)\end{bmatrix} = \begin{bmatrix} C_1\xi \\ D_{12}F_P\xi(k\tau) - D_{12}T_P^{-1}S_Pw_d(k)\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned} -\dot{x} &= Ax,\ k\tau < t < (k+1)\tau, \\ x(k\tau^-) &= (A_d + B_2F_P)x(k\tau) + B_2T_P^{-\frac{1}{2}}v(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1x \\ D_{12}F_Px(k\tau) + D_{12}T_P^{-\frac{1}{2}}v(k) \end{bmatrix}. \end{aligned}$$

Then we have the following.:

(a) The system $\mathbf{G}_{FIj}$ is equivalent to the interconnection of the system (4.176) and (G_{cj}^b, U_j^b).
(b) K stabilizes $\mathbf{G}_{FIj}$ if and only if it stabilizes $\bar{\mathbf{G}}^b$.

Next we need the following lemma.

Lemma 4.31 *(a)* $\| U_j^b v \|_{L^2\times l^2} = \| v \|_{L^2\times l^2} - x'(0^-)P(0)x(0^-)$ *for any* $v \in l^2(0,\infty;\mathbf{R}^{m_2})$.
(b) $< G_{cj}^b \delta_{\cdot N} w_0, U_j^b v >_{L^2\times l^2} = -\xi'(0^-)P(0)x(0^-)$ *for any* $w_0 \in \mathbf{R}^{m_1}$ *and* $v \in l^2(0,\infty;\mathbf{R}^{m_2})$.

Proof. (a) As in the proof of Lemma 4.29, we have

$$\frac{d}{dt}[x'(t)P(t)x(t)] = | z_c(t) |^2,\ k\tau < t < (k+1)\tau$$

and

$$x'(k\tau)P(k\tau^+)x(k\tau) - x'(k\tau^-)P(k\tau)x(k\tau^-) = | z_d(k) |^2 - | v(k) |^2 .$$

Then we have

$$\begin{aligned} & \int_0^\infty | z_c(t) |^2\, dt + \sum_{k=1}^{\infty}[| z_d(k) |^2 - | v(k) |^2] \\ = & \int_0^\infty \frac{d}{dt}[x'(t)P(t)x(t)]dt \\ = & -x'(0)P(0^+)x(0) \\ = & | v(0) |^2 - | z_d(0) |^2 - x'(0^-)P(0)x(0^-). \end{aligned}$$

(b) Consider the system G_{cj}^b with $w_d(N) = w_0$, $w_d(k) = 0$, $k \neq N$. Then $\xi(t) = 0$, $\forall t > N\tau$ and as in the proof of Lemma 4.29, we have

$$\frac{d}{dt}[\xi'(t)P(t)x(t)] = \zeta_c'(t)z_c(t),\ k\tau < t < (k+1)\tau$$

and at $t = k\tau$, $k \neq N$

$$\begin{aligned} & \xi'(k\tau)P(k\tau^+)x(k\tau) - \xi'(k\tau^-)P(k\tau)x(k\tau^-) \\ & \qquad = [D_{12}F_P(k)\xi(k\tau)]'z_d(k) = \zeta_d'(k)z_d(k). \end{aligned}$$

We also have

$$\begin{aligned} &\xi'(N\tau)P(N\tau^+)x(N\tau) - \xi'(N\tau^-)P(N\tau)x(N\tau^-) \\ &\qquad = -w_d'(N)S_P'(k)T_P^{-1}(k)T_P^{-\frac{1}{2}}(k)v(N) = \zeta_d'(N)z_d(N). \end{aligned}$$

Integrating $\frac{d}{dt}[\xi'(t)P(t)x(t)]$ from 0 to ∞, we have

$$\begin{aligned} \int_0^\infty \zeta_c'(t)z_c(t)dt + \sum_{k=1}^{\infty}\zeta_d'(k)z_d(k) &= -\xi'(0)P(0^+)x(0) \\ &= -\zeta_d'(0)z_d(0) - \xi'(0^-)P(0)x(0^-) \end{aligned}$$

where ξ and x are the states of $G_{cj}^b\delta_{\cdot N}w_0$ and $U_j^b v$, respectively. ∎

Let $u = Ky$ be an internally stabilizing controller and $\bar{G}^b$ the input-output operator of the closed-loop system (4.176) with $u = Ky$ given by

$$v = \bar{G}^b w_d.$$

Note that

$$\begin{aligned} G_{FI}\delta_{\cdot N}e_i &= G_{cj}^b\delta_{\cdot N}e_i + U_j^b\bar{G}^b\delta_{\cdot N}e_i, \\ (\bar{G}^b\delta_{\cdot N}e_i)(k) &= 0, \text{ for } k > N \end{aligned}$$

and by Lemma 4.31

$$\begin{aligned} \| G_{cj}^b\delta_{\cdot N}e_i + U_j^b\bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 &= \| G_{cj}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 + \| U_j^b\bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 \\ &\quad -2\xi_{Ni}'(0^-)P(0)x_{Ni}(0^-) \end{aligned}$$

where ξ_{Ni} and x_{Ni} are the states of $G_{cj}^b\delta_{\cdot N}e_i$ and $U_j^b\bar{G}^b\delta_{\cdot N}e_i$, respectively. Since $\lim_{N\to\infty}\xi_{Ni}(0^-) = 0$, we have

$$\begin{aligned} &\lim_{N\to\infty}\sum_{i=1}^{m_1} \| G_{cj}^b\delta_{\cdot N}e_i + U_j^b\bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 \\ = \;&\lim_{N\to\infty}\sum_{i=1}^{m_1} \| G_{cj}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 + \lim_{N\to\infty}\sum_{i=1}^{m_1} \| U_j^b\bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 . \end{aligned}$$

Since $\lim_{N\to\infty}\bar{G}^b\delta_{\cdot N}e_i = 0$, $\lim_{N\to\infty}x_{Ni}(0^-) = 0$. By Lemma 4.31

$$\lim_{N\to\infty} \| U_j^b\bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 = \lim_{N\to\infty} \| \bar{G}^b\delta_{\cdot N}e_i \|_{L^2\times l^2}^2 .$$

Hence

$$\| G_{FI} \|_2^2 = \| G_{cj}^b \|_2^2 + \| \bar{G}^b \|_2^2$$

and we have

$$\min_{K\in\mathbf{K}} \| G_{FI} \|_2^2 = \| G_{cj}^b \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G}^b \|_2^2 .$$

Thus the H$_2$-problem of the system $\mathbf{G}_{FIj}$ is reduced to the one for the system $\bar{\mathbf{G}}^b$. Since $u(k) = F_P x(k\tau)$ is stabilizing, $u(k) = [\, F_P \quad -T_P^{-1} S_P \,] y(k)$ internally stabilizes (4.172) and this yields $v = 0$ or $\bar{G}^b = 0$. Hence $u(k) = [\, F_P \quad -T_P^{-1} S_P \,] y(k)$ is the optimal controller for $\mathbf{G}_{FIj}$ and

$$\min_{K \in \mathbf{K}} \| G_{FI} \|_2^2 = \| G_{cj}^b \|_2^2 .$$

The controllability gramian for the forward system associated with G_{cj}^b is a unique τ-periodic nonnegative solution of

$$\begin{aligned} \dot{L}_o &= A' L_o + L_o A + C_1' C_1, \ k\tau < t < (k+1)\tau, \\ L_o(k\tau^+) &= (A_d + B_2 F_P)' L_o(k\tau)(A_d + B_2 F_P) + F_P' F_P \end{aligned}$$

which implies $L_o = P$. Hence by Theorem 4.1,

$$\| G_{cj}^b \|_2^2 = tr.[\bar{B}_1' P(0) \bar{B}_1 + S_P' T_P^{-2} S_P]$$

where $\bar{B}_1 = B_1 - B_2 T_P^{-1} S_P$.

Summarizing the above we have shown the following result.

Lemma 4.32 *Assume* **J1**, **J5** *and consider the H_2-problem for the system* $\mathbf{G}_{FIj}$. *Then*
(a) $\min_{K \in \mathbf{K}} \| G_{FI} \|_2^2 = \| G_{cj}^b \|_2^2 = tr.[\bar{B}_1' P(0) \bar{B}_1 + S_P' T_P^{-2} S_P]$.
(b) $K = [\, F_P \quad -T_P^{-1} S_P \,]$ *is optimal.*

Next we consider the H$_2$-problem for the system $\mathbf{G}_{DFj}$

$$\begin{aligned} -\dot{x} &= Ax, \ k\tau < t < (k+1)\tau, \\ x(k\tau^-) &= A_d x(k\tau) + B_1 w_d(k) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{4.177}$$

where D_{21} is a nonsingular matrix and we take controllers K_{DF} of the form (4.173). This is called the DF-problem. Here we assume **J1**, **J5** and **J6**, i.e.,

$$\mathbf{J6} : \ (A, A_d - B_1 D_{21}^{-1} C_2) \text{ is stable.}$$

As we see below, this problem is equivalent to the FI-problem.

Proposition 4.7 *A controller K_{DF} internally stabilizes $\mathbf{G}_{DFj}$ if and only if $K = K_{DF}[C_2 \ D_{21}]$ internally stabilizes $\mathbf{G}_{FIj}$. In this case $G_{DF} = G_{FI}$ where G_{DF} is the input-output operator of the closed-loop system $\mathbf{G}_{DFj}$ with $u = K_{DF} y$ defined by $z = G_{DF} w_d$.*

Proof. The proof follows from $u = K_{DF}y = K_{DF}[C_2\ D_{21}]\begin{bmatrix} x(k\tau) \\ w_d(k) \end{bmatrix}$. ∎

Consider the controller K_{DF}:

$$\begin{aligned} -\dot{p} &= Ap, \ k\tau < t < (k+1)\tau, \\ p(k\tau^-) &= A_d p(k\tau) + B_1 D_{21}^{-1}[y(k) - C_2 p(k\tau)] + B_2 u_{FI}(k), \\ u(k) &= u_{FI}(k), \\ u_{FI} &= K y_{FI}, \\ y_{FI}(k) &= \begin{bmatrix} p(k\tau) \\ D_{21}^{-1}(y(k) - C_2 p(k\tau)) \end{bmatrix}. \end{aligned} \tag{4.178}$$

Proposition 4.8 *The controller K internally stabilizes $\mathbf{G}_{FIj}$ if and only if K_{DF} given by (4.178) internally stabilizes $\mathbf{G}_{DFj}$. In this case $G_{FI} = G_{DF}$.*

Proof. Let $e = x - p$ where x and p are the states of $\mathbf{G}_{DFj}$ and (4.178) respectively. Then e satisfies

$$\begin{aligned} -\dot{e} &= Ae, \ k\tau < t < (k+1)\tau, \\ e(k\tau^-) &= (A_d - B_1 D_{21}^{-1} C_2) e(k\tau) \end{aligned}$$

which is exponentially stable. Moreover

$$\begin{aligned} p(k\tau^-) &= A_d p(k\tau) + B_1 D_{21}^{-1}[y(k) - C_2 p(k\tau)] + B_2 u_{FI}(k) \\ &= A_d p(k\tau) + B_1 \hat{w}_d(k) + B_2 u(k), \\ u(k) &= u_{FI}(k) = K \begin{bmatrix} x(k\tau) \\ w_d(k) \end{bmatrix} = K \begin{bmatrix} p(k\tau) \\ \hat{w}_d(k) \end{bmatrix} \end{aligned}$$

where $\hat{w}_d(k) = w_d(k) + D_{21}^{-1} C_2 e(k\tau)$. Hence

$$\begin{aligned} -\dot{p} &= Ap, \ k\tau < t < (k+1)\tau, \\ p(k\tau^-) &= A_d p(k\tau) + B_1 \hat{w}_d(k) + B_2 u(k), \\ u(k) &= K \begin{bmatrix} p(k\tau) \\ \hat{w}_d(k) \end{bmatrix}. \end{aligned} \tag{4.179}$$

Now suppose K stabilizes $\mathbf{G}_{FIj}$. Then $p \in L^2$, but $e \in L^2$ and hence $x \in L^2$. Thus K_{DF} stabilizes $\mathbf{G}_{DFj}$. Conversely suppose K_{DF} stabilizes $\mathbf{G}_{DFj}$. Then (4.179) is stable and K stabilizes $\mathbf{G}_{FIj}$. Finally z is given by

$$z = \begin{bmatrix} z_c \\ z_d \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix} = \begin{bmatrix} C_1(p+e) \\ D_{12} u_{FI}(k) \end{bmatrix}$$

subject to (4.179). Hence $G_{FI} = G_{DF}$. ∎

Now it is easy to obtain the solution of DF-problem. Since $K = [F_P \ \ -T_P^{-1}S_P]$ is optimal for $\mathbf{G}_{FIj}$, the optimal controller for $\mathbf{G}_{DFj}$ is given by

$$u(k) = [\,F_P \quad -T_P^{-1}S_P\,]\begin{bmatrix} p(k\tau) \\ D_{21}^{-1}[y(k) - C_2p(k\tau)] \end{bmatrix}$$

and (4.178) in this case

$$\begin{aligned} -\dot{p} &= Ap, \ k\tau < t < (k+1)\tau, \\ p(k\tau^-) &= \hat{A}p(k\tau) + \hat{B}y(k), \\ u(k) &= \hat{C}p(k\tau) + \hat{D}y(k) \end{aligned} \tag{4.180}$$

where

$$\begin{aligned} \hat{A} &= A_d - (B_1 - B_2T_P^{-1}S_P)D_{21}^{-1}C_2 + B_2F_P, \\ \hat{B} &= (B_1 - B_2T_P^{-1}S_P)D_{21}^{-1}, \\ \hat{C} &= F_P + T_P^{-1}S_PD_{21}^{-1}C_2, \\ \hat{D} &= -T_P^{-1}S_PD_{21}^{-1}. \end{aligned}$$

Lemma 4.33 *Assume* **J1**, **J5** *and* **J6** *and consider the H_2-problem for* $\mathbf{G}_{DFj}$. *Then*
(a) $\min_{K\in\mathbf{K}} \| G_{DF} \|_2^2 = \| G_{cj}^b \|_2^2$.
(b) The controller (4.180) is optimal.

Proof of the Main Result

Now we return to the H₂-problem for $\mathbf{G}_j$. By (4.170) we have

$$\min_{K\in\mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2$$

and the original H_2-problem was reduced to the H_2-problem for the backward system (4.171), which is a DF-problem. Since the conditions **J1**, **J5** and **J6** are satisfied for (4.171), we can apply Lemma 4.33 to obtain

$$\begin{aligned} \min_{K\in\mathbf{K}} \| \bar{G} \|_2^2 &= tr.\{T_2^{\frac{1}{2}}\hat{F}[I - C_2'T_{2Y}^{-1}C_2Y(0)]'Y(0)[I - C_2'T_{2Y}^{-1}C_2Y(0)]\hat{F}'T_2^{\frac{1}{2}} \\ &\qquad + T_2^{\frac{1}{2}}\hat{F}Y(0)C_2'T_{2Y}^{-1}C_2Y(0)\hat{F}'T_2^{\frac{1}{2}}\} \\ &= tr.T_2\hat{F}[I + Y(0)C_2'C_2]^{-1}Y(0)\hat{F}' \end{aligned} \tag{4.181}$$

and the optimal controller is given by

$$\begin{aligned} -\dot{\tilde{p}} &= A'\tilde{p}, \ k\tau < t < (k+1)\tau, \\ \tilde{p}(ih^-) &= (A_d + B_2\hat{F} + \hat{H}C_2 - B_2LC_2)'\tilde{p}(k\tau) - (\hat{F} - LC_2)'\tilde{y}(k), \\ \tilde{u}(k) &= (\hat{H} - B_2L)'\tilde{p}(k\tau) + L'\tilde{y}(k). \end{aligned}$$

Hence the forward controller (4.167) is optimal for the system $\bar{\mathbf{G}}_j$ and hence for the system $\mathbf{G}_j$ and

$$\min_{K \in \mathbf{K}} \| G \|_2^2 = \| G_c \|_2^2 + tr.T_2 \hat{F}[I + Y(0)C_2'C_2]^{-1}Y(0)\hat{F}'.$$

Now we express $\| G_c \|_2^2$ using the controllability gramian G_{cj} which is a unique τ-periodic nonnegative solution of

$$\begin{aligned} -\dot{L}_o &= A'L_o + L_oA + C_1'C_1, \ k\tau < t < (k+1)\tau, \\ L_o(k\tau) &= (A_d + B_2\hat{F})'L_o(k\tau^+)(A_d + B_2\hat{F}) + \hat{F}'\hat{F}. \end{aligned}$$

But X in Lemma 4.27 satisfies the equation above and hence $L_o = X$. Then by Theorem 4.1, we have (4.168) and the proof of Theorem 4.24 is complete.

4.5 Notes and References

Jump systems in the H_∞ context were first introduced in [67, 68] and used to solve H_∞ control and filtering problems for sampled-data systems. More general jump systems were then considered in [65] and the disturbance attenuation problems were solved.

This chapter is based on [37, 43] and is developed parallel to Chapters 2 and 3. Stability of jump systems is discussed within the H_∞ theory in [67, 81, 82], but independent developments as in Section 4.1.1 were not available. The disturbance attenuation problems for jump systems were studied in [65] and Section 4.1.4 gives a generalization in that initial and terminal conditions are allowed and treated symmetrically on finite horizons.

H_∞ control for jump systems were considered in [36, 37, 64, 68]. Section 4.2 generalizes these papers and allow for initial uncertainty and the output of the terminal state, gives the characterization of all suboptimal controllers and discusses the relation of three Riccati equations as in [38]. The differential game results in Section 4.2.2 are taken from [35]. The jump system (4.42) is not fully general although it suffices to consider sampled-data systems in Chapter 5. Section 4.2.5 discuss an extension to a fully general jump system.

The H_∞ filtering problem for continuous-time systems with sampled observation was considered in [67]. Section 4.3 derives H_∞ filtering for jump systems. We give the set of all suboptimal filters. We allow for nonzero initial conditions and the estimation of the terminal state.

The H_2 results in Section 4.4 are taken from [37]. H_2 control for jump systems is not discussed anywhere else. The reason seems to be that H_2 control for sampled-data system can be easily reduced to that of discrete-time systems.

5. Sampled-data Systems

In this chapter we consider sampled-data systems with zero-order hold. We first convert them to jump systems and then solve the H_∞ control problem with initial uncertainty and the H_2 control problem.

5.1 Jump System Approach

We shall show how to transform the sampled-data systems to jump systems. Then we apply the results in Chapter 4 on stability, H_2 and H_∞ norms, disturbance attenuation problems and quadratic control.

5.1.1 Transformation to Jump Systems

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12} \tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T),\ 0 \leq N\tau \leq T < (N+1)\tau \end{aligned} \tag{5.1}$$

with initial condition

$$x(0) = Hh,\ h \in \mathbf{R}^{n_1}$$

where $x \in \mathbf{R}^n$ is the state, $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbance, $\tilde{u} \in \mathbf{R}^{m_2}$ is the control input realized through a zero-order hold,

$$\tilde{u}(t) = u(k),\ k\tau < t \leq (k+1)\tau,$$

$(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1+m_2}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the sampled observation, τ is a sampling period, $C_1 \in \mathbf{R}^{p_1 \times n}$, $D_{12} \in \mathbf{R}^{m_2 \times m_2}$ and other matrices are of compatible dimensions. For the system $\mathbf{G}_s$ we introduce discrete-time controllers of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k) \end{aligned} \tag{5.2}$$

where $\hat{x} \in \mathbf{R}^{\hat{n}}$ and all matrices are of compatible dimensions. Since the system $\mathbf{G}_s$ is essentially a continuous-time system and the controller is a discrete-time system, we need two devices, a zero-order hold and a sampler, to connect these two systems.

We first express this system as a jump system of the form (4.42). Since the control $\tilde{u}(t)$ is constant between two sampling instants, i.e., $k\tau < t \leq (k+1)\tau$, we can introduce the following state space representation:

$$\dot{\bar{x}} = 0, \ \bar{x}(k\tau^+) = u(k), \ k\tau < t \leq (k+1)\tau.$$

Then clearly $\tilde{u}(t) = \bar{x}(t)$. Let

$$x_e(t) = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}(t)$$

be the new state variable. Then the sampled-data system $\mathbf{G}_s$ is equivalent to the following system with jumps (denoted by $\mathbf{G}_e$):

$$\begin{aligned} \dot{x}_e(t) &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} x_e(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t), \ k\tau < t < (k+1)\tau, \\ x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \ k = 0, 1, 2, ..., \\ z &= \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = \begin{bmatrix} [C_1 \ \ 0]\, x_e(t) \\ \sqrt{\tau} D_{12} u(k) \end{bmatrix}, \\ y(k) &= [C_2 \ \ 0]\, x_e(k\tau) + D_{21} w_d(k), \\ z_1 &= [F \ \ 0]\, x_e(T) \end{aligned} \tag{5.3}$$

and

$$x_e(0) = [H \ \ 0] \begin{bmatrix} h \\ 0 \end{bmatrix}.$$

Here $z_d = \sqrt{\tau} D_{12} u(k)$ comes from

$$\int_0^\infty | D_{12}\tilde{u}(t) |^2 = \sum_{k=0}^{\infty} \int_0^\tau | D_{12} u(k) |^2 = \sum_{k=0}^{\infty} | \sqrt{\tau} D_{12} u(k) |^2, u(\cdot) \in l^2.$$

Since the system $\mathbf{G}_e$ is a jump system, we can solve the H_2 and H_∞ control problems for the sampled-data systems using results in Chapter 4.

For the systems $\mathbf{G}_s$ and $\mathbf{G}_e$, we can easily obtain the following result.

Lemma 5.1 *If* (A, B_1, C_1) *is stabilizable and detectable for the system* $\mathbf{G}_s$, *then the jump system*

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right], \left[\begin{bmatrix} B_1 \\ 0 \end{bmatrix}, 0 \right], [[C_1 \ \ 0], 0] \right)$$

is stabilizable and detectable.

Proof. Since (A, B_1) is stabilizable, there exists a matrix K such that the system

$$\dot{\xi} = (A + B_1 K)\xi$$

is exponentially stable. Then the system

$$\begin{aligned} \dot{\xi}_e(t) &= \left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} [K \quad 0] \right) \xi_e(t) = \begin{bmatrix} A + B_1 K & B_2 \\ 0 & 0 \end{bmatrix} \xi_e(t), \\ & \qquad k\tau < t \le (k+1)\tau, \\ \xi_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \xi_e(k\tau) \end{aligned}$$

is obviously exponentially stable and hence we have the assertion. We can show the detectability in a similar manner. ∎

However, stabilizability and detectability of (A, B_2, C_2) does not imply that of

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right], \left[0, \begin{bmatrix} 0 \\ I \end{bmatrix} \right], [0, [C_2 \quad 0]] \right)$$

under the special sampling period. We shall show this in the next section.

5.1.2 Comments on the Sampling Period

Consider the system

$$\dot{x} = Ax + B_2 \tilde{u}, \quad y(k) = C_2 x(k\tau) \tag{5.4}$$

where

$$\tilde{u}(t) = u(k), \ k\tau < t \le (k+1)\tau$$

and assume that (A, B_2, C_2) is stabilizable and detectable in the usual sense. As we see in the previous section, the system (5.4) is equivalent to the following jump system

$$\begin{aligned} \dot{x}_e(t) &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} x_e(t), \ k\tau < t < (k+1)\tau, \\ x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \ k = 0, 1, 2, ..., \\ y(k) &= [C_2 \quad 0] \, x_e(k\tau) \end{aligned} \tag{5.5}$$

and moreover

$$\begin{aligned} x((k+1)\tau) &= e^{A\tau} x(k\tau) + \Pi(\tau) B_2 u(k), \\ y(k) &= C_2 x(k\tau) \end{aligned} \tag{5.6}$$

where $\Pi(t) = \int_0^t e^{Ar} dr$. Note that (5.5) is stabilizable and detectable if and only if (5.6) is stabilizable and detectable. We now introduce an important notion about a sampling period τ.

Definition 5.1 *The sampling period τ is called pathological (with respect to A) if A has two eigenvalues, say λ and $\tilde{\lambda}$, such that*

$$\lambda = \sigma + j\omega, \quad \tilde{\lambda} = \sigma + j\tilde{\omega}$$

with $\mid \omega - \tilde{\omega} \mid = k\frac{2\pi}{\tau}$ for some positive integer k. Otherwise τ is called nonpathological.

The following example [8] shows that if the sampling period is pathological, the stabilizability and detectability of (A, B_2, C_2) does not necessarily imply that of $(e^{A\tau}, \Pi(\tau)B_2, C_2)$.

Example 5.1 ([8]) Consider the sampled-data system with sampling period τ:

$$\begin{aligned} \dot{x} &= Ax + b_2\tilde{u}, \ A = \begin{bmatrix} 0 & 1 \\ -(\frac{2\pi}{\tau})^2 & 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ y(k) &= c_2 x(k\tau), \ c_2 = [0 \quad 1] \end{aligned} \tag{5.7}$$

(A, b_2, c_2) is obviously controllable and observable (and hence stabilizable and detectable). Note that eigenvalues of A are

$$\lambda = 0 + j\omega, \ 0 + j\tilde{\omega}, \ \omega = \frac{2\pi}{\tau}, \ \tilde{\omega} = -\frac{2\pi}{\tau}.$$

Since $\omega - \tilde{\omega} = 2\frac{2\pi}{\tau}$, the sampling period τ is pathological. Now

$$\begin{aligned} e^{At} &= \begin{bmatrix} \cos(\frac{2\pi}{\tau}t) & \frac{\tau}{2\pi}\sin(\frac{2\pi}{\tau}t) \\ -\frac{2\pi}{\tau}\sin(\frac{2\pi}{\tau}t) & \cos(\frac{2\pi}{\tau}t) \end{bmatrix}, \\ e^{A\tau} &= \begin{bmatrix} \cos(2\pi) & \frac{\tau}{2\pi}\sin(2\pi) \\ -\frac{2\pi}{\tau}\sin(2\pi) & \cos(2\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \Pi(\tau)b_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence the discrete-time system obtained from the sampled-data system (5.7) in neither stabilizable nor detectable.

However, if the sampling period is nonpathological, stabilizability and detectability are preserved [8].

Lemma 5.2 *Assume that the sampling period τ is nonpathological. Then (A, B_2, C_2) is stabilizable and detectable if and only if $(e^{A\tau}, \Pi(\tau)B_2, C_2)$ is stabilizable and detectable.*

Proof. Note that $e^{\lambda_i\tau}$ is an eigenvalue of $e^{A\tau}$ if λ_i is an eigenvalue of A. Using the Taylor expansion of $e^{s\tau} - e^{\lambda_i\tau}$ we can write

$$e^{s\tau} - e^{\lambda_i\tau} = g(s)(s - \lambda_i).$$

Hence

$$\begin{bmatrix} e^{A\tau} - e^{\lambda_i \tau} I \\ C_2 \end{bmatrix} = \begin{bmatrix} g(A) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda_i I \\ C_2 \end{bmatrix}.$$

Now we shall show that $g(A)$ is nonsingular. It is enough to show that 0 is not an eigenvalue of $g(A)$. Since the eigenvalues of $g(A)$ are $g(\lambda_k)$, we shall show $g(\lambda_k) \neq 0$ for any eigenvalue λ_k of A. If $\lambda_k \neq \lambda_i$, then $g(\lambda_k) \neq 0$ since otherwise

$$e^{\lambda_k \tau} - e^{\lambda_i \tau} = g(s)(\lambda_k - \lambda_i) = 0$$

which contradicts the nonpathological assumption of τ. Moreover, by direct calculation

$$g(\lambda_i) = \tau e^{\lambda_i \tau} \neq 0.$$

Hence

$$\text{rank} \begin{bmatrix} e^{A\tau} - e^{\lambda_i \tau} I \\ C_2 \end{bmatrix} = \text{rank} \begin{bmatrix} A - \lambda_i I \\ C_2 \end{bmatrix}$$

for any eigenvalue λ_i of A. Since $| e^{\lambda_i \tau} | \geq 1$ if and only if $Re\lambda_i \geq 0$, detectability is preserved. Considering the adjoint of the original system we can show that stabilizability is also preserved. ∎

5.1.3 Stability

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \ x(0) = Hh, \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12} \tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T), \ 0 \leq N\tau \leq T < (N+1)\tau \end{aligned}$$

and the discrete-time controllers (5.2)

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k) \end{aligned}$$

where we assume that (A, B_2, C_2) is stabilizable and detectable and the sampling period is nonpathological. Since the system $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the controller (5.2) is equivalent to the following jump system

$$\begin{aligned} \dot{\hat{x}} &= 0, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}\hat{x}(k\tau) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k\tau) + \hat{D}y(k), \end{aligned} \tag{5.8}$$

the closed-loop system $\mathbf{G}_s$ and (5.2) (and hence $\mathbf{G}_e$ and (5.8)) is given by

$$\begin{aligned} \dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}w, \ k\tau < t < (k+1)\tau, \\ x_{cl}(k\tau^+) &= A_{dcl}x_{cl}(k\tau) + B_{dcl}w_d(k), \qquad (5.9) \\ z_c(t) &= C_{cl}x_{cl}, \\ z_d(k) &= C_{dcl}x_{cl}(k\tau) + D_{dcl}w_d(k), \\ z_1 &= F_{cl}x_{cl}(T) \qquad (5.10) \end{aligned}$$

and

$$x_{cl}(0) = H_{cl}\begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \qquad (5.11)$$

which is the jump system of the form (4.4) where $x_{cl} = [x_e' \quad \hat{x}']'$ and

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_{dcl} &= \begin{bmatrix} I & 0 & 0 \\ \hat{D}C_2 & 0 & \hat{C} \\ \hat{B}C_2 & 0 & \hat{A} \end{bmatrix}, \\ B_{cl} &= \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, & B_{dcl} &= \begin{bmatrix} 0 \\ \hat{D}D_{21} \\ \hat{B}D_{21} \end{bmatrix}, \\ C_{cl} &= [C_1 \quad 0 \quad 0], & C_{dcl} &= [\sqrt{\tau}D_{12}\hat{D}C_2 \quad 0 \quad \sqrt{\tau}D_{12}\hat{C}], \\ D_{dcl} &= \sqrt{\tau}D_{12}D_{21}, & & \\ F_{cl} &= [F \quad 0 \quad 0], & H_{cl} &= [H \quad 0 \quad 0]. \end{aligned}$$

Hence we can consider stability, H_2, H_∞ norms and the disturbance attenuation problems of the sampled-data feedback systems using the system (5.9) and the results in Chapter 4.

Lyapunov Equations

By applying Propositions 4.2, 4.3 and Corollary 4.1 to the homogeneous system of (5.9), we have the following result.

Proposition 5.1 *The following statements are equivalent.*
(a) The feedback system (5.9) is exponentially stable.
(b) There exists a τ-periodic symmetric matrix $X(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$ such that

$$\begin{aligned} &(i) \quad c_1 I \le X(t) \le c_2 I, \ ^\forall t \ge 0 \text{ for some } c_i > 0, \ i = 1,2, \\ &(ii) \quad -\dot{X} = A_{cl}'X + XA_{cl} + I, \ k\tau < t < (k+1)\tau, \\ &\qquad\ X(k\tau^-) = A_{dcl}'X(k\tau)A_{dcl} + I. \end{aligned}$$

(c) There exists a symmetric matirx $Y(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$ and a

$0 < \delta < \tau$ *such that*

$$\begin{array}{ll} (i) & 0 < Y(t),\ ^\forall t \geq 0 \text{ and } c_1 I \leq Y(t),\ ^\forall t \geq \delta \text{ for some } c_1 > 0, \\ (ii) & Y(t) \leq c_2 I,\ 0 \leq\ ^\forall t < \infty \text{ for some } c_2 > 0, \\ (iii) & \dot{Y} = A_{cl} Y + Y A'_{cl} + I,\ k\tau < t < (k+1)\tau, \\ & Y(k\tau^+) = A_{dcl} Y(k\tau) A'_{dcl} + I, \\ & Y(0) = 0. \end{array}$$

(d) There exists a τ-periodic symmetric solution $Y_\tau(t)$ of (iii) in (c) without $Y(0) = 0$ such that $c_1 I \leq Y_\tau(t) \leq c_2 I$ for some c_1, $c_2 > 0$.

H_2 and H_∞ Norms

Now we assume that the sampled-data feedback system (5.9) is exponentially stable and $h = 0$. Then we can define its H_2-norm as in Definition 4.7 and calculate it using Theorem 4.1.

Proposition 5.2 *Let $\| G \|_2$ be the H_2-norm of the system (5.9). Then*

$$\| G \|_2^2 = \frac{1}{\tau} \int_0^\tau tr.B'_{cl} X(s) B_{cl}\ ds + tr.[B'_{dcl} X(0) B_{dcl} + D'_{dcl} D_{dcl}]$$

where X is a τ-periodic nonnegative solution of

$$\begin{aligned} -\dot{X} &= A'_{cl} X + X A_{cl} + C'_{cl} C_{cl},\ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A'_{dcl} X(k\tau) A_{dcl} + C'_{dcl} C_{dcl}. \end{aligned}$$

We can also define the H_∞-norm of the sampled-data feedback system (5.9) as in Definition 4.8.

Disturbance Attenuation Problems

Let G_T be the input-output operator of the sampled-data feedback system (5.9)-(5.11) on $[0, T]$. Then by Theorem 4.6 we have the following result.

Proposition 5.3 *The following statements are equivalent.*
(a) $\| G_T \| < \gamma$.
(b) There exists a nonnegative solution $X(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$, $t \in [0, T]$ to (4.29)-(4.33) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.
(b) There exists a nonnegative solution $Y(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$, $t \in [0, T]$ to (4.34)-(4.38) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.

Next we consider the system (5.9) and (5.11) on $[0, \infty)$. We assume that (A_{cl}, A_{dcl}) is exponentially stable. Let G be the input-output operator of the system (5.9) and (5.11). Then by Theorem 4.7 we have the following result.

Proposition 5.4 *The following statements are equivalent.*
(a) $\| G \| < \gamma$.
(b) There exists a τ-periodic nonnegative stabilizing solution $X(t)$, $t \in [0, \infty)$ to (4.29)-(4.31) satisfying (4.33) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.
(b) There exists a τ-periodic nonnegative stabilizing solution $Y(t)$, $t \in [0, \infty)$ to (4.34)-(4.37) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.

5.1.4 Quadratic Control

Consider the system

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \ \tilde{u}(t) = u(k), \ k\tau < t \le (k+1)\tau, \\ x(0) &= x_0 \end{aligned} \tag{5.12}$$

and the functional to be minimized

$$\begin{aligned} J_T(\tilde{u}; x_0) &= \int_0^T [\mid Cx(t) \mid^2 + \mid \tilde{u}(t) \mid^2] dt + \mid Fx(T) \mid^2, \\ & \qquad 0 \le N\tau \le T < (N+1)\tau \end{aligned} \tag{5.13}$$

where $x \in \mathbf{R}^n$, $\tilde{u} \in \mathbf{R}^{m_2}$, $C \in \mathbf{R}^{p_2 \times n}$ and other matrices are of compatible dimensions. Since the system (5.12) and the functional (5.13) are equivalent to the jump system

$$\begin{aligned} \dot{x}_e &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x_e, \ k\tau < t < (k+1)\tau, \\ x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \ x_e(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ z &= \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = \begin{bmatrix} [C \ \ 0] x_e \\ \sqrt{\tau} u(k) \end{bmatrix} \end{aligned} \tag{5.14}$$

and the functional

$$\tilde{J}_T(u; x_0) = \int_0^T \mid z_c(t) \mid^2 dt + \sum_{k=0}^{N} \mid z_d(k) \mid^2 + \mid [F \quad 0] x_e(T) \mid^2$$

we can apply Theorems 4.2 and 4.3. Let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \ X_1 \in \mathbf{R}^{n \times n}, \ X_{12} \in \mathbf{R}^{n \times m}, \ X_2 \in \mathbf{R}^{m \times m}$$

be the solution of the Riccati equation (4.12)-(4.14) with

$$T_2(k) = I + B_d' X(k\tau) B_d$$

replaced by

$$T_2(k) = \tau I + B_d' X(k\tau) B_d.$$

Then we obtain for $k\tau < t < (k+1)\tau$

$$\begin{aligned} -\dot{X}_1 &= A'X_1 + X_1 A + C'C, \\ -\dot{X}_{12} &= X_{12}A' + B'X_1, \\ -\dot{X}_2 &= B'X_{12} + X_{12}'B \end{aligned} \tag{5.15}$$

and at $t = k\tau$

$$\begin{aligned} X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\ X_{12}(k\tau^-) &= 0, \\ X_2(k\tau^-) &= 0 \end{aligned} \tag{5.16}$$

with

$$\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix} (T) = \begin{bmatrix} F'F & 0 \\ 0 & 0 \end{bmatrix} \tag{5.17}$$

and by Theorem 4.2 we have the following result.

Theorem 5.1 *There exists a unique nonnegative solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$, $X_1 \in \mathbf{R}^{n \times n}$, $X_{12} \in \mathbf{R}^{n \times m}$, $X_2 \in \mathbf{R}^{m \times m}$ *to the Riccati equation (5.15)-(5.17). Moreover, the state feedback law*

$$\bar{\tilde{u}}(t) = \bar{u}(k), \;\; \bar{u}(k) = -[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau) x(k\tau), \;\; k\tau < t \le (k+1)\tau$$

is optimal and

$$J_T(\bar{\tilde{u}}; x_0) = \tilde{J}_T(\bar{u}; x_0) = x_0' X_1(0^-) x_0.$$

Next we consider the infinite horizon problem

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \;\; x(0) = x_0, \\ J(\tilde{u}; x_0) &= \int_0^\infty [| \, Cx(t) \, |^2 + | \, \tilde{u}(t) \, |^2] dt \end{aligned}$$

where $\tilde{u} \in L^2(0, \infty; \mathbf{R}^{m_2})$ is admissible if its response $x \in L^2(0, \infty; \mathbf{R}^n)$ and $\lim_{t \to \infty} x(t) = 0$. This problem is again equivalent to

$$\tilde{J}(u; x_0) = \int_0^\infty | \, z_c(t) \, |^2 \, dt + \sum_{k=0}^{\infty} | \, z_d(k) \, |^2$$

subject to the jump system (5.14) where $u \in l^2(0, \infty; \mathbf{R}^{m_2})$ is admissible if its response $x_e \in L^2(0, \infty; \mathbf{R}^{n+m_2})$ and $\lim_{t \to \infty} x_e(t) = 0$.

Now we assume that (A, B) is stabilizable and the sampling period is nonpathological for the system (5.12). Then by Lemma 5.2

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ I \end{bmatrix}\right]\right)$$

is stabilizable and the condition **RJ** in Section 4.1.3 for the system (5.14) is satisfied. If we further assume that (C, A) is detectable, then by Lemma 5.1

$$\left([[C \quad 0], 0]\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right]\right)$$

is detectable. Hence we can apply Theorem 4.3 to the system (5.14) (and hence (5.12)). We say that $X = \begin{bmatrix} X_1 & X_{12} \\ X'_{12} & X_2 \end{bmatrix}$ is a stabilizing solution of (5.15) and (5.16) if

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ -[\tau I + X_2(k\tau)]^{-1}X'_{12}(k\tau) & 0 \end{bmatrix}\right)$$

is exponentially stable, which is equivalent to the stability of the system

$$\begin{aligned} \dot{x} &= Ax + B\hat{u}(t), \\ \hat{u}(t) &= -[\tau I + X_2(k\tau)]^{-1}X'_{12}(k\tau)x(k\tau), \;\; k\tau < t < (k+1)\tau \end{aligned}$$

and equivalently that of the discrete-time system

$$x((k+1)\tau) = \left\{e^{A\tau} - \Pi(\tau)B[\tau I + X_2(k\tau)]^{-1}X'_{12}(k\tau)\right\} x(k\tau).$$

Summing up we have the following result.

Theorem 5.2 *Suppose (C, A, B) is stabilizable and detectable and the sampling period τ is nonpathological. Then there exists a τ-periodic nonnegative stabilizing solution $X = \begin{bmatrix} X_1 & X_{12} \\ X'_{12} & X_2 \end{bmatrix}$ to (5.15) and (5.16). Moreover, the state feedback law*

$$\begin{aligned} \bar{\bar{u}}(t) &= \bar{u}(k), \\ \bar{u}(k) &= -[\tau I + X_2(0)]^{-1}X'_{12}(0)x(k\tau), \;\; k\tau < t \le (k+1)\tau \end{aligned}$$

is optimal and

$$J_T(\bar{\bar{u}}; x_0) = \tilde{J}_T(\bar{u}; x_0) = x'_0 X_1(0^-)x_0.$$

Example 5.2 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tilde{u}(t), \;\; \tilde{u}(t) = u(k), \;\; k < t \le k+1$$

and the functional

$$J(\tilde{u}; x_0) = \int_0^\infty [\| x_1(t) |^2 + | \tilde{u}(t) |^2] dt$$

For this system and the functional the assumptions of Theorem 5.2 are satisfied. Then there exists a periodic nonnegative stabilizing solution $X(t) = [X_{ij}(t)]$, $i, j = 1, 2, 3$ with period 1 of the Riccati equation (5.15) and (5.16). Figure 5.1 shows the periodic solution $X(t)$. Figure 5.2 shows the response of the closed-loop system with $x_1(0) = 1$ and $x_2(0) = 0$ to the optimal state feedback.

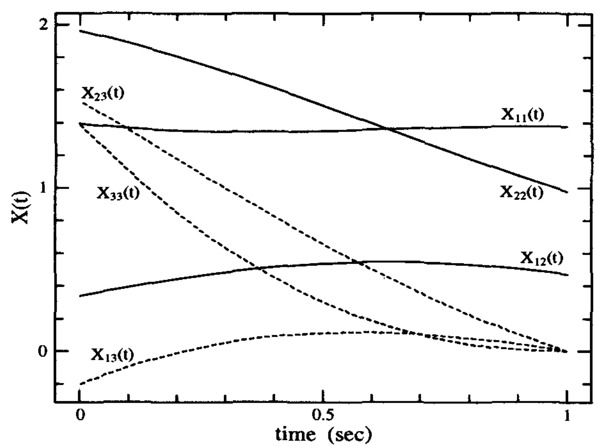

Figure 5.1: The periodic nonnegative solution $X(t)$

5.2 H$_\infty$ Control

Here we consider the H$_\infty$ control problem initial uncertainty. We apply the results in Section 4.2 to the jump systems obtained from the sampled-data systems.

5.2.1 Finite Horizon Problems

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12} \tilde{u}(t) \end{bmatrix}, \qquad (5.18) \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T),\ 0 \le N\tau \le T < (N+1)\tau \qquad (5.19) \end{aligned}$$

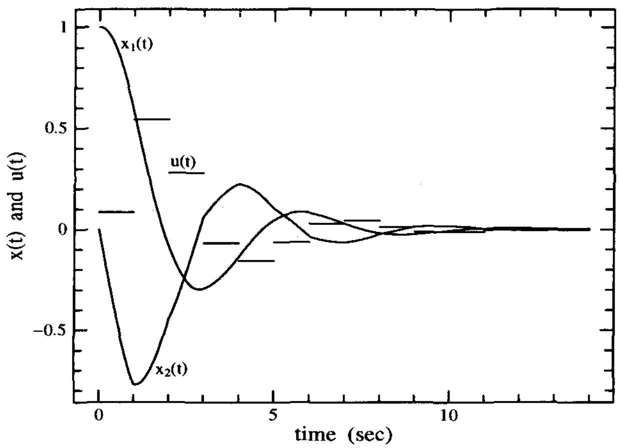

Figure 5.2: Simulation result

with initial condition

$$x(0) = Hh \tag{5.20}$$

and a discrete-time controller of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k). \end{aligned} \tag{5.21}$$

For the system $\mathbf{G}_s$ we assume

$$\mathbf{S1} : D_{12}' D_{12} = I, \ D_{21} D_{21}' = I.$$

Consider the sampled-data system $\mathbf{G}_s$ and a discrete-time controller $u = Ky$ of the form (5.21) on $[0, T]$. Define the input-output operator of the closed-loop system by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.$$

Then

$$\begin{aligned} G \ \in \ & \mathcal{L}(\mathbf{R}^{n_1} \times L^2(0,T;\mathbf{R}^{m_1}) \times l^2(0,N;\mathbf{R}^{m_{1d}}); \\ & \mathbf{R}^q \times L^2(0,T;\mathbf{R}^{p_1}) \times l^2(0,N;\mathbf{R}^{p_{1d}})). \end{aligned}$$

The H_∞-problem for the sampled-data system $\mathbf{G}_s$ is to find necessary and sufficient conditions for the existence of a discrete-time controller such that $\| G \| < \gamma$, i.e.,

$$| z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2 \times l^2}^2 \leq d^2 (| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2) \text{ for some } 0 < d < \gamma.$$

Such a controller is called γ-suboptimal. Since the sampled-data system $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the assumption **S1** implies the conditions **J1** for the system $\mathbf{G}_e$, we can apply Theorems 4.8 and 4.9 to the system $\mathbf{G}_e$ and hence for the system $\mathbf{G}_s$.

Remark 5.1 The standard way to solve the H$_\infty$ and H$_2$ problems for $\mathbf{G}_s$ with $D_{21} = 0$ is to use the lifting technique which converts periodic systems to discrete-time systems with infinite dimensional input and output spaces and to reduce the original problems to those for ordinary discrete-time systems [2, 3, 8, 25, 76, 88]. We shall show that their lifted system is directly obtained from $\mathbf{G}_e$. In fact for $k\tau < t \le (k+1)\tau$ we have

$$x_e(t) = e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(t-k\tau)} x_e(k\tau^+) + \int_{k\tau}^{t} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(t-r)} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(r)dr.$$

Since

$$e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}t} = \begin{bmatrix} e^{At} & \Pi(t)B_2 \\ 0 & I \end{bmatrix},$$

we have for $k\tau < t \le (k+1)\tau$

$$\begin{aligned} x_e(t) &= \begin{bmatrix} x \\ \bar{x} \end{bmatrix}(t) \\ &= \begin{bmatrix} e^{A(t-k\tau)}x(k\tau^+) + \Pi(t-k\tau)B_2u(k) + \int_{k\tau}^{t} e^{A(t-r)}B_1w(r)dr \\ u(k) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} x((k+1)\tau) &= e^{A\tau}x(k\tau^+) + \Pi(\tau)B_2u(k) + \int_{k\tau}^{(k+1)\tau} e^{A[(k+1)\tau-r]}B_1w(r)dr \\ &= e^{A\tau}x(k\tau) + \Pi(\tau)B_2u(k) + \int_0^{\tau} e^{A(\tau-s)}B_1w(s+k\tau)ds \\ &= e^{A\tau}x(k\tau) + \Pi(\tau)B_2u(k) + \int_0^{\tau} e^{A(\tau-s)}B_1\hat{w}_k(s)ds \end{aligned}$$

where $\hat{w}_k(s) = w(s+k\tau)$. We also have

$$\begin{aligned} z_c(t) &= C_1e^{A(t-k\tau)}x(k\tau) + C_1\int_0^{t-k\tau-s} e^{A(t-k\tau-s)}B_1\hat{w}_k(s)ds \\ &\quad +C_1\Pi(t-k\tau)B_2u(k), \\ z_d(k) &= \sqrt{\tau}D_{12}u(k). \end{aligned}$$

Hence the system $\mathbf{G}_s$ is equivalent to the following lifted system (denoted by $\tilde{\mathbf{G}}$)

$$\hat{x}(k+1) = e^{A\tau}\hat{x}(k) + \int_0^{\tau} e^{A(\tau-s)}B_1\hat{w}_k(s)ds + \Pi(\tau)B_2u(k),$$

$$\begin{aligned} z_c(t) &= C_1 e^{A(t-k\tau)}\hat{x}(k) + C_1 \int_0^{t-k\tau-s} e^{A(t-k\tau-s)} B_1 \hat{w}_k(s) ds \\ &\quad + C_1 \Pi(t-k\tau) B_2 u(k), \\ z_d(k) &= \sqrt{\tau} D_{12} u(k), \\ y(k) &= C_2 \hat{x}(k) + D_{21} w_d(k). \end{aligned}$$

Contrary to the discrete-time representation $\tilde{\mathbf{G}}$ of the sampled-data system $\mathbf{G}_s$, the jump system $\mathbf{G}_e$ is a natural state space representation of $\mathbf{G}_s$ in the following sense.

(a) The genuine control input to $\mathbf{G}_s$ is $u(k)$ rather than $\tilde{u}(t)$.
(b) Original signals and parameters of $\mathbf{G}_s$ are maintained in the system $\mathbf{G}_e$ [37, 65].
(c) The H_∞ and H_2 problems can be treated in a unified manner as in Chapters 1-3. Hence it is easy to introduce the theory to those who are not familiar with sampled-data systems.
(d) The jump system approach to sampled-data control can be easily extended to more general cases of delayed observation [45], a first-order hold [32, 34] and infinite dimensions [33] (see Chapter 6).

Let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \; \bar{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}' & Y_2 \end{bmatrix}$$

be the solutions of the Riccati equations (4.46)-(4.49) and (4.50)-(4.52), respectively with $T_2(k) = I + B_2'X(k\tau)B_2$ replaced by $T_2(k) = \tau I + B_2'X(k\tau)B_2$, where X_1, $Y_1 \in \mathbf{R}^{n\times n}$, X_{12}, $Y_{12} \in \mathbf{R}^{n\times m_2}$ and X_2, $Y_2 \in \mathbf{R}^{m_2\times m_2}$ and n and m_2 are the dimensions of x and u respectively. Then from the Riccati equation (4.46)-(4.49) we obtain

$$\begin{aligned} -\dot{X}_1 &= A'X_1 + X_1 A + C_1'C_1 + \frac{1}{\gamma^2} X_1 B_1 B_1' X_1, \\ -\dot{X}_{12} &= A'X_{12} + X_1 B_2 + \frac{1}{\gamma^2} X_1 B_1 B_1' X_{12}, \\ -\dot{X}_2 &= B_2'X_{12} + X_{12}'B_2 + \frac{1}{\gamma^2} X_{12}' B_1 B_1' X_{12} \end{aligned} \tag{5.22}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$, $k = 0, 1, , 2, ..$

$$\begin{aligned} X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\ X_{12}(k\tau^-) &= 0, \\ X_2(k\tau^-) &= 0 \end{aligned} \tag{5.23}$$

with

$$\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(T) = \begin{bmatrix} F'F & 0 \\ 0 & 0 \end{bmatrix} \tag{5.24}$$

and

$$H'X_1(0^-)H \le d^2 I \text{ for some } 0 < d < \gamma. \tag{5.25}$$

The second equation is written

$$\begin{aligned}
\dot{Y}_1 &= AY_1 + Y_1A' + B_1B_1' + \frac{1}{\gamma^2}Y_1C_1'C_1Y_1 + B_2Y_{12}' + Y_{12}B_2', \\
\dot{Y}_{12} &= AY_{12} + B_2Y_2 + \frac{1}{\gamma^2}Y_1C_1'C_1Y_{12}, \\
\dot{Y}_2 &= \frac{1}{\gamma^2}Y_{12}C_1'C_1Y_{12}
\end{aligned}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$, $k = 1, 2, ..$

$$\begin{aligned}
Y_1(k\tau^+) &= Y_1(k\tau) - Y_1(k\tau)C_2'(I + C_2Y_1(k\tau)C_2')^{-1}C_2Y_1(k\tau), \\
Y_{12}(k\tau^+) &= 0, \\
Y_2(k\tau^+) &= 0, \qquad\qquad k = 0, 1, ..
\end{aligned}$$

with

$$\begin{bmatrix} Y_1(0) & Y_{12}(0) \\ Y_{12}'(0) & Y_2(0) \end{bmatrix} = \begin{bmatrix} HH' & 0 \\ 0 & 0 \end{bmatrix}.$$

Since Y_{12} and Y_2 form a homogeneous system, we conclude $Y_{12} = 0$ and $Y_2 = 0$. Hence $\bar{Y}$ is of the form $\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$, where $Y \in \mathbf{R}^{n\times n}$ is the solution of

$$\dot{Y} = AY + YA + B_1B_1' + \frac{1}{\gamma^2}YC_1'C_1Y, \tag{5.26}$$
$$k\tau < t < (k+1)\tau,$$
$$Y(k\tau^+) = Y(k\tau) - Y(k\tau)C_2'(I + C_2Y(k\tau)C_2')^{-1}C_2Y(k\tau), \tag{5.27}$$
$$Y(0) = HH'. \tag{5.28}$$

Replacing Z by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ in (4.57), we obtain a γ-suboptimal controller

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}(t)\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \quad k\tau < t < (k+1)\tau, \qquad (5.29) \\
\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau^+) &= \begin{bmatrix} \hat{A}_d & 0 \\ \hat{C}_1 & 0 \end{bmatrix}(k)\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \begin{bmatrix} \hat{B}_1 \\ \hat{D}_{11} \end{bmatrix}(k)y(k) \\
&\quad + \begin{bmatrix} \hat{B}_2 \\ \hat{D}_{12} \end{bmatrix}(k)v(k), \\
u(k) &= [\hat{C}_1(k) \quad 0]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\
r(k) &= [\hat{C}_2(k) \quad 0]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \hat{D}_{21}(k)y(k), \\
v &= Qr, \ Q \in Q_\gamma
\end{aligned}$$

where

$$\begin{aligned}
\hat{A}_1(t) &= A + \frac{1}{\gamma^2} B_1 B_1' X_1(t), \\
\hat{A}_2(t) &= B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t), \\
\hat{A}_d(k) &= (I + W(k) Y(k\tau) C_2' C_2)^{-1}, \\
\hat{B}_1(k) &= \hat{A}_d(k) W(k) Y(k\tau) C_2', \\
\hat{B}_2(k) &= \frac{1}{\gamma} \hat{A}_d(k) W(k) Y(k\tau) X_{12}(k\tau) E^{-\frac{1}{2}}(k) \Xi^{-\frac{1}{2}}(k), \\
\hat{C}_1(k) &= -E^{-1}(k) X_{12}'(k\tau) \hat{A}_d(k), \\
\hat{C}_2(k) &= -(I + C_2 W(k) Y(k\tau) C_2')^{-\frac{1}{2}} C_2, \\
\hat{D}_{11}(k) &= -E^{-1}(k) X_{12}'(k\tau) \hat{A}_d(k) W(k) Y(k\tau) C_2', \\
\hat{D}_{12}(k) &= \frac{1}{\gamma} E^{-\frac{1}{2}}(k) \Xi^{\frac{1}{2}}(k), \\
\hat{D}_{21}(k) &= (I + C_2 W(k) Y(k\tau) C_2')^{-\frac{1}{2}}, \\
W(k) &= [I - \frac{1}{\gamma^2} Y(k\tau) X_1(k\tau^-)]^{-1}, \\
E(k) &= \tau I + X_2(k\tau), \\
\Xi(k) &= \gamma^2 I - E^{-\frac{1}{2}}(k) X_{12}'(k\tau) \hat{A}_d(k) W(k) Y(k\tau) X_{12}(k\tau) E^{-\frac{1}{2}}(k)
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
Q_\gamma = \{ & Q \in \mathcal{L}(l^2(0,N;\mathbf{R}^{p_2}); l^2(0,N;\mathbf{R}^{m_2})) : \\
& \text{Q is of the form (4.45) and } \| Q \| < \gamma \}.
\end{aligned}$$

Since

$$\begin{aligned}
\dot{\hat{x}}_2 &= 0, \ k\tau < t < (k+1)\tau, \\
\hat{x}_2(k\tau^+) &= \hat{C}_1(k)\hat{x}_1(k\tau) + \hat{D}_{11}(k) y(k) + \hat{D}_{12}(k) v(k),
\end{aligned}$$

we can rewrite (5.29) as

$$\begin{aligned}
\dot{\hat{x}} &= \hat{A}_1(t)\hat{x} + \hat{A}_2(t)\tilde{s}(t), \ k\tau < t < (k+1)\tau, \\
\hat{x}(k\tau^+) &= \hat{A}_d(k)\hat{x}(k\tau) + \hat{B}_1(k) y(k) + \hat{B}_2(k) v(k), \\
u(k) &= \hat{C}_1(k)\hat{x}(k\tau) + \hat{D}_{11}(k) y(k) + \hat{D}_{12}(k) v(k), \\
r(k) &= \hat{C}_2(k)\hat{x}(k\tau) + \hat{D}_{21}(k) y(k), \\
v &= Qr, \ Q \in Q_\gamma
\end{aligned} \tag{5.31}$$

where $\tilde{s}$ is given by

$$\tilde{s}(t) = u(k), \ k\tau < t \leq (k+1)\tau.$$

Summing up we have the following result.

Theorem 5.3 *Assume* **S1** *and consider the system* $\mathbf{G}_s$.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the following hold:*
(i) There exists a nonnegative solution $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t)$, $t \in [0,T]$, $X_1 \in \mathbf{R}^{n\times n}$, $X_{12} \in \mathbf{R}^{n\times m_2}$, $X_2 \in \mathbf{R}^{m_2\times m_2}$ *to the Riccati equation (5.22)-(5.25).*
(ii) There exists a nonnegative solution Y *to the Riccati equation (5.26)-(5.28).*
(iii) $\rho(\begin{bmatrix} X_1 Y \\ X_{12}Y \end{bmatrix}(t)) \le d^2$, $t \in [0,T]$, *for some* $0 < d < \gamma$.
(b) In this case the set of all γ*-suboptimal controllers of the form (4.45) is given by (5.31).*

We now convert the controller (5.31) to the usual discrete one. Let $S(\cdot,\cdot)$ be the state transition matrix of $\hat{A}_1$. Then $\hat{x}((k+1)\tau)$ in (5.31) is given by

$$\hat{x}((k+1)\tau) = S((k+1)\tau, k\tau)\hat{x}(k\tau^+) + \int_{k\tau}^{(k+1)\tau} S((k+1)\tau, r)A_2(r)\tilde{s}(r)dr.$$

Since $\tilde{s}(t)$, $k\tau < t \le (k+1)\tau$ is given by

$$\tilde{s}(t) = \hat{C}_1(k)\hat{x}(k\tau) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k),$$

we have

$$\hat{x}((k+1)\tau) = A_D(k)\hat{x}(k\tau) + B_{1D}(k)y(k) + B_{2D}(k)s(k)$$

where

$$\begin{aligned} A_D(k) &= S((k+1)\tau, k\tau)\hat{A}_d(k) + (\Theta\hat{C}_1)(k), \\ B_{1D}(k) &= S((k+1)\tau, k\tau)\hat{B}_1(k) + (\Theta\hat{D}_{11})(k), \\ B_{2D}(k) &= S((k+1)\tau, k\tau)\hat{B}_2(k) + (\Theta\hat{D}_{12})(k) \end{aligned}$$

and

$$\Theta(k) = \int_{k\tau}^{(k+1)\tau} S((k+1)\tau, r)\hat{A}_2(r)dr. \tag{5.32}$$

Hence the controller (5.31) is equivalent to the following discrete-time controller:

$$\begin{aligned} \hat{x}(k+1) &= A_D(k)\hat{x}(k) + B_{1D}(k)y(k) + B_{2D}(k)s(k), \\ u(k) &= \hat{C}_1(k)\hat{x}(k) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\ r(k) &= \hat{C}_2(k)\hat{x}(k) + \hat{D}_{21}(k)y(k), \\ v &= Qr,\ Q \in Q_\gamma^D \end{aligned} \tag{5.33}$$

where

$$Q_\gamma^D = \{Q \in \mathcal{L}(l^2(0,N;\mathbf{R}^{p_2}); l^2(0,N;\mathbf{R}^{m_2})) : \\ Q \text{ is of the form (5.21) and } \| Q \| < \gamma\}.$$

Hence we have the following result.

Theorem 5.4 *Assume* **S1** *and consider the system* $\mathbf{G}_s$.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the conditions (i)-(iii) in Theorem 5.3 hold.*
(b) In this case the set of all γ*-suboptimal controllers of the form (5.21) is given by (5.33).*

5.2.2 The Infinite Horizon Problem

Next we consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \;\; x(0) = Hh, \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned}$$

on $[0, \infty)$ and a controller $u = Ky$ of the form (5.21) where we assume **S1** and

S2 : (A, B_1, C_1) is stabilizable and detectable,
S3 : (A, B_2, C_2) is stabilizable and detectable,
S4 : The sampling period τ is nonpathological.

Assumptions **S1-S4** imply **J1-J4** for G_e. If the controller is IO-stabilizing (or internally stabilizing), then the closed-loop system is defined by

$$z = G\begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.$$

Then

$$G \in \mathcal{L}(\mathbf{R}^{n_1} \times L^2(0,\infty;\mathbf{R}^{m_1}) \times l^2(0,\infty;\mathbf{R}^{m_{1d}}); \\ L^2(0,\infty;\mathbf{R}^{p_1}) \times l^2(0,\infty;\mathbf{R}^{p_{1d}})).$$

The H_∞-problem on $[0, \infty)$ is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., an internally stabilizing discrete-time controller such that $\| G \| < \gamma$, i.e.,

$$\left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \le d^2\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2}\right) \text{ for some } 0 < d < \gamma.$$

Such a controller is called γ-suboptimal.

To give the solution of this problem, we first consider the stabilizing solutions of the Riccati equations (5.22), (5.23), (5.26) and (5.27). If $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ is a stabilizing solution of the Riccati equation (5.22) and (5.23), then

$$\left(\begin{bmatrix} A + \frac{1}{\gamma^2} B_1 B_1' X_1(t) & B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t) \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ -E^{-1}(k) X_{12}'(k\tau) & 0 \end{bmatrix} \right)$$

is exponentially stable. So the system

$$\begin{aligned} \dot{\xi} &= [A + \frac{1}{\gamma^2} B_1 B_1' X_1(t)]\xi + [B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t)]\tilde{v}(t), \\ \tilde{v}(k) &= -E^{-1}(k) X_{12}'(k\tau)\xi(k\tau), \ k\tau < t \le (k+1)\tau \end{aligned} \tag{5.34}$$

is exponentially stable. Similarly if Y is a stabilizing solution, then

$$\left(\begin{bmatrix} A + \frac{1}{\gamma^2} Y(t) C_1' C_1 & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I - Y(k\tau) C_2' (I + C_2 Y(k\tau) C_2')^{-1} C_2 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

is exponentially stable, which is equivalent to the exponential stability of the system

$$\begin{aligned} \dot{\xi} &= [A + \frac{1}{\gamma^2} Y(t) C_1' C_1]\xi, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^+) &= [I - Y(k\tau) C_2' (I + C_2 Y(k\tau) C_2')^{-1} C_2]\xi(k\tau). \end{aligned} \tag{5.35}$$

Remark 5.2 (a) If $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t)$ is a τ-periodic nonnegative stabilizing solution (5.22) and (5.23), then the exponential stability of the system (5.34) is equivalent to that of the following discrete-time system

$$\begin{aligned} \xi(k+1) &= [S((k+1)\tau, k\tau) - \Theta(k) E^{-1}(k) X_{12}(k\tau)]\xi(k) \\ &= [S(\tau, 0) - \Theta(0) E^{-1}(0) X_{12}(0)]\xi(k) \end{aligned}$$

where $S(\cdot, \cdot)$ is the state transition matrix of $A + \frac{1}{\gamma^2} B_1 B_1' X_1$ and $\Theta(k)$ is defined by (5.32).
(b) The exponential stability of the system (5.35) is equivalent to that of the following time-varying discrete-time system

$$\xi(k+1) = S_Y((k+1)\tau, k\tau)[I - Y(k\tau) C_2' (I + C_2 Y(k\tau) C_2')^{-1} C_2]\xi(k) \tag{5.36}$$

where $S_Y(\cdot, \cdot)$ is the state transition matrix of $A + \frac{1}{\gamma^2} Y C_1' C_1$. If $Y(t)$ is τ-periodic, then the system (5.36) becomes

$$\xi(k+1) = S_Y(\tau, 0)[I - Y(0) C_2' (I + C_2 Y(0) C_2')^{-1} C_2]\xi(k)$$

which is time-invariant.

Again we define Q_γ and Q_γ^D as

$$\begin{aligned} Q_\gamma &= \{Q \in \mathcal{L}(l^2(0,\infty;\mathbf{R}^{p_2}); l^2(0,\infty;\mathbf{R}^{m_2})) : \\ &\quad \text{Q is of the form (4.45) and internally stable with } \| Q \| < \gamma\}, \\ Q_\gamma^D &= \{Q \in \mathcal{L}(l^2(0,\infty;\mathbf{R}^{p_2}); l^2(0,\infty;\mathbf{R}^{m_2})) : \\ &\quad Q \text{ is of the form (5.21) and internally stable with } \| Q \| < \gamma\}. \end{aligned}$$

Then we have the following results.

Theorem 5.5 *Consider the system* $\mathbf{G}_s$ *on* $[0,\infty)$ *with the assumptions* **S1-S4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the following hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ *to the Riccati equation (5.22), (5.23) and (5.25).*
(ii) There exists a bounded nonnegative stabilizing solution Y *to the Riccati equation (5.26)-(5.28).*
(iii) $\rho(\begin{bmatrix} X_1 Y \\ X_{12} Y \end{bmatrix}(t)) \le d^2$, $t \in [0,\infty)$, *for some* $0 < d < \gamma$.
(b) The set of all γ*-suboptimal controllers of the form (4.45) is also given by (5.31) with* Q *internally stable.*
(c) In this case the set of all γ*-suboptimal controllers of the form (5.21) is given by (5.33).*
Moreover the $\lim_{n\to\infty} Y(t+n\tau)$ *exists (denoted by* Y_τ*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution to (5.26) and (5.27).*

Since the solution Y in (ii) is not τ-periodic, γ-suboptimal controllers (5.31) and (5.33) are in general time-varying. However applying Corollary 4.8 we also obtain τ-periodic controllers and time-invariant discrete-time controllers.

Theorem 5.6 *Consider the system* $\mathbf{G}_s$ *with* $h = 0$ *on* $[0,\infty)$ *and assume* **S1-S4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *on* $[0,\infty)$ *of the form (5.21) if and only if the following hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ *to the Riccati equation (5.22) and (5.23).*
(ii) There exists a τ*-periodic nonnegative stabilizing solution* Y_τ *to the Riccati equation (5.26) and (5.27).*
(iii) $\rho(\begin{bmatrix} X_1 Y_\tau \\ X_{12} Y_\tau \end{bmatrix}(t)) \le d^2$, $t \in [0,\tau)$, *for some* $0 < d < \gamma$.
(b) In this case the following controllers are γ*-suboptimal:*

$$\dot{\hat{x}} = \hat{A}_1(t) + \hat{A}_2(t)\tilde{s}(t), \; k\tau < t < (k+1)\tau,$$

$$\begin{aligned}\hat{x}(k\tau^+) &= \hat{A}_d(0)\hat{x}(k\tau) + \hat{B}_1(0)y(k) + \hat{B}_2(0)v(k), \\ u(k) &= \hat{C}_1(0)\hat{x}(k\tau) + \hat{D}_{11}(0)y(k) + \hat{D}_{12}(0)v(k), \\ r(k) &= \hat{C}_2(0)\hat{x}(k\tau) + \hat{D}_{21}(0)y(k), \\ v &= Qr, \; Q \in Q_\gamma\end{aligned} \tag{5.37}$$

where $\hat{A}_d(0)$, $\hat{B}_1(0)$ are defined by (5.30) with Y replaced by Y_τ. Moreover, controllers given by (5.37) with τ-periodic Q are τ-periodic.
(c) Discrete-time controllers given by (5.33) with Y replaced by Y_τ are γ-suboptimal. Moreover, if we restrict $Q \in Q_\gamma^D$ to be time-invariant, the controllers (5.33) are time-invariant.

Remark 5.3 Some comments on the comparison of the lifting technique and the jump system approach to sampled-data H_∞ control are found in [64, 77].

Example 5.3 Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}(t), \\ z &= \begin{bmatrix} [1 \quad 0]x \\ \tilde{u}(t) \end{bmatrix}, \\ y(k) &= [1 \quad 0]\, x(k) + w_d(k)\end{aligned}$$

where $\tilde{u}(t) = u(k)$, $k < t \leq k+1$. For this system all the assumptions **S1-S4** are satisfied. For all $\gamma \geq 2.1$, the conditions (i)-(iii) of Theorems 5.5 and 5.6 are satisfied. Figure 5.3 shows the periodic solution $X(t) = [X_{ij}(t)]$, $i, j = 1, 2, 3$ of the Riccati equation (5.22) and (5.23) with $\gamma = 2.1$ and period 1. Figure 5.4 shows the bounded nonnegative stabilizing solution $Y(t) = [Y_{ij}(t)]$, $i, j = 1, 2$ of the Riccati equation (5.26)-(5.28) which converges to a periodic solution. Figure 5.5 shows that the condition (iii) of both Theorems 5.5 and 5.6 are satisfied. In this case a central discrete-time controller is given by

$$\begin{aligned}\hat{x}(k+1) &= \begin{bmatrix} -0.3683 & 0.5812 \\ -0.0282 & 0.0313 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0.9707 \\ -0.7417 \end{bmatrix} y(k), \\ \hat{u}(k) &= [0.4982 \quad -0.7709]\, \hat{x}(k) - 0.3729 y(k).\end{aligned}$$

Figure 5.6 shows the simulation result of the closed-loop system with the central discrete-time controller where $\gamma = 2.1$ and the disturbances $w(t) = 10e^{-10t} \sin 10t$ and $w_d(k) = 0$.

5.3 H_2 Control

As in the previous section we apply the H_2 theory for jump systems to the sampled-data systems.

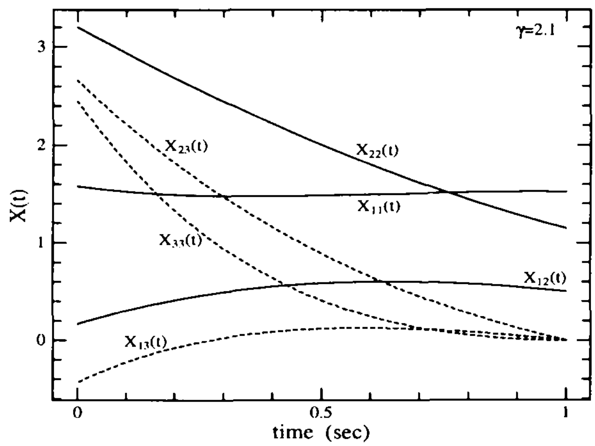

Figure 5.3: The periodic nonnegative solution $X(t)$

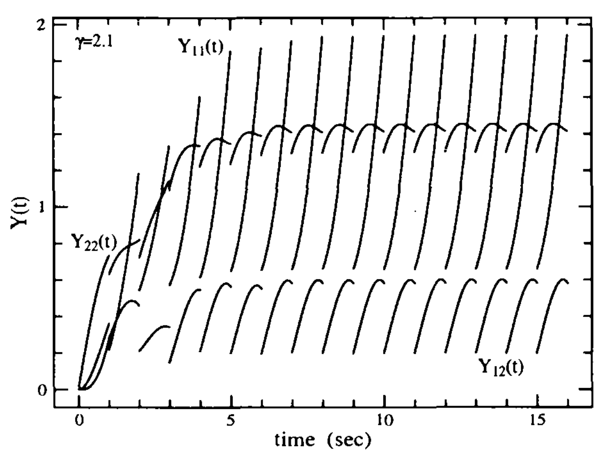

Figure 5.4: The bounded nonnegative solution $Y(t)$

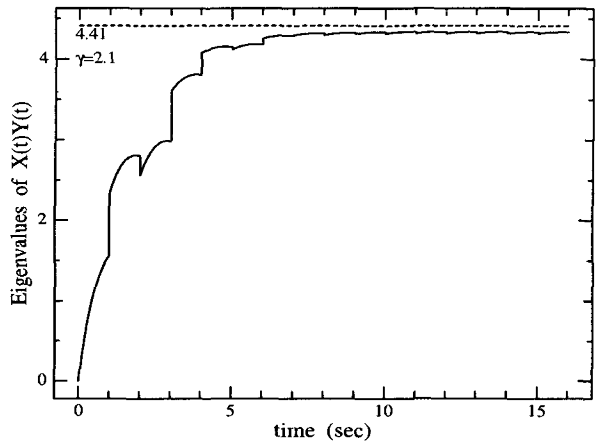

Figure 5.5: Eigenvalues of $X(t)Y(t)$

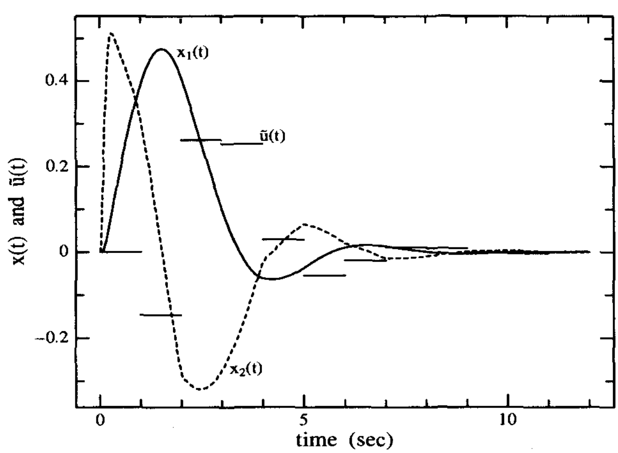

Figure 5.6: Simulation result

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{5.38}$$

and a discrete-time controller of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k). \end{aligned} \tag{5.39}$$

We assume **S1-S4**. To formulate the H_2-problem for $\mathbf{G}_s$ we introduce the following set of controllers

$$\mathbf{K} = \{K : K \text{ is of the form (5.39) and internally stabilizes the system } \mathbf{G}_s\}.$$

The H_2 control problem for the system $\mathbf{G}_s$ is to find an internally stabilizing controller which minimizes $\| G \|_2$, where G is the input-output operator of the closed-loop system defined by

$$z = G\begin{pmatrix} w \\ w_d \end{pmatrix}.$$

Since $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the assumptions **S1** $-$ **S4** imply the assumptions **J1** $-$ **J4** for $\mathbf{G}_e$, we can apply Theorem 4.24 to the system $\mathbf{G}_e$.

As in the H_∞ control problem, let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \; \bar{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}' & Y_2 \end{bmatrix}$$

be the solutions of the Riccati equations (4.163)-(4.166) respectively, with $T_2(k) = I + B_2' X(k\tau) B_2$ replaced by $T_2(k) = \tau I + B_2' X(k\tau) B_2$, where X_1, $Y_1 \in \mathbf{R}^{n\times n}$, X_{12}, $Y_{12} \in \mathbf{R}^{n\times m_2}$ and X_2, $Y_2 \in \mathbf{R}^{m_2\times m_2}$ and n and m_2 are the dimensions of x and u respectively. Then from the first Riccati equation (4.163) and (4.164), we obtain for $k\tau < t < (k+1)\tau$

$$-\begin{bmatrix} \dot{X}_1 & \dot{X}_{12} \\ \dot{X}_{12}' & \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}' X + X\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C_1'C_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.40}$$

and at $t = k\tau$, $k = 0, 1, 2, ..$

$$\begin{aligned} X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\ X_{12}(k\tau^-) &= 0, \\ X_2(k\tau^-) &= 0. \end{aligned} \tag{5.41}$$

The second Riccati equation (4.165) and (4.166) is written

$$\begin{aligned} \dot{Y}_1 &= AY_1 + Y_1A' + \frac{1}{\tau}B_1B_1' + B_2Y_{12}' + Y_{12}'B_2, \\ \dot{Y}_{12} &= AY_{12} + B_2Y_2', \\ \dot{Y}_2 &= 0 \end{aligned} \tag{5.42}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$

$$\begin{aligned} Y_1(k\tau^+) &= Y_1(k\tau) - Y_1(k\tau)C_2'(I + C_2Y_1(k\tau)C_2')^{-1}C_2Y_1(k\tau), \\ Y_{12}(k\tau^+) &= 0, \\ Y_2(k\tau^+) &= 0. \end{aligned} \tag{5.43}$$

Note that

$$\bar{Y}(t) = \lim_{n\to\infty} \hat{Y}(t + n\tau),$$

as we see in Remark 4.6 where $\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_{12} \\ \hat{Y}_{12}' & \hat{Y}_2 \end{bmatrix}$ is the solution of (5.42) and (5.43) with $\hat{Y}(0) = 0$. Since $\hat{Y}_{12}(t)$, $\hat{Y}_2(t)$ form a homogeneous system with $\hat{Y}_{12}(0) = 0$ and $\hat{Y}_2(0) = 0$, we conclude $\hat{Y}_{12}(t) = 0$ and $\hat{Y}_2(t) = 0$ for all $t \geq 0$ and $\bar{Y}$ has the form $\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$ where $Y \in \mathbf{R}^{n\times n}$ is the solution of

$$\dot{Y} = AY + YA' + \frac{1}{\tau}B_1B_1', \quad k\tau < t < (k+1)\tau, \tag{5.44}$$

$$Y(k\tau^+) = Y(k\tau) - Y(k\tau)C_2'(I + C_2Y(k\tau)C_2')^{-1}C_2Y(k\tau). \tag{5.45}$$

If $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ is a τ-periodic nonnegative stabilizing solution,

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ F_s & 0 \end{bmatrix} \right)$$

is exponentially stable where

$$F_s = -[\tau I + X_2(0)]^{-1}X_{12}'(0).$$

So the system

$$\dot{\xi} = A\xi + B_2\tilde{v}(t), \quad \tilde{v}(t) = F_s\hat{\xi}(k\tau), \quad k\tau < t \leq (k+1)\tau \tag{5.46}$$

is exponentially stable. Similarly if Y is a τ-periodic stabilizing solution,

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I + H_sC_2 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

is exponentially stable where

$$H_s = -Y(0)C_2'[I + C_2Y(0)C_2']^{-1},$$

which is equivalent to the exponentially stability of the system

$$\begin{aligned} \dot{\xi} &= A\xi, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^+) &= (I + H_s C_2)\xi(k\tau). \end{aligned} \tag{5.47}$$

Remark 5.4 The exponential stability of the systems (5.46) and (5.47) are equivalent respectively to that of the following time-invariant discrete-time systems:

$$\begin{aligned} \xi(k+1) &= (\tilde{A} + \tilde{B}_2 F_s)\xi(k), \ \tilde{A} = e^{A\tau}, \ \tilde{B}_2 = \int_0^\tau e^{At} dt B_2, \\ \xi(k+1) &= (\tilde{A} + \tilde{H}_s C_2)\xi(k), \ \tilde{H}_s = e^{A\tau} H_s. \end{aligned}$$

From (4.167) the controller given by

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \ k\tau < t < (k+1)\tau, \\ \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau^+) &= \begin{bmatrix} I + H_s C_2 & 0 \\ F_s(I + H_s C_2) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) - \begin{bmatrix} H_s \\ F_s H_s \end{bmatrix} y(k), \\ u(k) &= \begin{bmatrix} F_s(I + H_s C_2) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) - F_s H_s y(k) \end{aligned} \tag{5.48}$$

is optimal. Since

$$\begin{aligned} \dot{\hat{x}}_2 &= 0, \ k\tau < t < (k+1)\tau, \\ \hat{x}_2(k\tau^+) &= F_s(I + H_s C_2)\hat{x}_1(k\tau) - F_s H_s y(k), \end{aligned}$$

we can rewrite (5.48) as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2\tilde{v}(t), \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= (I + H_s C_2)\hat{x}(k\tau) - H_s y(k), \\ u(k) &= F_s(I + H_s C_2)\hat{x}(k\tau) - F_s H_s y(k) \end{aligned} \tag{5.49}$$

where $\tilde{v}$ is given by $\tilde{v}(t) = F_s(I + H_s C_2)\hat{x}(k\tau) - F_s H_s y(k)$, $k\tau < t \le (k+1)\tau$. The controller (5.49) is equivalent to the following discrete-time controller

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k\tau) + \hat{D}y(k) \end{aligned} \tag{5.50}$$

where

$$\begin{aligned} \hat{A} &= \tilde{A} + \tilde{B}_2 F_s + \tilde{H}_s C_2 + \tilde{B}_2 F_s H_s C_2, \\ \hat{B} &= -(\tilde{H}_s + \tilde{B}_2 F_s H_s), \\ \hat{C} &= F_s + F_s H_s C_2, \\ \hat{D} &= -F_s H_s. \end{aligned}$$

Finally the optimal value is given by

$$\begin{aligned}
&\min_{K\in\mathbf{K}} \| G \|_2^2 \\
= \;& \frac{1}{\tau}\int_0^\tau tr.\begin{bmatrix} B_1' & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(s)\begin{bmatrix} B_1 \\ 0 \end{bmatrix} ds \\
& +tr.[\tau I + X_2(0)]^{\frac{1}{2}}\begin{bmatrix} F_s & 0 \end{bmatrix}\begin{bmatrix} [I+Y(0)C_2'C_2]^{-1}Y(0) & 0 \\ 0 & 0 \end{bmatrix} \\
& \qquad \times \begin{bmatrix} F_s' \\ 0 \end{bmatrix}[\tau I + X_2(0)]^{\frac{1}{2}} \\
= \;& \frac{1}{\tau}\int_0^\tau tr.B_1'X_1(s)B_1 ds + tr.[\tau I + X_2(0)]F_s[I+Y(0)C_2'C_2]^{-1}Y(0)F_s'.
\end{aligned}$$

Since $Y(0^+) = [I + Y(0)C_2'C_2]^{-1}Y(0)$ we can rewrite $\min_{K\in\mathbf{K}} \| G \|_2^2$ as

$$\begin{aligned}
\min_{K\in\mathbf{K}} \| G \|_2^2 = \;& \frac{1}{\tau}\int_0^\tau tr.B_1'X_1(s)B_1 ds \\
& + tr.[\tau I + X_2(0)]F_sY(0^+)F_s'. \quad (5.51)
\end{aligned}$$

Summing up we have the following.

Theorem 5.7 *Assume* **S1-S4** *and consider the H_2-problem for* $\mathbf{G}_s$. *Then the controller (5.49) (and hence (5.50)) is optimal and the minimum H_2 norm is given by (5.51).*

We now compare our results with the known results in [2, 8, 50]. By (5.40) for $k\tau < t \le (k+1)\tau$ we have

$$\begin{aligned}
&\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t) \\
= \;& \int_t^{(k+1)\tau} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}'(r-t)}\begin{bmatrix} C_1'C_1 & 0 \\ 0 & 0 \end{bmatrix} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(r-t)} dr \\
& + e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}'((k+1)\tau - t)}\begin{bmatrix} X_1((k+1)\tau^-) & 0 \\ 0 & 0 \end{bmatrix} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}((k+1)\tau - t)}.
\end{aligned}$$

Since

$$e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}t} = \begin{bmatrix} e^{At} & \int_0^t e^{At}dtB_2 \\ 0 & I \end{bmatrix},$$

we have

$$\begin{aligned}
X_1(t) = \;& e^{A'((k+1)\tau - t)}X_1((k+1)\tau^-)e^{A((k+1)\tau - t)} \\
& + \int_t^{(k+1)\tau} e^{A'(r-t)}C_1'C_1e^{A(r-t)}dr,
\end{aligned}$$

$$\begin{aligned}
X_{12}(t) &= e^{A'((k+1)\tau - t)} X_1((k+1)\tau^-) \int_0^{(k+1)\tau - t} e^{As} ds B_2 \\
&\quad + \int_t^{(k+1)\tau} e^{A'(r-t)} C_1' C_1 \int_0^{r-t} e^{As} ds B_2 dr, \\
X_2(t) &= B_2' \int_0^{(k+1)\tau - t} e^{A's} ds X_1((k+1)\tau^-) \int_0^{(k+1)\tau - t} e^{As} ds B_2 \\
&\quad + \int_t^{(k+1)\tau} B_2' \int_0^{r-t} e^{A's} ds C_1' C_1 \int_0^{r-t} e^{As} ds B_2 dr
\end{aligned}$$

and

$$\begin{aligned}
X_1(k\tau) &= \tilde{A}' X_1((k+1)\tau^-) \tilde{A} + \tilde{C}_1' \tilde{C}_1, \\
X_{12}(k\tau) &= \tilde{A}' X_1((k+1)\tau^-) \tilde{B}_2 + \tilde{C}_1 \tilde{D}_{12}, \\
X_2(k\tau) &= \tilde{B}_2' X_1((k+1)\tau^-) \tilde{B}_2 + \int_0^\tau B_2' \int_0^t e^{A's} ds C_1' C_1 \int_0^t e^{As} ds B_2 dt
\end{aligned}$$

where

$$\begin{bmatrix} \tilde{C}_1' \\ \tilde{D}_{12}' \end{bmatrix} [\tilde{C}_1 \quad \tilde{D}_{12}] = \int_0^\tau e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}' t} \begin{bmatrix} C_1' \\ D_{12}' \end{bmatrix} [C_1 \quad D_{12}] e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} t} dt.$$

Hence the Riccati equation (5.40) and (5.41) is equivalent to

$$\begin{aligned}
X(t) &= e^{A'((k+1)\tau - t)} \tilde{X} e^{A((k+1)\tau - t)} + \int_t^{(k+1)\tau} e^{A'(r-t)} C_1' C_1 e^{A(r-t)} dr, \\
&\qquad k\tau \le t < (k+1)\tau, \\
\tilde{X} &= \tilde{A}' \tilde{X} \tilde{A} + \tilde{C}_1' \tilde{C}_1 \\
&\quad - (\tilde{A}' \tilde{X} \tilde{B}_2 + \tilde{C}_1' \tilde{D}_{12})(\tilde{D}_{12}' \tilde{D}_{12} + \tilde{B}_2' \tilde{X} \tilde{B}_2)^{-1} (\tilde{B}_2' \tilde{X} \tilde{A} + \tilde{D}_{12}' \tilde{C}_1)
\end{aligned}$$

where $\tilde{X} = X_1(k\tau^-) = X_1(0^-)$. Similarly the Riccati equation (5.44) and (5.45) is equivalent to

$$\begin{aligned}
Y(t) &= e^{A(t-k\tau)} Y(k\tau) e^{A'(t-k\tau)} + \int_{k\tau}^t e^{A(t-r)} (\frac{1}{\tau} B_1 B_1') e^{A'(t-r)} dr, \\
&\qquad k\tau < t \le (k+1)\tau, \\
Y((k+1)\tau) &= \tilde{A} Y(k\tau) \tilde{A}' + \tilde{B}_1 \tilde{B}_1' \\
&\quad - \tilde{A}' Y(k\tau) C_2' (I + C_2 Y(k\tau) C_2')^{-1} C_2 Y(k\tau) \tilde{A}
\end{aligned}$$

where $\tilde{B}_1 \tilde{B}_1' = \frac{1}{\tau} \int_0^\tau e^{At} B_1 B_1' e^{A't} dt$. We also have

$$\begin{aligned}
F_s &= -(\tilde{D}_{12}' \tilde{D}_{12} + \tilde{B}_2' \tilde{X} \tilde{B}_2)^{-1} (\tilde{B}_2' \tilde{X} \tilde{A} + \tilde{D}_{12}' \tilde{C}_1), \\
\tilde{H}_s &= \tilde{A} H_s = -\tilde{A} Y^0 C_2' (I + C_2 Y^0 C_2')^{-1} \; Y^0 = Y(0).
\end{aligned}$$

Remark 5.5 The optimal controller (5.50) is obtained via the two algebraic Riccati equations:

$$\begin{aligned}
\tilde{X} &= \tilde{A}'\tilde{X}\tilde{A} + \tilde{C}_1'\tilde{C}_1 \\
&\quad -(\tilde{A}'\tilde{X}\tilde{B}_2 + \tilde{C}_1'\tilde{D}_{12})(\tilde{D}_{12}'\tilde{D}_{12} + \tilde{B}_2'\tilde{X}\tilde{B}_2)^{-1}(\tilde{B}_2'\tilde{X}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1), \\
Y^0 &= \tilde{A}Y^0\tilde{A}' + \tilde{B}_1\tilde{B}_1' - \tilde{A}'Y^0C_2'(I + C_2Y^0C_2')^{-1}C_2Y^0\tilde{A}.
\end{aligned}$$

5.4 Notes and References

This chapter is based on [36, 37].

The transformation of sampled-data system into jump systems was introduced in [68]. The notion of pathological sampling periods and its related results can be found in [8]. The H_∞ problem for sampled-data systems is originally solved using the so-called lifting technique [3, 8, 25, 76, 88]. An advantage of the jump system approach is that original systems are maintained in the formulation and the results are regarded as an extension of those for continuous- and discrete-time systems. The jump system approach is adopted in [64, 68, 77]. Necessary and sufficient conditions for the existence of a γ-suboptimal controller are given in [68]. The equivalence of the jump system approach and the lifting technique is discussed in [64].

The H_2 problem in Section 5.3 can be solved using lifting or FR-operator approach [2, 8, 24, 50]. The solution using the jump system representation is found in [37]. The H_2 and H_∞ problems can be considered within the same framework if the jump system approach is adopted.

6. Further Developments

In this chapter we give some further developments in the theory of jump systems. One is an extension to infinite dimensions, which can describe sampled-data systems with first-order hold and of course sampled-data distributed parameter systems. We also introduce fuzzy jump systems which can represent sampled-data nonlinear systems.

6.1 Jump Systems in Infinite Dimensions

Consider the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ k\tau < t < (k+1)\tau, \ \tau > 0, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \ k = 0, 1, 2, .., \\ z &= \begin{bmatrix} z_c \\ z_d(k) \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned}$$

where $x \in H$, $u \in U$, $w \in W$, $w_d \in W_d$, $z_c \in Z_c$, $z_d \in Z_d$, $y \in Y$, A is the infinitesimal generator of a C_0-semigroup $S(t)$ in a separable Hilbert space H, the input and output spaces W, U, W_d, Z_c, Z_d, and Y are all separable Hilbert spaces and the operators B_1, A_d and so on are all bounded linear operators in appropriate spaces i.e., $B_1 \in \mathcal{L}(W, H)$, $A_d \in \mathcal{L}(H)$, etc. The inner products in Hilbert spaces are denoted by $< \cdot, \cdot >$ and the norm for vectors and operators are denoted by $| \cdot |$. The abstract system $\mathbf{G}_j$ is useful when we consider parabolic equations, hyperbolic equations, delay differential equations and neutral equations with sampled-data control or impulse control. The (mild) solution of $\mathbf{G}_j$ for a locally Bochner integrable w is defined in a piecewise manner as follows:

$$x(t) = S(t - k\tau)x(k\tau^+) + \int_{k\tau}^{t} S(t - r)B_1 w(r)dr, \ k\tau < t \leq (k+1)\tau.$$

It is left-continuous with jumps at $k\tau$. Let $S(t, \tau)$ be the fundamental solution of the homogeneous part of (6.8). Then it is τ-periodic i.e., $S(t + \tau, s + \tau) =$

$S(t,s)$ for any $t \geq s \geq 0$. We can express $x(t)$ with $x(0) = x_0$ as

$$\begin{aligned} x(t) &= \int_0^t S(t,s)B_1 w(s)ds + \sum_{j=0}^{k} S(t, j\tau^+)B_2 u(j) + S(t,0)x_0, \\ & \qquad 0 < k\tau < t \leq (k+1)\tau. \end{aligned}$$

We assume

H1 : $D_{12}^* D_{12} = I$,
H2 : $D_{21} D_{21}^* = I$,
H3 : $([A, A_d], [B_2, 0], [C_1, 0])$ is stabilizable and detectable,
H4 : $([A, A_d], [0, B_1], [0, C_2])$ is stabilizable and detectable.

In this section we give an extension of Theorems 4.10, 4.11 and 4.24 with $Hh = 0$ to infinite dimensions. we only give main results and omit their proofs although they are direct generalizations of those in Section 4.2 but are beyond the scope of this book. Instead we apply them to sampled-data systems with first-order hold.

6.1.1 $\mathbf{H}_\infty$ Control

Consider the system $\mathbf{G}_j$ and an internally stabilizing controller $u = Ky$ of the form

$$\begin{aligned} \dot{p} &= \hat{A}(t)p, \ k\tau < t < (k+1)\tau, \\ p(k\tau^+) &= \hat{A}_d(k)p(k\tau) + \hat{B}(k)y(k), \\ u(k) &= \hat{C}(k)p(k\tau) + \hat{D}(k)y(k) \end{aligned} \tag{6.1}$$

where $\hat{A}(t)$ generates an evolution operator in some Hilbert space $\hat{H}$ and all other operators are linear and bounded and their norms are bounded uniformly in t. Then

$$G \in \mathcal{L}(L^2(0,\infty;W) \times l^2(0,\infty;W_d); L^2(0,\infty;Z_c) \times l^2(0,\infty;Z_d)).$$

The H_∞-control problem for $\mathbf{G}_j$ is to find necessary and sufficient conditions for the existence of an internally stabilizing controller such that $\| G \| < \gamma$ (γ-suboptimal).

As in Section 4.2, we introduce the following Riccati equations:

$$\begin{aligned} -\dot{X} &= A^*X + XA + C_1^*C_1 + \frac{1}{\gamma^2}XB_1B_1^*X, \\ & \qquad k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A_d^*X(k\tau)A_d - (R_2^*T_2^{-1}R_2)(k), \\ \dot{Y} &= AY + YA^* + B_1B_1^* + \frac{1}{\gamma^2}YC_1^*C_1Y, \end{aligned} \tag{6.2}$$

$$
\begin{aligned}
&& k\tau < t < (k+1)\tau, \qquad (6.3)\\
Y(k\tau^+) &=& A_d Y(k\tau)A_d^* - (R_{2Y}^* T_{2Y}^{-1} R_{2Y})(k),\\
Y(0) &=& 0 \qquad (6.4)
\end{aligned}
$$

with

$$
\begin{aligned}
T_2(k) = I + B_2^* X(k\tau) B_2, \quad & R_2(k) = B_2^* X(k\tau) A_d,\\
T_{2Y}(k) = I + C_2 Y(k\tau) C_2^*, \quad & R_{2Y}(k) = C_2 Y(k\tau) A_d^*
\end{aligned}
$$

and the Riccati equation depending on X:

$$
\begin{aligned}
\dot{Z} &=& (A + \frac{1}{\gamma^2} B_1 B_1^* X) Z + Z (A + \frac{1}{\gamma^2} B_1 B_1^* X)^* + B_1 B_1^*,\\
&& k\tau < t < (k+1)\tau,\\
V_Z(k) &>& aI \text{ for some } a > 0, \qquad (6.5)\\
Z(k\tau^+) &=& A_d Z(k\tau) A_d^* - (R_{2Z}^* T_{2Z}^{-1} R_{2Z})(k) + (F_{1Z}^* V_Z F_{1Z})(k),\\
Z(0) &=& 0 \qquad (6.6)
\end{aligned}
$$

where

$$
\begin{aligned}
& T_{1Z}(k) = \gamma^2 I - T_2^{-\frac{1}{2}} R_2 Z(k\tau) R_2^* T_2^{-\frac{1}{2}}, && T_{2Z}(k) = I + C_2 Z(k\tau) C_2^*,\\
& R_{1Z}(k) = T_2^{-\frac{1}{2}} R_2 Z(k\tau) A_d^*, && R_{2Z}(k) = C_2 Z(k\tau) A_d^*,\\
& S_Z(k) = C_2 Z(k\tau) R_2^* T_2^{-\frac{1}{2}}, && V_Z(k) = [T_{1Z} + S_Z^* T_{2Z}^{-1} S_Z](k),\\
& F_{1Z}(k) = [V_Z^{-1}(R_{1Z} - S_Z^* T_{2Z}^{-1} R_{2Z})](k), &&\\
& F_{2Z}(k) = -[T_{2Z}^{-1}(R_{2Z} + S_Z F_{1Z})](k). &&
\end{aligned}
$$

An operator X is called a mild solution of (6.2) if it is right-continuous and satisfies

$$
\begin{aligned}
X(t)x &=& \int_t^s S^*(r-t)[C_1^* C_1 + \frac{1}{\gamma^2} X(r) B_1 B_1^* X(r)] S(r-t) x dr\\
&& + S^*(s-t) X(s) S(s-t) x, \; k\tau < t \le s < (k+1)\tau.
\end{aligned}
$$

Mild solutions of (6.3) and (6.5) are defined in a similar manner. As in Section 4.2, the solution X of (6.2) (Y of (6.3)) is called stable if $(A + \frac{1}{\gamma^2} B_1 B_1^* X, A_d - B_2 T_2^{-1}(0) R_2(0))$ is exponentially stable ($(A + \frac{1}{\gamma^2} Y C_1^* C_1, A_d - R_{2Y}^* T_{2Y}^{-1} C_2)$ is exponentially stable, respectively). Similarly the solution Z of (6.5) is called stable if $(A + \frac{1}{\gamma^2} B_1 B_1^* X, A_d + F_1^* T_2^{-\frac{1}{2}} R_2 + F_2^* C_2)$ is exponentially stable.

Theorem 6.1 *Consider the system* $\mathbf{G}_j$ *and assume* **H1-H4**.
(a) There exists a γ-suboptimal controller $u = Ky$ of the form (6.1) if and only if the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing mild solution X of (6.2).
(ii) For the solution X in (i), there exists a bounded nonnegative stabilizing

mild solution Z of (6.5) and (6.6).
(b) In this case the set of all γ-suboptimal controllers is given by

$$\begin{aligned}
\dot{p} &= [A + \frac{1}{\gamma^2} B_1 B_1^* X(t)]p, \ k\tau < t < (k+1)\tau, \\
p(k\tau^+) &= \hat{A}_d(k)p(k\tau) + \hat{B}_1(k)y(k) + \hat{B}_2(k)s(k), \\
u(k) &= \hat{C}(k)p(k\tau) + \hat{D}_1(k)y(k) + \hat{D}_2(k)s(k), \\
g(k) &= T_{2Z}^{-\frac{1}{2}}(k)[-C_2 p(k\tau) + y(k)], \\
s &= Qg, \ Q \in Q_\gamma
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}
&\hat{A}_d(k) = (A_d - B_2 T_2^{-1} R_2)\Psi(k), \\
&\hat{B}_1(k) = (A_d - B_2 T_2^{-1} R_2) Z(k\tau) C_2^* T_{2Z}^{-1}(k), \\
&\hat{B}_2(k) = \tfrac{1}{\gamma}([F_1^* + B_2 T_2^{-\frac{1}{2}}] V_Z^{\frac{1}{2}})(k), \qquad \hat{C}(k) = -T_2^{-1} R_2 \Psi(k), \\
&\hat{D}_1(k) = -T_2^{-1} R_2 Z(k\tau) C_2^* T_{2Z}^{-1}(k), \qquad \hat{D}_2(k) = \tfrac{1}{\gamma} T_2^{-\frac{1}{2}} V_Z^{\frac{1}{2}}(k), \\
&\Psi(k) = I - Z(k\tau) C_2^* T_{2Z}^{-1}(k) C_2
\end{aligned}$$

and

$$\begin{aligned}
Q_\gamma = \ &\{Q \in \mathcal{L}(l^2(0,\infty;\mathbf{R}^{p_2}); l^2(0,\infty;\mathbf{R}^{m_2})) : \\
&Q \text{ is of the form (6.1) and internally stable with } \| Q \| < \gamma\}.
\end{aligned}$$

Moreover, $Z_\tau(t) = \lim_{n\to\infty} Z(t + n\tau)$ exists and it is a τ-periodic nonnegative stabilizing mild solution of (6.5).

Theorem 6.2 *Consider the system $\mathbf{G}_j$ and assume* **H1-H4**.
(a) There exists an internally stabilizing controller $u = Ky$ of the form (6.1) such that $\| G \| < \gamma$ if and only if the following conditions hold:
(i) There exists a τ-periodic nonnegative stabilizing solution X of (6.2).
(ii) There exists a bounded nonnegative stabilizing solution Y of (6.3) and (6.4).
(iii) $\rho(XY) \le d^2$, $t \ge 0$, for some $0 < d < \gamma$, where ρ is the spectral radius.
(b) In this case the set of all γ-suboptimal controllers is given by (6.7) with $Z(k\tau) = [I - \frac{1}{\gamma^2} Y(k\tau) X(0^-)]^{-1} Y(k\tau)$.
Moreover, $Y_\tau(t) = \lim_{n\to\infty} Y(t + n\tau)$ exists and it is a τ-periodic nonnegative stabilizing mild solution of (6.3).

Note that necessary and sufficient conditions and τ-periodic controllers may be obtained in terms of X, Y_τ and Z_τ.

6.1.2 H_2 Control

Consider the system

$$\dot{x} = Ax + Bw, \ k\tau < t < (k+1)\tau,$$

$$\begin{aligned} x(k\tau^+) &= A_d x(k\tau) + B_d w_d(k), \\ z_c &= Cx, \\ z_d(k) &= C_d x(k\tau) + D_d w_d(k) \end{aligned} \tag{6.8}$$

where operators except A are linear and bounded. Assume that (A, A_d) is exponentially stable. We introduce H_2 norm of the system (6.8). Let T_{zw} and T_{zw_d} be the operators given by

$$z = \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = T_{zw} w = \begin{bmatrix} C \int_0^t S(t,r) B w(r) dr \\ C_d \int_0^{k\tau} S(k\tau, r) B w(r) dr \end{bmatrix}$$

and

$$z = \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = T_{zw_d} w_d = \begin{bmatrix} C \sum_{j=0}^{k} S(t, j\tau^+) B_d w_d(j) \\ 0 < k\tau < t \leq (k+1)\tau \\ C_d \sum_{j=0}^{k-1} S(k\tau, j\tau^+) B_d w_d(j) + D_d w_d(k) \end{bmatrix},$$

respectively. Let (e_i) and (f_j) be the orthonormal bases in W and W_d respectively. As in Section 4.1.2 we consider the impulse $w(t) = \delta(t-s)e_i$, $0 < s < \tau$. The resulting output will be denoted by $T_{zw}\delta(t-s)e_i$. We also consider the input w_d with $w_d(0) = f_j$ and $w_d(k) = 0$ for all $k \geq 1$. We denote its output by $T_{zw_d}\delta_{\cdot 0} f_j$. Then $T_{zw}\delta(t-s)e_i \in L^2(s, \infty; Z_c) \times l^2(0, \infty; Z_d)$ and $T_{zw_d}\delta_{\cdot 0} f_j \in L^2(0, \infty; Z_c) \times l^2(0, \infty; Z_d)$ where $l^2(0, \infty; Z_d)$ is the space of square summable vectors in Z_d. Define

$$\begin{aligned} P(s)x &= \int_s^\infty S^*(t,s) C^* C S(t,s) x ds, \\ P_d(s)x &= \sum_{k=1}^\infty S^*(k\tau, s) C_d^* C_d S(k\tau, s) x. \end{aligned}$$

Then $< Be_i, [P(s) + P_d(s)] Be_i > = \| T_{zw}\delta(\cdot - s) e_i \|^2_{L^2 \times l^2}$ and

$$< B_d f_j, [P(0^+) + P_d(0^+)] B_d f_j > + < D_d f_j, D_d f_j > = \| T_{zw_d} \delta_{\cdot 0} f_j \|^2_{L^2 \times l^2} .$$

In order to introduce the H_2 norm to (6.8) we assume one of the following conditions:

(i) B, B_d and D_d are Hilbert-Schmidt operators.
(ii) C, C_d and D_d are Hilbert-Schmidt operators.

Then in either case $B^*[P(s)+P_d(s)]B$ is a trace class operator ([22]) and

$$tr.B^*[P(s)+P_d(s)]B = \sum_{i=1}^{\infty} < Be_i, [P(s)+P_d(s)]Be_i >< \infty$$

where $tr.$ denotes the trace of operators. Now we define the H_2 norm of (6.8) as follows:

$$\| G \|_2^2 = \sum_{i=1}^{\infty} \frac{1}{\tau} \int_0^{\tau} \| T_{zw}\delta(\cdot - s)e_i \|_{L^2 \times l^2}^2 \, ds + \sum_{j=1}^{\infty} \| T_{zw_d}\delta_{\cdot 0} f_j \|_{L^2 \times l^2}^2 .$$

Consider

$$\begin{aligned} -\dot{L}_o &= A^* L_o + L_o A + C^* C, \ k\tau < t < (k+1)\tau, \\ L_o(k\tau^-) &= A_d^* L_o(k\tau) A_d + C_d^* C_d. \end{aligned} \tag{6.9}$$

Since (A, A_d) is stable, there exists a unique nonnegative τ-periodic mild solution L_o. It is called the observability gramian. Similarly there exists a unique nonnegative τ-periodic mild solution L_c of the equation

$$\begin{aligned} \dot{L}_c &= A L_c + L_c A^* + \frac{1}{\tau} B B^*, \ k\tau < t < (k+1)\tau, \\ L_c(k\tau^+) &= A_d L_c(k\tau) A_d^* + B_d B_d^*, \end{aligned} \tag{6.10}$$

which is called the controllability gramian. We can express $\| G \|_2$ in terms of L_o or L_c.

Theorem 6.3

$$\begin{aligned} \| G \|_2^2 &= \frac{1}{\tau} \int_0^{\tau} tr.B^* L_o(s) B ds + tr.[B_d^* L_o(0) B_d + D_d^* D_d] \\ &= tr.C_d L_c(0) C_d^* + \int_0^{\tau} tr.C L(s) C^* ds. \end{aligned} \tag{6.11}$$

Recall the jump system $\mathbf{G}_j$:

$$\begin{aligned} \dot{x} &= Ax + B_1 w, \ k\tau < t < (k+1)\tau, \\ x(k\tau^+) &= A_d x(k\tau) + B_2 u(k), \\ z &= \begin{bmatrix} z_c \\ z_d \end{bmatrix} = \begin{bmatrix} C_1 x \\ D_{12} u(k) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k). \end{aligned}$$

Now we introduce the H_2-control problem. Consider feedback controllers $u = Ky$ of the form

$$\begin{aligned} \dot{p} &= \hat{A} p, \ k\tau < t < (k+1)\tau, \\ p(k\tau^+) &= \hat{A}_d p(k\tau) + \hat{B} y(k), \\ u(k) &= \hat{C} p(k\tau) + \hat{D} y(k) \end{aligned} \tag{6.12}$$

where $\hat{A}$ is the infinitesimal generator of a C_0-semigroup in a separable Hilbert space $\hat{H}$ and other operators are linear and bounded. It is said to be internally stabilizing if the closed-loop system is exponentially stable. We assume one of the following:

(i) B_1 and D_{21} are Hilbert-Schmidt operators.
(ii) C_1 and D_{12} are Hilbert-Schmidt operators.

In either case the H_2-norm of the closed-loop system is well-defined. Our H_2-problem is to find an internally stabilizing controller K which minimizes $\| G \|_2$. Under the assumptions **H1-H4** we have the following result:

Lemma 6.1 *(a) Suppose* **H1** *and* **H3** *hold. Then there exists a unique τ-periodic nonnegative mild solution of the Riccati equation:*

$$\begin{aligned} \dot{X} &= A^*X + XA + C_1^*C_1, \ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A_d^*X(k\tau)A_d - (R_2^*T_2^{-1}R_2)(k), \end{aligned} \tag{6.13}$$

which is stable, i.e., $(A, A_d + B_2F)$ with $F = T_2^{-1}(0)R_2(0)$ is exponentially stable where $R_2(k) = B_2^*X(k\tau)A_d$ and $T_2(k) = I + B_2^*X(k\tau)B_2$. Under the condition (ii) X is a trace class operator and $tr.X(\cdot)$ is uniformly bounded.
(b) Suppose **H2** *and* **H4** *hold. Then there exists a unique τ-periodic nonnegative mild solution of the Riccati equation:*

$$\begin{aligned} \dot{Y} &= AY + YA^* + \frac{1}{\tau}B_1B_1^*, \ k\tau < t < (k+1)\tau, \\ Y(k\tau^+) &= A_dY(k\tau)A_d^* - (R_{2Y}^*T_{2Y}^{-1}R_{2Y})(k), \end{aligned} \tag{6.14}$$

which is stable, i.e., $(A, A_d + HC_2)$ with $H = -R_{2Y}^*(0)T_{2Y}^{-1}(0)$ is exponentially stable where $R_{2Y}(k) = C_2Y(k\tau)A_d^*$ and $T_{2Y}(k) = I + C_2Y(k\tau)C_2^*$. Under the condition (b) Y is a trace class operator and $tr.Y(\cdot)$ is uniformly bounded

Define the stabilizing controller based on the feedback gain F and the observer gain H:

$$\begin{aligned} \dot{p} &= Ap, \ k\tau < t < (k+1)\tau, \\ p(k\tau^+) &= (A_d + B_2F + HC_2 - B_2LC_2)p(k\tau) \\ &\quad -(H - B_2L)y(k), \\ u(k) &= (F - LC_2)p(k\tau) + Ly(k) \end{aligned} \tag{6.15}$$

where $L = FY(0)C_2'T_{2Y}^{-1}(0)$.

Theorem 6.4 *Consider the H_2-problem for $\mathbf{G}_j$. Then the controller (6.15) is optimal and*

$$\begin{aligned} \min_K \| G \|_2^2 &= \frac{1}{\tau}\int_0^\tau tr.B_1^*X(s)B_1\,ds \\ &\quad + tr.T_2(0)F[I + Y(0)C_2'C_2]^{-1}Y(0)F^*. \end{aligned} \tag{6.16}$$

6.1.3 Sampled-data Systems with First-order Hold

We apply the main results in Sections 6.1.1 and 6.1.2 to the sampled-data system $\mathbf{G}_s$ in Chapter 5:

$$\begin{aligned}\dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t),\\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t)\end{bmatrix},\\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k)\end{aligned}$$

where $\tilde{u}$ is realized through a first-order hold. Then

$$\tilde{u}(t) = u(k) + \frac{t - k\tau}{\tau}[u(k) - u(k-1)],\ k\tau < t < (k+1)\tau. \tag{6.17}$$

Let $\tilde{A}$ be the operator on $M_2 = \mathbf{R}^{2m_2} \times L^2(-\tau, 0; \mathbf{R}^{2m_2})$ given by

$$\begin{aligned}\tilde{A}\begin{bmatrix}\tilde{p}(0)\\ \tilde{p}\end{bmatrix} &= \begin{bmatrix} A_0\tilde{p}(0) + A_1\tilde{p}(-\tau)\\ d\tilde{p}/ds\end{bmatrix},\\ \mathcal{D}(\tilde{A}) &= \left\{\begin{bmatrix}\tilde{p}(0)\\ \tilde{p}(\cdot)\end{bmatrix} : \tilde{p}(\cdot) \text{ absolutely continuous,}\right.\\ &\qquad\qquad \left.\frac{d\tilde{p}}{ds} \in L^2(-\tau, 0; \mathbf{R}^{2m_2})\right\}\end{aligned}$$

where $A_0 = -A_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau}I & 0\end{bmatrix}$. Then it is the infinitesimal generator of a strongly continuous semigroup on M_2. Setting $\tilde{p}(t) = [p_1(t), p_2(t), p_1(t + \cdot), p_2(t+\cdot)]$, we can rewrite (6.17) as

$$\begin{aligned}\dot{\tilde{p}} &= \tilde{A}\tilde{p},\ \tilde{p}(0) = 0, \\ \tilde{p}(k\tau^+) &= \begin{bmatrix} 0 & 0 \\ 0 & I\end{bmatrix}\tilde{p}(k\tau) + \begin{bmatrix} B_d \\ 0\end{bmatrix} u(k),\ B_d = \begin{bmatrix} I \\ I\end{bmatrix}\end{aligned} \tag{6.18}$$

and $\tilde{u}(t) = p_2(t)$. Then except for an additional output z_d, $\mathbf{G}_s$ is equivalent to the system

$$\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{\tilde{p}}\end{bmatrix} &= \begin{bmatrix} A & A_2 & 0 \\ 0 & \tilde{A} & \end{bmatrix}\begin{bmatrix} x \\ \tilde{p}\end{bmatrix} + \begin{bmatrix} B_1 \\ 0\end{bmatrix} w(t),\ A_2 = [0 \quad B_2],\\ \begin{bmatrix} x(k\tau^+)\\ \tilde{p}(k\tau^+)\end{bmatrix} &= \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I\end{bmatrix}\begin{bmatrix} x(k\tau) \\ \tilde{p}(k\tau)\end{bmatrix} + \begin{bmatrix} 0 \\ B_d \\ 0\end{bmatrix} u(k),\\ z_c(t) &= [C_1 \quad 0 \quad D_{12} \quad 0]\begin{bmatrix} x(t) \\ \tilde{p}(t)\end{bmatrix},\\ z_d(k) &= D_d u(k),\\ y(k) &= [C_2 \quad 0 \quad 0]\begin{bmatrix} x(k\tau)\\ \tilde{p}(k\tau)\end{bmatrix} + D_{21} w_d(k).\end{aligned}$$

We assume the following:

(i) $D_d'D_d = I$, $D_{21}D_{21}' = I$.
(ii) (A, B_1, C_1) is stabilizable and detectable.
(iii) There exists a stabilizing feedback of the form

$$u(k) = K_1 x(k\tau) + K_2 u(k-1) + \int_{-\tau}^{0} K_3(s)\tilde{u}(k\tau + s)ds.$$

(iv) $([0, C_2], [A, I])$ is detectable.

Then the assumptions **H1-H4** are satisfied and we can apply Theorems 6.2-6.4. If we set $D_d'D_d = \epsilon I$ and let $\epsilon \to 0$, we obtain the limiting case G_s and (6.17). We now derive Riccati equation for the H_∞-control problem. For this purpose we set $X = [X_{ij}], i, j = 1, 2, 3$. Then $X_{13} \in \mathcal{L}(L^2(-\tau, 0; \mathbf{R}^{2m_2}), H)$ and we denote by $X_{13}(t, s)$ the kernel of $X_{13}(t)$ i.e.,

$$X_{13}(t)p(\cdot) = \int_{-\tau}^{0} X_{13}(t, s)p(s)ds, \quad p(\cdot) \in L^2(-\tau, 0; \mathbf{R}^{2m_2}).$$

Similarly we denote the kernel functions of X_{23}, X_{33} respectively by $X_{23}(t, s)$, $X_{33}(t, s, r)$. Using the definition of $\tilde{A}$ and (6.2) we obtain the following Riccati equation:

$$\begin{aligned}
-\dot{X}_{11}(t) &= A'X_{11}(t) + X_{11}(t)A + C_1'C_1 + \frac{1}{\gamma^2}X_{11}(t)B_1B_1'X_{11}(t), \\
X_{11}(k\tau^-) &= X_{11}(k\tau) - X_{12}(k\tau)\Phi(k)X_{21}(k\tau), \\
-\dot{X}_{12}(t) &= A'X_{12}(t) + X_{12}(t)A_0 + X_{13}(t, 0) + X_{11}(t)A_2 \\
&\quad + \frac{1}{\gamma^2}X_{11}(t)B_1B_1'X_{12}(t) + C_1'\begin{bmatrix} 0 & D_{12} \end{bmatrix}, \\
X_{12}(k\tau^-) &= 0, \\
-\frac{\partial}{\partial t}X_{13}(t, s) &= -\frac{\partial}{\partial s}X_{13}(t, s) + A'X_{13}(t, s) + \frac{1}{\gamma^2}X_{11}(t)B_1B_1'X_{13}(t), \\
X_{13}(t, -\tau) &= X_{12}(t)A, \qquad\qquad (6.19) \\
X_{13}(k\tau^-, s) &= X_{13}(k\tau, s) - X_{12}(k\tau)\Phi(k)X_{23}(k\tau), \\
-\dot{X}_{22}(t) &= A_0'X_{22}(t) + X_{22}(t)A_0 + A_2'X_{12}(t) \\
&\quad + X_{21}(t)A_2 + \frac{1}{\gamma^2}X_{21}(t)B_1B_1'X_{11}(t) + \begin{bmatrix} 0 & 0 \\ 0 & D_{12}'D_{12} \end{bmatrix}, \\
X_{22}(k\tau^-) &= 0, \\
-\frac{\partial}{\partial t}X_{23}(t, s) &= -\frac{\partial}{\partial s}X_{23}(t, s) + A_0'X_{23}(t, s) + A_2'X_{13}(t, s) \\
&\quad + X_{33}(t, 0, s) + \frac{1}{\gamma^2}X_{21}(t)B_1B_1'X_{13}(t),
\end{aligned}$$

$$\begin{aligned}
X_{23}(t,-\tau) &= X_{22}(t)A_1,\\
X_{23}(k\tau^-,s) &= 0,\\
-\frac{\partial}{\partial t}X_{33}(t,s,r) &= -(\frac{\partial}{\partial s}+\frac{\partial}{\partial r})X_{33}(t,s,r)+\frac{1}{\gamma^2}X_{31}(t,s)B_1B_1'X_{13}(t,r),\\
X_{33}(t,-\tau,r) &= A_1'X_{23}(t,r),\\
X_{33}(t,s,-\tau) &= X_{32}(t,s)A_1,\\
X_{33}(k\tau^-,s,r) &= X_{33}(k\tau,s,r)-X_{32}(k\tau,s)\Phi(k)X_{23}(k\tau,r)
\end{aligned}$$

where $T_{22}(k) = I + B_d'X_{22}(k\tau)B_d$ and $\Phi(k) = B_dT_{22}^{-1}(k)B_d'$. Setting $\bar{Y} = [Y_{ij}], i,j = 1,2,3$ we can derive a similar Riccati equation to (6.19). Since Y_{ij} form a homogeneous system, we seek for a solution with $Y_{ij} = 0$ and $Y = Y_{11}$ satisfying

$$\begin{aligned}
\dot{Y} &= AY + YA' + B_1B_1' + \frac{1}{\gamma^2}YC_1'C_1Y,\\
&\qquad k\tau < t \le (k+1)\tau, \qquad (6.20)\\
Y(k\tau^+) &= Y(k\tau)[I + C_2'C_2Y(k\tau)]^{-1}.
\end{aligned}$$

Since $([A,I],[0,C_2])$ is detectable, there exists an unique stabilizing solution of (6.20). The controller (6.7) is written as

$$\begin{aligned}
\dot{\hat{x}} &= [A+\frac{1}{\gamma^2}B_1B_1'X_{11}(t)]\hat{x} + [A_2+\frac{1}{\gamma^2}B_1B_1'X_{12}(t)]\hat{p}\\
&\quad +\frac{1}{\gamma^2}B_1B_1'\int_{-\tau}^{0}X_{13}(t,s)\hat{p}(t+s)ds,\\
\dot{\hat{p}} &= A_0\hat{p}(t)+A_1\hat{p}(t-\tau),\ k\tau < t \le (k+1)\tau, \qquad (6.21)\\
\hat{x}(k\tau^+) &= M_{11}\hat{x}(k\tau)+M_{12}y(k)+M_{13}s(k),\\
\hat{p}(k\tau^+) &= M_{21}\hat{x}(k\tau)+M_{22}y(k)+M_{23}s(k)\\
&\quad -\Phi(0)\int_{-\tau}^{0}X_{23}(k\tau,s)\hat{p}(k\tau+s)ds,\\
u(k) &= N_1\hat{x}(k\tau)+N_2y(k)+N_3s(k)\\
&\quad -T_{22}^{-1}(0)B_d'\int_{-\tau}^{0}X_{23}(k\tau,s)\hat{p}(k\tau+s)ds,\\
g(k) &= T_2^{-\frac{1}{2}}(k)[-C_2\hat{x}(k\tau)+y(k)],\\
s &= Qg, \quad \| Q \|_\infty < \gamma
\end{aligned}$$

where $W(k) = [I - \frac{1}{\gamma^2}Y(k\tau)X_{11}(0^-)]^{-1}$ and

$$\begin{aligned}
M_{11}(k) &= I - W(k)Y(k\tau)C_2'T_2^{-1}(k)C_2,\\
M_{12}(k) &= M_{11}(k)W(k)Y(k\tau)C_2',\\
M_{13}(k) &= -\frac{1}{\gamma}F_1'V^{\frac{1}{2}}(k),
\end{aligned}$$

$$
\begin{aligned}
M_{21}(k) &= -\Phi(0)X_{21}(k\tau)M_{11}(k),\\
M_{22}(k) &= M_{21}(k)W(k)Y(k\tau)C_2',\\
M_{23}(k) &= -\frac{1}{\gamma}B_dT_{22}^{-\frac{1}{2}}(0)V^{\frac{1}{2}}(k),\\
N_1(k) &= -T_{22}^{-1}(0)B_d'X_{21}(k\tau)M_{11}(k),\\
N_2(k) &= N_1(k)Y(k\tau)C_2',\\
N_3(k) &= \frac{1}{\gamma}T_{22}^{-\frac{1}{2}}(0)V^{\frac{1}{2}}(k),\\
T_1(k) &= \gamma^2 I - T_{22}^{-\frac{1}{2}}(0)B_d'X_{21}(k\tau)W(k)Y(k\tau)X_{12}(k\tau)B_dT_{22}^{-\frac{1}{2}}(0),\\
T_2(k) &= I + C_2W(k)Y(k\tau)C_2',\\
R_1(k) &= T_{22}^{-\frac{1}{2}}(0)B_d'X_{21}(k\tau)W(k)Y(k\tau),\\
R_2(k) &= C_2W(k)Y(k\tau),\\
S(k) &= C_2W(k)Y(k\tau)X_{12}(k\tau)B_dT_{22}^{-\frac{1}{2}}(0),\\
V(k) &= (T_1 + S'T_2^{-1}S)(k),\\
F_1(k) &= [V^{-1}(R_1 - S'T_2^{-1}R_2)](k).
\end{aligned}
$$

Summing up we have the following.

Theorem 6.5 *(a) There exists a γ-suboptimal controller for the system $\mathbf{G}_s$ if and only if the following hold:*
(i) There exists an τ-periodic nonnegative stabilizing mild solution X to the Riccati equation (6.19).
(ii) There exists a bounded nonnegative stabilizing mild solution Y to (6.20) with $Y(0) = 0$.
(iii) $\rho\left(\begin{bmatrix} X_{11}Y \\ X_{21}Y \\ X_{31}Y \end{bmatrix}(t)\right) \le d^2$ *for any $t \in [0,\infty)$ and for some $0 < d < \gamma$.*
(b) In this case the set of all γ-suboptimal controllers is given by (6.21).

We now derive two sets of Riccati equations for the H_2 problem. Using the definition of $\tilde{A}$ and (6.13) we obtain the following Riccati equation:

$$
\begin{aligned}
-\dot{X}_{11}(t) &= A'X_{11}(t) + X_{11}(t)A + C_1'C_1,\\
X_{11}(k\tau^-) &= X_{11}(k\tau) - X_{12}(k\tau)\Phi(k)X_{21}(k\tau),\\
-\dot{X}_{12}(t) &= A'X_{12}(t) + X_{12}(t)A_0 + X_{13}(t,0)\\
&\qquad + X_{11}(t)A_2 + C_1'\begin{bmatrix}0 & D_{12}\end{bmatrix},\\
X_{12}(k\tau^-) &= 0,\\
-\frac{\partial}{\partial t}X_{13}(t,s) &= -\frac{\partial}{\partial s}X_{13}(t,s) + A'X_{13}(t,s),\\
X_{13}(t,-\tau) &= X_{12}(t)A,\\
X_{13}(k\tau^-,s) &= X_{13}(k\tau,s) - X_{12}(k\tau)\Phi(k)X_{23}(k\tau),
\end{aligned}
$$

$$
\begin{aligned}
-\dot{X}_{22}(t) &= A_0'X_{22}(t) + X_{22}(t)A_0 + A_2'X_{12}(t) \\
&\quad + X_{21}(t)A_2 + \begin{bmatrix} 0 & 0 \\ 0 & D_{12}'D_{12} \end{bmatrix}, \\
X_{22}(k\tau^-) &= 0, \qquad\qquad (6.22) \\
-\frac{\partial}{\partial t}X_{23}(t,s) &= -\frac{\partial}{\partial s}X_{23}(t,s) + A_0'X_{23}(t,s) \\
&\quad + A_2'X_{13}(t,s) + X_{33}(t,0,s), \\
X_{23}(t,-\tau) &= X_{22}(t)A_1, \\
X_{23}(k\tau^-,s) &= 0, \\
-\frac{\partial}{\partial t}X_{33}(t,s,r) &= -(\frac{\partial}{\partial s} + \frac{\partial}{\partial r})X_{33}(t,s,r), \\
X_{33}(t,-\tau,r) &= A_1'X_{23}(t,r), \\
X_{33}(t,s,-\tau) &= X_{32}(t,s)A_1, \\
X_{33}(k\tau^-,s,r) &= X_{33}(k\tau,s,r) - X_{32}(k\tau,s)\Phi(k)X_{23}(k\tau,r)
\end{aligned}
$$

where $T_{22}(k) = I + B_d'X_{22}(k\tau)B_d$, $\Phi(k) = B_dT_{22}^{-1}(k)B_d'$. The Riccati equation (6.14) is reduced to

$$
\begin{aligned}
\dot{Y} &= AY + YA' + \frac{1}{\tau}B_1B_1', \qquad\qquad (6.23) \\
Y(k\tau^+) &= Y(k\tau)[I + C_2'C_2Y(k\tau)]^{-1}.
\end{aligned}
$$

Since $([A,I],[0,C_2])$ is detectable, there exists an unique stabilizing solution of (6.23). Hence from Theorem 6.4 we conclude that the Riccati equations (6.22) and (6.23) have unique τ-periodic nonnegative stabilizing solutions. The optimal controller is given by

$$
\begin{aligned}
\dot{\hat{x}}(t) &:= A\hat{x}(t) + A_2\hat{p}(t), \\
\hat{x}(k\tau^+) &= [I - Y(0)C_2'(I + C_2Y(0)C_2')^{-1}C_2]\hat{x}(k\tau) \\
&\quad + Y(0)C_2'(I + C_2Y(0)C_2')^{-1}y(k), \\
\dot{\hat{p}}(t) &:= A_0\hat{p}(t) + A_1\hat{p}(t-\tau), \\
\hat{p}(k\tau^+) &= -\Phi(0)[X_{21}(0)\hat{x}(k\tau) + \int_{-\tau}^{0} X_{23}(0,s)\hat{p}(k\tau+s)ds \\
&\quad + X_{21}(0)Y(0)C_2'(I + C_2Y(0)C_2')^{-1}y(k)], \\
u(k) &= -T_{22}^{-1}B_d'[X_{21}(0)\hat{x}(k\tau) + \int_{-\tau}^{0} X_{23}(0,s)\hat{p}(k\tau+s)ds \\
&\quad + X_{21}(0)Y(0)C_2'(I + C_2Y(0)C_2')^{-1}y(k)]
\end{aligned}
$$

and the optimal value is given by

$$
\min \| G_{z\tilde{w}} \|_2^2 = \frac{1}{\tau}\int_0^{\tau} tr.B_1'X(s)B_1\,ds
$$

$$+tr.T_{22}^{-1}B_d'X_{21}(0)(I+Y(0)C_2'C_2)^{-1}Y(0)X_{12}(0)B_d$$
$$= \frac{1}{\tau}\int_0^\tau tr.B_1'X(s)B_1\,ds + tr.T_{22}^{-1}B_d'X_{21}(0)Y(0^+)X_{12}(0)B_d.$$

6.2 Sampled-data Fuzzy Systems

Takagi-Sugeno fuzzy models are nonlinear systems described by a set of IF-THEN rules which gives a local linear representation of an underlying system. Such models can approximate a wide class of nonlinear systems. They can even describe exactly certain nonlinear systems. In this section we consider a sampled-data fuzzy systems and give a design method of output feedback controllers.

6.2.1 Sampled-data Fuzzy Systems

Consider the following IF-THEN rules:

$$\begin{array}{ll} \textbf{IF} & z_1 \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p \text{ is } M_{ip} \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + B_i \tilde{u}(t), \\ & y(k) = C_i x(k\tau),\ k = 0,1,2,\cdots,\ i = 1,\cdots,r \end{array} \tag{6.24}$$

where τ is a sampling period, $x(\cdot) \in \mathbf{R}^n$ is the state, $\tilde{u}(\cdot) \in \mathbf{R}^m$ is the control input realized through zero-order hold, i.e., $\tilde{u}(t) = u(k)$, $k\tau < t \le (k+1)\tau$, $y(\cdot) \in \mathbf{R}^q$ is the sampled observation, the matrices A_i, B_i and C_i are of appropriate dimensions, r is the number of IF-THEN rules, M_{ij} are the fuzzy sets and $z_1, \cdots, z_p$ are premise variables. $M_{ij}(z_j)$ denotes the grade of z_j being in the fuzzy set M_{ij}, $0 \le M_{ij}(z_j) \le 1$. We set $z = [\,z_1 \quad \cdots \quad z_p\,]$ and assume that z is a given function. Then the state equation and the output are defined as weighted linear systems

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^r \lambda_i(z(t))\{A_i x(t) + B_i \tilde{u}(t)\}, \\ y(k) &= \sum_{i=1}^r \lambda_i(z(k\tau)) C_i x(k\tau) \end{aligned} \tag{6.25}$$

where

$$\lambda_i(z) = \frac{w_i(z)}{\sum_{i=1}^r w_i(z)}, \quad w_i(z) = \prod_{j=1}^p M_{ij}(z_j).$$

We assume

$$w_i(z) \ge 0,\ i = 1,\cdots,r,\ \sum_{i=1}^r w_i(z) > 0,\ \forall z.$$

Hence $\lambda_i(z)$ satisfies

$$\lambda_i(z) \geq 0,\ i = 1, \cdots, r,\ \ \sum_{i=1}^{r} \lambda_i(z) = 1,\ \forall z.$$

First we consider the stabilization problem by state feedback controllers and assume that the following rules are given:

IF $z_1(k\tau)$ is M_{i1} and $\cdots$ and $z_p(k\tau)$ is M_{ip}
THEN $\tilde{u}(t) = u(k) = F_i x(k\tau),\ k\tau < t \leq (k+1)\tau,\ i = 1, \cdots, r.$

This says that the state feedback $u(k) = F_i x(k\tau)$ is suitable for the i-th system (6.24). Then the natural choice of the controller is the following:

$$\tilde{u}(t) = \sum_{i=1}^{r} \lambda_i(z(k\tau)) F_i x(k\tau),\ k\tau < t \leq (k+1)\tau \qquad (6.26)$$

where we use the same weights $\lambda_i(z)$ as in (6.25). Now consider the closed-loop system (6.25) with (6.26). As in Chapter 5 we express the closed-loop system (6.25) with (6.26) by the jump system

$$\begin{aligned} \dot{x}_c(t) &= \sum_{i=1}^{r} \lambda_i(z(t)) G_i x_c(t),\ k\tau < t < (k+1)\tau, \qquad (6.27) \\ x_c(k\tau^+) &= \sum_{i=1}^{r} \lambda_i(z(k\tau)) \hat{G}_i x_c(k\tau) \end{aligned}$$

where $x_c = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$, $G_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}$ and $\hat{G}_i = \begin{bmatrix} I & 0 \\ F_i & 0 \end{bmatrix}$. Now we give sufficient conditions for the exponential stability of (6.27) based on Lyapunov functions.

Theorem 6.6 *The fuzzy system (6.27) is exponentially stable, if there exists a bounded right continuous matrix $X(t) \geq a_1 I$, $a_1 > 0$, $t \geq 0$ that satisfies*

$$\begin{aligned} \dot{X} + G_i'X + XG_i &\leq -P_i < 0,\ i = 1, \cdots, r, \qquad (6.28) \\ \hat{G}_i'X(k\tau)\hat{G}_i - X(k\tau^-) &\leq -\hat{P}_i < 0,\ i = 1, \cdots, r \end{aligned}$$

where P_i and $\hat{P}_i$ are positive definite matrices.

Proof. Using the first inequality of (6.28), we obtain

$$\begin{aligned} & \dot{X} + \left(\sum_{i=1}^{r} \lambda_i(z(t)) G_i \right)' X + X \left(\sum_{i=1}^{r} \lambda_i(z(t)) G_i \right) \\ \leq & \sum_{i=1}^{r} \lambda_i(z(t)) [\dot{X} + G_i'X + XG_i] \\ \leq & -cI \end{aligned}$$

for some $c > 0$ with $P_i \geq cI$. At $t = k\tau$ we have

$$\begin{aligned}
&\left(\sum_{i=1}^{r} \lambda_i(z(k\tau))\hat{G}_i\right)' X(k\tau) \left(\sum_{i=1}^{r} \lambda_i(z(k\tau))\hat{G}_i\right) - X(k\tau^-) \\
= &\sum_{i=1}^{r} \lambda_i^2(z(k\tau))[\hat{G}_i' X(k\tau)\hat{G}_i - X(k\tau^-)] \\
&+ \sum_{i<j} \lambda_i(z(k\tau))\lambda_j(z(k\tau))[\hat{G}_i' X(k\tau)\hat{G}_j + \hat{G}_j' X(k\tau)\hat{G}_i - 2X(k\tau^-)]
\end{aligned}$$

and by the second inequality we also have

$$\begin{aligned}
&\hat{G}_i' X(k\tau)\hat{G}_j + \hat{G}_j' X(k\tau)\hat{G}_i - 2X(k\tau^-) \\
= &-(\hat{G}_i - \hat{G}_j)' X(k\tau)(\hat{G}_i - \hat{G}_j) \\
&\qquad + [\hat{G}_i' X(k\tau)\hat{G}_i + \hat{G}_j' X(k\tau)\hat{G}_j - 2X(k\tau^-)] \\
< &-(P_i + P_j).
\end{aligned}$$

Hence we obtain

$$\left(\sum_{i=1}^{r} \lambda_i(z(k\tau))\hat{G}_i\right)' X(k\tau) \left(\sum_{i=1}^{r} \lambda_i(z(k\tau))\hat{G}_i\right) - X(k\tau^-) \leq -\hat{c}I$$

for some $\hat{c} > 0$ with $\hat{P}_i \geq \hat{c}I$. As in the proof of Proposition 4.2, we can show the assertion. ∎

Now consider for (6.25) an observer of the form

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(z(t))\{A_i\hat{x}(t) + B_i\tilde{u}(t)\}, \qquad (6.29)\\
\hat{x}(k\tau^+) &= \hat{x}(k\tau) + K[y(k) - \hat{y}(k)]
\end{aligned}$$

where $\hat{y}(k)$ is given by

$$\hat{y}(k) = \sum_{i=1}^{r} \lambda_i(z(k\tau))C_i\hat{x}(k\tau).$$

We wish to find K such that $e(t) = x(t) - \hat{x}(t) \to 0$ exponentially as $t \to \infty$. We assume that the following rules are given concerning an observer of each subsystem in (6.24):

IF z_1 is M_{i1} and $\cdots$ and z_p is M_{ip}
THEN $\dot{\hat{x}}(t) = A_i\hat{x}(t) + B_i\tilde{u}(t), \; k\tau < t < (k+1)\tau,$
$\hat{x}(k\tau^+) = \hat{x}(k\tau) + K_i(y(k) - \hat{y}(k)), \; i = 1, \cdots, r.$

We propose the following observer gain:

$$K = \sum_{i=1}^{r} \lambda_i(z(k\tau))K_i. \tag{6.30}$$

Substituting (6.30) into (6.29), we have

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(z(t))\{A_i\hat{x}(t) + B_i\tilde{u}(t)\}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + \sum_{j=1}^{r} \lambda_j(z(k\tau))K_j[y(k) - \hat{y}(k)]. \end{aligned} \tag{6.31}$$

Subtracting (6.31) from (6.25) with $x(k\tau^+) = x(k\tau)$, we have the error system

$$\begin{aligned} \dot{e}(t) &= \sum_{j=1}^{r} \lambda_j(z(t))A_j e(t), \ k\tau < t < (k+1)\tau, \\ e(k\tau^+) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(z(k\tau))\lambda_j(z(k\tau))(I - K_jC_i)e(k\tau). \end{aligned} \tag{6.32}$$

We have a result similar to Theorem 6.6.

Theorem 6.7 *The error system (6.32) is exponentially stable, if there exists a bounded right continuous matrix $Y(t) \geq a_2 I$, $a_2 > 0$, $t \geq 0$ that satisfies*

$$\begin{aligned} \dot{Y} + A_i'Y + YA_i &\leq -Q_i < 0, \ i = 1, \cdots, r \\ (I - K_jC_i)'Y(k\tau)(I - K_jC_i) - Y(k\tau^-) &\leq -\hat{Q}_{ij} < 0, \ i, j = 1, \cdots, r \end{aligned} \tag{6.33}$$

where Q_i and $\hat{Q}_{ij}$ are positive definite matrices.

Next we consider the output feedback stabilization of the fuzzy system (6.25). Consider the controller based on the state feedback controller (6.26) and the observer (6.31):

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(z(t))\{A_i\hat{x}(t) + B_i\tilde{u}(t)\}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + \sum_{j=1}^{r} \lambda_j(z(k\tau))K_j[y(k) - \hat{y}(k)], \\ u(k) &= \sum_{i=1}^{r} \lambda_i(z(k\tau))F_i\hat{x}(k\tau). \end{aligned} \tag{6.34}$$

Then the closed-loop system (6.25) with the controller (6.34) is equivalent to

$$\begin{aligned} \dot{\tilde{x}}(t) &= \sum_{i=1}^{r} \lambda_i(z(t)) H_i \tilde{x}(t), \quad k\tau < t < (k+1)\tau, \\ \dot{\tilde{x}}(k\tau^+) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(z(k\tau))\lambda_j(z(k\tau)) \hat{H}_{ij}\tilde{x}(k\tau) \end{aligned} \tag{6.35}$$

where $\tilde{x} = \begin{bmatrix} x_c \\ e \end{bmatrix}$, $H_i = \begin{bmatrix} G_i & 0 \\ 0 & A_i \end{bmatrix}$, $\hat{H}_{ij} = \begin{bmatrix} \hat{G}_j & \hat{F}_j \\ 0 & I - K_j C_i \end{bmatrix}$ and $\hat{F}_j = \begin{bmatrix} 0 \\ -F_j \end{bmatrix}$.

Now we assume that $X(t)$ and $Y(t)$ satisfy the conditions of Theorems 6.6 and 6.7, respectively. Then we shall show that

$$\bar{X}(t) = \begin{bmatrix} X(t) & 0 \\ 0 & \eta Y(t) \end{bmatrix} \tag{6.36}$$

for sufficiently large $\eta > 0$ is a Lyapunov function for (6.35). In fact

$$\begin{aligned} \dot{\bar{X}} + H_i'\bar{X} + \bar{X}H_i &\leq -\begin{bmatrix} P_i & 0 \\ 0 & Q_i \end{bmatrix}, \ i = 1,..,r, \\ \hat{H}_{ij}'\bar{X}(k\tau)\hat{H}_{ij} - \bar{X}(k\tau^-) &\leq -\begin{bmatrix} P_j & -\hat{G}_j'X(k\tau)\hat{F}_j \\ -\hat{F}_j'X(k\tau)\hat{G}_j & \eta Q_{ij} - \hat{F}_j'X(k\tau)\hat{F}_j \end{bmatrix}, \\ & \qquad i, \ j = 1,..,r. \end{aligned}$$

Since

$$\begin{bmatrix} M_1 & M_{12} \\ M_{12}' & M_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ M_{12}'M_1^{-1} & I \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 - M_{12}'M_1^{-1}M_{12} \end{bmatrix} \begin{bmatrix} I & M_1^{-1}M_{12} \\ 0 & I \end{bmatrix}$$

for any matrices M_{12}, M_2 and nonsingular M_1,

$$\begin{bmatrix} P_j & -\hat{G}_j'X(k\tau)\hat{F}_j \\ -\hat{F}_j'X(k\tau)\hat{G}_j & \eta Q_{ij} - \hat{F}_j'X(k\tau)\hat{F}_j \end{bmatrix} > 0, \ i, \ j = 1,..,r$$

is equivalent to

$$\eta Q_{ij} - \hat{F}_j'X(k\tau)\hat{F}_j - \hat{F}_j'X(k\tau)\hat{G}_j(P_j)^{-1}\hat{G}_j'X(k\tau)\hat{F}_j > 0, \ i, \ j = 1,..,r.$$

This is always satisfied if we choose η sufficiently large. Hence if we can find a stabilizing feedback and an exponentially convergent observer, we can always construct a stabilizing output feedback controller. This is a generalization of the separation property in the linear theory.

Theorem 6.8 *Suppose there exist bounded right continuous matrices $X(t) \geq a_1 I$, $a_1 > 0$ and $Y(t) \geq a_2 I$, $a_2 > 0$ that satisfy (6.28) and (6.33), respectively. Then the fuzzy system (6.35) is exponentially stable.*

6.2.2 The Case with Premise Variable y

In Section 6.2.1, the premise variable z for the fuzzy system (6.25) is assumed given and unspecified. Here we consider the case where the premise variable z coincides with the observation y of the underlying system. To make the output (6.25) definite, we assume that $C_i = C$, $i = 1, 2, \cdots, r$. Then the sampled-data fuzzy model is described by the following IF-THEN rules:

$$\begin{array}{ll} \textbf{IF} & y_1(k) \text{ is } M_{i1} \text{ and } \cdots \text{ and } y_p(k) \text{ is } M_{ip} \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + B_i \tilde{u}(t), \; k\tau < t < (k+1)\tau, \\ & y(k) = Cx(k\tau), \; i = 1, \cdots, r. \end{array} \tag{6.37}$$

The state equation and the output are defined as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(y(k))\{A_i x(t) + B_i \tilde{u}(t)\}, \\ y(k) &= Cx(k\tau) \end{aligned} \tag{6.38}$$

and the output in (6.38) coincides with the premise variables. Now the observer (6.31) becomes

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(y(k))\{A_i \hat{x}(t) + B_i \tilde{u}(t)\}, \; k\tau < t \leq (k+1)\tau, \quad (6.39) \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + \sum_{i=1}^{r} \lambda_i(y(k)) K_i [y(k) - C\hat{x}(k\tau)]. \end{aligned}$$

From Theorem 6.7, we have the following.

Theorem 6.9 *The error system*

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{r} \lambda_i(y(k)) A_i e(t), \; k\tau < t < (k+1)\tau, \\ e(k\tau^+) &= \sum_{i=1}^{r} \lambda_i(y(k))(I - K_i C) e(k\tau) \end{aligned} \tag{6.40}$$

is exponentially stable, if there exists a bounded right continuous matrix $Y(t) \geq a_3 I$, $a_3 > 0$, $t \geq 0$ *that satisfies*

$$\begin{aligned} \dot{Y} + A_i' Y + Y A_i &\leq -Q_i < 0, \\ (I - K_i C)' Y(k\tau)(I - K_i C) - Y(k\tau^-) &\leq -\hat{Q}_i < 0, \; i = 1, \cdots, r \end{aligned} \tag{6.41}$$

where Q_i *and* $\hat{Q}_i$ *are positive definite matrices.*

Now consider the output stabilization of (6.38). As in Section 6.2.1, we take the following controller:

$$\begin{aligned}\dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(y(k))\{A_i\hat{x}(t) + B_i\tilde{u}(t)\}, \; k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + \sum_{i=1}^{r} \lambda_j(y(k))K_i[y(k) - C\hat{x}(k\tau)], \\ u(k) &= \sum_{i=1}^{r} \lambda_i(y(k))F_i\hat{x}(k\tau).\end{aligned} \tag{6.42}$$

Then the closed-loop system (6.38) and (6.42) is equivalent to the extended fuzzy system

$$\begin{aligned}\dot{\tilde{x}}(t) &= \sum_{i=1}^{r} \lambda_i(y(k))H_i\tilde{x}(t), \; k\tau < t < (k+1)\tau, \\ \dot{\tilde{x}}(k\tau^+) &= \sum_{i=1}^{r} \lambda_i(y(k))\hat{H}_{ij}\tilde{x}(k\tau)\end{aligned} \tag{6.43}$$

where $\tilde{x} = \begin{bmatrix} x_c \\ e \end{bmatrix}$, $H_i = \begin{bmatrix} G_i & 0 \\ 0 & A_i \end{bmatrix}$, $\hat{H}_i = \begin{bmatrix} \hat{G}_i & \hat{F}_i \\ 0 & I - K_iC \end{bmatrix}$ and $\hat{F}_i = \begin{bmatrix} 0 \\ -F_i \end{bmatrix}$. From Theorem 6.8, we obtain the following.

Theorem 6.10 *Suppose there exist bounded right continuous matrices $X(t) \geq a_1 I$, $a_1 > 0$ and $Y(t) \geq a_3 I$, $a_3 > 0$ that satisfy (6.28) and (6.41), respectively. Then the fuzzy system (6.43) is exponentially stable.*

6.2.3 The Case with Premise Variable x

In this section, we consider the sampled-data fuzzy models described by (6.25) with $z = x$. Then (6.25) becomes

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(x(t))\{A_i x(t) + B_i\tilde{u}(t)\}, \\ y(k) &= \sum_{i=1}^{r} \lambda_i(x(k\tau))C_i x(k\tau).\end{aligned} \tag{6.44}$$

This class is very general and can describe the largest class of nonlinear systems provided that x is given. However we assume that the only information available to us is the observation y. Then the design of observers is more difficult. In fact the observer (6.31) with $z = x$ is no longer feasible. However, if we replace z by $\hat{x}$ in (6.31), then we obtain a candidate of observers

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} \lambda_i(\hat{x}(t))\{A_i\hat{x}(t) + B_i\tilde{u}(t)\}, \; k\tau < t < (k+1)\tau, \tag{6.45}$$

$$\hat{x}(k\tau^+) = \hat{x}(k\tau) + \sum_{j=1}^{r} \lambda_j(\hat{x}(k\tau))K_j[y(k) - \hat{y}(k)]$$

where $\hat{y}(k)$ is given by

$$\hat{y}(k) = \sum_{i=1}^{r} \lambda_i(\hat{x}(k\tau))C_i\hat{x}(k\tau).$$

We can show that (6.45) is an observer for the following fuzzy system, which can be regarded as an approximation of (6.25):

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(t))\{A_i x(t) + B_i\tilde{u}(t)\}, \qquad (6.46) \\ y(k) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(k\tau))C_i x(k\tau). \end{aligned}$$

Consider the system (6.46) and the observer (6.45), and let $e = x - \hat{x}$. Then we have

$$\begin{aligned} \dot{e}(t) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(t))A_i e(t), \;\; k\tau < t < (k+1)\tau, \qquad (6.47) \\ e(k\tau^+) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(\hat{x}(k\tau))\lambda_j(\hat{x}(k\tau))(I - K_jC_i)e(k\tau). \end{aligned}$$

From Theorem 6.7, we have the following.

Theorem 6.11 *The error system (6.47) is exponentially stable, if there exists a bounded right continuous matrix $Y(t) \geq a_2 I$, $a_2 > 0$ that satisfies (6.33).*

Now we assume that there exists a bounded right continuous matrix $Y(t) \geq a_2 I$, $a_2 > 0$ that satisfies (6.33) and consider the asymptotic convergence of the observer (6.45) for the original fuzzy system (6.44). For this purpose, we set

$$\begin{aligned} f(x,u,z) &= \sum_{i=1}^{r} \lambda_i(z)\{A_i x + B_i u\}, \\ m(z) &= \sum_{i=1}^{r} \lambda_i(z)K_i, \\ n(z) &= \sum_{i=1}^{r} \lambda_i(z)C_i. \end{aligned}$$

Then (6.46) and (6.45) are written respectively as

$$\dot{x}(t) = f(x, \tilde{u}, \hat{x}) \tag{6.48}$$

and

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}, \tilde{u}, \hat{x}), \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + m(\hat{x}(k\tau))n(\hat{x}(k\tau))e(k\tau). \end{aligned} \tag{6.49}$$

Subtracting (6.49) from (6.48) with $x(k\tau^+) = x(k\tau)$ and setting $e = x - \hat{x}$ we obtain

$$\begin{aligned} \dot{e}(t) &= f(x, \tilde{u}, \hat{x}) - f(\hat{x}, \tilde{u}, \hat{x}), \\ e(k\tau^+) &= \{I - m(\hat{x}(k\tau))n(\hat{x}(k\tau))\}e(k\tau). \end{aligned} \tag{6.50}$$

Since there exists a bounded right continuous matrix $Y(t) > 0$ that satisfies (6.33) and the error system (6.50) coincides with the system (6.47), we have

$$\begin{aligned} e'\dot{Y}e + \psi_1'Ye + e'Y\psi_1 &\leq -\alpha \mid e \mid^2, \ \alpha > 0, \\ \phi_1'Y(k\tau)\phi_1 - e'Y(k\tau^-)e &\leq -\hat{\alpha} \mid e \mid^2, \ \hat{\alpha} > 0 \end{aligned} \tag{6.51}$$

for any $\tilde{u}$, x and $\hat{x}$ with $e = x - \hat{x}$, where $\psi_1 = f(x, \tilde{u}, \hat{x}) - f(\hat{x}, \tilde{u}, \hat{x})$, $\phi_1 = \{I - m(\hat{x})n(\hat{x})\}e$ and we have suppressed $k\tau$ in ϕ_1, e in the second inequality.

Now consider the fuzzy system (6.44), which is also written as

$$\dot{x}(t) = f(x, \tilde{u}, x)$$

and the observer (6.45). Let $\epsilon = x - \hat{x}$. Then

$$\begin{aligned} \dot{\epsilon}(t) &= f(x, \tilde{u}, x) - f(\hat{x}, \tilde{u}, \hat{x}) \\ &= \psi_1 + f(x, \tilde{u}, x) - f(x, \tilde{u}, \hat{x}), \\ \epsilon(k\tau^+) &= \epsilon(k\tau) - m(\hat{x}(k\tau))\{n(x(k\tau))x(k\tau) - n(\hat{x}(k\tau))\hat{x}(k\tau)\} \\ &= \bar{\phi}_1(k\tau) - m(\hat{x}(k\tau))\{n(x(k\tau)) - n(\hat{x}(k\tau))\}x(k\tau) \end{aligned}$$

where $\bar{\phi}_1 = \{I - m(\hat{x})n(\hat{x})\}\epsilon$. Using (6.51) we obtain

$$\begin{aligned} &\epsilon'\dot{Y}\epsilon + (\psi_1 + \psi_2)'Y\epsilon \\ &\qquad + \epsilon'Y(\psi_1 + \psi_2) \leq -\alpha \mid \epsilon \mid^2 + \psi_2'Y\epsilon + \epsilon'Y\psi_2, \\ &(\phi_1 - \phi_2)'Y(k\tau)(\phi_1 - \phi_2) \\ &\qquad - \epsilon'Y(k\tau^-)\epsilon \leq -\hat{\alpha} \mid \epsilon \mid^2 + \phi_2'Y(k\tau)\phi_2 \\ &\qquad\qquad - \phi_2'Y(k\tau)\phi_1 - \phi_1'Y(k\tau)\phi_2 \end{aligned}$$

where $\psi_2 = f(x, \tilde{u}, x) - f(x, \tilde{u}, \hat{x})$, $\phi_2 = m(\hat{x})\{n(x) - n(\hat{x})\}x$ and we have suppressed $k\tau$ in ϕ_i, $i = 1, 2$, ϵ in the second inequality. Hence (6.45) is an asymptotically convergent observer if

$$\begin{aligned} \psi_2' Y \epsilon + \epsilon' Y \psi_2 &\leq \beta \mid \epsilon \mid^2, \qquad (6.52) \\ \phi_2'(k\tau)Y(k\tau)\phi_2(k\tau) - \phi_2'(k\tau)Y(k\tau)\phi_1(k\tau) & \\ -\phi_1'(k\tau)Y(k\tau)\phi_2(k\tau) &\leq \hat{\beta} \mid \epsilon(k\tau) \mid^2 \end{aligned}$$

for some $\beta < \alpha$ and $\hat{\beta} < \hat{\alpha}$.

Theorem 6.12 *Assume that there exists a bounded right continuous matrix $Y(t) \geq a_2 I$, $a_2 > 0$ that satisfies (6.33). Then (6.45) is an observer for the fuzzy system (6.44) if there exist positive numbers $\beta < \alpha$ and $\hat{\beta} < \hat{\alpha}$ that satisfy (6.52).*

Now we consider a special case: $C_i = C$, $i = 1, 2, \cdots, r$. Then $\phi_2 = 0$ and (6.45) is an asymptotically convergent observer if

$$\psi_2' Y e + e' Y \psi_2 \leq \beta \mid e \mid^2 \qquad (6.53)$$

for some $\beta < \alpha$.

Corollary 6.1 *Suppose $C_i = C$, $i = 1, 2, \cdots, r$. (6.45) is an observer for the fuzzy system (6.44) if there exists a positive number $\beta < \alpha$ that satisfies (6.53).*

The condition (6.52) is rather restrictive. Instead, we may assume (6.52) locally. Then (6.45) becomes a local observer for (6.44).

Next consider the output stabilization of (6.46). We take the following controller:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(t))\{A_i \hat{x}(t) + B_i \tilde{u}(t)\}, \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{x}(k\tau) + \sum_{j=1}^{r} \lambda_j(\hat{x}(k\tau))K_j(y(k) - \hat{y}(k)), \qquad (6.54) \\ \tilde{u}(t) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(k\tau))F_i x(k\tau). \end{aligned}$$

In view of (6.35), the closed-loop system (6.46) and (6.54) is equivalent to

$$\begin{aligned} \dot{\tilde{x}}(t) &= \sum_{i=1}^{r} \lambda_i(\hat{x}(t))H_i \tilde{x}(t), k\tau < t < (k+1)\tau, \qquad (6.55) \\ \tilde{x}(k\tau^+) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i(\hat{x}(k\tau))\lambda_j(\hat{x}(k\tau))\hat{H}_{ij}\tilde{x}(k\tau). \end{aligned}$$

From Theorem 6.8, we obtain the following.

Theorem 6.13 *Suppose that there exist bounded right continuous matrices* $X(t) \geq a_1 I$, $a_1 > 0$ *and* $Y(t) \geq a_2 I$, $a_2 > 0$ *that satisfy (6.28) and (6.33), respectively. Then the fuzzy system (6.55) is exponentially stable.*

Under the assumptions of Theorem 6.13, we obtain similar inequalities to (6.51):

$$\begin{aligned} \tilde{x}'\dot{\bar{X}}\tilde{x} + \Psi_1'\bar{X}\tilde{x} + \tilde{x}'\bar{X}\Psi_1 &\leq -\gamma \mid \tilde{x} \mid^2, \ \gamma > 0, \quad (6.56) \\ \Phi_1'(k\tau)\bar{X}(k\tau)\Phi_1(k\tau) - \tilde{x}'(k\tau)\bar{X}(k\tau^-)\tilde{x}(k\tau) &\leq -\hat{\gamma} \mid \tilde{x}(k\tau) \mid^2, \ \hat{\gamma} > 0 \end{aligned}$$

for any $\tilde{u}$, x and $\hat{x}$ with $e = x - \hat{x}$ where

$$\tilde{x} = \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix}, \ \Psi_1 = \begin{bmatrix} f(x, \bar{x}, \hat{x}) \\ 0 \\ f(x, \bar{x}, \hat{x}) - f(\hat{x}, \bar{x}, \hat{x}) \end{bmatrix}, \ \Phi_1 = \begin{bmatrix} x \\ g(\hat{x})\hat{x} \\ \{I - m(\hat{x})n(\hat{x})\}e \end{bmatrix}$$

and $g(z) = \sum_{i=1}^{r} \lambda_i(z) F_i$.

Now we consider the output feedback stabilization of the original fuzzy system (6.44). The closed-loop system (6.44) with controller (6.54) is equivalent to

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \\ \dot{\epsilon} \end{bmatrix}(t) = \begin{bmatrix} f(x, \bar{x}, x) \\ 0 \\ f(x, \bar{x}, x) - f(\hat{x}, \bar{x}, \hat{x}) \end{bmatrix}, \\ \tilde{x}(k\tau^+) &= \begin{bmatrix} x(k\tau) \\ g(\hat{x}(k\tau))\hat{x}(k\tau) \\ \{I - m(\hat{x}(k\tau))n(\hat{x}(k\tau))\}\epsilon(k\tau) \\ -m(\hat{x}(k\tau))\{n(x(k\tau)) - n(\hat{x}(k\tau))\}x(k\tau) \end{bmatrix}. \end{aligned}$$

Using (6.56) we obtain

$$\begin{aligned} &\tilde{x}'\dot{\bar{X}}\tilde{x} + (\Psi_1 + \Psi_2)'\bar{X}\tilde{x} + \tilde{x}'\bar{X}(\Psi_1 + \Psi_2) \\ &\qquad \leq -\gamma \mid \tilde{x} \mid^2 + \Psi_2'\bar{X}(t)\tilde{x} + \tilde{x}'\bar{X}(t)\Psi_2, \\ &(\Phi_1 - \Phi_2)'\bar{X}(k\tau)(\Phi_1 - \Phi_2) - \tilde{x}'\bar{X}(k\tau^-)\tilde{x} \\ &\qquad \leq -\hat{\gamma} \mid \tilde{x} \mid^2 + \Phi_2'\bar{X}(k\tau)\Phi_2 - \Phi_2'\bar{X}(k\tau)\Phi_1 - \Phi_1'\bar{X}(k\tau)\Phi_2 \end{aligned}$$

where $\Psi_2 = \begin{bmatrix} f(x, \bar{x}, x) - f(x, \bar{x}, \hat{x}) \\ 0 \\ f(x, \bar{x}, x) - f(x, \bar{x}, \hat{x}) \end{bmatrix}$, $\Phi_2 = \begin{bmatrix} 0 \\ 0 \\ m(\hat{x})\{n(x) - n(\hat{x})\}x \end{bmatrix}$ and we have suppressed $k\tau$ in ϕ_i, $i = 1, 2$, $\tilde{x}$ in the second inequality. Hence the controller (6.54) stabilizes (6.44) if

$$\begin{aligned} \Psi_2'\bar{X}\tilde{x} + \tilde{x}'\bar{X}\Psi_2 &\leq \delta \mid \tilde{x} \mid^2, \quad (6.57) \\ \Phi_2'(k\tau)\bar{X}(k\tau)\Phi_2(k\tau) - \Phi_2'(k\tau)\bar{X}(k\tau)\Phi_1(k\tau) & \\ -\Phi_1'(k\tau)\bar{X}(k\tau)\Phi_2(k\tau) &\leq \hat{\delta} \mid \tilde{x}(k\tau) \mid^2 \end{aligned}$$

for some $\delta < \gamma$ and $\hat{\delta} < \hat{\gamma}$.

Theorem 6.14 *Suppose that the assumptions of Theorem 6.13 are satisfied so that (6.56) holds. Then the fuzzy system described by (6.44) and (6.54) is exponentially stable, if there exist positive numbers and $\delta < \gamma$ and $\hat{\delta} < \hat{\gamma}$ that satisfy (6.57).*

Consider the case $C_i = C$, $i = 1, 2, \cdots, r$. Then $\Phi_2 = 0$ and the controller (6.54) stabilizes (6.44) if

$$\Psi_2' \bar{X} \tilde{x} + \tilde{x}' \bar{X} \Psi_2 \leq \delta \mid \tilde{x} \mid^2 \text{ for some } \delta < \gamma. \tag{6.58}$$

Corollary 6.2 *Suppose $C_i = C$, $i = 1, 2, \cdots, r$ and that (6.56) holds. Then the fuzzy system described by (6.44) and (6.54) is exponentially stable, if there exists a positive number $\delta < \gamma$ that satisfies (6.58).*

If we assume (6.57) locally in $\tilde{x}$, we again obtain local stability.

Example 6.1 [74] Consider a nonlinear mass-spring-damper system

$$\begin{aligned} \ddot{\xi} &= -0.02\xi - 0.67\xi^3 - 0.1\dot{\xi}^3 + u, \\ y &= \xi. \end{aligned} \tag{6.59}$$

The nonlinear terms satisfy the following condition for $\xi \in [-0.3 \quad 0.3]$ and $\dot{\xi} \in [-0.3 \quad 0.3]$.

$$\begin{cases} -0.0603\xi \leq -0.67\xi^3 \leq 0 \cdot \xi, & \xi \geq 0, \\ 0 \cdot \xi \leq -0.67\xi^3 \leq -0.0603\xi, & \xi < 0, \end{cases}$$

$$\begin{cases} -0.009\dot{\xi} \leq -0.1\dot{\xi}^3 \leq 0 \cdot \dot{\xi}, & \dot{\xi} \geq 0, \\ 0 \cdot \dot{\xi} \leq -0.1\dot{\xi}^3 \leq -0.009\dot{\xi}, & \dot{\xi} < 0. \end{cases}$$

Hence they can be represented by the convex combination of the upper bound and the lower bound as

$$\begin{aligned} -0.67\xi^3 &= N_1^1(\xi) \cdot 0 \cdot \xi - (1 - N_1^1(\xi)) \cdot 0.0603\xi, \\ -0.1\dot{\xi}^3 &= N_2^1(\dot{\xi}) \cdot 0 \cdot \dot{\xi} - (1 - N_2^1(\dot{\xi})) \cdot 0.009\dot{\xi} \end{aligned} \tag{6.60}$$

where $N_1^1(\xi) \in [0 \quad 1]$, $N_2^1(\xi) \in [0 \quad 1]$. By solving the above equation, $N_1^1(\xi)$, $N_1^2(\xi)$, $N_2^1(\xi)$ and $N_2^2(\xi)$ representing zero and nonzero are obtained as follows:

$$N_1^1(\xi) = 1 - \frac{\xi^2}{0.09}, \quad N_1^2(\xi) = 1 - N_1^1(\xi) = \frac{\xi^2}{0.09},$$

$$N_2^1(\dot{\xi}) = 1 - \frac{\dot{\xi}^2}{0.09}, \quad N_2^2(\dot{\xi}) = 1 - N_2^1(\xi) = \frac{\dot{\xi}^2}{0.09}.$$

Using N_1^1, N_1^2, N_2^1 and N_2^2, the original nonlinear model (6.59) can be represented by the following continuous-time fuzzy model:

$$\begin{array}{ll}\textbf{IF} & \xi(t) \text{ is } M_{i1} \text{ and } \dot{\xi}(t) \text{ is } M_{i2} \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + bu(t), \\ & y(t) = cx(t),\ i = 1,2,3,4 \end{array} \tag{6.61}$$

where $M_{11} = M_{21} = N_1^1$, $M_{31} = M_{41} = N_1^2$, $M_{12} = M_{32} = N_2^1$, $M_{22} = M_{42} = N_2^2$, $x = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 1 \\ -0.01 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ -0.01 & -0.009 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 1 \\ -0.0703 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 1 \\ -0.0703 & -0.009 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $c = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

In the sampled-data case, we set the sampling period $\tau = 0.2$, and assume that the input $u = \tilde{u}$ is realized through a zero-order hold and the observation is taken at $k\tau$. Then (6.59) and (6.61) become

$$\begin{aligned} \ddot{\xi} &= -0.02\xi - 0.67\xi^3 - 0.1\dot{\xi}^3 + \tilde{u}, \\ y(k) &= \xi(k\tau) \end{aligned} \tag{6.62}$$

and

$$\begin{array}{ll}\textbf{IF} & \xi \text{ is } M_{i1} \text{ and } \dot{\xi} \text{ is } M_{i2} \\ \textbf{THEN} & \dot{x}(t) = A_i x(t) + b\tilde{u}(t), \\ & y(k) = cx(k\tau),\ k = 0,1,2,\cdots,\ i = 1,2,3,4, \end{array} \tag{6.63}$$

respectively. This is an example of the fuzzy system (6.44). The simulation result of (6.62)((6.59)) with $\tilde{u} = 0$, $x(0) = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$ is given in Figure 6.1.

First we design a state feedback controller. We take $Q = I_{3\times 3}$, $\hat{Q} = I_{3\times 3}$, $R = 1$, $A = \begin{bmatrix} A_1 & b \\ 0 & 0 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Solving the equation

$$\begin{aligned} -\dot{X} &= A'X + XA + Q,\ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= \hat{A}'X(k\tau)\hat{A} + \hat{Q}, -\hat{A}'X(k\tau)B(R + B'X(k\tau)B)^{-1}B'X(k\tau)\hat{A}, \end{aligned} \tag{6.64}$$

we obtain the τ-periodic solution $X(t)$ and the feedback gain

$$f_1 = \begin{bmatrix} -0.622 & -1.289 \end{bmatrix}$$

where

$$\begin{aligned} X(0) &= X(k\tau) = \begin{bmatrix} 12.097 & 10.265 & 1.812 \\ 10.265 & 19.901 & 3.754 \\ 1.812 & 3.754 & 1.913 \end{bmatrix}, \\ X(0^-) &= X(k\tau^-) = \begin{bmatrix} 11.970 & 7.930 & 0 \\ 7.930 & 16.062 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Similarly we obtain the feedback gains:

$$f_2 = [\,-0.619 \quad -1.328\,],\ f_3 = [\,-0.661 \quad -1.323\,],\ f_4 = [\,-0.658 \quad -1.361\,].$$

The τ-periodic solution $X(t)$ satisfies the conditions of Theorem 6.6, the closed-loop system (6.27) for these f_i is exponentially stable.

As for the observer gains, we take

$$k_1 = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix},\ k_2 = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix},\ k_3 = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix},\ k_4 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}.$$

Setting $Q = \hat{Q} = I_{2\times 2}$ and solving the equation

$$\begin{aligned} -\dot{Y} &= A_1'Y + YA_1 + Q,\ k\tau < t < (k+1)\tau, \\ Y(k\tau^-) &= (I - k_1c)'Y(k\tau)(I - k_1c) + \hat{Q}, \end{aligned}$$

we obtain the τ-periodic solution $Y(t)$ where

$$\begin{aligned} Y(0) &= Y(k\tau) = \begin{bmatrix} 3.695 & -2.198 \\ -2.198 & 3.924 \end{bmatrix}, \\ Y(0^-) &= Y(k\tau^-) = \begin{bmatrix} 3.475 & -2.898 \\ -2.898 & 4.745 \end{bmatrix}. \end{aligned}$$

The τ-periodic solution $Y(t)$ satisfies the conditions of Theorem 6.11 and the error system (6.47) for these k_i is exponentially stable.

We can show that $\bar{X}(t) = \begin{bmatrix} X(t) & 0 \\ 0 & \eta Y(t) \end{bmatrix}$ with $\eta = 12$ (and hence $\eta \geq 12$) assures the exponential stability of the fuzzy system (6.55). Thus we have obtained an output feedback stabilizing controller for the approximate fuzzy system. The simulation result of the approximating fuzzy system with $x(0) = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$, $\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is given in Figure 6.2. Next we apply the observer to the original nonlinear model (6.59). We calculate the left-hand-side of (6.53) to show that (6.45) is an observer for the original nonlinear system if x and $\hat{x}$ remain in a neighbourhood of the origin.

Similarly, we calculate the left-hand-saide of (6.58) to show that the closed-loop system (6.44) and (6.54) is exponentially stable in a neighbourhood of the origin. The simulation result of the original nonlinear system (6.62) with the same controller is given in Figure 6.3.

6.3 Notes and References

The H_2 and H_∞ results in Section 6.1 are taken from [34] and details of proofs are found in [33]. The application to sampled-data systems with first-order

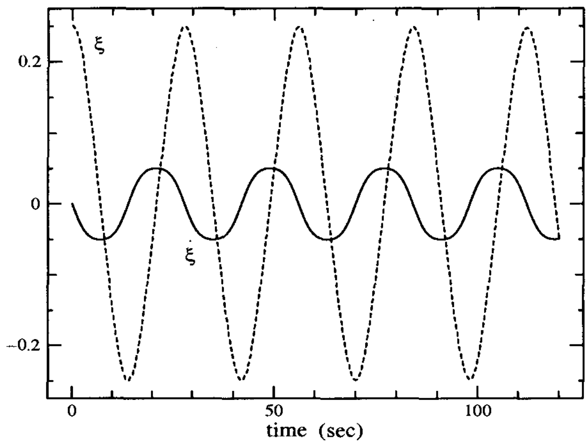

Figure 6.1: The trajectory of the state without control

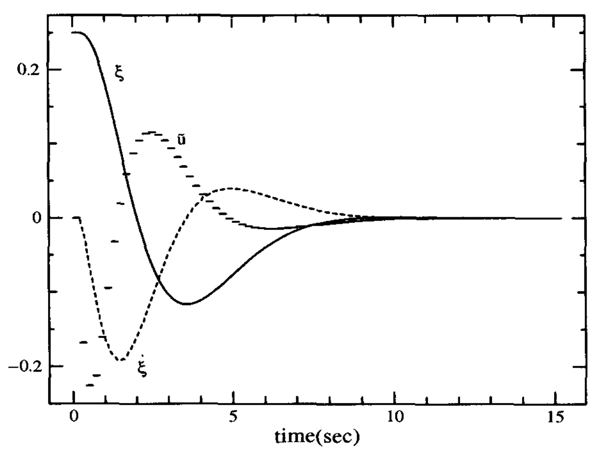

Figure 6.2: The trajectories of the state and control input of the approximating fuzzy model

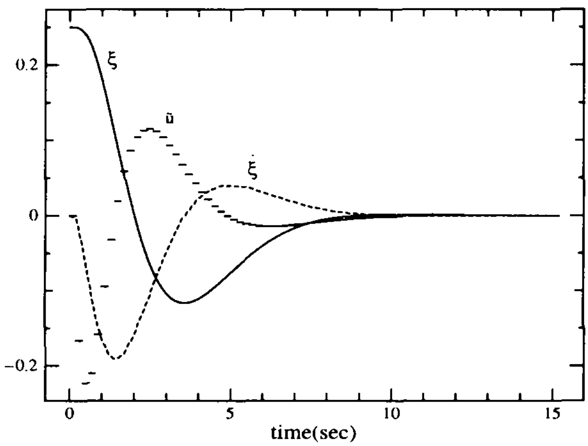

Figure 6.3: The trajectories of the state and control input of the original nonlinear model

hold is given in [32] while [33] contains some examples of distributed parameter systems. We have shown that the H_2 and H_∞ problems for sampled-data systems with first-order hold can be solved using an infinite dimensional jump system. But further works from computational point of view are necessary to implement the controllers given in this section. The H_∞ theory in infinite dimensions can be found in [31, 46, 47, 48, 70]. The basic results on semigroup can be found in [7, 9, 10, 17, 60, 73] and systems theory in infinite dimensions is given in [7, 9, 10].

Section 6.2 is concerned with the output stabilization of nonlinear sampled-data systems and is taken from [59]. This problem is very difficult in general but for a class of nonlinear systems described by fuzzy systems we have given a design method based on jump systems. The paper [59] is an extension of [89, 90] to sampled-data systems. Basic materials on Takagi-Sugeno fuzzy models can be found in [57, 72, 75, 92]. H_∞-control for fuzzy systems is found in [91].

Appendix A. Basic Results of Functional Analysis

We shall recall some basic definitions and results in functional analysis [10, 17, 54].

Definition A.1 *A nonnegative function, denoted by* $\| \cdot \|$, *on a linear space* V *is a norm if the following properties hold.*
(a) $\| x \| = 0$ *if and only if* $x = 0$.
(b) $\| x + y \| \leq \| x \| + \| y \|$ *for all* $x, y \in V$.
(c) $\| \alpha x \| = |\alpha| \, \| x \|$ *for all* $x \in V$ *and all* $\alpha \in \mathbf{C}$.

Here we assume that the scalar field is **C** of complex numbers, but we may replace it by **R** of real numbers.

Definition A.2 *(a) A sequence* $\{x_n\}$ *in a normed linear space* V *is a Cauchy sequence if* $\| x_m - x_n \| \to 0$ *as* $m, n \to \infty$.
(b) A normed linear space V *is complete if every Cauchy sequence has a limit in* V.
(c) A Banach space is a complete normed linear space.

Definition A.3 *(a) A subset* S *of a normed linear space* V *is closed if every convergent sequence in* S *has its limit in* S.
(b) Let S *be a subset of a normed linear space* V. *The smallest closed set containing* S *is the closure of* S.
(c) A subset S *of a normed linear space* V *is dense if its closure coincides with* V.
(d) A normed linear space is separable if it contains a dense set which is countable.

Definition A.4 *An inner product on a linear vector space* V *is a map* $< \cdot, \cdot >: V \times V \to \mathbf{C}$ *with the following properties.*
(a) $< \alpha x + \beta y, z > = \alpha < x, z > + \beta < y, z >$ *for all* $x, y \in V$.
(b) $< y, x > = < x,\bar{} y >$.
(c) $< x, x > \geq 0$ *and* $< x, x > = 0$ *if and only if* $X = 0$.

If we set $\| x \| = < x, x >^{\frac{1}{2}}$, then it is a norm.

Definition A.5 *A linear space with inner product is a Hilbert space if it is complete with respect to the norm induced by the inner product.*

Definition A.6 *Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be two normed linear spaces.*
(a) A map $T : \mathcal{D}(T) \subset X \to Y$ is a linear operator if $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all $x_1, x_2 \in \mathcal{D}(T)$ and all scalars α, β.
(b) A linear operator T is bounded if $\| Tx \|_Y \leq c \| x \|_X$ for any $x \in \mathcal{D}(T)$ for some $c > 0$.
(c) The set of bounded linear operators mapping X into Y is denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X) = \mathcal{L}(X, X)$.
(d) The induced norm of a bounded linear operator $T \in \mathcal{L}(X, Y)$ is defiend by

$$\| T \| = \sup_{0 \neq x \in X} \frac{\| Tx \|_Y}{\| x \|_X}.$$

Definition A.7 *A bounded linear functional on a normed linear space V is a bounded linear operator mapping $V \to \mathbf{C}$.*

Theorem A.1 *(Riesz Representaion Theorem) Let X be a Hilbert space. Then for every bounded linear functional f on X, there exists a unique vector $z \in X$ such that $f(x) = < x, z >$ for all $x \in X$. Moreover, $\| f \| = \| z \|$.*

Let X and Y be two Hilbert spaces and let $T \in \mathcal{L}(X, Y)$. Then by Theorem A.1 there exists a unique operator $T^* \in \mathcal{L}(Y, X)$ which satisfies $< Tx, y >_Y = < x, T^* y >_X$ for all $x \in X$ and $y \in Y$.

Definition A.8 *The operator T^* is the adjoint operator of T.*

Theorem A.2 *Let $T \in \mathcal{L}(X, Y)$ for some Hilbert spaces X, Y. Then $\| T^* \| = \| T \|$.*

Definition A.9 *Let X be a Hilbert space.*
(a) An operator $T \in \mathcal{L}(X)$ is self-adjoint if $T^* = T$.
(b) A self-adjoint operator $T \in \mathcal{L}(X)$ is nonnegative, positive and coercive respectively, if $< Tx, x > \geq 0$ for all $x \in X$, $< Tx, x >> 0$ for all $0 \neq x \in X$, and $< Tx, x > \geq \epsilon \| x \|^2$ for all $x \in X$ for some $\epsilon > 0$.

Theorem A.3 *Let X be a Hilbert space.*
(a) Let $T \in \mathcal{L}(X)$ with $\| T \| < 1$. Then $(I - T)^{-1}$ exists and is in $\mathcal{L}(X)$ with $\| (I - T)^{-1} \| \leq (1 - \| T \|)^{-1}$.
(b) Let $T \in \mathcal{L}(X)$ be coercive with $< Tx, x > \geq \epsilon \| x \|^2$. Then $T^{-1} \in L(X)$ with $\| T^{-1} \| \leq \frac{1}{\epsilon}$ and it is coercive.

Theorem A.4 *Consider a quadratic form* $q(x) = <Tx, x> + <x, h> + <h, x> + c$ *on a Hilbert space* X *for some* $h \in X$ *and* $c \in \mathbf{R}$. *If* $T(-T$ *respectively) is coercive,* $q(x)$ *has the minimum (maximum)* $c - <T^{-1}h, h>$ *at* $x = -T^{-1}h$

Definition A.10 *A sequence* $\{x_n\}$ *in a Hilbert space converges weakly to* x *if* $<x_n - x, y> \longrightarrow 0$ *as* $n \to \infty$ *for all* $y \in X$.

Theorem A.5 *(a) If* x_n *converges weakly to* x *in a Hilbert space, then* $\{x_n\}$ *is bounded and* $\| x \| \leq \liminf \| x_n \| < \infty$.
(b) Every bounded sequence in a Hilbert space contains a weakly convergent subsequence.

If x_n converges weakly to x and if $\| x_n \| \leq c$, then $\| x \| \leq c$.

Definition A.11 *A family of bounded linear operators* $S(t)$, $t \geq 0$ *on a Banach space* X *is called a strongly continuous semigroup (or* C_0*-semigroup) if the following holds:*
(a) $S(t+s) = S(t)S(s)$ *for any* $t,\ s \geq 0$.
(b) $S(0) = I$.
(c) $S(t)x \to x$ *in* X *as* $t \to 0$ *for all* $x \in X$.

There exists real numbers $M > 0$ and ω such that $\| S(t) \| \leq Me^{\omega t}$, $t \geq 0$.

Definition A.12 *The infinitesimal generator of a* C_0*-semigroup* $S(t)$ *is defined by*

$$Ax = \lim_{t \to 0} \frac{1}{x}(S(t)x - x)$$

whenever the limit exists. The domain of A*, denoted by* $\mathcal{D}(A)$*, is the set of all* $x \in X$ *for which the limit exists.*

$\mathcal{D}(A)$ is dense in X. If $x_0 \in \mathcal{D}(A)$, then $S(t)x \in \mathcal{D}(A)$ and

$$\frac{d}{dt}S(t)x_0 = AS(t)x_0.$$

Theorem A.6 *(Hille-Yoshida Theorem) A closed linear operator* A *with dense domain* $\mathcal{D}(A)$ *in a Banach space* X *is the infinitesimal generator of a* C_0*-semigroup* $S(t)$ *if and only if there exist real numbers* M *and* ω *such that for all real* $\lambda > \omega$, $(\lambda - A)^{-1} \in \mathcal{L}(X)$ *and*

$$\| (\lambda - A)^{-m} \| \leq \frac{M}{(\lambda - \omega)^m}, \quad m = 1, 2, ..$$

In this case $\| S(t) \| \leq Me^{\omega t}$.

A linear operator A is closed if $x_n \in \mathcal{D}(A) \to x$, $Ax_n \to y$ imply that $x \in \mathcal{D}(A)$ and $Ax = y$.

When we consider inhomogeneous systems in Banach spaces we need integration in Banach spaces. The extension of the Lebesgue integral to Banach spaces is called the Bochner integral [10, 17]. It requires the notions of strongly measurable functions and simple functions. The space of square integrable functions f on $[a, b]$ with values in X (strongly measurable functions such that $\| f(t) \|^2$ in Lebesgue integrable) is denoted by $L^2(a, b; X)$.

Definition A.13 *Let T be a bounded linear operator in a complex Banach space X. The resolvent set of T is the set of complex numbers λ such that $(\lambda I - T)^{-1} \in \mathcal{L}(X)$. The complement of the resolvent set, denoted by $\sigma(T)$, is called the spectrum of T. The spectral radius of T, denoted by $\rho(T)$, is defined by $\rho(T) = \sup_{\lambda \in \sigma(T)} | \lambda |$.*

Definition A.14 *Let X and Y be two separable Hilbert spaces. A bounded linear operator P mapping X into Y is called a Hilbert-Schmidt operator if $\sum_{i=1}^{\infty} \| Pe_i \|^2 < \infty$ for some orthonormal basis $\{e_i\}$ in X.*

A self-adjoint operator P on X, i.e., $P^* = P$ is nonnegative if $< Px, x > \geq 0$ for any $x \in X$. A nonnegative operator P on X is called a trace class operator if $\sum_{i=1}^{\infty} < Pe_i, e_i >$ converges for some orthonormal basis $\{e_i\}$.

If P is Hilbert-Schmidt, $\sum_{i=1}^{\infty} \| Pe_i \|^2$ converges for any orthonormal basis and is independent of $\{e_i\}$. If P is a nonnegative trace class of operator, $\sum_{i=1}^{\infty} < Pe_i, e_i >$ converges for any orthonormal basis and is independent of $\{e_i\}$. Its value is denoted by $tr.P$ and is called the trace of P.

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