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Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis

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To my wife Kaija and daughters Maritta and Emma for great support. Memories of Greg Hancock for his inspiration and advice on school finance research possibilities.

Vincent Blackburn

To my parents, Karen and Larry, and my siblings, Anson, Vail, Reid, and Adelle, for not only supporting me, but also inspiring me to always do more.

Shae Brennan

To Jerry Miner, John Rapp, and the memories of Bill Duncombe and Larry Hadley, four friends and colleagues who made a difference in my career and taught me the importance of giving back.

John Ruggiero

Preface

After working briefly in a management training program for a textile company, I decided to apply to the Ph.D. program in economics at Syracuse University. I was fortunate to obtain a research assistantship working for Jerry Miner on a project analyzing funding for public schools in New York. In addition to being kind and fatherly, Jerry provided me with an incredible opportunity to work on a real research project. I gained valuable research skills including a working knowledge of SAS programming. I also gained valuable experience maintaining and updating the data for the funded project for rural school districts. A lot of my modest success in academics can be traced to Jerry's supervision and guidance.

In the early 1990s Bill Duncombe approached Jerry about using the education data set to analyze consolidation of New York school districts. Bill did not have to include me on the research project but did so. In addition to cleaning and providing the data, Bill allowed me to estimate educational costs using the stochastic frontier. At the time, I was learning the topic of efficiency measurement using data envelopment analysis. The project with Bill and Jerry led to my first publication. More importantly, Bill taught me the process of research. In addition, Bill was one of the most valuable resources for my dissertation—I thanked him because he did more than one should reasonably expect. It took me many years to realize the profound influence Bill had on my career. Unfortunately, I never properly thanked him for his contributions. Bill passed away on May 11, 2013 after a brief battle with cancer. Thank you, Bill.

My interest in data envelopment analysis arose from a search on suitable research topics given the education data set that I maintained for my research assistantship. At the time I was more interested in finding a dissertation topic that would allow me to exploit my data set. Among the early papers that I found were Data Envelopment Analysis (DEA) papers analyzing education, and were Charnes, Cooper, and Rhodes (1981), Bessent, Bessent, Kennington, and Reagan (1982) and Färe, Grosskopf, and Weber (1989). The New York school data was richer than the ones used in these papers and my research topic was chosen. I obtained all of Rajiv Banker's working papers from Carnegie Mellon and all papers on DEA from both the microeconomics and the operations researchers. At the time, Bill and Jerry obtained funding to analyze technical and scale efficiency of school districts for a coalition of rural school districts. We used the Banker and Morey (1986) DEA models to allow for exogenous socioeconomic factors. I began my journey into the supporting literature on bureaucracies, including the work of Niskanen (1971, 1975) and Chubb and Moe (1990), and on educational production with Hanushek (1979, 1986) and Cohn and Geske (1990). Eventually, all of this work was synthesized in my dissertation on educational efficiency. In the process, I developed a competing model to control for exogenous factors of production (Ruggiero, 1996a). This was the genesis of over two decades of research on performance evaluation which included applications to analyze educational production of New York, Ohio, Illinois, Dutch, and Australian schools. These applications focused not only on the measurement of technical efficiency (Ruggiero, 1996b) but also the causes of efficiency (Duncombe, Miner, & Ruggiero, 1997), the measurement of education costs (Duncombe, Ruggiero, & Yinger, 1996; Ruggiero, 1999), adequacy (Ruggiero, 2007), and productivity (Johnson & Ruggiero, 2011; Brennan, Haelermans, & Ruggiero, 2014).

After I finished my dissertation, my goals were to finish up the research I had started on efficiency measurement and move to other applied econometric work. Nearly 20 years after publishing my first article on efficiency, I continue to work in this area.

I would like to acknowledge my coauthors who have written on the topic of education (in order of publication date): Bill Duncombe, Jerry Miner, Johnny Yinger, Stuart Bretschneider, Don Vitaliano, Lloyd Blanchard, Andy Johnson, Sarah Estelle, Jaye Flavin, Ryan Murphy, Carla Haelermans, and Shae Brennan. In addition to Jaye, Ryan, and Shae, Scott Knowles and Craig Letavec also coauthored papers with me as undergraduate students. My former dean Matt Shank encouraged me to work on scholarship with undergraduate students as part of my additional goals as an endowed professor. The results include two research articles (Blackburn, Brennan & Ruggiero, 2013; Brennan, Haelermans, & Ruggiero, 2014) and this book, with more projects in progress. The experience of working with Shae has been one of the best I have had as a professor. I would like to thank Trevor Collier and Paul Bobrowski for allowing me to continue with these goals; I am currently working on research projects with Nikki Mazza, Kara Colety, Kristen Broadbent, Lesley Chilton, and Kelli Marquardt. Also, I have benefited from useful discussions with Mariana Almeida, David Ausdenmoore, Leslie Douglas, Paulo Henrique, Matheus Lambertucci, and Marco Mendes. And I would be remiss not to acknowledge the support of Joyce Zanini, one of the best workers at the University of Dayton.

Dayton, OH, USA John Ruggiero

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Chapter 1
Introduction

1.1 Education Outcome Provision

A useful starting point for viewing education as a production process was the 1966 report "Equality of Educational Opportunity" for the U.S. Department of Education. This report, more widely referred to as the Coleman Report (1966), provided evidence that socioeconomic factors (student socioeconomic status) are the most important factors in determining educational outcome. While school resources and spending per pupil can positively impact outcomes, the empirical evidence suggests that parental background and student characteristics have a bigger effect. This finding largely explains why equalization of spending per pupil has not removed the large differences in test scores that are still observed.

Hanushek (1989) summarized approximately 20 years of educational production studies and concluded that differences in school spending do not explain variations in student performance. Family background, however, does explain the differences in outcomes. Hanushek further finds that students with wealthier and more educated parents perform better. Hanushek (1979, 1986) provides a useful foundation to analyze education as a production process whereby outcomes are function of school inputs and socioeconomic variables. In addition to Hanushek's work, Bridge, Judd, and Moock (1979) and Cohn and Geske (1990) provide useful discussions of the education production process.

Bradford, Malt, and Oates (1969) provided a two-stage model to analyze public sector production where intermediate outputs (e.g. instruction in mathematics, reading, etc.) are determined by school resources. In a second stage, the final outcomes of interest are functionally related to the intermediate outputs and the socioeconomic environment. Importantly for our work, these socioeconomic factors of production are exogenous even in the long-run. For purposes of measuring efficiency, it is important therefore to properly control for the socioeconomic environment.

One explanation of the finding that per student spending is not strongly correlated with student performance is that schools are relatively inefficient in the

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producing outcomes. If one views education in the context of political theory, one can find sufficient theoretical and empirical support for school inefficiency. Leibenstein (1966), Niskanen (1971, 1975), Migué and Bélanger (1974), Chubb and Moe (1990) and Wyckoff (1990) provide useful political frameworks for the existence of inefficiency. Instead of profit maximizing or cost minimizing firms, bureaucrats are inclined to either maximize their budgets (in which case we would see overprovision of the local public good) or the excess spending which could lead to inefficiency.

A few useful empirical tests of bureaucracy models were provided by De Borger, Kerstens, Moesen, and Vanneste (1994), Duncombe, Miner, and Ruggiero (1997) and Hayes, Razzolini, and Ross (1998). De Borger et al. (1994) analyzed efficiency of Belgian municipalities and found that a larger more educated population leads to more efficiency while wealthier municipalities tended to be less efficient. Duncombe et al. (1997) analyzed cost efficiency of New York schools and found similar results. The evidence suggested that efficiency was negatively related to district size, the percent of teachers that were tenured, district wealth and positively related to the percent of adults that were college educated. Hayes et al. (1998) measured the efficiency of police and fire services in Illinois and found similar results as the two previous studies; wealthier and less educated municipalities tended to be more inefficient. In addition, they find that increased competition does have a significant positive effect on efficiency.

1.2 Data Envelopment Analysis

With economic production theory as a basis, Farrell (1957) showed how efficiency can be estimated relative to a piecewise linear isoquant. Overall inefficiency was composed of technical and allocative parts. Technical inefficiency is observed when a given production possibility is not on the isoquant. As a result, the unit is using too many inputs to produce the observed output, leading to excess costs and lower profits. Allocative inefficiency results when the firm uses the wrong mix of inputs given exogenous input prices. Farrell provided the decomposition of the overall inefficiency into technical and allocative components relative to a piecewise linear isoquant.

Farrell and Fieldhouse (1962) extended Farrell's earlier work by relaxing the assumption of constant returns to scale by allowing decreasing returns. In addition, linear programming was suggested as a methodology that could solve for inefficiency. Boles (1971) extended the models to variable returns to scale and provided computer programs to estimate the efficiency. Afriat (1972) provided the formulation for technical efficiency measurement that was consistent with data envelopment analysis (DEA) with variable returns to scale and the Free Disposal Hull model. Färe, Grosskopf, and Lovell (1994) provide a useful theoretical framework for the production economic approach to efficiency measurement.

DEA is the term coined in the Operations Research literature by Charnes, Cooper, and Rhodes (1978) to measure the technical efficiency of a given observed decision making unit (DMU) assuming constant returns to scale for a multiple input, multiple output production correspondence. The model was extended by Banker, Charnes, and Cooper (1984) to allow variable returns to scale and showed that solutions to the constant returns to scale and variable returns to scale models allowed a decomposition into technical and scale components. Theoretical extensions useful for analyzing educational production were made by Banker and Morey (1986) which allowed nondiscretionary inputs. Alternative models to control for the socio-economic environment include Ray (1988, 1991) which used a second-stage ordinary least squares regression. McCarty and Yaisawarng (1993) extended this by using a Tobit regression in the second stage.[1] Ruggiero (1996a, 1998) provided a conditional technology that does not assume convexity with respect to the nondiscretionary variables. The resulting models of Ruggiero will be the basis for the public sector models used in this book.

1.3 Educational Production and Efficiency

While DEA can be applied to analyze any economic production environment, the motivation for analyzing technical efficiency was public sector models where prices were hard to obtain. In the case of education, for example, while input prices are readily available for the discretionary inputs, the prices of the outcomes are neither well defined nor observed. Charnes, Cooper, and Rhodes (1981) applied the constant returns to scale model to analyze program and managerial efficiency of Program Follow Through. Here, the authors consider two separate frontiers based recognizing possible differences in the production technology. However, some of the variables chosen as discretionary inputs include exogenous socio-economic characteristics (education of the mother, highest occupation of a family member, etc.)

Bessent, Bessent, Kennington, and Reagan (1982) analyzed the 167 elementary schools in the Houston Independent School District. Similar to the Charnes, Cooper and Rhodes paper, this paper used a number of non-controllable correlates (for example, percent of students paying full lunch price and percent of nonminority students) as discretionary inputs. Smith and Mayston (1987) illustrated DEA with English school authorities using a constant returns to scale model. The authors properly delineate the types of inputs into discretionary factors and exogenous environmental factors but treat them similarly within the DEA model. Färe, Grosskopf, and Weber (1989) measured efficiency of Missouri schools and used discretionary inputs and standardized tests. Lack of student background data prevented a more detailed analysis. Thanassoulis and Dunstan (1994) analyze

[1] Simar and Wilson (2007) criticize the two-stage models. Banker and Natarajan (2008) and McDonald (2009) prove the consistency of the OLS estimator in the second stage.

cohorts of students using DEA with targets for improvement. The DEA models used a pre-test to control for prior attainment and a socio-economic variable (percent of students not receiving free school lunches) as the discretionary inputs.

Ruggiero, Duncombe, and Miner (1995) measured the inefficiency of New York school districts and sought to explain the causes from the bureaucratic models. This paper incorporated the conditional convexity model developed by Ruggiero (1996a) to measure public sector efficiency while controlling for the environment. Ruggiero (1996b) used canonical regression to weight excess slack found in an analysis of New York schools. The results are suggestive that the positive relationship between school resources and student outcomes are clouded by inefficient behavior.[2] Ruggiero (1998) provided a cost framework whereby the cost efficiency of school districts could be measured. With education data inputs are typically reported at the aggregate level. In order to measure cost efficiency, the rather herculean assumption of identical prices had to be assumed. Nonetheless, the model allowed an analysis of expenditure data while controlling for the environment. Recognizing that learning takes place at the individual student level, Thanassoulis (1999) provided a detailed estimates of student targeted attainment. In addition, student role models (benchmarks) are identified. Silva Portela and Thanassoulis (2001) extend this model to provide a richer decomposition focusing on the factors of student underachievement. This detailed analysis allowed a sourcing of underachievement to the student and to the school.

Ruggiero and Vitaliano (1999) provided a comparative analysis of DEA and the stochastic frontier model applied to New York school districts. The results were consistent achieving a rank correlation of 0.86 between the measured efficiency. Ruggiero (2000) extended his 1996 model to allow various measures of returns to scale. In addition to the standard notion of scale economies for a given environment, a measure of environmental harshness was defined as the distance between two different frontiers. Sharing similarities to productivity indices, the environmental harshness index is defined with continuous nondiscretionary factors and reveals the additional resources necessary to overcome adverse socioeconomic conditions.

Ruggiero (2001) used DEA to show estimate the based cost of providing a given education. Previous consultants used rather simplistic methods to calculate the based cost but did not properly control for the socioeconomic environment or account for inefficiency. Ruggiero, Miner, and Blanchard (2002) showed that traditional public policy analyses focusing on equity provide misleading results. Differences in spending result not only from differences in outcomes, but also differences in efficiency and in environmental conditions. Ruggiero (2007b) extended DEA to provide a nonparametric measure of adequacy, defined as the expenditures necessary to achieve predefined goals.

[2] Of course, as pointed out in that paper, the assumption of monotonicity requires a positive relationship between discretionary inputs and outcomes. The results can best be considered suggestive.

DEA has also been extended to measure public sector productivity with applications to education. Ouellette and Vierstraete (2010), Johnson and Ruggiero (2011), Brennan, Haelermans, and Ruggiero (2014) and Essid, Ouellete, and Vigeant (2014) provided a Malmquist decomposition of productivity in the presence of fixed factors of production. Johnson and Ruggiero (2011) and Brennan et al. (2014) provide a decomposition based on the conditional technology developed in Ruggiero (1996a). The latter provided a variable returns to scale decomposition with an application to Dutch schools. Essid et al. (2014) analyzed Tunisian schools in the presence of quasi-fixed factors and extended the technique by bootstapping the results.

Haelermans, De Witte, and Blank (2012) and Haelermans and Ruggiero (2013) provide recent analyses of efficiency of Dutch schools. Haelermans and De Witte (2012), Haelermans et al. (2012) studied the allocation of resources and found that teachers are over-utilized while support staff and administrators are under-utilized. Haelermans and Ruggiero (2013) extended the conditional model of Ruggiero (1996a) to decompose overall efficiency into technical and allocative components subject to the operating environment. The results indicate that most inefficiency arises from technical efficiency even though allocative inefficiency is a large component.

1.4 Australian Education

Over the last two decades governments worldwide have increasingly sought to maximize the 'value for money' in school education. This need has stimulated the analysis of performance focused on measuring the efficiency and productivity of public school educational providers. Likewise in Australia increasing attention has been paid to the performance and public accountability of Commonwealth and State government funds devoted to school education. The need for school efficiency and productivity studies as measured by the academic performance of students in relation to money spent, while considering socio-demographic variables outside the control of schools has been recognized since 2008.

This marks the date of the introduction by the Commonwealth Government of the "My School" website developed by the Australian Curriculum and Reporting Agency (ACARA), NAPLAN (National Assessment Program Literacy and Numeracy), My School Test Scores, (2010). This website reports student test scores, student and family characteristics and financial variables for each school. This has enabled studies to be undertaken aimed at measuring the overall performance of each school and reporting the cost efficiency levels enhancing robust evaluations of public school funding policy. In addition the recent Gonski Inquiry Report (Review of Funding for Schooling-Final Report, December, 2011, Commonwealth Government, Canberra), into the funding of Australian schools has also increased the demand for well-constructed school efficiency, productivity and performance studies. However there is a lack of current studies that examine the effects of school and non-school inputs such as financial resources, teacher characteristics, family socio-economic status, and student composition on student outcomes in the context of Australian schools.

The school efficiency case studies in this book seek to address some of the requirements for the future research directions into school performance assessment as outlined in the Gonski report. The major objectives of these case studies are: (1) to identify the factors which account for performance differentials among schools, utilizing robust Data Envelopment Analysis (DEA) models; and (2) investigate whether there are schools that are consistently over or under performing, after taking account of a wide range of school, environmental and regional influences. The focus of these case studies is on New South Wales primary and secondary schools performance utilising the ACARA My School Test Scores using the DEA methods outlined in the recent work of Brennan, Haelermans and Ruggiero (2013). In investigating a different schooling system, as well as informing educational policy in New South Wales, this current study also provides an important robustness check for the DEA methodology, namely application to an entirely different country dataset.

1.4.1 New South Wales Secondary School System

NSW operates a centralised system of funding to government schools. Approximately 82.5 % of school recurrent resources are provided through the NSW Department of Education and Communities (DEC) state wide formula allocations. Commonwealth government allocations make up 13 %, this amount having grown since 2009 through increased Federal funding under the "Building the Education Revolution" and National Partnership programs (Keating, Annett, Burke, and O'Hanlon, 2011). School derived revenue makes up about 5 % of school funding. The expenditure that is incurred at the school level from these State and Commonwealth allocations is met through two basic methods: (1) Central allocations of resources (including staff) and funds that schools can utilise; and (2) direct central payments of school based costs. This is provided through two core mechanisms, centralised staffing allocations and via grants, which are either 'tied' or 'untied'.

All staff positions are centrally allocated upon the basis of formulae, with some capacity for variation based on negotiations between the school and the Department of Education and Communities personnel. Schools may seek additional staff if they have a budget surplus. Staffing constitutes about 81 % of the operational costs of a school, and the effective budget allocations using the same formula will vary due to the different salary steps of teachers. Low Socio Economic Status (SES) schools also receive allocations under the Priority Schools Funding Scheme. In addition Global funding allocations are calculated annually for each school at the beginning of each school year and at the commencement of Semester 2 and are intended to help schools meet operational costs.

Special factor loadings are also additional entitlements to compensate schools affected by specific circumstances such as urgent minor maintenance and geographic isolation, an important factor in the New South Wales environment. A Global Funding enhancement element also operates to take account of rural location and socio-economic considerations. Beyond the above allocations a range of services and grants are delivered by central and regional staff including school cleaning and maintenance and professional development programs. Additional equity and needs allocations are also delivered to schools mainly through the staffing formulae. The student population factors utilised include SES, English as a Second Language (ESL) and new arrivals, Indigenous, Isolated and Disability characteristics. School circumstances recognised include location, enrolment size (diseconomies of scale) and complexity. The disabilities and SES dimensions contribute the most.

1.4.2 Proposed School Funding and Staffing Reforms

As indicated above the NSW Government Primary and Secondary school system is currently a predominately homogenous one that is mainly funded from State Government resources across a common curriculum. Individual schools now have a very limited degree of control over decision making, over teacher hiring and firing and resource allocation processes. However following the election of a new State Government in NSW in March 2011, a new policy to devolve decision making to schools is being progressively implemented. Each school in NSW in 2015 will have control over some 70 % of their total school budget as well as control over hiring and firing teachers and other school personnel. This new policy called “Local Schools, Local Decisions”, commenced in the 2013 school year with 229 schools participating in the program with the balance of the remaining 2,000 schools being integrated into the program of decentralised school decision making by the start of the 2015 school year (NSW DEC, 2012).

The school efficiency and productivity case studies outlined in this book provides a platform for a series of “before” and “after” assessments to be made aimed at measuring any significant changes in school efficiency and productivity using robust Data Envelopment Analysis (DEA) modelling analytics. These studies will indicate the degree of success in achieving greater ‘Value for Money’ in NSW schooling arising from such budgetary and staffing devolutionary reforms. The studies will also give impetus to evaluating the implementation and effectiveness of similar school decentralisation policies across the other seven government school systems across Australia. Likewise similar studies could be undertaken for the non-government catholic and other independent school system authorities in the Australian States.

1.4.3 Budget and Staffing Decisions in Other Jurisdictions

An Independent Public Schools (IPS) initiative was launched by the West Australian Government in 2010 allowing government school principals and boards more power to hire staff, manage the budget and shape the curriculum. However unlike private schools they cannot charge fees and must accept all student enrolments. About one third of all government schools (almost 220 schools) in 2013 have the above powers. An evaluation of this scheme by the School of Education in the University of Melbourne was commissioned by the WA Education Department in 2012. However, this preliminary evaluation report indicated that, "in this early phase of the IPS development there is little evidence of change to student outcomes, such as enrolment or student achievements", (Evaluation of the Independent Public Schools Initiative, WA Government, May 2013). When interpreting results the report indicated that before joining the initiative the schools involved had better academic results than other government schools, but made no improvement under the independent model.

The state of Victoria in the mid 1990's was the first State to move to an 'autonomous' government school system. A recent Grattan Institute report titled 'The Myth of Markets in school education', (Ben Jensen et al., Grattan Institute, Melbourne, July 2013), indicated that: "Despite this greater autonomy, Victoria's performance on national and international assessments is not significantly different from NSW where school autonomy is much lower. Scores in ACARA NAPLAN School Test Scores follow similar trends." The remaining five government school jurisdictions across Australia are also currently considering the devolution of central office control of finance and staffing decisions to school principals and school councils.

1.4.4 Implementing 'Gonski' School Funding Reforms

The previous Labor Commonwealth government before the 7 September 2013 Federal Election reached agreement on the Gonski reform proposals up to 2019 with only four State government school systems, (New South Wales, Victoria, South Australia and Tasmania) and one Territory government, (ACT). The detail of the full agreement is contained in the National Education Reform Agreement (NERA), Council of Australian Governments (COAG, 2013). The Act to authorise this agreement was passed into law by the Commonwealth House of Representatives and the Senate with effect from 27 June 2013, (Australian Education Act, No. 67, 2013).

After the Federal election on 7 September 2013 the newly elected Liberal/ National coalition government secured agreement of the remaining two states (Queensland and Victoria), and the Northern Territory to their version of the 'Gonski' reforms covering these government sector schools, but only for the next

4 years to 2017 of the Commonwealth Budget Forward Estimates. The non-government Catholic and other Independent Schools across the eight Australian jurisdictions all signed up to the Australian Education Act with the previous Commonwealth Labor Government. This arrangement took effect from the 2014 school year. At the time of writing this section of the book the new Federal government has not yet elaborated on its final funding plan for school level education in Australia, nor has it resolved the inherent conflicts between the two groupings of State government sector schools as well as the future funding for the non-government school sector for years 5 and 6 of the Commonwealth's Forward Budget Estimates.

When considering the level of school education efficiency and productivity across Australia's government and non-government schools, determining how to remove conflicting school education legislation and school sector policies, improved oversight, accountability and value for money mechanisms seems to be an appropriate starting point. Increasing the participation of front-line school education professionals, practitioners, and researchers in the development of policies and legislation seems to be an appropriate second step. This type of increase in transparency, participation, and autonomy could be exchanged for increased levels of innovation, parental involvement, and accountability.

A recent Australian Senate Inquiry into Schools lends credibility to future DEA analyses across all Australian schools—both government and non-government indicating that "improving our schools and the outcomes for our students is not simply about spending more money: the way money is spent and what it is spent on matters. Decisions must be made about how to allocate finite resources and any increased funding must be expended strategically and directed to areas of most need-while maintaining fairness", (Australian Senate, Education, Employment and Workplace Relations Reference Committee, 2013, Chapter 2, paragraph2.37, p12).

1.5 Outline of Book

This book is structured in two parts. In Chaps. 2 and 3 we provide an overview of data envelopment analysis. We first consider the standard production technology and provide measurement of efficiency relative to the boundaries of the input and output sets. In our discussion, we include both input-oriented and output-oriented models and present measures of technical, scale and allocative efficiency. The results are illustrated with numerical examples and we provide complete SAS programming for efficiency analysis. In Chap. 3, we present the extension to the technologies that are conditional on environmental factors. The resulting DEA models useful for public sector production including education are presented. The models, complete with SAS code, are fully developed with illustrative examples. We provide measures for technical, scale and allocative efficiency conditional on the exogenous environment. We also present a measure of environmental harshness that is derived from solving the conditional and unconditional models.

The three-stage model to control for multiple exogenous factors is also presented. It is well-known that the nonparametric measures require a relatively large number of observations compared to the parametric models. This is exacerbated in the public sector models with multiple environmental factors. One solution is a three-stage model that uses a second stage regression to decrease the dimensionality by creating an overall index of environmental harshness. We present this model and provide a simulation with SAS code for illustrative purposes.

The second part of the book provides a complete empirical analysis of the technical, allocative and scale efficiency of the primary and secondary schools in Australia. In Chap. 4, we analyze the input oriented models and provide estimates not only of efficiency indices but also of educational costs. Given the availability of input prices, we are able to derive the minimum cost of providing the observed outcomes conditional on the socioeconomic environment. This allows us to measure the costs arising from having a relatively harsh operating environment. We further extend this analysis by determining the cost of providing an adequate education by finding the minimum expenditure per pupil necessary to meet pre-defined standards. This is achieved while controlling for the socio-economic environment.

For completeness, we provide output oriented measures of efficiency in Chap. 5. We are able to estimate the additional improvement in outcomes that could be possible without increasing school resources subject to the socioeconomic environment. Like the input-oriented measures, we also identify systematic improvements that could be achieved by reallocating school resources to exploit scale economies. In addition, we are able to identify the reduction in outcomes due to having a relatively harsh environment. In the last chapter, we focus on productivity in educational production by using multiple years of data. By evaluating production across time, we are able to measure productivity and its components. The decomposition of productivity allows us to measure changes in technical and scale efficiency, technical and scale efficiency change and the change in the relative harshness of the environment.

Chapter 2
Data Envelopment Analysis

2.1 Technology

Analysis of performance has economic production theory as its foundation. Firms employ inputs to produce output with an incentive to either maximize profits or minimize costs. Technically inefficient firms could either increase outputs (and therefore revenue) holding inputs constant or could decrease inputs (and hence costs) holding outputs constant. As a result, technically inefficient firms are neither profit maximizing nor cost minimizing. The seminal paper on technical efficiency measurement was Farrell (1957) which provided a decomposition of inefficiency into technical and allocative parts. From an input-oriented perspective, firms that are not operating on the isoquant associated with observed production are technically inefficient. Farrell provided a comprehensive measure of technical efficiency as the equiproportional reduction of all inputs holding output at current levels. Proportional reduction in observed inputs holds the input mix constant. Cost minimization, however, requires not only production on the isoquant but also the appropriate mix of inputs that depends on the associated input prices. Hence, if technically efficient firms are not using the allocatively efficient input mix they could still lower costs by adjusting input levels accordingly.

Farrell provided the formulation to handle a single output in the case of constant returns to scale. The paper also discussed decreasing returns to scale and the extension to multiple outputs. Farrell and Fieldhouse (1962) extended the approach as a linear program allowing increasing returns to scale. Afriat (1972) provided the formulation for technical efficiency measurement that was consistent with data envelopment analysis (DEA). The theoretical foundations of efficiency measurement are provided in Färe, Grosskopf, and Lovell (1994).

DEA is the term coined in the Operations Research literature by Charnes, Cooper, and Rhodes (1978) (CCR) to measure the technical efficiency of a given

V. Blackburn et al., *Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis*, DOI 10.1007/978-1-4899-7469-3_2,

observed decision making unit (DMU) assuming constant returns to scale. Their linear programming formulation allowed multiple inputs and multiple outputs. Banker, Charnes, and Cooper (1984) (BCC) extended the CCR model to allow variable returns to scale and showed that solutions to both CCR and BCC allowed a decomposition of CCR efficiency into technical and scale components.

In this section, we introduce the representation of the technology that serves as the basis for efficiency measurement. We assume that each decision making unit (DMU) uses a vector of m discretionary inputs $X = (x_1, \ldots, x_M)$ to produce a vector of s outputs $Y = (y_1, \ldots, y_S)$. Inputs and outputs for DMU_j ($j = 1, \ldots, n$) are given by $X_j = (x_{1j}, \ldots, x_{Mj})$ and $Y_j = (y_{1j}, \ldots, y_{Sj})$. Assuming variable returns to scale, the empirical production possibility set is given by:

$$\begin{aligned} \tau_V = \Big\{(Y,X) : &\sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ &\sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ &\sum_{i=1}^{n} \lambda_i = 1; \\ &\lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}. \end{aligned} \tag{2.1}$$

It is assumed that each of the n observed production points is feasible (i.e., standard measurement error and statistical noise does not exist). Further, we assume that any convex combination of the observed points is also feasible and outputs and inputs are freely disposable.

The technology is illustrated in Fig. 2.1, where we assume that four DMUs $A - D$ employ one input x_1 to produce one output y_1. Based on Eq. (2.1), a piecewise linear approximation of the technology emerges. Line segment AB, obtained by convex combinations of A and B, corresponds to increasing returns to scale; likewise, BC (CD) allows constant (decreasing) returns to scale. The variable returns to scale technology results by allowing only convex combinations. The shaded area in Fig. 2.1 reveals the feasible region as defined by Eq. (2.1).

An alternative way to represent the technology is with input requirement sets and isoquants.[1] Following Lovell (1993) we define the input set as

$$L_V(Y) = \{X : (Y,X) \in \tau_V\}. \tag{2.2}$$

[1] As is well known in the DEA literature, an inefficient DMU can be on the isoquant. Given that the Farrell measure projects a production possibility to the isoquant, it is possible that the resulting benchmark is not technically efficient. In this case, alternative projections could be used. See Färe and Lovell (1978), Ruggiero and Bretschneider (1998) or Ruggiero (2000) for examples.

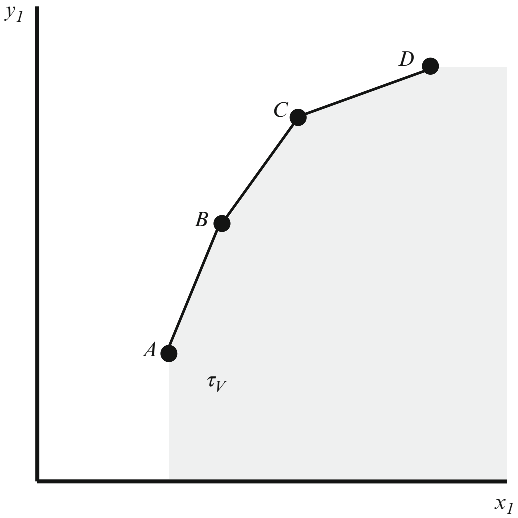


Fig. 2.1 Empirical Production Possibility Set τ_V

For each output vector Y we can define the isoquant for input set $L_V(Y)$ as

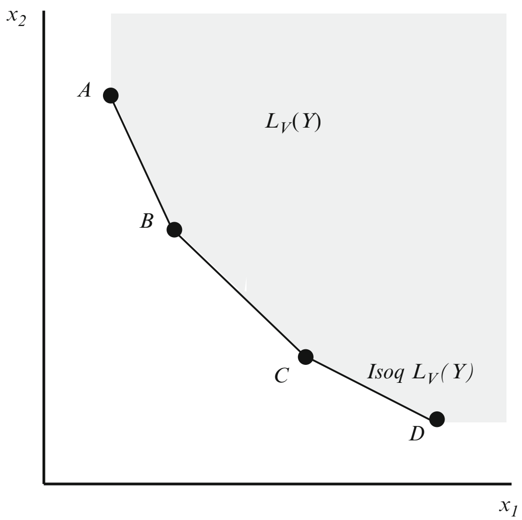
$$Isoq\ \ L_V(Y) = \{X : X \in L_V(Y),\ \lambda X \notin L_V(Y),\ \lambda \in [0, 1)\}. \tag{2.3}$$

The isoquant of the input set shows the boundary such that any equiproportional reduction in all inputs cannot produce the given output vector. As such, the isoquant represents the boundary between feasible and infeasible production. The input set and the associated isoquant are useful for illustrating the technology in input space. Following Färe et al. (1994), we specify the input set $L_V(Y)$ using a piecewise linear representation:

$$\begin{aligned} L_V(Y) = \{X : & \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \ \ s = 1, \ldots, S; \\ & \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \ \ m = 1, \ldots, M; \\ & \sum_{i=1}^{n} \lambda_i = 1; \\ & \lambda_i \geq 0, \ \ i = 1, \ldots, N\}. \end{aligned} \tag{2.4}$$

We illustrate the input set and the associated isoquant in Fig. 2.2, where we assume that four DMUs $A - D$ produce the same level of output Y using two inputs x_1 and x_2. Using Eq. (2.4), we obtain a piecewise linear $Isoq\ L_V(Y)$ defined with line segments AB, BC and CD. The resulting input set $L_V(Y)$ is shown with the shaded area.

Fig. 2.2 Input Requirement Set $L_V(Y)$ and Isoquant *Isoq* $L_V(Y)$



We can also define the technology using the output set

$$P_V(X) = \{Y : (Y, X) \in \tau_V\} \tag{2.5}$$

and its associated isoquant

$$Isoq\ \ P_V(X) = \left\{Y : Y \in P_V(X), \lambda^{-1}Y \notin P_V(X), \lambda \in \left[0, 1\right)\right\}. \tag{2.6}$$

The isoquant of the output set shows the boundary of the output set such that equiproportional expansion of outputs is not feasible without additional resources. The output set and its associated isoquant are useful for illustrating the technology in output space. Following Färe et al. (1994), we specify the input set $P_V(X)$ using a piecewise linear representation:

$$\begin{aligned} P_V(X) = \Big\{Y : &\sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ &\sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ &\sum_{i=1}^{n} \lambda_i = 1; \\ &\lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}. \end{aligned} \tag{2.7}$$

The output set and the associated isoquant are illustrated in Fig. 2.3. We assume that four *DMU*s $A - D$ use the same vector of inputs X to produce two outputs y_1 and y_2. Using Eq. (2.7), we obtain a piecewise linear *Isoq* $P_V(X)$ defined with line segments *AB*, *BC* and *CD*. The resulting output set $P_V(X)$ is shown with the shaded area.

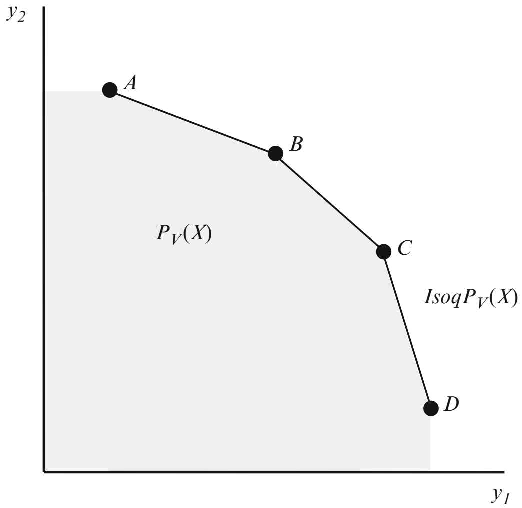


Fig. 2.3 Output Set $P_V(X)$ and Isoquant $Isoq\ P_V(X)$

2.2 Technical Efficiency Measurement

2.2.1 Input-Orientation

Using the technologies defined in Eqs. (2.1) and (2.4) we now consider input-oriented efficiency measures. In this subsection, we consider the Farrell (1957) measure of technical efficiency, defined for DMU_j with the following distance function:

$$D_V^I\left(Y_j, X_j\right) = \min\left\{\theta : \left(Y_j, \theta X_j\right) \in \tau_V\right\}. \tag{2.8}$$

The Farrell measure of technical efficiency projects observed production possibilities to the frontier subject to the constraint that the resulting benchmark projection is feasible, i.e. belongs to τ_V. Hence, efficiency is defined as the maximum equiproportional reduction in observed inputs relative to the variable returns to scale production technology. The technical efficiency measure $D_V^I(Y_j, X_j)$ for each DMU_j ($j = 1, \ldots, n$) is obtained via the solution to the following linear program introduced by Banker et al. (1984) for the multiple input multiple output production correspondence:

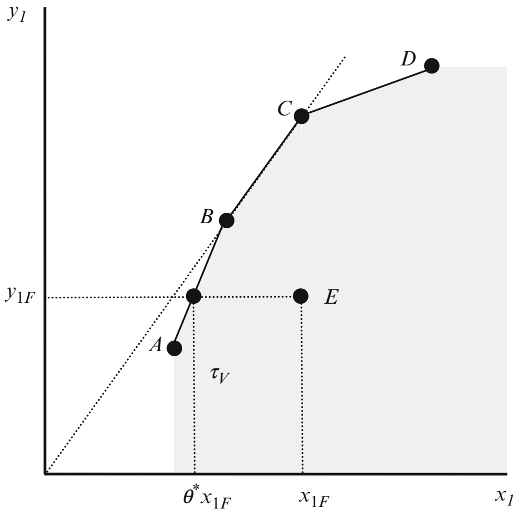


Fig. 2.4 Input-Oriented Technical Efficiency

$$
\begin{aligned}
& D_V^I\left(Y_j, X_j\right) = \min \theta \\
& \text{subject to} \\
& \qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
& \qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, \ldots, M; \\
& \qquad \sum_{i=1}^{n} \lambda_i = 1; \\
& \qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{2.9}
$$

The technical efficiency measure is illustrated in Fig. 2.4, where we extend Fig. 2.1 with an additional *DMU F* that is technically inefficient. *DMU F* is observed producing y_{1E} using an input level of x_{1E}. We note that a convex combination of *DMU*s *A* and *B* produces the same output level y_{1E} with less input $\theta^* x_{1E}$. In the solution of linear program 2.9 we find $D_V^I(Y_F, X_F) = \theta^*$. The results indicate that *DMU F* could produce its observed output level using θ^* of its observed input level. For *DMU*s $A-D$, we observe $D_V^I(Y_j, X_j) = 1$; *DMU*s $A-D$ are technically efficient.

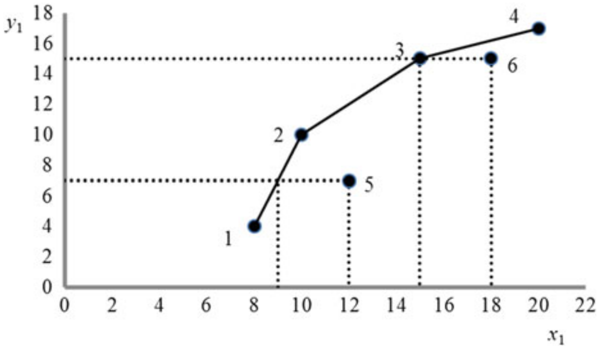


Fig. 2.5 Input-Oriented Technical Efficiency using Example 1 Data

Example 1 Consider the following data where we assume that six *DMU*s employ one input x_1 to produce one output y_1. Data are presented in the following chart.

DMU	x_1	y_1
1	8	4
2	10	10
3	15	15
4	20	17
5	12	7
6	18	15

The data for this example are illustrated in Fig. 2.5. As shown, only *DMU*s 1 – 4 are technically efficient. *DMU* 5 is observed producing 7 units of output using too much x_1. Based on the diagram, we see that *DMU* 5 could have produced the observed output using 9 units of input instead of the observed 12. The benchmark is an equally weighted convex combination of *DMU*s 1 and 2. As a result, the technical efficiency $D_V^I(Y_5, X_5) = \frac{9}{12} = 0.75$; *DMU* 5 should be able to produce the observed 7 units of output using 75 % of the observed 12 units of input. *DMU* 6 is observed using 18 units of the input to produce 15 units of output. Using the input-oriented projection, we observe that technically efficiency *DMU* 3 produces the same output using only 15 units of the input. As shown, *DMU* 3 is the benchmark for *DMU* 6 and $D_V^I(Y_6, X_6) = \frac{15}{18} = 0.8333$. *DMU* 6 is only 83.33 % efficient relative to *DMU* 3.

The SAS code used to measure technical efficiency for example 1 is as follows[2]:

[2] The SAS/OR 12.1 User's Guide Mathematical Programming Examples provides sample DEA code assuming constant returns to scale. The code presented in this book is similar to that code and was provided by SAS.

```
option nonotes;

data example1; input dmu x1 y1;
datalines;
1  8  4
2  10 10
3  15 15
4  20 17
5  12 7
6  18 15
;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   min Objective = theta;

   num k;
   num DVI {DMU};
   num benchmark_weight {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

   con input_con {r in x_Num}:
      sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];
```

(continued)

(continued)

```
    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVI [k] = theta.sol;
       for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
            Weight[j].sol else .);
       end;

    create data tech_eff from [dmu] DVI;
   create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

data benchmark; set benchmark;
  if benchmark_weight = . then delete;

proc print data = tech_eff;
proc print data = benchmark;
run;
```

The above code produces the following output:

```
                The SAS System
                 Obs     dmu        DVI
                  1       1       1.00000
                  2       2       1.00000
                  3       3       1.00000
                  4       4       1.00000
                  5       5       0.75000
                  6       6       0.83333

                 The SAS System
                                          benchmark_
            Obs     dmu     dmu_ref         weight
             1       1         1             1.0
             2       2         2             1.0
             3       3         3             1.0
             4       4         4             1.0
             5       5         1             0.5
             6       5         2             0.5
             7       6         3             1.0
```

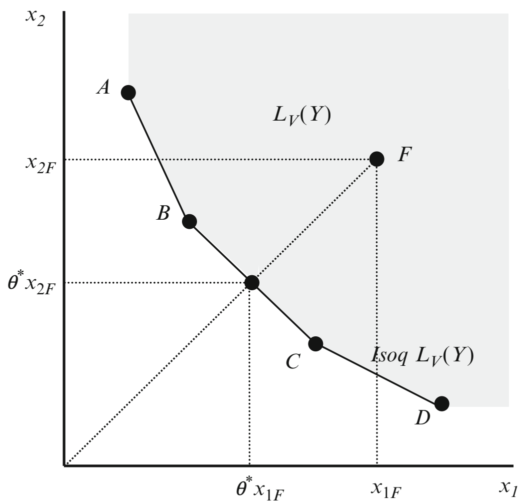


Fig. 2.6 Input-Oriented Efficiency Measurement and *Isoq* $L_V(Y)$

The results are consistent with the discussion above. *DMU*s $1-4$ are technically efficient while *DMU* 5 (6) is only 75 (83.33) percent efficient. In addition, each technically efficient *DMU* has itself as a benchmark. *DMU* 5's benchmark is an equally weighted convex combination of *DMU* 1 and 2 (here labeled as dmu_ref for *DMU* reference). *DMU* 6 is benchmarked against *DMU* 3.

We illustrate efficiency measure using $L_V(Y)$ and its associated isoquant *Isoq* $L_V(Y)$ by extending Fig. 2.2 to add an inefficient *DMU*. In Fig. 2.6, *DMU F* is assumed to produce the same level of output as *DMU*s $A - D$. Here, *DMU F* belongs to the input set but is not on the isoquant. *DMU F* is observed producing the output using inputs x_{1E} and x_{2E}. A convex combination of *DMU*s *B* and *C* produces the same output level with less of both inputs, $\theta^* x_{1E}$ and $\theta^* x_{2E}$. Importantly, the Farrell measure seeks the maximum equiproportional reduction in both inputs. In the solution of the linear program (Eq. (2.9)), $D_V^I(Y_E, X_E) = \theta^*$. The results indicate that *DMU F* could produce its observed output level using θ^* times as much of both inputs. For technically efficient *DMU*s $A-D$, $D_V^I(Y_j, X_j) = 1$.

Example 2 For example 2, we consider a two-input one-output production process where each of five *DMU*s $(1-5)$ produces one unit of output. Data are presented in the following chart. The data are shown in Fig. 2.7.

DMU	x_1	x_2	y_1
1	5	35	1
2	10	20	1
3	20	10	1
4	35	5	1
5	25	25	1

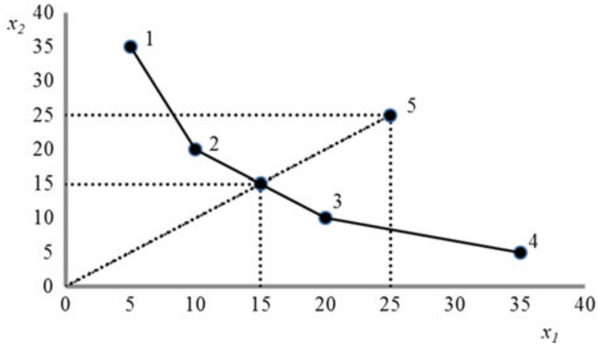


Fig. 2.7 Example 2 Data Input-Oriented Technical Efficiency

As illustrated, the piecewise linear unit isoquant is shown with *DMU*s $1-4$ efficiently producing the observed output. *DMU* 5 is observed producing inefficiently producing the output with 25 units of both inputs. An equally weighted convex combination of technically efficient *DMU*s 2 and 3 uses 15 units of each input. As a result, $D_V^I(Y_5, X_5) = \frac{15}{25} = 0.60$. Hence, DMU 5 is only 60 % efficient.

The SAS code to evaluate the efficiency of all *DMU*s is given below.

```
option nonotes;

data example2; input dmu x1 x2 y1;
datalines;
1  5  35 1
2  10 20 1
3  20 10 1
4  35 5  1
5  25 25 1
;

proc optmodel printlevel = 0;
  set x_num = 1..2;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example2
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

   var Weight {DMU} >= 0;
   var theta >= 0;
```

(continued)

(continued)

```
    min Objective = theta;

    num k;
    num DVI {DMU};
    num benchmark_weight {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVI [k] = theta.sol;
       for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
           Weight[j].sol else .);

    end;
    create data tech_eff from [dmu] DVI;
    create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

data benchmark; set benchmark;
  if benchmark_weight = . then delete;

proc print data = tech_eff;
proc print data = benchmark;

run;
```

The code after the data are inputted differs from the code for example 1 with the recognition that there are two inputs instead of one (set x_num = 1..2;). This code produces the following SAS output:

The SAS System

Obs	dmu	DVI
1	1	1.0
2	2	1.0
3	3	1.0
4	4	1.0
5	5	0.6

The SAS System

Obs	dmu	dmu_ref	benchmark_ weight
1	1	1	1.0
2	2	2	1.0
3	3	3	1.0
4	4	4	1.0
5	5	2	0.5
6	5	3	0.5

2.2.2 Output-Orientation

We now consider the output-oriented Farrell (1957) measure of technical efficiency using the technologies defined in Eqs. (2.1) and (2.7). We define output-oriented technical efficiency for DMU_j using Shephard's distance function:

$$D_V^O\left(Y_j, X_j\right) = \left(\max\left\{\theta : \left(\theta Y_j, X_j\right) \in \tau_V\right\}\right)^{-1}. \tag{2.10}$$

The Farrell measure of technical efficiency identifies the maximum equiproportional expansion of outputs possible subject to feasibility defined by the technology τ_V. The technical efficiency measure $D_V^O(Y_j, X_j)$ for each DMU_j $(j = 1, \ldots, n)$ is obtained via the solution to the following linear program:

$$\begin{aligned} & D_V^O\left(Y_j, X_j\right)^{-1} = \max\theta \\ & \text{subject to} \\ & \qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq \theta y_{sj}, \quad s = 1, \ldots, S; \\ & \qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_{mj}, \quad m = 1, \ldots, M; \\ & \qquad \sum_{i=1}^{n} \lambda_i = 1; \\ & \qquad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{2.11}$$

Fig. 2.8 Output-Oriented Technical Efficiency

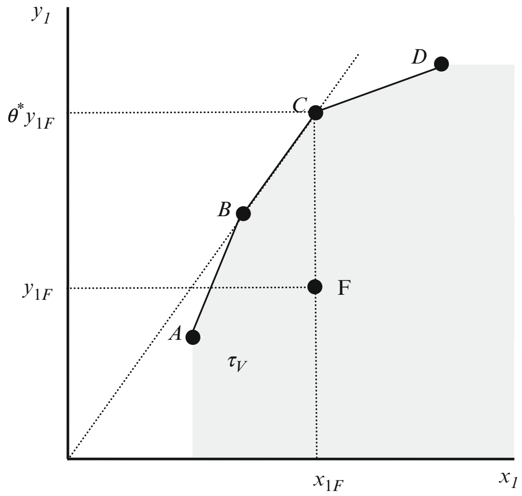
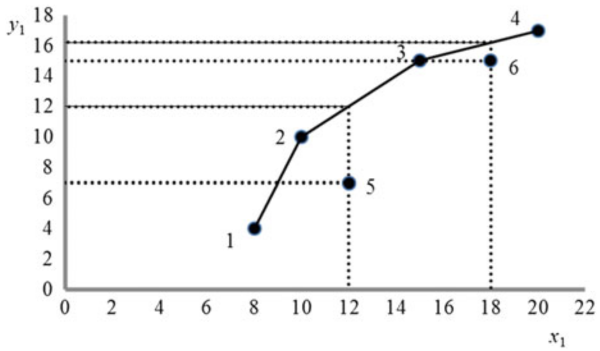


Fig. 2.9 Output-Oriented Technical Efficiency using Example 1 Data



To illustrate the output-oriented measure of technical efficiency, we extend Fig. 2.4. Technically inefficient *DMU F* is observed producing y_{1E} using an input level of x_{1E}. Holding the input level fixed, we seek the maximum expansion in output possible and obtain *DMU C* as the benchmark. DMU C uses the same input level but produces more output $\theta^* y_{1E}$. In the solution of the linear program (Eq. (2.11)) we find $D_V^O(Y_E, X_E) = (\theta^*)^{-1}$. The results indicate that *DMU F* is only producing $(\theta^*)^{-1}$ times as much output as it could be given its observed input usage. Like the input-oriented measures, *DMU*s $A - D$ are technically efficient with $D_V^O(Y_j, X_j)$ equal to unity (Fig. 2.8).

In Fig. 2.5, we illustrated the example 1 data and showed the input-oriented projections used to define the input –oriented technical efficiency measure. Figure 2.9 reproduces the data but shows the output-oriented projections. Two *DMU*s (5 and 6) are technically inefficient using the output-oriented measure of technical efficiency. *DMU* 5 is observed producing only 7 units of output using 12 units of input. The convex combination defined by $\lambda_2 = 0.6$ and $\lambda_3 = 0.4$ uses the same input level but produces 12 units of output. Thus, the convex combination produces 1.71 times as much output. As a result, $D_V^O(Y_5, X_5) = 0.5833$. Hence, *DMU* 5 is only

58.33 % efficient. For *DMU* 6, observed producing 15 units of output while employing 18 units of input, the relevant benchmark is defined by the convex combination $\lambda_3 = 0.4$ and $\lambda_4 = 0.6$. Consequently, we find that $D_V^O(Y_6, X_6) = 0.9259$, implying that *DMU* 6 could increase its output by 8 % given its input usage.

The SAS code used to measure output-oriented efficiency using example 1 data is provided below.

```
option nonotes;

data example1; input dmu x1 y1;
datalines;
1   8   4
2  10  10
3  15  15
4  20  17
5  12   7
6  18  15
;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   max Objective = theta;

   num k;
   num DVO {DMU};
   num benchmark_weight {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

   con input_con {r in x_Num}:
      sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];
```

(continued)

(continued)

```
    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVO [k] = 1/theta.sol;
       for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
Weight[j].sol else .);

    end;
    create data tech_eff from [dmu] DVO;
    create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

data benchmark; set benchmark;
   if benchmark_weight = . then delete;

proc print data = tech_eff;
proc print data = benchmark;

run;
```

The resulting SAS output:

```
                The SAS System
             Obs      dmu        DVO
              1        1       1.00000
              2        2       1.00000
              3        3       1.00000
              4        4       1.00000
              5        5       0.58333
              6        6       0.92593
                  The SAS System
                                          benchmark_
             Obs      dmu      dmu_ref      weight
              1        1         1           1.0
              2        2         2           1.0
              3        3         3           1.0
              4        4         4           1.0
              5        5         2           0.6
              6        5         3           0.4
              7        6         3           0.4
              8        6         4           0.6
```

2.3 Scale Efficiency Measurement

In order to measure scale efficiency and scale economies, we must first define the technology under the assumption of constant returns to scale. Under constant returns to scale, the empirical production possibility set is given by:

$$\tau_C = \Big\{(Y,X) : \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ \lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}. \tag{2.12}$$

Here, we obtain τ_C in Eq. (2.12) from τ_V in Eq. (2.1) by removing the convexity constraint. Likewise, we can define the input requirement set and the output set under constant returns to scale as

$$L_C(Y) = \Big\{X : \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ \lambda_i \geq 0, \quad i = 1, \ldots, N\Big\} \tag{2.13}$$

and

$$P_C(X) = \Big\{Y : \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ \lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}, \tag{2.14}$$

respectively. Importantly, the frontier defined by Eq. (2.12) allows not only convex combinations but a rescaling of inputs and outputs consistent with constant returns to scale. Necessarily, notions of economies of scale exist only along a production frontier. In order to insure that unit is operating on the variable returns to scale frontier, we can apply either model 2.6 (input-oriented) or 2.11 (output-oriented) to project units to the variable returns to scale frontier and remove any technical inefficiency.

We illustrate the empirical production possibility set based on Eq. (2.12) in Fig. 2.10, which provides the same data points shown in Fig. 2.1. Here, the

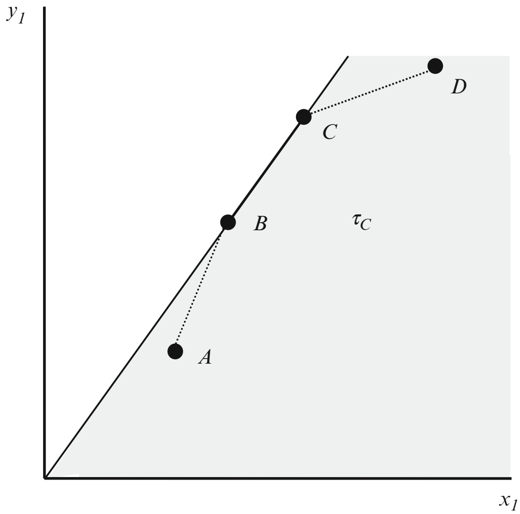


Fig. 2.10 Empirical Production Possibility Set τ_C

technology in Eq. (2.12) allows rescaling up or down of *DMU B* (or *DMU C*) along the ray from the origin. If the technology in fact was characterized by constant returns to scale at any level of input usage, then the shaded area would be the feasible region. As shown, *DMU*s *B* and *C* operate on the constant returns to scale portion of the VRS frontier. As a result, both *B* and *C* are maximizing average product. In DEA terms, *B* and *C* are scale efficient, operating at the most productive scale size (see Banker et al., 1984).

*DMU*s *A* and *D*, however, are not operating at most productive scale size. *DMU A* is observed operating under increasing returns to scale could reduce its input level the furthest compared to any point on *AB* while maintaining observed output. Likewise, *DMU D* is operating on the decreasing returns to scale portion of the VRS frontier. Both *DMU*s *A* and *D* are scale inefficient.

2.3.1 Input-Orientation

In order to measure scale efficiency, we need to project a given *DMU* not only to the VRS frontier using Eq. (2.4) but also to the CRS frontier defined in Eq. (2.12). We define the distance function projecting DMU_j to the boundary of the CRS technology with the following distance function:

$$D_C^I(Y_j, X_j) = \min\{\theta : (Y_j, \theta X_j) \in \tau_C\}. \quad (2.15)$$

This associated linear programming model due to Charnes et al. (1978) is given by:

$$
\begin{aligned}
&D_C^I\left(Y_j, X_j\right) = \min\theta_C \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta_C x_{mj}, \quad m = 1, \ldots, M; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{2.16}
$$

We use θ_C instead of θ to distinguish between the CRS and VRS projections. This model is obtained from Eq. (2.4) by removing the convexity constraint. If the true technology is characterized by variable returns to scale, Eq. (2.16) overestimates true technical inefficiency by projecting to a technically infeasible point if the relevant technically efficient benchmark is characterized by either increasing or decreasing returns to scale. If the technically efficient benchmark is operating under constant returns to scale, the solution of 2.16 is feasible as a solution to 2.4 and technical efficiency is not overestimated.

Banker et al. (1984) introduce the concept of most productive scale size consistent with technically efficient production on the constant returns to scale facet of the production frontier. Technically efficiency production that occurs on increasing (or decreasing) returns to scale facet is not most productive and hence, scale inefficient.[3] In such cases, the solution to (2.16) provides a composed measure of technical and scale inefficiency. A measure of scale efficiency for DMU_j is then obtained as the ratio of two distance functions:

$$
SE^I\left(Y_j, X_j\right) = \frac{D_C^I\left(Y_j, X_j\right)}{D_V^I\left(Y_j, X_j\right)}. \tag{2.17}
$$

A useful interpretation is that the variable returns to scale measure (the denominator) effectively removes technical inefficiency by projecting the unit to the variable returns to scale frontier.

In Fig. 2.11, we show the projection of inefficient DMU F to the technology τ_C defined by constant returns to scale. As discussed about Fig. 2.4, DMU F is technically inefficient because it could have reduced its input to $\theta^* x_{1E}$ via the solution to Eq. (2.4). The solution of Eq. (2.16) shows the minimum input level $\theta_C^* x_{1E}$ necessary to produce y_{1E} if the technology was characterized by constant returns to scale. However, under the assumption of variable returns to scale, the production possibility $(y_{1E}, \theta^* x_{1E})$ is technically efficient and operating on the

[3] Panzar and Willig (1977) provide a useful discussion of returns to scale in multiple output technologies.

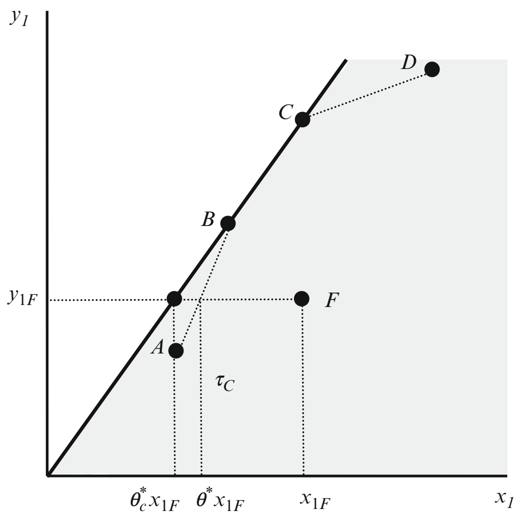


Fig. 2.11 Input-Oriented Projection $D_C^I(y_{1F}, x_{1F})$

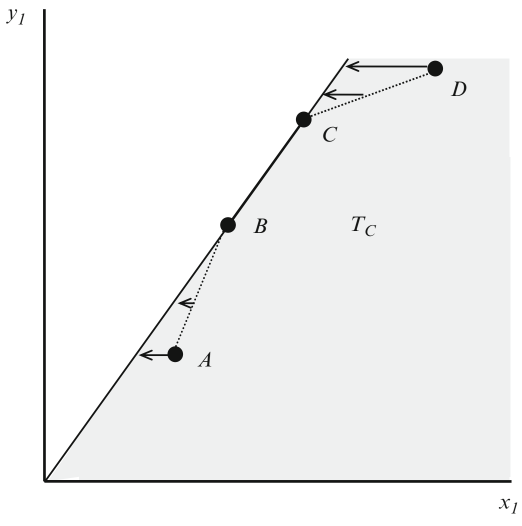


Fig. 2.12 Input Orientation and Scale Efficiency

increasing returns to scale portion of the VRS frontier. The difference in inputs $(\theta^* - \theta_C^*)x_{1E}$ is the amount of extra inputs necessary to produce y_{1E} given that returns are not constant, i.e. $(y_{1E}, \theta^* x_{1E})$ is not most productive scale size. The resulting scale efficiency of DMU F is $SE^I(Y_E, X_E) = \theta_C^*/\theta^* < 1$.

In Fig. 2.12 we show projections from the VRS to the CRS frontiers for four scale inefficient production possibilities. Along the increasing returns to scale portion of the frontier AB, we observe the distance between the VRS and CRS frontiers gets smaller as inputs increase. Along the decreasing returns portion CD the distance gets smaller as we inputs decrease. Hence, our scale efficiency measure increases as we get closer to most productive scale size. Along the constant returns to scale portion *BC* the distance function under CRS and VRS is the same, leading to a scale efficiency measure of unity.

Using data from example 1, we measure the distance functions under both constant and variable returns to scale and calculate the scale efficiency of each unit. The SAS code follows.

```
option nonotes;

data example1; input dmu x1 y1;
datalines;
1  8  4
2  10 10
3  15 15
4  20 17
5  12 7
6  18 15
;

* Projection to the VRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   min Objective = theta;

   num k;
   num DVI {DMU};
   num benchmark_weight_v {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

   con input_con {r in x_Num}:
      sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

   con weight_con: sum {j in DMU} Weight[j] = 1;
```

(continued)

(continued)

```
    do k = DMU;
       solve;
       DVI [k] = theta.sol;
       for {j in DMU} benchmark_weight_v[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_v from [dmu] DVI;
   create data benchmark_v from [dmu dmu_ref] benchmark_weight_v ;
quit;

data benchmark_v; set benchmark_v;
  if benchmark_weight_v = . then delete;

proc print data = benchmark_v;
title 'Benchmarks on the VRS Frontier';

* Projection to the CRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

    var Weight {DMU} >= 0;
    var theta >= 0;

    min Objective = theta;

    num k;
    num DCI {DMU};
    num benchmark_weight_c {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;
```

(continued)

(continued)

```
    con input_con {r in x_Num}:
        sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

    do k = DMU;
        solve;
        DCI [k] = theta.sol;
        for {j in DMU} benchmark_weight_c[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_c from [dmu] DCI;
    create data benchmark_c from [dmu dmu_ref] benchmark_weight_c ;
quit;

data benchmark_c; set benchmark_c;
  if benchmark_weight_c = . then delete;
proc print data = benchmark_c;
title 'Benchmarks on the CRS Frontier';
proc sort; by dmu;
proc means sum noprint; var benchmark_weight_c; by dmu; output out =
crs_sum_of_weights sum = sum_weights;

data crs_sum_of_weights; set crs_sum_of_weights;
 keep dmu sum_weights;


proc sort data = tech_eff_v; by dmu;
proc sort data = tech_eff_c; by dmu;
proc sort data = crs_sum_of_weights; by dmu;

data final; merge tech_eff_v tech_eff_c crs_sum_of_weights; by dmu;
  scale_eff = DCI/DVI;
  If scale_eff > 0.9999999 then rts_class = "CRS"; else if
    sum_weights < 1 then rts_class = "IRS"; else rts_class = "DRS";
proc print data = final;
title 'Final Results';

run;
```

The SAS output from the above code:

Benchmarks on the VRS Frontier

Obs	dmu	dmu_ref	benchmark_ weight_v
1	1	1	1.0
2	2	2	1.0
3	3	3	1.0
4	4	4	1.0
5	5	1	0.5
6	5	2	0.5
7	6	3	1.0

Benchmarks on the CRS Frontier

Obs	dmu	dmu_ref	benchmark_ weight_c
1	1	2	0.4
2	2	2	1.0
3	3	2	1.5
4	4	2	1.7
5	5	2	0.7
6	6	2	1.5

Final Results

Obs	dmu	DVI	DCI	sum_ weights	scale_ eff	rts_ class
1	1	1.00000	0.50000	0.4	0.50000	IRS
2	2	1.00000	1.00000	1.0	1.00000	CRS
3	3	1.00000	1.00000	1.5	1.00000	CRS
4	4	1.00000	0.85000	1.7	0.85000	DRS
5	5	0.75000	0.58333	0.7	0.77778	IRS
6	6	0.83333	0.83333	1.5	1.00000	CRS

The results provided by SAS are illustrated in Fig. 2.13. *DMU* 2 was the benchmark for all *DMU*s in the projections to the CRS frontier. Given that *DMU* 2 is technically efficient and operating on the CRS portion of the VRS frontier, its production is scaled down (for the increasing returns to scale *DMU*s) or up (for the decreasing returns to scale *DMU*s). The solution, however, is not unique.

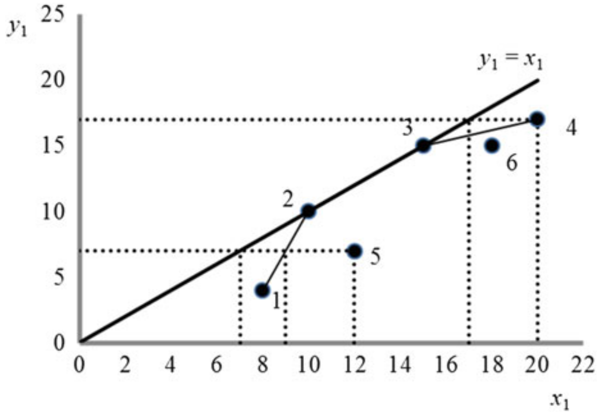


Fig. 2.13 Input-Oriented Scale Efficiency Using Example 1 Data

DMU 3, which is also most productive scale size, could have been rescaled as well. The input levels associated with CRS are determined by the line from the origin through *DMU*s 2 and 3 (the 45° line $y_1 = x_1$.)

Given the distance functions, we calculate the scale efficiency as the ratio of the distance functions. For *DMU* 1, which is technically efficiency, we find $SE^I(Y_1, X_1) = 0.5/1 = 0.5$. If constant returns to scale did exist, *DMU* 1 would be able to produce 4 units of output using only 4 units of input. This ratio of the distance functions is equivalent to the ratio of the input level (4) needed to produce the observed output (4) to the technically efficient input level (8). The results also reveal that *DMU*s 2 and 3 are both technically and scale efficient. *DMU* 6 is technically inefficient but would have been operating at most productive scale size after technically efficiency was eliminated via the input oriented model.

The results also provide information not only about scale efficiency but also about the returns to scale classification. A benchmark[4] is operating under constant returns to scale using the input oriented model if $S^I(Y_j, X_j) = 1$. If the benchmark is scale inefficient, $\sum_{j=1}^{n} \lambda_j^*$ obtained in the solution of (2.16) provides information on the scale class; if $\sum_{j=1}^{n} \lambda_j^* < 1$, the benchmark is operating on the increasing returns to scale portion of the frontier. Here, a most productive scale size production possibility is being scaled downward below the constant returns to scale frontier. If $\sum_{j=1}^{n} \lambda_j^* > 1$, the benchmark is operating under decreasing returns to scale because

[4] We refer only to the benchmark to reinforce the notion that returns to scale is not identified for technically inefficient units.

a most productive scale size production possibility is being rescaled beyond constant returns to scale. The results from example 1 show that the condition on the sum of weights only applies to the scale inefficient units. The solution of 2.16 for *DMU* 3 was obtained by rescaling *DMU* 2 up. But this solution is not unique. *DMU* 3 could have served as its own benchmark while obtaining the same objective value.

2.3.2 Output-Orientation

Alternatively, we can measure scale efficiency using the output oriented model. The measures are similar; we need to project each *DMU* to both the VRS and CRS frontiers. For completeness, we define the distance function projecting DMU_j to the boundary of the CRS technology using an output-orientation:

$$D_C^O\left(Y_j, X_j\right) = \left(\max\left\{\theta : \left(\theta Y_j, X_j\right) \in \tau_C\right\}\right)^{-1}. \tag{2.18}$$

This associated linear programming model to estimate this distance function is:

$$\begin{aligned}
&D_C^O\left(Y_j, X_j\right)^{-1} = \max \theta_C \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq \theta_C y_{sj}, \quad s = 1, \ldots, S; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_{mj}, \quad m = 1, \ldots, M; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned} \tag{2.19}$$

This model is the output-oriented equivalent model where the convexity constraint has been removed from Eq. (2.11). Similar to the procedure for the input-oriented model, we can estimate scale efficiency as the ratio of distance functions:

$$SE^O\left(Y_j, X_j\right) = \frac{D_C^O\left(Y_j, X_j\right)}{D_V^O\left(Y_j, X_j\right)}. \tag{2.20}$$

The output-oriented projection of inefficient *DMU* F to the technology τ_C defined by constant returns to scale is illustrated in Fig. 2.14. *DMU* F is technically inefficient because it could have increased its output to $\theta_C^* y_{1E}$ (with $\theta_C^* > 1$) holding input at the observed level x_{1E} in the solution to Eq. (2.19). The benchmark projection *DMU C* (from the solution to Eq. (2.11)) is operating under constant returns to scale and hence, $\lambda_C = 1$ in a solution to 2.19. Since returns to scale are

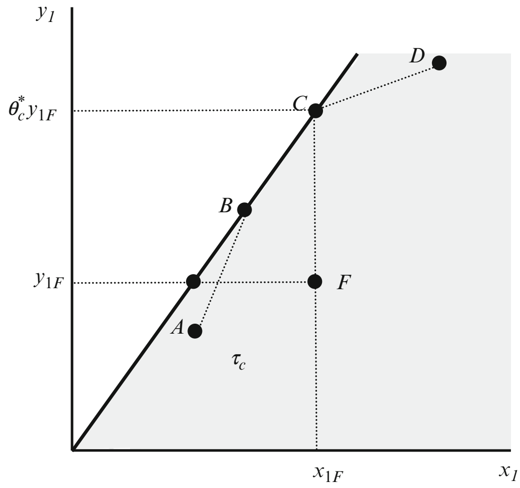


Fig. 2.14 Output-Oriented Projection $D_C^O(y_{1E}, x_{1F})$

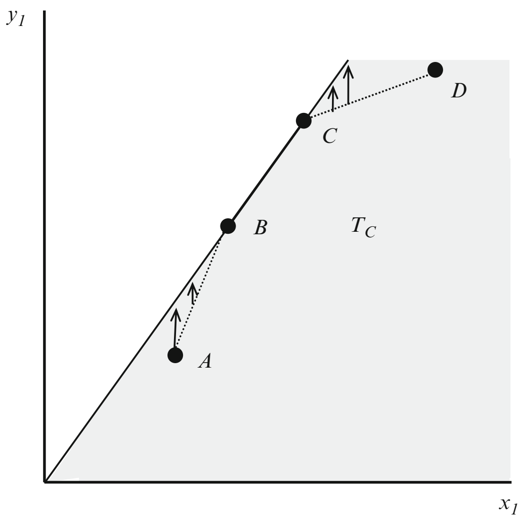


Fig. 2.15 Output Orientation and Scale Efficiency

classified on the frontier, after technical efficiency is eliminated, the resulting scale efficiency of *DMU F* is $SE^O(Y_E, X_E) = \theta_C^* / \theta_C^* = 1$. Also, for DMU EWe also infer from Fig. 2.14 that $SE^O(Y_j, X_j) < 1$ for *DMU*s *A* and *D* and that $SE^O(Y_j, X_j) = 1$ for most productive scale size *DMU*s *B* and *C*.

Like we showed in Fig. 2.12 for the input-oriented model, we show output-oriented projections from the VRS to the CRS frontiers for four scale inefficient production possibilities in Fig. 2.15. Along the increasing returns to scale portion of the frontier *AB*, we observe the distance between the VRS and CRS frontiers gets smaller as inputs increase. Along the decreasing returns portion *CD* the distance gets smaller as we inputs decrease. Hence, the scale efficiency measure using an output-oriented projection behaves the same as it does in the input-oriented case: scale efficiency increases as we get closer to most productive scale size.

We measure the scale efficiency using the output-oriented projections using the data from example 1. The SAS code to estimate the relevant distance functions, the scale efficiency and the returns to scale class follows.

```
option nonotes;
data example1; input dmu x1 y1;
datalines;
1  8  4
2  10 10
3  15 15
4  20 17
5  12 7
6  18 15
;
* Projection to the VRS frontier;
proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;
   var Weight {DMU} >= 0;
   var theta >= 0;
```

(continued)

(continued)

```
    max Objective = theta;

    num k;
    num DVO {DMU};
    num benchmark_weight_v {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVO [k] = 1/theta.sol;
       for {j in DMU} benchmark_weight_v[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_v from [dmu] DVO;
    create data benchmark_v from [dmu dmu_ref] benchmark_weight_v ;
quit;

data benchmark_v; set benchmark_v;
  if benchmark_weight_v = . then delete;

proc print data = benchmark_v;
title 'Benchmarks on the VRS Frontier';

* Projection to the CRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

    var Weight {DMU} >= 0;
    var theta >= 0;
```

(continued)

(continued)

```
    max Objective = theta;

    num k;
    num DCO {DMU};
    num benchmark_weight_c {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

    do k = DMU;
       solve;
       DCO [k] = 1/theta.sol;
       for {j in DMU} benchmark_weight_c[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_c from [dmu] DCO;
   create data benchmark_c from [dmu dmu_ref] benchmark_weight_c ;
quit;

data benchmark_c; set benchmark_c;
  if benchmark_weight_c = . then delete;
proc print data = benchmark_c;
title 'Benchmarks on the CRS Frontier';
proc sort; by dmu;
proc means sum noprint; var benchmark_weight_c; by dmu; output out =
crs_sum_of_weights sum = sum_weights;

data crs_sum_of_weights; set crs_sum_of_weights;
 keep dmu sum_weights;

proc sort data = tech_eff_v; by dmu;
proc sort data = tech_eff_c; by dmu;
proc sort data = crs_sum_of_weights; by dmu;

data final; merge tech_eff_v tech_eff_c crs_sum_of_weights; by dmu;
  scale_eff = DCO/DVO;
  If scale_eff > 0.9999999 then rts_class = "CRS"; else if
    sum_weights < 1 then rts_class = "IRS"; else rts_class = "DRS";
proc print data = final;
title 'Final Results';

run;
```

The SAS output from the above code:

```
           Benchmarks on the VRS Frontier

                                    benchmark_
        Obs     dmu     dmu_ref      weight_v

         1       1         1           1.0
         2       2         2           1.0
         3       3         3           1.0
         4       4         4           1.0
         5       5         2           0.6
         6       5         3           0.4
         7       6         3           0.4
         8       6         4           0.6

           Benchmarks on the CRS Frontier

                                    benchmark_
        Obs     dmu     dmu_ref      weight_c

         1       1         2           0.8
         2       2         2           1.0
         3       3         2           1.5
         4       4         2           2.0
         5       5         2           1.2
         6       6         2           1.8

                    Final Results

                                      sum_     scale_    rts_
       Obs    dmu      DVO       DCO     weights     eff     class

        1      1    1.00000   0.50000     0.8      0.50     IRS
        2      2    1.00000   1.00000     1.0      1.00     CRS
        3      3    1.00000   1.00000     1.5      1.00     CRS
        4      4    1.00000   0.85000     2.0      0.85     DRS
        5      5    0.58333   0.58333     1.2      1.00     CRS
        6      6    0.92593   0.83333     1.8      0.90     DRS
```

The data and projections reported in the SAS output above are illustrated in Fig. 2.16. Like the case for the input-oriented projections, *DMU* 2 serves as the benchmark for all *DMU*s in the projections to the CRS frontier. Based on the projections, we see that only DMU 1 is operating under increasing returns to scale. DMUs 2, 3 and 5 are operating under constant returns to scale. Only DMUs 2 and 3 are technically and scale efficient. DMU 5 is technically efficient; unlike the input-oriented projection, DMU 5 is projected to the CRS portion of the

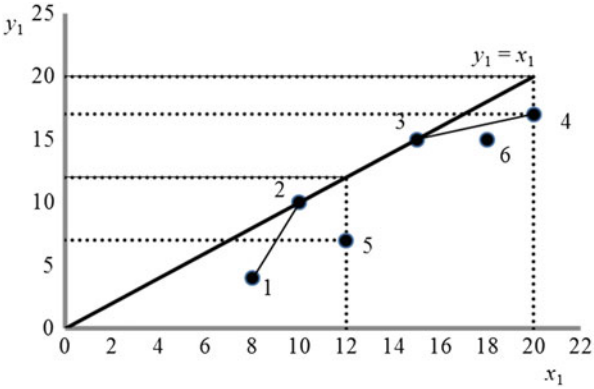


Fig. 2.16 Output-Oriented Scale Efficiency Using Example 1 Data

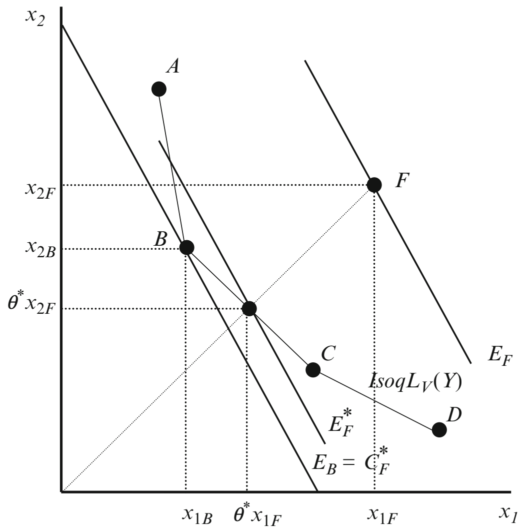
VRS frontier. As a result, DMU 5 is identified as technically inefficient but scale efficient in the output oriented model. The change in classification happens with DMU 6 as well; under the input-oriented model, it was projected to the constant returns to scale portion of the VRS frontier. In the output-oriented models, however, it is projected to the DRS portion.

Similar to the input projections, we note that the returns to scale classification is obtained from the scale efficiency measure and the sum of the weights in the solution of Eq. (2.19). A benchmark is operating under constant returns to scale using the input oriented model if $S^O(Y_j, X_j) = 1$. If the benchmark is scale inefficient, $\sum_{j=1}^{n} \lambda_j^*$ obtained in the solution of (2.19) provides information on the scale class; if $\sum_{j=1}^{n} \lambda_j^* < 1,$ the benchmark is operating on the increasing returns to scale portion of the frontier. Here, a most productive scale size production possibility is being scaled downward below the constant returns to scale frontier. If $\sum_{j=1}^{n} \lambda_j^* > 1$, the benchmark is operating under decreasing returns to scale because a most productive scale size production possibility is being rescaled beyond constant returns to scale.

2.4 Allocative Efficiency Measurement

To this point, we have presented DEA models employing distance functions to measure technical efficiency and returns to scale using only quantity measures of inputs and outputs. The Farrell measure of technical efficiency holds input and output mixes constant by seeking equiproportional reduction (expansion) of inputs (outputs). But a DMU that is operating technically efficiently might be choosing the wrong mix of inputs or outputs depending on the prevailing prices. Consequently,

Fig. 2.17 Allocative and Technical Efficiency



the allocatively inefficient *DMU* is spending above minimum costs.[5] Farrell (1957) provided a decomposition of overall efficiency into technical and allocative parts. In this section, we focus on cost efficiency when input prices are available.[6]

Maintaining the notation defined in Sect. 2.1, we now assume that each decision making unit uses a vector of m discretionary inputs $X=(x_1,\ldots,x_M)$ to produce a vector of s outputs $Y=(y_1,\ldots,y_S)$ while facing exogenous input prices $P=(p_1,\ldots,p_M)$. We represent the inputs, outputs and prices for DMU_j $(j=1,\ldots,n)$ as $X_j=(x_{1j},\ldots,x_{Mj})$, $Y_j=(y_{1j},\ldots,y_{Sj})$ and $P_j=(p_{1j},\ldots,p_{Mj})$. Given the observed input and the associated prices, observed expenditures E_j for DMU_j $(j=1,\ldots,n)$ can be calculated as $E_j=\sum_{l=1}^{M} p_{lj}x_{lj}$. Feasibility of a production possibility is still subject to the piecewise linear approximation defined in Sect. 2.1.

In Fig. 2.17, we consider five DMUs observed producing the same output levels. Here, DMUs $A-D$ are technically efficient, producing the producing output on the isoquant $Isoq\, L_V(Y)$. Projection possibility F is technically inefficient producing the same output with $D_V^I(Y_F,X_F)=\theta*$. WOLOG, assume that all of the DMUs face the same prices $P_F=(p_{1F},p_{2F})$. Given the observed prices, we show three isocost lines: E_F, E_F^* and C_F^*. The first isocost line $E_F=p_{1F}x_{1F}+p_{2F}x_{2F}$ shows the observed

[5] Much of the material in this section (and previous sections) is formally derived by Färe et al. (1994).

[6] The extension to revenue efficiency and output orientation is similar. In our empirical applications, we do not have output prices and hence, omit this discussion. See Färe et al. (1994) for a complete treatment.

expenditures of *DMU F*. There are two sources of increased spending above the minimum cost level. The first is due to technical inefficiency. The second isocost line $E_F^* = \theta^*(p_{1F}x_{1F} + p_{2F}x_{2F}) = \theta^* E_F$ is the spending that would result if *DMU F* were technically efficient, where θ^* is obtained in the solution of Eq. (2.9). Finally, the isocost line $C_F^* = p_{1F}x_{1B} + p_{2F}x_{2B} = E_B$ represents the minimum cost of producing the observed output given the observed prices. *DMU B* is the only feasible production possibility that can produce the observed output at the cost level C_F^*.

We can define the level of cost efficiency (*CE*) as the ratio of minimum cost to observed expenditures. For *DMU F*, $CE = C_F^*/E_F$ is a measure of cost efficiency. As shown, the cost inefficiency of *F* arises for two reasons; producing off of the isoquant (technical inefficiency) and using the wrong mix of inputs (allocative inefficiency) given the observed prices. Farrell (1957) provided a decomposition of the overall cost inefficiency into technical and scale components.

To obtain the measure cost efficiency, we first solve the following linear programming model for the minimum cost of producing the observed output for *DMU j* as:

$$
\begin{aligned}
C_j^* &= \min \sum_{m=1}^{M} p_{mj} x_m \\
&\text{subject to} \\
&\quad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
&\quad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\
&\quad \sum_{i=1}^{n} \lambda_i = 1; \\
&\quad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{2.21}
$$

We assume a variable returns to scale technology with the observed convexity constraint and obtain an optimal vector of inputs $(x_1^*, \ldots, x_M^*)$ for each *DMU* that minimizes the costs of production. Our measure of cost efficiency is then derived as the ratio of minimum costs to observed expenditures:

$$CE(Y_j, X_j) = \frac{C_j^*}{E_j}. \tag{2.22}$$

Finally, given our cost efficiency measure $CE(Y_j, X_j)$ and our technical efficiency measure $D_V^I(Y_j, X_j)$ we can obtain a measure of allocative efficiency as:

$$AE(Y_j, X_j) = \frac{CE(Y_j, X_j)}{D_V^I(Y_j, X_j)}. \tag{2.23}$$

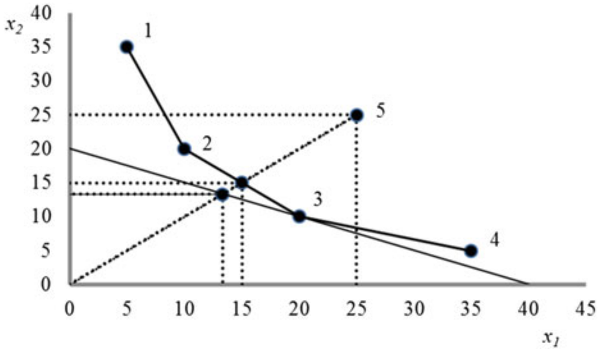


Fig. 2.18 Example 2 Data Technical and Allocative Efficiency

For DMU F in Fig. 2.17, we observe $AE(Y_F, X_F) = C_F^*/E_F^*$.

We extend the data in Example 2 to include input prices and observed expenditures:

DMU	x_1	x_2	p_1	p_2	y_1	*E*
1	5	35	5	10	1	375
2	10	20	5	10	1	250
3	20	10	5	10	1	200
4	35	5	5	10	1	225
5	25	25	5	10	1	375

These data are presented in Fig. 2.18, where we focus on the technical and allocative efficiency of *DMU* 5, which is observed producing the unit output with 25 units of both inputs. Given the observed prices of $p_1 = 5$ and $p_2 = 5$ *DMU* 5 is observed spending \$375. If *DMU* 5 eliminated its technical inefficiency, it could produce on the isoquant using 15 units of both inputs at a cost of \$225. This holds the input mix $x_2/x_1 = 1$ constant but does not minimize the cost of producing the output. As observed from the table and the SAS output, *DMU* 3 serves as a benchmark for *DMU* 5 (and all of the *DMU*s). *DMU* 3 produces the same output with expenditures of \$200 using an input mix of $x_2/x_1 = 0.5$ (the ratio of the p_1/p_2.) This example also illustrates the problem of using only technical efficiency as a benchmark. *DMU* 1 is technically efficient in production but spends the same amount on inputs as *DMU* 5. For DMU 5, we observe $D_V^I(Y_5, X_5) = \frac{15}{25}$, $CE(Y_5, X_5) = \frac{13.33}{25} = 0.5333$ and $AE(Y_5, X_5) = 0.89$.

The SAS code to estimate cost and allocative efficiency for all *DMU*s is provided below.

```
option nonotes;

data example2; input DMU x1 x2 p1 p2 y1 E;
datalines;
1   5   35   5  10  1   375
2  10   20   5  10  1   250
3  20   10   5  10  1  200
4  35    5   5  10   1  225
5  25   25  5  10   1  375
;

* estimating technical efficiency;

proc optmodel printlevel = 0;
  set x_num = 1..2;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example2
    into DMU = [dmu]
   {r in x_num} < X[dmu, r] = col("x"||r)>
   {s in y_num} < Y[dmu, s] = col("y"||s)>;

   var Weight {DMU} >= 0;
   var theta >= 0;
  min Objective = theta;
  num k;
  num DVI {DMU};
  num benchmark_weight {DMU,DMU};

  con output_con {s in y_Num}:
  sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

  con input_con {r in x_Num}:
    sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

  con weight_con: sum {j in DMU} Weight[j] = 1;
    do k = DMU;
      solve;
      DVI [k] = theta.sol;
     for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
       Weight[j].sol else .);
    end;
```

(continued)

(continued)

```
    create data tech_eff from [dmu] DVI;
   create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

* estimating minimum costs;

proc optmodel printlevel = 0;
  set x_num = 1..2;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num P {DMU, x_num};
  num Y {DMU, y_num};
  read data example2
     into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {r in x_num} < P[dmu, r] = col("p"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

 var Weight {DMU} >= 0;
 var xo{x_num} >= 0;

 num k;
 min Objective = sum{t in x_num} xo[t]*P[k,t];
 num C{DMU};
 num c_benchmark_weight {DMU,DMU};

 con output_con {s in y_Num}:
  sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

 con input_con {r in x_Num}:
  sum {j in DMU} X[j,r] * Weight[j] <= xo[r];

 con weight_con: sum {j in DMU} Weight[j] = 1;

 do k = DMU;
   solve;
   C[k] = sum{t in x_num} xo[t]*P[k,t];
   for {j in DMU} c_benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
      Weight[j].sol else .);
 end;
```

(continued)

(continued)

```
  create data cost from [dmu] C;
  create data c_benchmark from [dmu dmu_ref] c_benchmark_weight ;
quit;

data c_benchmark; set c_benchmark;
   if c_benchmark_weight = . then delete;

proc print data=c_benchmark;

proc sort data = example2; by dmu;
proc sort data = tech_eff; by dmu;
proc sort data = cost; by dmu;

data final; merge example2 tech_eff cost; by dmu;
  TE = DVI;
  CE = C/E;
  AE = CE/DVI;
proc print data = final; var DMU C TE AE CE;
run;
```

The resulting SAS output:

```
                       The SAS System

                                       c_benchmark_
          Obs     dmu     dmu_ref         weight

           1       1         3               1
           2       2         3               1
           3       3         3               1
           4       4         3               1
           5       5         3               1

                       The SAS System

      Obs     DMU      C       TE       AE          CE
       1       1      200     1.0     0.53333     0.53333
       2       2      200     1.0     0.80000     0.80000
       3       3      200     1.0     1.00000     1.00000
       4       4      200     1.0     0.88889     0.88889
       5       5      200     0.6     0.88889     0.53333
```

In this chapter, we defined technologies and presented the standard data envelopment models using piecewise linear frontiers. Linear programming models were developed to estimate technical, scale and allocative efficiency using an input-oriented framework. Output-oriented technical and scale efficiency measures were also presented. In addition, SAS code was provided for all measures and implemented using two illustrative data sets. In the next chapter, we extend these models to allow environmental control variables useful for public sector applications.

Chapter 3
DEA in the Public Sector

In the previous chapter, standard DEA models analyzing the performance of DMUs producing multiple outputs using multiple inputs were presented. These models provide a useful starting point for analyzing educational and other public sector production processes. One of the key distinguishing features of public sector publication is the presence of non-discretionary environmental factors of production that introduces heterogeneity among decision making units. It is well known, for example, that socioeconomic factors such as income, poverty, parental education etc. play a large role in the production of output. In fire services, the material of the houses (brick vs. wood) determines how successful firefighters will be in putting out fires. In health care, preexisting conditions and age of the patients could determine the success of a particular treatment.

In this chapter, we extend the DEA models from Chap. 1 to control for the environmental factors that are nondiscretionary. This requires a specification of the technology that includes not only the discretionary inputs and outputs but also the environmental factors. The models presented rely on conditional estimators. In the first section, we will redefine the technology to be conditional on the environment. The public sector DEA model developed by Ruggiero (1996a) to measure technical efficiency assuming conditional convexity will be presented. In applications with multiple nondiscretionary factors, this model suffers the curse of dimensionality by overestimating technical efficiency. Ruggiero (1998) provided a three-stage extension of this model using regression analysis in a second stage to develop a composite measure of the nondiscretionary factor.[1]

[1] Estelle, Johnson, and Ruggiero (2010) compare and contrast using OLS, tobit, fractional logit and nonparametric regression in the second stage. The results provide similar results.

V. Blackburn et al., *Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis*, DOI 10.1007/978-1-4899-7469-3_3,

We also present models to estimate returns to scale in the public sector. These measures, due to Ruggiero (2000) include not only traditional measures found in Chap. 2, but also a returns to environmental scale developed by Ruggiero (2000). Importantly, we are able to measure the additional resources necessary to achieve given outcomes because of a harsh environment. Alternatively, we can measure the shortfall of outcomes attributable to environmental harshness holding discretionary inputs constant. These models are important because it reveals that standard measures that do not include the nondiscretionary factors lead to biased efficiency measures that include the effect the environment has on the production process. Finally, we present the allocative efficiency model developed by Haelermans and Ruggiero (2013) that controls for socioeconomic variables.

3.1 Technology

We extend the representation of the technology from Chap. 2 by assuming that each DMU uses a vector of m discretionary inputs $X = (x_1, \ldots, x_M)$ to produce a vector of s outputs $Y = (y_1, \ldots, y_S)$ while facing an environment represented by a continuous variable z. Data for each DMU$_j$ $(j = 1, \ldots, n)$ are given by $X_j = (x_{1j}, \ldots, x_{Mj})$, $Y_j = (y_{1j}, \ldots, y_{Sj})$ and z_j. Assuming variable returns to scale, the empirical production possibility set conditional on z is given by:

$$\begin{aligned}\tau_V(z) = \Big\{\big(Y, X, z\big) : &\sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S;\\ &\sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M;\\ &\sum_{i=1}^{n} \lambda_i = 1;\\ &\lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N;\\ &\lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}.\end{aligned} \tag{3.1}$$

This technology allows variable returns to scale for any given level of the environmental variable z. This technology is conditional on the level of the nondiscretionary factor; if a unit has a more favorable environment represented by a higher level of z then the unit is not allowed in the spanning of the technology.[2]

We illustrate the conditional technology in Fig. 3.1, where we assume one input x_1 is employed to produce one output y_1 with heterogeneity introduced with nondiscretionary variable z. All DMUs are assumed technically efficient.

[2] In the multiple stage model presented later in the chapter, the assumption of a monotonically increasing relationship between output and the environmental factors is dropped.

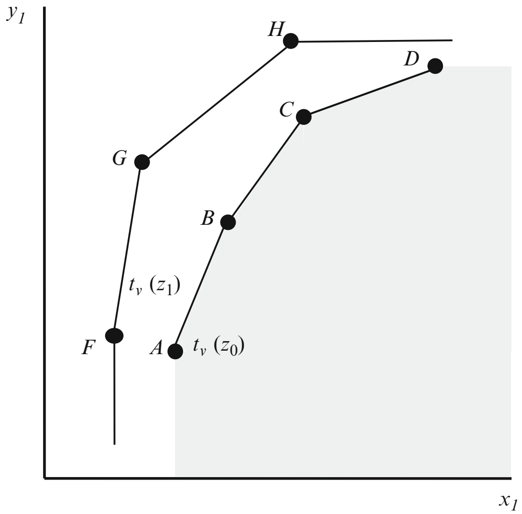


Fig. 3.1 Empirical Production Possibility Set $\tau_V(z)$

WOLOG, we further assume that we only observe two levels of the nondiscretionary variable z. DMUs $A-D$ face the harshest environment z_0 while DMUs $F-H$ have a more favorable environment z_1. As shown, DMUs $F-H$ are able to produce more output for a given level of discretionary input because of its more favorable environment. Alternatively, these DMUs can produce a given level of output with less of the discretionary inputs. The empirical production possibility set conditional on z_0 is defined by production possibilities $A-D$; DMUs with a more favorable environment are observed to be infeasible. Consequently, the shaded area shows the feasible region for those units with the harsher environment represented by z_0.

We can also define the technology with input and output sets conditional on the nondiscretionary factor and define the associated isoquants. The conditional input set and its isoquant are given as

$$L_V(Y,z) = \{X : (Y,X,z) \in \tau_V(\mathrm{z})\} \tag{3.2}$$

and

$$Isoq\ L_V(Y,z) = \{X : X \in L_V(Y,z), \lambda X \notin L_V(Y,z), \lambda \in [0,1)\}. \tag{3.3}$$

The isoquant of the conditional input set shows the boundary such that any equiproportional reduction in all discretionary inputs cannot produce the given output vector with feasibility defined subject to the nondiscretionary factor. The conditional input set and its isoquant is specified using a piecewise linear representation:

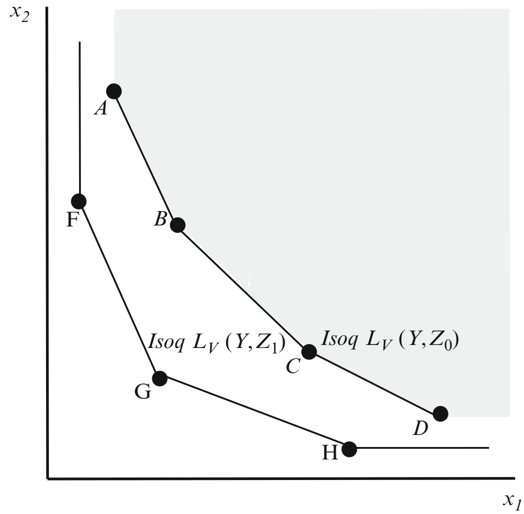


Fig. 3.2 Conditional Input Requirement Sets

$$
\begin{aligned}
L_V(Y,z) = \Big\{X : & \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\
& \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\
& \sum_{i=1}^{n} \lambda_i = 1; \\
& \lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N; \\
& \lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}.
\end{aligned}
\tag{3.4}
$$

We illustrate the two conditional input requirement sets and the associated isoquants in Fig. 3.2, where we assume that four DMUs $A-D$ produce the same level of output Y using two inputs x_1 and x_2 while facing the harsher environment with nondiscretionary input level z_0. DMUs F, G and H face a better environment with $z_1 > z_0$ and are able to produce the same level of output with less of both inputs. The conditional input set for nondiscretionary input level z_0 is shown with the shaded area; the associated isoquant $Isoq\ L_V(Y, z_0)$ is defined with line segments AB, BC and CD. These isoquants define the appropriate projections to evaluate technical efficiency.

We can also define the technology using the conditional output set

$$P_V(X,z) = \{Y : (Y,X,z) \in \tau_V(\text{z})\} \tag{3.5}$$

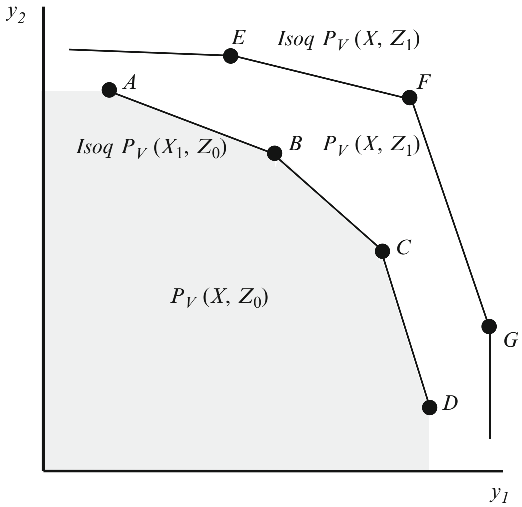


Fig. 3.3 Conditional Output sets

and its associated isoquant

$$Isoq\ \ P_V(X,z) = \left\{Y : Y \in P_V(X,z), \lambda X \notin P_V(X,z), \lambda \in [0,1)\right\}. \tag{3.6}$$

The isoquant of the conditional output set shows the boundary of the output set such that equiproportional expansion of outputs is not feasible without additional resources holding fixed the nondiscretionary factor z. A piecewise linear representation for a given nondiscretionary input level z is given by:

$$\begin{aligned} P_V(X,z) = \Big\{Y : &\sum_{i=1}^{n}\lambda_i y_{si} \geq y_s, \ \ s = 1, \ldots, S; \\ &\sum_{i=1}^{n}\lambda_i x_{mi} \leq x_m, \ \ m = 1, \ldots, M; \\ &\sum_{i=1}^{n}\lambda_i = 1; \\ &\lambda_i = 0 \text{ if } z_i > z, \ \ i = 1, \ldots, N; \\ &\lambda_i \geq 0, \ \ i = 1, \ldots, N\Big\}. \end{aligned} \tag{3.7}$$

We illustrate the two conditional output sets with their associated isoquants in Fig. 3.3. All DMUs produce two outputs using the same amount of inputs. Similar to the previous figures in this chapter, we assume that DMUs $A - D$ face the harsher environment with nondiscretionary input level z_0. DMUs F, G and H face a better environment with $z_1 > z_0$ and are therefore able to produce more of both outputs with the same input levels. The conditional output set for nondiscretionary input level z_0 is shown with the shaded area; the associated isoquant *Isoq* $P_V(X, z_0)$ is defined with line segments AB, BC and CD.

3.2 Technical Efficiency Measurement

3.2.1 Input-Orientation

Using the conditional technologies defined in Eqs. (3.1) and (3.4) we can define technical efficiency for DMU_j with the distance function:

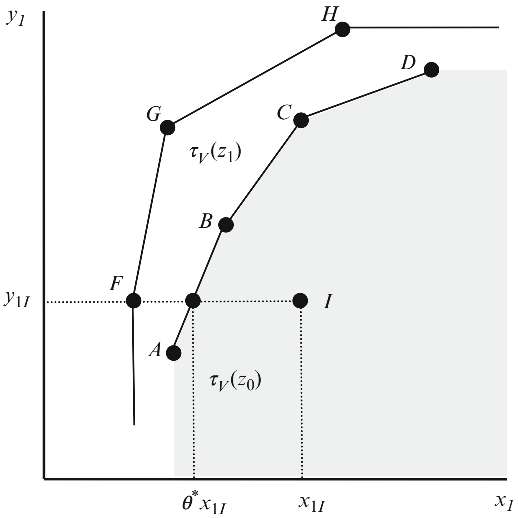
$$D_V^I\left(Y_j, X_j, z_j\right) = \min\left\{\theta : \left(Y_j, \theta X_j\right) \in \tau_V\left(z_j\right)\right\}. \tag{3.8}$$

The Farrell measure of technical efficiency projects observed production possibilities to the conditional frontier. This measure is consistent with the measure provided in Chap. 2 with additional restrictions to account for environmental heterogeneity. Ruggiero (1996a) showed that technical efficiency $D_V^I(Y_j, X_j, z_j)$ for each DMU_j ($j = 1, \ldots, n$) can be estimated with the following linear programming model:

$$\begin{aligned}
&D_V^I\left(Y_j, X_j, z_j\right) = \min \theta \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, \ldots, M; \\
&\qquad \sum_{i=1}^{n} \lambda_i = 1; \\
&\qquad \lambda_i = 0 \text{ if } z_i > z_j, \quad i = 1, \ldots, N; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned} \tag{3.9}$$

The technical efficiency measure is illustrated in Fig. 3.4, where we extend Fig. 3.1 by adding technically inefficient DMU I. DMU I is observed producing y_{1I} using an input level of x_{1I} while facing the harsher environment z_0.

Fig. 3.4 Input-Oriented Technical Efficiency



If we ignore the nondiscretionary input and use model (2.9), we obtain a measure of "technical efficiency" relative to benchmark DMU F. But this is not feasible given that DMU I has nondiscretionary factor z_0. Using model (3.9), we obtain a referent benchmark defined as a convex combination of DMUs A and B, both of whom face the same environment, and $D_V^I(Y_I, X_I, z_0) = \theta^*$. Solving model (3.9) for all DMUs properly identifies $A - D$ and $F - H$ as technically efficient with $D_V^I(Y_j, X_j, z_j) = 1$.

Example 1 Assume that ten DMUs employ one input x_1 to produce one output y_1 while facing an index of environmental harshness z. There are two levels for the nondiscretionary variable, $z = 1$ and $z = 2$, where a higher value is associated with a more favorable environment. Data are presented in the following chart.

DMU	x_1	y_1	z
1	8	4	1
2	10	10	1
3	15	15	1
4	20	17	1
5	12	7	1
6	5	4	2
7	7	10	2
8	12	15	2
9	17	17	2
10	12	7	2

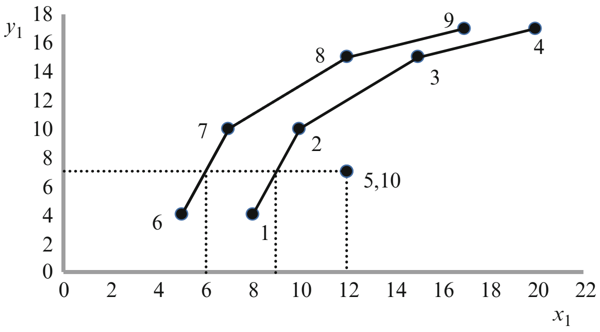


Fig. 3.5 Input-Oriented Technical Efficiency using Example 1 Data

The data are illustrated in Fig. 3.5 where it is assumed that DMUs 1 – 4 and 6 – 9 are technically efficient. DMUs 1 – 5 all face the same relatively harsh environment while DMUs 6 – 10 face the better one. DMU 5 is observed producing 7 units of output using too much x_1. DMUs 6 – 9 replicate the output levels of DMUs 1 – 4 but use an input level of three less. Finally, DMU 10 uses the same amount of discretionary input and produces the same level of output as DMU 5 but faces the more favorable environment.

To evaluate the inefficiency of DMU 5 we note that the relevant benchmark is an equally weighted convex combination of DMUs 1 and 2, which produces the observed 7 units of output using only 9 units of the discretionary input. Hence, $D_V^I(Y_5, X_5, 1) = \frac{9}{12} = 0.75$. DMU 10 on the other hand, has a more favorable environment and hence, should be able to produce the observed 7 units of output with only 6 units of the input. This results because the relevant benchmark for DMU 10 is an equally weighted convex combination of DMUs 6 and 7. Thus, $D_V^I(Y_{10}, X_{10}, 2) = \frac{6}{12} = 0.5$; DMU 10 should be able to produce the observed output using half as many inputs as it is observed using.

The SAS code used to measure technical efficiency for this example is as follows[3]:

[3] For programming purposes, we refer to the nondiscretionary input as z_1. In Sect. 3.5 we consider the multiple stage model when there are multiple nondiscretionary variables.

```
option nonotes;

data example1; input dmu x1 y1 z1;
datalines;
1  8  4  1
2  10 10 1
3  15 15 1
4  20 17 1
5  12 7  1
6  5  4  2
7  7  10 2
8  12 15 2
9  17 17 2
10 12 7  2
;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set z_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  num Z {DMU, z_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {t in z_num} < Z[dmu, t] = col("z"||t)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   min Objective = theta;

   num k;
   num DVI {DMU};
   num benchmark_weight {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

   con input_con {r in x_Num}:
      sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];
```

(continued)

(continued)

```
    con exoginput_con {t in z_Num, j in DMU}:
        if z[j,t] > z[k,t] then Weight[j] = 0;

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
        solve;
        DVI [k] = theta.sol;
        for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
               Weight[j].sol else .);

    end;
    create data tech_eff from [dmu] DVI;
    create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

data benchmark; set benchmark;
   if benchmark_weight = . then delete;

proc print data = tech_eff;
proc print data = benchmark;

run;
```

The above code produces the following output:

```
                 The SAS System

            Obs     dmu      DVI

              1       1     1.00
              2       2     1.00
              3       3     1.00
              4       4     1.00
              5       5     0.75
              6       6     1.00
              7       7     1.00
              8       8     1.00
              9       9     1.00
             10      10     0.50
```

(continued)

(continued)

The SAS System

Obs	dmu	dmu_ref	benchmark_ weight
1	1	1	1.0
2	2	2	1.0
3	3	3	1.0
4	4	4	1.0
5	5	1	0.5
6	5	2	0.5
7	6	6	1.0
8	7	7	1.0
9	8	8	1.0
10	9	9	1.0
11	10	6	0.5
12	10	7	0.5

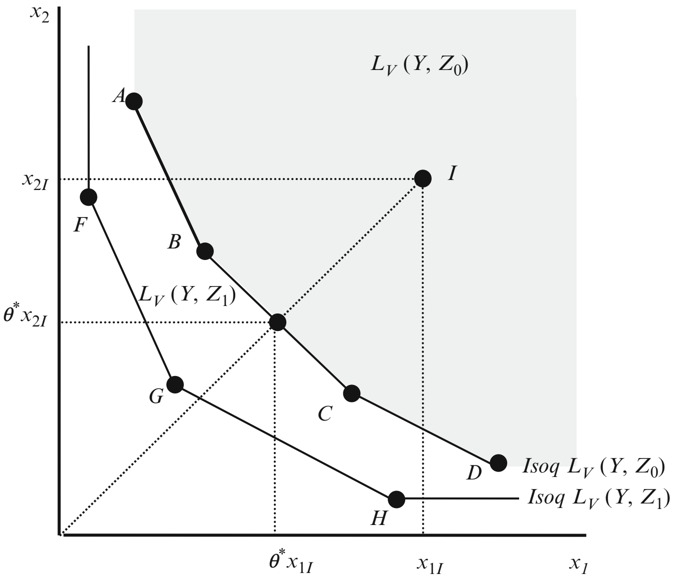


Fig. 3.6 Conditional Input Requirement sets

The results correctly indicate that DMU 5 (10) is technically inefficient with a benchmark defined as an equally weighted convex combination of DMUs 1 and 2 (6 and 7).

In the case of multiple inputs, we illustrate the Farrell measure of efficiency using $L_V(Y, z)$ and its associated isoquant $Isoq\ L_V(Y, z)$ in Fig. 3.6 by adding inefficient DMU I to Fig. 3.2. We assume DMU I faces the harsher environment with nondiscretionary factor $z_0 < z_1$ and therefore must use more of the discretionary inputs to achieve the given level of output. DMU I is observed using inputs (x_{1I}, x_{2I}) to produce the given output level but is technically inefficient producing off of the isoquant. Applying the conditional model (3.9), we observe that a benchmark consisting of convex combinations of DMUs B and C produces the same output level with less of both inputs ($\theta^* x_{1I}, \theta^* x_{2I}$) where $\theta^* < 1$ is obtained in the solution of the linear program. The model disallows DMUs $F - H$ from serving in the benchmark for DMU I because each as the more favorable environment. For all other DMUs, we observe $D_V^I(Y_j, X_j, z_j) = 1$; each of these decision making units is technically efficient relative to the appropriate isoquant defined by the environment.

3.2.2 Output-Orientation

The output-oriented measure of technical efficiency using the technologies defined in Eqs. (3.1) and (3.7) is defined using the following distance function:

$$D_V^O\left(Y_j, X_j, z_j\right) = \left(\max\left\{\theta : \left(\theta Y_j, X_j\right) \in \tau_V\left(z_j\right)\right\}\right)^{-1}. \tag{3.10}$$

This definition is consistent with the standard definition 2.10 but with the inclusion of the nondiscretionary factor. Hence, the measure is consistent with the Farrell measure identifying the maximum the maximum equiproportional expansion of outputs possible with feasibility defined relative to the conditional technology $\tau_V(z)$. The technical efficiency measure $D_V^O(Y_j, X_j, z_j)$ for each DMU$_j$ ($j = 1, \ldots, n$) is obtained via the solution to the following linear program:

$$\begin{aligned} & D_V^O\left(Y_j, X_j, z_j\right)^{-1} = \max\theta \\ & \text{subject to} \\ & \quad \sum_{i=1}^{n} \lambda_i y_{si} \geq \theta y_{sj}, \quad s = 1, \ldots, S; \\ & \quad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_{mj}, \quad m = 1, \ldots, M; \\ & \quad \sum_{i=1}^{n} \lambda_i = 1; \\ & \quad \lambda_i = 0 \text{ if } z_i > z_j, \quad i = 1, \ldots, N; \\ & \quad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{3.11}$$

The measure of technical efficiency is illustrated in Fig. 3.7 where we assume that DMU I uses x_{1I} to produce y_{1I} while facing the lower level z_0 of the nondiscretionary factor. DMU I could expand output by a factor of θ^* with a convex combination of

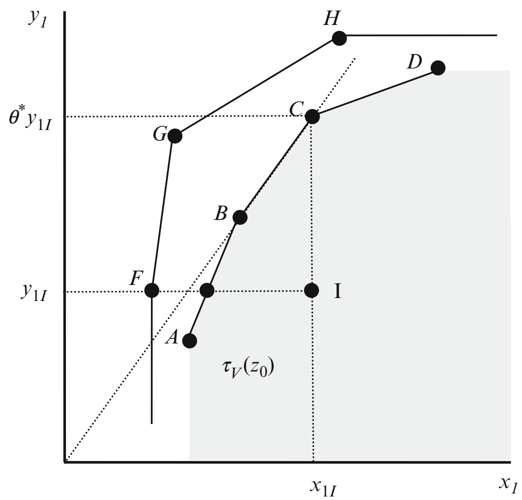


Fig. 3.7 Output-Oriented Technical Efficiency

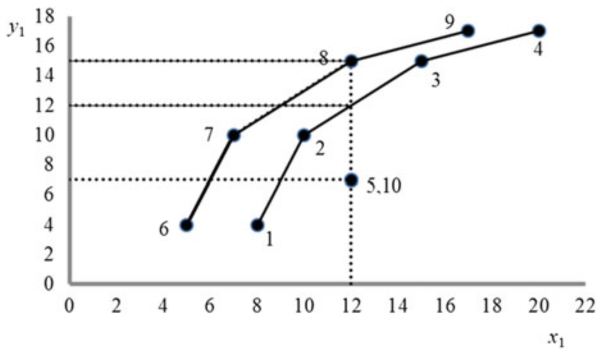


Fig. 3.8 Output-Oriented Technical Efficiency using Example 1 Data

production possibilities B and C as shown. Additional expansion is not possible because DMU I has a harsher environment with a nondiscretionary factor of z_0. From the solution of (3.11), we obtain $D_V^O(Y_I, X_I, z_0) = (\theta^*)^{-1}$. All other DMUs are producing on the isoquant under the assumption that DMUs A-D (F-H) face an environment defined by nondiscretionary input level z_0 (z_1). As a result, DMUs A-D and F-H are technically efficient with $D_V^O(Y_j, X_j, z_j) = 1$.

Returning to Example 1 data, we now consider the output oriented projections and measures of efficiency in Fig. 3.8. Like the input-oriented counterpart, we observe that DMUs 1 – 4 and 6 – 9 are technically efficient. DMUs 5 and 10 are observed employing 12 units of input while producing too little output $y_1 = 7$. Finally, we recall that DMU 5 (10) faces the harsher (more favorable) environment.

To evaluate the inefficiency of DMUs 5 and 10 we seek the maximum expansion of output consistent with the technology conditioned on the nondiscretionary factor. For DMU 5, the relevant benchmark is a convex combination of technically efficient production possibilities 2 and 3 with weights of 0.6 and 0.4, respectively. This convex combination produces 5 more units of output while using the same amount of inputs and facing the same environment. Consequently, $D_V^O(Y_5, X_5, 1)^{-1} = \frac{7}{12} = 0.58\overline{3}$, i.e., DMU 5 is only producing 58.33 % of the efficient output level. While DMU 10 uses the same discretionary input level and produces the same output as DMU 5, it faces the more favorable environment. The relevant benchmark is therefore not the same as the one for DMU 5. Instead, DMU 10 would be benchmarked against DMU 8, which uses the same level of the discretionary input while facing the same environment. DMU 8 produces 8 more units of output. Therefore, $D_V^O(Y_{10}, X_{10}, 2) = \frac{7}{15} = 0.4\overline{6}$. DMU 10 is observed producing only 46.67 % of the technically efficient output level.

The SAS code used to measure output-oriented efficiency using example 1 data is provided below.

```
option nonotes;

data example1; input dmu x1 y1 z1;
datalines;
1  8  4  1
2  10 10 1
3  15 15 1
4  20 17 1
5  12 7  1
6  5  4  2
7  7  10 2
8  12 15 2
9  17 17 2
10 12 7  2
;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set z_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  num Z {DMU, z_num};
```

(continued)

(continued)

```
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {t in z_num} < Z[dmu, t] = col("z"||t)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   max Objective = theta;

   num k;
   num DVO {DMU};
   num benchmark_weight {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

   con input_con {r in x_Num}:
      sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

   con exoginput_con {t in z_Num, j in DMU}:
      if z[j,t] > z[k,t] then Weight[j] = 0;

   con weight_con: sum {j in DMU} Weight[j] = 1;

   do k = DMU;
      solve;
      DVO [k] = 1/theta.sol;
      for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
           Weight[j].sol else .);

   end;
   create data tech_eff from [dmu] DVO;
   create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

data benchmark; set benchmark;
  if benchmark_weight = . then delete;

proc print data = tech_eff;
proc print data = benchmark;

run;
```

The resulting SAS output:

```
                    The SAS System

                 Obs     dmu      DVI

                   1       1     1.00000
                   2       2     1.00000
                   3       3     1.00000
                   4       4     1.00000
                   5       5     0.58333
                   6       6     1.00000
                   7       7     1.00000
                   8       8     1.00000
                   9       9     1.00000
                  10      10     0.46667

                    The SAS System

                                        benchmark_
        Obs     dmu     dmu_ref          weight

          1       1        1              1.0
          2       2        2              1.0
          3       3        3              1.0
          4       4        4              1.0
          5       5        2              0.6
          6       5        3              0.4
          7       6        6              1.0
          8       7        7              1.0
          9       8        8              1.0
         10       9        9              1.0
         11      10        8              1.0
```

3.3 Scale Efficiency Measurement

Similar to the case without a nondiscretionary factor of production, we can define standard measures of returns to scale for a given level of the nondiscretionary factor. One could view this as a production process with heterogeneity even in the long run. The conditional measures control for heterogeneity, allowing a discussion of returns to scale for a given level of the environmental variable. In addition, useful information is provided by the distances between any two frontiers. Ruggiero (2000) provided a measure of the returns to environmental scale revealing the additional discretionary resources needed to provide a given level of outputs because a DMU faces a harsher environment. In the output oriented case, the measure shows the loss of output due to a harsher environment.[4]

First, we consider the standard returns to scale measures for a given level of the nondiscretionary factor. We first define the empirical piecewise linear constant returns to scale technologies conditional on the nondiscretionary factor:

$$\begin{aligned}\tau_C(z) = \Big\{ \big(Y, X, z\big) : & \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ & \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ & \lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N; \\ & \lambda_i \geq 0, \quad i = 1, \ldots, N \Big\}.\end{aligned} \tag{3.12}$$

The constant returns to scale technology $\tau_C(z)$ is obtained from $\tau_V(z)$ in Eq. (2.1) by removing the convexity constraint. Similarly, we obtain the input requirement and output sets as:

$$\begin{aligned}L_C(Y, z) = \Big\{ X : & \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ & \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ & \lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N; \\ & \lambda_i \geq 0, \quad i = 1, \ldots, N \Big\}\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}P_C(X, z) = \Big\{ Y : & \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s, \quad s = 1, \ldots, S; \\ & \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ & \lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N; \\ & \lambda_i \geq 0, \quad i = 1, \ldots, N \Big\}\end{aligned} \tag{3.14}$$

respectively.

We illustrate the empirical production possibility set under the assumption of constant returns to scale using $\tau_C(z)$ from Eq. (3.12) in Fig. 3.9. The shaded area shows the empirical production possibility set $\tau_C(z_0)$ for the harsher environment z_0. Given z_0 we observe increasing returns to scale along AB, constant returns to scale along BC and decreasing returns to scale along CD. Hence, B and C and any convex

[4] The environmental scale measures can be defined using either a VRS or CRS technology. We follow Ruggiero (2000) and only consider VRS measures.

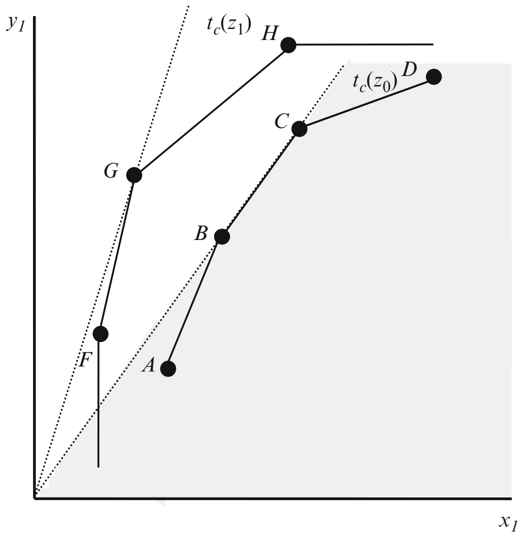


Fig. 3.9 Empirical Production Possibility Set $\tau_C(z)$

combination of the two are most productive scale size. With z_1 only DMU G is operating at most productive scale size; we observe increasing returns to scale along FG and decreasing returns to scale along GH. Interestingly, a given output level can correspond to different returns to scale classes depending on the level of the environment.

3.3.1 Input-Orientation

Standard measures of returns to scale using an input-orientation require projections to the VRS technology using (3.9) and the CRS technology. The distance function projecting DMU_j to the boundary of the CRS technology conditional on the nondiscretionary factor is given by:

$$D_C^I(Y_j, X_j, z_j) = \min\{\theta : (Y_j, \theta X_j) \in \tau_C(z_j)\} \tag{3.15}$$

and is estimated with the following linear program for each DMU_j ($j = 1,\ldots, n$):

$$\begin{aligned} &D_C^I(Y_j, X_j, z_j) = \min \theta_C \\ &\text{subject to} \\ &\quad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\ &\quad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta_C x_{mj}, \quad m = 1, \ldots, M; \\ &\quad \lambda_i = 0 \text{ if } z_i > z, \quad i = 1, \ldots, N; \\ &\quad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{3.16}$$

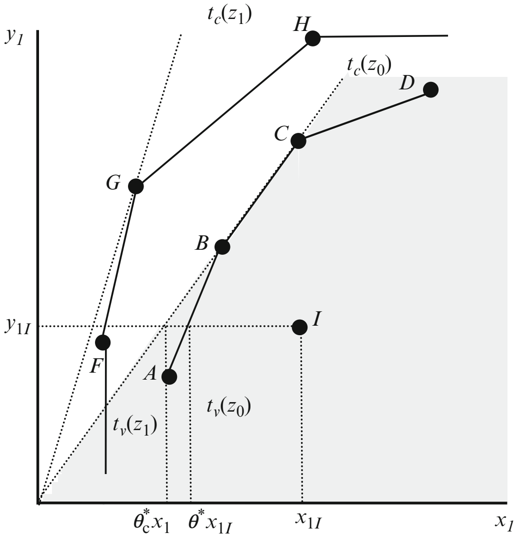


Fig. 3.10 Input-Oriented Projection $D_C^I(y_{1I}, x_{1I}, z_0)$

Using models (3.9) and (3.16), we obtain the standard measure of scale efficiency for DMU_j as the ratio of the two distance functions:

$$SE^I(Y_j, X_j, z_j) = \frac{D_C^I(Y_j, X_j, z_j)}{D_V^I(Y_j, X_j, z_j)}. \tag{3.17}$$

In Fig. 3.10, we show the projection of inefficient DMU I to the technology $\tau_C(z)$. DMU I is technically inefficient because it could have reduced its input to $\theta^* x_{1I}$ under variable returns to scale. Assuming constant returns to scale and solving Eq. (3.16), DMU I would be projected even further to an input level $\theta_C^* x_{1I}$, which is obtained via a rescaling of either production possibility B or C or a convex combination of the two. As illustrated, the technically efficient benchmark $(y_{1I}, \theta^* x_{1I})$ for DMU I is operating under increasing returns to scale; after controlling for the environment, the difference in inputs $(\theta^* - \theta_C^*)x_{1E}$ reveals the extra input level necessary to produce y_{1I} given that $(y_{1I}, \theta^* x_{1I})$ is not most productive scale size. The resulting scale efficiency of DMU I is $SE^I(Y_I, X_I, z_I) = \theta_C^*/\theta^* < 1$. Given that the projections to the VRS and CRS frontiers and the resulting measure of scale efficiency is defined for a given environment level similarly to the case without nondiscretionary factors, the returns to scale classification with respect to the discretionary inputs and outputs is the same.

Following Ruggiero (2000), we can evaluate returns to environmental scale $ES^I(Y_j, X_j, z_i)$ for each DMU_j $(j = 1, \ldots, n)$ as the ratio of two distance functions:

$$ES_V^I(Y_j, X_j, z_j) = \frac{D_V^I(Y_j, X_j)}{D_V^I(Y_j, X_j, z_j)} \leq 1, \tag{3.18}$$

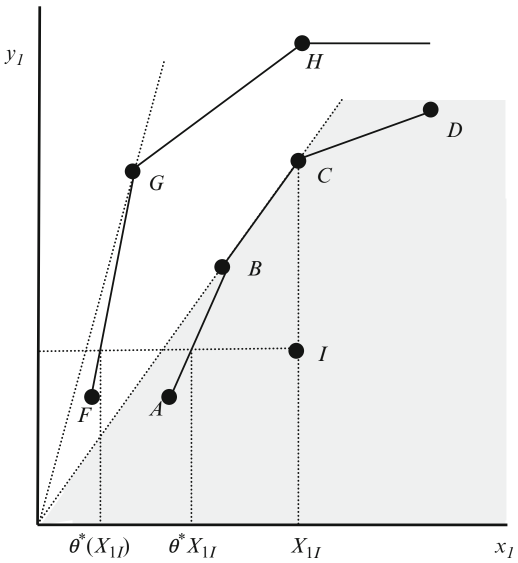


Fig. 3.11 Empirical Production Possibility Set $T_v(z)$

where the numerator (denominator) is the solution to model 2.9 (3.9).[5] Given that production does depend on the environment, the numerator excludes these variables and hence, projects a given unit to the frontier associated with the best environment. As a result, the environmental scale measure captures the influence the environment has on production by revealing how much each input could be proportionally reduced if the unit under evaluation had the most favorable environment.

We show our environmental scale measure with DMU I in Fig. 3.11. Given its environment, the technically efficient production plan for DMU I is $(y_{1I}, \theta^* x_{1I})$. If we remove the nondiscretionary factor constraint and solve model 2.9, we obtain production plan $(y_{1I}, \delta^* x_{1I})$, where $\delta^* = D_V^I(Y_I, X_I) < \theta^*$. The results indicate that DMU I would have been able to reduce its input level to $\delta^* x_{1I}$ if I faced the most favorable environment. In this case, the benchmark would have been a convex combination of DMUs F and G, both of which have the most favorable environment. This provides important policy information; in public production environments with nondiscretionary factors, we can measure the additional resources required to achieve a given level of output for DMUs that have a harsher environment.

The various input-oriented measures used in the public sector model are implemented in SAS for example 1 data.

[5] Alternatively, we could remove scale inefficiency from this measure by using the constant returns to scale projections.

```
option nonotes;

data example1; input dmu x1 y1 z1;
datalines;
1  8  4  1
2  10 10 1
3  15 15 1
4  20 17 1
5  12 7  1
6  5  4  2
7  7  10 2
8  12 15 2
9  17 17 2
10 12 7  2
;

* Projection to the VRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set z_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  num Z {DMU, z_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {t in z_num} < Z[dmu, t] = col("z"||t)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   min Objective = theta;

   num k;
   num DVI {DMU};
   num benchmark_weight_v {DMU,DMU};

   con output_con {s in y_Num}:
      sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;
```

(continued)

(continued)

```
    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

    con exoginput_con {t in z_Num, j in DMU}:
       if z[j,t] > z[k,t] then Weight[j] = 0;

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVI [k] = theta.sol;
       for {j in DMU} benchmark_weight_v[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_v from [dmu] DVI;
    create data benchmark_v from [dmu dmu_ref] benchmark_weight_v ;
quit;

data benchmark_v; set benchmark_v;
  if benchmark_weight_v = . then delete;

proc print data = benchmark_v;
title 'Benchmarks on the VRS Frontier';

* Projection to the VRS frontier without z1;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;

    var Weight {DMU} >= 0;
    var theta >= 0;
```

(continued)

(continued)

```
    min Objective = theta;

    num k;
    num DVI_noz {DMU};
    num benchmark_weight_v {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];


    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVI_noz [k] = theta.sol;
    end;

create data tech_eff_v_noz from [dmu] DVI_noz;

quit;

* Projection to the CRS frontier;

proc optmodel printlevel = 0;
   set x_num = 1..1;
   set y_num = 1..1;
   set z_num = 1..1;
   set <num> DMU;
   num X {DMU, x_num};
   num Y {DMU, y_num};
   num Z {DMU, z_num};
   read data example1
     into DMU = [dmu]
     {r in x_num} < X[dmu, r] = col("x"||r)>
     {s in y_num} < Y[dmu, s] = col("y"||s)>
     {t in z_num} < Z[dmu, t] = col("z"||t)>;

    var Weight {DMU} >= 0;
    var theta >= 0;

    min Objective = theta;
```

(continued)

(continued)

```
    num k;
    num DCI {DMU};
    num benchmark_weight_c {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

    con exoginput_con {t in z_Num, j in DMU}:
       if z[j,t] > z[k,t] then Weight[j] = 0;

    do k = DMU;
       solve;
       DCI [k] = theta.sol;
       for {j in DMU} benchmark_weight_c[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_c from [dmu] DCI;
    create data benchmark_c from [dmu dmu_ref] benchmark_weight_c ;
quit;

data benchmark_c; set benchmark_c;
  if benchmark_weight_c = . then delete;
proc print data = benchmark_c;
title 'Benchmarks on the CRS Frontier';
proc sort; by dmu;
proc means sum noprint; var benchmark_weight_c; by dmu; output out =
crs_sum_of_weights sum = sum_weights;

data crs_sum_of_weights; set crs_sum_of_weights;
 keep dmu sum_weights;

proc sort data = tech_eff_v; by dmu;
proc sort data = tech_eff_v_noz; by dmu;
proc sort data = tech_eff_c; by dmu;
proc sort data = crs_sum_of_weights; by dmu;
```

(continued)

(continued)

```
data final; merge tech_eff_v tech_eff_v_noz tech_eff_c
crs_sum_of_weights; by dmu;
 scale_eff = DCI/DVI;
 E = DVI_noz/DVI;
 If scale_eff > 0.9999999 then rts_class = "CRS"; else if
   sum_weights < 1 then rts_class = "IRS"; else rts_class = "DRS";
proc print data = final;
title 'Final Results';

run;
```

The SAS output from the above code[6]:

```
            Benchmarks on the VRS Frontier
                                      benchmark_
        Obs      dmu      dmu_ref      weight_v
          1        1         1            1.0
          2        2         2            1.0
          3        3         3            1.0
          4        4         4            1.0
          5        5         1            0.5
          6        5         2            0.5
          7        6         6            1.0
          8        7         7            1.0
          9        8         8            1.0
         10        9         9            1.0
         11       10         6            0.5
         12       10         7            0.5

            Benchmarks on the CRS Frontier
                                      benchmark_
        Obs      dmu      dmu_ref      weight_c
          1        1         2            0.4
          2        2         2            1.0
          3        3         2            1.5
          4        4         2            1.7
          5        5         2            0.7
          6        6         7            0.4
          7        7         7            1.0
          8        8         7            1.5
          9        9         7            1.7
         10       10         7            0.7
```

(continued)

[6] The output results are modified for formatting purposes.

(continued)

Final Results

dmu	DVI	DVI_noz	DCI	sum_ weights	scale_ eff	ES	rts_ class
1	1.00	0.625	0.50000	0.4	0.5000	0.6250	IRS
2	1.00	0.700	1.00000	1.0	1.0000	0.7000	CRS
3	1.00	0.800	1.00000	1.5	1.0000	0.8000	CRS
4	1.00	0.850	0.85000	1.7	0.8500	0.8500	DRS
5	0.75	0.500	0.58333	0.7	0.7778	0.6667	IRS
6	1.00	1.000	0.56000	0.4	0.5600	1.0000	IRS
7	1.00	1.000	1.00000	1.0	1.0000	1.0000	CRS
8	1.00	1.000	0.87500	1.5	0.8750	1.0000	DRS
9	1.00	1.000	0.70000	1.7	0.7000	1.0000	DRS
10	0.50	0.500	0.40833	0.7	0.8167	1.0000	IRS

Referring to Fig. 3.5 and the SAS output results we observe that only DMUs 2, 3 and 7 are operating at most productive scale size under constant returns to scale. DMU 7 has an average product of 1.43. DMUs 2 and 3 have an average product of 1 and are also most productive scale even with a lower average product because they have the highest average product of all units facing the harsher environment. All other technically efficient units are scale inefficient, operating under either increasing returns to scale (DMUs 1 and 6) or decreasing returns to scale (DMUs 4, 8 and 9.) Technically inefficient units 5 and 10 are both projected to the IRS portion of the VRS frontiers and hence, are also scale efficient after removing technical inefficiency.

Next, we consider the returns to environmental scale from Eq. (3.18). Production possibilities 6 - 10 all face the best environment; consequently $ES_V^I(Y_j, X_j, 2) = 1$ for each. This results because adding the constraint on the environmental factor does not change the solution. Production possibilities 1 – 5 all face the harsher environment. As a result, these DMUs have $ES_V^I(Y_j, X_j, 1) < 1$ indicating that each has to employ more of the discretionary input to achieve a given level of output. For example, technically efficient DMU 1 is observed producing $y_1 = 4$ using $x_1 = 8$. If DMU 1 remained efficient but had a more favorable environment, it could have produced the same output with $x_1 = 5$ using DMU 6 as a benchmark. As a result, $ES_V^I(4, 8, 1) = 5/8 = 0.625$. This indicates that DMU 1 needs to use 60 % more of the discretionary input than DMU 6 to produce four units of output given the harsher environment.

3.3.2 Output-Orientation

For completeness, we present the scale measures using the output oriented model. The distance function projecting DMU$_j$ to the boundary of the CRS technology conditional on the nondiscretionary input is given by:

$$D_C^O\left(Y_j,X_j,z_j\right)=\left(\max\left\{\theta:\left(\theta Y_j,X_j\right)\in\tau_C\left(z_j\right)\right\}\right)^{-1} \tag{3.19}$$

which can be estimated with the following linear programming model:

$$\begin{aligned}
&D_C^O\left(Y_j,X_j,z_j\right)^{-1}=\max\,\theta_C\\
&\text{subject to}\\
&\qquad\sum_{i=1}^{n}\lambda_i y_{si}\geq\theta_C y_{sj},\quad s=1,\ldots,S;\\
&\qquad\sum_{i=1}^{n}\lambda_i x_{mi}\leq x_{mj},\quad m=1,\ldots,M;\\
&\qquad\lambda_i=0\text{ if }z_i>z,\quad i=1,\ldots,N;\\
&\qquad\lambda_i\geq 0,\quad i=1,\ldots,N.
\end{aligned} \tag{3.20}$$

We seek the maximum equiproportional increase in all outputs subject to a CRS technology conditional on the nondiscretionary factor. Scale efficiency is then estimated as the ratio of the constant to the variable returns to scale distance functions:

$$SE^O\left(Y_j,X_j,z_j\right)=\frac{D_C^O\left(Y_j,X_j,z_j\right)}{D_V^O\left(Y_j,X_j,z_j\right)}. \tag{3.21}$$

The output-oriented projection of inefficient DMU I to the technology $\tau_C(z_I)$ defined by constant returns to scale is illustrated in Fig. 3.12. DMU I is technically inefficient relative to the variable returns to scale technology because it could have increased its output to $\theta_C^* y_{1I}$ holding input at the observed level x_{1I}, where $\theta_C^*=D_V^O(Y_I,X_I,z_I)$ is obtained in the solution to (3.11). If we remove the convexity constraint and solve (3.20) we obtain $\theta_C^*=D_C^O(Y_I,X_I,z_I)$. Hence, DMU C is the benchmark under both the CRS and the VRS models. Therefore, $SE^O(Y_I, X_I,z_I)=1$ indicating that DMU I is scale efficient and operating under constant returns to scale after technical inefficiency is eliminated. Using the output oriented projections, we observe that technically efficient DMUs A and F are scale inefficient operating under increasing returns to scale; DMUs B, C and G are operating under constant returns to scale while DMUs D and F are scale

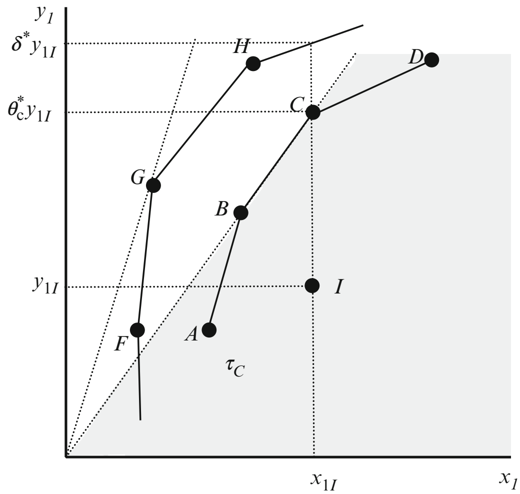


Fig. 3.12 Output-Oriented Projection $D_C^O(y_{1I}, x_{1I}, z_0)$

inefficient on the decreasing returns to scale portion of the frontier. As in the input oriented case, the output level associated with most productive scale size depends on the nondiscretionary input level.

We extend our measure of environmental scale to the output oriented model using projections with and without the conditional constraints. For each DMU$_j$ ($j = 1,\ldots,n$), we have:

$$ES_V^O(Y_j, X_j, z_j) = \frac{D_V^O(Y_j, X_j)}{D_V^O(Y_j, X_j, z_j)} \leq 1. \tag{3.22}$$

In the output-oriented case, environmental scale measures the reduction in output that can be attributed to having a less than favorable environment. Returning to Fig. 3.12, we observe $ES_V^O(Y_j, X_j, z_j) = 1$ for DMUs F, G and H who face the most favorable environment. If technically efficient, DMU I would have production plan $(\theta_C^* y_{1I}, x_{1I})$ given that I faces the harsher environment. If DMU I had the most favorable environment, the technically efficient production plan would be $(\delta^* y_{1I}, x_{1I})$, where $\delta^* = 1/D_V^O(Y_I, X_I)$. Hence, $ES_V^O(Y_I, X_I, z_I) = \frac{\theta_C^*}{\delta^*}$. The results indicate the shortfall in output that DMU I realizes due to the harsher environment.

The SAS code and results for the output oriented efficiency and scale measures is provided below.

```
option nonotes;

data example1; input dmu x1 y1 z1;
datalines;
1  8  4  1
2  10 10 1
3  15 15 1
4  20 17 1
5  12 7  1
6  5  4  2
7  7  10 2
8  12 15 2
9  17 17 2
10 12 7  2
;

* Projection to the VRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set z_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  num Z {DMU, z_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {t in z_num} < Z[dmu, t] = col("z"||t)>;

   var Weight {DMU} >= 0;
   var theta >= 0;

   max Objective = theta;

   num k;
   num DVO {DMU};
   num benchmark_weight_v {DMU,DMU};
```

(continued)

(continued)

```
    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

    con exoginput_con {t in z_Num, j in DMU}:
       if z[j,t] > z[k,t] then Weight[j] = 0;

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVO [k] = 1/theta.sol;
       for {j in DMU} benchmark_weight_v[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_v from [dmu] DVO;
    create data benchmark_v from [dmu dmu_ref] benchmark_weight_v ;
quit;

data benchmark_v; set benchmark_v;
  if benchmark_weight_v = . then delete;

proc print data = benchmark_v;
title 'Benchmarks on the VRS Frontier';


* Projection to the VRS frontier without z1;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;
```

(continued)

(continued)

```
    var Weight {DMU} >= 0;
    var theta >= 0;

    max Objective = theta;

    num k;
    num DVO_noz {DMU};
    num benchmark_weight_v {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

    con weight_con: sum {j in DMU} Weight[j] = 1;

    do k = DMU;
       solve;
       DVO_noz [k] = 1/theta.sol;
    end;

create data tech_eff_v_noz from [dmu] DVO_noz;

quit;

* Projection to the CRS frontier;

proc optmodel printlevel = 0;
  set x_num = 1..1;
  set y_num = 1..1;
  set z_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  num Z {DMU, z_num};
  read data example1
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {t in z_num} < Z[dmu, t] = col("z"||t)>;
```

(continued)

```
    var Weight {DMU} >= 0;
    var theta >= 0;

    max Objective = theta;

    num k;
    num DCO {DMU};
    num benchmark_weight_c {DMU,DMU};

    con output_con {s in y_Num}:
       sum {j in DMU} Y[j,s] * Weight[j] >= theta*Y[k,s] ;

    con input_con {r in x_Num}:
       sum {j in DMU} X[j,r] * Weight[j] <= X[k,r];

    con exoginput_con {t in z_Num, j in DMU}:
       if z[j,t] > z[k,t] then Weight[j] = 0;

    do k = DMU;
       solve;
       DCO [k] = 1/theta.sol;
       for {j in DMU} benchmark_weight_c[k,j] = (if Weight[j].sol > 1e-6
then Weight[j].sol else .);

    end;
    create data tech_eff_c from [dmu] DCO;
    create data benchmark_c from [dmu dmu_ref] benchmark_weight_c ;
quit;

data benchmark_c; set benchmark_c;
  if benchmark_weight_c = . then delete;
proc print data = benchmark_c;
title 'Benchmarks on the CRS Frontier';
proc sort; by dmu;
proc means sum noprint; var benchmark_weight_c; by dmu; output out =
crs_sum_of_weights sum = sum_weights;

data crs_sum_of_weights; set crs_sum_of_weights;
 keep dmu sum_weights;

proc sort data = tech_eff_v; by dmu;
proc sort data = tech_eff_v_noz; by dmu;
proc sort data = tech_eff_c; by dmu;
proc sort data = crs_sum_of_weights; by dmu;
```

(continued)

(continued)

```
data final; merge tech_eff_v tech_eff_v_noz tech_eff_c
crs_sum_of_weights; by dmu;
  scale_eff = DCO/DVO;
  E = DVO_noz/DVO;
  If scale_eff > 0.9999999 then rts_class = "CRS"; else if
    sum_weights < 1 then rts_class = "IRS"; else rts_class = "DRS";
proc print data = final;
title 'Final Results';

run;
```

The SAS output from the above code:

```
                Benchmarks on the VRS Frontier

                                             benchmark_
             Obs      dmu      dmu_ref        weight_v
              1        1          1              1.0
              2        2          2              1.0
              3        3          3              1.0
              4        4          4              1.0
              5        5          2              0.6
              6        5          3              0.4
              7        6          6              1.0
              8        7          7              1.0
              9        8          8              1.0
             10        9          9              1.0
             11       10          8              1.0

                Benchmarks on the CRS Frontier

                                             benchmark_
             Obs      dmu      dmu_ref        weight_c
              1        1          2            0.80000
              2        2          2            1.00000
              3        3          2            1.50000
              4        4          2            2.00000
              5        5          2            1.20000
              6        6          7            0.71429
              7        7          7            1.00000
              8        8          7            1.71429
              9        9          7            2.42857
             10       10          7            1.71429
```

(continued)

(continued)

Final Results

dmu	DVO	DVO_noz	DCO	sum_ weights	scale_ eff	E	rts_ class
1	1.0000	0.36364	0.5000	0.8000	0.500	0.3636	IRS
2	1.0000	0.76923	1.0000	1.0000	1.000	0.7692	CRS
3	1.0000	0.92593	1.0000	1.5000	1.000	0.9259	CRS
4	1.0000	1.00000	0.8500	2.0000	0.850	1.0000	DRS
5	0.5833	0.46667	0.5833	1.2000	1.000	0.8000	CRS
6	1.0000	1.00000	0.5600	0.7143	0.560	1.0000	IRS
7	1.0000	1.00000	1.0000	1.0000	1.000	1.0000	CRS
8	1.0000	1.00000	0.8750	1.7143	0.875	1.0000	DRS
9	1.0000	1.00000	0.7000	2.4286	0.700	1.0000	DRS
10	0.4667	0.46667	0.4083	1.7143	0.875	1.0000	DRS

The results reported above are illustrated in Fig. 3.8. DMUs 1 – 4 and 6 – 9 are technically efficient in the output oriented model. DMU 5 and 10 are both observed producing 7 units of output using 12 units of input. Given the environmental differences, DMU 10 should have produced 15 units of output and DMU 5 should have produced 12. Because the projection for DMU 5 is the same under VRS and CRS, it is scale efficient. We note the benchmark for DMU 5 under the CRS model is either DMU 2 or DMU 3 or a combination of the two, all of which are on the CRS portion of the frontier. DMU 10 is benchmarked against DMU 8 in both the CRS and VRS models; since DMU 8 is operating under DRS with a scale efficiency of 87.5 %, so too is DMU 10.

Finally, the differences between the frontiers reveal the effect of the nondiscretionary factor. For example, if technically efficient, DMU 5 would have produced 12 units of output. If DMU 5 had the most favorable environment (like DMU 10), it could produce 15 units of output. Hence, our measure of environmental scale is 0.8 (12/15); DMU 5 can only produce 80 % (12) of the output (15) of the most favorable environment given its observed input usage.

3.4 Allocative Efficiency Measurement

In this section, we extend the Farrell decomposition of overall inefficient into technical and allocative parts to the public sector models with nondiscretionary inputs. We assume that each decision making unit uses a vector of m discretionary inputs $X=(x_1,\ldots,x_M)$ to produce a vector of s outputs $Y=(y_1,\ldots,y_S)$ while facing nondiscretionary input z and exogenous input prices $P=(p_1,\ldots,p_M)$. We represent the inputs, outputs and prices for DMU$_j$ $(j=1,\ \ldots,\ n)$ as $X_j=(x_{1j},\ldots,x_{Mj})$, z_j, $Y_j=(y_{1j},\ldots,y_{Sj})$ and $P_j=(p_{1j},\ldots,p_{Mj})$. We also observe expenditures $E_j=\sum_{l=1}^{M}p_{lj}x_{lj}$ for each DMU j.

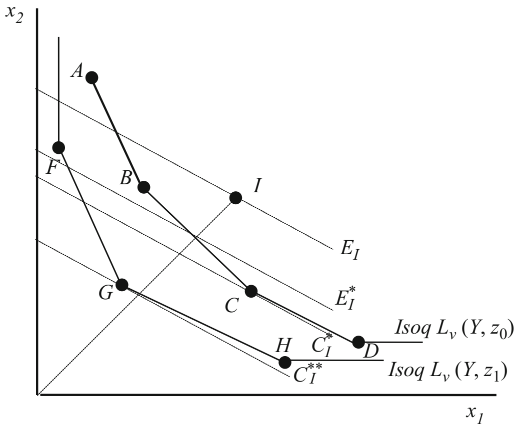


Fig. 3.13 Allocative and Technical Efficiency

In Fig. 3.13, we extend the allocative efficiency analysis from the previous chapter to include the nondiscretionary variable. Similar to Fig. 3.6, we observe two isoquants depending on the harshness of the environment. DMUs A – D are technically efficient producing output on the isoquant $Isoq\ L_V(Y, z_0)$; given the harsher environment, more inputs are required to produce the given output. DMUs F – H are also technically efficient, producing the same output level with a more favorable environment. The advantage of the better environment is the ability to efficiently produce a given output level with less discretionary inputs. For this diagram, we assume that technically inefficient DMU I faces the harsher environment with $D_V^I(Y_I, X_I, z_0) < 1$. Given the observed prices of DMU I, we superimpose four isocost lines. Isocost line labeled E_I shows the observed expenditures of DMU I.

The second isocost line, E_I^* reveals the expenditure level associated with the technically efficient benchmark for DMU I. Holding the input mix constant, we evaluate expenditures at the technically efficient input levels. We obtain $E_I^* = D_V^I(Y_I, X_I, z_0)\ x\ E_I$. This is not the minimum cost of producing the observed output given the environment faced by DMU I unless DMU I is choosing the correct input mix. In this case, DMU I is using too much x_2 relative to x_1; DMU I would achieve allocative efficiency by using the same input mix as DMU C. The isocost line $C_I^* = p_{1I}x_{1C} + p_{2I}x_{2C}$ represents the minimum cost for DMU I of producing the observed output given the observed prices and the harsher environment. The measure of cost efficiency (CE) is the ratio of minimum cost to observed expenditures. The measure $CE = C_I^*/E_I$ reveals the amount that observed expenditures could be reduced if DMU I simultaneously removed technical inefficiency and used the correct input mix. The last isocost line shows the minimum cost $C_I^{**} = p_{1I}x_{1G} + p_{2I}x_{2G}$ of producing the observed output using the input prices faced by DMU I assuming it had the most favorable environment. As shown, the cost minimizing input mix (DMU G's) depends on the harshness of the environment; if DMU I faced the better environment, the cost minimizing input mix requires more x_2 relative to x_1.

To obtain the measure cost efficiency, we solve the linear programming model conditional on the nondiscretionary input to obtain minimum cost for each DMU j as:

$$\begin{aligned} & C_j^* = \min \sum_{m=1}^{M} p_{mj} x_m \\ & \text{subject to} \\ & \qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\ & \qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\ & \qquad \sum_{i=1}^{n} \lambda_i = 1; \\ & \qquad \lambda_i = 0 \text{ if } z_i > z_j, \quad i = 1, \ldots, N; \\ & \qquad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{3.23}$$

We assume a variable returns to scale technology with the observed convexity constraint and obtain an optimal vector of inputs $(x_1^*, \ldots, x_M^*)$ for each DMU that minimizes the costs of production. Our measure of cost efficiency is then derived as the ratio of minimum costs to observed expenditures:

$$CE\left(Y_j, X_j, z_j\right) = \frac{C_j^*}{E_j}. \tag{3.24}$$

Given the technical and cost efficiency measures $D_V^I(Y_j, X_j, z_j)$ and $CE(Y_j, X_j, z_j)$ we obtain a measure of allocative efficiency as:

$$AE\left(Y_j, X_j, z_j\right) = \frac{CE\left(Y_j, X_j, z_j\right)}{D_V^I\left(Y_j, X_j, z_j\right)}. \tag{3.25}$$

We obtain the minimum cost of producing the observed output assuming the most favorable environment by solving the following linear program for each DMU j:

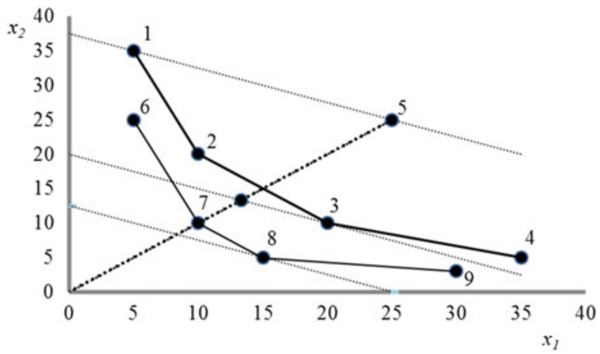


Fig. 3.14 Allocative and Technical Efficiency using Example 2 Data

$$
C_j^{**} = \min \sum_{m=1}^{M} p_{mj} x_m
$$

subject to

$$
\begin{aligned}
&\sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
&\sum_{i=1}^{n} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, M; \\
&\sum_{i=1}^{n} \lambda_i = 1; \\
&\lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned} \tag{3.26}
$$

Model (3.26) is obtained from model (3.23) by removing the conditional constraint on the nondiscretionary input.[7]

Consider the following data where nine DMUs produce one unit of output facing the same input prices. Output is produced using two discretionary inputs and one nondiscretionary input. The data are illustrated in Fig. 3.14.

[7] We could define an index of environmental harshness based on the ratio of minimum costs

DMU	x_1	x_2	z_1	p_1	p_2	y_1	E
1	5	35	1	5	10	1	375
2	10	20	1	5	10	1	250
3	20	10	1	5	10	1	200
4	35	5	1	5	10	1	225
5	25	25	1	5	10	1	375
6	5	25	2	5	10	1	275
7	10	10	2	5	10	1	150
8	15	5	2	5	10	1	125
9	30	3	2	5	10	1	180

All DMUs except DMU 5 are technically efficient. Only DMUs 3 and 8 are allocatively efficient. DMU 3 is using the proper input mix of $x_2/x_1 = 0.5$ and is allocatively efficient in producing the unit output given the harsher environment. Because DMU 3 is also technically efficient, the minimum cost in this stylized example is \$200. Cost efficiency for all of the units facing the harsher environment is then measured as \$200 divided by the observed expenditures. DMU 5 is technically inefficient with $D_V^I(Y_5, X_5, z_5) = 0.6$. If DMU 5 removed the technical inefficiency, it could reduce observed expenditures from \$375 to \$225. This is not the minimum cost because even after projecting DMU 5 to its isoquant, it is still using the wrong input mix. The cost efficiency of DMU 5 is $CE(Y_5, X_5, z_5) = \frac{200}{375} =$ 0.5333, leading to $AE(Y_5, X_5, z_5) = 0.89$. These results are identical for the allocative efficiency example presented in the last chapter. With the additional DMUs facing the more favorable environment, we uncover additional information. The minimum cost of producing the unit output is only \$125 for the units facing the best environment.[8] DMU 8 is using the only allocatively efficient units Hence, the units facing the harsher environment would need an additional \$75 to compensate for the operating conditions.

The SAS code to estimate cost and allocative efficiency for all DMUs in public sector production characterized by a nondiscretionary input is provided below.

[8] This variable is labeled C1 in the SAS code that follows.

```
option nonotes;



data example; input DMU x1 x2 y1 z1 p1 p2 E;
datalines;
1  5  35  1  1  5  10  375
2 10 20  1  1  5  10  250
3 20 10  1  1  5  10  200
4 35  5   1  1  5  10  225
5 25 25  1  1  5  10  375
6 5   25  1  2  5  10  275
7 10 10  1  2  5  10  150
8 15  5   1  2  5  10  125
9 30  3   1  2  5  10  180
;

* estimating technical efficiency;

proc optmodel printlevel = 0;
   set x_num = 1..2;
   set y_num = 1..1;
   set z_num = 1..1;
   set <num> DMU;
   num X {DMU, x_num};
   num Y {DMU, y_num};
   num Z {DMU, z_num};
   read data example
     into DMU = [dmu]
 {r in x_num} < X[dmu, r] = col("x"||r)>
 {s in y_num} < Y[dmu, s] = col("y"||s)>
 {t in z_num} < Z[dmu, t] = col("z"||t)>;

var Weight {DMU} >= 0;
var theta >= 0;

min Objective = theta;

num k;
num DVI {DMU};
num benchmark_weight {DMU,DMU};

con output_con {s in y_Num}:
sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;
```

(continued)

(continued)

```
con input_con {r in x_Num}:
sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

con exoginput_con {t in z_Num, j in DMU}:
if z[j,t] > z[k,t] then Weight[j] = 0;

con weight_con: sum {j in DMU} Weight[j] = 1;

do k = DMU;
  solve;
DVI [k] = theta.sol;
    for {j in DMU} benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
      Weight[j].sol else .);
end;

create data tech_eff from [dmu] DVI;
    create data benchmark from [dmu dmu_ref] benchmark_weight ;
quit;

* estimating minimum costs;

proc optmodel printlevel = 0;
   set x_num = 1..2;
   set y_num = 1..1;
   set z_num = 1..1;
   set <num> DMU;
   num X {DMU, x_num};
   num P {DMU, x_num};
   num Y {DMU, y_num};
   num Z {DMU, z_num};
   read data example
      into DMU = [dmu]
      {r in x_num} < X[dmu, r] = col("x"||r)>
      {r in x_num} < P[dmu, r] = col("p"||r)>
      {s in y_num} < Y[dmu, s] = col("y"||s)>
      {t in z_num} < Z[dmu, t] = col("z"||t)>;

var Weight {DMU} >= 0;
var xo{x_num} >= 0;

num k;
min Objective = sum{t in x_num} xo[t]*P[k,t];
num C{DMU};
num c_benchmark_weight {DMU,DMU};
```

(continued)

(continued)

```
con output_con {s in y_Num}:
sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

con input_con {r in x_Num}:
sum {j in DMU} X[j,r] * Weight[j] <= xo[r];

con weight_con: sum {j in DMU} Weight[j] = 1;

con exoginput_con {t in z_Num, j in DMU}:
if z[j,t] > z[k,t] then Weight[j] = 0;

do k = DMU;
solve;
C[k] = sum{t in x_num} xo[t]*P[k,t];
   for {j in DMU} c_benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
      Weight[j].sol else .);
end;
create data cost from [dmu] C;
   create data c_benchmark from [dmu dmu_ref] c_benchmark_weight ;
quit;

data c_benchmark; set c_benchmark;
  if c_benchmark_weight = . then delete;

proc print data=c_benchmark;


* estimating minimum costs most favorable environment;

proc optmodel printlevel = 0;
  set x_num = 1..2;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num P {DMU, x_num};
  num Y {DMU, y_num};
  read data example
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>
    {r in x_num} < P[dmu, r] = col("p"||r)>;
```

(continued)

(continued)

```
var Weight {DMU} >= 0;
var xo{x_num} >= 0;

num k;
min Objective = sum{t in x_num} xo[t]*P[k,t];

num C1{DMU};
num c1_benchmark_weight {DMU,DMU};

con output_con {s in y_Num}:
sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

con input_con {r in x_Num}:
sum {j in DMU} X[j,r] * Weight[j] <= xo[r];

con weight_con: sum {j in DMU} Weight[j] = 1;

do k = DMU;
  solve;
  C1[k] = sum{t in x_num} xo[t]*P[k,t];
   for {j in DMU} c1_benchmark_weight[k,j] = (if Weight[j].sol > 1e-6 then
     Weight[j].sol else .);
end;

create data cost1 from [dmu] C1;
create data c_benchmark from [dmu dmu_ref] c1_benchmark_weight ;
quit;

data c1_benchmark; set c1_benchmark;
   if c1_benchmark_weight = . then delete;

proc print data=c1_benchmark;

proc sort data = example; by dmu;
proc sort data = tech_eff; by dmu;
proc sort data = cost; by dmu;
proc sort data = cost1; by dmu;

data final; merge example tech_eff cost cost1; by dmu;
  TE = DVI;
  CE = C/E;
  AE = CE/DVI;
proc print data = final; var DMU C c1 TE AE CE;
run;
```

The resulting SAS output:

The SAS System

Obs	dmu	dmu_ref	c_benchmark_ weight
1	1	3	1
2	2	3	1
3	3	3	1
4	4	3	1
5	5	3	1
6	6	8	1
7	7	8	1
8	8	8	1
9	9	8	1

The SAS System

Obs	DMU	C	C1	TE	AE	CE
1	1	200	125	1.0	0.53333	0.53333
2	2	200	125	1.0	0.80000	0.80000
3	3	200	125	1.0	1.00000	1.00000
4	4	200	125	1.0	0.88889	0.88889
5	5	200	125	0.6	0.88889	0.53333
6	6	125	125	1.0	0.45455	0.45455
7	7	125	125	1.0	0.83333	0.83333
8	8	125	125	1.0	1.00000	1.00000
9	9	125	125	1.0	0.69444	0.69444

Ruggiero (1996a) provided the basis for analyzing public sector production. A limitation discussed in Ruggiero (1998) is the curse of dimensionality that arises when there are multiple nondiscretionary inputs. Without a large number of DMUs, multiple nondiscretionary factors cause efficiency estimates to be biased upward. Ruggiero (1998) provided a three-stage model to reduce the dimensions by creating an index of environmental harshness. In the first stage, a standard DEA model is applied using only discretionary inputs and outputs. The resulting index will capture not only unobserved inefficiency but also the effect the nondiscretionary inputs have on production. In the second stage, regression is applied to provide a weighting structure of the importance each nondiscretionary factor has. The predicted first stage index is then used in the third stage using any of the models discussed in this chapter. In the next section, we present the three-stage model and illustrate it using simulated data in SAS.

3.5 Multiple Nondiscretionary Inputs[9]

We extend our analysis by assuming that each decision making unit uses a vector of m discretionary inputs $X = (x_1, \ldots, x_M)$ to produce a vector of s outputs $Y = (y_1, \ldots, y_S)$ while facing a vector of r nondiscretionary inputs $Z = (z_1, \ldots, z_R)$. We define inputs and outputs for DMU$_j$ $(j = 1, \ldots, n)$ as $X_j = (x_{1j}, \ldots, x_{Mj})$, $Z_j = (z_{1j}, \ldots, z_{rj})$ and $Y_j = (y_{1j}, \ldots, y_{Sj})$. In the first stage, we apply the standard DEA model assuming variable returns to scale without the conditional constraint on the nondiscretionary factors. For each DMU$_j$ $(j = 1, \ldots, n)$ we estimate the first stage index (FS) as the solution to the following linear program:

$$
\begin{aligned}
& FS_j = \min \theta \\
& \text{subject to} \\
& \quad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\
& \quad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, \ldots, M; \\
& \quad \sum_{i=1}^{n} \lambda_i = 1; \\
& \quad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{3.27}
$$

This model is identical to (3.9) with the exclusion of the constraint on the nondiscretionary inputs. The resulting index is composed of inefficiency and the effect the nondiscretionary inputs have on the production process. In a second stage, the following regression is applied:

$$
FS_j = \alpha + \sum_{i=1}^{r} \beta_i z_{ij} + \varepsilon_{j.} \tag{3.28}
$$

Ray (1991) introduced the (3.27) and (3.28) as a two-stage model using OLS to estimate (3.28). Assuming that each nondiscretionary input is not correlated with inefficiency, the residual provides a measure of technical efficiency. However, the

[9] In this section, we only consider the case of measuring input-oriented technical efficiency. All models presented in this chapter can be easily extended to handle multiple nondiscretionary factors.

measure is not well-suited as an efficiency index because it is mean zero.[10] Ruggiero (1998) extended Ray's model by using (3.28) to derive a predicted index of environmental harshness:

$$\hat{z}_j = \alpha + \sum_{i=1}^{r} \beta_i z_{ij} \tag{3.29}$$

which is then used in a third stage using model (3.9). Hence, the second stage is used to provide weights of the relative importance of each nondiscretionary factor and combines them into a single measure of environmental harshness. For each DMU j ($j = 1, \ldots, n$) we estimate technical efficiency using the third stage model:

$$\begin{aligned} & D_V^I(Y_j, X_j, z_j) = \min \theta \\ & \text{subject to} \\ & \quad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, S; \\ & \quad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, \ldots, M; \\ & \quad \sum_{i=1}^{n} \lambda_i = 1; \\ & \quad \lambda_i = 0 \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\ & \quad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{3.30}$$

Ruggiero (1998) and Estelle et al. (2010) provide evidence using simulated data that the three-stage model works well.

We provide a simulation to illustrate the three-stage model. We use the data generating process found in Estelle et al. (2010) and assume that 500 DMUs use two discretionary inputs x_1 and x_2 to produce one output y_1 given two nondiscretionary inputs z_1 and z_2. Inefficiency was generated using $u \sim N(0, 0.2)$ with $e^{-|u|}$ is the measure of technical efficiency. We assume the following distributions for the discretionary and nondiscretionary inputs: $\ln x_i \sim N(0, 1)$ for $i = 1, 2$ and $\ln z_i \sim N$

[10] Other regression procedures have been considered. For example, McCarty and Yaisawarng (1993) used Tobit. Banker and Natarajan (2008) provided the conditions under which OLS provided consistent parameter estimates. McDonald (2009) argued that Tobit is inappropriate and recommend either OLS or fractional logit. Estelle et al. (2010) provided a Monte Carlo analysis using OLS, Tobit, fractional logit and nonparametric regression. The models provided nearly identical results. In this chapter, we only consider OLS.

$(0, 0.1)$ for $i = 1, 2$, respectively. Observed output was calculated assuming the following production function:

$$y_1 = e^{-|u|} z_1^2 z_2^{-3} x_1^{0.4} x_2^{0.6}. \tag{3.31}$$

The production function exhibits constant returns to scale with respect to the discretionary inputs. Therefore, we use a CRS model in both the first and second stage models. In addition, we do not assume monotonicity with respect to the nondiscretionary inputs; increases in the second nondiscretionary input lead to decreases in output. We use correlation and rank correlation to evaluate the performance of the model. The SAS code to generate the data and estimate efficiency is provided below.

```
option nonotes;

data gen_data;
  do i = 1 to 500;
    ln_x1 = rand('Normal',0,1);
    ln_x2 = rand('Normal',0,1);
    ln_z1 = rand('Normal',0,0.1);
    ln_z2 = rand('Normal',0,0.1);
    u = rand('Normal',0,0.2);
    TE = exp(-(abs(u)));
    x1 = exp(ln_x1);
    x2 = exp(ln_x2);
    z1 = exp(ln_z1);
    z2 = exp(ln_z2);
    y1 = te*(z1**2)*(z2**(-3))*(x1**0.4)*(x2**0.6);
   dmu = i;
    output;
   end;
keep dmu x1 x2 y1 z1 z2 te;

proc sort; by dmu;

proc optmodel printlevel = 0;
  set x_num = 1..2;
  set y_num = 1..1;
  set <num> DMU;
  num X {DMU, x_num};
  num Y {DMU, y_num};
  read data gen_data
    into DMU = [dmu]
    {r in x_num} < X[dmu, r] = col("x"||r)>
    {s in y_num} < Y[dmu, s] = col("y"||s)>;
```

(continued)

```
   var Weight {DMU} >= 0;
   var theta >= 0;

   min Objective = theta;

   num k;
   num FS {DMU};

   con output_con {s in y_Num}:
    sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

   con input_con {r in x_Num}:
    sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

   do k = DMU;
 solve;
    FS [k] = theta.sol;
  end;

 create data first_stage from [dmu] FS;
quit;


proc sort data = first_stage; by dmu;

data second_stage; merge first_stage gen_data; by dmu;

proc reg noprint; model fs = z1 z2;  output out = z_index p=aggz;

data z_index; set z_index;
 drop z1 z2;
data third_stage; set z_index;
 z1 = aggz;
 drop aggz;

proc optmodel printlevel = 0;
   set x_num = 1..2;
   set y_num = 1..1;
   set z_num = 1..1;
   set <num> DMU;
   num X {DMU, x_num};
   num Y {DMU, y_num};
   num Z {DMU, z_num};
```

(continued)

(continued)

```
  read data third_stage
    into DMU = [dmu]
      {r in x_num} < X[dmu, r] = col("x"||r)>
      {s in y_num} < Y[dmu, s] = col("y"||s)>
      {t in z_num} < Z[dmu, t] = col("z"||t)>;

      var Weight {DMU} >= 0;
      var theta >= 0;

  min Objective = theta;

  num k;
  num DVI {DMU};
  num benchmark_weight {DMU,DMU};

  con output_con {s in y_Num}:
   sum {j in DMU} Y[j,s] * Weight[j] >= Y[k,s] ;

 con input_con {r in x_Num}:
  sum {j in DMU} X[j,r] * Weight[j] <= theta*X[k,r];

 con exoginput_con {t in z_Num, j in DMU}:
  if z[j,t] > z[k,t] then Weight[j] = 0;

 do k = DMU;
  solve;
  DVI [k] = theta.sol;
  end;

 create data tech_eff from [dmu] DVI;
 quit;

proc sort data = tech_eff; by dmu;

data final; merge tech_eff gen_data; by dmu;
proc corr pearson spearman; var dvi te;
run;
```

The SAS results:

```
                      The SAS System

                    The CORR Procedure

                     2  Variables:    DVI      TE


                      Simple Statistics

Variable     N      Mean   Std Dev    Median    Minimum    Maximum

DVI         500   0.92852  0.09103    0.96832    0.60556    1.00000
TE          500   0.86387  0.09762    0.88561    0.52601    0.99878


                  Pearson Correlation Coefficients, N = 500
                          Prob > |r| under H0: Rho=0

                                  DVI             TE

                       DVI      1.00000       0.87831
                                                <.0001

                       TE       0.87831       1.00000
                                 <.0001


                 Spearman Correlation Coefficients, N = 500
                         Prob > |r| under H0: Rho=0

                                 DVI             TE

                      DVI      1.00000       0.81940
                                               <.0001

                      TE       0.81940       1.00000
                                <.0001
```

The results are consistent with the Monte Carlo results in Estelle et al. (2010): there is a relatively large (rank) correlation between true and estimated efficiency.

In the next chapter, we analyze the technical efficiency of Australian primary and secondary schools using the input-oriented models discussed above.

Chapter 4
Input-Oriented Efficiency Measures in Australian Schools

In this chapter we apply the DEA models presented in Chap. 3. We focus on the input-oriented models to measure technical, scale and input allocative efficiency of the primary and secondary schools in Australia. In addition, we also present a production based model of adequacy. In recent years, there has been a movement towards measuring adequacy of educational service provision, defined by Berne and Stiefel (1999) and Duncombe and Yinger (1999) as the minimum amount of resources necessary for a school to meet some pre-defined absolute standard of performance. Typically, this is defined as achieving minimum passing standards on standardized tests. In this chapter, we measure technical, allocative and scale efficiency of Australian schools using the models developed in Chapter 3. We also apply a model (Ruggiero 2007b) to measure the minimum expenditure necessary to provide an adequate education by projecting observations using the predefined adequacy standards instead of the observed outcomes. We use data from school year 2009–2010. In the following section, we discuss our data.

4.1 Data

The data for this study came from the Departmental Annual Financial Statements in the state of New South Wales (NSW). The original dataset contained detailed information on several inputs, outputs and socio-economic variables for all primary and secondary schools in NSW. For outputs we have data for 2008–2010 on standardized test scores for reading, writing, spelling, grammar, and numeracy in the third and fifth grades for primary schools and seventh and ninth grade (lower secondary) from the NAPLAN "My School" database—(the Commonwealth Government initiative from 2008 for each school in Australia). Given the high degree of correlation between outputs, we reduce the dimensionality by averaging test scores

V. Blackburn et al., *Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis*, DOI 10.1007/978-1-4899-7469-3_4,

Table 4.1 Variable description

Variable	Description
Outcomes[a]	
Primary schools	
y_1	Average third grade score
y_2	Average fifth grade score
Secondary schools	
y_1	Average seventh grade score
y_2	Average ninth grade score
Discretionary inputs	
x_1	Full time equivalent teachers
x_2	Full time equivalent (SASS)
x_3	Other expenses
Nondiscretionary inputs	
z_1	Index of community socio-educational advantage
z_2	Percent of limited english proficiency
z_3	Percent of aboriginal students
z_4	Percent of special education students

[a]For each grade, tests are given in Reading, Writing, Spelling, Grammar and Numeracy. We average across tests for each grade to obtain our outcome measures. Inputs are measured per student in the DEA models

in each given grade.[1] Given that there is no overlap between primary and secondary schools, we apply the models separately; for each application, we use two output measures.

For school inputs we use the number of full-time equivalent teachers and support personnel (SASS) and other school related expenses. Average salaries for teachers and support personnel are the prices used for the allocative efficiency model. We assume that the other expenses are available to all schools at the same price and normalize the price of other expenses to unity for all schools. In addition, because larger schools will necessary have more inputs, we measure our inputs on a per student basis.

We also include four socio-economic variables: the Index of Community Socio-Educational Advantage (ICSEA), the percent of students that have limited English proficiency, the percent of students that are aboriginal and the percent of students that require special education services. Given that we have multiple environmental variables, we use multiple-stage models to reduce the dimensionality by creating an aggregate index of environmental harshness. Variable descriptions are reported in Table 4.1.

Descriptive statistics for the primary (secondary) schools are reported in Table 4.2 (Table 4.3). We report mean and standard deviations for the variables

[1] The correlations between tests scores for the primary schools for grade 3 (5) were all above 0.94 (0.92). For the secondary schools, the correlations were above 0.92 (0.90) for the seventh (ninth) grade scores.

Table 4.2 Primary school descriptive statistics[a]

		Enrollment quintiles				
Variable	All schools	I	II	III	IV	V
N	1,341	269	271	267	268	266
y_1	397.9 (69.2)	358.9 (132.7)	398.7 (32.4)	401.9 (33.4)	410.9 (31.1)	419.4 (32.0)
y_2	479.6 (76.7)	434.9 (150.0)	481.0 (31.8)	484.0 (31.3)	494.9 (31.1)	503.8 (33.9)
x_1	18.27 (11.3)	4.39 (2.0)	11.76 (3.3)	17.42 (3.5)	22.95 (4.0)	35.09 (7.3)
x_2	3.39 (1.9)	1.49 (0.7)	2.60 (1.3)	3.57 (1.6)	4.14 (1.8)	5.18 (1.7)
x_3	749,386 (379,931)	287,998 (145,261)	565,199 (187,109)	748,819 (199,207)	897,746 (179,184)	1,254,718 (267,104)
p_1	110,856 (12,077)	121,845 (16,778)	110,656 (10,102)	108,457 (9,016)	107,801 (7,231)	105,434 (6,873)
p_2	78,906 (22,573)	78,114 (24,458)	85,122 (24,953)	82,578 (23,539)	77,271 (18.988)	71,335 (17,416)
z_1	996.88 (93.9)	963.10 (82.7)	978.31 (91.9)	990.60 (95.3)	1,017.08 (88.3)	1,035.88 (92.6)
z_2	0.226 (0.27)	0.076 (0.13)	0.179 (0.22)	0.232 (0.26)	0.246 (0.26)	0.399 (0.32)
z_3	0.072 (0.11)	0.101 (0.15)	0.088 (0.12)	0.080 (0.11)	0.055 (0.07)	0.036 (0.05)
z_4	0.044 (0.05)	0.042 (0.05)	0.052 (0.05)	0.053 (0.05)	0.043 (0.04)	0.032 (0.03)
Enrollment	310.67 (198.9)	69.42 (26.3)	182.00 (29.6)	282.75 (28.6)	399.58 (47.4)	624.18 (108.9)

[a]Mean (standard deviations) are reported. The price p_3 was set to unity

Table 4.3 Secondary school descriptive statistics[a]

Variable	All schools	Enrollment quintiles I	II	III	IV	V
N	371	75	74	74	74	74
y_1	530.9 (42.2)	506.1 (27.9)	516.7 (23.1)	536.7 (47.8)	549.9 (50.3)	545.3 (38.2)
y_2	566.4 (40.8)	544.5 (26.8)	548.5 (21.3)	572.9 (45.4)	585.3 (48.5)	580.9 (37.2)
x_1	59.5 (18.5)	35.21 (8.4)	52.10 (4.8)	60.62 (11.4)	66.47 (3.7)	83.80 (13.7)
x_2	12.7 (3.6)	8.71 (2.6)	11.93 (2.0)	12.93 (3.0)	13.73 (2.1)	16.46 (3.3)
x_3	2,324,665 (749,765)	1,721,904 (941,059)	2,113,226 (469,144)	2,308,895 (345,654)	2,572,923 (551,099)	2,914,524 (688,633)
p_1	106,419 (6,568)	106,259 (7,738)	106,652 (7,300)	106,537 (6,289)	107,645 (5,489)	105,006 (5,595)
p_2	64,029 (14,224)	68,630 (23,786)	64,510 (11,175)	61,221 (9,589)	64,028 (10,741)	61,694 (9,175)
z_1	979.006 (82.01)	924.137 (81.46)	957.92 (54.17)	994.33 (82.84)	1,005.29 (76.31)	1,014.09 (77.54)
z_2	0.28 (0.3)	0.19 (0.3)	0.25 (0.3)	0.30 (0.29)	0.28 (0.29)	0.38 (0.33)
z_3	0.065 (0.09)	0.138 (0.17)	0.069 (0.05)	0.047 (0.04)	0.043 (0.03)	0.028 (0.03)
z_4	0.043 (0.03)	0.062 (0.05)	0.052 (0.03)	0.041 (0.03)	0.035 (0.02)	0.027 (0.02)
Enrollment	779.23 (301.2)	348.07 (109.7)	623.01 (56.9)	778.17 (45.5)	951.13 (48.5)	1,186.55 (161.7)

[a]Mean (standard deviations) are reported. The price p_3 was set to unity

used in the empirical analyses of primary schools. We also provide mean and standard deviations by enrollment quintile. For the primary schools, we note that the average outcomes and inputs increase as the enrollment increases from quintile I (smallest schools) to quintile V (largest schools). In addition, the ICSEA (a measure of school advantage) also increases on average as enrollment increases. This suggests that larger schools tend to have a more favorable operating environment. However, we also observe that as we increase enrollment, the percent of limited English proficiency students also increases. Similar patters emerge in Table 4.3 for the secondary schools. In general, as the average size of the school increases, the discretionary inputs and the ICSEA increase. This holds true for the outcome measures as well, except for the highest enrollment quintile where there is a slight decrease in the average scores.

In the next section we present the input-oriented models to analyze technical, scale and allocative efficiency the primary and secondary schools.

4.2 Empirical Input-Oriented Models

4.2.1 Indexing the Socio-Economic Environment

Given that we have multiple nondiscretionary factors, we employ the multiple stage models to reduce the dimensionality. Primary and secondary schools are analyzed separately and provide the general models used for both. We assume that each of the schools use three inputs ($x_1, x_2,$ and x_3) to produce two outcomes (y_1 and y_2) given four nondiscretionary inputs (z_1, z_2, z_3 and z_4), all of which are described in Table 4.1. For each school j inputs, outputs and nondiscretionary variables are given by x_{ij} for ($i = 1, 2, 3$), y_{ij} for ($i = 1, 2$) and z_{ij} for ($i = 1,.., 4$). The first stage index (FS_j) for each school j is obtained as the solution to the following linear program:

$$
\begin{aligned}
&FS_j\left(Y_j, X_j\right) = \min \theta \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, 2; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, 2, 3; \\
&\qquad \sum_{i=1}^{n} \lambda_i = 1; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
\tag{4.1}
$$

Table 4.4 Primary school regression results[a] (N = 1,341)

Variable	Coefficient
Intercept	0.361** (0.045)
z_1*	0.520 (0.043)
z_2	−0.081** (0.011)
z_3	−0.176** (0.037)
z_4	−1.016*** (0.064)
R^2	0.464**

All calculations by authors
[a]Standard errors are reported in parentheses
*Measured in 000s
**Indicates significance at the 99 % level of confidence
***Indicates significance at the 95 % level of confidence

Table 4.5 Secondary school regression results[a] (N = 371)

Variable	Coefficient
Intercept	0.321** (0.100)
z_1*	0.572** (0.010)
z_2	−0.059** (0.020)
z_3	−0.364** (0.089)
z_4	−0.438*** (0.194)
R^2	0.365**

All calculations by authors
[a]Standard errors are reported in parentheses
*Measured in 000s
**Indicates significance at the 99 % level of confidence
***Indicates significance at the 95 % level of confidence

In the second stage, the following regression is applied:

$$FS_j = \alpha + \sum_{i=1}^{4} \beta_i z_{ij} + \varepsilon_{j.} \tag{4.2}$$

OLS was used to estimate the second stage regression. Our first nondiscretionary factor, the Index of Community Socio-Educational Advantage (ICSEA) provides a scale measuring the advantage of students in a given community. Higher values indicate more advantage. We expect the coefficient on this variable to be positive: higher levels of this index should lead to better outcomes for a given level of discretionary inputs. Therefore, schools with a higher index value should be closer to the overall best practice frontier, i.e. have a higher value of the first stage index. It is also expected that the coefficients on the other nondiscretionary factors (percent of limited English proficiency, percent of aboriginal students and percent of special educations) will have a negative coefficient: increases in these variables represent a harsher environment and hence, a lower value of the first stage index.

The regression results for primary (secondary) schools are reported in Table 4.4 (Table 4.5). For both the primary and secondary schools, the results are economically significant; the sign of the coefficients all match our expectations. Nearly half

(46.4 %) of the variation in the first stage index can be explained by the nondiscretionary factors for the primary schools. For the secondary schools, only 36.6 % of the variation can be explained. This implies that failure to control for the environment will lead to biased efficiency estimates; schools with a harsher environment will be deemed to be more inefficient than they actually are without controlling for the environment. All of the parameters for both regressions are significant at the 95 % level of confidence; all parameters except the coefficient on z_3 are significant at the 99 % level.

The predicted value of the first stage index captures the effect that the environment has on the production process. Importantly, the slope parameters provide weights to evaluate the importance each nondiscretionary factor has on the production process. For each school district, we calculate this predicted value as:

$$\hat{z}_j = \alpha + \sum_{i=1}^{4} \beta_i z_{ij}. \tag{4.3}$$

This index will be incorporated into the DEA models to control for the environment while estimating technical, allocative and scale efficiency.

4.2.2 Technical Efficiency

Given the overall index capturing the influence of the nondiscretionary variables on the production process, we can now measure the technical efficiency of primary school j with the following linear program:

$$\begin{aligned}
&D_V^I\left(Y_j, X_j, \hat{z}_j\right) = \min \theta \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{N} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, 2; \\
&\qquad \sum_{i=1}^{N} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, 2, 3; \\
&\qquad \sum_{i=1}^{N} \lambda_i = 1; \\
&\qquad \lambda_i = 0 \ \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned} \tag{4.4}$$

The solution of Eq. (4.4) provides the maximum equiproportional reduction in all inputs consistent with observed production conditional on the operating environment. Further, the solutions of Eqs. (4.1) and (4.4) provide a measure of the environmental harshness for school j, defined as the ratio of the two distance functions:

$$ES_V^I\left(Y_j, X_j, z_j\right) = \frac{FS_j\left(Y_j, X_j\right)}{D_V^I\left(Y_j, X_j, \hat{z}_j\right)} \leq 1, \tag{4.5}$$

4.2.3 Scale Efficiency

Removing the convexity constraint from Eq. (4.4) allows us to project each primary school j to the constant returns to scale isoquant while controlling for the aggregated index of the nondiscretionary inputs:

$$\begin{aligned}
&D_C^I\left(Y_j, X_j, \hat{z}_j\right) = \min \theta \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{N} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, 2; \\
&\qquad \sum_{i=1}^{N} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, 2, 3; \\
&\qquad \lambda_i = 0 \ \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned} \tag{4.6}$$

From the solutions of Eqs. (4.4) and (4.6) we can calculate the scale efficiency of each primary school j as the ratio of the distance functions:

$$SE^I\left(Y_j, X_j, \hat{z}_j\right) = \frac{D_C^I\left(Y_j, X_j, \hat{z}_j\right)}{D_V^I\left(Y_j, X_j, \hat{z}_j\right)}. \tag{4.7}$$

Here, the scale efficiency measure indicates the proximity each primary school is away from most productive scale size.

4.2.4 Allocative Efficiency

For inputs x_1 and x_2 (teachers and support personnel) we use average prices p_1 and p_2 as our price measures. Given that we use other expenses for x_3 we assume that the resources purchased are obtained competitively and that each school can purchase these at the same price. Given this assumption, we normalize the price $p_3 = 1$ for all schools. For school j we observe prices p_{1j}, p_{1j} and p_{3j} and define observed expenditures as $E_j = \sum_{l=1}^{3} p_{lj} x_{lj}$.

To obtain the measure cost efficiency, we solve the linear programming model conditional on the aggregate nondiscretionary input $\hat{z}_j$ to obtain minimum cost for each school j as:

$$
\begin{aligned}
&C_j^* = \min \sum_{m=1}^{3} p_{mj} x_m \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{N} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, 2; \\
&\qquad \sum_{i=1}^{N} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, 3; \\
&\qquad \sum_{i=1}^{N} \lambda_i = 1; \\
&\qquad \lambda_i = 0 \ \text{if} \ \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{4.8}
$$

From the solution of Eq. (4.8) we obtain an optimal vector of inputs $(x_{1j}^*, x_{2j}^*, x_{3j}^*)$ for each school j that minimizes the costs of production. We measure cost efficiency as the ratio of minimum costs to observed expenditures:

$$
CE(Y_j, X_j, \hat{z}_j) = \frac{C_j^*}{E_j}. \tag{4.9}
$$

Furthermore, given observed expenditures, we can define wasted expenditures as the difference between observed expenditures and the minimum cost of producing the observed outcomes:

$$
W(Y_j, X_j, \hat{z}_j) = E_j - C_j^*, \tag{4.10}
$$

which arises from technical and allocative inefficiency. Given the technical and cost efficiency measures $D_V^I(Y_j, X_j, \hat{z}_j)$ and $CE(Y_j, X_j, \hat{z}_j)$ we obtain a measure of allocative efficiency via the Farrell decomposition as:

$$
AE(Y_j, X_j, \hat{z}_j) = \frac{CE(Y_j, X_j, \hat{z}_j)}{D_V^I(Y_j, X_j, \hat{z}_j)}. \tag{4.11}
$$

We obtain the minimum cost of producing the observed output assuming the most favorable environment by solving the following linear program for each school j:

$$
\begin{aligned}
& C_j^{**} = \min \sum_{m=1}^{3} p_{mj} x_m \\
& \text{subject to} \\
& \qquad \sum_{i=1}^{N} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, \ldots, 2; \\
& \qquad \sum_{i=1}^{N} \lambda_i x_{mi} \leq x_m, \quad m = 1, \ldots, 3; \\
& \qquad \sum_{i=1}^{N} \lambda_i = 1; \\
& \qquad \lambda_i \geq 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{4.12}
$$

For this chapter, we define environmental costs as the difference between the minimum cost of production with and without controlling for the environment:

$$
\text{Environmental Costs} = C_j^{**} - C_j^{*} \tag{4.13}
$$

In the next section, we discuss the results for the primary schools.

4.3 Adequacy in a Production Context

Given the two outcomes selected in the empirical analysis for this chapter, we now define the adequate output levels as $Y^A = (y_1^A, y_2^A)$. We note that we can define the conditional input set and the isoquant associated with these outcome levels. For adequacy considerations, we are concerned with feasible projections not to the observed output isoquant but to the isoquant defined by the minimally acceptable levels. Given that we use a variable returns to scale technology conditional on the socio-economic environment, it is possible (and probable) that we cannot determine the minimum inputs and spending necessary to achieve adequacy. This will arise for schools with the harsher environments if there is a minimum environment associated with adequate output levels.

Following Ruggiero (2007a, 2007b) we solve the following linear program to determine the minimum expenditure (E_j^A) necessary to achieve the adequate outcomes for school j:

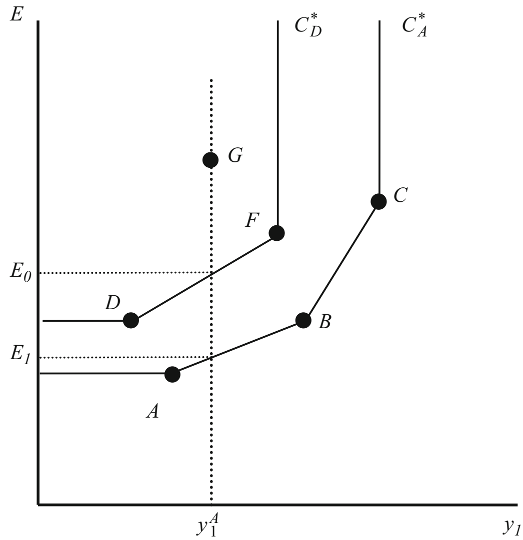


Fig. 4.1 Adequate expenditures

$$
\begin{aligned}
&E_j^A = \min E \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq y_s^A, \quad s = 1, 2; \\
&\qquad \sum_{i=1}^{n} \lambda_i E_i \leq E; \\
&\qquad \sum_{i=1}^{n} \lambda_i = 1; \\
&\qquad \lambda_i = 0 \ \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
\tag{4.14}
$$

We observe that we control for the environment using the aggregated index $\hat{z}_j$ calculated in the second stage regression. As shown, the model extends the basic DEA model by including the predefined adequate output levels instead of the observed levels in the output constraints.

We illustrate the estimation of the expenditures necessary to achieve the adequate outcomes in Fig. 4.1. Here we assume that five schools produce one outcome y_1 subject to two different operating environments. Schools A–C face the more favorable environment $(\hat{z}_1 > \hat{z}_0)$ and can therefore produce their outcomes at lower cost ($C_A^* < C_D^*$) than schools D, F and G can. The adequate level y_1^A of the outcome is also illustrated. We observe that schools A and D are not producing the adequate level because spending levels are insufficient. The solution to programming model (4.14) reveals that A would need to increase spending per pupil to E_1 in order to achieve the adequate outcome level. Schools D, F and G need to spend more ($E_0 > E_1$) given the harsher environment.

We also note that schools F and G, both of which face the harsher environment, are producing outcome levels at or above the adequate level. School G is observed to be inefficient, producing the adequate level of the outcome but doing so above minimum cost. Given the assumptions about resource prices, the excess spending observed for school G results from technical inefficiency. School F spends more than the minimum level but this is explained by production above the adequate level. The resulting policy implications are clear. Some schools (like A and D) need additional resources to meet the policy objective of meeting minimum outcome standards. Also, schools with harsher environments need additional resources to meet these goals. And, given that we seek minimum expenditure levels associated with the different socio-economic conditions, we have to control for inefficiency. We arbitrarily choose the median value of the two aggregate outcome measures as the desired adequate levels for illustrative purposes. Future adequacy analyses of Australian schools should identify the desired minimum standards. Based on our assumptions, we identify the adequate outcome levels $(y_1^A, y_2^A) = (405, 488)$ for the primary schools and $(y_1^A, y_2^A) = (522, 559)$ for the secondary schools.

4.4 Empirical Results

In Table 4.6 we report the results for the primary schools. Focusing on all schools, we note that the average spending per pupil was $10,766. Based on the DEA models, we estimate that the minimum cost of providing the observed outcomes is only $8,651 per pupil on average, leading to excess spending of over $2,100 per pupil. We also estimate that the cost of achieving the adequate outcome levels is $7,703 per pupil. In addition, the average school had to pay $1,381 extra to account for the environment. These environmental costs are estimated to be less than the waste associated with technical and allocative efficiency. The indices suggest that on average the primary schools are only 82.5 % cost efficient. Most of the inefficiency is due to technical inefficiency; on average schools are only 88.8 % efficient. Schools tend to be relatively scale efficient (96.7 %), operating close to most productive scale size. Further, schools tend to be using the relative correct of input mix, achieving a relatively high average allocative efficiency rating of 94.5 %.

Focusing on the enrollment quintiles, we observe that the smaller schools tend to spend more per pupil; the average school in the lowest enrollment quintile is spending $14,127 on average, which is approximately $5,500 more per pupil than the average school in the highest enrollment quintile. On average, these districts need to spend twice as much than the larger schools holding outcome provision constant. The driving factor for the environmental costs is the difference in the

Table 4.6 Primary school efficiency results[a]

Variable	All schools	Enrollment quintiles I	II	III	IV	V
Index						
Technical efficiency	0.888 (0.11)	0.815 (0.14)	0.858 (0.10)	0.890 (0.09)	0.922 (0.07)	0.954 (0.05)
Scale efficiency	0.967 (0.04)	0.958 (0.05)	0.955 (0.05)	0.967 (0.03)	0.976 (0.02)	0.981 (0.03)
Cost efficiency	0.825 (0.13)	0.702 (0.15)	0.788 (0.11)	0.839 (0.10)	0.881 (0.08)	0.919 (0.07)
Allocative efficiency	0.945 (0.06)	0.924 (0.10)	0.939 (0.05)	0.943 (0.05)	0.960 (0.03)	0.960 (0.04)
Environ. harshness	0.900 (0.12)	0.907 (0.12)	0.881 (0.14)	0.881 (0.13)	0.910 (0.10)	0.921 (0.09)
Expenditures/costs						
Observed expenditures	10,766 (3,395)	14,127 (4,692)	11,457 (2,788)	10,382 (2,224)	9,260 (1,573)	8,564 (1,171)
Tech. Eff. expenditures	9,389 (2,619)	11,243 (3,624)	9,809 (2,736)	9,214 (2,136)	8,504 (1,401)	8,154 (1,094)
Minimum costs	8,651 (2,329)	9,612 (3,347)	8,988 (2,541)	8,688 (2,127)	8,112 (1,362)	7,844 (1,060)
Wasted expenditures	2,114 (2,351)	4,515 (3,619)	2,469 (1,654)	1,694 (1,256)	1,148 (904)	720 (667)
Environmental costs	1,381 (2,114)	1,786 (3,064)	1,701 (2,404)	1,539 (1,972)	1,015 (1,280)	855 (974)
Adequacy	7,703 (1,780)	7,613 (1,566)	7,853 (1,882)	8,040 (2,241)	7,587 (1,641)	7,441 (1,428)

[a]Mean (standard deviations) are reported. Expenditures and costs are measured per pupil. All calculations by authors

ICSEA: smaller schools are operating in less advantageous environment. However, the differences in minimum cost are much less than the $5,500 per pupil difference in observed spending. The smaller schools are observed wasting approximately $4,500 per pupil; as a result, differences in average minimum costs between the smallest and largest schools are only $1,800 per pupil. Holding outcomes at the assumed adequate levels, we observe that the smaller schools in the lowest three quintiles need to pay more than the two larger quintiles. We note that we were not able to calculate adequacy results for 50 primary schools because many of the schools with the harshest environment do not meet our adequacy standards.

We note that the larger schools tend to be more technically efficient, with the highest quintile achieving an average technical efficiency rating of 95.4 %. The lowest quintile has the lowest technical efficiency rating of 81.5 % on average. In general, across all efficiency measures, we observe that smaller schools tend to be more inefficient, not only using relatively more inputs for the given output but also using the wrong mix of inputs given the observed input prices. The main cause of the excess spending by the smaller schools is the larger technical inefficiency relative to allocative efficiency.

For the secondary schools, we see similar results. We replicate the analysis in Table 4.6 for secondary schools and report the results in Table 4.7. The average minimum costs for all schools are nearly $11,000 per pupil, over $2,000 less than the observed expenditures per pupil. In order to meet our adequacy standards, schools would need to pay $9,851 per pupil on average.[2] On average, the technical efficiency of all schools is only 89.4 %. Like the primary schools, the secondary schools tend to be scale and allocative efficient. In terms of the enrollment quintiles, in general, we observe similar patterns. The larger schools tend to be more efficient technically, with the average efficiency of the lower quintile only 77.2 %. The average amount of technical efficiency increases as the quintiles increase, with the top two quintiles being relative efficient with an average rating of approximately 96.9 %. We observe similar patterns for the scale, cost and allocative efficiency as well. In terms of actual dollars, the smaller schools from the lowest quintile are observed wasting over $5,000 per pupil while the larger schools average under $800. Interestingly, the smaller schools also face the higher environmental costs, having to spend an additional $3,000 on average due to the harsher environment. In terms of achieving adequacy, the smaller schools need to pay over $1,000 more per student than the larger schools.

In Tables 4.8 and 4.9 we report individual results for school 1118. In Table 4.8, we compare the inputs, outputs, prices and efficiency results for school 1118 and the three schools that comprise the benchmark using the technical efficiency model under variable returns to scale.[3] We note that the benchmark schools (1297, 1553

[2] We were unable to calculate adequacy costs for 37 secondary schools; 23 of these schools were in the lowest enrollment quintile.

[3] The inputs used in the DEA models are measured per pupil.

Table 4.7 Secondary school efficiency results[a]

		Enrollment quintiles				
Variable	All schools	I	II	III	IV	V
Index						
Technical efficiency	0.894 (0.11)	0.772 (0.15)	0.852 (0.07)	0.910 (0.07)	0.968 (0.03)	0.969 (0.04)
Scale efficiency	0.960 (0.04)	0.947 (0.06)	0.959 (0.04)	0.959 (0.03)	0.963 (0.03)	0.970 (0.03)
Cost efficiency	0.848 (0.13)	0.722 (0.18)	0.800 (0.09)	0.860 (0.09)	0.929 (0.05)	0.931 (0.07)
Allocative efficiency	0.945 (0.06)	0.924 (0.10)	0.939 (0.05)	0.943 (0.05)	0.960 (0.03)	0.960 (0.04)
Environ. harshness	0.921 (0.10)	0.840 (0.17)	0.912 (0.079)	0.936 (0.06)	0.959 (0.03)	0.961 (0.04)
Expenditures/costs						
Observed expenditures	13,281 (4,194)	18,532 (6,285)	13,606 (1,686)	12,298 (1,904)	11,167 (919)	10,732 (1,123)
Tech. Eff. expenditures	11,618 (2,972)	14,211 (5,455)	11,556 (1,564)	11,107 (1,099)	10,808 (940)	10,372 (809)
Minimum costs	10,949 (2,839)	13,093 (5,381)	10,843 (1,496)	10,469 (1,194)	10,375 (990)	9,938 (713)
Wasted expenditures	2,331 (3,400)	5,438 (5,939)	2,763 (1,525)	1,829 (2,014)	791 (588)	794 (973)
Environmental costs	1,145 (2,460)	2,988 (4,814)	1,092 (1,135)	744 (851)	457 (454)	417 (546)
Adequacy	9,851 1,709	10,303 (2,229)	10,497 (2,104)	9,881 (1,704)	9,474 (997)	9,291 (1,106)

[a]Mean (standard deviations) are reported. Expenditures and costs are measured per pupil. All calculations by authors

Table 4.8 Technical efficiency analysis of school 1118

		Benchmark schools		
	1118	1297	1553	4034
Weight	–	0.164	0.018	0.819
x_1	10.65	4.54	4.54	23.00
x_2	2.02	1.71	1.61	3.48
x_3	\$507,977	\$300,043	\$258,548	\$1,012,959
y_1	414.20	423.00	382.40	443.60
y_2	507.40	499.20	488.40	526.40
z_1	1,109.38	1,002.77	1,001.66	1,076.40
z_2	0.048	0.049	0.059	0.051
z_3	0.016	0.029	0.069	0.010
z_4	0.059	0.049	0.040	0.075
Enrollment	186	103	101	493
$\hat{z}$	0.858	0.818	0.818	0.828
p_1	\$125,269	\$151,497	\$129,341	\$118,580
p_2	\$88,661	\$69,204	\$77,381	\$117,108
Index				
Tech. efficiency	0.807	1.000	1.000	1.000
Scale efficiency	0.972	1.000	0.971	1.000
Environ. Harsh.	1.000	1.000	1.000	1.000

All calculations by authors

and 4034) all have an aggregate nondiscretionary index $\hat{z}$ that is lower than school 1118's; as such each of these schools can serve as proper benchmarks with environments no better than school 1118's. This results because school 1118 has the highest level of the ICSEA index. School 4645 has the highest weight (0.819) and therefore has the most influence in determining the technical efficiency of school 1118.

School 1118 is only 80.7 % technically efficient, suggesting that it could decrease all of its inputs (measured per pupil) by nearly 20 % and still achieve the same level of output. All benchmark schools are technically efficient. Like school 1118, school 1553 is the only benchmark that is scale inefficient. Based on the solutions to the linear programs, all schools are operating on the most favorable environment.

In Table 4.9, we analyze the results of school 1118 and the two benchmark schools (4617 and 4642) found in the solution to the cost efficiency model. We observe that the school 1118 is not only 80.7 % technically efficient but it is also only 95.1 % allocatively inefficient. School 1118 is primarily operating off the frontier but is also choosing the wrong mix of inputs. The overall amount of waste is over \$2,500 per pupil. Both of the benchmark schools (4617 and 4645) are technically, scale and allocatively efficient and have environments that are worse than school 1118's. Both of the benchmark schools are larger and the primary

Table 4.9 Cost efficiency analysis of school 1118

		Benchmark schools	
Variable	1118	4617	4642
Weight	–	0.757	0.243
x_1	10.65	39.80	23.00
x_2	2.02	4.87	3.53
x_3	$507,977	$1,371,238	$672,217
y_1	414.20	424.60	420.00
y_2	507.40	515.80	481.20
z_1	1,109.38	1,033.2	1,019.94
z_2	0.048	0.041	0.077
z_3	0.016	0.016	0.041
z_4	0.059	0.030	0.032
Enrollment	186	811	469
$\hat{z}$	0.858	0.854	0.838
p_1	$125,269	$111,670	$92,992
p_2	$88,661	$84,694	$80,538
Index			
Tech. efficiency	0.807	1.000	1.000
Scale efficiency	0.972	1.000	1.000
Alloc. efficiency	0.951	1.000	1.000
Cost efficiency	0.768	1.000	1.000
Environ. Harsh.	1.000	0.969	0.993
Expenditures/costs			
Observed Expend.	$10,865	$7,679	$6,600
Tech. Eff. Expend.	$8,768	$7,679	$6,600
Minimum costs	$8,340	$7,679	$6,600
Wasted Expend.	$2,525	$0	$0
Environ. costs	368	383	223
Adequacy	6,640	6,640	6,642

All calculations by authors. Expenditures and minimum costs are measured per pupil

benchmark school 4617 (with a weight of 0.757) has a worse index of community socio-educational advantage but produces higher outcomes with lower per pupil levels of the school inputs. We also note that because school 1118 and its benchmark schools have similar environmental harshness indices the cost of achieving adequacy is virtually identical for all three schools.

Chapter 5
Output-Oriented Efficiency Measures in Australian Schools

In accordance with Chap. 4, in this chapter we apply the DEA models presented in Chap. 3. However this chapter will focus on the output-oriented models to measure technical and scale efficiency of the primary and secondary schools in Australia. We will again use data from school year 2009–2010 identical to the data discussed in Chap. 4. In the following section, we present the output-oriented models.

5.1 Empirical Output-Oriented Models

5.1.1 *Indexing the Socio-Economic Environment*

The model below (4.1) was applied separately to both primary and secondary schools. We use two outputs (y_1 and y_2) that are produced by the three inputs (x_1, x_2, and x_3) under four nondiscretionary inputs (z_1, z_2, z_3 and z_4), all defined in Table 4.1. The first stage index (FS_j) for each school j is found by the solution to the following linear program Eq. (5.1). The denotations: x_{ij} for ($i = 1, 2, 3$), y_{ij} for ($i = 1, 2$) and z_{ij} for ($i = 1,.., 4$) represent the inputs, outputs, and nondiscretionary variables for each school.

$$\begin{aligned} & FS_j\left(Y_j, X_j\right) = \max \theta \\ & \text{subject to} \\ & \qquad \sum_{i=1}^{n} \lambda_i y_{si} \geq \theta y_{sj}, \quad s = 1, 2; \\ & \qquad \sum_{i=1}^{n} \lambda_i x_{mi} \leq x_{mj}, \quad m = 1, 2, 3; \\ & \qquad \sum_{i=1}^{n} \lambda_i = 1; \\ & \qquad \lambda_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{5.1}$$

V. Blackburn et al., *Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis*, DOI 10.1007/978-1-4899-7469-3_5,

Table 5.1 Primary school regression results[a] (N = 1,341)

Variable	Coefficient
Intercept	0.326** (0.019)
z_1*	0.536** (0.018)
z_2	−0.004 (0.00)
z_3	0.013 (0.02)
z_4	−0.191** (0.03)
R^2	0.625**

All calculations by authors
[a] Standard errors are reported in parentheses
* Measured in 000s
** Indicates significance at the 99 % level of confidence

Table 5.2 Secondary school regression results[a] (N = 371)

Variable	Coefficient
Intercept	0.173** (0.049)
z_1*	0.652** (0.047)
z_2	0.025** (0.009)
z_3	0.106*** (0.043)
z_4	−0.417** (0.094)
R^2	0.577**

All calculations by authors
[a] Standard errors are reported in parentheses
* Measured in 000s
** Indicates significance at the 99 % level of confidence
*** Indicates significance at the 95 % level of confidence

In the second stage, the following OLS regression is applied:

$$FS_j = \alpha + \sum_{i=1}^{4} \beta_i z_{ij} + \varepsilon_{j.} \tag{5.2}$$

Depending on the socio economic measurement under examination, we will expect either positive or negative coefficients. For the Index of Community Socio-Educational Advantage (ICSEA), just as with the input-oriented measurements, we will expect that this will have a positive coefficient because higher levels of this index should lead to better outcomes. Higher values indicate more advantage, meaning the schools with higher FSI should be expected to have a higher ICSEA. This is opposite with the other nondiscretionary factors (percent of limited English proficiency, percent of aboriginal students and percent of special educations). When these measurements are higher the school's FSI should be lower, these coefficients should be negative (reference Table 4.3).

The regression results for both primary and secondary schools are reported above in Tables 5.1 and 5.2, respectively. Over half (0.625) of the variation in the FSI for the primary schools can be explained by the nondiscretionary inputs.

The coefficients, aside from z_3, matched our predictions; making z_1, z_2, and z_4 economically significant. This failure to be economically significant could be contributed to z_3 being highly correlated with another environmental variable.

For the regression on the secondary schools, we found that z_2 and z_3 were not economically significant, while z_1 and z_4 were. These insignificances could also be contributed to a high correlation with other variables. The nondiscretionary inputs for the secondary schools also explain slightly over half (0.577) of the variation in the FSI.

From these results we believe that without controlling for the environment some schools with a harsher environment will have a lower efficiency measurement than they deserve. To solve this we created the Eq. (5.3), which will be a predicted value of the FSI that captures the effect that the environment has on the data points. We found the $\hat{z}_j$ for each school district.

$$\hat{z}_j = \alpha + \sum_{i=1}^{4} \beta_i z_{ij}. \tag{5.3}$$

This index will be incorporated into the DEA models to control for the environment while estimating technical and scale efficiency.

5.1.2 Technical Efficiency

While using the predicted value of z, we can now use the modeling in Eq. (5.4) to find the technical efficiency of primary school j. This model will result in a maximum equiproportional expansion in all the inputs consistent with observed production conditional on the operating environment.

$$\begin{aligned} & D_V^I\left(Y_j, X_j, \hat{z}_j\right) = \max\theta \\ & \text{subject to} \\ & \qquad \sum_{i=1}^{N} \lambda_i y_{si} \geq \theta y_{sj}, \quad s = 1, 2; \\ & \qquad \sum_{i=1}^{N} \lambda_i x_{mi} \leq x_{mj}, \quad m = 1, 2, 3; \\ & \qquad \sum_{i=1}^{N} \lambda_i = 1; \\ & \qquad \lambda_i = 0 \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\ & \qquad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{5.4}$$

A ratio of the results in the models shown in Eqs. (5.1) and (5.4) will produce a measurement of the environmental harshness, shown in Eq. (5.5).

$$ES_V^I(Y_j, X_j, z_j) = \frac{FS_j(Y_j, X_j)}{D_V^I(Y_j, X_j, \hat{z}_j)} \leq 1, \tag{5.5}$$

5.1.3 Scale Efficiency

The model in Eq. (5.6) removes the convexity constraint from the Eq. (5.4). This allows us to project the point to the constant returns to scale isoquant, while controlling for the $\hat{z}$ variable.

$$\begin{aligned} &D_C^I(Y_j, X_j, \hat{z}_j) = \min \theta \\ &\text{subject to} \\ &\quad \sum_{i=1}^{N} \lambda_i y_{si} \geq y_{sj}, \quad s = 1, 2; \\ &\quad \sum_{i=1}^{N} \lambda_i x_{mi} \leq \theta x_{mj}, \quad m = 1, 2, 3; \\ &\quad \lambda_i = 0 \text{ if } \hat{z}_i > \hat{z}_j, \quad i = 1, \ldots, N; \\ &\quad \lambda_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{5.6}$$

To create the scale efficiency measurement of each school j we create a ratio of the solutions found in models Eqs. (5.4) and (5.6). The solution to function in Eq. (5.7) is the closeness each school is to the most productive scale size. This is shown in the following distance function:

$$SE^I(Y_j, X_j, \hat{z}_j) = \frac{D_C^I(Y_j, X_j, \hat{z}_j)}{D_V^I(Y_j, X_j, \hat{z}_j)}. \tag{5.7}$$

5.2 Empirical Efficiency Results

The primary schools' efficiency data is reported in Table 5.3. This index shows that overall the schools are 94.6 % technically efficient and operate at a 90.1 environmental harshness. Most of the inefficiency can be found in the scale efficiency measurement where the schools operate at 89.9 % efficiency to the most productive point. The index also shows that the quintiles grow progressively more efficient in every measurement, showing that the harsher the environment the less efficient a school will be.

The secondary school results are found in Table 5.4. This table shows the secondary schools having 97.1 % technical efficiency, along with 83.1 % environmental harshness. The environmental harshness is where the secondary schools lack in efficiency the most. They operate at 89.9 % scale efficiency. Just as the primary results show, as the quintiles increase as does the efficiency. Proving again that the harsher environment that a school has the less efficient it will be. In Table 5.5 an assessment of school 8288 in comparison with its benchmark school 8121 is shown.

Table 5.3 Primary school efficiency results[a]

Variable	All schools	Enrollment quintiles I	II	III	IV	V
Index						
Technical efficiency	0.946 (0.05)	0.921 (0.07)	0.937 (0.04)	0.945 (0.04)	0.958 (0.03)	0.971 (0.03)
Scale efficiency	0.899 (0.10)	0.857 (0.12)	0.870 (0.10)	0.897 (0.09)	0.922 (0.08)	0.950 (0.058)
Environ. harshness	0.901 (0.06)	0.912 (0.06)	0.891 (0.06)	0.890 (0.06)	0.899 (0.06)	0.913 (0.06)

[a] Mean (standard deviations) are reported. Expenditures and costs are measured per pupil. All calculations by authors

Table 5.4 Secondary school efficiency results[a]

Variable	All schools	Enrollment quintiles I	II	III	IV	V
Index						
Technical efficiency	0.971 (0.03)	0.968 (0.03)	0.962 (0.03)	0.966 (0.03)	0.982 (0.03)	0.977 (0.03)
Scale efficiency	0.899 (0.12)	0.758 (0.14)	0.862 (0.07)	0.921 (0.07)	0.982 (0.02)	0.976 (0.04)
Environ. harshness	0.831 (0.08)	0.755 (0.05)	0.792 (0.05)	0.840 (0.07)	0.868 (0.07)	0.904 (0.07)

[a] Mean (standard deviations) are reported. Expenditures and costs are measured per pupil. All calculations by authors

Table 5.5 Technical efficiency analysis of school 8288

	8288	Benchmark school 8121
Weight	–	1.000
x_1	0.18	67.90
x_2	0.08	15.77
x_3	$6,932	$5,307,005
y_1	466.20	655.60
y_2	554.20	693.80
z_1	1,098.88	1,066.30
z_2	1.104	0.762
z_3	0.000	0.003
z_4	0.000	0.000
Enrollment	211	977
$\hat{z}$	0.917	0.888
p_1	$110,853	$107,860
p_2	$50,070	$92,695
Index		
Tech. efficiency	0.799	1.000
Scale efficiency	0.380	1.000
Environ. Harsh.	0.920	0.926

All calculations by authors

The inputs, outputs, prices, and efficiency measurements are shown for both schools. School 8288 is 79.9 % efficient, but most of their inefficiency is accounted for in their 38 % scale efficiency measurement. The benchmark and their school have nearly the same environmental harshness measurement, only differing by 0.006.

Chapter 6
Productivity Measurement

In this chapter we extend our analysis to panel data to allow us to measure productivity changes. We analyze shifts in the frontier with distance functions to measure productivity. The standard decomposition of the Malmquist Productivity Index (MPI) to measure efficiency change, technical change, and scale efficiency change. Following Brennan, Haelermans and Ruggiero (2013), we further decompose the Malmquist productivity index for public sector production characterized by the influence of environmental variables. We derive decomposed measures of technical, efficiency, scale, and environmental change and apply this decomposition to the 2008–2009 and 2009–2010 school years for both primary and secondary Australian public schools. In the next sections, we redefine our technology to be time specific and present our measures of the public sector Malmquist Productivity Index and its components. Much of the modeling and discussion in this chapter is borrowed from Brennan, Haelermans and Ruggiero (2013).

6.1 Technology

We extend our description of the environment to be time specific. Under the environment characterized by index z^t in time t $(t = T, T + 1)$,[1] we assume that each of n production units use a vector $X^t = (x_1^t, \ldots, x_m^t)$ of m discretionary inputs to produce a vector $Y^t = (y_1^t, \ldots, y_s^t)$ of s outputs. Data for each producer j in time t for $t = T$, $T + 1$ are given by $X_j^t = (x_{1j}^t, \ldots, x_{mj}^t)$, $Y_j^t = (y_{1j}^t, \ldots, y_{sj}^t)$ and z_j^t.[2]

[1] We adopt the convention of referring to specific time periods with upper case T and $T + 1$ and use lower case t to generically refer to the index value.

[2] Like in the two previous chapters, we assume only one nondiscretionary input in our exposition. In the empirical analysis we use multiple nondiscretionary inputs and employ OLS in the second stage to derive an overall index of environmental harshness.

V. Blackburn et al., *Nonparametric Estimation of Educational Production and Costs using Data Envelopment Analysis*, DOI 10.1007/978-1-4899-7469-3_6,

The empirical production possibility set assuming variable returns to scale defined in time t for $t=T, T+1$ is given by:

$$\begin{aligned}\tau_V^t(z^t) = \Big\{\big(Y^t, X^t, z^t\big) : &\sum_{i=1}^{n} \lambda_i y_{si}^t \geq y_s^t, \ \ s = 1, \ldots, S; \\ &\sum_{i=1}^{n} \lambda_i x_{mi}^t \leq x_m^t, \ \ m = 1, \ldots, M; \\ &\sum_{i=1}^{n} \lambda_n = 1; \\ &\lambda_i = 0 \ \text{if} \ \mathrm{z}_i^t > z^t, \ \ i = 1, \ldots, N; \\ &\lambda_i \geq 0, \ \ i = 1, \ldots, N\Big\}.\end{aligned} \tag{6.1}$$

The technology is the same as the technology defined in Chap. 3 for a given time period; in addition to variable returns to scale, the technology is conditional on the level of the nondiscretionary input observed in time t. We continue assuming a higher value of z implies a more favorable environment. By removing the convexity constraint from Eq. (6.1) we are able to define the technology in each time period t $(t=T, T+1)$ under constant returns to scale. Equation (6.2) shows the CRS technology:

$$\begin{aligned}\tau_C^t(z^t) = \Big\{\big(Y^t, X^t, z^t\big) : &\sum_{i=1}^{n} \lambda_i y_{si}^t \geq y_s^t, \ \ s = 1, \ldots, S; \\ &\sum_{i=1}^{n} \lambda_i x_{mi}^t \leq x_m^t, \ \ m = 1, \ldots, M; \\ &\lambda_i = 0 \ \text{if} \ \mathrm{z}_i^t > z^t, \ \ i = 1, \ldots, N; \\ &\lambda_i \geq 0, \ \ i = 1, \ldots, N\Big\}.\end{aligned} \tag{6.2}$$

We illustrate the VRS Eq. (6.1) and CRS Eq. (6.2) technologies in Fig. 6.1 where we assume only one discretionary input is used in the production of the one output.[3] For simplicity we also only show data point A in each time period T and $T+1$, denoted as A^T and A^{T+1}, respectively. We employ two vertical axes for clarity purposes. Additionally, we show two levels (z_A^t and z_B^t) of the environmental variable z in time t with $z_A^t < z_B^t$ for t $(t=T, T+1)$. The z_A^t and z_A^{t+1} represent the observed levels of the environmental variable for producer A, which faces a harsher environment than producer B. We observe four variable returns to scale frontiers in Fig. 6.1 associated with technologies $\tau_V^T(z_A^T)$ and $\tau_V^T(z_B^T)$ in time T and technologies $\tau_V^{T+1}(z_A^{T+1})$ and $\tau_V^{T+1}(z_B^{T+1})$ in time period $T+1$ for the two different observed

[3] This figure is borrowed from Brennan, Haelermans and Ruggiero (2013).

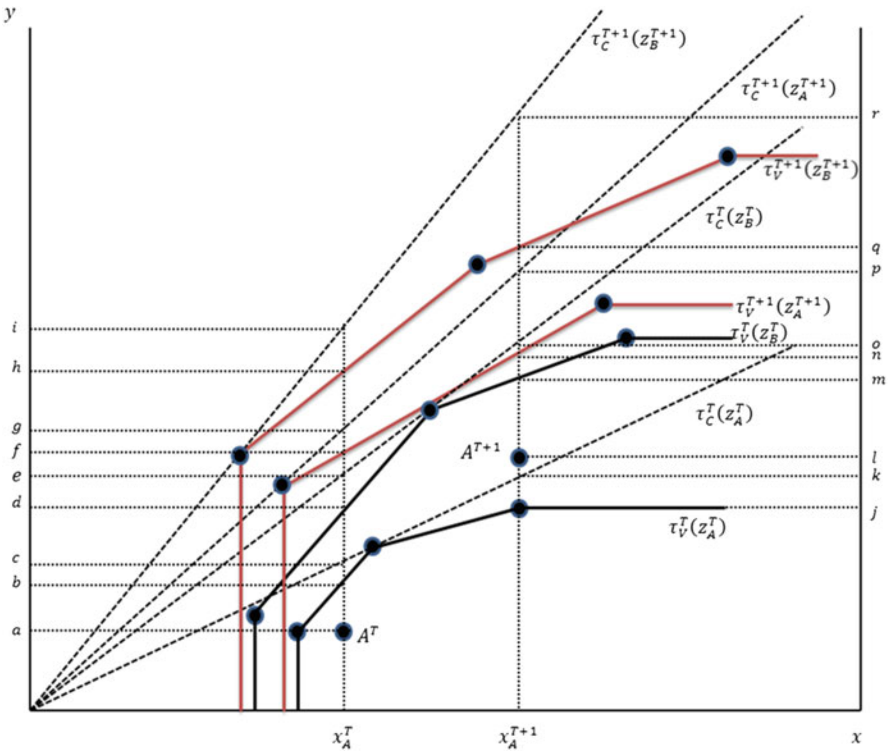


Fig. 6.1 VRS and CRS technologies across time. Reprinted from European Journal of Operational Research, 234/3, Brennan, Shae, Carla Haelermans, John Ruggiero, Nonparametric Estimation of Education Productivity Incorporating Nondiscretionary Inputs with an Application to Dutch Schools, page 809–818, 2014, with permission from Elsevier

levels of z. We also represent the associated CRS frontiers $\tau_C^T(z_A^T), \tau_C^T(z_B^T)$, $\tau_C^{T+1}(z_A^{T+1})$ and $\tau_C^{T+1}(z_B^{T+1})$ with dashed lines.

We also define the best-practice technologies for the units facing the most favorable environment, i.e., the unconditional technology that does not depend on the nondiscretionary index. For time period $t\,(t = T, T+1)$ under variable returns to scale:

$$\begin{aligned}\tau_V^t = \Big\{(Y^t, X^t) : &\sum_{i=1}^{n} \lambda_i y_{si}^t \geq y_s^t, \quad s = 1, \ldots, S;\\ &\sum_{i=1}^{n} \lambda_i x_{mi}^t \leq x_m^t, \quad m = 1, \ldots, M;\\ &\sum_{i=1}^{n} \lambda_i = 1;\\ &\lambda_i \geq 0, \quad i = 1, \ldots, N\Big\}\end{aligned} \tag{6.3}$$

and under constant returns to scale:

$$\tau_C^t = \Big\{(Y^t, X^t) : \sum_{i=1}^{n} \lambda_i y_{si}^t \geq y_s^t, \;\; s = 1, \ldots, S; \\ \sum_{i=1}^{n} \lambda_i x_{mi}^t \leq x_m^t, \;\; m = 1, \ldots, M; \\ \lambda_i \geq 0, \;\; i = 1, \ldots, N\Big\}. \tag{6.4}$$

These technologies provide the reference to project a given production possibility assuming the unit has the most favorable environment. In Fig. 6.1, we assume that $\tau_V^t = \tau_V^t(z_B^t)$ and $\tau_C^t = \tau_C^t(z_B^t)$ for $t = T, T+1$.

6.2 Malmquist Productivity Index

Productivity has a long history in the economic literature; measurement using parametric estimation of distance functions was provided by Caves et al. (1982). This was extended using nonparametric estimation by Färe et al. (1992). The nonparametric method allowed a further decomposition of the Malmquist Productivity Index (MPI) to analyze productivity in terms of changes in technical and scale efficiency and actual technical progress. Separating Malmquist productivity into the three components using the DEA approach was achieved by Fare and Grosskopf (1992), Färe et al. (1992, 1994) and Ray and Desli (1997) using an output orientation allowing variable returns to scale. Johnson and Ruggiero (2011) provided a decomposition assuming constant returns to scale useful for the public sector that measures how changes in the nondiscretionary operating environment impacts productivity. Brennan, Haelermans and Ruggiero (2013) extended the public sector decomposition to include the change in scale efficiency component. Johnson and Ruggiero's Environmental Malmquist Productivity Index (*EMPI*) is estimated using distance functions for producer *j* as:

$$EMPI\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}, Y_j^T, X_j^T, z_j^T\right) = \left[\frac{D_C^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_C^T\left(Y_j^T, X_j^T, z_j^T\right)} \; \frac{D_C^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_C^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}\right]^{\frac{1}{2}}. \tag{6.5}$$

All distance functions in Eq. (6.5) project to the constant returns to scale technologies defined in Eq. (6.2). We follow convention and use the geometric mean to avoid arbitrarily choosing a particular year as the benchmark technology. Following Brennan, Haelermans and Ruggiero (2013) the productivity index in Eq. (6.5) can be decomposed as:

$$EMPI\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}, Y_j^T, X_j^T, z_j^T\right) = \frac{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_V^T\left(Y_j^T, X_j^T, z_j^T\right)}$$

$$x\left[\frac{D_V^T\left(Y_j^T, X_j^T, z_j^T\right)}{D_V^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)} \frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}\right]^{\frac{1}{2}} \tag{6.6}$$

$$x\left[\frac{SE^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{SE^T\left(Y_j^T, X_j^T, z_j^T\right)} \frac{SE^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{SE^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}\right]^{\frac{1}{2}}.$$

This productivity index consists of the product of three terms:

$$\text{Technical Efficiency Change} = \frac{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_V^T\left(Y_j^T, X_j^T, z_j^T\right)}; \tag{6.7}$$

$$\text{Technical Change} = \left[\frac{D_V^T\left(Y_j^T, X_j^T, z_j^T\right)}{D_V^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)} \frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}\right]^{\frac{1}{2}}; \tag{6.8}$$

and

$$\text{Scale Efficiency Change} = \left[\frac{SE^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{SE^T\left(Y_j^T, X_j^T, z_j^T\right)} \frac{SE^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{SE^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}\right]^{\frac{1}{2}}. \tag{6.9}$$

Similar to Eq. (6.5) we use geometric means for the scale and technical change measurements to avoid arbitrarily choosing the time period or reference technology.

6.3 Technical Efficiency Change

The first component of the productivity index in Eq. (6.5) is the change in technical efficiency Eq. (6.7). Technical efficiency measures have already been defined in the previous chapters. Here, we will include definitions of the distance functions used to incorporate the time period references. We exclusively focus on output-oriented projections in this chapter.

Definition. $D_V^t(Y_j^t, X_j^t, z_j^t) = (\max\{\theta : (\theta Y_j^t, X_j^t, z_j^t) \in \tau_V^t(z_j^t)\})^{-1} \leq 1$ is the output-oriented measure of technical efficiency for $(X_j^t, Y_j^t, z_j^t) \in \tau_V^t(z_j^t)$ in time $t\,(t = T, T+1)$.

For each producer j in time t $(t = T, T+1)$ we solve the following linear programming model to obtain the measure of technical efficiency:[4]

$$
\begin{aligned}
& D_V^t\left(Y_j^t, X_j^t, z_j^t\right)^{-1} = Max\ \theta \\
& s.t. \\
& \qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^t, \quad k = 1, \ldots, s; \\
& \qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^t, \quad l = 1, \ldots, m; \\
& \qquad \sum_{i=1}^{n} \lambda_i = 1; \\
& \qquad \lambda_i = 0 \ \text{if} \ z_i^t > z_j^t, \quad i = 1, \ldots, n; \\
& \qquad \lambda_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
\tag{6.10}
$$

Returning to Fig. 6.1, we find the technical efficiency of A in time T $D_V^T\left(Y_A^T, X_A^T, z_A^T\right) = \frac{a}{b}$ and in time $T+1$ $D_V^{T+1}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right) = \frac{l}{m}$. The change in technical efficiency is then given as the ratio $\dfrac{D_V^{T+1}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right)}{D_V^T\left(Y_A^T, X_A^T, z_A^T\right)} = \dfrac{l}{m}\dfrac{b}{a}$. If the change in technical efficiency is greater (less) than unity, the unit became more (less) efficient and hence, more (less) productive.

6.4 Cross-Period Distance Functions

Before we turn to the technical and scale efficiency components of productivity, we first define the cross-period projections that are used in the calculation of both.[5] We are required to project the DMUs input-output combination in one period to the frontier in the other period. The cross-period projections are defined with the following distance functions:

[4] For each time period, model 6.10 is the same as model 3.11. The SAS code for estimation is provided in Chap. 3.

[5] Infeasibility can arise when projecting a given data point to the VRS technology in a different period. We report our results only for those units where cross-period projections are feasible. In addition, we only consider the shifts in the VRS frontiers for the technical change calculations.

$$D_V^t\left(Y_j^u,X_j^u,z_j^u\right) = \left(\max\left\{\theta : \left(\theta Y_j^u,X_j^u,z_j^u\right) \in \tau_V^t\left(z_j^u\right)\right\}\right)^{-1} \text{ for } t,u=T,T+1 \text{ and } t \neq u;$$
$$D_C^t\left(Y_j^u,X_j^u,z_j^u\right) = \left(\max\left\{\theta : \left(\theta Y_j^u,X_j^u,z_j^u\right) \in \tau_C^t\left(z_j^u\right)\right\}\right)^{-1} \text{ for } t,u=T,T+1 \text{ and } t \neq u;$$
$$D_V^t\left(Y_j^u,X_j^u\right) = \left(\max\left\{\theta : \left(\theta Y_j^u,X_j^u\right) \in \tau_V^t\right\}\right)^{-1} \text{ for } t,u=T,T+1 \text{ and } t \neq u;$$

and

$$D_C^t\left(Y_j^u,X_j^u\right) = \left(\max\left\{\theta : \left(\theta Y_j^u,X_j^u\right) \in \tau_C^t\right\}\right)^{-1} \text{ for } t,u=T,T+1 \text{ and } t \neq u.$$

The linear programming formulations associated with these eight distance functions for producer j in time $t,u=T,T+1$ and $t \neq u$ are as follows:

$$\begin{aligned}
&\left[D_V^t\left(Y_j^u,X_j^u,z_j^u\right)\right]^{-1} = Max\ \theta \\
&s.t. \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^u, \quad k=1,\ldots,s; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^u, \quad l=1,\ldots,m; \\
&\qquad \sum_{i=1}^{n} \lambda_i = 1; \\
&\qquad \lambda_i = 0 \text{ if } \mathrm{z}_i^t > z_j^u, \quad i=1,\ldots,n; \\
&\qquad \lambda_i \geq 0, \quad i=1,\ldots,n;
\end{aligned} \tag{6.11}$$

$$\begin{aligned}
&\left[D_C^t\left(Y_j^u,X_j^u,z_j^u\right)\right]^{-1} = Max\ \theta \\
&s.t. \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^u, \quad k=1,\ldots,s; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^u, \quad l=1,\ldots,m; \\
&\qquad \lambda_i = 0 \text{ if } \mathrm{z}_i^t > z_j^u, \quad i=1,\ldots,n; \\
&\qquad \lambda_i \geq 0, \quad i=1,\ldots,n;
\end{aligned} \tag{6.12}$$

$$
\begin{aligned}
&\left[D_V^t\left(Y_j^u, X_j^u\right)\right]^{-1} = Max\ \theta \\
&s.t. \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^u, \quad k = 1, \ldots, s; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^u, \quad l = 1, \ldots, m; \\
&\qquad \sum_{i=1}^{n} \lambda_i = 1; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, n;
\end{aligned}
\tag{6.13}
$$

and

$$
\begin{aligned}
&\left[D_C^t\left(Y_j^u, X_j^u\right)\right]^{-1} = Max\ \theta \\
&s.t. \\
&\qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^u, \quad k = 1, \ldots, s; \\
&\qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^u, \quad l = 1, \ldots, m; \\
&\qquad \lambda_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
\tag{6.14}
$$

The first two sets of distance functions (Eqs. (6.11) and (6.12)) project a given point in one time period to the frontier conditional on the environment in the other time period. The last two sets of distance functions (Eqs. (6.13) and (6.14)) project each point to the other time period's unconditional frontier. Referring to Fig. 6.1, we observe:

$$
\begin{aligned}
D_V^{T+1}\left(Y_A^T, X_A^T, z_A^T\right) &= \frac{a}{f} \\
D_V^{T}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right) &= \frac{l}{j} \\
D_C^{T+1}\left(Y_A^T, X_A^T, z_A^T\right) &= \frac{a}{g} \\
D_C^{T}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right) &= \frac{l}{k} \\
D_V^{T+1}\left(Y_A^T, X_A^T\right) &= \frac{a}{h} \\
D_V^{T}\left(Y_A^{T+1}, X_A^{T+1}\right) &= \frac{l}{m} \\
D_C^{T+1}\left(Y_A^T, X_A^T\right) &= \frac{a}{i} \\
D_C^{T}\left(Y_A^{T+1}, X_A^{T+1}\right) &= \frac{l}{o}.
\end{aligned}
$$

These measures are used in the technical change Eq. (6.8) and scale efficiency change Eq. (6.9) components of the productivity index. We now focus on the technical change measure.

6.5 Technical Change

Technical change measures the shift in the production frontier across time. Recognizing that frontier production is conditional on the environment, we can define technical change similar to Fare and Grosskopf (1992) as the ratio of distance functions:

Definition. $TC_V\left(Y_j^u, X_j^u, z_j^u\right) = \dfrac{D_V^T\left(Y_j^u, X_j^u, z_j^u\right)}{D_V^{T+1}\left(Y_j^u, X_j^u, z_j^u\right)}$ measures technical change for producer j observed in time u ($u = T, T+1$) under variable returns to scale.

For producer A in Fig. 6.1, we observe $TC_V\left(y_A^T, x_A^T, z_A^T\right) = \frac{f}{b} > 1$ and $TC_V\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right) = \frac{n}{j} > 1$. In our stylistic example, producer A experiences technical progress for the data point observed in both time periods. Using the geometric mean, we obtain the technical change component defined in Eq. (6.8). For producer A we observe:

$$\text{Technical Change} = \left[\frac{D_V^T\left(Y_A^T, X_A^T, z_A^T\right)}{D_V^{T+1}\left(Y_A^T, X_A^T, z_A^T\right)} \frac{D_V^T\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right)}{D_V^{T+1}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right)}\right]^{\frac{1}{2}}$$
$$= \left[\frac{f}{b}\frac{n}{j}\right]^{\frac{1}{2}} > 1.$$

The measure indicates that producer A experienced an improvement in productivity due to the expansion of its frontier from T to $T+1$. Brennan, Haelermans and Ruggiero (2013) provided a further decomposition of the technical change component to include the influence of the environment. In order to do so, we redefine the unconditional distance functions defined in Chap. 3 to include time references.

Definition. $D_V^t\left(Y_j^t, X_j^t\right) = \left(\max\left\{\theta : \left(\theta Y_j^t, X_j^t\right) \in \hat{\tau}_V^t\right\}\right)^{-1}$ is the output-oriented distance function projecting (X_j^t, Y_j^t, z_j^t) to the boundary of τ_V^t in time t ($t = T, T+1$).

The following linear programming model provides an empirical estimate of the above distance function for producer j in time t ($t = T, T+1$):

$$\begin{aligned}&\left[D_V^t\left(Y_j^t, X_j^t\right)\right]^{-1} = Max\ \theta\\&s.t.\\&\qquad \sum_{i=1}^{n}\lambda_i y_{ki}^{k} \geq \theta y_{kj}^{t},\ \ k = 1, \ldots, s;\\&\qquad \sum_{i=1}^{n}\lambda_i x_{li}^{t} \leq x_{lj}^{t},\ \ l = 1, \ldots, m;\\&\qquad \sum_{i=1}^{n}\lambda_i = 1;\\&\qquad \lambda_i \geq 0,\ \ i = 1, \ldots, n.\end{aligned} \tag{6.14}$$

Referring to Fig. 6.1, we find $D_V^T\left(y_A^T, x_A^T\right) = \frac{a}{d}$ and $D_V^{T+1}\left(y_A^{T+1}, x_A^{T+1}\right) = \frac{l}{q}$. Recalling our definition for environmental scale, for each time period t $(t = T, T+1)$ we can measure environmental harshness for each producer j as $E_V^t\left(Y_j^t, X_j^t, z_j^t\right) = \frac{D_V^t\left(Y_j^t, X_j^t\right)}{D_V^t\left(Y_j^t, X_j^t, z_j^t\right)}$. Returning to Fig. 6.1, we observe $E_V^T\left(y_A^T, x_A^T, z_A^T\right) = \frac{b}{d}$ and $E_V^{T+1}\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right) = \frac{n}{q}$. These measures capture the distance between the conditional and unconditional frontiers for each time period and represent the reduction in output that results from not having the most favorable environment *ceteris paribus*.

The Brennan, Haelermans and Ruggiero (2013) decomposition of the technical change component Eq. (6.8) is given as:

$$\begin{aligned}\text{Technical Change} &= \left[\frac{D_V^T\left(Y_j^T, X_j^T, z_j^T\right)}{D_V^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}\ \frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}\right]^{\frac{1}{2}}\\&= \left[\frac{D_V^T\left(Y_j^T, X_j^T\right)}{D_V^{T+1}\left(Y_j^T, X_j^T\right)}\ \frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}\right)}\right]^{\frac{1}{2}}\\&\quad x\left[\frac{E_V^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}{E_V^T\left(Y_j^T, X_j^T, z_j^T\right)}\ \frac{E_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{E_V^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}\right]^{\frac{1}{2}}.\end{aligned} \tag{6.15}$$

Technical change has two components according to Eq. (6.15):

$$\text{Technical Change MFE} = \left[\frac{D_V^T\left(Y_j^T, X_j^T\right)}{D_V^{T+1}\left(Y_j^T, X_j^T\right)}\ \frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}\right)}\right]^{\frac{1}{2}}; \tag{6.16}$$

and

$$\text{Change in Environmental Harshness} = \left[\frac{E_V^{T+1}\left(Y_j^T, X_j^T, z_j^T\right)}{E_V^T\left(Y_j^T, X_j^T, z_j^T\right)} \frac{E_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}{E_V^T\left(Y_j^{T+1}, X_j^{T+1}, z_j^{T+1}\right)}\right]^{\frac{1}{2}}. \tag{6.17}$$

Consider producer A in Fig. 6.1. For the technical change assuming the most favorable environment, we observe $\frac{D_V^T\left(Y_A^T, X_A^T\right)}{D_V^{T+1}\left(Y_A^T, X_A^T\right)} = \frac{h}{d}$, which indicates the increase in output that was possible for producer A's discretionary input-output (in time T) combination resulting from the shift in the technology across time under the assumption that A had the most favorable environment. Using the observed point in time $T+1$ instead, we find $\frac{D_V^T\left(Y_j^{T+1}, X_j^{T+1}\right)}{D_V^{T+1}\left(Y_j^{T+1}, X_j^{T+1}\right)} = \frac{q}{m}$. For the change in environmental harshness, we observe $\frac{E_V^{T+1}\left(Y_A^T, X_A^T, z_A^T\right)}{E_V^T\left(Y_A^T, X_A^T, z_A^T\right)} = \frac{f/h}{b/d}$. This measure reveals the degree to which the percentage shortfall in output due to the environment has changed using producer A's data from time period T. If $\frac{E_V^{T+1}\left(Y_A^T, X_A^T, z_A^T\right)}{E_V^T\left(Y_A^T, X_A^T, z_A^T\right)} > 1$, then the adverse effect of the environment on A's production has decreased from T to $T+1$ using the discretionary input level from time T. We also observe $\frac{E_V^{T+1}\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right)}{E_V^T\left(Y_A^{T+1}, X_A^{T+1}, z_A^{T+1}\right)} = \frac{n/q}{j/m}$. If this measure is less than unity, than the environment has a larger adverse impact on A's production using the data from time period $T+1$.

6.6 Scale Efficiency Change

The last component in the productivity index Eq. (6.5) is the change in scale efficiency Eq. (6.9). Extending our discussion from Chaps. 2 and 3, we include superscripts to represent each period under analysis and define the composed measure of technical and scale efficiency as the distance function from a given production possibility to the constant returns to scale technology conditional on the nondiscretionary index defined in Eq. (6.2):

Definition. $D_C^t(Y_j^t, X_j^t, z_j^t) = (\max\{\theta : (\theta Y_j^t, X_j^t, z_j^t) \in T_C^t(z_j^t)\})^{-1}$ is the output-oriented measure of technical and scale efficiency for $(X_j^t, Y_j^t, z_j^t) \in T_C^t(z_j^t)$ in time t $(t = T, T+1)$.

The following linear programming provides an estimate of technical and scale efficiency for producer j in time t ($t = T, T+1$):

$$\begin{aligned} &\left[D_C^t\left(Y_j^t, X_j^t, z_j^t\right)\right]^{-1} = Max\ \theta \\ &s.t. \\ &\qquad \sum_{i=1}^{n} \lambda_i y_{ki}^t \geq \theta y_{kj}^t, \quad k = 1, \ldots, s; \\ &\qquad \sum_{i=1}^{n} \lambda_i x_{li}^t \leq x_{lj}^t, \quad l = 1, \ldots, m; \\ &\qquad \lambda_i = 0 \text{ if } z_i^t > z_j^t, \quad i = 1, \ldots, n; \\ &\qquad \lambda_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{6.18}$$

For each period t, we obtain the measure of scale efficiency as the ratio of the distance function projecting the unit to the VRS frontier and the distance function projecting the unit to the CRS frontier.

Definition. $SE^t\left(Y_j^t, X_j^t, z_j^t\right) = \frac{D_C^t\left(Y_j^t, X_j^t, z_j^t\right)}{D_V^t\left(Y_j^t, X_j^t, z_j^t\right)} \leq 1$ is the scale efficiency for (X_j^t, Y_j^t, z_j^t) in time period t.

In Fig. 6.1 we observe $SE^T\left(y_A^T, x_A^T, z_A^T\right) = \frac{b}{c}$ in time T and $SE^{T+1}\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right) = \frac{n}{o}$ in time $T+1$. The scale efficiency measures are less than unity because producer A is not operating at most productive scale size in either time period. To compute the change in scale efficiency we also need to define the scale efficiency of each observation relative to the technology in the other period. From Eqs. (6.11) and (6.12), we obtain $SE^T\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right) = \frac{j}{k}$ and $SE^{T+1}\left(y_A^T, x_A^T, z_A^T\right) = \frac{d}{e}$. Using the technology in time period T we observe $\frac{SE^T\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right)}{SE^T\left(y_A^T, x_A^T, z_A^T\right)} = \frac{j}{k}\frac{c}{b}$. If this measure is greater than unity, A has chosen an input level that is closer to most productive scale size and hence, more scale efficient. Based on Fig. 6.1 it appears that the measure is less than unity since A moved farther away from the most productive scale size. Using the technology in time period $T+1$ we have $\frac{SE^{T+1}\left(y_A^{T+1}, x_A^{T+1}, z_A^{T+1}\right)}{SE^{T+1}\left(y_A^T, x_A^T, z_A^T\right)} = \frac{n}{o}\frac{e}{d}$. Using this benchmark technology, it appears that there is no change in scale efficiency. This results because the most productive scale size in time $T+1$ occurs at a higher input level. Using the geometric mean, we conclude that A has improved its scale efficiency.

We conclude by noting that the overall public sector productivity index (*EMPI*) consists of the following components: technical efficiency change, technical change assuming the most favorable environment, change in environmental harshness, and scale efficiency change. In the next section, we apply our measure to primary and secondary Australian schools using data from the 2008–2009 and 2009–2010 school years.

6.7 Application to Australian Schools

The data for this chapter's empirical analysis was described in Chap. 4. We measure the overall productivity index and its components using the variables defined Table 4.1. For the primary (secondary) schools, we use average third and fifth (seventh and ninth) grade scores on five tests: Reading, Writing, Spelling, Grammar and Numeracy. For both types of schools, we use full-time equivalent teachers and support staff and other expenses as our three discretionary inputs. We also use four nondiscretionary inputs: Index of Community Socio-Educational Advantage, percent of limited English proficiency, percent of Aboriginal students and the percent of special education students. Given the multiple nondiscretionary inputs, we construct an overall index using the multiple-stage model defined in Chap. 3.

Descriptive statistics for the 2009–2010 school years for the variables used in this analysis are reported in Tables 4.2 (primary schools) and 4.3 (secondary schools). We report the descriptive statistics for the relevant variables for each school year for both primary and secondary schools in Tables 6.1. The second-stage regressions for the 2009–2010 school year were reported in Tables 5.1 (primary schools) and 5.2 (secondary schools).[6]

Table 6.1 Descriptive statistics[a]

	Primary schools (N = 1,341)		Secondary schools (N = 371)	
Variable	2008–2009	2009–2010	2008–2009	2009–2010
y_1	400.2 (69.9)	397.9 (69.2)	529.8 (42.7)	530.9 (42.3)
y_2	479.5 (81.2)	479.6 (76.7)	570.5 (64.2)	566.4 (40.8)
x_1	0.061 (0.01)	0.061 (0.01)	0.081 (0.02)	0.082 (0.02)
x_2	0.014 (0.01)	0.014 (0.01)	0.018 (0.01)	0.018 (0.01)
x_3	3,124.8 (1,440.8)	2,913.0 (1,412.2)	3,568.2 (2,734.2)	3,407.8 (2,214.7)
z_1	0.840 (0.05)	0.852 (0.05)	0.789 (0.06)	0.807 (0.06)
Enrollment	309.40 (196.1)	310.67 (198.9)	773.70 (295.25)	776.23 (301.2)

[a] Mean (standard deviations) are reported. Discretionary inputs are measured per student. We report the aggregate nondiscretionary factor obtained from the second stage regression

[6] We use the second-stage regressions from the output oriented models (Chap. 5). The regressions are similar for both years and omit them from our empirical analysis. We also note that the overall indexes are highly correlated across years. This allows us to use the aggregated index across time periods.

Table 6.2 Average productivity results for primary schools

	All schools	Classification of environment		
		Harsh	Moderate	Favorable
Number of schools	1,135	378	379	378
EMPI	1.027	1.036	1.032	1.013
Efficiency change	0.988	0.979	0.996	0.989
Technical change	1.029	1.037	1.022	1.028
Technical change MFE	1.017	1.022	1.016	1.012
Change in Env. harshness	1.012	1.015	1.006	1.016
Scale efficiency change	1.010	1.019	1.014	0.997

All calculations by authors. Schools are classified in the last three columns according to the aggregate nondiscretionary factor from the 2009–2010 school year

Table 6.3 Average productivity results for secondary schools

	All schools	Classification of environment		
		Harsh	Moderate	Favorable
Number of schools	275	91	92	92
EMPI	1.034	1.041	1.035	1.027
Efficiency change	0.978	0.986	0.995	0.953
Technical change	1.045	1.031	1.024	1.081
Technical change MFE	1.019	1.025	1.023	1.010
Change in Env. HARSHNESS	1.026	1.005	1.001	1.070
Scale efficiency change	1.013	1.023	1.017	0.999

All calculations by authors. Schools are classified in the last three columns according to the aggregate nondiscretionary factor from the 2009–2010 school year

The main results of our productivity analysis are reported in Tables 6.2 and 6.3 for the primary and secondary schools, respectively. We note that we do not obtain results for approximately 15 % (206) of the primary schools and approximately 26 % (96) of the secondary schools due to infeasibilities from the variable returns to scale cross-period programming models. For the primary and secondary schools, we report the average results for all schools and for three categories of the environment defined by the aggregate measure of the nondiscretionary input (school year 2009–2010).

Focusing on the primary schools in Table 6.2, we observe that on average, productivity increased by 2.7 % from 2008–2009 to 2009–2010. The average productivity results for schools facing harsh to moderate environments were even higher (approximately 3.7 % improvement in productivity.) And, on average, the schools with the most favorable environment saw an improvement in productivity of about 1.3 %. On average, all schools (and in each environmental harshness category) became relatively less technically efficient. The improvement in primary school productivity arises from technical change; on average schools are able to realize an increase in output by approximately 3 % due to the shift in the technology. This pattern of technical progress is observed across environmental harshness

Table 6.4 Illustrative results

	School		
	8129	8142	8225
EMPI	1.086	1.030	1.172
Efficiency change	1.000	1.032	1.082
Technical change	1.016	1.000	1.041
Technical change MFE	1.011	1.018	0.994
Change in Env. harshness	1.006	0.983	1.047
Scale efficiency change	1.069	0.997	1.041

All calculations by authors

categories. The results also indicate that the environment had less of adverse impact in 2009–2010 than in 2008–2009 on average for all environmental harshness categories. Finally, we note a small improvement in productivity for the harshest and moderate classifications due to a movement closer to most productive scale size. On average, the schools with the most favorable environment did not see an improvement towards efficient scale size.

The results for the secondary schools are similar. On average, secondary schools saw an increase in productivity of about 3.7 %. Schools with the harshest environment had the highest gains (4.1 %) while the schools with the most favorable environment saw an improvement of only 2.7 %. On average, and across classifications, the Australian schools became less technically efficient. However, the schools with the harsh to moderate environments tended to move closer to the most productive scale size. In addition, technical change is the component that tends to explain the improvement in productivity. Overall, secondary schools (like the primary schools) are experiencing technical progress, less of an impact of the environment and a movement closer to most productive scale size.

We report illustrative results for three randomly chosen secondary schools in Table 6.4. School 8129 has a relatively harsh environment with an aggregate nondiscretionary input (0.779) in the 40th percentile. The school realized an improvement in productivity of 8.6 %. An analysis of the components reveals that this school's productivity improvements were achieved primarily by becoming more scale efficient. In addition, the school experienced technical progress but did not see any real improvements in technical efficiency nor in environmental harshness. School 8142 realized a 3 % increase in productivity, primarily due to an improvement in its technical efficiency. This school operates in a better socioeconomic environment than school 8129 with an aggregate nondiscretionary input level (0.803) in the 60th percentile. Finally, school 8225 had relatively the most favorable environment of the three with an aggregate index (0.854) in the 84th percentile. This school had a large increase in productivity (17.2 %) arising from a large improvement in technical and scale efficiency while experience technical progress largely due to an improvement in the environmental harshness.

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