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THE HANDBOOK OF Formulas and Tables for Signal Processing

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He has held positions as assistant, associate, and professor at the University of Rhode Island (1965–1983), professor and Chairman of the Engineering Department at the University of Denver (1983–1985), and professor (1985–) and Chairman (1985–1989) at the University of Alabama in Huntsville. Dr. Poularikas was a visiting scientist at MIT (1971–1972), and summer faculty fellow at NASA (1968, 1972), at Stanford University (1966), and at Underwater Systems Center (1971, 1973, 1974).

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Dr. Poularikas is a senior member of the IEEE, was a Fulbright scholar and was awarded the Outstanding Educator's Award by the IEEE Huntsville Section in 1990 and 1996. His main interest is in the area of signal processing.

PREFACE

The purpose of *The Handbook of Formulas and Tables for Signal Processing* is to include in a single volume the most important and most useful tables and formulas that are used by engineers and students involved in signal processing. This includes deterministic as well as statistical signal processing applications. The handbook contains a large number of standard mathematical tables, so it can also be used as a mathematical formulas handbook.

The handbook is organized into 45 chapters. Each contains tables, formulas, definitions, and other information needed for the topic at hand. Each chapter also contains numerous examples to explain how to use the tables and formulas. Some of the figures were created using MATLAB and MATHEMATICA.

The editor and CRC Press would be grateful if readers would send their opinions about the handbook, any error they may detect, suggestions for additional material for future editions, and suggestions for deleting material.

The handbook is testimony to the efforts of colleagues whose contributions were invaluable, Nora Konopka, Associate Editor at CRC Press, the commitment of the Editor-in-Chief of the series, Dr. Richard Dorf, and others. Special thanks go to Dr. Yunlong Sheng for contributing Chapter 42.

Alexander D. Poularikas
Huntsville, Alabama
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1

Fourier Series

1.1 Definitions and Series Formulas

1.1.1 A function is ***periodic*** if $f(t) = f(t + nT)$, where n is an integer and T is the period of the function.

1.1.2 The function $f(t)$ is ***absolutely integrable*** if $\int_a^b |f(t)|\,dt < \infty$.

1.1.3 An ***infinite series*** of function

$$f_1(t) + f_2(t) + \cdots + f_k(t) + \cdots = \sum_{k=1}^{\infty} f_k(t)$$

converges at a given value of t if its ***partial sums***

$$s_n(t) = \sum_{k=1}^{\infty} f_k(t), \quad (n = 1,2,3,\cdots)$$

have a finite limit $s(t) = \lim_{n\to\infty} s_n(t)$.

1.1.4 The series in 1.1.3 is ***uniformly convergent*** in [a,b] if, for any positive number ε, there exists a number N such that the inequality $|s(t) - s_n(t)| \le \varepsilon$ holds for all $n \ge N$ and for all t in the interval [a,b].

1.1.5 ***Complex form*** of the series:

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} |\alpha_n| e^{j(n\omega_o t + \varphi_n)}, \quad t_o \le t \le t_o + T$$

$$\alpha_n = \frac{1}{T}\int_{t_o}^{t_o+T} f(t)e^{-jn\omega_o t}\,dt, \qquad \omega_o = \frac{2\pi}{T}, \quad T = \text{period}$$

$$\alpha_n = |\alpha_n| e^{j\varphi_n} = |\alpha_n|\cos\varphi_n + j|\alpha_n|\sin\varphi_n, \qquad \alpha_{-n} = \alpha_n^*, \ t_o = \text{any real value.}$$

1.1.6 ***Trigonometric form*** of the series

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty}(A_n \cos n\omega_o t + B_n \sin n\omega_o t), \ \ A_o = 2\alpha_o = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\,dt$$

$$A_n = (\alpha_n + \alpha_n^*) = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\cos n\omega_o t\,dt, \ \ B_n = j(\alpha_n - \alpha_n^*) = \frac{2}{T}\int_{t_o}^{t_o+T} f(t)\sin n\omega_o t\,dt$$

$$f(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \varphi_n), \ \ C_n = (A_n^2 + B_n^2)^{1/2}, \ \varphi_n = -\tan^{-1}(B_n / A_n)$$

1.1.7 ***Parseval's*** formula

$$\frac{1}{T}\int_{t_o}^{t_o+T} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty}|\alpha_n|^2 = \frac{A_o^2}{4} + \sum_{n=1}^{\infty}\left(\frac{A_n^2}{2} + \frac{B_n^2}{2}\right) = \frac{A_o^2}{4} + \sum_{n=1}^{\infty}\frac{C_n^2}{2}$$

1.1.8 ***Sum*** of cosines

$$\frac{1}{2} + \cos t + \cos 2t + \cdots + \cos nt = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}$$

1.1.9 ***Truncated*** Fourier series

$$f_N(t) = \frac{A_o}{2} + \sum_{n=1}^{N}(A_n \cos n\omega_o t + B_n \sin n\omega_o t)$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(v)\frac{\sin\left[(2N+1)\omega_o \frac{t-v}{2}\right]}{\sin\left[\omega_o \frac{t-v}{2}\right]}\,dv$$

1.1.10 ***Sum*** and ***difference*** functions

$$p(t) = C_1 f(t) \pm C_2 h(t) = \sum_{n=-\infty}^{\infty}[C_1\beta_n \pm C_2\gamma_n]e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty}\alpha_n\, e^{jn\omega_o t},$$

C_1 = constant, C_2 = constant, β_n = Fourier expansion coefficients of $f(t)$, γ_n = Fourier expansion coefficients of $h(t)$, $\alpha_n = C_1\beta_n \pm C_2\gamma_n$, $f(t)$ and $h(t)$ are periodic with same period.

1.1.11 ***Product*** of two functions

$$p(t)=f(t)h(t)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(\beta_{n-m}\gamma_m)e^{jn\omega_o t}=\sum_{n=-\infty}^{\infty}\alpha_n\,e^{jn\omega_o t}$$

$$\alpha_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)h(t)e^{-jn\omega_o t}\,dt=\sum_{m=-\infty}^{\infty}(\beta_{n-m}\gamma_m)$$

β_n = Fourier expansion coefficients of $f(t)$, γ_n = Fourier expansion coefficients of $h(t)$, $f(t)$ and $h(t)$ are periodic with same period.

1.1.12 ***Convolution*** of two functions

$$g(t)=\frac{1}{T}\int_{-T/2}^{T/2}f(\tau)h(t-\tau)\,d\tau=\sum_{n=-\infty}^{\infty}\alpha_n\,e^{jn\omega_o t}=\sum_{n=-\infty}^{\infty}\beta_n\gamma_n\,e^{jn\omega_o t}$$

$\alpha_n=\beta_n\gamma_n$. β_n = Fourier expansion coefficients of $f(t)$, γ_n = Fourier expansion coefficients of $h(t)$, $f(t)$, and $h(t)$ are periodic with same period.

1.1.13 If $H(\omega)$ (***transfer function***) is the Fourier transform of the ***impulse response*** $h(t)$ of a linear time invariant system (LTI), then its output due to a periodic input function $f(t)$ is

$$y(t)=\frac{A_o}{2}H(0)+\sum_{n=1}^{\infty}\left|H(n\omega_o)\right|[A_n\cos[n\omega_o t+\varphi(n\omega_o)]+B_n\sin[n\omega_o t+\varphi(n\omega_o)]]$$

$$H(n\omega_o)=H_r(n\omega_o)+jH_i(n\omega_o)=[H_r^2(n\omega_o)+H_i^2(n\omega_o)]^{1/2}e^{j\varphi(n\omega_o)}$$

$$\varphi(n\omega_o)=\tan^{-1}[H_i(n\omega_o)/H_r(n\omega_o)]$$

$H_r(\cdot)$ and $H_i(\cdot)$ are real functions.

1.1.14 Lanczos ***smoothing*** factor

$$f_N(t)=\frac{A_o}{2}+\sum_{n=1}^{N}\frac{\sin(n\pi/N)}{n\pi/N}\,[A_n\cos n\omega_o t+B_n\sin n\omega_o t]$$

where A_0, A_n, and B_n are the trigonometric expansion Fourier series coefficients (see 1.1.6).

1.1.15 Fejé ***smoothing*** series

$$f_N(t)=\frac{A_o}{2}+\sum_{n=1}^{N}\frac{N-n}{N}\,[A_n\cos n\omega_o t+B_n\sin n\omega_o t]$$

where A_0, A_n, and B_n are the trigonometric expansion Fourier series coefficients (see 1.1.6).

1.1.16 ***Transformation*** from 2ℓ to 2π

If the period is 2ℓ, then the Fourier series of f(t) is

$$f(t)=\frac{A_o}{2}+\sum_{k=1}^{N}\left[A_k\cos\frac{\pi kt}{\ell}+B_k\sin\frac{\pi kt}{\ell}\right]$$

If we set $\pi t/\ell=x$ or $t=x\ell/\pi$, we obtain the equivalent series

$$\varphi(x) = f\left(\frac{x\ell}{\pi}\right) = \frac{a_o}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx]$$

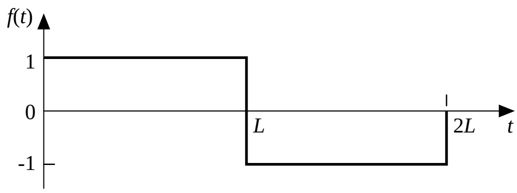
The above means: If $f(t)$ has period 2ℓ, then $\varphi(x) = f(x\ell/\pi)$ has a period 2π.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad k = 0,1,2,\cdots$$

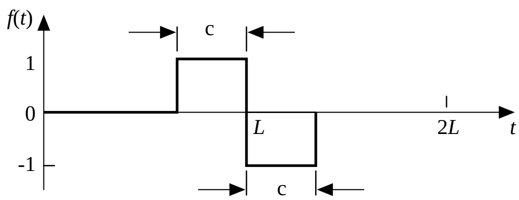
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \qquad k = 1,2,\cdots$$

1.1.17 Table of Fourier Series Expansions

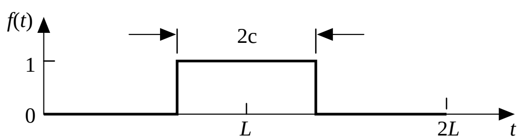
1. $f(t) = \frac{1}{\pi}\sum_{n=1,3,5\cdots} \frac{1}{n}\sin\frac{n\pi t}{L}$



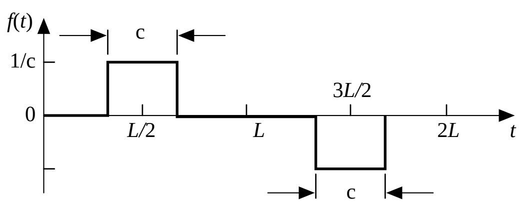
2. $f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\cos\frac{n\pi c}{L} - 1\right)\sin\frac{n\pi t}{L}$



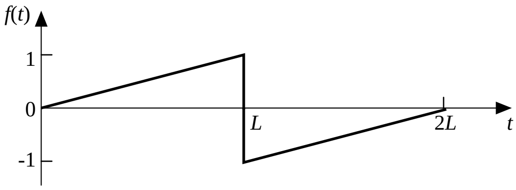
3. $f(t) = \frac{c}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin\frac{n\pi c}{L}\cos\frac{n\pi t}{L}$



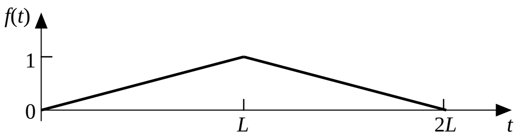
4. $f(t) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi}{2}\,\frac{\sin\frac{1}{2}n\pi c/L}{\frac{1}{2}n\pi c/L}\sin\frac{n\pi t}{L}$



5. $$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{L}$$

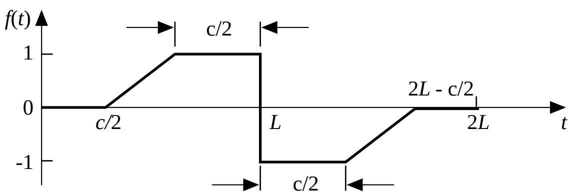


6. $$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5\cdots}\frac{1}{n^2}\cos\frac{n\pi t}{L}$$

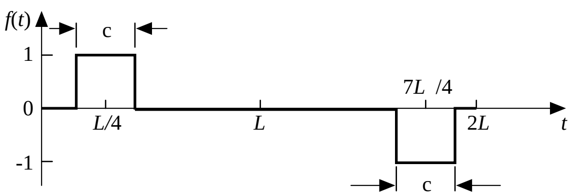


7. $$f(t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

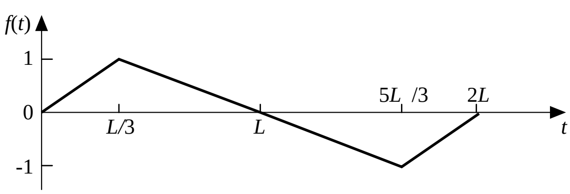
$$\left[1 + \frac{1+(-1)^n}{n\pi(1-2a)}\sin n\pi a\right]\sin\frac{n\pi t}{L}; \quad a = \frac{c}{2L}$$



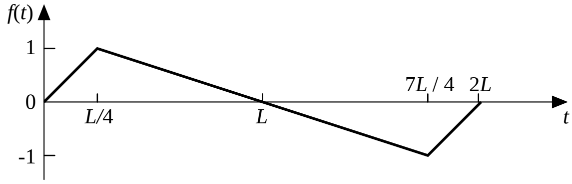
8. $$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{4}\sin n\pi a\sin\frac{n\pi t}{L}; \quad a = \frac{c}{2L}$$



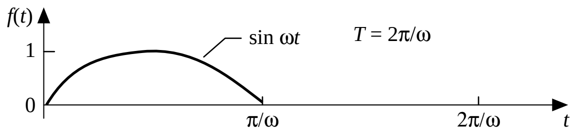
9. $$f(t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi t}{L}$$



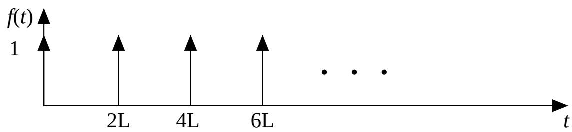
10. $f(t)=\frac{32}{3\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{4}\sin\frac{n\pi t}{L}$



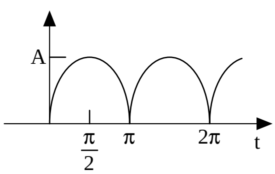
11. $f(t)=\frac{1}{\pi}+\frac{1}{2}\sin\omega t-\frac{2}{\pi}\sum_{n=2,4,6\cdots}\frac{1}{n^2-1}\cos n\omega t$



12. $f(t)=\frac{1}{2L}+\frac{1}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi t}{L}$



13. $f(t)=\frac{2A}{\pi}-\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nt$



1.2 Orthogonal Systems and Fourier Series

1.2.1 An infinite system of real functions $\varphi_o(t),\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t),\ldots$ is said to be **orthogonal** on an interval [a,b] if $\int_a^b \varphi_n(t)\varphi_m(t)\,dt=0$ for $n\neq m$ and n,m = 0,1,2,…. It is assumed that

$$\int_a^b \varphi_n^2(t)\,dt\neq 0 \quad \text{for} \quad n=0,1,2,\cdots.$$

1.2.2 The expansion of a function $f(t)$ in [a,b] is given by

$$f(t)=c_o\varphi_o(t)+c_1\varphi_1(t)+\cdots+c_n\varphi_n(t)+\cdots$$

$$c_n = \frac{\int_a^b f(t)\varphi_n(t)\,dt}{\int_a^b \varphi_n^2(t)\,dt} = \frac{\int_a^b f(t)\varphi_n(t)\,dt}{\|\varphi_n\|^2} \qquad n = 0,1,2,\cdots$$

1.2.3 ***Bessel's*** inequality

$$\int_a^b f^2(t)\,dt \ge \sum_{k=0}^{n} c_k^2 \|\varphi_k\|^2 \qquad n = \text{arbitrary}$$

1.2.4 ***Completeness*** of the system (1.2.1): A necessary and sufficient condition for the system (1.2.1) to be complete is that the Fourier series of any square integrable function $f(t)$ converges to $f(t)$ in the mean.

If the system (1.2.1) is complete, then every square integrable function $f(t)$ is completely determined (except for its values at a finite number of points) by its Fourier series.

1.2.5 The ***limits*** as $n \to \infty$ of the ***trigonometric*** integrals

$$\lim_{n\to\infty} \int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}\,dt = \lim_{n\to\infty} \int_{-T/2}^{T/2} f(t)\sin\frac{2\pi nt}{T}\,dt$$

1.2.6 ***Convergence*** in discontinuity: If $f(t)$ is the absolutely integrable function of period T, then at every point of discontinuity where $f(t)$ has a right-hand and left-hand derivative, the Fourier series of $f(t)$ converges to the value $[f(t+0) + f(t-0)]/2$.

1.3 Decreasing Coefficients of Trigonometric Series

1.3.1 ***Abel*** lemma: Let $u_o + u_1 + u_2 + \ldots + u_n + \ldots$ be a numerical series whose partial sums σ_n satisfy the condition $|\sigma_n| \le M$, where M is a constant. Then, if the positive numbers $\alpha_o, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ approach zero monotonically, the series $\alpha_o u_o + \alpha_1 u_1 + \ldots + \alpha_n u_n + \ldots$ converges, and the sum S satisfies the inequality $|S| \le M\alpha_o$.

1.3.2 The ***sum*** of sines

$$\sin t + \sin 2t + \sin 3t + \cdots + \sin nt = \frac{\cos\frac{t}{2} - \cos\left[\left(n+\frac{1}{2}\right)t\right]}{2\cos\frac{t}{2}}$$

$$1 + \frac{\cos t}{p} + \frac{\cos 2t}{p^2} + \cdots + \frac{\cos nt}{p^n} + \cdots = \frac{p(p-\cos t)}{p^2 - 2p\cos t + 1}$$

$$\frac{\sin t}{p} + \frac{\sin 2t}{p^2} + \cdots + \frac{\sin nt}{p^n} + \cdots = \frac{p\sin t}{p^2 - 2p\cos t + 1}$$

1.4 Operations on Fourier Series

1.4.1 ***Integration*** of Fourier series: If the absolutely integrable function $f(t)$ of period T is specified by its Fourier series (1.1.6) then

$$\int_a^b f(t)\,dt$$

can be found by term-by-term integration of the series.

1.4.2 ***Differentiation*** of Fourier series: If $f(t)$ is a continuous function of period T with absolutely integrable derivative, which may not exist at certain points, then the Fourier series of $df(t)/dt$ can be obtained from the Fourier series of $f(t)$ by term-by-term differentiation.

1.5 Two-Dimensional Fourier Series

1.5.1 Complex form

$$f(x,y) = \sum_{m,n=-\infty}^{\infty} c_{mn} e^{j\pi\left(\frac{mx}{l}+\frac{ny}{h}\right)} \qquad R\{-l \le x \le l, \quad -h \le y \le h\}$$

$$c_{mn} = \frac{1}{2l2h}\iint_R f(x,y)\, e^{-j\pi\left(\frac{mx}{l}+\frac{ny}{h}\right)} \qquad m,n = 0,\pm 1,\pm 2,\cdots$$

1.5.2 Trigonometric form

$$f(x,y) = \sum_{m,n=0}^{\infty}\left[A_{mn}\cos\frac{\pi mx}{l}\cos\frac{\pi ny}{h} + B_{mn}\sin\frac{\pi mx}{l}\cos\frac{\pi ny}{h} + C_{mn}\cos\frac{\pi mx}{l}\sin\frac{\pi ny}{h} + D_{mn}\sin\frac{\pi mx}{l}\sin\frac{\pi ny}{h}\right]$$

$$R\{-l \le x \le l, \quad -h \le y \le h\}$$

$$A_{mn} = \frac{1}{lh}\int_{-l}^{l}\int_{-h}^{h} f(x,y)\cos\frac{\pi mx}{l}\cos\frac{\pi ny}{h}\,dxdy$$

$$B_{mn} = \frac{1}{lh}\int_{-l}^{l}\int_{-h}^{h} f(x,y)\sin\frac{\pi mx}{l}\cos\frac{\pi ny}{h}\,dxdy$$

$$C_{mn} = \frac{1}{lh}\int_{-l}^{l}\int_{-h}^{h} f(x,y)\cos\frac{\pi mx}{l}\sin\frac{\pi ny}{h}\,dxdy$$

$$D_{mn} = \frac{1}{lh}\int_{-l}^{l}\int_{-h}^{h} f(x,y)\sin\frac{\pi mx}{l}\sin\frac{\pi ny}{h}\,dxdy$$

1.5.3 Trigonometric form with limits $-\pi \le x \le \pi$, $-\pi \le y \le \pi$

$$f(x,y) = \sum_{m,n=0}^{\infty} \lambda_{mn}[a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny$$

$$+ c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny]$$

$$R\{-\pi \le x \le \pi, \quad -\pi \le y \le \pi\}$$

$$a_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \cos mx \cos ny \, dxdy$$

$$b_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \sin mx \cos ny \, dxdy$$

$$c_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \cos mx \sin ny \, dxdy$$

$$d_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) \sin mx \sin ny \, dxdy$$

$$\lambda_{mn} = \begin{cases} \frac{1}{4} & m = n = 0 \\ \frac{1}{2} & m > 0,\ n = 0, \text{ or } m = 0,\ n > 0 \\ 1 & m > 0,\ n > 0 \end{cases}$$

$$m, n = 0,1,2,3,4 \cdots$$

1.5.4 Parseval's formula

$$\frac{1}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^2(x,y) \, dxdy = \sum_{m,n=0}^{\infty} \lambda_{mn}(a_{mn}^2 + b_{mn}^2 + c_{mn}^2 + d_{mn}^2)$$

Appendix 1

Examples

Example 1

Expand the function shown in Figure 1.1 in Fourier series and plot the results.

$$\alpha_n = \frac{1}{3.5} \int_{-0.5}^{3} f(t) e^{-jn\omega_o t} \, dt = \frac{1}{3.5} \left[\int_{-0.5}^{1} 1 \cdot e^{-jn\omega_o t} dt + \int_{1}^{3} 0 \cdot e^{-jn\omega_o t} \, dt \right]$$

$$= \frac{1}{3.5(-jn\omega_o)} e^{-jn\omega_o t} \Big|_{-0.5}^{1} = \frac{1}{-j3.5n\omega_o} (e^{-jn\omega_o} - e^{j0.5n\omega_o})$$

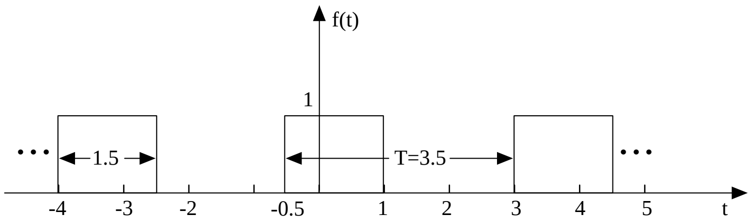


FIGURE 1.1

$$\alpha_o = \frac{1}{3.5}\int_{-0.5}^{1} dt = \frac{3}{7}, \qquad \omega_o = \frac{2\pi}{3.5}$$

$$f(t) = \alpha_o + \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_o t} = \frac{3}{7} + \sum_{n=1}^{\infty}\left\{\left[\frac{1}{-j3.5n\omega_o}(e^{-jn\omega_o} - e^{j0.5n\omega_o}) + \frac{1}{j3.5n\omega_o}(e^{jn\omega_o} - e^{-j0.5n\omega_o})\right]\cos n\omega_o t\right.$$

$$\left. + j\left[\frac{1}{-j3.5n\omega_o}\times(e^{-jn\omega_o} - e^{j0.5n\omega_o}) - \frac{1}{j3.5n\omega_o}(e^{jn\omega_o} - e^{-j0.5n\omega_o})\right]\sin n\omega_o t\right\}$$

$$= \frac{3}{7} + \sum_{n=1}^{\infty}\left[\frac{4}{3.5n\omega_o}[(\sin 0.75n\omega_o \cos 0.25n\omega_o)\cos n\omega_o t + (\sin 0.75n\omega_o \sin 0.25n\omega_o)\sin n\omega_o t]\right]$$

Figure 1.2 shows $f(t)$ for the cases $1 \le n \le 3$ (curve 1) and $1 \le n \le 10$ (curve 2). Figure 1.3 shows $f(t)$ for $10 \le n \le 50$, and Figure 1.4 shows $f(t)$ for $1 \le n \le 60$. Observe the Gibbs phenomenon in Figures 1.2 and 1.4.

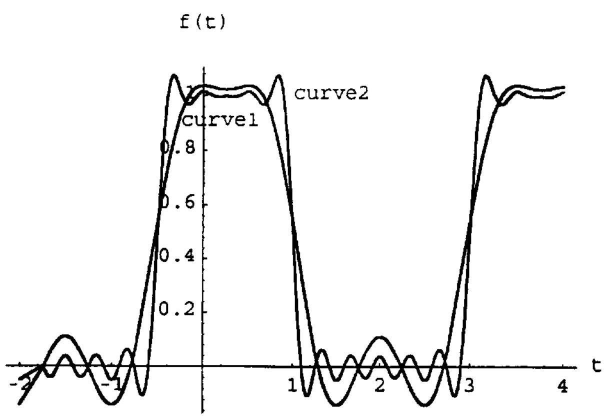


FIGURE 1.2

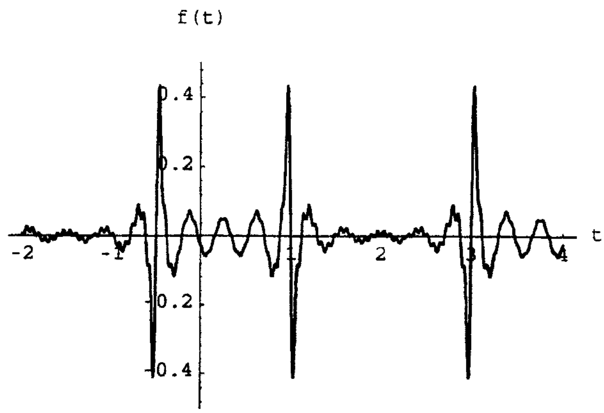


FIGURE 1.3

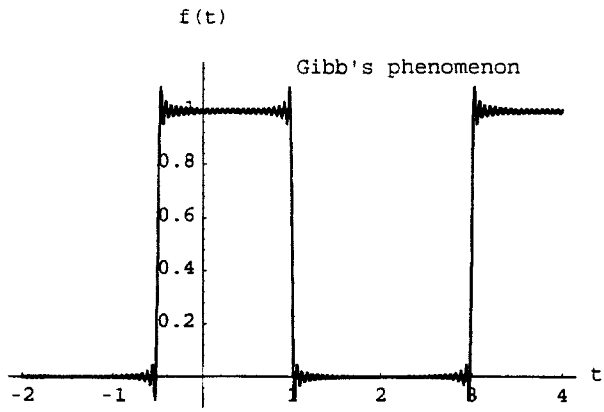


FIGURE 1.4

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2

Laplace Transforms

2.1 Definitions and Laplace Transform Formulae

2.1.1 One-Sided Laplace Transform

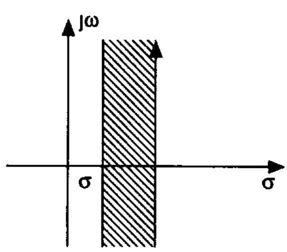
$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s = \sigma + j\omega$$

$f(t)$ = piecewise continuous and of exponential order

2.1.2 One-Sided Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds$$

where the integration is within the regions of convergence. The region of convergence is half-plane $\sigma < \text{Re}\{s\}$.



2.1.3 Two-Sided Laplace Transform

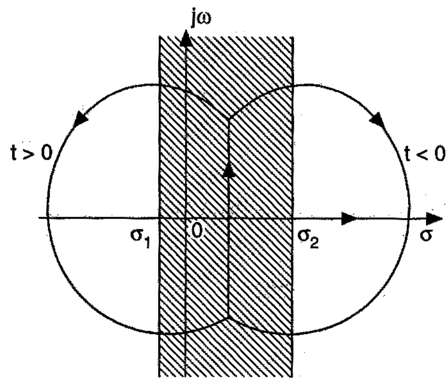
$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt \qquad s = \sigma + j\omega$$

$f(t)$ = piecewise continuous and of exponential order

2.1.4 Two-Sided Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds$$

where the integration is within the regions of convergence
which is a vertical strip $\sigma_1 < \mathrm{Re}\{s\} < \sigma_2$.



2.2 Properties

2.2.1 Properties of the Laplace Transform (one sided)

TABLE 2.1 Laplace Transform Properties

1. Linearity $L\{K_1 f_1(t) \pm K_2 f_2(t)\} = L\{K_1 f_1(t)\} \pm L\{K_2 f_2(t)\} = K_1 F_1(s) \pm K_2 F_2(s)$
2. Time derivative

$$L\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0+)$$

3. Higher time derivative

$$L\left\{\frac{d^n}{dt^n} f(t)\right\} = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f^{(1)}(0+) - \cdots - f^{(n-1)}(0+)$$

where $f^{(i)}(0+)$, $i = 1,2,\ldots,n-1$ is the i^{th} derivative of $f(\cdot)$ at $t = 0+$.

4. Integral with zero initial condition $L\left\{\int_0^t f(\xi)\,d\xi\right\} = \frac{F(s)}{s}$
5. Integral with initial conditions $L\left\{\int_{-\infty}^t f(\xi)\,d\xi\right\} = \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}$ where $f^{(-1)}(0+) = \lim_{t\to 0+}\int_{-\infty}^t f(\xi)\,d\xi$
6. Multiplication by exponential $L\{e^{\pm at} f(t)\} = F(s \mp a)$
7. Multiplication by t $L\{t\,f(t)\} = -\frac{d}{ds}F(s)$; $L\{t^n\ f(t)\} = (-1)^n \frac{d^n}{ds^s} F(s)$
8. Time shifting $L\{f(t \pm \lambda)u(t \pm \lambda)\} = e^{\pm s\lambda} F(s)$
9. Scaling $L\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$; $L\{f(ta)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$ $a > 0$
10. Time convolution $L\left\{\int_0^t f_1(t-\tau) f_2(\tau)\,d\tau\right\} \stackrel{\Delta}{=} L\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$
11. Frequency convolution

$$L\{f_1(t) f_2(t)\} = \frac{1}{2\pi j}\int_{x-j\infty}^{x+j\infty} F_1(z)F_2(s-z)\,dz = \frac{1}{2\pi j}\{F_1(s) * F_2(s)\}$$

where $z = x + jy$, and where x must be greater than the abscissa of absolute convergence for $f_1(t)$ over the path of integration.

12. Initial value $\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$ provided that this limit exists.
13. Final value $\lim_{t\to\infty} f(t) = \lim_{s\to 0+} sF(s)$ provided that sF(s) is analytic on the $j\omega$ axis and in the right half of the s plane
14. Division by t $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s')\,ds'$
15. $f(t)$ periodic $L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$ $f(t) = f(t + T)$

2.2.2 Methods of Finding the Laplace Transform

1. Direct method by solving (2.1.1).
2. Expand $f(t)$ in power series if such an expansion exists.
3. Differentiation with respect to a parameter.
4. Use of tables.

2.3 Inverse Laplace Transforms

2.3.1 Properties

1. Linearity $L^{-1}\{c_1F_1(s) \pm c_2F_2(s)\} = c_1 f_1(t) \pm c_2 f_2(t)$
2. Shifting $L^{-1}\{F(s-a)\} = e^{at} f(t)$
3. Time shifting $L^{-1}\{e^{-as}F(s)\} = f(t-a) \qquad t > a$
4. Scaling property $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right) \qquad a > 0$
5. Derivatives $L^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t) \qquad F^{(n)}(s) = \frac{d^n F(s)}{ds^n}$
6. Multiplication by s $L^{-1}\{sF(s) - f(0+)\} = L\{sF(s)\} - f(0+)L\{1\} = f^{(1)}(t) + f(0)\delta(t)$
7. Division by s $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t')\,dt'$
8. Convolution $L^{-1}\{F(s)H(s)\} = \int_0^t F(u)H(t-u)\,du = F(s) * H(s)$

2.3.2 Methods of Finding Inverse Laplace Transforms

1. ***Partial fraction method*:** Any rational function $P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials, with the degree of $P(s)$ less than that of $Q(s)$, can be written as the sum of rational functions, known as partial fractions, having the form $A/(as + b)^r$, $(As + B)/(as^2 + bs + c)^r$, $r = 1,2,\ldots$.
2. Expand $F(s)$ in inverse powers of s if such an expansion exists.
3. Differentiation with respect to a parameter.
4. Combination of the above methods.
5. Use of tables.
6. Complex inversion (see Appendix 1).

2.4 Relationship Between Fourier Integrals of Causal Functions and One-Sided Laplace Transforms

2.4.1 $F(\omega)$ from $F(s)$

$$F(\omega) = \int_0^{\infty} e^{-j\omega t} f(t)\,dt \qquad f(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

a) The region of convergence of $F(s)$ contains the $j\omega$ axis in its interior, $\sigma < 0$ (see 2.1.2)

$$F(\omega) = F(s)\big|_{s=j\omega}$$

b) If the axis $j\omega$ is outside the region of convergence of $F(s)$, $\sigma > 0$, then $F(\omega)$ does not exist; the function $f(t)$ has no Fourier transform.

c) Let $\sigma = 0$, $F(s)$ is analytic for $s > 0$, and has one singular point on the $j\omega$ axis, hence, $F(s) = \dfrac{1}{s - j\omega_o}$ or $F(s) = L\{e^{j\omega_o t}u(t)\}$. But $F\{e^{j\omega_o t}u(t)\} = \pi\delta(\omega - \omega_o) + \dfrac{1}{j\omega - j\omega_o}$ and there we obtain the correspondence

$$F(s) = \frac{1}{s - j\omega_o} \qquad F(\omega) = F(s)\big|_{s=j\omega} = \pi\delta(\omega - \omega_o) + F(s)\big|_{s=j\omega}$$

Also

$$F(s) = \frac{1}{(s - j\omega_o)^n} \qquad F(\omega) = \frac{\pi j^{n-1}}{(n-1)!}\delta^{(n-1)}(\omega - \omega_o) + F(s)\big|_{s=j\omega}$$

$\delta^{(n-1)}(\cdot)$ = the $(n-1)^{th}$ derivative.

d) $F(s)$ has n simple poles $j\omega_1, j\omega_2, \ldots, j\omega_n$ and no other singularities in the half plane Re $s \geq 0$. $F(s)$ takes the form $F(s) = G(s) + \sum_{n=1}^{n} \dfrac{a_n}{s - j\omega_n}$ where $G(s)$ is free of singularities for Re $s \geq 0$. The correspondence is

$$F(\omega) = G(s)\big|_{s=j\omega} + \sum_{n=1}^{n} \frac{a_n}{s - j\omega_n}\bigg|_{s=j\omega} + \pi \sum_{n=1}^{n} a_n \delta(\omega - \omega_n)$$

2.5 Table of Laplace Transforms

TABLE 2.2 Table of Laplace Operations

	$F(s)$	$f(t)$
1	$\int_0^{\infty} e^{-st} f(t)\,dt$	$f(t)$
2	$AF(s) + BG(s)$	$Af(t) + Bg(t)$
3	$sF(s) - f(+0)$	$f'(t)$
4	$s^n F(s) - s^{n-1} f(+0) - s^{n-2} f^{(1)}(+0) - \cdots - f^{(n-1)}(+0)$	$f^{(n)}(t)$
5	$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\,d\tau$
6	$\frac{1}{s^2} F(s)$	$\int_0^t \int_0^{\tau} f(\lambda)\,d\lambda\,d\tau$
7	$F_1(s)F_2(s)$	$\int_0^t f_1(t-\tau)\, f_2(\tau)\,d\tau = f_1 * f_2$
8	$-F'(s)$	$tf(t)$
9	$(-1)^n F^{(n)}(s)$	$t^n f(t)$
10	$\int_s^{\infty} F(x)\,dx$	$\frac{1}{t} f(t)$
11	$F(s-a)$	$e^{at} f(t)$
12	$e^{-bs} F(s)$	$f(t-b)$, where $f(t) = 0$; $t < 0$
13	$F(cs)$	$\frac{1}{c} f\left(\frac{t}{c}\right)$
14	$F(cs-b)$	$\frac{1}{c} e^{(bt)/c} f\left(\frac{t}{c}\right)$
15	$\frac{\int_0^a e^{-st} f(t)\,dt}{1-e^{-as}}$	$f(t+a) = f(t)$ periodic signal
16	$\frac{\int_0^a e^{-st} f(t)\,dt}{1+e^{-as}}$	$f(t+a) = -f(t)$
17	$\frac{F(s)}{1-e^{-as}}$	$f_1(t)$, the half-wave rectification of $f(t)$ in No. 16.
18	$F(s) \coth \frac{as}{2}$	$f_2(t)$, the full-wave rectification of $f(t)$ in No. 16.
19	$\frac{p(s)}{q(s)}, q(s) = (s-a_1)(s-a_2)\cdots(s-a_m)$	$\sum_1^m \frac{p(a_n)}{q'(a_n)} e^{a_n t}$
20	$\frac{p(s)}{q(s)} = \frac{\phi(s)}{(s-a)^r}$	$e^{at} \sum_{n=1}^{r} \frac{\phi^{(r-n)}(a)}{(r-n)!} \frac{t^{n-1}}{(n-1)!} + \cdots$

.

TABLE 2.3 Table of Laplace Transforms

	$F(s)$	$f(t)$
1	s^n	$\delta^{(n)}(t)$ n^{th} derivative of the delta function
2	s	$\frac{d\delta(t)}{dt}$
3	1	$\delta(t)$
4	$\frac{1}{s}$	1
5	$\frac{1}{s^2}$	t
6	$\frac{1}{s^n}\ (n=1,2,\cdots)$	$\frac{t^{n-1}}{(n-1)!}$
7	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
8	$s^{-3/2}$	$2\sqrt{\frac{t}{\pi}}$
9	$s^{-[n+(1/2)]}\ (n=1,2,\cdots)$	$\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}$
10	$\frac{\Gamma(k)}{s^k}\ (k\geq 0)$	t^{k-1}
11	$\frac{1}{s-a}$	e^{at}
12	$\frac{1}{(s-a)^2}$	te^{at}
13	$\frac{1}{(s-a)^n}\ (n=1,2,\cdots)$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
14	$\frac{\Gamma(k)}{(s-a)^k}\ (k\geq 0)$	$t^{k-1}e^{at}$
15	$\frac{1}{(s-a)(s-b)}$	$\frac{1}{(a-b)}(e^{at}-e^{bt})$
16	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{(a-b)}(ae^{at}-be^{bt})$
17	$\frac{1}{(s-a)(s-b)(s-c)}$	$-\frac{(b-c)e^{at}+(c-a)e^{bt}+(a-b)e^{ct}}{(a-b)(b-c)(c-a)}$
18	$\frac{1}{(s+a)}$	e^{-at} valid for complex a
19	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$
20	$\frac{1}{s^2(s+a)}$	$\frac{1}{a^2}(e^{-at}+at-1)$
21	$\frac{1}{s^3(s+a)}$	$\frac{1}{a^2}\left[\frac{1}{a}-t+\frac{at^2}{2}-\frac{1}{a}e^{-at}\right]$
22	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(e^{-at}-e^{-bt})$
23	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[1+\frac{1}{(a-b)}(be^{-at}-ae^{-bt})\right]$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
24	$\frac{1}{s^2(s+a)(s+b)}$	$\frac{1}{(ab)^2}\left[\frac{1}{(a-b)}(a^2e^{-bt}-b^2e^{-at})+abt-a-b\right]$
25	$\frac{1}{s^3(s+a)(s+b)}$	$\frac{1}{(ab)}\left[\frac{a^3-b^3}{(ab)^2(a-b)}+\frac{1}{2}t^2-\frac{(a+b)}{ab}t+\frac{1}{(a-b)}\left(\frac{b}{a^2}e^{-at}-\frac{a}{b^2}e^{-bt}\right)\right]$
26	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{1}{(b-a)(c-a)}e^{-at}+\frac{1}{(a-b)(c-b)}e^{-bt}+\frac{1}{(a-c)(b-c)}e^{-ct}$
27	$\frac{1}{s(s+a)(s+b)(s+c)}$	$\frac{1}{abc}-\frac{1}{a(b-a)(c-a)}e^{-at}-\frac{1}{b(a-b)(c-b)}e^{-bt}-\frac{1}{c(a-c)(b-c)}e^{-ct}$
28	$\frac{1}{s^2(s+a)(s+b)(s+c)}$	$\begin{cases}\frac{ab(ct-1)-ac-bc}{(abc)^2}+\frac{1}{a^2(b-a)(c-a)}e^{-at} \\ +\frac{1}{b^2(a-b)(c-b)}e^{-bt}+\frac{1}{c^2(a-c)(b-c)}e^{-ct}\end{cases}$
29	$\frac{1}{s^3(s+a)(s+b)(s+c)}$	$\begin{cases}\frac{1}{(abc)^3}[(ab+ac+bc)^2-abc(a+b+c)]-\frac{ab+ac+bc}{(abc)^2}t+\frac{1}{2abc}t^2 \\ -\frac{1}{a^3(b-a)(c-a)}e^{-at}-\frac{1}{b^3(a-b)(c-b)}e^{-bt}-\frac{1}{c^3(a-c)(b-c)}\varepsilon^{-ct}\end{cases}$
30	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
31	$\frac{s}{s^2+a^2}$	$\cos at$
32	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
33	$\frac{s}{s^2-a^2}$	$\cosh at$
34	$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2}(1-\cos at)$
35	$\frac{1}{s^2(s^2+a^2)}$	$\frac{1}{a^3}(at-\sin at)$
36	$\frac{1}{(s^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at-at\cos at)$
37	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$
38	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$
39	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
40	$\frac{s}{(s^2+a^2)(s^2+b^2)}(a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$
41	$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b}e^{at}\sin bt$
42	$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
43	$\frac{1}{[(s+a)^2+b^2]^n}$	$\frac{-e^{-at}}{4^{n-1}b^{2n}}\sum_{r=1}^{n}\binom{2n-r-1}{n-1}(-2t)^{r-1}\frac{d^r}{dt^r}[\cos(bt)]$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
44	$\dfrac{s}{[(s+a)^2+b^2]^n}$	$\left\{ \begin{aligned} & \frac{e^{-at}}{4^{n-1}b^{2n}} \left\{ \sum_{r=1}^{n} \binom{2n-r-1}{n-1} (-2t)^{r-1} \frac{d^r}{dt^r}[a\cos(bt)+b\sin(bt)] \right. \\ & \left. \quad -2b \sum_{r=1}^{n-1} r \binom{2n-r-2}{n-1} (-2t)^{r-1} \frac{d^r}{dt^r}[\sin(bt)] \right\} \end{aligned} \right.$
45	$\dfrac{3a^2}{s^3+a^3}$	$e^{-at} - e^{(at)/2}\left(\cos\dfrac{at\sqrt{3}}{2} - \sqrt{3}\sin\dfrac{at\sqrt{3}}{2}\right)$
46	$\dfrac{4a^3}{s^4+4a^4}$	$\sin at \cosh at - \cos at \sinh at$
47	$\dfrac{s}{s^4+4a^4}$	$\dfrac{1}{2a^2}(\sin at \sinh at)$
48	$\dfrac{1}{s^4-a^4}$	$\dfrac{1}{2a^3}(\sinh at - \sin at)$
49	$\dfrac{s}{s^4-a^4}$	$\dfrac{1}{2a^2}(\cosh at - \cos at)$
50	$\dfrac{8a^3s^2}{(s^2+a^2)^3}$	$(1 + a^2t^2)\sin at - \cos at$
51	$\dfrac{1}{s}\left(\dfrac{s-1}{s}\right)^n$	$L_n(t) = \dfrac{e^t}{n!}\dfrac{d^n}{dt^n}(t^n e^{-t})$ [$L_n(t)$ is the Laguerre polynomial of degree n]
52	$\dfrac{1}{(s+a)^n}$	$\dfrac{t^{(n-1)}e^{-at}}{(n-1)!}$ where n is a positive integer
53	$\dfrac{1}{s(s+a)^2}$	$\dfrac{1}{a^2}[1-e^{-at}-ate^{-at}]$
54	$\dfrac{1}{s^2(s+a)^2}$	$\dfrac{1}{a^3}[at-2+ate^{-at}+2e^{-at}]$
55	$\dfrac{1}{s(s+a)^3}$	$\dfrac{1}{a^3}\left[1-\left(\dfrac{1}{2}a^2t^2+at+1\right)e^{-at}\right]$
56	$\dfrac{1}{(s+a)(s+b)^2}$	$\dfrac{1}{(a-b)^2}\{e^{-at}+[(a-b)t-1]e^{-bt}\}$
57	$\dfrac{1}{s(s+a)(s+b)^2}$	$\dfrac{1}{ab^2}-\dfrac{1}{a(a-b)^2}e^{-at}-\left[\dfrac{1}{b(a-b)}t+\dfrac{a-2b}{b^2(a-b)^2}\right]e^{-bt}$
58	$\dfrac{1}{s^2(s+a)(s+b)^2}$	$\dfrac{1}{a^2(a-b)^2}e^{-at}+\dfrac{1}{ab^2}\left(t-\dfrac{1}{a}-\dfrac{2}{b}\right)+\left[\dfrac{1}{b^2(a-b)}t+\dfrac{2(a-b)-b}{b^3(a-b)^2}\right]e^{-bt}$
59	$\dfrac{1}{(s+a)(s+b)(s+c)^2}$	$\left\{ \begin{aligned} & \left[\frac{1}{(c-b)(c-a)}t+\frac{2c-a-b}{(c-a)^2(c-b)^2}\right]e^{-ct} \\ & \quad +\frac{1}{(b-a)(c-a)^2}e^{-at}+\frac{1}{(a-b)(c-b)^2}e^{-bt} \end{aligned} \right.$
60	$\dfrac{1}{(s+a)(s^2+\omega^2)}$	$\dfrac{1}{a^2+\omega^2}e^{-at}+\dfrac{1}{\omega\sqrt{a^2+\omega^2}}\sin(\omega t-\phi);\quad \phi=\tan^{-1}\left(\dfrac{\omega}{a}\right)$
61	$\dfrac{1}{s(s+a)(s^2+\omega^2)}$	$\dfrac{1}{a\omega^2}-\dfrac{1}{a^2+\omega^2}\left(\dfrac{1}{\omega}\sin\omega t+\dfrac{a}{\omega^2}\cos\omega t+\dfrac{1}{a}e^{-at}\right)$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
62	$\dfrac{1}{s^2(s+a)(s^2+\omega^2)}$	$\begin{cases} \dfrac{1}{a\omega^2}t - \dfrac{1}{a^2\omega^2} + \dfrac{1}{a^2(a^2+\omega^2)}e^{-at} \\ + \dfrac{1}{\omega^3\sqrt{a^2+\omega^2}}\cos(\omega t+\phi); \quad \phi = \tan^{-1}\left(\dfrac{a}{\omega}\right) \end{cases}$
63	$\dfrac{1}{[(s+a)^2+\omega^2]^2}$	$\dfrac{1}{2\omega^3}e^{-at}[\sin\omega t - \omega t\cos\omega t]$
64	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$
65	$\dfrac{1}{s^2(s^2-a^2)}$	$\dfrac{1}{a^3}\sinh at - \dfrac{1}{a^2}t$
66	$\dfrac{1}{s^3(s^2-a^2)}$	$\dfrac{1}{a^4}(\cosh at - 1) - \dfrac{1}{2a^2}t^2$
67	$\dfrac{1}{s^3+a^3}$	$\dfrac{1}{3a^2}\left[e^{-at} - e^{\frac{a}{2}t}\left(\cos\dfrac{\sqrt{3}}{2}at - \sqrt{3}\sin\dfrac{\sqrt{3}}{2}at\right)\right]$
68	$\dfrac{1}{s^4+4a^4}$	$\dfrac{1}{4a^3}(\sin at\ \cosh at - \cos at\ \sinh at)$
69	$\dfrac{1}{s^4-a^4}$	$\dfrac{1}{2a^3}(\sinh at - \sin at)$
70	$\dfrac{1}{[(s+a)^2-\omega^2]}$	$\dfrac{1}{\omega}e^{-at}\sinh\omega t$
71	$\dfrac{s+a}{s[(s+b)^2+\omega^2]}$	$\begin{cases} \dfrac{a}{b^2+\omega^2} - \dfrac{1}{\omega} + \sqrt{\dfrac{(a-b)^2+\omega^2}{b^2+\omega^2}}\, e^{-bt}\sin(\omega t+\phi); \\ \phi = \tan^{-1}\left(\dfrac{\omega}{b}\right) + \tan^{-1}\left(\dfrac{\omega}{a-b}\right) \end{cases}$
72	$\dfrac{s+a}{s^2[(s+b)^2+\omega^2]}$	$\begin{cases} \dfrac{1}{b^2+\omega^2}[1+at] - \dfrac{2ab}{(b^2+\omega^2)^2} + \dfrac{\sqrt{(a-b)^2+\omega^2}}{\omega(b^2+\omega^2)}e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\dfrac{\omega}{a-b}\right) + 2\tan^{-1}\left(\dfrac{\omega}{b}\right) \end{cases}$
73	$\dfrac{s+a}{(s+c)[(s+b)^2+\omega^2]}$	$\begin{cases} \dfrac{a-c}{(c-b)^2+\omega^2}e^{-ct} + \dfrac{1}{\omega}\sqrt{\dfrac{(a-b)^2+\omega^2}{(c-b)^2+\omega^2}}\, e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\dfrac{\omega}{a-b}\right) - \tan^{-1}\left(\dfrac{\omega}{c-b}\right) \end{cases}$
74	$\dfrac{s+a}{s(s+c)[(s+b)^2+\omega^2]}$	$\begin{cases} \dfrac{a}{c(b^2+\omega^2)} + \dfrac{(c-a)}{c[(b-c)^2+\omega^2]}e^{-ct} \\ -\dfrac{1}{\omega\sqrt{b^2+\omega^2}}\sqrt{\dfrac{(a-b)^2+\omega^2}{(b-c)^2+\omega^2}}\, e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\dfrac{\omega}{b}\right) + \tan^{-1}\left(\dfrac{\omega}{a-b}\right) - \tan^{-1}\left(\dfrac{\omega}{c-b}\right) \end{cases}$
75	$\dfrac{s+a}{s^2(s+b)^3}$	$\dfrac{a}{b^3}t + \dfrac{b-3a}{b^4} + \left[\dfrac{3a-b}{b^4} + \dfrac{a-b}{2b^2}t^2 + \dfrac{2a-b}{b^3}t\right]e^{-bt}$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
76	$\frac{s+a}{(s+c)(s+b)^3}$	$\frac{a-c}{(b-c)^3}e^{-ct}+\left[\frac{a-b}{2(c-b)}t^2+\frac{c-a}{(c-b)^2}t+\frac{a-c}{(c-b)^3}\right]e^{-bt}$
77	$\frac{s^2}{(s+a)(s+b)(s+c)}$	$\frac{a^2}{(b-a)(c-a)}e^{-at}+\frac{b^2}{(a-b)(c-b)}e^{-bt}+\frac{c^2}{(a-c)(b-c)}e^{-ct}$
78	$\frac{s^2}{(s+a)(s+b)^2}$	$\frac{a^2}{(b-a)^2}e^{-at}+\left[\frac{b^2}{(a-b)}t+\frac{b^2-2ab}{(a-b)^2}\right]e^{-bt}$
79	$\frac{s^2}{(s+a)^3}$	$\left[2-2at+\frac{a^2}{2}t^2\right]e^{-at}$
80	$\frac{s^2}{(s+a)(s^2+\omega^2)}$	$\frac{a^2}{(a^2+\omega^2)}e^{-at}-\frac{\omega}{\sqrt{a^2+\omega^2}}\sin(\omega t+\phi);\ \ \phi=\tan^{-1}\left(\frac{\omega}{a}\right)$
81	$\frac{s^2}{(s+a)^2(s^2+\omega^2)}$	$\left[\frac{a^2}{(a^2+\omega^2)}t-\frac{2a\omega^2}{(a^2+\omega^2)^2}\right]e^{-at}-\frac{\omega}{(a^2+\omega^2)}\sin(\omega t+\phi);$ $\phi=-2\tan^{-1}\left(\frac{\omega}{a}\right)$
82	$\frac{s^2}{(s+a)(s+b)(s^2+\omega^2)}$	$\frac{a^2}{(b-a)(a^2+\omega^2)}e^{-at}+\frac{b^2}{(a-b)(b^2+\omega^2)}e^{-bt}$ $-\frac{\omega}{\sqrt{(a^2+\omega^2)(b^2+\omega^2)}}\sin(\omega t+\phi);\ \ \phi=-\left[\tan^{-1}\left(\frac{\omega}{a}\right)+\tan^{-1}\left(\frac{\omega}{b}\right)\right]$
83	$\frac{s^2}{(s^2+a^2)(s^2+\omega^2)}$	$-\frac{a}{(\omega^2-a^2)}\sin(at)-\frac{\omega}{(a^2-\omega^2)}\sin(\omega t)$
84	$\frac{s^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}(\sin\omega t+\omega t\cos\omega t)$
85	$\frac{s^2}{(s+a)[(s+b)^2+\omega^2]}$	$\frac{a^2}{(a-b)^2+\omega^2}e^{-at}+\frac{1}{\omega}\sqrt{\frac{(b^2-\omega^2)^2+4b^2\omega^2}{(a-b)^2+\omega^2}}\,e^{-bt}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\frac{-2b\omega}{b^2-\omega^2}\right)-\tan^{-1}\left(\frac{\omega}{a-b}\right)$
86	$\frac{s^2}{(s+a)^2[(s+b)^2+\omega^2]}$	$\frac{a^2}{(a-b)^2+\omega^2}te^{-at}-2\left[\frac{a[(b-a)^2+\omega^2]+a^2(b-a)}{[(b-a)^2+\omega^2]^2}\right]e^{-at}$ $+\frac{\sqrt{(b^2-\omega^2)^2+4b^2\omega^2}}{\omega[(a-b)^2+\omega^2]}e^{-bt}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\frac{-2b\omega}{b^2-\omega^2}\right)-2\tan^{-1}\left(\frac{\omega}{a-b}\right)$
87	$\frac{s^2+a}{s^2(s+b)}$	$\frac{b^2+a}{b^2}e^{-bt}+\frac{a}{b}t-\frac{a}{b^2}$
88	$\frac{s^2+a}{s^3(s+b)}$	$\frac{a}{2b}t^2-\frac{a}{b^2}t+\frac{1}{b^3}[b^2+a-(a+b^2)e^{-bt}]$
89	$\frac{s^2+a}{s(s+b)(s+c)}$	$\frac{a}{bc}+\frac{(b^2+a)}{b(b-c)}e^{-bt}-\frac{(c^2+a)}{c(b-c)}e^{-ct}$
90	$\frac{s^2+a}{s^2(s+b)(s+c)}$	$\frac{b^2+a}{b^2(c-b)}e^{-bt}+\frac{c^2+a}{c^2(b-c)}e^{-ct}+\frac{a}{bc}t-\frac{a(b+c)}{b^2c^2}$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
91	$\dfrac{s^2+a}{(s+b)(s+c)(s+d)}$	$\dfrac{b^2+a}{(c-b)(d-b)}e^{-bt}+\dfrac{c^2+a}{(b-c)(d-c)}e^{-ct}+\dfrac{d^2+a}{(b-d)(c-d)}e^{-dt}$
92	$\dfrac{s^2+a}{s(s+b)(s+c)(s+d)}$	$\dfrac{a}{bcd}+\dfrac{b^2+a}{b(b-c)(d-b)}e^{-bt}+\dfrac{c^2+a}{c(b-c)(c-d)}e^{-ct}+\dfrac{d^2+a}{d(b-d)(d-c)}e^{-dt}$
93	$\dfrac{s^2+a}{s^2(s+b)(s+c)(s+d)}$	$\left\{\begin{array}{l}\dfrac{a}{bcd}t-\dfrac{a}{b^2c^2d^2}(bc+cd+db)+\dfrac{b^2+a}{b^2(b-c)(b-d)}e^{-bt}\\ +\dfrac{c^2+a}{c^2(c-b)(c-d)}e^{-ct}+\dfrac{d^2+a}{d^2(d-b)(d-c)}e^{-dt}\end{array}\right.$
94	$\dfrac{s^2+a}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(a+\omega^2)\sin\omega t-\dfrac{1}{2\omega^2}(a-\omega^2)\,t\cos\omega t$
95	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t\cos\omega t$
96	$\dfrac{s^2+a}{s(s^2+\omega^2)^2}$	$\dfrac{a}{\omega^4}-\dfrac{(a-\omega^2)}{2\omega^3}t\sin\omega t-\dfrac{a}{\omega^4}\cos\omega t$
97	$\dfrac{s(s+a)}{(s+b)(s+c)^2}$	$\dfrac{b^2-ab}{(c-b)^2}e^{-bt}+\left[\dfrac{c^2-ac}{b-c}t+\dfrac{c^2-2bc+ab}{(b-c)^2}\right]e^{-ct}$
98	$\dfrac{s(s+a)}{(s+b)(s+c)(s+d)^2}$	$\left\{\begin{array}{l}\dfrac{b^2-ab}{(c-b)(d-b)^2}e^{-bt}+\dfrac{c^2-ac}{(b-c)(d-c)^2}e^{-ct}+\dfrac{d^2-ad}{(b-d)(c-d)}te^{-dt}\\ +\dfrac{a(bc-d^2)+d(db+dc-2bc)}{(b-d)^2(c-d)^2}e^{-dt}\end{array}\right.$
99	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)}$	$\dfrac{b^2-a_1b+a_o}{b^2}e^{-bt}+\dfrac{a_o}{b}t+\dfrac{a_1b-a_o}{b^2}$
100	$\dfrac{s^2+a_1s+a_o}{s^3(s+b)}$	$\dfrac{a_1b-b^2-a_o}{b^3}e^{-bt}+\dfrac{a_o}{2b}t^2+\dfrac{a_1b-a_o}{b^2}t+\dfrac{b^2-a_1b+a_o}{b^3}$
101	$\dfrac{s^2+a_1s+a_o}{s(s+b)(s+c)}$	$\dfrac{a_o}{bc}+\dfrac{b^2-a_1b+a_o}{b(b-c)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{c(c-b)}e^{-ct}$
102	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)(s+c)}$	$\dfrac{a_o}{bc}t+\dfrac{a_1bc-a_o(b+c)}{b^2c^2}+\dfrac{b^2-a_1b+a_o}{b^2(c-b)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{c^2(b-c)}e^{-ct}$
103	$\dfrac{s^2+a_1s+a_o}{(s+b)(s+c)(s+d)}$	$\dfrac{b^2-a_1b+a_o}{(c-b)(d-b)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{(b-c)(d-c)}e^{-ct}+\dfrac{d^2-a_1d+a_o}{(b-d)(c-d)}e^{-dt}$
104	$\dfrac{s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)}$	$\dfrac{a_o}{bcd}-\dfrac{b^2-a_1b+a_o}{b(c-b)(d-b)}e^{-bt}-\dfrac{c^2-a_1c+a_o}{c(b-c)(d-c)}e^{-ct}-\dfrac{d^2-a_1d+a_o}{d(b-d)(c-d)}e^{-dt}$
105	$\dfrac{s^2+a_1s+a_o}{s(s+b)^2}$	$\dfrac{a_o}{b^2}-\dfrac{b^2-a_1b+a_o}{b}t\,e^{-bt}+\dfrac{b^2-a_o}{b^2}e^{-bt}$
106	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)^2}$	$\dfrac{a_o}{b^2}t+\dfrac{a_1b-2a_o}{b^3}+\dfrac{b^2-a_1b+a_o}{b^2}t\,\varepsilon^{-bt}+\dfrac{2a_o-a_1b}{b^3}e^{-bt}$
107	$\dfrac{s^2+a_1s+a_o}{(s+b)(s+c)^2}$	$\dfrac{b^2-a_1b+a_o}{(c-b)^2}e^{-bt}+\dfrac{c^2-a_1c+a_o}{(b-c)}t\,e^{-ct}+\dfrac{c^2-2bc+a_1b-a_o}{(b-c)^2}e^{-ct}$
108	$\dfrac{s^3}{(s+b)(s+c)(s+d)^2}$	$\left\{\begin{array}{l}\dfrac{b^3}{(b-c)(d-b)^2}e^{-bt}+\dfrac{c^3}{(c-b)(d-c)^2}e^{-ct}+\dfrac{d^3}{(d-b)(c-d)}t\,e^{-dt}\\ +\dfrac{d^2[d^2-2d(b+c)+3bc]}{(b-d)^2(c-d)^2}e^{-dt}\end{array}\right.$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
109	$\frac{s^3}{(s+b)(s+c)(s+d)(s+f)^2}$	$\begin{cases} \frac{b^3}{(b-c)(d-b)(f-b)^2}e^{-bt}+\frac{c^3}{(c-b)(d-c)(f-c)^2}e^{-ct} \\ +\frac{d^3}{(d-b)(c-d)(f-d)^2}e^{-dt}+\frac{f^3}{(f-b)(c-f)(d-f)}te^{-ft} \\ +\left[\frac{3f^2}{(b-f)(c-f)(d-f)} \right. \\ \left. +\frac{f^3[(b-f)(c-f)+(b-f)(d-f)+(c-f)(d-f)]}{(b-f)^2(c-f)^2(d-f)^2}\right]\varepsilon^{-dt} \end{cases}$
110	$\frac{s^3}{(s+b)^2(s+c)^2}$	$-\frac{b^3}{(c-b)^2}te^{-bt}+\frac{b^2(3c-b)}{(c-b)^3}e^{-bt}-\frac{c^3}{(b-c)^2}te^{-ct}+\frac{c^2(3b-c)}{(b-c)^3}e^{-ct}$
111	$\frac{s^3}{(s+d)(s+b)^2(s+c)^2}$	$\begin{cases} -\frac{d^3}{(b-d)^2(c-d)^2}e^{-dt}+\frac{b^3}{(c-b)^2(b-d)}te^{-bt} \\ +\left[\frac{3b^2}{(c-b)^2(d-b)}+\frac{b^3(c+2d-3b)}{(c-b)^3(d-b)^2}\right]e^{-bt}+\frac{c^3}{(b-c)^2(c-d)}te^{-ct} \\ +\left[\frac{3c^2}{(b-c)^2(d-c)}+\frac{c^3(b+2d-3c)}{(b-c)^3(d-c)^2}\right]e^{-ct} \end{cases}$
112	$\frac{s^3}{(s+b)(s+c)(s^2+\omega^2)}$	$\begin{cases} \frac{b^3}{(b-c)(b^2+\omega^2)}e^{-bt}+\frac{c^3}{(c-b)(c^2+\omega^2)}e^{-ct} \\ -\frac{\omega^2}{\sqrt{(b^2+\omega^2)(c^2+\omega^2)}}\sin(\omega t+\phi) \\ \phi=\tan^{-1}\left(\frac{c}{\omega}\right)-\tan^{-1}\left(\frac{\omega}{b}\right) \end{cases}$
113	$\frac{s^3}{(s+b)(s+c)(s+d)(s^2+\omega^2)}$	$\begin{cases} \frac{b^3}{(b-c)(d-b)(b^2+\omega^2)}e^{-bt}+\frac{c^3}{(c-b)(d-c)(c^2+\omega^2)}e^{-ct} \\ +\frac{d^3}{(d-b)(c-d)(d^2+\omega^2)}e^{-dt} \\ -\frac{\omega^2}{\sqrt{(b^2+\omega^2)(c^2+\omega^2)(d^2+\omega^2)}}\cos(\omega t-\phi) \\ \phi=\tan^{-1}\left(\frac{\omega}{b}\right)+\tan^{-1}\left(\frac{\omega}{c}\right)+\tan^{-1}\left(\frac{\omega}{d}\right) \end{cases}$
114	$\frac{s^3}{(s+b)^2(s^2+\omega^2)}$	$\begin{cases} -\frac{b^3}{b^2+\omega^2}te^{-bt}+\frac{b^2(b^2+3\omega^2)}{(b^2+\omega^2)^2}e^{-bt}-\frac{\omega^2}{(b^2+\omega^2)}\sin(\omega t+\phi) \\ \phi=\tan^{-1}\left(\frac{b}{\omega}\right)-\tan^{-1}\left(\frac{\omega}{b}\right) \end{cases}$
115	$\frac{s^3}{s^4+4\omega^4}$	$\cos(\omega t)\cosh(\omega t)$
116	$\frac{s^3}{s^4-\omega^4}$	$\frac{1}{2}[\cosh(\omega t)+\cos(\omega t)]$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
117	$\dfrac{s^3+a_2s^2+a_1s+a_o}{s^2(s+b)(s+c)}$	$\left\{\begin{array}{l} \dfrac{a_o}{bc}t-\dfrac{a_o(b+c)-a_1bc}{b^2c^2}+\dfrac{-b^3+a_2b^2-a_1b+a_o}{b^2(c-b)}e^{-bt} \\ \quad +\dfrac{-c^3+a_2c^2-a_1c+a_o}{c^2(b-c)}e^{-ct} \end{array}\right.$
118	$\dfrac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)}$	$\left\{\begin{array}{l} \dfrac{a_o}{bcd}-\dfrac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)}e^{-bt}-\dfrac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)}e^{-ct} \\ \quad -\dfrac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)}e^{-dt} \end{array}\right.$
119	$\dfrac{s^3+a_2s^2+a_1s+a_o}{s^2(s+b)(s+c)(s+d)}$	$\left\{\begin{array}{l} \dfrac{a_o}{bcd}t+\left[\dfrac{a_1}{bcd}-\dfrac{a_o(bc+bd+cd)}{b^2c^2d^2}\right]+\dfrac{-b^3+a_2b^2-a_1b+a_o}{b^2(c-b)(d-b)}\varepsilon^{-bt} \\ \quad +\dfrac{-c^3+a_2c^2-a_1c+a_o}{c^2(b-c)(d-c)}e^{-ct}+\dfrac{-d^3+a_2d^2-a_1d+a_o}{d^2(b-d)(c-d)}e^{-dt} \end{array}\right.$
120	$\dfrac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)}$	$\left\{\begin{array}{l} \dfrac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)}e^{-bt}+\dfrac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)}e^{-ct} \\ \quad +\dfrac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)}e^{-dt}+\dfrac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)}e^{-ft} \end{array}\right.$
121	$\dfrac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)(s+f)}$	$\left\{\begin{array}{l} \dfrac{a_o}{bcdf}-\dfrac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)(f-b)}e^{-bt}-\dfrac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)(f-c)}e^{-ct} \\ \quad -\dfrac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)(f-d)}e^{-dt}-\dfrac{-f^3+a_2f^2-a_1f+a_o}{f(b-f)(c-f)(d-f)}e^{-ft} \end{array}\right.$
122	$\dfrac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)(s+g)}$	$\left\{\begin{array}{l} \dfrac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)(g-b)}e^{-bt}+\dfrac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)(g-c)}e^{-ct} \\ \quad +\dfrac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)(g-d)}e^{-dt}+\dfrac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)(g-f)}e^{-ft} \\ \quad +\dfrac{-g^3+a_2g^2-a_1g+a_o}{(b-g)(c-g)(d-g)(f-g)}e^{-gt} \end{array}\right.$
123	$\dfrac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)^2}$	$\left\{\begin{array}{l} \dfrac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)^2}e^{-bt}+\dfrac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)^2}e^{-ct} \\ \quad +\dfrac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)}te^{-dt} \\ \quad +\dfrac{\begin{array}{l}a_o(2d-b-c)+a_1(bc-d^2) \\ +a_2d(db+dc-2bc)+d^2(d^2-2db-2dc+3bc)\end{array}}{(b-d)^2(c-d)^2}e^{-dt} \end{array}\right.$
124	$\dfrac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)^2}$	$\left\{\begin{array}{l} \dfrac{a_o}{bcd^2}-\dfrac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)^2}e^{-bt}-\dfrac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)^2}e^{-ct} \\ \quad -\dfrac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)}te^{-dt}-\dfrac{3d^2-2a_2d+a_1}{d(b-d)(c-d)}e^{-dt} \\ \quad -\dfrac{(-d^3+a_2d^2-a_1d+a_o)[(b-d)(c-d)-d(b-d)-d(c-d)]}{d^2(b-d)^2(c-d)^2}e^{-dt} \end{array}\right.$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
125	$\frac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)^2}$	$\frac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)^2}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)^2}e^{-ct}$ $+\frac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)^2}e^{-dt}+\frac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)}te^{-ft}$ $+\frac{3f^2-2a_2f+a_1}{(b-f)(c-f)(d-f)}e^{-ft}-\frac{(-f^3+a_2f^2-a_1f+a_o)[(b-f)(c-f)+(b-f)(d-f)+(c-f)(d-f)]}{(b-f)^2(c-f)^2(d-f)^2}e^{-ft}$
126	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
127	$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$
128	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$
129	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf}(a\sqrt{t})$
130	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}\,d\lambda$
131	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf}(a\sqrt{t})$
132	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}\,d\pi$
133	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2t}[b-a\operatorname{erf}(a\sqrt{t})]-be^{b^2t}\operatorname{erfc}(b\sqrt{t})$
134	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t}\operatorname{erfc}(a\sqrt{t})$
135	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}\operatorname{erf}(\sqrt{b-a}\sqrt{t})$
136	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}\left[\frac{b}{a}\operatorname{erf}(a\sqrt{t})-1\right]+e^{b^2t}\operatorname{erfc}(b\sqrt{t})$
137	$\frac{(1-s)^n}{s^{n+(1/2)}}$	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$ $\left[H_n(t)=\text{Hermite polynomial}=e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})\right]$
138	$\frac{(1-s)^n}{s^{n+(3/2)}}$	$-\frac{n!}{\sqrt{\pi}(2n+1)!}H_{2n+1}(\sqrt{t})$
139	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$	$ae^{-at}[I_1(at)+I_o(at)]$ $[I_n(t)=j^{-n}J_n(jt)$ where J_n is Bessel's function of the first kind]
140	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(1/2)(a+b)t}I_o\left(\frac{a-b}{2}t\right)$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
141	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k} (k \ge 0)$	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-(1/2)} e^{-(1/2)(a+b)t} I_{k-(1/2)}\left(\frac{a-b}{2}t\right)$
142	$\frac{1}{(s+a)^{1/2}(s+b)^{3/2}}$	$t e^{-(1/2)(a+b)t}\left[I_o\left(\frac{a-b}{2}t\right)+I_1\left(\frac{a-b}{2}t\right)\right]$
143	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}I_1(at)$
144	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}} \quad (k>0)$	$\frac{k}{t}e^{-(1/2)(a+b)t}I_k\left(\frac{a-b}{2}t\right)$
145	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2v}}{\sqrt{s}\sqrt{s+a}}$	$\frac{1}{a^v}e^{-(1/2)(at)}I_v\left(\frac{1}{2}at\right)$
146	$\frac{1}{\sqrt{s^2+a^2}}$	$J_o(at)$
147	$\frac{(\sqrt{s^2+a^2}-s)^v}{\sqrt{s^2+a^2}} \quad (v>-1)$	$a^v J_v(at)$
148	$\frac{1}{(s^2+a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-(1/2)} J_{k-(1/2)}(at)$
149	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t}J_k(at)$
150	$\frac{(s-\sqrt{s^2-a^2})^v}{\sqrt{s^2-a^2}} \quad (v>-1)$	$a^v I_v(at)$
151	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-(1/2)} I_{k-(1/2)}(at)$
152	$\frac{1}{s\sqrt{s+1}}$	$\text{erf}(\sqrt{t})$; $\text{erf}(y) \triangleq$ the error function $= \frac{2}{\sqrt{\pi}}\int_o^y e^{-u^2}du$
153	$\frac{1}{\sqrt{s^2+a^2}}$	$J_o(at)$; Bessel function of 1st kind, zero order
154	$\frac{1}{\sqrt{s^2+a^2}+s}$	$\frac{J_1(at)}{at}$; J_1 is the Bessel function of 1st kind, 1st order
155	$\frac{1}{[\sqrt{s^2+a^2}+s]^N}$	$\frac{N}{a^N}\frac{J_N(at)}{t}$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
156	$\frac{1}{s[\sqrt{s^2+a^2}+s]^N}$	$\frac{N}{a^N}\int_o^t \frac{J_N(au)}{u}du$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
157	$\frac{1}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)}$	$\frac{1}{a}J_1(at)$; J_1 is the Bessel function of 1st kind, 1st order
158	$\frac{1}{\sqrt{s^2+a^2}[\sqrt{s^2+a^2}+s]^N}$	$\frac{1}{a^N}J_N(at)$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
159	$\frac{1}{\sqrt{s^2-a^2}}$	$I_o(at)$; I_o is the modified Bessel function of 1st kind, zero order
160	$\frac{e^{-ks}}{s}$	$S_k(t)=\begin{cases}0 & \text{when } 0<t<k \\ 1 & \text{when } t>k\end{cases}$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
161	$\dfrac{e^{-ks}}{s^2}$	$\begin{cases} 0 & \text{when } 0<t<k \\ t-k & \text{when } t>k \end{cases}$
162	$\dfrac{e^{-ks}}{s^{\mu}} \quad (\mu>0)$	$\begin{cases} 0 & \text{when } 0<t<k \\ \dfrac{(t-k)^{\mu-1}}{\Gamma(\mu)} & \text{when } t>k \end{cases}$
163	$\dfrac{1-e^{-ks}}{s}$	$\begin{cases} 1 & \text{when } 0<t<k \\ 0 & \text{when } t>k \end{cases}$
164	$\dfrac{1}{s(1-e^{-ks})} = \dfrac{1+\coth\frac{1}{2}ks}{2s}$	$S(k,t) = \begin{cases} n & \text{when} \\ & (n-1)k<t<n\ \ k\,(n=1,2,\cdots) \end{cases}$
165	$\dfrac{1}{s(e^{+ks}-a)}$	$S_k(t) = \begin{cases} 0 & \text{when } 0<t<k \\ 1+a+a^2+\cdots+a^{n-1} & \\ \text{when } nk<t<(n+1)k\ (n=1,2,\cdots) & \end{cases}$
166	$\dfrac{1}{s}\tanh ks$	$\begin{cases} M(2k,t)=(-1)^{n-1} \\ \quad \text{when } 2k(n-1)<t<2nk \\ \quad (n=1,2,\cdots) \end{cases}$
167	$\dfrac{1}{s(1+e^{-ks})}$	$\begin{cases} \dfrac{1}{2}M(k,t)+\dfrac{1}{2}=\dfrac{1-(-1)^n}{2} \\ \quad \text{when } (n-1)k<t<nk \end{cases}$
168	$\dfrac{1}{s^2}\tanh ks$	$\begin{cases} H(2k,t) & [H(2k,t)=k+(r-k)(-1)^n \text{ where } t=2kn+r; \\ & 0\le r\le 2k;\ n=0,1,2,\cdots] \end{cases}$
169	$\dfrac{1}{s\sinh ks}$	$\begin{cases} 2S(2k,t+k)-2=2(n-1) \\ \quad \text{when } (2n-3)k<t<(2n-1)k\ \ (t>0) \end{cases}$
170	$\dfrac{1}{s\cosh ks}$	$\begin{cases} M(2k,t+3k)+1=1+(-1)^n \\ \quad \text{when } (2n-3)k<t<(2n-1)k\ \ (t>0) \end{cases}$
171	$\dfrac{1}{s}\coth ks$	$\begin{cases} 2S(2k,t)-1=2n-1 \\ \quad \text{when } 2k(n-1)<t<2kn \end{cases}$
172	$\dfrac{k}{s^2+k^2}\coth\dfrac{\pi s}{2k}$	$\lvert\sin kt\rvert$
173	$\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$	$\begin{cases} \sin t & \text{when } (2n-2)\pi<t<(2n-1)\pi \\ 0 & \text{when } (2n-1)\pi<t<2n\pi \end{cases}$
174	$\dfrac{1}{s}\,e^{-k/s}$	$J_o(2\sqrt{kt})$
175	$\dfrac{1}{\sqrt{s}}\,e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
176	$\dfrac{1}{\sqrt{s}}\,e^{k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$
177	$\dfrac{1}{s^{3/2}}\,e^{-k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$
178	$\dfrac{1}{s^{3/2}}\,e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
179	$\frac{1}{s^\mu} e^{-k/s} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} J_{\mu-1}(2\sqrt{kt})$
180	$\frac{1}{s^\mu} e^{k/s} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} I_{\mu-1}(2\sqrt{kt})$
181	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} \exp\left(-\frac{k^2}{4t}\right)$
182	$\frac{1}{s} e^{-k\sqrt{s}} \quad (k \ge 0)$	$\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
183	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \quad (k \ge 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$
184	$s^{-3/2} e^{-k\sqrt{s}} \quad (k \ge 0)$	$2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{k^2}{4t}\right) - k \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
185	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k \ge 0)$	$-e^{ak} e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
186	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k \ge 0)$	$e^{ak} e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$
187	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ e^{-(1/2)at} I_o\left(\frac{1}{2} a\sqrt{t^2 - k^2}\right) & \text{when } t > k \end{cases}$
188	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{(s^2+a^2)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ J_o(a\sqrt{t^2 - k^2}) & \text{when } t > k \end{cases}$
189	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{(s^2-a^2)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ I_o(a\sqrt{t^2 - k^2}) & \text{when } t > k \end{cases}$
190	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{(s^2+a^2)}} \quad (k \ge 0)$	$J_o(a\sqrt{t^2 + 2kt})$
191	$e^{-ks} - e^{-k\sqrt{s^2+a^2}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \frac{ak}{\sqrt{t^2-k^2}} J_1(a\sqrt{t^2 - k^2}) & \text{when } t > k \end{cases}$
192	$e^{-k\sqrt{s^2+a^2}} - e^{-ks}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \frac{ak}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2 - k^2}) & \text{when } t > k \end{cases}$
193	$\frac{a^\nu e^{-k\sqrt{s^2-a^2}}}{\sqrt{(s^2+a^2)}\left(\sqrt{s^2+a^2}+s\right)^\nu} \quad (\nu > -1)$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \left(\frac{t-k}{t+k}\right)^{(1/2)\nu} J_\nu(a\sqrt{t^2 - k^2}) & \text{when } t > k \end{cases}$
194	$\frac{1}{s} \log s$	$\Gamma'(1) - \log t \quad [\Gamma'(1) = -0.5772]$
195	$\frac{1}{s^k} \log s \quad (k > 0)$	$t^{k-1}\left\{\frac{\Gamma'(k)}{[\Gamma(k)]^2} \frac{\log t}{\Gamma(k)}\right\}$
196	$\frac{\log s}{s-a} \quad (a > 0)$	$e^{at}[\log a - \operatorname{Ei}(-at)]$

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
197	$\frac{\log s}{s^2+1}$	$\cos t \,\mathrm{Si}(t) - \sin t \,\mathrm{Ci}(t)$
198	$\frac{s \log s}{s^2+1}$	$-\sin t \,\mathrm{Si}(t) - \cos t \,\mathrm{Ci}(t)$
199	$\frac{1}{s}\log(1+ks) \quad (k>0)$	$-\mathrm{Ei}\left(-\frac{t}{k}\right)$
200	$\log \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$
201	$\frac{1}{s}\log(1+k^2 s^2)$	$-2\mathrm{Ci}\left(\frac{t}{k}\right)$
202	$\frac{1}{s}\log(s^2+a^2) \quad (a>0)$	$2\log a - 2\mathrm{Ci}(at)$
203	$\frac{1}{s^2}\log(s^2+a^2) \quad (a>0)$	$\frac{2}{a}[at\log a + \sin at - at\,\mathrm{Ci}(at)]$
204	$\log\frac{s^2+a^2}{s^2}$	$\frac{2}{t}(1-\cos at)$
205	$\log\frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1-\cosh at)$
206	$\arctan\frac{k}{s}$	$\frac{1}{t}\sin kt$
207	$\frac{1}{s}\arctan\frac{k}{s}$	$\mathrm{Si}(kt)$
208	$e^{k^2 s^2}\mathrm{erfc}(ks) \quad (k>0)$	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$
209	$\frac{1}{s}e^{k^2 s^2}\mathrm{erfc}(ks) \quad (k>0)$	$\mathrm{erf}\left(\frac{t}{2k}\right)$
210	$e^{ks}\mathrm{erfc}(\sqrt{ks}) \quad (k>0)$	$\frac{\sqrt{k}}{\pi\sqrt{t(t+k)}}$
211	$\frac{1}{\sqrt{s}}\mathrm{erfc}(\sqrt{ks})$	$\begin{cases} 0 & \text{when } 0<t<k \\ (\pi t)^{-1/2} & \text{when } t>k \end{cases}$
212	$\frac{1}{\sqrt{s}}e^{ks}\mathrm{erfc}(\sqrt{ks}) \quad (k>0)$	$\frac{1}{\sqrt{\pi(t+k)}}$
213	$\mathrm{erf}\left(\frac{k}{\sqrt{s}}\right)$	$\frac{1}{\pi t}\sin(2k\sqrt{t})$
214	$\frac{1}{\sqrt{s}}e^{k^2/s}\mathrm{erfc}\left(\frac{k}{\sqrt{s}}\right)$	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$
215	$-e^{as}\mathrm{Ei}(-as)$	$\frac{1}{t+a}; \quad (a>0)$
216	$\frac{1}{a} + se^{as}\mathrm{Ei}(-as)$	$\frac{1}{(t+a)^2}; \quad (a>0)$
217	$\left[\frac{\pi}{2} - \mathrm{Si}(s)\right]\cos s + \mathrm{Ci}(s)\sin s$	$\frac{1}{t^2+1}$

TABLE 2.3 Table of Laplace Transforms (continued)

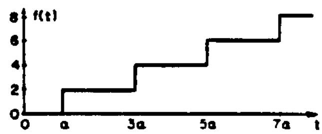
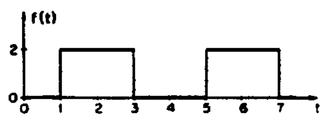
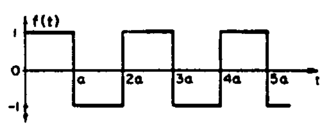
	$F(s)$	$f(t)$
218	$K_o(ks)$	$\begin{cases} 0 & \text{when } 0<t<k \\ (t^2-k^2)^{-1/2} & \text{when } t>k \end{cases}$ [$K_n(t)$ is Bessel function of the second kind of imaginary argument]
219	$K_o(k\sqrt{s})$	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$
220	$\frac{1}{s}e^{ks}K_1(ks)$	$\frac{1}{k}\sqrt{t(t+2k)}$
221	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s})$	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$
222	$\frac{1}{\sqrt{s}}e^{k/s}K_o\left(\frac{k}{s}\right)$	$\frac{2}{\sqrt{\pi t}}K_o(2\sqrt{2kt})$
223	$\pi e^{-ks}I_o(ks)$	$\begin{cases} [t(2k-t)]^{-1/2} & \text{when } 0<t<2k \\ 0 & \text{when } t>2k \end{cases}$
224	$e^{-ks}I_1(ks)$	$\begin{cases} \frac{k-t}{\pi k\sqrt{t(2k-t)}} & \text{when } 0<t<2k \\ 0 & \text{when } t>2k \end{cases}$
225	$\frac{1}{s\sinh(as)}$	$2\sum_{k=0}^{\infty}u[t-(2k+1)a]$ 
226	$\frac{1}{s\cosh s}$	$2\sum_{k=0}^{\infty}(-1)^k u(t-2k-1)$ 
227	$\frac{1}{s}\tanh\left(\frac{as}{2}\right)$	$u(t)+2\sum_{k=1}^{\infty}(-1)^k u(t-ak)$ square wave 
228	$\frac{1}{2s}\left(1+\coth\frac{as}{2}\right)$	$\sum_{k=0}^{\infty}u(t-ak)$ stepped function 

TABLE 2.3 Table of Laplace Transforms (continued)

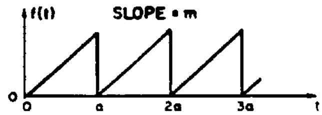
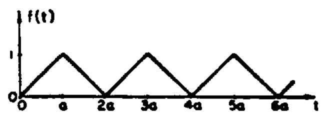
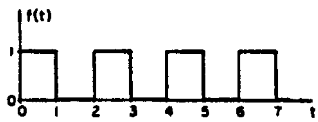
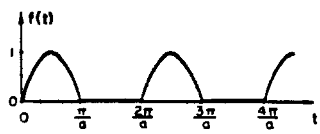
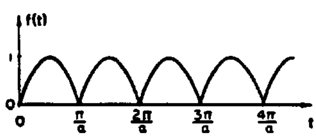
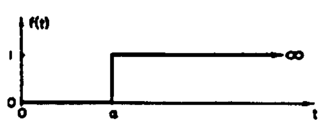
	$F(s)$	$f(t)$
229	$\frac{m}{s^2} - \frac{ma}{2s}\left(\coth\frac{as}{2} - 1\right)$	$mt - ma\sum_{k=1}^{\infty} u(t-ka)$ saw – tooth function 
230	$\frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$	$\frac{1}{a}\left[t + 2\sum_{k=1}^{\infty}(-1)^k (t-ka)\cdot u(t-ka)\right]$ triangular wave 
231	$\frac{1}{s(1+e^{-s})}$	$\sum_{k=0}^{\infty}(-1)^k u(t-k)$ 
232	$\frac{a}{(s^2+a^2)(1-e^{-\frac{\pi}{a}s})}$	$\sum_{k=0}^{\infty}\left[\sin a\left(t-k\frac{\pi}{a}\right)\right]\cdot u\left(t-k\frac{\pi}{a}\right)$ half – wave rectification of sine wave 
233	$\left[\frac{a}{(s^2+a^2)}\right]\coth\left(\frac{\pi s}{2a}\right)$	$[\sin(at)]\cdot u(t) + 2\sum_{k=1}^{\infty}\left[\sin a\left(t-k\frac{\pi}{a}\right)\right]\cdot u\left(t-k\frac{\pi}{a}\right)$ full – wave rectification of sine wave 
234	$\frac{1}{s}e^{-as}$	$u(t-a)$ 

TABLE 2.3 Table of Laplace Transforms (continued)

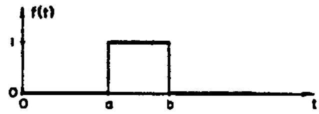
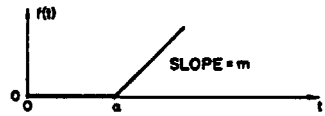
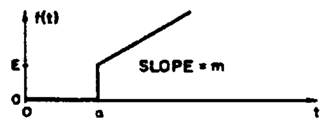
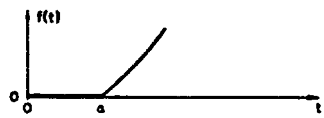
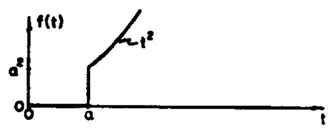
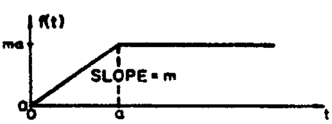
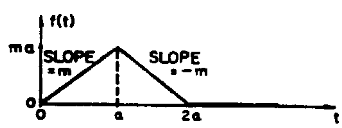
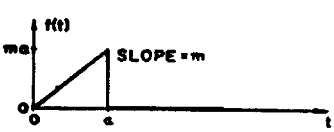
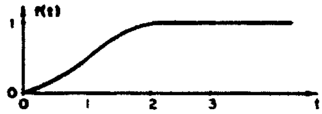
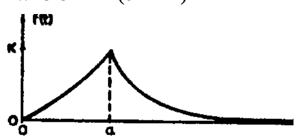
	$F(s)$	$f(t)$
235	$\frac{1}{s}(e^{-as} - e^{-bs})$	$u(t-a) - u(t-b)$ 
236	$\frac{m}{s^2} e^{-as}$	$m \cdot (t-a) \cdot u(t-a)$ 
237	$\left[\frac{ma}{s} + \frac{m}{s^2}\right] e^{-as}$	$mt \cdot u(t-a)$ or $\left[ma + m(t-a)\right] \cdot u(t-a)$ 
238	$\frac{2}{s^3} e^{-as}$	$(t-a)^2 \cdot u(t-a)$ 
239	$\left[\frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\right] e^{-as}$	$t^2 \cdot u(t-a)$ 
240	$\frac{m}{s^2} - \frac{m}{s^2} e^{-as}$	$mt \cdot u(t) - m(t-a) \cdot u(t-a)$ 
241	$\frac{m}{s^2} - \frac{2m}{s^2} e^{-as} + \frac{m}{s^2} e^{-2as}$	$mt - 2m(t-a) \cdot u(t-a) + m(t-2a) \cdot u(t-2a)$ 
242	$\frac{m}{s^2} - \left(\frac{ma}{s} + \frac{m}{s^2}\right) e^{-as}$	$mt - \left[ma + m(t-a)\right] \cdot u(t-a)$ 

TABLE 2.3 Table of Laplace Transforms (continued)

	$F(s)$	$f(t)$
243	$\frac{(1-e^{-s})^2}{s^3}$	$0.5t^2$ for $0 \le t < 1$ $1-0.5(t-2)^2$ for $0 \le t < 2$ 1 for $2 \le t$ 
244	$\left[\frac{(1-e^{-s})}{s}\right]^3$	$0.5t^2$ for $0 \le t < 1$ $0.75-(t-1.5)^2$ for $1 \le t < 2$ $0.5(t-3)^2$ for $2 \le t < 3$ 0 for $3 < t$ 
245	$\frac{b}{s(s-b)} + (e^{ba}-1)\left[\frac{1}{s+b} - \frac{s+\frac{b}{e^{ba}-1}}{s(s-b)}\right]e^{-as}$	$(e^{bt}-1)\cdot u(t) - (e^{bt}-1)\cdot u(t-a) + Ke^{-b(t-a)}\cdot u(t-a)$ where $K = (e^{ba}-1)$ 

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W. Magnus, F. Oberhettinger, and F. G. Tricom, *Tables of Integral Transforms, Vol. I*, McGraw-Hill Book Co., New York, NY, 1954.

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Appendix 1

Examples

1.1 Laplace Transformations

Example 2.1 (Inversion)

The inverse of $\dfrac{s^2+a}{s^2(s+b)}$ is found by partial expansion

$$\frac{s^2+a}{s^2(s+b)} = \frac{A}{s} + \frac{B}{s^2} + \frac{c}{s+b}; \qquad B = \left.\frac{s^2+a}{s+b}\right|_{s=0} = \frac{a}{b}, \qquad C = \left.\frac{s^2+a}{s^2}\right|_{s=-b} = \frac{b^2+a}{b^2}.$$

Hence

$$\frac{s^2+a}{s^2(s+b)} = \frac{A}{s} + \frac{a}{b}\frac{1}{s^2} + \frac{b^2+a}{b^2}\frac{1}{s+b}.$$

Set any value of s, e.g., $s = 1$, and solve for $A = -\dfrac{a}{b^2}$.
Hence

$$L^{-1}\left\{\frac{s^2+a}{s^2(s+b)}\right\} = -\frac{a}{b^2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{a}{b}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{b^2+a}{b^2}L^{-1}\left\{\frac{1}{s+b}\right\} = -\frac{a}{b^2}u(t) + \frac{a}{b}t + \frac{b^2+a}{b^2}e^{-bt}.$$

Example 2.2 (Differential equation)

To solve $y' + by = e^{-t}$ with $y(0) = 1$ we take the Laplace transform of both sides. Hence we obtain $sY(s) - y(0) + bY(s) = \frac{1}{s+1}$ or $Y(s) = \frac{1}{s+b} + \frac{1}{(s+1)(s+b)}$. The inverse transform is $y(t) = e^{-bt} + L^{-1}\{\frac{1}{-1+b}\frac{1}{s+1} + \frac{1}{1-b}\frac{1}{s+b}\} = e^{-bt} + \frac{1}{b-1}e^{-t} + \frac{1}{1-b}e^{-bt} = \frac{2-b}{1-b}e^{-bt} - \frac{1}{1-b}e^{-t}$

1.2 Inversion in the Complex Plane

When the Laplace transform $F(s)$ is known, the function of time can be found by (2.1.2), which is rewritten

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds$$

This equation applies equally well to both the two-sided and the one-sided transforms.

The path of integration is restricted to values of σ for which the direct transform formula converges. In fact, for the two-sided Laplace transform, the region of convergence must be specified in order to determine uniquely the inverse transform. That is, for the two-sided transform, the regions of convergence for functions of time that are zero for $t > 0$, zero for $t < 0$, or in neither category, must be distinguished. For the one-sided transform, the region of convergence is given by σ, where σ is the abscissa of absolute convergence.

The path of integration is usually taken as shown in Figure 2.1 and consists of the straight line *ABC* displaced to the right of the origin by σ and extending in the limit from $-j\infty$ to $+j\infty$ with connecting semicircles. The evaluation of the integral usually proceeds by using the Cauchy integral theorem (see Chapter 20), which specifies that

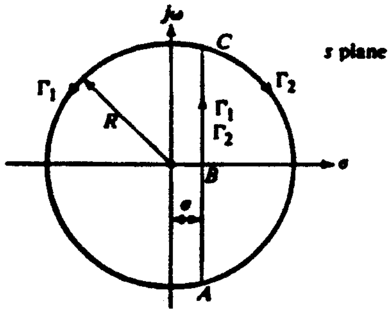


FIGURE 2.1 The Path of Integraton in the *s* Plane

$$f(t) = \frac{1}{2\pi j} \lim_{R\to\infty} \oint_{\Gamma_1} F(s) e^{st}\, ds$$

$$= \sum [\text{residues of } F(s)e^{st} \text{ at the singularities to the left of } ABC] \text{ for } t > 0$$

As we shall find, the contribution to the integral around the circular path with $R \to \infty$ is zero, leaving the desired integral along path *ABC*, and

$$f(t) = \frac{1}{2\pi j} \lim_{R\to\infty} \oint_{\Gamma_2} F(s) e^{st}\, ds$$

$$= -\sum [\text{residues of } F(s)e^{st} \text{ at the singularities to the right of } ABC] \text{ for } t < 0$$

Example 2.3

Use the inversion integral to find $f(t)$ for the function

$$F(s) = \frac{1}{s^2 + \omega^2}$$

Note that the inverse of the above formula is $\sin\omega t/\omega$.

Solution

The inversion integral is written in a form that shows the poles of the integrand

$$f(t) = \frac{1}{2\pi j} \oint \frac{e^{st}}{(s + j\omega)(s - j\omega)}\, ds$$

The path chosen is Γ_1 in Figure 2.1. Evaluate the residues

$$\text{Res}\left[(s - j\omega)\frac{e^{st}}{(s^2 + \omega^2)}\right]_{s=j\omega} = \left.\frac{e^{st}}{(s + j\omega)}\right|_{s=j\omega} = \frac{e^{j\omega t}}{2j\omega}$$

$$\text{Res}\left[(s + j\omega)\frac{e^{st}}{(s^2 + \omega^2)}\right]_{s=-j\omega} = \left.\frac{e^{st}}{(s - j\omega)}\right|_{s=-j\omega} = \frac{e^{-j\omega t}}{-2j\omega}$$

Therefore,

$$f(t) = \sum \text{Res} = \frac{e^{j\omega t} - e^{-j\omega t}}{-2j\omega} = \frac{\sin \omega t}{\omega}.$$

Example 2.4

Find $L^{-1}\{1/\sqrt{s}\}$.

Solution

The function $F(s) = 1/\sqrt{s}$ is a double-valued function because of the square root operation. That is, if s is represented in polar form by $re^{j\theta}$, then $re^{j(\theta+2\pi)}$ is a second acceptable representation, and $\sqrt{s} = \sqrt{re^{j(\theta+2\pi)}} = -\sqrt{re^{j\theta}}$, thus showing two different values for $\sqrt{s}$. But a double-valued function is not analytic and requires a special procedure in its solution.

The procedure is to make the function analytic by restricting the angle of s to the range $-\pi < \theta < \pi$ and by excluding the point $s = 0$. This is done by constructing a branch cut along the negative real axis, as shown in Figure 2.2. The end of the **branch cut**, which is the origin in this case, is called a **branch point**. Since a branch cut can never be crossed, this essentially ensures that $F(s)$ is single-valued. Now, however, the inversion integral becomes, for $t > 0$,

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi j}\int_{GAB} F(s)e^{st}\,ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds$$

$$= -\frac{1}{2\pi j}\left[\int_{BC} + \int_{\Gamma_2} + \int_{l-} + \int_{\gamma} + \int_{l+} + \int_{\Gamma_3} + \int_{FG}\right]$$

which does not include any singularity.

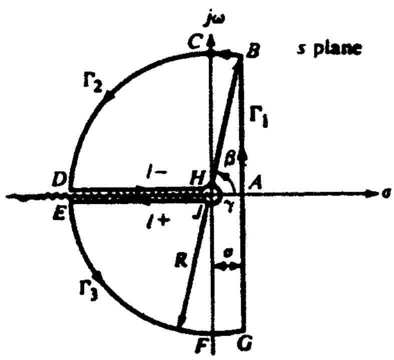


FIGURE 2.2 The Integration Contour for $L^{-1}\{1/\sqrt{s}\}$

First we will show that for $t > 0$ the integrals over the contours BC and CD vanish as $R \to \infty$, from which $\int_{\Gamma_2} = \int_{\Gamma_3} = \int_{BC} = \int_{FG} = 0$. Note from Figure 2.2 that $\beta = \cos^{-1}(\sigma/R)$ so that the integral over the arc BC is, since $\left|e^{j\theta}\right| = 1$,

$$|I| \le \int_{BC}\left|\frac{e^{\sigma t}e^{j\omega t}}{R^{1/2}e^{j\theta/2}}\,jRe^{j\theta}\,d\theta\right| = e^{\sigma t}R^{1/2}\int_{\beta}^{\pi/2} d\theta = e^{\sigma t}R^{1/2}\left(\frac{\pi}{2} - \cos^{-1}\frac{\sigma}{R}\right)$$

$$= e^{\sigma t}R^{1/2}\sin^{-1}\frac{\sigma}{R}$$

But for small arguments $\sin^{-1}(\sigma/R) = \sigma/R$, and in the limit as $R \to \infty$, $I \to 0$. By a similar approach, we find that the integral over CD is zero. Thus the integrals over the contours Γ_2 and Γ_3 are also zero as $R \to \infty$.

For evaluating the integral over γ, let $s = re^{j\theta} = r(\cos\theta + j\sin\theta)$ and

$$\int_\gamma F(s)e^{st}\,ds = \int_\pi^{-\pi} \frac{e^{r(\cos\theta + j\sin\theta)}}{\sqrt{r}\,e^{j\theta/2}}\, jre^{j\theta}\,d\theta$$

$$= 0 \text{ as } r \to 0$$

The remaining integrals are written

$$f(t) = -\frac{1}{2\pi j}\left[\int_{\ell-} F(s)e^{st}\,ds + \int_{\ell+} F(s)e^{st}\,ds\right]$$

Along path $l-$, let $s = -u$; $\sqrt{s} = j\sqrt{u}$, and $ds = -du$, where u and $\sqrt{u}$ are real positive quantities. Then

$$\int_{l-} F(s)e^{st}\,ds = -\int_\infty^0 \frac{e^{-ut}}{j\sqrt{u}}\,du = \frac{1}{j}\int_0^\infty \frac{e^{-ut}}{\sqrt{u}}\,du$$

Along path $l+$, $s = -u$; $\sqrt{s} = -j\sqrt{u}$ (not $+j\sqrt{u}$), and $ds = -du$. Then

$$\int_{l+} F(s)e^{st}\,ds = -\int_0^\infty \frac{e^{-ut}}{-j\sqrt{u}}\,du = \frac{1}{j}\int_0^\infty \frac{e^{-ut}}{j\sqrt{u}}\,du$$

Combine these results to find

$$f(t) = -\frac{1}{2\pi j}\left[\frac{2}{j}\int_0^\infty u^{-1/2}e^{-ut}\,du\right] = \frac{1}{\pi}\int_0^\infty u^{-1/2}e^{-ut}\,du$$

which is a standard form integral listed in most handbooks of mathematical tables, with the result

$$f(t) = \frac{1}{\pi}\sqrt{\frac{\pi}{t}} = \frac{1}{\sqrt{\pi t}} \qquad t > 0\,.$$

Example 2.5

Find the inverse Laplace transform of the given function with an infinite number of poles.

$$F(s) = \frac{1}{s(1 + e^{-s})}$$

Solution

The integrand in the inversion integral $e^{st}/s(1 + e^{-s})$ possesses simple poles at

$$s = 0 \text{ and } s = jn\pi, \quad n = \pm 1, \pm 3, + \cdots \quad \text{(odd values)}$$

These are illustrated in Figure 2.3. This means that the function $e^{st}/s(1 + e^{-s})$ is analytic in the s plane except at the simple poles at $s = 0$ and $s = jn\pi$. Hence, the integral is specified in terms of the residues in the various poles. We thus have:

For $s = 0$

$$\text{Res} = \left\{ \frac{se^{st}}{s(1+e^{-s})} \right\}\Bigg|_{s=0} = \frac{1}{2}$$

For $s = jn\pi$

$$\text{Res} = \left\{ \frac{(s - jn\pi)e^{st}}{s(1+e^{-s})} \right\}\Bigg|_{s=jn\pi} = \frac{0}{0}$$

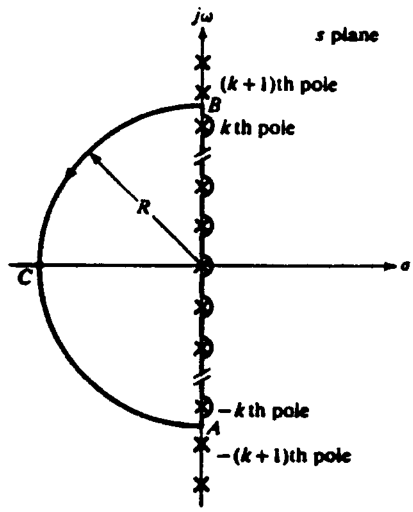


FIGURE 2.3 Illustrating Example 2.5, the Laplace Inversion for the Case of Infinitely Many Poles

The problem we now face in this evaluation is that

$$\text{Res} = \left\{ (s-a)\frac{n(s)}{d(s)} \right\}\Bigg|_{s=a} = \frac{0}{0}$$

where the roots of d(s) are such that $s = a$ cannot be factored. However, we have discussed such a situation in Chapter 20 for complex variables, and we have the following result

$$\frac{d[d(s)]}{ds}\Bigg|_{s=a} = \lim_{s \to a} \frac{d(s) - d(a)}{s-a} = \lim_{s \to a} \frac{d(s)}{s-a} \quad \text{since } d(a) = 0.$$

Combine this expression with the above equation to obtain

$$\text{Re}\, s\left\{ (s-a)\frac{n(s)}{d(s)} \right\}\Bigg|_{s=a} = \frac{n(s)}{\dfrac{d}{ds}[d(s)]}\Bigg|_{s=a}.$$

Therefore, we proceed as follows:

$$\text{Res} = \left\{ \frac{e^{st}}{s\frac{a}{ds}(1+e^{-s})} \right\}\Bigg|_{s=jn\pi} = \frac{e^{jn\pi t}}{jn\pi} \quad (n = \text{odd})$$

We obtain, by adding all of the residues,

$$f(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{e^{jn\pi t}}{jn\pi} \quad (n = \text{odd})$$

This can be rewritten as follows

$$f(t) = \frac{1}{2} + \left[\cdots + \frac{e^{-j3\pi t}}{-j3\pi} + \frac{e^{-j\pi t}}{-j\pi} + \frac{e^{j\pi t}}{j\pi} + \frac{e^{j3\pi t}}{j3\pi} + \cdots \right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2j\sin n\pi t}{jn\pi} \quad (n = \text{odd})$$

which we write, finally

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi t}{2k-1}$$

As a second approach to a solution to this problem, we will show the details in carrying out the contour integration for this problem. We choose the path shown in Figure 2.3 that includes semicircular hooks around each pole, the vertical connecting line from hook to hook, and the semicircular path as $R \to \infty$. Thus we have

$$f(t) = \frac{1}{2\pi j} \oint \frac{se^{st}\,ds}{s(1+e^{-s})}$$

$$= \frac{1}{2\pi j} \left[\underbrace{\int_{\text{BCA}}}_{I_1} + \underbrace{\int_{\text{Vertical connecting lines}}}_{I_2} + \underbrace{\Sigma \int_{\text{Hooks}}}_{I_3} - \Sigma\,\text{Res} \right]$$

We consider the several integrals:

Integral I_1. By setting $s = re^{j\theta}$ and taking into consideration that cos = –cosθ for $\theta > \pi/2$, the integral $I_1 \to 0$ as $r \to \infty$.

Integral I_2. Along the *Y*-axis, $s = jy$ and

$$I_2 = j\int_{\substack{-\infty \\ r\to 0}}^{\infty} \frac{e^{jyt}}{jy(1+e^{-jy})}\,dy$$

Note that the integrated is an odd function, whence $I_2 = 0$.

Integral I_3. Consider a typical hook at $s = jn\pi$. Since

$$\lim_{\substack{r\to 0\\ s\to jn\pi}} \left[\frac{(s - jn\pi)e^{st}}{s(1+e^{-s})} \right] = \frac{0}{0},$$

this expression is evaluated and yields $e^{jn\pi t}/jn\pi$. Thus, for all poles,

$$I_3 = \frac{1}{2\pi j} \int_{\substack{-\pi/2\\ r\to 0\\ s\to jn\pi}}^{\pi/2} \frac{e^{st}}{s(1+e^{-s})} ds$$

$$= \frac{j\pi}{2\pi j} \left[\sum_{\substack{n=-\infty\\ n \text{ odd}}}^{\infty} \frac{e^{jn\pi t}}{jn\pi} + \frac{1}{2} \right] = \frac{1}{2} \left[\frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\sin n\pi t}{n} \right]$$

Finally, the residues enclosed within the contour are

$$\text{Res} \frac{e^{st}}{s(1+e^{-s})} = \frac{1}{2} + \sum_{\substack{n=-\infty\\ n \text{ odd}}}^{\infty} \frac{e^{jn\pi t}}{jn\pi} = \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\sin n\pi t}{n}$$

which is seen to be twice the value around the hooks. Then when all terms are included

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\sin n\pi t}{n} = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi t}{2k-1}.$$

1.3 Complex Integration and the Bilateral Laplace Transform

We have discussed the fact that the region of absolute convergence of the unilateral Laplace transform is the region to the left of the abscissa of convergence. This is not true for the bilateral Laplace transform: the region of convergence must be specified to invert a function $F(s)$ obtained using the bilateral Laplace transform. This requirement is necessary because different time signals might have the same Laplace transform but different regions of absolute convergence.

To establish the region of convergence, we write the bilateral transform in the form

$$F_2(s) = \int_0^{\infty} e^{-st} f(t)\, dt + \int_{-\infty}^{0} e^{-st} f(t)\, dt$$

If the function $f(t)$ is of exponential order $(e^{\sigma_1 t})$, the region of convergence for $t > 0$ is $\text{Re}\{s\} > \sigma_1$ If the function $f(t)$ for $t < 0$ is of exponential order $\exp(\sigma_2 t)$, then the region of convergence is $\text{Re}\{s\} < \sigma_2$. Hence, the function $F_2(s)$ exists and is analytic in the vertical strip defined by

$$\sigma_1 < \text{Re}\{s\} < \sigma_2$$

Provided, of course, that $\sigma_1 < \sigma_2$. If $\sigma_1 > \sigma_2$, no region of convergence would exist and the inversion process could not be performed. This region of convergence is shown in Figure 2.4.

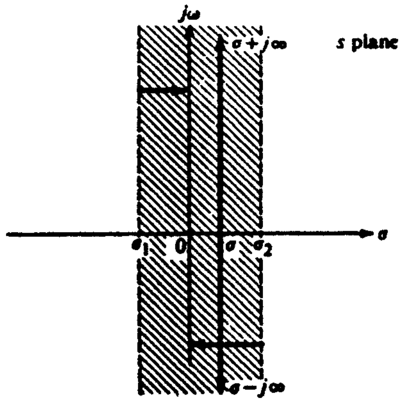


FIGURE 2.4 Region of Convergence for the Bilateral Laplace Transform

Example 2.6

Find the bilateral Laplace transform of the signals $f(t) = e^{-at}u(t)$ and $f(t) = -e^{-at}u(-t)$ and specify their regions of convergence.

Solution

Using the basic definition of the transform, we obtain

a. $$F_2(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt = \frac{1}{s+a}$$

and its region of convergence is

$$\text{Re}\{s\} > -a$$

For the second signal

b. $$F_2(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(s+a)t}\,dt = \frac{1}{s+a}$$

and its region of convergence is

$$\text{Re}\{s\} < -a$$

Clearly, the knowledge of the region of convergence is necessary to find the time function unambiguously.

Example 2.7

Find the function, if its Laplace transform is given by

$$F_2(s) = \frac{1}{(s-4)(s+1)(s+2)} \qquad -2 < \text{Re}\{s\} < -1$$

Solution

The region of convergence and the paths of integration are shown in Figure 2.5.

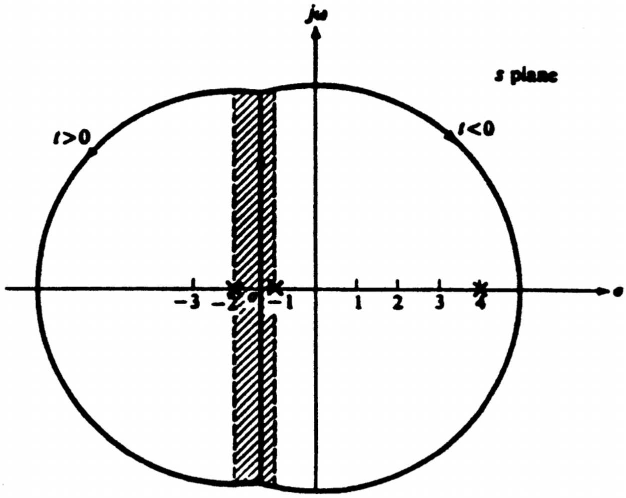


FIGURE 2.5 Illustrating Example 2.7

For $t > 0$, we close the contour to the left, we obtain

$$f(t) = \frac{3e^{st}}{(s-4)(s+1)}\bigg|_{s=-1} = \frac{1}{2}e^{-2t} \qquad t > 0$$

For $t < 0$, the contour closes to the right, and now

$$f(t) = \frac{3e^{st}}{(s-4)(s+2)}\bigg|_{s=-1} + \frac{3e^{st}}{(s+1)(s+2)}\bigg|_{s=4} = -\frac{3}{5}e^{-t} + \frac{e^{4t}}{10} \qquad t < 0$$

These examples confirm that we must know the region of convergence to find the inverse transform.

Poularikas A. D. "Fourier Transform"
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

3

Fourier Transform

3.1 One-Dimensional Fourier Transform

3.1.1 Definitions

3.1.1.1 Fourier Transform

$$F(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ft}\, dt = |F(f)| e^{j\phi(\omega)}$$

$$f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi ft}\, df$$

or

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = |F(\omega)| e^{j\phi(\omega)} = R(\omega) + jX(\omega)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

If $f(t)$ is piecewise continuous and absolutely integrable, then its Fourier transform is a bounded continuous function, bounded by $\int_{-\infty}^{\infty} |F(\omega)|\, d\omega$. If $F(\omega)$ is absolutely integrable $|F(\omega)|$ then its inverse is $f(t)$.

3.1.1.2

$f(t) = f_r(t) + j\, f_i(t)$ is complex, $f_r(t)$ and $f_i(t)$ are real functions

$$F(\omega)=\int_{-\infty}^{\infty}[f_r(t)\cos\omega t+f_i(t)\sin\omega t]dt-j\int_{-\infty}^{\infty}[f_r(t)\sin\omega t-f_i(t)\cos\omega t]dt$$

$$R(\omega)=\int_{-\infty}^{\infty}[f_r(t)\cos\omega t+f_i(t)\sin\omega t]dt$$

$$X(\omega)=-\int_{-\infty}^{\infty}[f_r(t)\sin\omega t-f_i(t)\cos\omega t]dt$$

$$f_r(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega)\cos\omega t-X(\omega)\sin\omega t]d\omega$$

$$f_i(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega)\sin\omega t+X(\omega)\cos\omega t]d\omega$$

$f(t)=\text{real}$ $(f_r(t) = f(t),\ f_i(t) = 0)$

$$R(\omega)=\int_{-\infty}^{\infty}f(t)\cos\omega t\,dt \qquad X(\omega)=-\int_{-\infty}^{\infty}f(t)\sin\omega t\,dt$$

$$R(\omega)=R(-\omega) \qquad X(-\omega)=-X(\omega) \qquad F(-\omega)=F^*(\omega)$$

$$f(t)=\frac{1}{\pi}\operatorname{Re}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}\,d\omega$$

$f(t)=j\,f_i(t)$ is purely imaginary $(f_r(t) = 0)$

$$R(\omega)=\int_{-\infty}^{\infty}f_i(t)\sin\omega t\,dt \qquad X(\omega)=\int_{-\infty}^{\infty}f_i(t)\cos\omega t\,dt$$

$$R(\omega)=-R(-\omega) \qquad X(\omega)=X(-\omega) \qquad F(-\omega)=-F^*(\omega)$$

$f(t)$ is even $[f(t)=f(-t)]$

$$R(\omega)=2\int_{o}^{\infty}f(t)\cos\omega t\,dt \qquad X(\omega)=0$$

$$f(t)=\frac{1}{\pi}\int_{o}^{\infty}R(\omega)\cos\omega t\,d\omega$$

$f(t)$ is odd $[f(t)=-f(-t)]$

$$R(\omega)=0 \qquad X(\omega)=-2\int_{o}^{\infty}f(t)\sin\omega t\,dt$$

$$f(t)=-\frac{1}{\pi}\int_{o}^{\infty}X(\omega)\sin\omega t\,d\omega$$

3.1.2 Properties

3.1.2.1 Properties of Fourier Transform

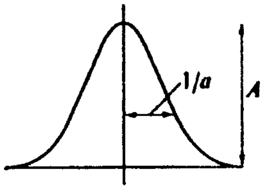
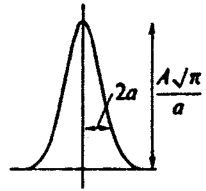
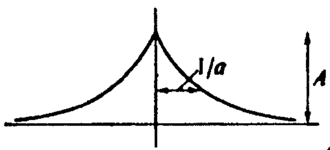
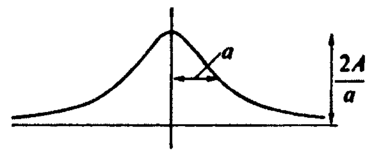
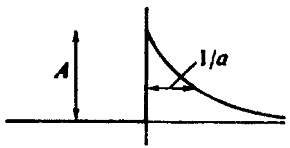
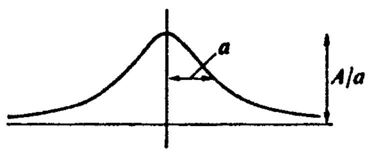
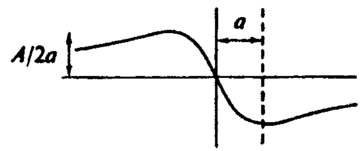
TABLE 3.1 Properties of Fourier Transform

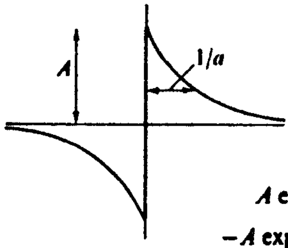
	Operation	$f(t)$	$F(\omega)$
1.	Transform-direct	$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
2.	Inverse transform	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$	$F(\omega)$
3.	Linearity	$af_1(t)+bf_2(t)$	$aF_1(\omega)+bF_2(\omega)$
4.	Symmetry	$F(t)$	$2\pi f(-\omega)$
5.	Time shifting	$f(t\pm t_o)$	$e^{\pm j\omega t_o}F(\omega)$
6.	Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
7.	Frequency shifting	$e^{\pm j\omega_o t}f(t)$	$F(\omega\mp\omega_o)$
8.	Modulation	$\begin{cases} f(t)\cos\omega_o t \\ f(t)\sin\omega_o t\end{cases}$	$\frac{1}{2}[F(\omega+\omega_o)+F(\omega-\omega_o)]$ $\frac{1}{2j}[F(\omega-\omega_o)-F(\omega+\omega_o)]$
9.	Time differentiation	$\frac{d^n}{dt^n}f(t)$	$(j\omega)^n F(\omega)$
10.	Time convolution	$f(t)*h(t)=\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$	$F(\omega)H(\omega)$
11.	Frequency convolution	$f(t)h(t)$	$\frac{1}{2\pi}F(\omega)*H(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\tau)H(\omega-\tau)\,d\tau$
12.	Autocorrelation	$f(t)\bigstar f^*(t)=\int_{-\infty}^{\infty} f(\tau)f^*(\tau-t)\,d\tau$	$F(\omega)F^*(\omega)=\lvert F(\omega)\rvert^2$
13.	Parseval's formula	$E=\int_{-\infty}^{\infty}\lvert f(t)\rvert^2\,dt$	$E=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert F(\omega)\rvert^2\,d\omega$
14.	Moments formula	$m_n=\int_{-\infty}^{\infty} t^n f(t)\,dt=\frac{F^{(n)}(0)}{(-j)^n}$ where	$F^{(n)}(0)=\left.\frac{d^n F(\omega)}{d\omega^n}\right\rvert_{\omega=0}, \quad n=0,1,2\cdots$
15.	Frequency differentiation	$\begin{cases}(-jt)f(t) \\ (-jt)^n f(t)\end{cases}$	$\frac{dF(\omega)}{d\omega}$ $\frac{d^n F(\omega)}{d\omega^n}$
16.	Time reversal	$f(-t)$	$F(-\omega)$
17.	Conjugate function	$f^*(t)$	$F^*(-\omega)$
18.	Integral $(F(0)=0)$	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{j\omega}F(\omega)$
19.	Integral $(F(0)\neq 0)$	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{j\omega}F(\omega)+\pi F(0)\delta(\omega)$

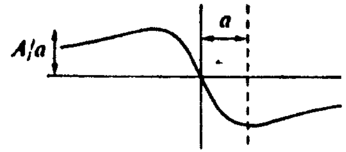
3.1.3 Tables

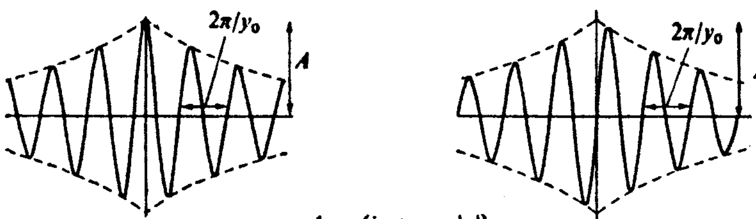
3.1.3.1 Graphical Representations of Some Fourier Transforms

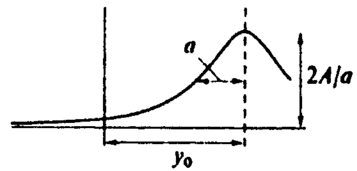
TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$)

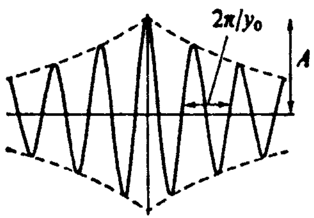
$f(x)$ $\left[f(x) = (1/2\pi)\int_{-\infty}^{+\infty} F(y)e^{+ixy}\,dy\right]$	$F(y)$ $\left[F(y) = \int_{-\infty}^{+\infty} f(x)e^{-ixy}\,dx\right]$	
 $A\exp(-a^2x^2)$ [Gaussian]	 $\frac{A\sqrt{\pi}}{a}\exp(-y^2/4a^2)$ [Gaussian]	(3.1)
 $A\exp(-a\|x\|)$	$\frac{2A}{a}\frac{a^2}{a^2+y^2}$ [Lorentzian] 	(3.2)
 $A\exp(-ax)$ [$x > 0$] 0 [$x < 0$]	  $A\left\{\frac{a-iy}{a^2+y^2}\right\}$	(3.3)

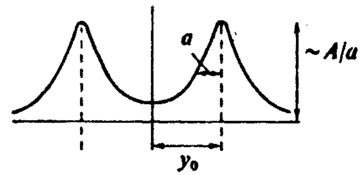
<table>
<tr><th>$f(x)$</th><th>$F(y)$</th><th></th></tr>
<tr><td>

$A \exp(-ax) \quad [x > 0]$
$-A \exp(-a|x|) \quad [x < 0]$</td><td>

$-2iA \dfrac{y}{a^2 + y^2}$</td><td>(3.4)</td></tr>
<tr><td>

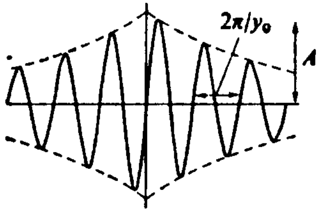
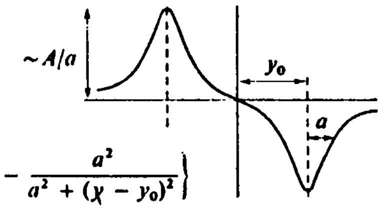
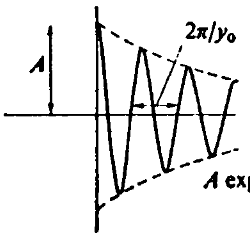
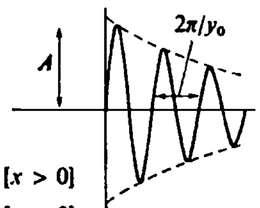
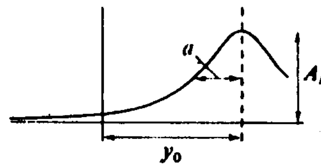
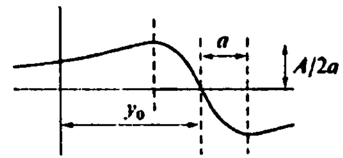
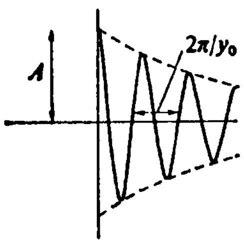
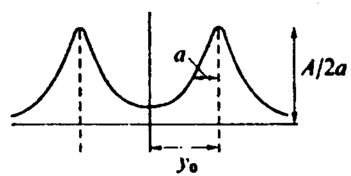
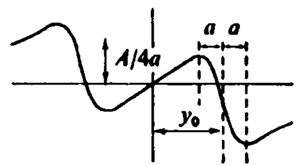
$A \exp(iy_0x - a|x|)$</td><td>

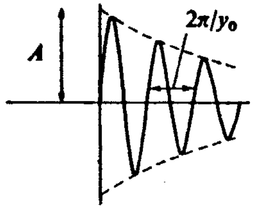
$\dfrac{2A}{a} \dfrac{a^2}{a^2 + (y - y_0)^2}$</td><td>(3.5)</td></tr>
<tr><td>

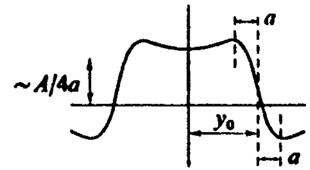
$A \cos y_0x \exp(-a|x|)$</td><td>

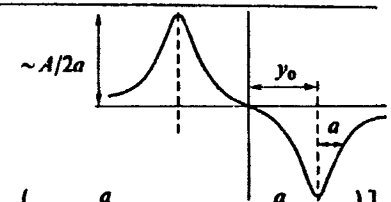
$\dfrac{A}{a}\left\{\dfrac{a^2}{a^2 + (y - y_0)^2} + \dfrac{a^2}{a^2 + (y + y_0)^2}\right\}$
$= \dfrac{A}{a}\left\{\dfrac{2a^2(a^2 + y_0^2 + y^2)}{(a^2 + y_0^2 - y^2)^2 + 4a^2y^2}\right\}$</td><td>(3.6)</td></tr>
</table>

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

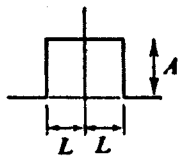
$f(x)$	$F(y)$	
 $A \sin y_0 x \exp(-a\|x\|)$	 $\frac{iA}{a}\left\{\frac{a^2}{a^2+(y+y_0)^2} - \frac{a^2}{a^2+(y-y_0)^2}\right\}$ $= \frac{iA}{a}\left\{\frac{-4a^2 y y_0}{(a^2+y_0^2-y^2)^2+4a^2y^2}\right\}$	(3.7)
 $A \exp(iy_0 x - ax)$ [$x > 0$] 0 [$x < 0$]	 $A\left\{\frac{a+i(y_0-y)}{a^2+(y_0-y)^2}\right\} = A\left\{\frac{1}{a+i(y-y_0)}\right\}$	(3.8)
 $A \cos y_0 x \exp(-ax)$ [$x > 0$] 0 [$x < 0$]	 $\frac{A}{2}\left[\left\{\frac{a}{a^2+(y+y_0)^2}+\frac{a}{a^2+(y-y_0)^2}\right\} + i\left\{\frac{y_0-y}{a^2+(y_0-y)^2} - \frac{y_0+y}{a^2+(y_0+y)^2}\right\}\right]$ $= A\left\{\frac{a(a^2+y_0^2+y^2) - iy(a^2+y^2-y_0^2)}{(a^2+y_0^2-y^2)^2+4a^2y^2}\right\}$	(3.9)

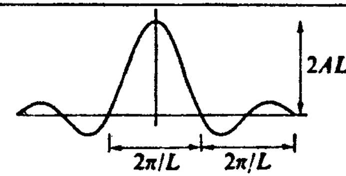
<table>
<tr><th>$f(x)$</th><th>$F(y)$</th><th></th></tr>
<tr><td>


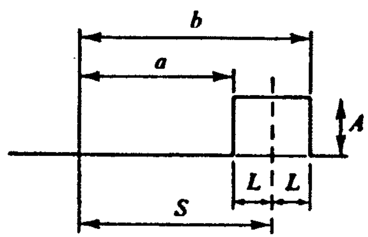
$A \sin y_0 x \exp(-ax) \quad [x > 0]$
$0 \quad [x < 0]$
</td><td>


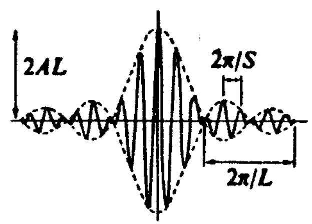


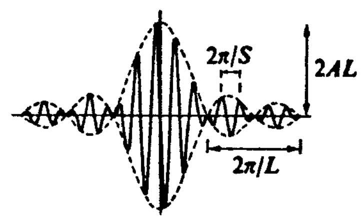
$$\frac{A}{2}\left[\left\{\frac{y_0 - y}{a^2 + (y_0 - y)^2} + \frac{y_0 + y}{a^2 + (y_0 + y)^2}\right\} + i\left\{\frac{a}{a^2 + (y_0 + y)^2} - \frac{a}{a^2 + (y_0 - y)^2}\right\}\right]$$

$$= Ay_0 \left\{\frac{1}{(a^2 + y_0^2 - y^2) + i2ay}\right\}$$
</td><td>(3.10)</td></tr>
<tr><td>


$A \quad [|x| < L]$
$0 \quad [|x| > L]$
</td><td>


$$2A\frac{\sin Ly}{y}$$
</td><td>(3.11)</td></tr>
<tr><td>


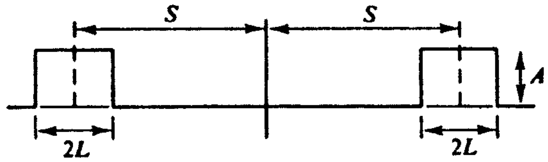
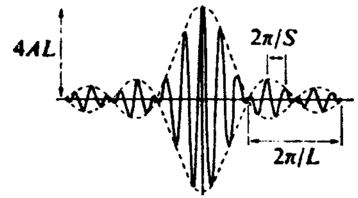
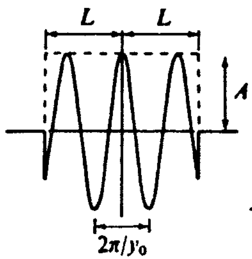
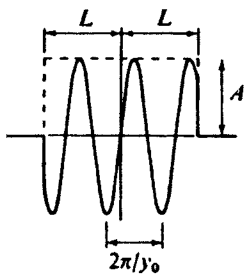
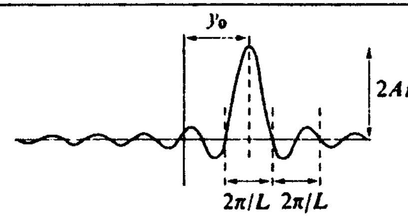
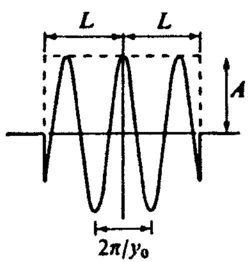
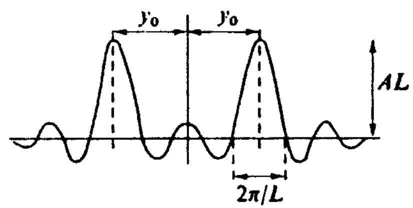
$A \quad [a < x < b]$
$0 \quad [x < a;\ x > b]$
</td><td>




$$2A\frac{\sin Ly}{y}\exp(-iSy) = A\left[\frac{(\sin by - \sin ay) - i(\cos ay - \cos by)}{y}\right]$$

$$= 2A\left[\frac{(\sin Ly \cos Sy) - i(\sin Ly \sin Sy)}{y}\right] = \frac{iA}{y}[\exp(-iby) - \exp(-iay)]$$
</td><td>(3.12)</td></tr>
</table>

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
 $A \quad [(S - L) < \lvert x \rvert < (S + L)]$ $0 \quad$ [otherwise]	 $4A \dfrac{\cos Sy \sin Ly}{y}$	(3.13)
 $A \exp(iy_0x) \quad [\lvert x \rvert < L]$ $0 \quad [\lvert x \rvert > L]$ 	 $2A \dfrac{\sin\{L(y_0 - y)\}}{(y_0 - y)}$	(3.14)
 $A \cos y_0x \quad [\lvert x \rvert < L]$ $0 \quad [\lvert x \rvert > L]$	 $A\left[\dfrac{\sin L(y - y_0)}{(y - y_0)} + \dfrac{\sin L(y + y_0)}{(y + y_0)}\right]$	(3.15)

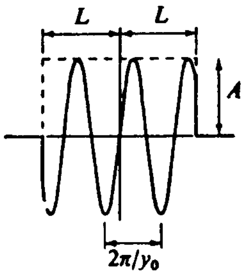
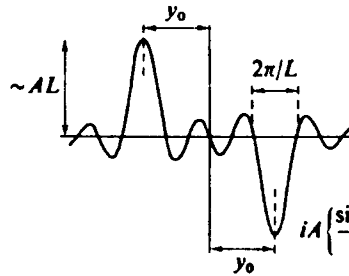
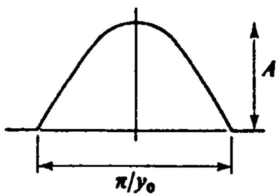
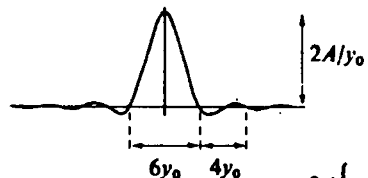
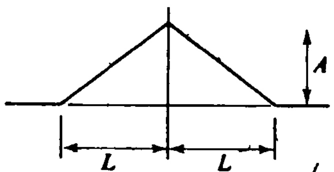
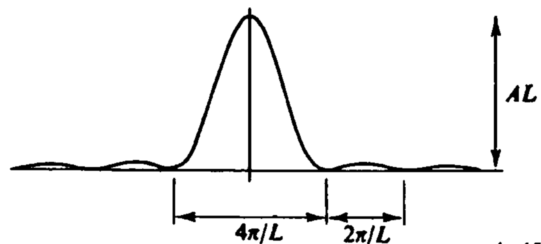
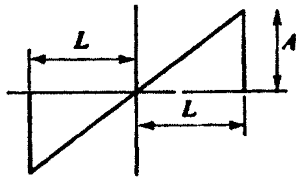
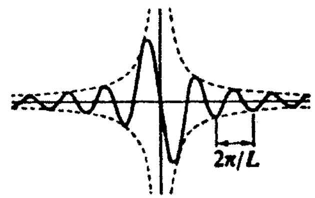
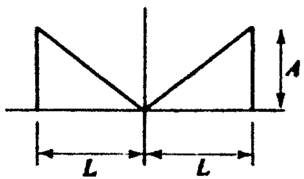
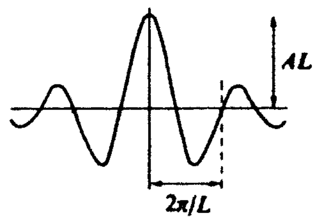
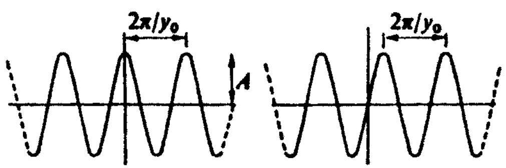
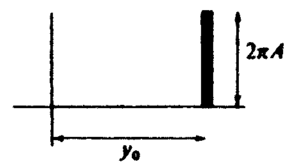
$f(x)$	$F(y)$	
 $A \sin y_0 x \quad [\lvert x\rvert < L]$ $0 \quad [\lvert x\rvert > L]$	 $iA\left\{\frac{\sin L(y+y_0)}{(y+y_0)} - \frac{\sin L(y-y_0)}{(y-y_0)}\right\}$	(3.16)
 $A \cos y_0 x \quad [\lvert x\rvert < (\pi/2y_0)]$ $0 \quad [\lvert x\rvert > (\pi/2y_0)]$	 $2A\left\{\frac{y_0}{y_0^2 - y^2}\right\}\cos\left(\frac{\pi y}{2y_0}\right)$ [See (2.52) with $L = \pi/2y_0$].	(3.17)
 $A\left(1 - \frac{\lvert x\rvert}{L}\right) \quad [\lvert x\rvert < L]$ $0 \quad [\lvert x\rvert > L]$	 $AL\left\{\frac{\sin(Ly/2)}{(Ly/2)}\right\}^2$	(3.18)

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
		
$\frac{Ax}{L}$ $[\lvert x\rvert < L]$ 0 $[\lvert x\rvert > L]$	$\frac{2iA}{y}\left\{\cos Ly - \frac{\sin Ly}{Ly}\right\}$	(3.19)
		
$\frac{A\lvert x\rvert}{L}$ $[\lvert x\rvert < L]$ 0 $[\lvert x\rvert > L]$	$2AL\left\{\frac{\sin Ly}{Ly} - 2\left(\frac{\sin (Ly/2)}{Ly}\right)^2\right\}$	(3.20)
		
$A\exp(iy_0x)$	$2\pi A\delta(y - y_0)$.	(3.21)

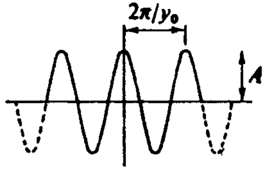
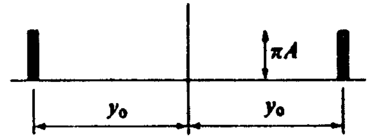
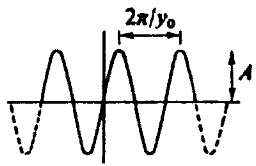
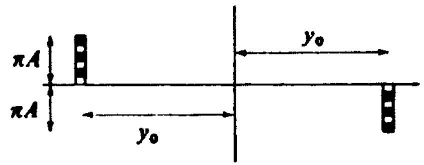
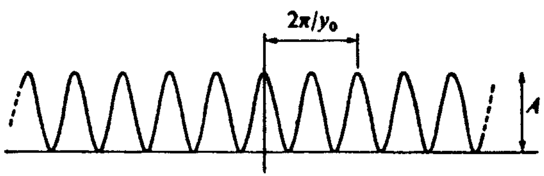
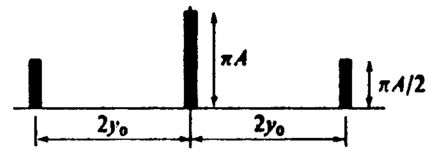
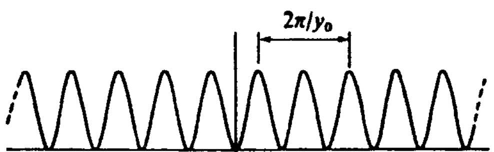
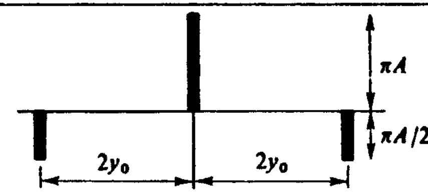
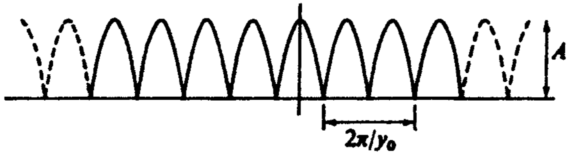
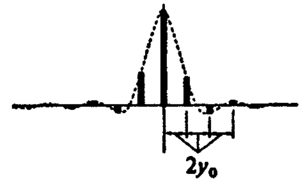
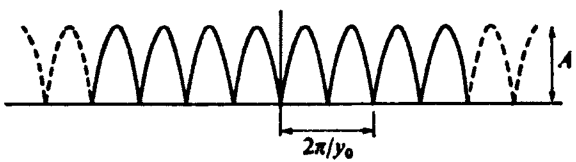
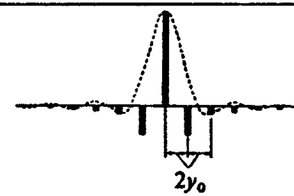
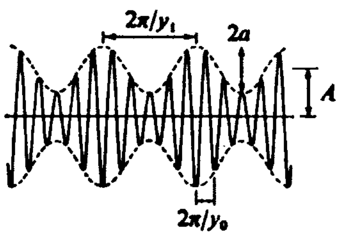
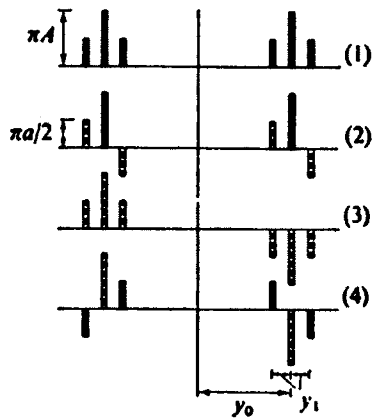
$f(x)$	$F(y)$	
 $A\cos y_0x$	 $\pi A\{\delta(y-y_0)+\delta(y+y_0)\}$	(3.22)
 $A\sin y_0x$	 $\pi iA\{\delta(y+y_0)-\delta(y-y_0)\}$	(3.23)
 $A\cos^2 y_0x$	 $\pi A\{\frac{1}{2}\delta(y+2y_0)+\delta(y)+\frac{1}{2}\delta(y-2y_0)\}$	(3.24)
 $A\sin^2 y_0x$	 $\pi A\{-\frac{1}{2}\delta(y+2y_0)+\delta(y)-\frac{1}{2}\delta(y-2y_0)\}$	(3.25)

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$
 $A\lvert\cos y_0 x\rvert$	 $\sum_{n=-\infty}^{n=+\infty} 4A\left\{\frac{y_0^2}{y_0^2 - y^2}\right\}\cos\left(\frac{\pi y}{2y_0}\right)\delta(y - 2ny_0)$ $[n = 0, \pm 1, \pm 2, \ldots]$ (3.26)
 $A\lvert\sin y_0 x\rvert$	 $\sum_{n=-\infty}^{n=+\infty} (-1)^n 4A\left\{\frac{y_0^2}{y_0^2 - y^2}\right\}\cos\left(\frac{\pi y}{2y_0}\right)\delta(y - 2ny_0)$ $[n = 0, \pm 1, \pm 2, \ldots]$ (3.27)
 $\cos y_0 x\{A + a\cos y_1 x\}$ ... (1) $\cos y_0 x\{A + a\sin y_1 x\}$ ... (2) $\sin y_0 x\{A + a\cos y_1 x\}$ ... (3) $\sin y_0 x\{A + a\sin y_1 x\}$ ... (4)	 $F(y)$ consists of delta functions as shown (3.28)

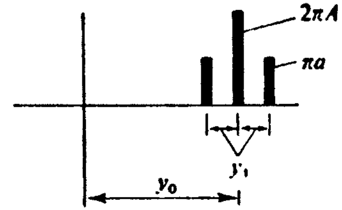
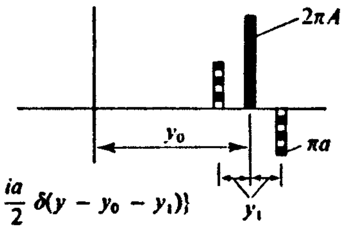
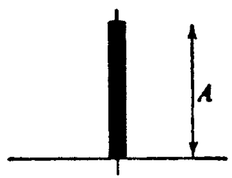
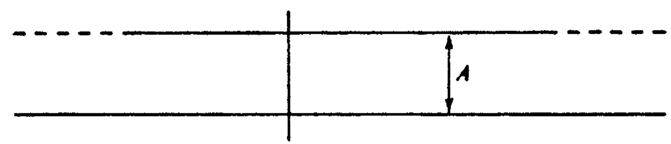
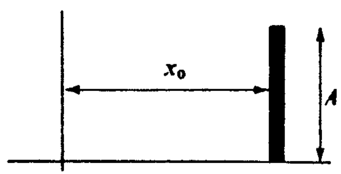
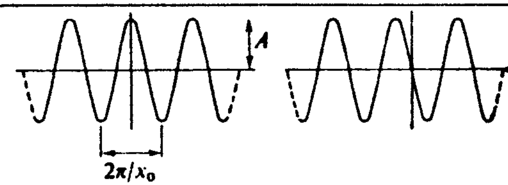
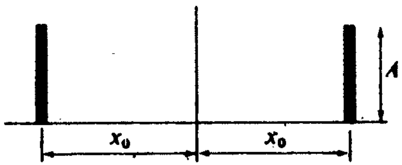
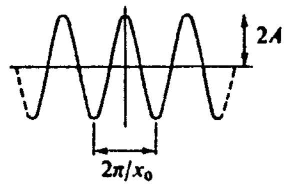
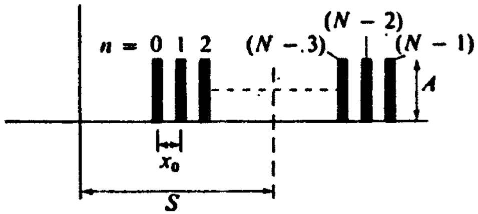
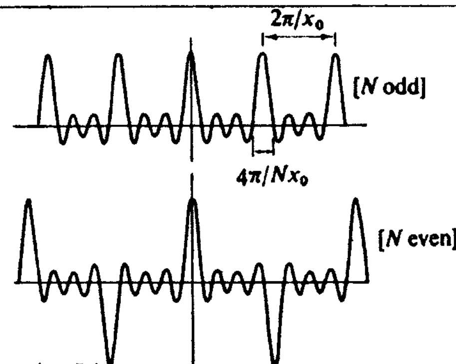
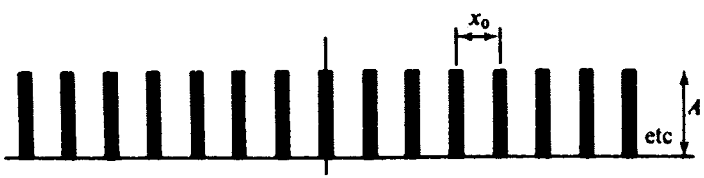
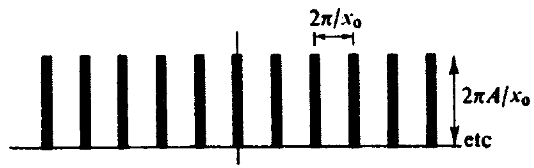
$f(x)$	$F(y)$	
$\exp(iy_0x)(A + a\cos y_1x)$	$2\pi\left\{A\delta(y - y_0) + \frac{a}{2}\delta(y - y_0 + y_1) + \frac{a}{2}\delta(y - y_0 - y_1)\right\}$ 	(3.29)
$\exp(iy_0x)(A + a\sin y_1x)$	 $2\pi\{A\delta(y - y_0) + \frac{ia}{2}\delta(y - y_0 + y_1) - \frac{ia}{2}\delta(y - y_0 - y_1)\}$	(3.30)
 $A\delta(x)$	 A	(3.31)
 $A\delta(x - x_0)$	 $A\exp(-ix_0y)$	(3.32)

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
 $A\{\delta(x - x_0) + \delta(x + x_0)\}$	 $2A \cos x_0 y$	(3.33)
 $\sum_{n=0}^{N-1} A\delta\left\{x - nx_0 - S + \frac{(N-1)x_0}{2}\right\}$ Set of N delta functions symmetrically placed about $x = S$.	 $A \frac{\sin(Nyx_0/2)}{\sin(yx_0/2)} \exp(-iSy)$ [Drawn for $S = 0$; $N = 7$ and $N = 8$]	(3.34)
 $\sum_{n=-\infty}^{+\infty} A\delta(x - nx_0)$	 $\sum_{n=-\infty}^{+\infty} \frac{2\pi A}{x_0} \delta\left(y - n\frac{2\pi}{x_0}\right)$	(3.35)

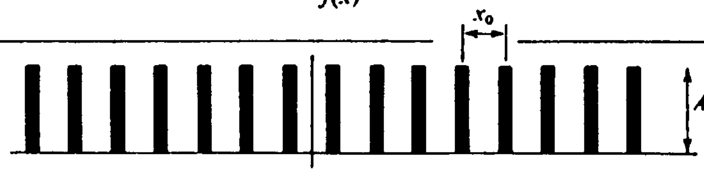
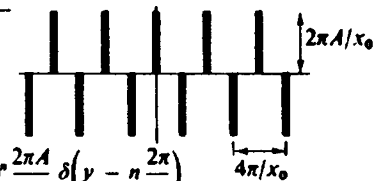
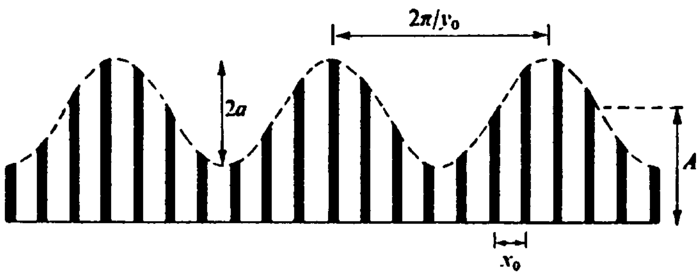
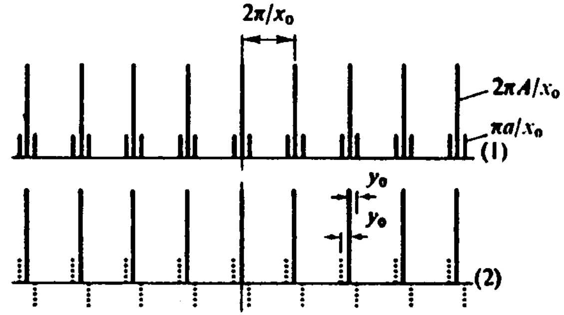
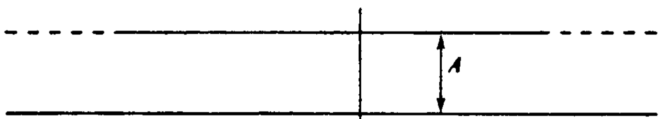
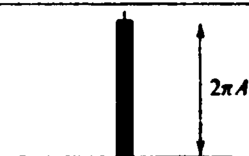
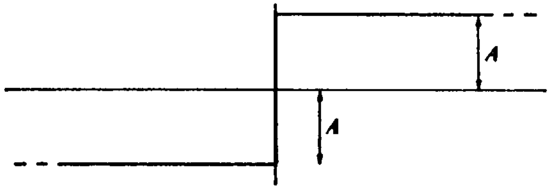
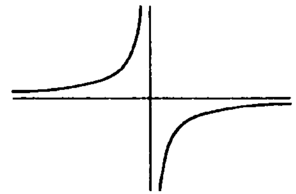
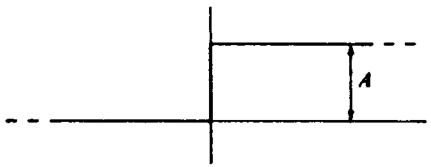
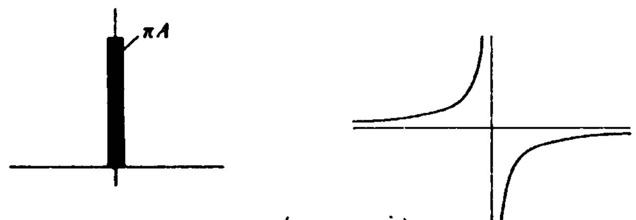
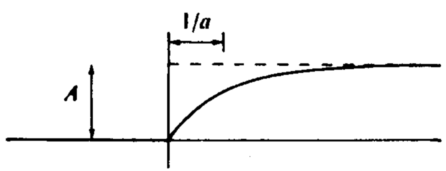
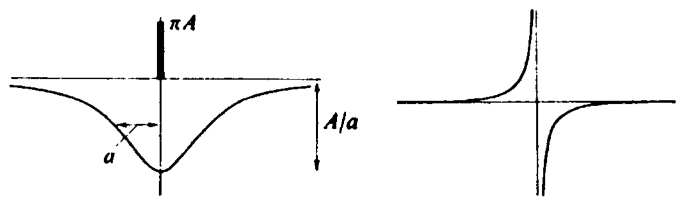
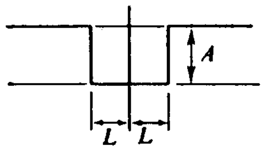
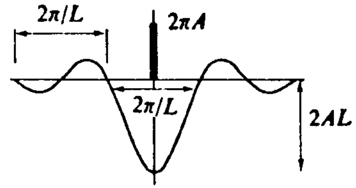
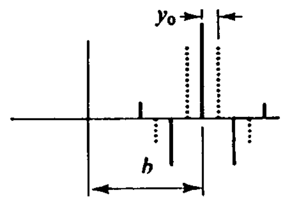
$f(x)$	$F(y)$	
 $\sum_{n=-\infty}^{+\infty} A\delta\left(x - \frac{x_0}{2} - nx_0\right)$	 $\sum_{n=-\infty}^{+\infty} (-1)^n \frac{2\pi A}{x_0}\delta\left(y - n\frac{2\pi}{x_0}\right)$	(3.36)
 $\sum \delta(x - nx_0)\{A + a\cos y_0x\}$ (1) $\sum \delta(x - nx_0)\{A + a\sin y_0x\}$ (2) $[n = 0, \pm 1, \pm 2, \ldots]$	 $\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y - n\frac{2\pi}{x_0}\right) + \frac{a}{2}\delta\left(y - n\frac{2\pi}{x_0} + y_0\right) + \frac{a}{2}\delta\left(y - \frac{2\pi}{x_0} - y_0\right)\right\}$ (1) $\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y - n\frac{2\pi}{x_0}\right) + \frac{ia}{2}\delta\left(y - n\frac{2\pi}{x_0} + y_0\right) - \frac{ia}{2}\delta\left(y - n\frac{2\pi}{x_0} - y_0\right)\right\}$ (2) $[n = 0, \pm 1, \pm 2, \ldots]$	(3.37)
 A	 $2\pi A\delta(y)$	(3.38)

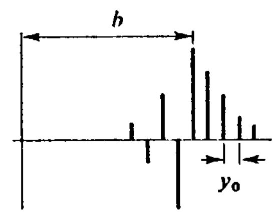
TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
 $+A$ $[x > 0]$, $-A$ $[x < 0]$ $[f(x) = A\,\text{sgn}(x)]$	 $-2iA\dfrac{1}{y}$	(3.39)
 A $[x > 0]$, 0 $[x < 0]$ $[f(x) = AU(x)]$	 $A\left\{\pi\delta(y) - \dfrac{i}{y}\right\}$	(3.40)
 $A\{1 - \exp(-ax)\}$ $[x > 0]$, 0 $[x < 0]$	 $\pi A\delta(y) - A\left\{\dfrac{a}{a^2 + y^2} + i\dfrac{a^2}{y(a^2 + y^2)}\right\}$	(3.41)

<table>
<tr><th>$f(x)$</th><th>$F(y)$</th><th></th></tr>
<tr><td>

$A \quad [|x| > L]$
$0 \quad [|x| < L]$</td><td>

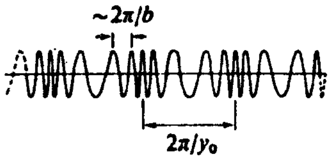
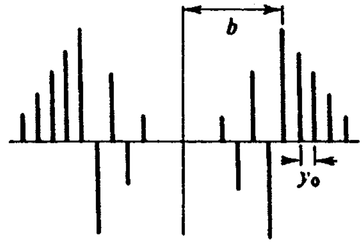
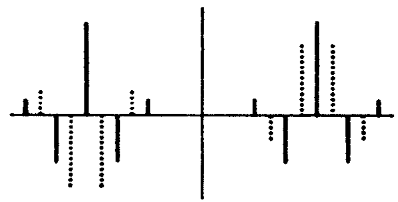
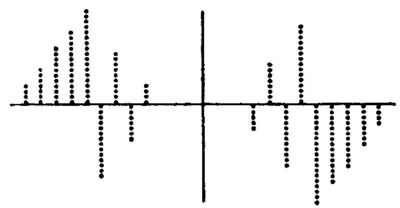
$$2\pi A\delta(y) - 2A\frac{\sin Ly}{y}$$</td><td>(3.42)</td></tr>
<tr><td>$A\exp\{i(a\cos y_0x + bx)\}$</td><td>

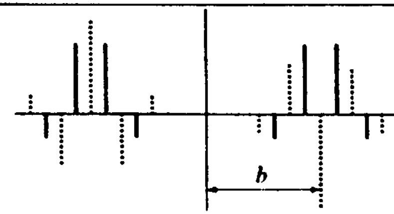
$$2\pi A\sum_{n=-\infty}^{+\infty}(i)^nJ_n(a)\delta(y - b - ny_0)$$</td><td>(3.43)</td></tr>
<tr><td>$A\exp\{i(a\sin y_0x + bx)\}$</td><td>

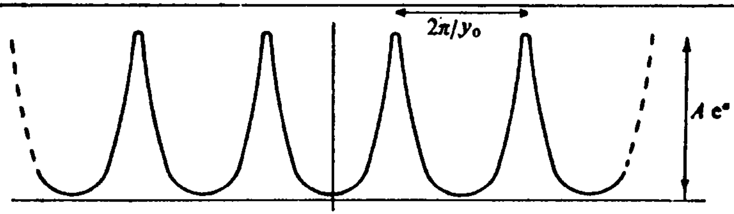
$$2\pi A\sum_{n=-\infty}^{+\infty}J_n(a)\delta(y - b - ny_0)\}$$</td><td>(3.44)</td></tr>
</table>

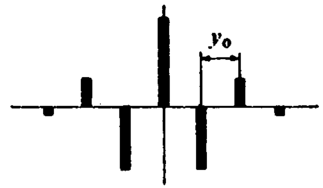
Note: $J_n(-a) = J_{-n}(a) = (-1)^nJ_n(a)$. See Appendix H for some properties of Bessel functions.

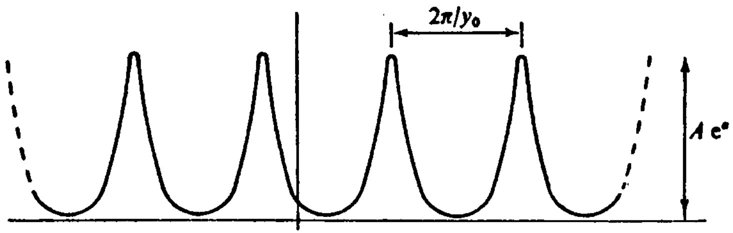
TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

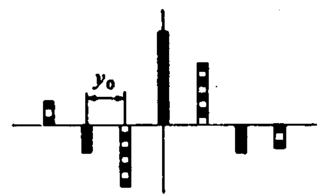
$f(x)$	$F(y)$
 $A\cos(a \sin y_0 x + bx)$	 $\pi A \sum_{n=-\infty}^{+\infty} \{J_n(a)\delta(y - b - ny_0) + J_n(a)\delta(y + b + ny_0)\}$ (3.45)
$A\cos(a \cos y_0 x + bx)$	 $\pi A \sum_{n=-\infty}^{+\infty} \{(+i)^n J_n(a)\delta(y - b - ny_0) + (-i)^n J_n(a)\delta(y + b + ny_0)\}$ (3.46)
$A\sin(a \sin y_0 x + bx)$	 $i\pi A \sum_{n=-\infty}^{+\infty} \{-J_n(a)\delta(y - b - ny_0) + J_n(a)\delta(y + b + ny_0)\}$ (3.47)

<table>
<tr><th>$f(x)$</th><th>$F(y)$</th></tr>
<tr><td>$A \sin(a \cos y_0 x + bx)$</td><td>

$$i\pi A \sum_{n=-\infty}^{+\infty} \{-(i)^n J_n(a)\delta(y - b - ny_0) + (-i)^n J_n(a)\delta(y + b + ny_0)\} \quad (3.48)$$</td></tr>
<tr><td>

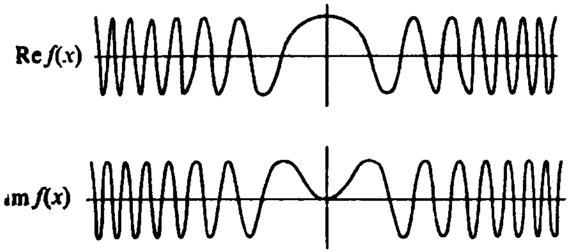
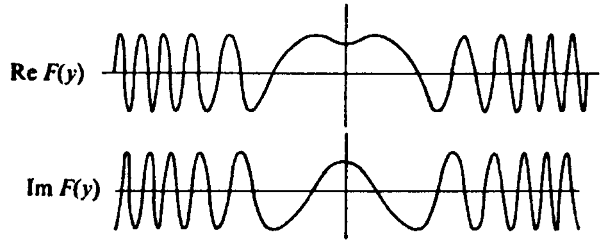
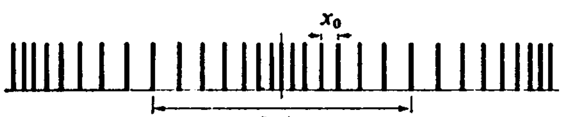
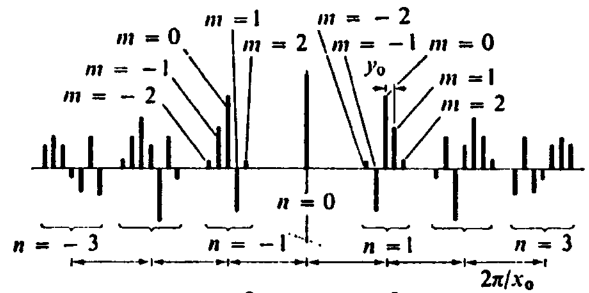
$A \exp(-a \cos y_0 x)$</td><td>

$$2\pi A \sum_{n=-\infty}^{+\infty} (-1)^n I_n(a)\delta(y - ny_0) \quad (3.49)$$</td></tr>
<tr><td>

$A \exp(-a \sin y_0 x)$</td><td>

$$2\pi A \sum_{n=-\infty}^{+\infty} (i)^n I_n(a)\delta(y - ny_0) \quad (3.50)$$</td></tr>
</table>

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
 $A \exp(\pm i a^2 x^2)$	 $\left(\frac{\pi}{2}\right)^{1/2} \frac{A(1 \pm i)}{a} \exp(\mp i y^2/4a^2)$	(3.51)
 $f(x) = A \sum_n \delta(x - n x_0 + a \sin y_0 x)$	 $F(y) = \frac{2\pi A}{x_0} \sum_{m,n} J_m\left(n \frac{2\pi a}{x_0}\right) \delta\left(y - n \frac{2\pi}{x_0} - m y_0\right)$ $(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$ $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$	(3.52)

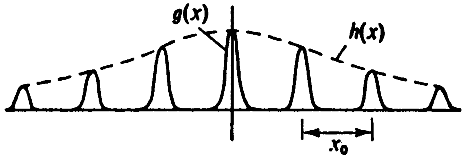
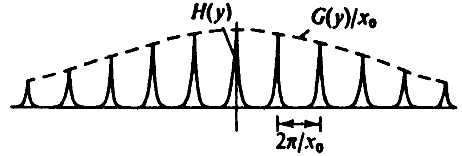
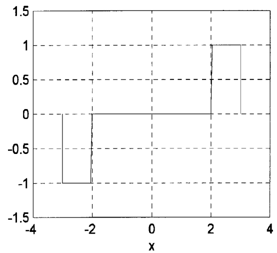
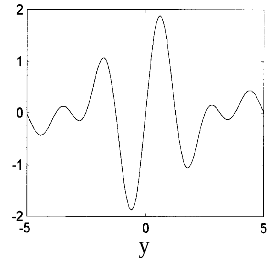
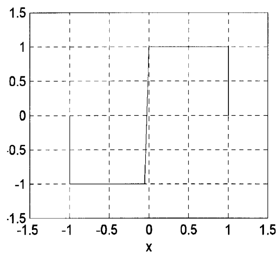
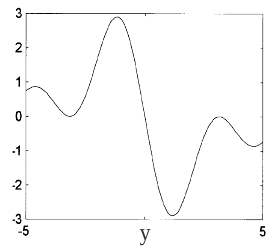
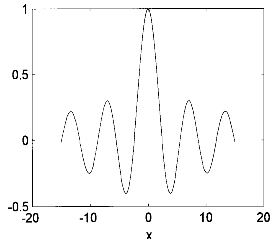
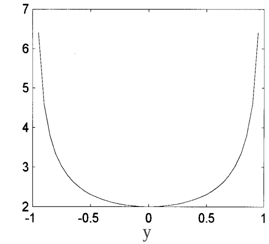
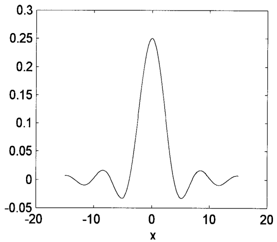
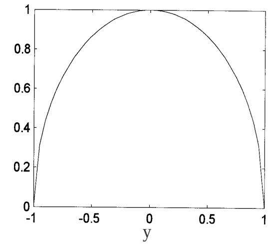
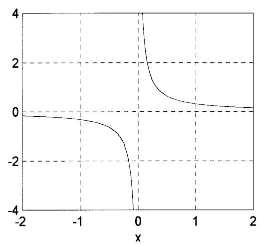
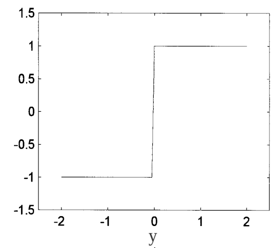
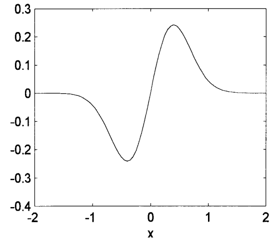
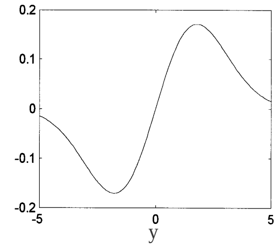
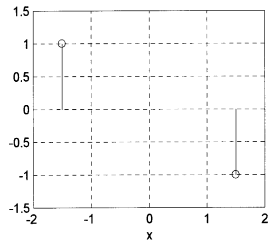
$f(x)$	$F(y)$	
 $f(x) = h(x) \sum_{n=-\infty}^{+\infty} g(x - nx_0)$	 $F(y) = \frac{1}{x_0} \sum_{n=-\infty}^{+\infty} \left\{ G\left(\frac{n2\pi}{x_0}\right) H\left(y - \frac{n2\pi}{x_0}\right) \right\}$	**3.53**
$f(x) = \sum_{n=-\infty}^{+\infty} h(nx_0) g(x - nx_0)$	$F(y) = \frac{1}{x_0} G(y) \sum_{n=-\infty}^{+\infty} H\left(y - \frac{n2\pi}{x_0}\right)$	**3.54**
 A s-a<x<s+a -A -s-a<x<-s+a	 $-2Aj \frac{\sin ay}{y} \sin ys$	**3.55**
 A 0<x<2a -A -2a<x<0	 $-4jA \sin ay \frac{\sin ay}{y}$	**3.56**

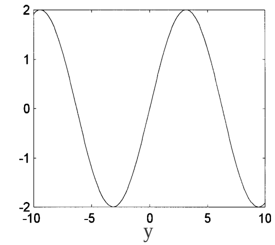
TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

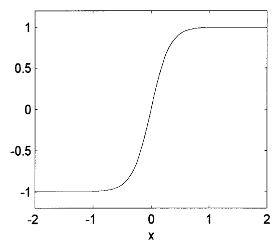
$f(x)$	$F(y)$	
 $J_0(x)$	 $\frac{2}{\sqrt{1-y^2}}$	**3.57**
 $\frac{J_1(x)}{2x}$	 $\sqrt{1-y^2}$	**3.58**
 $\frac{j}{\pi x}$	 $\text{sqn } y = \begin{cases} 1 & y>0 \\ 0 & y=0 \\ -1 & y<0 \end{cases}$	**3.59**

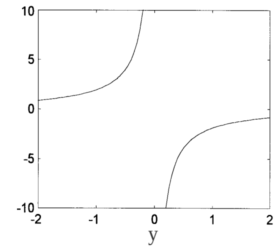
<table>
<tr><th>f(x)</th><th>F(y)</th><th></th></tr>
<tr><td>

$xe^{-\pi x^2}$</td><td>

$-j\,\frac{y}{2\pi}\,e^{-\frac{y^2}{4\pi}}$</td><td>3.60</td></tr>
<tr><td>

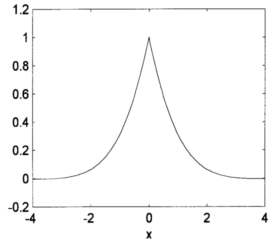
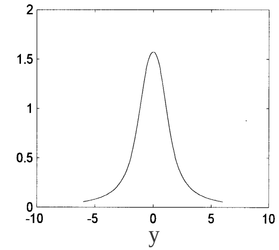
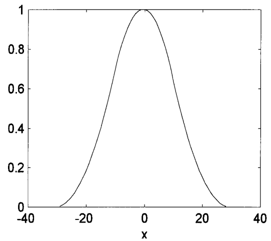
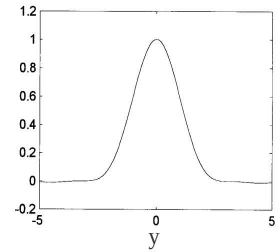
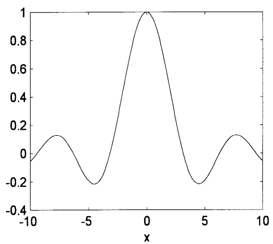
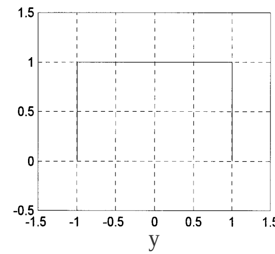
$\delta(x+a) - \delta(x-a)$</td><td>

$2j\sin\frac{y}{2}$</td><td>3.61</td></tr>
<tr><td>

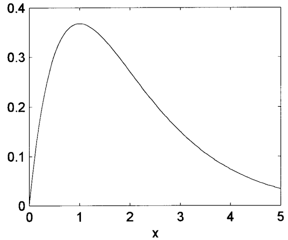
$\tanh\pi x$</td><td>

$-j\ \mathrm{cosech}\ \frac{y}{2}$</td><td>3.62</td></tr>
</table>

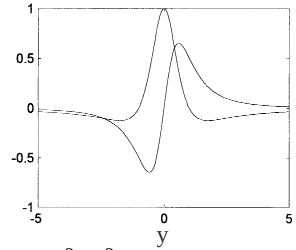
TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

$f(x)$	$F(y)$	
 $e^{-\lvert x\rvert}\ \frac{\sin x}{x}$	 $\tan^{-1}\frac{2}{y^2}$	**3.63**
 $p(x) * p(x) * p(x)$	 $\left(\frac{\sin y}{y}\right)^3$	**3.64**
 $\frac{\sin ax}{\pi x}$	 $p_a(x)$	**3.65**

f(x)	*F(y)*	

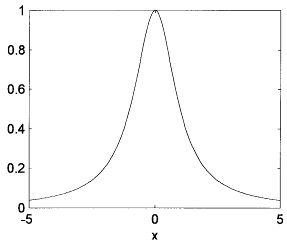


$x\,e^{-ax}$ a>0 $x \geq 0$

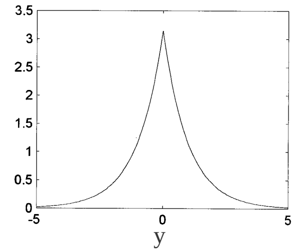


$$\frac{a^2 - y^2}{(a^2 + y^2)^2} - j\,\frac{2ay}{(a^2 + y^2)^2}$$

3.66

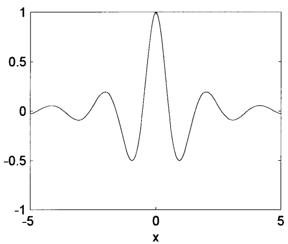


$$\frac{1}{a^2 + x^2}$$

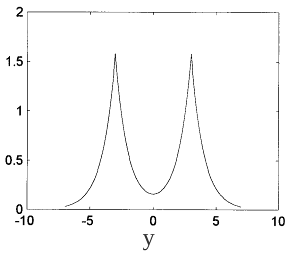


$$\frac{\pi}{a}\,e^{-a|y|}$$

3.67



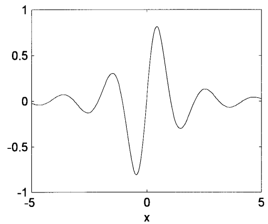
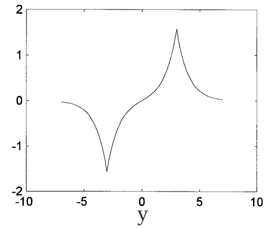
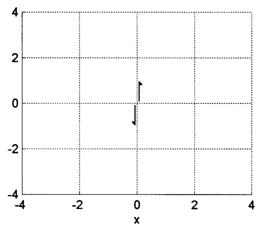
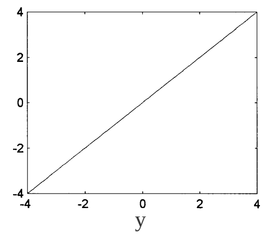
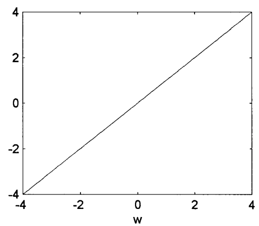
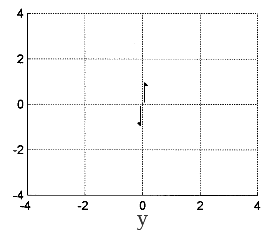
$$\frac{\cos bx}{a^2 + x^2}$$

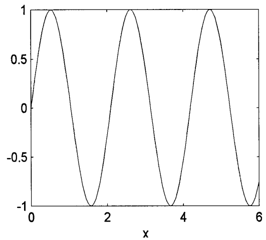
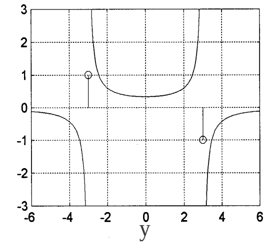
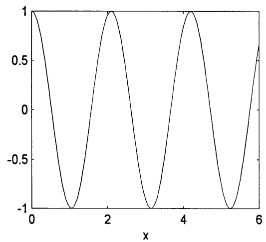
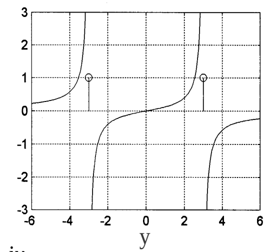
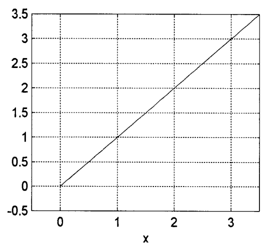
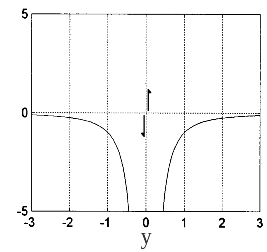


$$\frac{\pi}{2a}\left[\,e^{-a|y-b|} + e^{-a|y+b|}\,\right]$$

3.68

TABLE 3.2 Table of Fourier Transforms ($x = t$; $y = w$) (continued)

f(x)	*F(y)*	
 $\dfrac{\sin bx}{a^2 + x^2}$	 $\dfrac{\pi}{2aj}\left[e^{-a\lvert y-b\rvert} - e^{-a\lvert y+b\rvert} \right]$	**3.69**
 $\dfrac{d\delta(x)}{dx}$	 jy	**3.70**
 x	 $2\pi j \dfrac{d\delta(y)}{dy}$	**3.71**

f(x)	*F(y)*	
 $\sin w_0 x \quad x \geq 0$; $0 \quad x < 0$	 $\frac{w_0}{w_0^2 - y^2} - j\frac{\pi}{2}[\delta(y-w_0) - \delta(y+w_0)]$	**3.72**
 $\cos w_0 x \quad x \geq 0$; $0 \quad x < 0$	 $\frac{jy}{w_0^2 - y^2} + \frac{\pi}{2}[\delta(y-w_0) - \delta(y+w_0)]$	**3.73**
 $x \quad x \geq 0$; $0 \quad x < 0$	 $j\pi\frac{d\delta(y)}{dy} - \frac{1}{y^2}$	**3.74**

3.2 Two-Dimensional Continuous Fourier Transform

3.2.1 Definitions

3.2.1.1 Two-Dimensional Fourier Transform

$$\mathcal{F}\{f(x,y)\} = F(\omega_1,\omega_2) = \int\int_{-\infty}^{\infty} f(x,y)e^{-j(x\omega_1 + y\omega_2)}\,dx\,dy$$

$$\mathcal{F}^{-1}\{F(\omega_1,\omega_2)\} = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} F(\omega_1,\omega_2)e^{j(\omega_1 x + \omega_2 y)}\,d\omega_1\,d\omega_2$$

3.2.1.2 Properties of Two-Dimensional Fourier Transform

TABLE 3.3 Properties of Two-Dimensional Fourier Transform

Rotation	$f(\pm x, \pm y)$	$F(\pm\omega_1, \pm\omega_2)$
Linearity	$a_1 f_1(x,y) + a_2 f_2(x,y)$	$a_1 F_1(\omega_1,\omega_2) + a_2 F_2(\omega_1,\omega_2)$
Conjugation	$f^*(x,y)$	$F^*(-\omega_1,-\omega_2)$
Separability	$f_1(x)\, f_2(y)$	$F_1(\omega_1)\, F_2(\omega_2)$
Scaling	$f(ax,by)$	$\frac{1}{\lvert ab\rvert} F\left(\frac{\omega_1}{a}, \frac{\omega_2}{b}\right)$
Shifting	$f(x \pm x_o, y \pm y_o)$	$e^{\pm j(\omega_1 x_o + \omega_2 y_o)} F(\omega_1,\omega_2)$
Modulation	$e^{\pm j(\omega_{c1} x + \omega_{c2} y)} f(x,y)$	$F(\omega_1 \mp \omega_{c1}, \omega_2 \mp \omega_{c2})$
Convolution	$g(x,y) = h(x,y) ** f(x,y)$	$G(\omega_1,\omega_2) = H(\omega_1,\omega_2)F(\omega_1,\omega_2)$
Multiplication	$g(x,y) = h(x,y)\, f(x,y)$	$G(\omega_1,\omega_2) = \frac{1}{(2\pi)^2} H(\omega_1,\omega_2) ** F(\omega_1,\omega_2)$
Correlation	$c(x,y) = h(x,y)\ \mu\mu\ \ f(x,y)$	$G(\omega_1,\omega_2) = H(-\omega_1,-\omega_2)\, F(\omega_1,\omega_2)$
Inner Product	$I = \int\int_{-\infty}^{\infty} f(x,y)\, h^*(x,y)\, dx\, dy$	$I = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} F(\omega_1,\omega_2)\, H^*(\omega_1,\omega_2)\, d\omega_1\, d\omega_2$
Parseval Formula	$I = \int\int_{-\infty}^{\infty} \lvert f(x,y)\rvert^2\, dx\, dy$	$I = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} \lvert F(\omega_1,\omega_2)\rvert^2\, d\omega_1\, d\omega_2$

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Howell, K., Fourier transform, in *The Transforms and Application Handbook*, Edited by A. D. Poularikas, CRC Press Inc., Boca Raton, FL, 1996.

Papoulis, Athanasios, *The Fourier Integral and Its Applications*, McGraw-Hill Book Company, New York, NY, 1962.

Walker, James S., *Fourier Analysis*, Oxford University Press, New York, NY, 1988.

Appendix 1

Examples

1.1 Gibbs Phenomenon

Example 3.1

Let $U(\omega)$ be the spectrum of unit step function, and let the truncated spectrum $U_{\omega_o}(\omega) = U(\omega)$ for $|\omega| \le \omega_o$ and zero otherwise. We can also write the truncated spectrum as follows: $U_{\omega_o}(\omega) = U(\omega)\; p_{\omega_o}(\omega)$, where $p_{\omega_o}(\omega)$ is a unit pulse of $2\omega_o$ duration and centered at the origin. The approximate step function

$$u_a(t) = F^{-1}\{U(\omega)\, p_{\omega_o}(\omega)\} = F^{-1}\{U(\omega)\} * F^{-1}\{p_{\omega_o}(\omega)\} = u(t) * \frac{1}{\pi}\frac{\sin\omega_o t}{t} = \frac{1}{\pi}\int_{-\infty}^{t} \frac{\sin\omega_o \tau}{\tau}\, d\tau$$

and is shown in Figure 3.1.

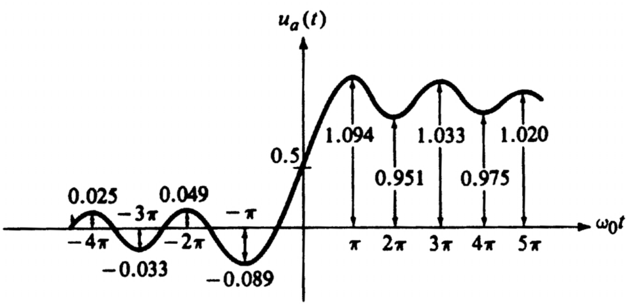

FIGURE 3.1

1.2 Special Functions

Example 3.2

$$F\{\mathrm{sgn}(t)\} = F\{\lim_{\varepsilon\to 0} e^{-\varepsilon|t|}\,\mathrm{sgn}(t)\} = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} e^{-\varepsilon|t|}\,\mathrm{sgn}(t) e^{-j\omega t}\, dt$$

$$= \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{o} e^{(\varepsilon-j\omega)t}\, dt + \int_{o}^{\infty} e^{-(\varepsilon+j\omega)t}\, dt\right] = \lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon - j\omega} + \frac{1}{\varepsilon + j\omega}\right) = \frac{2}{j\omega}$$

Example 3.3

$$F\{u(t)\} = F\{\tfrac{1}{2} + \tfrac{1}{2}\mathrm{sgn}(t)\} = \tfrac{1}{2}2\pi\delta(\omega) + \tfrac{1}{2}\tfrac{2}{j\omega} = \pi\delta(\omega) + \tfrac{1}{j\omega}$$

Example 3.4

$$F\sum_{n=-\infty}^{\infty}\delta(t - nT)\} \triangleq F\{comb_T(t) = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT)\right] e^{-j\omega t}\, dt$$

But $comb_T(t)$ is periodic with the period $\omega_o = 2\pi/T$, and can be expanded in the complex form of Fourier series $comb_T(t) =$ $comb_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t}$. Hence

$$COMB_{\omega_o}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(\omega - n\omega_o)t}\,dt = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_o)$$

Poularikas A. D. “Discrete-Time Fourier Transform, One- and Two-Dimensional”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

4

Discrete-Time Fourier Transform, One- and Two-Dimensional

4.1 One-Dimensional Discrete-Time Fourier Transform

4.1.1 Definitions

$$\mathscr{F}\{x(n)\} \doteq X(\omega) \doteq X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad -\pi \le \omega \le \pi$$

$X(\omega)$ = periodic with period 2π

$$x(n) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{-j\omega n}\, d\omega$$

The Fourier transform function appears as a function of $e^{j\omega}$ and, hence, in sequences we will use both representations for convenience.

4.1.2 Properties

TABLE 4.1 Properties of One-Dimensional Discrete-Time Fourier Transform

Properties	Sequence	Transform $[X(\omega) \equiv X(e^{j\omega})]$
Discrete-time Fourier transform	$x(n)$	$X(\omega)$
Linearity	$ax(n)+by(n)$	$aX(\omega)+bY(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Complex conjugation	$x^*(n)$	$X^*(-\omega)$
Reversal and complex conjugate	$x^*(-n)$	$X^*(\omega)$

TABLE 4.1 Properties of One-Dimensional Discrete-Time Fourier Transform (continued)

Properties	Sequence	Transform $[X(\omega) \equiv X(e^{j\omega})]$
Time shifting	$x(n \pm m)$	$e^{\pm j\omega m} X(\omega)$
Frequency shift	$e^{\pm j\omega_o n} x(n)$	$X(\omega \mp \omega_o)$
Modulation	$\cos \omega_o n \; x(n)$ $\sin \omega_o n \; x(n)$	$\frac{1}{2}[X(\omega + \omega_o) + X(\omega - \omega_o)]$ $\frac{1}{2j}[X(\omega + \omega_o) - X(\omega - \omega_o)]$
Convolution	$x(n) * h(n)$	$X(\omega) Y(\omega)$
Multiplication	$x(n) h(n)$	$\frac{1}{2\pi} X(\omega) * Y(\omega)$
Delta function	$\delta(n - n_o) = \begin{cases} 1, & n = n_o \\ 0, & \text{otherwise} \end{cases}$	$e^{-jn_o\omega}$
Frequency domain delta function	$e^{j\omega_o n}$	$\frac{1}{2\pi} \delta(\omega - \omega_o)$
Cosine function	$\cos \omega_o n$	$\pi\delta(\omega + \omega_o) + \pi\delta(\omega - \omega_o)$
Sine function	$\sin \omega_o n$	$\frac{\pi}{j}(\delta\omega + \omega_o) - \frac{1}{2}\delta(\omega - \omega_o)$
N sample step sequence	$u_N(n) = \begin{cases} 1, & n = 0, 1, 2, \cdots \\ 0, & \text{otherwise} \end{cases}$	$e^{-j\omega(N-1)/2} \dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$
Symmetric pulse	$p_N(n) = \begin{cases} 1, & \|n\| \le N \\ 0, & \|n\| \ge N \end{cases}$	$\dfrac{\sin[\omega(N + 1/2)]}{\sin(\omega/2)}$
Triangle sequence	$\wedge_N(n) = \begin{cases} N - \|n\|, & \|n\| \le N \\ 0, & \|n\| > N \end{cases}$	$\dfrac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$
Real sequence	$x(n)$	$X(\omega) = X^*(-\omega)$
Decomposition of real $x(n)$ in even $x_e(n)$ and $x_o(n)$ parts.	$\begin{cases} x(n) = x_e(n) + x_o(n) \\ x_e = \frac{1}{2}[x(n) + x(-n)] \\ x_o = \frac{1}{2}[x(n) - x(-n)] \end{cases}$	$X(\omega)$ $\mathrm{Re}\{X(\omega)\}$ $j\,\mathrm{Im}\{X(\omega)\}$
Decomposition of a complex sequence $x(n)$ into a conjugate symmetric part $x_e(n)$ and conjugate antisymmetric part $x_o(n)$	$\begin{cases} x(n) = x_e(n) + x_o(n) \\ x_e = \frac{1}{2}[x(n) + x^*(-n)] \\ x_o = \frac{1}{2}[x(n) - x^*(-n)] \end{cases}$	$X(\omega)$ $\mathrm{Re}\{X(\omega)\}$ $j\,\mathrm{Im}\{X(\omega)\}$
Decomposition of a complex transform $X(\omega)$	$\begin{cases} x(n) \\ \mathrm{Re}\{x(n)\} \\ j\,\mathrm{Im}\{x(n)\} \end{cases}$	$X(\omega) = X_e(\omega) + X_o(\omega)$ $X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$ $X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
Increasing sampling frequency by m; i.e., transforming a data sequence $x_1(n)$ padded with zeros $x_1(n)$ by a factor of M	$x(n) = \begin{cases} x_1(n), & \text{if } n/M = m \\ 0, & \text{otherwise} \end{cases}$	$X_1(M\omega)$
Reducing sampling frequency by M; i.e., decimating a sequence $x_1(n)$ by a factor of M	$\begin{cases} x(n) = x_1(Mn) \\ n = 0, \pm 1, \pm 2, \cdots \end{cases}$	$\dfrac{1}{M} \sum_{\ell=0}^{M-1} X_1\left(\omega - \dfrac{2\pi\ell}{M}\right)$
Parseval's theorem	$\begin{cases} \sum_{n=-\infty}^{\infty} x(n) h^*(n) \\ \sum_{n=-\infty}^{\infty} \|x(n)\|^2 \end{cases}$	$= \dfrac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) H^*(\omega)\, d\omega$ $= \dfrac{1}{2\pi} \int_{-\pi}^{\pi} \|X(\omega)\|^2\, d\omega$
Correlation	$x(n) \star h(n)$	$X(\omega) H(\omega)$

4.1.3 Finite Sequence

$$F_N(\omega)=\sum_{n=0}^{N-1} f(n)e^{-j\omega n}=e^{-j\omega(N-1)/2}\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

4.1.4 Approximation to Continuous-Time Fourier Transform

$$F(\omega_c)=\int_{-\infty}^{\infty} f(t)e^{-j\omega_c t}\,dt \qquad \omega_c = \text{frequency for continuous Fourier transform}$$

$$F(\omega_c)\cong\sum_{n=-\infty}^{\infty} T\,f(nt)e^{-j\omega_c nT} \qquad T = \text{sampling time such that } F(\omega_c)\cong 0 \text{ for all } |\omega_c|>\pi/T$$

4.2 Two-Dimensional Discrete-Time Fourier Transform

4.2.1 Definition

$$X(\omega_1,\omega_2)\doteq\sum\sum_{m,n=-\infty}^{\infty} x(m,n)e^{-j(m\omega_1+n\omega_2)} \qquad -\pi\le\omega_1,\omega_2\le\pi$$

$$x(m,n)=\frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi} X(\omega_1,\omega_2)e^{j(m\omega_1+n\omega_2)}$$

4.2.2 Properties of Two-Dimensional Discrete-Time Fourier Transform

TABLE 4.2 Properties of Two-Dimensional Discrete-Time Fourier Transform

Properties	Sequence	Transform
	$x(m,n), y(m,n), h(m,n), \ldots$	$X(\omega_1,\omega_2), Y(\omega_1,\omega_2), H(\omega_1,\omega_2), \ldots$
Linearity	$a_1x_1(m,n) + a_2x_2(m,n)$	$a_1X_1(\omega_1,\omega_2) + a_2X_2(\omega_1,\omega_2)$
Conjugation	$x^*(m,n)$	$X^*(-\omega_1,-\omega_2)$
Separability	$x_1(m)x_2(m)$	$X_1(\omega_1)X_2(\omega_2)$
Shifting	$x(m \pm m_0, n \pm n_0)$	$\exp[\pm j(m_0\omega_1 + n_0\omega_2)]X(\omega_1,\omega_2)$
Modulation	$\exp[\pm j(\omega_{01}m + \omega_{02}n)]x(m,n)$	$X(\omega_1 \mp \omega_{01},\omega_2 \mp \omega_{02})$
Convolution	$y(m,n) = h(m,n) ** x(m,n)$	$Y(\omega_1,\omega_2) = H(\omega_1,\omega_2)X(\omega_1,\omega_2)$
Multiplication	$h(m,n)x(m,n)$	$\left(\frac{1}{4\pi^2}\right)H(\omega_1,\omega_2)**X(\omega_1,\omega_2)$
Spatial correlation	$c(m,n) = h(m,n) \star\star\, x(m,n)$	$C(\omega_1,\omega_2) = H(-\omega_1,-\omega_2)X(\omega_1,\omega_2)$
Inner product	$I=\sum\sum_{m,n=-\infty}^{\infty} x(m,n)y^*(m,n)$	$I=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)Y^*(\omega_1,\omega_2)\,d\omega_1\,d\omega_2$
Energy conservation	$\mathsf{E}=\sum\sum_{m,n=-\infty}^{\infty} \lvert x(m,n)\rvert^2$	$\mathsf{E}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \lvert X(\omega_1,\omega_2)\rvert^2\,d\omega_1\,d\omega_2$
	$\sum\sum_{m,n=-\infty}^{\infty} \exp[j(m\omega_{01}+n\omega_{02})]$	$4\pi^2\delta(\omega_1-\omega_{01},\omega_2-\omega_{02})$

TABLE 4.2 Properties of Two-Dimensional Discrete-Time Fourier Transform (continued)

Properties	Sequence	Transform
	$\delta(m,n)$	$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp[-j(\omega_1 m+\omega_2 n)\,d\omega_1\,d\omega_2$
Differentiation	$-jmx(m,n)$	$\frac{\partial X(\omega_1,\omega_2)}{\partial\omega_1}$
	$-jnx(m,n)$	$\frac{\partial X(\omega_1,\omega_2)}{\partial\omega_2}$
	$mnx(m,n)$	$\frac{\partial^2 X(\omega_1,\omega_2)}{\partial\omega_1\partial\omega_2}$
Transportation	$x(m,n)$	$X(\omega_2,\omega_1)$
Reflection	$x(-m,n)$	$X(-\omega_1,\omega_2)$
	$x(m,-n)$	$X(\omega_1,-\omega_2)$
	$x(-m,-n)$	$X(-\omega_1,-\omega_2)$
Real and Imaginary	$\mathrm{Re}[x(m,n)]$	$\frac{1}{2}[X(\omega_1,\omega_2)+X^*(-\omega_1,-\omega_2)]$
Parts	$j\,\mathrm{Im}[x(m,n)]$	$\frac{1}{2}[X(\omega_1,\omega_2)-X^*(-\omega_1,-\omega_2)]$
	$\frac{1}{2}[x(m,n)+x^*(-m,-n)]$	$\mathrm{Re}[X(\omega_1,\omega_2)]$
	$\frac{1}{2}[x(m,n)-x^*(-m,-n)]$	$j\mathrm{Im}[X(\omega_1,\omega_2)]$
Real-valued sequence		$X(\omega_1,\omega_2) = X^*(-\omega_1,-\omega_2)$ $\mathrm{Re}[X(\omega_1,\omega_2)] = \mathrm{Re}[X(-\omega_1,-\omega_2)]$ $\mathrm{Im}[X(\omega_1,\omega_2)] = -\mathrm{Im}[X(-\omega_1,-\omega_2)]$ $\mathrm{Re}[X(\omega_1,\omega_2)],\ \lvert X(\omega_1,\omega_2)\rvert$: even (symmetric with respect to the origin) $\mathrm{Im}[X(\omega_1,\omega_2)]\,;\tan^{-1}\frac{\mathrm{Im}[X(\omega_1,\omega_2)]}{\mathrm{Re}[X(\omega_1,\omega_2)]}$: odd (antisymmetric with respect to the origin)
	$x(m,n)$: real and even	$X(\omega_1,\omega_2)$: real and even
	$x(m,n)$: real and odd	$X(\omega_1,\omega_2)$: pure imaginary and odd

References

Dudgeon, Dan E., and Russell M. Mersereau, *Multidimensional Digital Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1984.

Jain, Aril K., *Fundamentals of Digital Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.

Lim, Jae S., *Two-Dimensional Signal and Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1990.

Appendix 1

1.1 Two-Dimensional Discrete-Time Fourier Transform

Example

The transform of $x(m,n) = \frac{1}{5}\delta(m-1)\delta(n) + \frac{1}{5}\delta(m+1)\delta(n) + \frac{1}{5}\delta(m)\delta(n-1) + \frac{1}{5}\delta(m)\delta(n+1) + \frac{2}{5}\delta(m)\delta(n)$ which is shown in Figure 4.1 is $X(\omega_1,\omega_2) = \sum_{m,n=-\infty}^{\infty}\sum x(m,n)e^{-jm\omega_1 - jn\omega_2} = \frac{1}{5}e^{-j\omega_1} + \frac{1}{5}e^{j\omega_1} + \frac{1}{5}e^{-j\omega_2} + \frac{1}{5}e^{j\omega_2} + \frac{1}{2} = \frac{2}{5}\cos\omega_1 + \frac{2}{5}\cos\omega_2 + \frac{2}{5}$ and is shown in Figure 4.2.

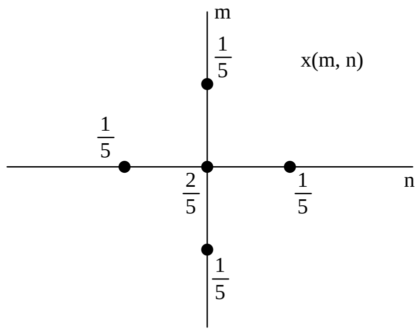

FIGURE 4.1

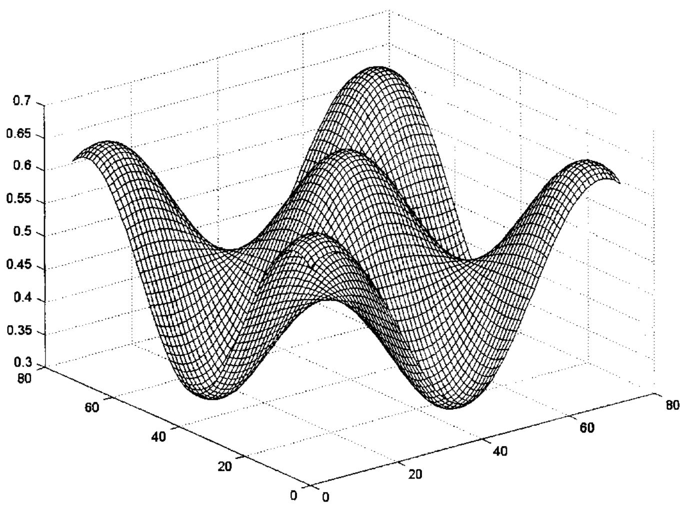

FIGURE 4.2

1.2 Two-Dimensional Discrete-Time Fourier Transform

Example

(Ideal Lowpass Filter). To find the inverse of $H(\omega_1,\omega_2)$ shown in Figure 4.3 we observe that it is equal to the multiplication of two pulse filters whose one direction extends from minus infinity to infinity. Hence (a and b are less than π)

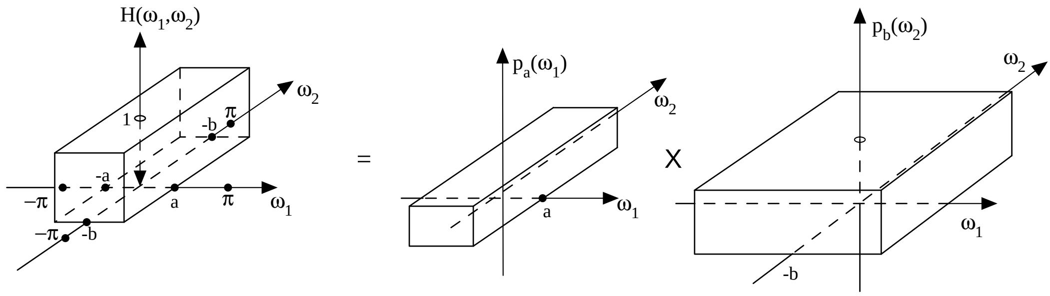

FIGURE 4.3

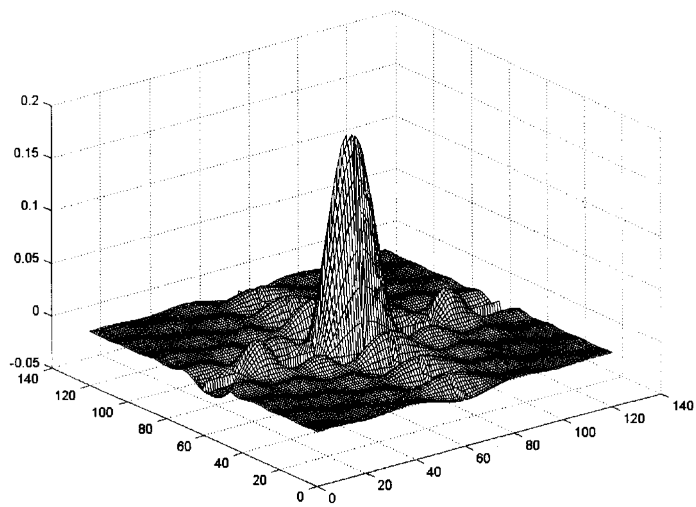

FIGURE 4.4

$$h(m,n) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2)e^{j\omega_1 m}e^{j\omega_2 n}\,d\omega_1\,d\omega_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_a(\omega_1)e^{j\omega_1 m}\,d\omega_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} p_b(\omega_2)e^{j\omega_2 n}\,d\omega_2$$

$$= \frac{1}{2\pi}\int_{-a}^{a} e^{j\omega_1 m}\,d\omega_1 \frac{1}{2\pi}\int_{-b}^{b} e^{j\omega_2 n}\,d\omega_2 = \frac{\sin am}{\pi m}\,\frac{\sin bn}{\pi n}$$

which is plotted in Figure 4.4.

Example

Let the ideal filter be of the form $H(\omega_1,\omega_2) = 1$ for $\omega_1^2 + \omega_2^2 \le \omega_c^2$ and zero otherwise (see Figure 4.5). Hence

$$h(m,n) = \frac{1}{(2\pi)^2} \iint\limits_{\substack{(\omega_1,\omega_2)\in \\ [\omega_1^1+\omega_2^2\le\omega_c^2]}} 1e^{j\omega_1 m}e^{j\omega_2 n}\,d\omega_1\,d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{r=0}^{\omega_c} r\,dr \int_{\theta=a}^{a+2\pi} e^{jr(m\cos\theta+n\sin\theta)}\,d\theta \qquad a = \text{real constant}$$

where we set $r\cos\theta = \omega_1$, $r\sin\theta = \omega_2$. Next change to: $m = q\cos\phi$ and $n = q\sin\phi$. Hence

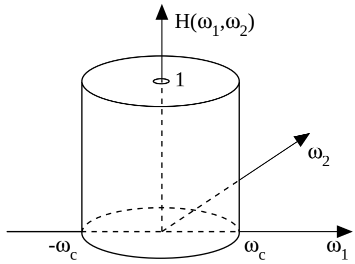

FIGURE 4.5

$$h(m,n) = \frac{1}{(2\pi)^2} \int_{r=0}^{\omega_c} r \int_{\theta=a}^{a+2\pi} e^{jrn\cos(\theta-\phi)}\, d\theta = \frac{1}{(2\pi)^2} \int_{r=0}^{\omega_c} r\, 2\pi J_o\left(r\sqrt{m^2+n^2}\right) dr = \frac{\omega_c}{2\pi\sqrt{m^2+n^2}} J_1\left(\omega_c\sqrt{m^2+n^2}\right)$$

Poularikas A. D. “Distributions, Delta Function ”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

5

Distributions, Delta Function

5.1 Test Function

5.1.1 A Test Function

$\varphi(t)$ is a real-valued function of the real independent variable that can be differentiated an arbitrary number of times, and which is identical to zero outside a finite interval.

Example 5.1

$$\varphi(t,a) = \text{test function} = \begin{cases} \exp\left[-a^2/(a^2-t^2)\right] & |t| < a \\ 0 & |t| \ge a \end{cases}$$

5.1.2 Properties of Test Functions

1. If $f(t)$ can be differentiated arbitrarily often, $\psi(t) = f(t)\varphi(t)$ = test function.
2. If $f(t)$ is zero outside a finite interval, $\psi(t) = \int_{-\infty}^{\infty} f(\tau)\varphi(t-\tau)\,d\tau$, $-\infty < t < \infty$, is a test function.
3. A sequence of test functions, $\{\varphi_n(t)\}$ $1 \le n < \infty$, converges to zero if all φ_n are identically zero outside some interval independent of n and each φ_n, as well as all of its derivatives, tend uniformly to zero.

Example 5.2

$$\varphi_n(t) = \varphi\left(t + \frac{1}{n}\right) - \varphi(t).$$

4. Test functions belong to a set D, where D is a linear vector space such that if $\varphi_1 \in D$ and $\varphi_2 \in D$, then $\varphi_1 + \varphi_2 \in D$ and $a\varphi_1 \in D$ for any number a.

5.2 Distributions

5.2.1 Definition

A ***distribution*** (or ***generalized*** function) $g(t)$ is a process of assigning our arbitrary test function $\varphi(t)$ a number $N_g[\varphi(t)]$. A distribution is also a ***functional***.

Example 5.3

$$\int_{-\infty}^{\infty} u(\tau)\varphi(t)\,dt = \int_{o}^{\infty} \varphi(t)\,dt = N_f[\varphi(t)]$$

and implies that $u(t)$ is a distribution that assigns a number to each $\varphi(t)$ equal to its area.

5.2.2 Properties

1. Linearity-homogeneity:

$$\int_{-\infty}^{\infty} g(t)[a_1\varphi_1(t) + a_2\varphi_2(t)]dt = a_1\int_{-\infty}^{\infty} g(t)\varphi_1(t)\,dt + a_2\int_{-\infty}^{\infty} g(t)\varphi_2(t)\,dt$$

2. Shifting: $\int_{-\infty}^{\infty} g(t-t_o)\varphi(t)\,dt = \int_{-\infty}^{\infty} g(t)\varphi(t+t_o)\,dt$

3. Scaling: $\int_{-\infty}^{\infty} g(at)\varphi(t)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} g(t)\varphi\left(\frac{t}{a}\right)dt$

4. Even distribution: $\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt = 0, \qquad \varphi(t) = \text{odd}$

5. Odd distribution: $\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt = 0, \qquad \varphi(t) = \text{even}$

6. Derivative: $\int_{-\infty}^{\infty} \frac{dg(t)}{dt}\varphi(t)\,dt = -\int_{-\infty}^{\infty} g(t)\frac{d\varphi(t)}{dt}\,dt$

7. nth derivative: $\int_{-\infty}^{\infty} \frac{d^n g(t)}{dt^n}\varphi(t)\,dt = (-1)^n\int_{-\infty}^{\infty} g(t)\frac{d^n\varphi(t)}{dt^n}\,dt$

8. Product with ordinary function: $\int_{-\infty}^{\infty} [g(t)f(t)]\varphi(t)\,dt = \int_{-\infty}^{\infty} g(t)[f(t)\varphi(t)]\,dt$

   provided that $f(t)\varphi(t)$ belongs to the set of test functions.

9. Convolution: $\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)\,d\tau\right]\varphi(t)\,dt = \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} g_2(t-\tau)\varphi(t)\,dt\right]d\tau$

Definition

A sequence of distributions $\{g_n(t)\}_1^\infty$ is said to converge to the distribution $g(t)$ if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} g_n(t)\varphi(t)\,dt = \int_{-\infty}^{\infty} g(t)\varphi(t)\,dt$$

for all φ belonging to the set of test functions.

10. Every distribution is the limit, in the sense of distributions, of a sequence of infinitely differentiable functions.
11. If $g_n(t) \to g(t)$ and $r_n(t) \to r(t)$ (r being a distribution), and the numbers $a_n \to a$, then $\frac{d}{dt} g_n(t) \to \frac{dg(t)}{dt}$, $g_n(t) + r_n(t) \to g(t) + r(t)$, $a_n g_n(t) \to ag(t)$
12. Any distribution $g(t)$ may be differentiated as many times as desired. The derivative of any distribution always exists, and it is a distribution.

5.3 One-Dimensional Delta Function

5.3.1 Definition

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt = \varphi(0), \qquad \varphi(t) \text{ is a testing function}$$

5.3.2 Properties

TABLE 5.1 Properties of Delta Function

Delta Function Properties
$\delta(at) = \frac{1}{\lvert a\rvert}\delta(t)$
$\delta\left(\frac{t-t_o}{a}\right) = \lvert a\rvert\,\delta(t-t_o)$
$\delta(at-t_o) = \frac{1}{\lvert a\rvert}\,\delta\left(t-\frac{t_o}{a}\right)$
$\delta(-t+t_o) = \delta(t-t_o)$
$\delta(-t) = \delta(t)$; $\quad \delta(t) =$ even function
$\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$
$\int_{-\infty}^{\infty} \delta(t-t_o) f(t) = f(t_o)$
$f(t)\delta(t) = f(0)\delta(t)$

TABLE 5.1 Properties of Delta Function (continued)

Delta Function Properties
$f(t)\delta(t-t_o) = f(t_o)\delta(t-t_o)$
$t\delta(t) = 0$
$\int_{-\infty}^{\infty} A\delta(t)\,dt = \int_{-\infty}^{\infty} A\delta(t-t_o)\,dt = A$
$f(t) * \delta(t) = \text{convolution} = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)\,d\tau = f(t)$
$\delta(t-t_1) * \delta(t-t_2) = \int_{-\infty}^{\infty} \delta(\tau-t_1)\delta(t-\tau-t_2)\,d\tau = \delta[t-(t_1+t_2)]$
$\sum_{n=-N}^{N} \delta(t-nT) * \sum_{n=-N}^{N} \delta(t-nT) = \sum_{n=-2N}^{2N} (2N+1-\|n\|)\delta(t-nT)$
$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} f(t)\,dt = -\frac{df(0)}{dt}$
$\int_{-\infty}^{\infty} \frac{d\delta(t-t_o)}{dt} f(t)\,dt = -\frac{df(t_o)}{dt}$
$\int_{-\infty}^{\infty} \frac{d^n\delta(t)}{dt^n} f(t)\,dt = (-1)^n \frac{d^n f(0)}{dt^n}$
$f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\delta(t) + f(0)\frac{d\delta(t)}{dt}$
$t\frac{d\delta(t)}{dt} = -\delta(t)$
$t^n \frac{d^m\delta(t)}{dt^m} = \begin{cases} (-1)^n\, n!\,\delta(t), & m = n \\ (-1)^n \frac{m!}{m-n!} \frac{d^{m-n}\delta(t)}{dt^{m-n}}, & m > n \\ 0, & m < n \end{cases}$
$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt}\,dt = 0, \quad \frac{d\delta(t)}{dt} = \text{odd function}$
$f(t) * \frac{d\delta(t)}{dt} = \frac{df(t)}{dt}$
$f(t)\frac{d^n\delta(t)}{dt^n} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{d^k f(0)}{dt^k} \frac{d^{n-k}\delta(t)}{dt^{n-k}}$
$\frac{\partial\delta(yt)}{\partial y} = -\frac{1}{y^2}\delta(t)$
$\delta(t) = \frac{du(t)}{dt}$
$\frac{d^n\delta(-t)}{dt^n} = (-1)^n \frac{d^n\delta(t)}{dt^n} \quad , \quad \left\{ \frac{d^n\delta(t)}{dt^n} \text{ is even if } n \text{ is even, and odd if } n \text{ is odd.} \right\}$
$(\sin at)\frac{d\delta(t)}{dt} = -a\delta(t)$

TABLE 5.1 Properties of Delta Function (continued)

Delta Function Properties
$\frac{d\delta(t)}{dt} = \frac{d^2u(t)}{dt^2}$
$-\delta(t) = \frac{du(-t)}{dt}$
$\delta(t-t_o) = \frac{du(t-t_o)}{dt}$
$\frac{d\,\mathrm{sgn}(t)}{dt} = 2\delta(t)$
$\delta[r(t)] = \sum_n \frac{\delta(t-t_n)}{\left\lvert \frac{dr(t_n)}{dt} \right\rvert}$ $\qquad t_n = \text{zeros of } r(t),\ \frac{dr(t_n)}{dt} \neq 0$
$\frac{d\,\delta[r(t)]}{dt} = \sum_n \frac{\frac{d\delta(t-t_n)}{dt}}{\frac{dr(t)}{dt}\left\lvert \frac{dr(t_n)}{dt} \right\rvert}$ $\qquad t_n = \text{zeros of } r(t),\ \frac{dr(t_n)}{dt} \neq 0,\ \frac{dr(t)}{dt} \neq 0$
$\delta(\sin t) = \sum_{n=-\infty}^{\infty} \delta(t-n\pi)$
$\delta(t^2-1) = \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t+1)$
$\delta(t^2-a^2) = \frac{1}{2a}[\delta(t+a)+\delta(t-a)]$
$\delta(t) = \lim_{\varepsilon \to o} \frac{e^{-t^2/\varepsilon}}{\sqrt{\varepsilon\pi}}$
$\delta(t) = \lim_{\omega \to \infty} \frac{\sin \omega t}{\pi t}$
$\delta(t) = \lim_{\varepsilon \to o} \frac{1}{\pi} \frac{\varepsilon}{t^2+\varepsilon^2}$
$\frac{d\delta(t)}{dt} = \lim_{\varepsilon \to 0} -\frac{2}{\pi} \frac{\varepsilon t}{(t^2+\varepsilon^2)^2}$
$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \omega t \, d\omega$
$\frac{df(t)}{dt} = \frac{d}{dt}[t\,u(t) - (t-1)u(t-1) - u(t-1)]$ $= t\delta(t) + u(t) - (t-1)\delta(t-1) - u(t-1) - \delta(t-1)$
$comb_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s), \qquad f(t)comb_{T_s}(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t-nT_s)$
$COMB_{\omega_s}(\omega) = \mathcal{F}\{comb_{T_s}(t)\} = \omega_s \sum_{n=-\infty}^{\infty} \delta(t-n\omega_s) \qquad \omega_s = \frac{2\pi}{T_s}$
$\lim_{a \to \infty} \int_{-a}^{a} e^{j\omega t}\, d\omega = 2\pi\delta(t)$

TABLE 5.1 Properties of Delta Function (continued)

Delta Function Properties
$\lim_{a\to\infty}\int_{-a}^{a}(j\omega)e^{j\omega t}\,d\omega = 2\pi\frac{d\delta(t)}{dt}$
$\lim_{a\to\infty}\int_{-a}^{a}(j\omega)^2 e^{j\omega t}\,d\omega = 2\pi\frac{d^2\delta(t)}{dt^2}$

The following examples will elucidate some of the delta properties and the use of the delta function.

5.4 Examples

Example 5.4

Equivalence of expressions involving the delta functions:

a) $(\cos t + \sin t)\,\delta(t) = \delta(t)$ b) $\cos 2t + \sin t\,\delta(t) = \cos 2t$

c) $1 + 2e^{-t}\,\delta(t-1) = 1 + 2e^{-1}\,\delta(t-1)$

Example 5.5

The values of the following integrals are:

$$\int_{-\infty}^{\infty}(t^2+4t+5)\delta(t)\,dt = 0^2+4\cdot 0+5 = 5,$$

$$\int_{-\infty}^{\infty}\frac{(1+\cos t)\delta(t)}{1+2e^t}\,dt = \frac{2}{1+2}$$

$$\int_{-\infty}^{\infty} t^2\sum_{k=1}^{n}\delta(t-k)\,dt = \sum_{k=1}^{n}k^2 = \tfrac{1}{6}[n(n+1)(2n+1)]$$

Example 5.6

The first derivative of the functions is:

$$\frac{d}{dt}(2u(t+1)+u(1-t)) = \frac{d}{dt}(2u(t+1)+u[-(t-1)]) = 2\delta(t+1)-\delta(t-1)$$

$$\frac{d}{dt}([2-u(t)]\cos t = \frac{d}{dt}(2\cos t - u(t)\cos t) = 2\sin t - \delta(t)\cos t + u(t)\sin t = (u(t)-2)\sin t - \delta(t)$$

$$\frac{d}{dt}\left(\left[u\left(t-\frac{\pi}{2}\right)-u(t-\pi)\right]\sin t\right) = \left[\delta\left(t-\frac{\pi}{2}\right)-\delta(t-\pi)\right]\sin t + \left[u\left(t-\frac{\pi}{2}\right)-u(t-\pi)\right]\cos t$$

$$= \delta\left(t-\frac{\pi}{2}\right)+[u\left(t-\frac{\pi}{2}\right)-u(t-\pi)]\cos t$$

Example 5.7

The values of the following integrals are:

$$\int_{-\infty}^{\infty} e^{2t}\sin 4t \frac{d^2\delta(t)}{dt^2}\,dt = (-1)^2 \frac{d^2}{dt^2}[e^{2t}\sin 4t]\Big|_{t=o} = 2\times 2\times 4 = 16$$

$$\int_{-\infty}^{\infty} (t^3+2t+3)\left(\frac{d\delta(t-1)}{dt} + 2\frac{d^2\delta(t-2)}{dt^2}\right)dt = \int_{-\infty}^{\infty} (t^3+2t+3)\,\frac{d\delta(t-1)}{dt}\,dt$$

$$+2\int_{-\infty}^{\infty} (t^3+2t+3)\,\frac{d^2\delta(t-2)}{dt^2}\,dt = (-1)(3t^2+2)\Big|_{t=1} + (-1)^2\,2\,(6t)\Big|_{t=2} = -5+24 = 19$$

Example 5.8

The values of the following integrals are:

$$\int_{o}^{4} e^{4t}\delta(2t-3)\,dt = \int_{o}^{4} e^{4t}\delta\left[2\left(t-\frac{3}{2}\right)\right] = \frac{1}{2}\int_{o}^{4} e^{4t}\delta\left(t-\frac{3}{2}\right)dt = \frac{1}{2}e^{4\frac{3}{2}} = \frac{1}{2}e^6$$

$$\int_{o}^{4} e^{4t}\delta(3-2t)\,dt = \int_{o}^{4} e^{4t}\delta[-(2t-3)]\,dt = \frac{1}{2}\int_{o}^{4} e^{4t}\delta(2t-3)\,dt = \frac{1}{2}e^6$$

5.5 Two-Dimensional Delta Function

5.5.1 Definitions

$$\delta(x,y) = \delta(x)\delta(y)$$

$$\delta(x-x_o, y-y_o) = \delta(x-x_o)\delta(y-y_o)$$

$$\iint_{-\infty}^{\infty} f(\xi,\eta)\,\delta(x-\xi)\delta(y-\eta)\,d\xi\,d\eta = f(x,y)$$

$$\iint_{A} \delta(x-a)\delta(y-b) = p_a(a,b) \qquad a,b \text{ not on the boundary of } A$$

$$p_A(x,y) = \begin{cases} 1 & x,y \in A \\ 0 & \text{otherwise} \end{cases}$$

5.5.2 Line Masses

The function $\varphi(y)\delta(x-a)$ can be interpreted as a line mass on the line $x = a$ of density $\varphi(y)$.

Example 5.9

$p_a(y)\delta(x)$ is a line mass on the y-axis with density one on the y-axis from $y = -a$ to $y = a$.

Example 5.10

$f(x,y) ** \delta(x) = \iint\limits_{-\infty}^{\infty} f(\xi,\eta)\,\delta(x-\xi)\,d\xi\,d\eta = \int\limits_{-\infty}^{\infty} f(x,\eta)\,d\eta$ which is the x profile of $f(x,y)$.

5.5.3 Line Mass on a Curve

$\delta[\alpha(x,y)]$ is a line mass on the curve $\alpha(x,y) = 0$ with density $\lambda(x,y) = \dfrac{1}{\sqrt{\alpha_x^2+\alpha_y^2}}$ where $\alpha_x = \dfrac{\partial\alpha(x,y)}{\partial x}$, $\alpha_y = \dfrac{\partial\alpha(x,y)}{\partial y}$.

5.5.4 Line Masses Along x- and y-Axes

The line masses have densities along the x- and y-directions given by $\dfrac{dm}{dx} = \dfrac{1}{|\alpha_x|}\left(\alpha_x = \dfrac{\partial\alpha(x,y)}{\partial x}\right)$ and $\dfrac{dm}{dy} = \dfrac{1}{|\alpha_y|}\left(\alpha_y = \dfrac{\partial\alpha(x,y)}{\partial y}\right)$, respectively. $\dfrac{dm}{ds} = \dfrac{dm}{\sqrt{dx^2+dy^2}} = \dfrac{1}{\sqrt{\left(\dfrac{dx}{dm}\right)^2+\left(\dfrac{dy}{dm}\right)^2}} = \dfrac{1}{\sqrt{\alpha_x^2+\alpha_y^2}}$ and hence $\delta[\alpha(x,y)] = \dfrac{1}{\sqrt{\alpha_x^2+\alpha_y^2}}\delta(s)$ and $s = \alpha(x,y) = 0$ is the curve of $\alpha(x,y)$.

5.5.5 Solution of $\alpha(x,y)$

If we solve $\alpha(x,y) = 0$ for x and denote i^{th} root with x_i then we may regard $\delta[\alpha(x,y)]$ as the line mass

$$\delta[\alpha(x,y)] = \sum_i \frac{1}{|\alpha_x|}\delta(x-x_i), \qquad \alpha_x = \frac{\partial\alpha(x,y)}{\partial x}$$

and similarly for the y solution

$$\delta[\alpha(x,y)] = \sum_i \frac{1}{|\alpha_y|}\delta(y-y_i), \qquad \alpha_y = \frac{\partial\alpha(x,y)}{\partial y}$$

Example 5.11

If $\delta[\sqrt{x^2+y^2}-r_o]$ then $\alpha(x,y) = \sqrt{x^2+y^2}-r_o$, $\alpha_x = x/\sqrt{x^2+y^2}$, $\alpha_y = y/\sqrt{x^2+y^2}$, $x_1,x_2 = \pm\sqrt{r_o^2-y^2}$, $y_1,y_2 = \pm\sqrt{r_o^2-x^2}$.

$$\delta[\alpha(x,y)] = \left|\frac{\sqrt{x^2+y^2}}{x^2}\right|\left[\delta(x-\sqrt{r_o^2-y^2})+\delta(x+\sqrt{r_o^2-y^2})\right]$$

$$= \frac{r_o}{\sqrt{r_o^2-y^2}}\left[\delta(x-\sqrt{r_o^2-y^2})+\delta(x+\sqrt{r_o^2-y^2})\right], \qquad |y|<r_o.\ \text{Also}$$

$$\delta[\alpha(x,y)] = \frac{r_o}{\sqrt{r_o^2-x^2}}\left[\delta(y-\sqrt{r_o^2-x^2})+\delta(y+\sqrt{r_o^2-x^2})\right].$$

Since $\alpha_x^2+\alpha_y^2 = 1$ and $r = \sqrt{x^2+y^2}$ then $\delta[\alpha(x,y)] = \delta(r-r_o)$ is a ring delta function with unit density along $r = r_o$.

Example 5.12

If $\delta(\alpha x + by + c)$, then $\alpha(x,y) = \alpha x + by + c$, $\alpha_x = a$, $\alpha_y = b$, $x = -\frac{b}{a}y - \frac{c}{a}$, $y = -\frac{a}{b}x - \frac{c}{b}$ and hence

$$\delta(\alpha x + by + c) = \frac{1}{|a|}\delta\left(x + \frac{b}{a}y + \frac{c}{a}\right) = \frac{1}{|b|}\delta\left(y + \frac{a}{b}y + \frac{c}{b}\right).$$

5.5.6 Transformation of Coordinates for $\delta(ax + by + c)$ (see Figure 5.1)

$x' = \cos\theta' + y\sin\theta'$, $y' = -x\sin\theta' + y\cos\theta'$, $\theta' = \tan^{-1}\left(\frac{b}{a}\right)$, $k = \sqrt{a^2 + b^2}$, $\cos\theta' = \frac{a}{k}$, $\sin\theta' = \frac{b}{k}$, $x = (ax' - by')/k$, $y = (bx' + ay')/k$.

$$\delta(ax + by + c) = \delta\left(\frac{a^2x' - aby'}{k} + \frac{b^2x' + aby'}{k} + c\right) = \delta(kx' + c) = \frac{1}{k}\delta(x' - x_o) \text{ where } x_o' = -c/k.$$

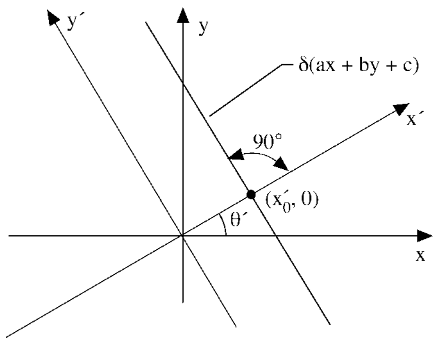

FIGURE 5.1

Example 5.13

$f(x,y)\delta(ax + by + c) = \frac{1}{k} f\left(\frac{ax_o' - b_1y'}{k}, \frac{bx_o' + ay'}{k}\right)\delta(x' - x_o')$ where $k = \sqrt{a^2 + b^2}$ and $c/k = -x_o'$.

The density along this line is $\frac{dm}{dy'} = \left(\frac{1}{k}\right) f\left(\frac{ax_o' - by'}{k}, \frac{bx_o' + ay'}{k}\right)$.

5.5.7 The Function $\delta(a_1x + b_1y + c_1, a_2x + b_2y + c_2)$: From (5.5.5)

$$\delta(a_1x + b_1y + c_1, a_2x + b_2y + c_2) = \frac{1}{|a_1|}\delta\left(x + \frac{b_1y}{a_1} + \frac{c_1}{a_1}\right)\delta\left(\frac{-a_2b_1y - a_2c_1}{a_1} + b_2y + c_2\right)$$

$$= \frac{1}{|a_1|}\delta\left(x + \frac{b_1}{a_1}y + \frac{c_1}{a_1}\right)\frac{|a_1|}{|a_1b_2 - a_2b_1|}\delta\left(y - \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

$$= \frac{1}{|a_1b_2 - a_2b_1|}\delta\left(x - \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right)\delta\left(y - \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

$$= \frac{1}{|D|}\delta(x - x_o, y - y_o)$$

5.5.8 The function $f(x,y)\delta(a_1x + b_1y + c_1,\ a_2x + b_2y + c_2)$

$f(x,y)\delta(a_1x+b_1y+c_1, a_2x+b_2y+c_2) = \frac{1}{|D|} f(x_o,y_o)\delta(x-x_o, y-y_o)$. See (5.5.7) for the values of D, x_o, and y_o.

5.5.9 $comb(a_1x + b_1y + c_1,\ a_2x + b_2y + c_2)$

$$comb(a_1x+b_1y+c_1,\ a_2x+b_2y+c_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(a_1x+b_2y+c_1-n)\delta(a_2x+b_2y+c_2-m)$$

$$= \frac{1}{|D|}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(x-x_o-\frac{b_2n}{D}+\frac{b_1m}{D}\right)\delta\left(y-y_o+\frac{a_2n}{D}-\frac{a_1m}{D}\right)$$

See (5.5.7) for the values of D, x_o, and y_o.

5.5.10 $comb(a_1x + b_1y,\ a_2x + b_2y)$

$$comb(a_1x+b_1y,\ a_2x+b_2y) = \frac{1}{|D|}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(x-\frac{b_2n}{D}+\frac{b_1m}{D}\right)\delta\left(y+\frac{a_2n}{D}-\frac{a_1m}{D}\right)$$

5.5.11 $f(x,y)\ comb(a_1x + b_1y + c_1,\ a_2x + b_2y + c_2)$

$$f(x,y)comb(a_1x+b_1y+c_1,\ a_2x+b_2y+c_2) = \frac{1}{|D|}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f\left(x_o+\frac{b_2n}{D}-\frac{b_1m}{D},\ y_o-\frac{a_2n}{D}+\frac{a_1m}{D}\right)$$

$$\times\delta\left(x-x_o-\frac{b_2n}{D}+\frac{b_1m}{D}\right)\delta\left(y-y_o+\frac{a_2n}{D}-\frac{a_1m}{D}\right)$$

5.5.12 $\delta[\alpha_1(x,y)]\ \delta[\alpha_2(x,y)]$

$$\delta[\alpha_1(x,y)]\delta[\alpha_2(x,y)] = \sum_i \frac{\delta(x-x_i)\delta(y-y_i)}{|\alpha_{1x}\alpha_{2y}-\alpha_{1y}\alpha_{2x}|}$$

where x_i, y_i are the coordinates of the intersections of the curves $\alpha_1(x,y)$ and $\alpha_2(x,y)$,

$$\alpha_{1x} = \frac{\partial\alpha_1(x,y)}{\partial x},\ \alpha_{1y} = \frac{\partial\alpha_1(x,y)}{\partial y},\ \alpha_{2x} = \frac{\partial\alpha_2(x,y)}{\partial x},\ \text{and}\ \alpha_{2y} = \frac{\partial\alpha_2(x,y)}{\partial y}.$$

Example 5.14

(See Figure 5.2)

$\delta[\alpha_1(x,y)]\ \delta[\alpha_2(x,y)] = \delta(\sqrt{x^2+y^2}-a)\ \delta(x-x_o)$. Intersect at $(x_o, \sqrt{a^2-x_o^2})$ and at $(x_o, -\sqrt{a^2-x_o^2})$ and $x_o \le a$. $\alpha_1(x,y) = \sqrt{x^2+y^2}-a$, $\alpha_2(x,y) = x - x_o$, $\partial\alpha_1/\partial x = x/\sqrt{x^2+y^2}$, $\partial\alpha_1/\partial y = y/\sqrt{x^2+y^2}$, $\partial\alpha_2/\partial x = 1$, and $\partial\alpha_2/\partial y = 0$.

Hence from (5.5.12)

$$\delta[\alpha_1(x,y)]\delta[\alpha_2(x,y)] = \frac{\sqrt{x^2+y^2}}{y}[\delta(x-x_o)\delta(y+\sqrt{a^2-x_o^2}) + \delta(x-x_o)\delta(y-\sqrt{a^2-x_o^2})]$$

and

$$\iint_{-\infty}^{\infty} \delta[\alpha_1(x,y)]\delta[\alpha_2(x,y)]\,dx\,dy = \iint_{-\infty}^{\infty} \frac{\sqrt{x^2+y^2}}{y}\delta(x-x_o)\delta(y+\sqrt{a^2-x_o^2})\,dx\,dy$$

$$+\iint_{-\infty}^{\infty} \frac{\sqrt{x^2+y^2}}{y}\delta(x-x_o)\delta(y-\sqrt{a^2-x_o^2})\,dx\,dy$$

$$= \frac{\sqrt{x_o^2+a^2-x_o^2}}{\sqrt{a^2-x_o^2}} + \frac{\sqrt{x_o^2+a^2-x_o^2}}{\sqrt{a^2-x_o^2}} = \frac{2a}{\sqrt{a^2-x_o^2}} \qquad \text{for } x_o < a.$$

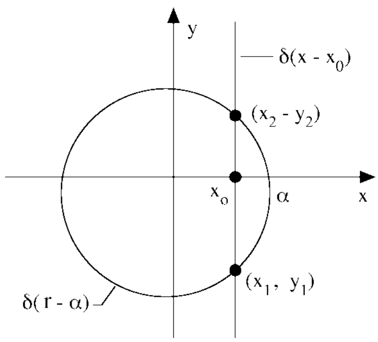

FIGURE 5.2

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6
The Z-Transform

6.1 One-Dimensional Z-Transform (positive-time sequences)

6.1.1 Definitions of One-Sided Z-Transform

$$Z\{f(nT)\} \triangleq F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$Z\{f(n)\} \triangleq F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

6.1.2 Radius of Convergence for One-Sided Z-Transform

$$|z| > \overline{\lim_{n\to\infty}} \sqrt[n]{|f(nT)|} = R$$

where $\overline{\lim}$ denotes the **greatest** limit points of $\overline{\lim_{n\to\infty}} |f(nT)|^{1/n}$.

6.1.3 Region of Convergence for One-Sided Z-Transform

The series will converge absolutely for all points in the z-plane that lie **outside** the circle of radius R.

For example, the region of convergence of $f(nT) = e^{-aTn}u(nT)$ is: $\left|z^{-n}\, e^{-aTn}\right|^{1/n} < 1$ or $|z| > e^{-aT}$.

6.1.4 Table of One-Sided Z-Transform Properties

TABLE 6.1 Z-Transform Properties for Positive-Time Sequences

1. Linearity

$$Z\{c_i\, f_i(nT)\} = c_i\, F_i(z) \qquad |z| > R_i, \quad c_i = \text{constants}$$

$$Z\left\{\sum_{i=0}^{\ell} c_i\, f_i(nT)\right\} = \sum_{i=0}^{\ell} c_i\, F_i(z) \qquad |z| > \max R_i$$

2. Shifting Property

$$Z\{f(nT-kT)\} = z^{-k} F(z) \qquad f(-nT) = 0 \quad n = 1, 2, \cdots$$

$$Z\{f(nT-kT)\} = z^{-k} F(z) + \sum_{n=1}^{k} f(-nT)\, z^{-(k-n)}$$

$$Z\{f(nT+kT)\} = z^{k} F(z) - \sum_{n=0}^{k-1} f(nT)\, z^{k-n}$$

$$Z\{f(nT+T)\} = z[F(z) - f(0)]$$

3. Time Scaling

$$Z\{a^{nT} f(nT)\} = F(a^{-T} z) = \sum_{n=0}^{\infty} f(nT)(a^{-T} z)^{-n} \qquad |z| > a^{T}$$

4. Periodic Sequence

$$Z\{f(nT)\} = \frac{z^N}{z^N - 1} F_{(1)}(z) \qquad |z| > R$$

N = number of time units in a period

R = radius of convergence of $F_{(1)}(z)$

$F_{(1)}(z)$ = Z-transform of the first period

5. Multiplication by n and nT

$$Z\{n\, f(nT)\} = -z \frac{dF(z)}{dz} \qquad |z| > R$$

$$Z\{nT\, f(nT)\} = -zT \frac{dF(z)}{dz} \qquad |z| > R$$

R = radius of convergence of $F(z)$

6. Convolution

$$Z\{f(nT)\} = F(z) \qquad |z| > R_1$$

$$Z\{h(nT)\} = H(z) \qquad |z| > R_2$$

$$Z\{f(nT) * h(nT)\} = F(z)\, H(z) \qquad |z| > \max(R_1, R_2)$$

7. Initial Value

$$f(0T) = \lim_{z\to\infty} F(z) \qquad |z| > R$$

8. Final Value

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1} (z-1) F(z) \qquad \text{if } f(\infty T) = \text{exists}$$

9. Multiplication by $(nT)^k$

$$Z\{n^k T^k\, f(nT)\} = -Tz \frac{d}{dz} Z\{(nT)^{k-1} f(nT)\} \qquad k > 0 \text{ and integer}$$

10. Complex Conjugate Signals

$$Z\{f(nT)\} = F(z) \qquad |z| > R$$

$$Z\{f^*(nT)\} = F^*(z^*) \qquad |z| > R$$

TABLE 6.1 Z-Transform Properties for Positive-Time Sequences (continued)

11. Transform of Product

$$Z\{f(nT)\} = F(z) \qquad |z| > R_f$$

$$Z\{h(nT)\} = H(z) \qquad |z| > R_h$$

$$Z\{f(nT)\,h(nT)\} = \frac{1}{2\pi j}\oint_C F(\tau)\,H\!\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau}, \qquad |z| > R_f\,R_h,\; R_f < |\tau| < \frac{|z|}{R_h}$$

counter-clockwise integration

12. Parseval's Theorem

$$Z\{f(nT)\} = F(z) \qquad |z| > R_f$$

$$Z\{h(nT)\} = H(z) \qquad |z| > R_h$$

$$\sum_{n=0}^{\infty} f(nT)\,h(nT) = \frac{1}{2\pi j}\oint_C F(z)\,H(z^{-1})\,\frac{dz}{z} \qquad |z| = 1 > R_f\,R_h$$

counter-clockwise integration

13. Correlation

$$f(nT) \bigstar\; h(nT) = \sum_{m=0}^{\infty} f(mT)\,h(mT - nT) = \frac{1}{2\pi j}\oint_C F(\tau)\,H\!\left(\frac{1}{\tau}\right)\tau^{n-1}\,d\tau, \qquad n \ge 1$$

Both $f(nT)$ and $h(nT)$ must exist for $|z| > 1$. The integration is counter-clockwise.

14. Transforms with Parameters

$$Z\left\{\frac{\partial}{\partial a} f(nT, a)\right\} = \frac{\partial}{\partial a} F(z, a)$$

$$Z\{\lim_{a \to a_o} f(nT, a)\} = \lim_{a \to a_o} F(z, a)$$

$$Z\left\{\int_{a_o}^{a_1} f(nT, a)\,da\right\} = \int_{a_o}^{a_1} F(z, a)\,da \qquad \text{finite interval}$$

6.1.5 Inverse Z-Transform (see Appendix)

1. Use tables.
2. Decompose the expression into simpler partial forms which are included in the tables
3. If the transform is decomposed into a product of partial series, the resulting object function is obtained as the convolution of the partial object function.
4. Use inversion integral.

6.2 One-Dimensional, Two-Sided Z-Transform

6.2.1 Definitions

$$Z_{II}\{f(nT)\} \triangleq F(z) = \sum_{n=-\infty}^{\infty} f(nT)\,z^{-n} \qquad R_+ < |z| < R_-$$

$$Z_{II}\{f(n)\} \triangleq F(z) = \sum_{n=-\infty}^{\infty} f(n)\,z^{-n} \qquad R_+ < |z| < R_-$$

6.2.2 Region of Convergence

Assuming that the algebraic expression for the Z-transform $F(z)$ is a rational function and that $f(nT)$ has finite amplitude, except possibly at infinities, the properties of the region of convergence are:

1. The ROC is a ring or disc in the z-plane and centered at the origin. $0 \le R_+ < |z| < R_- \le \infty$.
2. The Fourier transform converges also absolutely if and only if the ROC of the Z-transform of $f(nT)$ includes the unit circle.
3. No poles exist in the ROC.
4. The ROC of a finite sequence $\{f(nT)\}$ is the entire z-plane except possibly for $z = 0$ or $z = \infty$.
5. If $f(nT)$ is right-handed, $0 \le n < \infty$, the ROC extends outward from the outermost pole of $F(z)$ to infinity.
6. If $f(nT)$ is left-handed, $-\infty \le n < 0$, the ROC extends inward from the innermost pole of $F(z)$ to zero.
7. An infinite-duration two-sided sequence $\{f(nT)\}$ has a ring as its ROC bounded on the interior and exterior by a pole. The ring contains no poles.
8. The ROC must be a connected region.

6.2.3 Properties of Two-Sided Z-Transform

TABLE 6.2 Z-Transform Properties for Positive- and Negative-Time Sequences

1. Linearity

$$Z_{II}\left\{\sum_{i=0}^{\ell} c_i\, f_i(nT)\right\} = \sum_{i=0}^{\ell} c_i\, F_i(z) \qquad \max R_{i+} < |z| < \min R_{i-}$$

2. Shifting Property

$$Z_{II}\{f(nT \pm kT)\} = z^{\pm k} F(z) \qquad R_+ < |z| < R_-$$

3. Scaling

$$Z_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$Z_{II}\{a^{nT} f(nT)\} = F(a^{-T} z) \qquad |a^T| R_+ < |z| < |a^T| R_-$$

4. Time Reversal

$$Z_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$Z_{II}\{f(-nT)\} = F(z^{-1}) \qquad \tfrac{1}{R_-} < |z| < \tfrac{1}{R_+}$$

5. Multiplication by nT

$$Z_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$Z_{II}\{nT\, f(nT)\} = -zT \frac{dF(z)}{dz} \qquad R_+ < |z| < R_-$$

6. Convolution

$$Z_{II}\{f_1(nT) * f_2(nT)\} = F_1(z) F_2(z)$$

$$ROC\, F_1(z) \cup ROC\, F_2(z), \quad \max(R_{+f_1}, R_{+f_2}) < |z| < \min(R_{-f_1}, R_{-f_2})$$

7. Correlation

$$R_{f_1 f_2}(z) = Z_{II}\{f_1(nT) \bigstar f_2(nT)\} = F_1(z) F_2(z^{-1})$$

$$ROC\, F_1(z) \cup ROC\, F_2(z^{-1}), \quad \max(R_{+f_1}, R_{+f_2}) < |z| < \min(R_{-f_1}, R_{-f_2})$$

8. Multiplication by e^{-anT}

$$Z_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$Z_{II}\{e^{-anT} f(nT)\} = F(e^{aT} z) \qquad |e^{-aT}| R_+ < |z| < |e^{-aT}| R_-$$

TABLE 6.2 Z-Transform Properties for Positive- and Negative-Time Sequences (continued)

9. Frequency Translation

$$G(\omega)=Z_{II}\{e^{j\omega_o nT}f(nT)\}=G(z)\Big|_{z=e^{j\omega T}}=F(e^{j(\omega-\omega_o)T})=F(\omega-\omega_o)$$

ROC of $F(z)$ must include the unit circle

10. Product

$$Z_{II}\{f(nT)\}=F(z)\qquad R_{+f}<|z|<R_{-f}$$

$$Z_{II}\{h(nT)\}=H(z)\qquad R_{+h}<|z|<R_{-h}$$

$$Z_{II}\{f(nT)h(nT)\}=G(z)=\frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau},\qquad R_{+f}R_{+h}<|z|<R_{-f}R_{-h}$$

$$\max\left(R_{+f},\frac{|z|}{R_{-h}}\right)<|\tau|<\min\left(R_{-f},\frac{|z|}{R_{+h}}\right)$$

counter-clockwise integration

11. Parseval's Theorem

$$Z_{II}\{f(nT)\}=F(z)\qquad R_{+f}<|z|<R_{-f}$$

$$Z_{II}\{h(nT)\}=H(z)\qquad R_{+h}<|z|<R_{-h}$$

$$\sum_{n=-\infty}^{\infty}f(nT)h(nT)=\frac{1}{2\pi j}\oint_C F(z)H(z^{-1})\frac{dz}{z}\qquad R_{+f}R_{+h}<|z|=1<R_{-f}R_{-h}$$

$$\max\left(R_{+f},\frac{1}{R_{-h}}\right)<|z|<\min\left(R_{-f},\frac{1}{R_{+h}}\right)$$

counter-clockwise integration

12. Complex Conjugate Signals

$$Z_{II}\{f(nT)\}=F(z)\qquad R_{+f}<|z|<R_{-f}$$

$$Z_{II}\{f^*(nT)\}=F^*(z^*)\qquad R_{+f}<|z|<R_{-f}$$

6.3 Inverse Z-Transform

TABLE 6.3 Inverse Transforms of the Partial Fractions of $F(z)$

Partial Fraction Term	Inverse Transform Term If $F(z)$ Converges Absolutely for Some $|z|>|a|$
$\frac{z}{z-a}$	$a^k,\quad k\ge 0$
$\frac{z^2}{(z-a)^2}$	$(k+1)a^k,\quad k\ge 0$
$\frac{z^3}{(z-a)^3}$	$\frac{1}{2}(k+1)(k+2)a^k,\quad k\ge 0$
$\vdots$	$\vdots$
$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}(k+1)(k+2)\cdots(k+n-1)a^k,\quad k\ge 0$

TABLE 6.3 Inverse Transforms of the Partial Fractions of $F(z)$ (continued)

Partial Fraction Term	Inverse Transform Term If $F(z)$ Converges Absolutely for Some $\|z\| < \|a\|$
$\frac{z}{z-a}$	$-a^k, \quad k \le -1$
$\frac{z^2}{(z-a)^2}$	$-(k+1)a^k, \quad k \le -1$
$\frac{z^3}{(z-a)^3}$	$-\frac{1}{2}(k+1)(k+2)a^k, \quad k \le -1$
$\vdots$	$\vdots$
$\frac{z^n}{(z-a)^n}$	$-\frac{1}{(n-1)!}(k+1)(k+2)\cdots(k+n-1)a^k, \quad k \le -1$

TABLE 6.4 Inverse Transforms of the Partial Fractions of $F_i(z)$[1]

Elementary Transform Term $F_i(z)$	Corresponding Time Sequence: (I) $F_i(z)$ converges for $\|z\| > R_c$	Corresponding Time Sequence: (I) $F_i(z)$ converges for $\|z\| < R_c$
$\frac{1}{z-a}$	$a^{k-1}\big\|_{k\ge 1}$	$-a^{k-1}\big\|_{k\le 0}$
$\frac{z}{(z-a)^2}$	$ka^{k-1}\big\|_{k\ge 1}$	$-ka^{k-1}\big\|_{k\le 0}$
$\frac{z(z+a)}{(z-a)^3}$	$k^2a^{k-1}\big\|_{k\ge 1}$	$-k^2a^{k-1}\big\|_{k\le 0}$
$\frac{z(z^2+4az+a^2)}{(z-a)^4}$	$k^3a^{k-1}\big\|_{k\ge 1}$	$-k^3a^{k-1}\big\|_{k\le 0}$

1. The function must be a proper function.

TABLE 6.5 Total Square Integrals

A solution of the integral

$$I_n = \frac{1}{2\pi j}\oint_{\substack{\text{unit}\\ \text{circle}}} F(z,m)F(z^{-1},m)z^{-1}\,dz$$

is presented in this table.

Let

$$F(z,m) = \frac{B(z)}{A(z)}$$

where

$$A(z) = \sum_{r=0}^{n} a_r z^{n-r}, \qquad a_0 \ne 0$$

$$B(z)=\sum_{r=0}^{n} b_r z^{n-r}, \quad \text{and the } br\text{'s are bounded functions of } m, \quad 0 \le m \le 1$$

and the coefficients $a_r,\ 0 < r \le n$, and $b_r,\ 0 \le r \le n$ are not necessarily nonzero.

The integral I_n is equivalent to the ratio of two determinants as follows:

$$I_n = \frac{|\Omega_1|}{a_0|\Omega|}$$

where Ω is the following matrix:

$$\Omega \triangleq \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_0+a_2 & a_1+a_3 & a_2+a_4 & \cdots & a_{n-1} \\ a_2 & a_3 & a_0+a_1 & a_1+a_5 & \cdots & a_{n-2} \\ \vdots & & & & & \\ a_n & 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}$$

and Ω_1 is the maxtrix formed from Ω by replacing the first column by

$$\begin{bmatrix} \sum_{i=0}^{n} b_i^2 \\ 2\sum b_i b_{i+1} \\ 2\sum b_i b_{i+2} \\ \vdots \\ 2\sum b_i b_{i+n-1} \\ 2b_0 b_n \end{bmatrix}$$

Tabulated values of the integral I_n

$$I_n = \frac{1}{2\pi j}\oint_{\substack{\text{unit}\\ \text{circle}}} \frac{B(z)}{A(z)}\frac{B(z^{-1})}{A(z^{-1})} z^{-1}\, dz$$

Counter clockwise integration

1. $F(z)=\dfrac{b_0 z+b_1}{a_0 z+a_1}=\dfrac{B(z)}{A(z)}$

$$I_1 = \frac{(b_0^2+b_1^2)a_0 - 2b_0 b_1 a_1}{a_0(a_0^2-a_1^2)}$$

2. $F(z)=\dfrac{b_0 z^2+b_1 z+b_2}{a_0 z^2+a_1 z+a_2}$

$$I_2 = \frac{B_0 a_0 e_1 - B_1 a_0 a_1 + B_2(a_1^2 - a_2 e_1)}{a_0[(a_0^2-a_2^2)e_1 - (a_0 a_1 - a_1 a_2)a_1]}$$

$$B_0 = b_0^2 + b_1^2 + b_2^2$$

$$B_1 = 2(b_0 b_1 + b_1 b_2)$$

$$B_2 = 2b_0 b_2$$

$$e_1 = a_0 + a_2$$

3. $F(z)=\dfrac{b_0z^3+b_1z^2+b_2z+b_3}{a_0z^3+a_1z^2+a_2z+a_3}$

$$I_3=\frac{a_0B_0Q_0-a_0B_1Q_1+a_0B_2Q_2-B_3Q_3}{[(a_0^2-a_3^2)Q_0-(a_0a_1-a_2a_3)Q_1+(a_0a_2-a_1a_3)Q_2]a_0}$$

$B_0=b_0^2+b_1^2+b_2^2+b_3^2$

$B_1=2(b_0b_1+b_1b_2+b_2b_3)$

$B_2=2(b_0b_2+b_1b_3)$

$B_3=2b_0b_3$

$Q_0=(a_0e_1-a_3e_2)$

$Q_1=(a_0a_1-a_2a_3)$

$Q_2=(a_1e_2-a_2e_1)$

$Q_3=(a_1-a_3)(e_2^2-e_1^2)+a_0(a_0e_2-a_3e_1)$

$e_1=a_0+a_2$

$e_2=a_1+a_3$

4. $F(z)=\dfrac{b_0z^4+b_1z^3+b_2z^2+b_3z+b_4}{a_0z^4+a_1z^3+a_2z^2+a_3z+a_4}$

$$I_4=\frac{a_0B_0Q_0-a_0B_1Q_1+a_0B_2Q_2-a_0B_3Q_3+B_4Q_4}{a_0[(a_0^2-a_4^2)Q_0-(a_0a_1-a_3a_4)Q_1+(a_0a_2-a_2a_4)Q_2-(a_0a_3-a_1a_4)Q_3]}$$

$B_0=b_0^2+b_1^2+b_2^2+b_3^2+b_4^2,\qquad B_3=2(b_0b_3+b_1b_4)$

$B_1=2(b_0b_1+b_1b_2+b_2b_3+b_3b_4),\qquad B_4=2b_0b_4$

$B_2=2(b_0b_2+b_1b_3+b_2b_4)$

$Q_0=a_0e_1e_4-a_0a_3e_2+a_4(a_1e_2-e_3e_4)$

$Q_1=a_0a_1e_4-a_0a_2a_3+a_4(a_1a_2-a_3e_4)$

$Q_2=a_0a_1e_2-a_0a_2e_1+a_4(a_2e_3-a_3e_2)$

$Q_3=a_1(a_1e_2-e_3e_4)-a_2(a_1e_1-a_3e_3)+a_3(e_1e_4-a_3e_2)$

$Q_4^*=a_0[e_2(a_1a_4-a_0a_3)+e_5(a_0^2-a_4^2)]+(e_2^2-e_5^2)[a_1(a_1-a_3)+(a_0-a_4)(e_4-a_2)]$

$e_1=a_0+a_2$

$e_2=a_1+a_3$

$e_3=a_2+a_4$

$e_4=a_0+a_4$

$e_5=a_0+a_2+a_4$

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* For ease of calculations Q_4 and Q_3 of I_3 can be respectively written

$$Q_4=-a_4Q_0+a_3Q_1-a_2Q_2+a_1Q_3$$

$$Q_3=a_3Q_0-a_2Q_1+a_1Q_2$$

TABLE 6.6 Closed Forms of the Function $\sum_{n=0}^{\infty} n^r x^n, \quad x<1$

r	$\sum_{n=0}^{\infty} n^r x^n$
0	$\frac{1}{1-x}$
1	$\frac{x}{(1-x)^2}$
2	$\frac{x^2+x}{(1-x)^3}$
3	$\frac{x^3+4x^2+x}{(1-x)^4}$
4	$\frac{x^4+11x^3+11x^2+x}{(1-x)^5}$
5	$\frac{x^5+26x^4+66x^3+26x^2+x}{(1-x)^6}$
6	$\frac{x^6+57x^5+302x^3+57x^2+x}{(1-x)^7}$
7	$\frac{x^7+120x^6+1191x^5+2416x^4+1191x^3+120x^2+x}{(1-x)^8}$
8	$\frac{x^8+247x^7+4293x^6+15619x^5+\ldots+x}{(1-x)^9}$
9	$\frac{x^9+502x^8+14608x^7+88234x^6+156190x^5+\ldots+x}{(1-x)^{10}}$
10	$\frac{x^{10}+1013x^9+47840x^8+455192x^7+1310354x^6+\ldots+x}{(1-x)^{11}}$

The missing terms are apparent since the numerator polynomials are symmetric in the coefficients.

6.4 Positive-Time Z-Transform Tables

TABLE 6.7 Positive-Time Z-Transform Tables

Numbe r	Discrete Time-Function $f(n), n \geq 0$	z-Transform $F(z) = Z[f(n)], \lvert z\rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
1	$u(n) = \begin{cases} 1, & \text{for} \quad n \geq 0 \\ 0, & \text{otherwise} \end{cases}$	$\frac{z}{z-1}$
2	$e^{-\alpha n}$	$\frac{z}{z-e^{-\alpha}}$
3	n	$\frac{z}{(z-1)^2}$
4	n^2	$\frac{z(z+1)}{(z-1)^3}$

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Numbe r	Discrete Time-Function $f(n), n \ge 0$	z-Transform $F(z) = Z[f(n)], \lvert z \rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
5	n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	n^4	$\frac{z(z^3+11z^2+11z+1)}{(z-1)^5}$
7	n^5	$\frac{z(z^4+26z^3+66z^2+26z+1)}{(z-1)^6}$
8	n^{k*}	$(-1)^k D^k\left(\frac{z}{z-1}\right); \quad D = z\frac{d}{dz}$
9	$u(n-k)$	$\frac{z^{-k+1}}{z-1}$
10	$e^{-\alpha n} f(n)$	$F(e^{\alpha} z)$
11	$n^{(2)} = n(n-1)$	$2\frac{z}{(z-1)^3}$
12	$n^{(3)} = n(n-1)(n-2)$	$3!\frac{z}{(z-1)^4}$
13	$n^{(k)} = n(n-1)(n-2)\ldots(n-k+1)$	$k!\frac{z}{(z-1)^{k+1}}$
14	$n^{[k]} f(n), n^{[k]} = n(n+1)(n+2)\ldots(n+k-1)$	$(-1)^k z^k \frac{d^k}{dz^k}[F(z)]$
15	$(-1)^k n(n-1)(n-2)\ldots(n-k+1) f_{n-k+1}{}^{\leq}$	$zF^{(k)}(z),\ F^{(k)}(z) = \frac{d^k}{dz^k}F(z)$
16	$-(n-1) f_{n-1}$	$F^{(1)}(z)$
17	$(-1)^k (n-1)(n-2)\ldots(n-k) f_{n-k}$	$F^{(k)}(z)$
18	$nf(n)$	$-zF^{(1)}(z)$
19	$n^2 f(n)$	$z^2F^{(2)}(z) + zF^{(1)}(z)$
20	$n^3 f(n)$	$-z^3F^{(3)}(z) - 3z^2F^{(2)}(z) - zF^{(1)}(z)$
21	$\frac{c^n}{n!}$	$e^{c/z}$
22	$\frac{(\ln c)^n}{n!}$	$c^{1/z}$
23	$\binom{k}{n} c^n a^{k-n}, \quad \binom{k}{n} = \frac{k!}{(k-n)!n!}, \quad n \le k$	$\frac{(az+c)^k}{z^k}$
24	$\binom{n+k}{k} c^n$	$\frac{z^{k+1}}{(z-c)^{k+1}}$
25	$\frac{c^n}{n!}, \quad (n = 1,3,5,7,\ldots)$	$\sinh\left(\frac{c}{z}\right)$

* Table 6.6 represents entries for k up to 10.

† It may be noted that f_n is the same as $f(n)$.

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Numbe r	Discrete Time-Function $f(n), n \ge 0$	z-Transform $F(z) = Z[f(n)],\ \lvert z\rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
26	$\frac{c^n}{n!}, \quad (n = 0,2,4,6,\ldots)$	$\cosh\left(\frac{c}{z}\right)$
27	$\sin(\alpha n)$	$\frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}$
28	$\cos(\alpha n)$	$\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$
29	$\sin(\alpha n + \psi)$	$\frac{z^2 \sin\psi + z\sin(\alpha - \psi)}{z^2 - 2z\cos\alpha + 1}$
30	$\cosh(\alpha n)$	$\frac{z(z-\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$
31	$\sinh(\alpha n)$	$\frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$
32	$\frac{1}{n}, \quad n > 0$	$\ln\frac{z}{z-1}$
33	$\frac{1 - e^{-\alpha n}}{n}$	$\alpha + \ln\frac{z - e^{-\alpha}}{z-1}, \quad \alpha > 0$
34	$\frac{\sin\alpha n}{n}$	$\alpha + \tan^{-1}\frac{\sin\alpha}{z - \cos\alpha}, \quad \alpha > 0$
35	$\frac{\cos\alpha n}{n}, \quad n > 0$	$\ln\frac{z}{\sqrt{z^2 - 2z\cos\alpha + 1}}$
36	$\frac{(n+1)(n+2)\ldots(n+k-1)}{(k-1)!}$	$\left(1 - \frac{1}{z}\right)^{-k}, \quad k = 2,3,\ldots$
37	$\sum_{m=1}^{n} \frac{1}{m}$	$\frac{z}{z-1}\ln\frac{z}{z-1}$
38	$\sum_{m=0}^{n-1} \frac{1}{m!}$	$\frac{e^{1/z}}{z-1}$
39	$\begin{cases} \dfrac{(-1)^{(n-p)/2}}{2^n\left(\frac{n-p}{2}\right)!\left(\frac{n+p}{2}\right)!}, & \text{for } n \ge p \text{ and } n-p = \text{even} \\ = 0, & \text{for } n < p \text{ or } n-p = \text{odd} \end{cases}$	$J_p(z^{-1})$
40	$\begin{cases} \binom{\alpha}{n/k} b^{n/k}, & n = mk, \quad (m = 0,1,2,\ldots) \\ = 0 & n \ne mk \end{cases}$	$\left(\frac{z^k + b}{z^k}\right)^{\alpha}$
41	$a^n P_n(x) = \frac{a^n}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n$	$\frac{z}{\sqrt{z^2 - 2xaz + a^2}}$
42	$a^n T_n(x) = a^n \cos(n\cos^{-1} x)$	$\frac{z(z-ax)}{z^2 - 2xaz + a^2}$
43	$\frac{L_n(x)}{n!} = \sum_{r=0}^{\infty} \binom{n}{r}\frac{(-x)^r}{r!}$	$\frac{z}{z-1}e^{-x/(z-1)}$

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Number	Discrete Time-Function $f(n),\ n \ge 0$	z-Transform $F(z) = Z[f(n)],\ \lvert z\rvert > R$ $= \sum_{n=0}^{\infty} f(n) z^{-n}$
44	$\frac{H_n(x)}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{n-k} x^{n-2k}}{k!(n-2k)!2^k}$	$e^{-x/z-1/2z^2}$
45	$a^n P_n^m(x) = a^n (1-x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_n(x), \quad m = \text{integer}$	$\frac{(2m)!}{2^m m!} \frac{z^{m+1}(1-x^2)^{m/2} a^m}{(z^2 - 2xaz + a^2)^{m+1/2}}$
46	$\frac{L_n^m(x)}{n!} = \left(\frac{d}{dx}\right)^m \frac{L_n(x)}{n!}, \quad m = \text{integer}$	$\frac{(-1)^m z}{(z-1)^{m+1}} e^{-x/(z-1)}$
47	$-\frac{1}{n} Z^{-1}\left[z \frac{F'(z)}{F(z)} - \frac{G'(z)}{G(z)}\right]$, where $F(z)$ and $G(z)$ are rational	$\ln \frac{F(z)}{G(z)}$
48	$\frac{1}{m(m+1)(m+2)\ldots(m+n)}$	$(m-1)!z^m \left[e^{1/z} - \sum_{k=0}^{m-1} \frac{1}{k!z^k}\right]$
49	$\frac{\sin(\alpha n)}{n!}$	$e^{\cos\alpha/z} \cdot \sin\left(\frac{\sin\alpha}{z}\right)$
50	$\frac{\cos(\alpha n)}{n!}$	$e^{\cos\alpha/z} \cdot \cos\left(\frac{\sin\alpha}{z}\right)$
51	$\sum_{k=0}^{n} f_k g_{n-k}$	$F(z)G(z)$
52	$\sum_{k=0}^{n} k f_k g_{n-k}$	$-F^{(1)}(z)G(z),\ F^{(1)}(z) = \frac{dF(z)}{dz}$
53	$\sum_{k=0}^{n} k^2 f_k g_{n-k}$	$F^{(2)}(z)G(z)$
54	$\frac{\alpha^n + (-\alpha)^n}{2\alpha^2}$	$\frac{1}{\alpha^2} \frac{z^2}{z^2 - \alpha^2}$
55	$\frac{\alpha^n - \beta^n}{\alpha - \beta}$	$\frac{z}{(z-\alpha)(z-\beta)}$
56	$(n+k)^{(k)}$	$k!z^k \frac{z}{(z-1)^{k+1}}$
57	$(n-k)^{(k)}$	$k!z^{-k} \frac{z}{(z-1)^{k+1}}$
58	$\frac{(n \mp k)^{(m)}}{m!} e^{\alpha(n-k)}$	$\frac{z^{1\mp k} e^{m\alpha}}{(z - e^{\alpha})^{m+1}}$
59	$\frac{1}{n} \sin \frac{\pi}{2} n$	$\frac{\pi}{2} + \tan^{-1} \frac{1}{z}$
60	$\frac{\cos\alpha(2n-1)}{2n-1}, \quad n > 0$	$\frac{1}{4\sqrt{z}} \ln \frac{z + 2\sqrt{z}\cos\alpha + 1}{z - 2\sqrt{z}\cos\alpha + 1}$
61	$\frac{\gamma^n}{(\gamma-1)^2} + \frac{n}{1-\gamma} - \frac{1}{(1-\gamma)^2}$	$\frac{z}{(z-\gamma)(z-1)^2}$
62	$\frac{\gamma + a_0}{(\gamma-1)^2} \gamma^n + \frac{1+a_0}{1-\gamma} n + \left(\frac{1}{1-\gamma} - \frac{a_0 + 1}{(1-\gamma)^2}\right)$	$\frac{z(z+a_0)}{(z-\gamma)(z-1)^2}$

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Numbe r	Discrete Time-Function $f(n), n \ge 0$	z-Transform $F(z) = Z[f(n)],\ \lvert z\rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
63	$a^n \cos \pi n$	$\frac{z}{z+a}$
64	$e^{-\alpha n} \cos an$	$\frac{z(z - e^{-\alpha}\cos a)}{z^2 - 2ze^{-\alpha}\cos a + e^{-2\alpha}}$
65	$e^{-\alpha n} \sin(an + \psi)$	$\frac{z^2 \sinh \psi + ze^{-\alpha}\sinh(a-\psi)}{z^2 - 2ze^{-\alpha}\cosh a + e^{-2\alpha}}$
66	$\frac{\gamma^n}{(\gamma-\alpha)^2 + \beta^2} + \frac{(\alpha^2+\beta^2)^{n/2}\sin(n\theta+\psi)}{\beta[(\alpha-\gamma)^2+\beta^2]^{1/2}}$ $\quad \theta = \tan^{-1}\frac{\beta}{\alpha}$, $\psi = \tan^{-1}\frac{\beta}{\alpha-\gamma}$	$\frac{z}{(z-\gamma)[(z-\alpha)^2+\beta^2]}$
67	$\frac{n\gamma^{n-1}}{(\gamma-1)^3} - \frac{3\gamma^n}{(\gamma-1)^4} + \frac{1}{2}\left[\frac{n(n-1)}{(1-\gamma)^2} - \frac{4n}{(1-\gamma)^3} + \frac{6}{(1-\gamma)^4}\right]$	$\frac{z}{(z-\gamma)^2(z-1)^3}$
68	$\sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{(n+k-v)^{(k)}}{k!} e^{\alpha(n-v)}$	$\frac{z(z-1)^k}{(z-e^{\alpha})^{k+1}}$
69	$\frac{f(n)}{n}$	$\int_z^{\infty} p^{-1}F(p)\,dp + \lim_{n\to 0}\frac{f(n)}{n}$
70	$\frac{f_{n+2}}{n+1}, \quad f_0 = 0,\ f_1 = 0$	$z\int_z^{\infty} F(p)\,dp$
71	$\frac{1+a_0}{(1-\gamma)[(1-\alpha)^2+\beta^2]} + \frac{(\gamma+a_0)\gamma^n}{(\gamma-1)[(\gamma-\alpha)^2+\beta^2]}$ $+ \frac{[\alpha^2+\beta^2]^{n/2}[(a_0+\alpha)^2+\beta^2]^{1/2}}{\beta[(\alpha-1)^2+\beta^2]^{1/2}[(\alpha-\gamma)^2+\beta^2]^{1/2}}\sin(n\theta+\psi+\lambda),$ $\psi = \psi_1 + \psi_2, \quad \psi_1 = -\tan^{-1}\frac{\beta}{\alpha-1}, \quad \theta = \tan^{-1}\frac{\beta}{\alpha}$ $\lambda = \tan^{-1}\frac{\beta}{a_0+\alpha}, \quad \psi_2 = -\tan^{-1}\frac{\beta}{\alpha-\gamma}$	$\frac{z(z+a_0)}{(z-1)(z-\gamma)[(z-\alpha)^2+\beta^2]}$
72	$(n+1)e^{\alpha n} - 2ne^{\alpha(n+1)} + e^{\alpha(n-2)}(n-1)$	$\left(\frac{z-1}{z-e^{\alpha}}\right)^2$
73	$(-1)^n \frac{\cos \alpha n}{n}, \quad n > 0$	$\ln \frac{z}{\sqrt{z^2 + 2z\cos\alpha + 1}}$
74	$\frac{(n+k)!}{n!} f_{n+k}, \quad f_n = 0, \quad \text{for } 0 \le n < k$	$(-1)^k z^{2k} \frac{d^k}{dz^k}[F(z)]$
75	$\frac{f(n)}{n+h}, \quad h > 0$	$z^h \int_z^{\infty} p^{-(1+h)} F(p)\,dp$
76	$-na^n \cos\frac{\pi}{2}n$	$\frac{2a^2z^2}{(z^2+a^2)^2}$
77	$na^n \frac{1+\cos\pi n}{2}$	$\frac{2a^2z^2}{(z^2-a^2)^2}$
78	$a^n \sin\frac{\pi}{4}n \cdot \frac{1+\cos\pi n}{2}$	$\frac{a^2z^2}{z^4+a^4}$

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $F(z) = Z[f(n)],\ \lvert z\rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
79	$a^n\left(\frac{1+\cos \pi n}{2} - \cos\frac{\pi}{2}n\right)$	$\frac{2a^2 z^2}{z^4 - a^4}$
80	$\frac{P_n(x)}{n!}$	$e^{xz^{-1}} J_0(\sqrt{1-x^2}\, z^{-1})$
81	$\frac{P_n^{(m)}(x)}{(n+m)!}, \quad m > 0, \quad P_n^m = 0, \quad \text{for } n < m$	$(-1)^m e^{xz^{-1}} J_m(\sqrt{1-x^2}\, z^{-1})$
82	$\frac{1}{(n+\alpha)^{\beta}}, \quad \alpha > 0, \quad \operatorname{Re}\beta > 0$	$\Phi(z^{-1}, \alpha, \beta)$, where $\Phi(1, \beta, \alpha) = \zeta(\beta, \alpha)$ = generalized Rieman-Zeta function
83	$a^n\left(\frac{1+\cos \pi n}{2} + \cos\frac{\pi}{2}n\right)$	$\frac{2z^4}{z^4 - a^4}$
84	$\frac{c^n}{n}, \quad (n = 1,2,3,4,\ldots)$	$\ln z - \ln(z-c)$
85	$\frac{c^n}{n}, \quad n = 2,4,6,8,\ldots$	$\ln z - \frac{1}{2}\ln(z^2 - c^2)$
86	$n^2 c^n$	$\frac{cz(z+c)}{(z-c)^3}$
87	$n^3 c^n$	$\frac{cz(z^2 + 4cz + c^2)}{(z-c)^4}$
88	$n^k c^n$	$-\frac{dF(z/c)}{dz}, \quad F(z) = Z[n^{k-1}]$
89	$-\cos\frac{\pi}{2}n \sum_{i=0}^{(n-2)/4} \binom{n/2}{2i+1} a^{n-2-4i}(a^4 - b^4)^i$	$\frac{z^2}{z^4 + 2a^2 z^2 + b^4}$
90	$n^k f(n), \quad k > 0$ and integer	$-z\frac{d}{dz}F_1(z),\ F_1(z) = Z[n^{k-1} f(n)]$
91	$\frac{(n-1)(n-2)(n-3)\ldots(n-k+1)}{(k-1)!} a^{n-k}$	$\frac{1}{(z-a)^k}$
92	$\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$	$\left(1+\frac{1}{z}\right)^k$
93	$na^n \cos bn$	$\frac{[(z/a)^3 + z/a]\cos b - 2(z/a)^2}{[(z/a)^2 - 2(z/a)\cos b + 1]^2}$
94	$na^n \sin bn$	$\frac{(z/a)^3 \sin b - (z/a)\sin b}{[(z/a)^2 - 2(z/a)\cos b + 1]^2}$
95	$\frac{na^n}{(n+1)(n+2)}$	$\frac{z(a-2z)}{a^2}\ln\left(1-\frac{a}{z}\right) - \frac{2}{a}z$
96	$\frac{(-a)^n}{(n+1)(2n+1)}$	$2\sqrt{z/a}\tan^{-1}\sqrt{a/z} - \frac{z}{a}\ln\left(1+\frac{a}{z}\right)$

TABLE 6.7 Positive-Time Z-Transform Tables (continued)

Numbe r	Discrete Time-Function $f(n), n \ge 0$	z-Transform $F(z) = Z[f(n)],\ \lvert z \rvert > R = \sum_{n=0}^{\infty} f(n) z^{-n}$
97	$\frac{a^n \sin \alpha n}{n+1}$	$\frac{z\cos\alpha}{a}\tan^{-1}\frac{a\sin\alpha}{z-a\cos\alpha} + \frac{z\sin\alpha}{2a}\ln\frac{z^2-2az\cos\alpha+a^2}{z^2}$
98	$\frac{a^n \cos(\pi/2) n \sin\alpha(n+1)}{n+1}$	$\frac{z}{4a}\ln\frac{z^2+2az\sin\alpha+a^2}{z^2-2az\sin\alpha+a^2}$
99	$\frac{1}{(2n)!}$	$\cosh(z^{-1/2})$
100	$\binom{-\frac{1}{2}}{n}(-a)^n$	$\sqrt{z/(z-a)}$
101	$\binom{-\frac{1}{2}}{\frac{n}{2}} a^n \cos\frac{\pi}{2}n$	$\frac{z}{\sqrt{z^2-a^2}}$
102	$\frac{B_n(x)}{n!}$ $B_n(x)$ are Bernoulli polynomials	$\frac{e^{x/z}}{z(e^{1/z}-1)}$
103	$W_n(x) \triangleq$ Tchebycheff polynomials of the second kind	$\frac{z^2}{z^2-2xz+1}$
104	$\left\lvert \sin\frac{n\pi}{m} \right\rvert, \quad m = 1, 2, \ldots$	$\frac{z\sin\pi/m}{z^2-2z\cos\pi/m+1}\frac{1+z^{-m}}{1-z^{-m}}$
105	$Q_n(x) = \sin(n\cos^{-1}x)$	$\frac{z}{z^2-2xz+1}$

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Appendix 1

Examples

1.1 One-Sided Z-Transforms

Example 6.1

The radius of convergence of $f(nT) = e^{-anT}u(nT)$, a positive number, is:

$$\left|z^{-1}e^{-aT}\right| < 1 \qquad \text{or} \qquad |z| > e^{-aT}.$$

The Z-transform of $f(nT) = e^{-anT}u(nT)$ is

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} = \sum_{n=0}^{\infty} (e^{-aT}z^{-1})^n = \frac{1}{1-e^{-aT}z^{-1}}.$$

If $a = 0$

$$F(z) = \sum_{n=0}^{\infty} u(nT)z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

Example 6.2

The function $f(nT) = a^{nT} \cos nT\omega\, u(nT)$ has the Z-transform

$$F(z) = \sum_{n=0}^{\infty} a^{nT}\frac{e^{jnT\omega}+e^{-jnT\omega}}{2}z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}(a^T e^{jT\omega}z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty}(a^T e^{-jT\omega}z^{-1})^n$$

$$= \frac{1}{2}\frac{1}{1-a^T e^{jT\omega}z^{-1}} + \frac{1}{2}\frac{1}{1-a^T e^{-jT\omega}z^{-1}} = \frac{1-a^T z^{-1}\cos T\omega}{1-2a^T z^{-1}\cos T\omega + a^{2T}z^{-2}}.$$

The ROC is given by the relations

$$\left|a^T e^{jT\omega}z^{-1}\right| < 1 \quad \text{or} \quad |z| > \left|a^T\right|$$

$$\left|a^T e^{-jT\omega}z^{-1}\right| < 1 \quad \text{or} \quad |z| > \left|a^T\right|$$

Therefore the ROC is $|z| > \left|a^T\right|$.

Example 6.3

$$Z\{n\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}, \qquad n \ge 0$$

$$Z\{n^2\} = -z\frac{d}{dz}Z\{n\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}, \qquad n \ge 0$$

$$Z\{n^3\} = -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = \frac{z(z^2+4z+1)}{(z-1)^4} \qquad n = 0,1,2$$

Example 6.4

See Figure 6.1 for graphical representation of the complex integration ($n \geq 0$).

$$Z\{nT\} \risingdotseq H(z) = \frac{z}{(z-1)^2}T \quad |z| > 1, \quad Z\{e^{-nT}\} \risingdotseq F(z) = \frac{z}{z-e^{-T}} \quad |z| > e^{-T}$$

Hence (counter-clockwise integration)

$$Z\{nTe^{-nT}\} = \frac{1}{2\pi j}\oint_C T\frac{z}{\tau(\tau-e^{-T})\left(\frac{z}{\tau}-1\right)^2}\,d\tau .$$

The contour must have a radius $|\tau|$ of the value $e^{-T} < |\tau| < |z| = 1$ and we have from complex integration

$$Z\{nTe^{-nT}\} = \underset{\tau=e^{-T}}{Res}\left\{(\tau-e^{-T})T\frac{z\tau}{(\tau-e^{-T})(z-\tau)^2}\right\} = T\frac{ze^{-T}}{(z-e^{-T})^2}$$

But

$$Z\{nTe^{-nT}\} = -Tz\frac{d}{dz}\left(\frac{1}{1-e^{-T}z^{-1}}\right) = T\frac{ze^{-T}}{(z-e^{-T})^2}$$

and verifies the complex integration approach.

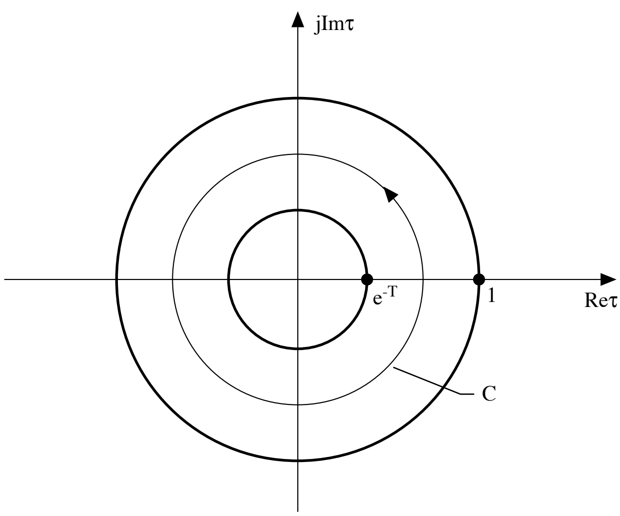

FIGURE 6.1

1.2 One-Sided Inverse Z-Transform

Example 6.5

a) If $F(z)=\dfrac{z^2+1}{(z-1)(z-2)}=A+\dfrac{Bz}{z-1}+\dfrac{Cz}{z-2}$ with $|z|>2$, then we obtain $A=\dfrac{0+1}{(0-1)(0-2)}=\dfrac{1}{2}$,

$B=\dfrac{1}{z}\dfrac{z^2+1}{(z-2)}\Big|_{z=1}=-2$ and $C=\dfrac{1}{z}\dfrac{z^2+1}{(z-1)}\Big|_{z=2}=\dfrac{5}{2}$. Hence $F(z)=\dfrac{1}{2}-2\dfrac{z}{z-1}+\dfrac{5}{2}\dfrac{z}{z-2}$ and

its inverse is $f(nT)=\dfrac{1}{2}\delta(nT)-2u(nT)+\dfrac{5}{2}(2)^n u(nT)$.

b) If $F(z)=\dfrac{z+1}{(z-1)(z-2)}=\dfrac{A}{z-1}+\dfrac{B}{z-2}$ then we obtain $A=\dfrac{z+1}{(z-2)}\Big|_{z=1}=-2$ and $B=\dfrac{z+1}{(z-1)}\Big|_{z=2}$

$=3$. Hence $F(z)=-2\dfrac{1}{(z-1)}+3\dfrac{1}{(z-2)}$ and its inverse is $f(nT)=-2u(nT-T)+3(2)^{n-1}u(nT-T)$

with $ROC\ |z|>2$.

Example 6.6

1. By Expansion
   If $F(z)$ has the region of convergence $|z|>5$, then

$$F(z)=\frac{5z}{(z-5)^2}=\frac{5z}{z^2-10z+25}$$

$$=5z^{-1}+50z^{-2}+375z^{-3}+\cdots=0\cdot 5^0 z^{-0}+1\cdot 5z^{-1}+2\cdot 5^2 z^{-2}+3\cdot 5^3 z^{-3}+\cdots\cdot$$

   Hence $f(nT)=n5^n$ $n=0,1,2,\ldots$ which sometimes is difficult to recognize using the expansion method.
2. By Fraction Expansion
   $F(z)=\dfrac{5z}{(z-5)^2}=\dfrac{Az}{z-5}+\dfrac{Bz^2}{(z-5)^2}$, $B=\dfrac{5}{z}\Big|_{z=5}=1$, $\dfrac{5\times 6}{(6-1)^2}=\dfrac{A\times 6}{6-5}+\dfrac{6^2}{(6-5)^2}$ or $A=-1$. Hence

   $F(z)=-\dfrac{z}{z-5}+\dfrac{z^2}{(z-5)^2}$ and $f(nT)=-(5)^n+(n+1)5^n=n5^n$, $n\ge 0$.
3. By Integration

$$\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-5)^2\frac{5z}{(z-5)^2}z^{n-1}\right]\Bigg|_{z=5}=5nz^{n-1}\Big|_{z=5}=n5^n,\qquad n\ge 0.$$

1.3 Two-Sided Z-Transform

Example 6.7

$$F(z) = Z_{II}\{e^{-|nT|}\} = \sum_{n=-\infty}^{-1} e^{nT} z^{-n} + \sum_{n=0}^{\infty} e^{-nT} z^{-n} = \sum_{n=-\infty}^{0} e^{nT} z^{-n} - 1 + \sum_{n=0}^{\infty} e^{-nT} z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-nT} z^{n} - 1 + \sum_{n=0}^{\infty} e^{-nT} z^{-n} = \frac{1}{1-e^{-nT}z} - 1 + \frac{1}{1-e^{-nT}z^{-1}}$$

The first sum (negative time) converges if $|e^{-T}z| <$ or $|z| < e^{T}$. The second sum (positive time) converges if $|e^{-T}z^{-1}| < 1$ or $e^{-T} < |z|$. Hence the region of convergence is $R_+ = e^{-T} < |z| < R_- = e^{T}$. The two poles of $F(z)$ are $z = e^{T}$ and $z = e^{-T}$.

Example 6.8

The Z-transform of $u(nT)$ is

$$F(z) = \frac{1}{1-z^{-1}} \qquad |z| > 1 = R_{+f}, \quad R_{-f} = \infty$$

and the Z-transform of $h(nT) = \exp(-|nT|)$ is

$$H(z) = \frac{1-e^{-2T}}{(1-e^{-T}z^{-1})(1-e^{-T}z)} \qquad R_{+h} = e^{-T} < |z| < e^{T} = R_{-h}.$$

But $R_{-f} = \infty$ and hence from the product property $1 \cdot \exp(-T) < |z| < \infty$. The contour must lie in the region $\max(1, |z|e^{-T}) < |\tau| < \min(-\infty, |z|e^{T})$. The pole-zero configuration and the contour are shown in Figure 6.2. If we choose $|z| > e^{T}$ then the contour is that shown in the figure.

Therefore, we obtain

$$Z_{II}\{u(nT)h(nT)\} \doteq G(z) = \frac{1}{2\pi j}\oint_C \frac{1}{1-\tau^{-1}} \frac{1-e^{-2T}}{\left(1-e^{-T}\frac{\tau}{z}\right)\left(1-e^{-T}\frac{z}{\tau}\right)} \frac{d\tau}{\tau}$$

The poles of $H(z/\tau)$ are at $\tau = z\exp(-T)$ and $\tau = z\exp(T)$. Hence the contour encloses the poles $\tau = 1$ and $\tau = z\exp(-T)$. Applying the residue theorem next, we obtain

$$G(z) = \frac{1}{1-e^{-T}z^{-1}} \qquad |z| > e^{-T}$$

which has the inverse function $g(nT) = e^{-nT}u(nT)$ as it was expected.

Example 6.9

The Z-transform of $f(nT) = \exp(-nT)u(nT)$ is $F(z) = 1/(1 - e^{-T}z^{-1})$ for $|z| > e^{-T}$. From the Parseval property we obtain (counter-clockwise integration)

$$\sum_{n=-\infty}^{\infty} f^2(nT) = \sum_{n=0}^{\infty} f^2(nT) = \frac{1}{2\pi j}\oint_C \frac{1}{1-e^{-T}z^{-1}} \frac{1}{1-e^{-T}z} \frac{dz}{z}$$

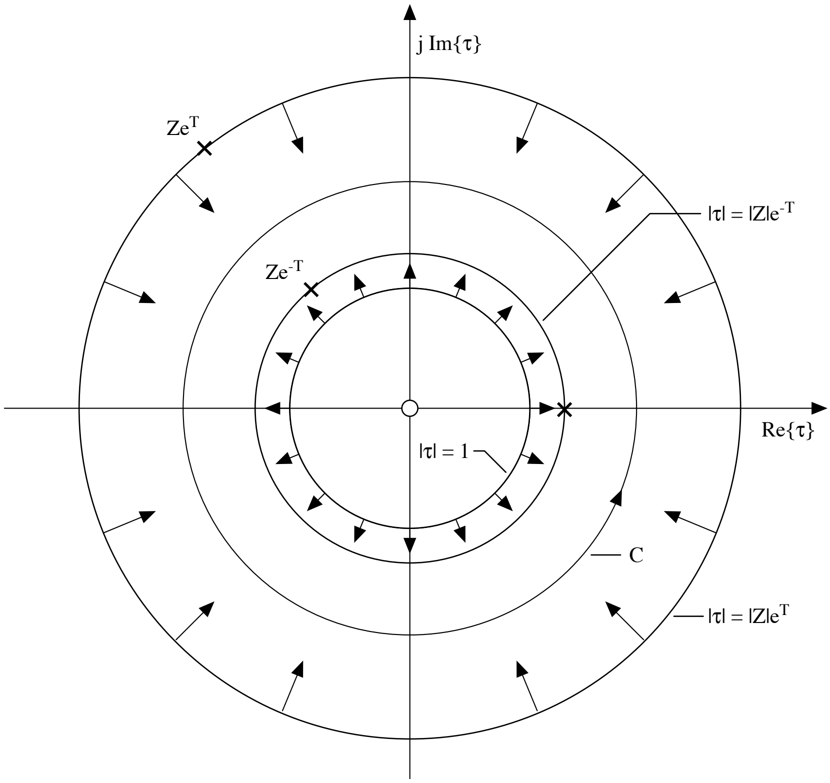

FIGURE 6.2

with $\max(e^{-T},0)<|z|<\min(\infty,e^{T})$. The contour encircles the pole at $z = e^{-T}$ so that

$$\sum_{n=0}^{\infty} f^2(nT) = Res\left[\frac{z-e^{-T}}{(z-e^{-T})(1-e^{-T}z)}\right]\Bigg|_{z=e^{-T}} = \frac{1}{1-e^{-2T}}$$

Also we find directly

$$\sum_{n=0}^{\infty} e^{-nT}e^{-nT} = \sum_{n=0}^{\infty} e^{-2nT} = (1+e^{-2T}+(e^{-2T})^2+\cdots) = \frac{1}{1-e^{-2T}} \,.$$

1.4 Two-Sided Inverse Z-Transform

Example 6.10

If $F(z)=[z(z+1)]/(z^2-2z+1)=(1+z^{-1})/(1-2z^{-1}+z^{-2})$ and the ROC is $|z|>1$, then

$$
\begin{array}{r|l}
 & 1+3z^{-1}+5z^{-2}+7z^{-3}+\cdots \\
\hline
1-2z^{-1}+z^{-2} & 1+z^{-1} \\
 & \underline{1-2z^{-1}+z^{-2}} \\
 & \quad 3z^{-1}-z^{-2} \\
 & \quad \underline{3z^{-1}-6z^{-2}+3z^{-3}} \\
 & \qquad\quad 5z^{-2}-3z^{-3} \\
 & \qquad\qquad \cdots
\end{array}
$$

and by continuing the division, we recognize that

$$
f(nT)=\begin{cases} 0 & n<0 \\ (2n+1) & n\ge 0 \end{cases}
$$

If $f(nT)$ is known to be zero for positive n, that the ROC is $|z|<1$, then

$$
\begin{array}{r|l}
 & z+3z^{2}+5z^{3}+\cdots \\
\hline
z^{-2}-2z^{-1}+1 & z^{-1}+1 \\
 & \underline{z^{-1}-2+z} \\
 & \quad 3-z \\
 & \quad \underline{3-6z+3z^{2}} \\
 & \qquad 5z-3z^{2} \\
 & \qquad\quad \cdots
\end{array}
$$

This series is recognized as

$$
f(nT)=\begin{cases} -(2n+1) & n<0 \\ 0 & n\ge 0 \end{cases}
$$

Example 6.11

To determine the inverse Z-transform of $F(z) = 1/(1-1.5z^{-1}+0.5z^{-2})$ if a) $ROC: |z| > 1$, b) $ROC: |z| < 0.5$, and c) $ROC: 0.5 < |z| < 1$, we proceed as follows:

$$
F(z)=\frac{z^2}{z^2-1.5z+0.5}=\frac{z^2}{(z-1)(z-\frac{1}{2})}=A+\frac{Bz}{z-1}+\frac{Cz}{z-\frac{1}{2}}
$$

or

$$
F(z)=2\frac{z}{z-1}-\frac{z}{z-\frac{1}{2}}
$$

a) $f(nT) = 2(1)^n - (\frac{1}{2})^n,\ n \geq 0$ since both poles are outside the region of convergence $|z| > 1$ (inside the unit circle).
b) $f(nT) = -2(1)^n u(-nT - T) + (\frac{1}{2})^n u(-nT - T),\ n \leq -1$ since both poles are outside the region of convergence (outside the circle $|z| = 0.5$).
c) Pole at $\frac{1}{2}$ provides the causal part, and the pole at 1 provides the anticausal. Hence

$$f(nT) = -2(1)^n u(-nT - T) - (\tfrac{1}{2})^n u(nT) \qquad -\infty < n < \infty$$

Example 6.12

Let

$$F(z) = \frac{1 - 0.8^2}{(1 - 0.8z)(1 - 0.8z^{-1})} \qquad 0.8 < |z| < 0.8^{-1}$$

For $n \geq 0$ the contour C encloses counter-clockwise only the pole $z = 0.8$ of the function $F(z)z^{n-1}$. Therefore

$$f(nT) = Res\{F(z)z^{n-1}\}\Big|_{z=0.8} = \frac{(1 - 0.8^2)z^n(z - 0.8)}{(1 - 0.8z)\,(z - 0.8)}\Bigg|_{z=0.8} = 0.8^n \qquad n \geq 0$$

For $n < 0$ only the pole $z = 1/0.8$ is outside C. Hence

$$f(nT) = Res\{F(z)z^{n-1}\}\Big|_{z=1/0.8} = -\frac{(1 - 0.8^2)0.8^{-1}z^n(z - 0.8^{-1})}{-(1 - 0.8^{-1})\,(z - 0.8)}\Bigg|_{z=0.8^{-1}} = 0.8^{-n} \qquad n \leq -1$$

The residue for a multiple pole of order k at z_o is given by

$$Res\{F(z)z^{n-1}\}\Big|_{z=z_o} = \lim_{z \to z_o} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_o)^k \, F(z)z^{n-1}\right]$$

1.5 Solution of Difference Equation

Example 6.13

The solution of the difference equation $y(n) - ay(n - 1) = u(n)$ with initial condition $y(-1) = 2$ and $|a| < 1$ proceeds as follows:

$$Y(z) - ay(-1) - az^{-1}Y(z) = \frac{z}{z - 1}$$

$$Y(z) = \frac{2a}{1 - az^{-1}} + \frac{z}{z - 1}\frac{1}{1 - az^{-1}} = \frac{2a}{1 - az^{-1}} + \frac{z^2}{(z - 1)(z - a)}$$

$$= \frac{2a}{1 - az^{-1}} + \frac{1}{1 - a}\frac{1}{1 - z^{-1}} + \frac{a}{a - 1}\frac{1}{1 - az^{-1}}$$

Hence, the inverse Z-transform gives

$$y(n) = \underbrace{2a \cdot a^n}_{\text{zero input}} + \underbrace{\frac{1}{1 - a} u(n) + \frac{a}{a - 1} a^n}_{\text{zero state}} = \underbrace{\frac{1}{1 - a} u(n)}_{\text{steady state}} + \underbrace{\frac{2a - 1}{a - 1} a^{n+1}}_{\text{transient}} \qquad n \geq 0$$

Poularikas A. D. “Windows ”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

7 Windows

7.1 Introductory Material

7.1.1 Introduction

N = number of samples

T = interval between samples

NT = total time duration of the signal

$\frac{1}{NT}$ = minimum spectral resolution (s^{-1})

DFT = discrete Fourier transform

Leakage = Spectral leakage takes place when the signal has frequencies other than those of the basis set. These other frequencies will exhibit non-zero properties on the entire basis set known as leakage.

7.2 Figures of Merit (see Table 7.1)

7.2.1 Equivalent Noise Bandwidth

$$ENBW = \frac{\sum_n w^2(nT)}{\left|\sum_n w(nT)\right|^2} = \text{equivalent noise bandwidth} =$$

the width of an equivalent ideal rectangular spectral response that will pass the same noise power as the window (filter) under test.

$w(nT)$ = window samples

TABLE 7.1 Figures of Merit for Shaped DFT Filters

Weighting		Figure of Merit								Overlap correlation (%)	
		Highest sidelobe level (dB)	Sidelobe falloff (dB/octave)	Coherent gain	Equivalent noise BW (bins)	3.0-dB BW (bins)	Scallop loss (dB)	Worst-case process loss (dB)	6.0-dB BW (bins)	75% OL	50% OL
Rectangle		–13	–6	1.00	1.00	0.89	3.92	3.92	1.21	75.0	50.0
Triangle		–27	–12	0.50	1.33	1.28	1.82	3.07	1.78	71.9	25.0
	$\alpha = 1.0$	–23	–12	0.64	1.23	1.20	2.10	3.01	1.65	75.5	31.8
$\cos^{\alpha}(x)$	$\alpha = 2.0$	–32	–18	0.50	1.50	1.44	1.42	3.18	2.00	65.9	16.7
Hann	$\alpha = 3.0$	–39	–24	0.42	1.73	1.66	1.08	3.47	2.32	56.7	8.5
	$\alpha = 4.0$	–47	–30	0.38	1.94	1.86	0.86	3.75	2.59	48.6	4.3
Hamming		–43	–6	0.54	1.36	1.30	1.78	3.10	1.81	70.7	23.5
Parabolic		–21	–12	0.67	1.20	1.16	2.22	3.01	1.59	76.5	34.4
Riemann		–26	–12	0.59	1.30	1.26	1.89	3.03	1.74	73.4	27.4
Cubic		–53	–24	0.38	1.92	1.82	0.90	3.72	2.55	49.3	5.0
	$\alpha = 0.25$	–14	–18	0.88	1.10	1.01	2.96	3.39	1.38	74.1	44.4
Tukey	$\alpha = 0.50$	–15	–18	0.75	1.22	1.15	2.24	3.11	1.57	72.7	36.4
	$\alpha = 0.75$	–19	–18	0.63	1.36	1.31	1.73	3.07	1.80	70.5	25.1
Bohman		–46	–24	0.41	1.79	1.71	1.02	3.54	2.38	54.5	7.4
	$\alpha = 2.0$	–19	–6	0.44	1.30	1.21	2.09	3.23	1.69	69.9	27.8
Poisson	$\alpha = 3.0$	–24	–6	0.32	1.65	1.45	1.46	3.64	2.08	54.8	15.1
	$\alpha = 4.0$	–31	–6	0.25	2.08	1.75	1.03	4.21	2.58	40.4	7.4
	$\alpha = 0.5$	–35	–18	0.43	1.61	1.54	1.26	3.33	2.14	61.3	12.6
Hamming	$\alpha = 1.0$	–39	–18	0.38	1.73	1.64	1.11	3.50	2.30	56.0	9.2
Poisson	$\alpha = 2.0$	none	–18	0.29	2.02	1.87	0.87	3.94	2.65	44.6	4.7
	$\alpha = 3.0$	–31	–6	0.42	1.48	1.34	1.71	3.40	1.90	61.6	20.2
Cauchy	$\alpha = 4.0$	–35	–6	0.33	1.76	1.50	1.36	3.83	2.20	48.8	13.2
	$\alpha = 5.0$	–30	–6	0.28	2.06	1.68	1.13	4.28	2.53	38.3	9.0

	α = 2.0	–40	–6	0.57	1.30	1.25	1.91	3.06	1.74	75.7	28.3
	α = 2.5	–50	–6	0.51	1.43	1.36	1.60	3.15	1.90	71.3	21.4
Taylor	α = 3.0	–60	–6	0.47	1.55	1.47	1.37	3.26	2.06	67.0	16.1
	α = 3.5	–70	–6	0.44	1.66	1.58	1.20	3.40	2.21	62.9	12.1
	α = 4.0	–80	–6	0.41	1.76	1.67	1.06	3.52	2.35	59.1	9.1
	α = 2.5	–42	–6	0.51	1.39	1.33	1.69	3.14	1.86	67.7	20.0
Gaussian	α = 3.0	–55	–6	0.43	1.64	1.55	1.25	3.40	2.18	57.5	10.6
	α = 3.5	–69	–6	0.37	1.90	1.79	0.94	3.73	2.52	47.2	4.9
Dolph-	α = 2.5	–50	0	0.53	1.39	1.33	1.70	3.12	1.85	69.6	22.3
Cheby-	α = 3.0	–60	0	0.48	1.51	1.44	1.44	3.23	2.01	64.7	16.3
shev	α = 3.5	–70	0	0.45	1.62	1.55	1.25	3.35	2.17	60.2	11.9
	α = 4.0	–80	0	0.42	1.73	1.65	1.10	3.48	2.31	55.9	8.7
Kaiser-	α = 2.0	–46	–6	0.49	1.50	1.43	1.46	3.20	1.99	65.7	16.9
Bessel	α = 2.5	–57	–6	0.44	1.65	1.57	1.20	3.38	2.20	59.5	11.2
	α = 3.0	–69	–6	0.40	1.80	1.71	1.02	3.56	2.39	53.9	7.4
	α = 3.5	–82	–6	0.37	1.93	1.83	0.89	3.74	2.57	48.8	4.8
	α = 3.0	–53	–6	0.47	1.56	1.49	1.34	3.27	2.07	63.0	14.2
Barcilon	α = 3.5	–58	–6	0.43	1.67	1.59	1.18	3.40	2.23	58.6	10.4
Temes	α = 4.0	–68	–6	0.41	1.77	1.69	1.05	3.52	2.36	54.4	7.6
Exact Blackman		–68	–6	0.46	1.57	1.52	1.33	3.29	2.13	62.7	14.0
Blackman		–58	–18	0.42	1.73	1.68	1.10	3.47	2.35	56.7	9.0
Minimum 3-sample Blackman-Harris		–71	–6	0.42	1.71	1.66	1.13	3.45	1.81	57.2	9.6
Minimum 4-sample Blackman-Harris		–92	–6	0.36	2.00	1.90	0.83	3.85	2.72	46.0	3.8
62-dB 3-sample Blackman-Harris		–62	–6	0.45	1.61	1.56	1.27	3.34	2.19	61.0	12.6
74-dB 4-sample Blackman-Harris		–74	–6	0.40	1.79	1.74	1.03	3.56	2.44	53.9	7.4
4-sample Kaiser–Bessel	α = 3.0	–69	–6	0.40	1.80	1.74	1.02	3.56	2.44	53.9	7.4

7.2.2 Coherent Gain

$$CG = \text{coherent gain} = \frac{1}{N}\sum_{n=0}^{N-1} w(nT) = \text{zero frequency gain (dc gain) of the window}$$

7.2.3 Processing Gain

$$PG = \frac{1}{ENBW} = \frac{\text{output signal-to-noise ratio}}{\text{input signal-to-noise ratio}}$$

7.2.4 Scalloping Loss

$$\text{scalloping loss} = \frac{\left|\sum_n w(nT)\exp\left(-j\frac{\pi}{N}n\right)\right|}{\sum_n w(nT)} =$$

$$= \frac{\text{coherent gain for a tone located half a bin from DFT sample point}}{\text{coherent gain for a tone located at a DFT sample point}}$$

$$= \text{ maximum reduction in } PG \text{ due to signal frequency}$$

7.2.5 Mainlobe Spectral Response

mainlobe spectral response = spectral interval between the peak gain and the –3.0 dB and –6.0 dB response level

7.2.6 Overlap Correlation

Correlation coefficients represent the degree of correlation of filter output points that are separated by 25% and 50% of the filter length. These terms are useful in quantifying the estimation uncertainty (or variance reduction) related to incoherent averaging of filter (window) data.

7.3 Window (Filter) Descriptions

7.3.1 Introduction

- $T = 1$
- $-\pi \le \omega \le \pi$ or $0 \le \omega \le 2\pi$
- $DFT\ bin = 2\pi/N$
- Windows are even (about the origin) sequences with an odd number of points.
- The right-most point of the window will be discarded.
- N will be taken to be even, and the total points will be *odd*, and hence

$$N = 2 \times (\text{total points}) = \text{even}$$

7.3.2 Rectangle (Dirichlet) Window

$$w(n) = 1.0 \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$W(\omega) = \sum_{n=-N/2}^{N/2} w(n)e^{-jn\omega}$$

To make it realizable shift the sequence by $N/2$ to the right. Hence we obtain

$$w(n) = 1 \qquad n = 0,1,\cdots,N-1$$

$$W(\omega) = \sum_{n=0}^{N-1} e^{-jn\omega} = e^{-j\frac{N-1}{2}\omega} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\frac{\omega}{2}}$$

Figure 7.1 shows the rectangular window and its amplitude spectrum $|W(\omega)|$.

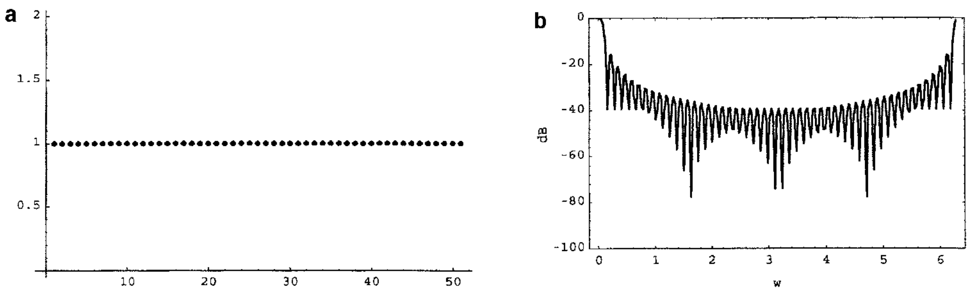

FIGURE 7.1 a) Rectangular window. b) Amplitude spectrum of rectangular window.

7.3.3 Triangle (Fejer, Bartlet) Window

$$w(n) = 1.0 - \frac{|n|}{N/2} \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

For DFT the window is

$$w(n) = \begin{cases} \dfrac{n}{N/2} & n = 0,1,\cdots,\tfrac{N}{2} \\ \dfrac{N-n}{N/2} & n = \tfrac{N}{2}+1,\cdots,N-1 \end{cases}$$

and its DFT is

$$W(\omega) = e^{-j(\frac{N}{2}-1)\omega}\left[\frac{\sin\left(\frac{N}{4}\omega\right)}{\sin\frac{\omega}{2}}\right]^2$$

since the symmetric function $w(n)$ is shifted by $\frac{N}{2}-1$ positions to produce the DFT sequence. Figure 7.2 shows the triangular window and its DFT.

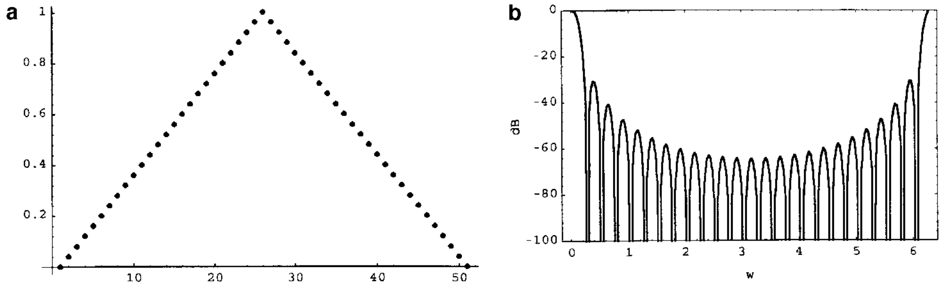

FIGURE 7.2 a) Triangular window. b) Amplitude spectrum of triangular window.

7.3.4 cos$^\alpha$ (t) Windows

$$w(n) = \cos^\alpha\left[\left(\frac{n}{N}\right)\pi\right] \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$w(n) = \sin^\alpha\left[\left(\frac{n}{N}\right)\pi\right] \qquad n = 0,1,\cdots,N-1$$

Common values of $\alpha : 1 \le \alpha \le 4$
Figures 7.3 through 7.5 show the $\cos^2\left(\frac{n\pi}{N}\right)$, $\cos^3\left(\frac{n\pi}{N}\right)$, and $\cos^4\left(\frac{n\pi}{N}\right)$ windows and their Fourier transform.

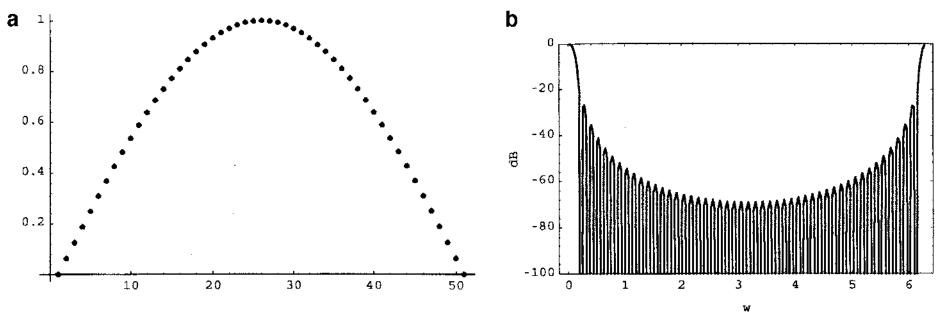

FIGURE 7.3 a) Cosine window with $\alpha = 1$. b) Amplitude spectrum of cosine window with $\alpha = 1$.

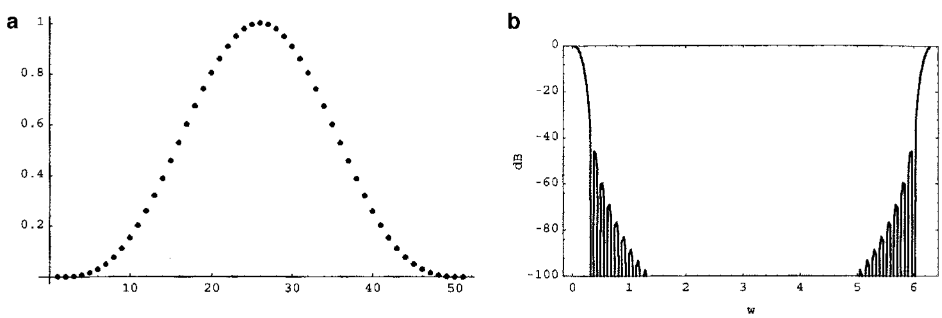

FIGURE 7.4 a) Cosine window with $\alpha = 3$. b) Amplitude spectrum of cosine window with $\alpha = 3$.

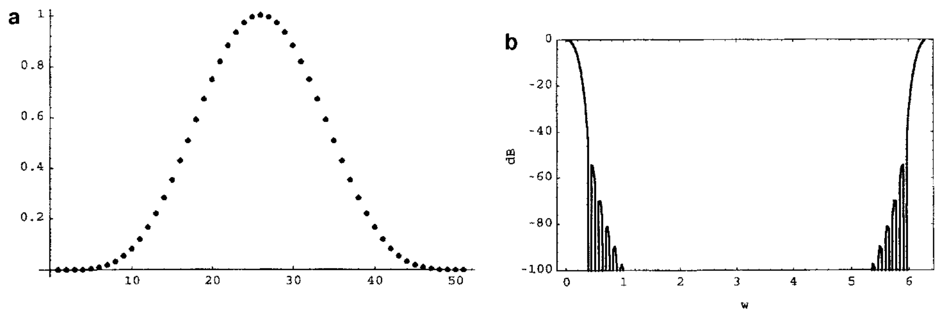

FIGURE 7.5 a) Cosine window with α = 4. b) Amplitude spectrum of cosine window with α = 4.

7.3.5 Hann Window

$$w(n) = \cos^2\left(\frac{n}{N}\pi\right) = \frac{1}{2}\left[1 + \cos\left(\frac{2n}{N}\pi\right)\right] \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$w(n) = \sin^2\left[\left(\frac{n}{N}\right)\pi\right] = \frac{1}{2}\left[1 - \cos\left[\left(\frac{2n}{N}\right)\pi\right]\right] \qquad n = 0,1,\cdots,N-1$$

DFT of the window is

$$W(\omega) = \frac{1}{2}D(\omega) + \frac{1}{4}\left[D\left(\omega - \frac{2\pi}{N}\right) + D\left(\omega + \frac{2\pi}{N}\right)\right]$$

$$D(\omega) = e^{j\frac{\omega}{2}}\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\frac{\omega}{2}} = \text{Dirichlet Kernel} \qquad -\pi \le \omega \le \pi$$

See Figure 7.6 for the Hann window.

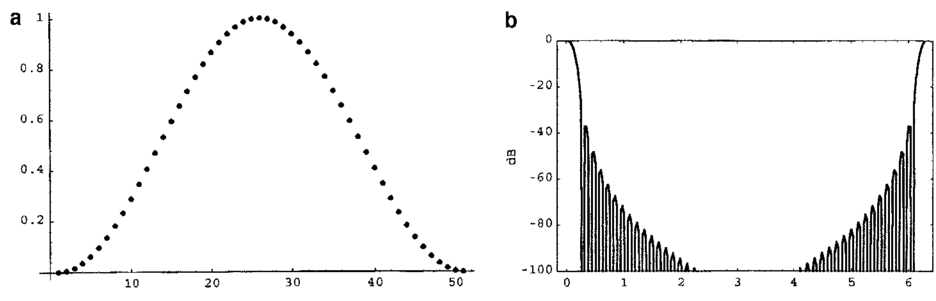

FIGURE 7.6 a) Hann window. b) Amplitude spectrum of Hann window.

7.3.6 Hamming Window

$$w(n) = \alpha + (1-\alpha)\cos\frac{2\pi}{N}n \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$W(\omega) = \alpha D(\omega) + \frac{1}{2}(1-\alpha)\left[D\left(\omega - \frac{2\pi}{N}\right) + D\left(\omega + \frac{2\pi}{N}\right)\right] \qquad -\pi \le \omega \le \pi$$

D = Dirichlet Kernel (see 7.3.5)

$$w(n) = 0.54 + 0.46\cos\frac{2\pi}{N}n \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$w(n) = 0.54 - 0.46\cos\frac{2\pi}{N}n \qquad n = 0,1,\cdots,N-1$$

Figure 7.7 depicts the Hamming window and its amplitude spectrum.

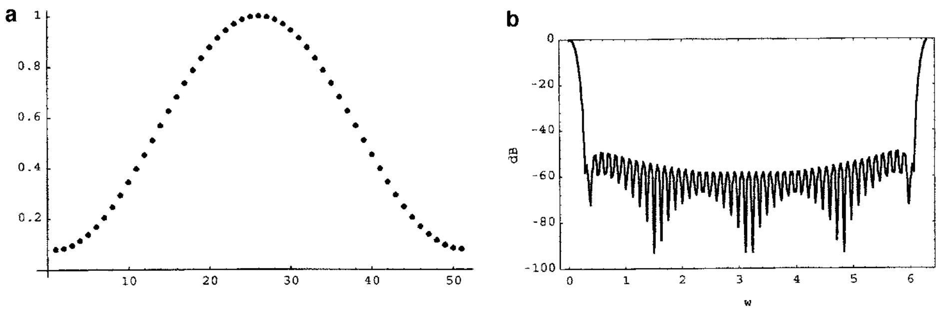

FIGURE 7.7 a) Hamming window. b) Amplitude spectrum of Hamming window.

7.3.7 Short Cosine Series Window

$$w(n) = \sum_{k=0}^{K/2} a(k)\cos\left[\left(\frac{2\pi}{N}\right)kn\right] \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$\sum_{k=0}^{K/2} a(k) = 1 \qquad \text{constraint}$$

$$W(\omega) = a(0)D(\omega) + \sum_{k=0}^{K/2} \frac{a(k)}{2}\left[D\left(\omega - k\frac{2\pi}{N}\right) + D\left(\omega + k\frac{2\pi}{N}\right)\right] \qquad -\pi \le \omega \le \pi$$

7.3.8 Blackman Window

$$a(0) = 0.42659071 \cong 0.42, \quad a(1) = 0.49656062 \cong 0.50, \quad a(2) = 0.07684867 \cong 0.08$$

$$w(n) = 0.42 + 0.5\cos\left(\frac{2\pi}{N}n\right) + 0.08\cos\left(\frac{2\pi}{N}2n\right) \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$w(n) = 0.42 + 0.5\cos\left(\frac{2\pi}{N}(n-25)\right) + 0.08\cos\left(\frac{2\pi}{N}2(n-25)\right) \qquad n = 0,1,\cdots,N-1$$

Figure 7.8 shows the characteristics of the Blackman window.

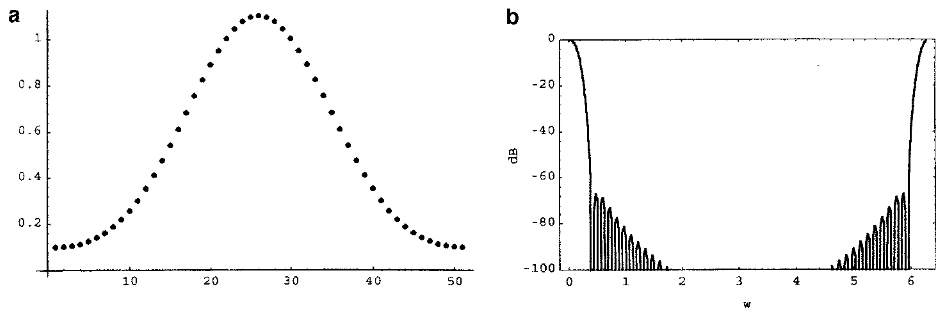

FIGURE 7.8 a) Blackman window. b) Amplitude spectrum of Blackman window.

7.3.9 Harris-Nutall Window

Table 7.2 gives the coefficients for short cosine series windows.

TABLE 7.2 Coefficients of Three- and Four-Term Harris-Nutall Windows

	3-Term (–61 dB)	3-Term (–67 dB)	4-Term (–74 dB)	4-Term (–94 dB)
a(0)	0.44959	0.42323	0.40217	0.35875
a(1)	0.49364	0.49755	0.49703	0.48829
a(2)	0.05677	0.07922	0.09392	0.14128
a(3)	0	0	0.00183	0.01168

Figures 7.9 and 7.10 show the Harris-Nutall window characteristics.

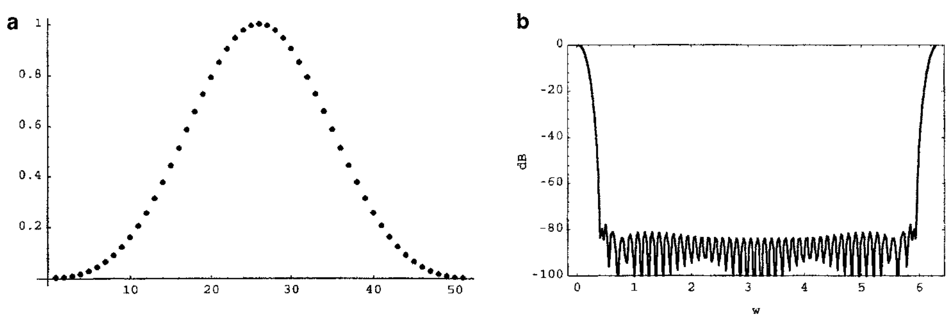

FIGURE 7.9 a) Harris-Nutall window (3-term). b) Amplitude spectrum of Harris-Nutall window (3-term).

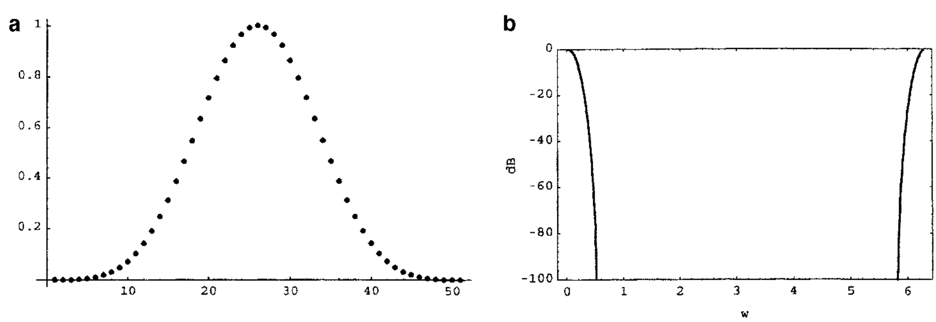

FIGURE 7.10 a) Harris-Nutall window (4-term). b) Amplitude spectrum of Harris-Nutall window (4-term).

7.3.10 Sampled Kaiser-Bessel Window

$$\text{Kaiser-Bessel spectrum window } = W(\omega) = \frac{\sinh\sqrt{\pi^2\alpha^2 - (\omega N/2)^2}}{\sqrt{\pi^2\alpha^2 - (\omega N/2)^2}} \qquad 0 \le \alpha \le 4$$

$$H_1(m) = \frac{\sinh(\pi\sqrt{\alpha^2 - m^2})}{\pi\sqrt{\alpha^2 - m^2}} \qquad \omega = m(2\pi/N)$$

$$c = H_1(0) + 2H_1(1) + 2H_1(2) + [2H_1(3)]$$

$$a(0) = \frac{H_1(0)}{c}, \qquad a(m) = \frac{2H_1(0)}{c}, \qquad m = 1,2,3$$

$$a(0) = 0.40243, \qquad a(1) = 0.49804, \qquad a(2) = 0.09831, \qquad a(3) = 0.00122$$

7.3.11 Parabolic (Riesz, Bochner, Parzen) Window

$$w(n) = 1.0 - \left(\frac{n}{N/2}\right)^2 \qquad 0 \le |n| \le \tfrac{N}{2}$$

$$w(n) = 1.0 - \left(\frac{n - \frac{N}{2}}{N/2}\right)^2 \qquad n = 0,1,2,\cdots,N-1$$

Figure 7.11 shows the parabolic window and its spectrum characteristics.

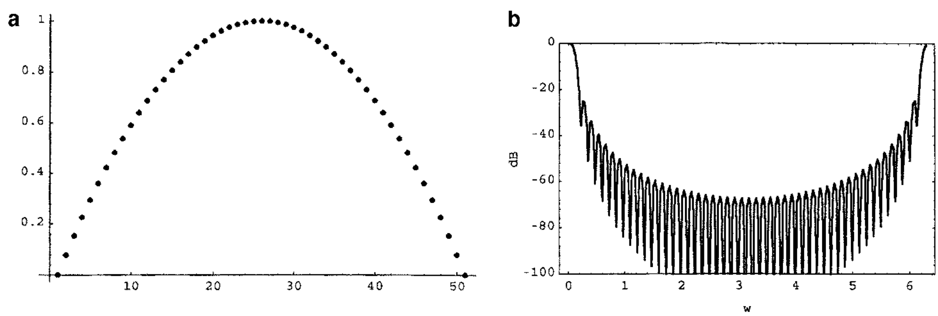

FIGURE 7.11 a) Parabolic window. b) Amplitude spectrum of Parabolic window.

7.3.12 Riemann Window

$$w(n) = \frac{\sin\frac{2\pi n}{N}}{\frac{2\pi n}{N}} \qquad 0 \le |n| \le \tfrac{N}{2}$$

$$w(n)=\frac{\sin\left(\frac{2\pi\left(n-\frac{N}{2}\right)}{N}\right)}{\frac{2\pi\left(n-\frac{N}{2}\right)}{N}} \qquad n=0,1,2,\cdots,N-1$$

Figure 7.12 shows the window's characteristics.

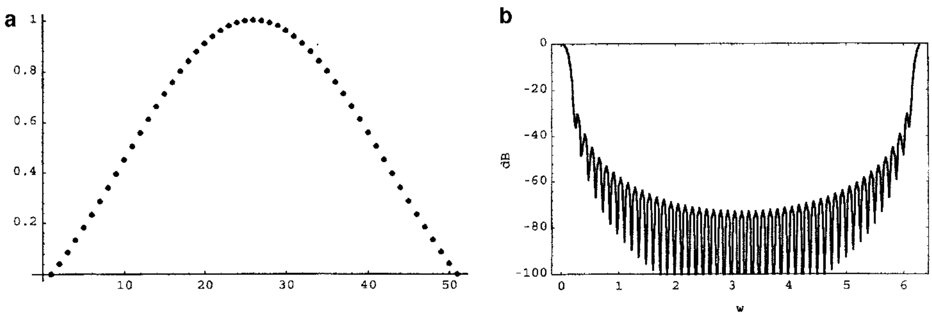

FIGURE 7.12 a) Riemann window. b) Amplitude spectrum of Riemann window.

7.3.13 de la Vallé-Poussin (Jackson, Parzen) Window

$$w(n)=\begin{cases} 1-6\left[\frac{n}{N/2}\right]^2\left[1-\frac{|n|}{N/2}\right] & 0\le |n|\le \frac{N}{4} \\ 2\left[1-\frac{|n|}{N/2}\right]^3 & \frac{N}{4}\le |n|\le \frac{N}{2} \end{cases}$$

Figure 7.13 shows the window and its frequency response.

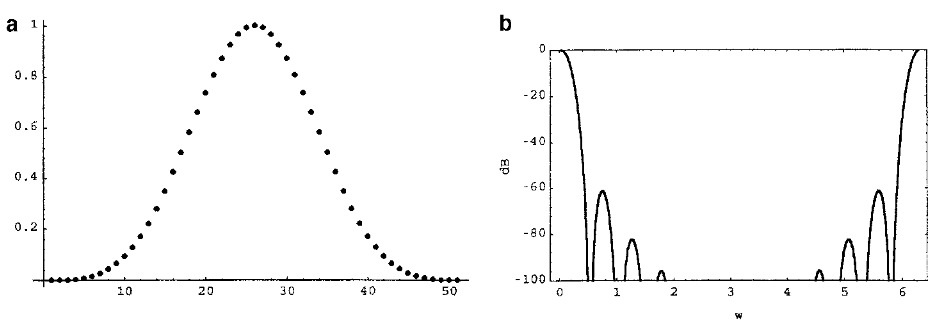

FIGURE 7.13 a) de la Vallé-Poussin window. b) Amplitude spectrum of de la Vallé-Poussin window.

7.3.14 Cosine Taper (Tukey) Window

The Tukey window is equal to one over $(1-\alpha/2)N$ points, with a cosine taper from one to zero for the remaining points $(\alpha/2)N$.

$$w(n)=\begin{cases} 1 & 0\le |n| \le \alpha\frac{N}{2} \\ 0.5\left(1+\cos\left[\pi\dfrac{n-\alpha(N/2)}{(1-\alpha)(N/2)}\right]\right) & \alpha\frac{N}{2}<|n|\le\frac{N}{2}\end{cases}$$

Figures 7.14 and 7.15 show the window and its frequency responses for α = 8/25 and α = 12/25.

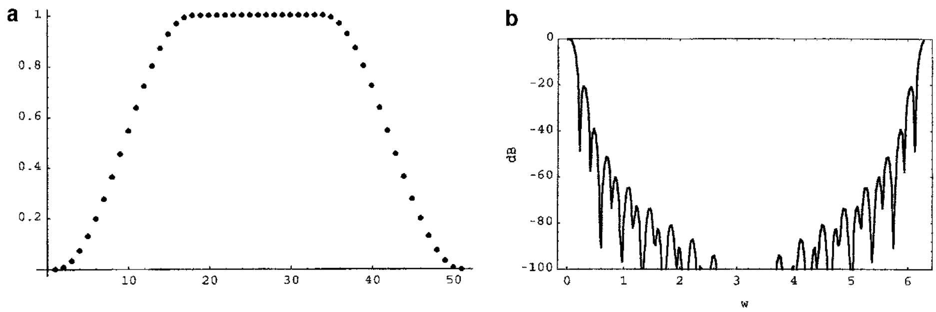

FIGURE 7.14 a) Tukey window with α = 8/25. b) Amplitude spectrum of Tukey window with α = 8/25.

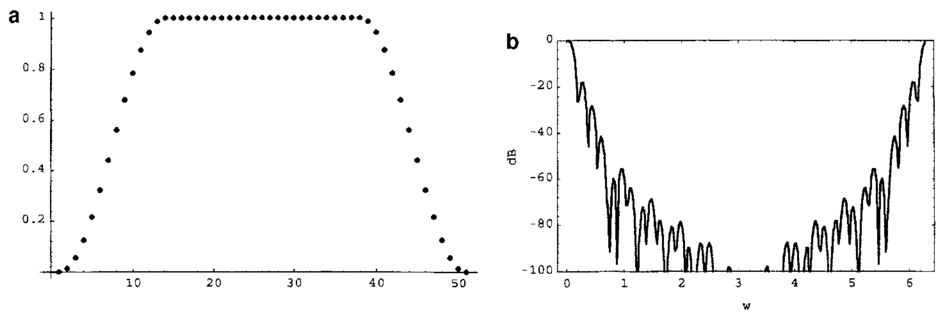

FIGURE 7.15 a) Tukey window with α = 12/25. b) Amplitude spectrum of Tukey window with α = 12/25.

7.3.15 Bohman Window

$$w(n)=\left(1-\frac{|n|}{N/2}\right)\cos\left(\pi\frac{|n|}{N/2}\right)+\frac{1}{\pi}\sin\left(\pi\frac{|n|}{N/2}\right),\qquad 0\le|n|\le\frac{N}{2}$$

Figure 7.16 shows the window and its spectrum.

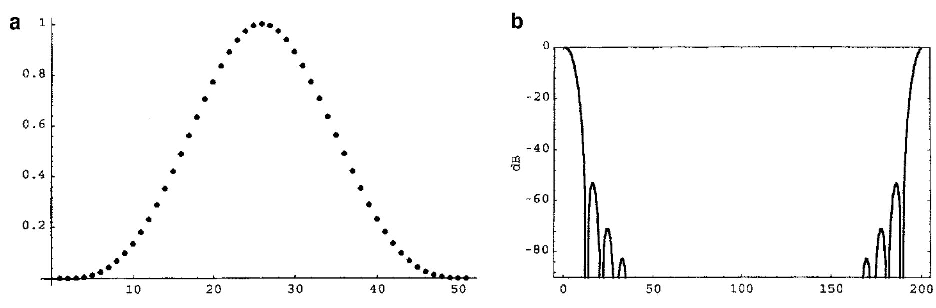

FIGURE 7.16 a) Bohman window. b) Amplitude spectrum of Bohman window.

7.3.16 Poisson Window

$$w(n) = \exp\left(-\alpha \frac{|n|}{N/2}\right), \qquad 0 \le |n| \le \tfrac{N}{2}$$

Figures 7.17 through 7.19 show the window and its spectrum with $\alpha = 1.5$, $\alpha = 3$, and $\alpha = 4$.

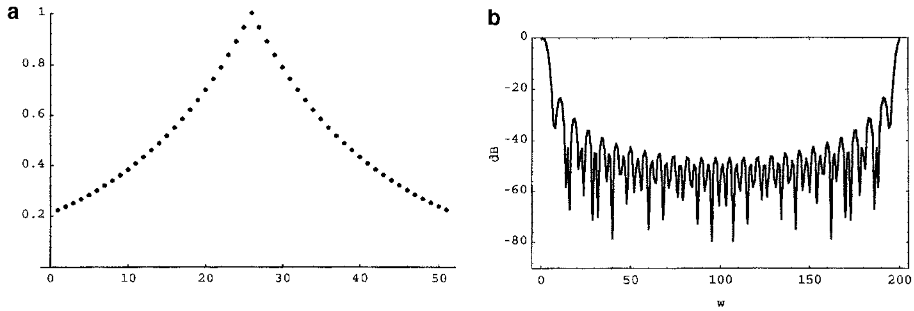

FIGURE 7.17 a) Poisson window with $\alpha = 1.5$. b) Amplitude spectrum of Poisson window with $\alpha = 1.5$.

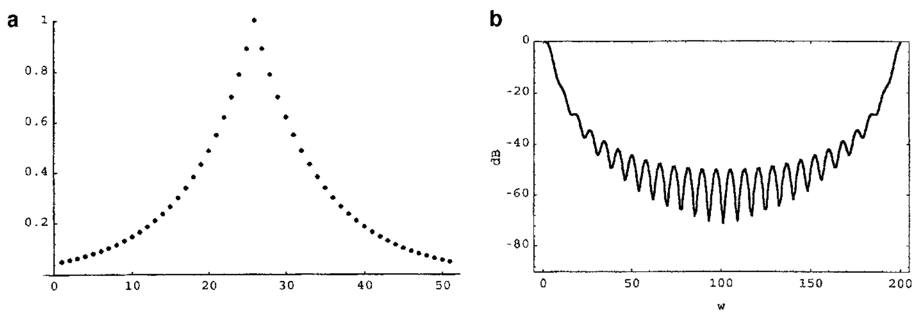

FIGURE 7.18 a) Poisson window with $\alpha = 3.0$. b) Amplitude spectrum of Poisson window with $\alpha = 3.0$.

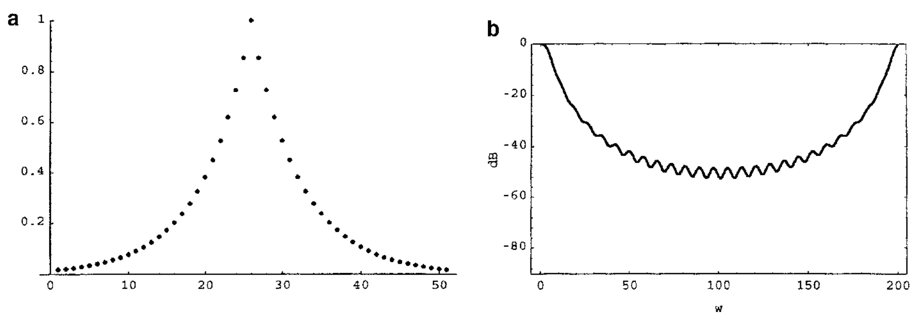

FIGURE 7.19 a) Poisson window with $\alpha = 4.0$. b) Amplitude spectrum of Poisson window with $\alpha = 4.0$.

7.3.17 Hann-Poisson Window

$$w(n) = 0.5\left[1 + \cos\left(\pi \frac{n}{N/2}\right)\right]\exp\left(-\alpha \frac{|n|}{N/2}\right), \qquad 0 \le |n| \le \tfrac{N}{2}$$

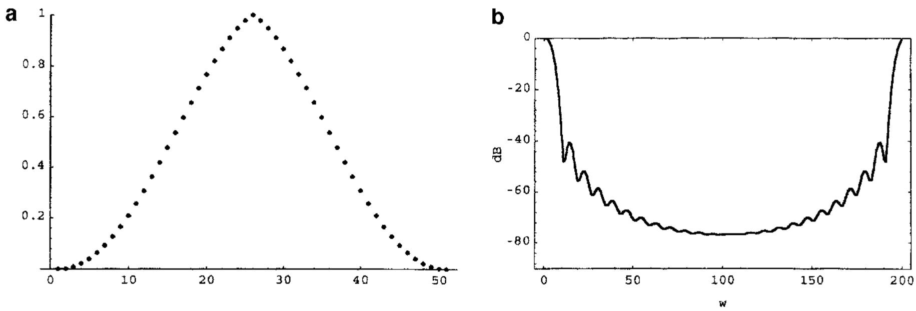

FIGURE 7.20 a) Hann-Poisson window with α = 0.5. b) Amplitude spectrum of Hann-Poisson window with α = 0.5.

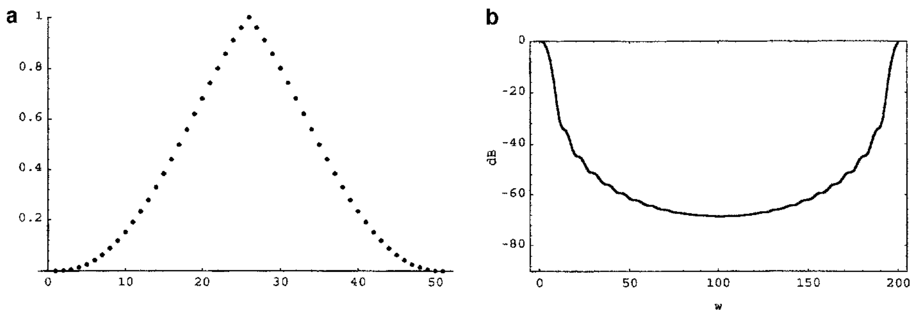

FIGURE 7.21 a) Hann-Poisson window with α = 1.0. b) Amplitude spectrum of Hann-Poisson window with α = 1.0.

Figures 7.20 and 7.21 show the window and its spectrum with α = 0.5 and α = 1.0, respectively.

7.3.18 Cauchy (Abel, Poisson) Window

$$w(n) = \frac{1}{1 + \left(\alpha \dfrac{n}{N/2}\right)^2}, \qquad 0 \le |n| \le \tfrac{N}{2}$$

Figures 7.22 through 7.24 show the window and its spectrum with α = 3.0, α = 4.0, and α = 6.0, respectively.

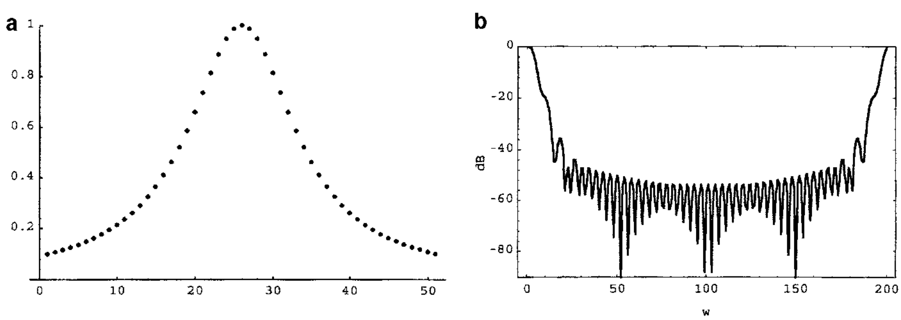

FIGURE 7.22 a) Cauchy window with α = 3.0. b) Amplitude spectrum of Cauchy window with α = 3.0.

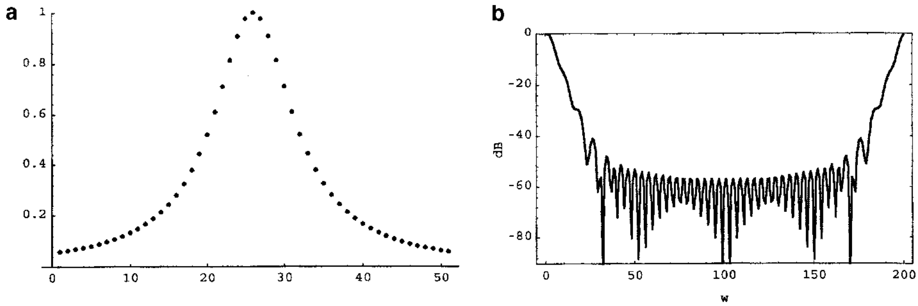

FIGURE 7.23 a) Cauchy window with α = 4.0. b) Amplitude spectrum of Cauchy window with α = 4.0.

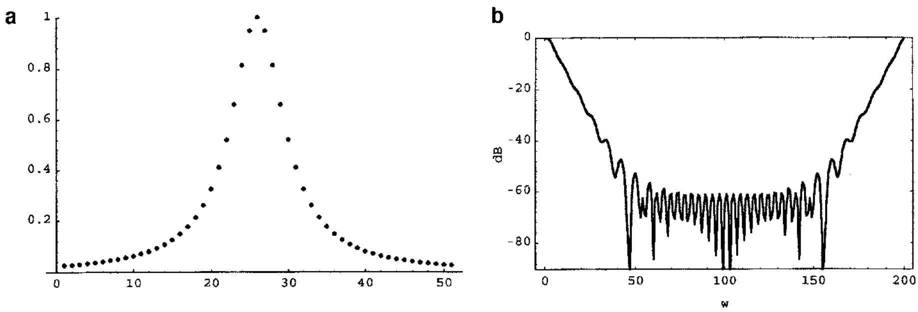

FIGURE 7.24 a) Cauchy window with α = 6.0. b) Amplitude spectrum of Cauchy window with α = 6.0.

7.3.19 Gaussian (Weierstrass) Window

$$w(n) = \exp\left[-\tfrac{1}{2}\left(\alpha \frac{n}{N/2}\right)^2\right], \qquad 0 \le |n| \le \tfrac{N}{2}$$

Figures 7.25 and 7.26 show the window and its spectrum for α = 2.5 and α = 3.5.

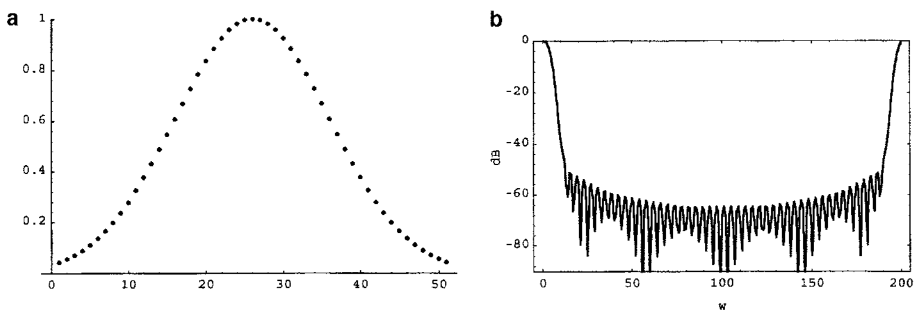

FIGURE 7.25 a) Gaussian window with α = 2.5. b) Amplitude spectrum of Gaussian window with α = 2.5.

7.3.20 Dolph-Chebyshev Window

$$W(k) = \frac{\cosh(N\cosh^{-1}(\beta\cosh(\pi k/N)))}{\cosh(N\cosh^{-1}(\beta))}$$

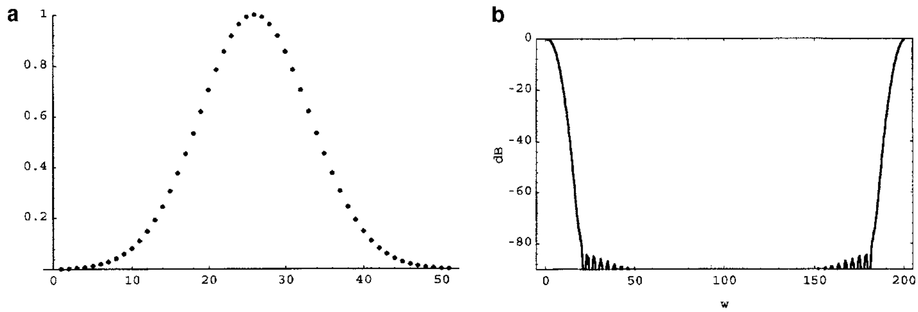

FIGURE 7.26 a) Gaussian window with α = 3.5. b) Amplitude spectrum of Gaussian window with α = 3.5.

where

$$\cosh^{-1}(X) = \ln\left(X + \sqrt{X^2 - 1.0}\right), \qquad |X| > 1.0$$

$$W(k) = \frac{\cos(N\cos^{-1}(\beta\cos(\pi k/N)))}{\cosh(N\cosh^{-1}(\beta))}$$

where

$$\cos^{-1}(X) = \frac{\pi}{2} - \tan^{-1}\left(\frac{X}{\sqrt{X^2 - 1.0}}\right), \qquad |X| \le 1.0$$

where β satisfies

$$\beta = \cosh\left(\frac{1}{N}\cosh^{-1} 10^{\alpha}\right)$$

and

$$w(n) = \sum_{k=0}^{N-1} W(k)\exp\left(j\frac{2\pi}{N}nk\right)$$

7.3.21 Kaiser-Bessel Window

$$w(n) = \frac{I_o\left[\pi\alpha\sqrt{1.0 - \left(\frac{n}{N/2}\right)^2}\right]}{I_o[\pi\alpha]} \qquad 0 \le |n| \le \tfrac{N}{2}$$

$$I_o(x) = \sum_{k=0}^{\infty}\left[\frac{\left(\frac{x}{2}\right)^k}{k!}\right]^2 = \text{zero-order modified Bessel function}$$

Figures 7.27 and 7.28 show the window and its spectrum for α = 2 and α = 3.

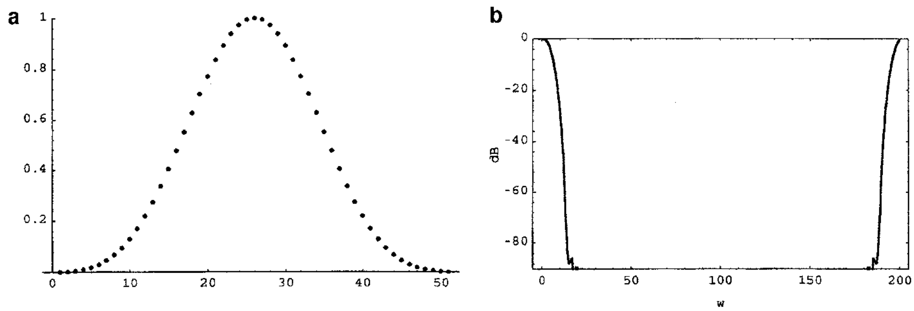

FIGURE 7.27 a) Kaiser-Bessel window with α = 3.0. b) Amplitude spectrum of Kaiser-Bessel window with α = 3.0.

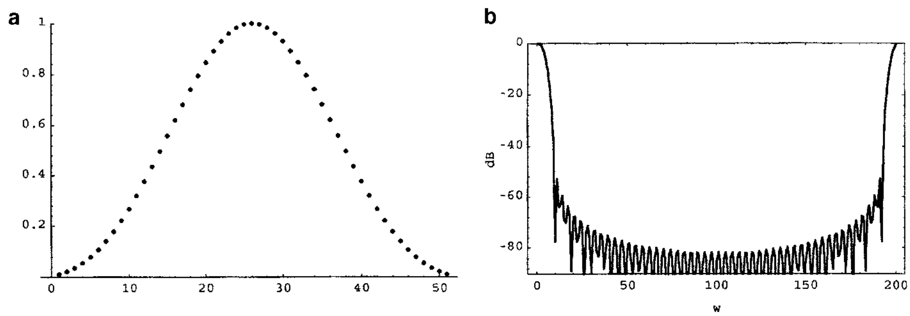

FIGURE 7.28 a) Kaiser-Bessel window with α = 2.0. b) Amplitude spectrum of Kaiser-Bessel window with α = 2.0.

7.3.22 Barcilon-Themes Window

$$W(k) = \frac{A\cos[y(k)] + B[y(k)/C]\sin[y(k)]}{(C + AB)[(y(k)/C)^2 + 1.0]}$$

where

$$A = \sinh C = \sqrt{10^{2\alpha} - 1.0}$$

$$B = \cosh C = 10^{\alpha}$$

$$C = \cosh^{-1}(10^{\alpha})$$

$$\beta = \cosh(C/N)$$

$$y(k) = N\cos^{-1}\left[\beta\cos\left(\frac{\pi k}{N}\right)\right]$$

7.3.23 Highest Sidelobe Level versus Worst-Case Processing Loss

Figure 7.29 shows the highest sidelobe level versus worst-case processing loss. Shaped DFT filters in the lower left tend to perform well.

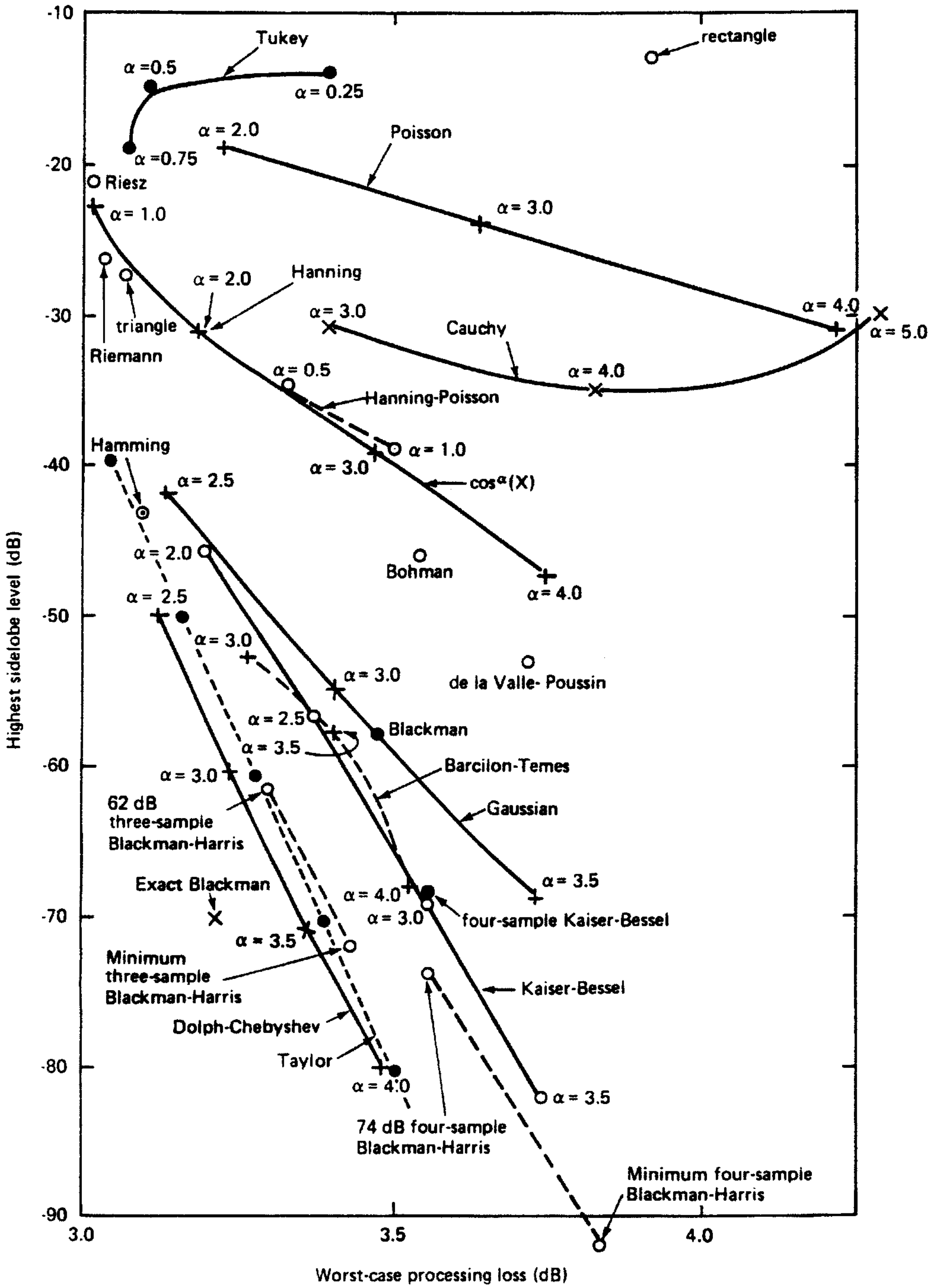

FIGURE 7.29 Highest sidelobe level versus worst-case processing loss. Shaped DFT filters in the lower left tend to perform well.

References

Harris, F. J., On the use of windows for harmonic analysis with the discrete fourier transforms, *Proc. IEEE*, 66, 55-83, January 1978.

Poularikas A. D. “Two-Dimensional Z-Transform”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

8

Two-Dimensional Z-Transform

8.1 The Z-Transform

8.1.1 Definition

$$X(z_1,z_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)z_1^{-n_1}z_2^{-n_2}$$

8.1.2 Relationship to Discrete-Time Fourier Transform

$$X(z_1,z_2)\Big|_{z_1=e^{j\omega_1},z_2=e^{j\omega_2}}=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}=X(\omega_1,\omega_2)$$

evaluated at $z_1=e^{j\omega_1}$ and $z_2=e^{j\omega_2}$.

8.1.3 Region of Convergence (ROC)

Points (z_1,z_2) for which

$$\sum_{n_1}\sum_{n_2}|x(n_1,n_2)||z_1|^{-n_1}|z_2|^{-n_2}<\infty$$

are located in the ROC. This implies that

$$|X(z_1,z_2)|<\infty$$

If (z_{01},z_{02}) point lies in the ROC, then all points (z_1,z_2) that satisfy

$$|z_1| \geq |z_{01}|, \quad |z_2| \geq |z_{02}|$$

also lie in the ROC.

For the first quadrant sequences, the boundary of the ROC must have nonpositive slope.

8.1.4 Sequences with Finite Support

$$X(z_1,z_2) = \sum_{n_1=N_1}^{M_1} \sum_{n_2=N_2}^{M_2} x(n_1,n_2) z_1^{-n_1} z_2^{-n_2}$$

The Z-transform converges for all finite values of z_1 and z_2, except possibly for $z_1 = 0$ and $z_2 = 0$.

8.1.5 Sequences with support on a wedge

If a sequence has support shown in Figure 8.1a, then its ROC is shown in Figure 8.1b (see Dudgeon and Mersereau, 1984). If the point (z_{01},z_{02}) belongs to the ROC, then

$$\ell n|z_1| \geq \ell n|z_{01}| \quad \text{and} \quad \ell n|z_2| \geq L\ell n|z_1| + \{\ell n|z_{02}| - L\ell n|z_{01}|\}$$

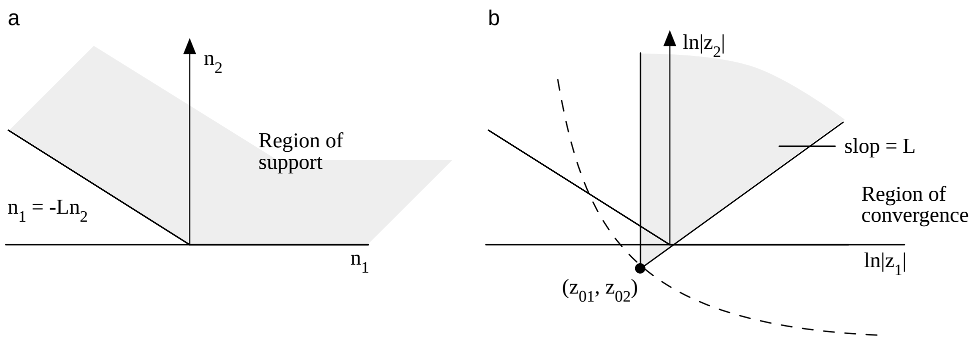

FIGURE 8.1

8.1.6 Sequences with Support on a Half-Plane

The boundary of the region of convergence is constrained to be a single-valued function of $|z_1|$ (or $\ell n|z_1|$) (see Dudgeon and Mersereau, 1984).

8.1.7 Sequences with Support Everywhere

- $x(n_1,n_2) = e^{-n_1^2-n_2^2}$ converges for all values of (z_1,z_2)
- $x(n_1,n_2) = 2^{|n_1|} 2^{|n_2|}$ will not converge for any value of (z_1,z_2)
- Often the z-transform of a sequence with support everywhere will converge in a region of finite area.
- A sequence with support everywhere can be split into four quadrant sequences, e. g.,

$$x(n_1,n_2) = x_1(n_1,n_2) + x_2(n_1,n_2) + x_3(n_1,n_2) + x_4(n_1,n_2)$$

where

$$x_1(n_1,n_2) = \begin{cases} x(n_1,n_2) & \text{for } n_1 > 0,\ n_2 > 0 \\ \frac{1}{2}x(n_1,n_2) & \text{for } n_1 = 0,\ n_2 > 0 \text{ or } n_1 > 0,\ n_2 = 0 \\ \frac{1}{4}x(n_1,n_2) & \text{for } n_1 = n_2 = 0 \\ 0 & \text{for } n_1 < 0 \text{ or } n_2 < 0 \end{cases}$$

Similarly are defined $x_2(n_1,n_2)$, $x_3(n_1,n_2)$, and $x_4(n_1,n_2)$, . The region of convergence of $x(n_1,n_2)$ is the *intersection* of the region of convergence of the four z-transforms of the four quadrants.

8.1.8 ROCs of Different Supports

Figure 8.2 shows the support of the function and its ROC (Lim, 1990).

8.2 Properties of the Z-Transform

8.2.1 Properties of the Z-Transform

TABLE 8.1 Properties of the 2-D Z-Transform

$$x(n_1,n_2) \longleftrightarrow X(z_1,z_2), \quad ROC: R_x$$

$$y(n_1,n_2) \longleftrightarrow Y(z_1,z_2), \quad ROC: R_y$$

1. Linearity

$$ax(n_1,n_2) + by(n_1,n_2) \longleftrightarrow aX(z_1,z_2) + bY(z_1,z_2), \quad ROC: \text{ at least } R_x \cap R_y$$

2. Convolution

$$x(n_1,n_2) * y(n_1,n_2) = \sum_{k_1}\sum_{k_2} x(n_1-k_1, n_2-k_2)\, y(k_1,k_2) \longleftrightarrow X(z_1,z_2)Y(z_1,z_2), \quad ROC: R_x \cap R_y$$

3. Separable Signals

$$x(n_1,n_2) = x_1(n_1)x_2(n_2) \longleftrightarrow X(z_1,z_2) = X_1(z_1)X_2(z_2), \quad ROC: |z_1| \in ROC\, X_1(z_1) \text{ and } |z_2| \in ROC\, X_2(z_2)$$

4. Shift Property

$$x(n_1 \pm m_1,\ n_2 \pm m_2) \longleftrightarrow X(z_1,z_2) = z_1^{\pm m_1} z_2^{\pm m_2} X(z_1,z_2),$$

$$ROC: R_x \text{ with possible exceptions } |z_1| = 0,\infty \text{ and } |z_2| = 0,\infty$$

5. Differentiation Property

$$-n_1 x(n_1,n_2) \longleftrightarrow z_1 \frac{\partial X(z_1,z_2)}{\partial z_1}, \quad ROC: R_x$$

$$-n_2 x(n_1,n_2) \longleftrightarrow z_2 \frac{\partial X(z_1,z_2)}{\partial z_2}, \quad ROC: R_x$$

$$n_1 n_2 x(n_1,n_2) \longleftrightarrow z_1 z_2 \frac{\partial^2 X(z_1,z_2)}{\partial z_1 \partial z_2}, \quad ROC: R_x$$

6. Modulation Property

$$w(n_1,n_2) = a^{n_1} b^{n_2} x(n_1,n_2) \longleftrightarrow X(a^{-1}z_1, b^{-1}z_2), \quad ROC: W(z_1,z_2) \text{ has the same as } X(z_1,z_2),$$

but scaled by $|a|$ in z_1 variable and by $|b|$ in the z_2 variable.

7. Conjugate Properties

$$x(n_1,n_2) = \text{complex function} \longleftrightarrow X(z_1,z_2)$$

TABLE 8.1 Properties of the 2-D Z-Transform (continued)

	$x^*(n_1,n_2) \longleftrightarrow X^*(z_1^*,z_2^*)$	
	$\mathrm{Re}\{x(n_1,n_2)\} \longleftrightarrow \frac{1}{2}[X(z_1,z_2)+X^*(z_1^*,z_2^*)]$	
	$\mathrm{Im}\{x(n_1,n_2)\} \longleftrightarrow \frac{1}{2j}[X(z_1,z_2)-X^*(z_1^*,z_2^*)]$	
		ROC: same as $X(z_1,z_2)$
8.	Reflection Properties	
	$x(n_1,n_2) \longleftrightarrow X(z_1,z_2)$	
	$x(-n_1,n_2) \longleftrightarrow X(z_1^{-1},z_2)$	
	$x(n_1,-n_2) \longleftrightarrow X(z_1,z_2^{-1})$	
	$x(-n_1,-n_2) \longleftrightarrow X(z_1^{-1},z_2^{-1})$	ROC : $\lvert z_1^{-1}\rvert$, $\lvert z_2^{-1}\rvert$ in R_x
9.	Multiplication Property	
	$x(n_1,n_2)\,y(n_1,n_2) \longleftrightarrow \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X\left(\frac{z_i}{\upsilon_1},\frac{z_2}{\upsilon_2}\right) Y(z_1,z_2)\,\frac{d\upsilon_1}{\upsilon_1}\,\frac{d\upsilon_2}{\upsilon_2}$	
10.	Parseval's Theorem	
	$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)\,y^*(n_1,n_2) \longleftrightarrow \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)\,Y^*\left(\frac{1}{z_1^*},\frac{1}{z_2^*}\right)\frac{dz_1}{z_1}\,\frac{dz_2}{z_2}$	
	Contours must: closed, counter-clockwise, encircle the origin, lie totally within *ROC*.	
11.	Initial Value Theorems	
	$x(n_1,n_2)=0$	$n_1<0,\ n_2<0$
	$\lim_{z_1\to\infty} X(z_1,z_2)=\sum_{n_2} x(0,n_2)\,z_2^{-n_2}$	
	$\lim_{z_2\to\infty} X(z_1,z_2)=\sum_{n_1} x(n_1,0)\,z_1^{-n_1}$	
	$\lim_{\substack{z_1\to\infty\\ z_2\to\infty}} X(z_1,z_2)=x(0,0)$	
12.	Linear Mapping of Variables	
	$x(n_1,n_2)=y(m_1,m_2)$	$n_1=I\,m_1+J\,m_2$, $n_2=Km_1+Lm_2$
		$I,\ J,\ K,\ L$ are integers
		$IL-KJ\neq 0$
	$X(z_1,z_2)=Y(z_1^I z_2^K, z_1^J z_2^L)$	ROC : $(\lvert z_1^I z_2^K\rvert$, $\lvert z_1^J z_2^L\rvert)$ in R_x

8.3. Inverse Z-Transform

8.3.1 Inverse Z-Transform

$$x(n_1,n_2)=\left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)\,z_1^{n_1-1}\,z_2^{n_2-1}\,dz_1\,dz_2$$

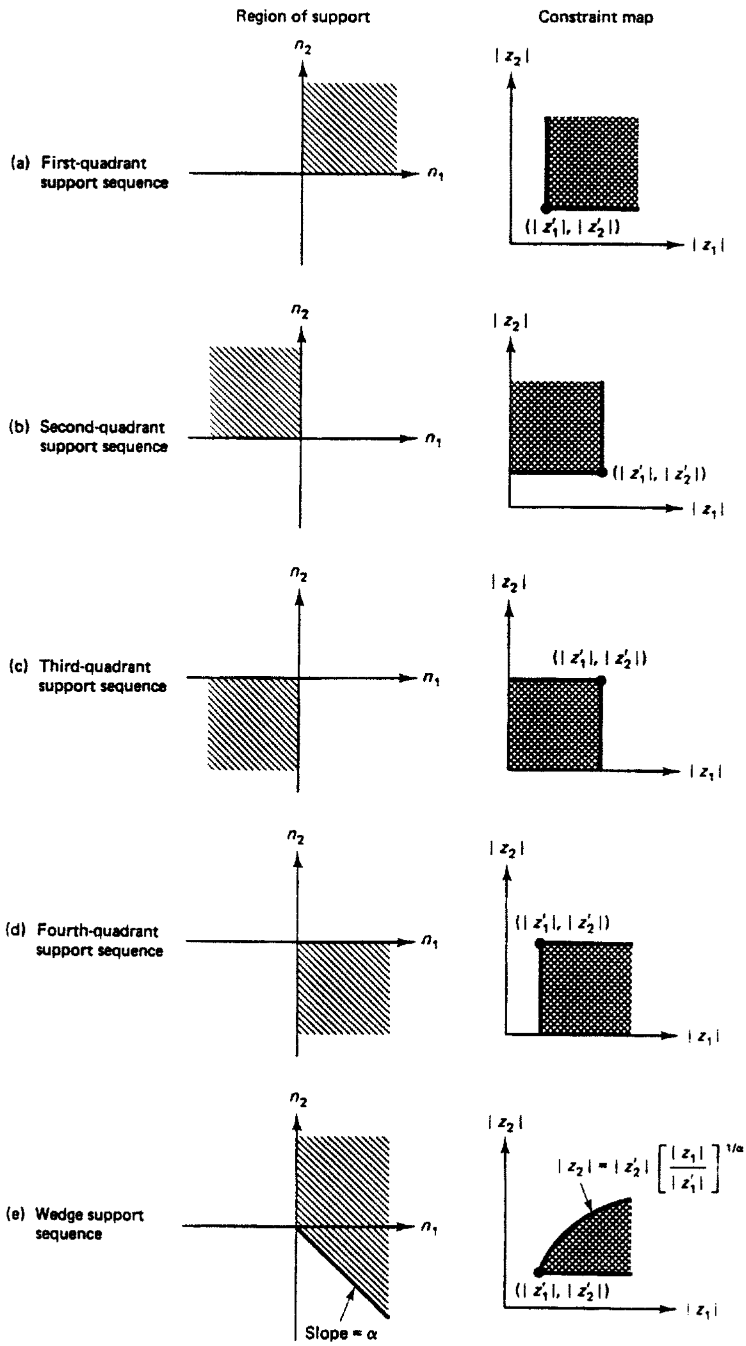

FIGURE 8.2

C_1, C_2 both in the ROC

C_1 counter-clockwise encircling the origin in the z_1 plane with z_2 fixed

C_2 counter-clockwise encircling the origin in the z_2 plane with z_1 fixed

8.4 System Function

8.4.1 System Function

$$\sum_{k_1}\sum_{k_2} a(k_1,k_2)y(n_1-k_1,n_2-k_2) = \sum_{k_1}\sum_{k_2} b(k_1,k_2)x(n_1-k_1,n_2-k_2)$$

$$a(k_1,k_2),\;\; b(k_1,k_2) \equiv \text{finite} - \text{extent sequences}$$

$$Y(z_1,z_2)\sum_{k_1}\sum_{k_2} a(k_1,k_2)z_1^{-k_1}\,z_2^{-k_2} = X(z_1,z_2)\sum_{k_1}\sum_{k_2} b(k_1,k_2)z_1^{-k_1}\,z_2^{-k_2}$$

or

$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{\sum_{k_1}\sum_{k_2} b(k_1,k_2)z_1^{-k_1}\,z_2^{-k_2}}{\sum_{k_1}\sum_{k_2} a(k_1,k_2)z_1^{-k_1}\,z_2^{-k_2}} = \frac{B(z_1,z_2)}{A(z_1,z_2)}$$

8.5 Stability Theorems

8.5.1 Theorem 8.5.1.1 (Shanks, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. This filter is stable if, and only if, $A(z_1,z_2) \neq 0$ for every point (z_1,z_2) such that $|z_1| \geq 1$ or $|z_2| \geq 1$.

8.5.2 Theorem 8.5.1.2 (Shanks, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. Then $H(z_1,z_2)$ is stable if, and only if, the following conditions are true:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| \geq 1,\;\; |z_2| = 1$
b) $A(z_1,z_2) \neq 0,\;\; |z_1| = 1,\;\; |z_2| \geq 1$

8.5.3 Theorem 8.5.1.3 (Huang, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first-quadrant recursive filter. The filter is stable if, and only if, $A(z_1,z_2)$ satisfies the following two conditions:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| \geq 1,\;\; |z_2| = 1$
b) $A(a,z_2) \neq 0,\;\; |z_2| \geq 1$ for any a such that $|a| \geq 1$

8.5.4 Theorem 8.5.1.4 (DeCarlo, 1977; Strintzis, 1977)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. The filter is stable if, and only if, $A(z_1,z_2)$ satisfies the following three conditions:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| = 1,\;\; |z_2| = 1$
b) $A(a,z_2) \neq 0,$ for $|z_2| \geq 1$ for any a such that $|a| = 1$
c) $A(z_1,b) \neq 0,$ for $|z_1| \geq 1$ for any b such that $|b| = 1$

References

DeCarlo, R., J. Murray, and R. Saeks, Multivariate Nyquist theory, *Int. J. Control*, 25, 657-75, 1977.
Dudgeon, Dan E. and Russell M. Mersereau, *Multidimensional Digital Signal Processing*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1984.
Huang, Thomas S., Stability of two-dimensional recursive filters, *IEEE Trans. Audio and Electroacoustics*, AU-20, 158-63, 1972.
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Shanks, John L., Sven Treitel, and James H. Justice, Stability and synthesis of two-dimensional recursive filters, *IEEE Trans. Audio and Electroacoustics*, AU-20, 115-28, 1972.
Strintzis, Michael G., Test of stability of multidimensional filters, *IEEE Trans. Circuits and Systems*, CAS-24, 432-37, 1977.

Appendix 1

Examples

1.1 Two-Dimensional Z-transforms

Example 8.1

The Z-transform of $x(n_1,n_2)=a^{n_1}b^{n_2}u(n_1,n_2)=a^{n_1}b^{n_2}u(n_1)u(n_2)$ is

$$X(z_1,z_2)=\sum_{n_1=-\infty}^{\infty}a^{n_1}u(n_1)z_1^{-n_1}\sum_{n_2=-\infty}^{\infty}b^{n_2}u(n_2)z_2^{-n_2}=\sum_{n_1=0}^{\infty}a^{n_1}z_1^{-n_1}\sum_{n_2=0}^{\infty}b^{n_2}z_2^{-n_2}$$

$$=\frac{1}{1-az_1^{-1}}\frac{1}{1-bz_2^{-1}}$$

with region of convergence (ROC) $|az_1^{-1}|<1$ and $|bz_2^{-1}|<1$, or $|z_1|>|a|$ and $|z_2|>|b|$ (see Figure 8.3).

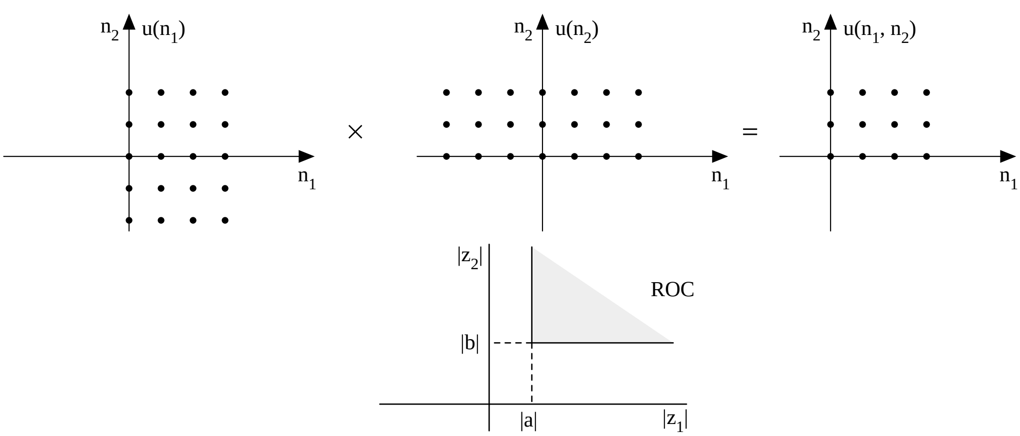


FIGURE 8.3

Example 8.2

The Z-transform of $x(n_1,n_2) = a^{n_1}\,\delta(n_1 - n_2)u(n_1,n_2)$ is

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a^{n_1}\,u(n_1)u(n_2)\delta(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,\delta(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_2=0}^{\infty} a^{n_2}\,z_1^{-n_2}\,z_2^{-n_2} = \frac{1}{1-az_1^{-1}\,z_2^{-1}}$$

From last summation the ROC is $\left|az_1^{-1}\,z_2^{-1}\right| < 1$ or $|a| < |z_1|\,|z_2|$ equivalently

$$\ell n|a| < \ell n|z_1| + \ell n|z_2|$$

The function and its ROC are shown in Figure 8.4.

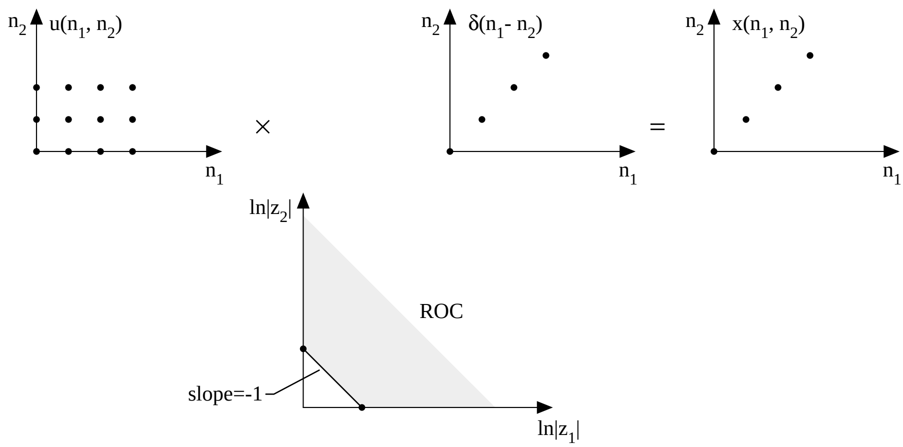


FIGURE 8.4

Example 8.3

The Z-transform of $x(n_1,n_2) = a^{n_1}\,u(n_1,n_2)u(n_1 - n_2)$ is

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a^{n_1}\,u(n_1,n_2)u(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,u(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty} a^{n_1}\,z_1^{-n_1} = \frac{1-\left(z_2^{-1}\right)^{n_1+1}}{1-z_2^{-1}} = \frac{1}{\left(1-az_1^{-1}\right)\left(1-az_1^{-1}z_2^{-1}\right)}$$

$$\text{ROC}: \quad |z_1| > |a|\ , \quad |z_1\,z_2| > |a|$$

The function and its ROC are shown in Figure 8.5.

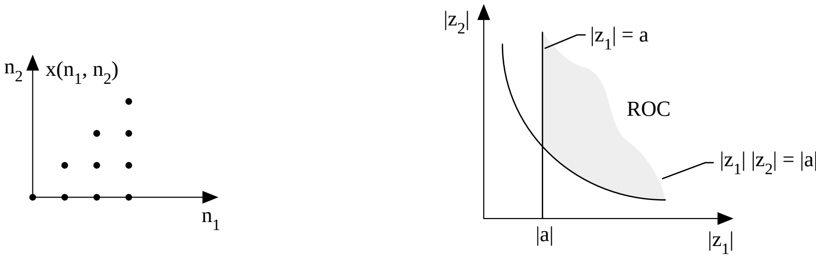


FIGURE 8.5

Example 8.4 (inverse integration)

If $X(z_1,z_2)$ has a region of convergence in the 2-D unit surface,

$$x(n_1,n_2)=\left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} \frac{z_1^{n_1}\, z_2^{n_2}}{1-\frac{1}{2}z_1^{-1}-\frac{1}{4}z_2^{-2}}\, dz_1\, dz_2$$

$$=\left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} \frac{(z_2-\frac{1}{4})^{-1}\, z_1^{n_1}\, z_2^{n_2}}{z_1-[\frac{1}{2}z_2/(z_2-\frac{1}{4})]}\, dz_1\, dz_2$$

$$=\frac{2\pi j}{(2\pi j)^2}\oint_{C_2} (z_2-\tfrac{1}{4})^{-1}\, \frac{(\frac{1}{2})^{n_1}\, z_2^{n_1}}{(z_2-\frac{1}{4})^{n_1}}\, z_2^{n_2}\, dz_2 = (\tfrac{1}{2})^{n_1}\, (\tfrac{1}{4})^{n_2}\, \frac{(n_1+n_2)!}{n_1!\, n_2!}, \quad n_1,n_2 \ge 0$$

Poularikas A. D. “Analytical Methods”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

9

Analytical Methods

9.1. Binomial Theorem and Binomial Coefficients; Arithmetic and Geometric Progressions; Arithmetic, Geometric, Harmonic, and Generalized Means

Binomial Theorem

9.1.1 $$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

(n is a positive integer)

Binomial Coefficients

9.1.2 $$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

9.1.3 $$\binom{n}{k} = \binom{n}{n-k} = (-1)^k\binom{k-n-1}{k}$$

9.1.4 $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

9.1.5 $\binom{n}{0} = \binom{n}{n} = 1$

9.1.6 $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

9.1.7 $1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$

9.1.8 **The Multinomial Theorem**

$$(a_1 + \cdots + a_m)^n = \sum_{k_1+\cdots+k_m=n} \frac{n!}{k_1!\cdots k_m!} a_1^{k_1} \cdots a_m^{k_m}$$

9.1.9 **Factorials and Binomial Coefficients**

$n! = 1 \cdot 2 \cdot 3 \cdots n, \qquad 0! = 1$ (factorials)

$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1), \qquad (2n)!! = 2 \cdot 4 \cdot 6 \cdots 2n$ (semifactorials)

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ (binomial coefficients)

$n! \sim \sqrt{2\pi}\, n^{n+1/2} e^{-n}$ as $n \to \infty$ (Stirling's formula)

9.1.10 $\binom{n}{n-k} = \binom{n}{k}, \qquad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k}$

9.1.11 $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$

9.1.12 $\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0} = \binom{m+n}{k}$

9.1.13 $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, \qquad \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

9.1.14 $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0$

9.1.15 $\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$

9.1.16 $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

9.1.17 $\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}$

9.1.18 $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \qquad m = 0, 1, 2, \cdots$

9.1.19 $\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \quad n = 0,1,2,\cdots$

9.1.20 $\sum_{k=0}^{n} \binom{n}{k}\binom{s+k}{m}(-1)^k = (-1)^n \binom{s}{m-n}$

9.1.21 **Cauchy product**: Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ have absolute convergence. Then

$$\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]$$

9.1.22 $\binom{-1/2}{n} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$

9.1.23 $\binom{-2k-1}{n} = (-1)^n \frac{(n+2k)!}{(2k!)n!}, \quad k = 0,1,2,\cdots$

9.1.24 **Sum of Arithmetic Progression to *n* Terms**

$$a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = na + \frac{1}{2}n(n-1)d = \frac{n}{2}(a+l),$$

last term in series = $l = a + (n-1)d$

9.1.25 **Sum of Geometric Progression to *n* Terms**

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} = s_n = a/(1-r), \quad (-1 < r < 1)$$

9.1.26 **Arithmetic Mean of *n* Quantities *A***

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

9.1.27 **Geometric Mean of *n* Quantities *G***

$$G = (a_1 a_2 \cdots a_n)^{1/n} \qquad (a_k > 0,\ k = 1,2,\cdots n)$$

9.1.28 **Harmonic Mean of *n* Quantities *H***

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \qquad (a_k > 0,\ k = 1,2,\cdots n)$$

Generalized Mean

9.1.29 $M(t) = \left(\frac{1}{n}\sum_{k=1}^{n} a_k^t\right)^{1/t}$

9.1.30 $M(t) = 0 \qquad (t < 0,$ some a_k zero)

9.1.31 $\lim_{t\to\infty} M(t) = \max, \qquad (a_1, a_2, \cdots, a_n) = \max a$

9.1.32 $\lim_{t=-\infty} M(t) = \min, \qquad (a_1, a_2, \cdots, a_n) = \min a$

9.1.33 $\lim_{t\to 0} M(t) = G$

9.1.34 $M(1) = A$

9.1.35 $M(-1) = H$

9.2 Inequalities

Relation Between Arithmetic, Geometric, Harmonic, and Generalized Means

9.2.1 $A \ge G \ge H$, equality if and only if $a_1 = a_2 = \cdots = a_n$

9.2.2 $\min a < M(t) < \max a$

9.2.3 $\min a < G < \max a$; equality holds if all a_k are equal, or $t < 0$ and an a_k is zero.

9.2.4 $M(t) < M(s)$ if $t < s$ unless all a_k are equal, or $s < 0$ and an a_k is zero.

Triangle Inequalities

9.2.5 $|a_1| - |a_2| \le |a_1 + a_2| \le |a_1| + |a_2|$

9.2.6 $\left|\sum_{k=1}^{n} a_k\right| \le \sum_{k=1}^{n} |a_k|$

9.2.7 **Chebyshev's Inequality**

If $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$

$b_1 \ge b_2 \ge b_3 \ge \cdots \ge b_n$

$$n\sum_{k=1}^{n} a_k b_k \ge \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

9.2.8 **Holder's Inequality for Sums**

If $\frac{1}{p} + \frac{1}{q} = 1, \qquad p > 1, \quad q > 1$

$$\sum_{k=1}^{n} |a_k b_k| \le \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q};$$

equality holds if and only if $|b_k| = c|a_k|^{p-1}$ (c = constant > 0). If $p = q = 2$, we get Cauchy's inequality

9.2.9 **Cauchy's Inequality**

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \le \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \quad \text{(equality for } a_k = cb_k,\ c \text{ constant)}.$$

9.2.10 **Holder's Inequality for Integrals**

If $\frac{1}{p} + \frac{1}{q} = 1, \qquad p > 1, \quad q > 1$

$$\int_a^b |f(x)g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \left[\int_a^b |g(x)|^q\,dx\right]^{1/q}$$

equality holds if and only if $|g(x)| = c|f(x)|^{p-1}$ (c = constant > 0). If $p = q = 2$, we get Schwarz's inequality.

9.2.11 **Schwarz's Inequality**

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \le \int_a^b |f(x)|^2\,dx \int_a^b |g(x)|^2\,dx$$

9.2.12 **Minkowski's Inequality for Sums**

If $p > 1$ and $a_k, b_k > 0$ for all k,

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p},$$

equality holds if and only if $b_k = ca_k$ (c = constant > 0).

9.2.13 **Minkowski's Inequality for Integrals**

If $p > 1$,

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}$$

equality holds if and only if $g(x) = cf(x)$ (c = constant > 0).

9.3 Numbers

9.3.1 Number Systems

N ***natural number***, $N = \{0,1,2,3,\cdots\}$ sometimes 0 is omitted.

Z ***integers***, (Z^+ positive integers), $Z = \{0,\pm1,\pm2,\pm3,\cdots\}$

Q ***rational numbers***, $Q = \{p/q : p,q \in z,\ q \neq 0\}$, Q is ***countable***, i.e., there exists a one-to-one correspondence between Q and N

R ***real numbers***, R = {real numbers} is not countable. Real numbers which are not rational are called ***irrational***. Every irrational number can be represented by an infinite non-periodic decimal expansion. ***Algebraic numbers*** are solutions to an equation of the form$a_n x^n + \ldots + a_o = 0$, $a_k \in z$. ***Transcendental*** are those numbers in **R** which are not algebraic. (Example: 4/7 is rational; $\sqrt{5}$ is algebraic and irrational; e and π are transcendentals.)

C ***Complex numbers***, $C = \{x + jy : x,y \in R\}$ where $j^2 = -1$.

9.3.2 The Supreme Axiom

For any non-empty bounded subset S of R there exist unique numbers $G = \sup S$ and $g = \inf S$ such that:

1. $g \le x \le G$, all $x \in S$
2. For any $\varepsilon > 0$ there exists $x_1 \in S$ and $x_2 \in S$ such that

$$x_1 > G - \varepsilon \text{ and } x_2 < g + \varepsilon$$

9.3.3 Theorems on Prime Numbers

1. For every positive integer n exists a prime factor of $n!+1$ exceeding n.
2. Every prime factor of $p_1p_2\ldots p_n+1$, where $p_1,p_2,\ldots,p_n$ are prime, differs from each of $p_1,p_2,\ldots,p_n$.
3. There are infinitely many primes. (Euclid)
4. For every positive integer $n \ge 2$, there exists a string of n consecutive composite integers.
5. If a and b are relatively prime, then the arithmetic sequence $an+b$, $n = 1,2,\ldots$, contains an infinite number of primes. (Lejeune-Dirichlet)

The following conjectures have not been proved.

6. Every even number ≥ 6 is the sum of two odd primes (the Goldbach conjecture).
7. There exists infinitely many prime twins. Prime twins are pairs like (3,5), (5,7), and (2087,2089).

9.3.4 Unique Factorization Theorem

Every integer > 1 is either a prime or a product of uniquely determined primes.

9.3.5 The Function $\pi(x)$

The function value $\pi(x)$ is the number of primes which are less than or equal to x. Asymptotic behavior:

$$\pi(x) \sim \frac{x}{\ell nx} \quad \text{as} \quad x \to \infty$$

9.3.6 Least Common Multiple (LCM)

Let $[a_1,\ldots,a_n]$ denote the *least common multiple* of the integers $a_1,\cdots,a_n$. One method of finding that number is: Prime number factorize $a_1,\cdots,a_n$. Then form the product of these primes raised to the greatest power in which they appear.

Example: Determine $A = [18,24,30]$. $18 = 2 \cdot 3^2$, $24 = 2^3 \cdot 3$, $30 = 2 \cdot 3 \cdot 5$. Thus, $A = 2^3 \cdot 3^2 \cdot 5 = 360$.

9.3.7 Greatest Common Divisor (GCD)

Let (a,b) denote the *greatest common divisor* of a and b. If $(a,b) = 1$ the numbers are *relatively prime*. One method (Euclid's algorithm) of finding (a,b) is:

Assuming $a > b$ and dividing a by b yields $a = q_1b_1 + r_1$, $0 \le r_1 < b$. Dividing b by r_1 gives $b = q_2r_1 + r_2$, $0 \le r_2 < r_1$. Continuing like this, let r_k be the first remainder which equals 0. Then $(a,b) = r_{k-1}$

Example: Determine (112,42). By the above algorithm, $112 = 2 \cdot 42 + 28$, $42 = 1 \cdot 28 + 14$, $28 = 2 \cdot 14 + 0$. Thus, $(112,42) = 14$. ***Note:*** $(a,b) \cdot [a,b] = ab$

9.3.8 Modulo

If m, n, and p are integers, then m and n are *congruent modulo p*, $m = n \bmod(p)$, if $m - n$ is a multiple of p, i.e., m/p and n/p have equal remainders.

$$m_1 = n_1 \text{mod}(p), \quad m_2 = n_2 \text{mod}(p) \Rightarrow$$

$$\text{(i)} \ \ m_1 + m_2 = (n_1 \pm n_2)\text{mod}(p) \qquad \text{(ii)} \ \ m_1 m_2 = (n_1 n_2)\text{mod}(p)$$

9.3.9 Diophantine Equations

A *Diophantine* equation has integer coefficients and integer solutions. As an example, the equation $ax + by = c$, $a,b,c \in \mathbf{Z}$ (a), has integer solutions x and y if and only if (a,b) divides c. In particular, $ax + by = 1$ is solvable $\Leftrightarrow (a,b) = 1$. If x_o, y_o is a particular solution of (a), then the general solution is $x = x_o + nb/(a,b)$, $y = y_o - na/(a,b)$, $n \in \mathbf{Z}$.

9.3.10 Mersenne Numbers

$$M_n = 2^n - 1$$

9.3.11 Mersenne Primes

If $2^p - 1$ is prime, then p is prime.

9.3.12 Fermat Primes

If $2^p + 1$ is prime, then p is a power of 2.

9.3.13 Fibonacci Numbers

$$F_1 = 1 \qquad F_2 = 1 \qquad F_{n+2} = F_n + F_{n+1} \qquad n \geq 1$$

9.3.14 Decimal and Binary Systems

$$x = x_m B^m + x_{m-1} B^{m-1} + \cdots + x_o B^o + x_{-1} B^{-1} + \cdots = (x_m x_{m-1} \cdots x_o .\, x_{-1}, \cdots)$$

$B > 1$ is base; x_i is one of the numbers $0,1\cdots,B-1$.

Example: 36.625_{10}, $x = 3 \cdot 10^1 + 6 \cdot 10^0 + 6 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3} = (36.625)_{10}$

$x = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} = (100100.101)_2$

9.3.15 Conversion Bases

1. $B \to 10$: $x_B = (x_m x_{m-1} \cdots x_o .\, x_{-1}, \cdots)_B$

   $x_{10} = x_m B^m + x_{m-1} B^{m-1} + \cdots + x_o B^o + x_{-1} B^{-1} + \cdots$

2. $10 \to B$: (**Example**: $x_{10} = 12545.6789$ to x_8)

$\frac{12545}{8} = 1568 + \frac{1}{8}$	$R_1 = 1$	$0.6789 \times 8 = 5.4312 \quad I_1 = 5$
$\frac{1568}{8} = 196 + \frac{0}{8}$	$R_2 = 0$	$0.4312 \times 8 = 3.4496 \quad I_2 = 3$
$\frac{196}{8} = 24 + \frac{4}{8}$	$R_3 = 4$	$0.4496 \times 8 = 3.5968 \quad I_3 = 3$
$\frac{24}{8} = 3 + \frac{0}{8}$	$R_4 = 0$	$0.5968 \times 8 = 4.7744 \quad I_4 = 4$
$\frac{3}{8} = 0 + \frac{3}{8}$	$R_5 = 3$	etc.

Therefore $x_8 = (30401.5334)_8$

9.3.16 Binary System

Addition: $0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 10$

Multiplication: $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1$

9.3.17 Hexadecimal System

Digits: 0,1,2,3,4,5,6,7,8,9, $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, and $F = 15$

Addition Table

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**A**	**B**	**C**	**D**	**E**	**F**	
1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	**1**
2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	**2**
3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	**3**
4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	**4**
5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	**5**
6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	**6**
7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	**7**
8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	**8**
9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	**9**
A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	**A**
B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	**B**
C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	**C**
D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	**D**
E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	**E**
F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E	**F**
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**A**	**B**	**C**	**D**	**E**	**F**	

E.g., B + 6 = 11

Multiplication Table

	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	**1**
2	2	4	6	8	A	C	E	10	12	14	16	18	1A	1C	1E	**2**
3	3	6	9	C	F	12	15	18	1B	1E	21	24	27	2A	2D	**3**
4	4	8	C	10	14	18	1C	20	24	28	2C	30	34	38	3C	**4**
5	5	A	F	14	19	1E	23	28	2D	32	37	3C	41	46	4B	**5**
6	6	C	12	18	1E	24	2A	30	36	3C	42	48	4E	54	5A	**6**
7	7	E	15	1C	23	2A	31	38	3F	46	4D	54	5B	62	69	**7**
8	8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	**8**
9	9	12	1B	24	2D	36	3F	48	51	5A	63	6C	75	7E	87	**9**
A	A	14	1E	28	32	3C	46	50	5A	64	6E	78	82	8C	96	**A**
B	B	16	21	2C	37	42	4D	58	63	6E	79	84	8F	9A	A5	**B**
C	C	18	24	30	3C	48	54	60	6C	78	84	90	9C	A8	B4	**C**
D	D	1A	27	34	41	4E	5B	68	75	82	8F	9C	A9	B6	C3	**D**
E	E	1C	2A	38	46	54	62	70	7E	8C	9A	A8	B6	C4	D2	**E**
F	F	1E	2D	3C	4B	5A	69	78	87	96	A5	B4	C3	D2	E1	**F**
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**A**	**B**	**C**	**D**	**E**	**F**	

E.g., B · 6 = 42

9.3.18 Special Numbers in Different Number Bases

B = 2: $\pi = 11.001001\ 000011\ 111101\ 101010\ 100010\ 001000\ 010110\ 100011\cdots$

$e = 10.101101\ 111110\ 000101\ 010001\ 011000\ 101000\ 101011\ 101101\cdots$

$\gamma = 0.100100\ 111100\ 010001\ 100111\ 111000\ 110111\ 110110\ 110110\cdots$

$\sqrt{2} = 1.011010\ 100000\ 100111\ 100110\ 011001\ 111111\ 001110\ 111100\cdots$

$\text{In}2 = 0.101100\ 010111\ 001000\ 010111\ 111101\ 111101\ 000111\ 001111\cdots$

B = 3: $\pi = 10.010211\ 012222\cdots$

$e = 2.201101\ 121221\cdots$

$\gamma = 0.120120\ 210100\cdots$

$\sqrt{2} = 1.102011\ 221222\cdots$

$\text{In}2 = 0.200201\ 022012\cdots$

B = 12: $\pi = 3.184809\ 493B91\cdots$

$e = 2.875236\ 069821\cdots$

$\gamma = 0.6B1518\ 8A6760\cdots$

$\sqrt{2} = 1.4B7917\ 0A07B8\cdots$

$\text{In}2 = 0.839912\ 483369\cdots$

B = 8: $\pi = 3.110375\ 524210\ 264302\cdots$

$e = 2.557605\ 213050\ 535512\cdots$

$\gamma = 0.447421\ 477067\ 666061\cdots$

$\sqrt{2} = 1.324047\ 463177\ 167462\cdots$

$\ln 2 = 0.542710\ 277574\ 071736\cdots$

B = 16: $\pi = 3.243F6A\ 8885A3\cdots$

$e = 2.B7E151\ 628AED\cdots$

$\gamma = 0.93C467\ E37DB0\cdots$

$\sqrt{2} = 1.6A09E6\ 67F3BC\cdots$

$\ln 2 = 0.B17217\ F7D1CF\cdots$

9.4 Complex Numbers $(j^2 = -1)$

9.4.1 Rectangular Form

$z = x + jy,\ z^* = x - jy = \text{conjugate},\ |z| = [zz^*]^{1/2} = [x^2 + y^2]^{1/2} = \text{modulus},$

$|z_1 - z_2|$ = distance between the points z_1 and z_2, $x = \text{Re}\{z\}$, $y = \text{Im}\{z\}$,

$\theta = \tan^{-1}(y/x) + n\pi, \quad (n = 0 \text{ if } x > 0, \quad n = 1 \text{ if } x < 0)$

9.4.2 Polar Form

$$z = x + jy = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x} + n\pi \qquad (n = 0 \text{ if } x > 0,\ n = 1 \text{ if } x < 0)$$

9.4.3 De Moivre's and Euler Formulas

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta, \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

9.5 Algebraic Equations

9.5.1 Algebraic Equation

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_o = 0 \quad (a_i = \text{complex numbers}) \ \ n^{th} \text{degree equation}$$

9.5.2 Zeros and Roots

If $P(z) = (z - r)^m Q(z)$, r = *zero* of multiplicity m, also a *root* of multiplicity m. If r is a root of multiplicity $m(m \geq 1)$ of Eq. (9.5.1) $P(z) = 0$, then r is a root of multiplicity $m - 1$, of the equation $P'(z) = 0$.

9.5.3 Factor Formula

$P(z)$ contains the factor $z - r \Leftrightarrow P(r) = 0$. $P(z)$ contains the factor $(z - r)^m \Leftrightarrow P'(r) = P(r) = \cdots = P^{m-1}(r) = 0$, then r is a root of multiplicity m–1 of the equation $P'(z) = 0$

9.5.4 Fundamental Theorem of Algebra

Eq. (9.5.1) $P(z) = 0$ of degree n has n roots (including multiplicity). If the roots are $r_1,\cdots,r_n$ then $P(z) = a_n(z - r_1)\cdots(z - r_n)$.

9.5.5 Relationship Between Roots and Coefficients

If $r_1,\cdots,r_n$ are the roots of Eq. (9.5.1) then

$$\begin{cases} r_1 + r_2 + \cdots + r_n = -\dfrac{a_{n-1}}{a_n} \\ r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = \displaystyle\sum_{i<j} r_i r_j = \dfrac{a_{n-2}}{a_n} \\ \cdots \\ r_1 r_2 \cdots r_n = (-1)^n \dfrac{a_o}{a_n} \end{cases}$$

9.5.6 Equations with Real Coefficients

Assume that all a_i of Eq. (9.5.1) are real.

1. If r is a non-real root of Eq. (9.5.1), then so is $\bar{r}$ (conjugate of r), i.e., $P(r) = 0 \Rightarrow P(\bar{r}) = 0$.
2. $P(z)$ can be factorized into real polynomials of degree, at most two.
3. If all a_i are integers and if $r = p/q$ (p and q having no common divisor) is a rational root of Eq. (9.5.1), then p divides a_o and q divides a_n.
4. The number of positive real roots (including multiplicity) of Eq. (9.5.1) either equals the number of sign changes of the sequence $a_0,a_1,\cdots,a_n$ or equals this number minus an even number. If all roots of the equation are real, the first case always applies. (Descartes' rule of signs)

9.5.7 Quadratic Equations

$ax^2 + bx + c = 0 \qquad x^2 + px + q = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$b^2 - 4ac > 0 \Rightarrow$ two unequal real roots

$b^2 - 4ac < 0 \Rightarrow$ two unequal complex roots $(\pm\sqrt{-d} = \pm j\sqrt{d})$

$b^2 - 4ac = 0 \Rightarrow$ the roots are real and equal

The expression $b^2 - 4ac$ is called the *discriminant*.

Let x_1 and x_2 be roots of the equation $x^2 + px + q = 0$. Then

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}$$

9.5.8 Cubic Equations

The equation $az^3 + bz^2 + cz + d = 0$ is by the substitution $z = x - b/3a$ reduced to

$$x^3 + px + q = 0.$$

Set
$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2.$$

Then the cubic equation has (1) one real root if $D > 0$, (2) three real roots of which at least two are equal if $D = 0$, and (3) three distinct real roots if $D < 0$. Put

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{D}}, \qquad v = \sqrt[3]{-\frac{q}{2} - \sqrt{D}}.$$

The roots of the cubic equation are

$$x_1 = u + v \qquad x_{2,3} = -\frac{u+v}{2} \pm \frac{u-v}{2} j\sqrt{3} \qquad (\textit{Cardano's formula})$$

If x_1, x_2, x_3 are roots of the equation $x^3 + rx^2 + sx + t = 0$ then

$$\begin{cases} x_1 + x_2 + x_3 = -r \\ x_1x_2 + x_1x_3 + x_2x_3 = s \\ x_1x_2x_3 = -t \end{cases}$$

9.5.9 Binomic Equations

A *binomic equation* is of the form

$$z^n = c, \quad c = \text{complex number}$$

1. Special case $n = 2$: $z^2 = a + jb.$
   Roots:

$$z = \pm\sqrt{a + jb} = \begin{cases} \pm\left[\sqrt{\dfrac{r+a}{2}} + j\sqrt{\dfrac{r-a}{2}}\right], & b \geq 0 \\ \pm\left[\sqrt{\dfrac{r+a}{2}} - j\sqrt{\dfrac{r-a}{2}}\right], & b \leq 0 \end{cases}, \qquad r = \sqrt{a^2 + b^2}$$

2. General case: Solution in *polar* form: Set $c = re^{i\theta}$

$$z^n = c = re^{i(\theta + 2k\pi)}$$

   Roots:

$$z = \sqrt[n]{r}\, e^{j(\theta+2k\pi)/n} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right),$$

$$k = 0, 1, \cdots, n-1$$

9.6 Differentiation

9.6.1 $\frac{d}{dx}(cu) = c\frac{du}{dx}$, c constant

9.6.2 $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

9.6.3 $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

9.6.4 $\frac{d}{dx}(u/v) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

9.6.5 $\frac{d}{dx}u(v) = \frac{du}{dv}\,\frac{dv}{dx}$

9.6.6 $\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)}$, where parentheses in exponents mean number of differentiations.

9.6.7 $\frac{d^n}{dx^n}(u^n v^m) = u^{n-1}v^{m-1}\left(nv\frac{du}{dx} + mu\frac{dv}{dx}\right)$

9.6.8 $\frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}}\left(nv\frac{du}{dx} - mu\frac{dv}{dx}\right)$

9.6.9 $f(x) = u(x)^a\,v(x)^b\,w(x)^c$

$$\frac{\frac{df(x)}{dx}}{f(x)} = a\frac{\frac{du(x)}{dx}}{u(x)} + b\frac{\frac{dv(x)}{dx}}{v(x)} + c\frac{\frac{dw(x)}{dx}}{w(x)}$$

9.6.10 $\frac{d^2}{dx^2}u(v(x)) = \frac{d^2u}{dv^2}\left(\frac{du}{dx}\right)^2 + \frac{du}{dv}\,\frac{d^2v(x)}{dx^2}$, $\qquad \frac{d^2 f}{dx^2} = \frac{d^2 f}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{df}{dy}\cdot\frac{d^2y}{dx^2}$

9.6.11 $\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} + \int_{u(x)}^{v(x)}\frac{\partial}{\partial x}f(x,t)\,dt$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad \frac{d}{dx}\int_x^a f(x)\,dt = -f(x)$$

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

9.7 Functions

9.7.1 Definitions

$f(x) = f(-x) \equiv$ even; $\qquad f(x) = -f(-x) \equiv$ odd; $\qquad f(x) = f(x+T) \equiv$ periodic

$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \quad [f(x_1) \ge f(x_2)] \equiv$ increasing [decreasing]

Convex (concave): if for any two points the chord lies above (below) the curve.
Inflection point: the point at which the curve changes from convex to concave (or vice versa)
Local maximum (minimum): A function has a local maximum (minimum) at $x = a$ if there is a neighborhood U such that $f(x) \le f(a)$ $[f(x) \ge f(a)]$ for all $x \in U \cap D_f$ (domain of the function).

Strictly increasing: $\frac{df(x)}{dx} > 0$

Increasing: $\frac{df(x)}{dx} \ge 0$; **Constant:** $\frac{df(x)}{dx} = 0$

Decreasing: $\frac{df(x)}{dx} \le 0$; **Strictly Decreasing:** $\frac{df(x)}{dx} < 0$

Stationary (critical) point: $\left.\frac{df(x)}{dx}\right|_{x=x_o} = 0$

Convex: $\frac{d^2 f(x)}{dx^2} \ge 0$; **Concave:** $\frac{d^2 f(x)}{dx^2} \le 0$

Inflection point: $\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_o} = 0$ and $\frac{d^2 f(x)}{dx^2}$ changes sign at x_o

Jensen's inequality: If $f(x)$ is convex and $a_1 + a_2 + \dots + a_n = 1$, $a_i > 0$, then $f(a_1x_1 + \dots + a_nx_n) \le a_1 f(x_1) + \dots + a_nf(x_n)$.

Continuous at x_o: $\lim_{x\to x_o} f(x) = f(x_o)$

Uniformly continuous in *I*: If for any $\varepsilon > 0$ there exists a $\delta > 0$ for all $|f(x_1) - f(x_2)| < \varepsilon$ for all $x_1, x_2 \in I$ such that $|x_1 - x_2| < \delta$.

9.8 Limits, Maxima and Minima

9.8.1 Limits

1. $\lim_{x\to a}(f(x) \pm g(x)) = f(a) \pm g(a)$
2. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}, \quad g(a) \ne 0$
3. $\lim_{x\to a} h(f(x)) = h(f(a))$ $(h(t) = \text{continuous})$
4. $\lim_{x\to a} f(x)g(x) = f(a)g(a)$
5. $\lim_{x\to a} f(x)^{g(x)} = f(a)^{g(a)} \quad (f(a) > 0)$
6. $f(x) \le g(x) \Rightarrow f(a) \le g(a)$

9.8.2 l'Hospital's Rules

1. $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}}$, if the latter limit exists.

2. $\lim_{x\to \infty}\frac{f(x)}{g(x)} = \lim_{x\to \infty}\frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}}$, if the latter limit exists.

9.8.3 Not Well-Defined Forms

$$\frac{0}{0}; \quad \frac{\infty}{\infty}; \quad 0\cdot\infty; \quad [0^+]^0; \quad \infty^0; \quad 1^\infty; \quad \infty-\infty$$

9.8.4 Limits

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e; \quad \lim_{x\to\infty} x^{1/x} = 1; \quad \lim_{m\to\infty}\frac{a^m}{m!} = 0, \quad \lim_{x\to 0}\frac{\sin ax}{x} = a$$

$$\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1; \quad \lim_{x\to\infty}\frac{\ell nx}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/2\sqrt{x}} = 0$$

9.8.5 Function of Two Variables

The function $f(x,y)$ has a maximum or minimum for those values of (x_o,y_o) for which

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0\,,$$

and for which $\begin{vmatrix} \partial^2 f/\partial x\partial y & \partial^2 f/\partial x^2 \\ \partial^2 f/\partial y^2 & \partial^2 f/\partial x\partial y \end{vmatrix} < 0$

(a) $f(x,y)$ has a maximum

$$\text{if } \frac{\partial^2 f}{\partial x^2} < 0 \text{ and } \frac{\partial^2 f}{\partial y^2} < 0 \text{ at } (x_o, y_o)$$

(b) $f(x,y)$ has a minimum

$$\text{if } \frac{\partial^2 f}{\partial x^2} > 0 \text{ and } \frac{\partial^2 f}{\partial y^2} > 0 \text{ at } (x_o, y_o)$$

9.9 Integrals

9.9.1 Primitive Function

$F(x)$ if a p*rimitive* function of $f(x)$ on an interval I if $dF(x)/dx = f(x)$ for all $x \in I$.

$$F(x) = \int f(x)\,dx$$

9.9.2 Integration Properties

Linearity: $$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

Integration by Parts: $$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

Substitution:

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt, \quad [t = g(x)]$$

$$\int f(g(x))g'(x)\,dx = F(g(x))$$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$$

$f(x)$ odd $\Rightarrow F(x)$ even

$f(x)$ even $\Rightarrow F(x)$ odd (if $F(0) = 0$)

9.9.3 Useful Integrals

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} \quad (a \neq -1) \qquad \int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x \qquad \int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = -\sin x \qquad \int \frac{dx}{\sin^2 x} = -\cot x$$

$$\int \frac{dx}{\cos^2 x} = \tan x \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} \quad (a > 0) \qquad \int \frac{dx}{\sqrt{a + x^2}} = \ln\left|x + \sqrt{x^2 + a}\right|$$

$$\int \sinh x\,dx = \cosh x \qquad \int \cosh x\,dx = \sinh x$$

9.9.4 Integrals of Rational Algebraic Functions (constants of integration are omitted)

1. $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} \qquad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|$

The following formulas are useful for evaluating

$$\int \frac{P(x)\,dx}{(ax^2 + bx + x)^n}$$

where $P(x)$ is a polynomial and $n > 1$ is an integer.

3. $$\int \frac{dx}{(ax^2+bx+c)} = \frac{2}{(4ac-b^2)^{1/2}} \arctan \frac{2ax+b}{(4ac-b^2)^{1/2}}, \qquad (b^2-4ac<0)$$

4. $$= \frac{1}{(b^2-4ac)^{1/2}} \ln \left| \frac{2ax+b-(b^2-4ac)^{1/2}}{2ax+b+(b^2-4ac)^{1/2}} \right|, \qquad (b^2-4ac>0)$$

5. $$= \frac{-2}{2ax+b}, \qquad (b^2-4ac=0)$$

6. $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a} \ln\left|ax^2+bx+c\right| - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}$$

7. $$\int \frac{dx}{(a+bx)(c+dx)} = \frac{1}{ad-bc} \ln \left| \frac{c+dx}{a+bx} \right| \qquad (ad \neq bc)$$

8. $$\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab} \arctan \frac{bx}{a}$$

9. $$\int \frac{x\,dx}{a^2+b^2x^2} = \frac{1}{2b^2} \ln\left|a^2+b^2x^2\right|$$

10. $$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab} \ln \left| \frac{a+bx}{a-bx} \right|$$

11. $$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3} \arctan \frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}$$

12. $$\int \frac{dx}{(x^2-a^2)^2} = \frac{-x}{2a^2(x^2-a^2)} + \frac{1}{4a^3} \ln \left| \frac{a+x}{a-x} \right|$$

9.9.5 Integrals of Irrational Algebraic Functions

1. $$\int \frac{dx}{[(a+bx)(c+dx)]^{1/2}} = \frac{2}{(-bd)^{1/2}} \arctan \left[\frac{-d(a+bx)}{b(a+dx)} \right]^{1/2} \qquad (bd<0)$$

2. $$= \frac{-1}{(-bd)^{1/2}} \arcsin \left(\frac{2bdx+ad+bc}{bc-ad} \right) \qquad (b>0,\ d<0)$$

3. $$= \frac{2}{(bd)^{1/2}} \ln\left|[bd(a+bx)]^{1/2} + b(c+dx)^{1/2}\right| \qquad (bd>0)$$

4. $$\int \frac{dx}{(a+bx)^{1/2}(c+dx)} = \frac{2}{[d(bc-ad)]^{1/2}} \arctan \left[\frac{d(a+bx)}{(bc-ad)} \right]^{1/2} \qquad (d(ad-bc)<0)$$

5. $$= \frac{2}{[d(ad-bc)]^{1/2}} \ln \left| \frac{d(a+bx)^{1/2} - [d(ad-bc)]^{1/2}}{d(a+bx)^{1/2} + [d(ad-bc)]^{1/2}} \right| \qquad (d(ad-bc)>0)$$

6. $$\int [(a+bx)(c+dx)]^{1/2}$$

$$= \frac{(ad-bc)+2b(c+dx)}{4bd}[(a+bx)(c+dx)]^{1/2} - \frac{(ad-bc)^2}{8bd}\int \frac{dx}{[(a+bx)(c+dx)]^{1/2}}$$

7. $$\int \left[\frac{c+dx}{a+bx}\right]^{1/2} dx = \frac{1}{b}[(a+bx)(c+dx)]^{1/2} - \frac{(ad-bc)}{2b}\int \frac{dx}{[(a+bx)(c+dx)]^{1/2}}$$

8. $$\int \frac{dx}{(ax^2+bx+c)^{1/2}} = a^{-1/2}\ln\left|2a^{1/2}(ax^2+bx+c)^{1/2}+2ax+b\right| \qquad (a>0)$$

9. $$= a^{-1/2}\operatorname{arcsin h}\frac{(2ax+b)}{(4ac-b^2)^{1/2}} \qquad (a>0, \quad 4ac>b^2)$$

10. $$= a^{-1/2}\ln|2ax+b| \qquad (a>0, \quad b^2=4ac)$$

11. $$= -(-a)^{-1/2}\arcsin\frac{(2ax+b)}{(b^2-4ac)^{1/2}}$$

$$(a<0, \quad b^2>4ac, \quad |2ax+b|<(b^2-4ac)^{1/2})$$

12. $$\int (ax^2+bx+c)^{1/2}\,dx = \frac{2ax+b}{4a}(ax^2+bx+c)^{1/2} + \frac{4ac-b^2}{8a}\int \frac{dx}{(ax^2+bx+c)^{1/2}}$$

13. $$\int \frac{dx}{x(ax^2+bx+c)^{1/2}} = -\int \frac{dt}{(a+bt+ct^2)^{1/2}} \quad \text{where } t=1/x$$

14. $$\int \frac{x\,dx}{(ax^2+bx+c)^{1/2}} = \frac{1}{a}(ax^2+bx+c)^{1/2} - \frac{b}{2a}\int \frac{dx}{(ax^2+bx+c)^{1/2}}$$

15. $$\int \frac{dx}{(x^2\pm a^2)^{1/2}} = \ln\left|x+(x^2\pm a^2)^{1/2}\right|$$

16. $$\int (x^2\pm a^2)^{1/2}\,dx = \frac{x}{2}(x^2\pm a^2)^{1/2} \pm \frac{a^2}{2}\ln\left|x+(x^2\pm a^2)^{1/2}\right|$$

17. $$\int \frac{dx}{x(x^2+a^2)^{1/2}} = -\frac{1}{a}\ln\left|\frac{a+(x^2+a^2)^{1/2}}{x}\right|$$

18. $$\int \frac{dx}{x(x^2-a^2)^{1/2}} = \frac{1}{a}\arccos\frac{a}{x}$$

19. $$\int \frac{dx}{(a^2-x^2)^{1/2}} = \arcsin\frac{x}{a}$$

20. $$\int (a^2-x^2)^{1/2}\,dx = \frac{x}{2}(a^2-x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

21. $$\int \frac{dx}{x(a^2-x^2)^{1/2}} = -\frac{1}{a}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|$$

22. $\int \frac{dx}{(2ax - x^2)^{1/2}} = \arcsin \frac{x-a}{a}$

23. $\int (2ax - x^2)^{1/2}\, dx = \frac{x-a}{a}(2ax - x^2)^{1/2} + \frac{a^2}{2} \arcsin \frac{x-a}{a}$

24. $\int \frac{dx}{(ax^2 + b)(cx^2 + d)^{1/2}} = \frac{1}{[b(ad - bc)]^{1/2}} \arctan \frac{x(ad - bc)^{1/2}}{[b(cx^2 + d)]^{1/2}}$ $(ad > bc)$

25. $= \frac{1}{2[b(bc - ad)]^{1/2}} \ln \left| \frac{[b(cx^2 + d)]^{1/2} + x(bc - ad)^{1/2}}{[b(cx^2 + d)]^{1/2} - x(bc - ad)^{1/2}} \right|$ $(bc > ad)$

9.9.6 Exponential, Logarithmic, and Trigonometric Functions

1. $\int R(e^{ax})\, dx$ Substitution: $e^{ax} = t$, $x = \frac{1}{a}\ln t$, $dx = \frac{dt}{at}$

2. $\int P(x) e^{ax}\, dx = [\text{integration by parts}] = \frac{1}{a} P(x) e^{ax} - \frac{1}{a}\int P'(x) e^{ax}\, dx$, etc.
($P(x)$ polynomial)

3. $\int x^a (\ln x)^n\, dx = \begin{cases} [\text{integration by parts}] = \frac{x^a + 1}{a+1}(\ln x)^n - \\ \frac{n}{a+1}\int x^a (\ln x)^{n-1} dx, \text{ etc.} (a \neq -1) \\ \frac{(\ln x)^{n+1}}{n+1} \quad (a = -1) \end{cases}$

or $t = \ln x$ and use #2

4. $\int \frac{1}{x} f(\ln x)\, dx$ Substitution: $\ln x = t$, $\frac{dx}{x} = dt$

5. $\int f(\sin x) \cos x\, dx$ Substitution: $\sin x = t$, $\cos x\, dx = dt$

6. $\int f(\cos x) \sin x\, dx$ Substitution: $\cos x = t$, $-\sin x\, dx = dt$

7. $\int f(\tan x)\, dx$ Substitution: $\tan x = t$, $dx = \frac{dt}{1 + t^2}$

8. $\int R(\cos x, \sin x)\, dx$ Substitution:

$\tan \frac{x}{2} = t$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

9. $\int \sin^n x\, dx$ $(n \geq 1)$ n odd: Use $\sin^2 x = 1 - \cos^2 x$ and #6

n even: Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ etc.

10. $\int \cos^n x\,dx \quad (n \geq 1)$ $\quad n$ odd : Use $\cos^2 x = 1 - \sin^2 x$ and #5

n even : Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ etc.

11. $\int P(x) \begin{Bmatrix} \cos x \\ \sin x \end{Bmatrix} dx$ = Integration by parts, differentiating the polynomial #2.

($P(x)$ polynomial)

12. $\int P(x) e^{ax} \cos bx\,dx = \mathrm{Re} \int P(x) e^{(a+jb)x}\,dx$. Use #2.

13. $\int P(x) e^{ax} \sin bx\,dx = \mathrm{Im} \int P(x) e^{(a+jb)x}\,dx$. Use #2.

14. $\int x^n \arctan x\,dx =$ [integration by parts]

$$= \frac{x^{n+1}}{n+1} \arctan x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2}\,dx$$

15. $\int x^n \arcsin x\,dx =$ [integration by parts]

$$= \frac{x^{n+1}}{n+1} \arcsin x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx$$

16. $\int f(\arcsin x)\,dx$ $\quad$ Substitution: $\arcsin x = t,\ x = \sin t$

17. $\int f(\arctan x)\,dx$ $\quad$ Substitution: $\arctan x = t,\ x = \tan t$

9.9.7 Definite Integrals

$$I = \lim_{\max|x_i - x_{i-1}| \to 0} \sum_{i=1}^{m} f(\xi_i)(x_i - x_{i-1}) = \int_a^b f(x)\,dx$$

is the *definite integral* of $f(x)$ over (a,b) in the sense of Riemann integration.

9.9.8 Mean Value Theorem

$$\int_a^b f(x)\,dx = f(\xi)(b-a), \qquad \int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$$

where 1) $f(x)$, $g(x)$ are continuous in $[a,b]$, and 2) $g(x)$ does not change sign.

9.9.9 Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

9.9.10 Cauchy Principal Value

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx\right)$$

9.9.11 Convergence Test

$$\int_a^b g(x) \text{ converges} \Rightarrow \int_a^b f(x)\,dx \text{ converges}; \qquad 0 \le f(x) \le g(x)$$

$$\int_a^b |f(x)| \text{ converges} \Rightarrow \int_a^b f(x)\,dx \text{ converges}$$

9.9.12 Stieltjes Integral

The *Riemann-Stieltjes integral* of $f(x)$ with respect to $g(x)$ over the bounded interval $[a,b]$ is defined as

$$\int_a^b f(x)\,dg(x) = \lim_{\max|x_i - x_{i-1}|} \sum_{i=1}^m f(\xi_i)[g(x_i) - g(x_{i-1})]$$

for an arbitrary sequence of partitions

$$a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \cdots < \xi_m < x_m = b$$

The limit exists whenever $g(x)$ is of bounded variation and $f(x)$ is continuous on $[a,b]$. (A real function $f(x)$ is of *bounded variation* in (a,b) if and only if there exists a real number M such that $\sum_{i=1}^m |f(x_i) - f(x_{i-1})| < M$ for all partitions $a = x_0 < x_1 < \cdots < x_m = b$. If $f(x)$ and $g(x)$ are of bounded variation so is $f(x) + g(x)$ and $f(x)g(x)$.)

9.9.13 Properties of Stieltjes Integrals

1. $\int_a^b fdg = -\int_b^a fdg \qquad \int_a^b fdg = \int_a^c fdg + \int_c^b fdg$

2. $\int_a^b (f_1 + f_2)\,dg = \int_a^b f_1dg + \int_a^b f_2dg$ and $\int_a^b fd(g_1 + g_2) = \int_a^b f\,dg_1 + \int_a^b fdg_2$

3. $\int_a^b (\alpha f)\,dg = \int_b^a fd(\alpha g) = \alpha \int_a^b fdg$

4. $\int_a^b f\,dg = fg\Big|_a^b - \int_a^b g\,df$

5. $\left|\int_a^b fdg\right| \le \int_b^a |f|dg, \quad g(x) = \text{nondecreasing}$

6. $$\int_a^b f dg \le \int_a^b F dg, \quad g(x) = \text{nondecreasing}, \quad f(x) \le F(x)$$

7. $$\int_a^b f(x)\, dg(x) = \int_a^b f(x) \frac{dg(x)}{dx} dx, \quad g(x) = \text{continuous}$$

9.10 Sequences and Series

9.10.1 Convergence

A sequence of real or complex numbers $s_0, s_1, s_2, \ldots,$ converges if and only if, for every positive real number ε, there exists a real integer N such that $m > N$, $n > N$ implies $|s_n - s_m| < \varepsilon$.

9.10.2 Test of Convergence

An infinite series, $a_0 + a_1 + \ldots,$ of real positive terms converges if there exists a real number N such $n > N$ implies one or more of the following:

1. $a_n \le M_n$ and/or $\frac{a_{n+1}}{a_n} \le \frac{m_{n+1}}{m_n}$ where $m_0 + m_1 + \ldots$ is a convergence comparison series with real positive terms.
2. At least one of the quantities

$$\frac{a_{n+1}}{a_n}, \quad \sqrt[n]{a_n}, \quad n\left(\frac{a_{n+1}}{a_n} - 1\right) + 2, \quad \left[n\left(\frac{a_{n+1}}{a_n} - 1\right) + 1\right] \ln n + 2 \quad \text{has an upper bound } A < 1.$$

3. $a_n \le f(n)$, where $f(x)$ is a real positive decreasing function whose (improper) integral $\int_{N+1}^{\infty} f(x)\, dx$ exists.
4. An infinite series, $a_0 + a_1 + \ldots,$ of real terms converges
   a) If successive terms are alternatively positive and negative (alternating series), decrease in absolute value, and $\lim_{n\to\infty} a_n = 0$.
   b) If the sequence $s_0, s_1, s_2, \ldots,$ of the partial sums is bounded and monotonic.
5. Given a decreasing sequence of real positive numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$ the infinite series $\alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2, \ldots$ converges
   a) If the series $a_0 + a_1 + a_2 + \ldots,$ converges (Abel's test)
   b) If $\lim_{n\to\infty} a_n = 0$ and $\sum_{k=0}^{n} a_k$ is bounded for all n. (Dirichlet's test)
6. If $\{a_n\}$ is decreasing and $a_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. (Leibniz's test)

Example

1. $$\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n \begin{cases} \text{conv. for } |x| < 2 \\ \text{div. for } |x| > 2 \end{cases} \quad \text{(Root test)}$$

2. $$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p} \quad \begin{cases} \text{conv. for } p > 1 \\ \text{div. for } p \le 1 \end{cases} \quad \text{(integral set)}$$

3. $\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$ conv.

comparison test : $1-\cos\frac{1}{n}=\frac{1}{2n^2}+O\left(\frac{1}{n^4}\right)$ and $\sum_{1}^{\infty}\frac{1}{n^2}$ conv.

4. $\sum_{1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ conv. (Leibniz' test)

5. $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}e^{jkx}$ conv. for $x\neq 2m\pi$. (Dirichlet's test)

9.11 Absolute and Relative Errors

If x_0 is an approximation to the true value of x, then

9.11.1 The *absolute error* of x_0 is $\Delta x = x_0 - x$, $x - x_0$ is the correction to x.

9.11.2 The *relative error* x_0 is $\delta x=\frac{\Delta x}{x}\approx\frac{\Delta x}{x_0}$

9.11.3 The *percentage error* is 100 times the relative error. The absolute error of the sum or difference of several numbers is at most equal to the sum of the absolute errors of the individual numbers.

9.11.4 If $f(x_1,x_2,\ldots,x_n)$ is a function of $x_1,x_2,\ldots,x_n$ and the absolute error in $x_i (i = 1,2,\ldots,n)$ is Δx_i, then the absolute error in f is

$$\Delta f\approx\frac{\partial f}{\partial x_1}\Delta x_1+\frac{\partial f}{\partial x_2}\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}\Delta x_n$$

9.11.5 The relative error of the product or quotient of several factors is at most equal to the sum of the relative errors of the individual factors.

9.11.6 If $y = f(x)$, the relative error $\delta y=\frac{\Delta y}{y}\approx\frac{f'(x)}{f(x)}\Delta x$

Approximate Values:

If $|\varepsilon|<<1,\ |\eta|<<1,\ b<<a,$

9.11.7 $(a+b)^k\approx a^k+ka^{k-1}b$

9.11.8 $(1+\varepsilon)(1+\eta)\approx 1+\varepsilon+\eta$

9.11.9 $\frac{1+\varepsilon}{1+\eta}\approx 1+\varepsilon-\eta$

9.12 Convergence of Infinite Series

9.12.1 A sequence $s_0(x)$, $s_1(x)$,… of real or complex function *converges uniformly* on a set S of values of x if and only if for every positive real number ε there exists a real number N independent of x such that $m > N$, $n > N$ implies $|s_n(x)-s_m(x)|<\varepsilon$ for x in S. (Cauchy's test)

9.12.2 An infinite series $a_0(x) + a_1(x) + a_2(x) + \ldots$ of real or complex functions converges uniformly and absolutely on every set S of values of x such that $|a_n(x)|\leq M_n$ for all n, where $M_0 + M_1 + M_2 + \ldots$ is a convergent comparison series of real positive terms. (Weierstrass' test)

9.12.3 Given a decreasing sequence of real positive terms $\alpha_0, \alpha_1, \alpha_2, \ldots$, the infinite series $\alpha_0 a_0(x) + \alpha_1 a_1(x) + \alpha_2 a_2(x) + \ldots$ converges uniformly on a set S of values x

1. If the infinite series $a_0(x) + a_1(x) + a_2(x) + \ldots$ converges uniformly on S (Abel's test)

2. If $\lim_{n\to\infty} \alpha_n = 0$ and there exists a real number $A \geq \left|\sum_{k=0}^{n} a_k(x)\right|$ for all n and all x in S. (Dirichlet's test)

9.12.4 Assume (i) $f_n(x) \to f(x)$ pointwise on $[a,b]$

(ii) $|f_n(x)| < M$, all n and $x \in [a,b]$

(iii) $f_n(x), f(x)$ integrable

Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx \quad \text{(Arzela's theorem)}$$

9.12.5 Assume (i) $\{f_n(x)\}_1^\infty$ increasing, i.e., $f_n(x) \leq f_{n+1}(x)$, all n, x (or decreasing)

(ii) $f_n(x) \to f(x)$ pointwise on $[a,b]$

(iii) $f_n(x), f(x)$ continuous on $[a,b]$

Then the convergence is uniform. (Dini's theorem)

9.13 Series of Functions

9.13.1 Representation of Functions by Infinite Series and Integrals

A function $f(x)$ is often represented by a corresponding infinite series

$$\sum_{k=1}^{\infty} \alpha_k \varphi_k(x)$$

because

1. A sequence of partial sums may yield useful numerical approximations to $f(x)$.
2. It may be possible to describe operations on $f(x)$ in terms of simpler operations on the functions $\varphi_k(x)$ or on the coefficients α_k (transform methods). The functions $\varphi_k(x)$ and the coefficients α_k may have an intuitive (physical) meaning.

9.13.2 Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x + a_1 x + a_2 x^2 + \cdots, \quad a_n = \frac{1}{n!}\frac{d^n f(0)}{dx^n}$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad a_n = \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}$$

where x and a can be real or complex.

9.13.3 Taylor's Expansion

Given a real function $f(x)$ such that $\dfrac{d^n f(x)}{dx^n} \doteq f^{(n)}$ exists $\quad a \le x < b$

$$\left.\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \cdots \\ &\qquad + \frac{1}{(n-1)!} f^{n-1}(a)(x-a)^{n-1} + R_n(x) \\ &\text{with } |R_n(x)| \le \frac{|x-a|^n}{n!} \sup_{a<\xi<x} |f^{(n)}(\xi)| \end{aligned}\right\} \; (a \le x < b)$$

9.13.4 Taylor-Series Expansion

Given a function $f(x)$ such that all derivatives $f^{(k)}(x)$ exist and $\lim_{n\to\infty} R_n(x) = 0$ for $a \le x < b$,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k, \quad (a \le x < b)$$

and the series converges uniformly to $f(x)$ for $a \le x < b$. (Taylor-series expansion of $f(x)$ about $x = a$.)

9.13.5 Order Concepts

a) $f(x) = O(x^a)$ as $x \to 0$ means: $f(x) = x^a F(x)$, where $F(x)$ is bounded in a neighborhood of $x = 0$.
b) $f(x) = o(x^a)$ as $x \to 0$ means: $f(x)/x^a \to 0$ as $x \to 0$.

$$O(x^5) \pm O(x^5) = O(x^5); \qquad O(x^3) \pm O(x^4) = O(x^3);$$
$$x^2 O(x^3) = O(x^5) = x^5 O(1); \qquad O(x^2)\,O(x^3) = O(x^5)$$

c) $\displaystyle f(x) - \sum_{k=0}^{n-1} a_k x^{-k} = O(x^{-n})$ as $x \to \infty$

Example $\quad f(x) = e^{-4x^2}, \; a = 0, \; n = 6$

$$f(x) = [t = -4x^2] = e^t = 1 + t + \frac{t^2}{2} + O(t^3) = 1 - 4x^2 + 8x^4 + O(x^6)$$

Example $\quad f(x) = e^x \sin x, \; a = 0, \; n = 5$

$$f(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)\right)\left(x - \frac{x^3}{6} + O(x^5)\right) = x + x^2 + \frac{x^3}{3} + O(x^5)$$

Example $\quad f(x) = e^{(x-1)^2}, \; a = 0, \; n = 3$

$$f(x) = e^{x^2-2x+1} = ee^{x^2-2x} = (t = x^2 - 2x) = ee^t = e\left(1 + t + \frac{t^2}{2} + O(t^3)\right)$$

$$= e\left(1 - 2x + 3x^2\right) + O(x^3)$$

Example $\quad \dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{d}{dx}(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + \cdots$

9.13.6 Multiple Taylor Expansion

$$f(x,y) = f(0,0) + x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y} + \cdots + \frac{1}{n!}\left[x^n \frac{\partial^n f(0,0)}{\partial x^n} + \cdots\right]$$

$$+\frac{1}{(n+1)!}\left[x^{n+1}\frac{\partial^{n+1} f(0,0)}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f(0,0)}{\partial x^n \partial y} + \cdots\right]$$

$$f(x,y) = f(0,0) + x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y} + \frac{1}{2!}\left[x^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2xy\frac{\partial^2 f(0,0)}{\partial x \partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right]$$

$$+\frac{1}{3!}\left[x^3\frac{\partial^3 f(0,0)}{\partial x^3} + 3x^2 y\frac{\partial^3 f(0,0)}{\partial x^2 \partial y} + 3xy^2\frac{\partial^2 f(0,0)}{\partial x \partial y^2} + y^3\frac{\partial^3 f(0,0)}{\partial y^3}\right] + \cdots$$

9.14 Sums and Series

9.14.1 $$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k \qquad (-1 < x < 1)$$

9.14.2 $$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad (-1 < x < 1)$$

9.14.3 $$\sum_{k=1}^{\infty}\frac{x^k}{k} = -\ln(1-x) \qquad (-1 \le x < 1)$$

9.14.4 $$\sum_{k=1}^{n} e^{kx} = e^x \cdot \frac{e^{nx}-1}{e^x-1} = \frac{\sinh\frac{nx}{2}}{\sinh\frac{x}{2}}\, e^{(n+1)x/2} \qquad (x \ne 0)$$

9.14.5 $$\sum_{k=0}^{\infty} e^{-kx} = \frac{1}{1-e^{-x}} \qquad (x > 0)$$

9.14.6 $$\sum_{k=1}^{n} e^{jkx} = e^{jx}\cdot\frac{1-e^{jnx}}{1-e^{jx}} = \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\cdot e^{j(n+1)x/2} \qquad (x \ne 2m\pi)$$

9.14.7 $$\sum_{k=1}^{n}\sin kx = \mathrm{Im}\sum_{k=0}^{n} e^{ikx} = \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \qquad (\text{cf. } 9.14.6)$$

9.14.8 $$\sum_{k=0}^{n}\cos kx = \mathrm{Re}\sum_{k=0}^{n} e^{jkx} = \frac{\cos\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \qquad (\text{cf. } 9.14.6)$$

9.14.9 $$\sum_{k=1}^{n-1} r^k \sin kx = \operatorname{Im}\sum_{k=0}^{n-1} (re^{jx})^k = \frac{r\sin x(1-r^n\cos nx)-(1-r\cos x)r^n \sin nx}{1-2r\cos x + r^2}$$

9.14.10 $$\sum_{k=0}^{n-1} r^k \cos kx = \operatorname{Re}\sum_{k=0}^{n-1} (re^{jx})^k = \frac{(1-r\cos x)(1-r^n\cos nx)+r^{n+1}\sin x \sin nx}{1-2r\cos x + r^2}$$

9.14.11 $$\sum_{k=1}^{n-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}$$

9.14.12 $$e = \sum_{k=0}^{\infty}\frac{1}{k!} = 2.7182818284\cdots \qquad \text{(transcendental)}$$

9.14.13 $$\pi = 4\arctan 1 = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 3.1415926535\cdots \qquad \text{(transcendental)}$$

9.14.14 $$\ln 2 = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = \sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k} = 0.69315\cdots \qquad \text{(transcendental)}$$

9.14.15 $$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577215665\cdots \qquad \text{(Euler's constant, irrational)}$$

9.14.16 $$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \qquad (-1<x<1)$$

9.14.17 $$\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x,\quad 0! = 1,\quad R_n(x) = \frac{e^{\theta x}}{n!}x^n \qquad (0<\theta<1) \qquad -\infty<x<\infty$$

9.14.18 $$\sum_{k=0}^{\infty}\frac{(x\ln a)^k}{k!} = a^x \qquad -\infty<x<\infty$$

9.14.19 $$\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \sinh x \qquad -\infty<x<\infty$$

9.14.20 $$\sum_{k=0}^{\infty}\frac{x^{2n}}{(2n)!} = \cosh x \qquad -\infty<x<\infty$$

9.14.21 $$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} = \ln(1+x) \qquad -1<x<1$$

9.14.22 $$\ln a + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{x}{a}\right)^k = \ln(a+x) \qquad -a<x<a$$

9.14.23 $$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{1+x}\right)^k = \ln(1+x)$$

9.14.24 $$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x, \quad R_{2k+1}(x) = (-1)^k \frac{\cos\theta x}{(2n+1)!} x^{2k+1}, \qquad 0<\theta<1, \qquad -\infty < x < \infty$$

9.14.25 $$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x, \quad R_{2k}(x) = (-1)^k \frac{\cos\theta x}{(2k)!} x^{2k}, \qquad 0<\theta<1, \qquad -\infty < x < \infty$$

9.14.26 $$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$$

9.14.27 $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

9.14.28 $$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

9.14.29 $$\sum_{k=0}^{n} k^2\binom{n}{k} = (n^2+n)2^{n-2}$$

9.14.30 $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

9.14.31 $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

9.14.32 $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

9.15 Lagrange's Expansion

9.15.1 If $y = f(x)$, $y_0 = f(x_0)$, $f'(x_0) \neq 0$, then

$$x = x_0 + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!}\left[\frac{d^{k-1}}{dx^{k-1}}\left\{\frac{x-x_0}{f(x)-y_0}\right\}^k\right]_{x=x_0}$$

9.15.2 $$g(x) = g(x_0) + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!}\left[\frac{d^{k-1}}{dx^{k-1}}\left(g'(x)\left\{\frac{x-x_0}{f(x)-y_0}\right\}^k\right)\right]_{x=x_0}$$

where $g(x)$ is any function indefinitely differentiable.

9.16 Orthogonal Polynomial

9.16.1 Gram-Schmidt Orthogonalization Process

If the system $\{f_i\}$is linear independent, we can replace by an equivalent orthonormal system ($\|f\|$ = norm = $\langle f, f\rangle^{1/2}$ = inner product)

$$\varphi_1 = \frac{f_1}{\|f_1\|};\ \varphi_2 = \frac{f_2 - \langle f_2, \varphi_1\rangle\varphi_1}{\|f_2 - \langle f_2, \varphi_1\rangle\varphi_1\|};\ \varphi_k = \frac{f_k - \sum_{i=1}^{k-1}\langle f_k, \varphi_i\rangle\varphi_i}{\left\|f_k - \sum_{i=1}^{k-1}\langle f_k, \varphi_i\rangle\varphi_i\right\|}, \qquad k = 1, 2, \cdots, n$$

9.16.2 Gram-Schmidt Orthogonalization Matrix Form Process

$$d_k = \begin{vmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & \cdots & \langle f_1, f_k\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \cdots & \langle f_2, f_k\rangle \\ \vdots & & & \\ \langle f_{k-1}, f_1\rangle & \langle f_{k-1}, f_2\rangle & \cdots & \langle f_{k-1}, f_k\rangle \\ f_1 & f_2 & \cdots & f_k \end{vmatrix}, \qquad \varphi_k = \frac{d_k}{\langle d_k, d_k\rangle^{1/2}}$$

Example If $f_1(x) = 1$ and $f_2(x) = x$ and $f_3(x) = x^2$ in the range (–1,1), we obtain

$$\varphi_0(x) = \frac{1}{\left[\int_{-1}^{1} dx\right]^{1/2}} = \frac{1}{\sqrt{2}}; \qquad d_1(x) = \begin{vmatrix} \langle 1,1\rangle & \langle 1,x\rangle \\ 1 & x \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & x \end{vmatrix} = 2x$$

$$\varphi_1(x) = \frac{2x}{\|2x\|} = \frac{2x}{\left[\int_{-1}^{1} 4x^2 dx\right]^{1/2}} = \frac{x}{[2/3]^{1/2}} = \sqrt{\frac{3}{2}}\,x;$$

$$d_2(x) = \begin{vmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ 1 & x & x^2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 1 & x & x^2 \end{vmatrix} = \frac{4}{3}x^2 - \frac{4}{3}; \qquad \varphi_2(x) = \frac{\frac{4}{3}x^2 - \frac{4}{9}}{\left\|\frac{4}{3}x^2 - \frac{4}{9}\right\|} = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$$

9.16.3 Recurrence Formula

If $\varphi_n(x) = k_n x^n + k_{n-1}x^{n-1} + \ldots$, then $\{\varphi_n\}$ satisfy

$$\varphi_{n+1} - (A_n x + B_n)\varphi_n + C_n\varphi_{n-1} = 0 \qquad n = 0, 1, 2, \cdots$$

$$A_n = \frac{k_{n+1}}{k_n}, \quad C_n = \frac{A_n}{A_{n-1}}, \quad C_o = 0$$

9.16.4 Christoffel-Darboux Formula

If $K_n(x,y) = \sum_{k=0}^{n} \varphi_k(x)\varphi_k(y)$ then

$$K_n(x,y)=\sum_{k=0}^{n}\varphi_k(x)\varphi_k(y)=\frac{k_n}{k_{n+1}}\left[\frac{\varphi_n(y)\varphi_{n+1}(x)-\varphi_n(x)\varphi_{n+1}(y)}{x-y}\right]$$

In the limit as $y \to x$

$$K_n(x,y)=\sum_{k=0}^{n}\varphi_k^2(x)\geq 0$$

9.16.5 Weierstrass Approximation Theorem

If $f(x)$ is a continuous function defined on the closed and bounded interval $[a,b]$, then it is possible to find a polynomial $p(x)$ such that $|f(x)-p(x)|<\varepsilon$ for all $x \in [a,b]$, $\varepsilon > 0$.

9.16.6 Zeros of the Orthogonal Polynomials

The polynomial of $\varphi_n(x)$ has n real, simple zeros, all in the interval (a,b).

9.16.7 Zeros of $\varphi_n(x)$ and $\varphi_{n+1}(x)$

The zeros of $\varphi_n(x)$ and $\varphi_{n+1}(x)$ alternate on the interval (a,b), and $\varphi_n(x)$ and $\varphi_{n+1}(x)$ do not vanish simultaneously.

9.16.8 Minimization of a Function

Among all polynomials of degree n, there is precisely one for which $\|f(x)-p_n(x)\|$ is minimized. It is given by

$$p_n(x)=\sum_{k=0}^{n}\alpha_k\varphi_k(x), \qquad \alpha_k=\langle f,\varphi_k\rangle=\int_a^b w(x)\,f(x)\varphi_k(x)\,dx$$

$$\left(\int_a^b w(x)\,f^2(x)\,dx<\infty, \qquad w(x)=\text{weighting function}\right)$$

9.16.9 Zeros of $\varphi_n(x)$ and $\varphi_m(x)$

If $m > n$, then between any two zeros of $\varphi_n(x)$, $\varphi_m(x)$ has to vanish at least once.

9.16.10 Zeros in an Interval

Let (α,β) be a subinterval of the finite and bounded interval (a,b). Then, for sufficiently large n, $\varphi_n(x)$ vanishes at least once in (α,β).

9.17 Completeness of Orthonormal Polynomials

9.17.1 Hilbert Space

The space $L_2(w)$ consisting of real function $f(x)$ for which $\int_a^b w(x)\,f^2(x)\,dx<\infty$ holds, with the inner product

$$\langle f,g\rangle = \int_a^b w(x)\, f(x) g(x)\, dx$$

and the norm $\|f\| = \int_a^b [w(x)\, f^2(x)\, dx]^{1/2} = \langle f, f\rangle^{1/2}$ is a Hilbert space.

9.17.2 Completeness

An orthonormal set $\{\varphi_n\}$ is complete in $L_2(w)$ if for every $f(x)$ in $L_2(w)$ we have $\|f\|^2 = \sum_{n=0}^{\infty} \alpha_k^2$, $\alpha_k = \langle f, \varphi_k\rangle$. (*Parseval's* inequality)
The set of orthonormal polynomials $\{\varphi_n\}$ is complete in the space $L_2(w)$.

9.17.3 Closed Space

The set $\{\varphi_n\}$ in a Hilbert space is closed if from $\langle f, \varphi_k\rangle = 0,\ n \geq 0,$ we conclude that $f = 0$.

9.17.4 Countably Infinite

An orthonormal set $\{\varphi_n\}$ in $L_2(w)$ is countably infinite.

Poularikas A. D. “Signals and Their Characterization”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC,1999

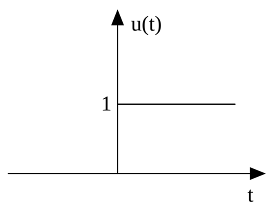
10

Signals and Their Characterization

10.1 Common One-Dimensional Continuous Signals

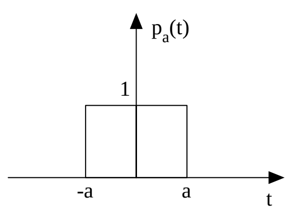
10.1.1 Step Function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



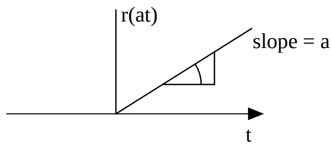
10.1.2 Pulse Function

$$p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$$



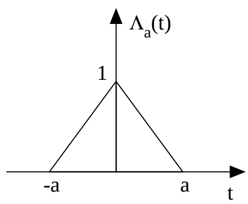
10.1.3 Ramp Function

$$r(at) \doteq ar(t) = \begin{cases} at & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



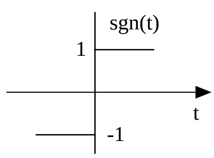
10.1.4 Triangular Function

$$\wedge_a(t) = \begin{cases} 1 - \dfrac{|t|}{a} & |t| \leq a \\ 0 & \text{otherwise} \end{cases}$$



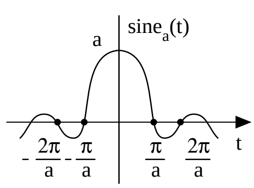
10.1.5 Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



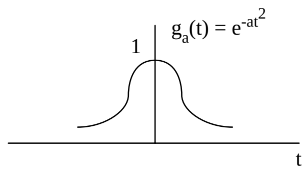
10.1.6 Sinc Function

$$\operatorname{sinc}_a(t) = \frac{\sin at}{t} \qquad -\infty < t < \infty$$



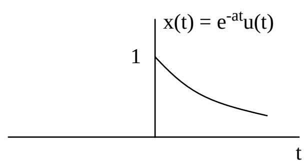
10.1.7 Gaussian Function

$$g_a(t) = e^{-at^2} \qquad -\infty < t < \infty$$



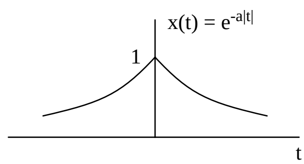
10.1.8 Exponential Function

$$x(t) = e^{-at}u(t) \qquad 0 \le t < \infty$$



10.1.9 Double Exponential Function

$$x(t) = e^{-a|t|} \qquad -\infty < t < \infty$$

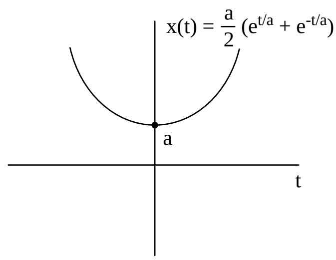


10.1.10 Exponentially Decaying Cosine Function

$$x(t) = \begin{cases} e^{-at}\cos\omega_o t\, u(t) & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

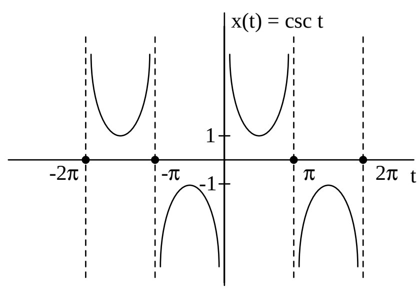
10.1.11 Hyperbolic Cosine Function

$$x(t) = \frac{a}{2}(e^{t/a} + e^{-t/a}) = a\cosh\frac{t}{a} \qquad -\infty < t < \infty$$



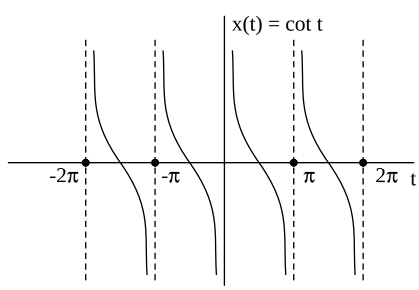
10.1.12 Cosecant Function

$$x(t) = \csc t = \frac{1}{\sin t} \qquad -\infty < t < \infty$$



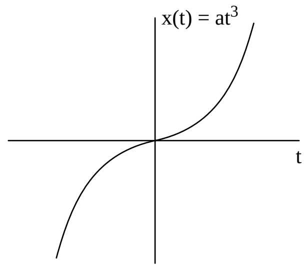
10.1.13 Cotangent Function

$$x(t) = \cot t = \frac{\cos t}{\sin t} \qquad -\infty < t < \infty$$



10.1.14 Cubical parabola

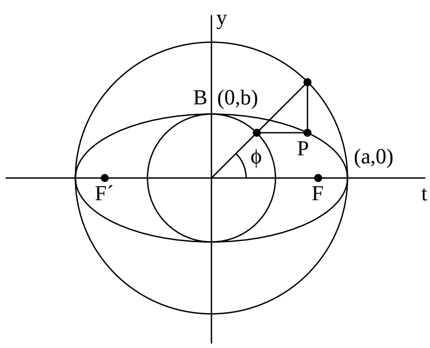
$$x(t) = at^3 \qquad a > \qquad -\infty < t < \infty$$



10.1.15 Ellipse

$$\frac{t^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad t = a\cos\phi, \quad y = b\sin\phi$$

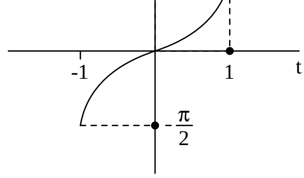
$$BF' = BF = a, \qquad PF' + PF = 2a$$



10.1.16 Arcsine Function

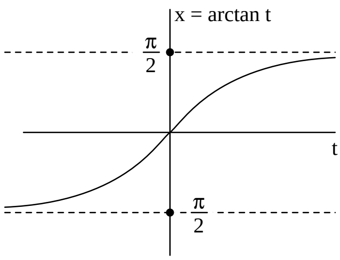
$$x = \arcsin t \qquad -1 \le t \le 1$$

$$\sin x = t$$



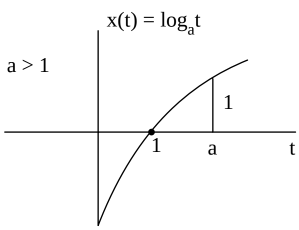
10.1.17 Arctangent Function

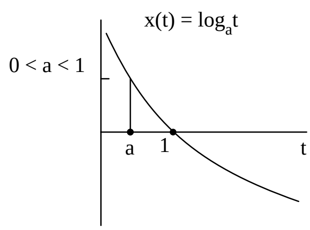
$$x = \arctan t \qquad -\infty < t < \infty$$



10.1.18 Logarithmic Function

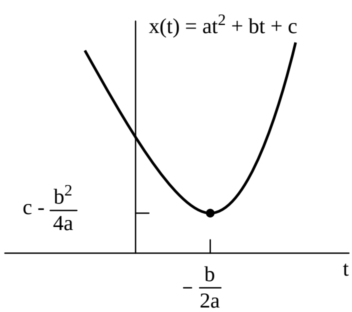
$$x(t) = \log_a t \qquad 0 < t < \infty$$





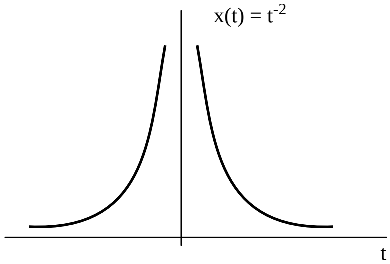
10.1.19 Parabola Function

$$x(t) = at^2 + bt + c \qquad a > 0$$



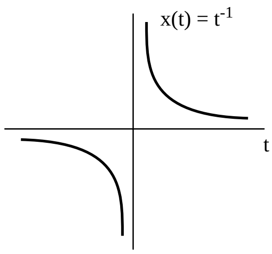
10.1.20 Power Function

$$x(t) = t^{-2} \qquad -\infty < t < \infty$$



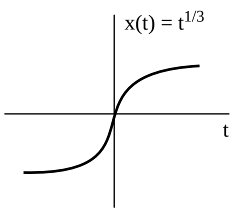
10.1.21 Equilateral Hyperbola

$$x(t) = t^{-1} \qquad -\infty < t < \infty$$



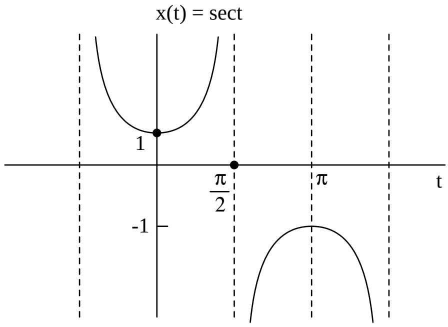
10.1.22 Cubical Parabola

$$x(t) = t^{1/3} \qquad -\infty < t < \infty$$



10.1.23 Secant Function

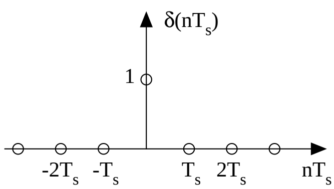
$$x(t) = \sec t = \frac{1}{\cos t} \qquad -\infty < t < \infty$$



10.2 One-Dimensional Discrete Signal

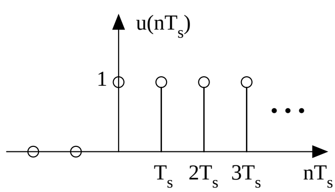
10.2.1 Unit Sample Sequence

$$\delta(nT_s) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$



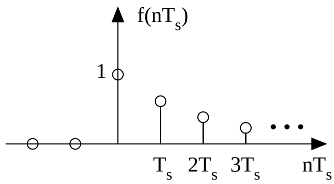
10.2.2 Unit Step Sequence

$$u(nT_s) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



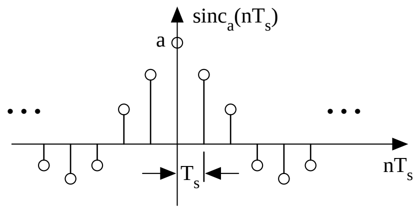
10.2.3 Real Exponential Sequence

$$f(nT_s) = \begin{cases} e^{-anT_s} & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad a = \text{positive constant}$$



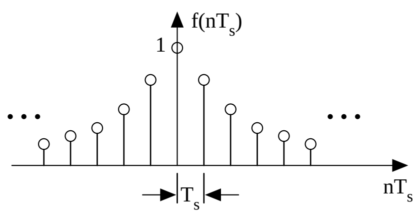
10.2.4 Sinc Function

$$\operatorname{sin}c_a(nT_s) = \frac{\sin anT_s}{nT_s} \qquad -\infty < n < \infty$$



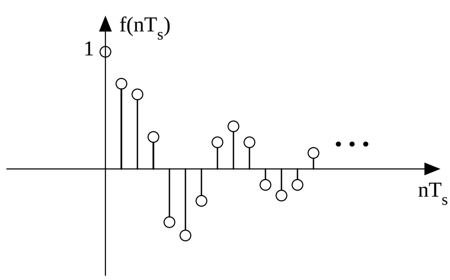
10.2.5 Double Exponential

$$f(nT_s) = e^{-a|nT_s|} \qquad -\infty < n < \infty$$



10.2.6 Exponentially Decaying Cosine Function

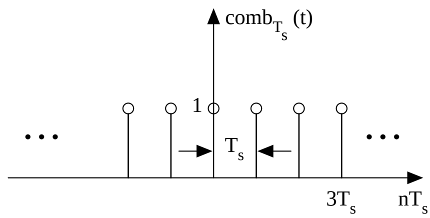
$$f(nT_s) = e^{-anT_s}\cos\omega_o nT_s\; u(nT_s) \quad \text{for} \;\; n \ge 0$$



and zero otherwise. a and ω_o are positive constants.

10.2.7 Comb Function

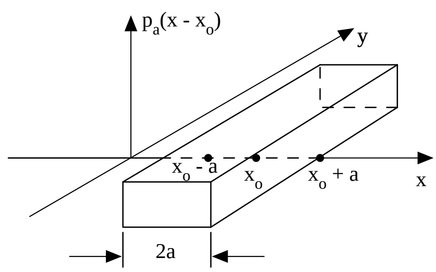
$$comb_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



10.3 Two-Dimensional Continuous Signal

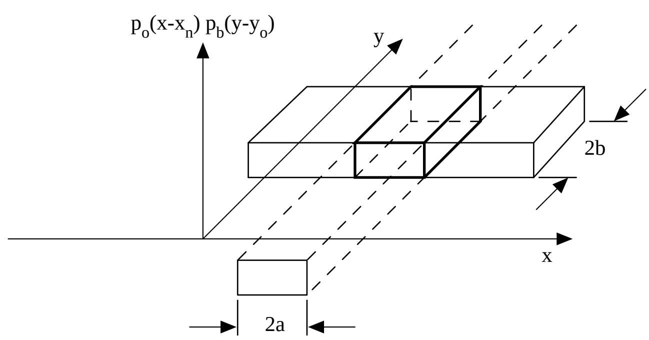
10.3.1 The Rectangle Function

$$f(x,y) = p_a(x - x_o) = \begin{cases} 1 & x_o - a \le x \le x_o + a \\ 0 & \text{otherwise} \end{cases}$$



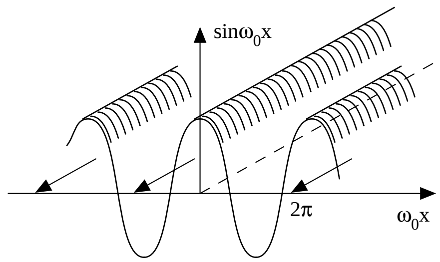
10.3.2 The Pulse Function

$$f(x,y) = p_a(x - x_o)\, p_b(y - y_o) = \begin{cases} 1 & x_o - a \le x \le x_o + a,\ \ y_o - b \le y \le y_o + b \\ 0 & \text{otherwise} \end{cases}$$



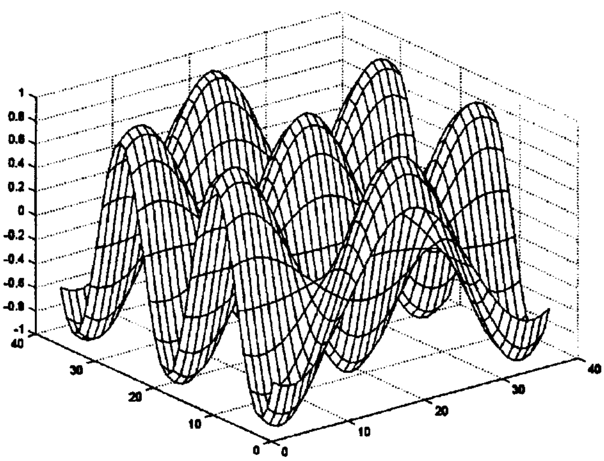
10.3.3 The Cosine Function

$$f(x,y) = \cos\omega_0 x$$



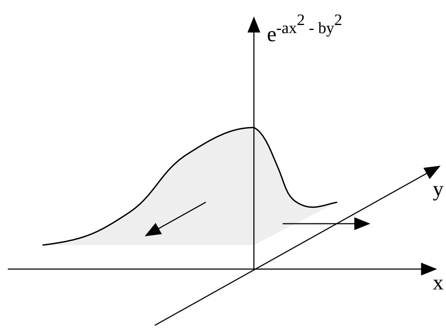
10.3.4 The Cosine-Cosine Function

$$f(x,y) = \cos\omega_1 x \cos\omega_2 x$$



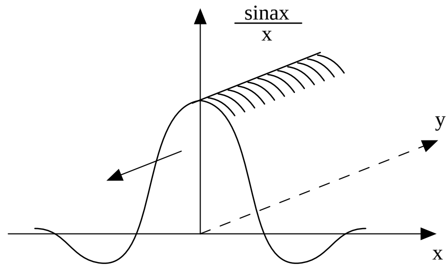
10.3.5 The Gaussian Function

$$f(x,y) = e^{-ax^2 - by^2}$$



10.3.6 The Sinc Function

$$f(x,y) = \frac{\sin ax}{x}$$



Poularikas A. D. “Discrete Fourier Transform”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999

11

Discrete Fourier Transform

11.1 Definitions

11.1.1 Discrete Fourier Transform of Sequences

$$X(k) = DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x(n) W^{kn}, \qquad k = 0,1,2,\cdots,N-1$$

$$x(n) = IDFT\{X(k)\} = \tfrac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} = \sum_{k=0}^{N-1} X(k) W^{-kn}, \qquad n = 0,1,2,\cdots,N-1$$

where $W = e^{-j\frac{2\pi}{N}}$. The spectrum has symmetry at N/2 or at frequency $\frac{2\pi}{N} \times \frac{N}{2} = \pi$ radians and is periodic every N or every 2π radians. Both $x(n)$ and $x(k)$ are periodic with period N.

11.1.2 Discrete Fourier Transform of Sampled Functions

$$X(k) = T \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = T \sum_{n=0}^{N-1} x(n) W^{kn}, \qquad k = 0,1,2,\cdots,N-1$$

$$x(n) = \frac{1}{NT} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} = \frac{1}{NT} \sum_{k=0}^{N-1} X(k) W^{-kn}, \qquad n = 0,1,2,\cdots,N-1$$

where $x(n) \equiv x(nT)$ and $X(k) \equiv X(k2\pi/NT)$. The spectrum has symmetry at N/2 or at $\frac{2\pi}{NT} \times \frac{N}{2} = \pi/T$ radians and is periodic every N or every 2π/T radians. Both $X(k)$ and $x(n)$ are periodic because of the definition by their transform.

$$X(k) = X(k + Ni) \qquad i = 0,\pm 1,\pm 2,\cdots$$

$$x(n) = x(n + Ni) \qquad i = 0,\pm 1,\pm 2,\cdots$$

11.2 Properties

11.2.1 Table 11.1 Presents the Properties of Discrete Fourier Transform.

TABLE 11.1 Discrete Fourier Transform Properties

Sequence (period N)	DFT (period N)
$x(n)$, $x_1(n)$, $x_2(n)$	$X(k)$, $X_1(k)$, $X_2(k)$
$ax_1(n) + bx_2(n)$ (linearity)	$aX_1(k) + bX_2(k)$
$X(n)$ (symmetry)	$Nx(-k)$
$x(n-m)$ (time shifting)	$W^{km}X(k)$
$W^{-\ell n}x(n)$ (frequency shifting)	$X(k-\ell)$
$\sum_{m=0}^{N-1} x_1(m)\,x_2(n-m)$ (periodic time convolution)	$X_1(k)X_2(k)$
$x_1(n)x_2(n)$ (periodic frequency convolution)	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)\,X_2(k-\ell)$
$x^*(n)$ (conjugate)	$X^*(-k)$
$x^*(-n)$ (Hermitian conjugate)	$X^*(k)$
$\mathrm{Re}\{x(n)\}$ (real part)	$X_e(k) = \frac{1}{2}[X(k) + X^*(-k)]$
$j\,\mathrm{Im}\{x(n)\}$ (imaginary part)	$X_o(k) = \frac{1}{2}[X(k) - X^*(-k)]$
$x_e(k) = \frac{1}{2}[x(n) + x^*(-n)]$ (even)	$\mathrm{Re}\{X(k)\}$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$ (odd)	$j\,\mathrm{Im}\{X(k)\}$
$x(n)$ (real)	$X(k) = X^*(-k)$
$x(n)$ (real)	$\mathrm{Re}\{X(k)\} = \mathrm{Re}\{X(-k)\}$
$x(n)$ (real)	$\mathrm{Im}\{X(k)\} = -\mathrm{Im}\{X(-k)\}$
$x(n)$ (real)	$\lvert X(k)\rvert = \lvert X(-k)\rvert$
$x(n)$ (real)	$\arg\{X(k)\} = -\arg\{X(-k)\}$
$x_e(k) = \frac{1}{2}[x(n) + x(-n)]$ (even and real)	$\mathrm{Re}\{X(k)\}$
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (odd and real)	$j\,\mathrm{Im}\{X(k)\}$

11.3 Convolutions

11.3.1 Cyclic Convolution

$$y(n) = \sum_{m=0}^{N-1} x(m)\,h([n-m]_N) = x(n) *_N h(n) \qquad n = 0,1,2,\cdots,N-1$$

where $y(n)$, $h(n)$ are sequences of length N. The symbol $[n-m]_N$ indicates the residue of $n-m$ evaluated modulo N, and $*_N$ denotes ***cyclic*** convolution of length N.

11.3.2 DFT of Cyclic Convolution (see Section 11.3.1)

$$Y(k) = X(k)\,H(k) \qquad k = 0,1,2,\cdots,N-1$$

$$y(n) = IDFT\{X(k)\,H(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,H(k)\,W^{-kn} \qquad n = 0,1,2,\cdots,N-1$$

11.3.3 Polynomial of $X(z)$ of $x(n)$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^n$$

11.3.4 Noncylic Convolution of $x(n)$ and $h(n)$

$$Y(z) = X(z)H(z)$$

The elements of the output sequence $y(n)$ is of $M + N - 1$ length and can be found as the coefficients of the polynomial $Y(z)$. M is the length of $h(n)$, and N is the length of $x(n)$.

11.3.5 Cyclic Convolution

$$Y(z) = H(z)H(z) \bmod (Z^N - 1)$$

The above indicates that the product polynomial is reduced modulo the polynomial $(Z^N - 1)$.

Example 11.1: Cyclic Convolution

$$x(n) = \{1,2,-1\},\ h(n) = \{5,1,4\},\ Y(z) = X(z)H(z) = (1+2z-z^2)(5+z+4z^2)$$

$$= 5+10z-5z^2+z+2z^2-z^3+4z^2+8z^3-4z^4 \bmod(z^3-1) = 5+11z+z^2+7z^3-4z^4$$

$$= 5+11z+z^2+7z^0z^3-4zz^3 = 12+7z+z^2,\ \ y(n) = \{12,7,1\}.$$

Example 11.2: Noncyclic Convolution

From 3.5.1 $Y(z) = 5 + 11z + z^2 + 7z^3 - 4z^4$, $y(n) = \{5,11,1,7,-4\}$. Length of $y(n)$ is $3 + 3 - 1 = 5$.

11.4 Fast Fourier Transform Programs

Program 11.1: Radix-2 DIF FFT

```
CC =============================================================== CC
CC                                                                  CC
CC     Subroutine CFFT2DF (X,Y,M):                                  CC
CC          A Cooley-Tukey in-place, radix-2 complex FFT program    CC
CC          Decimation-in-frequency, cos/sin in second loop         CC
CC                                                                  CC
CC     Input/output:                                                CC
CC          X    Array of real part of input/output (length > = N)  CC
CC          Y    Array of imaginary part of input/output (length > = N) CC
CC          M    Transform length is N = 2**M                       CC
CC                                                                  CC
CC     Author:                                                      CC
CC          H. V. Sorensen, University of Pennsylvania, Dec. 1984   CC
CC               Internet address: hvs@ee.upenn.edu                 CC
CC                                                                  CC
CC        This program may be used and distributed freely provided  CC
CC        this header is included and left intact.                  CC
CC                                                                  CC
```

```
CC ======================================================================= CC
            SUBROUTINE CFFT2DF (X,Y,M)
            Real X(1), Y(1)
C ------------ Main FFT loops --------------------------------------------------------------------------CC
          N = 2**M
          N2 = N
          DO 10 K = 1, M
                  N1 = N2
                  N2 = N2/2
                  E = 6.283185307179586/N1
                  A = 0
                  DO 20 J = 1, N2
                          C = cos(A)
                          S = sin(A)
                          A = J*E
                          DO 30 I = J, N, N1
                                  L = I + N2
                                  XT = X(I) – X(L)
                                  X(I) = X(I) + X(L)
                                  YT = Y(I) – Y(L)
                                  Y(I) = Y(I) + Y(L)
                                  X(L) = C*XT + S*YT
                                  Y(L) = C*YT – S*XT
30                        CONTINUE
20                CONTINUE
10        CONTINUE
C ------------ Digit reverse counter -----------------------------------------------------------------CC
100       J = 1
          DO 104 I = 1, N-1
                  IF (I . GE . J) GOTO 101
                           XT = X(J)
                           X(J) = X(I)
                           X(I) = XT
                           XT = Y(J)
                           Y(J) = Y(I)
                           Y(I) = XT
101               K = N/2
102               IF (K.GE.J) GOTO 103
                           J = J - K
                           K = K/2
                           GOTO 102
103               J = J + K
104       CONTINUE
          RETURN
          END
```

Program 11.2: Radix-2 DIT FFT

```
CC ======================================================================= CC
CC                                                                         CC
CC        Subroutine CFFT2DF (X,Y,M):                                      CC
CC                  A Cooley-Tukey in-place, radix-2 complex FFT program   CC
```

```
CC            Decimation-in-time, cos/sin in second loop                          CC
CC                                                                                CC
CC    Input/output:                                                               CC
CC          X    Array of real part of input/output (length > = N)                CC
CC          Y    Array of imaginary part of input/output (length > = N)           CC
CC          M    Transform length is N = 2**M                                     CC
CC                                                                                CC
CC    Author:                                                                     CC
CC          H. V. Sorensen, University of Pennsylvania, Dec. 1984                 CC
CC                         Internet address: hvs@ee.upenn.edu                     CC
CC                                                                                CC
CC          This program may be used and distributed freely provided              CC
CC          this header is included and left intact.                              CC
CC                                                                                CC
CC ============================================================================= CC
      SUBROUTINE CFFT2DF (X,Y,M)
      Real X(1), Y(1)
      N = 2**M
C ------------Digit reverse counter ---------------------------------------------CC
100   J = 1
      DO 104 I = 1, N-1
         IF (I . GE . J) GOTO 101
            XT = X(J)
            X(J) = X(I)
            X(I) = XT
            XT = Y(J)
            Y(J) = Y(I)
            Y(I) = XT
101      K = N/2
102      IF (K.GE.J) GOTO 103
            J = J – K
            K = K/2
            GOTO 102
103      J = J + K
104   CONTINUE
C ------------Main FFT loops ----------------------------------------------------CC
      N1 = 1
      DO 10 K = 1, M
         N2 = N1
         N1 = N2*2
         E = 6.283185307179586/N1
         A = 0
         DO 20 J = 1, N2
            C = cos (A)
            S = sin (A)
            A = J*E
            DO 30 I = J, N, N1
               L = I + N2
               X(T) = C*X(L) + S*Y(L)
               Y(T) = C*Y(L) – S*X(L)
               X(L) = X(I) – XT
```

```
                    X(I) = X(I) + XT
                    Y(L) = Y(I) * YT
                    Y(I) = Y(I) + YT
30              CONTINUE
20          CONTINUE
10      CONTINUE
        RETURN
        END
```

Program 11.3: Split-Radix FFT Without Table Look-up

```
CC ================================================================== CC
CC                                                                    CC
CC      Subroutine CFFTSR (X,Y,M):                                    CC
CC          An in-place, split-radix complex FFT program              CC
CC          Decimation-in-frequency, cos/sin in second loop           CC
CC          and is computed recursively                               CC
CC                                                                    CC
CC      Input/output:                                                 CC
CC          X    Array of real part of input/output (length > = N)    CC
CC          Y    Array of imaginary part of input/output (length > = N) CC
CC          M    Transform length is N = 2**M                         CC
CC                                                                    CC
CC      Calls:                                                        CC
CC          CSTAGE, CBITREV                                           CC
CC                                                                    CC
CC      Author:                                                       CC
CC          H. V. Sorensen, University of Pennsylvania, Dec. 1984     CC
CC                    Internet address: hvs@ee.upenn.edu              CC
CC      Modified:                                                     CC
CC          H. V. Sorensen, University of Pennsylvania, July 1987     CC
CC                                                                    CC
CC      Reference:                                                    CC
CC          Sorensen, Heideman, Burrus: "On computing the split-radix CC
CC          FFT," IEEE Trans. ASSP, Vol. ASSP-34, No. 1, pp. 152-156, CC
CC          Feb. 1986                                                 CC
CC                                                                    CC
CC          This program may be used and distributed freely provided  CC
CC          this header is included and left intact.                  CC
CC                                                                    CC
CC ================================================================== CC
        SUBROUTINE CFFTSR (X,Y,M)
        Real X(1), Y(1)
        N = 2**M
C -------------L shaped butterflies--------------------------------------CC
        N2 = 2*N
        DO 10 K = 1, M–1
            N2 = N2/2
            N4 = N2/4
            CALL CSTAGE (N, N2, N4, X(1), X(N4+1), X(2*N4+1), X(3*N4+1),
    $                        Y(1), Y(N4+1), Y(2*N4+1), Y(3*N4+1) )
```

```
10        CONTINUE
C ------------Length two butterflies---------------------------------------------------------------------------CC
          IS = 1
          ID = 4
20                DO 30 I1          = IS, N, ID
                            T1        = X(I1)
                            X(I1)     = T1 + X(I1+1)
                            X(I1+1)   = T1 – X(I1+1)
                            T1        = Y(I1)
                            Y(I1)     = TI + Y(I1+1)
                            Y(I1+1)   = TI – Y(I1+1)
30                CONTINUE
                  IS = 2*ID – 1
                  ID = 4*ID
             IF (IS.LT.N) GOTO 20
C ------------Digit reverse counter ----------------------------------------------------------------------------CC
          CALL CBITREV (X,Y,M)
          RETURN
          END
CC ======================================================================= CC
CC                                                                             CC
CC        Subroutine CSTAGE – the workhorse of the CFFTSR                      CC
CC                 computes one stage of a complex split-radix length N         CC
CC                 transform.                                                  CC
CC                                                                             CC
CC        Author:                                                              CC
CC                 H. V. Sorensen, University of Pennsylvania, July 1987       CC
CC                                                                             CC
CC                 This program may be used and distributed freely provided    CC
CC                 this header is included and left intact.                    CC
CC                                                                             CC
CC ======================================================================= CC
          SUBROUTINE CSTAGE (N, N2, N4, X1, X2, X3, X4, Y1, Y2, Y3, Y4)
          REAL X1(1), X2(1), X3(1), X4(1), Y1(1), Y2(1), Y3(1), Y4(1)
          N8 = N4/2
C ------------zero butterfly--------------------------------------------------------------------------------------CC
          IS = 0
          ID = 2*N2
10                DO 20 I1= IS+1, N, ID
                            T1        = X1(I1) - X3(I1)
                            X1(I1)    = X1(I1) + X3(I1)
                            T2        = Y2(I1) – Y4(I1)
                            Y2(I1)    = Y2(I1) + Y4(I1)
                            X3(I1)    = TI + T2
                            T2        = T1 – T2
                            T1        = X2(I1) – X4(I1)
                            X2 (I1)   = X2(I1) + X4(I1)
                            X4(I1)    = T2
                            T2        = Y1(I1) – Y3(I1)
                            Y1(I1)    = Y1(I1) + Y3(I1)
                            Y3(I1)    = T2 – T1
```

```
                    Y4(I1)   = T2 + T1
30            CONTINUE
              IS = 2*ID – N2
              ID = 4*ID
        IF (IS. LT. N) GOTO 10
C
        IF (N8–1) 100, 100, 60
C -------------N/8 butterfly ---------------------------------------------------------------------------------------C
30      IS = 0
        ID = 2*N2
40            DO 50 I1 = IS + 1 + N8, N, ID
                    T1     = X1(I1) – X3(I1)
                    X1(I1) = X1(I1) + X3(I1)
                    T2     = X2(I1) – X4(I1)
                    X2(I1) = X2(I1) + X4(I1)
                    T3     = Y1(I1) – Y3(I1)
                    Y1(I1) = Y1(I1) + Y3(I1)
                    T4     = Y2(I1) – Y4(I1)
                    Y2(I1) = Y2(I1) + Y4(I1)
                    T5     = (T4 – T1)*0.707106778
                    T1     = (T4 + T1)*0.707106778
                    T4     = (T3 – T2)*0.707106778
                    T2     = (T3 + T2)*0.707106778
                    X3(I1) = T4 + T1
                    Y3(I1) = T4 – T1
                    X4(I1) = T5 + T2
                    Y4(I1) = T5 – T2
50            CONTINUE
              IS = 2*ID – N2
              ID = 4*ID
        IF (IS . LT. N–1) GOTO 40
C
        IF (N8–1) 100, 100, 60
C -------------General butterfly. Two at a time ---------------------------------------------------------------C
60      E = 6.283185307179586/N2
        SS1 = SIN(E)
        SD1 = SS1
        SD3 = 3 . *SD1–4 . *SD1**3
        SS3 = SD3
        CC1 = COS (E)
        CD1 = CC1
        CD3 = 4 .* CD1**3–3 . *CD1
        CC3 = CD3
        DO 90 J = 2, N8
              IS = 0
              ID = 2*N2
              JN = N4 – 2*J + 2
70                  DO 80 I1 = IS + J, N+J, ID
                          T1     = X1(I1) – X3(I1)
                          X1(I1) = X1(I1) + X3(I1)
                          T2     = X2(I1) – X4(I1)
```

```
                  X2(I1) = X2(I1) + X4(I1)
                  T3     = Y1(I1) – Y3(I1)
                  Y1(I1) = Y1(I1) + Y3(I1)
                  T4     = Y2(I1) – Y4(I1)
                  Y2(I1) = Y2(I1) + Y4(I1)
                  T5     = (T1 – T4)
                  T1     = (T1 + T4)
                  T4     = (T2 – T3)
                  T2     = (T2 + T3)
                  X3(I1) = T1*CC1 + T4*SS1
                  Y3(I1) = –T4*CC1 - T1*SS1
                  X4(I1) = T5*CC3 + T2*SS3
                  Y4(I1) = T2*CC3 – T5*SS3
                  I2     = I1 +JN
                  T1     = X1(I2) – X3(I2)
                  X1(I2) = X1(I2) + X3(I2)
                  T2     = X2(I2) – X4(I2)
                  X2(I2) = X2(I2) + X4(I2)
                  T3     = Y1(I2) – Y3(I2)
                  Y1(I2) = Y1(I2) + Y3(I2)
                  T4     = Y2(I2) – Y4(I2)
                  Y2(I2) = Y2(I2) + Y4(I2)
                  T5     = (T1 – T4)
                  T1     = (T1 + T4)
                  T4     = (T2 – T3)
                  T2     = (T2 + T3)
                  X3(I2) = T1*SS1 – T4*CC1
                  Y3(I2) = –T4*SS1 – T1*CC1
                  X4(I2) = –T5*SS3 – T2*CC3
                  Y4(I2) = T2*SS3 + T5*CC3
80             CONTINUE
               IS = 2*ID – N2
               ID = 4*ID
            IF (IS . LT. N–1) GOTO 40
C
            T1   = CC1*CD1 – SS1*SD1
            SS1  = CC1*SD1 + SS1*CD1
            CC1  = T1
            T3   = CC3*CD3 – SS3*SD3
            SS3  = CC3*SD3 + SS3*CD3
            CC3  = T3
90      CONTINUE
100     RETURN
        END
CC ============================================================== CC
CC                                                                 CC
CC      Subroutine CBITREV (X,Y,M):                                CC
CC           Bit reverses the array X of length 2**M. It generates a      CC
CC           table ITAB (minimum length is SQRT (2**M) if M is even       CC
CC           or SQRT(2*2**M) if M is odd). ITAB need only be generated    CC
CC           once for a given transform length.                           CC
```

```
CC                                                                        CC
CC      Author:                                                           CC
CC              H. V. Sorensen, University of Pennsylvania, Aug. 1987     CC
CC                              Arpa address: hvs@ee.upenn.edu            CC
CC                                                                        CC
CC      Reference:                                                        CC
CC              D. Evans, Tran. ASSP, Aol. ASSP–35, No. 8, pp. 1120–1125, CC
CC              Aug. 1987                                                 CC
CC                                                                        CC
CC              This program may be used and distributed freely provided  CC
CC              this header is included and left intact.                  CC
CC                                                                        CC
CC ====================================================================== CC
        SUBROUTINE CBITREV (X,Y,M)
        DIMENSION X(1), Y(1), ITAB (256)
C -------------Initialization of ITAB array ---------------------------------C
        M2 = M/2
        NBIT = 2**M2
        IF (2*M2 . NE . M) M2 = M2 + 1
        ITAB (1) = 0
        ITAB (2) = 1
        IMAX = 1
        DO 10 LBSS = 2, M2
              IMAX = 2 * IMAX
        DO 10 I = 1, IMAX
              ITAB (I) = 2 * ITAB(I)
              ITAB (I + 2\IMAX) = 1 + ITAB(I)
10      CONTINUE
C -------------The actual bit reversal---------------------------------------C
        DO 20 K = 2, NBIT
              JO  = NBIT * ITAB (K) + 1
              I   = K
              J   + JO
              DO 20 L = 2, ITAB (K) + 1
                    T1   = X(I)
                    X(I) = X(J)
                    X(J) = T1
                    T1   = Y(I)
                    Y(I) = Y(J)
                    Y(J) = T1
                    I    = I + NBIT
                    J    = JO + ITAB(L)
20      CONTINUE
        RETURN
        END
```

Program 11.4: Split–Radix with Table Look–up

```
CC ====================================================================== CC
CC                                                                        CC
CC      Subroutine CTFFTSR (X,Y,M,CT1, CT3, ST1, ST3, ITAB):              CC
CC              An in–place, split–radix–2 complex FFT program            CC
```

```
CC              Decimation–in–frequency, cos/sin in third loop                          CC
CC              and is looked–up in table. Tables CT1, CT3, ST1, ST3                    CC
CC              have to have length > = N/8–1. The bit reverser uses partly table look–up. CC
CC                                                                                      CC
CC      Input/output:                                                                   CC
CC              X       Array of real part of input/output (length > = N)               CC
CC              Y       Array of imaginary part of input/output (length > = N)          CC
CC              M       Transform length is N = 2**M                                    CC
CC              CT1     Array of cos( ) table (length > = N/8–1)                        CC
CC              CT3     Array of cos( ) table (length > = N/8–1)                        CC
CC              ST1     Array of sin( ) table (length > = N/8–1)                        CC
CC              ST3     Array of sin( ) table (length > = N/8–1)                        CC
CC              ITAB  Array of bit reversal indices (length > = sqrt (2*N))             CC
CC                                                                                      CC
CC      Calls:                                                                          CC
CC              CTSTAG                                                                  CC
CC                                                                                      CC
CC      Note:                                                                           CC
CC              TINIT must be called before this program!!                              CC
CC                                                                                      CC
CC      Author:                                                                         CC
CC              H. V. Sorensen, University of Pennsylvania, Dec. 1984                   CC
CC                              Internet address: hvs@ee.upenn.edu                      CC
CC      Modified:                                                                       CC
CC              Hinrik Sorensen, University of Pennsylvania, July 1987                  CC
CC                                                                                      CC
CC              This program may be used and distributed freely provided                CC
CC              this header is included and left intact.                                CC
CC                                                                                      CC
CC ================================================================================= CC
        SUBROUTINE CTFFTSR (X,Y,M, CT1, CT3, SR1, ST3, ITAB)
        Real X(1), Y(1), CT1(1), CT3(1), ST1(1), ST3(1)
        INTEGER ITAB (1)
        N = 2**M
C -------------L shaped butterflies------------------------------------------------------C
        ITS = 1
        N2 = 2*N
        DO 10 K = 1, M–1
              N2 = N2/2
              N4 = N2/4
              CALL CTSTAG (N, N2, N4, ITS, X(1), X(N4+1), X(2*N4+1), X(3*N4+1),
        $                           Y(1), Y(N4+1), Y(2*N4+1), Y(3*N4+1),
        $                           CT1, CT2, ST1, ST3)
              ITS = 2 * ITS
10      CONTINUE
C -------------Length two butterflies---------------------------------------------------CC
        IS = 1
        ID = 4
20            DO 30 I1= IS, N, ID
                    T1        = X(I1)
                    X(I1)     = TI + X(I1+1)
```

```
                        X(I1+1)= TI – X(I1+1)
                        T1      = Y(I1)
                        Y(I1)   = T1 + Y(I1+1)
                        Y(I1+1)= T1 – Y(I1+1)
30                CONTINUE
                  IS = 2*ID – 1
                  ID = 4*ID
          IF (IS.LT.N) GOTO 20
C -------------Digit reverse counter -------------------------------------------------------------------------------CC
          M2 = M/2
          NBIT = 2**M2
          DO 50 K = 2, NBIT
                 JO  = NBIT * ITAB (K) + 1
                 I    = K
                 J    = JO
                 DO 40 L = 2, ITAB (K) + 1
                        T1   = X(I)
                        X(I)  = X(J)
                        X(J)  = T1
                        T1   = Y(I)
                        Y(I)  = Y(J)
                        Y(J)  = T1
                        I      = I + NBIT
                        J      = JO + ITAB(L)
40        CONTINUE
50        CONTINUE
          RETURN
          END
CC ======================================================================= CC
CC                                                                                                            CC
CC        Subroutine CTSTAG – the workhorse of the CTFFTSR                                       CC
CC                  computes one stage of a length N split–radix transform                     CC
CC                                                                                                            CC
CC        Author:                                                                                             CC
CC                  H. V. Sorensen, University of Pennsylvania, July 1987                       CC
CC                                                                                                            CC
CC                  This program may be used and distributed freely provided                    CC
CC                  this header is included and left intact.                                          CC
CC                                                                                                            CC
CC ======================================================================= CC
          SUBROUTINE CTSTAG (N, N2, N4, X1, X2, X3, X4, Y1, Y2, Y3, Y4
        $                                       CT1, CT3, ST1, ST3)
          REAL X1(1), X2(1), X3(1), X4(1), Y1(1), Y2(1), Y3(1), Y4(1)
          REAL CT1(1), CT3(1), ST1(1), ST3(1)
          N8 = N4/2
C -------------zero butterfly-----------------------------------------------------------------------------------------CC
          IS = 0
          ID = 2*N2
10                  DO 20 I1       = IS+1, N, ID
                         T1       = X1(I1) – X3(I1)
                         X1(I1) = X1(I1) + X3(I1)
```

```
                        T2     = Y2(I1) – Y4(I1)
                        Y2(I1) = Y2(I1) + Y4(I1)
                        X3(I1) = TI + T2
                        T2     = T1 – T2
                        T1     = X2(I1) – X4(I1)
                        X2 (I1)= X2(I1) + X4(I1)
                        X4(I1) = T2
                        T2     = Y1(I1) – Y3(I1)
                        Y1(I1) = Y1(I1) + Y3(I1)
                        Y3(I1) = T2 – T1
                        Y4(I1) = T2 – T1
20               CONTINUE
                 IS = 2*ID – N2
                 ID = 4*ID
         IF (IS. LT. N) GOTO 10
C
         IF (N4–1) 100, 100, 30
C -------------N/8 butterfly ------------------------------------------------------------------------------------------C
30       IS = 0
         ID = 2*N2
40               DO 50 I1     = IS + 1 + N8, N, ID
                        T1     = X1(I1) – X3(I1)
                        X1(I1) = X1(I1) + X3(I1)
                        T2     = X2(I1) – X4(I1)
                        X2(I1) = X2(I1) + X4(I1)
                        T3     = Y1(I1) – Y3(I1)
                        Y1(I1) = Y1(I1) + Y3(I1)
                        T4     = Y2(I1) – Y4(I1)
                        Y2(I1) = Y2(I1) + Y4(I1)
                        T5     = (T4 – T1)*0.707106778
                        T1     = (T4 + T1)*0.707106778
                        T4     = (T3 – T2)*0.707106778
                        T2     = (T3 + T2)*0.707106778
                        X3(I1) = T4 + T1
                        Y3(I1) = T4 – T1
                        X4(I1) = T5 + T2
                        Y4(I1) = T5 – T2
50               CONTINUE
                 IS = 2*ID – N2
                 ID = 4*ID
         IF (IS . LT. N–1) GOTO 40
C
         IF (N8–1) 100, 100, 60
C -------------General butterfly. Two at a time ---------------------------------------------------------------------C
60       IS =1
         ID = N2*2
70               DO 90 I = IS, N, ID
                        IT = 0
                        JN = 1 + N4
70                      DO 80 J = 1, N8–1
                              IT        = IT + ITS
```

```
                I1     = 1 + J
                T1     = X1(I1) – X3(I1)
                X1(I1) = X1(I1) + X3(I1)
                T2     = X2(I1) – X4(I1)
                X2(I1) = X2(I1) + X4(I1)
                T3     = Y1(I1) – Y3(I1)
                Y1(I1) = Y1(I1) + Y3(I1)
                T4     = Y2(I1) – Y4(I1)
                Y2(I1) = Y2(I1) + Y4(I1)
                T5     = (T1 – T4)
                T1     = (T1 + T4)
                T4     = (T2 – T3)
                T2     = (T2 + T3)
                X3(I1) = T1*CT1(IT) – T4*ST1(IT)
                Y3(I1)  = –T4*CT1(IT) – T1*ST1(IT)
                X4(I1) = T5*CT3 (IT) + T2*ST3(IT)
                Y4(I1) = T2*CT3(IT) – T5*ST3(IT)
                I2     = JN – J
                T1     = X1(I2) – X3(I2)
                X1(I2) = X1(I2) + X3(I2)
                T2     = X2(I2) – X4(I2)
                X2(I2) = X2(I2) + X4(I2)
                T3     = Y1(I2) – Y3(I2)
                Y1(I2) = Y1(I2) + Y3(I2)
                T4     = Y2(I2) – Y4(I2)
                Y2(I2) = Y2(I2) + Y4(I2)
                T5     = (T1 – T4)
                T1     = (T1 + T4)
                T4     = (T2 – T3)
                T2     = (T2 + T3)
                X3(I2) = T1*ST1(IT) – T4*CT1(IT)
                Y3(I2) = –T4*ST1 (IT)– T1*CT1(IT)
                X4(I2) = –T5*ST3(IT) – T2*CT3(IT)
                Y4(I2) = T2*ST3(IT) + T5*CT3(IT)
80            CONTINUE
90         CONTINUE
           IS = 2*ID – N2
           ID = 4*ID
      IF (IS . LT. N–1) GOTO 70
100   RETURN
      END
CC ================================================================ CC
CC                                                                    CC
CC    Subroutine TINIT:                                               CC
CC          Initialize sin/cos and bit reversal tables                CC
CC                                                                    CC
CC    Author:                                                         CC
CC          H. V. Sorensen, University of Pennsylvania, July 1987     CC
CC                                                                    CC
CC          This program may be used and distributed freely provided  CC
CC          this header is included and left intact.                  CC
```

```
CC                                                                          CC
CC ======================================================================== CC
      SUBROUTINE TINIT (M, CT1, CT3, ST1, ST3, ITAB)
      REAL CT1(1), CT3(1), ST1(1), ST3(1)
      INTEGER ITAB(1)
C -------------Sin/cos table ---------------------------------------------C
      N = 2**M
      ANG = 6.283185307179586/N
      DO 10 I = 1, N/8–1
         CT1(I) = COS(ANG*I)
         CT3(I) = COS(ANG*I*3)
         ST1(I) = SIN(ANG*I)
         ST3(I) = SIN(ANG*I*3)
10    CONTINUE
C -------------Bit reversal table -----------------------------------------C
      M2 = M/2
      NBIT = 2**M2
      IF (2*M2 . NE . M) M2 = M2 + 1
      ITAB(1) = 0
      ITAB(2) = 1
      IMAX = 1
      DO 30 LBSS = 2, M2
         IMAX = 2*IMAX
         DO 20 I = 1, IMAX
            ITAB(I) = 2*ITAB (I)
            ITAB(I + IMAX) = 1 + ITAB(I)
20       CONTINUE
30    CONTINUE
      RETURN
      END
```

Program 11.5: Inverse Split–Radix FFT

```
CC ======================================================================== CC
CC                                                                          CC
CC    Subroutine ICSRFFT (X,Y,M):                                           CC
CC         An in–place, inverse split–radix–2 complex FFT program           CC
CC         Decimation–in–frequency, cos/sin in second loop                  CC
CC         and is computed recursively                                      CC
CC                                                                          CC
CC    Input/output:                                                         CC
CC         X    Array of real part of input/output (length > = N)           CC
CC         Y    Array of imaginary part of input/output (length > = N)      CC
CC         M    Transform length is N = 2**M                                CC
CC                                                                          CC
CC    Calls:                                                                CC
CC         ICSTAGE, CBITREV                                                 CC
CC                                                                          CC
CC    Author:                                                               CC
CC         H. V. Sorensen, University of Pennsylvania, Dec. 1984            CC
CC                          Arpa address: hvs@ee.upenn.edu                  CC
CC    Modified:                                                             CC
```

```
CC              H. V. Sorensen, University of Pennsylvania, July 1987              CC
CC                                                                                 CC
CC      Reference:                                                                 CC
CC              Sorensen, Heideman, Burrus: “On computing the split–radix          CC
CC              FFT,” IEEE Trans. ASSP, Vol. ASSP–34, No. 1, pp. 152–156,          CC
CC              Feb. 1986                                                          CC
CC                                                                                 CC
CC              This program may be used and distributed freely provided           CC
CC              this header is included and left intact.                           CC
CC                                                                                 CC
CC ============================================================================== CC
        SUBROUTINE ICSRFFT (X,Y,M)
        Real X(1), Y(1)
        N = 2**M
C ------------L shaped butterflies-----------------------------------------------CC
        N2 = 2*N
        DO 10 K = 1, M–1
            N2 = N2/2
            N4 = N2/4
            CALL ICSTAGE (N, N2, N4, X(1), X(N4+1), X(2*N4+1), X(3*N4+1),
        $                         Y(1), Y(N4+1), Y(2*N4+1), Y(3*N4+1) )
10      CONTINUE
C ------------Length two butterflies---------------------------------------------CC
        IS = 1
        ID = 4
20              DO 30 I1      = IS, N, ID
                    R1        = X(I1)
                    X(I1)     = RI + X(I1+1)
                    X(I1+1)= RI – X(I1+1)
                    R1        = Y(I1)
                    Y(I1)     = RI + Y(I1+1)
                    Y(I1+1)  = RI – Y(I1+1)
30              CONTINUE
                IS = 2*ID – 1
                ID = 4*ID
         IF (IS.LT.N) GOTO 20
C ------------Digit reverse counter ----------------------------------------------C
        CALL CBITREV (X,Y,M)
C ------------Divide by N---------------------------------------------------------C
        DO 40 I = 1, N
            X(I) = X(I)/N
            Y(I) = Y(I)/N
40      CONTINUE
        RETURN
        END
CC ============================================================================== CC
CC                                                                                 CC
CC      Subroutine ICSTAGE – the workhorse of the ICFFTSR                          CC
CC              computes one stage of a complex split–radix length N               CC
CC              transform.                                                         CC
CC                                                                                 CC
```

```
CC      Author:                                                            CC
CC              H. V. Sorensen, University of Pennsylvania, July 1987      CC
CC                                                                         CC
CC              This program may be used and distributed freely provided   CC
CC              this header is included and left intact.                   CC
CC                                                                         CC
CC =========================================================================== CC
        SUBROUTINE ICSTAGE (N, N2, N4, X1, X2, X3, X4, Y1, Y2, Y3, Y4)
        REAL X1(1), X2(1), X3(1), X4(1), Y1(1), Y2(1), Y3(1), Y4(1)
        N8 = N4/2
C -------------zero butterfly---------------------------------------------------CC
        IS = 0
        ID = 2*N2
10            DO 20 I1      = IS+1, N, ID
                    R1      = X1(I0) – X3(I0)
                    X1(I0) = X1(I0) + X3(I0)
                    R2      = X2(I0) – X4(I0)
                    X2(I0)  = X2(I0) + X4(I0)
                    S1      = Y1(I0) – Y3(I0)
                    Y1(I0)  = Y1(I0) – Y3(I0)
                    S2      = Y2(I0) – Y4(I0)
                    Y2(I0)  = Y2(I0) + Y4(I0)
                    X3(I0)  = R1 – S2
                    X4(I0)  = R1 + S2
                    Y4(I0)  = S1 – R2
                    Y3(I0)= R2 + S1
20            CONTINUE
              IS = 2*ID – N2
              ID = 4*ID
        IF (IS. LT. N) GOTO 10
C
        IF (N4–1) 100, 100, 30
C -------------N/8 butterfly ----------------------------------------------------C
30      IS = 0
        ID = 2*N2
40            DO 50 I0 = IS + 1 + N8, N, ID
                    R1      = X1(I0) – X3(I0)
                    X1(I0) = X1(I0) + X3(I0)
                    R2      = X2(I0) – X4(I0)
                    X2(I0) = X2(I0) + X4(I0)
                    S1      = Y1(I0) – Y3(I0)
                    Y1(I0) = Y1(I0) + Y3(I0)
                    S2      = Y2(I0) – Y4(I0)
                    Y2(I0)  = Y2(I0) + Y4(I0)
                    S3      = R1 – S2
                    R1      = R1 + S2
                    S2      = R2 – S1
                    R2      = R2 + S1
                    X3(I0)  = (S3 – R2)*0.707106778
                    Y3(I0)  = (R2 + S3)*0.707106778
                    X4(I0)  = (S2 – R1)*0.707106778
```

```
                 Y4(I0)  = (S2 + R1)*0.707106778
50            CONTINUE
              IS = 2*ID – N2
              ID = 4*ID
          IF (IS . LT. N–1) GOTO 40
C
          IF (N8–1) 100, 100, 60
C -------------General butterfly. Two at a time ------------------------------------------------------------------C
60        E = 6.283185307179586/N2
          SS1 = SIN(E)
          SD1 = SS1
          SD3 = 3 . *SD1–4 . *SD1**3
          SS3 = SD3
          CC1 = COS (E)
          CD1 = CC1
          CD3 = 4 .* CD1**3–3 . *CD1
          CC3 = CD3
          DO 90 J = 2, N8
              IS = 0
              ID = 2*N2
              JN = N4 – 2*J + 2
70                 DO 80 I0 = IS + J, N+J, ID
                      R1     = X1(I0) – X3(I0)
                      X1(I0) = X1(I0) + X3(I0)
                      R2     = X2(I0) – X4(I0)
                      X2(I0) = X2(I0) + X4(I0)
                      S1     = Y1(I0) – Y3(I0)
                      Y1(I0) = Y1(I0) + Y3(I0)
                      S2     = Y2(I0) – Y4(I0)
                      Y2(I0)  = Y2(I0) + Y4(I0)
                      S3     = (R1 – S2)
                      R1     = (R1 + S2)
                      S2     = (R2 – S1)
                      R2     = (R2 + S1)
                      X3(I0) = S3*CC1 + R2*SS1
                      Y3(I0) = R2*CC1 – S3*SS1
                      X4(I0) = R1*CC3 + S2*SS3
                      Y4(I0) = –S2*CC3 – R1*SS3
                      I1     = I0 +JN
                      R1     = X1(I1) – X3(I1)
                      X1(I1) = X1(I1) + X3(I1)
                      R2     = X2(I1) – X4(I0)
                      X2(I1) = X2(I1) + X4(I1)
                      S1     = Y1(I1) – Y3(I1)
                      Y1(I1) = Y1(I1) + Y3(I1)
                      S2     = Y2(I1) – Y4(I1)
                      Y2(I1) = Y2(I1) + Y4(I1)
                      S3     = R1 – S2
                      R1     = R1 + S1
                      S2     = R2 – S1
                      R2     = R2 + S1
```

```
                  X3(I1) = S3*SS1 – R2*CC1
                  Y3(I1) = R2*SS1 + S3*CC1
                  X4(I1) = –R1*SS3 – S2*CC3
                  Y4(I1) = S2*SS3 + R1*CC3
80             CONTINUE
               IS = 2*ID – N2
               ID = 4*ID
          IF (IS . LT. N–1) GOTO 70
C
          T1   = CC1*CD1 – SS1*SD1
          SS1  = CC1*SD1 + SS1*CD1
          CC1  = T1
          T3   = CC3*CD3 – SS3*SD3
          SS3  = CC3*SD3 + SS3*CD3
          CC3  = T3
90    CONTINUE
100   RETURN
      END
CC ======================================================================== CC
CC                                                                          CC
CC    Subroutine CBITREV (X,Y,M):                                           CC
CC          Bit reverses the array X of length 2**M. It generates a         CC
CC          table ITAB (minimum length is SQRT (2**M) if M is even          CC
CC          or SQRT(2*2**M) if M is odd). ITAB need only be generated       CC
CC          once for a given transform length.                              CC
CC                                                                          CC
CC    Author:                                                               CC
CC          H. V. Sorensen, University of Pennsylvania, Aug. 1987           CC
CC                             Arpa address: hvs@ee.upenn.edu               CC
CC                                                                          CC
CC    Reference:                                                            CC
CC          D. Evans, Tran. ASSP, Aol. ASSP–35, No. 8, pp. 1120–1125,       CC
CC          Aug. 1987                                                       CC
CC                                                                          CC
CC          This program may be used and distributed freely provided        CC
CC          this header is included and left intact.                        CC
CC                                                                          CC
CC ======================================================================== CC
      SUBROUTINE CBITREV (X,Y,M)
      DIMENSION X(1), Y(1), ITAB (1)
C ------------- Initialization of ITAB array ---------------------------------C
      M2 = M/2
      NBIT = 2**M2
      IF (2*M2 . NE . M) M2 = M2 + 1
      ITAB (1) = 0
      ITAB (2) = 1
      IMAX = 1
      DO 10 LBSS = 2, M2
          IMAX = 2 * IMAX
          DO 10 I = 1, IMAX
               ITAB (I) = 2 * ITAB(I)
```

```
                ITAB (I + IMAX) = 1 + ITAB(I)
10      CONTINUE
C -------------The actual bit reversal-----------------------------------------------------------------------------C
        DO 20 K = 2, NBIT
              JO = NBIT * ITAB (K) + 1
              I   = K
              J   = JO
              DO 20 L = 2, ITAB (K) + 1
                    T1    = X(I)
                    X(I)  = X(J)
                    X(J)  = T1
                    T1    = Y(I)
                    Y(I)  = Y(J)
                    Y(J)  = T1
                    I     = I + NBIT
                    J     = JO + ITAB(L)
20      CONTINUE
        RETURN
        END
```

Program 11.6: Prime Factor FFT

```
CC ================================================================= CC
CC                                                                     CC
CC      Subroutine PFA (X, Y, N, M, NI):                               CC
CC            A prime factor FFT program. In–place and in–order.       CC
CC            Length N transform with M factors in array NI            CC
CC                  N = NI(1) * NI(2)* . . . *NI(M)                    CC
CC            Factors are implemented for NI = 2,3,4,5,7               CC
CC                                                                     CC
CC      Input/output:                                                  CC
CC            X     Array of real part of input/output (length > = N)  CC
CC            Y     Array of imaginary part of input/output (length > = N) CC
CC            N     Transform length                                   CC
CC            M     Number of factors in NI                            CC
CC            NI    Array with factors of N (length > = M)             CC
CC                                                                     CC
CC      Author:                                                        CC
CC            C. S. Burrus, Rice University, August 1987               CC
CC                                                                     CC
CC            This program may be used and distributed freely provided CC
CC            this header is included and left intact.                 CC
CC                                                                     CC
CC ================================================================= CC
        SUBROUTINE PFA (X, Y, N, M, NI)
        INTEGER NI(4), I (16), IP (16), LP (16)
        REAL X(1), Y(1)
        DATA C31, C32   /     –0.86602540, –1.50000000  /
        DATA C51, C52   /     –0.95105652, –1.53884180  /
        DATA C53, C54   /     –0.36327126, 0.55901699  /
        DATA C55        /     –1.25  /
        DATA C71, C72   /     –1.16666667, –0.79015647  /
```

```
      DATA C73, C74   /      0.055854267, 0.7343022  /
      DATA C75, C76   /      0.44095855, –0.87484229  /
      DATA C77, C78   /      0.53396936, 0.87484229  /
C -------------Nested loops -----------------------------------------------------------------------------------------C
      DO 10 K = 1, M
           N1 = NI(K)
           N2 = N/NI
           L = 1
           N3 = N2 – N1*(N2/N1)
           DO 15 J = 2, N1
                L = L + N3
                IF (L.GT.N1) L = L – N1
                LP(J) = L
15         CONTINUE
C
           DO 20 J = 1, N, N1
                IT   = J
                I(1) = J
           IP(1) = J
           DO 30 L = 2, N1
                IT = IT + N2
                IF (IT.GT.N) IT = IT – N
                I(L) = IT
                IP(LP(L)) = IT
30         CONTINUE
           GOTO (20, 102, 103, 104, 105, 20, 107), N1
C -------------WFTA N = 2 ---------------------------------------------------------------------------------------C
102        R1       = X(I(1))
           X(I(1))  = R1 + X(I(2))
           X(I(2))  = R1 – X(I(2))
           R1       = Y(I(1))
           Y(IP(1)) = R1 + Y(I(2))
           Y(IP(2)) = R1 – Y(I(2))
           GOTO 20
C -------------WFTA N = 3 ---------------------------------------------------------------------------------------C
103        R2       = X(I(2)) – X(I(3)) * C31
           R1       = X(I(2)) + X(I(3))
           X(I(1))  = X(I(1)) + R1
           R1       = X(I(1)) + R1 * C32
           S2       = Y(I(2)) – Y(I(3)) * C31
           S1       = Y(I(2)) + Y(I(3))
           Y(I(1))  = Y(I(1)) + S1
           S1       = Y(I(1)) + S1*C32
           X(IP(2)) = R1 –S2
           X(IP(3)) = R1 + S2
           Y(IP(2)) = S1 + R2
           Y(IP(3)) = S1 – R2
           GOTO 20
C -------------WFTA N = 4 ---------------------------------------------------------------------------------------C
104        R1       = X(I(1)) + X(I(3))
           T1       = X(I(1)) – X(I(3))
```

```
      R2       = X(I(2)) + X(I(4))
      X(IP(1)) = R1 + R2
      X(IP(3)) = R1 – R2
      R1       = Y(I(1)) + Y(I(3))
      T2       = Y(I(1)) – Y(I(3))
      R2       = Y(I(2)) + Y(I(4))
      Y(IP(1)) = R1 + R2
      Y(IP(3)) = R1 – R2
      R1       = X(I(2)) – X(I(4))
      R2       = Y(I(2)) – Y(I(4))
      X(IP(2)) = T1 + R2
      X(IP(4)) = T1 – R2
      Y(IP(2)) = T2 – R1
      Y(IP(4)) = T2 + R1
      GOTO 20
C -------------WFTA N = 5 ------------------------------------------------------------------------------------------C
105   R1       = X(I(2)) + X(I(5))
      R4       = X(I(2)) – X(I(5))
      R3       = X(I(3)) + X(I(4))
      R2       = X(I(3)) – X(I(4))
      T        = (R1 – R3) * C54
      R1       = R1 + R3
      X(I(1))  = X(I(1)) + R1
      R1       = X(I(1)) + R1 * C55
      R3       = R1 – T
      R1       = R1 + T
      T        = (R4 + R2) * C51
      R4       = T + R4 * C52
      R2       = T + R2 * C53
      S1       = Y(I(2)) + Y(I(5))
      S4       = Y(I(2)) – Y(I(5))
      S3       = Y(I(3)) + Y(I(4))
      S2       = Y(I(3)) – Y(I(4))
      T        = (S1 – S3) * C54
      S1       = S1 + S3
      Y(I(1))  = Y(I(1)) + S1
      S1       = Y(I(1)) + S1 * C55
      S3       = S1 – T
      S1       = S1 + T
      T        = (S4 + S2) * C51
      S4       = T + S4 * C52
      S2       = T + S2 * C53
      X(IP(2))= R1 + S2
      X(IP(5))= R1 – S2
      X(IP(3))= R3 – S4
      X(IP(4))= R3 + S4
      Y(IP(2))= S1 – R2
      Y(IP(5))= S1 + R2
      Y(IP(3))= S3 + R4
      Y(IP(4))= S3 – R4
      GOTO 20
```

```
C --------------WFTA N = 7 ----------------------------------------------------------------------------------------C
107             R1        = X(I(2)) + X(I(7))
                R6        = X(I(2)) – X(I(7))
                S1        = Y(I(2)) + Y(I(7))
                S6        = Y(I(2)) – Y(I(7))
                R2        = X(I(3)) + X(I(6))
                R5        = X(I(3)) – X(I(6))
                S2        = Y(I(3)) + Y(I(6))
                S5        = Y(I(3)) – Y(I(6))
                R3        = X(I(4)) + X(I(5))
                R4        = X(I(4)) – X(I(5))
                S3        = Y(I(4)) + Y(I(5))
                S4        = Y(I(4)) – Y(I(5))
                T3        = (R1 – R2) * C74
                T         = (R1 – R3) * C72
                R1        = R1 + R2 + R3
                X(I(1))   = X(I(1)) + R1
                R1        = X(I(1)) + R1 * C71
                R2        = (R3 – R2) * C73
                R3        = R1 – T + R2
                R2        = R1 – R2 – T3
                R1        = R1 + T + T3
                T         = (R6 – R5) * C78
                T3        = (R6 + R4) * C76
                R6        = (R6 + R5 + R4) * C75
                R5        = (R5 + R4) * C77
                R4        = R6 – T3 + R5
                R5        = R6 – R5 – T
                R6        = R6 + T3 + T
                T3        = (S1 – S2) * C74
                T         = (S1 – S3) * C72
                S1        = S1 + S2 + S3
                Y(I(1))   = Y(I(1)) + S1
                S1        = Y(I(1)) + S1 * C71
                S2        = (S3 – S2) * C73
                S3        = S1 – T + S2
                S2        = S1 – S2 – T3
                S1        = S1 + T + T3
                T         = (S6 – S5) * C78
                T3        = (S6 + S4) * C76
                S6        = (S6 + S5 – S4) * C75
                S5        = (S5 + S4) * C77
                S4        = S6 – T3 + S5
                S5        = S6 – S5 – T
                S6        = S6 + T3 + T
                X(IP(2))  = R3 + S4
                X(IP(7))  = R3 – S4
                X(IP(3))  = R1 + S6
                X(IP(6))  = R1 – S6
                X(IP(4))  = R2 – S5
                X(IP(5))  = R2 + S5
```

```
            Y(IP(4))  = S2 + R5
            Y(IP(5))  = S2 – R5
            X(IP(2))  = S3 – R4
            Y(IP(7))  = S3 +R4
            Y(IP(3))  = S1 – R6
            Y(IP(6))  = S1 + R6
20        CONTINUE
10      CONTINUE
        RETURN
        END
```

Program 11.7: Real–Valued Split–Radix FFT

```
CC ==================================================================== CC
CC                                                                      CC
CC      Subroutine RSRFFT (X,M):                                        CC
CC              A real–valued, in–place, inverse split–radix FFT program CC
CC              Decimation–in–frequency, cos/sin in second loop         CC
CC              and is computed recursively                             CC
CC              Output in order:                                        CC
CC                  [ Re(0), Re(1),......, R(N/2), Im(N/2–1),....Im(1) ] CC
CC                                                                      CC
CC      Input/output:                                                   CC
CC              X     Array of input/output (length > = N)              CC
CC              M     Transform length is N = 2**M                      CC
CC                                                                      CC
CC      Calls:                                                          CC
CC              RSTAGE, RBITREV                                         CC
CC                                                                      CC
CC      Author:                                                         CC
CC              H. V. Sorensen, University of Pennsylvania, Oct. 1985   CC
CC                              Arpa address: hvs@ee.upenn.edu          CC
CC      Modified:                                                       CC
CC              F. Bonzanigo,    ETH–Zurich,                July 1986   CC
CC              H. V. Sorensen, University of Pennsylvania,  July 1987  CC
CC              H. V. Sorensen, University of Pennsylvania,  July 1987  CC
CC                                                                      CC
CC      Reference:                                                      CC
CC              H. V. Sorensen, Jones, Heideman, Burrus: “Real–valued fast Fourier CC
CC              transform algorithms,” IEEE Trans. ASSP, Vol. ASSP–35, No. 6, CC
CC              pp. 849–864, June 1987                                  CC
CC                                                                      CC
CC              This program may be used and distributed freely provided CC
CC              this header is included and left intact.                CC
CC                                                                      CC
CC ==================================================================== CC
        SUBROUTINE RSRFFT (X, M)
        Real X(2)
        N = 2**M
C ------------Digit reverse counter ---------------------------------------C
        CALL CBITREV (X,M)
```

```
C -------------Length two butterflies-----------------------------------------------------------------------------CC
      IS = 1
      ID = 4
50         DO 60 I0        = IS, N, ID
              T1          = X(I0)
              X(I0)       = TI + X(I0+1)
              X(I0+1)     = TI – X(I0+1)
60         CONTINUE
           IS = 2*ID – 1
           ID = 4*ID
      IF (IS.LT.N) GOTO 50
C -------------L shaped butterflies--------------------------------------------------------------------------------C
      N2 = 2
      DO 70 K = 2, M
           N2 = N2*2
           N4 = N2/4
           CALL RSTAGE (N, N2, N4, X(1), X(N4+1), X(2*N4+1), X(3*N4+1))
70    CONTINUE
      RETURN
      END
CC ====================================================================== CC
CC                                                                       CC
CC    Subroutine RSTAGE – the workhorse of the RFFT                      CC
CC            computes one stage of a real–valued split–radix length N   CC
CC            transform.                                                 CC
CC                                                                       CC
CC    Author:                                                            CC
CC            H. V. Sorensen, University of Pennsylvania, March 1987     CC
CC                                                                       CC
CC            This program may be used and distributed freely provided   CC
CC            this header is included and left intact.                   CC
CC                                                                       CC
CC ====================================================================== CC
      SUBROUTINE RSTAGE (N, N2, N4, X1, X2, X3, X4)
      DIMENSION X1(1), X2(1), X3(1), X4(1)
      N8 = N2/8
      IS = 0
      ID = 2*N2
10         DO 20 I1      = IS+1, N, ID
              T1       = X4(I1) + X3(I1)
              X4(I1) = X4(I1) – X3(I1)
              X3(I1) = X1(I1) – T1
              X1(I1) = X1(I1) + T1
20         CONTINUE
           IS = 2*ID – N2
           ID = 4*ID
      IF (IS. LT. N) GOTO 10
C
      IF (N4–1) 100, 100, 30
30    IS = 0
      ID = 2*N2
```

```
40          DO 50 I2 = IS + 1 + N8, N, ID
                  T1     = (X3(I2) + X4(I2)) * 0.707106811865475
                  T2     = (X3(I2) – X4(I2)) * 0.707106811865475
                  X4(I2) = X2 (I2) – T1
                  X3(I2) = –X2(I2) – T1
                  X2(I2) = X1(I2) – T2
                  X1(I2) = X1(I2) + T2
50          CONTINUE
            IS = 2*ID – N2
            ID = 4*ID
      IF (IS . LT. N) GOTO 40
C
      IF (N8–1) 100, 100, 60
60    E = 2. * 3.1415926535897323/N2
      SS1 = SIN(E)
      SD1 = SS1
      SD3 = 3 . *SD1 – 4 . *SD1**3
      SS3 = SD3
      CC1 = COS (E)
      CD1 = CC1
      CD3 = 4 . CD1**3 – 3 . *CD1
      CC3 = CD3
      DO 90 J = 2, N8
            IS = 0
            ID = 2*N2
            JN = N4 – 2*J + 2
70                DO 80 I1 = IS + J, N, ID
                        I2     = I1 + JN
                        T1     = X3(I1)*CC1 + X3(I2)*SS1
                        T2     = X3(I2)*CC1 – X3(I1)*SS1
                        T3     = X4(I1)*CC3 + X4(I2)*SS3
                        T4     = X4(I2)*CC3 – X4(I1)*SS3
                        T5     = T1 = T3
                        T3     = T1 – T3
                        T1     = T2 + T4
                        T4     = T2 – T4
                        X3(I1) = T1 – X2(I2)
                        X4(I2) = T1 + X2(I2)
                        X3(I2) = –X2(I1) – T3
                        X4(I1) = X2(I1) – T3
                        X2(I2) = X1(I1) – T5
                        X1(I1) = X1(I1) + T5
                        X2(I1) = X2(I2) + T4
                        X1(I2) = X2(I2) – T4
80                CONTINUE
                  IS = 2*ID – N2
                  ID = 4*ID
            IF (IS . LT. N–1) GOTO 70
C
            T1          = CC1*CD1 – SS1*SD1
            SS1         = CC1*SD1 + SS1*CD1
```

```
          CC1       = T1
          T3        = CC3*CD3 – SS3*SD3
          SS3       = CC3*SD3 + SS3*CD3
          CC3       = T3
90      CONTINUE
C
100     RETURN
        END
CC ======================================================================= CC
CC                                                                          CC
CC      Subroutine RBITREV (X,Y,M):                                         CC
CC           Bit reverses the array X of length 2**M. It generates a        CC
CC           table ITAB (minimum length is SQRT (2**M) if M is even         CC
CC           or SQRT(2*2**M) if M is odd). ITAB need only be generated      CC
CC           once for a given transform length.                             CC
CC                                                                          CC
CC      Author:                                                             CC
CC           H. V. Sorensen, University of Pennsylvania, Aug. 1987          CC
CC                         Arpa address: hvs@ee.upenn.edu                   CC
CC                                                                          CC
CC      Reference:                                                          CC
CC           D. Evans, Tran. ASSP, Aol. ASSP–35, No. 8, pp. 1120–1125,      CC
CC           Aug. 1987                                                      CC
CC                                                                          CC
CC           This program may be used and distributed freely provided       CC
CC           this header is included and left intact.                       CC
CC                                                                          CC
CC ======================================================================= CC
        SUBROUTINE RBITREV (X,M)
        DIMENSION X(1), ITAB (256)
C ------------- Initialization of ITAB array ------------------------------C
        M2 = M/2
        NBIT = 2**M2
        IF (2*M2 . NE . M) M2 = M2 + 1
        ITAB (1) = 0
        ITAB (2) = 1
        IMAX = 1
        DO 10 LBSS = 2, M2
          IMAX = 2 * IMAX
          DO 10 I = 1, IMAX
            ITAB (I) = 2 * ITAB(I)
            ITAB (I + IMAX) = 1 + ITAB(I)
10      CONTINUE
C ------------- The actual bit reversal -----------------------------------C
        DO 20 K = 2, NBIT
          JO = NBIT * ITAB (K) + 1
          I = K
          J = JO
          DO 20 L = 2, ITAB (K) + 1
            T1    = X(I)
            X(I)  = X(J)
```

```
                X(J)  = T1
                I     = I + NBIT
                J     = JO + ITAB(L)
20      CONTINUE
        RETURN
        END
```

Program 11.8: Inverse Real–Valued Split–Radix FFT

```
CC ============================================================ CC
CC                                                              CC
CC      Subroutine IRSRFFT (X,M):                               CC
CC           An inverse real–valued, in–place, inverse split–radix FFT program   CC
CC           Decimation–in–frequency, cos/sin in second loop    CC
CC           and is computed recursively                        CC
CC           Symmetric input in order:                          CC
CC                [ Re(0), Re(1),......, R(N/2), Im(N/2–1),....Im(1) ]   CC
CC                                                              CC
CC      Input/output:                                           CC
CC           X     Array of input/output (length > = N)         CC
CC           M     Transform length is N = 2**M                 CC
CC                                                              CC
CC      Calls:                                                  CC
CC           IRSTAGE, RBITREV                                   CC
CC                                                              CC
CC      Author:                                                 CC
CC           H. V. Sorensen, University of Pennsylvania, Oct. 1985   CC
CC                          Arpa address: hvs@ee.upenn.edu      CC
CC      Modified:                                               CC
CC           F. Bonzanigo,    ETH–Zurich,                  Sept. 1986   CC
CC           H. V. Sorensen,  University of Pennsylvania,  Mar. 1987    CC
CC                                                              CC
CC      Reference:                                              CC
CC           H. V. Sorensen, Jones, Heideman, Burrus: "Real–valued fast Fourier   CC
CC           transform algorithms," IEEE Trans. ASSP, Vol. ASSP–35, No. 6,   CC
CC           pp. 849–864, June 1987                             CC
CC                                                              CC
CC           This program may be used and distributed freely provided   CC
CC           this header is included and left intact.           CC
CC                                                              CC
CC ============================================================ CC
        SUBROUTINE IRSRFFT (X, M)
        Real X(1)
        N = 2**M
C ----------L shaped butterflies------------------------------------C
        N   = 2**M
        N2  = 2*N
        DO 70 K = 1, M – 1
              N2 = N2/2
              N4 = N2/4
              CALL IRSTAGE (N, N2, N4, X(1), X(N4+1), X(2*N4+1), X(3*N4+1))
10      CONTINUE
```

```
C -------------Length two butterflies-----------------------------------------------------------------------------CC
          IS = 1
          ID = 4
70             DO 60 I1= IS, N, ID
                    T1         = X(I1)
                    X(I1)      = TI + X(I1+1)
                    X(I1+1)  = TI – X(I1+1)
60             CONTINUE
               IS = 2*ID – 1
               ID = 4*ID
          IF (IS.LT.N) GOTO 70
C -------------Digit reverse counter ------------------------------------------------------------------------------C
          CALL CBITREV (X,M)
C -------------Divide by N---------------------------------------------------------------------------------------C
          DO 99 I = 1, N
               X(I) = X(I)/N
99        CONTINUE
          RETURN
          END
CC ========================================================================== CC
CC                                                                            CC
CC        Subroutine IRSTAGE – the workhorse of the IRFFT                     CC
CC                 computes one stage of an inverse real–valued split–radix length N     CC
CC                 transform.                                                 CC
CC                                                                            CC
CC        Author:                                                             CC
CC                 H. V. Sorensen, University of Pennsylvania, March 1987     CC
CC                                                                            CC
CC                 This program may be used and distributed freely provided   CC
CC                 this header is included and left intact.                   CC
CC                                                                            CC
CC ========================================================================== CC
          SUBROUTINE IRSTAGE (N, N2, N4, X1, X2, X3, X4)
          DIMENSION X1(1), X2(1), X3(1), X4(1)
          N8 = N2/8
          IS = 0
          ID = 2*N2
10             DO 20 I1= IS+1, N, ID
                    T1      = X4(I1) – X3(I1)
                    X1(I1) = X1(I1) + X3(I1)
                    X2(I1)= 2*X2(I1)
                    T2      = 2*X4(I1)
                    X4(I1)= T1 + T2
                    X3(I1)= T1 – T2
20             CONTINUE
               IS = 2*ID – N2
               ID = 4*ID
          IF (IS. LT. N) GOTO 10
C
          IF (N4–1) 100, 100, 30
30        IS = 0
```

```
      ID = 2*N2
40          DO 50 I2 = IS + 1 + N8, N, ID
                 T1      = (X2(I1) – X1(I1)) * 1.4142135623730950488
                 T1      = (X4(I1) + X3(I1)) * 1.4142135623730950488
                 X1(I1)= X1(I1) + X2 (I1)
                 X2(I1) = X4(I2) – X3(I1)
                 X3(I1)= – T2 – T1
                 X4(I1)= –T2 + T1
50          CONTINUE
            IS = 2*ID – N2
            ID = 4*ID
      IF (IS . LT. N) GOTO 40
C
      IF (N8–1) 100, 100, 60
60    E = 6.283185307179586/N2
      SS1 = SIN(E)
      SD1 = SS1
      SD3 = 3 . *SD1–4 . *SD1**3
      SS3 = SD3
      CC1 = COS (E)
      CD1 = CC1
      CD3 = 4 . CD1**3–3 . *CD1
      CC3 = CD3
      DO 90 J = 2, N8
            IS = 0
            ID = 2*N2
            JN = N4 – 2*J + 2
70                DO 80 I0 = IS + J, N, ID
                        I2      = I1 + JN
                        T1      = X1(I1) – X2(I2)
                        X1(I1) = X1(I1) + X2(I2)
                        T2      = X1(I2) – X2(I1)
                        X1(I2)= X2(I1) + X1(I2)
                        T3      = X4(I2) + X3(I1)
                        X2(I2) = X4(I2) – X3(I1)
                        T4      = X4(I1) + X3(I2)
                        X2(I1) = X4(I1) – X3(I2)
                        T5      = T1 – T4
                        T1      = T1 + T4
                        T4      = T2 – T3
                        T2      = T2 + T3
                        X3(I1) = T5*CC1 + T4*SS1
                        X3(I2) = T4*CC1 + T5*SS1
                        X4(I1) = T1*CC3 – T2*SS3
                        X4(I2) = T2*CC3 + T1*SS3
80                CONTINUE
                  IS = 2*ID – N2
                  ID = 4*ID
            IF (IS . LT. N–1) GOTO 70
C
            T1    = CC1*CD1 – SS1*SD1
```

```
             SS1  = CC1*SD1 + SS1*CD1
             CC1 = T1
             T3    = CC3*CD3 – SS3*SD3
             SS3  = CC3*SD3 + SS3*CD3
             CC3 = T3
90        CONTINUE
C
100       RETURN
          END
CC ===================================================================== CC
CC                                                                       CC
CC        Subroutine RBITREV (X, M):                                     CC
CC                Bit reverses the array X of length 2**M. It generates a CC
CC                table ITAB (minimum length is SQRT (2**M) if M is even CC
CC                or SQRT(2*2**M) if M is odd). ITAB need only be generated CC
CC                once for a given transform length.                     CC
CC                                                                       CC
CC        Author:                                                        CC
CC                H. V. Sorensen, University of Pennsylvania, Aug. 1987  CC
CC                                 Arpa address: hvs@ee.upenn.edu        CC
CC                                                                       CC
CC        Reference:                                                     CC
CC                D. Evans, Tran. ASSP, Aol. ASSP–35, No. 8, pp. 1120–1125, CC
CC                Aug. 1987                                              CC
CC                                                                       CC
CC                This program may be used and distributed freely provided CC
CC                this header is included and left intact.               CC
CC                                                                       CC
CC ===================================================================== CC
          SUBROUTINE RBITREV (X, M)
          DIMENSION X(1), ITAB (256)
C -------------Initialization of ITAB array --------------------------------C
          M2 = M/2
          NBIT = 2**M2
          IF (2*M2 . NE . M) M2 = M2 + 1
          ITAB (1) = 0
          ITAB (2) = 1
          IMAX = 1
          DO 10 LBSS = 2, M2
             IMAX = 2 * IMAX
             DO 10 I = 1, IMAX
                ITAB (I) = 2 * ITAB(I)
                ITAB (I + IMAX) = 1 + ITAB(I)
10        CONTINUE
C -------------The actual bit reversal-----------------------------------------C
          DO 20 K = 2, NBIT
             JO  = NBIT * ITAB (K) + 1
             I    = K
             J    = JO
             DO 20 L = 2, ITAB (K) + 1
                T1   = X(I)
```

```
                X(I) = X(J)
                X(J) = T1
                I    = I + NBIT
                J    = JO + ITAB(L)
20          CONTINUE
      RETURN
      END
```

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12
Analog Filter Approximations

12.1 Filter Definitions

12.1.1 Normalized *Ideal Low-pass Filter* (see Figure 12.1a)

$$H(j\omega) = e^{-j\omega} \qquad 0 \le |\omega| \le 1$$

$$= 0 \qquad |\omega| > 1$$

12.1.2 Filter Transfer Function

$$H(j\omega) = |H(j\omega)| e^{j\theta(\omega)}$$

$$\theta(\omega) = Arg\, H(j\omega)$$

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = \text{group delay}$$

$$\omega_c = \text{cutoff frequency at which } |H(j\omega_c)|^2 = \frac{1}{2}$$

or

$$20 \log |H(j\omega)|_{\omega=\omega_c} = -3\, dB$$

$$A(\omega) = -10 \log |H(j\omega)|^2 \ \ (= \text{attenuation})\ \ dB$$

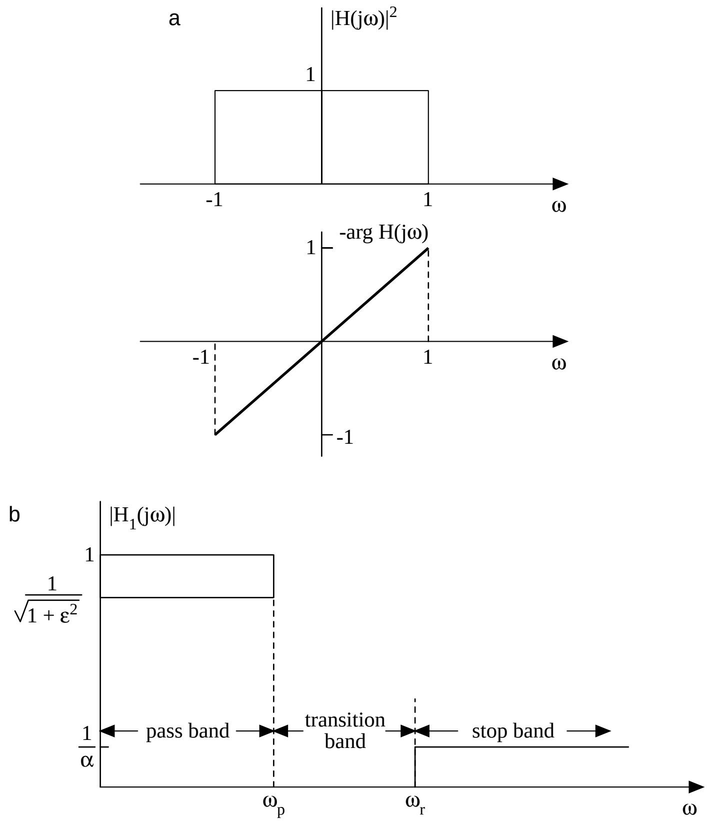

FIGURE 12.1

$$|H(j\omega)| = \left[H(j\omega)H^*(j\omega)\right]^{1/2} = \left[H(j\omega)H(-j\omega)\right]^{1/2} = \left[H(s)H(-s)\big|_{s=j\omega}\right]^{1/2}$$

$$|H(j\omega)|^2 = \frac{K(\omega^2 + z_1^2)(\omega^2 + z_2^2)\cdots}{(\omega^2 + p_1^2)(\omega^2 + p_2^2)\cdots}$$

Complex poles and zeros occur in conjugate pairs. Both the numerator and denominator polynomials of the magnitude squared function of a transfer function are polynomials of ω^2 with real coefficients, and these polynomials are greater than zero for all ω.

12.2 Butterworth Approximation

12.2.1 Definition of Butterworth Low-Pass Filter

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\dfrac{\omega}{\omega_c}\right)^{2n}}; \qquad |H(j\omega_c)|^2 = \frac{1}{2}$$

$$10\log|H(j\omega)|^2\Big|_{\omega=\omega_c} = -3.01 \cong -3.0\,dB$$

Normalized

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}}; \qquad |H(j1)|^2 = \frac{1}{2}$$

12.3 Properties of Butterworth Approximation

12.3.1 $|H(j0)|^2 = 1; \qquad |H(j1)|^2 = \frac{1}{2}; \qquad |H(j\infty)|^2 = 0$

$$-10\log|H(j1)|^2 = -10\log 0.5 = 3.01 \cong 3.0\,dB$$

12.3.2 $|H(j\omega)|^2$ monotonically decreasing for $\omega \geq 0$. Its maximum value is at $\omega = 0$.

12.3.3 The first (2*n* – 1) derivatives of an n[th] -order low-pass Butterworth filter are zero at $\omega = 0$ *maximally flat* magnitude).

12.3.4 The high-frequency roll-off of an n[th] -order filter is 20*n dB/decade*

$$-10\log|H(j\omega)|^2 = -\log\frac{1}{1+\omega^{2n}} \cong -\log\frac{1}{\omega^{2n}} = 10\log\omega^{2n} = 20\,n\log\omega\ dB$$

12.4 Transfer Function of Butterworth Approximation

12.4.1 $|H(j\omega)|^2 = H(s)H(-s)\big|_{s=j\omega} = \dfrac{1}{1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}} = \dfrac{1}{1+(-1)^n\left(\dfrac{s}{\omega_c}\right)^{2n}}$

Poles:

$$1+(-1)^n\left(\frac{s}{\omega_c}\right)^{2n} = 0 \quad \text{or} \quad s_k = \omega_c e^{j\pi(1-n+2K)/2n}, \qquad K = 0,1,\cdots,2n-1$$

12.4.2 Stable Function

Left-half-plane poles are used

$$s_K = \omega_c\left[-\sin\frac{(2K+1)\pi}{2n} + j\cos\frac{(2K+1)\pi}{2n}\right], \qquad K = 0,1,\cdots,n-1$$

12.4.3 Transfer Function

$$H(s) = (-1)^n \prod_{K=0}^{n-1} \frac{s_K}{s-s_K}$$

12.4.4 Butterworth Normalized Low-Pass Filter

Table 12.1 gives the Butterworth polynomials (ω_c = 1) to be used for normalized filters.

TABLE 12.1 Butterworth Normalized and Factored Polynomials

n	Butterworth Polynomials
1	$s + 1$
2	$s^2 + 1.41421s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 0.51764s + 1)(s^2 + 1.41421s + 1)(s^2 + 1.93185s + 1)$
7	$(s + 1)(s^2 + 0.44504s + 1)(s^2 + 1.24798s + 1)(s^2 + 1.80194s + 1)$
8	$(s^2 + 0.39018s + 1)(s^2 + 1.11114s + 1)(s^2 + 1.66294s + 1)(s^2 + 1.96157s + 1)$
9	$(s + 1)(s^2 + 0.34730s + 1)(s^2 + s + 1)(s^2 + 1.53209s + 1)(s^2 + 1.87939s + 1)$
10	$(s^2 + 0.31287s + 1)(s^2 + 0.90798s + 1)(s^2 + 1.41421s + 1)(s^2 + 1.78201s + 1)(s^2 + 1.97538s + 1)$

12.4.5 Butterworth Filter Specifications (see also Figure 12.1)

A_p = maximum passband attenuation

f_p = passband edge frequency

Maximum allowable attenuation in the stopband

f_r = stopband edge frequency

$$A_p = 10\log\left[1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n}\right] \quad \text{(see also 12.3.4)}$$

$$A_r = 10\log\left[1 + \left(\frac{\omega_r}{\omega_c}\right)^{2n}\right]$$

$$\omega_p = 2\pi f_p$$

$$\omega_r = 2\pi f_r$$

Solve A_p and A_r to find

$$n = \frac{\left|\log[(10^{0.1A_p} - 1)/(10^{0.1A_r} - 1)]\right|}{\left|\log(\omega_p/\omega_r)\right|}$$

$$k = \text{selectivity parameter} = \frac{\omega_p}{\omega_r} = \frac{f_p}{f_r} < 1$$

$$d = \text{discrimination factor} = \left(\frac{(10^{0.1A_p} - 1)}{(10^{0.1A_r} - 1)}\right)$$

Note: a) larger values of k imply steeper roll off, b) smaller d values imply greater difference between A_p and A_r

$$n \geq \frac{|\log d|}{|\log k|} \quad \text{(accept next higher integer to noninteger } n\text{)}$$

$$\omega_c = \frac{\omega_p}{(10^{0.1A_p} - 1)^{1/2n}}$$

$$\omega_c = \frac{\omega_r}{(10^{0.1A_r} - 1)^{1/2n}} \equiv \text{meets stopband attenuation exactly and exceeds the requirement of passband specification}$$

Figure 12.2 shows magnitude-squared characteristics of the Butterworth low-pass filter.

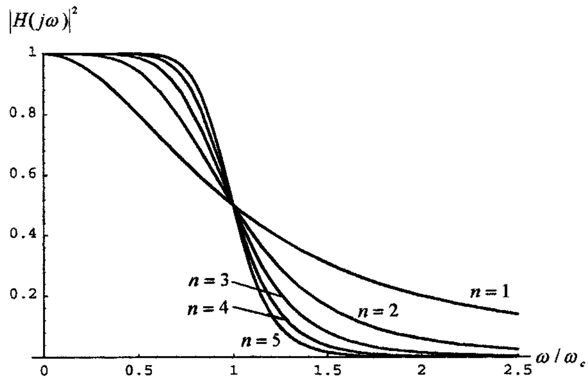

FIGURE 12.2

Example 12.1 Butterworth Filter Design

Filter requirements: a) no more than 1.5 dB deviation from ideal filter at 1300 Hz; b) at least 35 dB for frequencies above 6000 Hz.

Solution:

$$A_p = 1.5 dB \qquad \omega_p = 2\pi \times 1300 \text{ rad s}^{-1}$$

$$A_r = 35 dB \qquad \omega_r = 2\pi \times 6000 \text{ rad s}^{-1}$$

$$d = \frac{\sqrt{10^{0.1A_p} - 1}}{\sqrt{10^{0.1A_r} - 1}} = \frac{\sqrt{10^{0.15} - 1}}{\sqrt{10^{3.5} - 1}} = \frac{0.6423}{56.2252} = 1.1424 \times 10^{-2}$$

$$n \geq \frac{|\log d|}{|\log k|} = \frac{1.9422}{0.6576} = 2.953 \qquad \Rightarrow \qquad n = 3$$

$$s_k = -\sin\frac{(2K+1)\pi}{2n} + j\cos\frac{(2K+1)\pi}{2n} \qquad K = 0, 1, \cdots, n-1$$

$$s_o = -\sin\frac{\pi}{6} + j\cos\frac{\pi}{6} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$s_1 = -\sin\frac{3\pi}{6} + j\cos\frac{3\pi}{6} = -1$$

$$s_2 = -\sin\frac{5\pi}{6} + j\cos\frac{5\pi}{6} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$H(s) = (-1)^n \prod_{K=0}^{n-1} \frac{s_k}{s - s_k} = -\frac{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{s - \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} \frac{-1}{s - (-1)} \frac{-\frac{1}{2} - j\frac{\sqrt{3}}{2}}{s - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{(s+1)(s^2 + s + 1)} = \text{normalized}$$

$$\omega_c = \omega_p (10^{0.1A_p} - 1)^{-1/2n} = 2\pi \times 1300 (10^{0.15} - 1)^{-1/6} = 9416 \text{ rad s}^{-1}$$

$$H\left(\frac{s}{\omega_c}\right) = \frac{1}{\left(\frac{s}{9461} + 1\right)\left[\left(\frac{s}{9461}\right)^2 + \frac{s}{9461} + 1\right]}$$

12.5 Chebyshev Filter Approximation

12.5.1 Definition of Chebyshev Filters (equi-ripple passband)

$$C_o(\omega) = 1 \text{ and } C_1(\omega) = \omega. \quad |H(j\omega)|^2 = \frac{1}{1 + \varepsilon C_n^2(\omega)} = \text{normalized}$$

$$C_n(\omega) = \text{Chebyshev polynomials} = \cos(n \cos^{-1}\omega) \qquad 0 \le \omega \le 1$$

$$= \cosh(n \cosh^{-1}\omega) \qquad \omega > 1$$

ε = *ripple factor*

If we set $u = \cos^{-1}\omega$, then $C_n(\omega) = \cos nu$ and thus

$$C_o(\omega) = \cos 0 = 1, C_1(\omega) = \cos u = \cos(\cos^{-1}\omega) = \omega, \quad C_2(\omega) = \cos 2u = 2\cos^2 u - 1 = 2\omega^2 - 1,$$

$$C_3(\omega) = \cos 3u = 4\cos^3 u - 3\cos u = 4\omega^3 - 3\omega, \text{ etc.}$$

12.5.2 Recursive Formula for Chebyshev Polynomials

From $\cos[(n+1)u] = 2\cos nu \cos u - \cos[(n-1)u]$, we get

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega) \qquad n = 0,1,2,\cdots$$

with $C_o(\omega) = 1$ and $C_1(\omega) = \omega$. Figure 12.3 shows the first five Chebyshev polynomials.

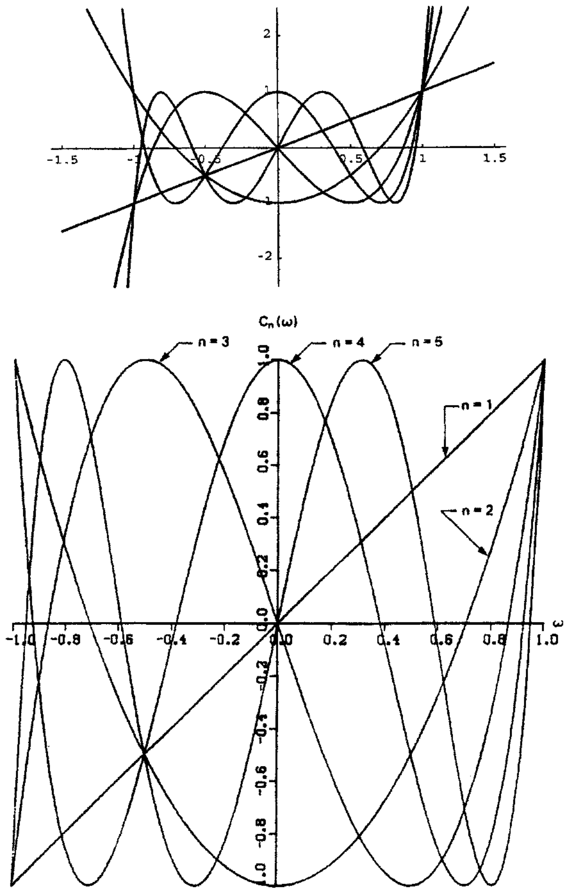

FIGURE 12.3 Chebyshev polynomials.

12.5.3 Table 12.2 gives the first ten Chebyshev polynomials

TABLE 12.2 Chebyshev Polynomials $C_n(\omega)$

n	Chebyshev Polynomials $C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

12.5.4 Properties of the Chebyshev Polynomials

1. For any n

$$0 \le |C_n(\omega)| \le 1 \quad \text{for} \quad 0 \le |\omega| \le 1$$
$$|C_n(\omega)| > 1 \quad \text{for} \quad |\omega| > 1$$

2. $C_n(1) = 1$ for any n
3. $|C_n(\omega)|$ increases monotonically for $|\omega| > 1$
4. $C_n(\omega)$ is an even (odd) polynomial if n is even (odd)
5. $|C_n(0)| = 0$ for odd n
6. $|C_n(0)| = 1$ for even n

12.5.5 Chebyshev Magnitude Response Properties

1. $|H(j\omega)|_{\omega=0} = 1$ when n is odd

$$= \frac{1}{\sqrt{1+\varepsilon^2}} \quad \text{when } n \text{ is even}$$

2. Since $C_n(1) = 1$ for any n

$$|H(j1)| = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \text{for any } n$$

3. $|H(j\omega)|$ decreases monotonically

12.5.6 Pole Location of Chebyshev Filters

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon C_n^2(\omega)} = \left.\frac{1}{1+\varepsilon C_n^2(-js)}\right|_{s=j\omega}$$

$$s = \sigma + j\omega$$

$$\sigma_K = \pm\sin\left[(2K+1)\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right]$$

$$\omega_K = \cos\left[(2K+1)\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right] \qquad K = 0,1,\cdots,2n-1$$

$$\frac{\sigma_K}{\sinh^2 y} + \frac{\omega_K}{\cosh^2 y} = 1 \qquad \text{an ellipse on the } \sigma-\omega \text{ plane}$$

$$y = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}$$

12.5.7 Design Relations of Chebyshev Filters

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon C_n^2\left(\frac{\omega}{\omega_p}\right)}$$

$$|H(j\omega_p)|^2 = \left.\frac{1}{1+\varepsilon C_n^2\left(\frac{\omega}{\omega_p}\right)}\right|_{\omega=\omega_p} = \frac{1}{1+\varepsilon^2}$$

$$A_p = 10\log(1+\varepsilon^2)$$

$$\varepsilon = \sqrt{10^{0.1A_p}-1}$$

$$A_r = 10\log\left[1+\varepsilon^2 C_n^2\left(\frac{\omega_r}{\omega_p}\right)\right]$$

$$= 10\log\left[1+\varepsilon^2\cosh^2\left[n\cosh^{-1}\left(\frac{\omega_r}{\omega_p}\right)\right]\right]$$

$$n \ge \frac{\cosh^{-1}\left(\frac{10^{0.1A_r}-1}{\varepsilon^2}\right)^{1/2}}{\cosh^{-1}\left(\frac{\omega_r}{\omega_p}\right)}$$

$$k = \frac{\omega_p}{\omega_r} = \frac{f_p}{f_r}, \qquad d = \left(\frac{10^{0.1A_p}-1}{10^{0.1A_r}-1}\right)^{1/2}$$

or

$$n \ge \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{k}\right)}$$

Left-Hand Poles for the Transfer Function

$$s_K = \sin\left[(2K+1)\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right] + j\cos\left[(2K+1)\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon}\right]$$

$$H(s) = -\prod_{K=0}^{n-1}\frac{s_K}{s-s_K}, \quad n \text{ odd}$$

$$H(s) = \frac{1}{\sqrt{1+\varepsilon^2}} \prod_{K=0}^{n-1} \frac{s_K}{s - s_K}, \quad n \text{ even}$$

For non-normalized transfer function set s/ω_p in place of s

$$\left|H(j\omega_c)\right|^2 = \frac{1}{2} = \frac{1}{1+\varepsilon^2 C_n^2(\omega_c)}, \qquad 3-dB \text{ cutoff}$$

$$\omega_c = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)$$

Example 12.2 (Chebyshev Filter Design):

Filter requirements: a) ripple not to exceed 2 dB up to ω_p; b) 50 dB rejection above 5 ω_p.

Solution

$$A_p \le 2\,dB \quad \text{at} \quad \omega = \omega_p$$

$$A_r \ge 50\,dB \quad \text{at} \quad \omega = \omega_r = 5\omega_p$$

$$\varepsilon = (10^{0.1A_p} - 1)^{1/2} = (10^{0.2} - 1)^{1/2} = 0.765$$

$$k = \frac{\omega_p}{\omega_r} = \frac{\omega_p}{5\omega_p} = 0.2$$

$$d = \frac{0.765}{(10^{0.1A_r} - 1)^{1/2}} = \frac{0.765}{(10^5 - 1)^{1/2}} = 2.42 \times 10^{-3}$$

$$n \ge \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = \frac{\ln(1/d + \sqrt{1/d^2 - 1})}{\ln(1/k + \sqrt{1/k^2 - 1})} = \frac{2.718}{2.312} = 2.91$$

$$\text{accept } n = 3$$

From 12.5.6

$$y = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon} = \frac{1}{n}\ell n\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}\right) = 0.361$$

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0.3689 \qquad \cosh y = \frac{e^y + e^{-y}}{2} = 1.0659$$

$$s_0 = \sin\left(\frac{\pi}{6}\right)(0.3689) + j\cos\left(\frac{\pi}{6}\right)(1.0659) = -0.1844 + j0.9231$$

$$s_1 = \sin\left(\frac{\pi}{2}\right)(0.3689) + j\cos\left(\frac{\pi}{2}\right)(1.0659) = -0.3689$$

$$s_2 = -\sin\left(\frac{\pi}{6}\right)(0.3689) + j\cos\left(\frac{5\pi}{6}\right)(1.0659) = -0.1844 - j0.9231 = s_0^*$$

$$H(s) = \frac{0.3289}{(s + 0.3689)(s^2 + 0.3689s + 0.8861)}$$

To denormalize $H(s)$ we set $\omega_p = 2\pi f_p$ we set s/ω_p in place of s. Table 12.3 gives the denominator or the normalized Chebyshev low-pass filters. Figure 12.4 shows the third-order filter with $\omega_p = 2\pi \times 2$.

TABLE 12.3 Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters

n	0.1-dB Ripple (ε = 0.15262)
1	$s + 6.55220$
2	$s^2 + 2.37236s + 3.31403$
3	$(s + 0.96941)(s^2 + 0.96941s + 1.68975)$
4	$(s^2 + 0.52831s + 1.33003)(s^2 + 1.27546s + 0.62292)$
5	$(s + 0.53891)(s^2 + 0.33307s + 1.19494)(s^2 + 0.87198s + 0.63592)$
6	$(s^2 + 0.22939s + 1.12939)(s^2 + 0.62670s + 0.69637)(s^2 + 0.85608s + 0.26336)$
7	$(s + 0.37678)(s^2 + 0.16768s + 1.09245)(s^2 + 0.46983s + 0.75322)(s^2 + 0.67893s + 0.33022)$
8	$(s^2 + 0.12796s + 1.06949)(s^2 + 0.36440s + 0.79889)(s^2 + 0.54536s + 0.41621)(s^2 + 0.64330s + 0.14561)$
9	$(s + 0.29046)(s^2 + 0.10088s + 1.05421)(s^2 + 0.29046s + 0.83437)$
	$\cdot (s^2 + 0.44501s + 0.49754)(s^2 + 0.54589s + 0.20134)$
10	$(s^2 + 0.08158s + 1.04351)(s^2 + 0.23675s + 0.86188)(s^2 + 0.36874s + 0.56799)$
	$\cdot (s^2 + 0.46464s + 0.27409)(s^2 + 0.51506s + 0.09246)$

n	0.2-dB Ripple (ε = 0.21709)
1	$s + 4.60636$
2	$s^2 + 1.92709s + 2.35683$
3	$(s + 0.81463)(s^2 + 0.81463s + 1.41363)$
4	$(s^2 + 0.44962s + 1.19866)(s^2 + 1.08548s + 0.49155)$
5	$(s + 0.46141)(s^2 + 0.28517s + 1.11741)(s^2 + 0.74658s + 0.55839)$
6	$(s^2 + 0.19705s + 1.07792)(s^2 + 0.53835s + 0.64491)(s^2 + 0.73540s + 0.21190)$
7	$(s + 0.32431)(s^2 + 0.14433s + 1.05566)(s^2 + 0.40441s + 0.71644)(s^2 + 0.58439s + 0.29343)$
8	$(s^2 + 0.11028s + 1.04183)(s^2 + 0.31407s + 0.77124)(s^2 + 0.47004s + 0.38855)(s^2 + 0.55445s + 0.11795)$
9	$(s + 0.25057)(s^2 + 0.08702s + 1.03263)(s^2 + 0.25057s + 0.81278)$
	$\cdot (s^2 + 0.38389s + 0.47596)(s^2 + 0.47092s + 0.17976)$
10	$(s^2 + 0.44461s + 0.07513)(s^2 + 0.40109s + 0.25677)(s^2 + 0.31830s + 0.55066)$
	$\cdot (s^2 + 0.20436s + 0.84455)(s^2 + 0.07042s + 1.02619)$

n	0.5-dB Ripple (ε = 0.34931)
1	$s + 2.86278$
2	$s^2 + 1.42562s + 1.51620$
3	$(s + 0.62646)(s^2 + 0.62646s + 1.14245)$
4	$(s^2 + 0.35071s + 1.06352)(s^2 + 0.84668s + 0.35641)$
5	$(s + 0.36232)(s^2 + 0.22393s + 1.03578)(s^2 + 0.58625s + 0.47677)$
6	$(s^2 + 0.15530s + 1.02302)(s^2 + 0.42429s + 0.59001)(s^2 + 0.57959s + 0.15610)$
7	$(s + 0.25617)(s^2 + 0.11401s + 1.01611)(s^2 + 0.31944s + 0.67688)(s^2 + 0.46160s + 0.25388)$
8	$(s^2 + 0.08724s + 1.01193)(s^2 + 0.24844s + 0.74133)(s^2 + 0.37182s + 0.35865)(s^2 + 0.43859s + 0.08805)$
9	$(s + 0.19841)(s^2 + 0.06891s + 1.00921)(s^2 + 0.19841s + 0.78937)$
	$\cdot (s^2 + 0.30398s + 0.45254)(s^2 + 0.37288s + 0.15634)$
10	$(s^2 + 0.05580s + 1.00734)(s^2 + 0.161934s + 0.82570)(s^2 + 0.25222s + 0.53181)$
	$\cdot (s^2 + 0.31781s + 0.23791)(s^2 + 0.35230s + 0.05628)$

n	1-dB Ripple (ε = 0.50885)
1	$s + 1.96523$
2	$s^2 + 1.09773s + 1.10251$
3	$(s + 0.49417)(s^2 + 0.49417s + 0.99421)$
4	$(s^2 + 0.27907s + 0.98651)(s^2 + 0.67374s + 0.27940)$
5	$(s + 0.28949)(s^2 + 0.17892s + 0.98832)(s^2 + 0.46841s + 0.42930)$

TABLE 12.3 Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters (continued)

n	1-dB Ripple (ε = 0.50885)
6	$(s^2 + 0.12436s + 0.99073)(s^2 + 0.33976s + 0.55772)(s^2 + 0.46413s + 0.12471)$
7	$(s + 0.20541)(s^2 + 0.09142s + 0.99268)(s^2 + 0.25615s + 0.65346)(s^2 + 0.37014s + 0.23045)$
8	$(s^2 + 0.07002s + 0.99414)(s^2 + 0.19939s + 0.72354)(s^2 + 0.29841s + 0.34086)(s^2 + 0.35110s + 0.07026)$
9	$(s + 0.15933)(s^2 + 0.05533s + 0.99523)(s^2 + 0.15933s + 0.77539)$
	$\cdot (s^2 + 0.24411s + 0.43856)(s^2 + 0.29944s + 0.14236)$
10	$(s^2 + 0.04483s + 0.99606)(s^2 + 0.13010s + 0.81442)(s^2 + 0.20263s + 0.52053)$
	$\cdot (s^2 + 0.25533s + 0.22664)(s^2 + 0.28304s + 0.04500)$

n	1.5-dB Ripple (ε = 0.64229)
1	$s + 1.55693$
2	$s^2 + 0.92218s + 0.92521$
3	$(s + 0.42011)(s^2 + 0.42011s + 0.92649)$
4	$(s^2 + 0.23826s + 0.95046)(s^2 + 0.57521s + 0.24336)$
5	$(s + 0.24765)(s^2 + 0.15306s + 0.96584)(s^2 + 0.40071s + 0.40682)$
6	$(s^2 + 0.10650s + 0.97534)(s^2 + 0.29097s + 0.54233)(s^2 + 0.39747s + 0.10932)$
7	$(s + 0.17603)(s^2 + 0.07834s + 0.98147)(s^2 + 0.21951s + 0.64225)(s^2 + 0.31720s + 0.21924)$
8	$(s^2 + 0.06003s + 0.98561)(s^2 + 0.17094s + 0.71501)(s^2 + 0.25583s + 0.33233)(s^2 + 0.30177s + 0.06173)$
9	$(s + 0.13667)(s^2 + 0.04745s + 0.98852)(s^2 + 0.13664s + 0.76867)$
	$\cdot (s^2 + 0.20934s + 0.43185)(s^2 + 0.25679s + 0.13565)$
10	$(s^2 + 0.03845s + 0.99063)(s^2 + 0.11159s + 0.80900)(s^2 + 0.17381s + 0.51510)$
	$\cdot (s^2 + 0.21901s + 0.22121)(s^2 + 0.24277s + 0.03958)$

n	2-dB Ripple (ε = 0.76478)
1	$s + 1.30756$
2	$s^2 + 0.80382s + 0.82306$
3	$(s + 0.36891)(s^2 + 0.36891s + 0.88610)$
4	$(s^2 + 0.20978s + 0.92868)(s^2 + 0.50644s + 0.22157)$
5	$(s + 0.21831)(s^2 + 0.13492s + 0.95217)(s^2 + 0.35323s + 0.39315)$
6	$(s^2 + 0.09395s + 0.96595)(s^2 + 0.25667s + 0.53294)(s^2 + 0.35061s + 0.09993)$
7	$(s + 0.15533)(s^2 + 0.06913s + 0.97462)(s^2 + 0.19371s + 0.63539)(s^2 + 0.27991s + 0.21239)$
8	$(s^2 + 0.05298s + 0.98038)(s^2 + 0.15089s + 0.70978)(s^2 + 0.22582s + 0.32710)(s^2 + 0.26637s + 0.05650)$
9	$(s + 0.12063)(s^2 + 0.04189s + 0.98440)(s^2 + 0.12063s + 0.76455)$
	$\cdot (s^2 + 0.18482s + 0.42773)(s^2 + 0.22671s + 0.13153)$
10	$(s^2 + 0.03395s + 0.98730)(s^2 + 0.09853s + 0.80567)(s^2 + 0.15347s + 0.51178)$
	$\cdot (s^2 + 0.19338s + 0.21788)(s^2 + 0.21436s + 0.03625)$

n	2.5-dB Ripple (ε = 0.88220)
1	$(s + 1.13353)$
2	$(s^2 + 0.71525s + 0.75579)$
3	$(s + 0.32995)(s^2 + 0.32995s + 0.85887)$
4	$(s^2 + 0.18796s + 0.91386)(s^2 + 0.45378s + 0.20676)$
5	$(s + 0.19577)(s^2 + 0.12099s + 0.94284)(s^2 + 0.31677s + 0.38382)$
6	$(s^2 + 0.08429s + 0.95953)(s^2 + 0.23028s + 0.52651)(s^2 + 0.31456s + 0.09350)$
7	$(s + 0.13941)(s^2\ 0.06204s + 0.96992)(s^2 + 0.17384 + 0.63070)(s^2 + 0.25120s + 0.20769)$
8	$(s^2 + 0.04756s + 0.97680)(s^2 + 0.13054s + 0.70620)(s^2 + 0.20269s + 0.32352)(s^2 + 0.23909s + 0.05292)$
9	$(s + 0.10829)(s^2 + 0.03761s + 0.98157)(s^2 + 0.10829s + 0.76173)$
	$\cdot (s^2 + 0.16591s + 0.42490)(s^2 + 0.20352s + 0.12870)$
10	$(s^2 + 0.19245s + 0.03396)(s^2 + 0.17361s + 0.21560)(s^2 + 0.13778s + 0.50949)$
	$\cdot (s^2 + 0.08846s + 0.80338)(s^2 + 0.03048s + 0.98502)$

TABLE 12.3 Factors of the Denominator Polynomials Normalized Chebyshev Low-Pass Filters (continued)

n	3-dB Ripple (ε = 0.99763)
1	(s + 1.00238)
2	(s^2 + 0.64490s + 0.70795)
3	(s + 0.29862) (s^2 + 0.29862s + 0.83917)
4	(s^2 + 0.17034s + 0.90309) (s^2 + 0.41124s + 0.19598)
5	(s + 0.17753) (s^2 + 0.10970s + 0.93603) (s^2 + 0.28725s + 0.37701)
6	(s^2 + 0.07646s + 0.95483) (s^2 + 0.20889s + 0.52182) (s^2 + 0.28535s + 0.08880)
7	(s + 0.12649) (s^2 0.05629s + 0.96648) (s^2 + 0.15773 + 0.62726) (s^2 + 0.22792s + 0.20425)
8	(s^2 + 0.04316s + 0.97417) (s^2 + 0.12290s + 0.70358) (s^2 + 0.18393s + 0.32089) (s^2 + 0.21696s + 0.05029)
9	(s + 0.09827) (s^2 + 0.03413s + 0.97950) (s^2 + 0.09827s + 0.75966) · (s^2 + 0.15057s + 0.42283) (s^2 + 0.18470s + 0.12664)
10	(s^2 + 0.02766s + 0.98335) (s^2 + 0.08028s + 0.80171) (s^2 + 0.12504s + 0.50782) · (s^2 + 0.15757s + 0.21393) (s^2 + 0.17466s + 0.03229)

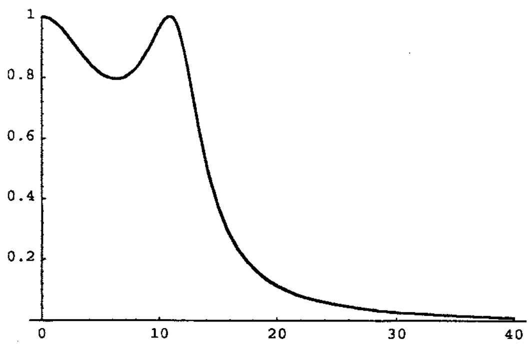

FIGURE 12.4 $f_p = 2H_z$; $\omega_p = 12.5664$

12.6 Inverse-Chebyshev Approximation

12.6.1 Definition

The inverse-Chebyshev filter is flat in the passband and equi-ripple in the stopband.

12.6.2 The Magnitude-Squared Transfer Function

$$|H(j\omega)|^2 = \frac{\varepsilon^2 C_n^2(\omega_r/\omega)}{1+\varepsilon^2 C_n^2(\omega_r/\omega)}$$

$C_n(\omega)$ = Chebyshev polynomial; ω_r = stopband edge frequency

12.6.3 Attenuation

$$A(\omega) = 10\log\left(1+\frac{1}{\varepsilon^2 C_n^2(\omega_r/\omega)}\right) dB$$

ε = ripple factor calculated at $\omega = \omega_r$

$$A_r(\omega) = 10\log\left(1+\frac{1}{\varepsilon^2 C_n^2(1)}\right), \quad C_n^2(1) = 1$$

$$\varepsilon = \frac{1}{\sqrt{10^{0.1A_r} - 1}}$$

12.6.4 Filter Order

$$n \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$$

$$k = \frac{f_p}{f_r}, \quad d = \left(\frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)^{1/2}$$

12.6.5 Poles and Zeros

$$H(s)H(-s) = \frac{\varepsilon^2 C_n^2(j\omega_r/s)}{1 + \varepsilon^2 C_n^2(j\omega_r/s)}$$

Zeros

$$C_n(j\omega_r/s) = 0 = \cos(n\cos^{-1}(j\omega_r/s))$$

$$\cos^{-1}(j\omega_r/s) = m\pi/2n, \quad m \text{ odd}$$

$$s_m = zeros = j\omega_r \sec(m\pi/2n), \quad m = 1,3,\cdots,2n-1$$

Poles

$$1 + \varepsilon^2 C_n^2(j\omega_r/s) = 0$$

same poles as in 12.5.6 except that $-s$ is replaced by $1/s$.

Denormalization is accomplished with respect o stopband edge frequency ω_r.

12.7 Elliptic Filters

12.7.1 Square Magnitude Response Function for Elliptic Filters

$$|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 R_n^2(\omega)}$$

$R_n(\omega)$ = rational function; ε = ripple factor

12.7.2 Properties of the Rational Function $R_n(\omega)$

1. $R_n(\omega)$ = even for n even. $R_n(\omega)$ = odd for n odd.
2. The zeros of $R_n(\omega)$ are in the range $|\omega| < 1$.
   The poles of $R_n(\omega)$ are in the range $|\omega| > 1$.
3. The function $R_n(\omega)$ oscillates between ±1 in the passband.
4. $R_n(\omega)$ = 1 at ω = 1.
5. $R_n(\omega)$ oscillates between $\pm 1/d$ and infinity in the stopband, where d is given in 12.5.7.

12.7.3 The Rational Normalized Function $R_n(\omega)$ with Respect to Center Frequency $\omega_0 = 1$

$$R_n(\omega) = \omega \prod_{i=1}^{(n-1)/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \qquad \text{for } n \text{ odd}$$

$$R_n(\omega) = \prod_{i=1}^{n/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \qquad \text{for } n \text{ even}$$

12.7.4 Steps to Calculate the Elliptic Filter

1. Find the selectivity factor k

$$k = \frac{\omega_p}{\omega_r}, \quad \omega_p = \text{passband frequency}, \ \omega_r = \text{stopband frequency}$$

2. Define

$$q_o = \frac{1}{2} \frac{1 - (1 - k^2)^{1/4}}{1 + (1 - k^2)^{1/4}}$$

3. Find the expression

$$q = q_o + 2q_o^5 + 15q_o^9 + 150q_o^{13}$$

4. Find d

$$d = \left(\frac{10^{0.1A_p} - 1}{10^{0.1A_r} - 1} \right)^{1/2}$$

5. Find the filter order n

$$n \geq \frac{\log(16/d^2)}{\log(1/q)}$$

6. Calculate ε

$$\varepsilon = \left(10^{0.1A_p} - 1\right)^{1/2}$$

$$A_p = 10 \log(1 + \varepsilon^2)$$

7. Define

$$\beta = \frac{1}{2n} \ell n \frac{(1 + \varepsilon^2)^{1/2} + 1}{(1 + \varepsilon^2)^{1/2} - 1}$$

8. Calculate

$$a = \frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)}\sinh[(2m+1)\beta]}{1+\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cosh(2m\beta)}$$

9. Define

$$U = \left[(1+ka^2)\left(1+\frac{a^2}{k}\right)\right]^{1/2}$$

10. Calculate

$$\omega_i = \frac{2q^{1/4}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)}\sin[(2m+1)\pi\ell/n]}{1+2\sum_{m=1}^{\infty}(-1)^m q^{m^2}\cos(2m\pi\ell/n)}$$

$\ell = i-\frac{1}{2},\ i=1,2,\cdots,\frac{n}{2},\ n=\text{even},\ \ell=i,\ i=1,2,\cdots,(n-1)/2,\ n=\text{odd}$

11. Define

$$V_i = \left[(1-k\omega_i^2)(1-\frac{\omega_i^2}{k})\right]^{1/2}$$

12. Set

$$a_i = \frac{1}{\omega_i^2}$$

13. Set

$$b_i = \frac{2aV_i}{1+a^2\omega_i^2}$$

14. Set

$$c_i = \frac{(aV_i)^2+(\omega_i U)^2}{(1+a^2\omega_i^2)^2}$$

15. Find

$$H_o = a\prod_{i=1}^{(n-1)/2}\frac{c_i}{a_i} \qquad \text{for } n = \text{odd}$$

$$H_o = \frac{1}{\sqrt{1+\varepsilon^2}} \prod_{i=1}^{n/2} \frac{c_i}{a_i} \qquad \text{for } n = \text{even}$$

$$H(s) = H_o \prod_{i=1}^{n/2} \frac{s^2 + a_i}{s^2 + b_i s + c_i} \qquad \text{for } n = \text{even}$$

$$H(s) = \frac{H_o}{s+a} \prod_{i=1}^{(n-1)/2} \frac{s^2 + a_i}{s^2 + b_i s + c_i} \qquad \text{for } n \text{ odd}$$

12.7.5 Unormalized Transfer Function

Replace s in the 15th step above (in Section 12.7.4) with s/ω_0 where $\omega_o = \sqrt{\omega_p \omega_r}$.

Note: Summations in steps 8 and 10 above converge fast, and up to four or five terms of the series will provide good accuracy.

12.8 Elliptic Filters (Second Approach*)

12.8.1 Transfer Function

$$H(s) = \frac{H_o}{D_o(s)} \prod_{i=1}^{r} \frac{s^2 + a_{oi}}{s^2 + b_{1i} s + b_{oi}}$$

$$r = \begin{cases} \dfrac{n-1}{2} & \text{for odd } n \\ \dfrac{n}{2} & \text{for even } n \end{cases}$$

$$D_o(s) = \begin{cases} s + \sigma_o & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$

12.8.2 Steps of Implementation

Given

ω_p = passband frequency; ω_r = stopband frequency;
A_p = maximum passband loss (dB); A_r = minimum stopband loss (dB)
k = selectivity factor = ω_p/ω_r

Steps

1. $k^1 = \sqrt{1-k^2}$

2. $q_o = \dfrac{1}{2}\left(\dfrac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)$

3. $q = q_o + 2q_o^5 + 15q_o^9 + 150q_o^{13}$

* Antoniou (1993)

4. $D = \dfrac{10^{0.1A_r} - 1}{10^{0.1A_p} - 1}$

5. $n \geq \dfrac{\log 16D}{\log(1/q)}$

6. $\Lambda = \dfrac{1}{2n} \ell n \dfrac{10^{0.05A_p} + 1}{10^{0.05A_p} - 1}$

7. $$\sigma_o = \left| \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh[(2m+1)\Lambda]}{1 + 2\sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh 2m\Lambda} \right|$$

8. $W = \sqrt{(1 + k\sigma_o^2)\left(1 + \dfrac{\sigma_o^2}{k}\right)}$

9. $$\Omega_i = \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sin\left[\dfrac{(2m+1)\pi\mu}{n}\right]}{1 + 2\sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos\left[\dfrac{2m\pi\mu}{n}\right]}$$

$$\mu = \begin{cases} i & \text{for odd } n \\ i - \dfrac{1}{2} & \text{for even } n \end{cases} \qquad i = 1, 2, \cdots, r$$

10. $V_i = \sqrt{(1 - k\Omega_i^2)\left(1 - \dfrac{\Omega_i^2}{k}\right)}$

11. $a_{oi} = \dfrac{1}{\Omega_i^2}$

$$b_{oi} = \frac{(\sigma_o V_i)^2 + (\Omega_i W)^2}{(1 + \sigma_o^2 \Omega_i^2)^2}$$

$$b_{1i} = \frac{2\sigma_o V_i}{1 + \sigma_o^2 \Omega_i^2}$$

12. $$H_o = \begin{cases} \sigma_o \prod_{i=1}^{r} \dfrac{b_{oi}}{a_{oi}} & \text{for odd } n \\ 10^{-0.05A_p} \prod_{i=1}^{r} \dfrac{b_{oi}}{a_{oi}} & \text{for even } n \end{cases}$$

The series in steps 7 and 9 converge rapidly, and three to four terms are sufficient for most purposes.

Example 12.3 Requirements for an Elliptic Filter:

$$\omega_p = \sqrt{0.9} \text{ rad/s}, \ \ \omega_r = 1/\sqrt{0.9} \text{ rad/s}, \ \ A_p = 0.1 \text{ dB}, \ \text{ and } \ A_r \geq 50.0 \text{ dB}$$

Results from steps 1 through 5 above:

$$k = 0.9, \ k' = 0.43589, \ q_o = 0.10233, \ q = 0.102352, \ D = 4{,}293{,}090, \ n \geq 7.92 \ \text{ or } \ n = 8$$

The coefficients for $H(s)$ are found from the rest of the steps and are:

i	a_{oi}	b_{oi}	b_{1i}
1	1.434825×10	2.914919×10^{-1}	8.711574×10^{-1}
2	2.231643	6.123726×10^{-1}	4.729136×10^{-1}
3	1.320447	8.397386×10^{-1}	1.825141×10^{-1}
4	1.128832	9.264592×10^{-1}	4.471442×10^{-2}

$H_o = 2.876332 \times 10^{-3}$

Hence from 12.8.1

$$H(s) = 2.87633 \times 10^{-3} \left(\frac{s^2 + 14.34825}{s^2 + 0.8711574s + 0.2914919} \right) \left(\frac{s^2 + 2.231643}{s^2 + 0.4729136s + 0.6123726} \right)$$

$$\times \left(\frac{s^2 + 1.320447}{s^2 + 0.1825141s + 0.8397386} \right) \left(\frac{s^2 + 1.128832}{s^2 + 0.04471442s + 0.9264592} \right)$$

and $|H(j\omega)|$ is plotted in Figure 12.5.

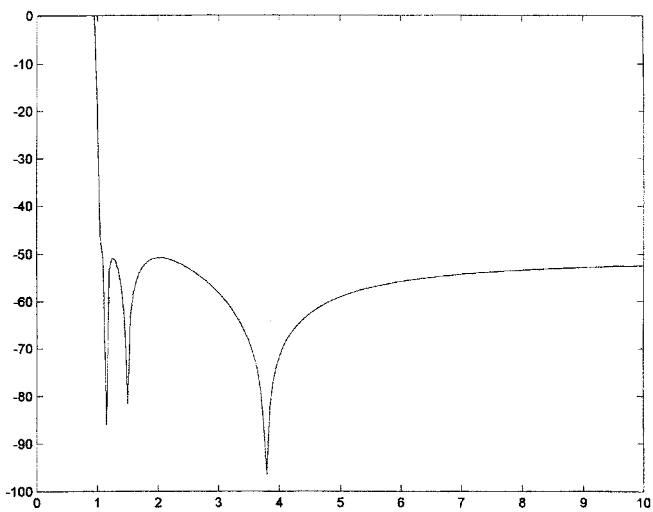

FIGURE 12.5

12.9 Transformations

12.9.1 Lowpass to Lowpass

$$\text{Set } s \to \frac{s}{\omega_p}, \quad \omega_p = \text{new passband frequency}$$

12.9.2 Lowpass to Highpass

$$\text{Set } s \to \frac{\omega_p}{s}, \quad \omega_p = \text{new passband frequency}$$

12.9.3 Lowpass to Bandpass

$$\text{Set } s \to \frac{s^2 + \omega_m^2}{Bs}$$

ω_m = geometric mean of the upper band edge frequency ω_u and the lower band edge frequency $\omega_\ell = \sqrt{\omega_u \omega_\ell}$

$B = \omega_u - \omega_\ell$ = filter bandwidth

12.9.4 Lowpass to Bandstop

$$s \to \frac{Bs}{s^2 + \omega_m^2}$$

ω_m and B are the same as in 12.9.3

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13

Sine and Cosine Transforms

13.1 Fourier Cosine Transform (FCT)

13.1.1 Definitions of FCT

$$F_c\{f(t)\} = F_c(\omega) = \int_0^\infty f(t)\cos\omega t\,dt \qquad \omega \ge 0$$

$$F_c^{-1}\{F_c(\omega)\} = f(t) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega t\,dt \qquad t \ge 0$$

The sufficient conditions for the inversion formula are that $f(t)$ be absolutely integrable in $[0,\infty)$ and that $f(t)$ be piece-wise continuous in each bounded subinterval of $[0,\infty)$. At the point t_o where $f(t)$ has a jump discontinuity $f(t) = [f(t_o+0) + f(t_o-0)]/2$.

13.1.2 Properties of the FCT

13.1.2.1 Transform of Derivatives

$$F_c\{f''(t)\} = \int_0^\infty f''(t)\cos\omega t\,dt = -\omega^2 F_c(\omega) - f'(0)$$

$f(t)$ and $f'(t)$ vanish as $t \to \infty$ and are continuous in $[0,\infty)$.

If $f(t)$ and $f'(t)$ have a jump discontinuity at t_o of d and d', respectively,

$$F_c\{f''(t)\} = -\omega^2 F_c(\omega) - f'(0) - \omega d \sin \omega t_o - d' \cos \omega t_o$$

$$d = f(t_o + 0) - f(t_o - 0), \qquad d' = f'(t_o + 0) - f'(t_o - 0)$$

13.1.2.2 Scaling

$$F_c\{f(at)\} = \frac{1}{a} F_c\left(\frac{\omega}{a}\right) \qquad a > 0$$

13.1.2.3 Shifting in t-domain

If $f_e(t) = f(|t|)$ and $f(t)$ is piece-wise continuous and absolutely integrable $[0,\infty)$, then

$$F_c\{f_e(t+a) + f_e(t-a)\} = 2F_c(\omega)\cos a\omega \qquad a > 0$$

$$F_c\{f(t+a) + f(|t-a|)\} = 2F_c(\omega)\cos a\omega$$

13.1.2.4 Shifting in the ω-domain

$$F_c(\omega + \beta) = F_c\{f(t)\cos\beta t\} - F_s\{f(t)\sin\beta t\} \qquad \beta > 0$$

where $\beta > 0$. F_s means the sine transform.

$$F_c(\omega - \beta) = F_c\{f(t)\cos\beta t\} + F_s\{f(t)\sin\beta t\} \qquad \beta > 0$$

$$F_c\{f(t)\cos\beta t\} = \frac{1}{2}\left[F_c(\omega+\beta) + F_c(\omega-\beta)\right]$$

$$F_c\{f(at)\cos\beta t\} = \frac{1}{2a}\left[F_c\left(\frac{\omega+\beta}{a}\right) + F_c\left(\frac{\omega-\beta}{a}\right)\right] \qquad a, \beta > 0$$

13.1.2.4.1 Differentiation in the ω -Domain

$$F_c^{(2n)}(\omega) = F_c\{(-1)^n t^{2n} f(t)\}$$

13.1.2.5 Asymptotic Behavior

$$\lim_{\omega\to\infty} F_c(\omega) = 0$$

13.1.2.6 Integration in the t-Domain

$$F_c\left\{\int_t^\infty f(\tau)\,d\tau\right\} = \frac{1}{\omega} F_s(\omega), \qquad F_s(\cdot) = \text{sine transform}$$

13.1.2.7 Convolution

For $f_e(t) = f(|t|)$ and $g_e(t) = g(|t|)$ then

$$f_e(t) * g_e(t) = \int_0^\infty f(\tau)[g(t+\tau) + g(|t-\tau|)]\,d\tau$$

$$F_c\left\{\int_0^\infty f(\tau)[g(t+\tau) + g(|t-\tau|)]\,d\tau\right\} = 2F_c(\omega)G_c(\omega)$$

13.1.3 Examples of FCT

Pulse Function

$$F_c\{p_a(t-1)\} = \int_0^\infty p_a(t-a)\cos\omega t\,dt = \int_0^{2a} \cos\omega t\,dt = \frac{\sin 2a\omega}{\omega}$$

Lambda Function

$$f(t) = \begin{cases} t/a \quad , & 0 < t < a \\ (2a-t)/a, & a < t < 2a \\ 0 \quad , & t > 2a \end{cases}$$

$$F_c\{f(t)\} = \int_0^a \frac{t}{a}\cos\omega t\,dt + \int_0^{2a} \frac{2a-t}{a}\cos\omega t\,dt = \frac{1}{a\omega^2}(2\cos a\omega - \cos 2a\omega - 1)$$

Inverse Function

$$f(t) = \frac{1}{t} \qquad\qquad t > a$$

$$F_c\{f(t)\} = \int_a^\infty \frac{1}{t}\cos\omega t\,dt = \int_{a\omega}^\infty \frac{1}{\tau}\cos\tau d\tau = -Ci(a\omega)$$

$$Ci(y) = -\int_y^\infty \frac{1}{\tau}\cos\tau\,d\tau = \text{cosine integral function}$$

Exponential Function

$$f(t) = e^{-at} \qquad a > 0,\ \ t \ge 0$$

$$F_c\{f(t)\} = \int_0^\infty e^{-at}\cos\omega t\,dt = \frac{a}{a^2+\omega^2}$$

Decaying Cosine

$$f(t) = e^{-bt}\cos at, \qquad a,b > 0,\ \ t \ge 0$$

$$F_c\{f(t)\} = \int_0^\infty e^{-bt}\cos at\,\cos\omega t\,dt = \frac{b}{2}\left[\frac{1}{b^2+(a-\omega)^2} + \frac{1}{b^2+(a+\omega)^2}\right]$$

by setting $\cos at = \dfrac{e^{jat}+e^{-jat}}{2}$ and $\cos\omega t = \dfrac{e^{j\omega t}+e^{-j\omega t}}{2}$.

13.2 Fourier Sine Transform (FST)

13.2.1 Definition FST

$$F_s(\omega) = F_s\{f(t)\} = \int_0^\infty f(t)\sin\omega t\,dt \qquad \omega > 0$$

$$f(t) = F_s^{-1}\{F_s(\omega)\} = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega \qquad t \ge 0$$

13.2.2 Properties of FST

13.2.2.1 Transforms of Derivatives

$$F_s\{f''(t)\} = -\omega^2 F_s(\omega) + \omega f(0)$$

$$F_s\{f'(t)\} = -\omega\, F_c(\omega)$$

13.2.2.2 Scaling

$$F_s\{f(at)\} = \frac{1}{a} F_s(\omega / a) \qquad a > 0$$

13.2.2.3 Shifting in t-Domain

$$f_e(t) = f(|t|) \qquad f_0(t) = \frac{t}{|t|} f(|t|)$$

$$F_s\{f_0(t+a) + f_0(t-a)\} = 2F_s(\omega)\cos a\omega$$

$$F_c\{f_0(t+a) - f_0(t-a)\} = 2F_s(\omega)\sin a\omega \qquad a > 0$$

13.2.2.4 Shifting in ω-Domain

$$F_s(\omega+\beta) = F_s\{f(t)\cos\beta t\} + F_c\{f(t)\sin\beta t\}$$

$$F_s\{f(t)\cos\beta t\} = \frac{1}{2}\left[F_s(\omega+\beta) + F_s(\omega-\beta)\right]$$

$$F_s\{f(at)\cos\beta t\} = \frac{1}{2a}\left[F_s\left(\frac{\omega+\beta}{a}\right) + F_s\left(\frac{\omega-\beta}{a}\right)\right]$$

$$F_s\{f(at)\sin\beta t\} = -\frac{1}{2a}\left[F_c\left(\frac{\omega+\beta}{a}\right) - F_c\left(\frac{\omega-\beta}{a}\right)\right]$$

13.2.2.5 Differentiation in the ω-Domain

$$F_s^{(2n)}(\omega) = F_s\{(-1)^n t^{2n} f(t)\}$$

$$F_s^{(2n+1)}(\omega) = F_c\{(-1)^n t^{2n+1} f(t)\}$$

13.2.2.6 Asymptotic Behavior

$$\lim_{\omega\to\infty} F_s(\omega) = 0$$

13.2.2.7 Integration in the t-Domain

$$F_s\left\{\int_o^t f(\tau)\,d\tau\right\} = \frac{1}{\omega} F_c(\omega)$$

13.2.2.8 Integration in the ω-Domain

$$F_c^{\pm 1}\left\{\int_\omega^\infty F_s(\beta)\,d\beta\right\} = \frac{1}{t}f(t)$$

13.2.2.9 The Convolution Property

$$2F_s(\omega)G_c(\omega) = F_s\left\{\int_0^\infty f(\tau)[g(|t-\tau|) - g(t+\tau)]\,d\tau\right\}$$

13.2.3 Examples of FST

Pulse Function

$$F_s\left\{p_{a/2}\left(t-\frac{a}{2}\right)\right\} = \int_0^a \sin\omega t\,dt = \frac{1-\cos\omega a}{\omega} \qquad a > 0$$

Lambda Function

$$f(t) = \begin{cases} t/a \quad , & 0 < t < a \\ (2a-t)/a, & a < t < 2a \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$F_s\{f(t)\} = \int_0^a \frac{t}{a}\sin\omega t\,dt + \int_a^{2a}\frac{2a-t}{a}\sin\omega t\,dt = \frac{1}{a\omega^2}(2\sin a\omega - \sin 2a\omega)$$

Inverse Function

$$f(t) = \frac{1}{t} \qquad t > a$$

$$F_s\{f(t)\} = \int_a^\infty \frac{1}{t}\sin\omega t\,dt = \int_{a\omega}^\infty \frac{1}{\tau}\sin\tau\,d\tau = -si(a\omega)$$

$$si(y) = -\int_y^\infty \frac{\sin x}{x}dx = \int_0^y \frac{\sin x}{x}dx - \int_0^\infty \frac{\sin x}{x}dx = Si(y) - \frac{\pi}{2}$$

Exponential Function

$$f(t) = e^{-at} \qquad a > 0,\ t \ge 0$$

$$F_s\{f(t)\} = \int_0^\infty e^{-at}\sin\omega t\,dt = \frac{\omega}{a^2+\omega^2}$$

by setting $\sin\omega t = (e^{j\omega t} - e^{-j\omega t})/2j$

Decaying Cosine

$$f(t) = e^{-bt}\cos at, \qquad a,b > 0,\ t \ge 0$$

$$F_s\{f(t)\} = \int_0^\infty e^{-bt}\cos at \,\sin\omega t\,dt = \frac{1}{2}\left[\frac{\omega - a}{b^2 + (\omega - a)^2} + \frac{\omega + a}{b^2 + (\omega + a)^2}\right]$$

by setting $\cos at = (e^{jat} + e^{-jat})/2$ and $\sin\omega t = (e^{j\omega t} - e^{-j\omega t})/2j$.

13.3 Discrete Cosine Transform (DCT)

13.3.1 Transform Kernel

$$K_c(\omega,t) = \cos\omega t$$

$$K_c(m,n) = K_c(\omega_m, t_n) = \cos(2\pi mn\,\Delta f\,\Delta t)$$

$$\omega_m = 2\pi m\,\Delta f = \text{sampled angular frequency}$$

$$t_n = \text{sampled time}$$

$$\Delta f, \Delta t = \text{sample intervals of frequency and time}$$

$$m,n = \text{positive integers}$$

If we set

$$\Delta f\,\Delta t = 1/2N$$

$$K_c(m,n) = \cos(\pi mn/N)$$

13.3.2 Discrete Cosine Transform (DCT)

If a finite duration signal is divided into N intervals of Δt each, there exists $N + 1$ sample points. If these $N + 1$ points are represented by vector $\underline{x}$, the DCT $\underline{X}_c$ is

$$\underline{X}_c = [C]\underline{x}, \qquad [C] = (N+1)\times(N+1) \text{ matrix}$$

where

$$[C]_{mn} = \text{matrix element of } [C] = \sqrt{2/N}\left[k_m k_n \cos\frac{mn\pi}{N}\right] \qquad m,n = 0,1,\ldots,N$$

$$k_i = 1 \text{ for } i \ne 0 \text{ or } N, \qquad k_i = \frac{1}{\sqrt{2}} \text{ for } i = 0 \text{ or } N$$

$$X_c(m) = \text{element of } \underline{X}_c = \sqrt{2/N}\sum_{n=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right)x(n)$$

$$x(n) = \text{element of } \underline{x} = \sqrt{2/N}\sum_{m=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right)X_c(m)$$

13.4 Discrete Sine Transform (DST)

13.4.1 Discrete Sine Transform (DST)

$$[S]_{mn} = \text{matrix element of } [S] = \sqrt{2/N}\sin\left(\frac{mn\pi}{N}\right) \qquad m,n = 1,2,\ldots,N-1$$

$$\underline{X}_s = [S]\underline{x}, \qquad [S] = (N-1)\times(N-1) \text{ matrix (boundary points are zero)}$$

$$\underline{x} = N-1 \text{ data vector}, \ \underline{X}_s = N-1 \text{ DST vector}$$

$$X_s(m) = \text{element of } \underline{X}_s = \sqrt{2/N}\sum_{n=1}^{N-1}\sin\left(\frac{mn\pi}{N}\right)x(n)$$

$X_s(m), x(n)$ DST pairs

$$x(n) = \text{element of } \underline{x} = \sqrt{2/N}\sum_{m=1}^{N-1}\sin\left(\frac{mn\pi}{N}\right)X_s(m)$$

13.5 Properties of DCT and DST

13.5.1 The Unitary Property

If $\underline{c}_m$ denotes the m^th^ column vector of matrix $[C]$ then

$$\underline{c}_m^T\underline{c}_n = 0 \qquad \text{for } m \neq n$$

$$\underline{c}_m^T\underline{c}_n = 1 \qquad \text{for } m = n \neq 0$$

$$\underline{c}_m^T\underline{c}_n = 1 \qquad \text{for } m = n = 0$$

or

$$\underline{c}_m^T\underline{c}_n = \delta_{mn}, \qquad \delta_{mn} = \text{Kronecker delta}$$

The same applies for matrix [*S*].

13.5.2 Inverse Transformation

$$[C]^{-1} = [C], \quad [S]^{-1} = [S]$$

These are **unitary** symmetric matrices.

13.5.3 Scaling

$$\Delta f\,\Delta t = 1/2N \qquad \text{or} \qquad \Delta f = 1/2N\,\Delta t$$

A change of Δt to $a\Delta t$ changes Δf to $\Delta f/a$, provided N remains the same. If we set $T = N\Delta t$, the time duration of the data, then

$$\Delta f = 1/2T.$$

13.6 FCT and FST Algorithm Based on FFT

13.6.1 FCT of Real Data Sequence

$$\{x(n),\ n = 0, 1, \ldots, N\} = \text{sequence with } N+1 \text{ points}$$

$$X_c(m) = \sqrt{2/N} \sum_{n=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right) x(n)$$

$$k_n = 1 \quad \text{for } n \neq 0 \ \ or \ \ N$$

$$k_n = 1/\sqrt{2} \quad \text{for } n = 0 \ \ or \ \ N$$

Construct an even or symmetric sequence using the sequence as follows:

$$\begin{aligned} s(n) &= x(n), && 0 < n < N \\ &= 2x(n), && n = 0, N \\ &= x(2N - n), && N < n \le 2N - 1 \end{aligned}$$

DFT of $\{s(n)\}$ is

$$S_F(m) = 2\left[x(0) + (-1)^m x(N) + \sum_{n=1}^{N-1} \cos\left(\frac{mn\pi}{N}\right) x(n)\right]$$

The $(N + 1)$ – point DCT of $\{x(n)\}$ is the same as the 2N-point DFT of the sequence $\{s(n)\}$. Hence the DCT of $\{x(n)\}$ can be computed using a 2N-point FFT of $\{s(n)\}$.

13.6.2 FST of Real Data Sequence

Let $\{x(n),\ n = 1, \ldots, N-1\}$ be an $(N - 1)$ – point data sequence. Its DST is (see 13.4.1)

$$X_s(m) = \sqrt{2/N} \sum_{n=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) x(n)$$

Construct a $(2N - 1)$ – point odd or skew-symmetric sequence $\{s(n)\}$ using $\{x(n)\}$.

$$\begin{aligned} s(n) &= x(n), && 0 < n < N \\ &= 0, && n = 0, N \\ &= -x(2N - n), && N < n \le 2N - 1 \end{aligned}$$

The $2N$-point DFT of $\{s(n)\}$ is

$$S_F(m) = -2j \sum_{n=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) x(n)$$

13.7 Fourier Cosine Transform Pairs

13.7.1 Fourier Cosine Transform Properties

TABLE 13.1 Properties of FCT

	$f(t)$	$F_c(\omega)=\int_o^\infty f(t)\cos\omega t\,dt \quad \omega>0$
1.	$F_c(t)$	$(\pi/2)f(\omega)$
2.	$f(at) \quad a>0$	$(1/a)F_c(\omega/a)$
3.	$f(at)\cos bt \quad a,b>0$	$(1/2a)\left[F_c\left(\frac{\omega+b}{a}\right)+F_c\left(\frac{\omega-b}{a}\right)\right]$
4.	$f(at)\sin bt \quad a,b>0$	$(1/2a)\left[F_s\left(\frac{\omega+b}{a}\right)-F_s\left(\frac{\omega-b}{a}\right)\right]$
5.	$t^{2n}f(t)$	$(-1)^n\frac{d^{2n}}{d\omega^{2n}}F_c(\omega)$
6.	$t^{2n+1}f(t)$	$(-1)^n\frac{d^{2n+1}}{d\omega^{2n+1}}F_s(\omega)$
7.	$\int_o^\infty f(r)[g(t+r)+g(\lvert t-r\rvert)]dr$	$2F_c(\omega)G_c(\omega)$
8.	$\int_t^\infty f(r)\,dr$	$(1/\omega)F_s(\omega)$
9.	$f(t+a)-f_o(t-a)$	$2F_s(\omega)\sin a\omega \quad a>0$
10.	$\int_o^\infty f(r)[g(t+r)-g_o(t-r)]dr$	$2F_s(\omega)G_s(\omega)$

13.7.2 Fourier Cosine Transform Pairs (see Section 13.9 for notation and definitions)

TABLE 13.2 Fourier Cosine Transform Pairs

	$f(t)$	$F_c(\omega) \quad \omega>0$
1.	$(1/\sqrt{t})$	$\sqrt{(\pi/2)}\,(1/\omega)^{1/2}$
2.	$(1/\sqrt{t})[1-U(t-1)]$	$(2\pi/\omega)^{1/2}\,C(\omega)$
3.	$(1/\sqrt{t})U(t-1)$	$(2\pi/\omega)^{1/2}\,[1/2-C(\omega)]$
4.	$(t+a)^{-1/2} \quad \lvert\arg a\rvert<\pi$	$(\pi/2\omega)^{1/2}\{\cos a\omega[1-2C(a\omega)]+\sin a\omega[1-2S(a\omega)]\}$
5.	$(t-a)^{-1/2}U(t-a)$	$(\pi/2\omega)^{1/2}[\cos a\omega-\sin a\omega]$
6.	$a(t^2+a^2)^{-1} \quad a>0$	$(\pi/2)\exp(-a\omega)$
7.	$t(t^2+a^2)^{-1} \quad a>0$	$-1/2[e^{-a\omega}\,\overline{\mathrm{Ei}}(a\omega)+e^{a\omega}\,\mathrm{Ei}(a\omega)]$
8.	$(1-t^2)(1+t^2)^{-2}$	$(\pi/2)\omega\exp(-\omega)$
9.	$-t(t^2-a^2)^{-1} \quad a>0$	$\cos a\omega\mathrm{Ci}(a\omega)+\sin a\omega\mathrm{Si}(a\omega)$
10.	$1 \quad 0<t<a$ $0 \quad a<t<\infty$	$\frac{1}{\omega}\sin(a\omega)$

TABLE 13.2 Fourier Cosine Transform Pairs (continued)

	$f(t)$	$F_c(\omega) \quad \omega > 0$
11.	$1 \quad 0<t<a$; $1/t \quad a<t<\infty$	$-\mathrm{Ci}(a\omega)$
12.	$\dfrac{[(a^2+t^2)^{1/2}+a]^{1/2}}{[a^2+t^2]^{1/2}} \quad a>0$	$\left(\dfrac{2\omega}{\pi}\right)^{-1/2} e^{-a\omega}$
13.	$\dfrac{1}{(a+jt)^{\nu}}+\dfrac{1}{(a-jt)^{\nu}}, \quad a>0,\ \nu>0$	$\pi[\Gamma(\nu)]^{-1}\omega^{\nu-1}e^{-a\omega}$
14.	$e^{-at} \quad \mathrm{Re}\, a>0$	$a(a^2+\omega^2)^{-1}$
15.	$(1+t)e^{-t}$	$2(1+\omega^2)^{-2}$
16.	$\sqrt{t}\, e^{-at} \quad \mathrm{Re}\, a>0$	$\dfrac{\sqrt{\pi}}{2}(a^2+\omega^2)^{-3/4}\cos[3/2\tan^{-1}(\omega/a)]$
17.	$e^{-at}/\sqrt{t} \quad \mathrm{Re}\, a>0$	$\sqrt{(\pi/2)}(a^2+\omega^2)^{-1/2}[(a^2+\omega^2)^{1/2}+a]^{1/2}$
18.	$t^n e^{-at} \quad \mathrm{Re}\, a>0$	$n![a/(a^2+\omega^2)]^{n+1}\sum_{2m=0}^{n+1}(-1)^m\binom{n+1}{2m}\left(\dfrac{\omega}{a}\right)^{2m}$
19.	$\exp(-at^2)/\sqrt{t}, \quad \mathrm{Re}\, a>0$	$\pi(\omega/8a)^{1/2}\exp(-\omega^2/8a)\, I_{-1/4}(-\omega^2/8a)$
20.	$t^{2n}\exp(-a^2t^2), \quad \lvert\arg a\rvert<\pi/4$	$(-1)^n\sqrt{\pi}\,2^{-n-1}a^{-2n-1}$ $\cdot\exp[-(\omega/2a)^2]\mathrm{He}_{2n}(2^{-1/2}\omega/a)$
21.	$t^{-3/2}\exp(-a/t), \quad \mathrm{Re}\, a>0$	$(\pi/a)^{1/2}\exp[-(2a\omega)^{1/2}]\cos(2a\omega)^{1/2}$
22.	$t^{-1/2}\exp(-a/\sqrt{t}), \quad \mathrm{Re}\, a>0$	$(\pi/2\omega)^{1/2}[\cos(2a\sqrt{\omega})-\sin(2a\sqrt{\omega})]$
23.	$e^{-at^2} \quad \mathrm{Re}\, a>0$	$\dfrac{\sqrt{\pi}}{2}\dfrac{1}{\sqrt{a}}e^{-\omega^2/(4a)}$
24.	$t^{\nu-1}e^{-at} \quad \mathrm{Re}\, a>0, \quad \mathrm{Re}\,\nu>0$	$\Gamma(\nu)(a^2+\omega^2)^{-\nu/2}\cos[\nu\tan^{-1}(\omega/a)]$
25.	$t^{-1/2}\ln t$	$-(\pi/2\omega)^{1/2}[\ln(4\omega)+\mathrm{C}+\pi/2]$
26.	$(t^2-a^2)^{-1}\ln t, \quad a>0$	$(\pi/2\omega)\{\sin(a\omega)[\mathrm{ci}(a\omega)-\ln a]$ $-\cos(a\omega)[\mathrm{si}(a\omega)-\pi/2]\}$
27.	$t^{-1}\ln(1+t)$	$(1/2)\{[\mathrm{ci}(\omega)]^2+[\mathrm{si}(\omega)]^2\}$
28.	$\exp(-t/\sqrt{2})\sin(\pi/4+t/\sqrt{2})$	$(1+\omega^4)^{-1}$
29.	$\exp(-t/\sqrt{2})\cos(\pi/4+t/\sqrt{2})$	$\omega^2(1+\omega^4)^{-1}$
30.	$\ln\dfrac{a^2+t^2}{1+t^2}, \quad a>0$	$(\pi/\omega)[\exp(-\omega)-\exp(-a\omega)]$
31.	$\ln[1+(a/t)^2], \quad a>0$	$(\pi/\omega)[1-\exp(-a\omega)]$
32.	$\ln(1+e^{-at}), \quad \mathrm{Re}\, a>0$	$\dfrac{1}{2}a\dfrac{1}{\omega^2}-\dfrac{1}{2}\dfrac{\pi}{\omega}\mathrm{csc}h\left(\pi\dfrac{\omega}{a}\right)$
33.	$t^{-1}e^{-t}\sin t$	$(1/2)\tan^{-1}(2\omega^{-2})$
34.	$t^{-2}\sin^2(at)$	$\begin{cases}(\pi/2)(a-\omega/2) & \omega<2a\\ 0 & \omega>2a\end{cases}$
35.	$\left(\dfrac{\sin t}{t}\right)^n \quad n=2,3,\ldots$	$\begin{cases}\dfrac{n\pi}{2^n}\displaystyle\sum_{r>0}^{r<(\omega+n)/2}\dfrac{(-1)^r(\omega+n-2r)^{n-1}}{r!(n-r)!}, & 0<\omega<n\\ 0 & n\le\omega\end{cases}$

TABLE 13.2 Fourier Cosine Transform Pairs (continued)

	$f(t)$	$F_c(\omega) \quad \omega > 0$
36.	$\exp(-\beta t^2)\cos at, \quad \mathrm{Re}\,\beta > 0$	$(1/2)(\pi/\beta)^{1/2}\exp\left(-\frac{a^2+\omega^2}{4\beta}\right)\cosh\left(\frac{a\omega}{2\beta}\right)$
37.	$(a^2+t^2)^{-1}(1-2\beta\cos t+\beta^2)^{-1}$ $\quad \mathrm{Re}\,a>0, \ \|\beta\|<1$	$(1/2)(\pi/a)(1-\beta^2)^{-1}(e^a-\beta)^{-1}$ $\cdot(e^{a-a\omega}+\beta e^{a\omega}), \ 0\le\omega<1$
38.	$\sin(at^2), \quad a>0$	$(1/4)(2\pi/a)^{1/2}\left[\cos\left(\frac{\omega^2}{4a}\right)-\sin\left(-\frac{\omega^2}{4a}\right)\right]$
39.	$\sin[a(1-t^2)], \quad a>0$	$-(1/2)(\pi/a)^{1/2}\cos[a+\pi/4+\omega^2/(4a)]$
40.	$\cos(at^2), \quad a>0$	$(1/4)(2\pi/a)^{1/2}\left[\cos\left(\frac{\omega^2}{4a}\right)+\sin\left(\frac{\omega^2}{4a}\right)\right]$
41.	$\cos[a(1-t^2)], \quad a>0$	$(1/2)(\pi/a)^{1/2}\sin[a+\pi/4+\omega^2/(4a)]$
42.	$\frac{\sin at}{t}, \quad a>0$	$\begin{cases}\frac{\pi}{2} & \omega<a\\ \frac{\pi}{4} & \omega=a\\ 0 & \omega>a\end{cases}$
43.	$e^{-\beta t}\sin at, \quad a>0, \ \mathrm{Re}\,\beta>0$	$\frac{\frac{1}{2}(a+\omega)}{\beta^2+(a+\omega)^2}+\frac{\frac{1}{2}(a-\omega)}{\beta^2+(a-\omega)^2}$
44.	$\frac{\sin t}{te^{-t}}$	$\frac{1}{2}\tan^{-1}\left(\frac{2}{\omega^2}\right)$
45.	$\frac{\sin^2(at)}{t^2}, \quad a>0$	$\begin{cases}\frac{\pi}{2}(a-\frac{1}{2}\omega) & \omega<2a\\ 0 & 2a<\omega\end{cases}$
46.	$\frac{1-\cos at}{t^2}, \quad a>0$	$\begin{cases}\frac{\pi}{2}(a-\omega) & \omega<a\\ 0 & a<\omega\end{cases}$
47.	$e^{-\beta t}\cos at, \quad \mathrm{Re}\,\beta>\|\mathrm{Im}\,a\|$	$\frac{\beta}{2}\left[\frac{1}{\beta^2+(a-\omega)^2}+\frac{1}{\beta^2+(a+\omega)^2}\right]$
48.	$\begin{cases}\frac{\cos[b(a^2-t^2)^{1/2}]}{(a^2-t^2)^{1/2}} & 0<t<a\\ 0 & a<t<\infty\end{cases}$	$\frac{\pi}{2}J_o[a(b^2+\omega^2)^{1/2}]$
49.	$\frac{\tan^{-1}\left(\frac{t}{a}\right)}{t}$	$-\frac{\pi}{2}Ei(-a\omega)$
50.	$e^{-t^2}He_{2n}(2t)$	$\frac{\sqrt{\pi}}{2}(-1)^n e^{-\omega^2/4}He_{2n}(\omega)$
51.	$e^{-t^2/2}[He_n(t)]^2$	$\sqrt{\frac{\pi}{2}}\ n!\ e^{-\omega^2/2}L_n(\omega^2)$
52.	$J_o(at), \quad a>0$	$\begin{cases}(a^2-\omega^2)^{-1/2} & 0<\omega<a\\ \infty & \omega=a\\ 0 & a<\omega<\infty\end{cases}$

13.8 Fourier Sine Transform Pairs

13.8.1 Fourier Sine Transform Properties

TABLE 13.3 Fourier Sine Transform Properties

	$f(t)$	$F_s(\omega)=\int_o^{\infty} f(t)\sin\omega t\,dt \quad \omega>0$
1.	$F_s(t)$	$(\pi/2)f(\omega)$
2.	$f(at),\ \ a>0$	$(1/a)F_s(\omega/a)$
3.	$f(at)\cos bt,\ \ a,b>0$	$(1/2a)\left[F_s\left(\frac{\omega+b}{a}\right)+F_s\left(\frac{\omega-b}{a}\right)\right]$
4.	$f(at)\sin bt,\ \ a,b>0$	$-(1/2a)\left[F_c\left(\frac{\omega+b}{a}\right)-F_c\left(\frac{\omega-b}{a}\right)\right]$
5.	$t^{2n}f(t)$	$(-1)^n\frac{d^{2n}}{d\omega^{2n}}F_s(\omega)$
6.	$t^{2n+1}f(t)$	$(-1)^{n+1}\frac{d^{2n+1}}{d\omega^{2n+1}}F_c(\omega)$
7.	$\int_o^{\infty} f(r)\int_{\|t-r\|}^{t+r} g(s)\,ds\,dr$	$(2/\omega)F_s(\omega)G_s(\omega)$
8.	$f_o(t+a)+f_o(t-a)$	$2F_s(\omega)\cos a\omega$
9.	$f_e(t-a)-f_e(t+a)$	$2F_c(\omega)\sin a\omega$
10.	$\int_o^{\infty} f(r)[g(\|t-r\|)-g(t+r)]\,dr$	$2F_s(\omega)G_c(\omega)$

13.8.2 Fourier Sine Transform Pairs

TABLE 13.4 Fourier Sine Transform Pairs

	$f(t)$	$F_s(\omega)$
1.	$1/t$	$\pi/2$
2.	$1/\sqrt{t}$	$(\pi/2\omega)^{1/2}$
3.	$1/\sqrt{t}\,[1-U(t-1)]$	$(2\pi/\omega)^{1/2}S(\omega)$
4.	$(1/\sqrt{t}\,)U(t-1)$	$(2\pi/\omega)^{1/2}[1/2-S(\omega)]$
5.	$(t+a)^{-1/2},\ \ \|\arg a\|<\pi$	$\begin{cases}(\pi/2\omega)^{1/2}\{\cos a\omega[1-2S(a\omega)] \\ \quad -\sin a\omega[1-2C(a\omega)]\}\end{cases}$
6.	$(t-a)^{-1/2}U(t-a)$	$(\pi/2\omega)^{1/2}(\sin a\omega+\cos a\omega)$
7.	$t(t^2+a^2)^{-1},\quad a>0$	$(\pi/2)\exp(-a\omega)$
8.	$t(a^2-t^2)^{-1},\quad a>0$	$-(\pi/2)\cos a\omega$
9.	$t(t^2+a^2)^{-2},\quad a>0$	$(\pi\omega/4a)\exp(-a\omega)$
10.	$a^2[t(t^2+a^2)]^{-1},\quad a>0$	$(\pi/2)[1-\exp(-a\omega)]$
11.	$t(4+t^4)^{-1}$	$(\pi/4)\exp(-\omega)\sin\omega$

TABLE 13.4 Fourier Sine Transform Pairs (continued)

	$f(t)$	$F_s(\omega)$
12.	$\begin{cases} 1 & 0<t<a \\ 0 & a<t<\infty \end{cases}$	$\dfrac{1-\cos a\omega}{\omega}$
13.	$\dfrac{\left[\sqrt{t^2+a^2}-a\right]^{1/2}}{(t^2+a^2)^{1/2}}$	$\pi\dfrac{e^{-a\omega}}{\sqrt{2\omega}}$
14.	e^{-at}, $\operatorname{Re} a>0$	$\omega(a^2+\omega^2)^{-1}$
15.	te^{-at}, $\operatorname{Re} a>0$	$(2a\omega)(a^2+\omega^2)^{-2}$
16.	$t(1+at)e^{-at}$, $\operatorname{Re} a>0$	$(8a^3\omega)(a^2+\omega^2)^{-3}$
17.	$e^{-at}\sqrt{t}$, $\operatorname{Re} a>0$	$\sqrt{(\pi/2)}\,(a^2+\omega^2)^{-1/2}[(a^2+\omega^2)^{1/2}-a]^{1/2}$
18.	$t^{-3/2}e^{-at}$, $\operatorname{Re} a>0$	$(2\pi)^{1/2}[(a^2+\omega^2)^{1/2}-a]^{1/2}$
19.	$\exp(-at^2)$, $\operatorname{Re} a>0$	$-j(1/2)(\pi/a)^{1/2}\exp(-\omega^2/4a)\operatorname{Erf}\left(\dfrac{j\omega}{2\sqrt{a}}\right)$
20.	$t\exp(-t^2/4a)$, $\operatorname{Re} a>0$	$2a\omega\sqrt{(\pi a)}\exp(-a\omega^2)$
21.	$t^{-3/2}\exp(-a/t)$, $\lvert\arg a\rvert<\pi/2$	$(\pi/a)^{1/2}\exp[-(2a\omega)^{1/2}]\sin(2a\omega)^{1/2}$
22.	$t^{-3/4}\exp(-a\sqrt{t})$, $\lvert\arg a\rvert<\pi/2$	$-(\pi/2)(a/\omega)^{1/2}[J_{1/4}(a^2/8\omega)$ $\cdot\cos(\pi/8+a^2/8\omega)+Y_{1/4}(a^2/8\omega)$ $\cdot\sin(\pi/8+a^2/8\omega)]$
23.	$e^{-t/2}(1-e^{-t})^{-1}$	$-\frac{1}{2}\tanh(\pi\omega)$
24.	$t^{-1}e^{-at^2}$, $\lvert\arg a\rvert<\pi/2$	$\frac{1}{2}\pi\operatorname{Erf}\left(\dfrac{\omega}{2\sqrt{a}}\right)$
25.	$t^{-1}\ln t$	$-(\pi/2)[C+\ln\omega]$
26.	$t(t^2-a^2)^{-1}\ln t$, $a>0$	$-(\pi/2)\{\cos a\omega[\operatorname{Ci}(a\omega)-\ln a]$ $+\sin a\omega[\operatorname{Si}(a\omega)-\pi/2]\}$
27.	$t^{-1}\ln(1+a^2t^2)$, $a>0$	$-\pi\operatorname{Ei}(-\omega/a)$
28.	$\ln\dfrac{t+a}{\lvert t-a\rvert}$, $a>0$	$(\pi/\omega)\sin a\omega$
29.	$t^{-1}\sin^2(at)$, $a>0$	$\begin{cases} \pi/4 & 0<\omega<2a \\ \pi/8 & \omega=2a \\ 0 & \omega>2a \end{cases}$
30.	$t^{-2}\sin^2(at)$, $a>0$	$(1/4)(\omega+2a)\ln\lvert\omega+2a\rvert$ $+(1/4)(\omega-2a)\ln\lvert\omega-2a\rvert-(1/2)\omega\ln\omega$
31.	$t^{-2}[1-\cos at]$, $a>0$	$(\omega/2)\ln\lvert(\omega^2-a^2)/\omega^2\rvert$ $+(a/2)\ln\lvert(\omega+a)/(\omega-a)\rvert$
32.	$\sin(at^2)$, $a>0$	$(\pi/2a)^{1/2}\{\cos(\omega^2/4a)C[\omega/(2\pi a)^{1/2}]$ $+\sin(\omega^2/4a)S[\omega/(2\pi a)^{1/2}]\}$
33.	$\cos(at^2)$, $a>0$	$(\pi/2a)^{1/2}\{\sin(\omega^2/4a)C[\omega/(2\pi a)^{1/2}]$ $-\cos(\omega^2/4a)S[\omega/(2\pi a)^{1/2}]\}$

TABLE 13.4 Fourier Sine Transform Pairs (continued)

	$f(t)$	$F_s(\omega)$
34.	$\tan^{-1}(a/t),\quad a>0$	$(\pi/2\omega)[1-\exp(-a\omega)]$
35.	$\dfrac{\sin at}{t},\quad a>0$	$\frac{1}{2}\ln\left\lvert\dfrac{\omega+a}{\omega-a}\right\rvert$
36.	$\dfrac{\sin \pi t}{1-t^2}$	$\begin{cases}\sin\omega & 0\le\omega\le\pi\\ 0 & \pi\le\omega\end{cases}$
37.	$\sin\left(\dfrac{a^2}{t}\right),\quad a>0$	$\left(\dfrac{\pi}{2}\right)\dfrac{a}{\sqrt{\omega}}\,J_1(2a\sqrt{\omega})$
38.	$\tan^{-1}(t/a),\quad a>0$	$\dfrac{\pi}{2}\,\dfrac{e^{-a\omega}}{\omega}$
39.	$\tan^{-1}(2a/t),\quad \mathrm{Re}\,a>0$	$\dfrac{\pi}{\omega}\,e^{-a\omega}\sinh(a\omega)$
40.	$\mathrm{Erfc}(at),\quad a>0$	$(1-e^{-\omega^2/4a^2})/\omega$
41.	$J_o(at),\quad a>0$	$\begin{cases}0 & 0<\omega<a\\ 1/\sqrt{\omega^2-a^2} & a<\omega<\infty\end{cases}$
42.	$J_o(at)/t,\quad a>0$	$\begin{cases}\sin^{-1}(\omega/a) & 0<\omega<a\\ \pi/2 & a<\omega<\infty\end{cases}$

13.9 Notations and Definitions

1. $f(t)$ Piece-wise smooth and absolutely integrable function on the positive real line.
2. $F_c(\omega)$ The Fourier cosine transform of $f(t)$.
3. $F_s(\omega)$ The Fourier sine transform of $f(t)$.
4. $f_o(t)$ The odd extension of the function f over the entire real line.
5. $f_e(t)$ The even extension of the function f over the entire real line.
6. $C(\omega)$ is defined as the integral:

$$(2\pi)^{-1/2}\int_o^{\omega} t^{-1/2}\cos t\,dt\,.$$

7. $S(\omega)$ is defined as the integral:

$$(2\pi)^{-1/2}\int_o^{\omega} t^{-1/2}\sin t\,dt\,.$$

8. $\mathrm{Ei}(x)$ is the exponential integral function defined as:

$$-\int_{-x}^{\infty} t^{-1}e^{-t}\,dt, \qquad \lvert\arg(x)\rvert<\pi\,.$$

9. $\overline{\mathrm{E}}\mathrm{i}(x)$ is defined as $(1/2)[(\mathrm{Ei}(x+j0)+\mathrm{Ei}(x-j0)]$.
10. $\mathrm{Ci}(x)$ is the cosine integral function defined as:

$$-\int_x^{\infty} t^{-1} \cos t \, dt \ .$$

11. Si(x) is the sine integral function defined as:

$$\int_o^x t^{-1} \sin t \, dt$$

12. $I_v(z)$ is the modified Bessell function of the first kind defined as:

$$\sum_{m=0}^{\infty} \frac{(z/2)^{v+2m}}{m!\Gamma(v+m+1)}, \qquad |z| < \infty, \qquad |\arg(z)| < \pi \ .$$

13. $He_n(x)$ is the Hermite polynomial function defined as

$$(-1)^n \exp(x^2/2) \frac{d^n}{dx^n}[\exp(-x^2/2)] \ .$$

14. C is the Euler constant defined as

$$\lim_{m \to \infty} \left[\sum_{n=1}^{m} (1/n) - \ln m \right] = 0.5772156649...$$

15. ci(x) and si(x) are related Ci(x) and Si(x) by the equations:

$$\mathrm{ci}(x) = -\mathrm{Ci}(x), \qquad \mathrm{si}(x) = \mathrm{Si}(x) - \pi/2 \ .$$

16. Erf(x) is the error function defined by

$$(2/\sqrt{\pi}) \int_o^x \exp(-t^2) \, dt \ .$$

17. $J_v(x)$ and $Y_v(x)$ are the Bessel functions for the first and second kind, respectively,

$$J_v(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{v+2m}}{m!\Gamma(v+m+1)}$$

and

$$Y_v(x) = \mathrm{cosec}\{v\pi[J_v(x) \cos v\pi - J_{-v}(x)]\} \ .$$

18. $U(t)$ is the Heaviside step function defined as

$$U(t) = 0 \qquad t < 0$$
$$U(t) = 1 \qquad t > 0$$

19. $\binom{m}{n}$ is the binomial coefficient defined as $\frac{m!}{n!(m-n)!}$.

20. $\Gamma(x)$ is the Gamma function defined as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt\ .$$

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14

The Hartley Transform

14.1 Introduction to the Hartley Transform

14.1.1 Definition of the Pair with Symmetry and the Use of ω (units: rad s^{-1})

$$H_x(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\,\text{cas}(\omega t)\,dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_x(\omega)\,\text{cas}(\omega t)\,d\omega$$

14.1.2 Definition of the Pair with Use of *f* (units: s^{-1})

$$H_x(f) = \int_{-\infty}^{\infty} x(t)\,\text{cas}(2\pi ft)\,dt$$

$$x(t) = \int_{-\infty}^{\infty} H_x(f)\,\text{cas}(2\pi ft)\,df$$

$$\text{cas}(\omega t) = \cos(\omega t) + \sin(\omega t)$$

$$\text{cas}(\omega t) = \sqrt{2}\sin\left(\omega t + \frac{\pi}{4}\right)$$

$$\text{cas}(\omega t) = \sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

$$\omega = 2\pi f$$

14.1.3 Odd and Even Parts of the Hartley Transform

$$H_x^e(f) = \frac{H_x(f) + H_x(-f)}{2} = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\,dt$$

$$H_x^o(f) = \frac{H_x(f) - H_x(-f)}{2} = \int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\,dt$$

14.1.4 Properties of the cas Function

TABLE 14.1 Trigonometric Properties of the cas Function

The cas function	$\text{cas}\xi = \cos\xi + \sin\xi$
The cas function	$cas\xi = \frac{1}{2}[(1+j)\exp(-j\xi) + (1-j)\exp(j\xi)]$
The complementary cas function	$\text{cas}'\xi = \text{cas}(-\xi) = \cos\xi - \sin\xi$
The complementary cas function	$\sqrt{2}\,\sin\left(\xi + \frac{3\pi}{4}\right) = \sqrt{2}\,\cos\left(\xi + \frac{\pi}{4}\right)$
Relation to cos	$\cos\xi = \frac{1}{2}[\text{cas}\xi + \text{cas}(-\xi)]$
Relation to sin	$\sin\xi = \frac{1}{2}[\text{cas}\xi - \text{cas}(-\xi)]$
Reciprocal relation	$\text{cas}\xi = \frac{\csc\xi + \sec\xi}{\sec\xi\csc\xi}$
Product relation	$\text{cas}\xi = \cot\xi\sin\xi + \tan\xi\cos\xi$
Function product relation	$\text{cas}\tau\ \text{cas}\nu = \cos(\tau-\nu) + \sin(\tau+\nu)$
Quotient relation	$\text{cas}\xi = \frac{\cot\xi\sec\xi + \tan\xi\csc\xi}{\csc\xi\sec\xi}$
Double angle relation	$\text{cas}2\xi = \text{cas}^2\xi - \text{cas}^2(-\xi)$
Indefinite integral relation	$\int \text{cas}(\tau)\,d\tau = -\text{cas}(-\tau) = -\text{cas}'\tau$
Derivative relation	$\frac{d}{dt}\text{cas}\,\tau = \text{cas}(-\tau) = \text{cas}'\tau$
Angle-sum relation	$\text{cas}(\tau+\nu) = \cos\tau\,\text{cas}\nu + \sin\tau\,\text{cas}'\nu$
Angle-difference relation	$\text{cas}(\tau-\nu) = \cos\tau\,\text{cas}'\nu + \sin\tau\,\text{cas}\nu$
Function-sum relation	$\text{cas}\tau + \text{cas}\nu = 2\text{cas}\,\frac{1}{2}(\tau+\nu)\cos\frac{1}{2}(\tau-\nu)$
Function-difference relation	$\text{cas}\tau - \text{cas}\nu = 2\text{cas}'\,\frac{1}{2}(\tau+\nu)\sin\frac{1}{2}(\tau-\nu)$

14.1.5 Signs of the cas Function

TABLE 14.2 Signs of the cas Function

Quadrant	cas
1st	+
2nd	+ and –
3rd	–
4th	+ and –

14.1.6 Variations of the cas Function

TABLE 14.3 Variations of the cas Function

Quadrant	cas
1st	$+1 \to +1$ with a maximum at $\pi/4$
2nd	$+1 \to -1$
3rd	$-1 \to -1$ with a maximum at $5\pi/4$
4th	$-1 \to +1$

14.1.7 Values of the cas at Special Angles

TABLE 14.4 Values of the cas at Special Angles

Angle	cas
$0° = 0$	0
$30° = \frac{\pi}{6}$	$\frac{1}{2}(\sqrt{3}+1)$
$45° = \frac{\pi}{4}$	$\sqrt{2}$
$60° = \frac{\pi}{3}$	$\frac{1}{2}(1+\sqrt{3})$
$90° = \frac{\pi}{2}$	1
$120° = \frac{2\pi}{3}$	$\frac{1}{2}(-1+\sqrt{3})$
$150° = \frac{5\pi}{6}$	$\frac{1}{2}(1-\sqrt{3})$
$180° = \pi$	-1
$270° = \frac{3\pi}{2}$	-1

14.2 Relationship with the Fourier Transform

14.2.1 Relationship to Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f}\,dt = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\,dt$$

$$= H_x^e(f) - jH_x^o(f) = \frac{H_x(f)+H_x(-f)}{2} - j\,\frac{H_x(f)-H_x(-f)}{2}$$

$$\mathrm{Re}\{X(f)\} = H_x^e(f), \qquad \mathrm{Im}\{X(f)\} = -H_x^o(f)$$

$$H_x(f) = H_x^e(f) + H_x^o(f) = \mathrm{Re}\{X(f)\} - \mathrm{Im}\{X(f)\}$$

***Note*:** The Fourier transform is the even part of the Hartley transform minus j times the odd part. The Hartley transform is the real part of the Fourier transform minus the imaginary part.

Example (Exponential Function):

$$H_x(f)=\int_{-\infty}^{\infty}e^{-at}u(t)\,\mathrm{cas}(2\pi ft)\,dt=\int_{o}^{\infty}e^{-at}\,\mathrm{cas}(2\pi ft)\,dt=\int_{o}^{\infty}e^{-at}\,\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\,dt$$

$$+\int_{o}^{\infty}e^{-at}\,\frac{e^{j2\pi ft}-e^{-j2\pi ft}}{2j}\,dt=\frac{1}{2}\frac{1}{a-j2\pi ft}+\frac{1}{2}\frac{1}{a+j2\pi ft}+\frac{1}{2j}\frac{1}{a-j2\pi ft}$$

$$-\frac{1}{2j}\frac{1}{a+j2\pi ft}=\frac{a+2\pi f}{a^2+4\pi^2f^2}=\frac{a}{a^2+4\pi^2f^2}+\frac{2\pi f}{a^2+4\pi^2f^2}=H_x^e(f)+H_x^o(f)$$

$$X(f)=F\{e^{-at}u(t)\}=\frac{a}{a^2+4\pi^2f^2}-j\frac{2\pi f}{a^2+4\pi^2f^2}$$

Example (Delta Function):

$$H_x(f)=\int_{-\infty}^{\infty}\delta(t-t_o)\,\mathrm{cas}(2\pi ft)\,dt=\mathrm{cas}2\pi ft_o$$

Example (Pulse Function):

$$H_x(f)=\int_{-\infty}^{\infty}p_a(t)\,\mathrm{cas}(2\pi ft)\,dt=\int_{-a}^{a}\mathrm{cas}(2\pi ft)\,dt=\int_{-a}^{a}\frac{e^{j2\pi ft}+e^{-j2\pi ft}}{2}\,dt$$

$$+\int_{-a}^{a}\frac{e^{j2\pi ft}-e^{-j2\pi ft}}{2j}\,dt=\frac{\sin 2\pi fa}{\pi f}$$

14.3 Power and Phase Spectra

14.3.1 Power Spectrum

If $X(f)$ is the Fourier transform of $x(t)$, then

$$P_x(f)=\text{power spectrum}=X(f)X^*(f)=[\mathrm{Re}\{X(f)\}]^2+[\mathrm{Im}\{X(f)\}]^2$$

From (14.2.1)

$$P_x(f)=[H_x^e(f)]^2+[-H_x^o(f)]^2=\left[\frac{H_x(f)+H_x(-f)}{2}\right]^2+\left[\frac{H_x(f)-H_x(-f)}{2}\right]^2$$

$$=\frac{[H_x(f)]^2+[H_x(-f)]^2}{2}$$

14.3.2 Phase Spectrum

If $X(f)$ is the Fourier transform of $x(t)$, then (see ([14.2.1])

$$\mathrm{pha}\{X(f)\}=\tan^{-1}\left[\frac{\mathrm{Im}\{X(f)\}}{\mathrm{Re}\{X(f)\}}\right]=\tan^{-1}\left[-\frac{H_x^o(f)}{H_x^e(f)}\right]=\tan^{-1}\left[\frac{H_x(-f)-H_x(f)}{H_x(f)+H_x(-f)}\right]$$

14.4 Properties of the Hartley Transform

14.4.1 Linearity

$$\int_{-\infty}^{\infty}[ax_1(t)+bx_2(t)]\text{cas}(2\pi ft)\,dt = aH_{x_1}(f)+bH_{x_2}(f)$$

where $H_{x_1}(f)$ and $H_{x_2}(f)$ are the Hartley transforms of $x_1(t)$ and $x_2(t)$.

14.4.2 Power and Phase Spectra (see 14.3.1 and 14.3.2)

$$P_x(f)=\frac{[H_x(f)]^2+[H_x(-f)]^2}{2},\ \text{phase}\{X(f)\}=\tan^{-1}\left[\frac{H_x(-f)-H_x(f)}{H_x(f)+H_x(-f)}\right]$$

14.4.3 Scaling

If $H_x(f)$ is the Hartley transform of $x(t)$, then

$$\int_{-\infty}^{\infty}x(at)\text{cas}(2\pi ft)\,dt=\frac{1}{|a|}\int_{-\infty}^{\infty}x(t')\text{cas}\left(\frac{2\pi ft'}{a}\right)dt'=\frac{1}{|a|}H_x\left(\frac{f}{a}\right)$$

Similarly

$$\int_{-\infty}^{\infty}x\left(\frac{t}{a}\right)\text{cas}(2\pi ft)\,dt=|a|H_x(af)$$

14.4.4 Function Reversal

If $x(t)$ and $H_x(f)$ are a Hartley transform pair, then

$$\int_{-\infty}^{\infty}x(-t)\text{cas}(2\pi ft)\,dt=H_x(-f)\qquad \text{(see 14.4.3 with a = –1)}$$

14.4.5 Shifting

$$H_x(f)=\int_{-\infty}^{\infty}x(t-t_o)\text{cas}(2\pi ft)\,dt=\int_{-\infty}^{\infty}x(t')\text{cas}[2\pi f(t'+t_o)]\,dt'$$

$$=\cos(2\pi ft_o)H_x(f)+\cos(2\pi ft_o)H_x(-f)$$

where we set $t-t_o=t'$ and expanded $\text{cas}[2\pi f(t'+t_o)]$.

14.4.6 Modulation

The Hartley transform of $x(t)\cos 2\pi f_o t$ is

$$H_x(f)=\int_{-\infty}^{\infty}x(t)\cos(2\pi f_o t)\text{cas}(2\pi ft)\,dt=\int_{-\infty}^{\infty}x(t)\cos(2\pi f_o t)\cos(2\pi ft)\,dt+\int_{-\infty}^{\infty}x(t)\cos(2\pi f_o t)\sin(2\pi ft)\,dt$$

$$=\tfrac{1}{2}H_x(f-f_o)+\tfrac{1}{2}H_x(f+f_o)$$

Example:

$$\int_{-\infty}^{\infty} p_a(t)\cos(2\pi f_o t)\operatorname{cas}(2\pi ft)\,dt = \frac{1}{2}\frac{\sin[2\pi(f-f_o)a]}{\pi(f-f_o)} + \frac{1}{2}\frac{\sin[2\pi(f+f_o)a]}{\pi(f+f_o)}$$

See (14.2.1), the third example.

14.4.7 Convolution

If $g(t) = x_1(t) * x_2(t)$, where $*$ stands for convolution, then [see (14.2.1)]

$$F\{g(t)\} = G(f) = X_1(f)X_2(f) = [H_{x_1}^e(f) - jH_{x_1}^o(f)][H_{x_2}^e(f) - jH_{x_2}^o(f)]$$

After separating and replacing the even and odd Hartley transform with their Hartley transform equivalent, e.g., $H_{x_1}^e(f) = [H_{x_1}(f) + H_{x_1}(-f)]/2$, we obtain

$$G(f) = \operatorname{Re}\{G(f)\} - j\operatorname{Im}\{G(f)\}$$

$$\begin{aligned} H_g(f) &= \operatorname{Re}\{G(f)\} - \operatorname{Im}\{G(f)\} \\ &= \tfrac{1}{2}[H_{x_1}(f)H_{x_2}(f) + H_{x_1}(-f)H_{x_2}(f) + H_{x_1}(f)H_{x_2}(-f) - H_{x_1}(-f)H_{x_2}(-f)] \end{aligned}$$

Note:

1. If $x_1(t)$ is even and $x_2(t)$ is odd, or $x_1(t)$ is odd and $x_2(t)$ is even, then

$$\int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\operatorname{cas}(2\pi ft)\,dt = H_{x_1}(f)H_{x_2}(f)$$

2. If $x_1(t)$ is odd, then $\int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\operatorname{cas}(2\pi ft)\,dt = H_{x_1}(f)H_{x_2}(-f)$
3. If $x_2(t)$ is odd, then $\int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\operatorname{cas}(2\pi ft)\,dt = H_{x_1}(-f)H_{x_2}(f)$
4. If both functions are odd, then $\int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\operatorname{cas}(2\pi ft)\,dt = -H_{x_1}(f)H_{x_2}(f)$

14.4.8 Autocorrelation

$$\int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} x(\tau) \star x_2(\tau - t)\,d\tau]\operatorname{cas}(2\pi ft)dt = \tfrac{1}{2}[H_x^2(f) + H_x^2(-f)] = [H_x^e(f)]^2 + [H_x^o(f)]^2$$

14.4.9 Product

The Hartley transform of the product $x_1(t)\,x_2(t)$ is given by

$$\begin{aligned} &\tfrac{1}{2}[H_{x_1}(f) * H_{x_2}(f) + H_{x_1}(-f) * H_{x_2}(f) + H_{x_1}(f) * H_{x_2}(-f) - H_{x_1}(-f) * H_{x_2}(-f)] \\ &= H_{x_1}^e(f) * H_{x_2}^e(f) - H_{x_1}^o(f) * H_{x_2}^o(f) + H_{x_1}^e(f) * H_{x_2}^o(f) + H_{x_1}^o(f) * H_{x_2}^e(f) \end{aligned}$$

14.4.10 Derivative of x(t)

$$F\left\{\frac{dx(t)}{dt}\right\} = j2\pi f\, X(f) = j2\pi f\, \mathrm{Re}\{X(f)\} + 2\pi f\, \mathrm{Im}\{X(f)\}$$

Hence

$$\begin{aligned} H_{x'(t)} &= 2\pi f\, \mathrm{Im}\{X(f)\} + 2\pi f\, \mathrm{Re}\{X(f)\} = 2\pi f[-H_x^o(f) + H_x^e(f)] \\ &= -2\pi f\, H_x(-f) \end{aligned}$$

By revision we can write

$$\int_{-\infty}^{\infty} \frac{d^n x(t)}{dt^n} \mathrm{cas}\, 2\pi ft\, dt = \left(\mathrm{cas}' \frac{n\pi}{2}\right)(2\pi f)^n H_x[(-1)^n f]$$

14.4.11 Second Moment

$$H(f) = \int_{-\infty}^{\infty} x(t)[\cos 2\pi ft + \sin 2\pi ft]\, dt$$

Differentiating both sides with respect to f and then setting $f = 0$, we easily obtain

$$\left.\frac{d^2 H(f)}{df^2}\right|_{f=0} = -4\pi^2 \int_{-\infty}^{\infty} t^2 x(t)\, \mathrm{cas}\, 2\pi ft\, dt$$

14.4.12 Fourier Transform from Hartley Transform

$$X(f) = \frac{1}{\sqrt{2}} e^{-j\pi/4} H(f) + \frac{1}{\sqrt{2}} e^{j\pi/4} H(-f)$$

14.4.13 Hartley Transform from Fourier Transform

$$H(f) = \frac{1}{\sqrt{2}} e^{j\pi/4} X(f) + \frac{1}{\sqrt{2}} e^{-j\pi/4} X^*(-f)$$

14.4.14 Hartley Transform Properties

TABLE 14.5 Hartley Transform Properties

Property	$x(t)$	$H(f) = \int_{-\infty}^{\infty} x(t)\, \mathrm{cas}\, 2\pi ft\, dt$
Linearity	$ax_1(t) + bx_2(t)$	$aH_{x_1}(f) + bH_{x_2}(f)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} H_x\left(\frac{f}{a}\right)$
	$x(t/a)$	$\lvert a\rvert H_x(af)$
Reversal	$x(-t)$	$H_x(-f)$

TABLE 14.5 Hartley Transform Properties (continued)

Property	$x(t)$	$H(f)=\int_{-\infty}^{\infty} x(t)\,\text{cas}\,2\pi ft\,dt$
Shift	$x(t-t_o)$	$\cos 2\pi f t_o H_x(f)+\sin 2\pi f t_o H_x(-f)$
Modulation	$\cos 2\pi f_o t\, x(t)$	$\frac{1}{2}H_x(f-f_o)+\frac{1}{2}H_x(f+f_o)$
Convolution	$x_1(t)*x_2(t)$	$\frac{1}{2}[H_{x_1}(f)H_{x_2}(f)+H_{x_1}(-f)H_{x_2}(f)$ $+H_{x_1}(f)H_{x_2}(-f)-H_{x_1}(-f)H_{x_2}(-f)]$
Autocorrelation	$x(t)\bigstar x(t)$	$\frac{1}{2}[H_x^2(f)+H_x^2(-f)]=[H_x^e(f)]^2+[H_x^o(f)]^2$
Product	$x_1(t)x_2(t)$	$\frac{1}{2}[H_{x_1}(f)*H_{x_2}(f)+H_{x_1}(-f)*H_{x_2}(f)$ $+H_{x_1}(f)*H_{x_2}(-f)-H_{x_1}(-f)*H_{x_2}(-f)]$ $=H_{x_1}^e(f)*H_{x_2}^e(f)-H_{x_1}^o(f)*H_{x_2}^o(f)$ $+H_{x_1}^e(f)*H_{x_2}^o(f)+H_{x_1}^o(f)*H_{x_2}^e(f)$
Derivative	$\frac{dx(t)}{dt}$	$-2\pi f\,H_x(-f)$
	$\frac{d^n x(t)}{dt^n}$	$\left(\text{cas}'\frac{n\pi}{2}\right)(2\pi f)^n H_x[(-1)^n f]$
FT from HT	$X(f)=\frac{1}{\sqrt{2}}e^{-j\pi/4}H(f)+\frac{1}{\sqrt{2}}e^{j\pi/4}H(-f)$	
HT from FT	$H(f)=\frac{1}{\sqrt{2}}e^{j\pi/4}X(f)+\frac{1}{\sqrt{2}}e^{-j\pi/4}X^*(-f)$	
Power spectrum	$P_x(f)=\lvert X(f)\rvert^2=\frac{[H_x(f)]^2+[H_x(-f)]^2}{2}$	
Phase spectrum	$\text{Phas}\,X(f)=\tan^{-1}\left[\frac{H_x(-f)-H_x(f)}{H_x(f)+H_x(-f)}\right]$	

14.5 Examples of Hartley Transforms

14.5.1 Example (Gaussian Function)

$$F\{e^{-at^2}\}=\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2 f^2}{a}},\qquad a>0.$$

Since HT is the even part of FT implies that

$$H_x(f)=\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2 f^2}{a}}.$$

If $a=\pi$, then $H_x(f)=e^{-\pi^2 f^2}$, which is identical with the function.

14.5.2 Example (Shifted Gaussian)

$$F\{e^{-a(t-t_o)^2}\}=e^{-j\omega t_o}F\{e^{-at^2}\}=\sqrt{\frac{\pi}{a}}\,e^{-j\omega t_o}e^{-\frac{\pi^2 f^2}{a}}=\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2 f^2}{a}}(\cos 2\pi f t_o - j\sin 2\pi f t_o).$$

Since the $H_x(f) = \text{Re}\{X(f)\} - \text{Im}\{X(f)\}$ (see 14.2.1), we obtain $H_x(f) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 f^2}{a}} \text{cas}\, 2\pi f t_o$.

14.5.3 Example (Modulated Gaussian):

$$F\{e^{-\pi t^2} \cos \pi t\} = F\left\{e^{-\pi t^2} \frac{e^{j\pi t} + e^{-j\pi t}}{2}\right\} = \int_{-\infty}^{\infty} e^{-\pi t^2} \frac{e^{-j2\pi(f-\frac{1}{2})t} + e^{-j2\pi(f+\frac{1}{2})t}}{2} dt = \tfrac{1}{2} e^{-\pi(f-\frac{1}{2})^2} + \tfrac{1}{2} e^{-\pi(f+\frac{1}{2})^2} .$$

Since this is an even function implies from (14.2.1) that $H_x(f) = F\{e^{-\pi t^2} \cos \pi t\} = X(f)$.

14.5.4 Example (Truncated Cosine)

If $x(t) = \cos \omega_o t\, p_a(t)$, then

$$F\{x(t)\} = \int_{-a}^{a} e^{-j2\pi f t}\, \tfrac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})\, dt = \tfrac{1}{2} \int_{-a}^{a} e^{-j(2\pi f - \omega_o)t}\, dt + \tfrac{1}{2} \int_{-a}^{a} e^{-j(2\pi f + \omega_o)t} dt$$

$$= \frac{\sin[(2\pi f - \omega_o)a]}{2\pi f - \omega_o} + \frac{\sin[(2\pi f + \omega_o)a]}{2\pi f + \omega_o}.$$

Since $X(f)$ is even, then $H_x(f) = X(f)$.

14.5.5 Example (Signum)

If $x(t) = \text{sgn}(t)$, then

$$F\{\text{sgn}(t)\} = \frac{1}{j\pi f} = -j \frac{1}{\pi f} .$$

Hence, $H_x(f) = -\text{Im}\{X(f)\} = \dfrac{1}{\pi f}$.

14.5.6 Example (Unit Step)

If $x(t) = u(t)$, then

$$F\{u(t)\} = \tfrac{1}{2}\delta(f) - j \frac{1}{2\pi f} \cdot .$$

Hence, $H_x(f) = \text{Re}\{X(f)\} - \text{Im}\{X(f)\} = \tfrac{1}{2}\delta(f) + \dfrac{1}{2\pi f}$.

14.5.7 Example (Cosine)

If $x(t) = \cos \omega_o t$, the $F\{\cos \omega_o t\} = \tfrac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$. Since the delta functions are even, this implies the $H_x(f) = \tfrac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$.

14.5.8 Example (Shifted Exponential)

If $x(t)=e^{-a(t-t_o)}u(t-t_o)$, then

$$F\{e^{-a(t-t_o)}u(t-t_o)\}=\int_{-\infty}^{\infty}e^{-a(t-t_o)}u(t-t_o)e^{-j2\pi ft}\,dt=\int_{-\infty}^{\infty}e^{-a(\tau)}u(\tau)e^{-j2\pi ft_o}e^{-j2\pi f\tau}\,d\tau$$

$$=e^{-j2\pi ft_o}\int_{o}^{\infty}e^{-(a+j2\pi f)\tau}\,d\tau=e^{-j2\pi ft_o}\frac{1}{a+j2\pi f}.$$

Hence, $H_x(f)=\text{Re}\{X(f)\}-\text{Im}\{X(f)\}=\dfrac{(a+2\pi f)\cos 2\pi ft_o+(a-2\pi f)\sin 2\pi ft_o}{a^2+(2\pi f)^2}.$

14.5.9 Example

Because the $F\{p_a(t)\}=\dfrac{2\sin\omega a}{\omega}$, the symmetry property of Fourier transform (see 3.1.2) gives $F\left\{\dfrac{2\sin at}{t}\right\}=2\pi\,p_a(\omega)$ or equivalently $F\left\{\dfrac{\sin at}{\pi t}\right\}=p_a(\omega)$. Hence, on the f axis $F\left\{\dfrac{\sin at}{\pi t}\right\}=p_{a/2\pi}(f)$, and since the pulse is an even function $H_x(f)=p_{a/2\pi}(f)$.

14.5.10 Example (Shifted Pulse)

The FT of $x(t)=p_a(t-a)$ is

$$X(f)=\int_{o}^{2a}e^{-j2\pi ft}\,dt=\frac{(\cos 2\pi fa-j\sin 2\pi fa)\sin 2\pi fa}{\pi f}.$$

Hence, $H_x(f)=\text{Re}\{X(f)\}-\text{Im}\{X(f)\}=\text{cas}2\pi fa\dfrac{\sin 2\pi fa}{\pi f}.$

14.5.11 Example

The FT of $x(t)=te^{-at}u(t),\ a>0$, is

$$X(f)=\int_{o}^{\infty}te^{-at}e^{-j2\pi ft}\,dt=\int_{o}^{\infty}t\,e^{-(a+j2\pi f)t}\,dt=\frac{1}{(a+j2\pi f)(a+j2\pi f)}.$$

Hence, $H_x(f)=\dfrac{a^2-(2\pi f)^2+4a\pi f}{[a^2+(2\pi f)^2]^2}.$

14.5.12 Example

The FT of $x(t)=\sin\pi t\,p(t)$ is

$$X(f)=\int_{-1}^{1}\sin\pi t\,e^{-j2\pi ft}\,dt=\int_{-1}^{1}\frac{e^{j\pi t}-e^{-j\pi t}}{2j}e^{-j2\pi ft}\,dt=-j\frac{\sin(2\pi f-\pi)}{2\pi f-\pi}+j\frac{\sin(2\pi f+\pi)}{2\pi f+\pi}..$$

Hence, $H_x(f)=-\text{Im}\{X(f)\}=\dfrac{\sin(2\pi f-\pi)}{2\pi f-\pi}-\dfrac{\sin(2\pi f+\pi)}{2\pi f+\pi}.$

14.6 Hartley Series

14.6.1 Orthonormal Set of Functions

The set $\{\phi_n(t)\} = \frac{1}{\sqrt{2\pi}}\operatorname{cas}(n2\pi f_o t), \quad n = 0, \pm 1, \pm 2, \cdots; \; f_o = 2\pi / \mathrm{T}; \, T = \text{period},$ is orthonormal in the interval

$$t_o \le t \le t_o + \mathrm{T}, \qquad (\phi_i, \phi_k) = \begin{cases} 1 & i = k \\ 0 & i \ne k \end{cases}.$$

14.6.2 Expansion of a Periodic Function

$$x(t + \mathrm{T}) = x(t) \text{ for all } t$$

$$x(t) = \sum_{i=-\infty}^{\infty} \gamma_i \, \phi_i(t)$$

$$\phi_i(t) = \operatorname{cas}(i 2\pi f_o t)$$

$$\gamma_i = \frac{1}{T}\int_{T_o}^{t_o+\mathrm{T}} x(t) \operatorname{cas}(i 2\pi f_o t)\, dt$$

14.6.3 Relation to Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn2\pi f_o t} = \text{complex Fourier series}$$

$$\alpha_n = \frac{1}{T}\int_{T_o}^{t_o+\mathrm{T}} x(t) e^{-jn2\pi f_o t}\, dt$$

$$\alpha_{-n}^* = \alpha_n$$

$$\gamma_k = \begin{cases} \operatorname{Re}\{\alpha_k\} - \operatorname{Im}\{\alpha_k\} & k \ne 0 \\ \alpha_k & k = 0 \end{cases}$$

$$\alpha_k = E\{\gamma_k\} - jO\{\gamma_k\}$$

$$E\{\gamma_k\} = \text{even part} = \frac{\gamma_k + \gamma_{-k}}{2}; \qquad O\{\gamma_k\} = \text{odd part} = \frac{\gamma_k - \gamma_{-k}}{2}$$

14.7 Tables of Fourier and Hartley Transforms

TABLE 14.6 Table of Fourier and Hartley Transform Pairs

$x(t)$	$X(t)$	$H_x(f)$
$u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$	$T\frac{\sin \pi T f}{\pi T f} = T \operatorname{sinc} Tf$	Because $x(t)$ is even, $H_x(f) = X(f)$
$\beta e^{-\alpha t}\, u(t)$	$\frac{\beta}{j\omega + \alpha}$	$\frac{\beta(\alpha + 2\pi f)}{\alpha^2 + (2\pi f)^2}$

TABLE 14.6 Table of Fourier and Hartley Transform Pairs (continued)

$x(t)$	$X(t)$	$H_x(f)$
$1-2\frac{\lvert t\rvert}{T},\ \lvert t\rvert<\frac{T}{2}$	$\frac{T}{2}\text{sinc}^2\left(\frac{Tf}{2}\right)=\frac{1-\cos\pi f T}{T\pi^2 f^2}$	Because $x(t)$ is even, $H_x(f)=X(f)$
$e^{-\alpha^2 t^2}$	$\frac{\sqrt{\pi}}{\alpha}e^{-(\pi^2 f^2/\alpha^2)}$	Because $x(t)$ is even, $H_x(f)=X(f)$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	Because $x(t)$ is even, $H_x(f)=X(f)$
$e^{-\alpha t}\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(\alpha+j2\pi f)^2+\omega_o^2}$	$\frac{\omega_o(\alpha^2+\omega_o^2-4\pi^2 f^2+4\pi f\alpha)}{(\alpha^2+\omega_o^2-4\pi^2 f^2)^2+(4\pi f\alpha)^2}$
$e^{-\alpha t}\cos(\omega_o t)u(t)$	$\frac{\alpha+j2\pi f}{(\alpha+j2\pi f)^2+\omega_o^2}$	$\frac{(\alpha-2\pi f)(\alpha^2+\omega_o^2-4\pi^2 f^2)+(4\pi f\alpha)(\alpha+2\pi f)}{(\alpha^2+\omega_o^2-4\pi^2 f^2)^2+(4\pi f\alpha)^2}$
$\frac{1}{\beta-\alpha}(e^{-\alpha t}-e^{-\beta t})u(t)$	$\frac{1}{(\alpha+j2\pi f)(\beta+j2\pi f)}$	$\frac{\alpha\beta-2\pi f(\alpha+\beta+2\pi f)}{[\alpha\beta-(2\pi f)^2]^2+[2\pi f(\alpha+\beta)]^2}$
$\cos\omega_o t\left[u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)\right]$	$\frac{T}{2}\left[\frac{\sin\pi T(f-f_o)}{\pi T(f-f_o)}+\frac{\sin\pi T(f+f_o)}{\pi T(f+f_o)}\right]$	Because $x(t)$ is even, $H_x(f)=X(f)$
$K\delta(t)$	K	K
K	$K\delta(f)$	$K\delta(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\frac{1}{2}\delta(f)+\frac{1}{2\pi f}$
$\text{sgn}\,t=\frac{t}{\lvert t\rvert}$	$\frac{1}{j\pi f}$	$\frac{1}{\pi f}$
$\cos\omega_o t$	$\frac{1}{2}[\delta(f-f_o)+\delta(f+f_o)]$	Because $x(t)$ is even, $H_x(f)=X(f)$
$\sin\omega_o t$	$\frac{-j}{2}[\delta(f-f_o)-\delta(f+f_o)]$	$\frac{1}{2}[\delta(f-f_o)-\delta(f+f_o)]$
$\sum_{-\infty}^{\infty}\delta(t-nT)$	$\frac{1}{T}\sum_{-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)$	Because $x(t)$ is even, $H_x(f)=X(f)$
$\sum_{-\infty}^{\infty}\alpha_n e^{jn2\pi f_o t}$	$\sum_{-\infty}^{\infty}\alpha_n\delta\left(f-\frac{n}{T}\right)$	$\sum_{-\infty}^{\infty}\gamma_n\delta\left(f-\frac{n}{T}\right)$
$Ae^{j\omega_o t}$	$A\delta(f-f_o)$	$H_x(f)=X(f)$ ($\delta(f)$ is even)
$t\,u(t)$	$\frac{j}{4\pi}\delta'(f)-\frac{1}{4\pi^2 f^2}$	$\frac{-1}{4\pi}\delta'(f)-\frac{1}{4\pi^2 f^2}$
$e^{-\alpha(t-t_o)^2}$	$\sqrt{\frac{\pi}{a}}\,e^{-j\omega t_o}\,e^{-\frac{\pi^2 f^2}{a}}$	$\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2 f^2}{a}}\text{cas}2\pi f t_o$
$e^{-\pi t^2}\cos\pi t$	$\frac{1}{2}e^{-\pi(f-\frac{1}{2})^2}+\frac{1}{2}e^{-\pi(f+\frac{1}{2})^2}$	$H_x(f)=X(f)$
$\cos\omega_o t\,p_a(t)$	$\left\{\frac{\sin[(2\pi f-\omega_o)a]}{2\pi f-\omega_o}+\frac{\sin(2\pi f+\omega_o)}{2\pi f+\omega_o}\right.$	$H_x(f)=X(f)$

TABLE 14.6 Table of Fourier and Hartley Transform Pairs (continued)

$x(t)$	$X(t)$	$H_x(f)$
$e^{-\alpha(t-t_o)}u(t)$	$e^{-j2\pi f t_o}\dfrac{1}{a+j2\pi f}$	$\dfrac{(\alpha+2\pi f)\cos 2\pi f t_o}{\alpha^2+(2\pi f)^2}+\dfrac{(\alpha-2\pi f)\sin 2\pi f t_o}{\alpha^2+(2\pi f)^2}$
$p_a(t)$	$\dfrac{\sin 2\pi f a}{\pi f}$	$H_x(f)=X(f)$
$\dfrac{\sin at}{\pi t}$	$p_{a/2\pi}(f)$	$H_x(f)=X(f)$
$p_a(t-a)$	$e^{-j2\pi f a}\dfrac{\sin 2\pi f a}{\pi f}$	$\text{cas}2\pi f a\dfrac{\sin 2\pi f a}{\pi f}$
$te^{-t}u(t)$	$\dfrac{1}{(a+j2\pi f)(a+j2\pi f)}$	$\dfrac{a^2+4a\pi f-(2\pi f)^2}{[a^2+(2\pi f)^2]^2}$
$\sin \pi t p(t)$	$-j\dfrac{\sin(2\pi f-\pi)}{2\pi f-\pi}+j\dfrac{\sin(2\pi f+\pi)}{2\pi f+\pi}$	$\dfrac{\sin(2\pi f-\pi)}{2\pi f-\pi}-\dfrac{\sin(2\pi f+\pi)}{2\pi f+\pi}$

14.8 Two-Dimensional Hartley Transform

14.8.1 Two-Dimensional

$$H_f(u,v)=\iint_{-\infty}^{\infty} f(x,y)\,\text{cas}[2\pi(ux+vy)]\,dx\,dy$$

$$f(x,y)=\iint_{-\infty}^{\infty} H_f(u,v)\,\text{cas}[2\pi(ux+vy)]\,du\,dv$$

14.8.2 Relation to Fourier Transform

$$F(u,v)=\iint_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy$$

$$=R_f(u,v)-jI_f(u,v)$$

$$f(x,y)=\iint_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

$$R_f(u,v)=\text{Re}\{F(u,v)\}=\text{even}$$

$$I_f(u,v)=\text{Im}\{F(u,v)\}=\text{odd}$$

$$H_f(u,v)=R_f(u,v)-I_f(u,v)$$

14.9 Discrete Hartley Transform

14.9.1 The Discrete Hartley Transform (DHT)

$$H_h(k\Omega_n) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h(nT)cas(k\Omega_h nT)$$

$$h(nT) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} H_h(k\Omega_n)cas(k\Omega_n nT)$$

$$\Omega_h = \frac{2\pi}{NT} = \text{frequency resolution (rad s}^{-1})$$

Similarly the discrete Fourier transform is given by

$$F(k\Omega_f) = \sum_{n=0}^{N-1} f(nT)e^{-jk\Omega_f nT}$$

$$f(nT) = \frac{1}{N} \sum_{k=0}^{N-1} F(k\Omega_f)e^{jk\Omega_f nT}$$

$$\Omega_f = \frac{2\pi}{NT} = \text{frequency resolution (rad s}^{-1})$$

14.9.2 Relation to Discrete Fourier Transform

$$H(k\Omega_h) = \text{Re}\{F(k\Omega_f)\} - \text{Im}\{F(k\Omega_f)\}$$

Note:

- DHT avoids complex arithmetic
- DHT requires half the memory storage
- DHT requires $N \log_2 N$ real operation instead of $N \log_2 N$ complex operations required for DFT
- DHT fewer operation may help in truncation and rounding errors
- DHT is its own inverse

14.9.3 A C Program for Fast Hartley Transforms

TABLE 14.7

```
/******************************************************************************
/* Program FHT.C ***************************************************************
/*
/* This FHT algorithm utilizes an efficient permutation algorithm
/* developed by David M. W. Evans. Additional /* details may be found
/* in: IEEE Transaction on Acoustics, Speech, and Signal Processing,
/* vol. ASSP-35, n. 8, pp. 1120-1125. August 1987.
/*
```

```
/* This FHT algorithm, authored by Lakshmikantha S. Prabhu, is
/* optimized for the SPARC RISC platform. Additional details may
/* be found in his M.S.E.E. thesis referenced below.
/*
/* L. S. Prabhu, "A Complexity-Based Timing Analysis of Fast
/* Real Transform Algorithms," Master's Thesis, University of
/* Arkansas, Fayetteville, AR, 72701-1201, 1993.
/*********************************************************************************/
/* This program assumes a maximum array length of 2^M = N where                 */
/* M=9 and N=512.                                                               */
/* See line 52 if the array length is increased.                                */
# include <stdio.h>
# include <math.h>
# define M 3
# define N 8
float * myFht () ;
main ()
{
/* Read the integer values 1, . . . , N into the vector X(N).                   */
        int i;
        float X[N];
        for ( i = 0 ; i < N; i++)
           X[i] = i+1 ;
        for ( i = 0; i < N; i++)
          printf ("%f\n", X[i]) ;
          myFht(X,N,M) ;
          printf("\n")
        for ( i = 0 ; i < N; i++)
          printf("%d: %f\n , i,X[i]/N;
/* It is assumed that the user divides by the integer N.                        */
}
float*
myFht (x,n,m)
float* x;
int n,m;
{
int i, j, k, kk, 1, 10, 11, 12, 13, 14, 15, m1, n1, n2, NN, s;
int diff = 0, diff2, gamma, gamma2=2, n2_2, n2_4, n_2, n_4,n_8, n_16;
int itemp, ntemp, phi, theta_by_2;
float ee, temp1, temp2, xtemp1, xtemp2;
double ccl, cc2, ss1, ss2;
double since [257] ;
/*********************************************************************************/
/* digit reverse counter.                                                       */
/*********************************************************************************/
int powers_of_2[16] , seed[256] ;
int firstj, log2_n, log2_seed_size ;
int group_no, nn, offset ;
log2_n = m >> 1 ;
nn = 2 << {log2_n -1) ;
if ( (m % 2) == 1)
```

```
  log2_n = log2_n + 1 ;
  seed[0] = 0 ; seed[1] = 1 ;
  for (log2_seed_size = 2; log2_seed_size < = log2_n; log2_seed_size++)
     {
     for ( i = 0; i < 2 << (log2_seed_size -2); i++)
        {
        seed [i] = 2 * seed [i]
        for (k = 1; k < 2; k++)
           seed [ i + k * (2 << (log2_seed_size - 1) >> 1] = seed [i] ;
        }
     }
  for (offset = 1; offset < nn; offset++)
     {
     firstj - nn * seed [offset];
     i = offset ; j = firstj ;
     xtemp = x[i] ;
     x[i] = x[j] ;
     x[j] = xtemp ;
     for ( group_no = 1; group_no < seed [offset] ; group_no++)
        {
        i = i + nn ; j = firstj + seed [group_no] ;
        xtemp = x[i];
        x[i] = x[j] ;
        x[j] = xtemp ;
        }
     }
j = 0
n1 = n - 1 ;
n_16 = n >> 4 ;
n_8 = n >> 3;
n_4 = n >> 2;
n_2 = n >> 1;
/*******************************************************************************/
/* Start the transform computation with 2-point butterflies.                  */
/*******************************************************************************/
for (i = 0 ; i < n ; i + -2)
    {
   s = 1i + 1
   xtemp = x[i] ;
   x[i] = x[s] ;
   x[s] = xtemp - x[s] ;
   }
/*******************************************************************************/
/* Now the 4-point butterflies.                                                */
/*******************************************************************************/
for (i = 0 ; i < n ; i + -2)
    {
   xtemp = x[i] ;
   x[i] += x[i+2] ;
   x[i+2] = xtemp - x[i+2]
   xtemp = x[i+1]
```

```
   x[i+1] += x[i+3] ;
   x[i+3] += xtemp - x[i+3] ;
   }
/*******************************************************************************/
/* Sine table initialization.                                                   */
/*******************************************************************************/
NN = n_4;
sine[0] = 0
sine [n_16] = 0.382683432 ;
sine [n_8] = 0.707106781 ;
sine [3*n_16] = 0.923879533 ;
sine -n_4] = 1.000000000 ;
h_sec_b = 0.509795579 ;
diff = n_16 ;
theta_by_2 = n_4 >> 3 ;
j = 0 ;
while (theta_by_2 >= 1)
    {
    for ( i = 0 ; i <= n_4 ; i += diff)
       {
       sine[j + theta_by_2] = h_sec_b * (sine[j] + sine[j + diff]) ;
       j = j + diff
       }
    j = 0
    diff = diff >> 1 ;
    theta_by_2 = theta_by_2 >> 1 ;
    h_sec_b = 1 / sqrt)2 + 1/h_sec_b) ;
    }
/*******************************************************************************/
/* Other butterflies.                                                           */
/*******************************************************************************/
for (i = 3 ; i < m ; i ++)
   {
   diff = 1 ; gamma = 0 ;
   ntemp = 0 ; phi = 2 << (m-1) >> 1 ;
   ss1 = sine [phi] ;
   ccl = sine [n_4 - phi] ;
   n2 = 2 << (i-1) ;
   n2_2 = n2 >> 1 ;
   n2_4 = n2 >> 2 ;
   gamma2 = n2_4 ;
   diff2 - gamma2 + gamma2 - 1 ;
   itemp = n2_4 ;
   k = 0 ;
/*******************************************************************************/
/* Initial section of stages 3,4,... for which sines & cosines are not required. */
/*******************************************************************************/
for (k = 0 ; k < (2 << (m-i) >>1) ; k+ +)
   {
   10 = gamma ;
   11 = 10 + n2_2 ;
```

```
   13 = gamma2 ;
   14 = gamma2 + n2_2 ;
   15 = 11 + itemp ;
   x0 = x[10] ;
   x1 = x[11] ;
   x3 = x[13] ;
   x5 = x[15] ;
   x[10] = x0 + x1 ;
   x[11] = x0 - x1 ;
   x[13] = x3 + x5 ;
      x[14] = x3 - x5 ;
      gamma = gamma + n2 ;
      gamma2 = gamma2 + n2 ;
      }
   gamma = diff ;
   gamma2 = diff2 ;
/*********************************************************************************/
/* Next sections of stages 3,4,...                                                */
/*********************************************************************************/
   for (k = 0 ; k < (2 << (m-i) >>1) ; k+ +)
      {
      for (k = 0 ; k < (2 << (m-i) >>1) ; k+ +)
         {
         10 = gamma ;
         11 = 10 + n2_2 ;
         13 = gamma2 ;
         14 = 13+ n2_2 ;
         x0 = x[10] ;
         x1 = x[11] ;
         x3 = x[13] ;
         x4 = x[14] ;
         x[10] = x0 + x1 * cc1 + x4 * ss1 ;
         x[10] = x0 + x1 * cc1 + x4 * ss1 ;
         x[13] = x3 - x4 * cc1 + x1 * ss1 ;
         x[14] = x3 + x4 * cc1 + x1 * ss1 ;
         gamma = gamma + n2 ;
         gamma2 = gamma2 + n2 ;
         }
      itemp = 0 ;
      phi = phi + (2 << (m-i) >> 1) ;
      ntemp = (phi < n_4) ? 0 : n_4 ;
      ss1 = sine [phi - ntemp] ;
      cc1 = sine [n_4 - phi + ntemp] ;
      diff++ ; diff2- ;
      gamma = diff ;
      gamma2 = diff2 ;
      }
   }
}
```

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15

The Hilbert Transform

15.1 The Hilbert Transform

15.1.1 Definition of Hilbert Transform

$$\upsilon(t) = \mathrm{H}\{x(t)\} = \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{x(\eta)}{\eta - t} d\eta = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{x(\eta)}{t - \eta} d\eta$$

$$x(t) = \mathrm{H}^{-1}\{\upsilon(t)\} = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{\upsilon(\eta)}{\eta - t} d\eta = \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\upsilon(\eta)}{t - \eta} d\eta$$

where P stands for the Cauchy principal value of the integral.

Convolution form representation

$$v(t) = x(t) * \frac{1}{\pi t}$$

$$x(t) = -\upsilon(t) * \frac{1}{\pi t}$$

Fourier transform of $v(t)$ and $x(t)$ and $1/\pi t$ (see Table 3.1.3)

$$V(\omega) = X(\omega)[-j\,\text{sgn}(\omega)]$$

$$\mathsf{F}^{-1}\{-j\,\text{sgn}(\omega)\,X(\omega)\} = \upsilon(t)$$

$$\mathsf{F}\left\{\frac{1}{\pi t}\right\} = -j\,\text{sgn}(\omega)$$

Example

If $x(t) = \cos\omega t,$ then

$$\begin{aligned}\mathsf{H}\{\cos\omega t\} &= \upsilon(t)\\ &= \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\cos\omega\eta}{\eta - t}\,d\eta\\ &= \frac{-1}{\pi} P\int_{-\infty}^{\infty} \frac{\cos[\omega(y+t)]}{y}\,dy\\ &= \frac{-1}{\pi}\left\{\cos\omega t\,P\int_{-\infty}^{\infty}\frac{\cos\omega y}{y}\,dy - \sin\omega t\,P\int_{-\infty}^{\infty}\frac{\sin\omega y}{y}\,dy\right\}\\ &= \sin\omega t.\end{aligned}$$

The result is due to the fact that $\cos\omega y/y$ is an odd function and $P\int_{-\infty}^{\infty}\frac{\sin\omega y}{y}\,dy = \pi.$

Example

If $x(t) = p_a(t)$ then

$$v(t) = \mathsf{H}\{p_a(t)\} = \frac{-1}{\pi} P\int_{-a}^{t-\varepsilon}\frac{d\eta}{\eta - t} - \frac{1}{\pi t} P\int_{t+\varepsilon}^{a}\frac{d\eta}{\eta - t}$$

$$= \lim_{\varepsilon\to 0}\left[-\frac{1}{\pi}\ln(\eta - t)\Big|_{-a}^{t-\varepsilon} - \frac{1}{\pi}\ln(\eta - t)\Big|_{t+\varepsilon}^{a}\right] = \upsilon(t) = \frac{1}{\pi}\ln\left|\frac{t+a}{t-a}\right|$$

Example

If $x(t) = a$ then

$$a\mathsf{H}\{1\} = a\lim_{a\to\infty}\frac{1}{\pi}\ln\left|\frac{t+a}{t-a}\right| = 0\,.$$

Hence, if x_o = *constant* is the mean value of a function, then $x(t) = x_o + x_1(t)$. Therefore $\mathsf{H}\{x_o + x_1(t)\}$ = $\mathsf{H}\{x_1(t)\}$. This implies that the Hilbert transform cancels the mean value or the DC term in electrical engineering terminology.

15.1.2 Analytic Signal

A complex signal whose imaginary part is the Hilbert transform of its real part is called the *analytic signal*.

$$\psi(z) = \psi(t,\tau) = x(t,\tau) + j\upsilon(t,\tau), \quad x \text{ and } \upsilon \text{ are real functions}$$

$$z = t + j\tau$$

$$\upsilon(t,\tau) = \mathsf{H}\{x(t,\tau)\}$$

The function $\psi(z) = x(t,\tau) + j\upsilon(t,\tau)$ is analytic if the Cauchy-Riemann conditions

$$\frac{\partial x}{\partial t} = \frac{\partial \upsilon}{\partial \tau} \quad \text{and} \quad \frac{\partial x}{\partial \tau} = -\frac{\partial \upsilon}{\partial t}$$

are satisfied.

Example

The real and imaginary parts of the analytic function

$$\psi(z) = 1/(\alpha - jz) = \frac{\alpha + \tau}{(\alpha+\tau)^2 + t^2} + j\frac{t}{(\alpha+\tau)^2 + t^2}$$

satisfy Cauchy-Riemann conditions and, hence, they are Hilbert transform pairs.

$$x(t) = \frac{\psi(t) + \psi^*(t)}{2} \qquad \upsilon(t) = \frac{\psi(t) - \psi^*(t)}{2j} \quad (\tau = 0)$$

15.2 Spectra of Hilbert Transformation

15.2.1 One-Sided Spectrum of the Analytic Signal

$$x(t) = x_e(t) + x_o(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}$$

$$X(\omega) = X_r(\omega) + jX_i(\omega) = \int_{-\infty}^{\infty} x_e(t)\cos\omega t\,dt + j\left(-\int_{-\infty}^{\infty} x_o(t)\sin\omega t\,dt\right)$$

$$V(\omega) = V_r(\omega) + jV_i(\omega) = \text{Spectrum of the Hilbert transform}$$

$$V_r(\omega) = -j\,\text{sgn}(\omega)[jX_i(\omega)] = \text{sgn}(\omega)\,X_i(\omega) \text{ (see also 15.1.1)}$$

$$V_i(\omega) = -\text{sgn}(\omega)\,X_r(\omega)$$

Example

$\mathsf{H}\{\cos\omega t\} = \sin\omega t$, $\mathsf{H}\{\sin\omega t\} = -\cos\omega t$ and, therefore,

$$\mathsf{H}\{e^{j\omega t}\} = \sin\omega t - j\cos\omega t = -j\,\text{sgn}(\omega)e^{j\omega t} = \text{sgn}(\omega)e^{j(\omega t - \frac{\pi}{2})}$$

Note: The operator $-j\,\text{sgn}(\omega)$ provides a $\pi/2$ phase lag for all positive frequencies and $\pi/2$ lead for all negative frequencies.

15.2.2 Fourier Spectrum of the Analytic Signal

$$\mathsf{H}\{x(t)\} = \upsilon(t); \quad \mathsf{F}\{x(t)\} = X(\omega); \quad \mathsf{F}\{\upsilon(t)\} = -j\,\text{sgn}(\omega)\,X(\omega)$$

$$\mathsf{F}\{\psi(t)\} = x(t) + j\upsilon(t) = \Psi(\omega) = X(\omega) + jV(\omega) = [1 + \operatorname{sgn}(\omega)]X(\omega)$$

$$1 + \operatorname{sgn}(\omega) = \begin{cases} 2 & \omega > 0 \\ 1 & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

Note: The spectrum of the analytic signal is twice that of its Fourier transform at the positive frequency range $0 < \omega < \infty$.

Example

If $\psi(t) = \frac{1}{1+t^2} + j\frac{t}{1+t^2}$ then $\mathsf{F}\{\psi(t)\} = [1 + \operatorname{sgn}(\omega)]\pi e^{-|\omega|}$ where

$$\mathsf{H}\{1/(1+t^2)\} = t/(1+t^2) \text{ and } \mathsf{F}\{1/(1+t^2)\} = \pi e^{-|\omega|}.$$

15.3 Hilbert Transform and Delta Function

15.3.1 Complex Delta Function

If we define $2\cdot 1(f) = 1(f) + \operatorname{sgn}(f)$, then the function (see Fourier transform properties [symmetry] and function, Chapter 3).

$$\psi_\delta(t) = \int_{-\infty}^{\infty} 2\cdot 1(f)e^{j\omega t}\,df = \int_{-\infty}^{\infty} 1(f)e^{j\omega t}\,df + \int_{-\infty}^{\infty} \operatorname{sgn}(f)e^{j\omega t}\,df$$

$$= \delta(t) + j\frac{1}{\pi t}$$

15.3.2 Hilbert Transform of the Delta Function

From (15.3.1) implies

$$\mathsf{H}\{\delta(t)\} = \frac{1}{\pi t}$$

15.4 Hilbert Transform of Periodic Signals

15.4.1 Hilbert Transform of Period Functions

A periodic function can be written in trigonometric form

$$x_p(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \varphi_n), \quad \omega_o = 2\pi/T, \quad T = \text{period}$$

Therefore we obtain

$$v_p(t) = \mathsf{H}\{x_p(t)\} = \sum_{n=1}^{\infty} C_n \sin(n\omega_o t + \varphi_n)$$

because the Hilbert transform of a constant is zero (see 15.1.1).

A periodic function can also be written in *complex* form

$$x_p(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_o t}$$

Therefore,

$$v_p(t) = \mathsf{H}\{x_p(t)\} = \sum_{n=-\infty}^{\infty} \alpha_n \mathsf{H}\{e^{jn\omega_o t}\} = \sum_{n=-\infty}^{\infty} -j\,\text{sgn}(n) e^{jn\omega_o t}$$

15.5 Hilbert Transform Properties and Pairs

15.5.1 Hilbert Transform Properties

TABLE 15.1 Properties of the Hilbert transformation

No.	Name	Original or Inverse Hilbert Transform	Hilbert Transform
1	Notations	$x(t)$ or $\mathsf{H}^{-1}[\upsilon]$	$\upsilon(t)$ or $\hat{x}(t)$ or $\mathsf{H}[\upsilon]$
2	Time domain definitions	$\begin{cases} x(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\upsilon(\eta)}{\eta - t}\,d\eta \\ x(t) = \dfrac{-1}{\pi t} * \upsilon(t) \end{cases}$ or	$\upsilon(t) = \dfrac{-1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{x(\eta)}{\eta - t}\,d\eta$ $\upsilon(t) = \dfrac{1}{\pi t} * x(t)$
3	Change of symmetry	$x(t) = x_{1e}(t) + x_{2o}(t)^*$;	$\upsilon(t) = \upsilon_{1o}(t) + \upsilon_{2e}(t)$
4	Fourier spectra		
	$x(t) = \overset{F}{\Longleftrightarrow} X(\omega) = X_e(\omega) + jX_o(\omega)$;		$\upsilon(t) = \overset{F}{\Longleftrightarrow} V(\omega) = V_e(\omega) + jV_o(\omega)$
	$X(\omega) = j\,\text{sgn}(\omega) V(\omega)$;		$V(\omega) = -j\,\text{sgn}(\omega) X(\omega)$
	For even functions the Hilbert transform is odd:		
	$X_e(\omega) = 2\displaystyle\int_o^{\infty} x_{1e}(t)\cos(\omega t)\,dt$		$\upsilon_o(t) = 2\displaystyle\int_o^{\infty} X_e(\omega)\sin(\omega t)\,df$
	For odd functions the Hilbert transform is even:		
	$X_o(\omega) = -2\displaystyle\int_o^{\infty} x_{2o}(t)\sin(\omega t)\,dt$		$\upsilon_e(t) = 2\displaystyle\int_o^{\infty} X_o(\omega)\cos(\omega t)\,df$
5	Linearity	$ax_1(t) + bx_2(t)$	$a\upsilon_1(t) + b\upsilon_2(t)$
6	Scaling and	$x(at)$; $a > 0$	$\upsilon(at)$
	time reversal	$x(-at)$	$-\upsilon(-at)$
7	Time shift	$x(t-a)$	$\upsilon(t-a)$
8	Scaling and time shift	$x(bt-a)$	$\upsilon(bt-a)$
			Fourier image
9	Iteration	$\mathsf{H}[x(t)] = \upsilon(t)$	$-j\,\text{sgn}(\omega) X(\omega)$
		$\mathsf{H}[\mathsf{H}[x]] = -x(t)$	$[-j\,\text{sgn}(\omega)]^2 X(\omega)$
		$\mathsf{H}[\mathsf{H}[\mathsf{H}[x]]] = -\upsilon(t)$	$[-j\,\text{sgn}(\omega)]^3 X(\omega)$
		$\mathsf{H}[\mathsf{H}[\mathsf{H}[\mathsf{H}[x]]]] = x(t)$	$[-j\,\text{sgn}(\omega)]^4 X(\omega)$

e = even; o = odd

TABLE 15.1 Properties of the Hilbert transformation (continued)

No.	Name	Original or Inverse Hilbert Transform	Hilbert Transform
			First option
10	Time derivatives	$\dot{x}(t) = \frac{-1}{\pi t} * \dot{\upsilon}(t)$	$\dot{\upsilon}(t) = \frac{1}{\pi t} * \dot{x}(t)$
			Second option
		$\dot{x}(t) = \left[\frac{d}{dt}\frac{-1}{\pi t}\right] * \upsilon(t)$	$\dot{\upsilon}(t) = \left[\frac{d}{dt}\frac{1}{\pi t}\right] * x(t)$
11	Convolution	$\begin{cases} x_1(t) * x_2(t) = \\ -\upsilon_1(t) * \upsilon_2(t) \end{cases}$	$x_1(t) * \upsilon_2(t) =$ $\upsilon_1(t) * x_2(t)$
12	Autoconvolution equality	$\int x(\tau)x(t-\tau)\,d\tau = -\int \upsilon(\tau)\upsilon(t-\tau)\,d\tau$ for $\tau = 0$ energy equality	
13	Multiplication by t	$t\,x(t)$	$t\upsilon(t) - \int_{-\infty}^{\infty} x(\tau)\,d\tau$
14	Multiplication of signals with non-overlapping spectra	$x_1(t)$ (low-pass signal)	$x_2(t)$ (high-pass signal)
		$x_1(t)x_2(t)$	$x_1(t)\upsilon_2(t)$
15	Analytic signal	$\psi(t) = x(t) + j\mathrm{H}[x(t)]$	$\mathrm{H}[\psi(t)] = -j\,\psi(t)$
16	Product of analytic signals	$\psi(t) = \psi_1(t)\psi_2(t)$	$\mathrm{H}[\psi(t)] = \psi_1(t)\mathrm{H}[\psi_2(t)]$ $= \mathrm{H}[\psi_1(t)]\psi_2(t)$
17	Nonlinear transformations	$x(x)$	$\upsilon(x)$
17a	$y = \frac{c}{bt+a}$	$x_1(t) = x\left[\frac{c}{bt+a}\right]$	$\upsilon_1(t) = \upsilon\left[\frac{c}{bt+a}\right] - \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{x(t)}{t}\,dt$
17b	$y = a + \frac{b}{t}$	$x_1(t) = x\left[a + \frac{b}{t}\right]$	$\upsilon_1(t) = \frac{b}{a}\left\{\upsilon\left[a + \frac{b}{t}\right] - \upsilon(a)\right\}$
	Notice that the nonlinear transformation may change the signal $x(t)$ of finite energy to a signal $x_1(t)$ of infinite energy. P is the Cauchy Principal Value.		
18	Asymptotic value as $t \Rightarrow \infty$ for even functions of finite support:		
	$x_e(t) = x_e(-t)$		$\lim_{t\Rightarrow\infty} \lvert\upsilon_o(t)\rvert = \frac{1}{\pi t}\int_s x_e(t)\,dt^a$

[a] S is support of $x_e(t)$

15.5.2 Iteration

- Iteration of the HT two times yields the original signal with reverse sign.
- Iteration of the HT four times restores the original signal
- In Fourier frequency domain, n-time iteration translates the n-time multiplication by $-j\text{sgn}(\omega)$

15.5.3 Parseval's Theorem

$$v(t) = \mathrm{H}\{x(t)\}$$

$$\mathcal{F}\{\upsilon(t)\} = V(\omega) = -j\,\text{sgn}(\omega)\,X(\omega)$$

$$|V(\omega)|^2 = |-j\,\text{sgn}(\omega)\,X(\omega)|^2 = |X(\omega)|^2$$

since

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(\omega)|^2\,df = \text{energy of } x(t)$$

$$E_\upsilon = \int_{-\infty}^{\infty} |V(\omega)|^2\,df = \int_{-\infty}^{\infty} |X(\omega)|^2\,df = E_x$$

15.5.4 Orthogonality

$$\int_{-\infty}^{\infty} \upsilon(t)\,x(t)\,dt = 0$$

15.5.5 Fourier Transform of the Autoconvolution of the Hilbert Pairs

$$\mathcal{F}\{x(t) * x(t)\} = X^2(\omega)$$

$$\mathcal{F}\{\upsilon(t) * \upsilon(t)\} = [-j\,\text{sgn}(\omega)\,X(\omega)]^2 = -X^2(\omega)$$

$$x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\,x(t-\tau)\,d\tau = -\int_{-\infty}^{\infty} \upsilon(\tau)\,\upsilon(t-\tau)\,d\tau = -\upsilon(t) * \upsilon(t)$$

$$x_1(t) * x_2(t) = -\upsilon_1(t) * \upsilon_2(t)$$

15.5.6 Hilbert Transform Pairs

TABLE 15.2 Selected Useful Hilbert Pairs

No.	Name	Function	Hilbert Transform
1	sine	$\sin(\omega t)$	$-\cos(\omega t)$
2	cosine	$\cos(\omega t)$	$\sin(\omega t)$
3	Exponential	$e^{j\omega t}$	$-j\,\text{sgn}(\omega)\,e^{j\omega t}$
4	Square pulse	$\prod_{2a}(t)$	$\frac{1}{\pi}\ln\left\|\frac{t+a}{t-a}\right\|$
5	Bipolar pulse	$\prod_{2a}(t)\,\text{sgn}(t)$	$-\frac{1}{\pi}\ln\left\|1-(a/t)^2\right\|$
6	Double triangle	$t\prod_{2a}(t)\,\text{sgn}(t)$	$-\frac{1}{\pi}\ln\left\|1-(a/t)^2\right\|$
7	Triangle, tri(t)	$1-\|t/a\|,\ \|t\| \le a$	$\frac{-1}{\pi}\left\{\ln\left\|\frac{t-a}{t+a}\right\| + \frac{t}{a}\ln\left\|\frac{t^2}{t^2-a^2}\right\|\right\}$
		$0,\ \|t\| > a$	
8	One-sided triangle		$\frac{1}{\pi}\left\{(1-t/a)\ln\left\|\frac{t}{t-a}\right\| + 1\right\}$

TABLE 15.2 Selected Useful Hilbert Pairs (continued)

No.	Name	Function	Hilbert Transform
9	Trapezoid		$\frac{-1}{\pi}\left\{\frac{b}{b-a}\ln\left\|\frac{(a+t)(b-t)}{(a-t)(b+t)}\right\|+\frac{t}{b-a}\ln\left\|\frac{a^2-t^2}{b^2-t^2}\right\|+\ln\left\|\frac{(a-t)}{(a+t)}\right\|\right\}$
10	Cauchy pulse	$\frac{a}{a^2+t^2}$	$\frac{t}{a^2+t^2}$
11	Gaussian pulse	$e^{-\pi t^2}$	$2\int_0^\infty e^{-\pi f^2}\sin(\omega t)\,df;\ \omega=2\pi f$
12	Parabolic pulse	$1-(t/a)^2,\ \|t\|\le a$	$\frac{-1}{\pi}\left\{\left[1-(t/a)^2\right]\ \ln\left\|\frac{t-a}{t+a}\right\|-\frac{2t}{a}\right\}$
13	Symmetric exponential	$e^{-a\|t\|}$	$2\int_0^\infty \frac{2a}{a^2-\omega^2}\sin(\omega t)\,df$
14	Antisymmetric exponential	$\operatorname{sgn}(t)e^{-a\|t\|}$	$-2\int_0^\infty \frac{2a}{a^2-\omega^2}\cos(\omega t)\,df$
15	One-sided exponential	$1(t)e^{-a\|t\|}$	$2\int_0^\infty \frac{a\sin(\omega t)-\omega\cos(\omega t)}{a^2-\omega^2}\,df$
16	Sinc pulse	$\frac{\sin(at)}{at}$	$\frac{\sin^2(at/2)}{(at/2)}=\frac{1-\cos(at)}{at}$
17	Video test pulse	$\begin{cases}\cos^2(\pi t/2a); & \|t\|\le a\\ 0, & \|t\|>a\end{cases}$	$2\int_0^\infty \frac{2a^2}{4a^2-\omega^2}\frac{\sin[\pi\omega/(2a)]}{\omega}\sin(\omega t)\,df$
18	{Spectra of $a(t)$ and $\cos(\omega_0 t)$ overlapping	$a(t)\cos(\omega_0 t)$	$\left[a(t)*\frac{\sin(\omega_0 t)}{\pi t}\right]\sin(\omega_0 t)+\left[a(t)*\frac{\cos(\omega_0 t)}{\pi t}\right]\cos(\omega_0 t)$
19	Bedrosian's theorem	$a(t)\cos(\omega_0 t)$	$a(t)\sin(\omega_0 t)$
20	A constant	a	zero

Hyperbolic Functions: Approximation by Summation of Cauchy Functions (see Hilbert Pairs No. 10 and 45)

No.	Name	Function	Hilbert Transform
21	Tangent hyp.	$\tanh(t)=2\sum_{\eta=0}^{\infty}\frac{t}{(\eta+0.5)^2\pi^2+t^2}$	$-2\pi\sum_{\eta=0}^{\infty}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$
22	Part of finite energy of tanh	$\operatorname{sgn}(t)-\tanh(t)$	$\pi\delta(t)+2\pi\sum_{\eta=0}^{\infty}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$
23	Cotangent hyp.	$\coth(t)=\frac{1}{t}+2\sum_{\eta=1}^{\infty}\frac{t}{(\eta\pi)^2+t^2}$	$-\pi\delta(t)+2\pi\sum_{\eta=1}^{\infty}\frac{\eta}{(\eta\pi)^2+t^2}$
24	Secans hyp.	$\operatorname{sech}(t)=-2\pi\sum_{\eta=0}^{\infty}(-1)^{(\eta-1)}\frac{(\eta+0.5)}{(\eta+0.5)^2\pi^2+t^2}$	$-2\sum_{\eta=0}^{\infty}(-1)^{(\eta-1)}\frac{t}{(\eta+0.5)^2\pi^2+t^2}$
25	Cosecans hyp.	$\operatorname{cosech}(t)=\frac{1}{t}-2\sum_{\eta=1}^{\infty}(-1)^{(\eta-1)}\frac{t}{(\eta\pi)^2+t^2}$	$-\pi\delta(t)+2\pi\sum_{\eta=1}^{\infty}(-1)^{(\eta-1)}\frac{\eta}{(\eta\pi)^2+t^2}$

TABLE 15.2 Selected Useful Hilbert Pairs (continued)

No.	Name	Function	Hilbert Transform
		Hyperbolic Functions by Inverse Fourier Transformation; $\omega = 2\pi f$	
26		$\operatorname{sgn}(t) - \tanh(at/2)$ $\operatorname{Re} a > 0$	$2\int_0^\infty \left[\frac{2\pi}{a\sinh(\pi\omega/a)} - \frac{2}{\omega}\right]\cos(\omega t)\,df$
27		$\coth(t) - \operatorname{sgn}(t)$	$2\int_0^\infty \left[\frac{2\pi}{a}\coth(\pi\omega/a) - \frac{2}{\omega}\right]\cos(\omega t)\,df$
28		$\operatorname{sech}(at/2)$	$2\int_0^\infty \frac{2\pi}{a\cosh(\pi\omega/(2a))}\sin(\omega t)\,df$
29		$\operatorname{cosech}(at/2)$	$-2\int_0^\infty \frac{2\pi}{a}\tanh(\pi\omega/(2a))\cos(\omega t)\,df$
30		$\operatorname{sech}^2(at/2)$	$2\int \frac{2\pi\omega}{a\sinh(\pi\omega/(2a))}\sin(\omega t)\,df$

Delta Distribution, $1/(\pi t)$ Distribution and its Derivatives: Derivation Using Successive Iteration and

No.	Operation	Differentiation If $x(t) \overset{H}{\Longleftrightarrow} v(t)$ then $\dot{x}(t) \overset{H}{\Longleftrightarrow} \dot{v}(t)$ $x(t)$	Iteration $H[v(t)] = HH[u(t)] = -x(t)$ $v(t)$
31		$\delta(t)$	$1/(\pi t)$
32	Iteration	$1/(\pi t)$	$-\delta(t)$
33	Differentiation	$\dot{\delta}(t)$	$-1/(\pi t^2)$
34	Iteration	$1/(\pi t^2)$	$\dot{\delta}(t)$
35	Differentiation	$\ddot{\delta}(t)$	$2/(\pi t^3)$
36	Iteration	$1/(\pi t^3)$	$-0.5\ddot{\delta}(t)$
37	Differentiation	$\dddot{\delta}(t)$	$-6/(\pi t^4)$
38	Iteration	$1/(\pi t^4)$	$(1/6)\dddot{\delta}(t)$
39		$x(t)\delta(t)$	$x(0)/(\pi t)$
		The procedure could be continued.	
		Equality of Convolution	
40		$\delta(t) * \delta(t) * \delta(t)$	$\frac{1}{\pi t} * \frac{1}{\pi t} = -\delta(t)$
41		$\dot{\delta}(t) * \delta(t) = \dot{\delta}(t)$	$\frac{1}{\pi t^2} * \frac{1}{\pi t} = \dot{\delta}(t)$
42		$\dot{\delta}(t) * \dot{\delta}(t) = \ddot{\delta}(t)$	$\frac{1}{\pi t^2} * \frac{1}{\pi t^2} = -\ddot{\delta}(t)$
43		$\dddot{\delta}(t) * \delta(t) = \dddot{\delta}(t) = \ddot{\delta}(t) * \dot{\delta}(t)$	$\frac{6}{\pi t^4} * \frac{1}{\pi t} = \dddot{\delta}(t) = \frac{2}{\pi t^3} * \frac{1}{\pi t^2}$
		Approximating Functions of Distributions (see No. 31 to 37 of this table)	
		$x(t)$	$v(t)$
44		$\int \delta(a,t)\,dt = \frac{1}{\pi}\tan^{-1}(t/a)$	$\int \theta(a,t)\,dt = \frac{\ln(a^2+t^2)}{2\pi}$

TABLE 15.2 Selected Useful Hilbert Pairs (continued)

No.	Name	Function	Hilbert Transform
45		$\delta(a,t) = \frac{1}{\pi}\frac{a}{a^2+t^2}$	$\theta(a,t) = \frac{1}{\pi}\frac{t}{a^2+t^2}$
46		$\dot{\delta}(a,t) = \frac{1}{\pi}\frac{-2at}{(a^2+t^2)^2}$	$\dot{\theta}(a,t) = \frac{1}{\pi}\frac{a^2-t^2}{(a^2+t^2)^2}$
47		$\ddot{\delta}(a,t) = \frac{1}{\pi}\frac{6at^2-2a^2}{(a^2+t^2)^3}$	$\ddot{\theta}(a,t) = \frac{1}{\pi}\frac{2t^2-6at^2}{(a^2+t^2)^3}$
48		$\dddot{\delta}(a,t) = \frac{1}{\pi}\frac{24a^3t-24at^2}{(a^2+t^2)^4}$	$\dddot{\theta}(a,t) = \frac{1}{\pi}\frac{-6t^2+36a^2t^2-6a^4}{(a^2+t^2)^4}$

Derivation Using Successive Iteration and Differentiation (see the information above No. 31)

Trigonometric Expressions

No.	Operation	$x(t)$	$v(t)$
49		$\frac{\sin(at)}{t}$	$\frac{1-\cos(at)}{t} = \frac{2\sin^2(at/2)}{t}$
50	Iteration	$\frac{\cos(at)}{t}$	$-\pi\delta(t) + \frac{\sin(at)}{t}$
51	Differentiation	$\frac{\sin(at)}{t^2}$	$-a\pi\delta(t) + \frac{1-\cos(at)}{t^2}$
52	Iteration	$\frac{\cos(at)}{t^2}$	$\pi\dot{\delta}(t) - \frac{a}{t} + \frac{\sin(at)}{t^2}$
53	Differentiation	$\frac{\sin(at)}{t^3}$	$\pi a\dot{\delta}(t) - \frac{a^2}{2t} + \frac{1-\cos(at)}{t^3}$
54	Iteration	$\frac{\cos(at)}{t^3}$	$-\frac{\pi}{2}\ddot{\delta}(t) + \frac{\pi a^2}{2}\delta(t) - \frac{a}{t^2} + \frac{\sin(at)}{t^3}$

Selected Useful Hilbert Pairs of Periodic Signals

No.	Name	$x_p(t)$	$\nu_p(t)$
55	Sampling sequence	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{1}{T}\sum_{n=-\infty}^{\infty}\cos[(\pi/T)(t-nT)]$
56	Even square wave	$\text{sgn}[\cos(\omega t)],\ \omega = 2\pi/T$	$(2/\pi)\ln\lvert\tan(\omega t/2+\pi/4)\rvert$
57	Odd square wave	$\text{sgn}[\sin(\omega t)],\ \omega = 2\pi/T$	$(2/\pi)\ln\lvert\tan(\omega t/2)\rvert$
58	Squared cosine	$\cos^2(\omega t)$	$0.5\sin(2\omega t)$
59	Squared sine	$\sin^2(\omega t)$	$-0.5\sin(2\omega t)$
60	Cube cosine	$\cos^3(\omega t)$	$\frac{3}{4}\sin(\omega t)+\frac{1}{4}\sin(3\omega t)$
61	Cube sine	$\sin^3(\omega t)$	$\frac{-3}{4}\cos(\omega t)+\frac{1}{4}\cos(3\omega t)$
62		$\cos^4(\omega t)$	$\frac{1}{2}\sin(2\omega t)+\frac{1}{8}\sin(4\omega t)$
63		$\sin^4(\omega t)$	$-\frac{1}{2}\sin(2\omega t)+\frac{1}{8}\sin(4\omega t)$
64		$\cos^5(\omega t)$	$\frac{5}{8}\sin(2\omega t)+\frac{5}{16}\sin(3\omega t)+\frac{1}{16}\sin(5\omega t)$
65		$\cos^6(\omega t)$	$\frac{15}{32}\sin(2\omega t)+\frac{6}{32}\sin(4\omega t)+\frac{1}{32}\sin(6\omega t)$
66		$\cos(at+\varphi)\cos(bt+\Psi)$ $0<a<b$ φ,Ψ = constants	$\cos(at+\varphi)\sin(bt+\Psi)$

TABLE 15.2 Selected Useful Hilbert Pairs (continued)

No.	Name	Function	Hilbert Transform
67	Fourier Series	$X_o + \sum_{n=1}^{\infty} X_n \cos(n\omega t + \varphi_n)$	$\sum_{n=1}^{\infty} X_n \sin(n\omega t + \varphi_n)$
68	Any periodic function	x_T = generating function	$x_T(t) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} \cot[(\pi/T)(t-kT)]$
		$x_T(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT)$	

15.6 Differentiation of Hilbert Pairs

15.6.1 Differentiation Pairs

$$\mathsf{H}\{\dot{x}(t)\} = \dot{v}(t)$$

$$\mathsf{H}\left\{\frac{d^n x(t)}{dt^n}\right\} = \frac{d^n v(t)}{dt^n}$$

Example

$$\mathsf{H}\{\delta(t)\} = \frac{1}{\pi t};\ \ \mathsf{H}\{\dot{\delta}(t)\} = \frac{d}{dt}\left(\frac{1}{\pi t}\right) = -\frac{1}{\pi t^2}$$

15.6.2 Derivative of Convolution

$$\mathsf{H}\{x(t)\} = \mathsf{H}\left\{\frac{-1}{\pi t} * v(t)\right\} \Rightarrow v(t) = \frac{1}{\pi t} * x(t)$$

$$\mathsf{H}\{\dot{x}(t)\} = \mathsf{H}\left\{-\frac{d}{dt}\left(\frac{1}{\pi t}\right) * v(t)\right\} \Rightarrow \dot{v}(t) = \frac{d}{dt}\left(\frac{1}{\pi t}\right) * x(t) \quad \text{(see 15.6.1 and 15.5.5)}$$

$$= \left(-\frac{-1}{\pi t^2}\right) * x(t) = \frac{1}{\pi t^2} * v(t)$$

$$\mathsf{H}\{\dot{x}(t)\} = \mathsf{H}\left\{-\frac{1}{\pi t} * \dot{v}(t)\right\} \Rightarrow \dot{v}(t) = \frac{1}{\pi t} * \dot{x}(t)$$

15.6.3 Fourier Transform of Hilbert Transform

$$v(t) = \frac{1}{\pi t} * x(t), \qquad \mathsf{F}\{v(t)\} = -j\,\mathrm{sgn}(\omega)\,X(\omega)$$

$$\mathsf{F}\{\dot{v}(t)\} = j\omega[-j\,\mathrm{sgn}(\omega)\,X(\omega)] = \omega\,\mathrm{sgn}(\omega)\,X(\omega)$$

15.7 Hilbert Transform of Hermite Polynomials

15.7.1 Hermite Polynomials and their Hilbert Transform

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \qquad n = 0,1,2,\cdots, \ -\infty < t < \infty$$

$$H_n(t) = 2t\,H_{n-1}(t) - 2(n-1)\,H_{n-2}(t) \qquad n = 1,2,\cdots$$

$$\mathsf{F}\{e^{-t^2}\} = \sqrt{\pi}\, e^{-\pi^2 f^2} = \sqrt{\pi}\, e^{-\omega^2/4}$$

$$v(t) = \mathsf{H}\{x(t)\} \doteq \mathsf{H}\{e^{-t^2}\} = \mathsf{F}^{-1}\{V(\omega)\} = \int_{-\infty}^{\infty} -j\,\mathrm{sgn}(\omega)\sqrt{\pi}\, e^{-\pi^2 f^2} e^{j\omega t}\, df$$

$$= 2\sqrt{\pi}\int_{-\infty}^{\infty} e^{-\pi^2 f^2} \sin\omega t\, df$$

$$\mathsf{H}\{2\,te^{-t^2}\} = -2\sqrt{\pi}\int_{-\infty}^{\infty} \omega e^{-\pi^2 f^2} \cos\omega t\, df$$

15.7.2 Table of Hilbert Transform of Hermite Polynomials

TABLE 15.3 Hilbert Transform of Weighted Hermite Polynomials [Notation: $x = \exp(-t^2)$]

	Hermite Polynomial	Hilbert Transform	Energy
n	$H_n x$	$\mathsf{H}(H_n x)$	E
0	$(1)\,x$	$2\sqrt{\pi}\int_0^{\infty} \exp(-\pi^2 f^2)\sin(\omega t)\,df$	$\sqrt{\pi/2}$
1	$(2t)\,x$	$-2\sqrt{\pi}\int_0^{\infty} \omega\exp(-\pi^2 f^2)\cos(\omega t)\,df$	$\sqrt{\pi/2}$
2	$(4t^2-2)\,x$	$-2\sqrt{\pi}\int_0^{\infty} \omega^2\exp(-\pi^2 f^2)\sin(\omega t)\,df$	$3\sqrt{\pi/2}$
3	$(8t^3-12t)\,x$	$2\sqrt{\pi}\int_0^{\infty} \omega^3\exp(-\pi^2 f^2)\cos(\omega t)\,df$	$15\sqrt{\pi/2}$
4	$(16t^4-48t^2+12)\,x$	$2\sqrt{\pi}\int_0^{\infty} \omega^4\exp(-\pi^2 f^2)\sin(\omega t)\,df$	$105\sqrt{\pi/2}$
5	$(32t^5-160t^3+120t)\,x$	$-2\sqrt{\pi}\int_0^{\infty} \omega^5\exp(-\pi^2 f^2)\cos(\omega t)\,df$	$945\sqrt{\pi/2}$
n	$H_n x = (-1)^n[2tH_{n-1}(t)$ $-2(n-1)H_{n-2}(t)]$	$(-1)^n 2\sqrt{\pi}\int_0^{\infty} \omega^n\exp(-\pi^2 f^2)\sin\left(\omega t + \frac{n\pi}{2}\right)df$	

Energy = $\int_{-\infty}^{\infty} x^2 H_n^2\, dt = \int_{-\infty}^{\infty} [\mathsf{H}(xH_n)]^2\, dt = 1\times 3\times 5\times\cdots\times(2n-1)\times\sqrt{\pi/2}, \quad n \ge 1$

15.7.3 Hilbert Transform of Orthonormal Hermite Functions (see Chapter 22)

$$h_n(t) = (2^n n!)^{-1/2} \pi^{-1/2} e^{-t^2} H_n(t) \qquad n = 0,1,2,\cdots$$

$$\mathsf{H}\{h_n(t)\} = \nu_n(t)$$

$$= \left[\frac{2(n-1)!}{n!}\right]^{1/2} \left[t\,\nu_{n-1}(t) - \frac{1}{\pi}\int_{-\infty}^{\infty} h_{n-1}(\tau)\,d\tau\right] - (n-1)\left[\frac{(n-2)!}{n!}\right]^{1/2} \nu_{n-2}(t)$$

15.7.4 Hilbert Transform of Orthonormal Hermite Functions

TABLE 15.4 Hilbert Transforms of Orthonormal Hermite Functions (Energy = 1).

Notations: $h_o(t), h_1(t), \cdots \Rightarrow h_o, h_1, \cdots;$ $\quad \upsilon_o(t), \cdots \Rightarrow \upsilon_o, \upsilon_1, \cdots$

$$g(t) = \int_0^{\infty} e^{-2\pi^2 f^2} \sin(2\pi f t)\,df; \qquad a = \pi^{-0.25} e^{-t^2/2}; \qquad b = \pi^{0.25}$$

Hermite Functions $h_n(t)$	Hilbert Transforms $\upsilon_n(t)$
	Recurrent Notation
$h_0 = a$	$\upsilon_0 = 2\sqrt{2}\ bg(t)$
$h_1 = \sqrt{2}\ th_0$	$\upsilon_1 = \sqrt{2}\left[t\upsilon_0 \frac{\sqrt{2}\,b}{\pi}\right]$
$h_2 = th_1 - \sqrt{1/2}\,h_0$	$\upsilon_2 = t\upsilon_1 - \sqrt{1/2}\ \upsilon_0$
$h_3 = \sqrt{2/3}\,[th_2 - h_1]$	$\upsilon_3 = \sqrt{2/3}\left[t\upsilon_2 - \frac{b}{\pi} - \upsilon_1\right]$
$h_4 = \sqrt{1/2}\ th_3 - \sqrt{3/4}\ h_2$	$\upsilon_4 = \sqrt{1/2}\ t\upsilon_3 - \sqrt{3/4}\ \upsilon_2$
$h_5 = \sqrt{2/5}\ th_4 - \sqrt{4/5}\ h_3$	$\upsilon_5 = \sqrt{2/5}\left[t\upsilon_4 - \frac{\sqrt{3}\,b}{2\pi}\right] - \sqrt{4/5}\ \upsilon_3$
$h_n = \sqrt{\frac{2(n-1)!}{n!}}\ th_{n-1} +$	$\upsilon_n = \sqrt{\frac{2(n-1)!}{n!}}\ [t\upsilon_{n-1}$
$(n-1)\sqrt{\frac{(n-2)!}{n!}}\ h_{n-2}$	$-\frac{1}{\pi}\int h_{n-1}(\tau)\,d\tau] - (n-1)\sqrt{\frac{(n-2)!}{n!}}\ \upsilon_{n-2}$
	Nonrecurrent Notation
$h_0 = a1$	$2\sqrt{2}\ bg(t)$
$h_1 = \sqrt{2}\ at$	$2b[2tg(t) - \pi^{-1}]$
$h_2 = \frac{a}{\sqrt{8}}(4t^2 - 2)$	$2b[(2t^2 - 1)g(t) - t\pi^{-1}]$
$h_3 = \frac{a}{\sqrt{48}}(8t^3 - 12t)$	$\sqrt{8/3b}\left[(2t^3 - 3t)g(t) - \frac{t^2}{\pi} + \frac{1}{2\pi}\right]$
$h_4 = \frac{a}{\sqrt{384}}(16t^4 - 48t^2 + 12)$	$\sqrt{4/3b}\left[(2t^4 - 6t^2 + 1.5)g(t) - \frac{t^3}{\pi} + \frac{2t}{2\pi}\right]$

TABLE 15.4 Hilbert Transforms of Orthonormal Hermite Functions (Energy = 1). (continued)

Notations: $h_o(t), h_1(t), \cdots \Rightarrow h_o, h_1, \cdots;$ $\upsilon_o(t), \cdots \Rightarrow \upsilon_o, \upsilon_1, \cdots$

$$g(t) = \int_0^\infty e^{-2\pi^2 f^2} \sin(2\pi f t)\, df; \qquad a = \pi^{-0.25} e^{-t^2/2}; \qquad b = \pi^{0.25}$$

Hermite Functions $h_n(t)$	Hilbert Transforms $\upsilon_n(t)$
$h_5 = \frac{a}{\sqrt{3840}}(32t^5 - 160t^3 + 120t)$	$\sqrt{8/15}\, b \left[(2t^5 - 10t^3 + 7.5) g(t) - \frac{(t^4 - 4t^2) + 1.75}{\pi} \right]$
………	………
$h_n(t) = \frac{a}{\sqrt{2^n n!}} H_n(t),$	$H_n(t) = 2t H_{n-1}(t) - 2(n-1) H_{n-2}(t)$

n	0	1	2	3	4	5	…
$\int_{-\infty}^{\infty} h_n(\tau)\, d\tau$	$\sqrt{2}\, b$	0	b	0	$\sqrt{3/4}\, b$	0	…

15.8 Hilbert Transform of Product of Analytic Signals

15.8.1 Hilbert Transform of Product of Analytic Signals:

From

$$\mathrm{H}\{\psi(t)\} = \mathrm{H}\{x(t) + j\upsilon(t)\} = \mathrm{H}\{x(t) + j\mathrm{H}\{x(t)\}\} = \mathrm{H}\{x(t)\} - jx(t)$$
$$= \upsilon(t) - jx(t) = -j(x(t) + j\upsilon(t)) = -j\psi(t)$$

we obtain $\mathrm{H}\{\psi_1(t)\psi_2(t)\} = -j\psi_1(t)\psi_2(t) = \psi_1(t)\mathrm{H}\{\psi_2(t)\} = \psi_2(t)\mathrm{H}\{\psi_1(t)\}$
since the product can be considered as an analytic function $\psi(t)$.

15.8.2 The n^th Product of an Analytic Signal

$$\mathrm{H}\{\psi^2(t)\} = \psi(t)\mathrm{H}\{\psi(t)\} = -j\psi^2(t)$$

$$\mathrm{H}\{\psi^n(t)\} = \psi^{n-1}(t)\mathrm{H}\{\psi(t)\} = -j\psi^n(t)$$

Example

Because $\mathrm{H}\{(1-jt)^{-1}\} = -j(1-jt)^{-2}$, we obtain

$$\mathrm{H}\{(1-jt)^{-2}\} = (1-jt)^{-1}(-j(1-jt)^{-1}) = -j(1-jt)^{-2}$$

15.9 Hilbert Transform of Bessel Functions

15.9.1 Hilbert Transform of Bessel Function:

$$\mathrm{H}\{J_n(t)\} = \hat{J}_n(t) = \frac{1}{\pi}\int_0^\pi \sin(t\sin\varphi - n\varphi)\, d\varphi = \sum_{n=0}^{\infty} \frac{\hat{J}_n^{(n)}(t=0)}{n!} t^n$$

$$\hat{J}_0(t) = \frac{1}{\pi}\int_0^1 \frac{2}{(1-\omega^2)^{1/2}}\sin\omega t\,d\omega$$

$$\psi_0(t) = J_0(t) + j\,\hat{J}_0(t)$$

$$\hat{J}_0(0) = \frac{1}{\pi}\int_0^1 \frac{2\,d\omega}{(1-\omega^2)^{1/2}}\sin(0) = 0, \qquad \hat{J}_0^{(1)}(t) = \frac{1}{\pi}\int_0^1 \frac{2\,\omega\,d\omega}{(1-\omega^2)^{1/2}}\cos(0) = \frac{2}{\pi}$$

The parenthesis in the exponent indicates number of differentiations with respect to time.

15.9.2 Hilbert Transform Pairs of Bessel Functions:

TABLE 15.5 Hilbert Transform of Bessel Functions of the First Kind

Bessel Function	Fourier Transform	Hilbert Transform
$J_n(t)$	$C_n(f)$	$\hat{J}_n(t) = \mathrm{H}[J_n(t)]$
$J_0(t)$	$C_0 = \frac{2}{(1-\omega^2)^{0.5}}; \quad \lvert\omega\rvert < 1$ $= 0; \quad \lvert\omega\rvert > 0$	$\frac{1}{\pi}\int_0^1 C_0(f)\sin(\omega t)\,d\omega$
$J_1(t)$	$C_1 = -j\omega C_0$	$-\frac{1}{\pi}\int_0^1 \lvert C_1(f)\rvert\cos(\omega t)\,d\omega$
$J_2(t)$	$C_2 = -(2\omega^2 - 1)C_0$	$-\frac{1}{\pi}\int_0^1 \lvert C_2(f)\rvert\sin(\omega t)\,d\omega$
$J_3(t)$	$C_3 = j(4\omega^3 - 3\omega)C_0$	$\frac{1}{\pi}\int_0^1 \lvert C_3(f)\rvert\cos(\omega t)\,d\omega$
$J_4(t)$	$C_4 = (8\omega^4 - 8\omega^2 + 1)C_0$	$\frac{1}{\pi}\int_0^1 \lvert C_4(f)\rvert\sin(\omega t)\,d\omega$
$J_5(t)$	$C_5 = -j(16\omega^5 - 20\omega^3 + 5\omega)C_0$	$-\frac{1}{\pi}\int_0^1 \lvert C_5(f)\rvert\cos(\omega t)\,d\omega$
$J_6(t)$	$C_6 = -(32\omega^6 - 48\omega^4 + 18\omega^2 - 1)C_0$	$-\frac{1}{\pi}\int_0^1 \lvert C_6(f)\rvert\sin(\omega t)\,d\omega$
...	...	...
$J_n(t)$	$C_n = (-j)^n 2^{n-1} T_n(\omega) C_0$	$\frac{(-1)^{n/2}}{\pi}\int_0^1 \lvert C_n(f)\rvert\sin(\omega t)\,d\omega$ for $n = 0,2,4,\ldots$ $\frac{(-1)^{(n+1)/2}}{\pi}\int_0^1 \lvert C_n(f)\rvert\cos(\omega t)\,d\omega$ for $n = 1,3,5,\ldots$

$T_n(\omega) = \cos[n\cos^{-1}(\omega)]$ is the Chebyshev polynomial

15.10 Instantaneous Amplitude, Phase, and Frequency

15.10.1 Instantaneous Angular Frequency

$$\psi(t) = x(t) + j\upsilon(t) = A(t)e^{j\varphi(t)} = A(t)\cos\varphi(t) + A(t)\sin\varphi(t)$$

$$A(t) = \sqrt{x^2(t) + \upsilon^2(t)}, \quad \varphi(t) = \tan^{-1}\frac{\upsilon(t)}{x(t)}$$

$$\dot{\varphi}(t) = \Omega(t) = 2\pi F(t) \equiv \text{instantaneous angular frequency}$$

$$F(t) = \text{instantaneous frequency} = \frac{\Omega(t)}{2\pi} = \frac{\dot{\varphi}(t)}{2\pi}$$

$$\Omega(t) = \frac{d}{dt}\tan^{-1}\frac{\upsilon(t)}{x(t)} = \frac{x(t)\dot{\upsilon}(t) - \upsilon(t)\dot{x}(t)}{x^2(t) + \upsilon^2(t)}$$

15.11 Hilbert Transform and Modulation

15.11.1 Modulated Signal (see 15.10.1)

$$\psi(t) = A_o\,\gamma(t)e^{j\Phi_0}\,e^{j\Omega_0 t}$$

$$\psi_x(t) = x(t) + j\hat{x}(t)$$

$$x(t) = \frac{\psi_x(t) + \psi_x^*(t)}{2}$$

15.11.2 Instantaneous Amplitude and Angular Frequency (see 15.10.1)

$$A(t) = \frac{m}{2}\left|\psi_x(t)\right| = \frac{m}{2}[x^2(t) + \hat{x}^2(t)]^{1/2}$$

$$\omega_x(t) = \pm\frac{d}{dt}\tan^{-1}\left[\frac{\hat{x}(t)}{x(t)}\right]$$

15.11.3 High-Frequency Analytic Signals ($\Phi_0 = 0$)

$$\psi_{upper}(t) = \text{upper sideband} = \psi_x(t)e^{j\Omega_0 t}$$

$$\psi_{lower}(t) = \text{lower sideband} = \psi_x^*(t)e^{j\Omega_0 t}$$

$$x_{SSB}(t) = x(t)\cos\Omega_0 t \mp \hat{x}(t)\sin\Omega_0 t$$

where $x(t)\cos(\Omega_0 t)$ and $\hat{x}(t)\sin\Omega_0 t$ represent double sideband (DSB) compressed carrier AM signals.

15.12 Hilbert Transform and Transfer Functions of Linear Systems

15.12.1 Causal Systems

$$H(s) = A(\alpha,\omega) + jB(\alpha,\omega), \qquad \sigma = \alpha + j\omega$$

$$A(\omega) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{B(\lambda)}{\lambda - \omega}d\lambda$$

$$B(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda - \omega} d\lambda$$

15.12.2 Minimum Phase Transfer Function

$$H(j\omega) = H_{\varphi}(j\omega) H_{ap}(j\omega)$$

$$H_{\varphi}(j\omega) = \text{minimum phase transfer function}$$

$$H_{ap}(j\omega) = \text{all-pass transfer function}$$

$$H_{\varphi}(j\omega) = |H(j\omega)| e^{j\varphi(\omega)} = A_{\varphi}(\omega) + jB_{\varphi}(\omega)$$

$H_{\varphi}(j\omega)$ has all the zeros lying in the left half-plane of the s-plane. The minimum phase transfer function is analytic and its real and imaginary parts for a Hilbert pair

$$\mathsf{H}\{A(\omega)\} = -B_{\varphi}(\omega)$$

15.13 The Discrete Hilbert Filter

15.13.1 Discrete Hilbert Filter

$$H(k) = \begin{cases} -j & k = 1, 2, \cdots, \frac{N}{2} - 1 \\ 0 & k = 0 \text{ and } k = \frac{N}{2} \\ j & k = \frac{N}{2} + 1, \frac{N}{2} + 2, \cdots, N - 1 \end{cases} \qquad (N = \text{even})$$

$$H(k) = -j\,\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k), \qquad k = 0, 1, \cdots, N - 1 \quad (N = \text{even})$$

15.13.2 Impulse Response of the Hilbert Filter

$$h(i) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{jw} = \frac{1}{N}\sum_{k=0}^{N-1} -j\,\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k) e^{jw}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \sin(w) = \frac{2}{N}\sin^2\left(\frac{\pi i}{2}\right)\cot\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, \cdots, N - 1, \; w = \frac{2\pi k i}{N} \quad (N \text{ even})$$

15.13.3 DHT of a Sequence $x(i)$ in the Form of Convolution

$$\upsilon(i) = -x(i) \otimes h(i) = -x(i) \otimes \frac{2}{N}\sin^2\left(\frac{\pi i}{2}\right)\cot\left(\frac{\pi i}{N}\right), \qquad i = 0, 1, \cdots, N - 1$$

$\otimes$ = circular convolution

$$\upsilon(i) = \sum_{r=0}^{N-1} h(i - r) x(r), \qquad i = 0, 1, \cdots, N - 1 \quad (N \text{ even})$$

15.13.4 DHT of a Sequence $x(i)$ via DFT

$$\mathsf{F}_D\{x(i)\} = X(k)$$

$$V(k) = -j\,\mathrm{sgn}\left(\frac{N}{2} - k\right)\mathrm{sgn}(k)\,X(k)$$

$$\upsilon(i) = \mathsf{F}_D^{-1}\{V(k)\}, \qquad i,k = 0,1,2,\cdots,N-1 \ (N \text{ even})$$

$\mathsf{F}_D \equiv$ discrete Fourier transform, $\mathsf{F}_D^{-1} \equiv$ inverse discrete Fourier transform

15.13.5 Discrete Hilbert Filter when N is odd

$$H(k) = \begin{cases} -j & k = 1,2,\cdots,\frac{N-1}{2} \\ 0 & k = 0 \\ j & k = \frac{N}{2}+1, \frac{N}{2}+2,\cdots,N-1 \end{cases}$$

$$h(i) = \frac{2}{N}\sum_{k=1}^{(N-1)/2} \sin(2\pi ki/N), \qquad i = 0,1,\cdots,N-1$$

Also

$$h(i) = \frac{1}{N}\left[1 - \frac{\cos(\pi i)}{\cos(\pi i/N)}\cot\left(\frac{\pi i}{N}\right)\right]$$

15.14 Properties of Discrete Hilbert Transform

15.14.1 Parseval's Theorem

$$E\{x(i)\} = \sum_{i=0}^{N-1} |x(i)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

$$E\{x(i)\} \neq E\{\upsilon(i)\}$$

The reason is that the DC term (average value of $x(i)$) is eliminated in the DHT.

$$x_{DC} = \frac{1}{N}\sum_{i=0}^{N-1} x(i) = X(0)$$

15.14.2 Discrete Hilbert Transform

$$\mathsf{H}_D\{x_{AC}(i)\} = \upsilon(i)$$

$$x_{AC}(i) = x(i) - x_{DC}$$

where $x_{AC}(i)$ is the alternating part of $x(i)$.

15.14.3 Energies (powers) of x_{AC} and $\upsilon(i)$

$$\sum_{i=0}^{N-1} |x_{AC}(i)|^2 = \sum_{i=1}^{N-1} |\upsilon(i)|^2 + \frac{|X(N/2)|^2}{N} \qquad (N \text{ even})$$

where the special term $X\left(\frac{N}{2}\right)$ is zero, the two energies are equal.

Example

If $x(i) = \delta(i)$ and $N = 8$ we obtain (see 15.13.3)

$$\upsilon(i) = -\delta(i) \otimes \tfrac{1}{4}\sin^2(\pi i/2)\cot(\pi i/N)$$

Figure 15.1 shows the desired components and transforms. The $x_{DC} = 1/8 = 0.125$ and the energies are: $E\{x(i)\} = 1,\ E\{x_{AC}(i)\} = 1 - \frac{1^2}{N} = 0.875,$ and $E\{\upsilon(i)\} = 1 - \frac{1^2}{N} - \frac{1^2}{N} = 1 - \frac{2}{8} = 0.75.$

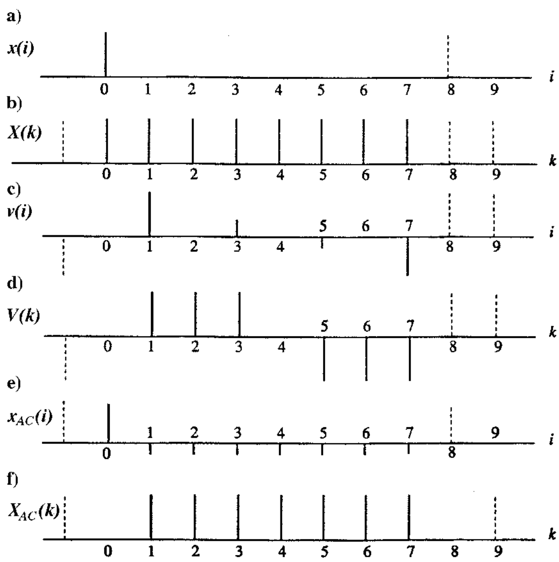

FIGURE 15.1 (a) The sequence $x(i)$ consisting of a single sample $\delta(i)$, (b) its spectrum $X(k)$ given by the DFT, (c) the samples of the discrete Hilbert transform, (d) the corresponding spectrum $V(k)$, (e) the samples of the AC component of $x(i)$, and (f) the corresponding spectrum $X_{AC}(k)$.

15.14.4 Shifting Property:

$$\mathsf{F}_D\{x(i \pm m)\} = e^{\pm j2\pi mk/N} X(k)$$

See 15.13.4

$$\upsilon(i) = \mathsf{F}_D^{-1}\{-j\,\mathrm{sgn}\left(\frac{N}{2} - k\right)\mathrm{sgn}(k)e^{\pm j2\pi mk/N} X(k)\}$$

15.14.5 Linearity:

$$\mathsf{H}_D\{ax_1(i)+bx_2(i)\}=a\upsilon_1(i)+b\upsilon_2(i)$$

15.14.6 Complex Analytic Discrete Sequence:

$$\psi(i)=x(i)+j\upsilon(i),\qquad \upsilon(i)=\mathsf{H}_D\{x(i)\}$$

$$\mathsf{H}_D\{\psi(i)\}=X(k)+j[-j\,\mathrm{sgn}\left(\frac{N}{2}-k\right)\mathrm{sgn}(k)]X(k),\qquad k=0,1,\cdots,N-1\quad (N\text{ even})$$

15.15 Hilbert Transformers (continuous)

15.15.1 Hilbert Transformer (quadratic filter)

$$H(jf)=\mathsf{F}\left\{\frac{1}{\pi t}\right\}=|H(f)|e^{j\varphi(f)}=-j\,\mathrm{sgn}\,f$$

$$H(jf)=\begin{cases}-j & f>0\\ 0 & f=0\\ j & f<0\end{cases}$$

$$\varphi(f)=\arg H(jf)=-\frac{\pi}{2}\mathrm{sgn}\,f$$

15.15.2 Phase-Splitter Hilbert Transformers

Analog Hilbert transformers are mostly implemented in the form of a phase splitter consisting of two parallel all-pass filters with a common input pot and separated output ports, each having the following transfer function respectively.

$$Y_1(jf)=e^{j\varphi_1(f)},\qquad Y_2(jf)=e^{j\varphi_2(f)}$$

with

$$\delta(f)=\varphi_1(f)-\varphi_2(f)=-\pi/2\quad\text{for all}\quad f>0$$

15.15.3 All-Pass Filters

$$H(j\omega)=\frac{R-jX(\omega)}{R+jX(\omega)}\qquad \omega=2\pi f$$

$$\varphi(\omega)=\arg\{(R-jX(\omega))^2\}=\tan^{-1}\left[\frac{-2RX(\omega)}{R^2-X^2(\omega)}\right]$$

See Figure 15.2a.

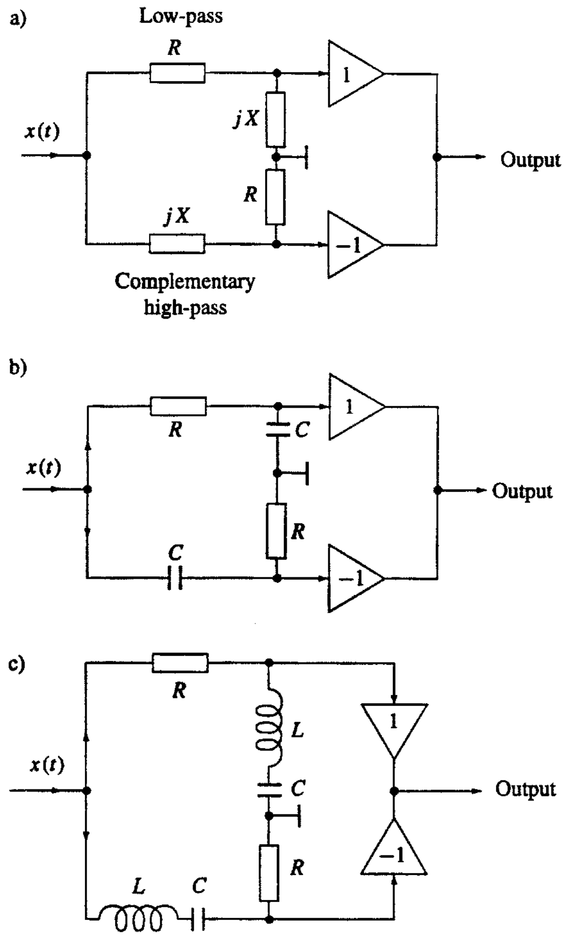

FIGURE 15.2 An all-pass consisting of (a) a low-pass and a complementary high-pass, (b) a first-order RC low-pass and complementary CR high-pass, and (c) a second-order RLC low-pass and complementary RLC high-pass.

If $X(\omega) = \dfrac{1}{\omega C}$, then (see Figure 15.2b)

$$\varphi(y) = \tan^{-1}\left[\frac{-2y}{1-y^2}\right], \qquad y = \omega RC = \omega\tau$$

If $X(\omega) = \omega L - 1/\omega C$, then (see Figure 15.2c)

$$\varphi(y) = \tan^{-1}\left[\frac{2(1-y^2)qy}{(1-y^2)^2 - q^2y^2}\right], \qquad y = \omega/\omega_r, \qquad \omega_r = 1/\sqrt{LC}$$

$$q = \omega_r RC = R\sqrt{C/L}$$

15.15.4 Design Hilbert Phase Splitters

Example

Filter with two first-order all-pass filters in each branch. The phase function for the first branch is (see Figure 15.3)

$$\varphi_1(f) = \tan^{-1}\left[\frac{-2y}{y^2-1}\right] + \tan^{-1}\left[\frac{-2ay}{a^2y^2-1}\right], \qquad y = 2\pi fRC$$

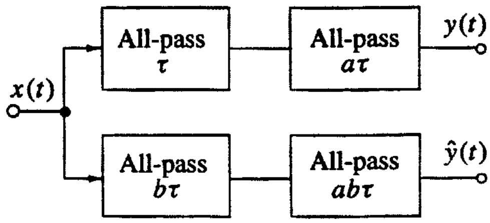

FIGURE 15.3 Phase Hilbert splitter with two all-pass filters.

Find a to get the best linearity of $\varphi_1(f)$ in the logarithmic scale. Small changes of a introduce a trade-off between the RMS phase error and the pass-band of the Hilbert transformer. Find shift parameter b to yield the minimum RMS phase error

$$\varphi_2(f) = \tan^{-1}\left[\frac{2by}{b^2y^2-1}\right] + \tan^{-1}\left[\frac{2aby}{a^2b^2y^2-1}\right]$$

Figure 15.4 shows an example with a = 0.08 and b = 0.24 giving the normalized edge frequencies $y_1 = 1.6$ and $y_2 = 30$ $(f_2/f_1 = 18.75,$ or more than 4 octaves) with $\varepsilon_{RMS} = 0.016$.

15.16 Digital Hilbert Transformers

15.16.1 Digital Hilbert Transformers

Ideal discrete-time Hilbert transformer is defined as an all-pass with a pure imaginary transfer function.

$$H(e^{j\psi}) = H_r(\psi) + jH_i(\psi)$$

$$H_r(\psi) = 0 \qquad \text{for all } f$$

$$H(e^{j\psi}) = jH_i(\psi) = \begin{cases} -j & 0 < \psi < \frac{\pi}{2} \\ 0 & \psi = 0,\ |\psi| = \pi \\ j & -\pi < \psi < 0 \end{cases}$$

Equivalent Notation

$$H(e^{j\psi}) = -j\,\mathrm{sgn}(\sin\psi) = -\mathrm{sgn}(\sin\psi)e^{j\pi/2} = |H(\psi)|e^{j\arg H(\psi)}$$

$$|H(\psi)| = |\mathrm{sgn}(\sin\psi)| = \begin{cases} 1 & 0 < |\psi| < \pi \\ 0 & \psi = 0,\ |\psi| = \pi \end{cases}$$

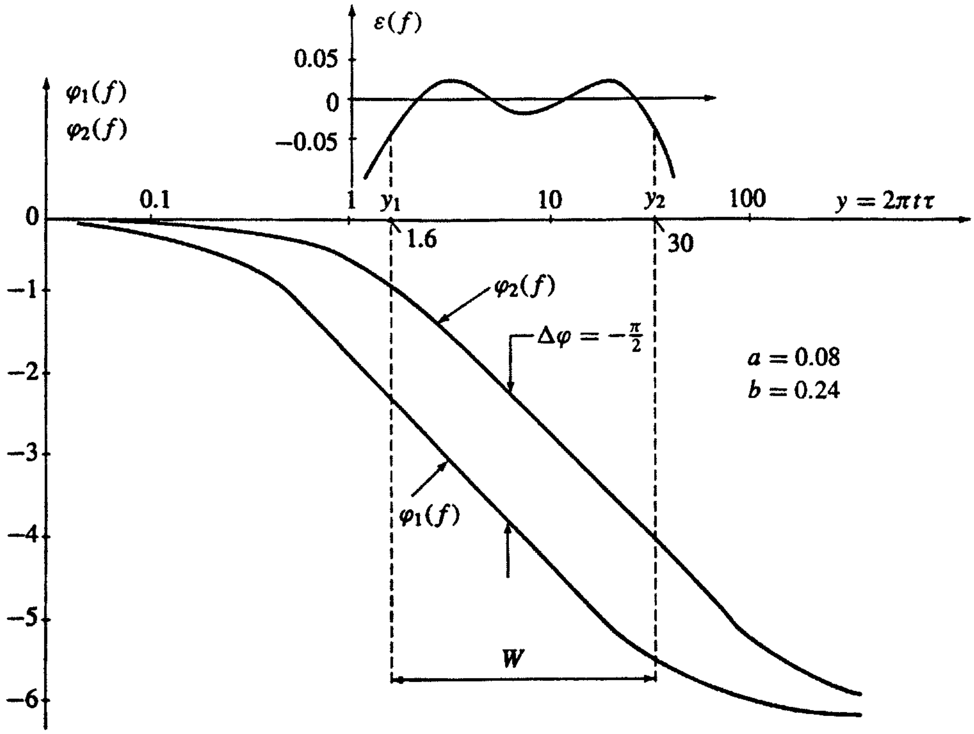

FIGURE 15.4 The phase functions and the phase error of the Hilbert transformer of Figure 15.3.

$$\arg[H(\psi)] = -\frac{\pi}{2}\operatorname{sgn}(\sin\psi)$$

$$\psi = 2\pi f_n, \qquad f_n = f/f_s, \qquad f_s = \text{sampling frequency}$$

Noncausal impulse response of the ideal Hilbert transformer is

$$h(i) = \frac{2}{\pi i}\sin^2\left(\frac{i\pi}{2}\right) \qquad i = 0, \pm 1, \pm 2, \cdots$$

15.16.2 Ideal Hilbert Transformer With Linear Phase Term

$$H(e^{j\psi}) = \begin{cases} -je^{j\psi\tau} & 0 < \psi < \pi \\ 0 & \psi = 0, \ |\psi| = \pi \\ je^{-j(\psi - 2\pi)\tau} & \pi < \psi < 2\pi \end{cases}$$

$$h(i) = \frac{2}{\pi}\frac{\sin^2\left[\frac{\pi}{2}(i-\tau)\right]}{i-\tau} \qquad i = 0, \pm 1, \pm 2, \cdots$$

$$h(i) = -h(-i) \qquad i = 0, 1, 2, \cdots$$

15.16.3 FIR Hilbert Transformers:

Figure 15.5 shows a noncausal impulse response Hilbert transformer and its truncated and shifted version so that a causal one is generated.

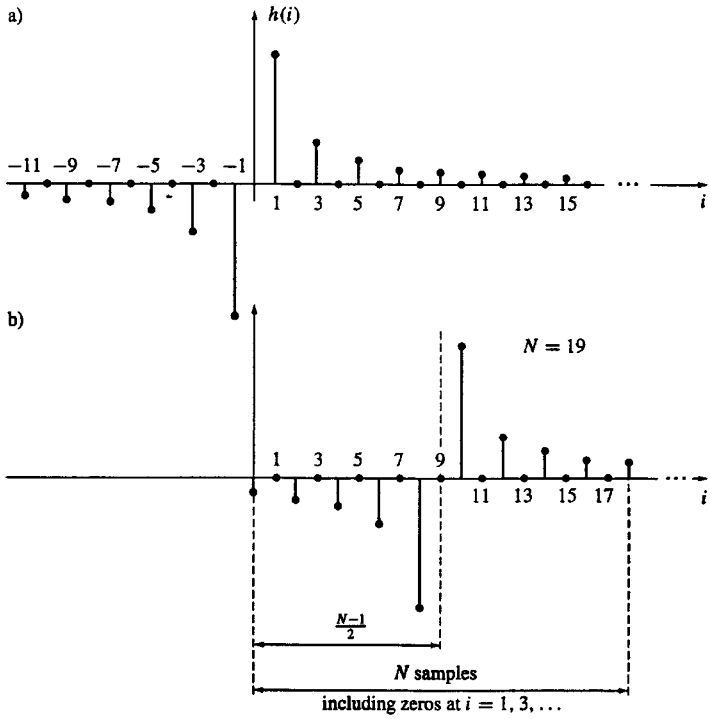

FIGURE 15.5 Impulse responses of (a) the ideal discrete time Hilbert transformer (see 15.16.1) and (b) a FIR Hilbert transformer given by the truncation and shifting of the impulse response shown in (a).

Causal Filter Impulse Response

$$H(i_1) = \sum_{i_1=0}^{N-1} h_1(i_1) z^{-i_1}$$

$$h_1\left(i + \frac{N-1}{2}\right) = h(i) \qquad i_1 = i + \tfrac{N-1}{2}, \quad i = -\tfrac{N-1}{2}, \cdots 0, \cdots, \tfrac{N-1}{2}$$

Transfer function =

$$H(e^{j\psi}) = e^{-j\psi\frac{N-1}{2}} \sum_{i=-\frac{N-1}{2}}^{N-1} h(i) e^{-j\psi i} = e^{-j\psi\frac{N-1}{2}} \sum_{i=1}^{\frac{N-1}{2}} - j2h(i)\sin(\psi i), \quad \psi = \frac{2\pi f}{f_s}$$

Amplitude of Hilbert Transformer (see Figure 15.6)

$$G(e^{j\psi}) = -\sum_{i=1}^{\frac{N-1}{2}} 2h(i)\sin(\psi i)$$

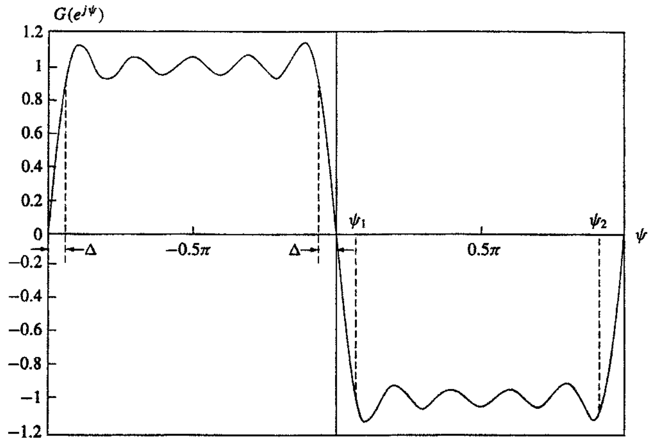

FIGURE 15.6 The $G(e^{j\psi})$ function of an FIR Hilbert transformer (amplitude).

Normalized Dimensionless Pass-band Hilbert Transformer

$$W_\psi = \psi_2 - \psi_1 = \pi - 2\Delta, \quad \psi_1, \psi_2 = \text{edge frequencies}$$

$$W_f[H_z] = \frac{\pi - 2\Delta}{2\pi} f_s$$

15.17 IIR Hilbert Transformers

15.17.1 IIR Ideal Hilbert Transformer (see Figure 15.7)

$$H_{HB}(z) = 1 + z^{-1} G(z^2) \equiv \text{ideal half} - \text{band filter} \qquad \text{(see Figure 15.7a)}$$

$$G(z^2) = \text{all} - \text{pass filter with unit magnitude}$$

$$H_H(z) = z^{-1} G(-z^2) \equiv \text{ideal IIR Hilbert transformer}$$

$$F(z) = z^{-1} G(z^2), \quad z = e^{j\psi} \qquad \text{(see Figure 15.7b)}$$

$$F(e^{j\psi}) = e^{-j\psi} e^{j\Phi_G(\psi)} = e^{j\Phi(\psi)}$$

$$\Phi(\psi) = 0.5\pi[\text{sgn}(\sin(2\psi)) - \text{sgn}\,\psi]$$

$$\Phi_G(\psi) = \Phi(\psi) + \psi \qquad \text{(see Figure 15.7e)}$$

$$H_H(e^{j\psi}) = e^{-j\psi} e^{j\Phi_G(0.5\pi+\psi)}, \quad z^2 = e^{j2\psi}, \; -z^2 = e^{j2(0.5\pi+\psi)}$$

$$\arg\{z^{-1}G(-z^2)\} = -\psi + \Phi_G(0.5\pi + \psi) \qquad \text{(see Figure 15.7g)}$$

IIR Hilbert transformer has an equi-ripple phase function and exact amplitude. A noncausal transfer function may have the form

$$H(z) = z^{-1} \sum_{i=1}^{N} \frac{1 - a_i z^2}{z^2 - a_i}$$

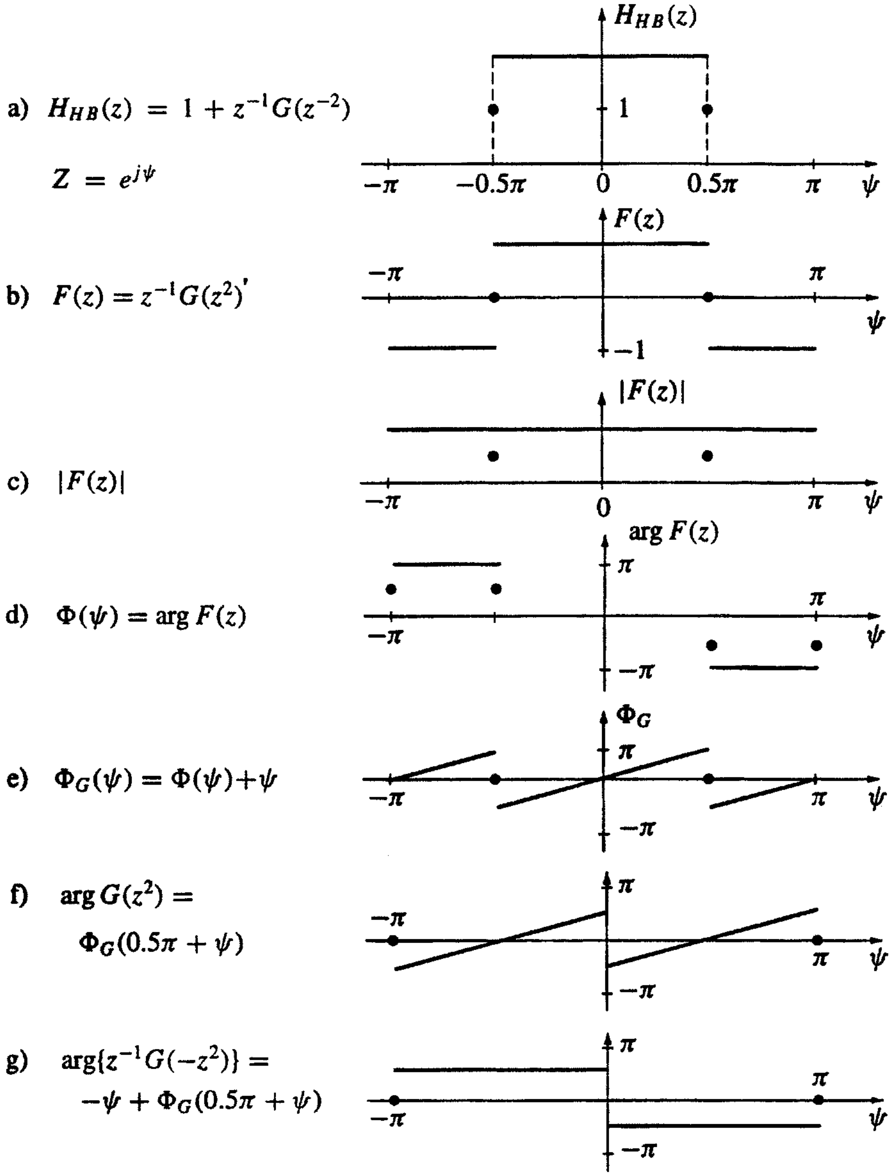

FIGURE 15.7 Step-by-step derivation of the IIR transfer function of a Hilbert transformer $Z^{-1}G(-z^2)$, starting from the transfer function of the ideal half-band filter given by $1 + Z^1G(z^2)$

Example

Let $\psi_1 = 0.02\pi \equiv$ low-frequency edge, $\psi_2 = 0.98\pi$ = high-frequency edge $(\Delta = 0.02\pi)$, phase equiripple amplitude $|\Delta\Phi| \le 0.01\pi$. Because $\delta = \sin(0.5\Delta\Phi)$, $\delta = 0.0157$. Using the procedure from Ansari (1985), we find $a(1) = 5.36078$, $a(2) = 1.2655$, $a(3) = 0.94167$, and $a(4) = 0.53239$. Inserting a_i's, in $H(z)$ above, we find the phase function.

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16

The Radon and Abel Transform

16.1 The Radon Transform

16.1.1 Definition in two dimensions (see Figure 16.1)

$$\mathrm{R}\{f(\bar{r})\} = \breve{f}(\bar{\xi};p) = \iint_{-\infty}^{\infty} f(\bar{r})\delta(p-\bar{\xi}\cdot\bar{r})\,dx\,dy$$

$$\bar{\xi} = (\xi_1,\xi_2), \quad \bar{r} = (x_1,x_2)$$

$$\bar{\xi}\cdot\bar{r} = \xi_1 x + \xi_2 y = x\cos\phi + y\cos\phi \equiv \text{line in the } (x,y) \text{ plane}$$

Radon space = surface of a half-cylinder (see Figure 16.2)

$$p = \xi_1 x + \xi_2 y$$

$$\bar{\xi} = \text{unit vector} = (\cos\phi, \sin\phi)$$

$$\bar{r} = \text{vector}$$

16.1.2 Other Interpretation

$$M(p,\bar{\xi})\iint_{\bar{\xi}\cdot\bar{r}<p} f(x,y)\,dx\,dy = \iint f(x,y)u(p-\bar{\xi}\cdot\bar{r})\,dx\,dy$$

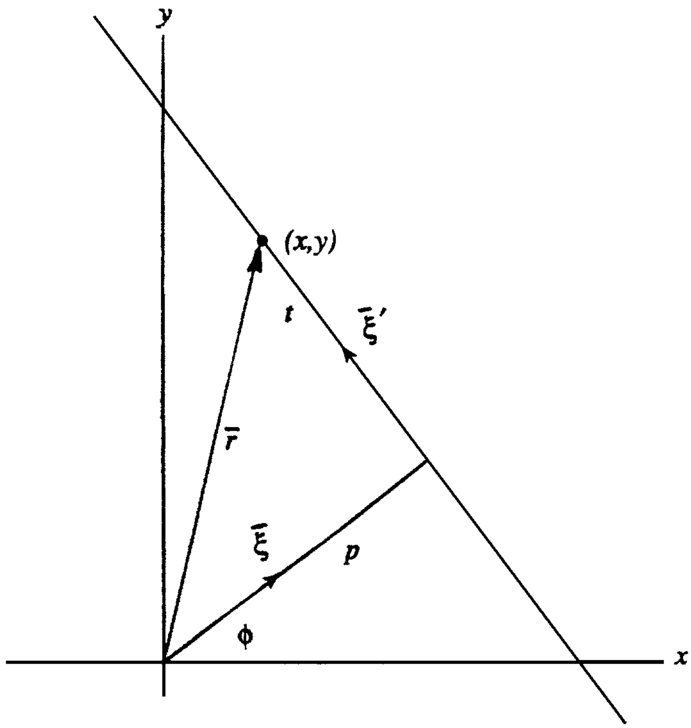

FIGURE 16.1 Coordinates in feature space used to define the Radon transform. The equation of the line is given by $p = x\cos\phi + y\sin\phi$.

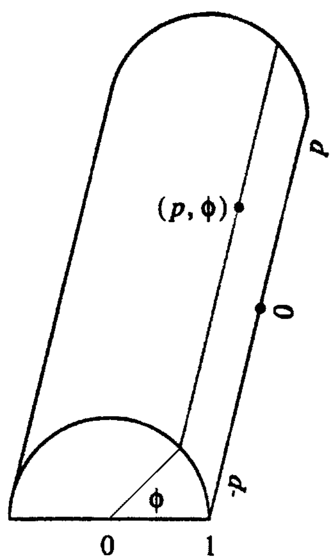

FIGURE 16.2 Coordinates in Radon space on the surface of a cylinder.

$$u(\cdot) = \text{unit step function}$$

$$\frac{\partial u(p)}{\partial p} = \delta(p)$$

$$M(p,\overline{\xi}) = \text{total mass in the region } \overline{\xi}\cdot\overline{r} < p$$

$$\frac{\partial M(p,\overline{\xi})}{\partial p} = \iint_{-\infty}^{\infty} f(\overline{r})\delta(p-\overline{\xi}\cdot\overline{r})\,dx\,dy = \mathrm{R}\{f(x,y)\}$$

16.1.3 Sample of Radon Transform

If the transform is found for only selected values of these variables, we call the result a *sample* of the Radon transform.

Example (see Figure 16.3)

The equation of line is $x = p$ and $\phi = 0$. Hence (see 16.1.2)

$$\check{f}(p) = \frac{\partial M(p)}{\partial p} = \frac{\partial}{\partial p}\left[2\int_0^p \sqrt{1-x^2\,dx}\right] = 2\sqrt{1-p^2}$$

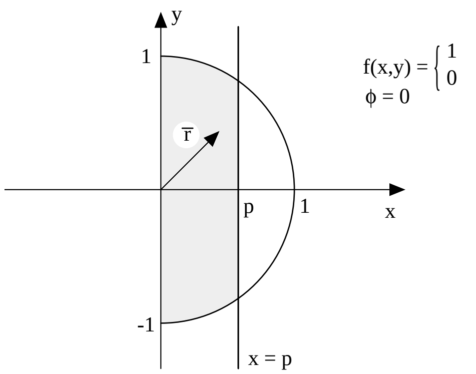

FIGURE 16.3 The sample Radon transform.

16.1.4 Rotated Coordinate System (see Figure 16.4)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} p \\ t \end{bmatrix} \equiv \text{transformation}$$

$$\mathrm{R}\{f(x,y)\} = \check{f}(p,\phi) = \int_{-\infty}^{\infty} f(p\cos\phi - t\sin\phi,\ p\sin\phi + t\cos\phi)\,dt \quad \text{(integration along } t\text{)}$$

$$\check{f}_\phi(p) = \int_{-\infty}^{\infty} f_\phi(p,t) \equiv \text{integral of } f_\phi(p,t) \text{ with respect to } t \text{ for fixed } \phi;$$

$$f_\phi(p,t) = f(p\cos\phi - t\sin\phi,\ p\sin\phi + t\cos\phi)$$

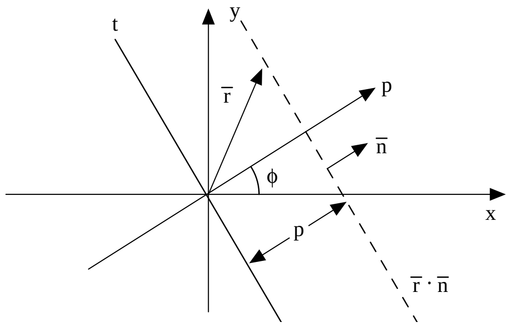

FIGURE 16.4 Rotated coordinate system.

Functions of p for various values of $\phi = \Phi$ are called *projections* of $f(x,y)$ at an angle Φ.

Example (see Figure 16.5)

The function is $f(x,y) = 1$, $0 \le r \le 1$, (or $f(x,y) = cyl_1(r)$) and zero otherwise. When $p = 1$, the length t = 0 and when $p = p,\ t = 2\sqrt{1-p^2}$. As ϕ varies, t assumes values in the range $-1 \le p \le p$. Hence

$$\check{f}(p,\phi) = \mathrm{R}\{f(x,y)\} = \int_0^{2\sqrt{1-p^2}} dt = \begin{cases} 2\sqrt{1-p^2} & |p| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The function $\check{f}(p,\phi)$ is plotted in Figure 16.5.

16.2 Transforms Between Spaces

16.2.1 Central-Slice Theorem

If $f(\bar{r})$ is a function of n variables

$$\mathrm{F}_1\mathrm{R}\{f\} = \mathrm{F}_1\{\check{f}\} = \mathrm{F}_n\{f\} = F(u,\upsilon) = F(q,\phi)$$

Example

$$\mathrm{F}\{f(x,y) = e^{-x^2-y^2}\} = F(u,\upsilon) = \int_0^\infty dr\, re^{-r^2} \int_0^{2\pi} d\theta\, e^{[-j2\pi qr\cos(\theta-\phi)]}$$

where we let $x = r\cos\theta,\ y = r\sin\theta,\ u = q\cos\phi$, and $\upsilon = q\sin\phi$. The integral over θ is $2\pi J_0(2\pi qr)$. The remaining integral is a Hankel transform of order zero. Hence $F(q,\phi) = 2\pi\int_0^\infty re^{-r^2} J_0(2\pi qr)\,dr =$ $\pi e^{-\pi^2 q^2}$ and therefore

$$\check{f}(p,\xi) = \pi\int_{-\infty}^{\infty} e^{-\pi^2 q^2} e^{j2\pi qp}\, dq = \sqrt{\pi}\, e^{-p^2}\ .$$

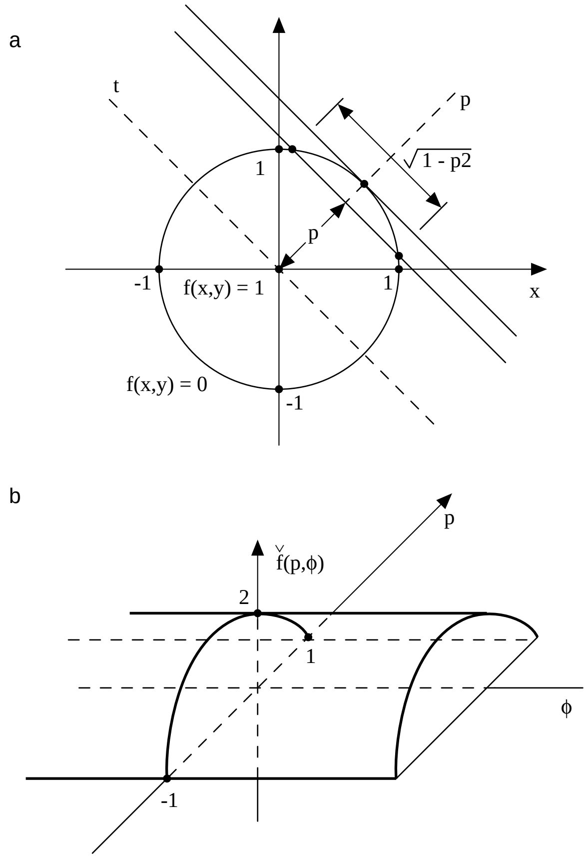

FIGURE 16.5 Radon transform of a cylinder function.

16.3 Basic Properties of the Radon Transform

16.3.1 Notation

$$\xi_1 = \cos\phi,\ \xi_2 = \sin\phi,\ \xi_1^2 + \xi_1^2 = 1,\ \ \check{f}(p,\xi_1,\xi_2) = \iint_{-\infty}^{\infty} f(x,y)\delta(p - x\xi_1 - y\xi_2)\,dx\,dy$$

$$F(u,\upsilon) = \iint_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+\upsilon y)}\,dx\,dy$$

16.3.2 Linearity

$$\mathsf{R}\{af + bg\} = a\check{f} + b\check{g}$$

16.3.3 Similarity

If

$$\mathsf{R}\{f(x,y)\} = \check{f}(p,\xi_1,\xi_2),$$

then

$$\mathsf{R}\{f(ax,by)\} = \frac{1}{|ab|}\check{f}\left(p,\frac{\xi_1}{a},\frac{\xi_2}{b}\right)$$

$$\mathsf{F}\{f(ax,by)\} = \frac{1}{|ab|}F\left(\frac{u}{a},\frac{\upsilon}{b}\right)$$

16.3.4 Symmetry

$$\check{f}(ap,a\bar{\xi}) = \frac{1}{|a|}\check{f}\left(p,\bar{\xi}\right)$$

$$\check{f}(-p,-\bar{\xi}) = \check{f}\left(p,\bar{\xi}\right) \equiv \text{even homogeneous function}$$

$$\check{f}(p,s\bar{\xi}) = \frac{1}{|s|}\check{f}\left(\frac{p}{s},\bar{\xi}\right)$$

16.3.5 Shifting

$$\mathsf{R}\{f(x-a,y-b)\} = \check{f}(p-a\xi_1-b\xi_2,\bar{\xi})$$

$$\mathsf{F}\{f(x-a,y-b)\} = e^{-j2\pi(au+b\upsilon)}F(u,\upsilon)$$

16.3.6 Differentiation

$$\mathsf{R}\left\{\frac{\partial f}{\partial x}\right\} = \xi_1\left\{\frac{\partial\check{f}(p,\bar{\xi})}{\partial p}\right\},\quad \mathsf{R}\left\{\frac{\partial f}{\partial y}\right\} = \xi_2\left\{\frac{\partial\check{f}(p,\bar{\xi})}{\partial p}\right\},\quad \mathsf{R}\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \xi_1^2\left\{\frac{\partial^2\check{f}(p,\bar{\xi})}{\partial p^2}\right\}$$

$$\mathsf{R}\left\{\frac{\partial^2 f}{\partial y^2}\right\} = \xi_2^2\left\{\frac{\partial^2\check{f}(p,\bar{\xi})}{\partial p^2}\right\},\quad \mathsf{R}\left\{\frac{\partial^2 f}{\partial x\partial y}\right\} = \xi_1\xi_2\left\{\frac{\partial^2\check{f}(p,\bar{\xi})}{\partial p^2}\right\}$$

16.3.7 Convolution

$$\check{f}(p,\bar{\xi}) = \mathsf{R}\{g(x,y) ** h(x,y)\} = \check{g} * \check{h} = \int_{-\infty}^{\infty}\check{g}(\tau,\bar{\xi})\check{h}(p-\tau,\bar{\xi})\,d\tau$$

16.3.8 Linear Transformation

$$A = n\times n \text{ nonsingular matrix},\quad \bar{y} = A\bar{x},\quad \bar{x} = A^{-1}\bar{y} \doteq B\bar{y},\quad A^{-1} = B,$$

$$\bar{\xi}\cdot\bar{x} = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$$

$$\mathsf{R}\{f(A\bar{x})\} = \mathsf{R}\{f(B^{-1}\bar{x})\} = \int f(A\bar{x})\delta(p - \bar{\xi}\cdot\bar{x})\,d\bar{x} = |\det B|\int f(\bar{y})\delta(p - \bar{\xi}\cdot B\bar{y})\,d\bar{y}$$

$$= |\det B|\int f(\bar{y})\delta(p - B^T\bar{\xi}\cdot\bar{y})\,d\bar{y} = |\det B|\,\check{f}(p, B^T\bar{\xi})$$

If $B^{-1} = B^T = A =$ orthogonal with $|\det B| = 1$, then

$$\mathsf{R}\{f(A\bar{x})\} = \check{f}(p, A\bar{\xi}), \qquad A\bar{\xi} = \text{unit vector}$$

If $A = cI$ with c real, $B = A^{-1} = c^{-1}I$, then

$$\mathsf{R}\{f(c\bar{x})\} = \frac{1}{|c|^n}\check{f}\left(p, \frac{\bar{\xi}}{c}\right) = \frac{1}{|c|^{n-1}}\check{f}\left(cp, \bar{\xi}\right)$$

If $B^T\bar{\xi}$ is not a unit vector, we define s equal to the magnitude of the vector $B^T\bar{\xi}$, and hence $\bar{\mu} = \frac{B^T\bar{\xi}}{s}$ is a unit vector. Hence we obtain

$$|\det B|\,\check{f}(p, B^T\bar{\xi}) = |\det B|\,\check{f}(p, s\bar{\mu}) = \frac{|\det B|}{s}\check{f}\left(\frac{p}{s}, \bar{\mu}\right)$$

and

$$\mathsf{R}\{f(B^{-1}\bar{x})\} = \frac{|\det B|}{s}\check{f}\left(\frac{p}{s}, \bar{\mu}\right), \quad s = \left|B^T\bar{\xi}\right|$$

16.3.9 Examples:

Example

$$\bar{\xi} = (\cos\phi, \sin\phi), \quad A = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix}, \quad \begin{bmatrix} u \\ \upsilon \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \xi_1 x + \xi_2 y \\ -\xi_2 x + \xi_1 y \end{bmatrix}.$$

A is orthogonal (see 16.3.7) and $u^2 + \upsilon^2 = x^2 + y^2$, $u = \xi_1 x + \xi_2 y$.

$$\mathsf{R}\{f(x,y) = e^{-x^2-y^2}\} = \mathsf{R}\{f(A\bar{x})\} = \mathsf{R}\{f(u,\upsilon)\} = \int\int_{-\infty}^{\infty} e^{-u^2-\upsilon^2}\,\delta(p-u)\,du\,d\upsilon$$

$$= e^{-p^2}\int_{-\infty}^{\infty} e^{-\upsilon^2}\,d\upsilon = \sqrt{\pi}\;e^{-p^2}$$

(See 16.2.1)

Also, $\mathsf{R}\{e^{-x^2-y^2-z^2}\} = \left(\sqrt{\pi}\right)^{3-1} e^{-p^2} = \pi e^{-p^2}$

Example

Because $\mathsf{R}\{f(\bar{x}) = e^{-x^2-y^2}\} = \check{f}(p, \bar{\xi}) = \sqrt{\pi}\;e^{-p^2}$ we can set

$$B=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad B^{-1}=\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}, \quad |\det B| = |ab|, \quad B^T\bar{\xi}=\begin{bmatrix} a\cos\phi \\ b\sin\phi \end{bmatrix} \text{ (not a unit vector),}$$

$s=\left|B^T\bar{\xi}\right|=(a^2\cos^2\phi+b^2\sin^2\phi)^{1/2}$ and from 16.3.7 $\mathsf{R}\{e^{-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\}=\dfrac{|ab|\sqrt{\pi}}{s}e^{-(p/s)^2}$.

Example (see Figure 16.4)

If $(p,t)\to(u,\upsilon)$ from the transformation $[\bar{\xi}=(\cos\phi,\sin\phi)]$

$$\begin{bmatrix} u \\ \upsilon \end{bmatrix}=A\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

we obtain $x=u\cos\phi-\upsilon\sin\phi$, and $y=u\sin\phi+\upsilon\cos\phi$. Therefore the RT of a function in a unit disk is

$$\mathsf{R}\{f(x,y)\}=\check{f}(p,\phi)=\int_{disk} f(x,y)\delta(p-x\cos\phi-y\sin\phi)\,dx\,dy$$

$$=\int_{disk} f(u\cos\phi-\upsilon\sin\phi,\ u\sin\phi+\upsilon\cos\phi)\delta(p-u)\,du\,d\upsilon$$

$$=\int_{-\sqrt{1-p^2}}^{\sqrt{1-p^2}} f(p\cos\phi-\upsilon\sin\phi,\ p\sin\phi+\upsilon\cos\phi)\,d\upsilon$$

Example

Because $\delta(x,y)=\delta(p,t)$, then $\mathsf{R}\{\delta(x,y)=\int_{-\infty}^{\infty}\delta(p,t)\,dt=\delta(p)$ (see Figure 16.6 and Section 16.1.3).

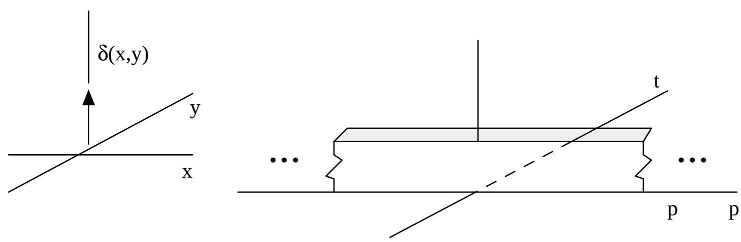

FIGURE 16.6 Delta function.

Example

$$\mathsf{R}\{f(x,y)=\delta(x-a,y-b)=\delta(x-a)\delta(x-b)\}=\int\int_{-\infty}^{\infty}\delta(x-a)\delta(x-b)\delta(p-\bar{\xi}\cdot\bar{r})\,dx\,dy$$

$$=\int\int_{-\infty}^{\infty}\delta(x-a)\delta(x-b)\delta(p-\xi_1 x-\xi_2 y)\,dx\,dy=\delta(p-\xi_1 a-\xi_2 b)=\delta(p-p_0)$$

$p_0=\ a\cos\phi+b\sin\phi$. Therefore as impulse function at the (x,y) point creates an impulse function along a sinusoidal curve in the Radon space (p,ϕ) which is $p=a\cos\phi+b\sin\phi=\sqrt{a^2+b^2}\ \cos[\phi-\tan^{-1}(b/a)]$ (see Figure 16.7)

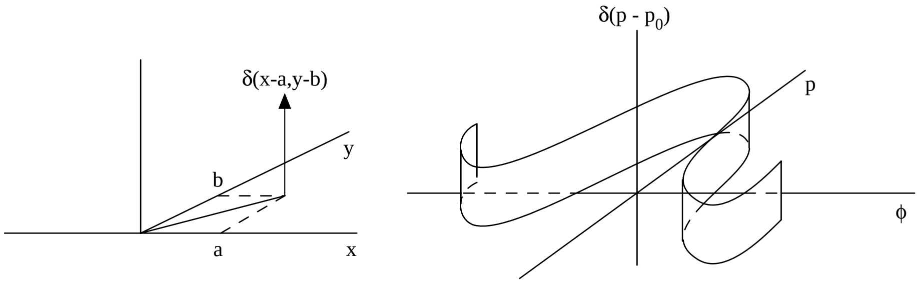

FIGURE 16.7 Displaced delta function.

Example

The Radon transform of a finite-extent delta function

$$f(x,y)=\begin{cases}\delta(x-p_0,0) & |y|<L/2\\ 0 & |y|\ge L/2\end{cases}$$

is given by (see 16.1.3) and Chapter 5 on delta function properties

$$\check{f}(p,\phi)=\int_{-L/2}^{L/2}\delta(p\cos\phi-t\sin\phi-p_0)\,dt=\begin{cases}L\delta(p-p_0) & \phi=2n\pi\\ L\delta(p+p_0) & \phi=(2n+1)\pi\end{cases}$$

If ϕ is different than multiple π (see Chapter 5 on delta function properties)

$$\check{f}(p,\phi)=\int_{p=t}\delta(p\cos\phi-t\sin\phi-p_0)\,dt=\begin{cases}\dfrac{1}{|\sin\phi|} & |p-p_0\cos\phi|\le\left|\frac{L}{2}\sin\phi\right|\\ 0 & \text{otherwise}\end{cases}$$

The inequality can be derived from the geometry of Figure 16.8b. The region of support is shown in Figure 16.8c, and the transform is shown in Figure 16.8d.

If $f(x,y)$ is a finite length delta function (see Figure 16.9) such that

$$f(x,y)=\begin{cases}\delta[p-(x\cos\theta_0+y\sin\theta_0)] & x\cos\theta_0+y\sin\theta_0\in(-\frac{L}{2},\frac{L}{2})\\ 0 & \text{otherwise}\end{cases}$$

then

$$\check{f}(p,\phi)=\begin{cases}L\delta(p-p_0) & \phi=\phi_0+2n\pi\\ L\delta(p+p_0) & \phi=\phi_0+(2n+1)\pi\\ \dfrac{1}{|\sin(\phi-\phi_0)|} & |p-p_0\cos(\phi-\phi_0)|\le\frac{L}{2}|\sin(\phi-\phi_0)|\\ 0 & \text{otherwise}\end{cases}$$

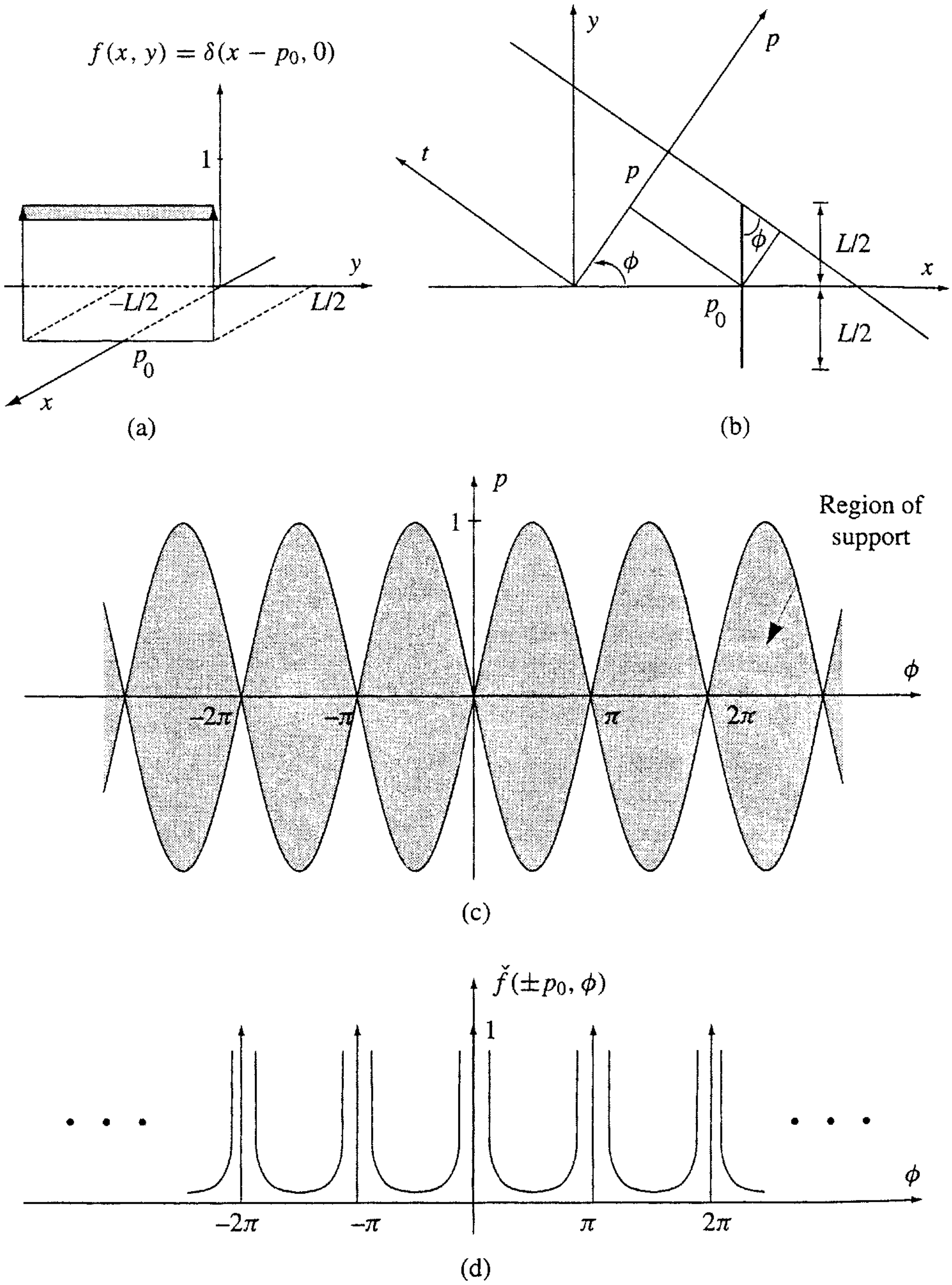

FIGURE 16.8 Radon transform of finite-extended delta function.

Example

Find the Radon transform of the cylinder defined in the Example above displaced at the point (x_o,y_o) as shown in Figure 16.10a. The solution follows immediately from the solution above combined with the shifting property. Also, the solution can be deduced from the geometry in Figure 16.10a. When $d = 1$, the length $t = 0$; also, for p such that the line of integrated passes through the cylinder,

$$t = 2\sqrt{1-[p-r_0\cos(\phi_0-\phi)]^2}\,.$$

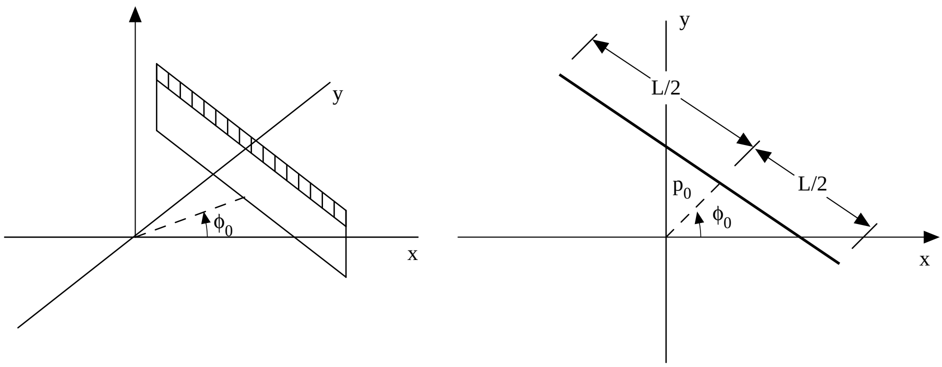

FIGURE 16.9 Finite delta in general direction.

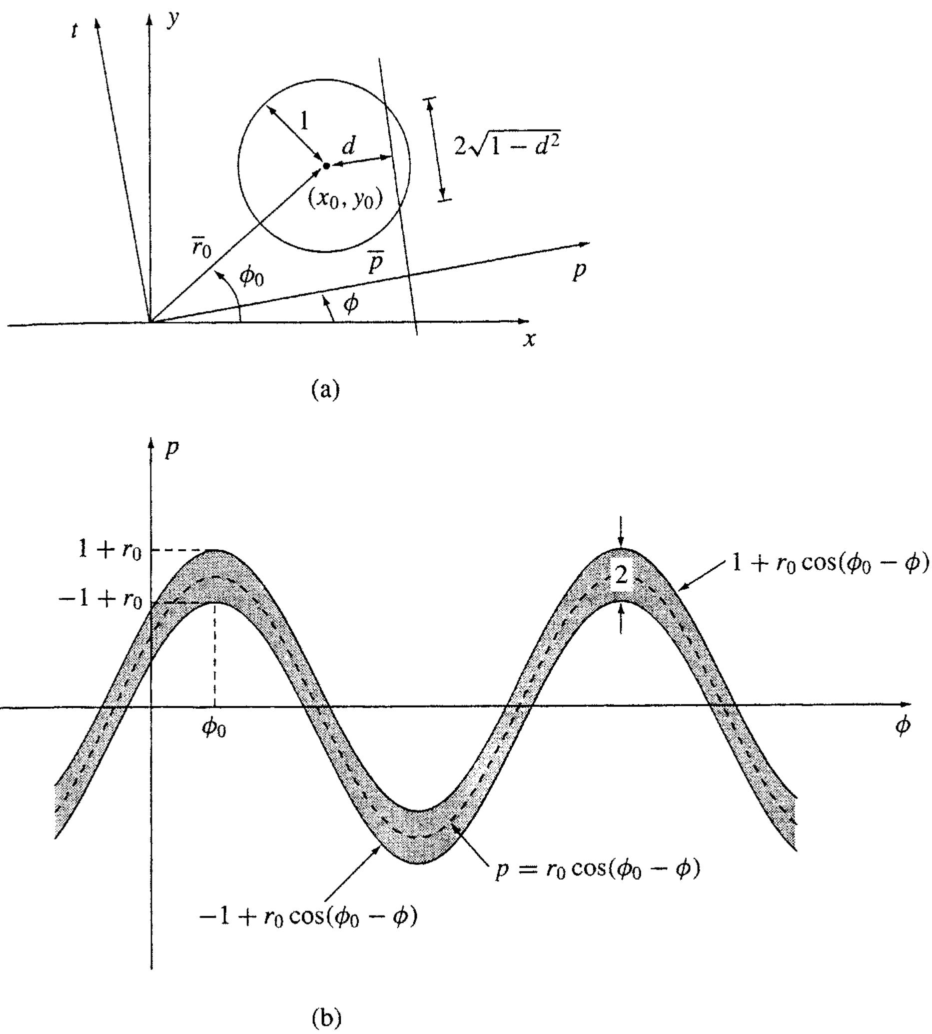

FIGURE 16.10 Displaced cylinder function and region of support of the transform.

Further, when ϕ varies, the values p can assume follow from the geometry. The transform is

$$\check{f}(p,\phi) = \begin{cases} 2\sqrt{1-[p-r_0\cos(\phi_0-\phi)]^2}\,, & -1+r_0\cos(\phi_0-\phi) \le p \\ & \le 1+r_0\cos(\phi_0-\phi) \\ 0, & \text{otherwise} \end{cases}$$

Figure 16.10b shows the (sinusoidal) region of support of the transform.

16.4 Additional Properties of Radon Transforms

16.4.1 Transform of Derivatives

$$\mathrm{R}\left\{\frac{\partial f}{\partial x_k}\right\} = \xi_k \left\{\frac{\partial \check{f}(p,\bar{\xi})}{\partial p}\right\}, \qquad f(x_1,\cdots,x_n)$$

$$\mathrm{R}\left\{\frac{\partial^2 f}{\partial x_k \partial y_k}\right\} = \xi_\ell \xi_k \frac{\partial^2 \check{f}(p,\bar{\xi})}{\partial p^2}$$

Example

$$\mathrm{R}\{\nabla^2 f(\bar{x})\} = |\bar{\xi}| \frac{\partial^2 \check{f}(p,\bar{\xi})}{\partial p^2} = \frac{\partial^2 \check{f}(p,\bar{\xi})}{\partial p^2}$$

Example

$$\mathrm{R}\{\nabla^2 f(\bar{x},t)\} = \mathrm{R}\left\{\frac{\partial^2 \check{f}(\bar{x},t)}{\partial t^2}\right\} \quad \text{or} \quad \frac{\partial^2 \check{f}(p,\bar{\xi})}{\partial p^2} = -\frac{\partial^2 \check{f}(p,\bar{\xi})}{\partial t^2}$$

16.4.2 Derivative of the Radon Transform

$$\frac{\partial}{\partial \eta_j}\delta(p-\bar{\eta}\cdot\bar{x}) = -x_j \frac{\partial}{p}\delta(p-\bar{\eta}\cdot\bar{x})$$

$$\frac{\partial \check{f}}{\partial \xi_k} = \left[\frac{\partial}{\partial \eta_k}\mathrm{R}\{f(\bar{x})\}\right]_{\bar{\eta}=\bar{\xi}} = -\frac{\partial}{\partial p}\mathrm{R}\{x_k f(\bar{x})\}$$

Example

$$\mathrm{R}\{f(x,y) = e^{-x^2-y^2}\} = \check{f}(p,\bar{\xi}) = \sqrt{\pi}\, e^{-p^2}.$$

Set $\bar{\eta} = s\bar{\xi}$ $(s = (\eta_1^2 + \eta_2^2)^{1/2})$ to obtain $\check{f}(p,\bar{\eta}) = \check{f}(p, s\bar{\xi}) = \frac{\sqrt{\pi}}{s} e^{-p^2/s^2}$ (scaling relation). But $\frac{\partial}{\partial \eta_k}$ = $\frac{\partial s}{\partial \eta_k}\frac{\partial}{\partial s}$ $(k = 1,2)$, and thus

$$\frac{\partial \check{f}}{\partial \eta_k} = \sqrt{\pi}\,(\eta_k | s)\,\frac{\partial}{\partial s}(s^{-1}e^{-p^2/s^2}) = \sqrt{\pi}\,\frac{\eta_k}{s^5}(2p^2 - s^2)e^{-p^2/s^2}$$

To find the derivative $\partial \check{f}/\partial \xi_k$ we calculate the above expression at $\bar{\eta} = \bar{\xi}$ or equivalent setting s = 1. Hence,

$$\frac{\partial \check{f}}{\partial \xi_k} = \sqrt{\pi}\,\xi_k(2p^2 - 1)e^{-p^2}$$

For 2-D function we also have

$$\frac{\partial^{\ell+k}\check{f}(p,\bar{\xi})}{\partial \xi_1^{\ell}\partial \xi_2^{k}} = \left(-\frac{\partial}{\partial p}\right)^{\ell+k} \mathrm{R}\{x^{\ell} y^{k} f(x,y)\}$$

16.5 Hermite Polynomials and Radon Transforms

16.5.1 Hermite Polynomials

From 16.4.1

$$\mathrm{R}\left\{\left(\frac{\partial}{\partial x}\right)^{\ell}\left(\frac{\partial}{\partial y}\right)^{k} f(x,y)\right\} = (\cos\phi)^{\ell}(\sin\phi)^{k}\left(\frac{\partial}{\partial p}\right)^{\ell+k}\check{f}(p,\bar{\xi})$$

and hence

$$\mathrm{R}\left\{H_{\ell}(x)H_k(y)e^{-x^2-y^2} = (-1)^{\ell+k}\left(\frac{\partial}{\partial x}\right)^{\ell}\left(\frac{\partial}{\partial y}\right)^{k} e^{-x^2-y^2}\right\}$$

$$= (-1)^{\ell+k}(\cos\phi)^{\ell}(\sin\phi)^{k}\left(\frac{\partial}{\partial p}\right)^{\ell+k}\sqrt{\pi}\,e^{-p^2} = (-1)^{\ell+k}\sqrt{\pi}\,(\cos\phi)^{\ell}(\sin\phi)^{k}\,e^{-p^2}H_{\ell+k}(p)$$

Example

$$x = \tfrac{1}{2}H_1$$

and hence

$$\mathsf{R}\{x e^{-x^2-y^2}\} = \mathsf{R}\{\tfrac{1}{2} H_1(x) e^{-x^2-y^2}\}$$

$$= (-1)^{1+0} \frac{1}{2}\left((\cos\phi)^1 (\sin\phi)^0 \frac{\partial}{\partial p} \sqrt{\pi}\; e^{-p^2} \right) = p\sqrt{\pi}\; e^{-p^2} \cos\phi$$

16.6 Inversion of Radon Transforms

16.6.1 2-D Inverse Radon Transform

$$f(x,y) = \frac{-1}{2\pi^2} P \int_0^{\pi} d\phi \int_{-\infty}^{\infty} \frac{\check{f}(p,\bar{\xi})}{p - \bar{\xi}\cdot\bar{x}}\, dp$$

P stands for principal value of the integral.

16.7 N-Dimensional Radon Transform

16.7.1 N-Dimensional Radon Transform with its Properties

TABLE 16.1 Properties the N-Dimensional Radon Transform

1	Linearity	$\mathsf{R}\{a_1 f_1 + a_2 f_2\} = a_1 \check{f}_1 + a_2 \check{f}_2$
2	Homogeneity	$\check{f}(\alpha\bar{\xi}, \alpha p) = \lvert\alpha\rvert^{-1} \check{f}(\bar{\xi}, p)$
3	Shifting	$\check{f}_{\bar{a}}(\bar{\xi}, p) = \check{f}(\bar{\xi}, p + \bar{\xi}\cdot\bar{a}), \quad f_{\bar{a}}(\bar{x}) = f(\bar{x} + \bar{a})$
4	Coordinate transformation	$\check{f}_A(\bar{\xi}, p) = \lvert\det A\rvert\, \check{f}(A^T\bar{\xi}, p)$ $f_A(\bar{x}) = f(A^{-1}\bar{x})$, A is nonsingular matrix
5	Differentiation	$\mathsf{R}\left\{\sum_{k=1}^{n} a_k \frac{\partial f}{\partial x_k}\right\} = \sum_{k=1}^{n} a_k \xi_k \frac{\partial}{\partial p} \check{f}(\bar{\xi}, p)$
6	Convolution	$\check{f}(\bar{\xi}, p) = \int_{-\infty}^{\infty} \check{f}_1(\bar{\xi}, t)\, \check{f}_2(\bar{\xi}, p - t)\, dt$ where $f(\bar{x}) = f_1(\bar{x}) *^n f_2(\bar{x})$
7	Relationship to Fourier transform of $f(\bar{x})$	with $F(\bar{\xi}) = \int f(\bar{x}) e^{-j\bar{\xi}\cdot\bar{x}}\, d\bar{x}$ $\check{f}(\bar{\xi}, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha\bar{\xi}) e^{j\alpha p}\, d\alpha$ $F(\bar{\xi}) = \int_{-\infty}^{\infty} \check{f}(\bar{\xi}, p) e^{-jp}\, dp$
8	Inverse Radon transform	n = odd $\quad f(\bar{x}) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{\Gamma} \check{f}_p^{(n-1)}(\bar{\xi}, \bar{\xi}\cdot\bar{x})\, \omega(\bar{\xi})\, dp$ n = even $\quad f(\bar{x}) = \frac{(-1)^{n/2}(n-1)!}{(2\pi)^n} \int_{\Gamma} \left[\int_{-\infty}^{\infty} \check{f}(\bar{\xi}, p)(p - \bar{\xi}\cdot\bar{x})^{-n}\, dp\right] \omega(\bar{\xi})$

TABLE 16.1 Properties the N-Dimensional Radon Transform (continued)

where $$\omega(\bar{\xi}) = \sum_{k=1}^{n} (-1)^{k-1} \xi_k \, d\xi_1 \cdots d\xi_{k-1} d\xi_{k+1} \cdots d\xi_n$$ Γ is any surface enclosing the origin in $\bar{\xi}$ space and $\check{f}_p^{(n-1)}$ is the $(n-1)^{th}$ derivative of $\check{f}(\bar{\xi}, p)$ with respect to p.

16.8 Abel Transforms

16.8.1 Definition of Abel Transform

$$g(x) = \int_o^x \frac{f(y)}{(x-y)^{\alpha}} dy = f(x) * \frac{1}{x^{\alpha}} \qquad x > 0,\ y < x,\ 0 < \alpha < 1$$

16.8.2 Laplace Transform of Abel Transformation

$$G(s) = F(s)K(s)$$

$$F(s) = \frac{G(s)}{K(s)} = [sG(s)]\left[\frac{1}{sK(s)}\right] = sG(s)H(s)$$

16.8.3 Inverse Solution

$$f(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x (x-y)^{\alpha-1} g(y)\, dy$$

$$f(x) = \frac{\sin \alpha\pi}{\pi} \left[\frac{g(0+)}{x^{1-\alpha}} + \int_0^x \frac{g'(y)}{(x-y)^{1-\alpha}} dy \right]$$

16.8.4 Abel Transform Pairs

$$\hat{f}_1(x) = \mathsf{A}_1\{f_1(r); x\} = \int_0^x \frac{f_1(r)\, dr}{(x^2 - r^2)^{1/2}}, \qquad x > 0$$

$$\hat{f}_2(x) = \mathsf{A}_2\{f_2(r); x\} = \int_x^{\infty} \frac{f_2(r)\, dr}{(r^2 - x^2)^{1/2}}, \qquad x > 0$$

$$\hat{f}_3(x) = \mathsf{A}_3\{f_3(r); x\} = 2\int_x^{\infty} \frac{r f_3(r)\, dr}{(r^2 - x^2)^{1/2}}, \qquad x > 0$$

$$\hat{f}_4(x) = \mathsf{A}_4\{f_4(r); x\} = 2\int_0^x \frac{r f_4(r)\, dr}{(x^2 - r^2)^{1/2}}, \qquad x > 0$$

16.8.5 Inverse of the Four Types of Abel Transform (see 16.8.3)

$$f_1(r)=\frac{2}{\pi}\frac{d}{dr}\int_0^r\frac{x\hat{f}_1(x)\,dx}{(r^2-x^2)^{1/2}}=\frac{2\hat{f}_1(0)}{\pi}+\frac{2r}{\pi}\int_0^r\frac{\hat{f}_1'(x)\,dx}{(r^2-x^2)^{1/2}}$$

$$f_2(r)=-\frac{2}{\pi}\frac{d}{dr}\int_r^\infty\frac{x\hat{f}_2(x)\,dx}{(x^2-r^2)^{1/2}}=-\frac{2r}{\pi}\int_r^\infty\frac{\hat{f}_2'(x)\,dx}{(x^2-r^2)^{1/2}}$$

$$f_3(r)=-\frac{1}{\pi r}\frac{d}{dr}\int_r^\infty\frac{x\hat{f}_3(x)\,dx}{(x^2-r^2)^{1/2}}=-\frac{1}{\pi}\int_r^\infty\frac{\hat{f}_3'(x)\,dx}{(x^2-r^2)^{1/2}}$$

$$f_4(r)=\frac{1}{\pi r}\frac{d}{dr}\int_0^r\frac{x\hat{f}_4(x)\,dx}{(r^2-x^2)^{1/2}}=\frac{\hat{f}_4(0)}{\pi r}+\frac{1}{\pi}\int_0^r\frac{\hat{f}_4'(x)\,dx}{(r^2-x^2)^{1/2}}$$

16.8.6 Relationships Among the Four Types of Abel Transforms

$$\mathsf{A}_3\{f(r)\}=2\mathsf{A}_2\{r\,f(r)\}$$

$$\mathsf{A}_4\{f(r)\}=2\mathsf{A}_1\{r\,f(r)\}$$

$$\mathsf{A}_4\{r^{-1}f_1(r)\}=2\,\hat{f}_1(x)$$

$$\mathsf{A}_3\{r^{-1}f_2(r)\}=2\,\hat{f}_2(x)$$

$$f_1(r)=\mathsf{A}_1^{-1}\{\hat{f}_1(x)\}=\frac{2}{\pi}\frac{d}{dr}\mathsf{A}_1\{x\,\hat{f}_1(x)\}$$

$$f_2(r)=\mathsf{A}^{-1}\{\hat{f}_2(x)\}=-\frac{2}{\pi}\frac{d}{dr}\mathsf{A}_2\{x\,\hat{f}_2(x)\}$$

Example

$$\hat{f}_1(x)=\mathsf{A}_1\{(a-r)\}=\int_0^x\frac{(a-r)\,dr}{(x^2-r^2)^{1/2}}=a\int_0^x\frac{dr}{(x^2-r^2)^{1/2}}-\int_0^x\frac{r\,dr}{(x^2-r^2)^{1/2}}$$

$$=a\sin^{-1}\frac{r}{x}\Big|_0^x-\frac{1}{2}\int_0^{y^2}\frac{dy}{(x^2-y)^{1/2}}=\frac{\pi a}{2}-\frac{1}{2}\left[\frac{2}{-1}(x^2-y)^{1/2}\Big|_0^{x^2}\right]=\frac{\pi a}{2}-x$$

(see 16.8.4)

$$f_1(r)=\mathsf{A}_1^{-1}\{\hat{f}_1(x)\}=\frac{2}{\pi}\frac{\pi a}{2}+\frac{2r}{\pi}\int_0^r\frac{-dx}{(r^2-x^2)^{1/2}}=a-r$$

16.9 Inverse Radon Transform

16.9.1 Back Projection

$$f(x,y)=2\mathsf{B}\{\check{f}(t,\phi)\}$$

where $\mathrm{B}\{F(p,\phi)\} = \int_o^{\pi} F(x\cos\phi + y\sin\phi, \phi)\, d\phi$ is the back projection operation which is defined by replacing p by $x\cos\phi + y\sin\phi$ and integrating over the angle ϕ. The over bar defines the Hilbert transform of the derivative of some function g as follows

$$\bar{g}(t) = -\frac{1}{4\pi}\mathrm{H}\{g_p(p);\ p \to t\} \quad \text{for} \quad n = 2$$

16.9.2 Backprojection of the Filtered Projection

$$f = 2\mathrm{B}\{\mathrm{F}^{-1}\mathrm{F}\,\bar{\check{f}}\} = \frac{2}{4\pi^2}\mathrm{B}\left\{\mathrm{F}^{-1}\left\{\mathrm{F}\frac{\partial}{\partial p}\left[\frac{1}{p} * \check{f}(p,\phi)\right]\right\}\right\}$$

$$= \frac{1}{2\pi^2}\mathrm{B}\left\{\mathrm{F}^{-1}\left\{(j2\pi k)\left[\mathrm{F}\left\{\frac{1}{p}\right\}\right]\left[\mathrm{F}\{\check{f}(p,\phi)\}\right]\right\}\right\}$$

$$= \frac{1}{2\pi^2}\mathrm{B}\{\mathrm{F}^{-1}\{(j2\pi k)(j\pi gnk)\mathrm{F}\{\check{f}(p,\phi)\}\}\} = \mathrm{B}\{\mathrm{F}^{-1}\{|k|\mathrm{F}\{\check{f}(p,\phi)\}\}\}$$

If we define $F(s,\phi) = \mathrm{F}^{-1}\{|k|\mathrm{F}\{\check{f}(p,\phi)\}\} = \mathrm{F}^{-1}\{|k|\mathrm{F}\{\tilde{\check{f}}(k,\phi)\}$ then the $f(x,y)$ is found by the backprojection of F,

$$f(x,y) = \mathrm{B}\{F(s,\phi)\} = \int_0^{\pi} F(x\cos\phi + y\sin\phi, \phi)\, d\phi$$

16.9.3 Convolution Method

If $\mathrm{F}\{g\} = |k|w(k)$, $w(k)$ is a window function, then $g(s) = \mathrm{F}^{-1}\{|k|w(k)\}$ and then $F(s,\phi)$ of 16.9.2 becomes

$$F(s,\phi) = \mathrm{F}^{-1}\{\mathrm{F}\{g\}\mathrm{F}\{\check{f}\}\} = \check{f} * g = \int_{-\infty}^{\infty} \check{f}(p,\phi)\, g(s-p)\, dp$$

16.9.4 Frequency Space Implementation

From 16.9.2 we obtain

$$F(s,\phi) = \mathrm{F}^{-1}\{|k|w(k)\mathrm{F}\{\check{f}(p,\phi)\}\} = \mathrm{F}^{-1}\{|k|w(k)\,\tilde{\check{f}}(k,\phi)\}$$

Figure 16.11 shows a flow chart of filtered backprojection and convolution.

16.9.5 Filter of Backprojection

$$b(x,y) = \mathrm{B}\{\check{f}(p,\phi)\} = \int_0^{\pi} \check{f}(x\cos\phi + y\sin\phi, \phi)\, d\phi$$

$$\tilde{g}(u,\upsilon) = |g|w(u,\upsilon), \qquad q = \sqrt{u^2 + \upsilon^2}, \qquad w(u,\upsilon) = FT \text{ of window}$$

$$f(x,y) = g(x,y) ** b(x,y)$$

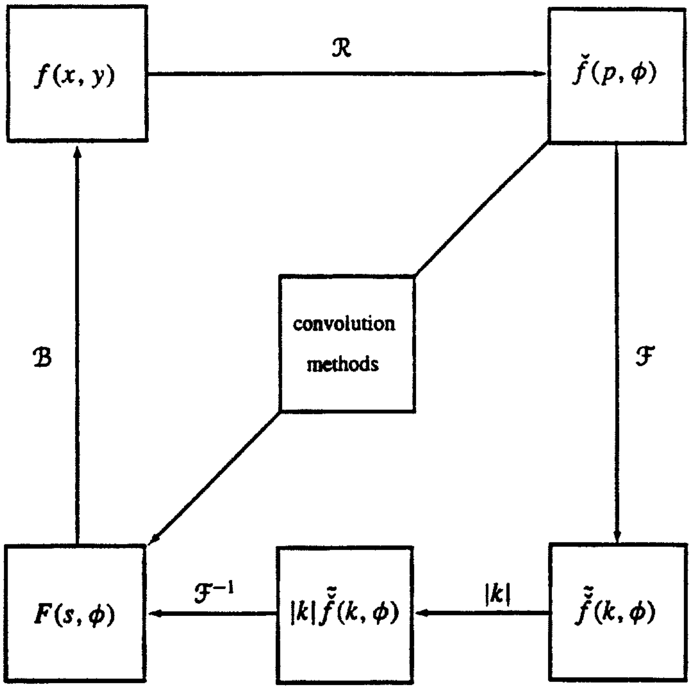

FIGURE 16.11 Filtered backprojection, convolution.

16.10 Tables of Abel and Radon Pairs

16.10.1 Abel and Radon Pairs

Just to remind the user of these tables, the sinc function is defined by

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

and the characteristic function for the unit disk, designated by $\chi(r)$ is defined by

$$\chi(r) = \begin{cases} 1, & \text{for } 0 \le r \le 1 \\ 0, & \text{for } r > 1 \end{cases}$$

The complete elliptic integral of the first kind is designated by $F(\frac{1}{2}\pi, t)$, and the complete elliptic integral of the second kind is designated by $E(\frac{1}{2}\pi, t)$. A good source of these is the tabulation by Gradshteyn et al. The constant $C(n)$ in the table for A_3 is $C(n) = 2\int_0^{\pi/2} \cos^n x\,dx$, with $n \ge 1$; Bessel functions of the first kind J_ν, and second kind N_ν (Neumann functions) conform to the standard definitions in Gradshteyn et al. In these tables, $a > 0$ and $b > 0$.

TABLE 16.2 Abel and Radon Transforms

Abel Transforms A_1

$f(r)$	$A_1\{f(r);x\}$
$\chi(r/a)$	$\sin^{-1}\left(\frac{a}{x}\right),\ x>a$
$\delta(r-a)$	$(x^2-a^2)^{-1/2},\ x>a$
$(a^2-r^2)^{-1/2}$	$a^{-1}F\left(\frac{\pi}{2},\frac{x}{a}\right),\ x<a$
$(a^2-r^2)^{1/2}$	$a\,E\left(\frac{\pi}{2},\frac{x}{a}\right),\ x<a$
$r^2(a^2-r^2)^{-1/2}$	$a\left[F\left(\frac{\pi}{2},\frac{x}{a}\right)-E\left(\frac{\pi}{2},\frac{x}{a}\right)\right],\ x<a$
$a-r$	$\frac{1}{2}\pi a-x,\ x<a$
$\cos br$	$\frac{1}{2}\pi J_0(bx)$
$r\sin br$	$\frac{1}{2}\pi x J_1(bx)$
$rJ_0(br)$	$b^{-1}\sin bx$
$J_\nu(br)$	$\frac{1}{2}\pi\left[J_{\frac{\nu}{2}}\left(\frac{bx}{2}\right)\right]^2$
$r^{\nu+1}J_\nu(br)$	$\pi^{\frac{1}{2}}(2b)^{-\frac{1}{2}}x^{\nu+\frac{1}{2}}J_{\nu+\frac{1}{2}}(bx)$

Abel Transforms A_2

$f(r)$	$A_2\{f(r);x\}$
$\chi(r/a)$	$\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right),\ x<a$
$\delta(r-a)$	$(a^2-x^2)^{-1/2},\ x<a$
$(a^2-r^2)^{-1/2}\chi(r/a)$	$a^{-1}F(\frac{1}{2}\pi,t),\ x<a$
$(a^2-r^2)^{1/2}\chi(r/a)$	$a[F(\frac{1}{2}\pi,t)-E(\frac{1}{2}\pi,t)],\ x<a$
$r^2(a^2-r^2)^{-1/2}\chi(r/a)$	$a\,E(\frac{1}{2}\pi,t),\ x<a$
$(a-r)\chi(r/a)$	$\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)-\sqrt{a^2-x^2},\ x<a$
$\sin br$	$\frac{1}{2}\pi J_0(bx)$
$r\cos br$	$-\frac{1}{2}\pi x J_1(bx)$
$rJ_0(br)$	$b^{-1}\cos bx$

Note: $t=a^{-1}\sqrt{a^2-x^2}$.

TABLE 16.2 Abel and Radon Transforms (continued)

Abel Transforms A_3	
$f(r)$	$A_3\{f(r);\, x\}$
$(a^2-r^2)^{-1/2}\chi(r/a)$	$\pi\chi(r/a)$
$\chi(r/a)$	$2(a^2-x^2)^{1/2}\chi(x/a)$
$(a^2-r^2)^{1/2}\chi(r/a)$	$\frac{1}{2}\pi(a^2-x^2)\chi(x/a)$
$(a^2-r^2)\chi(r/a)$	$\frac{4}{3}(a^2-x^2)^{3/2}\chi(x/a)$
$(a^2-r^2)^{\frac{3}{2}}\chi(r/a)$	$\frac{3\pi}{8}(a^2-x^2)^2\chi(x/a)$
$(a^2-r^2)^2\chi(r/a)$	$\frac{15}{16}(a^2-x^2)^{5/2}\chi(x/a)$
$(a^2-r^2)^{\frac{n-1}{2}}\chi(r/a)$	$C(n)(a^2-x^2)^{\frac{n}{2}}\chi(x/a)$
$(a^2+r^2)^{-1}$	$\pi(a^2+x^2)^{-1/2}$
$(a^2+r^2)^{-\frac{3}{2}}$	$2(a^2+x^2)^{-1}$
e^{-r^2}	$\sqrt{\pi}\, e^{-x^2}$
$r^2e^{-r^2}$	$\frac{1}{2}\sqrt{\pi}\,(2x^2+1)e^{-x^2}$
$\operatorname{sinc} 2ar$	$\frac{1}{2a}J_0(2\pi ax)$
$\cos br$	$-\pi x J_1(bx)$
$J_0(br)$	$2b^{-1}\cos bx$
$r^{-1}J_1(br)$	$2(bx)^{-1}\sin bx$
$r^{-1}J_\nu(br)$	$-\pi J_{\frac{\nu}{2}}\left(\frac{bx}{2}\right)N_{\frac{\nu}{2}}\left(\frac{bx}{2}\right)$
$r^{-1}N_\nu(br)$	$\frac{1}{2}\pi\left[J_{\frac{\nu}{2}}\left(\frac{bx}{2}\right)\right]^2-\frac{1}{2}\pi\left[N_{\frac{\nu}{2}}\left(\frac{bx}{2}\right)\right]^2$

Radon Transforms	
$f(x,y)$	$\check{f}(p,\phi)$
$e^{-x^2-y^2}$	$\sqrt{\pi}\, e^{-p^2}$
$(x^2+y^2)e^{-x^2-y^2}$	$\frac{1}{2}\sqrt{\pi}\,(2p^2+1)e^{-p^2}$
$x e^{-x^2-y^2}$	$\sqrt{\pi}\, e^{-p^2}\cos\phi$
$y e^{-x^2-y^2}$	$\sqrt{\pi}\, e^{-p^2}\sin\phi$
$x^2 e^{-x^2-y^2}$	$\frac{1}{2}\sqrt{\pi}\,(2p^2\cos^2\phi+\sin^2\phi)e^{-p^2}$
$y^2 e^{-x^2-y^2}$	$\frac{1}{2}\sqrt{\pi}\,(2p^2\sin^2\phi+\cos^2\phi)e^{-p^2}$
$\exp\left[-\left(\frac{x}{a}\right)^2-\left(\frac{y}{a}\right)^2\right]$	$\frac{\lvert ab\rvert\sqrt{\pi}}{s}\exp\left[-\left(\frac{p}{s}\right)^2\right]$
$\delta(x-a)\delta(y-b)$	$\delta(p-p_0)$
$\chi(r)$	$2(1-p^2)^{\frac{1}{2}}\chi(p)$
$x^2\chi(r)$	$(1-p^2)^{\frac{1}{2}}[2p^2\cos^2\phi+\frac{2}{3}(1-p^2)\sin^2\phi)]$

TABLE 16.2 Abel and Radon Transforms (continued)

Radon Transforms	
$f(x,y)$	$\check{f}(p,\phi)$
$y^2\chi(r)$	$(1-p^2)^{\frac{1}{2}}[2p^2\sin^2\phi+\frac{2}{3}(1-p^2)\cos^2\phi)]$
$(x^2+y^2)\chi(r)$	$\frac{2}{3}(1-p^2)^{\frac{1}{2}}(2p^2+1)$

The following notation issued in the above table,

$$s=(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}},\ r=\sqrt{a^2+y^2},\ p_o=a\cos\phi+b\sin\phi.$$

Formulas for Radon transforms involving Hermite polynomials, Laguerre polynomials, and Zernike polynomials appear in Chapters 22, 23, and 26, respectively.

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17
The Hankel Transform

17.1 The Hankel Transform

17.1.1 Definition of the ν^{th} Order Hankel Transform

$$F_\nu(s) \equiv \mathcal{H}a_\nu\{f(r)\} = \int_0^\infty r f(r) J_\nu(sr)\,dr, \qquad r = \sqrt{x^2+y^2}$$

$$f(r) \equiv \mathcal{H}a_\nu^{-1}\{F_\nu(s)\} = \int_0^\infty s F_\nu(s) J_\nu(sr)\,ds$$

17.1.2 The Zero-Order Hankel Transform

$$F(s) \equiv \mathcal{H}a_0\{f(r)\} = \int_0^\infty r f(r) J_0(sr)\,dr$$

$$f(r) \equiv \mathcal{H}a_0^{-1}\{F(s)\} = \int_0^\infty s F(s) J_0(sr)\,ds$$

17.1.3 Relation to Fourier Transform with Function of Circular Symmetry

$$\mathcal{F}\{f(\sqrt{x^2+y^2})\} = F(u,v)$$

$$F(u,v) = 2\pi F(s) = 2\pi F(\sqrt{u^2+v^2})$$

with $\mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y)\exp[-j(xu+yv)]\,dx\,dy.$

Example

$\mathcal{F}\{\exp[-a(x^2+y^2)]\} = \frac{\pi}{a}\exp[-(u^2+v^2)/4a]$ and, therefore,

$$F(s)=\frac{1}{2\pi}F(u,v)=\frac{1}{2a}\exp[-s^2/4a],\quad s^2=(u^2+v^2),\quad a>0$$

17.2 Properties of Hankel Transform

17.2.1 Derivatives

$$F_v(s)=Ha_v\{f(r)\}$$

$$G_v(s)=Ha_v\{f'(r)\}=s\left[\frac{v+1}{2v}F_{v-1}(s)-\frac{v-1}{2v}F_{v+1}(s)\right]$$

$$Ha_v\left\{\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}-\left(\frac{v}{r}\right)^2 f(r)\right\}=-s^2Ha_v\{f(r)\}$$

Note: $\frac{d}{dr}[rJ_v(sr)]=\frac{sr}{2v}[(\nu+1)J_{v-1}(sr)-(\nu-1)J_{v+1}(sr)]$

Example

$$F\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{df(r)}{dr}\right]\right\}=-(u^2+v^2)F(u,v).$$

But from 17.1.3

$$F\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{df(r)}{dr}\right]=\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}\right\}=-2\pi s^2F(s)$$

and hence

$$Ha_0\left\{\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}\right\}=-s^2F(s)=-s^2Ha_0\{f(r)\}$$

17.2.2 Similarity

$$Ha_v\{f(ar)\}=\frac{1}{a^2}F_v\left(\frac{s}{a}\right)$$

Example (see 17.1.3)

$$F\left\{f(a\sqrt{x^2+y^2})\right\}=F\left\{f(\sqrt{(ax)^2+(ay)^2})\right\}=\iint f(\sqrt{(ax)^2+(ay)^2})\exp[-jux-jvy]\,dx\,dy$$

$$=\frac{1}{a^2}\iint f(\sqrt{t^2+\tau^2})\exp[-j\tfrac{u}{a}t-j\tfrac{v}{a}\tau]\,dt\,d\tau=\frac{1}{a^2}2\pi F\left(\frac{s}{a}\right).$$

Hence

$$Ha_0\{f(ar)\}=\frac{1}{a^2}F\left(\frac{s}{a}\right)$$

17.2.3 Division by r

1. $\mathcal{H}a_v\{r^{-1}f(r)\} = \frac{s}{2v}\left[F_{v-1}(s) + F_{v+1}(s)\right]$

2. $\mathcal{H}a_v\{r^{v-1}\frac{d}{dr}[r^{1-v}f(r)]\} = sF_{v-1}(s)$

3. $\mathcal{H}a_v\left\{r^{-v-1}\frac{d}{dr}\left[r^{v+1}f(r)\right]\right\} = sF_{v+1}(s)$

17.2.4 Parseval's Theorem

$$F_v(s) = \mathcal{H}a_v\{f(r)\}, \quad G_v(s) = \mathcal{H}a_v\{g(r)\}$$

1. $\int_0^\infty F_v(s)G_v(s)s\,ds = \int_0^\infty rg(r)\,f(r)\,dr$

2. $\int_0^\infty F_v(s)G_v^*(s)s\,ds = \int_0^\infty rf(r)g^*(r)\,dr$ for complex signals

3. $\int_0^\infty r|f(r)|^2\,dr = \int_0^\infty s|F(s)|^2\,ds$

17.2.5 Convolution Identity

$$\mathcal{F}_{(2)}\left\{\iint_{-\infty}^{\infty} f_1(\sqrt{x_1^2+y_1^2})\,f_2(\sqrt{(x-x_1)^2+(y-y_1)^2})\,dx_1\,dy_1\right\} = 4\pi^2 F_1(s)F_2(s)$$

Hence $\mathcal{H}a_0\{f_1(r)**f_2(r)\} = \frac{1}{2\pi}\mathcal{F}_{(2)}\{f_1(r)**f_2(r)\} = 2\pi F_1(s)F_2(s)$

Also $\mathcal{H}a_0\{2\pi f_1(r)\,f_2(r)\} = F_1(s)**F_2(s)$

17.2.6 Moment

$$m_n = \int_0^\infty r^n f(r)\,dr$$

But

$$J_0(sr) = 1 - \left(\frac{sr}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{sr}{2}\right)^4 - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{sr}{2}\right)^{2n}$$

hence

$$F(s) = \mathcal{H}a_0\{f(r)\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{s}{2}\right)^{2n}\int_0^\infty r^{2n+1}f(r)\,dr = \sum_{n=0}^{\infty}\frac{(-1)^n m_{2n+1}}{(n!)^2 2^{2n}}s^{2n}$$

17.3 Examples of Hankel Transform

17.3.1 Example

From

$$\int_0^a rJ_0(sr)\,dr = \int_0^a \frac{1}{s}\frac{d}{dr}[rJ_1(sr)] = [aJ_1(as)]/s$$

implies that $Ha_0\{p_a(r)\} = \frac{a}{s} J_1(as)$, where $p_a(r) = 1$ for $r < a$ and zero otherwise.

17.3.2 Example

From $\int_0^a J_0(sr)dr = \frac{1}{s}$, $s > 0$ we obtain $Ha_0\{1/r\} = 1/s$.

17.3.3 Example

From $\int_0^a r\delta(r-a)\,J_0(sr)\,dr = aJ_0(as)$ we obtain $Ha_0\{\delta(r-a)\} = aJ_0(as)$, $s > 0$ and because of symmetry $Ha_0\{aJ_0(ar)\} = \delta(s-a)$, $a > 0$

17.3.4 Example

If $f_1(r) = f_2(r) = [J_1(ar)]/r$ then from 17.2.5 $Ha_0\{2\pi J_1^2(ar)/r^2\} = \frac{1}{a^2}\,p_a(s)**p_a(s)$ where

$$p_a(s)**p_a(s) = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)a^2.$$

Hence

$$Ha_0\{2\pi J_1^2(ar)/r^2\} = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)p_{2a}(s),$$

where $p_{2a}(s) = 1$ for $|s| \le 2a$ and 0 otherwise.

17.3.5 Example

From the relationship

$$\int_0^a rJ_0(br)\,J_0(sr)\,dr = a[bJ_1(ab)\,J_0(as) - sJ_0(ab)\,J_1(as)]/(b^2 - s^2)$$

we find

$$Ha_0\{J_0(br)\,p_a(r)\} = [abJ_1(ab)\,J_0(as) - asJ_0(ab)\,J_1(as)]/(b^2 - s^2)\,.$$

17.3.6 Example

From $\delta(s-a)**\delta(s-a) = 4a^2/(s\sqrt{4a^2 - s^2})$ for $s < 2a$ and equals zero for $s > 2a$, 17.2.5 and 17.3.3 we obtain $Ha_0\{J_0(ar)\,J_0(ar)\} = 2p_{2a}(s)/(\pi s\sqrt{4a^2 - s^2})$.

17.3.7 Example

From $p_a(s) ** \delta(s-a) = 2a\cos^{-1}(s/2a)$ for $s \le 2a$ and cycles zero for $s > 2a$, from $\mathcal{H}a_0\{J_0(ar)\} = \delta(s-a)/a$; from $\mathcal{H}a_0\{J_1(ar)/r\} = p_a(s)/a$ and 17.2.5 we obtain $\mathcal{H}a_0\{J_0(ar)J_1(ar)/r\} = p_{2a}(s)\ \cos^{-1}(s/2a)/(a\pi)$.

17.3.8 Example

$\mathcal{H}a_v\{r^v u(a-r)\} = \int_0^a r^{v+1} J_v(sr)\,dr = \frac{1}{s^{v+2}}\int_0^{as} x^{v+1} J_v(x)\,dx,\ a > 0$ where $u(a-r) = 1$ for $r \le a$ and 0 for $r > a$ is the unit step function. But $\int x^v J_{v-1}(x)\,dx = x^v J_v(x) + C$ (see 25.3.2) and, hence,

$$\mathcal{H}a_v\{r^v h(a-r)\} = \frac{(as)^{v+1}}{s^{v+2}} J_{v+1}(as) = a^{v+1} J_{v+1}(as),\ \ a > 0,\ \ v > -1/2.$$

17.3.9 Example

$\mathcal{H}a_0\{e^{-ar}\} = L\{rJ_0(sr)\} = -\frac{d}{da}[(s^2+a^2)^{-1/2}] = a/[s^2+a^2]^{3/2},\ \ a > 0$

17.4 Relation to Fourier Transform

17.4.1 Relationship Between Fourier and Hankel Transform

If $F(s)$ is the Hankel transform of $f(r)$, then

1. $2\pi F(\sqrt{u^2+v^2}) = F(u,v) = \mathcal{F}\{f\sqrt{x^2+y^2}\}$
2. $\Phi(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\varphi(x)\,dx$

   $\varphi(x) = \int_{-\infty}^{\infty} f(\sqrt{x^2+y^2})\,dy$

   $\mathcal{F}\{\varphi(x)\} = 2\pi F(s)\big|_{s=\omega}$

17.4.2 Example

If $f(r) = p_a(r)$, then

$$\varphi(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy = 2\sqrt{a^2-x^2}$$

for $|x| < a$, and $\varphi(x) = 0$ for $|x| > a$. But $\mathcal{H}a_0\{p_a(r)\} = aJ_1(as)/s$ (see 17.3.1) and, hence,

$$\mathcal{F}\{2\sqrt{a^2-x^2}\ p_a(x)\} = 2\pi J_1(a\omega)/\omega\ .$$

17.4.3 Example

If $f(r) = p_a(r)/\sqrt{a^2-x^2}$, then

$$\varphi(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-(x^2+y^2)}} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

for $|x| < a$ and equals zero for $|x| > a$.
Hence

$$\Phi(\omega) = \int_{-a}^{a} \pi e^{-j\omega x} d = \frac{2\pi \sin a\omega}{\omega} = 2\pi F(s)\Big|_{s=\omega}$$

which implies that $Ha_0\{p_a(r)/\sqrt{a^2-r^2}\} = \sin as / s.$

17.5 Hankel Transforms of Order Zero

Table 17.1 lists the Hankel transforms of some particular functions for the important special case ν = 0. Table 17.2 lists Hankel transforms of general order ν. In these tables, $u(x)$ is the unit step function, I_ν and K_ν are modified Bessel functions, $\boldsymbol{L}_0$ and $\boldsymbol{H}_0$ are Struve functions, and Ker and Kei are Kelvin functions as defined in Abramowitz and Stegun.

TABLE 17.1 Hankel Transform of Order Zero

$f(r)$	$F_0(s) = Ha_0\{f(r);s\}$
Algebraic function	
$1/r$	$1/s$
$1/r^\mu, \quad \frac{1}{2} < \mu < 2$	$2^{1-\mu} \dfrac{\Gamma\left(1-\frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)} \dfrac{1}{s^{2-\mu}}$
$\dfrac{1}{(a^2+r^2)^{1/2}}, \quad \mathrm{Re}(a) > 0$	$\dfrac{e^{-as}}{s}$
$\begin{cases} \dfrac{1}{(a^2-r^2)^{1/2}}, & 0 < r < a \\ 0, & a < r < \infty \end{cases}$	$\dfrac{\sin(as)}{s}$
$\dfrac{1}{(r^2+a^2)^{3/2}}, \quad \mathrm{Re}(a) > 0$	$a^{-1}e^{-as}$
$\dfrac{1}{r(r+a)}$	$\dfrac{\pi}{2}[\mathbf{H}_0(as) - Y_0(as)]$
$\dfrac{1}{r^2+a^2}$	$K_0(as)$
$\dfrac{1}{r(r^2+a^2)}$	$\dfrac{\pi}{2a}[I_0(as) - \mathbf{L}_0(as)]$
$\dfrac{1}{1+r^4}$	$-\mathrm{Kei}(s)$
$\dfrac{1}{a^4+r^4} \quad \lvert\arg a\rvert < \pi/4$	$-a^{-2}\mathrm{Kei}(as)$

TABLE 17.1 Hankel Transform of Order Zero (continued)

$f(r)$	$F_0(s) = Ha_0\{f(r);s\}$
$\frac{r^2}{1+r^4}$	$\mathrm{Ker}(s)$
$\frac{1}{\sqrt{r^4+a^4}}$	$K_0(as/\sqrt{2})\,J_0(as/\sqrt{2})$
Exponential function	
$e^{-ar}, \quad \mathrm{Re}(a)>0$	$\frac{s}{(s^2+a^2)^{3/2}}$
$\frac{e^{-ar}}{r}$	$\frac{1}{(s^2+a^2)^{1/2}}$
$(1-e^{-ar})r^{-2}, \quad \mathrm{Re}(a)>0$	$\sinh^{-1}(a/s)$
$r^{n-\frac{1}{2}}\,e^{-ar}, \quad \mathrm{Re}(a)>0$	$\frac{n!}{(a^2+s^2)^{\frac{1}{2}n+\frac{1}{2}}}\,P_n\left[\frac{a}{(a^2+s^2)^{1/2}}\right]$
$e^{-a^2r^2}$	$\frac{e^{-s^2/4a^2}}{2a^2}$
Trigonometric function	
$\frac{\sin ar}{r}, \quad a>0$	$\begin{cases} \frac{1}{(a^2-s^2)^{1/2}}, & 0<s<a \\ 0 & a<s<\infty \end{cases}$
$\frac{\sin ar}{r^2}, \quad a>0$	$\begin{cases} \frac{1}{2}\pi, & 0<s<a \\ \sin^{-1}(a/s), & a<s<\infty \end{cases}$
$\frac{\sin ar}{b^2+r^2}, \quad a>0 \quad \mathrm{Re}(b)>0$	$\frac{\pi}{2}e^{-ab}\,I_0(sb), \quad 0<s<a$
$\frac{\cos ar}{r}, \quad a>0$	$\begin{cases} 0, & 0<s<a \\ (s^2-a^2)^{-1/2}, & a<s<\infty \end{cases}$
$\frac{1-\cos ar}{r^2}, \quad a>0$	$\begin{cases} \cosh^{-1}(a/s), & 0<s<a \\ 0, & a<s<\infty \end{cases}$
$\frac{\cos ar}{b^2+r^2}, \quad a>0 \quad \mathrm{Re}(b)>0$	$\cosh(ab)K_0(bs), \quad a<s<\infty$
$\cos(a^2r^2/2), \quad a>0$	$a^{-2}\sin(a^{-2}s^2/2)$
Other functions	
$\frac{1-J_0(ar)}{r^2}, \quad a>0$	$\begin{cases} \ln\frac{a}{s}, & s<a \\ 0, & s>a \end{cases}$
$\frac{J_1(ar)}{r}, \quad a>0$	$\begin{cases} a^{-1}, & 0<s<a \\ 0, & a<s<\infty \end{cases}$

TABLE 17.2 Hankel Transform of Order ν^{th}

$f(r)$	$F_\nu(s) = Ha_\nu\{f(r);s\}$
Algebraic functions	
$1/r, \quad \mathrm{Re}(\nu) > -1$	$1/s$
$1/r^\mu, \quad \frac{1}{2} < \mu < \nu + 2$	$\dfrac{2^{1-\mu}\Gamma\left(\frac{\nu+2-\mu}{2}\right)}{s^{2-\mu}\Gamma\left(\frac{\nu+\mu}{2}\right)}$
$\begin{cases} r^\nu, & 0<r<1,\ \mathrm{Re}(\nu)>-1 \\ 0, & 1<r<\infty \end{cases}$	$\dfrac{J_{\nu+1}(s)}{s}$
$\dfrac{r^\nu}{r^2+a^2}, \quad \mathrm{Re}(a)>0,\ -1<\mathrm{Re}(\nu)<\frac{3}{2}$	$a^\nu K_\nu(as)$
$r^\nu(a^2-r^2)^\mu u(a-r), \quad \mathrm{Re}(\nu)>-1,$ $0<r<a, \quad \mathrm{Re}(\mu)>-1$	$2^\mu\Gamma(\mu+1)s^{-\mu-1}a^{\nu+\mu+1}J_{\nu+\mu+1}(as)$
$\dfrac{r^\nu}{(r^2+a^2)^{\nu+\frac{1}{2}}}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}\, s^{\nu-1}}{2^\nu e^{as}\Gamma(\nu+\frac{1}{2})}$
$\dfrac{r^\nu}{(r^2+a^2)^{\mu+1}}, \quad \mathrm{Re}(a)>0$ $-1<\mathrm{Re}(\nu)<2\,\mathrm{Re}(\mu)+\frac{3}{2}$	$\dfrac{a^{\nu-\mu}s^\mu K_{\nu-\mu}(as)}{2^\mu\Gamma(\mu+1)}$
$\dfrac{r^\nu}{(a^2-r^2)^{\nu+\frac{1}{2}}}u(a-r), \quad 0<r<a,$ $\lvert\mathrm{Re}(\nu)\rvert<\frac{1}{2}$	$\pi^{-1/2}2^{-\nu}\Gamma(\frac{1}{2}-\nu)s^{\nu-1}\sin(as)$
$\dfrac{r^\nu}{(r^4+4a^4)^{\nu+\frac{1}{2}}}, \quad \lvert\arg(a)\rvert<\pi/4$ $\lvert\mathrm{Re}(\nu)\rvert>-\frac{1}{2}$	$\dfrac{s^\nu\sqrt{\pi}\,J_\nu(as)K_\nu(as)}{a^{2\nu}2^{3\nu}e^{as}\Gamma(\nu+\frac{1}{2})}$
$\dfrac{r^{\nu+2}}{(r^4+4a^4)^{\nu+\frac{1}{2}}}, \quad \lvert\arg(a)\rvert<\pi/4$ $\lvert\mathrm{Re}(\nu)\rvert>\frac{1}{6}$	$\dfrac{\sqrt{\pi}\,s^\nu J_{\nu-1}(as)K_{\nu-1}(as)}{2^{3\nu-1}a^{2\nu-1}\Gamma(\nu+\frac{1}{2})}$
Exponential function	
$\dfrac{e^{-ar}}{r}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>-1$	$s^{-\nu}(s^2+a^2)^{-\frac{1}{2}}[(a^2+s^2)^{\frac{1}{2}}-a]^\nu$
$\dfrac{e^{-ar}}{r^2}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>0$	$\nu^{-1}s^{-\nu}[(a^2+s^2)^{\frac{1}{2}}-a]^\nu$
$r^\nu e^{-ar}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>-1$	$\dfrac{1}{\sqrt{\pi}}2^{\nu+1}\Gamma(\nu+\frac{3}{2})as^\nu\dfrac{1}{(a^2+s^2)^{\nu+\frac{3}{2}}}$
$r^{\nu-1}e^{-ar}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>-\frac{1}{2}$	$\dfrac{1}{\sqrt{\pi}}2^\nu\Gamma(\nu+\frac{1}{2})s^\nu\dfrac{1}{(a^2+s^2)^{\nu+\frac{1}{2}}}$
$r^\nu e^{-ar^2}, \quad \mathrm{Re}(a)>0,\ \mathrm{Re}(\nu)>-1$	$\dfrac{s^\nu}{(2a)^{\nu+1}}\exp\left(-\dfrac{s^2}{4a}\right)$
Trigonometric functions	
$\dfrac{\sin ar}{r}, \quad a>0, \quad \mathrm{Re}(\nu)>-2$	$\cos(\pi\nu/2)s^\nu(a^2-s^2)^{-\frac{1}{2}}[a+(a^2-s^2)^{\frac{1}{2}}]^{-\nu}, \quad 0<s<a$ $(s^2-a^2)^{-\frac{1}{2}}\sin[\nu\sin^{-1}(a/s)], \quad a<s<\infty$

TABLE 17.2 Hankel Transform of Order ν^{th} (continued)

$f(r)$	$F_\nu(s) = Ha_\nu\{f(r);s\}$
$\frac{\sin ar}{r^2}, \quad a>0, \quad \mathrm{Re}(\nu) > -1$	$\nu^{-1}\sin(\nu\pi/2)s^\nu[a+(a^2-s^2)^{\frac{1}{2}}]^{-\nu}, \quad 0<s<a$ $\nu^{-1}\sin[\nu\sin^{-1}(a/s)], \quad a<s<\infty$
$r^\nu \sin ar, \quad a>0, \quad -\frac{3}{2} < \mathrm{Re}(\nu) < -\frac{1}{2}$	$\frac{-2^{\nu+1}\sin\nu\pi\,\Gamma(\nu+\frac{3}{2})as^\nu}{\sqrt{\pi}\,(a^2-s^2)^{-\nu-\frac{3}{2}}}, \quad 0<s<a$ $\frac{-2^{\nu+1}\Gamma(\nu+\frac{3}{2})as^\nu}{\sqrt{\pi}\,(s^2-a^2)^{-\nu-\frac{3}{2}}}, \quad a<s<\infty$
$r^{\nu-1}\sin ar, \quad a>0, \quad -1<\mathrm{Re}(\nu)<\frac{1}{2}$	$\frac{\sqrt{\pi}\;2^\nu\;s^\nu}{\Gamma(\frac{1}{2}-\nu)\;(a^2-s^2)^{-\nu-\frac{1}{2}}}, \quad 0<s<a$ $0, \quad a<s<\infty$
$r^\nu \cos ar, \quad a>0, \quad -1<\mathrm{Re}(\nu)<-\frac{1}{2}$	$\frac{2^{1+\nu}\sqrt{\pi}\;a\,s^\nu}{\Gamma(-\frac{1}{2}-\nu)\;(a^2-s^2)^{-\nu-\frac{3}{2}}}, \quad 0<s<a$ $0, \quad a<s<\infty$
Other functions	
$\frac{J_{\nu-1}(ar)}{r}, \quad a>0, \quad \mathrm{Re}(\nu)>-1$	$0, \quad 0<s<a$ $a^{\nu-1}s^{-\nu}, \quad a<s<\infty$
$\frac{J_\nu(ar)}{r^2}, \quad a>0, \quad \mathrm{Re}(\nu)>0$	$\frac{1}{2\nu}\frac{s^\nu}{a^\nu}, \quad 0<s<a$ $\frac{1}{2\nu}\frac{a^\nu}{s^\nu}, \quad a\le s<\infty$
$\frac{J_{\nu+1}(ar)}{r}, \quad a>0, \quad \mathrm{Re}(\nu)>-\frac{3}{2}$	$a^{-\nu-1}s^\nu, \quad 0<s<a$ $0, \quad a<s<\infty$

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18

The Mellin Transform

18.1 The Mellin Transform

18.1.1 Definition

$$\mathrm{M}\{f(t);s\} = F(s) = \int_0^\infty f(t)t^{s-1}\,dt$$

Example

$$\mathrm{M}\{e^{-at}u(t)\} = \int_0^\infty e^{-at}\,t^{s-1}\,dt = a^{-s}\Gamma(s)$$

18.1.2 Relation to Laplace Transform

By letting $t = e^{-x}$, $dt = -e^{-x}\,dx$, the transform becomes

$$\mathrm{M}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(e^{-x})e^{-sx}dx = \mathrm{L}\,\{f(e^{-x})\}$$

18.1.3 Relation to Fourier Transform

By setting $s = \sigma + j2\pi\beta$ in 18.1.2 we obtain

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})e^{-ax}e^{-j2\pi\beta x}\,dx$$

which implies that

$$\mathrm{M}\{f(t);s = \sigma + j2\pi\beta\} = \mathrm{F}\{f(e^{-x})e^{-ax};\beta\}$$

where

$$\mathrm{F}\{f(x);\beta\} = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\beta x}\, dx = \text{ Fourier Transform}$$

18.1.4 Inversion Formula

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) t^{-s}\, ds$$

where c is within the strip of analyticity $a < \mathrm{Re}\, s < b$.

18.2 Properties of Mellin Transform

18.2.1 Scaling Property

$$\mathrm{M}\{f(at);s\} = \int_0^{\infty} f(at) t^{s-1}\, dt = a^{-s} \int_0^{\infty} f(x) x^{s-1}\, dx = a^{-s} F(s)$$

18.2.2 Multiplication by t^a

$$\mathrm{M}\{t^a f(t);s\} = \int_0^{\infty} f(t) t^{(s+a)-1}\, dt = F(s+a)$$

18.2.3 Raising the Independent Variable to a Real Power

$$\mathrm{M}\{f(t^a);s\} = \int_0^{\infty} f(t^a) t^{s-1}\, dt = \int_0^{\infty} f(x) x^{\frac{s}{a}-\frac{1}{a}} \left(\frac{1}{a} x^{\frac{1}{a}-1}\, dx \right) = a^{-1} F\left(\frac{s}{a} \right),\ a > 0$$

18.2.4 Inverse of Independent Variable

$$\mathrm{M}\{t^{-1} f(t^{-1});s\} = F(1-s)$$

18.2.5 Multiplication by ln t

$$\mathrm{M}\{\ln t\, f(t);s\} = \frac{d}{ds} F(s)$$

18.2.6 Multiplication by a Power of ln t

$$\mathrm{M}\{(\ln t)^k\, f(t);s\} = \frac{d^k}{ds^k} F(s)$$

18.2.7 Derivative

$$\mathcal{M}\left\{\frac{d^k}{ds^k} f(t);s\right\} = (-1)^k (s-k)_k F(s-k)$$

$$(s-k)_k \equiv (s-k)(s-k+1)\cdots(s-1) = \frac{(s-1)!}{(s-k-1)!} = \frac{\Gamma(s)}{\Gamma(s-k)}$$

18.2.8 Derivative Multiplied by Independent Variable

$$\mathcal{M}\left\{t^k \frac{d^k}{ds^k} f(t);s\right\} = (-1)^k (s)_k F(s) = (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)} F(s),\ (s)_k = s(s+1)\cdots(s+k-1)$$

Example

$$\mathcal{M}\left\{t^2 \frac{d^2 f(t)}{dt^2} + t\frac{df(t)}{dt};s\right\} = s^2 F(s)$$

18.2.9 Convolution

$$\mathcal{M}\{f(t)g(t);s\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(z)G(s-z)\,dz$$

18.2.10 Multiplicative Convolution

$$\mathcal{M}\{f\vee g\} = \mathcal{M}\left\{\int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u};s\right\} = F(s)G(s)$$

$$\mathcal{M}^{-1}\{F(s)G(s)\} = \int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u}$$

Properties of the Multiplicative Convolution

$$\int_0^\infty f\left(\frac{t}{u}\right)g(u)\frac{du}{u}$$

1. $f\vee g = g\vee f$ commutative
2. $(f\vee g)\vee h = f\vee(g\vee h)$ associative
3. $f\vee\delta(t-1) = f$ unit element
4. $\left(t\frac{d}{dt}\right)^k (f\vee g) = \left[\left(t\frac{d}{dt}\right)^k f\right]\vee g = f\vee\left[\left(t\frac{d}{dt}\right)^k g\right]$
5. $(\ln t)(f\vee g) = [(\ln t)f]\vee g + f\vee[(\ln t)g]$
6. $\delta(t-a)\vee f = a^{-1}f(a^{-1}t)$

$$\delta(t-p) \vee \delta(t-p') = \delta(t-pp'), \quad p, p' > 0$$

$$\frac{d^k \delta(t-1)}{dt^k} \vee f = \left(\frac{d}{ds}\right)^k (t^k f)$$

18.2.11 Parseval's Formulas

$$\int_0^\infty f(t) g(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathrm{M}\{f;s\} \mathrm{M}\{g;1-s\} \, ds$$

$$\int_0^\infty f(t) g^*(t) t^{2r+1} \, dt = \int_{-\infty}^\infty \mathrm{M}\{f\}(\beta) \mathrm{M}^*\{g\}(\beta) \, d\beta$$

where $\mathrm{M}\{f\}(\beta) = \int_0^\infty f(t) t^{2\pi j\beta + r} \, dt$

18.3 Examples of Mellin Transform

18.3.1 Example

$$\mathrm{M}\{t^a u(t-t_0)\} = \int_{t_0}^\infty t^{a+s-1} \, dt = -\frac{t_o^{a+s}}{a+s}, \quad \mathrm{Re}\{s\} < -a$$

18.3.2 Example

$$\mathrm{M}\left\{\frac{1}{1+t}; s\right\} = \int_0^\infty \frac{1}{1+t} t^{s-1} \, dt.$$

Setting $t+1 = \dfrac{1}{1-x}$ we obtain: $x = \dfrac{t}{t+1}$, $dx = \dfrac{dt}{(1+t)^2}$. Hence ,

$$\mathrm{M}\{f;s\} = \int_0^1 (1-x) \frac{x^{s-1}}{(1-x)^{s-1}} \frac{dx}{(1-x)^2} = \int_0^1 x^{s-1} (1-x)^{-s} \, dx = B(s, 1-s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

$0 < \mathrm{Re}\{s\} < 1.$

18.3.3 Example

From $\int_o^1 (1-u)^{m-1} u^{s-1} \, du = \dfrac{\Gamma(m)\Gamma(s)}{\Gamma(m+s)}$, $\mathrm{Re}\{s\} > 0$, $\mathrm{Re}\{m\} > 0$, with the setting $u = x/(1+x)$, we obtain

$$\int_0^\infty \frac{x^{s-1}}{(1+x)^{m+s}} \, dx = \frac{\Gamma(m)\Gamma(s)}{\Gamma(m+s)} .$$

Hence,

$$\mathrm{M}\{(1+t)^{-a}; s\} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}, \quad 0 < \mathrm{Re}\{s\} < \mathrm{Re}\{a\} .$$

18.3.4 Example

Using 18.2.3 and 18.3.3 we obtain

$$\mathrm{M}\{(1+t^a)^{-b};s\} = \frac{\Gamma(s/a)\Gamma\left(b-\frac{s}{a}\right)}{a\Gamma(b)}, \ 0 < \mathrm{Re}\{s\} < \mathrm{Re}\{ab\}.$$

18.3.5 Example

$$\mathrm{M}\{\delta(t-t_o);s\} = \int_0^\infty \delta(t-t_0)t^{s-1}\,dt = t_0^{s-1} \qquad \text{(see 5.3.1)}$$

18.3.6 Example

From 18.3.1 and 18.3.5 $\mathrm{M}\{t^a u(t-t_0)\} = -\frac{t_0^{a+s}}{a+s}$ and hence,

$$\mathrm{M}\left\{\frac{df}{dt};s\right\} = -(s-1)F(s-1) = (s-1)\frac{t_0^{a+s-1}}{a+s-1} = -a\frac{t_0^{a+s-1}}{a+s-1} + t_0^{a+s-1}$$

$$= \mathrm{M}\{au(t-t_0)t^{a+s-1};s\} + \mathrm{M}\{t_0^a\,\delta(t-t_0);s\}$$

18.4 Special Functions Frequency Occurring in Mellin Transforms

18.4.1 Definition

The gamma function $\Gamma(s)$ is defined on the complex half-plane $\mathrm{Re}(s) > 0$ by the integral

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$$

18.4.2 Analytic Continuation

The analytical continued gamma function is holomorphic in the whole plane except at the points $s = -n, n = 0,1,2,\cdots$, where it has a simple pole.

18.4.3 Residues at the Poles

$$\mathrm{Re}\,s_{s=-n}(\Gamma(s)) = \frac{(-1)^n}{n!}$$

18.4.4 Relation to the Factorial

$$\Gamma(n+1) = n!$$

18.4.5 Functional Relations

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(2s) = \pi^{-1/2} 2^{2s-1} \Gamma(s)\Gamma(s+1/2)$$

(Legendre's duplication formula)

$$\Gamma(ms) = m^{ms-1/2} (2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma(s+k/m), \quad m = 2,3,\cdots,$$

(Gauss-Legendre's multiplication formula)

$$\Gamma(s) \sim \sqrt{2\pi}\; s^{s-1/2} \exp\left[-s\left(1+\frac{1}{12s}\right) + O(s^{-2})\right], \quad s \to \infty, \; |\arg(s)| < \pi$$

(Stirling asymptotic formula)

18.4.6 The Beta Function

Definition: $B(x,y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$

Relation to the gamma function: $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

18.4.7 The psi Function (logarithmic derivative of the gamma function)

Definition: $\psi(s) \equiv \dfrac{d}{ds} \ln \Gamma(s)$

$$= -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{s+n}\right)$$

Euler's constant γ, also called C, is defined by

$$\gamma \equiv -\Gamma'(1)/\Gamma(1)$$

and has value $\gamma \cong 0.577 ....$

18.4.8 Riemann's Zeta Function

$$\zeta(z,q) \equiv \sum_{n=0}^{\infty} \frac{1}{(q+n)^z}, \; \mathrm{Re}(z) > 1, \; q \neq 0, -1, -2, \cdots$$

$$\zeta(z) \equiv \sum_{n=0}^{\infty} \frac{1}{n^z}, \; \mathrm{Re}(z) > 1$$

The function $\zeta(z)$ is analytic in the whole complex z-plane except in $z = 1$ where it has a simple pole with residue equal to +1.

18.5 Tables of Mellin Transform

18.5.1 Tables of Mellin Transform

TABLE 18.1 Some Standard Mellin Transform Pairs

Original Function	Mellin Transform	
$f(t),\ t>0$	$M[f;s]\equiv\int_0^\infty f(t)t^{s-1}\,dt$	Strip of holomorphy
Algebraic Functions		
$u(t-a)t^b,\ a>0$	$-\frac{a^{b+s}}{b+s}$	$\mathrm{Re}(s)<-\mathrm{Re}(b)$
$(u(t-a)-u(t))t^b$	$-\frac{a^{b+s}}{b+s}$	$\mathrm{Re}(s)>-\mathrm{Re}(b)$
$(1+t)^{-1}$	$\frac{\pi}{\sin(\pi s)}$	$0<\mathrm{Re}(s)<1$
$(a+t)^{-1},\ \lvert\arg a\rvert<\pi$	$\pi a^{s-1}\csc(\pi s)$	$0<\mathrm{Re}(s)<1$
$(1+t)^{-a}$	$\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}$	$0<\mathrm{Re}(s)<\mathrm{Re}(a)$
$(1-t)^{-1}$	$\pi\cot(\pi s)$	$0<\mathrm{Re}(s)<1$
$(a-t)^{-1},\ a>0$	$\pi a^{s-1}\cot(\pi s)$	$0<\mathrm{Re}(s)<1$
$u(1-t)(1-t)^{b-1},\ \mathrm{Re}(b)>0$	$\frac{\Gamma(s)\Gamma(b)}{\Gamma(s+b)}$	$\mathrm{Re}(s)>0$
$u(t-1)(t-1)^{-b}$	$\frac{\Gamma(b-s)\Gamma(1-b)}{\Gamma(1-s)}$	$\mathrm{Re}(s)<\mathrm{Re}(b)<1$
$(t^2+a^2)^{-1},\ \mathrm{Re}(a)>0$	$\frac{1}{2}\pi a^{s-2}\csc\left(\frac{\pi s}{2}\right)$	$0<\mathrm{Re}(s)<2$
$(t^n+a),\ \lvert\arg a\rvert<\pi$	$(\pi/n)\csc(\pi s/n)a^{s/n-1}$	$0<\mathrm{Re}(s)<n$
$\begin{cases} t^\nu & 0<t<1 \\ 0 & t>1 \end{cases}$	$(s+\nu)^{-1}$	$\mathrm{Re}(s)>-\mathrm{Re}(\nu)$
$\begin{cases} (1-t^h)^{\nu-1} & 0<t<1 \\ 0 & t\geq 0 \\ h>0 & \mathrm{Re}(\nu)>0 \end{cases}$	$h^{-1}\frac{\Gamma(\nu)\Gamma\left(\frac{s}{h}\right)}{\Gamma\left(\nu+\frac{s}{h}\right)}$	$\mathrm{Re}(s)>0$
$(1-t^a)(1-t^{na})^{-1}$	$\frac{\pi}{na}\sin\left(\frac{\pi}{n}\right)\csc\left(\frac{\pi s}{na}\right)\csc\left(\frac{\pi s+\pi a}{na}\right)$	$0<\mathrm{Re}(s)<(n-1)a$
Exponential Functions		
$e^{-pt},\ p>0$	$p^{-s}\Gamma(s)$	$\mathrm{Re}(s)>0$
$(e^t-1)^{-1}$	$\Gamma(s)\zeta(s)$	$\mathrm{Re}(s)>1$
	$(\zeta(s)=\text{zeta function})$	
$(e^{at}+1)^{-1},\ \mathrm{Re}(a)>0$	$a^{-s}\Gamma(s)(1-2^{1-s})\zeta(s)$	$\mathrm{Re}(s)>0$
$(e^{at}-1)^{-1},\ \mathrm{Re}(a)>0$	$a^{-s}\Gamma(s)\zeta(s)$	$\mathrm{Re}(s)>1$
$(e^{-at})(1-e^{-t})^{-1},\ \mathrm{Re}(a)>0$	$\Gamma(s)\zeta(s,a)$	$\mathrm{Re}(s)>1$
$(e^t-1)^{-2}$	$\Gamma(s)[\zeta(s-1)-\zeta(s)]$	$\mathrm{Re}(s)>2$

TABLE 18.1 Some Standard Mellin Transform Pairs (continued)

Original Function	Mellin Transform	
$f(t),\ t>0$	$M[f;s] \equiv \int_0^\infty f(t)t^{s-1}\,dt$	Strip of holomorphy
$e^{-at^h},\ \text{Re}(a)>0,\ h>0$	$h^{-1}a^{-s/h}\Gamma(s/h)$	$\text{Re}(s)>0$
$t^{-1}e^{-t^{-1}}$	$\Gamma(1-s)$	$-\infty<\text{Re}(s)<1$
e^{-t^2}	$\frac{1}{2}\Gamma(s/2)$	$0<\text{Re}(s)<\infty$
$1-e^{-at^h},\ \text{Re}(a)>0,\ h>0$	$-h^{-1}a^{-s/h}\Gamma(s/h)$	$-h<\text{Re}(s)<0$
e^{jat}	$a^{-s}\Gamma(s)e^{j\pi(s/2)}$	$0<\text{Re}(s)<1$
Logarithmic Functions		
$\ln(1+t)$	$\dfrac{\pi}{s\sin(\pi s)}$	$-1<\text{Re}(s)<0$
$\ln(1+at),\ \lvert\arg a\rvert<\pi$	$\pi s^{-1}a^{-s}\csc(\pi s)$	$-1<\text{Re}(s)<0$
$u(p-1)\ln(p-t)$	$-p^s s^{-1}[\psi(s+1)+p^{-1}\ln\gamma]$	$\text{Re}(s)>0$
$t^{-1}\ln(1+t)$	$\dfrac{\pi}{(1-s)\sin(\pi s)}$	$0<\text{Re}(s)<1$
$\ln\left\lvert\dfrac{1+t}{1-t}\right\rvert$	$(\pi/s)\tan(\pi s/2)$	$-1<\text{Re}(s)<1$
$\begin{cases}\ln t & 0<t<a\\ 0 & t>a\end{cases}$	$s^{-1}a^{-s}(\ln a-s^{-1})$	$\text{Re}(s)>0$
$\begin{cases}t^{\nu}\ln t & 0<t<1\\ 0 & 1<t<\infty\end{cases}$	$-(s+\nu)^{-2}$	$\text{Re}(s)>-\text{Re}(\nu)$
$e^{-t}(\ln t)^n$	$\dfrac{d^n\Gamma(s)}{ds^n}$	$\text{Re}(s)>0$
$u(t-1)\sin(a\ln t)$	$\dfrac{a}{s^2+a^2}$	$\text{Re}(s)<-\lvert\text{Im}(a)\rvert$
$u(1-t)\sin(-a\ln t)$	$\dfrac{a}{s^2+a^2}$	$\text{Re}(s)>\lvert\text{Im}(a)\rvert$
$(u(t)-u(t-p))\ln(p/t),\ p>0$	$\dfrac{p^s}{s^2}$	$\text{Re}(s)>0$
Trigonometric Functions		
$\sin(at),\ a>0$	$a^{-s}\Gamma(s)\sin(\pi s/2)$	$-1<\text{Re}(s)<1$
$e^{-at}\sin(pt),\ \text{Re}(a)>\lvert\text{Im}\,\beta\rvert$	$(a^2+\beta^2)^{-s/2}\Gamma(s)\sin\left(s\tan^{-1}\dfrac{\beta}{a}\right)$	$\text{Re}(s)>-1$
$\sin^2(at),\ a>0$	$-2^{-s-1}a^{-s}\Gamma(s)\cos(\pi s/2)$	$-2<\text{Re}(s)<0$
$\cos(at),\ a>0$	$a^{-s}\Gamma(s)\cos(\pi s/2)$	$0<\text{Re}(s)<1$
$\tan^{-1}(t)$	$\dfrac{-\pi}{2s\cos(\pi s/2)}$	$-1<\text{Re}(s)<0$
$\text{co}\tan^{-1}(t)$	$\dfrac{\pi}{2s\cos(\pi s/2)}$	$0<\text{Re}(s)<1$

TABLE 18.1 Some Standard Mellin Transform Pairs (continued)

Original Function	Mellin Transform	
$f(t),\ t>0$	$M[f;s] \equiv \int_0^\infty f(t)t^{s-1}\,dt$	Strip of holomorphy
Other Functions		
$J_\nu(at),\ a>0$	$\dfrac{2^{s-1}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)}{a^s\Gamma\left(\frac{\nu}{2}-\frac{s}{2}+1\right)}$	$-\mathrm{Re}(\nu)<\mathrm{Re}(s)<3/2$
$\sin at\, J_\nu(at),\ a>0$	$\dfrac{2^{\nu-1}\Gamma\left(\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{s}{2}\right)}{a^s\Gamma(1+\nu-s)\Gamma\left(1-\frac{\nu}{2}-\frac{s}{2}\right)}$	$-1<\mathrm{Re}(\nu)<\mathrm{Re}(s)<1/2$
$\delta(t-p),\ p>0$	p^{s-1}	whole plane
$\sum_{n=1}^{\infty}\delta(t-pn),\ p>0$	$p^{s-1}\zeta(1-s)$	$\mathrm{Re}(s)<0$
$J_\nu(t)$	$\dfrac{2^{s-1}\Gamma(s+\nu)/2}{\Gamma[(1/2)(\nu-s)+1]}$	$-\nu<\mathrm{Re}(s)<3/2$
$\sum_{n=-\infty}^{\infty}p^{-nr}\delta(t-p^n)$, $p>0,\ r=\text{real}$	$\dfrac{1}{\ln p}\sum_{n=-\infty}^{\infty}\delta\left(\beta-\dfrac{n}{\ln p}\right)$, $\beta=\mathrm{Im}(s)$	$s=r+j\beta$
t^b	$\delta(b+s)$	none (analytic functional)

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19

Time-Frequency Transformations

19.1 The Wigner Distribution

19.1.1 Definition of WD in Time Domain

$$WD_{x,g}(t;t,f)=\int_{-\infty}^{\infty}e^{-j2\pi f\tau}x\left(t+\frac{\tau}{2}\right)g*\left(t-\frac{\tau}{2}\right)d\tau$$

$$WD_x(t;t,f)=WD_{x,x}(t;t,f)=\int_{-\infty}^{\infty}e^{-j2\pi f\tau}x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)d\tau$$

19.1.2 Definition of WD in Frequency Domain

$$WD_{X,G}(f;f,t)=\int_{-\infty}^{\infty}e^{j2\pi t\nu}X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)d\nu$$

$$WD_X(f;f,t)=\int_{-\infty}^{\infty}e^{j2\pi t\nu}X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)d\nu$$

$$=\int\int_{-\infty}^{\infty}AF_x(t;\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu \quad \text{[see 19.1.3 for } AF_x(t;\tau,\nu)]$$

$WD_{X,G}(f;f,t) = WD_{x,g}(t;t,f)$ which means that the WD of the spectra of two signals can be determined from that of time functions by an interchange of frequency and time variables.

19.1.3 Definition of Ambiguity Function (AF)

$$AF_x(t;\tau,\nu) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi\nu t}\,dt$$

$$AF_X(t;\tau,\nu) = \int_{-\infty}^{\infty} X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)e^{j2\pi\tau f}\,df$$

$$= \int\int_{-\infty}^{\infty} WD_x(t;f)e^{-j2\pi(\nu t-\tau f)}\,dt\,df$$

19.1.4 Example [WD of the Delta Function $x(t) = \delta(t - t_o)$]

$$WD_x(t;f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\,\delta\left(t+\frac{\tau}{2}-t_0\right)\delta\left(t-\frac{\tau}{2}-t_0\right)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\,\delta\left(\frac{2t+\tau-2t_0}{2}\right)\delta\left(\frac{2t-\tau-2t_0}{2}\right)d\tau = 4e^{-j2\pi f 2(t-t_0)}\,\delta[4(t-t_0)] = \delta(t-t_0)$$

19.1.5 Example [WD of the Delta Function $X(f) = \delta(f - f_0)$]

$$WD_X(f;t,f) = \int_{-\infty}^{\infty} e^{j2\pi t\nu}\,\delta\left(f+\frac{\nu}{2}-f_0\right)\delta\left(f-\frac{\nu}{2}-f_0\right)d\nu = \delta(f-f_0)$$

19.1.6 Example [WD of the Linear FM Signal $x(t) = e^{j\pi\alpha t^2}$, α = sweep rate]

$$WD_x(t;t,f) = \int_{-\infty}^{\infty} e^{j\pi\alpha(t+\frac{\tau}{2})^2}e^{-j\pi\alpha(t-\frac{\tau}{2})^2}e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} e^{-j2\pi(f-\alpha t)\tau}\,d\tau = \delta(f-\alpha t)$$

19.1.7 Example [WD of the Gaussian Function $x(t) = \frac{1}{\sqrt{\sigma}}e^{-\pi(t/\sigma)^2}$]

$$WD_x(t;t,f) = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{-\pi[(t+\frac{\tau}{2})/\sigma]^2}e^{-\pi[(t-\frac{\tau}{2})/\sigma]^2}e^{-j2\pi f\tau}\,d\tau$$

$$= \frac{1}{\sigma}e^{-2\pi(t/\sigma)^2}\int_{-\infty}^{\infty} e^{-\pi\tau^2/(2\sigma^2)}e^{-j2\pi f\tau}\,d\tau = \sqrt{2}\;e^{-2\pi[(t/\sigma)^2+(\sigma f)^2]}$$

19.2 Properties of the Wigner Distribution

19.2.1 Conjugation

$$WD^*_{g,x}(t;t,f) = \int_{-\infty}^{\infty} e^{j2\pi f\tau}g^*(t+\tfrac{\tau}{2})x(t-\tfrac{\tau}{2})\,d\tau \quad \text{set } \tau = -\tau \text{ to find}$$

$$WD_{x,g}(t;t,f) = WD^*_{g,x}(t;t,f)$$

19.2.2 Real-Valued

From 19.2.1 $\quad WD_x(t;t,f) = WD_x^*(t;t,f) \equiv \text{real} \;\; (WD_{x,x} \equiv WD_x)$

19.2.3 Even in Frequency (real function)

$$WD_{x^*}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} x^*\left(t+\frac{\tau}{2}\right) x\left(t-\frac{\tau}{2}\right) d\tau$$

$$= -\int_{\infty}^{-\infty} e^{-j2\pi(-f)\tau} x^*\left(t-\frac{\tau}{2}\right) x\left(t+\frac{\tau}{2}\right) d\tau = WD_x(t;t,-f)$$

Also

$$WD_x(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\, x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) d\tau$$

$$= -\int_{\infty}^{-\infty} e^{-j2\pi(-f)\tau}\, x^*\left(t+\frac{\tau}{2}\right) x\left(t-\frac{\tau}{2}\right) dt = WD_{x^*}(t;t,-f)$$

19.2.4 Time Shift

If $x_s(t) = x(t-t_0)$ and $g_s(t) = g(t-t_0)$, then

$$WD_{x_s,g_s}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\, x\left(t-t_0+\frac{\tau}{2}\right) g^*\left(t-t_0-\frac{\tau}{2}\right) d(t-t_0) = WD_{x,g}(t-t_0;t-t_0,f)$$

19.2.5 Frequency Shift

If $x_s(t) = x(t)\, e^{j2\pi f_0 t}$ and $g_s(t) = g(t)\, e^{j2\pi f_0 t}$ are modulated, then

$$WD_{x_s,g_s}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} x\left(t+\frac{\tau}{2}\right) e^{j2\pi f_0 (t+\frac{\tau}{2})} g^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f_0 (t-\frac{\tau}{2})}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)\tau}\, x\left(t+\frac{\tau}{2}\right) g^*\left(t-\frac{\tau}{2}\right) d\tau = WD_{x,g}(t;t,f-f_0)$$

19.2.6 Time and Frequency Shifts

If $x_{sm}(t) = x(t-t_0)\, e^{j2\pi f_0 (t-t_0)}$ and $g_{sm}(t) = g(t-t_0)\, e^{j2\pi f_0 (t-t_0)}$, then

$$WD_{x_{sm} g_{sm}}(t;t,f) = WD_{x,g}(t;t-t_0, f-f_0)$$

19.2.7 Ordinates

$$WD_{x,g}(t;0,0) = \int_{-\infty}^{\infty} x\left(\frac{\tau}{2}\right) g^*\left(-\frac{\tau}{2}\right) d\tau$$

19.2.8 Sum of Functions

$$W_{g+x}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\left[x\left(t+\frac{\tau}{2}\right)g\left(t+\frac{\tau}{2}\right)\right]\left[x^*\left(t-\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)\right]d\tau$$

$$= WD_x(t;t,f) + WD_{x,g}(t;t,f) + WD_{g,x}(t;t,f) + WD_g(t;t,f)$$

$$= WD_x(t;t,f) + WD_g(t;t,f) + 2\,\mathrm{Re}\{W_{x,g}(t;t,f)\} \quad \text{since } W^*_{x,g} = W_{gx}$$

19.2.9 Multiplication by *t*

$$WD_{tx,g}(t;t,f) + WD_{x,tg}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\left(t+\frac{\tau}{2}\right)x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)d\tau$$

$$+\int_{-\infty}^{\infty} e^{-j2\pi f\tau}x\left(t+\frac{\tau}{2}\right)\left(t-\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)d\tau$$

$$= 2t\int_{-\infty}^{\infty} e^{-j2\pi f\tau}x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)d\tau = 2tWD_{x,g}(t;t,f)$$

19.2.10 Multiplication by *f*

$$2f\,WD_{x,g}(t;t,f) = WD_{fx,g}(t;t,f) + WD_{x,fg}(t;t,f)$$

19.2.11 Multiplication by *t* and *f*

$$2f\,t\,WD_{x,g}(t;t,f) = WD_{tx,fg}(t;t,f) + WD_{fx,tg}(t;t,f)$$

19.2.12 Fourier Transform

Since for a specific t the WD is the Fourier transform of $x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)$ implies that

$$x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right) = \mathcal{F}^{-1}\{WD_{x,g}(t;t,f)\} = \int_{-\infty}^{\infty} e^{j2\pi f\tau}WD_{x,g}(t;t,f)\,df$$

19.2.13 Time Marginal

In 19.2.12 set $t+\frac{\tau}{2} = t_1$ and $t-\frac{\tau}{2} = t_2$ which implies that $t = \frac{t_1+t_2}{2}$ and $\tau = t_1 - t_2$, and hence

$$\int_{-\infty}^{\infty} e^{j2\pi f(t_1-t_2)}WD_{x,g}\left(t;\frac{t_1+t_2}{2},f\right)df = x(t_1)g^*(t_2).$$

If we further set $t_1 = t_2 = t$, we obtain

1. $\int_{-\infty}^{\infty} WD_{x,g}(t;t,f)\,df = x(t)g^*(t).$

If x = g, we obtain

2. $\int_{-\infty}^{\infty} WD_x(t;t,f)\,df = |x(t)|^2.$

19.2.14 Recovery of Time Function

If we set $t_1 = t$ and $t_2 = 0$ the 19.2.13 becomes

$$\int_{-\infty}^{\infty} e^{j2\pi ft} WD_{x,g}\left(t;\frac{t}{2},f\right)df = \mathsf{F}^{-1}\left\{WD_{x,g}\left(t;\frac{t}{2},f\right)\right\} = x(t)g^*(0)$$

$$\int_{-\infty}^{\infty} e^{j2\pi ft} WD_{x}\left(t;\frac{t}{2},f\right)df = \mathsf{F}^{-1}\left\{WD_{x,g}\left(t;\frac{t}{2},f\right)\right\} = x(t)x^*(0)$$

which indicate that we can retrieve the function from its WD within a constant.

19.2.15 Frequency Marginal

From 19.1.2

$$\begin{aligned}\int_{-\infty}^{\infty} WD_{X,G}(f;t,f)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi t\nu} X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)d\nu\, dt \\ &= \int_{-\infty}^{\infty} X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)d\nu \int_{-\infty}^{\infty} e^{j2\pi t\nu}\, dt \\ &= \int_{-\infty}^{\infty} \delta(\nu) X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)d\nu = X(f)G^*(f)\end{aligned}$$

For $X(f) = G(f)$ implies that $\int_{-\infty}^{\infty} WD_X(f;t,f)\,dt = |X(f)|^2 = \int_{-\infty}^{\infty} WD_x(t;t,f)\,dt.$

19.2.16 Total Energy

From 19.2.15

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WD_{X,G}(f;f,t)\,dt\,df = \int_{-\infty}^{\infty} X(f)G^*(f)\,df$$

and for $X(f) = G(f)$ we obtain

$$\int\int_{-\infty}^{\infty} WD_X(f;f,t)\,dt\,df = \int_{-\infty}^{\infty} |X(f)|^2\,df = \|X(f)\| = E_x$$

where $\|\cdot\|$ is known as the norm. We can also write

$$\int\int_{-\infty}^{\infty} WD_x(f;t,f)\,dt\,df = \int_{-\infty}^{\infty} |X(f)|^2\,df.$$

19.2.17 Time Moments

$$\int\int_{-\infty}^{\infty} t^n WD_{x,g}(t;t,f)\,dt\,df = \int\int\int_{-\infty}^{\infty} t^n x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau\,dt\,df$$

$$= \int\int_{-\infty}^{\infty} t^n x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)\delta(\tau)\,d\tau\,dt = \int_{-\infty}^{\infty} t^n\, x(t)g^*(t)\,dt.$$

Similarly we obtain

$$\int\int_{-\infty}^{\infty} t^n WD_x(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} t^n |x(t)|^2\,dt$$

19.2.18 Frequency Moments

$$\int\int_{-\infty}^{\infty} f^n WD_{X,G}(f;f,t)\,dt\,df = \int\int\int_{-\infty}^{\infty} f^n X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)e^{j2\pi t\nu}\,d\nu\,dt\,df$$

$$= \int\int_{-\infty}^{\infty} f^n X\left(f+\frac{\nu}{2}\right)G^*\left(f-\frac{\nu}{2}\right)\delta(\nu)\,d\nu\,df = \int_{-\infty}^{\infty} f^n X(f)G^*(f)\,df.$$

Because of 19.1.2 we also write

$$\int\int_{-\infty}^{\infty} f^n WD_{x,g}(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} f^n X(f)G^*(f)\,df$$

similarly we write

$$\int\int_{-\infty}^{\infty} f^n WD_x(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} f^n |X(f)|^2\,df$$

19.2.19 Scale Covariance

If $y_1(t) = \sqrt{|a|}\; x(at)$ and $y_2(t) = \sqrt{|a|}\; g(at)$, then

$$WD_{y_1,y_2}(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\sqrt{|a|}\; x\left(at+\frac{a\tau}{2}\right)\sqrt{|a|}\; g^*\left(at-\frac{a\tau}{2}\right)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi\frac{f}{a}r}x\left(at+\frac{r}{2}\right)g^*\left(at-\frac{r}{2}\right)dr = WD_{x,g}(t;at,f/a)$$

19.2.20 Convolution Covariance

If $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$, then

$$WD_y(t;t,f) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h\left(t + \frac{t'}{2} - \alpha\right) x(\alpha)\, d\alpha \right] \left[\int_{-\infty}^{\infty} h^*\left(t - \frac{t'}{2} - \gamma\right) x^*(\gamma)\, d\gamma \right] e^{-j2\pi f t'}\, dt'.$$

Set $\alpha = \tau + p/2$, $\gamma = \tau - p/2$ and $t' = q + p$ in the above equation. Hence we obtain

$$WD_y(t;t,f) = \iiint_{-\infty}^{\infty} h\left((t-\tau) + \frac{q}{2}\right) h^*\left((t-\tau) - \frac{q}{2}\right) x(\tau + \tfrac{p}{2})\, x^*\left(\tau - \frac{p}{2}\right) e^{-j2\pi f(q+p)}\, dq\, d\tau\, dp$$

$$= \int_{-\infty}^{\infty} WD_h(t;t-\tau,f)\, WD_x(t;\tau,f)\, d\tau$$

The above indicates that convolution of two signals in the time domain produces a WD that is their convolution of their WD's. If

$$x_c(t) = x(t) * h_1(t) = \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau)\, d\tau$$

and

$$g_c(t) = g(t) * h_2(t) = \int_{-\infty}^{\infty} g(\tau) h_2(t-\tau)\, d\tau,$$

then

$$WD_{x_c,g_c}(t;t,f) = \int_{-\infty}^{\infty} WD_{x,g}(\tau;\tau,f)\, WD_{h_1,h_2}(t;t-\tau,f)\, d\tau$$

19.2.21 Modulation Covariance

If $y(t) = h(t)x(t)$, then

$$\mathcal{F}\{y(t)\} = Y(f) = \mathcal{F}\{h(t)x(t)\} = \int_{-\infty}^{\infty} H(f - f')\, X(f')\, df'.$$

Hence from 19.2.20 and 19.1.2 we obtain

$$WD_Y(f;f,t) = WD_y(t;t,f) = \int_{-\infty}^{\infty} WD_h(t;t,f-f')\, WD_x(t;t,f')\, df'$$

The above indicates that WD of the product of two functions is equal to the convolution of their WD's in the frequency domain. If $x_m(t) = x(t)m_1(t)$ and $g_m(t) = g(t)m_2(t)$, then

$$WD_{x_m,g_m}(t;t,f) = \int_{-\infty}^{\infty} WD_{x,g}(t;t,\eta) WD_{m_1,m_2}(t;t,f-\eta)\, d\eta.$$

19.2.22 Finite Time Support

$WD_{x,g}(t;t,f) = 0$ for $t \notin (t_1, t_2)$ if $x(t)$ or $g(t)$ is zero in $t \notin (t_1, t_2)$.

19.2.23 Finite Frequency Support

$WD_{x,g}(t;t,f) = 0$ for $f \notin (f_1, f_2)$ if $X(f)$ and $G(f)$ are zero for $f \notin (f_1, f_2)$.

19.2.24 Instantaneous Frequency

$$\int_{-\infty}^{\infty} f WD_x(t;t,f)\,df \Bigg/ \int_{-\infty}^{\infty} WD_x(t;t,f)\,df = \frac{1}{2\pi}\frac{d\arg\{x(t)\}}{dt}$$

Proof: Write $x(t)$ in its polar form $x(t) = A(t)e^{j2\pi\phi(t)}$ where $A(t) > 0$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f WD_x(t;t,f)\,df &= \int_{-\infty}^{\infty} f \int_{-\infty}^{\infty} A\left(t+\frac{\tau}{2}\right)e^{j2\pi\phi(t+\frac{\tau}{2})} A\left(t-\frac{\tau}{2}\right)e^{-j2\pi\phi(t-\frac{\tau}{2})} e^{-j2\pi f\tau}\, d\tau\, df \\
&= \int_{-\infty}^{\infty} A\left(t+\frac{\tau}{2}\right) A\left(t-\frac{\tau}{2}\right) e^{-j2\pi[\phi(t-\frac{\tau}{2})-\phi(t+\frac{\tau}{2})]} \left[\int_{-\infty}^{\infty} f e^{-j2\pi f\tau}\, df\right] d\tau \\
&= \int_{-\infty}^{\infty} A\left(t+\frac{\tau}{2}\right) A\left(t-\frac{\tau}{2}\right) e^{-j2\pi[\phi(t-\frac{\tau}{2})-\phi(t+\frac{\tau}{2})]} \frac{1}{j2\pi}\frac{\partial}{\partial\tau}\delta(\tau)\, d\tau \\
&= \frac{1}{j2\pi}\frac{\partial}{\partial\tau}\left[A\left(t+\frac{\tau}{2}\right) A\left(t-\frac{\tau}{2}\right) e^{-j2\pi[\phi(t-\frac{\tau}{2})-\phi(t+\frac{\tau}{2})]}\right]\Bigg|_{\tau=0} \\
&= \frac{1}{j2\pi}\left[\frac{\dot{A}(t)A(t)}{2} - \frac{A(t)\dot{A}(t)}{2} + A^2(t)\, j2\pi\dot{\phi}(t)\right] = A^2(t)\dot{\phi}(t)
\end{aligned}$$

From 19.2.13 $\int_{-\infty}^{\infty} WD_x(t;t,f)\,df = |x(t)|^2 = A^2(t)$ and the assertion above is shown to be correct.

19.2.25 Group Delay

$$\int_{-\infty}^{\infty} t\, WD_x(t;t,f)\,dt \Bigg/ \int_{-\infty}^{\infty} WD_x(t;t,f)\,dt = -\frac{1}{2\pi}\frac{d}{df}\arg\{X(f)\}$$

This property is the dual of property 19.2.24. The proof is similar to that of 19.2.24 except that the signal's Fourier transform is expressed in polar form, and the frequency-domain formulation of the WD 19.1.2 is used.

19.2.26 Fourier Transform of X(t)

$y(t) = X(t)$ where $\mathcal{F}\{y(t)\} = \mathcal{F}\{X(t)\} = x(-f)$ then

$$WD_y(t;t,f) = \int_{-\infty}^{\infty} X\left(t+\frac{\tau}{2}\right)X^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} X\left(t+\frac{\nu}{2}\right)X^*\left(t-\frac{\nu}{2}\right)e^{j2\pi(-f)\nu}\,d\nu = WD_x(t;-f,t)$$

where from 19.1.2 we see that *t* substitutes *f* and *f* substitutes *t*.

Example

If $x(t) = p_a(t)$ is a pulse with a 2a width centered at the origin, its WD is:

$$WD_x(t;t,f) = \frac{\sin[4\pi(a-|t|)f]}{\pi f}p_a(t).$$

Therefore if $y(t) = X(t) = \dfrac{\sin 2\pi at}{\pi t}$ implies that

$$WD_y(t;t,f) = WD_x(t;-f,t) = \frac{\sin[4\pi(a-|-f|)t]}{\pi t}p_a(f).$$

19.2.27 Frequency Localization

If $X(f) = \delta(f-f_0)$, then from 19.1.2 we obtain

$$WD_X(f;f,t) = \int_{-\infty}^{\infty} e^{j2\pi t\nu}\delta\left(f+\frac{\nu}{2}-f_0\right)\delta\left(f-\frac{\nu}{2}-f_0\right)d\nu$$

$$= 2\int_{-\infty}^{\infty} e^{j4\pi tw}\delta(t+w-f_0)\delta(f-w-f_0)\,dw = e^{j4\pi t(f-f_0)}\delta[2(f-f_0)] = \delta(f-f_0)$$

19.2.28 Time Localization

If $x(t) = \delta(t-t_0)$, then from 19.1.2 we obtain

$$WD_x(t;t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau}\,\delta\left(t+\frac{\tau}{2}-t_0\right)\delta\left(t-\frac{\tau}{2}-t_0\right)dt = \delta(t-t_0)$$

by following steps similar to those in 19.2.27.

19.2.29 Linear Chirp Localization

If $X(f) = e^{-j\pi cf^2}$ then from 19.1.5 we obtain

$$WD_X(f;f,t) = \int_{-\infty}^{\infty} e^{j2\pi t\nu}e^{-j\pi c(f+\frac{\nu}{2})^2}e^{-j\pi c(f-\frac{\nu}{2})^2}\,d\nu = \int_{-\infty}^{\infty} e^{j2\pi\nu(t-cf)}\,d\nu = \delta(t-cf)$$

19.2.30 Chirp Convolution

If $y(t) = \int_{-\infty}^{\infty} x(t-\tau)\sqrt{|c|}\;e^{+j\pi c\tau^2}\,d\tau$, then (see FT property) $\mathcal{F}\{y(t)\} = X(f)\sqrt{j}\;e^{-j\pi f^2/c}$ and from 19.1.5 we obtain

$$WD_Y(f;f,t)=\int_{-\infty}^{\infty}X\left(f+\frac{\nu}{2}\right)\sqrt{j}\;e^{-j\pi(f+\frac{\nu}{2})^2/c}\,X^*\left(f-\frac{\nu}{2}\right)\sqrt{j}\;e^{-j\pi(f-\frac{\nu}{2})^2/c}\,e^{j2\pi t\nu}\,d\nu$$

$$=\int_{-\infty}^{\infty}X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)e^{j2\pi\nu(t-\frac{f}{c})}\,d\nu=WD_x\left(t;t-\frac{f}{c},f\right)$$

19.2.31 Chirp Multiplication

If $y(t)=x(t)e^{j\pi ct^2}$, then $WD_y(t;t,f)=WD_x(t;t,f-ct)$.

Proof: If $z(t)=e^{j\pi ct^2}$, then

$$WD_z(t;t,f)=\int_{-\infty}^{\infty}e^{+j\pi c(t+\frac{\tau}{2})^2}e^{-j\pi c(t-\frac{\tau}{2})^2}e^{-j2\pi f\tau}\,d\tau=\int_{-\infty}^{\infty}e^{-j2\pi(f-ct)\tau}\,d\tau=\delta(f-ct).$$

Hence, from 19.2.21 we obtain $WD_y(t;t,f)=\int_{-\infty}^{\infty}WD_y(t;t,f')\delta[(f-f')-ct]df'=WD_x(t;t,f-ct)$.

Example

If $y(t)=p_a(t)e^{j\pi ct^2}$, then

$$WD_y(t;t,f)=WD_x(t;t,f-ct)=\frac{\sin[4\pi(a-|t|)(f-ct)]}{\pi(f-ct)}p_a(t),$$

where $p_a(t)$ is a pulse of 2a width centered at the origin (see 10.1.2). Observe that the WD is a sine function centered along the chirp's linear instantaneous frequency, $f=ct$ in the time-frequency plane.

19.2.32 Moyal's Formula

1. $$\int\int_{-\infty}^{\infty} WD_x(t;t,f)WD_y^*(t;t,f)\,dt\,df=\int\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x\left(t+\frac{u}{2}\right)x^*\left(t-\frac{u}{2}\right)e^{-j2\pi fu}\,du\right]$$

$$\left[\int_{-\infty}^{\infty}y^*\left(t+\frac{t'}{2}\right)y\left(t-\frac{t'}{2}\right)e^{j2\pi ft'}\,dt'\right]dt\,df$$

$$=\iiint_{-\infty}^{\infty}x\left(t+\frac{u}{2}\right)y^*\left(t+\frac{t'}{2}\right)x^*\left(t-\frac{u}{2}\right)y\left(t-\frac{t'}{2}\right)\delta(t'-u)\,du\,dt\,dt'$$

$$=\iint_{-\infty}^{\infty}\left[x\left(t+\frac{u}{2}\right)y^*\left(t+\frac{u}{2}\right)\right]\left[x\left(t-\frac{u}{2}\right)y^*\left(t-\frac{u}{2}\right)\right]^*du\,dt$$

$$=\left|\int_{-\infty}^{\infty}x(t)y^*(t)\,dt\right|^2$$

2. Similarly: $$\iint_{-\infty}^{\infty}WD_{x_1,x_2}(t;t,f)WD^*_{y_1,y_2}(t;t,f)\,dt\,df=\int_{-\infty}^{\infty}x_1(t)y_1^*(t)\,dt\int_{-\infty}^{\infty}x_2(t)y_2^*(t)\,dt$$

3. Similarly: $$\iint_{-\infty}^{\infty}WD_x(t;t,f)WD_x^*(t;t,f)\,dt\,df=\left|\int_{-\infty}^{\infty}|x(t)|^2\,dt\right|^4$$

19.2.33 Energy of the Time Functions in a Time Range $t_1 < t < t_2$

From 19.2.13 we obtain

$$\int_{-t_1}^{t_2}\left[\int_{-\infty}^{\infty}[WD_x(t;t,f)\,df\right]dt=\int_{t_1}^{t_2}|x(t)|^2\,dt$$

19.2.34 Energy of the Function Spectrum in the Range $f_1 < f < f_2$

From 19.2.15 we obtain

$$\int_{f_1}^{f_2}\left[\int_{-\infty}^{\infty}[WD_x(t;t,f)\,dt\right]df=\int_{f_1}^{f_2}|X(f)|^2\,df \quad \text{where } \mathcal{F}\{x(f)\}=X(f)$$

19.2.35 Time-Domain Windowing; the Pseudo-Wigner Distribution (PWD)

If $x_t(\tau)=x(\tau)w_x(\tau-t)$ and $g_t(\tau)=g(\tau)w_g(\tau-t)$, where w_x and w_g and are windows, then from 19.2.2 $WD_{x_t,g_t}(\tau;\tau,f) = \int_{-\infty}^{\infty}WD_{x,g}(\tau;\tau,\eta)WD_{w_x,w_g}(\tau;\tau-t,f-\eta)\,d\eta$ where t appears as a parameter that indicates the position of the window. For $\tau=t$ we obtain

$$WD_{x_t,g_t}(\tau;\tau,f)\Big|_{\tau=t}=\int_{-\infty}^{\infty}WD_{x,g}(t;t,\eta)WD_{w_x,w_g}(t;0,f-\eta)\,d\eta.$$

Hence we can write $\tilde{W}D_{x,g}(t;t,f)=WD_{x_t,g_t}(\tau;\tau,f)\Big|_{\tau=t}$ which is a function of t and f that resembles, but in general is <u>not</u> a WD. This distribution is called <u>pseudo-Wigner distribution</u> (PWD).

19.2.36 Analytic Signals

If $x_a(t)=x(t)+j\hat{x}(t)$ is the analytic signal and $\hat{x}(t)$ is the Hilbert transform of $x(t)$, then

$$X_a(f)=\mathcal{F}\{x_a(t)\}=\begin{cases}2X(f) & f>0\\ X(0) & f=0\\ 0 & f<0\end{cases}$$

From 19.2.23 $WD_{x_a}(t;t,f)=0$ for $\omega<0.$ From 19.1.2 and 19.2.1 and the definition of WD we obtain

$$WD_{X_a}(f;f,t)=\int_{-\infty}^{\infty}X_a\left(f+\frac{\nu}{2}\right)X_a^*\left(f-\frac{\nu}{2}\right)e^{j2\pi\nu t}\,d\nu$$

$$=4\int_{-\infty}^{\infty}e^{j2\pi\nu t}X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)d\nu$$

$$=4\int_{-2f}^{2f}X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)e^{j2\pi\nu t}\,d\nu \text{ for } f>0.$$

From the definition of WD in 19.1.2 we obtain its inverse Fourier transform as follows:

$$\int_{-\infty}^{\infty} e^{-j2\pi\nu t} WD_X(f;f,t)\,dt = X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)$$

and, hence, the above equation becomes

$$WD_{X_a}(f;f,t) = 4\int_{-\infty}^{\infty} WD_X(f;f,t)\int_{-2f}^{2f} e^{j2\pi\nu(t-\tau)}\,d\nu\,d\tau = \frac{4}{\pi}\int_{-\infty}^{\infty} WD_X(f;f,t)\frac{\sin 4\pi f(t-\tau)}{t-\tau}\,d\tau$$

$$= \frac{4}{\pi}\int_{-\infty}^{\infty} WD_X(f;f,t-\tau)\frac{\sin 4\pi f\tau}{\tau}\,d\tau = \frac{4}{\pi}\int_{-\infty}^{\infty} WD_x(t;t-\tau,f)\frac{\sin 4\pi f\tau}{\tau}\,d\tau$$

for f > 0 and 0 for f < 0. The last equation indicates that the WD of the analytic signal at a fixed frequency value f > 0 can be obtained by considering $WD_x(t;t,f)$ for this frequency value as a function of time and passing this time function through an ideal low-pass filter with cut-off frequency 2f. This means that $WD_{x_a}(t;t,f)$ with f fixed is a time function whose highest frequency is at most 2f.

19.3 Instantaneous Frequency and Group Delay

19.3.1 Instantaneous Frequency $f_x(t)$ of $x(t)$

$$f_x(t) = \frac{1}{2\pi}\frac{d}{dt}\arg x(t)$$

19.3.2 Group Delay of Linear Time-Invariant Filters

$$\tau_h(f) = -\frac{1}{2\pi}\frac{d}{dt}\arg H(f)$$

$$H(f) = \mathcal{F}\{h(t)\}, \quad h(t) = \text{impulse response of the filter}$$

19.4 Table of WD Properties

19.4.1 WD Properties and Ideal Time-Frequency Representations

TABLE 19.1 Table of WD Properties and Ideal Properties

Property Name	WD Property	Ideal
1. Conjugation	$WD_{x,g}(t;t,f) = WD^*_{g,x}(t;t,f)$	same
2. Real-Valued	$WD_x(t;t,f) = WD^*_x(t;t,f) \equiv \text{real } (WD_{x,x} \equiv WD_x)$	
3. Even in Frequency	$WD_{x^*}(t;t,f) = WD_x(t;t,-f)$	
(real $x(t)$)	$WD_x(t;t,f) = WD_{x^*}(t;t,-f)$	
4. Time Shift	$WD_{x_s,g_s}(t;t,f) = WD_{x,g}(t;t-t_0,f)$	
	$x_s(t) = x(t-t_0), \quad g_s(t) = g(t-t_0)$	

TABLE 19.1 Table of WD Properties and Ideal Properties (continued)

Property Name	WD Property	Ideal
5. Frequency Shift	$WD_{x_s,g_s}(t;t,f)=WD_{x,g}(t;f-f_0)$ $x_s(t)=x(t)e^{j2\pi f_0 t},\quad g_s(t)=g(t)\,e^{j2\pi f_0 t}$	
6. Time and Frequency Shift	$WD_{x_{sm},g_{sm}}(t;t,f)=WD_{x,g}(t;t-t_0,f-f_0)$ $x_{sm}(t)=x(t-t_0)e^{j2\pi f_0(t-t_0)},$ $g_{sm}(t)=g(t-t_0)e^{j2\pi f_0(t-t_0)}$	
7. Ordinates	$WD_{x,g}(t;0,0)=\int_{-\infty}^{\infty}x\left(\frac{\tau}{2}\right)g^*\left(-\frac{\tau}{2}\right)d\tau$	
8. Sum of Functions	$WD_{x+g}(t;t,f)=WD_x(t;t,f)+WD_g(t;t,f)$ $+2\,\mathrm{Re}\{WD_{x,g}(t;t,f)\}$ $WD_y(t;t,f)=\sum_{i=1}^{N}W_{x_i}(t;t,f)$ $+2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\mathrm{Re}\{W_{x_i x_k}(t;t,f)\}$ $y(t)=\sum_{i=1}^{N}x_i(t),$ N auto terms $+\frac{N(N-1)}{2}$ cross term $WD_y(t;t,f)=\sum_{i=1}^{N}WD_x(t;t-t_i,f-f_i)$ $+2\sum_{i=1}^{N-1}\sum_{m=i+1}^{N}WD_x\left(t;t-\frac{t_i+t_m}{2},f-\frac{f_i+f_m}{2}\right)$ $\times\cos 2\pi\left[(f_i-f_m)t-(t_i-t_m)f+\frac{f_i+f_m}{2}(t_i-t_m)\right]$	
9. Multiplication by t	$WD_{tx,g}(t;t,f)+WD_{x,tg}(t;t,f)=2t\,WD_{x,g}(t;t,f)$	
10. Multiplication by f	$WD_{fx,g}(t;t,f)+WD_{x,fg}(t;t,f)=2f\,W_{x,g}(t;t,f)$	
11. Multiplication by t and f	$WD_{tx,fg}(t;t,f)+WD_{fx,tg}(t;t,f)=2f\,t\,WD_{x,g}(t;t,f)$	
12. Fourier Transform	$\mathcal{F}^{-1}\{WD_{x,g}(t;t,f)\}=x\left(t+\frac{\tau}{2}\right)g^*\left(t-\frac{\tau}{2}\right)$	
13. Time Marginal	$\int_{-\infty}^{\infty}WD_{x,g}(t;t,f)\,df=x(t)g^*(t)$ $\int_{-\infty}^{\infty}WD_x(t;t,f)\,df=\lvert x(t)\rvert^2\quad (x(t)=g(t))$	
14. Recovery of Time Function	$\mathcal{F}^{-1}\left\{WD_{x,g}\left(t;\frac{t}{2},f\right)\right\}=x(t)g^*(0)$ $\mathcal{F}^{-1}\left\{WD_x\left(t;\frac{t}{2},f\right)\right\}=x(t)x^*(0)$	
15. Frequency Marginal	$\int_{-\infty}^{\infty}WD_{X,G}(f;t,f)dt=X(f)G^*(f)$ $\int_{-\infty}^{\infty}W_X(f;t,f)\,dt=\lvert X(f)\rvert^2=\int_{-\infty}^{\infty}WD_x(f;t,f)\,dt.$	

TABLE 19.1 Table of WD Properties and Ideal Properties (continued)

Property Name	WD Property	Ideal
16. Total Energy	$\iint_{-\infty}^{\infty} WD_{X,G}(f;f,t)\,dt\,df = \int_{-\infty}^{\infty} X(f)G^*(f)\,df$ $\iint_{-\infty}^{\infty} WD_X(f;f,t)\,dt\,df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$ $\iint_{-\infty}^{\infty} WD_x(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$	
17. Time Moments	$\iint_{-\infty}^{\infty} t^n WD_{x,g}(t;t,f)\,dt\,df = \iint_{-\infty}^{\infty} t^n x(t)g^*(t)\,dt$ $\iint_{-\infty}^{\infty} t^n WD_x(t;t,f)\,dt\,df = \iint_{-\infty}^{\infty} t^n \lvert x(t)\rvert^2\,dt$	
18. Frequency Moments	$\iint_{-\infty}^{\infty} f^n WD_{X,G}(f;f,t)\,dt\,df = \int_{-\infty}^{\infty} f^n X(f)G^*(f)\,df$ $\iint_{-\infty}^{\infty} f^n WD_{x,g}(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} f^n X(f)G^*(f)\,df$ $\iint_{-\infty}^{\infty} f^n WD_x(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} f^n \lvert X(f)\rvert^2\,df$	
19. Scale Covariance	$WD_{y_1,y_2}(t;t,f) = WD_{x,g}(t;at,f/a)$ $y_1(t) = \sqrt{\lvert a\rvert}\;x(at), \quad y_2(t) = \sqrt{\lvert a\rvert}\;g(at)$	
20. Convolution Covariance	$WD_y(t;t,f) = \int_{-\infty}^{\infty} WD_h(t;t-\tau,f)\,WD_x(t;\tau,f)\,d\tau$ $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$ $WD_{x_c,g_c}(t;t,f) =$ $\int_{-\infty}^{\infty} WD_{x,g}(\tau;\tau,f)\,WD_{h_1,h_2}(t;t-\tau,f)\,d\tau$ $x_c(t) = \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau,$ $g_c(t) = \int_{-\infty}^{\infty} g(\tau)h_2(t-\tau)d\tau$	
21. Modulation Covariance	$WD_Y(f;f,t) = WD_y(t;t,f)$ $= \int_{-\infty}^{\infty} WD_h(t;t,f-f')\,WD_x(t;t,f')\,df'$ $Y(f) = \mathcal{F}\{h(t)x(t)\} = \int_{-\infty}^{\infty} H(f-f')X(f')\,df'$	

TABLE 19.1 Table of WD Properties and Ideal Properties (continued)

Property Name	WD Property	Ideal
	$WD_{x_m,g_m}(t;t,f) = \int_{-\infty}^{\infty} WD_{x,g}(t;t,\eta)\, WD_{m_1,m_2}(t;t,f-\eta)\,d\eta$	
22. Finite Time Support	$WD_{x,g}(t;t,f) = 0$ for $t \notin (t_1,t_2)$ if $x(t)$ or $g(t)$ is zero in $t \notin (t_1,t_2)$	
23. Finite Frequency Support	$WD_{x,g}(t;t,f) = 0$ for $f \notin (f_1,f_2)$ if $X(f)$ and $G(f)$ are zero for $f \notin (f_1,f_2)$	
24. Instantaneous Frequency	$\dfrac{\int_{-\infty}^{\infty} f\,WD_x(t;t,f)\,df}{\int_{-\infty}^{\infty} WD_x(t;t,f)\,df} = \dfrac{1}{2\pi}\dfrac{d}{dt}\arg\{x(t)\}$	
25. Group Delay	$\dfrac{\int_{-\infty}^{\infty} t\,WD_x(t;t,f)\,dt}{\int_{-\infty}^{\infty} WD_x(t;t,f)\,dt} = -\dfrac{1}{2\pi}\dfrac{d}{df}\arg\{X(f)\}$	
26. Fourier Transform of $X(t)$	$WD_y(t;t,f) = WD_x(t;-f,t)$ $y(t) = X(t), \quad \mathcal{F}\{y(t)\} = x(-f)$	
27. Frequency Localization	$WD_X(f;f,t) = \delta(f-f_0)$ $X(f) = \delta(f-f_0)$	
28. Time Localization	$WD_x(t;t,f) = \delta(t-t_0)$ $x(t) = \delta(t-t_0)$	
29. Linear Chirp Localization	$WD_X(f;f,t) = \delta(t-cf)$ $X(f) = e^{-j\pi cf^2}$	
30. Chirp Convolution	$WD_Y(f;f,t) = WD_x\left(t;t-\dfrac{f}{c},f\right)$ $y(t) = \int_{-\infty}^{\infty} x(t-\tau)\sqrt{\lvert c\rvert}\, e^{j\pi c\tau^2}\,d\tau$ $\mathcal{F}\{y(t)\} = X(f)\sqrt{j}\, e^{-j\pi f^2/c}$	
31. Chirp Multiplication	$WD_y(t;t,f) = WD_x(t;t,f-ct)$ $y(t) = x(t)e^{j\pi ct^2}$	
32. Moyal's Formula	$\iint_{-\infty}^{\infty} WD_x(t;t,f)\,WD_y^*(t;t,f)\,dt\,df = \left\lvert \int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt \right\rvert^2$ $\iint_{-\infty}^{\infty} WD_{x_1,x_2}(t;t,f)\,WD_{y_1,y_2}(t;t,f)\,dt\,df = \int_{-\infty}^{\infty} x_1(t)y_1^*(t)\,dt \int_{-\infty}^{\infty} x_2(t)y_2^*(t)\,dt$	

TABLE 19.1 Table of WD Properties and Ideal Properties (continued)

Property Name	WD Property	Ideal
	$\iint_{-\infty}^{\infty} WD_x(t;t,f)\,WD_x^*(t;t,f)\,dt\,df = \left\| \int_{-\infty}^{\infty} \|x(t)\|^2\,dt \right\|^4$	
33. Energy of the Time Functions in a Time Range $t_1 < t < t_2$	$\int_{t_1}^{t_2} \left[\int_{-\infty}^{\infty} [WD_x(t;t,f)\,df \right] dt = \int_{t_1}^{t_2} \|x(t)\|^2\,dt$	
34. Energy of the Function Spectrum in the Range $f_1 < f < f_2$	$\int_{f_1}^{f_2} \left[\int_{-\infty}^{\infty} [WD_x(t;t,f)\,dt \right] df = \int_{f_1}^{f_2} \|X(f)\|^2\,df$ $\mathcal{F}\{x(t)\} = X(f)$	
35. Positivity		$W_x(t;t,f) \ge 0$ $-\infty < t < \infty$ $-\infty < f < \infty$
36. Time-Domain Windowing; the Pseudo-Wigner Distribution (PWD)	$\widetilde{WD}_{x,g}(t;t,f) = W_{x_t,g_t}(\tau;\tau,f)\big\|_{\tau=t} = PWD$ $WD_{x_t,g_t}(\tau;\tau,f) =$ $\int_{-\infty}^{\infty} W_{x,g}(\tau;\tau,\eta)\,WD_{w_x,w_g}(\tau;\tau-t,f-\eta)\,d\eta$ $x_t(\tau) = x(\tau)\,w_x(\tau-t), \quad g_t(\tau) = g(\tau)\,w_g(\tau-t)$	
37. Analytic Signals	$WD_{X_a}(f;f,t) =$ $\frac{4}{\pi}\int_{-\infty}^{\infty} WD_X(t;t-\tau,f)\frac{\sin 4\pi f\tau}{\tau}\,d\tau, \quad f>0$ $X_a(f) = \mathcal{F}\{x_a(t)\} = \begin{cases} 2X(f) & f>0 \\ X(0) & f=0 \\ 0 & f<0 \end{cases}$	
38. Hyperbolic Shift		$T_y(t;t,f) = T_x\left(t;t-\frac{c}{f},f\right)$ $Y(f) = \exp\left(-j2\pi c \ln\frac{f}{f_r}\right)X(f)$ for $f > 0$
39. Hyperbolic Localization		$T_x(t;t,f) = \frac{1}{f}\delta\left(t-\frac{c}{f}\right), \quad f>0$ $X_c(f) = \frac{1}{\sqrt{f}}e^{-j2\pi c \ln\frac{f}{f_r}}, \quad f>0$

19.5 Tables of Wigner Distribution and Ambiguity Function

19.5.1 Table Signals with Closed-Form Wigner Distributions (WD) and Ambiguity Functions (AF) (See Table 19.2)

TABLE 19.2* Signals with Closed-Form Equations for Their Wigner Distribution and Ambiguity Function

Signal, $x(t)$	Fourier Transform, $X(f)$	Wigner Distribution, $WD_x(t;t,f)$	Ambiguity Function $AF_x(t;\tau,\nu)$
$\delta(t-t_i)$	$e^{-j2\pi f t_i}$	$\delta(t-t_i)$	$e^{-j2\pi\nu t_i}\,\delta(\tau)$
$e^{j2\pi f_i t}$	$\delta(f-f_i)$	$\delta(f-f_i)$	$e^{j2\pi f_i \tau}\,\delta(\nu)$
$e^{+j\pi\alpha t^2}$	$\frac{1}{\sqrt{-j\alpha}}e^{-j\pi f^2/\alpha}$	$\delta(f-\alpha t)$	$\delta(\nu-\alpha t)$
$\frac{1}{\sqrt{j\alpha}}e^{j\pi t^2/\alpha}$	$e^{-j\pi\alpha f^2}$	$\delta(t-\alpha f)$	$\delta(\tau-\alpha\nu)$
$e^{j\pi(\alpha t^2+2f_i t+c)}$	$\frac{1}{\sqrt{-j\alpha}}e^{j\pi[c-(f-f_i)^2/\alpha]}$	$\delta(f-f_i-\alpha t)$	$\delta(\nu-\alpha\tau)e^{j2\pi f_i\tau}$
$\frac{1}{\sqrt{\sigma}}e^{-\pi(t/\sigma)^2}$	$\sqrt{\sigma}\,e^{-\pi(\sigma f)^2}$	$\sqrt{2}\,e^{-2\pi[(t/\sigma)^2+(\sigma f)^2]}$	$\frac{1}{\sqrt{2}}e^{-(\pi/2)[(\tau/\sigma)^2+(\sigma\nu)^2]}$
$\frac{1}{\sqrt{\sigma}}e^{-\pi(t/\sigma)^2}e^{j\pi\alpha t^2}$	$\frac{1}{\sqrt{\sigma[\sigma^{-2}-j\alpha]}}\exp\left[-\pi f^2\,\frac{\sigma^{-2}+j\alpha}{\sigma^{-4}+j\alpha^2}\right]$	$\sqrt{2}\,e^{-2\pi[(t/\sigma)^2+(\sigma)^2(f-\alpha t)^2]}$	$\frac{1}{\sqrt{2}}e^{-(\pi/2)[(\tau/\sigma)^2+(\sigma)^2(\nu-\alpha\tau)^2]}$
$\frac{1}{\sqrt{\sigma}}e^{-\pi[(t-t_i)/\sigma]^2}e^{j2\pi f_i t}$	$\sqrt{\sigma}e^{-\pi\sigma^2(f-f_i)^2}e^{-j2\pi(f-f_i)t_i}$	$\sqrt{2}\;e^{-2\pi[[(t-t_i)/\sigma]^2+\sigma^2(f-f_i)^2]}$	$\frac{1}{\sqrt{2}}e^{-(\pi/2)[(\tau/\sigma)^2+(\sigma\nu)^2]}\,e^{j2\pi(f_i\tau-t_i\nu)}$
$p_a(t)$	$\frac{\sin(2\pi af)}{\pi f}$	$\frac{\sin[4\pi(a-\|t\|)f]}{\pi f}p_a(t)$	$\frac{\sin[\pi\nu(2a-\|\tau\|)]}{\pi f}p_{2a}(\tau)$
$\frac{\sin(2\pi at)}{\pi t}$	$p_a(f)$	$\frac{\sin[4\pi(a-\|f\|)t]}{\pi t}p_a(t)$	$\frac{\sin[\pi\tau(2a-\|\nu\|)]}{\pi\tau}p_{2a}(\nu)$
$e^{j\pi\alpha t^2}p_a(t)$	$\frac{1}{\sqrt{-j\alpha}}\int e^{-j\frac{\pi}{\alpha}(f-\beta)^2}\,\frac{\sin(2\pi a\beta)}{\pi\beta}d\beta$	$\frac{\sin[4\pi(a-\|t\|)(f-\alpha t)]}{\pi(f-\alpha t)}p_a(t)$	$\frac{\sin[\pi(\nu-\alpha\tau)(2a-\|\tau\|)]}{\pi(\nu-\alpha\tau)}p_{2a}(\tau)$
$u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$	$\frac{\delta(f)}{2}-\frac{j}{2\pi f}$	$\frac{\sin(4\pi ft)}{\pi f}u(t)$	$\left[\frac{\delta(\nu)}{2}-\frac{j}{2\pi\nu}\right]e^{-j\pi\nu\|\tau\|}$
$e^{-\sigma t}u(t)$	$\frac{1}{\sigma+j2\pi f}$	$e^{-2\sigma t}\,\frac{\sin 4\pi ft}{\pi f}u(t)$	$\frac{e^{-(\sigma+j\pi\nu)\|\tau\|}-1}{2\sigma+j2\pi\nu}$
$h_n(t),\;\; n=0,1,\cdots$	$(-j)^n\,h_n(f)$	$2e^{-2\pi(t^2+f^2)}\,L_n(4\pi(t^2+f^2))$	$e^{-\pi(\tau^2+\nu^2)/2}\,L_n(\pi(\tau^2+\nu^2))$

TABLE 19.2* Signals with Closed-Form Equations for Their Wigner Distribution and Ambiguity Function (continued)

Signal, $x(t)$	Fourier Transform, $X(f)$	Wigner Distribution, $WD_x(t;t,f)$	Ambiguity Function $AF_x(t;\tau,\nu)$
$\cos(2\pi f_i t)$	$[\delta(f+f_i)+\delta(f-f_i)]/2$	$[\delta(f+f_i)+\delta(f-f_i)+2\delta(f)\cos(4\pi f_i t)]/4$	$[\delta(\nu+2f_i)+\delta(\nu-2f_i)+2\delta(\nu)\cos(2\pi f_i \tau)]/4$
$\sin(2\pi f_i t)$	$j[\delta(f+f_i)-\delta(f-f_i)]/2$	$[\delta(f+f_i)+\delta(f-f_i)-2\delta(f)\cos(4\pi f_i t)]/4$	$-[\delta(\nu+2f_i)+\delta(\nu-2f_i)-2\delta(\nu)\cos(2\pi f_i \tau)]/4$
$\delta(t-t_i)+\delta(t-t_m)$	$e^{-2\pi f t_i}+e^{-j2\pi f t_m}$	$\delta(t-t_i)+\delta(t-t_m)$ $+2\delta\left(t-\frac{t_i+t_m}{2}\right)\cos(2\pi(t_i-t_m)f)$	$[e^{-2\pi\nu t_i}+e^{-j2\pi\nu t_m}]\delta(\tau)$ $+[e^{-j\pi(t_i+t_m)\nu}[\delta(\tau-(t_i-t_m))+\delta(\tau+(t_i-t_m))]$
$e^{2\pi f_i t}+e^{j2\pi f_m t}$	$\delta(f-f_i)+\delta(f-f_m)$	$\delta(f-f_i)+\delta(f-f_m)$ $+2\delta\left(f-\frac{f_i+f_m}{2}\right)\cos(2\pi(f_i-f_m)t)$	$[e^{2\pi f_i\tau}+e^{j2\pi f_m\tau}]\delta(\nu)$ $+[e^{j\pi(f_i+f_m)\tau}[\delta(\nu-(f_i-f_m))+\delta(\nu+(f_i-f_m))]$
$\sum_k c_k e^{2\pi k f_0 t}$	$\sum_k c_k \delta(f-kf_0)$	$\sum_k \lvert c_k\rvert^2 \delta(f-kf_0)$ $+\sum_k\sum_{m\neq k} c_k c_m^* \delta\left(f-\frac{k+m}{2}f_0\right)e^{j2\pi(k-m)f_0 t}$	$\sum_k \lvert c_k\rvert^2 e^{j2\pi k f_0\tau}\delta(\tau)$ $+\sum_k\sum_{m\neq k} c_k c_m^* e^{j\pi(k+m)f_0\tau}\delta(\nu-(k-m)f_0)$

* $\sigma >; a,\alpha,c\in\Re$, $\operatorname{sgn}(t)\begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases}$, $p_a(t)=\begin{cases}1 & -a<\lvert t\rvert<a\\ 0 & \text{otherwise}\end{cases}$, $h_n(t)=\frac{2^{1/4}}{\sqrt{n!}}e^{-\pi t^2}H_n(2\sqrt{\pi}t)$

$H_n(t)=(-1)^n\exp(t^2/2)\frac{d^n}{dt^n}\exp(-t^2/2)=n^{th}$order Hermite Polynomial, $L_n(t)=\frac{1}{n!}e^t\frac{d^n}{dt^n}(t^n e^{-t})=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-t)^k=n^{th}$order Laguerre polynomials

19.6 Effects of WD and AF on Functions

19.6.1 Effects of WD and AF on Functions (See Table 19.3)

19.7 Other Time-Frequency Representation (TFRs)

19.7.1 Cohen's Class

$$C_x(t;t,f;\Psi_c) = \iint_{-\infty}^{\infty} \varphi_c(t-t',\tau)\, x\left(t'+\frac{\tau}{2}\right) x^*\left(t'-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, dt'\, d\tau$$

$$= \iint_{-\infty}^{\infty} \Phi_c(f-f',\nu)\, X\left(f'+\frac{\nu}{2}\right) X^*\left(f'-\frac{\nu}{2}\right) e^{j2\pi t\nu}\, df'\, d\nu$$

$$= \iint_{-\infty}^{\infty} \psi_c(t-t', f-f')\, WD_x(t;t',f')\, dt'\, df'$$

$$= \iint_{-\infty}^{\infty} \Psi_c(\tau,\nu)\, AF_x(t;\tau,\nu)\, e^{j2\pi(t\nu - f\tau)}\, d\tau\, d\nu$$

$$C_x(t;t,f;\Psi_c) = \iint_{-\infty}^{\infty} \Gamma_c(f-f_1, f-f_2)\, X(f_1)\, X^*(f_2)\, e^{j2\pi t(f-f_2)}\, df_1\, df_2$$

where

$$\varphi_c(t,\tau) = \iint_{-\infty}^{\infty} \Phi_c(f,\nu)\, e^{j2\pi(f\tau+\nu t)}\, df\, d\nu = \int_{-\infty}^{\infty} \Psi_c(\tau,\nu)\, e^{j2\pi\nu t}\, d\nu$$

$$\varphi_c(t,\tau) \leftrightarrow \Psi_c(f,\nu)$$

$$\psi_c(t,f) = \iint_{-\infty}^{\infty} \Psi_c(\tau,\nu)\, e^{j2\pi(\nu t - f\tau)}\, d\tau\, d\nu = \int_{-\infty}^{\infty} \Phi_c(f,\nu)\, e^{j2\pi\nu t}\, d\nu$$

$$\psi_c(t,f) \leftrightarrow \Psi_c(\tau,\nu)$$

$$\Gamma_c(f_1,f_2) = \Phi_c\left(\frac{f_1+f_2}{2}, f_2 - f_1\right)$$

TABLE 19.3 Effects of signal operations of the Wigner distribution and ambiguity function. Here $\sigma > 0,\ a,\ \alpha,\ c \in \Re$, and $\text{sgn}(a)$

Signal, $y(t)$	Fourier Transform, $Y(f)$	Wigner Distribution, $WD_y(t;t,f)$	Ambiguity Function $AF_y(t;\tau,\nu)$
$Ax(t)$	$AX(f)$	$\lvert A\rvert^2\, WD_x(t;t,f)$	$\lvert A\rvert^2\, AF_x(t;\tau,\nu)$
$x(-t)$	$X(-f)$	$WD_x(t;-t,-f)$	$AF_x(t;-\tau,-\nu)$
$\sqrt{\lvert a\rvert}\, x(at)$	$\frac{1}{\sqrt{\lvert a\rvert}} X(f/a)$	$WD_x(t;at,f/a)$	$AF_x(t;a\tau,\nu/a)$
$\sqrt{\lvert a\rvert}\, X(at)$	$\frac{1}{\sqrt{\lvert a\rvert}} x(-f/a)$	$WD_x(t;-f/a,at)$	$AF_x(t;-\nu/a,a\tau)$
$x(t) = \pm x(\pm t)$	$X(f) = \pm X(\pm f)$	$\pm 2AF_x(t;2t,2f)$	$\pm \frac{1}{2} WD_x(t;\tau/2,\nu/2)$
$x^*(t)$	$X^*(-f)$	$WD_x(t;t,-f)$	$AF_x^*(t;\tau,-\nu)$
$x(t-t_i)e^{j2\pi f_i t}$	$X(f-f_i)e^{-j2\pi(f-f_i)t_i}$	$WD_x(t;t-t_i,f-f_i)$	$AF_x(t;\tau,\nu)e^{j2\pi(f_i\tau - t_i\nu)}$
$x(t)h(t)$	$\int X(f')H(f-f')\,df'$	$\int WD_x(t;t,f')WD_h(t;t,f-f')\,df'$	$\int AF_x(t;\tau,\nu')AF_h(t;\tau,\nu-\nu')\,d\nu'$
$\int x(t')h(t-t')\,dt'$	$X(f)H(f)$	$\int WD_x(t;t',f)WD_h(t;t-t',f)\,dt'$	$\int AF_x(t;\tau',\nu)AF_h(t;\tau-\tau',\nu)\,d\tau'$
$x(t)e^{j\pi\alpha t^2}$	$\frac{1}{\sqrt{-j\alpha}}\int X(f-f')e^{-j\pi f'^2/\alpha}\,df'$	$WD_x(t;t,f-\alpha t)$	$AF_x(t;\tau,\nu-\alpha\tau)$
$\int \sqrt{\lvert\alpha\rvert}\, e^{j\pi\alpha u^2} x(t-u)\,du$	$\sqrt{j\,\text{sgn}(\alpha)}\, X(f)e^{-j\pi f^2/\alpha}$	$WD_x(t;t-f/\alpha,f)$	$AF_x(t;\tau-\nu/\alpha,\nu)$

$\sum_{i=0}^{N-1} x\left(t-\left(i-\frac{N-1}{2}\right)T_r\right),$ $T_r > 0$	$X(f)\frac{\sin(\pi T_r N f)}{\sin(\pi T_r f)}$	$\sum_{i=0}^{N-1} WD_x\left(t;t-\left(i-\frac{N-1}{2}\right)T_r, f\right)$ $+2\sum_{i=0}^{N-2}\sum_{m=i+1}^{N-1} WD_x\left(t;t-\left(\frac{(i+m)-(N-1)}{2}\right)T_r, f\right)$ $\times\cos[2\pi T_r(i-m)f]$	$\sum_{i=-N+1}^{N-1} AF_x(t;\tau-nT_r,\nu)\frac{\sin\pi\nu T_r(N-\lvert n\rvert)}{\sin(\pi\nu T_r)}$
$\sum_{i=1}^{N} x(t-t_i)e^{j2\pi f_i t}$	$\sum_{i=1}^{N} X(f-f_i)e^{-j2\pi(f-f_i)t_i}$	$\sum_{i=1}^{N} WD_x(t;t-t_i, f-f_i)$ $+2\sum_{i=1}^{N-1}\sum_{m=i+1}^{N} WD_x\left(t;t-\frac{(t_i+t_m)}{2}, f-\frac{(f_i+f_m)}{2}\right)$ $\times\cos 2\pi\left[(f_i-f_m)t-(t_i-t_m)f+\frac{(f_i+f_m)}{2}(t_i-t_m)\right]$	$AF_x(t;\tau,\nu)\sum_{i=1}^{N} e^{j2\pi(f_i\tau-\nu t_i)}$ $+\sum_{i=1}^{N}\sum_{\substack{m=1\\ m\neq 1}}^{N} AF_x(t;\tau-(t_i-t_m),\nu-(f_i-f_m))$ $\exp\left[j2\pi\left(\frac{f_i+f_m}{2}\right)\tau-\frac{t_i+t_m}{2}\nu+(f_i-f_m)\frac{t_i+t_m}{2}\right]$

19.7.2 Choi-Williams Distribution

The function x is a function of time and $CWD_x(t,f;\sigma)$ means that the distribution is a function of t and f with a parameter σ.

$$CWD_x(t,f;\sigma)=\sqrt{\frac{\sigma}{4\pi}}\iint_{-\infty}^{\infty}\frac{1}{|\tau|}\exp\left(\frac{-\sigma}{4}\left[\frac{t-t'}{\tau}\right]^2\right)x\left(t'+\frac{\tau}{2}\right)x^*\left(t'-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,dt'\,d\tau$$

$$=\sqrt{\frac{\sigma}{4\pi}}\iint_{-\infty}^{\infty}\frac{1}{|\nu|}\exp\left(\frac{-\sigma}{4}\left[\frac{f-f'}{\nu}\right]^2\right)X\left(f'+\frac{\nu}{2}\right)X^*\left(f'-\frac{\nu}{2}\right)e^{j2\pi t\nu}\,df'\,d\nu$$

$$=\sqrt{\frac{\sigma}{4\pi}}\iiint_{-\infty}^{\infty}\frac{1}{|u|}\exp\left(\frac{-\sigma}{4}\left[\frac{t-t'}{u}\right]^2\right)e^{-j2\pi(f-f')u}\,WD_x(t',f')\,du\,dt'\,df'$$

$$=\iint_{-\infty}^{\infty}e^{-(2\pi\tau\nu)^2/\sigma}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$$

$$=\sqrt{\frac{\sigma}{4\pi}}\iint_{-\infty}^{\infty}\frac{1}{|f_1-f_2|}\exp\left(\frac{-\sigma}{4}\left[\frac{f-(f_1+f_2)/2}{f_1-f_2}\right]^2\right)X(f_1)X^*(f_2)e^{j2\pi t(f_1-f_2)}\,df_1\,df_2$$

where $WD_x(t,f)\equiv WD_x(t;t,f)$ and $AF_x(\tau,\nu)=AF_x(t;\tau,\nu)$.

19.7.3 Table of Time-Frequency Representations of Cohen's Class

Cohen's-class TFRs. Here,

$$rect_a(t)=\begin{cases}1, & |t|<|a|\\ 0, & |t|>|a|\end{cases}, \qquad AF_x(\tau,\nu)$$

is the ambiguity function, and $\tilde{\mu}(\tilde{\tau},\tilde{\nu};\alpha,r,\beta,\gamma)=\tilde{\tau}^2(\tilde{\nu}^2)^\alpha+(\tilde{\tau})^\alpha\tilde{\nu}^2+2r(((\tilde{\tau},\tilde{\nu})^\beta)^\gamma)^2$. Functions with lower- and uppercase letters, e.g., $\gamma(t)$ and $\Gamma(f)$, are Fourier transform pairs. The subscript x implies a function of time.

TABLE 19.4

Cohen's-Class Distribution	Formula
Ackroyd	$ACK_x(t,f)=Re\{x^*(t)X(f)e^{j2\pi ft}\}$
Affine-Cohen Subclass	$AC_x(t,f;S_{AC})=\iint\frac{1}{\lvert\tau\rvert}S_{AC}\left(\frac{t-t'}{\tau}\right)x\left(t'+\frac{\tau}{2}\right)x^*\left(t'-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,dt'\,d\tau$
Born-Jordan	$BJD_x(t,f)=\iint\frac{\sin(\pi\tau\nu)}{\pi\tau\nu}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
	$=\int\frac{1}{\tau}\left[\int_{t-\lvert\tau\rvert/2}^{t+\lvert\tau\rvert/2}x\left(t'+\frac{\tau}{2}\right)x^*\left(t'-\frac{\tau}{2}\right)dt'\right]e^{-j2\pi f\tau}\,d\tau$

TABLE 19.4 (continued)

Cohen's-Class Distribution	Formula
Butterworth	$BUD_x(t,f;M,N)=\iint\left(1+\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\nu}{\nu_0}\right)^{2N}\right)^{-1}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
Choi-Williams (Exponential)	$CWD_x(t,f;\sigma)=\iint e^{-(2\pi\tau\nu)^2/\sigma}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
	$=\iint\sqrt{\frac{\sigma}{4\pi}}\frac{1}{\lvert\tau\rvert}\exp\left[-\frac{\sigma}{4}\left(\frac{t-t'}{\tau}\right)^2\right]x\left(t'+\frac{\tau}{2}\right)x^*\left(t'-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,dt'\,d\tau$
Cone Kernel	$CKD_x(t,f)=\iint g(\tau)\lvert\tau\rvert\frac{\sin(\pi\tau\nu)}{\pi\tau\nu}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
Cumulative Attack Spectrum	$CAS_x(t,f)=\left\lvert\int_{-\infty}^{t}x(\tau)e^{-j2\pi f\tau}\,d\tau\right\rvert^2$
Cumulative Decay Spectrum	$CDS_x(t,f)=\left\lvert\int_{t}^{\infty}x(\tau)e^{-j2\pi f\tau}\,d\tau\right\rvert^2$
Generalized Exponential	$GED_x(t,f)=\iint\exp\left[-\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\nu}{\nu_0}\right)^{2N}\right]AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
Generalized Rectangular	$GRD_x(t,f)=\iint rect_1\left(\lvert\tau\rvert^{M/N}\lvert\nu\rvert/\sigma\right)AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
Generalized Wigner	$GWD_x(t,f;\tilde{\alpha})=\int x(t+(\tfrac{1}{2}+\tilde{\alpha})\tau)\,x^*(t-(\tfrac{1}{2}-\tilde{\alpha})\tau)\,e^{-j2\pi f\tau}\,d\tau$
Levin	$LD_x(t,f)=-\frac{d}{dt}\left\lvert\int_{t}^{\infty}x(\tau)e^{-j2\pi f\tau}\,d\tau\right\rvert^2$
Margineau-Hill	$MH_x(t,f)=Re\{x(t)X^*(f)e^{-j2\pi ft}\}$
Multiform Tiltable Kernel	$MT_x(t,f;S)=\iint S\left(\tilde{\mu}\left(\frac{\tau}{\tau_0},\frac{\nu}{\nu_0};\alpha,r,\beta,\gamma\right)^{2\lambda}\right)AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
	$S_{MTED}(\beta)=e^{-\pi\beta},\quad S_{MTBUD}(\beta)=[1+\beta]^{-1}$
Nutall	$ND_x(t,f)=\iint\exp\left\{-\pi\left[\left(\frac{\tau}{\tau_0}\right)^2+\left(\frac{\nu}{\nu_0}\right)^2+2r\left(\frac{\tau\nu}{\tau_0\nu_0}\right)\right]\right\}AF_x(\tau,\nu)e^{j2\pi(t\nu-f\tau)}\,d\tau\,d\nu$
Page	$PD_x(t,f)=2Re\left\{x^*(t)e^{j2\pi tf}\int_{-\infty}^{t}x(\tau)e^{-j2\pi f\tau}\,d\tau\right\}$
Pseudo Wigner	$PWD_x(t,f;\Gamma)=\int x\left(t+\frac{\tau}{2}\right)x^*\left(t'-\frac{\tau}{2}\right)\gamma\left(\frac{\tau}{2}\right)\gamma^*\left(-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau$
	$=\int WD_\gamma(0,f-f')WD_x(t,f')\,df'$

TABLE 19.4 (continued)

Cohen's-Class Distribution	Formula
Reduced Interference	$RID_x(t,f) = \iint \frac{1}{\lvert\tau\rvert} s\left(\frac{t-t'}{\tau}\right) x\left(t'+\frac{\tau}{2}\right) x^*\left(t'-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, dt'\, d\tau$ with $S(\beta)\in\Re,\ S(0)=1,\ \frac{d}{d\beta}S(\beta)\Big\rvert_{\beta=0}=0,\ \{S(\alpha)=0 \text{ for } \lvert\alpha\rvert>\frac{1}{2}\}$
Rihaczek	$RD_x(t,f) = x(t)X^*(f)e^{-j2\pi tf}$
Smoothed Pseudo Wigner	$SPWD_x(t,f;\Gamma,s) = \int s(t-t')PWD_x(t',f;\Gamma)\,dt'$ $= \iint s(t-t')WD_\gamma(0,f-f')WD_x(t',f')\,dt'\,df'$
Spectrogram	$SPEC_x(t,f;\Gamma) = \left\lvert \int x(\tau)\gamma^*(\tau-t)e^{-j2\pi f\tau}\,d\tau \right\rvert^2 = \left\lvert \int X(f')\Gamma^*(f'-f)e^{j2\pi tf'}\,df' \right\rvert^2$
Wigner	$WD_x(t,f) = \int x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)e^{-j2\pi f\tau}\,d\tau = \int X\left(f+\frac{\nu}{2}\right)X^*\left(f-\frac{\nu}{2}\right)e^{j2\pi t\nu}\,d\nu$

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19.8 Kernels of Cohen's Class

19.8.1 Kernels of Cohen's Shift Covariant Class (See Table 19.5)

19.9 Affine and Hyperbolic TFRs

19.9.1 For the affine and hyperbolic classes of TFRs see Mixed Time-Frequency Signal Transformations by G. Faye Boudreaux-Bartels in *The Transforms and Applications Handbook*, edited by Alexander D. Poularikas, CRC Press, 1996.

19.10 Wigner Distribution of Discrete-Time Signals

19.10.1 WD of Discrete-Time Signals x(n) and g(n), ($\omega \in \boldsymbol{R}$, $\boldsymbol{n} \in \boldsymbol{Z}$)

$$WD_{x,g}(n,\omega) \equiv WD_{x,g}(n;n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2k\omega} \qquad \lvert\omega\rvert < \tfrac{\pi}{2T}$$

$$WD_{x,g}(n,\omega) = 2T\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2\omega kT} \qquad \lvert\omega\rvert < \tfrac{\pi}{2T}$$

$$WD_x(n,\omega) \equiv W_{x,x}(n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)x^*(n-k)e^{-j2k\omega} \qquad \lvert\omega\rvert < \tfrac{\pi}{2T}$$

TABLE 19.5 Kernels of Cohen's Shift Covariant Class of Time-Frequency Representations (TFRs) Defined in Table 19.4. (Here, $\tilde{\mu}(\tilde{\tau}, \tilde{\nu}; \alpha, r, \beta, \gamma) = \left((\tilde{\tau})^2((\tilde{\nu})^2)^\alpha + ((\tilde{\tau})^2)^\alpha(\tilde{\nu})^2 + 2r((\tilde{\tau}\tilde{\nu})^\beta)^\gamma\right)$. Function with lower case and upper case letters, e.g., $\gamma(t)$ and $\Gamma(f)$, indicate Fourier transform pairs.)

TFR	$\psi_C(t, f)$	$\Psi_C(\tau, \nu)$	$\varphi_C(t, \tau)$	$\Phi_C(f, \nu)$
AC	$\int \frac{1}{\lvert\tau\rvert} s_{AC}\left(\frac{t}{\tau}\right) e^{-j2\pi f\tau}\, d\tau$	$S_{AC}(\tau\nu)$	$\frac{1}{\lvert\tau\rvert} s_{AC}\left(\frac{t}{\tau}\right)$	$\frac{1}{\lvert\nu\rvert} S_{AC}\left(-\frac{f}{\nu}\right)$
ACK	$2\cos(4\pi tf)$	$\cos(\pi\tau\nu)$	$\frac{\delta(t+\tau/2)+\delta(t-\tau/2)}{2}$	$\frac{\delta(f-\nu/2)+\delta(f+\nu/2)}{2}$
BJD		$\frac{\sin(\pi\tau\nu)}{\pi\tau\nu}$	$\begin{cases} \frac{1}{\lvert\tau\rvert}, & \lvert t/\tau\rvert < 1/2 \\ 0, & \lvert t/\tau\rvert > 1/2 \end{cases}$	$\begin{cases} \frac{1}{\lvert\nu\rvert}, & \lvert f/\nu\rvert < 1/2 \\ 0, & \lvert f/\nu\rvert > 1/2 \end{cases}$
BUD		$\left(1+\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\nu}{\nu_0}\right)^{2N}\right)^{-1}$		
CWD	$\sqrt{\frac{\sigma}{4\pi}} \int \frac{1}{\lvert\beta\rvert} \exp\left[-\frac{\sigma}{4}\left(\frac{t}{\beta}\right)^2\right] e^{-j2\pi f\beta}\, d\beta$	$e^{-(2\pi\tau\nu)^2/\sigma}$	$\sqrt{\frac{\sigma}{4\pi}} \frac{1}{\lvert\tau\rvert} \exp\left[-\frac{\sigma}{4}\left(\frac{t}{\tau}\right)^2\right]$	$\sqrt{\frac{\sigma}{4\pi}} \frac{1}{\lvert\nu\rvert} \exp\left[-\frac{\sigma}{4}\left(\frac{f}{\nu}\right)^2\right]$
CKD		$g(\tau)\lvert\tau\rvert \frac{\sin(\pi\tau\nu)}{\pi\tau\nu}$	$\begin{cases} g(\tau), & \lvert t/\tau\rvert < 1/2 \\ 0, & \lvert t/\tau\rvert > 1/2 \end{cases}$	
CAS		$\left[\frac{1}{2}\delta(\nu) + \frac{1}{j\nu}\right] e^{-j\pi\lvert\tau\rvert\nu}$		
CDS		$\left[\frac{1}{2}\delta(-\nu) - \frac{1}{j\nu}\right] e^{j\pi\lvert\tau\rvert\nu}$		
GED		$\exp\left[-\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\nu}{\nu_0}\right)^{2N}\right]$	$\frac{\nu_0}{2\sqrt{\pi}}\left\lvert\frac{\tau_0}{\tau}\right\rvert^M \exp\left[\frac{-\nu_0^2\tau_0^{2M}t^2}{4\tau^{2M}}\right]$ N = 1 only	$\frac{\tau_0}{2\sqrt{\pi}}\left\lvert\frac{\nu_0}{\nu}\right\rvert^N \exp\left[\frac{-\tau_0^2\nu_0^{2N}f^2}{4\nu^{2N}}\right]$ M = 1 only
GRD		$\begin{cases} 1, & \left\lvert \lvert\tau\rvert^{M/N}\lvert\nu\rvert/\sigma \right\rvert < 1 \\ 0, & \left\lvert \lvert\tau\rvert^{M/N}\lvert\nu\rvert/\sigma \right\rvert > 1 \end{cases}$	$\frac{\sin\left(2\pi\lvert\sigma\rvert t/\lvert\tau\rvert^{M/N}\right)}{\pi t}$	

TABLE 19.5 Kernels of Cohen's Shift Covariant Class of Time-Frequency Representations (TFRs) defined in Table 19.4. (Here, $\tilde{\mu}(\tilde{\tau},\tilde{\nu};\alpha,r,\beta,\gamma) = \left((\tilde{\tau})^2((\tilde{\nu})^2)^{\alpha} + ((\tilde{\tau})^2)^{\alpha}(\tilde{\nu})^2 + 2r((\tilde{\tau}\tilde{\nu})^{\beta})^{\gamma}\right)$. Function with lower case and upper case letters, e.g., $\gamma(t)$ and $\Gamma(f)$, indicate Fourier transform pairs.) (continued)

TFR	$\psi_C(t,f)$	$\Psi_C(\tau,\nu)$	$\varphi_C(t,\tau)$	$\Phi_C(f,\nu)$
GWD	$\frac{1}{\lvert\tilde{\alpha}\rvert}e^{j2\pi t f/\tilde{\alpha}}$	$e^{j2\pi\tilde{\alpha}\tau\nu}$	$\delta(t+\tilde{\alpha}\tau)$	$\delta(f-\tilde{\alpha}\nu)$
LD		$e^{j\pi\lvert\tau\rvert\nu}$	$\delta(t+\lvert\tau\rvert/2)$	
MH	$2\cos(4\pi t f)$	$\cos(\pi\tau\nu)$	$\frac{\delta(t+\tau/2)+\delta(t-\tau/2)}{2}$	$\frac{\delta(f-\nu/2)+\delta(f+\nu/2)}{2}$
MT		$S\left(\tilde{\mu}\left(\frac{\tau}{\tau_0},\frac{\nu}{\nu_0};\alpha,r,\beta,\gamma\right)^{2\lambda}\right)$		
ND		$\exp\left[-\pi\tilde{\mu}\left(\frac{\tau}{\tau_0},\frac{\nu}{\nu_0};0,r,1,1\right)\right]$		
PD		$e^{-j\pi\lvert\tau\rvert\nu}$	$\delta(t-\lvert\tau\rvert/2)$	$\left[\delta\left(f+\frac{\nu}{2}\right)+\delta\left(f-\frac{\nu}{2}\right)+j\frac{\nu}{\pi(f^2-\nu^2/4)}\right]\Big/2$
PWD	$\delta(t)\,WD_{\gamma}(0,f)$	$\gamma(\tau/2)\gamma^*(-\tau/2)$	$\delta(t)\gamma(\tau/2)\gamma^*(-\tau/2)$	$WD_{\gamma}(0,f)$
RGWD	$\frac{1}{\lvert\tilde{\alpha}\rvert}\cos(2\pi t f/\tilde{\alpha})$	$\cos(2\pi\tilde{\alpha}\tau\nu)$	$\frac{\delta(t+\tilde{\alpha}\tau)+\delta(t-\tilde{\alpha}\tau)}{2}$	$\frac{\delta(f-\tilde{\alpha}\nu)+\delta(f+\tilde{\alpha}\nu)}{2}$
RID	$\int\frac{1}{\lvert\beta\rvert}s\left(\frac{t}{\beta}\right)e^{-j2\pi f\beta}d\beta$	$S(\tau\nu)$	$\frac{1}{\lvert\tau\rvert}s\left(\frac{t}{\tau}\right)$	$\frac{1}{\lvert\nu\rvert}S\left(-\frac{f}{\nu}\right)$
		$S(\beta)\in\Re,\ S(0)=1,\ \frac{d}{d\beta}S(\beta)\Big\vert_{\beta=0}=0$	$s(\alpha)=0,\ \lvert\alpha\rvert>\frac{1}{2}$	$s(\alpha)=0,\ \lvert\alpha\rvert>\frac{1}{2}$
RD	$2e^{-j4\pi tf}$	$e^{-j2\pi\tau\nu}$	$\delta(t-\tau/2)$	$\delta(f+\nu/2)$
SPWD	$s(t)\,WD_{\gamma}(0,f)$	$S(\nu)\gamma(\tau/2)\gamma^*(-\tau/2)$	$s(t)\gamma(\tau/2)\gamma^*(-\tau/2)$	$S(\nu)\,WD_{\gamma}(0,f)$
SPEC	$WD_{\gamma}(-t,-f)$	$AF_{\gamma}(-\tau,-\nu)$	$\gamma\left(-t-\frac{\tau}{2}\right)\gamma^*\left(-t+\frac{\tau}{2}\right)$	$\Gamma\left(-f-\frac{\nu}{2}\right)\Gamma^*\left(-f+\frac{\nu}{2}\right)$
WD	$\delta(t)\delta(f)$	1	$\delta(t)$	$\delta(f)$

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19.10.2 WD Using the Signal Spectrums

$$WD_{X,G}(\omega,n) \equiv WD_{X,G}(\omega;\omega,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} X(\omega+\xi)G^*(\omega-\xi)e^{j2n\xi}\,d\xi$$

$$WD_{X,G}(\omega,n) = WD_{x,g}(n,\omega)$$

$$X(\omega) = \mathsf{F}_d\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x(n)e^{-jn2\pi f}$$

$$x(n) \equiv \mathsf{F}_d^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{jn\omega}\,d\omega = \int_{-1/2}^{1/2} X(f)e^{jn2\pi f}\,df$$

19.11 WD Properties Involving Discrete-Time Signals

19.11.1 Periodicity

$WD_{x,g}(n,\omega) = WD_{x,g}(n,\omega+\pi)$ for all (n,ω). Observe that a factor of 2 is added on the exponential $e^{j2k\omega}$ so that the frequency components occur at ω rather than 2ω.

19.11.2 Symmetry

$$WD_{x,g}(n,\omega) = WD^*_{g,x}(n,\omega)$$

$$WD_x(n,\omega) = WD^*_x(n,\omega) = \text{real}$$

$$WD_x(n,\omega) = WD_{x^*}(n,-\omega)$$

19.11.3 Time Shift

$$WD_{x(n-k),g(n-k)}(n,\omega) = WD_{x,g}(n-k,\omega)$$

19.11.4 Modulation by $e^{jn\omega}$

If $y_1(n) = x(n)e^{jn\omega_c}$ and $y_2(n) = g(n)e^{jn\omega_c}$, then

$$WD_{y_1,y_2}(n,\omega) = 2\sum_{k=-\infty}^{\infty} e^{-j2k\omega}\, x(n+k)e^{jn\omega_c}\, e^{jk\omega_c}\, g^*(n-k)e^{-jn\omega_c}\, e^{jk\omega_c}$$

$$= 2\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2k(\omega-\omega_c)} = WD_{x,g}(n,\omega-\omega_c)$$

19.11.5 Inner Product

$$WD_{x,g}(0,0) = 2\sum_{k=-\infty}^{\infty} x(k)g^*(-k)$$

19.11.6 Sum Formula

$$WD_{x+g}(n,\omega) = 2\sum_{k=-\infty}^{\infty}[x(n+k)+g(n+k)][x^*(n-k)+g^*(n-k)]e^{-j2k\omega}$$

$$= 2\sum_{k=-\infty}^{\infty}[x(n+k)x^*(n-k)+x(n+k)g^*(n-k)+g(n+k)x^*(n-k)$$

$$+g(n+k)g^*(n-k)]e^{-j2k\omega}$$

$$= WD_x(n,\omega)+WD_g(n,\omega)+2Re\{WD_{x,g}(n,\omega)\}$$

19.11.7 Multiplication by n

$$2\sum_{k=-\infty}^{\infty}(n+k)x(n+k)g^*(n-k)e^{-j2k\omega}+2\sum_{k=-\infty}^{\infty}x(n+k)(n-k)g^*(n-k)e^{-j2k\omega}$$

$$= 2n\sum_{k=-\infty}^{\infty}x(n+k)g^*(n-k)e^{-j2k\omega}$$

Hence, $2n\,WD_{x,g}(n,\omega) = WD_{nx,g}(n,\omega)+WD_{x,ng}(n,\omega)$

19.11.8 Multiplication by $e^{j2\omega}$

$$2\sum_{k=-\infty}^{\infty}x(n-1+k)g^*(n+1-k)e^{j2k\omega} = 2\sum_{r=-\infty}^{\infty}x(n+r)g^*(n-r)e^{j2r\omega}\,e^{j2\omega} \quad \text{where we set } k-1=r.$$

Hence, $WD_{x(n-1),g(n+1)}(n,\omega) = e^{j2\omega}\,WD_{x,g}(n,\omega).$

19.11.9 Inverse Transform in Time

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{jk\omega}\,WD_{x,g}\left(n,\frac{\omega}{2}\right)d\omega = 2x(n+k)g^*(n-k)$$

where ω was substituted with ω/2. Hence, the WD evaluated at ω/2 can be considered as the Fourier transform of the sequence $2x(n+k)g^*(n-k)$. Set ω/2 = ω in the above equation with $n+k=n_1$ and $n-k=n_2$, we obtain

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j(n_1-n_2)\omega}\,WD_{x,g}\left(\frac{n_1+n_2}{2},\omega\right)d\omega = x(n_1)g^*(n_2)$$

Similarly

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j(n_1-n_2)\omega}\,WD_{x}\left(\frac{n_1+n_2}{2},\omega\right)d\omega = x(n_1)x^*(n_2)$$

19.11.10 Inverse Transform of the Product $x(n)g^*(n)$

Setting $n_1 = n_2 = n$ in 19.11.9

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\omega)\,d\omega = x(n)g^*(n)$$

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x}(n,\omega)\,d\omega = |x(n)|^2$$

The last equation shows that the integral over one period of WD in its frequency variable is equal to the instantaneous signal power.

19.11.11 Recovery

Set $n_1 = 2n$ and $n_2 = 0$ in 19.11.9 we obtain

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j2n\omega}\, WD_{x,g}(n,\omega)\,d\omega = x(2n)g^*(0)$$

Also

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j2n\omega}\, WD_{x}(n,\omega)\,d\omega = x(2n)x^*(0)$$

Set $n_1 = 2n-1$ and $n_2 = 1$ in 19.11.9 we obtain

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j2(n-1)\omega}\, WD_{x,g}(n,\omega)\,d\omega = x(2n-1)g^*(1)$$

19.11.12 Inner Product of Signals

Summing 19.11.10 over n we obtain

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\omega)\,d\omega = \sum_{n=-\infty}^{\infty} x(n)g^*(n)$$

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2} WD_{x}(n,\omega)\,d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

19.11.13 Moyal's Formula

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_{n=-\infty}^{\infty} WD_{x_1,g_1}(n,\omega)WD^*_{x_2,g_2}(n,\omega)\,d\omega$$

$$=\left[\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)\right]\left[\sum_{n=-\infty}^{\infty} g_1(n)g_2^*(n)\right]^* + \left[\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)e^{-jn\pi}\right]\left[\sum_{n=-\infty}^{\infty} g_1(n)g_2^*(n)e^{-jn\pi}\right]^*$$

$$=\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)\,d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\omega)G_2^*(\omega)\,d\omega\right]^*$$

$$+\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega-\pi)\,d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\omega)G_2^*(\omega-\pi)\,d\omega\right]^*$$

19.11.14 Time-Limited Signals

If $x(n)$ and $g(n)$ are time-limited (finite duration signal $x(n)=g(n)=0,\ n_b<n<n_a$, then

$$WD_{x,g}(n,\omega)=0, \qquad n_b<n<n_a$$

19.11.15 Sampled Analog Signals

$$WD_x(n,\omega)=2T\sum_{m=-\infty}^{\infty} x(n+m)x^*(n-m)e^{-j2\omega mT} \qquad |\omega|<\tfrac{\pi}{2T}$$

$$=2T\sum_{r=-\infty}^{\infty} x(2r)x^*[2(n-r)]e^{-j2\omega(2r-n)T}$$

$$=2T\sum_{r=-\infty}^{\infty} x(2r-1)x^*[2(n-r)+1]e^{-j2\omega(2r-1-n)T} \qquad |\omega|<\tfrac{\pi}{2T}$$

Example

If $x(n)=1$ for $|n|<N$ and zero otherwise, then

$$WD_x(n,\omega)=\begin{cases} 2\ \dfrac{\sin[2\omega(N-|n|+\frac{1}{2})]}{\sin\omega} & |n|<N \\ 0\ \ \text{otherwise} & |n|\ge N\end{cases}$$

19.11.16 Multiplication in the Time Domain

If x and g modulate the carriers m_x and m_g, respectively, we obtain: $x_m(n)=x(n)m_x(n)$ and $g_m(n)$ = $g(n)m_g(n)$. The WD is

$$WD_{x_n,g_m}(n,\omega)=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\xi)WD_{m_x,m_g}(n,\omega-\xi)\,d\xi$$

19.11.17 Pseudo-Wigner Distribution (PWD)

$$x_n(\nu) = x(\nu)w_x(\nu - n), \quad g_n(\nu) = g(\nu)w_g(\nu - n)$$

$$PWD_{x_n,g_n}(\nu,\omega) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(\nu,\xi)WD_{w_x,w_g}(\nu - n,\omega - \xi)d\xi$$

The discrete PWD is given by

$$PWD_{x,g}(n,\omega) = PWD_{x_n,g_n}(\nu,\omega)\big|_{\nu=n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\xi)WD_{w_x,w_g}(0,\omega - \xi)d\xi$$

19.11.18 Analytic Signals

If $x_a(n) = x(n) + j\hat{x}(n)$ where $x(n)$ is real and $\hat{x}(n)$ is the discrete Hilbert transform of $x(n)$, defined by

$$\hat{x}(n) = \mathsf{H}_d\{x(n)\} = \sum_{m\neq n} x(m)\frac{\sin^2 \pi(m-n)/2}{\pi(m-n)/2}$$

$$X_a(\omega) = \begin{cases} 2X(\omega) & 0 < \omega < \pi \\ X(\omega) & \omega = 0 \\ 0 & -\pi < \omega < 0 \end{cases}$$

19.11.19 Central Moments

If $\sum_n k(n) = m_0 > 0,$ then the central moments are found by minimization of the expression

$$i(n_0) = \sum_n (n - n_0)^2 k(n) / m_0$$

which gives $n_0 = n_k = \sum_n nk(n) / m_0$ and the minimum is

$$i(n_k) = \sum_n (n - n_k)^2 k(n) / m_0 = \sum_n n^2 k(n) / m_0 - n_k^2$$

19.12 Table of WD of Discrete-Time Function

19.12.1 Table of WD of Discrete-Time Functions

TABLE 19.6 WD Properties of Discrete-Time Signals $WD_{x,g}(n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2k\omega}$

Property Name	Property
1. Periodicity	$WD_{x,g}(n,\omega) = WD_{x,g}(n,\omega+\pi)$

TABLE 19.6 WD Properties of Discrete-Time Signals $WD_{x,g}(n,\omega)=2\sum_{k=-\infty}^{\infty}x(n+k)g^*(n-k)e^{-j2k\omega}$ (continued)

	Property Name	Property
2.	Symmetry	$WD_{x,g}(n,\omega)=WD^*_{g,x}(n,\omega);\; WD_x(n,\omega)=WD^*_x(n,\omega)=\text{real};$
		$WD_x(n,\omega)=WD_{x^*}(n,-\omega)$
3.	Time Shift	$WD_{x(n-k),g(n-k)}(n,\omega)=WD_{x,g}(n-k,\omega)$
4.	Modulation by $e^{jn\omega}$	$WD_{y_1,y_2}(n,\omega)=2\sum_{k=-\infty}^{\infty}x(n+k)g^*(n-k)e^{-j2k(\omega-\omega_c)}=WD_{x,g}(n,\omega-\omega_c)$
		$y_1(n)=x(n)\exp(jn\omega_c),\quad y_2(n)=g(n)\exp(jn\omega_c)$
5.	Inner Product	$WD_{x,g}(0,0)=2\sum_{k=-\infty}^{\infty}x(k)g^*(-k);\qquad WD_x(0,0)=2\sum_{k=-\infty}^{\infty}x(k)x^*(-k)$
6.	Sum Formula	$WD_{x+g}(n,\omega)=WD_x(n,\omega)+WD_g(n,\omega)+2Re\{WD_{x,g}(n,\omega)\}$
7.	Multiplication by n	$2n\,WD_{x,g}(n,\omega)=WD_{nx,g}(n,\omega)+WD_{x,ng}(n,\omega)$
8.	Multiplication by $e^{j2\omega}$	$WD_{x(n-1),g(n+1)}(n,\omega)=e^{j2\omega}\,WD_{x,g}(n,\omega).$
9.	Inverse Transform in Time	$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{jk\omega}WD_{x,g}\left(n,\frac{\omega}{2}\right)d\omega=2x(n+k)g^*(n-k)$
10.	Inverse Transform of the Product	$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j(n_1-n_2)\omega}\,WD_{x,g}\left(\frac{n_1+n_2}{2},\omega\right)d\omega=x(n_1)g^*(n_2)$
		$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}WD_{x,g}(n,\omega)\,d\omega=x(n)g^*(n)$
		$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}WD_x(n,\omega)\,d\omega=\lvert x(n)\rvert^2$
11.	Recovery of signals	$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j2n\omega}\,WD_{x,g}(n,\omega)\,d\omega=x(2n)g^*(0)$
		$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j2n\omega}\,WD_x(n,\omega)\,d\omega=x(2n)x^*(0)$
		$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}e^{j2(n-1)\omega}\,WD_{x,g}(n,\omega)\,d\omega=x(2n-1)g^*(1)$
12.	Inner Product of Signals	$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}WD_{x,g}(n,\omega)\,d\omega=\sum_{n=-\infty}^{\infty}x(n)g^*(n)$
		$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}WD_x(n,\omega)\,d\omega=\sum_{n=-\infty}^{\infty}\lvert x(n)\rvert^2$
13.	Moyal's Formula	$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\sum_{n=-\infty}^{\infty}WD_{x_1,g_1}(n,\omega)\,WD^*_{x_2,g_2}(n,\omega)\,d\omega$

TABLE 19.6 WD Properties of Discrete-Time Signals $WD_{x,g}(n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2k\omega}$ (continued)

Property Name		Property
		$= \left[\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)\right]\left[\sum_{n=-\infty}^{\infty} g_1(n)g_2^*(n)\right]^*$
		$+\left[\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)e^{-jn\pi}\right]\left[\sum_{n=-\infty}^{\infty} g_1(n)g_2^*(n)e^{-jn\pi}\right]^*$
		$= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\omega)G_2^*(\omega)d\omega\right]^*$
		$+\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega-\pi)d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\omega)G_2^*(\omega-\pi)d\omega\right]^*$
14.	Time-Limited Signals	If $x(n) = g(n) = 0$, for $n_b < n < n_a$, then $WD_{x,g}(n,\omega) = 0$, $\quad n_b < n < n_a$
15.	Sampled Analog Signals	$WD_x(n,\omega) = 2T\sum_{m=-\infty}^{\infty} x(n+m)x^*(n-m)e^{-j2\omega mT} \qquad \lvert\omega\rvert < \frac{\pi}{2T}$
		$= 2T\sum_{r=-\infty}^{\infty} x(2r)x^*[2(n-r)]e^{-j2\omega(2r-n)T}$
		$= 2T\sum_{r=-\infty}^{\infty} x(2r-1)x^*[2(n-r)+1]e^{-j2\omega(r-1-n)T} \qquad \lvert\omega\rvert < \frac{\pi}{2T}$
16.	Multiplication in the Time Domain:	If $x_m(n) = x(n)m_x(n)$ and $g_m(n) = g(n)m_g(n)$, then
		$WD_{x_m,g_m}(n,\omega) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\xi)WD_{m_x,m_g}(n,\omega-\xi)d\xi$
17.	Pseudo-Wigner Distribution (PWD):	If $x_n(\nu) = x(\nu)w_x(\nu-n), \quad g_n(\nu) = g(\nu)w_g(\nu-n)$
		$PWD_{x_n,g_n}(\nu,\omega) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(\nu,\xi)WD_{w_x,w_g}(\nu-n,\omega-\xi)d\xi$
		$PWD_{x,g}(n,\omega) \equiv$ discrete $PWD = PWD_{x_n,g_n}(\nu,\omega)\big\rvert_{\nu=n}$
		$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} WD_{x,g}(n,\xi)WD_{w_x,w_g}(0,\omega-\xi)d\xi$
18.	Analytic Signals	If $x_a(n) = x(n) + j\hat{x}(n)$ and $\hat{x}(n) = \mathsf{H}_d\{x(n)\}$ = Hilbert transform =
		$\sum_{m\neq n} x(m)\frac{\sin^2 \pi(m-n)/2}{\pi(m-n)/2}, \qquad X_a(\omega) = \begin{cases} 2X(\omega) & 0<\omega<\pi \\ X(\omega) & \omega = 0 \\ 0 & -\pi<\omega<0 \end{cases}$
19.	Central Moments	If $\sum_n k(n) = m_0 > 0$, then

TABLE 19.6 WD Properties of Discrete-Time Signals $WD_{x,g}(n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)g^*(n-k)e^{-j2k\omega}$ (continued)

Property Name	Property
	$i(n_k) = \text{central moment} = \sum_n (n-n_0)^2\, k(n)/m_0$
	$= \sum_n n^2 k(n)/m_0 - n_k^2$ where
	$n_0 = n_k = \sum_n nk(n)/m_0$

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20

Functions of a Complex Variable

20.1 Basic Concepts

Complex variable: $z = x + jy$

Complex conjugate: $z^* = x - jy$, $(z_1 + z_2)^* = z_1^* + z_2^*$, $(z_1 z_2)^* = z_1^* z_2^*$, $(z_1 - z_2)^* = z_1^* - z_2^*$,

$$(z_1 / z_2)^* = z_1^* / z_2^*,\quad \mathrm{Re}\{z\} = \frac{z + z^*}{2},\quad \mathrm{Im}\{z\} = \frac{z - z^*}{2j}$$

Polynomial: If z_0 is the root of a polynomial equation $a_n z^n + \cdots + a_1 z + a_0 = 0$ then z_0^* is also a root.

Absolute value: $|z|^2 = x^2 + y^2 = zz^*$, $|z| = \sqrt{x^2 + y^2}$, $|z| = |z^*|$, $z^{-1} = z^* / |z|^2$, $z \neq 0$,

$$|z_1 z_2| = |z_1|\,|z_2|,\quad |z_1 / z_2| = |z_1| / |z_2| \quad z_2 \neq 0$$

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$,

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

Lagrange's identity:

$$\left|\sum_{j=1}^{n} a_j b_j\right|^2 = \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 - \sum_{1\le j<k\le n} |a_j b_k^* - a_k b_j^*|^2$$

Polar form:

$$z = x + jy = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \text{ where } \cos\theta = \frac{x}{|z|},\ \sin\theta = \frac{y}{|z|}$$

$$\theta = \arg\{z\} = \{\theta + 2\pi n;\ n = 0, \pm 1, \pm 2, \cdots\} \equiv \arg\{z\} = \theta \bmod 2\pi$$

$$\arg\{z_1 z_2\} = \arg\{z_1\} + \arg\{z_2\} \bmod 2\pi$$

$$\arg\{z_1 / z_2\} = \arg\{z_1\} - \arg\{z_2\} \bmod 2\pi$$

DeMoivre identity: $(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta = e^{jn\theta}$

20.2 Roots of Complex Numbers

20.2.1 Roots of Complex Numbers

If $z^n = a$ then

$$z = a^{1/n} = \sqrt[n]{|a|}\left(\cos\frac{\theta + 2\pi k}{n} + j\sin\frac{\theta + 2\pi k}{n}\right),$$

$$k = 0, 1, 2, \cdots, n-1 \text{ (true also for negative integer)}$$

Example

$$(-1)^{2/3} = [(-1)^2]^{1/3} = 1^{1/3} = \left\{1, \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}\right\}.$$

Roots when *n* and *m* are relatively prime

If $z^{n/m} = a$ then,

$$z = (a^m)^{1/n} = a^{m/n} = (a^{1/n})^m = \sqrt[n]{|a|^m}\left(\cos\frac{m\theta + 2\pi k}{n} + j\sin\frac{m\theta + 2\pi k}{n}\right).$$

20.3 Functions, Continuity, and Analyticity

20.3.1 Continuous Function

A function $W(z)$ is *continuous* at a point $z = \lambda$ of R_z if, for each number $\varepsilon > 0$, however small, there exists another number $\delta > 0$ such that whenever

$$|z - \lambda| < \delta \quad \text{then} \quad |W(z) - W(\lambda)| < \varepsilon$$

20.3.2 Limit of Sum and Difference

If $\lim_{z \to z_0} W_1(z) = a$ and $\lim_{z \to z_0} W_2(z) = b$, then

$$\lim_{z \to z_0} [W_1(z) \pm W_2(z)] = a + b$$

20.3.3 Limit of Product and Ratio

$$\lim_{z \to z_0} W_1(z)W_2(z) = ab, \quad \lim_{z \to z_0} \frac{W_1(z)}{W_2(z)} = \frac{a}{b}$$

Example

Let D be the punctured disc $0 < |z| < 1$, and let $W(z) = z^{*2}$. Then if z_1 and $z_2 \in D$, we obtain $|W(z_1) - W(z_2)| = |z_1^{*2} - z_2^{*2}| = |z_1^* + z_2^*|\,|z_1^* - z_2^*| \le (|z_1| + |z_2|)\ |z_1 - z_2| \le 2|z_1 - z_2|$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $|z_1 - z_2| < \delta$, then $|W(z_1) - W(z_2)| < \varepsilon$. Clearly, choosing $\delta = \varepsilon/2$ we obtain $|W(z_1) - W(z_2)| < 2\varepsilon/2 = \varepsilon$ which shows that $f(z)$ is uniformly continuous.

20.3.4 Analytic Function

A function $W(z)$ is *analytic* at a point z if, for each number $\varepsilon > 0$, however small, there exists another number $\delta > 0$ such that whenever

$$|z - \lambda| < \delta \quad \text{then} \quad \left| \frac{W(z) - W(\lambda)}{z - \lambda} - \frac{dW(\lambda)}{dz} \right| < \varepsilon$$

20.3.5 Cauchy-Riemann Conditions

The function $W(z) = u(x,y) + j\upsilon(x,y)$ is analytic at a point if $\frac{\partial u}{\partial x} = \frac{\partial \upsilon}{\partial y}$ and $\frac{\partial \upsilon}{\partial x} = -\frac{\partial u}{\partial y}$.

20.3.6 Domain of Analyticity

The set of all points where a function is analytic is called the domain of analyticity. A function whose domain of analyticity is the whole complex plane is called <u>entire</u>.

20.3.7 Rules of Differentiation

If F, G, and H are analytic in D, then

$$\frac{d}{dz}(F(z) + G(z)) = \frac{dF(z)}{dz} + \frac{dG(z)}{dz}, \qquad \frac{d}{dz}\left[\frac{F(z)}{G(z)}\right] = \frac{F'(z)G(z) - F(z)G'(z)}{G^2(z)}$$

$$\frac{d}{dz}H(z) = \frac{d}{dz}(F(z)G(z)) = F'(z)G(z) + F(z)G'(z).$$

20.4 Power Series

20.4.1 Power Series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

20.4.2 Convergence of Power Series

If (20.4.1) converges at some point z_1 and diverges at some point z_2, then it converges absolutely for all z such that $|z-z_0| < |z_1 - z_0|$, and diverges for all z such that $|z-z_0| > |z_2 - z_0|$.

20.4.3 Radius of Convergence

$$R = \frac{1}{\lim_{n\to\infty} \sup_{k\ge n} \sqrt[k]{|a_k|}}$$

20.4.4 Cauchy-Hadamard Rule

If $R = 0$, (20.4.1) converges only for $z = z_0$. If $R = \infty$, (20.4.1) converges absolutely for all z. If $0 < R < \infty$, (20.4.1) converges absolutely if $|z-z_0| < R$ and diverges if $|z-z_0| > R$.

20.4.5 Uniform Convergence

If $0 < r < R$ then (20.4.1) converges uniformly in the set $|z-z_0| \le r$.

20.4.6 Representation of a Function

When (20.4.1) converges to a complex number $W(z)$ for each point z in a set S, we say that the series represents the function W in S.

20.4.7 Analyticity of Power Series

In the interior of its circle of convergence, the power series $W(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is an analytic function.

20.4.8 Infinite Differentiable

In the interior of its circle of convergence, a power series is infinitely differentiable.

20.5 Exponential, Trigonometric, and Hyperbolic Functions

20.5.1 Complex Exponential Function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for} \quad z \in C$$

20.5.2 Complex Sine Function

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{for} \quad z \in C \quad (R = \infty)$$

20.5.3 Complex Cosine Function

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \text{for} \quad z \in C \quad (R = \infty)$$

20.5.4 Euler's Formula

$$e^{jz} = \cos z + j\sin z, \quad \cos z = \frac{e^{jz} + e^{-jz}}{2}, \quad \sin z = \frac{e^{jz} - e^{-jz}}{2j}$$

20.5.5 Periodic

A function is periodic in D if there exists a non-zero constant ω, called period, such that $f(z+\omega) = f(z)$.

20.5.6 Trigonometric Functions

$$\tan z = \frac{\sin z}{\cos z} \quad \text{for} \quad z \in C - \left\{\frac{\pi}{2} + n\pi : n \in Z\right\}; \qquad \cot z = \frac{\cos z}{\sin z} \quad \text{for} \quad z \in C - \{n\pi : n \in Z\}$$

$$\sec z = \frac{1}{\cos z} \quad \text{for} \quad z \in C - \left\{\frac{\pi}{2} + n\pi : n \in Z\right\}; \qquad \csc z = \frac{1}{\sin z} \quad \text{for} \quad z \in C - \{n\pi : n \in Z\}$$

20.5.7 Hyperbolic Functions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2} \quad \text{for} \quad z \in C$$

$$\tanh z = \frac{\sinh z}{\cosh z} \quad \text{for} \quad z \in C - \{(n + \tfrac{1}{2})\pi j : n \in Z\}; \qquad \coth z = \frac{\cosh z}{\sinh z} \quad \text{for} \quad z \in C - \{n\pi j : n \in Z\}$$

$$\mathrm{sec}\, hz = \frac{1}{\cosh z} \quad \text{for} \quad z \in C - \{(n + \tfrac{1}{2})\pi j : n \in Z\}; \qquad \mathrm{csc}\, hz = \frac{1}{\sinh z} \quad \text{for} \quad z \in C - \{n\pi j : n \in Z\}$$

20.5.8 Other Hyperbolic Relations

$\tanh' z = \mathrm{sech}^2 z$; $\coth' z = -\mathrm{csch}^2 z$; $\mathrm{sech}' z = -\mathrm{sech}\, z \tanh z$; $\cosh' z = -\cosh z \coth z$; $\cosh z = \cos jz$; $\sinh z = -j\sin jz$; $\sin(x+jy) = \sin x \cosh y + j\cos x \sinh y$; $\cos(x+jy) = \cos x \cosh y - j\sin x \sinh y$; $|\sin z|^2 = \sin^2 x + \sinh^2 y = -\cos^2 x + \cosh^2 y$; $|\cos z|^2 = \cos^2 x + \sinh^2 y = -\sin^2 x + \cosh^2 y$; $|\sin x| \le |\sin z|$; $|\cos x| \le |\cos z|$; $|\sin z| \le \cosh y$ and $|\sin z| \ge |\sinh y|$; $\cos\left(\frac{\pi}{2} - z\right) = \sin z$; $\cos(\pi - z) = -\cos z$; $\tan(\pi + z) = \tan z$;

$\sin\left(\frac{\pi}{2}-z\right)=\cos z$; $\sin(\pi - z) = \sin z$; $\cot\left(\frac{\pi}{2}-z\right)=\tan z$; $\tan z = (\sin 2x + j\sinh 2y)/(\cos 2x + \cosh 2y)$;

$\cosh^2 z - \sinh^2 z = 1$; $\cosh 2z = \cosh^2 z + \sinh^2 z$; $\sinh 2z = 2\sinh z \cosh z$; $\sinh\left(\frac{j\pi}{2}-z\right)= j\cosh z$.

20.6 Complex Logarithm

20.6.1 Definitions

Determine all complex numbers q such that $e^q = z$. Hence if $z \in C - \{0\}$, we define $\ell n z = \{q : e^q = z\}$.

20.6.2 If $z = r(\cos\theta + j\sin\theta)$, then $z = |z|\, e^{j\theta} = e^{\ell n|z| + j\arg z}$ $(\arg z = \theta)$. Also

$$\ell n z = \ell n|z| + j\arg z$$

20.6.3 Principal Value

$$Ln z = \ell n|z| + j\mathrm{Arg} z, \quad -\pi < \mathrm{Arg} z \le \pi$$

20.6.4 Additional Properties

$e^{1nz} = z$; $\ell n e^z = z \bmod 2\pi j$; $\ell n z_1 z_2 = 1nz_1 + \ell n z_2 \bmod 2\pi j$; $\ell n(z_1/z_2) = 1nz_1 - 1nz_2 \bmod 2\pi j$; $\ell n z^n =$ $n\ell n z \bmod 2\pi j$ for all $n \in Z$.

20.6.5 Principal Value

The principal value of $z^a = e^{a\mathrm{Log} z}$ (z^a has many distinct elements).

Example

$j^j = e^{j\log j} = \left\{\exp\left[j\left(j\frac{\pi}{2}+2\pi jk\right)\right] : k \in Z\right\} = \{e^{-\frac{\pi}{2}-2\pi k} : k \in Z\}$. Hence the principal value of j^j is $e^{-\pi/2}$.

20.6.6 Other Relationships of Principal Values in General

$$z^{a_1}z^{a_2} = z^{a_1+a_2};\ (z_1z_2)^a \ne z_1^a z_2^a;\ (z_1/z_2)^2 \ne (z_1^a/z_2^a);\ \mathrm{Log} z^a \ne a\mathrm{Log} z;\ (z^a)^b \ne z^{ab}$$

20.7 Integration

20.7.1 Definition

The sum $\sum_{s=1}^{n} W_s \Delta z_s$ with overall values of s from a to b, and taking the limit $\Delta z_s \to 0$ and $n \to \infty$, we obtain the integral $I = \int_a^b W(z)\,dz$ where the path of integration in the z-plane must be specified.

Example

The integration of $W(z) = 1/z$ over a circle centered at the origin is given by

$$I = \oint \frac{1}{z} dz = \oint \frac{1}{re^{j\theta}} jre^{j\theta} d\theta = \oint j\, d\theta = \int_1 j\, d\theta + \int_2 j\, d\theta = j\pi + j\pi = 2j\pi .$$

where $\oint$ integrates counterclockwise, $\int_1$ integrates from 0 to π, and $\int_2$ integrates from $-\pi$ to 0.

Example

The integration of $W(z) = 1/z^2,\ W(z) = 1/z^3, \cdots, W(z) = 1/z^n$ around a contour encircling the origin is equal to zero.

Example

Find the value of the integral $\int_0^{z_0} z\, dz$ from the point (0,0) to $(2, j4)$.

Solution

Because z is an analytic function along any path, then

$$\int_0^{z_0} z\, dz = \left. \frac{z^2}{2} \right|_0^{2+j4} = -6 + j8$$

Equivalently, we could write

$$\int_0^{z_0} z\, dz = \int_0^2 x\, dx - \int_0^4 y\, dy + j\int_0^4 x\, dy = \left. \frac{x^2}{2} \right|_0^2 - \left. \frac{y^2}{2} \right|_0^4 + jxy\Big|_0^4 = 2 - \frac{16}{2} + j2 \times 4 = -6 + j8$$

20.7.2 Properties of Integration

1. $\int_C [kW(z) + \ell G(z)]dz = k\int_C W(z)dz + \ell \int_C G(z)dz$, k and ℓ are complex numbers

2. $\int_C W(z)dz = -\int_{C'} W(z)dz$, C' has opposite orientation to C

3. $\int_C W(z)dz = \int_{C_1} W(z)dz + \int_{C_2} W(z)dz$, $C = C_1 + C_2$

4. $\left| \int_C W(z)dz \right| \le ML$ if $|f(t)| \le M$ for $a \le t \le b$ and L is the length of the contour C.

5. $\int_C W(z)dz = \int_a^b W(C(t))C'(t)dt = F(b) - F(a)$, $F'(t) = W(C(t))C'(t)$

20.7.3 Cauchy First Integral Theorem

Given a region of the complex plane within which $W(z)$ is analytic and any closed curve that lies entirely within this region, then

$$\oint_C W(z)dz = 0$$

where the contour C is taken counterclockwise.

20.7.4 Corollary 1

If the contour C_2 completely encloses C_1, and if $W(z)$ is analytic in the region between C_1 and C_2, and also on C_1 and C_2, then

$$\oint_{C_1} W(z)\,dz = \oint_{C_2} W(z)\,dz$$

The integration is done in a counterclockwise direction.

20.7.5 Corollary 2

If $W(z)$ has a finite number n of isolated singularities within a region G bounded by a curve C, then

$$\oint_C W(z)\,dz = \sum_{s=1}^{N} \oint_{C_s} W(z)\,dz$$

(see Figure 20.1.)

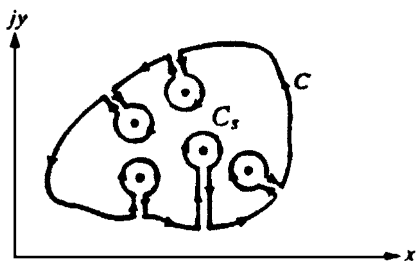

FIGURE 20.1 A countour enclosing n isolated singularities.

20.7.6 Corollary 3

The integral $\int_A^B W(z)\,dz$ depends only upon the end points A and B, and does not depend on the path of integration, provided that this path lies entirely within the region in which $W(z)$ is analytic.

20.7.7 The Cauchy Second Integral Theorem

If $W(z)$ is the function $W(z) = f(z)/(z - z_0)$ and the contour encloses the singularity at z_0, then

$$\oint_C \frac{f(z)}{z - z_0} dz = j2\pi f(z_0)$$

or

$$f(z_0) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0 d_z} dz$$

20.7.8 Derivative of an Analytic Function $W(z)$

The derivative of an analytic function is also analytic, and consequently itself possesses a derivative. Let C be a contour within and upon which $W(z)$ is analytic. Then a is a point inside the contour (the prime indicates first-order derivative)

$$W'(a) = \lim_{|h|\to 0} \frac{W(a+h) - W(a)}{h}$$

and can be shown that

$$W'(a) = \frac{1}{2\pi j} \oint_C \frac{W(z)\,dz}{(z-a)^2}$$

where the contour C is taken in a counterclockwise direction. Proceeding, it can be shown that

$$W^{(n)}(a) = \frac{n!}{2\pi j} \oint_C \frac{W(z)\,dz}{(z-a)^{n+1}}$$

The exponent (n) indicates the nth derivative and the contour is taken counterclockwise.

Example

$$\int_C \frac{\sin z}{(z-\pi)^3} dz = \pi\, j\, W''(\pi) = 0 \text{ where } C \text{ was the circle } |z| = 4.$$

20.7.9 Cauchy's Inequality

If $W(z)$ is analytic in the disk $|z-a| < R$ and if $|W(z)| \le M$ in this disk, then

$$\left|W^{(n)}(a)\right| \le \frac{Mn!}{R^n}$$

(see 20.7.8).

20.7.10 Liouville's Theorem

A bounded entire function $W(z)$ is identically constant (a function whose domain of analyticity is the whole complex plane is called _entire_).

20.7.11 Taylor's Theorem

If $W(z)$ is analytic in the disk $|z - z_0| < R,$ then

$$W(z) = \sum_{n=0}^{\infty} \frac{W^{(n)}(z_0)}{n!}(z - z_0)^n \text{ whenever } |z - z_0| < R$$

Example

The Taylor series of $\ell n z$ around $z_0 = 1$ is found by first identifying its derivatives $\ell n z,\ z^{-1}, -z^{-2}, 2z^{-3},$ $-3 \times 2 z^{-4},$ etc. In general $d^r \ell n z / dz^r = (-1)^{r+1}(r-1)!z^{-r}$ $(r = 1, 2, \cdots).$
Evaluating the derivatives at $z = 1$ we obtain,

$$\ell n z = 0 + (z-1) - \frac{(z-1)^2}{2!} + 2!\frac{(z-1)^3}{3!} - 3!\frac{(z-1)^4}{4!} + \cdots = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}(z-1)^r}{r!}$$

which is valid $|z-1| < 1.$

20.7.12 Maclaurin Series

When $z_0 = 0$ in (20.7.11) the series is known as a Maclaurin series.

20.7.13 Cauchy Product

The Cauchy product of two Taylor series

$$\sum_{i=0}^{\infty} a_i (z-z_0)^i \quad \text{and} \quad \sum_{i=0}^{\infty} b_i (z-z_0)^i$$

is defined to be the series

$$\sum_{i=0}^{\infty} c_i (z-z_0)^i \quad \text{where} \quad c_i = \sum_{i=0}^{k} a_{k-i} b_i .$$

20.7.14 Product of Taylor Series

Let f and g be analytic functions with Taylor series $f(z) = \sum_{i=0}^{\infty} a_i (z-z_0)^i$ and $g(z) = \sum_{i=0}^{\infty} b_i (z-z_0)^i$ around the point z_0. Then the Taylor series for $f(z)g(z)$ around z_0 is given by the Cauchy product of these two series.

Example

$$\sin z \cos z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right) = z - \left(\frac{1}{3!} + \frac{1}{2!} \right) z^3$$

$$+ \left(\frac{1}{5!} + \frac{1}{3!}\frac{1}{2!} + \frac{1}{4!} \right) z^5 - \left(\frac{1}{7!} + \frac{1}{5!}\frac{1}{2!} + \frac{1}{3!}\frac{1}{4!} + \frac{1}{6!} \right) z^7 + \cdots = z - \frac{4}{3!} z^3 + \frac{16}{5!} z^5 - \frac{64}{7!} z^7 + \cdots$$

20.7.15 Taylor Expansions

1. $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for} \quad z_0 = 0$
2. $\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \quad \text{for} \quad z_0 = 0$
3. $\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \quad \text{for} \quad z_0 = 0$
4. $\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{(z-j)^k}{(1-j)^{k+1}} \quad \text{for} \quad z_0 = j$
5. $\ln(1-z) = \sum_{k=1}^{\infty} \frac{-z^k}{k!} \quad \text{for} \quad z_0 = 0$

20.8 The Laurent Expansion

20.8.1 Laurent Theorem

Let C_1 and C_2 be two concentric circles, as shown in Figure 20.2, with their center at a. The function $f(z)$ is analytic with the ring and $(a + h)$ is any point in it. From the figure and Cauchy's theorem we obtain

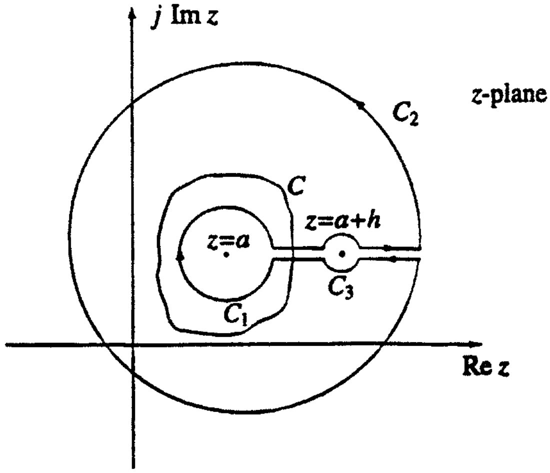

FIGURE 20.2 Explaining Laurent's theorem.

$$\frac{1}{2\pi j}\oint_{C_2}\frac{f(z)dz}{(z-a-h)}+\frac{1}{2\pi j}\oint_{C_1}\frac{f(z)dz}{(z-a-h)}+\frac{1}{2\pi j}\oint_{C_3}\frac{f(z)dz}{(z-a-h)}=0$$

where the first contour is counterclockwise and the last two are clockwise. The above equation becomes

$$f(a+h)=\frac{1}{2\pi j}\oint_{C_2}\frac{f(z)dz}{(z-a-h)}-\frac{1}{2\pi j}\oint_{C_1}\frac{f(z)dz}{(z-a-h)}$$

where both the contours are taken counterclockwise. For the C_2 contour $h<|(z-a)|$ and for the C_1 contour $h>|(z-a)|$. Hence we expand the above integral as follows:

$$f(a+h)=\frac{1}{2\pi j}\oint_{C_2}f(z)\left\{\frac{1}{(z-a)}+\frac{h}{(z-a)^2}+\cdots+\frac{h^n}{(z-a)^{n+1}}+\frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right\}dz$$

$$+\frac{1}{2\pi j}\oint_{C_1}f(z)\left\{\frac{1}{h}+\frac{z-a}{h^2}+\cdots+\frac{(z-a)^{n+1}}{h^{n+1}}+\frac{(z-a)^{n+1}}{h^{n+1}(z-a-h)}\right\}dz$$

From Taylor's theorem it was shown that the integrals of the last term in the two brackets tend to zero as n tends to infinity. Therefore, we have

$$f(a+h)=a_0+a_1h+a_2h^2+\cdots+\frac{b_1}{h}+\frac{b_2}{h^2}+\cdots \tag{20.1}$$

where

$$a_n=\frac{1}{2\pi j}\oint_{C_2}\frac{f(z)dz}{(z-a)^{n+1}}\qquad b_n=\frac{1}{2\pi j}\oint_{C_1}(z-a)^{n-1}f(z)dz$$

The above expansion can be put in more convenient form by substituting $h = z - a$, which gives

$$f(z)=c_0+c_1(z-a)+c_2(z-a)^2+\cdots+\frac{d_1}{(z-a)}+\frac{d_2}{(z-a)^2}+\cdots+\frac{d_n}{(z-a)^n}+\cdots \tag{20.2}$$

Because $z = a + h$, it means that z now is any point within the ring-shaped space between C_1 and C_2 where $f(z)$ is analytic. Equation (20.2) is the Laurent's expansion of $f(z)$ at a point $z + h$ within the ring. The coefficients c_n and d_n are obtained from (20.1) by replacing a_n, b_n, z by c_n, d_n, ζ, respectively. Here ζ is the variable on the contours, and z is inside the ring. When $f(z)$ has a simple pole at $z = a$, there is only one term, namely, $d_1/(z-a)$. If there exists an nth-order term, there are n terms of which the last is $d_n/(z-a)^n$; some of the d_n's may be zero.

If m is the highest index of the inverse power of $f(z)$ in (20.2) it is said that $f(z)$ has a pole of order m at $z = a$. Then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{m} \frac{d_n}{(z-a)^n}$$

The coefficient d_1 is the *residue* at the pole.

If the series in the *inverse powers of (z – a)* in (20.2) does not terminate, the function $f(z)$ is said to have an *essential singularity* at $z = a$. Thus

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{m} \frac{d_n}{(z-a)^n}$$

The coefficient d_1 is the *residue* of the singularity.

Example

Find the Laurent expansion of $f(z) = 1/[(z-a)(z-b)^n]\,(n \geq 1,\ a \neq b \neq 0)$ near each pole.

Solution

First remove the origin to $z = a$ by the transformation $\zeta = (z-a)$. Hence we obtain

$$f(z) = \frac{1}{\zeta}\frac{1}{(\zeta + c)^n} = \frac{1}{c^n \zeta}\frac{1}{\left(1 + \dfrac{\zeta}{c}\right)^n}, \quad c = a - b$$

If $|\zeta/c| < 1$ then we have

$$f(z) = \frac{1}{c^n\zeta}\left[1 - \frac{n\zeta}{c} + \frac{n(n+1)}{2!}\frac{\zeta^2}{c^2} - \cdots\right]$$

$$= \left[-\frac{n}{c^{n+1}} + \frac{n(n+1)\zeta}{2!\,c^{n+2}} - \cdots\right] + \frac{1}{c^n\zeta}$$

which is the Laurent series expansion near the pole at $z = a$. The residue is $1/c^n = 1/(a-b)^n$.

For the second pole set $\zeta = (z - a)$ and expand as above to find

$$f(z) = -\left(\frac{1}{c^{n+1}} + \frac{\zeta}{z^{n+2}} + \frac{\zeta^2}{c^{n+3}} + \cdots\right) - \left(\frac{1}{c^n\zeta} + \frac{\zeta}{c^{n-1}\zeta^2} + \cdots + \frac{1}{c\zeta^n}\right)$$

The second part of the expansion is the principal expansion near $z = b$, and the residue is $-1/c^n = -1/(a-b)^n$.

Example

Prove that

$$f(z) = \exp\left[\frac{x}{2}\left(z - \frac{1}{2}\right)\right] = J_0(x) + zJ_1(x) + z^2 J_2(x) + \cdots + z^n J_n(x) + \cdots$$

$$-\frac{1}{z} J_1(x) + \frac{1}{z^2} J_2(x) - \cdots + \frac{(-1)^n}{z^n} J_n(x) + \cdots$$

where

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

Solution

The function $f(z)$ is analytic except the point $z = a$. Hence by the Laurent's theorem we obtain

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

where

$$a_n = \frac{1}{2\pi j}\oint_{C_2} \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z^{n+1}}, \qquad b_n = \frac{1}{2\pi j}\oint_{C_1} \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right] z^{n-1}\, dz$$

where the contours are circles with center at the origin and are taken counterclockwise. Set C_2 equal to a circle of unit radius and write $z = \exp(j\theta)$. Then we have

$$a_n = \frac{1}{2\pi j}\int_0^{2\pi} e^{x\sin\theta} e^{-jn\theta}\, j d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

because the last integral vanishes, as can be seen by writing $2\pi - \varphi$ for θ. Thus $a_n = J_n(x)$, and $b_n = (-1)^n a_n$, because the function is unaltered if $-z^{-1}$ is substituted for z, so that $b_n = (-1)^n J_n(x)$.

20.9 Zeros and Singularities

20.9.1 Zero of Order m

A point z_0 is a *zero* of order m of $f(z)$ if $f(z)$ is analytic at z_0 and $f(z)$ and its $m - 1$ derivatives vanish at z_0, but $f^{(m)}(z_0) \neq 0$.

20.9.2 Essential Singularity

A function has an *essential singularity* at $z = z_0$ if its Laurent expansion about the point z_0 contains an infinite number of terms in inverse powers of $(z - z_0)$.

20.9.3 Nonessential Singularity (pole of order m)

A function has a *nonessential singularity* or *pole of order m* if its Laurent expansion can be expressed in the form

$$W(z)=\sum_{n=-m}^{\infty} a_n(z-z_0)^n$$

Note that the summation extends from $-m$ to infinity and not from minus infinity to infinity; that is, the highest inverse power of $(z - z_0)$ is m.

An alternative definition that is equivalent to this but somewhat simpler to apply is the following: If $\lim_{z\to z_0}[(z-z_0)^m W(z)] = c,$ a nonzero constant (here m is a positive number), then $W(z)$ is said to possess a pole of order m at z_0. The following examples illustrate these definitions:

Example

1. $\exp(1/z)$ has an essential singularity at the origin.
2. $\cos z/z$ has a pole of order 1 at the origin.
3. Consider the function

$$W(z)=\frac{e^z}{(z-4)^2(z^2+1)}$$

Note that functions of this general type exist frequently in the Laplace inversion integral. Because e^z is regular at all finite points of the z-plane, the singularities of $W(z)$ must occur at the points for which the denominator vanishes; that is, for

$$(z-4)^2(z^2+1)=0 \qquad \text{or} \qquad z=4,+j,-j$$

By the second definition above, it is easily shown that $W(z)$ has a second-order pole at $z = 4$, and first-order poles at the two points $+j$ and $-j$. That is,

$$\lim_{z\to 4}(z-4)^2\left[\frac{e^z}{(z-4)^2(z^2+1)}\right]=\frac{e^4}{17}\neq 0$$

$$\lim_{z\to j}(z-j)\left[\frac{e^z}{(z-4)^2(z^2+1)}\right]=\frac{e^j}{(j-4)^2 2j}\neq 0$$

20.9.4 Picard's Theorem

A junction with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.

Example

The zeros of $\sin(1 - z^{-1})$ are given by $1 - z^{-1} = n\pi$ or at $z = 1/(1 - n\pi)$ for $n=0,\pm 1,\pm 2,\cdots.$ Furthermore, the zeros are simple because the derivative at these points is

$$\frac{d}{dz}\sin(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}}=\frac{1}{z^2}\cos(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}}=(1-n\pi)^2\cos\pi\neq 0$$

The only singularity of $\sin(1 - z^{-1})$ appears at $z = 0$. Since zero is the limit point of the sequence $(1-n\pi)^{-1},\ n=1,2,\cdots,$ we observe that this function has a zero in every neighborhood of the origin. Hence $z=0$ is not a pole. This point is not a removable singularity because $\sin(1 - z^{-1})$ does not approach 0 as $z \to 0$ $(\sin(1 - z_p^{-1}) = 1$ for $z_p = \left(1 - 2p\pi - \frac{\pi}{2}\right)^{-1}$, $p = 1, 2, \cdots)$. Hence by elimination z = 0 is an essential singularity.

20.10 Theory of Residues

20.10.1 Residue

$$\frac{1}{2\pi} \oint_C W(z)\,dz = \text{residue of } W(z) \text{ at } z_0 \text{ singularity} \equiv \text{Res}(W)$$

20.10.2 Theorem

If the $\lim_{z \to z_0} [(z - z_0)W(z)]$ is finite, this limit is the residue of $W(z)$ at $z = z_0$. If the limit is not finite, then $W(z)$ has a pole of at least second order at $z = z_0$ (it may possess an essential singularity here). If the limit is zero, then $W(z)$ is regular at $z = z_0$.

Example

Evaluate the following integral

$$\frac{1}{2\pi j} \oint_C \frac{e^{zt}}{(z^2 + \omega^2)} dz$$

when the contour C encloses both first-order poles at $z = \pm j\omega$. Note that this is precisely the Laplace inversion integral of the function $1/(z^2 + \omega^2)$.

Solution

This involves finding the following residues

$$\text{Res}\left(\frac{e^{zt}}{(z^2 + \omega^2)}\right)_{z = j\omega} = \frac{e^{j\omega t}}{2j\omega} \qquad \text{Res}\left(\frac{e^{zt}}{(z^2 + \omega^2)}\right)_{z = -j\omega} = \frac{e^{-j\omega t}}{2j\omega}$$

Hence,

$$\frac{1}{2\pi j} \oint_C \frac{e^{zt}}{(z^2 + \omega^2)} dz = \sum \text{Res}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j\omega}\right) = \frac{\sin \omega t}{\omega}$$

A slight modification of the method for finding residues of simple poles

$$\text{Res}W(z_0) = \lim_{z \to z_0} [(z - z_0)W(z)]$$

makes the process even simpler. This is specified by the following theorem.

20.10.3 Theorem

Suppose that $f(z)$ is analytic at $z = z_0$ and suppose that $g(z)$ is divisible by $z - z_0$ but not by $(z - z_0)^2$. Then

$$\text{Res}\left[\frac{f(z)}{g(z)}\right]_{z = z_0} = \frac{f(z_0)}{g'(z_0)} \quad \text{where } g'(z) = \frac{dg(z)}{dz}$$

Example

If

$$W(z) = \frac{e^z}{(z-4)^2(z^2+1)}$$

then we take

$$f(z) = \frac{e^z}{(z-4)^2}, \qquad g(z) = z^2+1$$

thus, $g'(z) = 2z$ and the previous result follows immediately with

$$\text{Res}\left[\frac{e^z}{(z-4)^2(z^2+1)}\right] = \frac{e^j}{(j-4)^2 2j}$$

20.10.4 Residue of Pole Order n

If $W(z) = f(z)/(z-z_0)^n$ where $f(z)$ is analytic at $z = z_0$

$$\text{Res}(W(z))\Big|_{z=z_0} = \frac{1}{2\pi j}\oint W(z)\,dz = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n W(z)]\Big|_{z=z_0}$$

20.10.5 Residue with Nonfactorable Denominator

Sometimes the function takes the form

$$W(z) = \frac{f(z)}{zg(z)}$$

where the numerator and denominator are prime to each other, $g(z)$ has not zero at $z = 0$ and cannot be factored readily. The residue due to the pole at zero is given by

$$\text{Res } W(z) = \frac{f(z)}{g(z)}\Big|_{z=0} = \frac{f(0)}{g(0)}$$

If $z = a$ is the zero of $g(z)$, then the residue at $z = a$ is given by

$$\text{Res W(z)} = \frac{f(a)}{ag'(a)}$$

If there are N poles of $g(z)$, then the residues at all simple poles of $W(z)$ is given by

$$\sum \text{Res} = \frac{f(z)}{g(z)}\Big|_{z=0} + \sum_{m=1}^{N}\left[f(z)/z\frac{dg(z)}{dz}\right]_{z=a_m}$$

If $W(z)$ takes the form $W(z) = f(z)/[h(z)g(z)]$ and the simple poles to the two functions are not common, then the residues at all simple poles are given by

$$\sum \text{Res} = \sum_{m=1}^{N} \frac{f(a_m)}{h(a_m)g'(a_m)} + \sum_{r=1}^{R} \frac{f(b_r)}{h'(b_r)g(b_r)}$$

Example

Find the sum of the residues $e^{2z}/\sin mz$ at the first N + 1 pole on the negative axis.

Example

The simple poles occur at $z = -n\pi/m,\ n = 0,1,2,\cdots.$ Thus

$$\sum \text{Res} = \sum_{n=0}^{N} \left[\frac{e^{2z}}{m\cos mz}\right]_{z=-n\pi/m} = \frac{1}{m}\sum_{n=0}^{N} (-1)^n e^{-2n\pi/m}$$

Example

Find the sum of the residues of $e^{2z}/(z\cosh mz)$ at the origin of the first N pole on each side of it.

Solution

The zeros of coshmz are $z = -j(n+1/2)\pi/m,\ n$ integral. Because coshmz has no zero at z = 0, then we obtain

$$\sum \text{Res} = 1 + \sum_{n=-N}^{N-1} \left[\frac{e^{2z}}{mz\sinh mz}\right]_{z=-(n+\frac{1}{2})\pi j/m}$$

20.11 Aids to Complex Integration

20.11.1 Integration of an Arc (R → ∞) Theorem

If AB is the arc of a circle of radius $|z| = R$ for which $\theta_1 \le \theta \le \theta_2$ and if $\lim_{R\to\infty}(zW(z)) = k,$ a constant that may be zero, then

$$\lim_{R\to\infty}\int_{AB} W(z)\,dz = jk(\theta_2 - \theta_1)$$

20.11.2 Integration of an Arc (r → 0); Theorem

If AB is the arc of a circle of radius $|z - z_0| = r$ for which $\varphi_1 \le \varphi \le \varphi_2$ and if $\lim_{z\to z_0}[(z - z_0)W(z)] = k,$ a constant that may be zero, then

$$\lim_{r\to 0}\int_{AB} W(z)\,dz = jk(\varphi_2 - \varphi_1)$$

where r and φ are introduced polar coordinates, with the point $z = z_0$ as origin.

20.11.3 Maximum Value Over a Path, Theorem

If the maximum value of $W(z)$ along a path C (not necessarily closed) is M, the maximum value of the integral of $W(z)$ along C is Ml, where l is the length of C. When expressed analytically, this specifies that

$$\left|\int_C W(z)\,dz\right| \le Ml$$

20.11.4 Jordan's Lemma

If $t < 0$ and

$$f(z) \to 0 \quad \text{as } z \to \infty$$

then

$$\int_C e^{tz} f(z)\,dz \to 0 \quad \text{as } r \to \infty$$

where C is the arc shown in Figure 20.3.

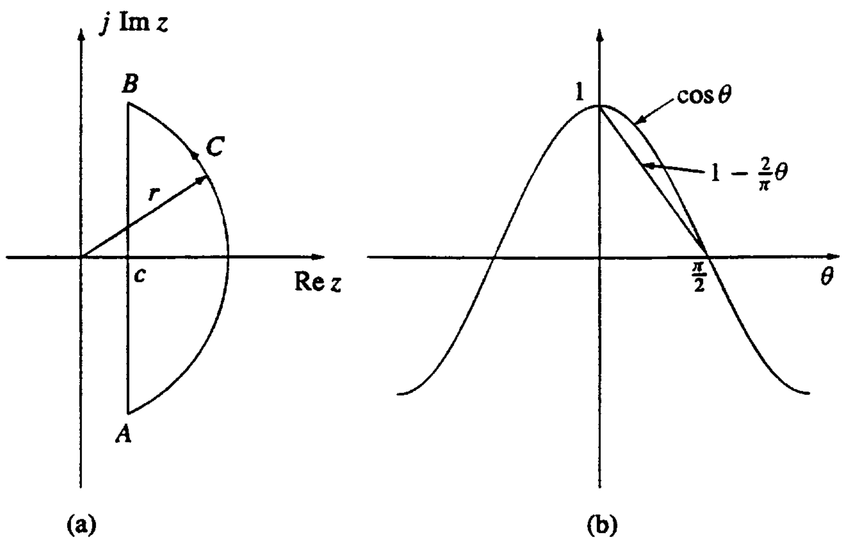

FIGURE 20.3

20.11.5 Theorem (Mellin 1)

Let

a. $\phi(z)$ be analytic in the strip $\alpha < x < \beta$, both alpha and beta being real
b. $\int_{x-j\infty}^{x+j\infty} |\phi(z)|\,dz = \int_{-\infty}^{\infty} |\phi(x+jy)|\,dy$ converges
c. $\phi(z) \to 0$ uniformly as $|y| \to \infty$ in the strip $\alpha < x < \beta$
d. θ = real and positive: If

$$f(\theta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \theta^{-z} \phi(z)\,dz \qquad (20.11.5.1)$$

then

$$\phi(z) = \int_0^{\infty} \theta^{z-1} f(\theta)\,d\theta \qquad (20.11.5.2)$$

20.11.6 Theorem (Mellin 2)

For θ real and positive, $\alpha < \mathrm{Re}\, z < \beta$, let $f(\theta)$ be continuous or piecewise continuous, and integral (20.11.5.2) be absolutely convergent. Then (20.11.5.1) follows from (20.11.5.2).

20.11.7 Theorem (Mellin 3)

If in (20.11.5.1) and (20.11.5.2) we write $\theta = e^{-t}$, t being real, and in (20.11.5.2) put p for z and $g(t)$ for $f(e^{-t})$, we get

$$g(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{zt}\phi(z)\,dz$$

$$\phi(p) = \int_0^\infty e^{-pt} g(t)\,dt$$

20.11.8 Transformation of Contour

To evaluate formally the integral

$$I = \int_0^a \cos xt\,dt$$

we set $\upsilon = xt$ that gives $dx = d\upsilon / t$ and, thus,

$$I = \frac{1}{t}\int_0^{at} \cos\upsilon\; d\upsilon = \frac{\sin at}{t}$$

Regarding this as a contour integral along the real axis for $x = 0$ to a, the change to $\upsilon = xt$ does not change the real axis. However, the contour is unaltered except in length.

Let t be real and positive. If we set $z = \zeta t$ or $\zeta = z/t$, the contour in the ζ-plane is identical in type with that in the z-plane. If it were a circle of radius r in the z-plane, the contour in the ζ-plane would be a circle of radius *r/t*. When t is complex $z = r_1 e^{j\theta_1}$, $z = r_2 e^{j\theta_2}$, so $\zeta = (r_1/r_2)e^{j(\theta_1-\theta_2)}$, r_1 and θ_1 being variables, while r_2 and θ_2 are fixed. If $z = jy = |z|e^{j\theta_1} = |z|e^{j\pi/2}$ and if the phase of t was $\theta_2 = \pi/4$ then the contour in the ζ-plane would be a straight line at 45 degrees with respect to the real axis. In effect, any figure in the z-plane transforms into a similar figure in the ζ-plane, whose orientation and dimensions are governed by the factor $1/t = e^{-j\theta_2}/r_2$.

Example

Make the transformation $z = \zeta t$ to the integral $I = \oint_C e^{z/t}\frac{dz}{z}$, where C is a circle of radius r_0 around the same origin.

Solution

$dz/z = d\zeta/\zeta$ so $I = \oint_{C'} e^{\zeta}\frac{d\zeta}{\zeta}$, where C' is a circle around the origin of radius $r_0 / r(r = |t|)$.

Example

Discuss the transformation $z = (\zeta - a)$, a being complex and finite.

Solution

This is equivalent to a shift of the origin to point $z = -a$. Neither the contour nor the position of the singularities is affected in relation to each other, so the transformation can be made without any alteration in technique.

Example

Find the new contour due to transformation $z = \zeta^2$ if the contour was the imaginary axis, $z = jy$.

Solution

Choosing the positive square root we have $\zeta = (jy)^{1/2}$ above and $\zeta = (-jy)^{1/2}$ below the origin. Because

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = e^{j\pi/4} \quad \text{and} \quad \sqrt{-j} = (e^{-j\pi/2})^{1/2} = e^{-j\pi/4}$$

the imaginary axis of the z-plane transforms to that in Figure 20.4.

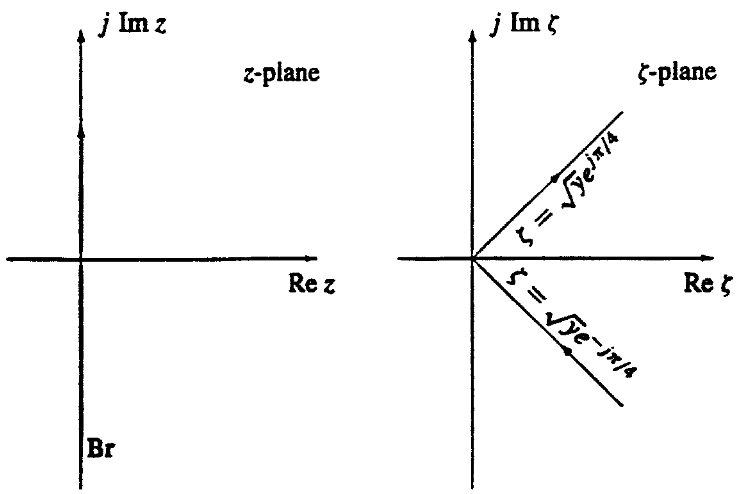

FIGURE 20.4

Example

Evaluate the integral $\int_C \frac{dz}{z}$, where C is a circle of radius 4 units around the origin, under the transformation $z = \zeta^2$.

Solution

The integral has a pole at $z = 0$ and its value is $2\pi j$. If we apply the transformation $z = \zeta^2$ then $dz = 2\,\zeta d\zeta$. Also, $\zeta = \sqrt{z} = \sqrt{r}\, e^{j\theta/2}$ if we choose the positive root. From this relation we observe that as the z traces a circle around the origin, the ζ traces a half-circle from 0 to π. Hence, the integral becomes

$$2\int_C \frac{d\zeta}{\zeta} = 2\int_0^{\pi} \frac{\rho j e^{j\theta}}{\rho e^{j\theta}} d\theta = 2\pi j$$

as we expected.

20.12 Bromwich Contour

20.12.1 Definition of the Bromwich Contour

The Bromwich contour takes the form

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{zt} F(z)\, dz$$

where $F(z)$ is a function of z, all of whose singularities lie on the left of the path, and t is the time, which is always real and positive, $t > 0$

20.12.2 Finite Number of Poles

Let us assume that $F(z)$ has n poles at $p_1, p_2, \cdots, p_n$ and no other singularities; this case includes the important case of *rational transforms*. To utilize the Cauchy's integral theorem, we must express $f(t)$ as an integral along a closed contour. Figure 20.5 shows such a situation. We know from Jordan's lemma that if $F(z) \to 0$ as $|z| \to \infty$ on the contour C then for $t > 0$

$$\lim_{R\to\infty} \int_C e^{tz} F(z)\,dz \to 0, \quad t > 0$$

and because

$$\int_{c-jy}^{c+jy} e^{tz} F(z)\,dz \to \int_{\text{Br}} e^{tz} F(z)\,dz, \quad y \to \infty$$

we conclude that $f(t)$ can be written as a limit,

$$f(t) \xrightarrow[R\to\infty]{} \frac{1}{2\pi j} \oint_C e^{zt} F(z)\,dz$$

of an integral along the closed path as shown in Figure 20.5. If we take R large enough to contain all the poles of $F(z)$, then the integral along C is independent of R. Therefore we write

$$f(t) = \frac{1}{2\pi j} \oint_C e^{zt} F(z)\,dz$$

Using Cauchy's theorem it follows that

$$\oint_C e^{zt} F(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} e^{zt} F(z)\,dz$$

where C_k's are the contours around each pole.

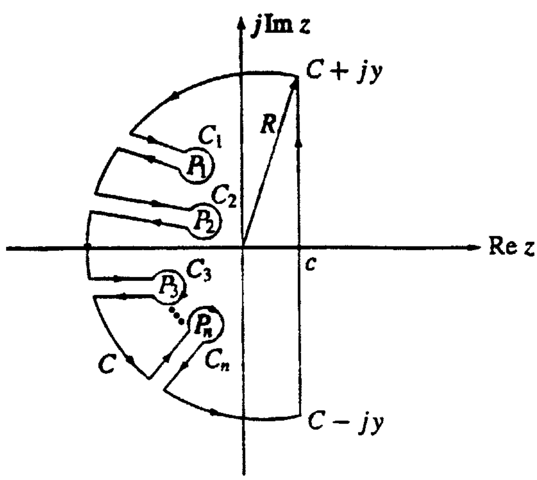

FIGURE 20.5

20.12.3 Simple Poles

$$f(t) = \sum_{k=1}^{n} F_k(z_k) e^{z_k t}, \quad t > 0$$

$$F_k(z_k) = F(z)(z - z_k)\Big|_{z=z_k}$$

20.12.4 Multiple Pole of *m* + 1 Multiplicity

$$\int_{C_k} e^{zt} F(z)\, dz = \int_{C_k} \frac{e^{zt} F_k(z)}{(z - z_k)^{m+1}}\, dz = \frac{2\pi j}{m!} \frac{d^m}{dz^m} [e^{zt} F_k(z)]\Big|_{z=z_k}$$

20.12.5 Infinitely Many Poles

(See Figure 20.6.) If we can find circular arcs with radii tending to infinity such that

$$F(z) \to 0 \text{ as } z \to \infty \qquad \text{on } C_n$$

Applying Jordan's lemma to the integral along those arcs, we obtain

$$\int_{C_n} e^{zt} F(z)\, dz \xrightarrow[n\to\infty]{} 0, \quad t > 0$$

and with C'_n the closed curve, consisting of C_n and the vertical line $\operatorname{Re} z = c$, we obtain

$$f(t) = \lim_{n\to\infty} \frac{1}{2\pi j} \int_{C'_n} e^{zt} F(z)\, dz, \quad t > 0$$

Hence, for simple poles $z_1, z_2, \cdots, z_n$ of $F(z)$ we obtain

$$f(t) = \sum_{k=1}^{\infty} F_k(z_k) e^{z_k t}$$

where $F_k(z) = F(z)(z - z_k)$.

Example

Find $f(t)$ from its transformed value $F(z) = 1/(z \cosh az)$, $a > 0$

Solution

The poles of the above function are

$$z_0 = 0, \quad z_k = \pm j \frac{(2k-1)\pi}{2a}, \quad k = 1, 2, 3, \cdots$$

We select the arcs C_n and their radii are $R_n = jn\pi$. It can be shown that $1/\cosh az$ is bounded on C_n and, therefore, $1/(\cosh az) \to 0$ as $z \to \infty$ on C_n. Hence,

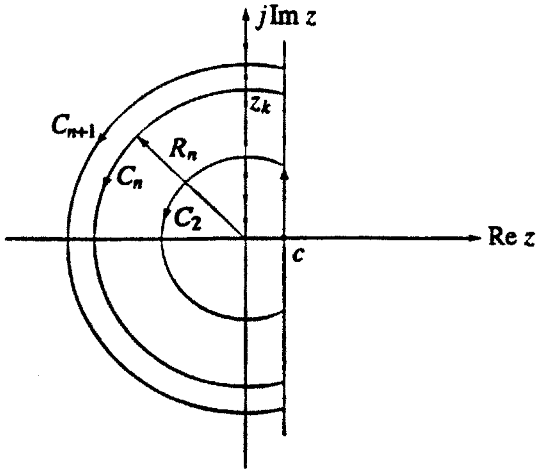

FIGURE 20.6

$$zF(z)\Big|_{z=0} = 1, \quad (z - z_k)F(z)\Big|_{z=z_k} = \frac{(-1)^k 2}{(2k-1)\pi}$$

and from (20.12.5) we obtain

$$f(t) = 1 + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} e^{z_k t} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} e^{-z_k t}$$

$$= 1 + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \cos\frac{(2k-1)\pi t}{2a}$$

20.13 Branch Points and Branch Cuts

20.13.1 Definition of Branch Points and Branch Cuts

The singularities that have been considered are those points at which $|W(z)|$ ceases to be finite. At a branch point the absolute value of $W(z)$ may be finite but $W(z)$ is not single valued, and hence is not regular. One of the simplest functions with these properties is

$$W_1(z) = z^{1/2} = \sqrt{r}\, e^{j\theta/2}$$

which takes on two values for each value of z, one the negative of the other depending on the choice of θ. This follows because we can write an equally valid form for $z^{1/2}$ as

$$W_2(z) = \sqrt{r}\, e^{j(\theta+2\pi)/2} = -\sqrt{r}\, e^{j\theta/2} = -W_1(z)$$

Clearly, $W_1(z)$ is not continuous at points on the positive real axis because

$$\lim_{\theta\to 2\pi}(\sqrt{r}\, e^{j\theta/2}) = -\sqrt{r} \quad \text{while} \quad \lim_{\theta\to 0}(\sqrt{r}\, e^{j\theta/2}) = \sqrt{r}$$

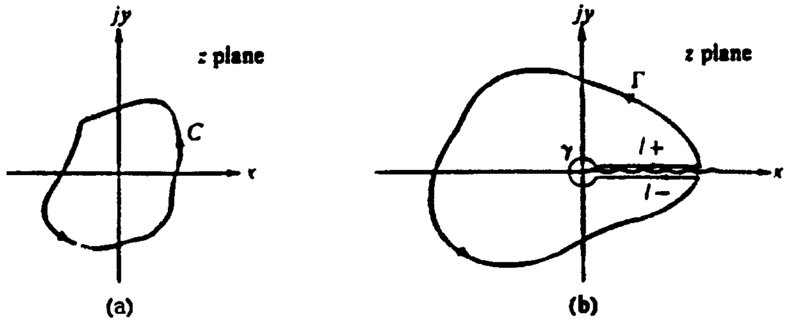

FIGURE 20.7

Hence, $W'(z)$ does not exist when z is real and positive. However, the branch $W_1(z)$ is analytic in the region $0 \le \theta \le 2\pi,\ r \to 0$. The part of the real axis where $x \ge 0$ is called a *branch cut* for the branch $W_1(z)$ and the branch is analytic except at points on the cut. Hence, the cut is a boundary introduced so that the corresponding branch is single valued and analytic throughout the open region bounded by the cut.

Suppose that we consider the function $W(z) = z^{1/2}$ and contour C, as shown Figure 20.7a, which encloses the origin. Clearly, after one complete circle in the positive direction enclosing the origin, θ is increased by 2π, given a value of $W(z)$ that changes from $W_1(z)$ to $W_2(z)$; that is, the function has changed from one branch to the second. To avoid this and to make the function analytic, the contour C is replaced by a contour Γ, which consists of a small circle γ surrounding the branch point, a semi-infinite cut connecting the small circle and C, and C itself (as shown in Figure 20.7b). Such a contour, which avoids crossing the branch cut, ensures that $W(z)$ is singled valued. Because $W(z)$ is single valued and excludes the origin, we would write for this composite contour C

$$\int_C W(z)\,dz = \int_\Gamma + \int_{l-} + \int_\gamma + \int_{l+} = 2\pi j \sum \text{Res}$$

The evaluation of the function along the various segments of C proceeds as before.

Example

If $0 < a < 1$, show that

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}$$

Solution

Consider the integral

$$\oint_C \frac{z^{a-1}}{1+z}\,dz = \int_\Gamma + \int_{l-} + \int_\gamma + \int_{l+} = I_1 + I_2 + I_3 + I_4 = \sum \text{Res}$$

which we will evaluate using the contour shown in Figure 20.8. Under the conditions

$$\left|\frac{z^a}{1+z}\right| \to 0 \quad \text{as } |z| \to 0 \quad \text{if } a > 0$$

$$\left|\frac{z^a}{z+1}\right| \to 0 \quad \text{as } |z| \to \infty \quad \text{if } a < 0$$

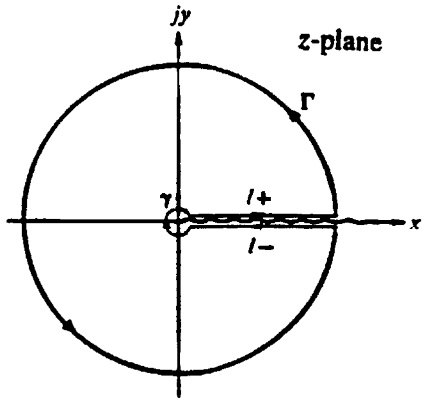

FIGURE 20.8

the integral becomes by (20.11.1)

$$\int_{\Gamma} \to 0, \quad \int_{l-} = -e^{2\pi ja} \int_0^{\infty}$$

by (20.11.2)

$$\int_{\gamma} \to 0, \qquad \int_{l+} = 1\int_0^{\infty}$$

Thus

$$(1-e^{2\pi ja}) \int_0^{\infty} \frac{x^{a-1}}{1+x} dx = 2\pi j \sum \text{Res}$$

Further, the residue at the pole $z = -1$, which is enclosed, is

$$\lim_{z=e^{j\pi}} (1+z) \frac{z^{a-1}}{1+z} = e^{j\pi(a-1)} = -e^{\pi ja}$$

Therefore,

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = 2\pi j \frac{e^{j\pi a}}{e^{j\pi a}-1} = \frac{\pi}{\sin \pi a}$$

If, for example, we have the integral $(1/2\pi j)\int_{\text{Br}_1} \frac{e^{zt}\, dz}{z^{\upsilon+1}}$ to evaluate with $\text{Re}\,\upsilon > -1$ and t real and positive, we observe that the integral has a branch point at the origin if υ is a nonintegral constant. Because the integral vanishes along the arcs as $R \to \infty$, the equivalent contour can assume the form depicted in Figure 20.9a and marked Br_2. For the contour made up of Br_1, Br_2, the arc is closed and contains no singularities and, hence, the integral around the contour is zero. Because the arcs do not contribute any value, provided $\text{Re}\,\upsilon > -1$, the integral along Br_1 is equal to that along Br_2, both being described positively. The angle γ between the barrier and the positive real axis may have any value

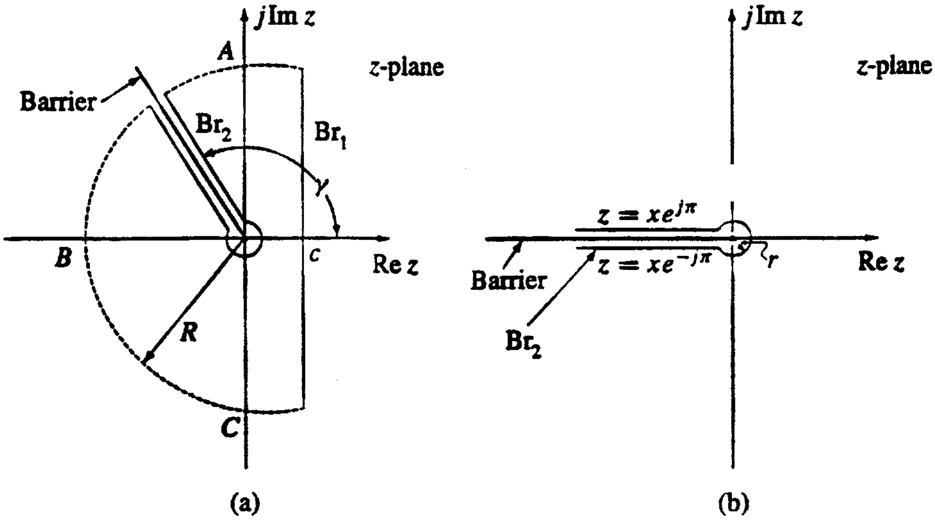

FIGURE 20.9

between $\pi/2$ and $3\pi/2$. When the only singularity is a branch point at the origin, the contour of Figure 20.9b is an approximate one.

Example

Evaluate the integral $I = \frac{1}{2\pi j}\int_{\mathrm{Br}_2} \frac{e^{zt}\,dz}{\sqrt{z}}$, where Br$_2$ is the contour shown in Figure 20.9b.

Solution

1. Write $z = e^{j\theta}$ on the circle. Hence we get

$$I_1 = \frac{1}{2\pi j}\int_{-\pi}^{\pi} \frac{e^{re^{j\theta}}\,d(re^{j\theta})}{\sqrt{r}\;e^{j\theta/2}} = \frac{\sqrt{r}}{2\pi}\int_{-\pi}^{\pi} e^{r(\cos\theta + j\sin\theta) + j\theta/2}\,d\theta$$

2. On the line below the barrier $z = x\exp(-j\pi)$ where $x = |x|$. Hence the integral becomes

$$I_2 = \frac{1}{2\pi j}\int_{\infty}^{r} \frac{e^{xe^{-j\pi}}\,d(xe^{-j\pi})}{\sqrt{x}\;e^{-j\pi/2}} = \frac{1}{2\pi}\int_{r}^{\infty} e^{-x}x^{-1/2}\,dx$$

3. On the line above the barrier $z = x\exp(j\pi)$ and, hence,

$$I_3 = \frac{1}{2\pi j}\int_{r}^{\infty} \frac{e^{xe^{j\pi}}\,d(xe^{j\pi})}{\sqrt{x}\;e^{j\pi/2}} = \frac{1}{2\pi}\int_{r}^{\infty} e^{-x}x^{-1/2}\,dx$$

Hence we have

$$I_2 + I_3 = \frac{1}{\pi}\int_{r}^{\infty} e^{-x}x^{-1/2}\,dx$$

As $r \to 0,\; I_1 \to 0$ and, hence,

$$I = I_1 + I_2 + I_3 = \frac{1}{\pi}\int_{0}^{\infty} e^{-x}x^{-1/2}\,dx = \frac{\sqrt{\pi}}{\pi} = \frac{1}{\sqrt{\pi}}$$

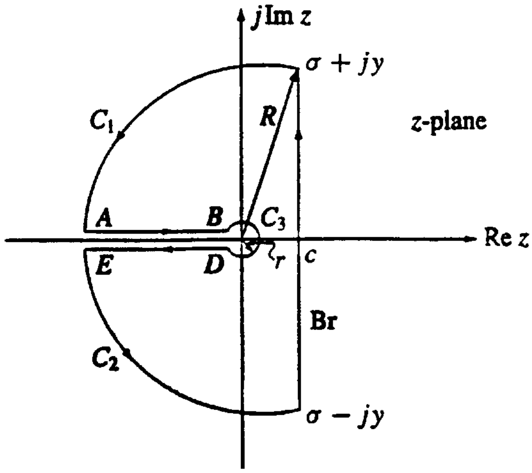

FIGURE 20.10

Example

Evaluate the integral $f(t) = \int_{\mathrm{Br}} \frac{e^{zt} e^{-a\sqrt{z}}}{\sqrt{z}} dz,\ a > 0$ (see Figure 20.10).

Solution

The origin is a point branch and we select the negative axis as the barrier. We select the positive value of $\sqrt{z}$ when z takes positive real values in order that the integral vanishes as z approaches infinity in the region $\mathrm{Re}\, z > \gamma$, where γ indicates the region of convergence, $\gamma \le c$. Hence we obtain

$$z = re^{j\theta} \qquad -\pi < \theta \le \pi \qquad \sqrt{z} = \sqrt{r}\ e^{j\theta/2}$$

The curve $C = \mathrm{Br} + C_1 + C_2 + C_3$ encloses a region with no singularities and, therefore, Cauchy's theorem applies (the integrand is analytic in the region). Hence,

$$\int_C e^{zt} \frac{e^{-a\sqrt{z}}}{\sqrt{z}} dz = 0$$

It is easy to see that the given function converges to zero as R approaches infinity and, therefore, the integration over $C_1 + C_2$ does not contribute any value. For z on the circle we obtain

$$\left| \frac{e^{zt} e^{-a\sqrt{z}}}{\sqrt{z}} \right| \le \frac{e^{rt}}{\sqrt{r}}$$

Therefore, for fixed $t > 0$ we obtain

$$\left| \int_{C_3} e^{zt} \frac{e^{-a\sqrt{z}}}{\sqrt{z}} dz \right| \le 2\pi r \frac{e^{rt}}{\sqrt{r}} = \lim_{r \to 0} 2\pi r \frac{e^{rt}}{\sqrt{r}} = 0$$

because

$$\left| \int_C f(z)\, dz \right| \le ML$$

where L is the length of the contour and $|f(z)| < M$ for z on C.

On AB, $z=-x$, $\sqrt{z}=j\sqrt{x}$, and on DE, $z=-x$, $\sqrt{z}=-j\sqrt{x}$. Therefore, we obtain

$$\int_{AB+DC} e^{zt}\frac{e^{-a\sqrt{z}}}{\sqrt{z}}dz \xrightarrow[R\to\infty]{r\to 0} -\int_{\infty}^{0} e^{-xt}\frac{e^{ja\sqrt{x}}}{j\sqrt{x}}dx - \int_{0}^{\infty} e^{-xt}\frac{e^{-ja\sqrt{x}}}{-j\sqrt{x}}dx$$

But from (20.12.1)

$$\int_{\mathrm{Br}} \frac{e^{zt}e^{-a\sqrt{z}}}{\sqrt{z}}dz = 2\pi j f(t)$$

and, hence,

$$f(t)+\frac{1}{2\pi j}\int_0^{\infty} e^{-xt}\frac{e^{ja\sqrt{x}}+e^{-ja\sqrt{x}}}{j\sqrt{x}}dx = 0$$

If we set $x = y^2$ we have

$$\int_0^{\infty} e^{-xt}\frac{\cos a\sqrt{x}}{\sqrt{x}}dx = 2\int_0^{\infty} e^{-y^2 t}\cos ay\,dy$$

But (see Fourier transform of Gaussian function, see Table 3.2),

$$2\int_0^{\infty} e^{-y^2 t}\cos ay\,dy = \sqrt{\frac{x}{t}}\,e^{-a^2/4t}$$

and hence

$$f(t)=\frac{1}{\sqrt{\pi t}}\,e^{-a^2/4t}$$

Example

Evaluate the integral $f(t)=\dfrac{1}{2\pi j}\displaystyle\int_C \frac{e^{zt}\,dz}{\sqrt{z^2-1}}$ where C is the contour shown in Figure 20.11.

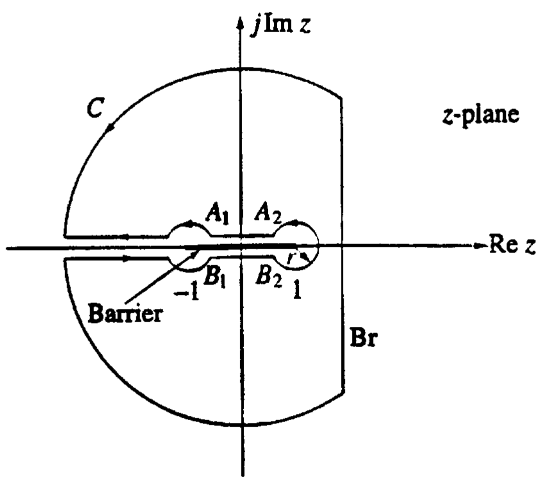

FIGURE 20.11

Solution

The Br contour is equivalent to the dumbbell-type contour shown in Figure 20.11 $B_1B_2A_2A_1B_1$. Set the phase along the line A_2A_1 equal to zero (it can also be set equal to π). Then on A_2A_1 $z = x$ from +1 to –1. Hence we have

$$I_1 = \frac{1}{2\pi j}\int_1^{-1} \frac{e^{xt}\,dx}{\sqrt{x^2-1}} = \frac{1}{2\pi}\int_{-1}^{1} \frac{e^{xt}\,dx}{\sqrt{1-x^2}}, \quad |x|<1$$

By passing around the $z = -1$ point the phase changes by π and hence on B_1B_2 $z = x\exp(2\pi j)$. The change by 2π is due to the complete transversal of the contour that contains two branch points. Hence we obtain

$$I_2 = -\frac{1}{2\pi j}\int_{-1}^{1} \frac{e^{xt}\,dx}{\sqrt{x^2-1}} = \frac{1}{2\pi}\int_{-1}^{1} \frac{e^{xt}dx}{\sqrt{1-x^2}}, \quad |x|<1$$

Changing the origin to –1, we set $\zeta = z + 1$ or $z = \zeta - 1$, which gives

$$I_3 = \frac{e^{-t}}{2\pi j}\int \frac{e^{\zeta t}\,d\zeta}{\sqrt{[(\zeta-2)\zeta]}}$$

On the small circle with $z = -1$ as center, $\zeta = r\exp(j\theta)$ and we get

$$I_3 = \frac{e^{-t}}{2\pi}\int_{\pi}^{-\pi} \frac{e^{rt(\cos\theta + j\sin\theta)+(j\theta/2)}\sqrt{r}\,d\theta}{\sqrt{re^{j\theta}-2}}$$

When $\theta = 0$ the integrand has the value $+\sqrt{r}\,e^{rt}/\sqrt{r-2}$, and for $\theta = 2\pi$ the value is $-\sqrt{r}\,e^{rt}/\sqrt{r-2}$. Therefore, the integrand changes sign in rounding the branch point at $z = -1$. Similarly for the branch point at $z = 1$, where the change is from – to +. As $r \to 0$, $I_3 \to 0$, and thus I_3 vanishes. The same is true for the branch point at $z = -1$. Therefore, by setting $x = \cos\theta$ we obtain

$$I = I_1 + I_2 = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{xt}}{\sqrt{1-x^2}}dx = \frac{1}{\pi}\int_0^{\pi} e^{t\cos\theta}d\theta = \frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{\infty}\frac{(t\cos\theta)^k}{k!}d\theta$$

$$= \frac{1}{\pi}\left[\pi + \pi\frac{1}{2}\frac{t^2}{2!} + \pi\frac{3}{4}\frac{1}{2}\frac{t^4}{4!} + \pi\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{t^6}{6!} + \cdots\right]$$

$$= 1 + \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} + \frac{t^6}{2^2 4^2 6^2} + \cdots = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}t\right)^{2k}}{(k!)^2} = I_0(t)$$

where $I_0(t)$ is the modified Bessel function of the first kind and zero order.

Example

Evaluate the integral $I = \int_C \frac{e^{zt}}{\sqrt{z^2+1}}dz$ where C is the close contour shown in Figure 20.12.

Solution

The Br contour is equal to the dumbbell-type contour as shown in Figure 20.12, $ABGDA = C_1$. Hence we have

$$f(t) = \frac{1}{2\pi j}\int_{C_1}\frac{e^{zt}}{\sqrt{z^2+1}}dz$$

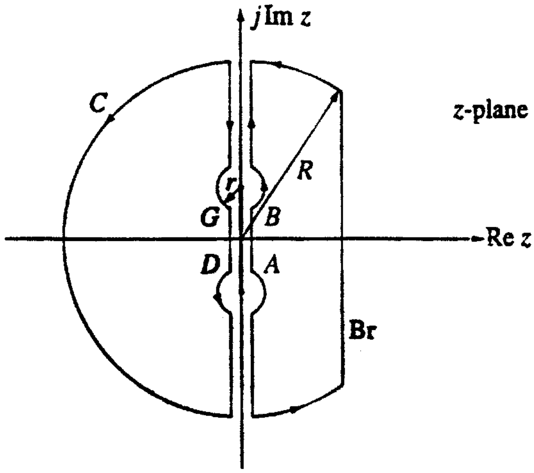

FIGURE 20.12

But

$$\left|\frac{e^{zt}}{\sqrt{z^2+1}}\right| < \frac{e^{rt}}{\sqrt{r}\ \sqrt{2-r}}$$

on the circle on the $+j$ branch point and therefore for $t > 0$

$$\left|\int \frac{e^{zt}}{\sqrt{z^2+1}}dz\right| < \frac{2\pi\sqrt{r}\,e^{rt}}{\sqrt{2-r}} \to 0 \quad \text{as} \quad r \to 0$$

We obtain similar results for the contour around the $-j$ branch point. However,

$$\text{on } AB, z = j\omega,\ \sqrt{1+z^2} = \sqrt{1-\omega^2}\ ;\ \text{on } GD,\ z = j\omega,\ \sqrt{1+z^2} = -\sqrt{1-\omega^2}$$

and, therefore, for $t > 0$ we obtain

$$f(t) = \frac{j}{2\pi j}\int_{-1}^{1}\frac{e^{j\omega t}}{\sqrt{1-\omega^2}}d\omega + \frac{j}{2\pi j}\int_{1}^{-1}\frac{e^{j\omega t}}{-\sqrt{1-\omega^2}}d\omega = \frac{1}{\pi}\int_{-1}^{1}\frac{\cos\omega t}{\sqrt{1-\omega^2}}d\omega$$

If we set $\omega = \sin\theta$

$$f(t) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(t\sin\theta)\,d\theta = J_0(t)$$

where $J_0(t)$ is the Bessel function of the first kind.

20.14 Evaluation of Definite Integrals

20.14.1 Evaluation of the Integrals of Certain Periodic Functions (0 to 2π)

An integral of the form

$$I = \int_0^{\pi/2} F(\cos\theta, \sin\theta)\, d\theta$$

where the integral is a *rational function* of cosθ and sinθ finite on the range of integration, and can be integrated by setting $z = \exp(j\theta)$,

$$\cos\theta = \frac{1}{2}(z + z^{-1}), \qquad \sin\theta = \frac{1}{2j}(z - z^{-1})$$

The above integral takes the form

$$I = \int_C F(z)\, dz$$

where $F(z)$ is a rational function of z finite on C, which is a circle of radius unity with center at the origin.

Example

If $0 < a < 1$, find the value of the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$$

Solution

Transforming the above integral, it becomes

$$I = \int_C \frac{dz}{j(1 - az)(z - a)}$$

The only pole inside the unit circle is at a. Therefore, by residue theory we have

$$I = 2\pi j \lim_{z \to a} \frac{z - a}{j(1 - az)(z - a)} = \frac{2\pi}{1 - a^2}$$

20.14.2 Evaluation of Integrals with Limits –∞ and +∞

We can now evaluate the integral $I = \int_{-\infty}^{\infty} F(x)\, dx$ provided that the function $F(z)$ satisfies the following properties:

1. It is analytic when the imaginary part of z is positive or zero (except at a finite number of poles).
2. It has no poles on the real axis.
3. As $|z| \to \infty$, $zF(z) \to 0$ uniformly for all values of arg z such that $0 \le \arg z \le \pi$, provided that when x is real, $xF(x) \to 0$ as $x \to \pm\infty$, in such a way that $\int_0^{\infty} F(x)\, dx$ and $\int_{-\infty}^{0} F(x)\, dx$ both converge.

The integral is given by

$$I = \int_C F(z)\, dz = 2\pi j \sum \text{Res}$$

where the contour is the real axis and a semicircle having its center in the origin and lying above the real axis.

Example

Evaluate the integral $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$.

Solution

The integral becomes

$$I = \int_C \frac{dz}{(z^2+1)^3} = \int_C \frac{dz}{(z+j)^3(z-j)^3}$$

which has one pole at j of order three. Hence we obtain

$$I = \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{1}{(z+j)^3}\right]\Bigg|_{z=j} = -j\frac{3}{16}$$

Example

Evaluate the integral $I = \int_0^{\infty} \frac{dx}{x^2+1}$.

Solution

Consider the integral

$$I = \int_C \frac{dz}{z^2+1}$$

where C is the contour of the real axis and the upper semicircle. From $z^2 + 1 = 0$ we obtain $z = \exp(j\pi/2)$ and $z = \exp(-j\pi/2)$. Only the pole $z = \exp(j\pi/2)$ exists inside the contour. Hence we obtain

$$2\pi j \lim_{z \to e^{j\pi/2}} \left(\frac{z - e^{j\pi/2}}{(z-e^{j\pi/2})(z-e^{-j\pi/2})} \right) = \pi$$

Therefore, we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\int_0^{\infty} \frac{dx}{x^2+1} = \pi \quad \text{or} \quad I = \frac{\pi}{2}$$

20.14.3 Certain Infinite Integrals Involving Sines and Cosines

If $F(z)$ satisfies conditions (1), (2), and (3), of 20.14.2 and if $m > 0$, then $F(z)e^{jmz}$ also satisfies the same conditions. Hence $\int_0^{\infty} [F(x)e^{jmx} + F(-x)e^{-jmx}]dx$ is equal to $2\pi j \sum \text{Res}$, where $\sum \text{Res}$ means the sum of the residues of $F(z)e^{jmz}$ at its poles in the upper half-plane. Therefore,

1. If $F(x)$ is an even function, that is, $F(x) = -F(-x)$, then

$$\int_0^{\infty} F(x)\cos mx\, dx = \pi j \sum \text{Res}.$$

2. If $F(x)$ is an odd function, that is, $F(x) = -F(-x)$, then

$$\int_o^{\infty} F(x)\sin mx\,dx = \pi \sum \text{Res}$$

Example

Evaluate the integral $I = \int_0^{\infty} \frac{\cos x}{x^2 + a^2} dx, \;\; a > 0.$

Solution

Consider the integral

$$I_1 = \int_C \frac{e^{jz}}{z^2 + a^2} dz$$

where the contour is the real axis and the infinite semicircle on the upper side with respect to the real axis. The contour encircles the pole *ja*. Hence,

$$I_1 = \int_C \frac{e^{jz}}{z^2 + a^2} dz = 2\pi j \frac{e^{jja}}{2ja} = \frac{\pi}{a} e^{-a}$$

However,

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{jz}}{z^2 + a^2} dz = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx + j\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + a^2} dx = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx$$

because the integrand of the third integral is odd and therefore is equal to zero. From the last two equations we find that

$$I = \int_0^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a}$$

because the integrand is an even function.

Example

Evaluate the integral $I = \int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} dx, \;\; k > 0$ and $a > 0.$

Solution

Consider the integral

$$I_1 = \int_C \frac{ze^{jaz}}{z^2 + b^2} dz$$

where C is the same type of contour as in the previous example. Because there is only one pole at $z = jb$ in the upper half of the z-plane, then

$$I_1 = \int_{-\infty}^{\infty} \frac{ze^{jaz}}{z^2 + b^2} dz = 2\pi j \frac{jbe^{jajb}}{2jb} = j\pi e^{-ab}$$

Because the integrand $x\sin ax/(x^2 + b^2)$ is an even function, we obtain

$$I_1 = j\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2 + b^2} dx = j\pi e^{-ab} \;\; \text{or} \;\; I = \frac{\pi}{2} e^{-ab}$$

Example

Show that $I_1 = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx = -\pi e^{-2\pi}$.

Integrals of the Form $\int_0^{\infty} x^{\alpha-1} f(x)\, dx,\ 0 < \alpha < 1$:

It can be shown that the above integral has the value

$$I = \int_0^{\infty} x^{\alpha-1} f(x)\, dx = \frac{2\pi j}{1 - e^{j2\pi\alpha}} \sum_{k=1}^{N} \text{Res}[z^{\alpha-1} f(x)]\big|_{z=z_k}$$

where $f(z)$ has N singularities and $z^{\alpha-1}f(z)$ has a branch point at the origin.

Example

Evaluate the integral $I = \int_0^{\infty} \frac{x^{-1/2}}{x+1} dx$.

Solution

Because $x^{-1/2} = x^{1/2-1}$ it is implied that $\alpha = 1/2$. From the integrand we observe that the origin is a branch point and the $f(x) = 1/(x + 1)$ has a pole at –1. Hence from the previous example we obtain

$$I = \frac{2\pi j}{1 - e^{j2\pi/2}} \text{Res}\left[\frac{z^{-1/2}}{z+1}\right]\Bigg|_{z=-1} = \frac{2\pi j}{j(1 - e^{j\pi})} = \pi$$

We can also proceed by considering the integral $I = \int_C \frac{z^{-1/2}}{z+1} dz$. Because $z = 0$ is a branch point we choose the contour C as shown in Figure 20.13. The integrand has a simple pole at $z = -1$ inside the contour C. Hence the residue at $z = -1 = \exp(j\pi)$ and is

$$\text{Res}\big|_{z=-1} = \lim_{z \to -1} (z+1) \frac{z^{-1/2}}{z+1} = e^{-j\frac{\pi}{2}}$$

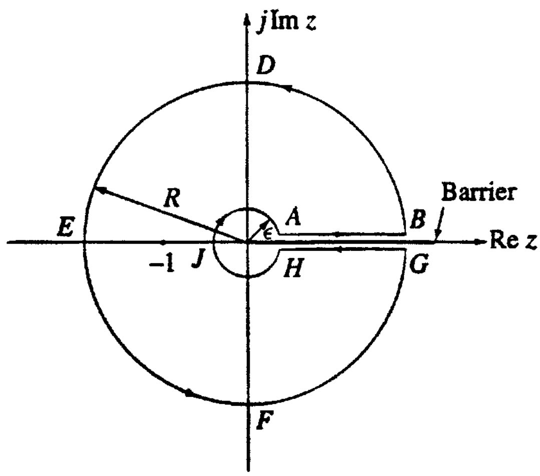

FIGURE 20.13

Therefore we write

$$\oint_C \frac{z^{-1/2}}{z+1} dz = \int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = e^{-j\pi/2}$$

The above integrals take the following form:

$$\int_\varepsilon^R \frac{x^{-1/2}}{x+1} dx + \int_0^{2\pi} \frac{(Re^{j\theta})^{-1/2} jRe^{j\theta} d\theta}{1+Re^{j\theta}} + \int_R^\varepsilon \frac{(xe^{j2\pi})^{-1/2}}{1+xe^{j2\pi}} dx + \int_{2\pi}^0 \frac{(\varepsilon e^{j\theta})^{-1/2} j\varepsilon e^{j\theta} d\theta}{1+\varepsilon e^{j\theta}} = j2\pi e^{-j\pi/2}$$

where we have used $z = x\exp(j2\pi)$ for the integral along GH, because the argument of z is increased by 2π in going around the circle $BDEFG$.

Taking the limit as $\varepsilon \to 0$ and $R \to \infty$ and noting that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{-1/2}}{x+1} dx + \int_\infty^0 \frac{e^{-j2\pi} x^{-1/2}}{x+1} dx = j2\pi e^{-j\pi/2}$$

or

$$(1-e^{-j\pi}) \int_0^\infty \frac{x^{-1/2}}{x+1} dx = j2\pi e^{-j\pi/2} \qquad \text{or} \qquad \int_0^\infty \frac{x^{-1/2}}{x+1} dx = \frac{j2\pi(-j)}{2} = \pi$$

20.14.4 Miscellaneous Definite Integrals

The following examples will elucidate some of the approaches that have been used to find the values of definite integrals.

Example

Evaluate the integral $I = \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} dx, \;\; a > 0.$

Solution

We write (see Figure 20.14)

$$\oint_C \frac{dz}{z^2 + a^2} = \int_{AB} + \int_{BDA} = 2\pi j \sum \text{Res}$$

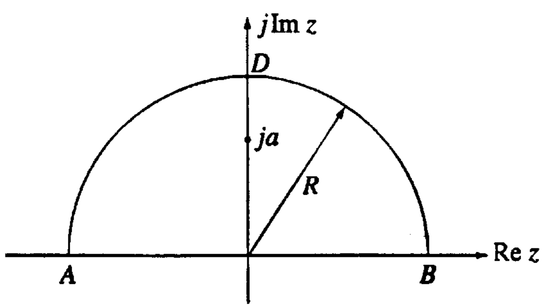

FIGURE 20.14

As $R \to \infty$

$$\int_{BDA} \frac{dz}{z^2 + a^2} = \int_0^{\pi} \frac{R\, je^{j\theta} d\theta}{R^2 e^{j2\theta} + a^2} \xrightarrow[R\to\infty]{} 0$$

and, therefore, we have

$$\int_{AB} \frac{dx}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = 2\pi j \frac{z - ja}{z^2 + a^2}\bigg|_{z=ja} = 2\pi j \frac{1}{2ja} = \frac{\pi}{a}$$

Example

Evaluate the integral $I = \int_0^{\infty} \frac{\sin ax}{x} dx.$

Solution

Because $\sin az/z$ is analytic near $z = 0$, we indent the contour around the origin as shown in Figure 20.15. With a positive we write

$$\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{1}{2} \int_{ABCD} \frac{\sin az}{z} dz = \frac{1}{4j} \int_{ABCD} \left[\frac{e^{jaz}}{z} - \frac{e^{-jaz}}{z} \right] dz$$

$$= \frac{1}{4j} \int_{ABCDA} \frac{e^{jaz}}{z} dz - \frac{1}{4j} \int_{ABCFA} \frac{e^{-jaz}}{z} dz = \frac{1}{4j} \left[2\pi j \frac{1}{1} - 0 \right] = \frac{\pi}{2}$$

because the lower contour does not include any singularity. Because $\sin ax$ is an odd function of a and $\sin 0 = 0$, we obtain

$$\int_0^{\infty} \frac{\sin x}{x} dx = \begin{cases} \dfrac{\pi}{2} & a > 0 \\ 0 & a = 0 \\ -\dfrac{\pi}{2} & a < 0 \end{cases}$$

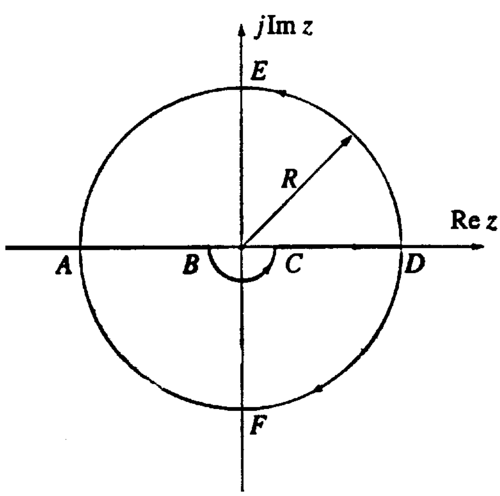

FIGURE 20.15

Example

Evaluate the integral $I = \int_0^\infty \frac{dx}{1+x^3}$.

Solution

Because the integrand $f(x)$ is odd, we introduce the ln z. Taking a branch cut along the positive real axis we obtain

$$\ln z = \ln r + j\theta, \qquad 0 \le \theta < 2\pi$$

the discontinuity of ln z across the cut is (see Figure 20.16a)

$$\ln z_1 - \ln z_2 = -2\pi j$$

Therefore, if $f(z)$ is analytic along the real axis and the contribution around an infinitesimal circle at the origin is vanishing, we obtain

$$\int_0^\infty f(x)dx = -\frac{1}{2\pi j}\int_{ABC} f(z)\ln(z)\,dz$$

If further $f(z) \to 0$ as $|z| \to \infty$, the contour can be completed with *CDA* (see Figure 20.16b). If $f(z)$ has simple poles of order one at points z_k, with residues Res(f, z_k), we obtain

$$\int_0^\infty f(x)dx = -\sum_k \text{Res}(f, z_k)\ln z_k$$

Hence, because $z^3 + 1 = 0$ has poles at $z_1 = e^{j\pi/3}$, $z_2 = e^{j\pi}$, $z_3 = e^{j5\pi/3}$, then the integral is given by

$$I = \int_0^\infty \frac{dx}{x^3+1} = -\left[\frac{j\pi/3}{3e^{2j\pi/3}} + \frac{j\pi}{3e^{j2\pi}} + \frac{j5\pi/3}{3e^{j10\pi/3}}\right] = \frac{2\pi\sqrt{3}}{9}$$

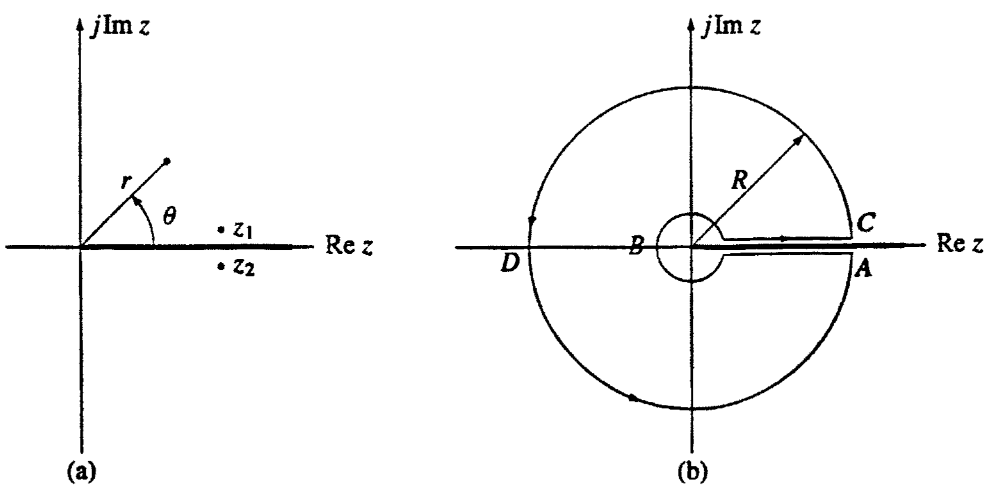

FIGURE 20.16

Example

Show that $\int_0^\infty \cos ax^2\, dx = \int_0^\infty \sin ax^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}, \quad a > 0.$

Solution

We first form the integral

$$F = \int_0^\infty \cos ax^2\, dx + j\int_0^\infty \sin ax^2\, dx = \int_0^\infty e^{jax^2}\, dx$$

Because exp(jaz^2) is analytic in the entire z-plane, we can use Cauchy's theorem and write (see Figure 20.17)

$$F = \int_{AB} a^{jaz^2}\, dz = \int_{AC} e^{jaz^2}\, dz + \int_{CB} e^{jaz^2}\, dz$$

Along the contour CB we obtain

$$\left| -\int_0^{\pi/4} e^{jR^2\cos 2\theta - R^2 \sin 2\theta} jRe^{j\theta}\, d\theta \right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R\, d\theta = \frac{R}{2}\int_0^{\pi/2} e^{-R^2 \sin 2\phi}\, d\phi$$

$$\le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\phi/\pi}\, d\phi = \frac{\pi}{4R}(1 - e^{-R^2})$$

where the transformation $2\theta = \phi$ and the inequality $\sin\phi \ge 2\phi/\pi$ were used $(0 \le \phi \le \pi/2)$. Hence, as R approaches infinity the contribution from CB contour vanishes. Hence,

$$F = \int_{AB} e^{jaz^2}\, dz = e^{j\pi/4}\int_0^\infty a^{-ar^2}\, dr = \frac{1+j}{\sqrt{2}}\frac{1}{2}\sqrt{\frac{\pi}{a}}$$

from which we obtain the desired result.

Example

Evaluate the integral $I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}\,(1+x^2)}$

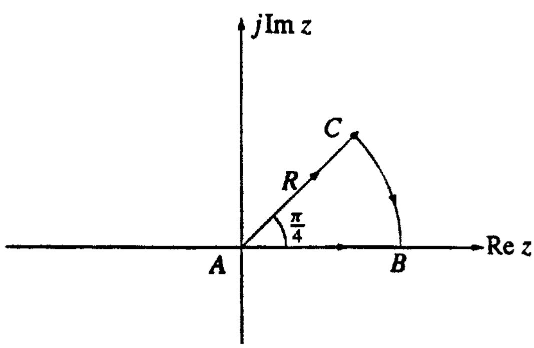

FIGURE 20.17

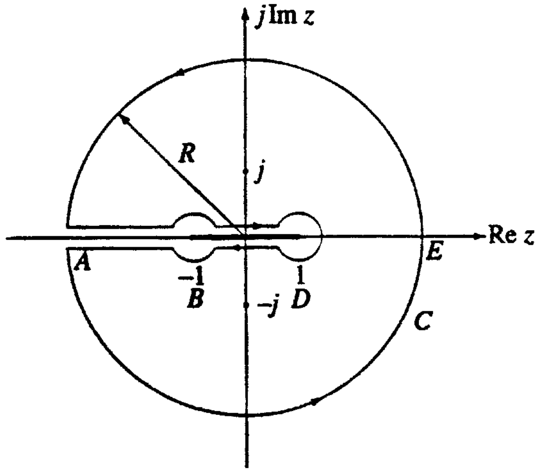

FIGURE 20.18

Solution

Consider the integral

$$I = \oint_C \frac{dx}{\sqrt{1-z^2}\,(1+z^2)}$$

whose contour C is that shown in Figure 20.18 On the top side of the branch cut we obtain *I*, and from the bottom we also get *I*. The contribution of the integral on the outer circle as R approaches infinity vanishes. Hence, due to two poles we obtain

$$2I = 2\pi j\left[\frac{1}{2j\sqrt{2}} + \frac{1}{2j\sqrt{2}}\right] = \pi\sqrt{2} \quad \text{or} \quad I = \frac{\sqrt{2}}{2}\pi$$

Example

Evaluate the integral $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{e^{bx}+1}dx, \quad a,b>0.$

Solution

From Figure 20.19 we find

$$I = \int_C \frac{e^{az}}{e^{bz}+1}dz = \int_C \frac{e^{az/b}}{e^{z}+1}dz = 2\pi j\sum \text{Res}$$

There is an infinite number of poles at $z = j\pi/b$, residue is $-\exp(j\pi a/b)$; at $z = 3j\pi/b$, residue is $-\exp(3j\pi a/b)$and so on. The sum of residue forms a geometric series and because we assume a small imaginary part of a, $|\exp(j2\pi a/b)| < 1$. Hence, by considering the common factor $\exp(j\pi a/b)$, we obtain

$$I = -\frac{2\pi}{b} j \frac{e^{j\pi a/b}}{1-e^{j2\pi a/b}} = \frac{1}{b}\frac{\pi}{b\sin(\pi a/b)}$$

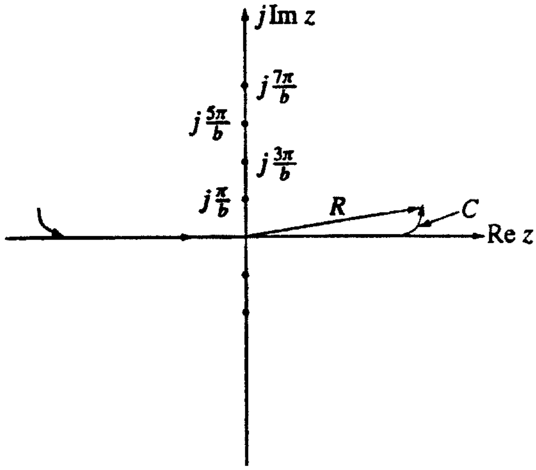

FIGURE 20.19

The integral is of the form $\int e^{j\omega x} f(x)dx$ whose evaluation can be simplified by Jordan's lemma

$$\int_C e^{j\omega x} f(x)dx = 0$$

for the contour semicircle C at infinity for which $\text{Im}(\omega x) > 0$, provided $\left|f(Re^{j\theta})\right| < \varepsilon(R) \to 0$ as $R \to \infty$ (note that the bound on $|f(x)|$ must be independent of θ).

Example

A relaxed RL series circuit with an input voltage source $\upsilon(t)$ is described by the equation $Ldi/dt + Ri = \upsilon(t)$. Find the current in the circuit using the inverse Fourier transform when the input voltage is a delta function.

Solution

The Fourier transform of the differential equation with delta input voltage function is

$$Lj\omega I(\omega) + RI(\omega) = 1 \quad \text{or} \quad I(\omega) = \frac{1}{R + j\omega L}$$

and hence

$$i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{R + j\omega L} d\omega$$

If $t < 0$ the integral is exponentially small for $\text{Im}\omega \to -\infty$. If we complete the contour by a large semicircle in the lower ω-plane, the integral vanishes by Jordan's lemma. Because the contour does not include any singularities, $i(t) = 0$, $t < 0$. For $t > 0$, we complete the contour in the upper ω-plane. Similarly no contribution exists from the semicircle. Because there is only one pole at $\omega = jR/L$ inside the contour the value of the integral is

$$i(t) = 2\pi j \frac{1}{2\pi}\frac{1}{jL} e^{j(jR/L)t} = \frac{1}{L} e^{-\frac{R}{L}t}$$

which is known as the *impulse response of the system.*

20.15 Principal Value of an Integral

20.15.1 Cauchy Principal Value

Refer to the limiting process employed in a previous example in section 20.14 for the integral $\int_0^\infty \frac{\sin ax}{x}\,dx$, which can be written in the form

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{jx}}{x}\,dx = j\pi$$

The limit is called the *Cauchy principal value* of the integral in the equation

$$\int_{-\infty}^{\infty}\frac{e^{jx}}{x}\,dx = j\pi$$

In general, if $f(x)$ becomes infinite at a point $x = c$ inside the range of integration, and if

$$\lim_{\varepsilon\to 0}\int_{-R}^{R} f(x)\,dx = \lim_{\varepsilon\to 0}\left[\int_{-R}^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{R} f(x)\,dx\right]$$

and if the separate limits on the right also exist, then the integral is convergent and the integral is written as $P\int$ where P indicates the principal value. Whenever each of the integrals

$$\int_{-\infty}^{0} f(x)\,dx \qquad \int_{0}^{\infty} f(x)\,dx$$

has a value, here $R \to \infty$, the principal value is the same as the integral. For example, if $f(x) = x$, the principal value of the integral is zero, although the value of the integral itself does not exist.

As another example, consider the integral

$$\int_a^b \frac{dx}{x} = \log\frac{b}{a}$$

If a is negative and b is positive, the integral diverges at $x = 0$. However, we can still define

$$P\int_a^b \frac{dx}{x} = \lim_{\varepsilon\to 0}\left[\int_a^{-\varepsilon}\frac{dx}{x} + \int_{\varepsilon+}^{b}\frac{dx}{x}\right] = \lim_{\varepsilon\to 0}\left(\log\frac{\varepsilon}{-a} + \log\frac{b}{\varepsilon}\right) = \log\frac{b}{|a|}$$

This principal value integral is unambiguous. The condition that the same value of ε must be used in both sides is essential; otherwise, the limit could be almost anything by taking the first integral from a to $-\varepsilon$ and the second from k to b and making these two quantities tend to zero in a suitable ratio.

If the complex variables were used, we could complete the path by a semicircle from $-\varepsilon$ to $+\varepsilon$ about the origin, either above or below the real axis. If the upper semicircle were chosen, there would be a contribution $-j\pi$, whereas if the lower semicircle were chosen, the contribution to the integral would be $+j\pi$. Thus, according to the path permitted in the complex plane we should have

$$\int_a^b \frac{dz}{z} == \log\frac{b}{|a|} \pm j\pi$$

The principal value is the mean of these alternatives.

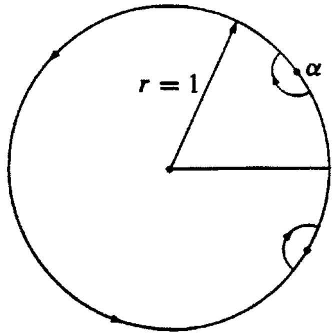

FIGURE 20.20

If a path in the complex plane passes through a simple pole at $z = a$, we can define a principal value of the integral along the path by using a hook of small radius ε about the point a and then making ε tend to zero, as already discussed. If we change the variable z to ζ, and $dz/d\zeta$ is finite and not equal to zero at the pole, this procedure will define an integral in the ζ-plane, but the values of the integrals will be the same. Suppose that the hook in the z-plane cuts the path at $a-\varepsilon$ and $a+\varepsilon'$, where $|\varepsilon| = |\varepsilon'|$, and in the ζ-plane the hook cuts the path at $\alpha - k$ and $\alpha + k'$. Then, if k and k' tend to zero so that $\varepsilon/\varepsilon' \to 1$, k and k' will tend to zero so that $k/k' \to 1$.

To illustrate this discussion, suppose we want to evaluate the integral

$$I = \int_0^{\pi} \frac{d\theta}{a - b\cos\theta}$$

where a and b are real and $a > b > 0$. A change of variable by writing $z = \exp(j\theta)$ transforms the integral to (where a new constant α is introduced)

$$I = \int_0^{\pi} \frac{2e^{j\theta}\, d\theta}{2ae^{j\theta} - b(e^{j2\theta}+1)} = -\frac{1}{j}\int_C \frac{2\,dz}{bz^2 - 2az + b} = -\frac{1}{j}\int_C \frac{2\,dz}{b(z-\alpha)\left(z - \frac{1}{\alpha}\right)}$$

where the path of integration is around the unit circle. Because the contour would pass through the poles, hooks are used to isolate the poles as shown in Figure 20.20. Because no singularities are closed by the path, the integral is zero. The contributions of the hooks are $-j\pi$ times the residue, where the residues are

$$-\frac{1}{j}\frac{\frac{2}{b}}{\alpha - \frac{1}{\alpha}} \qquad -\frac{1}{j}\frac{\frac{2}{b}}{\frac{1}{\alpha} - \alpha}$$

These are equal and opposite and cancel each other. Therefore, the principal value of the integral around the unit circle is zero. This approach for finding principal values succeeds only at simple poles.

20.16 Integral of the Logarithmic Derivative

Of importance in the study of mapping from z-plane to $W(z)$-plane is the integral of the logarithmic derivative. Consider, therefore, the function

$$F(z) = \log W(z)$$

Then

$$\frac{dF(z)}{dz} = \frac{1}{W(z)} \frac{dW(z)}{dz} = \frac{W'(z)}{W(z)}$$

The function to be examined is the following:

$$\int_C \frac{dF(z)}{dz} dz = \int_C \frac{W'(z)}{W(z)} dz$$

The integrand of this expression will be analytic within the contour C except for the points at which $W(z)$ is either zero or infinity.

Suppose that $W(z)$ has a pole of order n at z_0. This means that $W(z)$ can be written

$$W(z) = (z - z_0)^n g(z)$$

with n positive for a zero and n negative for a pole. We differentiate this expression to get

$$W'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$$

and so

$$\frac{W'(z)}{W(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

For n positive, $W'(z)/W(z)$ will possess a pole of order one. Similarly, for n negative $W'(z)/W(z)$ will possess a pole of order one, but with a negative sign. Thus, for the case of n positive or negative, the contour integral in the positive sense yields

$$\int_C \frac{W'(z)}{W(z)} dz = \pm \int_C \frac{n}{z - z_0} dz + \int_z \frac{g'(z)}{g(z)} dz$$

But because $g(z)$ is analytic at the point z_0, then $\int_C [g'(z)/g(z)]dz = 0,$ and since

$$\int_C \frac{W'(z)}{W(z)} dz = \pm 2\pi jn$$

Thus the existence of a zero of $W(z)$ introduces a contribution $2\pi jn_z$ to the contour integral, where n_z is the multiplicity of the zero of $W(z)$ at z_0. Clearly, if a number of zeros of $W(z)$ exist, the total contribution to the contour integral is $2\pi jN$, where N is the weighted value of the zeros of $W(z)$ (weight 1 to a first-order zero, weight 2 to a second-order zero, and so on).

For the case where n is negative, which specifies that $W(z)$ has a pole of order n at z_0, then in the last equation n is negative and the contribution to the contour integral is now $-2\pi jn_p$ for each pole of $W(z)$; the total contribution is $-2\pi jP$, where P is the weighted number of poles. Clearly, because both zeros and poles of $F(z)$ cause poles of $W'(z)/W(z)$ with opposite signs, then the total value of the integral is

$$\int_C \frac{W'(z)}{W(z)} dz = \pm 2\pi j(N - P)$$

Note further that

$$\int_C W'(z)dz = \int_C \frac{dW(z)}{dz}dz = \int d[\log W(z)] = \int d[\log|W(z)| + j\arg W(z)]$$

$$= \log|W(z)|\Big|_o^{2\pi} + j[\arg W(2\pi) - \arg W(0)]$$

$$= 0 + j[\arg W(2\pi) - \arg W(0)]$$

so that

$$[\arg W(0) - \arg W(2\pi)] = 2\pi(N - P)$$

This relation can be given simple graphical interpretation. Suppose that the function $W(z)$ is represented by its pole and zero configuration on the z-plane. As z traverses the prescribed contour on the z-plane, $W(z)$ will move on the $W(z)$-plane according to its functional dependence on z. But the left-hand side of this equation denotes the total change in the phase angle of $W(z)$ as z traverses around the complete contour. Therefore the number of times that the moving point representing $W(z)$ revolves around the origin in the $W(z)$-plane as z moves around the specified contour is given by $N - P$.

The foregoing is conveniently illustrated graphically. Figure 20.21a shows the prescribed contour in the z-plane, and Figure 20.21b shows a possible form for the variation of $W(z)$. For this particular case, the contour in the x-plane encloses one zero and no poles; hence, $W(z)$ encloses the origin once in the clockwise direction in the $W(z)$-plane.

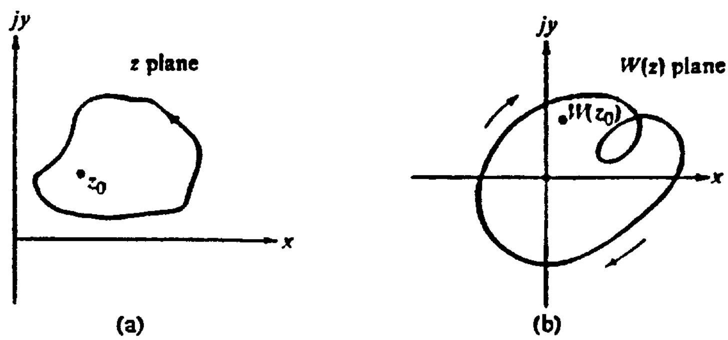

FIGURE 20.21

On the other hand, if the contour includes a pole but no zeros, it can be shown by a similar argument that any point in the interior of the z-contour must correspond to a corresponding point outside the $W(z)$-contour in the $W(z)$-plane. This is manifested by the fact that the $W(z)$-contour is traversed in a counterclockwise direction. With both zeros and poles present, the situation depends on the value of N and P.

Of special interest is the locus of the network function that contains no poles in the right-hand plane or on the $j\omega$-axis. In this case the frequency locus is completely traced as z varies along the ω-axis from $-j\infty$ to $+j\infty$. To show this, because $W(z)$ is analytic along this path, $W(z)$ can be written for the neighborhood of a point z_0 in a Taylor series

$$W(z) = \alpha_o + \alpha_1(z - z_o) + \alpha_2(z - z_o)^2 + \cdots$$

For the neighborhood $z \to \infty$, we examine $W'(z)$, where $z' = 1/z$. Because $W(z)$ does not have a pole at infinity, then $W(z')$ does not have a pole at zero. Therefore, we can expand $W(z')$ in a Maclaurin series.

$$W(z') = \alpha_0 + \alpha_1 z' + \alpha_2 (z')^2 + \cdots$$

which means that

$$W(z') = \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots$$

But as z approaches infinity, $W(\infty)$ approaches infinity. In a real network function when z* is written for z, then $W(z^*) = W^*(z)$. This condition requires that $\alpha_0 = a_0 + j0$ be a real number irrespective of how z approaches infinity; that is, as z approaches infinity, $W(z)$ approaches a fixed point in the $W(z)$-plane. This shows that a z varies around the specified contour in the z-plane, W varies from $W(-j\infty)$ to $W(+j\infty)$ as z varies along the imaginary axis. However, $W(-j\infty) = W(+j\infty)$ from the above, which thereby shows that the locus is completely determined. This is illustrated in Figure 20.22.

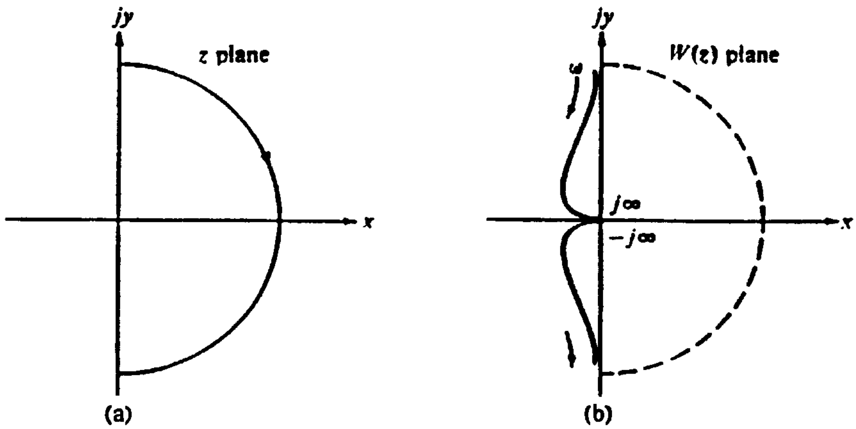

FIGURE 20.22

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21
Legendre Polynomials

21.1 Legendre Polynomials

21.1.1 Definition

$$P_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!\, t^{n-2k}}{2^n\, k!(n-k)!(n-2k)!}$$

$$[n/2] = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$$

21.1.2 Generating Function

$$w(t,s) = \frac{1}{\sqrt{1-2st+s^2}} = \begin{cases} \sum_{n=0}^{\infty} P_n(t)\, s^n & |s|<1 \\ \sum_{n=0}^{\infty} P_n(t)\, s^{-n-1} & |s|>1 \end{cases} \qquad \text{generating function}$$

$$w(-t,-s) = w(t,s)$$

21.1.3 Rodrigues Formula

$$P_n(t) = \frac{1}{2^n\, n!} \frac{d^n}{dt^n}(t^2-1)^n \qquad n = 0,1,2\cdots$$

21.1.4 Recursive Formulas

1. $(n+1)P_{n+1}(t) - (2n+1)\,t\,P_n(t) + n\,P_{n-1}(t) = 0 \qquad n = 1,2,\cdots$

2. $P'_{n+1}(t) - t\,P'_n(t) = (n+1)P_n(t) \qquad (P'(t) \doteq \text{derivative of } P(t) \qquad n = 0,1,2,\cdots$

3. $t P_n'(t) - P_{n-1}'(t) = n P_n(t)$ $\qquad n = 1, 2, \cdots$

4. $P_{n+1}'(t) - P_{n-1}'(t) = (2n+1) P_n(t)$ $\qquad n = 1, 2, \cdots$

5. $(t^2 - 1) P_n'(t) = n t P_n(t) - n P_{n-1}(t)$

6. $P_0(t) = 1 \qquad P_1(t) = t$

TABLE 21.1 Legendre Polynomials

$P_0 = 1$

$P_1 = t$

$P_2 = \frac{3}{2} t^2 - \frac{1}{2}$

$P_3 = \frac{5}{2} t^3 - \frac{3}{2} t$

$P_4 = \frac{35}{8} t^4 - \frac{30}{8} t^2 + \frac{3}{8}$

$P_5 = \frac{63}{8} t^5 - \frac{70}{8} t^3 + \frac{15}{8} t$

$P_6 = \frac{231}{16} t^6 - \frac{315}{16} t^4 + \frac{105}{16} t^2 - \frac{5}{16}$

$P_7 = \frac{429}{16} t^7 - \frac{693}{16} t^5 + \frac{315}{16} t^3 - \frac{35}{16} t$

Figure 21.1 shows a few Legendre functions.

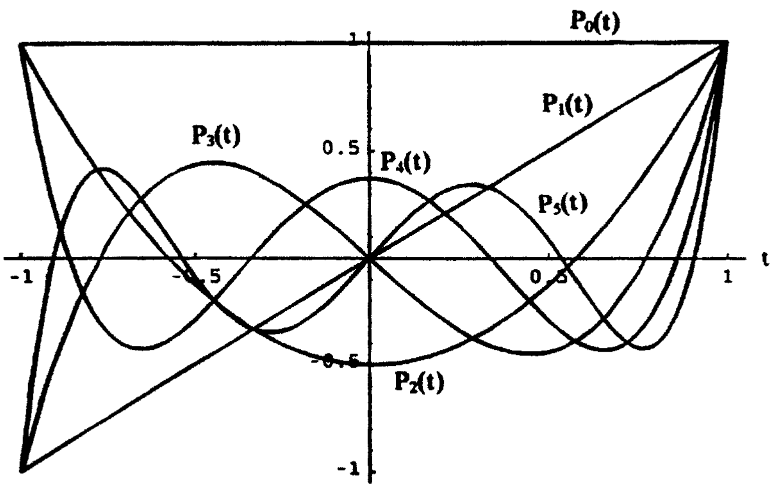

FIGURE 21.1

21.1.5 Legendre Differential Equation

If $y = P_n(x)$ $(n = 0, 1, 2, \cdots)$ is a solution to the second-order DE

$$(1 - t^2) y'' - 2ty' + n(n+1) y = 0$$

For $t = \cos\varphi$: $\qquad \dfrac{1}{\sin\varphi} \dfrac{d}{d\varphi}\left(\sin\varphi \dfrac{dy}{d\varphi}\right) + n(n+1) y = 0$

Example

From (21.1.4.4) and t = 1 implies $0 = nP_n(1) - nP_{n-1}(1)$ or $P_n(1) = P_{n-1}(1)$. For $n = 1,\ P_1(1) = P_0(1) = 1$. For $n = 2,\ P_2(1) = P_1(1) = 1$ and so forth. Hence $P_n(1) = 1$.

21.1.6 Integral Representation

1. Laplace integral: $$P_n(t) = \frac{1}{\pi}\int_0^{\pi} [t + \sqrt{t^2 - 1}\cos\varphi]^n \, d\varphi$$
2. Mehler-Dirichlet formula: $$P_n(\cos\theta) = \frac{2}{\pi}\int_0^{\theta} \frac{\cos(n + \frac{1}{2})\psi}{\sqrt{2\cos\psi - \cos\theta}} \, d\psi \qquad 0 < \theta < \pi,\ n = 0,1,2,\ldots$$
3. Schläfli integral: $$P_n(t) = \frac{1}{2\pi j}\int_C \frac{(z^2 - 1)^n}{2^n (z - t)^{n+1}} \, dz$$

   where C is any regular, simple, closed curve surrounding t.

21.1.7 Complete Orthonormal System

$$\{[\tfrac{1}{2}(2n+1)]^{1/2} P_n(t)\}$$

The Legendre polynomials are orthogonal in [–1,1]

$$\int_{-1}^{1} P_n(t) P_m(t) \, dt = 0$$

$$\int_{-1}^{1} [P_n(t)]^2 \, dt = \frac{2}{2n+1} \qquad n = 0,1,2\cdots$$

and therefore the set

$$\varphi_n(t) = \sqrt{\frac{2n+1}{2}}\, P_n(t) \qquad n = 0,1,2\cdots$$

is orthonormal.

21.1.8 Asymptotic Representation:

$$P_n(\cos\theta) \cong \sqrt{\frac{2}{\pi n \sin\theta}} \sin\left[\left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right], \qquad n \to \infty, \quad \delta \le \theta \le \pi - \delta$$

δ = fixed positive number

21.1.9 Series Expansion

If $f(t)$ is integrable in [–1,1] then

$$f(t) = \sum_{n=0}^{\infty} a_n P_n(t) \qquad -1 < t < 1$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)\,dt \qquad n = 0,1,2\cdots$$

For even $f(t)$, the series will contain term $P_n(t)$ of even index; if $f(t)$ is odd, the term of odd index only.

If the real function $f(t)$ is piecewise smooth in (–1,1) and if it is square integrable in (–1,1), then the series converges to $f(t)$ at every continuity point of $f(t)$. If there is a discontinuity at t then the series converges at $[f(t+0)+f(t-0)]/2$.

21.1.10 Change of Range

If a function $f(t)$ is defined in $[a,b]$, it is sometimes necessary in the applications to expand the function in a series in the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$t = \frac{2}{b-a}\left[x - \frac{b+a}{2}\right], \qquad a < b, \qquad \left[x = \frac{b-a}{2}t + \frac{b+a}{2}\right]$$

transform the interval $[a,b]$ of the x-axis into the interval [–1,1] of the t-axis. It is, therefore, sufficient to consider

$$f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right] = \sum_{n=0}^{\infty} a_n P_n(t)$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right]P_n(t)\,dt$$

The above equation can also be accomplished as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n X_n(t)$$

$$X_n(t) = \frac{1}{n!(b-a)^n}\frac{d^n (t-a)^n (t-b)^n}{dt^n}$$

$$a_n = \frac{2n+1}{b-a}\int_{b}^{a} f(t)X_n(t)\,dt$$

Example

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} 0 & -1 \le t < a \\ 1 & a < t \le 1 \end{cases}$$

Then

$$a_n = \frac{2n+1}{2}\int_{a}^{1} P_n(t)\,dt$$

Using (21.1.4.4), and noting that $P_n(1) = 1$ we obtain

$$a_n = -\tfrac{1}{2}[P_{n+1}(a) - P_{n-1}(a)], \qquad a_0 = \tfrac{1}{2}(1-a)$$

which leads to the expansion

$$f(t) \cong \frac{1}{2}(1-a) - \frac{1}{2}\sum_{n=1}^{\infty}[P_{n+1}(a) - P_{n-1}(a)]P_n(t), \qquad -1 < t < 1$$

Example

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 < t \le 1 \end{cases}$$

The function is an odd function and, therefore, $f(t)P_n(t)$ is an odd function of $P_n(t)$ with even index. Hence a_n are zero for $n = 0,2,4,\ldots$. For odd index n, the product $f(t)P_n(t)$ is even and hence

$$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(t)P_n(t)\,dt = 2\left(n+\frac{1}{2}\right)\int_{0}^{1} P_n(t)\,dt \qquad n = 1,3,5,\cdots$$

Using (21.1.4.4) and setting $n = 2k+1$, $k = 0,1,2\ldots$ we obtain

$$a_{2k+1} = (4k+3)\int_0^1 P_{2k+1}(t)\,dt = \int_0^1 [P'_{2k+2}(t) - P'_{2k}(t)]\,dt$$

$$= [P_{2k+2}(t) - P_{2k}(t)]\Big|_0^1 = P_{2k}(0) - P_{2k+2}(0)$$

where we have used the property $P_n(1) = 1$ for all n. But

$$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$$

and, thus, we have

$$a_{2k+1} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)}{2^{2k+2}[(k+1)!]^2} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[1+\frac{2k+1}{2k+2}\right]$$

$$= \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!}$$

The expansion is

$$f(t) = \sum_{n=0}^{\infty}\frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!}P_{2k+1}(t) \qquad -1 \le t \le 1$$

21.1.11 Expansion of Polynomials

If $q_m(t)=\sum_{k=0}^{m} c_k x^k$ is an arbitrary polynomial, then $q_m(t)=c_0P_0(t)+c_1P_1(t)+\cdots+c_mP_m(t)$ where $c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} q_m(t)P_n(t)\,dt = 0,\ n=0,1,2\cdots$. If $q_m(t)$ is a polynomial of degree m and $m<r$, then $\int_{-1}^{1} q_m(t)P_r(t)\,dt = 0,\ m<r.$

Example

To find $P_{2n}(0)$ we use the summation

$$P_{2n}(t)=\frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n}\frac{(-1)^k(2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!}t^{2k}$$

with k = 0. Hence

$$P_{2n}(0)=\frac{(-1)^n(2n-1)!}{2^{2n-1}(n-1)!n!}=\frac{(-1)^n 2n[(2n-1)!]}{2^{2n}n[(n-1)!]n!}=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$$

Example

To evaluate $\int_0^1 P_m(t)\,dt$ for $m\neq 0$ we must consider the two cases: m = odd and m = even.

(a) m = even and $m\neq 0$

$$\int_0^1 P_m(t)\,dt=\frac{1}{2}\int_{-1}^{1}P_m(t)\,dt=\frac{1}{2}\int_{-1}^{1}P_m(t)\cdot 1\,dt=\frac{1}{2}\int_{-1}^{1}P_m(t)P_0(t)\,dt=0$$

The result is due to the orthogonality principle.

(b) m = odd and $m\neq 0$. From the relation (see Table 21.2)

$$\int_{-1}^{1}P_m(t)\,dt=\frac{1}{2m+1}[P_{m-1}(t)-P_{m+1}(t)]$$

with t = 0 we obtain

$$\int_0^1 P_m(t)\,dt=\frac{1}{2m+1}[P_{m-1}(0)-P_{m+1}(0)]$$

Using the results of the previous example, we obtain

$$\int_0^1 P_m(t)\,dt=\frac{1}{2m+1}\left[\frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^{m-1}\left[\left(\frac{m-1}{2}\right)!\right]^2}-\frac{(-1)^{\frac{m+1}{2}}(m+1)!}{2^{m+1}\left[\left(\frac{m+1}{2}\right)!\right]^2}\right]$$

$$=\frac{(-1)^{\frac{m-1}{2}}(m-1)!(2m+1)(m+1)}{(2m+1)2^{m+1}\left(\frac{m+1}{2}\right)!\left(\frac{m+1}{2}\right)\left(\frac{m-1}{2}\right)!}=\frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^m\left(\frac{m+1}{2}\right)!\left(\frac{m-1}{2}\right)!}\qquad m=\text{odd}$$

21.2 Legendre Functions of the Second Kind (Second Solution)

21.2.1 Second Kind:

1. $Q_0 = \frac{1}{2}\ln\frac{1+t}{1-t}, \quad |t| < 1;$

2. $Q_1(t) = \frac{1}{2}t\ln\frac{1+t}{1-t} - 1, \quad |t| < 1;$

3. $Q_{n+1}(t) = \frac{2n+1}{n+1}tQ_n(t) - \frac{n}{n+1}Q_{n-1}(t), \quad n = 1,2,\cdots$

4. $Q_n(t) = P_n(t)Q_0(t) - \sum_{k=0}^{[\frac{1}{2}(n-1)]}\frac{2n-4k-1}{(2k+1)(n-k)}P_{n-2k-1}(t), \quad |t| < 1, \quad n = 1,2,\cdots$
   for $[\frac{1}{2}(n-1)]$ see 21.1.1.

21.2.2 Recursions

$Q_n(t)$ satisfies all the recurrence relations of $P_n(t)$.

21.2.3 Property

$$\frac{1}{x-t} = \sum_{n=0}^{\infty}(2n+1)P_n(t)Q_n(x)$$

21.2.4 Newman Formula

$$Q_n(t) = \frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{t-x}dx, \qquad n = 0,1,2\cdots$$

21.3 Associated Legendre Polynomials

21.3.1 Definition

If m is a positive integer and $-1 \le t \le 1$, then

$$P_n^m(t) = (1-t^2)^{m/2}\frac{d^m P_n(t)}{dt^m} \qquad m = 1,2,\cdots,n$$

where $P_n^m(t)$ is known as the *associated Legendre function* or *Ferrers' functions*.

21.3.2 Rodrigues Formula

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n, \qquad m = 1,2,\cdots,n; \quad n+m \ge 0$$

21.3.3 Properties

1. $P_n^{-m}(t) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(t)$

2. $P_n^0(t) = P_n(t)$

3. $(n-m+1)P_{n+1}^m(t) - (2n+1)t\,P_n^m(t) + (n+m)P_{n-1}^m(t) = 0$

4. $(1-t^2)^{1/2}\,P_n^m(t) = \dfrac{1}{2n+1}[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$

5. $(1-t^2)^{1/2}\,P_n^m(t) = \dfrac{1}{2n+1}[(n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(t)]$

6. $P_n^m(t) = 2mt(1-t^2)^{-1/2}\,P_n^m(t) - [n(n+1) - m(m-1)]P_n^{m-1}(t)$

7. $P_n^{m+1}(t) = (t^2-1)^{-1/2}[(n-m)\,tP_n^m(t) - (n+m)P_{n-1}^m(t)]$

8. $P_{n+1}^m(t) = P_{n-1}^m(t) + (2n+1)\,(t^2-1)^{1/2}\;P_n^{m-1}(t)$

9. $P_n^m(t) = (t^2-1)^{m/2}\dfrac{d^m P_n(t)}{dt^m}$

10. $\displaystyle\int_{-1}^{1} P_n^m(t)P_k^m(t)\,dt = 0 \qquad k \neq n$

11. $\displaystyle\int_{-1}^{1} [P_n^m(t)]^2\,dt = \frac{2(n+m)!}{(2n+1)(n-m)!}$

21.3.4 Differential Equation

$$(1-t^2)\frac{d^2 P_n^m(t)}{dt^2} - 2t\frac{d\,P_n^m(t)}{dt} + \left[n(n+1) - \frac{m^2}{(1-t^2)}\right]P_n^m(t) = 0$$

21.3.5 Schlafli Formula

$$P_n^m(t) = \frac{(n+m)!}{2\pi jn!}(1-t^2)^{m/2}\oint_C \frac{(x^2-1)^n}{2^n(x-t)^{n+m+1}}\,dx$$

where C is any regular closed curve surrounding the point t and taking it counterclockwise.

21.4 Bounds for Legendre Polynomials

21.4.1 Stieltjes Theorem

$$\left|P_n(\cos\gamma)\right| \le \sqrt{2}\,\frac{4}{\sqrt{\pi}}\,\frac{1}{\sqrt{n}\,\sqrt{\sin\gamma}}, \qquad 0 < \gamma < \pi, \quad n = 1,2,\cdots$$

21.4.2 Second Stieltjes Theorem

$$\left|P_n(t) - P_{n+2}(t)\right| < \frac{4}{\sqrt{\pi}\,\sqrt{n+2}}$$

21.4.3 $\left|\frac{dP_n(t)}{dt}\right| < \frac{2}{\sqrt{\pi}}\frac{\sqrt{n}}{1-t^2}, \qquad |t|<1, \quad n=1,2,\cdots$

21.4.4 $\left|P_{n+1}(t)+P_n(t)\right| < \frac{6\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{1-t}}, \qquad |t|<1$

21.5 Table of Legendre and Associate Legendre Functions

TABLE 21.2 Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind]

1.	$\frac{1}{\sqrt{1-2tx+x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n \qquad |t|\le 1 \qquad |x|<1$	
2.	$P_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!\,t^{n-2k}}{2^n k!(n-k)!(n-2k)!} \qquad [n/2]=\frac{n}{2} \quad n=\text{even}; \quad [n/2]=(n-1)/2 \quad n=\text{odd}$	
3.	$P_0(t)=1$	
4.	$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$	$n=1,2,\cdots$
5.	$P_{2n+1}(0)=0$	$n=0,1,2\cdots$
6.	$P_{2n}(-t)=P_{2n}(t) \qquad P_{2n+1}(-t)=-P_{2n+1}(t)$	$n=0,1,2\cdots$
7.	$P_n(-t)=(-1)^n P_n(t)$	$n=0,1,2\cdots$
8.	$P_n(1)=1 \quad n=0,1,2\cdots; \quad P_n(-1)=(-1)^n$	$n=0,1,2\cdots$
9.	$P_n(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$ = Rodrigues formula,	$n=0,1,2\cdots$
10.	$(n+1)P_{n+1}(t)-(2n+1)tP_n(t)+nP_{n-1}(t)=0$	$n=1,2,\cdots$
11.	$P'_{n+1}(t)-2tP'_n(t)+P'_{n-1}(t)-P_n(t)=0$	$n=1,2,\cdots$
12.	$P'_{n-1}(t)=P_n(t)+2tP'_n(t)-P'_{n+1}(t)$	
13.	$P'_{n+1}(t)=P_n(t)+2tP'_n(t)-P'_{n-1}(t)$	
14.	$P'_{n+1}(t)-tP'_n(t)=(n+1)P_n(t)$	
15.	$tP'_n(t)-P'_{n-1}(t)=nP_n(t)$	$n=1,2,\cdots$
16.	$P'_{n+1}(t)-P'_{n-1}(t)=(2n+1)P_n(t)$	$n=1,2,\cdots$
17.	$(1-t^2)P'_n(t)=nP_{n-1}(t)-ntP_n(t)$	
18.	$|P_n(t)|<1, \qquad -1<t<1$	
19.	$P_{2n}(t)=\frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n}\frac{(-1)^k(2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!}t^{2k}$	$n=0,1,2\cdots$
20.	$(1-t^2)P'_n(t)=(n+1)[tP_n(t)-P_{n+1}(t)]$	$n=0,1,2\cdots$
21.	$\int_{-1}^{1} P_n(t)\,dt=0$	$n=1,2,\cdots$
22.	$|P_n(t)|\le 1$	$|t|\le 1$

TABLE 21.2 Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

23.	$\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0$	$n \neq m$
24.	$\int_{-1}^{1} [P_n(t)]^2\,dt = \frac{2}{2n+1}$	$n = 0,1,2\cdots$
25.	$\frac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \frac{m(m-2)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+1)}$	$m, s =$ even
26.	$\frac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \frac{(m-1)(m-3)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+2)}$	$m, s =$ odd
27.	$\int_{-1}^{1} t\,P_n(t)P_{n-1}(t)\,dt = \frac{2n}{4n^2-1}$	$n = 1,2,\cdots$
28.	$\int_{-1}^{1} P_n(t)P'_{n+1}(t)\,dt = 2$	$n = 0,1,2\cdots$
29.	$\int_{-1}^{1} t\,P'_n(t)P_n(t)\,dt = \frac{2n}{2n+1}$	$n = 0,1,2\cdots$
30.	$\int_{-1}^{1} (1-t^2)\,P'_n(t)P'_k(t)\,dt = 0$	$k \neq n$
31.	$\int_{-1}^{1} (1-t)^{-1/2}\,P_n(t)\,dt = \frac{2\sqrt{2}}{2n+1}$	$n = 0,1,2\cdots$
32.	$\int_{-1}^{1} t^2\,P_{n+1}(t)P_{n-1}(t)\,dt = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$	$n = 1,2,\cdots$
33.	$\int_{-1}^{1} (t^2-1)\,P_{n+1}(t)P'_n(t)\,dt = \frac{2n(n+1)}{(2n+1)(2n+3)}$	$n = 1,2,\cdots$
34.	$\int_{-1}^{1} t^n\,P_n(t)\,dt = \frac{2^{n+1}(n!)^2}{(2n+1)!}$	$n = 0,1,2\cdots$
35.	$\int_{-1}^{1} t^2\,[P_n(t)]^2\,dt = \frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3}+\frac{n^2}{2n-1}\right]$	$n = 0,1,2\cdots$
36.	$P_n^m(t) = (1-t^2)^{m/2}\,\frac{d^m}{dt^m}P_n(t)$	$m > 0$
37.	$P_n^m(t) = \frac{1}{2^n\,n!}(1-t^2)^{m/2}\,\frac{d^{n+m}}{dt^{n+m}}[(t^2-1)^n]$	$m + n \geq 0$
38.	$P_n^{-m}(t) = (-1)^m\,\frac{(n-m)!}{(n+m)!}P_n^m(t)$	
39.	$P_n^0(t) = P_n(t), \quad P_n^m(t) = 0$ for $m > n$	
40.	$(n-m+1)P_{n+1}^m(t) - (2n+1)\,t\,P_n^m(t) + (n+m)P_{n-1}^m(t) = 0$	
41.	$(1-t^2)^{1/2}\,P_n^m(t) = \frac{1}{2n+1}[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$	

TABLE 21.2 Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

42.	$(1-t^2)^{1/2} P_n^m(t) = \frac{1}{2n+1}[(n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+2)P_{n+1}^{m-1}(t)]$	
43.	$P_n^{m+1}(t) = 2mt(1-t^2)^{-1/2} P_n^m(t) - [n(n+1)-m(m-1)]P_n^{m-1}(t)$	
44.	$\int_{-1}^{1} P_n^m(t) P_k^m(t)\,dt = 0$	$k \neq n$
45.	$\int_{-1}^{1} [P_n^m(t)]^2\,dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$	
46.	$P_n^m(-t) = (-1)^{n+m} P_n^m(t)$	
47.	$P_n^m(\pm 1) = 0$	$m > 0$
48.	$P_{2n}^1(0) = 0 \qquad P_{2n+1}^1(0) = \frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$	
49.	$P_n^m(0) = 0$	$n+m = \text{odd}$
	$P_n^m(0) = (-1)^{(n-m)/2}\frac{(n+m)!}{2^n[(n-m)/2]![(n+m)/2]!}$	$n+m = \text{even}$
50.	$\int_{-1}^{1} P_n^m(t) P_n^k(t)\,(1-t^2)^{-1}\,dt = 0$	$k \neq m$
51.	$\int_{-1}^{1} (1-t^2)^{-1/2} P_{2m}(t)\,dt = \left[\frac{\Gamma(\frac{1}{2}+m)}{m!}\right]^2$	
52.	$\int_{-1}^{1} t(1-t^2)^{-1/2} P_{2m+1}(t)\,dt = \frac{\Gamma(\frac{1}{2}+m)\Gamma(\frac{3}{2}+m)}{m!(m+1)!}$	
53.	$\int_{t}^{1} P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(t) - P_{n+1}(t)]$	
54.	$\int_{0}^{1} t^q P_n(t)\,dt = \Gamma(q+1)\sum_{k=0}^{n}\frac{(-1)^k\,\Gamma(n+k+1)}{2^k\,k!\,\Gamma(n-k+1)(q+k+2)}$	$q > -1$
55.	$\int_{0}^{1} t^{-1/2} P_n(t)\,dt = \begin{cases} \frac{2(-1)^{n/2}}{2n+1} & n = \text{even} \\ \frac{2(-1)^{(n-1)/2}}{2n+1} & n = \text{odd} \end{cases}$	
56.	$\int_{0}^{1} t^{1/2} P_n(t)\,dt = \begin{cases} \frac{2(-1)^{(n+2)/2}}{(2n-1)(2n+3)} & n = \text{even} \\ \frac{2(-1)^{(n+3)/2}}{(2n-1)(2n+3)} & n = \text{odd} \end{cases}$	
57.	$Q_0 = \frac{1}{2}\ln\frac{1+t}{1-t}, \qquad \lvert t\rvert < 1$	

TABLE 21.2 Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

58.	$Q_1(t) = \frac{1}{2} t \ln \frac{1+t}{1-t} - 1 = t Q_0(t) - 1, \quad \lvert t \rvert < 1$
59.	$Q_{n+1}(t) = \frac{2n+1}{n+1} t Q_n(t) - \frac{n}{n+1} Q_{n-1}(t), \quad n = 1, 2, \cdots$
60.	$Q_n(t) = P_n(t) Q_0(t) - \sum_{k=0}^{[\frac{1}{2}(n-1)]} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(t), \quad \lvert t \rvert < 1$
61.	$Q_n(t) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(x)}{t-x} dx \qquad n = 0, 1, 2 \cdots$
62.	$Q'_{n+1}(t) - 2t Q'_n(t) + Q'_{n-1}(t) - Q_n(t) = 0$
63.	$Q'_{n+1}(t) - t Q'_n(t) - (n+1) Q_n(t) = 0$
64.	$Q'_{n+1}(t) - Q'_{n-1}(t) = (2n+1) Q_n(t)$
65.	$Q_0(-t) = -Q_0(t), \quad Q_n(-t) = (-1)^{n+1} Q_n(t), \quad n = 1, 2, \cdots$

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22

Hermite Polynomials

22.1 Hermite Polynomials

22.1.1 Rodrigues Formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n} \qquad n = 0,1,2,\cdots, \ -\infty < t < \infty$$

The first few Hermite polynomials are:

$$H_0(t) = 1, \ H_1(t) = 2t, \ H_2(t) = 4t^2 - 2, \ H_3(t) = 8t^3 - 12t, \ H_4(t) = 16t^4 - 48t + 12,$$

$$H_5(t) = 32t^5 - 160t^3 + 120t, \text{ etc.}$$

(See Figure 22.1.)

22.1.2 Expansion Formula

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2t)^{n-2k}, \qquad [n/2] = \text{largest integer} \le n/2$$

22.1.3 Generating Function

$$w(t,x) = e^{2tx - x^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n w}{\partial x^n}\right]_{x=0} x^n, \quad |x| < \infty$$

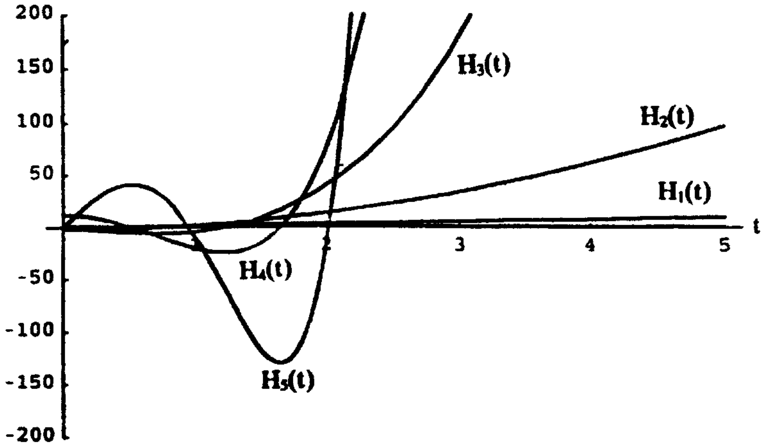

FIGURE 22.1

22.1.4 Even and Odd n

$$H_n(-t) = (-1)^n H_n(t);\ H_{2n}(0) = (-1)^n \frac{(2n)!}{n!},\ \ H_{2n+1}(0) = 0$$

22.2 Recurrence Relation

22.2.1 Recurrence Relation

1. $H_{n+1}(t) - 2t\,H_n(t) + 2n\,H_{n-1}(t) \qquad n = 1,2,\cdots$
2. $H_n'(t) = 2n\,H_{n-1}(t) \qquad n = 1,2,\cdots$

   $H_{n+1}'(t) = 2(n+1)\,H_n(t) \qquad n = 1,2,\cdots$
3. $H_{n+1}(t) - 2t\,H_n(t) + H_n'(t) = 0 \qquad n = 0,1,2,\cdots$

22.2.2 Hermite Differential Equation

$$H_n''(t) - 2t\,H_n'(t) + 2n\,H_n(t) = 0 \qquad n = 0,1,2\cdots$$

which implies that the Hermite polynomials are the solution of the second-order ordinary differential equation.

22.3 Integral Representation

$$H_n(t) = \frac{(-j)^n\, 2^n\, e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 + j2tx} x^n\, dx \qquad n = 0,1,2\cdots$$

$$e^{-t^2/2} H_n(t) = \frac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y)\, dy \qquad n = 0,1,2\cdots$$

$$e^{-t^2/2}H_{2m}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y^2/2} H_{2m}(y) \cos ty\, dy$$

$$e^{-t^2/2}H_{2m+1}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y^2/2} H_{2m+1}(y) \sin ty\, dy \qquad m = 0,1,2\cdots$$

22.4 Hermite Series

22.4.1 Orthogonality Property

$$\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t)\, dt = 0 \quad \text{if} \quad m \neq n$$

and

$$\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t)\, dt = 2^n n! \sqrt{\pi} \qquad n = 0,1,2\cdots$$

22.4.2 Orthonormal Hermite Polynomials

$$\varphi_n(t) = (2^n n! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_n(t) \qquad n = 0,1,2,\cdots, \ -\infty < t < \infty$$

22.4.3 Hermite Series

$$f(t) = \sum_{n=0}^{\infty} C_n H_n(t) \qquad -\infty < t < \infty$$

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t)\, dt \qquad n = 0,1,2,\cdots$$

where $f(t)$ is piecewise smooth in every finite interval and $\int_{-\infty}^{\infty} e^{-t^2} f^2(t)\, dt < \infty$.

Example

$f(t) = t^4 = \sum_{n=0}^{\infty} C_{2n} H_{2n}(t)$, since $f(t)$ is even.

$$C_{2n} = \frac{1}{2^{2n}(2n)! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^4 H_{2n}(t)\, dt$$

$$= \frac{1}{2^{2n}(2n)! \sqrt{\pi}} \int_{-\infty}^{\infty} t^4 \frac{d^{2n}}{dt^{2n}}(e^{-t^2})\, dt = \frac{1}{2^{2n}(2n)! \sqrt{\pi}} \frac{(4)!}{(4-2n)!} \int_{-\infty}^{\infty} e^{-t^2} t^{4-2n}\, dt$$

$$= \frac{1}{2^{2n}(2n)! \sqrt{\pi}} \frac{(4)!}{(4-2n)!} \Gamma\left(4 - n + \frac{1}{2}\right)$$

22.5 Properties of the Hermite Polynomials

TABLE 22.1 Properties of the Hermite Polynomial

1.	$H_n(t) = (-1)^n e^{t^2} \dfrac{d^n e^{-t^2}}{dt^n}$	$n = 0,1,2\cdots$
2.	$H_n(t) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k n!}{k!(n-2k)!}(2t)^{n-2k}$	$[n/2]$ = largest integer $\le n/2$
3.	$e^{2tx-x^2} = \sum_{n=0}^{\infty} H_n(t)\dfrac{x^n}{n!}$	
4.	$H_{2n}(0) = (-1)^n \dfrac{(2n)!}{n!}$	
5.	$H_{2n+1}(0) = 0, \quad H'_{2n}(0) = 0, \quad H'_{2n+1}(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$	
6.	$H_n(-t) = (-1)^n H_n(t)$	
7.	$H_{2n}(t)$ = even functions, $\quad H_{2n+1}(t)$ = odd functions	
8.	$H_{n+1}(t) - 2t H_n(t) + 2n H_{n-1}(t) = 0$	$n = 1,2,\cdots$
9.	$H'_n(t) = 2n H_{n-1}(t)$	$n = 1,2,\cdots$
10.	$H_{n+1}(t) - 2t H_n(t) + H'_n(t) = 0$	$n = 0,1,2\cdots$
11.	$H''_n(t) - 2t H'_n(t) + 2n H_n(t) = 0$	$n = 0,1,2\cdots$
12.	$H_n(t) = \dfrac{(-j)^n 2^n e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+j2tx} x^n \, dx$	$n = 0,1,2\cdots$
13.	$e^{-t^2/2} H_n(t) = \dfrac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y)\, dy$ = integral equation	
14.	$e^{-t^2/2} H_{2m}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m}(y) \cos ty \, dy$	
15.	$e^{-t^2/2} H_{2m+1}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m+1}(y) \sin ty \, dy$	
16.	$\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t)\, dt = 0 \quad$ if $\quad m \ne n$	
17.	$\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t)\, dt = 2^n n! \sqrt{\pi}$	$n = 0,1,2\cdots$
18.	$f(t) = \sum_{n=0}^{\infty} C_n H_n(t)$ $C_n = \dfrac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t)\, dt$	$-\infty < t < \infty$
19.	$\int_{-\infty}^{\infty} t^k e^{-t^2} H_n(t)\, dt = 0 \qquad k = 0,1,\cdots n-1$	

TABLE 22.1 Properties of the Hermite Polynomial (continued)

20.	$\int_{-\infty}^{\infty} t^2 e^{-t^2} H_n^2(t)\,dt = \sqrt{\pi}\; 2^n n!\left(n+\frac{1}{2}\right)$	
21.	$\int_{0}^{\infty} x^n e^{-x^2} H_n(tx)\,dx = \frac{\sqrt{\pi}\; n!}{2} P_n(t)$	
22.	$\int_{-\infty}^{\infty} e^{-2t^2} H_n^2(t)\,dt = 2^{n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)$	
23.	$\frac{d^m H_n(t)}{d t^m} = \frac{2^m n!}{(n-m)!} H_{n-m}(t)$	$m < n$
24.	$\int_{-\infty}^{\infty} e^{-a^2t^2} H_{2n}(t)\,dt = \frac{(2n)!}{n!}\frac{\sqrt{\pi}}{a}\left(\frac{1-a^2}{a^2}\right)^n$	$a > 0$
25.	$\int_{-\infty}^{\infty} e^{-t^2+2bt} H_n(t)\,dt = \sqrt{\pi}\;(2b)^n e^{b^2}$	$n = 0,1,2\cdots$

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23

Laguerre Polynomials

23.1 Laguerre Polynomials

23.1.1 Definition

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n! t^k}{(k!)^2 (n-k)!} \qquad n = 0, 1, 2, \cdots, \; 0 \le t < \infty$$

Figure 23.1 shows several Laguerre polynomials.

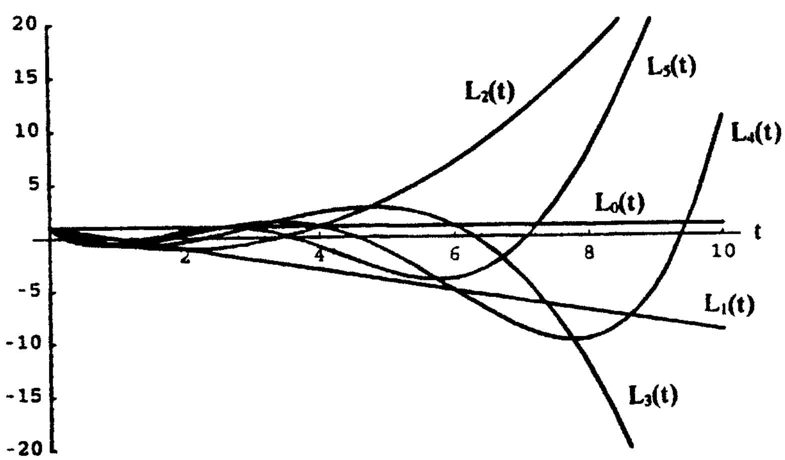

FIGURE 23.1

23.1.2 Rodrigues Formula

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) \qquad n = 0,1,2,\cdots$$

23.1.3 Laguerre Polynomials

$$L_0(t) = 1, \quad L_1(t) = -t+1, \quad L_2(t) = \tfrac{1}{2!}(t^2 - 4t + 2),$$

$$L_3(t) = \tfrac{1}{3!}(-t^3 + 9t^2 - 18t + 6), \quad L_4(t) = \tfrac{1}{4!}(t^4 - 16t^3 + 72t^2 - 96t + 24)$$

23.1.4 Generating Function

$$w(t,x) = (1-x)^{-1}\exp\left[-\frac{tx}{1-x}\right] = \sum_{n=0}^{\infty} L_n(t)x^n \qquad |x| < 1,\ 0 \le t < \infty$$

23.2 Recurrence Relations

23.2.1 Recurrence Relations

1. $L_n'(t) - L_{n-1}'(t) + L_{n-1}(t) = 0$ $\qquad n = 1,2,\cdots$
2. $L_{n+1}'(t) = L_n'(t) - L_n(t)$ $\qquad n = 1,2,\cdots$
3. $L_{n-1}'(t) = L_n'(t) + L_{n-1}(t)$ $\qquad n = 1,2,\cdots$
4. $(n+1)L_{n+1}'(t) + (t-1-2n)L_n'(t) + L_n(t) + nL_{n-1}'(t) = 0$ $\qquad n = 1,2,\cdots$
5. $tL_n'(t) = nL_n(t) - nL_{n-1}(t)$ $\qquad n = 1,2,\cdots$

23.2.2 Laguerre Equation

$$tL_n''(t) + (1-t)L_n'(t) + nL_n(t) = 0$$

23.3 Laguerre Series

23.3.1 Orthogonality Relation

$$\int_0^\infty e^{-t}L_n(t)L_m(t)\,dt = 0 \qquad n \ne m$$

23.3.2 Orthonormal Functions

$$\varphi_n(t) = e^{-t/2}L_n(t) \qquad n = 0,1,2\cdots$$

23.3.3 Laguerre Series

$$f(t) = \sum_{n=0}^{\infty} C_n L_n(t) \qquad 0 \le t < \infty$$

$$C_n = \int_0^\infty e^{-t} f(t) L_n(t) dt \qquad n = 0,1,2\cdots$$

23.4 Associated Laguerre Polynomials (or Generalized)

23.4.1 Definitions

For a real $a > -1$ the general Laguerre polynomials are defined by the formula

$$L_n^a(t) = e^t \frac{t^{-a}}{n!} \frac{d^n}{dt^n}(e^{-t} t^{n+a}) \qquad n = 0,1,2\cdots$$

Using Leibniz's formula

$$L_n^a(t) = \sum_{k=0}^{n} \frac{\Gamma(n+a+1)}{\Gamma(k+a+1} \frac{(-t)^k}{k!(n-k)!} \qquad 0 \le t < \infty$$

For $a = 0$, $L_n^a(t)$ become $L_n(t)$

23.4.2 Polynomials

$$L_0^a(t) = 1, \;\; L_1^a(t) = 1 + a - t, \;\;\; L_2^a(t) = \tfrac{1}{2}[(1+a)(2+a) - 2(2+a)t + t^2], \ldots$$

23.5 Recurrence Relations

23.5.1 Recurence Relations

1. $L_n^{a\prime}(t) - L_{n-1}^{a\prime}(t) + L_{n-1}^a(t) = 0 \qquad n = 1,2,\cdots$
2. $t L_n^{a\prime}(t) = n L_n^a(t) - (n+a) L_{n-1}^a(t) \qquad n = 1,2,\cdots$

23.6 Laguerre Series

23.6.1 Orthogonality Relations

$$\int_0^\infty e^{-t} t^a L_m^a(t) L_n^a(t) dt = 0 \qquad n \ne m \quad a > -1$$

$$\int_0^\infty e^{-t} t^a [L_n^a(t)]^2 \, dt = \frac{\Gamma(n+a+1)}{n!} \qquad a > -1 \quad n = 0,1,2\cdots$$

23.6.2 Orthonormal Functions

$$\varphi_n^a(t) = \left[\frac{n!}{\Gamma(n+a+1)}\right]^{1/2} e^{-t/2} t^{a/2} L_n^a(t) \qquad n = 0,1,2\cdots 0 \le t < \infty$$

23.6.3 Series

The Laguerre series is given by

$$f(t) = \sum_{n=0}^{\infty} C_n L_n^a(t) \qquad 0 \le t < \infty$$

$$C_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} e^{-t} t^a f(t) L_n^a(t)\, dt \qquad n = 0,1,2\cdots$$

Example

The function t^b can be expanded in series

$$t^b = \sum_{n=0}^{\infty} C_n L_n^a(t) \qquad b > -\tfrac{1}{2}(a+1)$$

$$C_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} t^{b+a} e^{-t} L_n^a(t)\, dt = \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} e^{-t} t^{b+a} \frac{e^t t^{-a}}{n!} \frac{d^n}{dt^n}(t^{n+a} e^{-t})\, dt$$

$$= \frac{1}{\Gamma(n+a+1)} \int_0^{\infty} t^b \frac{d^n}{dt^n}(t^{n+a} e^{-t})\, dt = \frac{(-1)^n b(b-1)\cdots(b-n+1)}{\Gamma(n+a+1)} \int_0^{\infty} e^{-t} t^{b+a}\, dt$$

$$= (-1)^n \frac{\Gamma(b+1)}{\Gamma(n+b+1)\Gamma(b-n+1)} \int_0^{\infty} e^{-t} t^{(b+a+1)-1}\, dt = (-1)^n \frac{\Gamma(b+1)\Gamma(b+a+1)}{\Gamma(n+b+1)\Gamma(b-n+1)}$$

The steps to find C_n were: a) substitution of (23.4.1), b) integration by parts n times, c) multiplication numerator and denominator by $\Gamma(b - n + 1)$. In particular if $b = m =$ *positive integer*

$$t^m = \Gamma(m+a+1)\, m! \sum_{n=0}^{m} \frac{(-1)^n L_n^a(t)}{\Gamma(n+a+1)(m-n)!} \qquad 0 \le t < \infty, \quad a > -1 \quad \text{and} \quad m = 0,1,2\cdots$$

If $a = 0$ we obtain the expansion

$$t^m = \Gamma(m+1)\, m! \sum_{n=0}^{m} \frac{(-1)^m L_n(t)}{n!(m-n)!}$$

Example

The function $f(t) = e^{-bt}$, with $b > -\tfrac{1}{2}$ and $t > 0$, is expanded as follows

$$C_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} e^{-(b+1)t} t^a L_n^a(t)\, dt = \frac{1}{\Gamma(n+a+1)} \int_0^{\infty} e^{-bt} \frac{d^n}{dt^n}(e^{-t} t^{n+a})\, dt$$

$$= \frac{b^n}{\Gamma(n+a+1)} \int_0^{\infty} e^{-(b+1)t} t^{n+a}\, dt = \frac{b^n}{(b+1)^{n+a+1}} \qquad n = 0,1,2\cdots$$

and thus

$$e^{-bt} = (b+1)^{-a-1} \sum_{n=0}^{\infty} \left(\frac{b}{b+1} \right)^n L_n^a(t) \qquad 0 \le t < \infty$$

For $a = 0$

$$e^{-bt} = (b+1)^{-1} \sum_{n=0}^{\infty} \left(\frac{b}{b+1} \right)^n L_n(t) \qquad 0 \le t < \infty$$

23.7 Tables of Laguerre Polynomials

TABLE 23.1 Properties of the Laguerre Polynomials

1.	$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n! t^k}{(k!)^2 (n-k)!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \binom{n}{k} t^k$	$n = 0,1,2,\cdots,\ 0 \le t < \infty$
2.	$L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t})$	$n = 0,1,2\cdots$
3.	$L_0(t) = 1,\quad L_1(t) = -t+1,\quad L_2(t) = \frac{1}{2!}(t^2 - 4t + 2),$	
	$L_3(t) = \frac{1}{3!}(-t^3 + 9t^2 - 18t + 6),\quad L_4(t) = \frac{1}{4!}(t^4 - 16t^3 + 72t^2 - 96t + 24)$	
	$L_n(0) = 1,\quad L_n'(0) = -n,\quad L_n''(0) = \frac{1}{2} n(n-1)$	
4.	$(n+1) L_{n+1}(t) + (t-1-2n) L_n(t) + n L_{n-1}(t) = 0$	$n = 1,2,3,\cdots$
5.	$L_n'(t) - L_{n-1}'(t) + L_{n-1}(t) = 0$	$n = 1,2,3,\cdots$
6.	$(n+1) L_{n+1}'(t) + (t-1-2n) L_n'(t) + L_n(t) + n L_{n-1}'(t) = 0$	$n = 1,2,3,\cdots$
7.	$L_{n+1}'(t) = L_n'(t) - L_n(t)$	
8.	$t L_n'(t) = n L_n(t) - n L_{n-1}(t)$	$n = 1,2,3,\cdots$
9.	$t L_n''(t) + (1-t) L_n'(t) + n L_n(t) = 0,$ Laguerre differential equation	
10.	$w(t,x) = (1-x)^{-1} \exp\left[-\frac{tx}{1-x} \right] = \sum_{n=0}^{\infty} L_n(t) x^n$ generating function	
11.	$\int_0^{\infty} e^{-t} L_n(t) L_k(t)\, dt = 0 \qquad k \ne n$	
12.	$\int_0^{\infty} e^{-t} [L_n(t)]^2\, dt = 1$	
13.	$f(t) = \sum_{n=0}^{\infty} C_n L_n(t)$	$0 \le t < \infty$
	$C_n = \int_0^{\infty} e^{-t} f(t) L_n(t)\, dt$	$n = 0,1,2\cdots$
14.	$L_n^m(t) = (-1)^m \frac{d^m}{dt^m} [L_{n+m}(t)]$	$m = 0,1,2\cdots$
15.	$L_n^m(t) = \sum_{k=0}^{n} \frac{(-1)^k (n+m)! t^k}{(n-k)!(m+k)!k!}$	$m = 0,1,2\cdots$
16.	$(n+1) L_{n+1}^m(t) + (t-1-2n-m) L_n^m(t) + (n+m) L_{n-1}^m(t) = 0$	
17.	$t L_n^{m\prime}(t) - n L_n^m(t) + (n+m) L_{n-1}^m(t) = 0$	

TABLE 23.1 Properties of the Laguerre Polynomials (continued)

18. $L_n^m(t) = \frac{1}{n!} e^t t^{-m} \frac{d^n}{dt^n}(e^{-t} t^{n+m}) =$ Rodrigues formula

19. $L_{n-1}^m(t) + L_n^{m-1}(t) - L_n^m(t) = 0$

20. $L_n^{m\prime}(t) = -L_{n-1}^{m+1}(t)$

21. $L_n^m(0) = \frac{(n+m)!}{n!\,m!}$

22. $\int_0^\infty e^{-t} t^k L_n(t)\,dt = \begin{cases} 0 & k < n \\ (-1)^n n! & k = n \end{cases}$

23. $\int_0^t L_k(x) L_n(t-x)\,dx = \int_0^t L_{n+k}(x)\,dx = L_{n+k}(t) - L_{n+k+1}(t)$

24. $\int_t^\infty e^{-x} L_n^m(x)\,dx = e^{-t}[L_n^m(t) - L_{n-1}^m(t)] \qquad m = 0,1,2.\cdots$

25. $\int_0^t (t-x)^m L_n(x)\,dx = \frac{m!\,n!}{(m+n+1)!}\, t^{m+1} L_n^{m+1}(t) \qquad m = 0,1,2.\cdots$

26. $\int_0^1 x^a (1-x)^{b-1} L_n^a(tx)\,dx = \frac{\Gamma(b)\Gamma(n+a+1)}{\Gamma(n+a+b+1)} L_n^{a+b}(t) \qquad a > -1 \;,\quad b > 0$

27. $\int_0^\infty e^{-t} t^a L_n^a(t) L_k^a(t)\,dt = 0 \qquad k \neq n \;,\quad a > -1$

28. $\int_0^\infty e^{-t} t^a [L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!} \qquad a > -1$

29. $\int_0^\infty e^{-t} t^{a+1} [L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!}(2n+a+1) \qquad a > -1$

30. $L_n^{-1/2}(t) = \frac{(-1)^n}{2^{2n} n!} H_{2n}(\sqrt{t}\,)$

31. $L_n^{1/2}(t) = \frac{(-1)^n}{2^{2n+1} n!} \frac{H_{2n+1}(\sqrt{t}\,)}{\sqrt{t}}$

32. $f(t) = \sum_{n=0}^\infty C_n L_n^m(t)$

$$C_n = \frac{n!}{\Gamma(n+m+1)} \int_0^\infty e^{-t} t^m f(t) L_n^m(t)\,dt$$

33. $\varphi_n^m = \left[\frac{n!}{\Gamma(n+m+1)}\right]^{1/2} e^{-t/2} t^{m/2} L_n^m(t),$ orthonormal sequence, $n = 0,1,2\cdots$

34. $t^p = p! \sum_{n=0}^p \binom{p}{n} (-1)^n L_n(t)$

35. $e^{-at} = (a+1)^{-1} \sum_{n=0}^\infty \left(\frac{a}{a+1}\right)^n L_n(t) \qquad a > -\frac{1}{2}$

36. $\int_0^\infty \frac{e^{-tx}}{x+1}\,dx = \sum_{n=0}^\infty \frac{L_n(t)}{n+1}$

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24

Chebyshev Polynomials

24.1 Chebyshev Polynomials

24.1.1 Definitions

$$T_0(t) = 1, \qquad T_n(t) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2t)^{n-2k} \qquad -1 < t < 1$$

$$U_n(t) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2t)^{n-2k}, \quad \text{second kind}$$

$[n/2] = n/2$ for n even and $[n/2] = (n-1)/2$ for n odd.
Figure 24.1 shows several polynomials.

24.2 Recurrence Relations

24.2.1 Recurrence

$$T_{n+1}(t) - 2t\,T_n(t) + T_{n-1}(t) = 0$$

$$U_{n+1}(t) - 2t\,U_n(t) + U_{n-1}(t) = 0$$

24.3 Orthogonality Relations

24.3.1 Relations

$$\int_{-1}^{1} (1-t^2)^{-1/2}\, T_n(t) T_k(t)\, dt = 0 \qquad k \neq n$$

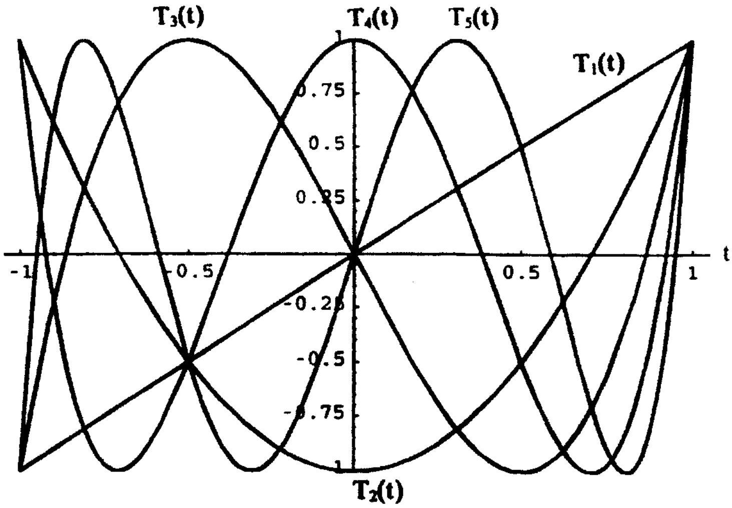

FIGURE 24.1

$$\int_{-1}^{1} (1-t^2)^{-1/2} U_n(t) U_k(t)\,dt = 0 \qquad k \neq n$$

24.4 Differential Equations

24.4.1 For $T_n(t)$: $(1-t^2)y'' - ty' + n^2 y = 0$. For $U_n(t)$: $(1-t^2)y'' - 3ty' + n(n+2)y = 0$.

24.5 Generating Function

$$\frac{1-st}{1-2st+s^2} = \sum_{n=0}^{\infty} T_n(t) s^n$$

24.6 Rodrigues Formula

$$T_n(t) = \frac{(-2)^n n!}{(2n)!} \sqrt{1-t^2}\, \frac{d^n}{dt^n} (1-t^2)^{n-\frac{1}{2}}$$

24.7 Table of Chebyshev Properties

TABLE 24.1 Properties of the Chebyshev Polynomials

1.	$(1-t^2)\dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + n^2 y = 0 \quad ; \quad y(t) = T_n(t)$
2.	$T_n(t) = \dfrac{n}{2} \sum_{k=0}^{[n/2]} \dfrac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2t)^{n-2k}, \qquad n = 1, 2 \cdots, \quad [n/2] = \text{largest integer } \leq n/2$

TABLE 24.1 Properties of the Chebyshev Polynomials (continued)

3.	$T_n(t) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-t^2}\,\frac{d^n}{dt^n}(1-t^2)^{n-\frac{1}{2}}$, Rodrigues formula
4.	$T_n(t) = \cos(n\cos^{-1} t)$
5.	$\frac{1-st}{1-2st+s^2} = \sum_{n=0}^{\infty} T_n(t)s^n$, generating formula
6.	$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t)$
7.	$\int_{-1}^{1} \frac{T_n(t)T_m(t)}{\sqrt{(1-t^2)}}dt = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \neq 0 \\ \pi & n = m = 0 \end{cases}$
8.	$T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_{2n}(0) = (-1)^n$, $T_{2n+1}(0) = 0$

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25

Bessel Functions

25.1 Bessel Functions of the First Kind

25.1.1 Definition of Integer Order

$$J_n(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{n+2k}}{k!(n+k)!}, \qquad -\infty<t<\infty,\ \ n=0,1,2,\ldots$$

$$J_{-n}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k-n}}{k!(k-n)!}=\sum_{k=n}^{\infty}\frac{(-1)^k(t/2)^{2k-n}}{k!(k-n)!}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}(t/2)^{2m+n}}{m!(m+n)!}$$

$$J_{-n}(t)=(-1)^n J_n(t)$$

$$J_0(0)=1,\quad J_n(0)=0 \qquad n\neq 0$$

Figure 25.1 shows several Bessel functions of the first kind.

25.1.2 Definition of Nonintegral Order

$$J_v(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k+v}}{k!\Gamma(k+v+1)} \qquad v\geq 0$$

$$J_{-v}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k-v}}{k!\Gamma(k-v+1)} \qquad v\geq 0$$

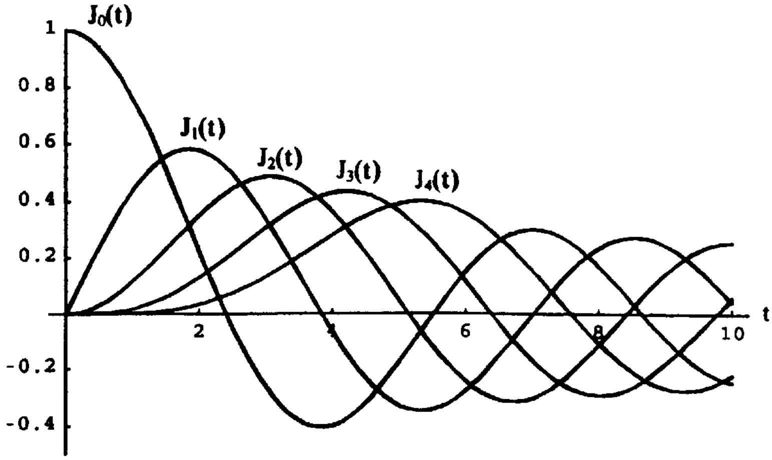

FIGURE 25.1

The two functions $J_{-v}(t)$ and $J_v(t)$ are linear independent for noninteger values of v and they do not satisfy any generating-function relation. The functions $J_{-v}(0) = \infty$ and $J_v(0)$ remain finite. Both these functions share most of the properties of $J_n(t)$ and $J_{-n}(t)$.

25.1.3 Generating Function

$$w(t,x) \doteq e^{\frac{1}{2}t(x-\frac{1}{x})} = \sum_{n=-\infty}^{\infty} J_n(t)\, x^n \qquad x \neq 0$$

25.1.4 Differential Equation

$$y'' + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right)y = 0 \qquad n = 0,1,2,\cdots$$

has solution the function $y = J_n(t)$

25.2 Recurrence Relation

25.2.1 Recurrence Relations

1. $$\frac{d}{dt}[t^v J_v(t)] = \frac{d}{dt}\sum_{k=0}^{\infty} \frac{(-1)^k (t)^{2k+2v}}{2^{2k+v}\, k!\,\Gamma(k+v+1)} = t^v \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k+(v-1)}}{k!\,\Gamma(k+v)}$$
$$= t^v J_{v-1}(t)$$

Similarly,

2. $$\frac{d}{dt}[t^{-v} J_v(t)] = -t^{-v} J_{v+1}(t)$$

Differentiate (1) and (2) and divide by t^v and t^{-v}, respectively, we find

3. $J_v'(t) + \frac{v}{t} J_v(t) = J_{v-1}(t)$

4. $J_v'(t) - \frac{v}{t} J_v(t) = -J_{v+1}(t)$

Set $v = 0$ in (4) to obtain

5. $J_0'(t) = -J_1(t)$

Add and subtract (3) and (4) to find, respectively, the relations

6. $2J_v'(t) = J_{v-1}(t) - J_{v+1}(t)$

7. $\frac{2v}{t} J_v(t) = J_{v-1}(t) + J_{v+1}(t)$

The above relation is known as the *three-term recurrence formula*. Repeated operations result in

8. $\left(\frac{d}{t\,dt}\right)^m [t^v J_v(t)] = t^{v-m} J_{v-m}(t); \qquad \left[\left(\frac{d}{t\,dt}\right)^2 y = \frac{1}{t}\frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dt}\right)\right]$

9. $\left(\frac{d}{t\,dt}\right)^m [t^{-v} J_v(t)] = (-1)^m t^{-v-m} J_{v+m}(t) \qquad m = 1,2,\ldots$

Example

We proceed to find the following derivative

$$\frac{d}{dt}[t^v J_v(at)] = \frac{d}{dt}\left[\left(\frac{u}{a}\right)^v J_v(u)\right] = \frac{d}{du}\left[\frac{u^v}{a^v} J_v(u)\right]\frac{du}{dt}$$

$$= a^{-v}\frac{d}{du}[u^v J_v(u)]a = a^{1-v}[u^v J_{v-1}(u)]$$

$$= a^{1-v}[(at)^v J_{v-1}(at)] = at^v J_{v-1}(at)$$

where (1) was used.

25.3 Integral Representation

25.3.1 Integral Representation

Set $x = \exp(-j\varphi)$ in (25.1.3), multiply both sides by $\exp(jn\varphi)$ and integrate the results from 0 to π. Hence

1. $\int_0^\pi e^{j(n\varphi - t\sin\varphi)}\,d\varphi = \sum_{k=-\infty}^{\infty} J_k(t)\int_0^\pi e^{j(n-k)\varphi}\,d\varphi$

Expand on both sides the exponentials in Euler's formula; equate the real and imaginary parts and use the relation

2. $\int_0^\pi \cos(n-k)\varphi\,d\varphi = \begin{cases} 0 & k \neq 0 \\ \pi & k = n \end{cases}$

to find that all terms of the infinite sum vanish except for $k = n$. Hence we obtain

3. $J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - t\sin\varphi)\,d\varphi \qquad n = 0,1,2,\ldots$

when $n = 0$, we find

4. $J_0(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t\sin\varphi)\,d\varphi$

For Bessel function with nonintegral order, the Poisson formula is

5. $J_v(t) = \frac{(t/2)^v}{\sqrt{\pi}\,\Gamma(v+\frac{1}{2})}\int_{-1}^{1} (1-x^2)^{v-\frac{1}{2}} e^{jtx}\,dx \qquad v > -\frac{1}{2},\ t > 0$

Set $x = \cos\theta$ to obtain

6. $J_v(t) = \frac{(t/2)^v}{\sqrt{\pi}\,\Gamma(v+\frac{1}{2})}\int_{0}^{\pi} \cos(t\cos\theta)\sin^{2v}\theta\,d\theta \qquad v > -\frac{1}{2},\ t > 0$

25.3.2 Integrals Involving Bessel Functions

Directly integrate (25.2.1.1) and (25.2.1.2) to find

$$\int t^{\nu} J_{\nu-1}(t)\,dt = t^{\nu} J_{\nu}(t) + C$$

$$\int t^{-\nu} J_{\nu+1}(t)\,dt = -t^{-\nu} J_{\nu}(t) + C$$

where C is the constant of integration.

Example

We apply the integration procedure to find

$$\begin{aligned}\int t^2 J_2(t)\,dt &= \int t^3[t^{-1}J_2(t)]\,dt = -\int t^3 \frac{d}{dt}[t^{-1}J_1(t)]\,dt \\ &= -t^2 J_1(t) + 3\int t\,J_1(t)\,dt = -t^2 J_1(t) - 3\int t[-J_1(t)]\,dt \\ &= -t^2 J_1(t) - 3\int t\left[\frac{d}{dt}J_0(t)\right]dt = -t^2 J_1(t) - 3t\,J_0(t) + 3\int J_0(t)\,dt\end{aligned}$$

The last integral has no closed solution.

Example

If $a > 0$ and $b > 0$, then (see 25.3.1.4)

$$\begin{aligned}\int_0^{\infty} e^{-at} J_0(bt)\,dt &= \int_0^{\infty} e^{-at}\,dt\,\frac{2}{\pi}\int_0^{\pi/2} \cos(bt\sin\varphi)\,d\varphi \\ &= \frac{2}{\pi}\int_0^{\pi/2} d\varphi \int_0^{\infty} e^{-at}\cos(bt\sin\varphi)\,dt = \frac{2}{\pi}\int_0^{\pi/2} \frac{a\,d\varphi}{a^2 + b^2\sin^2\varphi} = \frac{1}{\sqrt{a^2+b^2}}\end{aligned}$$

The usual method to find definite integrals involving Bessel functions is to replace the Bessel function by its series representation.

Example

$$I=\int_0^\infty e^{-at}t^p J_p(bt)\,dt \qquad p>-\frac{1}{2},\ \ a>0,\ \ b>0$$

$$=\sum_{k=0}^\infty \frac{(-1)^k (b/2)^{2k+p}}{k!\Gamma(k+p+1)}\int_0^\infty e^{-at}t^{2k+2p}\,dt$$

$$=b^p\sum_{k=0}^\infty \frac{(-1)^k\Gamma(2k+2p+1)}{2^{2k+p}k!\Gamma(k+p+1)}(a^2)^{-(p+\frac{1}{2})-k}(b^2)^k$$

where the last integral is in the form of a gamma function. But we know that

$$\binom{-r}{k}=(-1)^k\binom{r+k-1}{k},\qquad \binom{n}{k}=\binom{n}{n-k}$$

$$\binom{n+1}{k+1}=\binom{n}{k+1}+\binom{n}{k}\qquad 0\le k\le n-1$$

and thus we obtain

$$\frac{(-1)^k\Gamma(2k+2p+1)}{2^{2k+p}k!\Gamma(k+p+1)}=\frac{(-1)^k 2^p\Gamma\left(p+k+\frac{1}{2}\right)}{\sqrt{\pi}k!}$$

$$=\frac{(-1)^k}{\sqrt{\pi}}2^p\Gamma\left(p+\frac{1}{2}\right)\binom{p+k-\frac{1}{2}}{k}=\frac{2^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}}\binom{-\left(p+\frac{1}{2}\right)}{k}$$

Therefore, we obtain

$$I=\int_0^\infty e^{-at}t^p J_p(bt)\,dt=\frac{(2b)^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}}\sum_{k=0}^\infty\binom{-\left(p+\frac{1}{2}\right)}{k}(a^2)^{-(p+(1/2))-k}(b^2)^k$$

$$=\frac{(2b)^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{p+\frac{1}{2}}}\qquad p>-\frac{1}{2},\ \ a>0,\ \ b>0$$

Setting $p=0$ in this equation we find

$$\int_0^\infty e^{-at}J_0(bt)\,dt=\frac{1}{[a^2+b^2]^{1/2}}\qquad a>0,\quad b>0$$

Set a = 0+ in this equation to obtain

$$\int_0^\infty J_0(bt)\,dt = \frac{1}{b} \qquad b > 0$$

By assuming the real part to approach zero and writing a as pure imaginary, the equation before the previous one becomes

$$\int_0^\infty e^{-jat} J_0(bt)\,dt = \begin{cases} \dfrac{1}{(b^2 - a^2)^{1/2}} & b > a \\ \dfrac{-j}{(a^2 - b^2)^{1/2}} & b < a \end{cases}$$

The above integral, by equating real and imaginary parts, becomes

$$\int_0^\infty \cos(at) J_0(bt)\,dt = \frac{1}{(b^2 - a^2)^{1/2}} \qquad b > a$$

$$\int_0^\infty \sin(at) J_0(bt)\,dt = \frac{1}{(a^2 - b^2)^{1/2}} \qquad b < a$$

Example

To evaluate the integral $\int_0^b t\,J_0(at)\,dt$, we proceed as follows:

$$\int_0^\infty t\,J_0(at)\,dt = \int_0^\infty \frac{1}{a}\frac{d}{dt}[tJ_1(at)]\,dt$$

$$= \frac{1}{a}[t\,J_1(at)]\Big|_{t=0}^{b} = \frac{b}{a} J_1(ab) \qquad a \neq 0$$

where (25.2.1.1) with $\nu = 1$ was used.

25.4 Fourier-Bessel Series

25.4.1 Fourier-Bessel Series

$f(t) = \sum_{n=1}^\infty c_n J_\nu(t_n t) \quad 0 < t < a, \quad \nu > -\frac{1}{2},$ and $t_n \; (n = 1, 2, \cdots)$ are solutions of $J_\nu(t_n t) = 0 \quad n = 1, 2, \cdots.$

25.4.2 Product Property

$$\int_0^a t\,J_\nu(\alpha t)\,J_\nu(\beta t)\,dt = \frac{a\beta\,J_\nu(\alpha a)\,J'_\nu(\beta a) - a\alpha\,J_\nu(\beta a)\,J'_\nu(a\alpha)}{\alpha^2 - \beta^2}$$

25.4.3 Orthogonality

Setting $\alpha = t_n$, $\beta = t_m$ in (25.4.2), we obtain the orthogonality relation $\int_0^a t\,J_\nu(t_m t)\,J_\nu(t_n t)\,dt = 0$ since t_n and t_m are the roots of $J_\nu(t_n a)$ and $J_\nu(t_m a)$.

25.4.4 $t_m = t_n$

$$\int_0^a t[J_\nu(t_n t)]^2\,dt = \frac{a^2}{2}[J_{\nu+1}(t_n a)]^2,$$

which is found from (25.4.2) by limiting process $t_m \to t_n$ (using L' Hopital's rule and treating t_m as the variable).

25.4.5 Fourier Bessel Constants

Multiply (25.4.1) by $t\,J_\nu(t_m t)$, integrate from 0 to a, and use (25.4.4) to find

$$c_n = \frac{2}{a^2[J_{\nu+1}(t_n a)]^2}\int_0^a t\,f(t)\,J_\nu(t_n t)\,dt, \quad n = 1,2,\cdots.$$

Note: $f(t)$ must be piecewise continuous in the interval $(0,a)$ and $\int_0^a \sqrt{t}\,|f(t)|\,dt < \infty.$

Example

Find the Fourier-Bessel series for the function

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

corresponding to the set of functions $\{J_1(t_n t)\}$ where t_n satisfies $J_1(2t_n) = 0$ $(n = 1,2,3,\cdots)$.

Solution

We write the solution

$$f(t) = \sum_{n=1}^{\infty} c_n J_1(t_n t) \qquad 0 < t < 2$$

where

$$c_n = \frac{1}{2[J_2(2t_n)]^2}\int_0^2 tf(t)J_1(t_n t)\,dt$$

$$= (\cdot)\int_0^1 t^2 J_1(t_n t)\,dt \qquad (\text{let } r = t_n t)$$

$$= (\cdot)\frac{1}{t_n^3}\int_0^{t_n} r^2 J_1(r)\,dr \qquad (\text{apply (25.2.1.1)})$$

$$= (\cdot)\frac{1}{t_n^3}\int_0^{t_n} \frac{d}{dr}[r^2 J_2(r)]\,dr$$

$$= \frac{1}{2[J_2(2t_n)]^2 t_n^3}\,t_n^2 J_2(t_n) = \frac{J_2(t_n)}{2t_n[J_2(2t_n)]^2} \quad n = 1,2,3,...$$

Example

To express the function $f(t) = 1$ on the open interval $0 < t < a$ as an infinite series of Bessel functions of zero order, we proceed as follows (see 25.4.5):

$$c_n = \frac{2}{a^2[J_1(t_n a)]^2} \int_0^a t \cdot 1 \cdot J_0(t_n t)\, dt$$

$$= \frac{2}{a^2[J_1(t_n a)]^2} \int_0^a \frac{1}{t_n} \frac{d}{dt}[tJ_1(t_n t)]\, dt \qquad \text{(see (25.2.1.1))}$$

$$= \frac{2}{a^2 t_n [J_1(t_n a)]^2} [tJ_1(t_n t)]\Big|_{t=0}^{a} = \frac{2}{a t_n J_1(t_n a)}$$

Hence, the expression is

$$1 = 2\sum_{n=1}^{\infty} \frac{J_n(t_n t)}{t_n J_1(t_n a)} \qquad 0 < t < a$$

25.5 Properties of Bessel Function

TABLE 25.1 Properties of Bessel Functions

1.	$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!(n+k)!}$	$-\infty < t < \infty, \quad n = 0,1,2,3\cdots$
2.	$J_{-n}(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (t/2)^{2m+n}}{m!(m+n)!}$	$-\infty < t < \infty, \quad n = 0,1,2,3,\cdots$
3.	$J_{-n}(t) = (-1)^n J_n(t), \qquad J_n(-t) = (-1)^n J_n(t) = J_{-n}(t)$	$n = 0,1,2,3,\cdots$
4.	$J_0(0) = 1, \qquad J_n(0) = 0 \qquad n \neq 0$	
5.	$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}$	$\nu \geq 0, \quad \nu = \text{noninteger}$
6.	$J_{-\nu}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k-\nu}}{k!\Gamma(k-\nu+1)}$	$\nu \geq 0, \quad \nu = \text{noninteger}$
7.	$\frac{d}{dt}[t^\nu J_\nu(t)] = t^\nu J_{\nu-1}(t)$	
8.	$\frac{d}{dt}[t^\nu J_\nu(at)] = at^\nu J_{\nu-1}(at)$	
9.	$\frac{d}{dt}[t^{-\nu} J_\nu(t) = -t^{-\nu} J_{\nu+1}(t)$	
10.	$\frac{d^2 J_\nu(t)}{dt^2} = \frac{1}{2^2}[J_{\nu-2}(t) - 2J_\nu(t) + J_{\nu+2}(t)]$	
11.	$J'_\nu(t) + \frac{\nu}{t} J_\nu(t) = J_{\nu-1}(t)$	

TABLE 25.1 Properties of Bessel Functions (continued)

12. $J'_\nu(t) - \frac{\nu}{t} J_\nu(t) = -J_{\nu+1}(t), \quad J'_0(t) = -J_1(t)$

13. $t\,J'_\nu(t) = -\nu\,J_\nu(t) + t\,J_{\nu-1}(t)$

14. $2\,J'_\nu(t) = J_{\nu-1}(t) - J_{\nu+1}(t)$

15. $\frac{2\nu}{t} J_\nu(t) = J_{\nu-1}(t) + J_{\nu+1}(t)$

16. $\left(\frac{d}{t\,dt}\right)^m [t^\nu J_\nu(t)] = t^{\nu-m} J_{\nu-m}(t) \qquad m = 1, 2, 3, \cdots$

17. $\left(\frac{d}{tdt}\right)^m [t^{-\nu} J_\nu(t)] = (-1)^m t^{-\nu-m} J_{\nu+m}(t) \qquad m = 1, 2, 3, \cdots$

18. $J'_1(0) = \frac{1}{2}, \qquad J'_n(0) = 0 \qquad n > 1$

19. $J_n(t+r) = \sum_{k=-\infty}^{\infty} J_k(t) J_{n-k}(r)$

20. $J_0(2t) = [J_0(t)]^2 + 2\sum_{k=1}^{\infty} (-1)^k [J_k(t)]^2$

21. $|J_0(t)| \le 1, \qquad |J_n(t)| \le \frac{1}{\sqrt{2}} \qquad n = 1, 2, 3, \cdots$

22. $e^{jt\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(t) e^{jn\theta}$

23. $\cos(t\sin\theta) = J_0(t) + 2\sum_{n=1}^{\infty} J_{2n}(t)\cos(2n\theta)$

24. $\cos(t\cos\theta) = J_0(t) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(t)\cos(2n\theta)$

25. $\sin(t\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(t)\sin[(2n-1)\theta]$

26. $\sin(t\cos\theta) = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(t)\cos[(2n+1)\theta]$

27. $\cos t = J_0(t) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(t)$

28. $\sin t = 2\sum_{n=1}^{\infty} (-1)^n J_{2n-1}(t)$

29. $J_\nu(t) J_{1-\nu}(t) + J_{-\nu}(t) J_{\nu-1}(t) = \frac{2\sin\nu\pi}{\pi t}$ Lommel's formula

30. $\frac{d}{dt}[tJ_\nu(t) J_{\nu+1}(t)] = t[[J_\nu(t)]^2 - [J_{\nu+1}(t)]^2]$

31. $\frac{d}{dt}[t^2 J_{\nu-1}(t) J_{\nu+1}(t)] = 2t^2 J_\nu(t) J'_\nu(t)$

32. $J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t, \qquad J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}}\cos t$

33. $J_{1/2}(t) J_{-1/2}(t) = \frac{\sin 2t}{\pi t}, \qquad [J_{1/2}(t)]^2 + [J_{-1/2}(t)]^2 = \frac{2}{\pi t}$

34. $[J_0(t)]^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n!)^4} \left(\frac{t}{2}\right)^{2n}$

35. $J_n(t) = \frac{1}{\pi} \int_0^{\pi} \cos(n\varphi - t\sin\varphi)\, d\varphi$

36. $J_0(t) = \frac{1}{\pi} \int_0^{\pi} \cos(t\sin\varphi)\, d\varphi$

37. $J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_{-1}^{1} (1-x^2)^{\nu - \frac{1}{2}} e^{jtx}\, dx, \quad \nu > -\frac{1}{2} \quad t > 0$

38. $J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi} \cos(t\cos\theta)\sin^{2\nu}\theta\, d\theta \qquad \nu > -\frac{1}{2} \quad t > 0$

39. $\int t^\nu J_{\nu-1}(t)\, dt = t^\nu J_\nu(t) + C \qquad C = \text{constant}$

40. $\int t^{-\nu} J_{\nu+1}(t)\, dt = -t^{-\nu} J_\nu(t) + C \qquad C = \text{constant}$

41. $[1 + (-1)^n] J_n(t) = \frac{2}{\pi} \int_0^{\pi} \cos n\varphi \cos(t\sin\varphi)\, d\varphi \qquad n = 0, 1, 2, \cdots$

42. $J_{2k}(t) = \frac{1}{\pi} \int_0^{\pi} \cos 2k\varphi \cos(t\sin\varphi)\, d\varphi \qquad k = 0, 1, 2, \cdots$

43. $J_{2k+1}(t) = \frac{1}{\pi} \int_0^{\infty} \sin[(2k+1)\varphi] \sin(t\sin\varphi)\, d\varphi \qquad k = 0, 1, 2, \cdots$

44. $\int_0^{\pi} \cos[(2k+1)\varphi] \cos(t\sin\varphi)\, d\varphi = 0 \qquad k = 0, 1, 2, \cdots$

45. $\int_0^{\pi} \sin 2k\varphi \sin(t\sin\varphi)\, d\varphi = 0 \qquad k = 0, 1, 2, \cdots$

46. $J_0(t) = \frac{2}{\pi} \int_0^{1} \frac{\cos tx}{\sqrt{1-x^2}}\, dx$

47. $\frac{2\sin t}{t} = \sqrt{\frac{2\pi}{t}}\, J_{1/2}(t)$

48. $\int t\, J_0(t)\, dt = tJ_1(t) + C$

49. $\int t^2 J_0(t)\, dt = t^2 J_1(t) + t\, J_0(t) - \int J_0(t)\, dt + C$

50. $\int t^3 J_0(t)\, dt = (t^3 - 4t)\, J_1(t) + 2t^2 J_0(t) + C$

51. $\int J_1(t)\, dt = -J_0(t) + C$

52. $\int t\, J_1(t)\, dt = -tJ_0(t) + \int J_0(t)\, dt + C$

53. $\int t^2 J_1(t)\,dt = 2t\,J_1(t) - t^2 J_0(t) + C$

54. $\int t^3 J_1(t)\,dt = 3t^2 J_1(t) - (t^3 - 3t)\,J_0(t) - 3\int J_0(t)\,dt + C$

55. $\int J_3(t)\,dt = -J_2(t) - 2t^{-1} J_1(t) + C$

56. $\int t^{-1} J_1(t)\,dt = -J_1(t) + \int J_0(t)\,dt + C$

57. $\int t^{-2} J_2(t)\,dt = -\frac{2}{3t^2} J_1(t) - \frac{1}{3} J_1(t) + \frac{1}{3t} J_0(t) + \frac{1}{3}\int J_0(t)\,dt + C$

58. $\int J_0(t)\cos t\,dt = t\,J_0(t)\cos t + t\,J_1(t)\sin t + C$

59. $\int J_0(t)\sin t\,dt = t\,J_0(t)\sin t - t\,J_1(t)\cos t + C$

60. $\int_0^\infty e^{-at} t^\nu J_\nu(bt)\,dt = \frac{(2b)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}(a^2 + b^2)^{\nu + \frac{1}{2}}} \qquad \nu > -\frac{1}{2}, \quad a > 0, \quad b > 0$

61. $\int_0^\infty e^{-at} J_0(bt)\,dt = \frac{1}{(a^2 + b^2)^{1/2}} \qquad a > 0,\ \ b > 0$

62. $\int_0^\infty J_0(bt)\,dt = \frac{1}{b} \qquad b > 0$

63. $\int_0^\infty J_{n+1}(t) = \int_0^\infty J_{n-1}(t)\,dt \qquad n = 1, 2, \cdots$

64. $\int_0^\infty J_n(at)\,dt = \frac{1}{a} \quad a > 0$

65. $\int_0^\infty t^{-1} J_n(t)\,dt = \frac{1}{n} \qquad n = 1, 2, \cdots$

66. $\int_0^\infty e^{-at} t^{\nu+1} J_\nu(bt)\,dt = \frac{2^{\nu+1}\Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{\pi}} \frac{ab^\nu}{(a^2 + b^2)^{\nu + \frac{3}{2}}} \qquad \nu > -1,\ a > 0,\ b > 0$

67. $\int_0^\infty t^2 e^{-at} J_0(bt)\,dt = \frac{2a^2 - b^2}{(a^2 + b^2)^{5/2}} \qquad a > 0,\ b > 0$

68. $\int_0^\infty e^{-at^2} t^{\nu+1} J_p(bt)\,dt = \frac{b^\nu e^{-b^2/4a}}{(2a)^{\nu+1}} \qquad \nu > -1,\ a > 0,\ b > 0$

69. $\int_0^\infty e^{-at^2} t^{\nu+3} J_\nu(bt)\,dt = \frac{b^\nu}{2^{\nu+1} a^{\nu+2}} \left(\nu + 1 - \frac{b^2}{4a}\right) e^{-b^2/4a} \qquad \nu > -1,\ a > 0,\ b > 0$

70. $\int_0^\infty t^{-1} \sin t\, J_0(bt)\,dt = \arcsin\left(\frac{1}{b}\right) \qquad b > 1$

TABLE 25.1 Properties of Bessel Functions (continued)

71. $\int_0^{\pi/2} J_0(t\cos\varphi)\cos\varphi\, d\varphi = \frac{\sin t}{t}$

72. $\int_0^{\pi/2} J_1(t\cos\varphi)\, d\varphi = \frac{1-\cos t}{t}$

73. $\int_0^{\infty} e^{-t\cos\varphi} J_0(t\sin\varphi) t^n\, dt = n! P_n(\cos\varphi) \qquad 0 \le \varphi < \pi$

$P_n(t) = nth$ Legendre polynomial

74. $\int_0^{\infty} t(t^2+a^2)^{-1/2} J_0(bt)\, dt = \frac{e^{-ab}}{b} \qquad a \ge 0,\ b > 0$

75. $\int_0^{\infty} \frac{J_\nu(t)}{t^m} dt = \frac{\Gamma((\nu+1-m)/2)}{2^m \Gamma((\nu+1+m)/2)} \qquad m > \frac{1}{2},\ \nu - m > -1$

76. $\frac{1}{8}(1-t^2) = \sum_{n=1}^{\infty} \frac{J_0(k_n t)}{k_n^3 J_1(k_n)} \qquad 0 \le t \le 1,\ J_0(k_n) = 0 \quad n = 1, 2, \cdots$

77. $t^\nu = 2\sum_{n=1}^{\infty} \frac{J_\nu(k_n t)}{k_n J_{\nu+1}(k_n)} \qquad 0 < t < 1,\ J_\nu(k_n) = 0,\ n = 1, 2, \cdots$

78. $t^{\nu+1} = 2^2(\nu+1)\sum_{n=1}^{\infty} \frac{J_{\nu+1}(k_n t)}{k_n^2 J_{\nu+1}(k_n)} \qquad 0 < t < 1,\ \nu > -1/2,\ J_\nu(k_n) = 0,\ n = 1, 2, \cdots$

TABLE 25.2

					$J_0(x)$					
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.0000	.9975	.9900	.9776	.9604	.9385	.9120	.8812	.8463	.8075
1	.7652	.7196	.6711	.6201	.5669	.5118	.4554	.3980	.3400	.2818
2	.2239	.1666	.1104	.0555	.0025	–.0484	–.0968	–.1424	–.1850	–.2243
3	–.2601	–.2921	–.3202	–.3443	–.3643	–.3801	–.3918	–.3992	–.4026	–.4018
4	–.3971	–.3887	–.3766	–.3610	–.3423	–.3205	–.2961	–.2693	–.2404	–.2097
5	–.1776	–.1443	–.1103	–.0758	–.0412	–.0068	.0270	.0599	.0917	.1220
6	.1506	.1773	.2017	.2238	.2433	.2601	.2740	.2851	.2931	.2981
7	.3001	.2991	.2951	.2882	.2786	.2663	.2516	.2346	.2154	.1944
8	.1717	.1475	.1222	.0960	.0692	.0419	.0146	–.0125	–.0392	–.0653
9	–.0903	–.1142	–.1376	–.1577	–.1768	–.1939	–.2090	–.2218	–.2323	–.2403
10	–.2459	–.2490	–.2496	–.2477	–.2434	–.2366	–.2276	–.2164	–.2032	–.1881
11	–.1712	–.1528	–.1330	–.1121	–.0902	–.0677	–.0446	–.0213	.0020	.0250
12	.0477	.0697	.0908	.1108	.1296	.1469	.1626	.1766	.1887	.1988
13	.2069	.2129	.2167	.2183	.2177	.2150	.2101	.2032	.1943	.1836
14	.1711	.1570	.1414	.1245	.1065	.0875	.0679	.0476	.0271	.0064
15	–.0142	–.0346	–.0544	–.0736	–.0919	–.1092	–.1253	–.1401	–.1533	–.1650

When $x > 15.9$,

$$J_0(x) \cong \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\sin\left(x + \frac{1}{4}\pi\right) + \frac{1}{8x}\sin\left(x - \frac{1}{4}\pi\right)\right\}$$

$$\cong \frac{.7979}{\sqrt{x}}\left\{\sin(57.296x + 45°) + \frac{1}{8x}\sin(57.296x - 45°)\right\}$$

$J_1(x)$										
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0499	.0995	.1483	.1960	.2423	.2867	.3290	.3688	.4059
1	.4401	.4709	.4983	.5220	.5419	.5579	.5699	.5778	.5815	.5812
2	.5767	.5683	.5560	.5399	.5202	.4971	.4708	.4416	.4097	.3754
3	.3391	.3009	.2613	.2207	.1792	.1374	.0955	.0538	.0128	–.0272
4	–.0660	–.1033	–.1386	–.1719	–.2028	–.2311	–.2566	–.2791	–.2985	–.3147
5	–.3276	–.3371	–.3432	–.3460	–.3453	–.3414	–.3343	–.3241	–.3110	–.2951
6	–.2767	–.2559	–.2329	–.2081	–.1816	–.1538	–.1250	–.0953	–.0652	–.0349
7	–.0047	.0252	.0543	.0826	.1096	.1352	.1592	.1813	.2014	.2192
8	.2346	.2476	.2580	.2657	.2708	.2731	.2728	.2697	.2641	.2559
9	.2453	.2324	.2174	.2004	.1816	.1613	.1395	.1166	.0928	.0684
10	.0435	.0184	–.0066	–.0313	–.0555	–.0789	–.1012	–.1224	–.1422	–.1603
11	–.1768	–.1913	–.2039	–.2143	–.2225	–.2284	–.2320	–.2333	–.2323	–.2290
12	–.2234	–.2157	–.2060	–.1943	–.1807	–.1655	–.1487	–.1307	–.1114	–.0912
13	–.0703	–.0489	–.0271	–.0052	.0166	.0380	.0590	.0791	.0984	.1165
14	.1334	.1488	.1626	.1747	.1850	.1934	.1999	.2043	.2066	.2069
15	.2051	.2013	.1955	.1879	.1784	.1672	.1544	.1402	.1247	.1080

When $x > 15.9$,

$$J_1(x) \cong \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\sin\left(x - \frac{1}{4}\pi\right) + \frac{3}{8x}\sin\left(x + \frac{1}{4}\pi\right)\right\}$$

$$\cong \frac{.7979}{\sqrt{x}}\left\{\sin(57.296x - 45)^\circ + \frac{3}{8x}\sin(57.296x + 45)^\circ\right\}$$

$J_2(x)$										
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0012	.0050	.0112	.0197	.0306	.0437	.0588	.0758	.0946
1	.1149	.1366	.1593	.1830	.2074	.2321	.2570	.2817	.3061	.3299
2	.3528	.3746	.3951	.4139	.4310	.4461	.4590	.4696	.4777	.4832
3	.4861	.4862	.4835	.4780	.4697	.4586	.4448	.4283	.4093	.3879
4	.3641	.3383	.3105	.2811	.2501	.2178	.1846	.1506	.1161	.0813

When $0 \le x < 1$,

$$J_2(x) \cong \frac{x^2}{8}\left(1 - \frac{x^2}{12}\right)$$

$J_3(x)$										
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0000	.0002	.0006	.0013	.0026	.0044	.0069	.0122	.0144
1	.0196	.0257	.0329	.0411	.0505	.0610	.0725	.0851	.0988	.1134
2	.1289	.1453	.1623	.1800	.1981	.2166	.2353	.2540	.2727	.2911
3	.3091	.3264	.3431	.3588	.3754	.3868	.3988	.4092	.4180	.4250
4	.4302	.4333	.4344	.4333	.4301	.4247	.4171	.4072	.3952	.3811

When $0 \le x < 1$,

$$J_3(x) \cong \frac{x^3}{48}\left(1 - \frac{x^3}{16}\right)$$

$J_4(x)$										
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	..0000	.0000	.0000	.0001	.0002	.0003	.0006	.0010	.0016
1	.0025	.0036	.0050	.0068	.0091	.0118	.0150	.0188	.0232	.0283
2	.0340	.0405	.0476	.0556	.0643	.0738	.0840	.0950	.1067	.1190
3	.1320	.1456	.1597	.1743	.1891	.2044	.2198	.2353	.2507	.2661
4	.2811	.2958	.3100	.3236	.3365	.3484	.3594	.3693	.3780	.3853

When $0 \le x < 1$, $\quad J_4(x) \cong \frac{x^4}{384}\left(1-\frac{x^2}{20}\right)$

TABLE 25.3 Zeros of $J_o(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$, $J_4(x)$, $J_5(x)$

m	$j_{0,m}$	$j_{1,m}$	$j_{2,m}$	$j_{3,m}$	$j_{4,m}$	$j_{5,m}$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178
6	18.0711	19.6159	21.1170	22.5827	24.0190	25.4303
7	21.2116	22.7601	24.2701	25.7482	27.1991	28.6266
8	24.3525	25.9037	27.4206	28.9084	30.3710	31.8117
9	27.4935	29.0468	30.5692	32.0649	33.5371	34.9888
10	30.6346	32.1897	33.7165	35.2187	36.6990	38.1599

25.6 Bessel Functions of the Second Kind

25.6.1 Definition

$$Y_\nu(t) = \frac{(\cos\nu\pi)J_\nu(t) - J_{-\nu}(t)}{\sin\nu\pi} \qquad \nu \ne n$$

The above function is the second independent solution to the differential equation $t^2y'' + ty' + (t^2 - \nu^2)y = 0$.

25.6.2 Recurrence Relations

1. $\frac{d}{dt}\left[t^\nu Y_\nu(t)\right] = t^\nu Y_{\nu-1}(t)$
2. $\frac{d}{dt}\left[t^{-\nu} Y_\nu(t)\right] = -t^{-\nu} Y_{\nu+1}(t)$
3. $Y_{\nu-1}(t) + Y_{\nu+1}(t) = \frac{2\nu}{t} Y_\nu(t)$
4. $Y_{\nu-1}(t) - Y_{\nu+1}(t) = 2Y'_\nu(t)$
5. $Y_{-n}(t) = (-1)^n Y_n(t) \qquad n = 0,1,2,\cdots,\ \nu \to n$

25.6.3 Approximations

1. $Y_o(t) \cong \frac{2}{\pi}\ln t, \qquad t \to 0+$

2. $Y_\nu(t) \cong -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{t}\right)^\nu, \qquad \nu > 0,\ t \to 0+$

3. $Y_\nu(t) \cong \sqrt{\frac{2}{\pi t}}\sin\left[t - \frac{(\nu+\frac{1}{2})\pi}{2}\right], \qquad t \to \infty$

25.7 Modified Bessel Function

25.7.1 Definition

$$I_\nu(t) = j^{-\nu} J_\nu(jt) = \sum_{m=0}^{\infty} \frac{(t/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)} \qquad \nu \neq \text{integer},\ n = 0,1,2,\cdots$$

$$I_{-n}(t) = I_n(t)$$

$$I_0(0) = 1; \quad I_\nu(0) = 0 \qquad \nu > 0$$

25.7.2 Recurrence Relations

1. $\frac{d}{dt}\left[t^\nu I_\nu(t)\right] = t^\nu I_{\nu-1}(t)$

2. $\frac{d}{dt}\left[t^{-\nu} I_\nu(t)\right] = t^{-\nu} I_{\nu+1}(t)$

3. $I'_\nu(t) + \frac{\nu}{t} I_\nu(t) = I_{\nu-1}(t)$

4. $I'_\nu(t) - \frac{\nu}{t} I_\nu(t) = I_{\nu+1}(t)$

5. $I_{\nu-1}(t) + I_{\nu+1}(t) = 2I'_\nu(t)$

6. $I_{\nu-1}(t) - I_{\nu+1}(t) = \frac{2\nu}{t} I_\nu(t)$

25.7.3 Integral Representation

$$I_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}} e^{-xt}\,dx \qquad \nu > -\tfrac{1}{2},\ t > 0$$

25.7.4 Expansion Form

$$I_n(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} J_{n+m}(t), \qquad n = 0,1,2,\cdots$$

25.7.5 Asymptotic Formulas

1. $I_\nu(t) \cong \dfrac{(t/2)^\nu}{\Gamma(\nu+1)}, \qquad t \to 0+, \quad \nu \neq -1, -2, -3, \cdots$

2. $I_\nu(t) \cong \dfrac{e^t}{\sqrt{2\pi t}}, \qquad \nu \geq 0, \quad t \to \infty$

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26

Zernike Polynomials

26.1 Zernike Polynomials

26.1.1 Definition

$$V_{nl}(x,y) = V_{nl}(r\cos\theta, r\sin\theta) = V_{nl}(r,\theta) = R_{nl}(r)e^{jl\theta}$$

n = nonnegative integer, $n \geq 0$; l = integer subject to constraints: $n - |l|$ = even and $|l| \leq n$; r = length of vector from origin to (x,y) point; θ = angle between r and x axis in counterclockwise direction.

26.1.2 Orthogonality Property

1. $$\iint_{x^2+y^2\leq 1} V_{nl}^*(r,\theta)V_{mk}(r,\theta)r\,dr\,d\theta = \frac{\pi}{n+1}\delta_{mn}\delta_{k\ell}$$

where δ_{ij} is the Kronecker symbol. The real valued radial polynomials satisfy the orthogonality relation

2. $$\int_0^1 R_{nl}(r)R_{ml}(r)r\,dr = \frac{1}{2(n+1)}\delta_{mn}$$

The radial polynomials are given by

3. $$R_{n\pm|l|}(r) = \frac{1}{\left(\frac{n-|l|}{2}\right)!\,r^m}\left[\frac{d}{d(r^2)}\right]^{\frac{n-|l|}{2}}\left[(r^2)^{\frac{n+|l|}{2}}(r^2-1)^{\frac{n-|l|}{2}}\right]$$

$$= \sum_{s=0}^{\frac{n-|l|}{2}}(-1)^s\frac{(n-s)!}{s!\left(\frac{n+|l|}{2}-s\right)!\left(\frac{n-|l|}{2}-s\right)!}r^{n-2s}$$

For all permissible values of n and $|l|$

4. $R_{n\pm|l|}(1) = 1, \qquad R_{n|l|}(r) = R_{n(-|l|)}(r)$

Table 26.1 gives the explicit form of the function $R_{n(|l|)}(r)$.

TABLE 26.1 The Radial Polynomials $R_{n(|l|)}(r)$ for $|l| \le 8,\ n \le 8$

n / $\lvert l \rvert$	0	1	2	3	4	5	6	7	8
0	1		$2r^2-1$		$6r^4-6r^2+1$		$20r^6-30r^4+12r^2-1$		$70r^8-140r^6+90r^4-20r^2+1$
1		r		$3r^3-2r$		$10r^5-12r^3+3r$		$35r^7-60r^5+30r^3-4r$	
2			r^2		$4r^4-3r^2$		$15r^6-20r^4+6r^2$		$56r^8-105r^6+60r^4-10r^2$
3				r^3		$5r^5-4r^3$		$21r^7-30r^5+10r^3$	
4					r^4		$6r^6-5r^4$		$28r^8-42r^6+15r^4$
5						r^5		$7r^7-6r^5$	
6							r^6		$8r^8-7r^6$
7								r^7	
8									r^8

26.1.3 Relation to Bessel Function

A relation between radial Zernike polynomials and Bessel functions of the first kind is given by

$$\int_0^1 R_{n|l|}(r)\,J_n(vr)\,r\,dr = (-1)^{\frac{n-|l|}{2}}\,\frac{J_{n+1}(v)}{v}$$

26.1.4 Real Zernike Polynomials

$$U_{nl} = \frac{1}{2}[V_{nl} + V_{n(-l)}] = R_{nl}(r)\cos l\theta \qquad l \neq 0$$

$$U_{n(-l)} = \frac{1}{2j}[V_{nl} - V_{n(-l)}] = R_{nl}(r)\sin l\theta \qquad l \neq 0$$

$$V_{n0} = R_{n0}(r)$$

Figure 26.1 shows the function U_{nl} for a few radial modes.

26.2 Expansion in Zernike Polynomials

26.2.1 Zernike Series

If $f(x,y)$ is a piecewise continuous function, we can expand this function in Zernike polynomials in the form

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} A_{nl}\,V_{nl}(x,y), \qquad n-|l| = \text{even}, \quad |l| \leq n$$

Multiplying by $V_{nl}^{*}(x,y)$, integrating over the unit circle and taking into consideration the orthogonality property, we obtain

$$A_{nl} = \frac{n+1}{\pi}\int_0^{2\pi}\int_0^1 V_{nl}^{*}(r,\theta)\,f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \frac{n+1}{\pi}\iint_{x^2+y^2\leq 1} V_{nl}^{*}(x,y)\,f(x,y)\,dx\,dy = A_{n(-l)}^{*}$$

with restrictions of the values of n and l as shown above. A_{nl}'s are also known as *Zernike moments.*

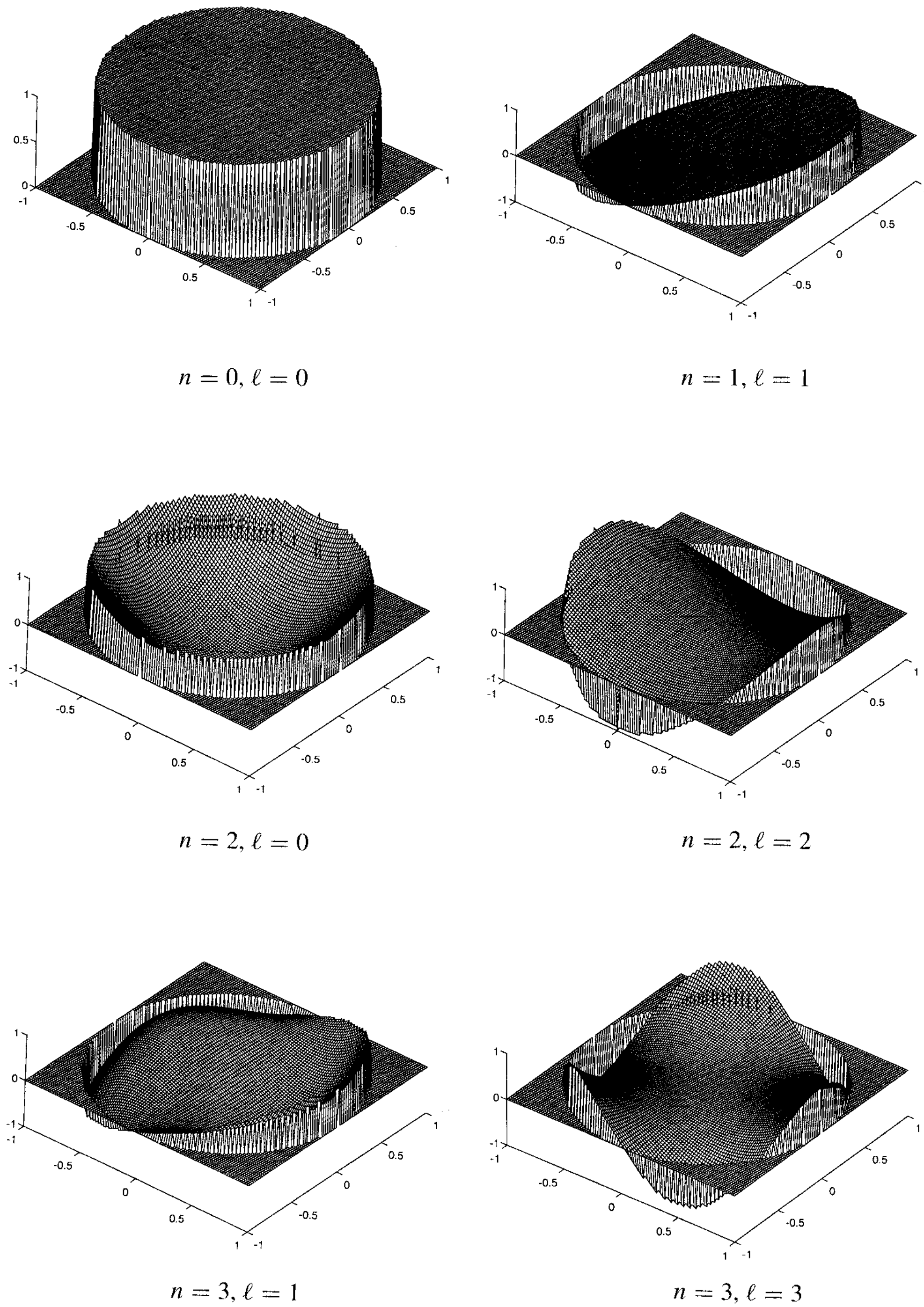
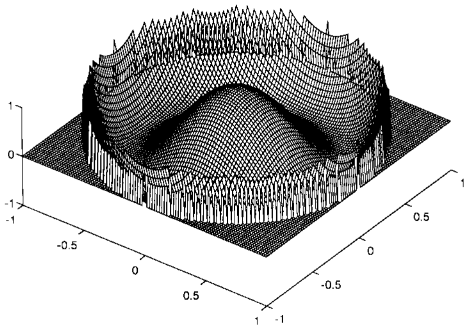
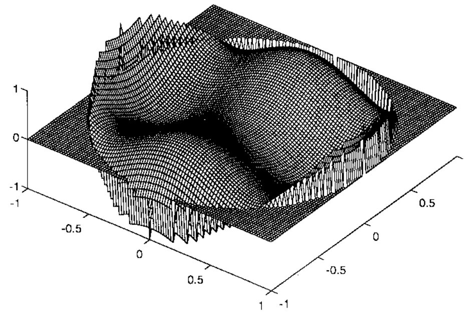


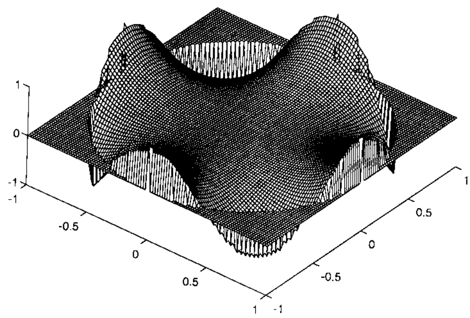
FIGURE 26.1



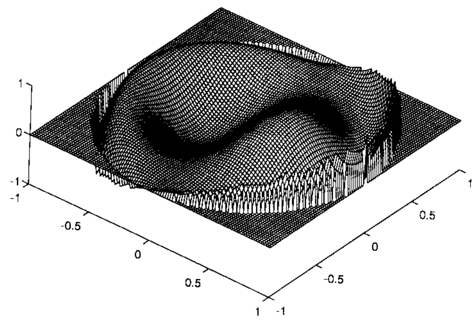
$n = 4, \ell = 0$



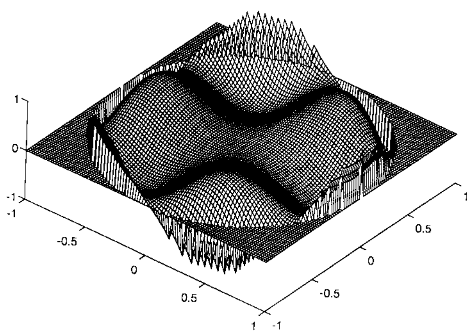
$n = 4, \ell = 2$



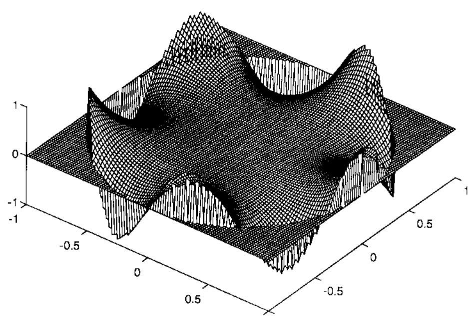
$n = 4, \ell = 4$



$n = 5, \ell = 1$



$n = 5, \ell = 3$



$n = 5, \ell = 5$

FIGURE 26.1(continued)

Example

Expand the function $f(x,y) = x$ in Zernike polynomials.

Solution

We write $f(r\cos\theta, r\sin\theta) = r\cos\theta$ and observe that r has exponent (degree) one. Therefore, the values of n will be 0,1 and since $n-|l|$ must be even, l will take 0,1 and –1 values. We then write

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} A_{nl}\, R_{nl}(r) e^{jl\theta}$$

$$= \sum_{n=0}^{1} (A_{n(-1)}\, R_{n(-1)}(r) e^{-j\theta} + A_{n0}\, R_{n0}(r) + A_{n1}\, R_{n1}(r) e^{j\theta})$$

$$= A_{00}\, R_{00}(r) + A_{1(-1)}\, R_{1(-1)}(r) e^{-j\theta} + A_{11}\, R_{11}(r) e^{j\theta}$$

where three terms were dropped because they did not obey the relation $n-|l|$ = even. From (26.1.2.4) $R_{1(-1)}(r) = R_{11}(r)$ and hence we obtain

$$A_{00} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{1} R_{00}(r)\, r\cos\theta\, r\,dr\,d\theta = 0$$

$$A_{1(-1)} = \frac{2}{\pi}\int_0^{2\pi}\int_0^{1} R_{11}(r)\, r\cos\theta\, e^{-j\theta}\, r\,dr\,d\theta = \frac{1}{2}$$

$$A_{11} = \frac{2}{\pi}\int_0^{2\pi}\int_0^{1} R_{11}(r)\, r\cos\theta\, e^{j\theta}\, r\,dr\,d\theta = \frac{1}{2}$$

Therefore, the expansion becomes

$$f(x,y) = \frac{1}{2} r e^{j\theta} + \frac{1}{2} r e^{-j\theta} = r\cos\theta = R_{11}(r)\cos\theta = x$$

as was expected.

26.2.2 Expansion of Real Functions

1. $$f(x,y) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} (C_{nl}\cos l\theta + S_{nl}\sin l\theta) R_{nl}(r)$$

where n – l is even and l < n and $f(x,y)$ is real. Observe that l takes only a positive value. The unknown constants are found from

2. $$\begin{bmatrix} C_{nl} \\ S_{nl} \end{bmatrix} = \frac{2n+2}{\pi}\int_0^{1}\int_0^{2\pi} r\,dr\,d\theta\, f(r\cos\theta, r\sin\theta) R_{nl}(r) \begin{bmatrix} \cos l\theta \\ \sin l\theta \end{bmatrix} \qquad l \neq 0$$

3. $$C_{n0} = A_{n0} = \frac{1}{\pi}\int_0^{1}\int_0^{2\pi} r\,dr\,d\theta\, f(r\cos\theta, r\sin\theta) R_{nl}(r) \qquad l = 0$$

4. $S_{n0} = 0$

If the function is axially symmetric only the cosine terms are needed. The connection between real and complex Zernike coefficients are:

5. $C_{nl} = 2\,\mathrm{Re}\{A_{nl}\}$

$S_{nl} = -2\,\mathrm{Im}\{A_{nl}\}$

$A_{nl} = (C_{nl} - jS_{nl})/2 = (A_{n(-l)})^*$

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27

Special Functions

27.1 The Gamma and Beta Functions

27.1.1 Gamma Function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{Re}\{z\} > 0$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad x > 0$$

The gamma function converges for all $x > 0$.

27.1.2 Incomplete Gamma Function

$$\gamma(x,\tau) = \int_0^\tau t^{x-1} e^{-t}\, dt \qquad x > 0, \quad \tau > 0$$

27.1.3 Beta Function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \qquad x > 0, \quad y > 0$$

The beta function is related to gamma function as follows:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

27.1.4 Properties of $\Gamma(x)$

Setting $x + 1$ in place of x we obtain

1. $$\Gamma(x+1) = \int_0^\infty t^{x+1-1} e^{-t}\,dt = \int_0^\infty t^x e^{-t}\,dt$$

$$= -\int_0^\infty t^x d(e^{-t}) = -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x\,t^{x-1} e^{-t}\,dt$$

$$= x\Gamma(x)$$

From the above relation, we also obtain

2. $$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

3. $$\Gamma(x) = (x-1)\Gamma(x-1)$$

4. $$\Gamma(-x) = \frac{\Gamma(x-1)}{-x} \qquad x \neq 0,1,2,\cdots$$

5. From (27.1.1) with $x = 1$, we find that $\Gamma(1) = 1$. Using (27.1.4.1) we obtain

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1\cdot 1 = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2\cdot 1$$

6. $$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n! \qquad n = 0,1,2,\cdots$$

7. $$\Gamma(n) = (n-1)! \qquad n = 0,1,2,\cdots$$

To find $\Gamma(\frac{1}{2})$ we first set $t = u^2$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty 2e^{-u^2}\,du \qquad (t = u^2)$$

Hence, its square value is

$$\Gamma^2\left(\frac{1}{2}\right) = \left[\int_0^\infty 2e^{-x^2}\,dx\right]\left[\int_0^\infty 2e^{-y^2}\,dy\right]$$

$$= 4\int_0^\infty \left[\int_0^\infty e^{-y^2}\,dy\right] e^{-x^2}\,dx = 4\int_0^{\pi/2}\left[\int_0^\infty e^{-r^2} r\,dr\right]d\theta$$

$$= 4\frac{\pi}{2}\cdot\frac{1}{2} = \pi$$

and thus

8. $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Next, let's find the expression for $\Gamma(n+\frac{1}{2})$ for integer positive value of n. From (27.1.4.3) we obtain

9. $$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n+1}{2}-1\right)\Gamma(\frac{2n+1}{2}-1) = \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right)$$
$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\Gamma\left(\frac{2n-3}{2}\right)$$

If we proceed to apply (27.1.4.3), we finally obtain

10. $$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^n}$$

Similarly we obtain

11. $$\Gamma\left(n+\frac{3}{2}\right) = \frac{(2n+1)(2n-1)(2n-3)\cdots(3)(1)\sqrt{\pi}}{2^{n+1}}$$

12. $$\Gamma\left(n-\frac{1}{2}\right) = \frac{(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^{n-1}}$$

Example

Applying (27.1.4.3) we find

$$2^n\Gamma(n+1) = 2^n n\Gamma(n) = 2^n n(n-1)\Gamma(n-1) = \cdots = 2^n n(n-1)(n-2)\cdots 2\cdot 1$$
$$= 2^n n! = (2\cdot 1)(2\cdot 2)(2\cdot 3)\cdots(2\cdot n) = 2\cdot 4\cdot 6\cdots 2n$$

If $n-1$ is substituted in place of n, we obtain

$$2\cdot 4\cdot 6\cdots(2n-2) = 2^{n-1}\Gamma(n)$$

27.1.5 Duplication Formula

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\;\Gamma(2z)$$

27.1.6 Graph of Gamma Function

Figure 27.1 shows the gamma function.

27.1.7 Definition of Beta Function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad x>0,\quad y>0$$

$$B(x,y) = \int_0^{\pi/2} 2\sin^{2x-1}\theta\cos^{y-1}\theta\,d\theta$$

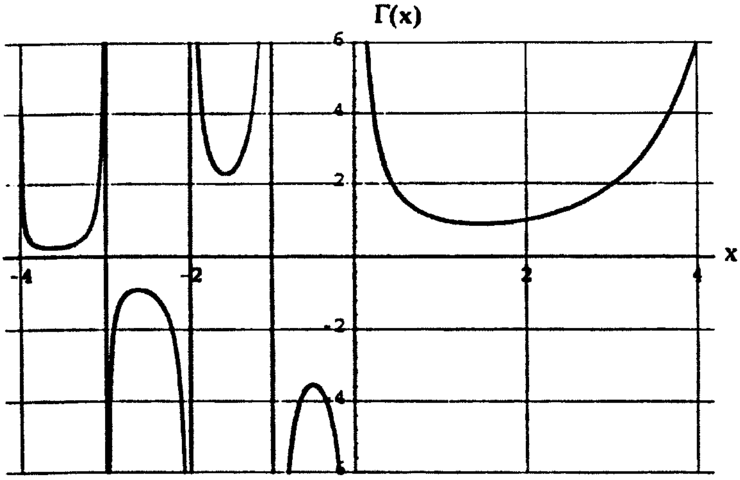

FIGURE 27.1 The gamma function.

$$B(x,y)=\int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}\,du \qquad x>0,\quad y>0$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$B(p,1-p)=\frac{\pi}{\sin p\pi} \qquad 0<p<1$$

Example

$$I=\int_o^\infty x^{-1/2}(1+x)^{-2}\,dx=B\left(\frac{1}{2},\frac{3}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}=\frac{\sqrt{\pi}\,(\sqrt{\pi}/2)}{1}=\frac{\pi}{2}$$

27.1.8 Properties of Beta Function

1. $B(x,y)=B(x+1,y)+B(x,y+1)$
2. $B(x,y)=B(y,x)$

27.1.9 Table of Gamma and Beta Function Relations

TABLE 27.1 Gamma and Beta Function Relations

$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt$	$x>0$
$\Gamma(x)=\int_0^\infty 2u^{2x-1}e^{-u^2}\,du$	$x>0$

TABLE 27.1 Gamma and Beta Function Relations (continued)

$\Gamma(x) = \int_0^1 \left[\log\left(\frac{1}{r}\right)\right]^{x-1} dr$	$x > 0$
$\Gamma(x) = \frac{\Gamma(x+1)}{x}$	$x \neq 0, -1, -2, \cdots$
$\Gamma(x) = (x-1)\Gamma(x-1)$	$x \neq 0, -1, -2, \cdots$
$\Gamma(-x) = \frac{\Gamma(1-x)}{-x}$	$x \neq 0, 1, 2, \cdots$
$\Gamma(n) = (n-1)!$	$n = 1, 2, 3, \cdots \quad , \quad 0! =$
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$	
$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots\cdot(2n-1)\sqrt{\pi}}{2^n}$	$n = 1, 2, \cdots$
$\Gamma\left(n+\frac{3}{2}\right) = \frac{(2n+1)(2n-1)(2n-3)\cdots(3)(1)\sqrt{\pi}}{2^{n+1}}$	$n = 1, 2, \cdots$
$\Gamma\left(n-\frac{1}{2}\right) = \frac{(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^{n-1}}$	$n = 1, 2, \cdots$
$\Gamma(n+1) = \frac{2\cdot 4\cdot 6\cdots\cdot 2n}{2^n}$	$n = 1, 2, \cdots$
$\Gamma(2n) = 1\cdot 3\cdot 5\cdots\cdot(2n-1)\Gamma(n)2^{1-n}$	$n = 1, 2, \cdots$
$\frac{\Gamma(2n)}{\Gamma(n)} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\ 2^{1-2n}}$	$n = 1, 2, \cdots$
$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}$	$x \neq 0, \pm 1, \pm 2, \cdots$
$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} + h$	$n = 1, 2, \cdots \quad , \quad 0 < \frac{h}{n!} < \frac{1}{12n}$
$\int_0^\infty t^a e^{-btc}\, dt = \frac{\Gamma\left(\frac{a+1}{c}\right)}{c\, b^{(a+1)/c}}$	$a > -1, \quad b > 0, \quad c > 0$
$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$	$x > 0, \quad y > 0$
$B(x,y) = \int_0^{\pi/2} 2\sin^{2x-1}\theta\cos^{2y-1}\theta\, d\theta$	$x > 0, \quad y > 0$
$B(x,y) = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}\, du$	$x > 0, \quad y > 0$
$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$	
$B(x,y) = B(y,x)$	
$B(x, 1-x) = \frac{\pi}{\sin x\pi}$	$0 < x < 1$
$B(x,y) = B(x+1,y) + B(x,y+1)$	$x > 0, \quad y > 0$

TABLE 27.1 Gamma and Beta Function Relations (continued)

$B(x,n+1)=\frac{1\cdot 2\cdots n}{x(x+1)\cdots(x+n)}$	$x>0$

27.1.10 Table of the Gamma Function

TABLE 27.2 $\Gamma(x)$, $1 \le x \le 1.99$

x	0	1	2	3	4	5	6	7	8	9
1.0	1.0000	.9943	.9888	.9835	.9784	.9735	.9698	.9642	.9597	.9555
.1	.9514	.9474	.9436	.9399	.9364	.9330	.9298	.9267	.9237	.9209
.2	.9182	.9156	.9131	.9108	.9085	.9064	.9044	.9025	.9007	.8990
.3	.8975	.8960	.8946	.8934	.8922	.8912	.8902	.8893	.8885	.8879
.4	.8873	.8868	.8864	.8860	.8858	.8857	8856	.8856	.8857	.8859
.5	.8862	.8866	.8870	.8876	.8882	.8889	.8896	.8905	.8914	.8924
.6	.8935	.8947	.8859	.8972	.8986	.9001	.9017	.9033	.9050	.9068
.7	.9086	.9106	.9126	.9147	.9168	.9191	.9214	.9238	.9262	.9288
.8	.9314	.9341	.9368	.9397	.9426	.9456	.9487	.9518	.9551	.9584
.9	.9618	.9652	.9688	.9724	.9761	.9799	.9837	.9877	.9917	.9958

27.2 Error Function

27.2.1 Error Function

$$erf\, z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

27.2.2 Coerror Function

$$erfc\, z=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt=1-erf\, z$$

27.2.3 Series Expansion

$$erf\, z=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)} \qquad |z|<\infty$$

27.2.4 Symmetry Relation

$$erf\,(-z)=-erf\, z,\quad erf\, z^*=(erf\, z)^*$$

27.3 Sine and Cosine Integrals

27.3.1 Sine Integral

$$Si(z)=\int_0^z\frac{\sin t}{t}\,dt$$

27.3.2 Cosine Integral

$$Ci(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt \qquad |\arg z| < \pi$$

27.3.3

$$si(z) = Si(z) - \frac{\pi}{2}$$

27.3.4 Auxiliary Functions

$$f(z) = Ci(z)\sin z - si(z)\cos z, \qquad g(z) = -Ci(z)\cos z - si(z)\sin z$$

27.3.5 Series Expansion

$$Si(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}, \qquad Si(z) = \pi \sum_{n=0}^{\infty} J_{n+\frac{1}{2}}^2\left(\frac{z}{2}\right)$$

$$Ci(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n(2n)!}$$

27.3.6 Symmetry Relation

$$Si(-z) = -Si(z), \quad Si(z^*) = [Si(z)]^*, \quad Ci(-z) = Ci(z) - j\pi,$$

$$0 < \arg z < \pi, \quad Ci(z^*) = [Ci(z)]^*$$

27.3.7 Value at Infinity

$$\lim_{x \to \infty} Si(x) = \frac{\pi}{2}$$

27.4 Fresnel Integrals

27.4.1 Fresnel Integrals

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt, \quad S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$$

27.4.2 Extrema

$C(x)$ has extrema at $x = \pm\sqrt{2n+1}$, $S(x)$ has extrema at $x = \pm\sqrt{2n}$ $(n = 0,1,2\cdots)$

27.4.3 Values at Infinity

$$C(\infty) = S(\infty) = \frac{1}{2}$$

27.4.4 Series Expansion

$$C(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n}}{(2n)!(4n+1)}z^{4n+1},\quad S(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!(4n+3)}z^{4n+3}$$

27.4.5 Symmetry Relation

$$C(-z)=-C(z),\quad S(-z)=-S(z),\quad C(jz)=jC(z),\quad S(iz)=-jS(z),$$

$$C(z^*)=[C(z)]^*,\quad S(z^*)=[S(z)]^*$$

27.4.6 Relation to Error Function

$$C(z)+jS(z)=\frac{1+j}{2}erf\left[\frac{\sqrt{\pi}}{2}(1-j)z\right]$$

27.5 Exponential Integrals

27.5.1 $$E_1(x)=\int_x^{\infty}\frac{e^{-t}}{t}dt=\int_1^{\infty}\frac{e^{-xt}}{t}dt$$

27.5.2 $$E_n(x)=\int_1^{\infty}\frac{e^{-xt}}{t^n}dt\qquad (x>0,\ n=0,1,2,\cdots)$$

$$E_{n+1}(x)=\frac{1}{n}[e^{-x}-xE_n(x)]\qquad (n=1,2,\cdots)$$

27.5.3 $$Ei(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt\qquad (x>0,\ \text{Cauchy P.V.})$$

27.5.4 $$\ell i(x)=\int_0^x\frac{dt}{\ln t}=Ei(\ln x)\qquad (x>1,\ \text{Cauchy P.V.})$$

27.5.5 Series Expansions

$$E_1(x)=-\gamma-\ln x-\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n\,n!}\qquad (x>0)$$

$$Ei(x)=\gamma+\ln x+\sum_{n=1}^{\infty}\frac{x^n}{n\,n!}\qquad (\gamma = \text{Euler's constant})$$

27.5.6 Special Values

$$E_n(0)=\frac{1}{n-1}\ \ (n>1),\qquad E_0(x)=\frac{e^{-x}}{x}$$

27.5.7 Derivatives

$$\frac{dE_n(x)}{dx} = -E_{n-1}(x) \qquad (n = 1, 2, \cdots)$$

27.6 Elliptic Integrals

27.6.1 Elliptic Integrals of the First Kind

$$F(k, \varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} \qquad (k^2 < 1, \quad x = \sin \varphi)$$

27.6.2 Elliptic Integrals of the Second Kind

$$E(k, \varphi) = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta = \int_0^x \sqrt{\left(\frac{1 - k^2 t^2}{1 - t^2} \right)}\, dt \qquad (k^2 < 1, \quad x = \sin \varphi)$$

27.6.3 Elliptic Integrals of the Third Kind

$$\pi(k, n, \varphi) = \int_0^{\varphi} \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

$$= \int_0^x \frac{dt}{(1 + mt^2)\sqrt{(1-t^2)(1-k^2 t^2)}} \qquad (k^2 < 1, \quad x = \sin \varphi)$$

27.6.4 Complete Elliptic Integrals

$$K = K(k) = F\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \qquad (k^2 < 1)$$

$$E = E(k) = E\left(k, \frac{\pi}{2}\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \qquad (k^2 < 1)$$

27.6.5 Legendre's Relation

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2} \qquad (k' = \sqrt{1 - k^2})$$

27.6.6 Differential Equations

$$k(1 - k^2)\frac{d^2 K}{dk^2} + (1 - 3k^2)\frac{dK}{dk} - kK = 0$$

$$k(1 - k^2)\frac{d^2 E}{dk^2} + (1 - k^2)\frac{dE}{dk} + kE = 0$$

27.6.7 Table of Complete Elliptic Integrals

TABLE 27.3 Numerical tables of complete elliptic integrals $k = \sin a$ (a in degrees)

a	K	E	a	K	E	a	K	E
0°	1.5708	1.5708	50°	1.9356	1.3055	81°.0	3.2553	1.0338
1	1.5709	1.5707	51	1.9539	1.2963	81.2	3.2771	1.0326
2	1.5713	1.5703	52	1.9729	1.2870	81.4	3.2995	1.0314
3	1.5719	1.5697	53	1.9927	1.2776	81.6	3.3223	1.0302
4	1.5727	1.5689	54	2.0133	1.2681	81.8	3.3458	1.0290
5	1.5738	1.5678	55	2.0347	1.2587	82.0	3.3699	1.0278
6	1.5751	1.5665	56	2.0571	1.2492	82.2	3.3946	1.0267
7	1.5767	1.5649	57	2.0804	1.2397	82.4	3.4199	1.0256
8	1.5785	1.5632	58	2.1047	1.2301	82.6	3.4460	1.0245
9	1.5805	1.5611	59	2.1300	1.2206	82.8	3.4728	1.0234
10	1.5828	1.5589	60	2.1565	1.2111	83.0	3.5004	1.0223
11	1.5854	1.5564	61	2.1842	1.2015	83.2	3.5288	1.0213
12	1.5882	1.5537	62	2.2132	1.1920	83.4	3.5581	1.0202
13	1.5913	1.5507	63	2.2435	1.1826	83.6	3.5884	1.0192
14	1.5946	1.5476	64	2.2754	1.1732	83.8	3.6196	1.0182
15	1.5981	1.5442	65	2.3088	1.1638	84.0	3.6519	1.0172
16	1.6020	1.5405	65.5	2.3261	1.1592	84.2	3.6852	1.0163
17	1.6061	1.5367	66.0	2.3439	1.1545	84.4	3.7198	1.0153
18	1.6105	1.5326	66.5	2.3622	1.1499	84.6	3.7557	1.0144
19	1.6151	1.5283	67.0	2.3809	1.1453	84.8	3.7930	1.0135
20	1.6200	1.5238	67.5	2.4001	1.1408	85.0	3.8317	1.0127
21	1.6252	1.5191	68.0	2.4198	1.1362	85.2	3.8721	1.0118
22	1.6307	1.5141	68.5	2.4401	1.1317	85.4	3.9142	1.0110
23	1.6365	1.5090	69.0	2.4610	1.1272	85.6	3.9583	1.0102
24	1.6426	1.5037	69.5	2.4825	1.1228	85.8	4.0044	1.0094
25	1.6490	1.4981	70.0	2.5046	1.1184	86.0	4.0528	1.0086
26	1.6557	1.4924	70.5	2.5273	1.1140	86.2	4.1037	1.0079
27	1.6627	1.4864	71.0	2.5507	1.1096	86.4	4.1574	1.0072
28	1.6701	1.4803	71.5	2.5749	1.1053	86.6	4.2142	1.0065
29	1.6777	1.4740	72.0	2.5998	1.1011	86.8	4.2744	1.0059
30	1.6858	1.4675	72.5	2.6256	1.0968	87.0	4.3387	1.0053
31	1.6941	1.4608	73.0	2.6521	1.0927	87.2	4.4073	1.0047
32	1.7028	1.4539	73.5	2.6796	1.0885	87.4	4.4811	1.0041
33	1.7119	1.4469	74.0	2.7081	1.0844	87.6	4.5609	1.0036
34	1.7214	1.4397	74.5	2.7375	1.0804	87.8	4.6477	1.0031
35	1.7312	1.4323	75.0	2.7681	1.0764	88.0	4.7427	1.0026
36	1.7415	1.4248	75.5	2.7998	1.0725	88.2	4.8478	1.0021
37	1.7522	1.4171	76.0	2.8327	1.0686	88.4	4.9654	1.0017
38	1.7633	1.4092	76.5	2.8669	1.0648	88.6	5.0988	1.0014
39	1.7748	1.4013	77.0	2.9026	1.0611	88.8	5.2527	1.0010
40	1.7868	1.3931	77.5	2.9397	1.0574	89.0	5.4349	1.0008
41	1.7992	1.3849	78.0	2.9786	1.0538	89.1	5.5402	1.0006
42	1.8122	1.3765	78.5	3.0192	1.0502	89.2	5.6579	1.0005
43	1.8256	1.3680	79.0	3.0617	1.0468	89.3	5.7914	1.0004
44	1.8396	1.3594	79.5	3.1064	1.0434	89.4	5.9455	1.0003
45	1.8541	1.3506	80.0	3.1534	1.0401	89.5	6.1278	1.0002

TABLE 27.3 Numerical tables of complete elliptic integrals $k = \sin a$ (a in degrees)

a	K	E	a	K	E	a	K	E
46	1.8691	1.3418	80.2	3.1729	1.0388	89.6	6.3509	1.0001
47	1.8848	1.3329	80.4	3.1928	1.0375	89.7	6.6385	1.0001
48	1.9011	1.3238	80.6	3.2132	1.0363	89.8	7.0440	1.0000
49	1.9180	1.3147	80.8	3.2340	1.0350	89.9	7.7371	1.0000

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28

Asymptotic Expansions

28.1 Introduction

28.1.1 Order O ( )

$f(x)=O(g(x))$ as $x\to x_0$ if there exists a constant A such that $|f(x)|\le A|g(x)|$ for all values of x in some neighborhood of x_0.

28.1.2 Order o( )

$f(x)=o(g(x))$ as $x\to x_0$ if $\lim_{x\to x_0}|f(x)/g(x)|=0$

28.1.3 Order One

$f(x)\sim g(x)$ as $x\to x_0$ if $\lim_{x\to x_0}|f(x)/g(x)|=1$

28.1.4 Examples

1. $\sin x=O(1),\ x\to\infty,$
2. $(1+x^2)^{-1}=O(1),\ x\to 0,$
3. $(1+x^2)^{-1}=o(x^{-1}),\ x\to\infty,$
4. $(1+x^2)^{-1}=O(x^{-2}),\ x\to\infty,$
5. $(1+x^2)^{-1}\sim x^{-2},\ x\to\infty,$
6. $(1+x^2)^{-1}=x^{-2}+o(x^{-2}),\ x\to\infty,$
7. $(1+x^2)^{-1}=x^{-2}+o(x^{-3}),\ x\to\infty,$
8. $(1+x^2)^{-1}=x^{-2}+O(x^{-4}),\ x\to\infty,$
9. $(1+x^2)^{-1}=x^{-2}-x^{-4}+O(x^{-6}),\ x\to\infty,$
10. $n/(n+1)\sim 1,\ n\to\infty,$
11. $\sin x\sim x,\ x\to 0,$
12. $\cos x=1+O(x^2),\ x\to 0,$
13. $\sqrt{n^2+1}\sim n,\ n\to\infty,$
14. $\sqrt{n^2+1}=n+o(1),\ n\to\infty,$
15. $\sqrt{n^2+1}=n+O(n^{-1}),\ n\to\infty,$
16. $(n/e)^n=O(n!),\ n\to\infty,$

17. $\sum_{n=1}^{\infty} x^n = O((1-x)^{-1}),\ x \to 1-,$

18. $\sum_{n=1}^{\infty} n^p x^n = O((1-x)^{-p}),\ x \to 1-,$

19. $\int_2^x \frac{dy}{y} = O(\log x),\ x \to \infty$

28.1.5 Asymptotic Sequence

$\{\varphi_n(x)\}_{n=0}^{\infty}$ is an asymptotic sequence for some fixed point x_0 if for each fixed n we have $\varphi_{n+1}(x) = o(\varphi_n(x))$ as $x \to x_0$.

Example

The sequence $1, x, x^2, \cdots$ is an asymptotic sequence for 0.

28.1.6 Asymptotic Series

An asymptotic series for a given function $f(x)$ at x_0 for each fixed integer n is given by

$$f(x) = a_0\varphi_0(x) + \cdots + a_n\varphi_n(x) + o(\varphi_n(x)) \text{ as } x \to x_0$$

or

$$f(x) \cong \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(x) \quad \text{as} \quad x \to x_0$$

28.1.7 Example

If $f(x) = \int_0^\infty \frac{e^{-t}\,dt}{x+t}$ with x large and positive, then integration by parts yields: $f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^n \frac{n!}{x^{n+1}} + (-1)^n (n+1)! \int_0^\infty \frac{e^{-t}\,dt}{(x+t)^{n+2}}$. The remainder term is

$$|R_n(x)| = (n+1)! \int_0^\infty \frac{e^{-t}\,dt}{(x+t)^{n+2}} = \frac{(n+1)!}{x^{n+1}} \int_0^\infty \frac{e^{-xy}}{(1+y)^{n+2}}\,dy \le \frac{(n+1)!}{x^{n+1}} \int_0^\infty e^{-xy}\,dy = \frac{(n+1)!}{x^{n+2}}$$

Hence, we terminate the expansion of $f(x)$ after the *nth* term (ignoring the remainder); the error is of the order of $\varphi_{n+1} = o(\varphi_n(x))$. Hence we write

$$f(x) \cong \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\nu!}{x^{n+1}} \text{ as } x \to \infty.$$

28.2 Sums

28.2.1 Bernoulli's Numbers B_n (n = 1,2,...):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{z^n}{n!} B_n$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2n+1} = 0\ \ (n = 1,2,3,\cdots),\ B_{2n} = (-1)^{n+1}(2n)!\frac{\zeta(2n)}{(2\pi)^{2n}}2\ \ (n = 1,2,3,\cdots),$$

where

$$\zeta(2n) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}}\ \ (n = 1,2,\cdots).$$

Also: $B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730},\ B_{14} = \frac{7}{6},\ B_{16} = -\frac{3617}{510},$ $B_{18} = \frac{43867}{798},\ B_{20} = -\frac{174611}{330}$

28.2.2 First-Form of Euler-Maclaurin Sum Formula

If $f(x)$ is continuously differentiable on the interval $[1,n]$ then

$$\sum_{\nu=1}^{\infty} f(\nu) = \int_1^n f(x)\,dx + \frac{1}{2}[f(1) + f(n)] + \int_1^n \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx$$

where [x] is the greatest integer contained in x (e.g., $[2] = 2,\ [4.114] = 4,\ [0.315] = 0$).

Example

If $f(x) = \frac{1}{x}$ then

$$\sum_{\nu=1}^{n} \frac{1}{\nu} = \int_1^n \frac{dx}{x} + \frac{1}{2}\left(1 + \frac{1}{n}\right) - \int_1^n \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x^2} = \ln n + \frac{1}{2} + \frac{1}{2n} - \int_1^n \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x^2}.$$

The integral on the right is $o(1)$ as $n \to \infty$ since $(x - [x] - \frac{1}{2})$ is less than $\frac{1}{2}$ in an absolute value and hence $\sum_{\nu=1}^{n} \frac{1}{\nu} = \ln n + O(1)$ as $n \to \infty$.

28.2.3 Euler's Constant

$$\gamma = \frac{1}{2} - \int_1^{\infty} \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x^2} = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = 0.5772\ldots$$

which comes from the last integral of the last example where the integral is split into two integrals $\int_1^n (\cdot)\,dx + \int_n^{\infty} (\cdot)\,dx$.

28.2.4 Second-Form Euler-Maclaurin Sum Formula

If $f(x)$ is $2k+1$ times continuously differentiable in $[1,n]$ then $\sum_{\nu=1}^{\infty} f(\nu) = \int_1^n f(x)\,dx + \frac{1}{2}(f(1)+f(n))$

$+ \frac{B_2}{2!}(f'(n)-f'(1)) + \frac{B_4}{4!}(f'''(n)-f'''(1)) + \cdots + \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n)-f^{(2k-1)}(1)) + \int_1^n P_{2k+1}(x)\,f^{(2k+1)}(x)\,dx.$

$P_{2k}(x) = (-1)^{k+1}\sum_{n=1}^{\infty}\frac{2\cos 2n\pi x}{(2n\pi)^{2k}}$ and $P_{2k+1}(x) = (-1)^{k+1}\sum_{n=1}^{\infty}\frac{2\sin 2n\pi x}{(2n\pi)^{2k+1}}$, $k=1,2,\cdots$, and B_i's are the Bernoulli's numbers (see 28.2.1).

28.3 Stirling's Formula

28.3.1 Stirling's Formula

$$\ln(n!) = \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{B_2}{1\cdot 2}\frac{1}{n} + \frac{B_4}{3\cdot 4}\frac{1}{n^3} + \cdots + \frac{B_{2k}}{2k(2k-1)}\frac{1}{n^{2k-1}} + O(n^{-2k}) \text{ as } n\to\infty.$$

28.3.2 Stirling's Formula

$$\ln(n!) = (n+\tfrac{1}{2})\ln n - n + \ln\sqrt{2\pi} + o(1) \text{ as } n\to\infty$$

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ as } n\to\infty$$

28.4 Sums of Powers

28.4.1 Sum of Powers

$$1^p + 2^p + \cdots + (n-1)^p = \frac{1}{p+1}\sum_{\nu=0}^{p}\binom{p+1}{\nu}B_\nu\, n^{p+1-\nu}$$

1. (p = 1) $\quad 1+2+3+\cdots+n = \frac{n^2}{2}+\frac{n}{2}$
2. (p = 2) $\quad 1^2+2^2+3^2+\cdots+n^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$
3. (p = 3) $\quad 1^3+2^3+3^3+\cdots+n^3 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$
4. (p = 4) $\quad 1^4+2^4+3^4+\cdots+n^4 = \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

28.5 Laplace Method for Integrals

28.5.1 Laplace Theorem

a) If $h(y)$ is real valued and continuous,
b) $h(0) = 0$ and $h(y) < 0$ for $y \neq 0$,
c) there are numbers $\alpha > \beta$ such that $h(y) \le -\alpha$ when $|y| \ge \beta$,
d) there is a neighborhood of $y = 0$ in which $h(y)$ is twice differentiable and $h''(0) < 0$, then

$$G(x) = \int_{-\infty}^{\infty} e^{xh(y)}\, dy \cong \left[\frac{2\pi}{-xh''(o)}\right]^{1/2} \quad \text{as } x \to \infty.$$

Example

To the Stirling formula $n! = \int_0^\infty e^{-t} t^n\, dt$ we replace n with continuous variable x and hence $x! = \Gamma(x + 1) = \int_0^\infty e^{-t} t^x\, dt$. Since the integrand has a maximum at $t = x$ we make the substitution $t = x(y + 1)$ to bring it to the above standard form. Hence $x! = x^{x+1} e^{-x} \int_{-1}^{\infty} \exp[x(\log(1+y) - y)]dy$, and thus $h(y) = \log(1+y) - y$ which satisfies all the conditions above. Therefore, $x! = \Gamma(x+1) \cong \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ as $x \to \infty$, since $h''(0) = -1$.

28.6 The Method of Stationary Phase

28.6.1 Theorem

If the function $r(t)$ is continuous and the derivative of the function $\mu(t)$ vanishes at only a single point $t = t_0$ in the interval $(-\infty,\infty) : \mu'(t_0) = 0,\ \mu''(t_0) \neq 0,$ then for sufficiently large k,

$$\int_{-\infty}^{\infty} r(t) e^{jk\mu(t)} dt \cong e^{jk\mu(t_0)} r(t_0) \sqrt{\frac{2\pi j}{k\mu''(t_0)}}\,.$$

Example

In the relation $J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jx\sin t}\, dt,\ r(t) = \frac{1}{2\pi},\ k = x$ and $\mu(t) = \sin t$. In the interval $(-\pi,\pi)$, $\mu'(t) = \cos t = 0$ for $t = t_1 = \pi/2$ and $t = t_2 = -\pi/2$. Since $\mu(t_1) = 1,\ \mu''(t_1) = -1,\ \mu(t_2) = -1$ and $\mu''(t_2) = 1,$ we conclude that

$$J_0(x) \cong \frac{e^{jx}}{2\pi}\sqrt{\frac{2\pi j}{x(-1)}} + \frac{e^{-jx}}{2\pi}\sqrt{\frac{2\pi j}{x}} = \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}\left(e^{jx - j\frac{\pi}{4}} + e^{-jx + j\frac{\pi}{4}}\right) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$$

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Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC,1999

29

Nonrecursive Filters (Finite Impulse Response, Fir)

29.1 Properties of Nonrecursive Filters

29.1.1 Causal Filter

$$H(z)=\sum_{n=0}^{N-1} h(nT_s)z^{-n}$$

$$H(e^{j\omega T_s})=M(\omega)e^{j\theta(\omega)}=\sum_{n=0}^{N-1} h(nT_s)e^{j\omega nT_s}$$

$$M(\omega)=\left|H(e^{j\omega T_s})\right|,\;\;\theta(\omega)=\arg H(e^{j\omega T_s})$$

$$T_s = \text{sampling time}$$

29.1.2 Phase and Group Delays

$$\tau_p=-\frac{\theta(\omega)}{\omega},\quad \tau_g=-\frac{d\theta(\omega)}{d\omega}$$

29.1.3 Constant Phase and Group Delays

$$\theta(\omega) = -\tau\omega = \tan^{-1} \frac{-\sum_{n=0}^{N-1} h(nT_s)\sin\omega nT_s}{\sum_{n=0}^{N-1} h(nT_s)\cos\omega nT_s},$$

$$\tau = \frac{(N-1)T_s}{2}, \quad h(nT_s) = h[(N-1-n)T_s] = \text{symmetrical}$$

Impulse response must be symmetrical about the midpoint between samples $(N-2)/2$ and $N/2$ for even N and about samples $(N-1)/2$ for odd N.

29.1.4 Constant Group Delay

$\theta(\omega) = \theta_0 - \tau\omega$; with $\theta_0 = \pm\pi/2$ we must have

$$\tau = \frac{(N-1)T_s}{2} \text{ and } h(nT_s) = -h[(N-1-n)T_s] = \text{antisymmetrical}$$

29.1.5 Frequency Response of Constant-Delay Nonrecursive Filters

$h(nT_s)$	N	$H(e^{j\omega T_s})$
Symmetrical	Odd	$e^{-j\omega(N-1)T_s/2} \sum_{k=0}^{(N-1)/2} a_k \cos\omega kT_s$
	Even	$e^{-j\omega(N-1)T_s/2} \sum_{k=1}^{N/2} b_k \cos\left[\omega\left(k-\frac{1}{2}\right)T_s\right]$
Antisymmetrical	Odd	$e^{-j[\omega(N-1)T_s/2-\frac{\pi}{2}]} \sum_{k=1}^{(N-1)/2} a_k \sin\omega kT_s$
	Even	$e^{-j[\omega(N-1)T_s/2-\frac{\pi}{2}]} \sum_{k=1}^{N-2} b_k \sin\left[\omega\left(k-\frac{1}{2}\right)T_s\right]$
$a_0 = h\left[\frac{(N-1)T_s}{2}\right]$,	$a_k = 2h\left[\left(\frac{N-1}{2}-k\right)T_s\right]$,	$b_k = 2h\left[\left(\frac{N}{2}-k\right)T_s\right]$

29.2 Fourier Series Design

29.2.1 **FIR Filter** is periodic function of ω with period $\omega_s = 2\pi/T_s$.

29.2.2 Fourier Series

$$H(e^{j\omega T_s}) = \sum_{n=-\infty}^{\infty} h(nT_s)e^{-j\omega nT_s}$$

$$h(nT_s) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T_s}) e^{j\omega n T_s}\, d\omega$$

29.2.3 Z-Transform Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h(nT_s) z^{-n} \qquad (z = e^{j\omega T_s})$$

29.2.4 Noncausal Finite-Order Filter

$$h(nT_s) = 0 \quad \text{for} \quad |n| > \frac{N-1}{2}$$

$$H(z) = h(0) + \sum_{n=1}^{(N-1)/2} [h(-nT_s) z^n + h(nT_s) z^{-n}] = \text{noncausal}$$

29.2.5 Causal Finite-Order Filter

$$H'(z) = z^{-(N-1)/2} H(z)$$

Example

Design a low-pass filter with a frequency response

$$H(e^{j\omega T_s}) = \begin{cases} 1 \text{ for } |\omega| \le \omega_c \\ 0 \text{ for } \omega_c < |\omega| \le \dfrac{\omega_s}{2}, \end{cases} \qquad \omega_s = \text{sampling frequency}$$

Solution

From (29.2.2)

$$h(nT_s) = \frac{1}{\omega_s} \int_{-\omega_c}^{\omega_c} e^{j\omega n T_s}\, d\omega = \frac{1}{n\pi} \sin \omega_c n T_s$$

From (29.2.4) and (29.2.5)

$$H(z) = z^{-(N-1)/2} \sum_{n=0}^{(N-1)/2} \frac{a_n}{2} (z^n + z^{-n}), \quad a_0 = h(0), \ a_n = 2h(nT_s)$$

For example, it may be requested that $\omega_c = \text{cutoff frequency} = 10\,\text{rad}\,\text{s}^{-1}$ and the sampling frequency $\omega_s = 30\,\text{rad}\,\text{s}^{-1}$. This implies that $T_s = 2\pi/30$, $z = e^{j\omega T_s} = \exp(j\omega 2\pi/30)$ and N is taken to be a relatively small number such as N = 21, 41, 51.

29.3 Window Functions in FIR Filters

29.3.1 Window Functions (see also Chapter 7)

$$H(z) = Z\{h(nT_s)\} = \sum_{n=-\infty}^{\infty} h(nT_s) z^{-n}$$

$$W(z) = Z\{w(nT_s)\} = \sum_{n=-\infty}^{\infty} w(nT_s) z^{-n}, \qquad w(nT_s) = \text{window function}$$

$$H_w(z) = Z\{w(nT_s) h(nT_s)\}$$

29.3.2 The Fourier Transform of the Windowed Filter

$$H_w(e^{j\omega T_s}) = \frac{T_s}{2\pi} \int_0^{2\pi/T_s} H(e^{j\xi T_s}) W(e^{j(\omega-\xi)T_s}) d\xi$$

In the ξ-domain

$$H(e^{j\xi T_s}) = \begin{cases} 1 & \text{for} \quad 0 \le |\xi| \le \omega_c \\ 0 & \text{for} \quad \omega_c < |\xi| \le \dfrac{\omega_s}{2} \end{cases}$$

and let $W(e^{j\xi T_s})$ be real and assume

$$W(e^{j\xi T_s}) = 0 \quad \text{for} \;\; \omega_m \le |\xi| \le \frac{\omega_s}{2}$$

29.3.3 Properties of Window Function $w(nT_s)$

1. $w(nT_s)$ for $|n| > \dfrac{N-1}{2}$
2. For odd N, it must be symmetrical about sample $n = 0$
3. Width of main lobe: $k\omega_s / N$, $k = \text{constant}$
4. Sidelobes give Gibbs oscillations in the amplitude response of the filter

29.4 Windows Frequently Used

29.4.1 Rectangular

$$w_R(nT_s) = \begin{cases} 1 & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$W(e^{j\omega T_s}) = \sum_{n=-(N-1)/2}^{(N-1)/2} e^{-j\omega n T_s} = \frac{\sin(\omega N T_s / 2)}{\sin(\omega T_s / 2)}$$

Lobe Widths

$$W(e^{j\omega T_s}) = 0 \text{ at } \omega = m\omega_s / N, \quad m = \pm 1, \pm 2, \cdots$$

$$\text{Main lobe width} = 2\omega_s / N$$

Ripple Ratio

$$r = \frac{100(\text{maximum side-lobe amplitude})}{\text{main-lobe amplitude}} \%$$

29.4.2 Hann and Hamming Windows

$$w_H(nT_s) = \begin{cases} \alpha + (1-\alpha)\cos\dfrac{2\pi n}{N-1} & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha = 0.5 \text{ Hann window}, \ \alpha = 0.54 \text{ Hamming window}$$

29.4.3 Blackman Window

$$w_B(nT_s) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1} & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Example

Design a low-pass filter with a frequency response

$$H(e^{j\omega T_s}) = \begin{cases} 1 \ \text{for } |\omega| \le \omega_c \\ 0 \ \text{for } \omega_c < |\omega| \le \dfrac{\omega_s}{2} \end{cases}$$

where ω_s is the sampling frequency using the window approach.

Solution

$$h(nT_s) = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} e^{j\omega nT_s}\, d\omega = \frac{1}{n\pi}\sin\omega_c nT_s$$

$$H'_w(z) = z^{-(N-1)/2}\sum_{n=0}^{(N-1)/2}\frac{a'_n}{2}(z^n + z^{-n}), \quad a'_0 = w(0)h(0), \ a'_n = 2w(nT_s)h(nT_s)$$

$$|M(\omega)| = \left|\sum_{n=0}^{(N-1)/2} a'_n \cos\omega nT_s\right|.$$

Any of the above windows can be used.

29.4.4 Dolph-Chebyshev Window

$$w_{DC}(nT_s) = \frac{1}{N}\left[\frac{1}{r} + 2\sum_{i=1}^{(N-1)/2} T_{N-1}\left(x_0 \cos\frac{i\pi}{N}\right)\cos\frac{2n\pi i}{N}\right], \quad n = 0,1,2,\cdots,(N-1)/2$$

r = required ripple ratio (see 29.4.1) and $x_0 = \cosh\left(\frac{1}{N-1}\cosh^{-1}\frac{1}{r}\right)$

$T_{N-1}(x)$ is the $(N-1)^{th}$-order Chebyshev polynomial and is given by

$$T_{N-1}(x) = \begin{cases} \cos((N-1)\cos^{-1} x) & \text{for } |x| \le 1 \\ \cosh(\cosh^{-1} x) & \text{for } |x| > 1 \end{cases}$$

Properties

a) An arbitrary ripple ratio can be achieved,
b) The main-lobe width is controlled by choosing N,
c) With N fixed, the main-lobe width is the smallest that can be achieved for a given ripple ratio, and
d) All the side lobes have the same amplitude.

29.4.5 Kaiser Window

$$w_K(nT) = \begin{cases} \dfrac{I_o(\beta)}{I_o(\alpha)} & \text{for } |n| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

α = independent parameter, $\beta = \alpha\sqrt{1 - \left(\frac{2n}{N-1}\right)^2}$

$I_0(x) =$ zero order modified Bessel function of the first kind (see Chapter 25)

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

$$W_K(e^{j\omega T_s}) = w_k(0) + 2\sum_{n=1}^{(N-1)/2} w_K(nT_s)\cos\omega nT_s$$

29.4.6 Window Parameters

Window	Main-lobe width	Ripple ratio in %		
		N = 11	N = 21	N = 101
Rectangular	$2\omega_s/N$	22.34	21.89	21.70
Hann	$4\omega_s/N$	2.62	2.67	2.67
Hamming	$4\omega_s/N$	1.47	0.93	0.74
Blackman	$6\omega_s/N$	0.08	0.12	0.12

29.4.7 Filter Specifications (see Figure 29.1)

$$A_p = \text{passband ripple} = 20\log\frac{1+\delta}{1-\delta} \quad \text{(in dB)}$$

$$A_a = \text{stopband attenuation} = -20\log\delta \quad \text{(in dB)}$$

$$B_t = \text{transition width (rad s}^{-1}\text{)}$$

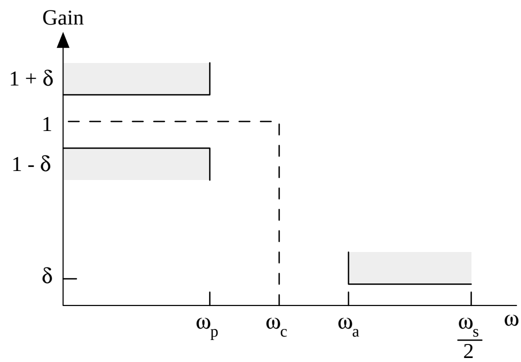

FIGURE 29.1

Steps for Design

1. Determine $h(nT_s)$ using Fourier-series (see 29.2.2), $\omega_c = \frac{1}{2}(\omega_p + \omega_a)$ (see Figure 29.1)
2. Choose δ such that $A_p \le A'_p$ and $A_a \ge A'_a$ where A'_p and A'_p are the desired passband ripple and stopband attenuation, respectively. Choose

$$\delta = \min(\delta_1, \delta_2) \text{ where } \delta_1 = 10^{-0.05A'_a}, \quad \delta_2 = \frac{10^{0.05A'_p} - 1}{10^{0.05A'_p} + 1}$$

3. Find $A_a = -20\log[\min(\delta_1, \delta_2)]$
4. Choose parameter α as follows:

$$\alpha = \begin{cases} 0 & \text{for } A_a \le 21 \\ 0.5842(A_a - 21)^{0.4} + 0.07886(A_a - 21) & \text{for } 21 < A_a \le 50 \\ 0.1102(A_a - 8.7) & \text{for } A_a > 50 \end{cases}$$

5. Choose D as follows:

$$D = \begin{cases} 0.9222 & \text{for } A_a \le 21 \\ \dfrac{A_a - 7.95}{14.36} & \text{for } A_a > 21 \end{cases}$$

Then select the lowest *odd* value of N satisfying the inequality

$$N \ge \frac{\omega_s D}{B_t} + 1$$

6. Use Kaiser window (see 29.4.5)
7. Form $H'_w(z) = z^{-(N-1)/2} H_w(z), \quad H_w(z) = Z\{w_K(nT_s)h(nT_s)\}$

Example

Design a lowpass filter satisfying the following specifications:

a) Maximum passband ripple to be 0.1 dB in the range 0 to 2 rad s^{-1}
b) Minimum stopband attenuation to be 35 dB in the range from 3 to 4.5 rad s^{-1}
c) Sampling frequency ω_s = 10 rad s^{-1}

Solution

1. $h(nT_s) = \dfrac{\sin \omega_c nT_s}{n\pi}, \quad \omega_c = \tfrac{1}{2}(2+3) = 2.5 \text{ rad s}^{-1}$

2. $\delta_1 = 10^{-0.05\times 35} = 0.0178, \quad \delta_2 = \dfrac{10^{0.05\times 0.1} - 1}{10^{0.05\times 0.1} + 1} = 5.7564 \times 10^{-3},$

   $\min(\delta_1, \delta_2) = 5.7564 \times 10^{-3}$

3. $A_a = -20 \log(5.7564 \times 10^{-3}) = 44.797 \text{ dB}$

4. $\alpha = 0.5842(44.797 - 21)^{0.4} + 0.07886(44.797 - 21) = 3.9524$

   $D = (44.797 - 7.95)/14.36 = 2.5660$

5. $N \geq \dfrac{10(2.566)}{1} + 1 = 26.66 \text{ or } N = 27$

6. $H'_w(z) = z^{-(N-1)/2} \sum_{n=0}^{(N-1)/2} \dfrac{a'_n}{2}(z^n + z^{-n}), \quad a'_0 = w_K(0)h(0), \quad a'_n = 2w_K(nT_s)h(nT_s)$

29.5 Highpass FIR Filter

29.5.1 Transition Width (see Figure 29.2)

$$B_t = \omega_p - \omega_a$$

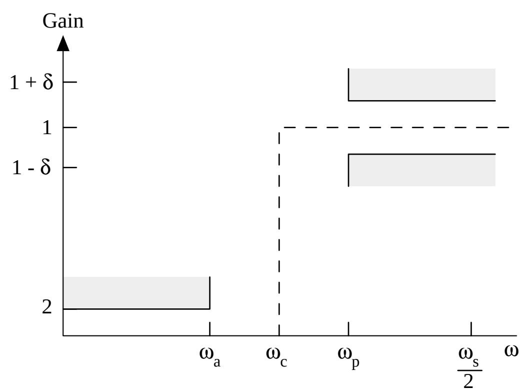

FIGURE 29.2

29.5.2 Ideal Frequency Response

$$H(e^{j\omega T_s}) = \begin{cases} 0 & \text{for } |\omega| \le \omega_c \\ 1 & \text{for } \omega_c < |\omega| \le \dfrac{\omega_s}{2} \end{cases}$$

$$\omega_c = (\omega_a + \omega_p)/2$$

29.6 Bandpass FIR Filter

29.6.1 Transition Width

$$B_t = \min\{(\omega_{p1} - \omega_{a1}),\ (\omega_{a2} - \omega_{p2})\}$$

29.6.2 Ideal Frequency Response (see Figure 29.3)

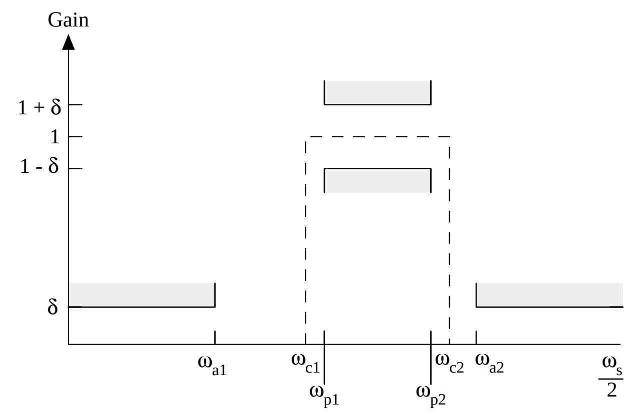

FIGURE 29.3

$$H(e^{j\omega T_s}) = \begin{cases} 0 & \text{for } 0 \le |\omega| \le \omega_{c1} & \omega_{c1} = \omega_{p1} - \dfrac{B_t}{2} \\ 1 & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} & \omega_{c2} = \omega_{p2} + \dfrac{B_t}{2} \\ 0 & \text{for } \omega_{c2} < |\omega| \le \dfrac{\omega_s}{2} & \end{cases}$$

29.7 Bandstop FIR Filter

29.7.1 Transition Width

$$B_t = \min\{(\omega_{a1} - \omega_{p1}),\ (\omega_{p2} - \omega_{a2})\}$$

29.7.2 Ideal Frequency Response (see Figure 29.4)

$$H(e^{j\omega T_s}) = \begin{cases} 1 & \text{for } 0 \le |\omega| \le \omega_{c1} \\ 0 & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 1 & \text{for } \omega_{c2} < |\omega| \le \dfrac{\omega_s}{2} \end{cases}$$

$$\omega_{c1} = \omega_{p1} + \frac{B_t}{2}, \quad \omega_{c2} = \omega_{p2} - \frac{B_t}{2}$$

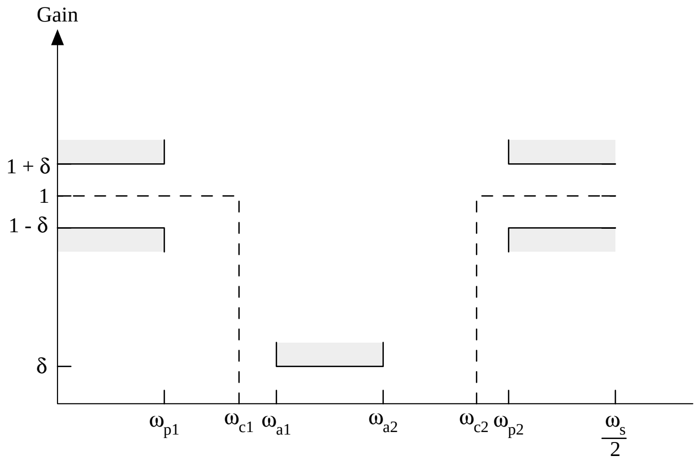

FIGURE 29.4

Example

Design a lowpass filter satisfying the following specifications:

a) Minimum stopband attenuation for $0 \le \omega \le 200$ to be 45 dB
b) Maximum passband ripple for $400 < \omega < 600$ to be 0.2 dB
c) Minimum stopband attenuation for $700 \le \omega \le 1000$ to be 45 dB
d) Sampling frequency $\omega_s = 2000$ rad s^{-1}

Solution

$B_t = \min\{(400 - 200), (700 - 600)\} = 100,, \quad \omega_{c1} = 400 - 50 = 350, \quad \omega_{c2} = 600 + 50 = 650,$

$$h(nT_s) = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T_s})e^{j\omega nT_s}\,d\omega = \frac{1}{\omega_s}\int_{o}^{\omega_s/2}[H(e^{j\omega T_s})e^{j\omega nT_s}$$

$$+ H(e^{-j\omega T_s})e^{-j\omega nT_s}]d\omega = \frac{1}{\omega_s}\int_{\omega_{c1}}^{\omega_{c2}} 2\cos\omega nT_s\,d\omega = \frac{1}{\pi n}(\sin\omega_{c2}nT_s - \sin\omega_{c1}nT_s)$$

$$T_s = 2\pi/2000, \quad \delta_1 = 10^{-0.05(45)} = 5.6234\times10^{-3}, \quad \delta_2 = \frac{10^{0.05(0.2)} - 1}{10^{0.05(0.2)} + 1} = 1.1512\times10^{-2},$$

$\min(\delta_1,\delta_2) = 5.6234\times10^{-3}$, $A_a = 45dB$, $\alpha = 3.9754$, $D = 2.580$, $N = 53$ and continue as Example 4.2.1.

Poularikas A. D. “Recursive Filters”
The Handbook of Formulas and Tables for Signal Processing.
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC,1999

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Recursive Filters (Infinite Impulse Response, IIR)

30.1 Introduction

30.1.1 Realizable Filter

The transfer function must a) be a rational function of z with real coefficients, b) have poles that lie within the unit circle of the z plane, and c) have the degree of the numerator equal to or less than that of the denominator polynomial.

30.2 Invariant-Impulse-Response Method

30.2.1 Steps to be taken

1. Deduce $h_A(t) =$ impulse response of the analog filter $= L^{-1}\{H_A(s)\},\ h(0+) = \lim_{s\to\infty} s\{H_A(s)\}$
2. Replace t by nT_s in $h_A(t)$
3. Form the Z-transform of $h_A(nT_s)$

30.2.2 Conditions

If $H_A(\omega) \cong 0$ for $|\omega| \geq \omega_s/2$ and $h(0+) = 0$, then

$$T_s H_D(e^{j\omega T_s}) = H_A(\omega) \qquad \text{for } |\omega| \leq \frac{\omega_s}{2}$$

Simple poles

$$H_A(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}, \qquad h_A(t) = \mathcal{L}^{-1}\{H_A(s)\} = \sum_{i=1}^{N} A_i\, e^{p_i t}$$

$$A_i = [(s - p_i) H_A(s)]\big|_{s=p_i}$$

$$h_A(nT_s) = \sum_{i=1}^{N} A_i\, e^{p_i n T_s}$$

$$H_D(z) = \mathcal{Z}\{h_A(nT_s)\} = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} (e^{p_i T_s} z^{-1})^n = \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T_s} z^{-1}} = \sum_{i=1}^{N} A_i \frac{z}{z - e^{p_i T_s}}$$

30.2.3 Procedure of Impulse-Invariant of IIR Filters

1. Obtain the transfer function $H_A(s)$ for the desired analog prototype filter (see Chapter 12)
2. For $i = 1, 2, \cdots, N$ determine the poles of p_i and $H_A(s)$ and compute the coefficients

$$A_i = [(s - p_i) H_A(s)]\big|_{s=p_i}$$

3. Use A_i's from step 2 to generate the digital filter system function

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - \exp(p_i T_s) z^{-1}}$$

Example

The normalized transfer function of a second-order Butterworth analog filter with a 3-dB cutoff frequency at 3000 Hz. The sampling frequency is f_s = 30,000 s^{-1}.

Solution

$$H_A(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1} = \text{normalized analog filter} = \frac{1}{(s - p_1)(s - p_2)}$$

$$\left[p_1 = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2},\ p_2 = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \right], \qquad \omega_c = 2\pi 3000 = 6000\pi,$$

$$H_A\left(\frac{s}{\omega_c}\right) = \text{un-normalized filter} = \frac{1}{\left(\frac{s}{\omega_c} - p_1\right)\left(\frac{s}{\omega_c} - p_2\right)} = \frac{\omega_c^2}{(s - \omega_c p_1)(s - \omega_c p_2)}$$

$$= \frac{A_1}{s - \omega_c p_1} + \frac{A_2}{s - \omega_c p_2}, \qquad A_1 = \frac{\omega_c^2}{\omega_c p_1 - \omega_c p_2} = \frac{\omega_c}{p_1 - p_2}, \qquad A_2 = \frac{\omega_c^2}{\omega_c p_2 - \omega_c p_1} = \frac{\omega_c}{p_2 - p_1}.$$

But $T_s = 1/f_s = 1/30,000,\ \omega_c T_s = \pi/5,$ and hence

$$H_D(z) = \frac{\omega_c}{p_1 - p_2}\frac{1}{1-\exp[\omega_c T_s p_1]z^{-1}} + \frac{\omega_c}{p_2 - p_1}\frac{1}{1-\exp[\omega_c T_s p_2]z^{-1}}$$

$$= \frac{(-j\sqrt{2}\ \omega_c/2)}{1-\exp[\omega_c T_s p_1]z^{-1}} + \frac{(j\sqrt{2}\ \omega_c/2)}{1-\exp[\omega_c T_s p_2]z^{-1}}.$$

Hence $\left|H_D(e^{j\omega T_s})\right| = \left|\frac{(-j\sqrt{2}\ \omega_c/2)}{1-\exp[\omega_c T_s p_1]e^{-j\omega T_s}} + \frac{(j\sqrt{2}\ \omega_c/2)}{1-\exp[\omega_c T_s p_2]e^{-j\omega T_s}}\right|$

30.3 Modified-Invariant-Impulse-Response

30.3.1 Analog Transfer Function

$$H_A(s) = H_0\frac{N(s)}{D(s)} = H_0\frac{\prod_{i=1}^{M}(s-s_i)}{\prod_{i=1}^{N}(s-p_i)},\quad M \le N;$$

$$H_A(s) = H_0\frac{H_{A1}(s)}{H_{A2}(s)},\quad H_{A1}(s) = \frac{1}{D(s)},\quad H_{A2}(s) = \frac{1}{N(s)}$$

Conditions: $h_{A1}(0+) = 0,\ \ h_{A2}(0+) = 0,\ \ M, N \ge 2,\ H_{A1}(\omega) = H_{A2}(\omega) \cong 0$ for $|\omega| \ge \frac{\omega_s}{2}$

30.3.2 Digital Filter

$$H_D(z) = H_0\frac{H_{D1}(z)}{H_{D2}(z)},\ \ H_D(e^{j\omega T_s}) = H_0\frac{H_{D1}(e^{j\omega T_s})}{H_{D2}(e^{j\omega T_s})} \cong H_A(\omega)\ \text{ for }\ |\omega| \le \frac{\omega_s}{2}$$

30.3.3 Zeroes and poles of $H_A(s)$

$$H_{D1}(z) = \sum_{i=1}^{N}\frac{A_i z}{z - e^{p_i T_s}} = \frac{N_1(z)}{D_1(z)},\ \ H_{D2}(z) = \sum_{i=1}^{M}\frac{B_i z}{z - e^{T_s s_i}} = \frac{N_2(z)}{D_2(z)}, H_D(z) = H_0\frac{N_1(z)D_2(z)}{N_2(z)D_1(z)},$$

30.3.4 Stability

If any pole of $H_D(z)$ is located outside the unit circle it can be replaced by their reciprocals without changing the shape of the loss characteristics, although a constant vertical shift will be introduced.

If any pole is on the unit circle, its magnitude can be reduced slightly.

30.4 Matched-Z-Transform Method

30.4.1 Matched-Z-Transform Method

$$H_D(z) = (z+1)^L \, H_0 \frac{\prod_{i=1}^{M} (z - e^{s_i T_s})}{\prod_{i=1}^{N} (z - e^{p_i T_s})}, \quad z = e^{j\omega T_s}$$

Values of L

Filter Type	Lowpass	Highpass	Bandpass	Bandstop
All pole	N	0	N/2	0
Elliptic				
N odd	1	0		
N even	0	0	0 for N/2 even	0

30.5 Bilinear-Transformation Method

30.5.1 Analog Integrator

$$H_{AI}(s) = \frac{1}{s}, \quad h_{AI}(t) = \begin{cases} 1 & t \geq 0+ \\ 0 & t \leq 0- \end{cases}$$

30.5.2 Digital Integrator

$$y(nT_s) - y(nT_s - T_s) = \frac{T_s}{2}[x(nT_s - T_s) + x(nT_s)]$$

$$Y(z) - z^{-1}Y(z) = \frac{T_s}{2}[z^{-1}X(z) + X(z)]$$

30.5.3 Transfer Function

$$H_{DI}(z) = \frac{Y(z)}{X(z)} = \frac{T_s}{2}\left(\frac{z+1}{z-1}\right)$$

30.5.4 Bilinear Transformation

$$H_{DI}(z) = H_{AI}(s)\Big|_{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}$$

30.5.5 Analog Filter Transfer Function

$$H_A(s) = \frac{\sum_{i=0}^{N} a_i s^{N-i}}{s^N + \sum_{i=0}^{N} b_i z^{N-i}} = \frac{\sum_{i=0}^{N} a_i \left[\frac{1}{H_{AI}(s)}\right]^{N-i}}{\left[\frac{1}{H_{AI}(s)}\right]^{N} + \sum_{i=0}^{N} b_i \left[\frac{1}{H_{AI}(s)}\right]^{N-i}} \qquad \text{(see 30.5.1)}$$

30.5.6 Discrete-Time Transfer Function

$$H_D(z) = \frac{\sum_{i=0}^{N} a_i \left[\frac{1}{H_{DI}(s)}\right]^{N-i}}{\left[\frac{1}{H_{DI}(z)}\right]^{N} + \sum_{i=0}^{N} b_i \left[\frac{1}{H_{DI}(z)}\right]^{N-i}} = H_A(s)\Big|_{s=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}$$

by replacing $H_{AI}(s)$ by $H_{DI}(z)$.

30.5.7 Mapping Properties of Bilinear Transformation

$$z = \frac{\frac{2}{T_s} + s}{\frac{2}{T_s} - s}$$

a) The open right-half s-plane is mapped onto the region exterior to the unit circle $|z| = 1$ of the z-plane;
b) The j axis of the s-plane is mapped onto the unit circle $|z| = 1$ of the z-plane;
c) The open left-half s-plane is mapped onto the interior of the unit circle $|z| = 1$;
d) The origin of the s-plane maps onto point (1,0) of the z-plane;
e) The positive and negative j axes of the s-plane map onto the upper and lower semicircles of $|z| = 1$, respectively;
f) The maxima and minima of $|H_A(\omega)|$ are preserved in $|H_D(e^{j\Omega T_s})|$;
g) If $m_1 \le |H_A(\omega)| \le m_2$ in $\omega_1 \le \omega \le \omega_2$, then $m_1 \le |H_D(e^{j\Omega T_s})| \le m_2$ for a corresponding frequency $\Omega_1 \le \Omega \le \Omega_2$;
h) Passbands or stopbands in the analog filter translate into passbands or stopbands in the digital filter;
i) A stable analog filter will yield a stable digital filter.

30.5.8 The Warping Effect

From (30.5.6) $H_D(e^{j\Omega T_s}) = H_A(s)$ provided that $\omega = \frac{2}{T_s}\tan\frac{\Omega T_s}{2}$. For $\Omega < 0.3/T_s$, $\omega \cong \Omega$ and hence both filters have the same frequency response. Figure 30.1 shows the warping effect.

Note: If prescribed passband and stopband edges $\tilde{\Omega}_1, \tilde{\Omega}_2, \cdots, \tilde{\Omega}_i$ are to be achieved in the digital filter, the analog filter must be prewarped before application to ensure that

$$\omega_i = \frac{2}{T_s}\tan\frac{\tilde{\Omega}_i T_s}{2}$$

and hence $\Omega_i = \tilde{\Omega}_i$.

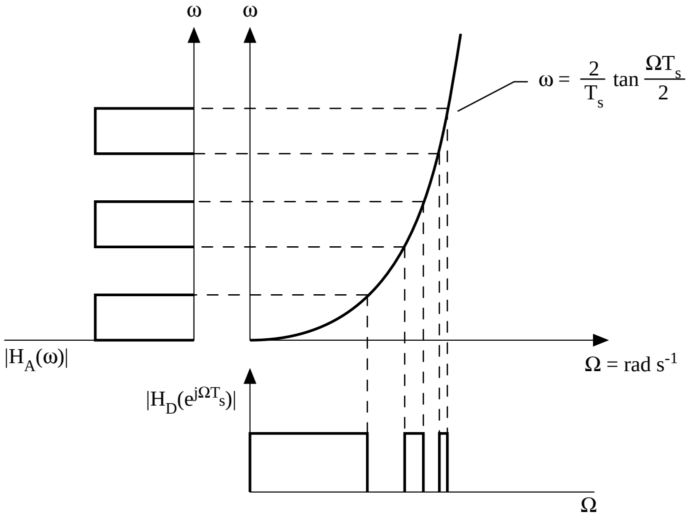

FIGURE 30.1

Note: The phase response of the derived digital filter is nonlinear, although the analog filter has linear phase.

Example

The second-order Butterworth analog filter

$$H_A(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\,\omega_c s + \omega_c^2}$$

with 3-dB cutoff frequency of 3 kHz and sampling rate of 30,000 samples per second is transformed to an IIR filter using bilinear transformation as follows:

$$\omega_c = 2\pi 3000 = 6000\pi, \;\; s = 2(1 - z^{-1})/[T_s(1 + z^{-1})], \text{ and } T_s = 1/30,000$$

$$H_D(z) = \frac{(6000\pi)^2}{(60,000)^2\left[\dfrac{1 - z^{-1}}{1 + z^{-1}}\right]^2 + 12000\pi \times 60,000\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right) + (6000\pi)^2}$$

$$= \frac{0.063964 + 0.127929 z^{-1} + 0.063964 z^{-2}}{1 - 1.168261 z^{-1} + 0.424118 z^{-2}}$$

30.6 Digital-Filter Transformations

30.6.1 Constantimides Transformations are given in Table 30.1.

TABLE 30.1 Table of IIR Digital-Filter Transformations

Type	Transformation	α, k
LP to LP	$z = \dfrac{\bar{z} - \alpha}{1 - \alpha \bar{z}}$	$\alpha = \dfrac{\sin[(\Omega_p - \omega_p)T_s/2]}{\sin[(\Omega_p + \omega_p)T_s/2]}$
LP to HP	$z = -\dfrac{\bar{z} - \alpha}{1 - \alpha \bar{z}}$	$\alpha = \dfrac{\cos[(\Omega_p - \omega_p)T_s/2]}{\cos[(\Omega_p + \omega_p)T_s/2]}$
LP to BP	$z = -\dfrac{\bar{z}^2 - \frac{2\alpha k}{k+1}\bar{z} + \frac{k-1}{k+1}}{1 - \frac{2\alpha k}{k+1}\bar{z} + \frac{k-1}{k+1}\bar{z}^2}$	$\alpha = \dfrac{\cos[(\omega_{p2} + \omega_{p1})T_s/2]}{\cos[(\omega_{p2} - \omega_{p1})T_s/2]}$
		$k = \tan\dfrac{\Omega_p T_s}{2} \cot\dfrac{(\omega_{p2} - \omega_{p1})T_s}{2}$
LP to BS	$z = \dfrac{\bar{z}^2 - \frac{2\alpha}{1+k}\bar{z} + \frac{1-k}{1+k}}{1 - \frac{2\alpha}{1+k}\bar{z} + \frac{1-k}{1+k}\bar{z}^2}$	$\alpha = \dfrac{\cos[(\omega_{p2} + \omega_{p1})T_s/2]}{\cos[(\omega_{p2} - \omega_{p1})T_s/2]}$
		$k = \tan\dfrac{\Omega_p T_s}{2} \tan\dfrac{(\omega_{p2} - \omega_{p1})T_s}{2}$

$z = re^{j\omega T_s}$, $\bar{z} = Re^{j\Omega T_s}$, Ω = digital frequency, ω = analog frequency.

30.6.2 Transformation Applications

1. Obtain a lowpass normalized transfer function $H_N(z)$ using any approximation method.
2. Determine the passband edge Ω_p in $H_N(z)$.
3. Form $H(\bar{z})$ using $H(\bar{z}) = H_N(z)\big|_{z=f(\bar{z})}$.

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31

Recursive Filters Satisfying Prescribed Specifications

31.1 Design Procedure

31.1.1 Design Is Accomplished in Two Steps

1. A normalized lowpass filter transfer function is transformed into a denormalized filter (lowpass, highpass, etc. transfer function) employing the standard analog-filter transformation.
2. A bilinear transformation is applied.

31.1.2 Passband Stopband Edges

If ω_i are analog passband and stopband edges of an analog filter, the corresponding passband and stopband edges in the derived digital filter are given by

$$\Omega_i = \frac{2}{T_s}\tan^{-1}\frac{\omega_i T_s}{2}, \quad i = 1, 2, \cdots$$

31.1.3 Prewarping

If desired passband and stopband edges Ω_{di} are to be achieved, the analog filter must be prewarped before the application of the bilinear transformation to ensure $\omega_i = \frac{2}{T_s}\tan\frac{\Omega_{di}T_s}{2}$, so that $\Omega_i = \Omega_{di}$.

31.1.4 Loss Amplitude

$A_N(\omega) = 20\log\frac{1}{|H_N(\omega)|}$ = loss amplitude of an analog normalized lowpass filter $H_N(\omega)$

$$0 \le A_N(\omega) \le A_p \quad \text{for} \quad 0 \le |\omega| \le \omega_p$$

$$A_N(\omega) \ge A_a \quad \text{for} \quad \omega_a \le |\omega| < \infty$$

(see Figure 31.1)

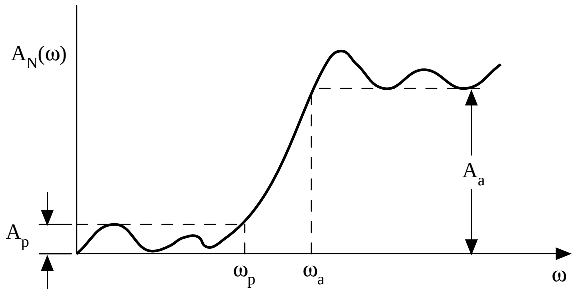

FIGURE 31.1

31.1.5 Transformation

Step 1. Form $H_t(\bar{s}) = H_N(s)\big|_{s=f(\bar{s})}$

Step 2. Form $H_D(s) = H_t(\bar{s})\Big|_{\bar{s}=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)}$

TABLE 31.1 Transformation of Analog Filters

Type	Transformation, $s = f(\bar{s})$
LP to LP	$s = \lambda\bar{s}$
LP to HP	$\lambda / \bar{s}$
LP to BP	$s = \frac{1}{B}\left(\bar{s} + \frac{\omega_0^2}{\bar{s}}\right)$
LP to BS	$s = \frac{B\bar{s}}{\bar{s}^2 + \omega_0^2}$

The parameters λ, ω_0 and B and order $H_N(s)$ must be chosen appropriately (see Section 31.2).

31.2 Analog Filters

31.2.1 Butterworth Filters

$A_N(\omega) = \text{normalized loss} = 10\,\log(1 + \omega^{2n})$

$$n \ge \frac{\log D}{2\,\log(1/K)},\;\; \omega_p = (10^{0.1\,A_p} - 1)^{1/2n},\;\; \omega_a = (10^{0.1\,A_a} - 1)^{1/2n},\;\; D = \frac{10^{0.1\,A_a} - 1}{10^{0.1\,A_p} - 1}$$

LP	$K = K_o$
HP	$K = 1/K_o$
BP	$K = \begin{cases} K_1 & \text{if } K_c \ge K_B \\ K_2 & \text{if } K_c < K_B \end{cases}$
BS	$K = \begin{cases} \frac{1}{K_2} & \text{if } K_c \ge K_B \\ \frac{1}{K_1} & \text{if } K_c < K_B \end{cases}$

$$K_A = \tan\frac{\Omega_{dp2}T_s}{2} - \tan\frac{\Omega_{dp1}T_s}{2}, \qquad K_B = \tan\frac{\Omega_{dp1}T_s}{2}\tan\frac{\Omega_{dp2}T_s}{2}$$

$$K_C = \tan\frac{\Omega_{da1}T_s}{2}\tan\frac{\Omega_{da2}T_s}{2}, \qquad K_1 = \frac{K_A\tan(\Omega_{da1}T_s/2)}{K_B - \tan^2(\Omega_{da1}T_s/2)}$$

$$K_2 = \frac{K_A\tan(\Omega_{da2}T_s/2)}{\tan^2(\Omega_{da2}T_s/2) - K_B}, \qquad K_0 = \frac{\tan(\Omega_{dp}T_s/2)}{\tan(\Omega_{da}T_s/2)}$$

$$A_N(\omega_p) = A_p = 10\log(1+\omega_p^{2n}), \quad A_N(\omega_a) = A_a = 10\log(1+\omega_a^{2n})$$

31.2.2 Chebyshev Filter

$A_N(\omega) = \text{normalized loss} = 10\log[1+\varepsilon^2 T_n^2(\omega)]$
$T_n(\omega) = \cosh(n\cosh^{-1}\omega)$ for $\omega_p \le \omega < \infty$, $\varepsilon^2 = 10^{0.1A_p} - 1$, $\omega_p = 1$

$$D = \frac{10^{0.1A_a}-1}{10^{0.1A_p}-1}, \quad n \ge \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)}, \quad \omega_p = 1, \quad K_0 = \frac{\tan(\Omega_{dp}T_s/2)}{\tan(\Omega_{da}T_s/2)}$$

LP	$K = K_0$
HP	$K = 1/K_0$
BP	$K = \begin{cases} K_1 & \text{if } K_c \ge K_B \\ K_2 & \text{if } K_c < K_B \end{cases}$
BS	$K = \begin{cases} \dfrac{1}{K_2} & \text{if } K_c \ge K_B \\ \dfrac{1}{K_1} & \text{if } K_c < K_B \end{cases}$

31.2.3 Elliptic Filters

$$n \ge \frac{\log 16D}{\log(1/q)}, \quad k' = \sqrt{1-k^2}, \quad q_0 = \frac{1}{2}\left(\frac{1-\sqrt{k'}}{1+\sqrt{k'}}\right)$$

$$q = q_0 + 5q_0^5 - 5q^9 + 10q^{13}, \quad \omega_p = \sqrt{k},, \quad K_0 = \frac{\tan(\Omega_{dp}T_s/2)}{\tan(\Omega_{da}T_s/2)}$$

	k	ω_p
LP	K_0	$\sqrt{K_0}$
HP	$1/K_0$	$1/\sqrt{K_0}$
BP	K_1 if $K_c \ge K_B$	$\sqrt{K_1}$
	K_2 if $K_c < K_B$	$\sqrt{K_2}$

BS	$\frac{1}{K_2}$ if $K_c \geq K_B$	$1/\sqrt{K_2}$
	$\frac{1}{K_1}$ if $K_c < K_B$	$1/\sqrt{K_1}$

31.3 Design Formulas

31.3.1 Lowpass and Highpass Filters (see Figure 31.2)

LP	$\omega_a \leq \frac{\omega_p}{K_0}$	$K_0 = \frac{\tan(\Omega_{dp} T_s / 2)}{\tan(\Omega_{da} T_s / 2)}$
	$\lambda = \frac{\omega_p T_s}{2\tan(\Omega_{dp} T_s / 2)}$	
HP	$\omega_a \leq \omega_p K_0$	
	$\lambda = \frac{2\omega_p \tan(\Omega_{dp} T_s / 2)}{T_s}$	

$$A_D(\Omega) = 20\log \frac{1}{\left|H_D(e^{j\Omega T_s})\right|} = \text{loss characteristics}$$

$$0 \leq A_D \leq A_p \quad \text{for} \quad 0 \leq |\Omega| \leq \Omega_p$$

$$A_D \geq A_a \quad \text{for} \quad \Omega_a \leq |\Omega| \leq \frac{\omega_s}{2}$$

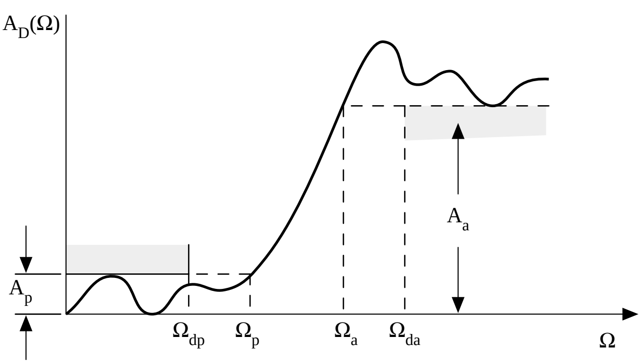

FIGURE 31.2

31.3.2 Bandpass and Bandstop Filters (see Figure 31.3)

BP	$\omega_0 = 2\sqrt{K_B}\,/\,T_s$
	$\omega_a \le \begin{cases} \omega_p / K_1 & \text{if } K_c \ge K_B \\ \omega_p / K_2 & \text{if } K_c < K_B \end{cases}$
	$B = 2K_A/(T_s\omega_p)$
BS	$\omega_0 = 2\sqrt{K_B}\,/\,T_s$
	$\omega_a \le \begin{cases} \omega_p K_2 & \text{if } K_c \ge K_B \\ \omega_p K_1 & \text{if } K_c < K_B \end{cases}$
	$B = \dfrac{2K_A\omega_p}{T_s}$

$$K_A = \tan\frac{\Omega_{dp2}T_s}{2} - \tan\frac{\Omega_{dp1}T_s}{2}, \qquad K_B = \tan\frac{\Omega_{dp1}T_s}{2}\tan\frac{\Omega_{dp2}T_s}{2}$$

$$K_C = \tan\frac{\Omega_{da1}T_s}{2}\tan\frac{\Omega_{da2}T_s}{2}, \qquad K_1 = \frac{K_A\tan(\Omega_{da1}T_s/2)}{K_B - \tan^2(\Omega_{da1}T_s/2)}$$

$$K_2 = \frac{K_A\tan(\Omega_{da2}T_s/2)}{\tan^2(\Omega_{da2}T_s/2) - K_B}$$

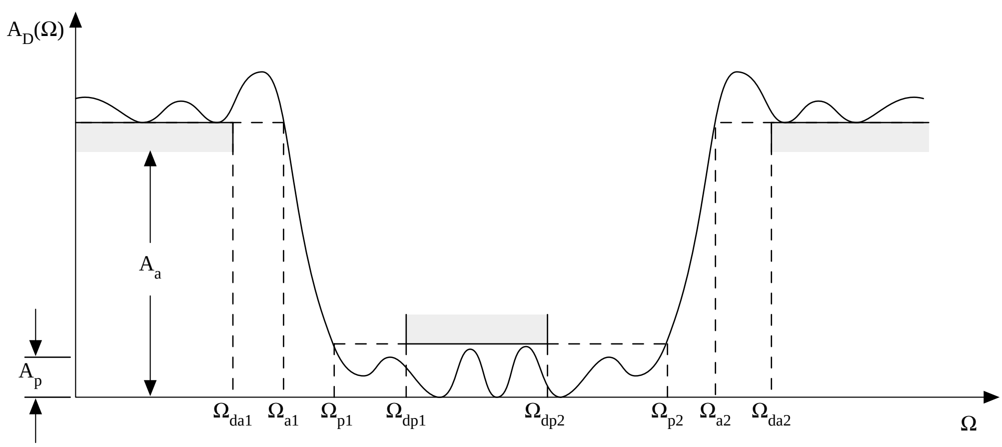

FIGURE 31.3

31.4 Examples

31.4.1 Example

Design a highpass filter using a Butterworth approximation satisfying the following specifications: $A_p = 1$ dB, $A_a = 15$dB, $\Omega_{dp} = 3.5$ rad s^{-1}, $\Omega_{da} = 1.5$ rad s^{-1} and $\omega_s = 10$ rad s^{-1}.

Solution

$\omega_s = 2\pi / T_s$ or $T_s = 2\pi/10 = 0.2\pi$. From (31.2.1)

$$K_0 = \frac{\tan(3.52\pi/10\times 2)}{\tan(1.52\pi/10\times 2)} = 3.85184; \qquad D = (10^{0.1\times 15} - 1)/(10^{0.1\times 1} - 1) = 118.268718;$$

$$n \geq \frac{\log D}{2\times \log K_0} = 1.7697 \text{ which implies } n = 2; \qquad \omega_p = (10^{0.1\times 1} - 1)^{1/4} = 0.713335.$$

From 31.3.1 $\lambda = \dfrac{2\omega_p \tan(\Omega_{dp} T_s/2)}{T_s} = 4.456334$. From (31.1.5) and Chapter 12 (12.4.4),

$$H_t(\bar{s}) = H_N(s)\Big|_{s=\lambda/\bar{s}} = \frac{1}{s^2 + 1.41421 s + 1}\Bigg|_{s=\lambda/\bar{s}} = \frac{\bar{s}^2}{\bar{s}^2 + 1.41421\lambda\bar{s} + \lambda^2}.$$

From (31.1.5) step #2

$$H_D(z) = H_t(\bar{s})\Big|_{\bar{s}=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)} = \frac{z^2 - 2z + 1}{4.939888z^2 + 1.919992z + 0.980104}.$$

Figure 31.4 shows the $\left|H_D(e^{j\omega T_s})\right|$ versus ω.

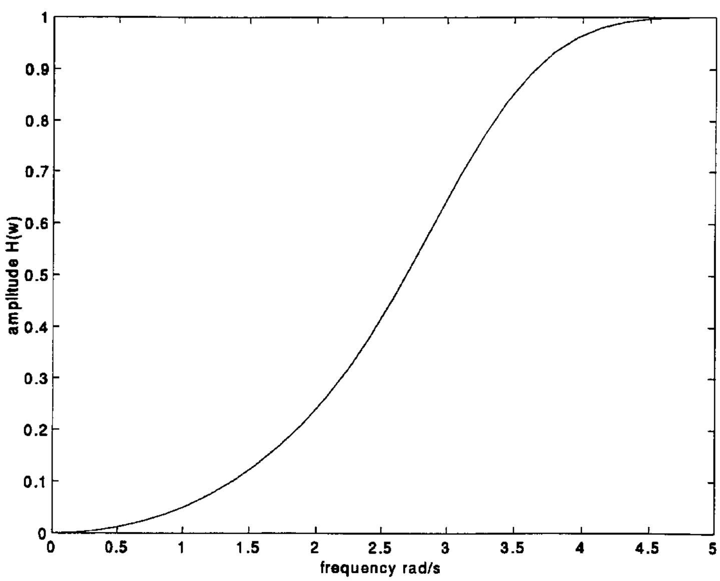

FIGURE 31.4

31.4.2 Example

Design a highpass filter using a Chebyshev approximation satisfying the following specifications: A_p = 1 dB, $A_a = 15\text{dB}$, $\Omega_{dp} = 3.5$ rad s^{-1}, $\Omega_{da} = 1.5$ rad s^{-1} and $\omega_s = 10$ rad s^{-1}.

Solution

$\omega_s = 2\pi/T_s$ or $T_s = 2\pi/10 = 0.2\pi$. From (31.2.2) $K_0 = 3.85184$ (see Example 31.4.1), D = 118.268718 (Example 31.4.1),

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1} K_0} = \frac{\ln(\sqrt{D}+\sqrt{D-1})}{\ln(K_0+\sqrt{K_0^2-1})} = \frac{3.07963}{2.02439} = 1.521$$

or $n = 2$, $\omega_p = 1$. From (31.3.1) $\lambda = 2\omega_p \tan(\Omega_{dp}T_s/2)/T_s = 6.247183$. From (31.1.5) $s = \lambda/\bar{s}$ and Chapter 12, Table 12.3 of 0.1 dB ripple,

$$H_t(\bar{s}) = H_N(s)\Big|_{s=\lambda/\bar{s}} = \frac{1}{s^2+2.37236s+3.31403}\Bigg|_{s=\lambda/\bar{s}} = \frac{\bar{s}^2}{3.31403\bar{s}^2+2.37236\lambda\bar{s}+\lambda^2}.$$

From (31.1.5) step #2

$$H_D(z) = H_t(\bar{s})\Bigg|_{\bar{s}=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)} = \frac{z^2-2z+1}{7.911168z^2+1.075619z+1.400869}.$$

31.4.3 Example

Design a Chebyshev bandstop filter satisfying the following specifications: $A_p = 0.5$ dB, $A_a = 15$ dB, $\Omega_{dp1} = 350$ rad s^{-1}, $\Omega_{dp2} = 700$ rad s^{-1}, $\Omega_{da1} = 430$ rad s^{-1}, $\Omega_{da2} = 600$ rad s^{-1}, $\omega_s = 3000$ rad s^{-1}, $T_s = 2\pi/3000$

Solution

From (31.3.2) and (31.2.2)

$$K_B = \tan\frac{\Omega_{dp1}T_s}{2}\tan\frac{\Omega_{dp2}T_s}{2} = 0.34563,$$

$$K_C = \tan\frac{\Omega_{da1}T_s}{2}\tan\frac{\Omega_{da2}T_s}{2} = 0.35122, \qquad K_C > K_B,\ \omega_0 = 2\sqrt{K_B}/T_s = 561.40606,$$

$$K_A = \tan\frac{\Omega_{dp2}T_s}{2} - \tan\frac{\Omega_{dp1}T_s}{2} = 0.51654,$$

$$B = \frac{2K_A\omega_p}{T_s} = 2\times 0.51654\times 1/T_s = 493.2594$$

$$D = (10^{0.1A_a}-1)/(10^{0.1A_p}-1) = 250.968,$$

$$K_2 = \frac{K_A \tan(\Omega_{da2} T_s/2)}{\tan^2(\Omega_{da2} T_s/2) - K_B} = 5.28362$$

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1} K_2} = \frac{\ln(\sqrt{D} + \sqrt{D-1})}{\ln(K_2 + \sqrt{K_2^2 - 1})} = 1.47$$

or $n = 2$. From Table 12.3 of Chapter 12 and $A_p = 0.5$ ripple,

$$H(s) = \frac{1}{s^2 + 1.42562s + 1.51620}\,.$$

From (31.1.5) Table 31.1,

$$H_t(\bar{s}) = H(s)\Big|_{s=B\bar{s}/(\bar{s}^2+\omega_0^2)} = \frac{\bar{s}^4 + 2\omega_0^2\bar{s}^2 + \omega_0^4}{1.5\bar{s}^4 + 1.42562B\bar{s}^3 + (B^2 + 1.5162 \times 2\omega_0^2)\bar{s}^2 + 1.42562B\omega_0^2\bar{s} + 1.5162\omega_0^4}$$

From (31.1.5) step #2

$$H_D(z) = H_t(\bar{s})\Big|_{\bar{s}=\frac{2}{T_s}\left(\frac{z-1}{z+1}\right)} = \frac{\text{Num.}}{\text{Den.}}$$

where

$$\text{Num:}\quad \left(1 + \frac{b_1T_s^2}{4}\right)z^4 - 4z^3 + \left(6 - 2\frac{b_1T_s^2}{4} + \frac{b_1T_s^2}{16}\right)z^2 + \left(\frac{b_2T_s^2 2}{16} - 4\right)z + \left(1 + \frac{b_1T_s^2}{4} + \frac{b_2T_s^2}{16}\right)$$

$$\text{Den:}\quad a_1z^4 - 4a_1z^3 + 6a_1z^2 - 4a_1z + a_1 + \frac{a_2T_s}{2}z^4 - 2\frac{a_2T_s}{2}z^3 + 2\frac{a_2T_s}{2}z - \frac{a_2T_s}{2} + \frac{a_3T_s}{2}z^4$$

$$-2\frac{a_3T_s}{4}z^2 + \frac{a_3T_s}{2}$$

$b_1 = 2\omega_o^2,\quad b_2 = \omega_o^4$

$a_1 = 1.51620,\quad a_2 = 1.42562B,\quad a_3 = B^2 + 1.51620 \times 2\omega_o^2$

$a_4 = 1.42562\omega_o^2,\quad a_5 = 1.51620\omega_o^4$

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32

Statistics

32.1 Estimation

32.1.1 Definitions

32.1.1.1 Sample Data

The experimental values $x(0), x(1), \cdots, x(N-1)$ of the r.v. $X(0), X(1), \cdots, X(N-1)$.

32.1.1.2 Statistic

A function of the observations alone, e.g., $t(x(0), x(1), \cdots, x(N-1))$

32.1.1.3 Estimator-Estimate

The rule or method of estimation is called an estimator, and the value to which it gives rise in a particular is called the estimate.

32.1.1.4 Consistency

Any statistic that converges stochastically to a parameter θ is called a consistent estimator of that parameter θ. $P\{|t_n - \theta| < \varepsilon\} > 1 - N$ for $n > N$ and any positive ε and η, $t \equiv$ statistic (see 32.1.1.2).

32.1.1.5 Biased and Unbiased Estimator

Any statistic whose mathematical expectation is equal to a parameter θ is called the unbiased estimator of the parameter θ. Otherwise, the statistic is said to be biased. For example, $E\{t\} = \theta$, t is an unbiased estimator of θ.

Example 1

$$E\left\{\frac{1}{N}\sum_{n=0}^{N-1} x(n)\right\} = \frac{1}{N}\sum_{n=0}^{N-1} E\{x(n)\} = \frac{N\mu}{N} = \mu,$$

which shows that the sample mean

$$\bar{x} = \left[\sum_{n=0}^{N-1} x(n)\right] \Big/ N$$

is an unbiased estimator of the population mean if it exists.

Example 2

$$E\left\{\sum_{n=0}^{N-1}(x(n)-\bar{x})^2\right\}=E\left\{\sum_{n=0}^{N-1}\left[x(n)-\sum_{n=0}^{N-1}\frac{x(n)}{N}\right]^2\right\}=E\left\{\frac{N-1}{N}\sum_{n=1}^{N-1}x^2(n)-\frac{1}{N}\sum_{n\neq k}\sum x(n)x(k)\right\}$$

$$=(N-1)\mu_2'-(N-1)\mu_2'=(N-1)\mu_2$$

and hence

$$(1/N)\sum_{n=0}^{N-1}(x(n)-\bar{x})^2$$

has the mean value $[(N-1)/N]\mu_2$ which is biased (depends on N). $s^2\equiv$ unbiased sample variance $=[1/(N-1)]\sum_{n=0}^{N-1}(x(n)-\bar{x})^2$. For μ_i' and μ_i see Chapter 34 Table 34.1. $\bar{x}=$ sample mean.

32.1.1.6 Minimum Variance Unbiased (MVU) Estimator

If the estimator exists whose variance is equal to Cramer-Rao lower bound (CRLB) for each value of the parameter θ then it is an MVU estimator.

32.1.1.7 Vector Parameter

$$\underline{\theta}=[\theta_1,\theta_2,\cdots,\theta_p]^T$$

32.1.1.8 Vector Unbiased Estimator

$$E\{\hat{\theta}_i\}=\theta_i,\quad E\{\hat{\underline{\theta}}\}=[E\{\hat{\theta}_1\},E\{\hat{\theta}_2\},\cdots,E\{\hat{\theta}_p\}]^T=\underline{\theta}$$

32.1.1.9 Likelihood Function

$$L(x(0),x(1),\cdots,x(N-1);\theta)\equiv L(\underline{x};\theta)=p(x(0);\theta),p(x(1);\theta)\cdots p(x(N-1);\theta)\equiv p(\underline{x};\theta)$$

$$\int\cdots\int p(\underline{x};\theta)d\underline{x}=1,\quad \int\cdots\int\frac{\partial p(\underline{x};\theta)}{\partial\theta}d\underline{x}=0,\quad E\left\{\frac{\partial\ln p(\underline{x};\theta)}{\partial\theta}\right\}=\int\cdots\int\frac{1}{p(\underline{x};\theta)}\frac{\partial p(\underline{x};\theta)}{\partial\theta}p(\underline{x};\theta)\,d\underline{x}=0$$

$$E\left\{\left(\frac{\partial\ln p(\underline{x};\theta)}{\partial\theta}\right)^2\right\}=-E\left\{\left(\frac{\partial^2\ln p(\underline{x};\theta)}{\partial\theta^2}\right)\right\},\quad d\underline{x}=dx(0)dx(1)\cdots dx(N-1)$$

32.1.1.10 Efficient Estimator

If an estimator is unbiased (see 32.1.1.5) and attains the (CRLB), it is said to be efficient.

32.1.2 Cramer-Rao Lower Bound

32.1.2.1 Cramer-Rao Lower Bound (scalar parameter) (CRLB)

$\operatorname{var}(\hat\theta) \equiv E\{[\hat\theta - \tau(\theta)]^2\} \geq [\tau'(\theta)]^2 \Big/ E\left\{\left(\frac{\partial \ln p(\underline{x};\theta)}{\partial\theta}\right)^2\right\} = [\tau'(\theta)]^2 \Big/ \left[-E\left\{\left(\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right)\right\}\right]$, $\hat\theta =$ unbiased estimator, $\tau(\theta) = \int\cdots\int \theta p(\underline{x};\theta)\,d\underline{x}$, $\tau'(\theta) = d\tau(\theta)/d\theta$, $L =$ (see 32.1.1.7) $=$ $p(x(0);\theta)$ $p(x(1);\theta)$ $p(x(N-1);\theta) \equiv p(\underline{x};\theta)$, $d\underline{x} = dx(0)\,dx(1)\cdots dx(N-1)$

Note: $\tau(\theta) = \theta$, $\operatorname{var}(\hat\theta) \geq 1 \Big/ \left[-E\left\{\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right\}\right]$

32.1.2.2 Attainment of CRLB

Attainment is possible if and only if $\frac{\partial \ln p(\underline{x};\theta)}{\partial\theta} = I(\theta)\,(t(\underline{x}) - \theta)$ and $\hat\theta = t(\underline{x})$ is an MVU estimator. *Minimum variance*: $1/I(\theta)$, $t(\underline{x}) = t(x(0), x(1), \cdots, x(N-1))$, $I(\theta) =$ a function of parameter θ. *General formula*: $\frac{\partial \ln p(\underline{x};\theta)}{\partial\theta} = I(\theta)\,[t - \tau(\theta)]$, $\operatorname{var}(t) = \tau'(\theta)/I(\theta)$, $I(\theta) =$ independent of observations.

Example 1

$p(\underline{x};\theta) = (1/\sqrt{2\pi})^N \exp\left(-\sum_{i=0}^{N-1} (x(i) - \theta)^2 \quad 2\right)$, where $X(0), X(1), \cdots, X(N-1)$ is a random sample from a normal distribution with mean θ and variance 1, $N(\theta,1)$. Since the maximum of $p(\underline{x};\theta)$ and $\ln p(\underline{x};\theta)$ are the same, $d\ln p(\underline{x};\theta)/d\theta = 0 = \sum_{i=0}^{N-1} (x(i) - \theta)$ or $\theta = t(x(0), x(1), \cdots, x(N-1)) = \sum_{i=0}^{N-1} x(i)/N$ maximizes $p(\underline{x};\theta)$.

$\hat\theta = t(x(0), \cdots, x(N-1)) = (1/N)\sum_{i=0}^{N-1} x(i) = \bar{x}$ is the maximum likelihood estimator of the mean θ.

Example 2

$x(n) = s(n;\theta) + v(n)$, $n = 0, 1, \cdots, N-1$, $s(n;\theta) =$ deterministic signals with unknown parameter θ, $v(n) =$ white Gaussian with variance σ^2, $x(n) =$ observation samples.

$$p(\underline{x};\theta) = [1/(2\pi\sigma^2)^{N/2}]\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} [x(n) - s(n;\theta)]^2\right],$$

$$\partial \ln p(\underline{x};\theta)/\partial\theta = (1/\sigma^2)\sum_{n=0}^{N-1} (x(n) - s(n;\theta))\frac{\partial s(n;\theta)}{\partial\theta},$$

$$\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2} = (1/\sigma^2)\sum_{n=0}^{N-1}\left[[x(n) - s(n;\theta)]\frac{\partial^2 s(n;\theta)}{\partial\theta^2} - \left(\frac{\partial s(n;\theta)}{\partial\theta}\right)^2\right],$$

$$E\left\{\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right\} = -(1/\sigma^2)\sum_{n=0}^{N-1}\left(\frac{\partial s(n,\theta)}{\partial\theta}\right)^2, \qquad \mathrm{var}\,\hat{\theta} \ge -1/E\left\{\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right\} = \sigma^2 \Big/ \sum_{n=0}^{N-1}\left(\frac{\partial s(n;\theta)}{\partial\theta}\right)^2$$

Example 3

$x(n) = A + v(n),\ v(n) =$ white Gaussian with variance σ^2,

$$p(\underline{x};A) = \prod_{n=0}^{N-1}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}(x(n)-A)^2\right] = \frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n)-A)^2\right],$$

$$\frac{\partial \ln p(\underline{x};A)}{\partial A} = \frac{\partial}{\partial A}\left[-\ln\left[(2\pi\sigma^2)^{N/2}\right] - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n)-A)\right] = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x(n)-A) = \frac{N}{\sigma^2}(\bar{x}-A),$$

$\bar{x} =$ sample mean, $\dfrac{\partial^2 \ln p(\underline{x};A)}{\partial A^2} = -\dfrac{N}{\sigma^2}$, $\mathrm{var}(\hat{A}) \ge \sigma^2/N$. Since $\dfrac{N}{\sigma^2}(\bar{x}-A)$ is similar to (32.1.2.2) is minimum variance unbiased estimator.

32.1.2.3 CRLB Value

$$\mathrm{var}(\hat{\theta}) = 1/I(\theta)$$

32.1.2.4 Fisher Information

$$I(\theta) = -E\left\{\frac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right\}$$

32.1.2.5 Transformation of Parameter

If $\alpha = g(\theta)$, then the CRLB is

$$\mathrm{var}(\hat{\alpha}) \ge \frac{\left(\dfrac{\partial g(\theta)}{\partial\theta}\right)^2}{-E\left\{\dfrac{\partial^2 \ln p(\underline{x};\theta)}{\partial\theta^2}\right\}}$$

Example 1

From 32.1.2.2 Example 3, $\alpha = g(A) = A^2$ then $\mathrm{var}(\hat{A}^2) \ge \dfrac{(2A)^2}{N/\sigma^2}$

32.1.2.6 CRLB-Vector Parameter

$\underline{\theta} = [\theta_1\ \theta_2 \cdots \theta_p]^T$, $\mathrm{var}(\hat{\theta}_i) \ge [I^{-1}(\underline{\theta})]_{ii}$, $I(\underline{\theta}) =$ Fisher information matrix, $[\ ]_{ii} =$ the ii element of the matrix,

$$[I(\underline{\theta})]_{ij} \equiv I(\underline{\theta})_{ij} = -E\left[\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial\theta_i\,\partial\theta_j}\right] = E\left\{\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial\theta_i}\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial\theta_j}\right\},$$

$i = 1,\cdots,p;\ j = 1,\cdots,p.\ C_{\hat{\theta}} - I^{-1}(\underline{\theta}) \ge \underline{0}$, $C_{\hat{\theta}} =$ covariance matrix of any unbiased estimator $\underline{\hat{\theta}}$.

Example 1

$x(n) = A + Bn + v(n),\ n = 0,1,\cdots,N-1,\ v(n) =$ *WGN*, parameter unknown A and B.
Hence $\underline{\theta} = [A\ B]^T$,

$$I(\underline{\theta})_{11} = -E\left\{\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial A^2}\right\} = -N/\sigma^2,\quad I(\underline{\theta})_{22} = -E\left\{\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial B^2}\right\} = -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} n^2$$

$$= -\frac{1}{\sigma^2}\frac{N(N-1)(2N-1)}{6},\quad I(\underline{\theta})_{12} = I(\underline{\theta})_{21} = -E\left\{\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial A\,\partial B}\right\} = -E\left\{\frac{\partial^2 \ln p(\underline{x};\underline{\theta})}{\partial B\,\partial A}\right\} = -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} n$$

$$= -\frac{1}{\sigma^2}\frac{N(N-1)}{2},\quad p(\underline{x};\underline{\theta}) = \frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n)-A-Bn)^2\right],$$

$$I^{-1}(\underline{\theta}) = \sigma^2\begin{bmatrix} \frac{2(2N-1)}{N(N+1)} & -\frac{6}{N(N+1)} \\ -\frac{6}{N(N+1)} & \frac{12}{N(N^2-1)} \end{bmatrix},\quad \operatorname{var}(\hat{A}) \ge \frac{2(2N-1)\sigma^2}{N(N+1)},\quad \operatorname{var}(\hat{B}) \ge \frac{12\sigma^2}{N(N^2-1)}$$

32.1.2.7 CRLB Value-Vector Parameter

$C_{\hat{\underline{\theta}}} = I^{-1}(\underline{\theta})$ if and only if $\dfrac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}} = I(\underline{\theta})(\underline{t}(\underline{x}) - \underline{\theta}),\ \underline{t} \equiv$ p-dimensional function; $I = p \times p$ matrix,

$\hat{\underline{\theta}} = \underline{t}(\underline{x}) \equiv$ MVU estimator with covariance matrix $I^{-1}(\underline{\theta})$.

Example 1

From Example 32.1.2.6.1,

$$\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial \underline{\theta}} = \begin{bmatrix} \frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial A} \\ \frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial B} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x(n)-A-Bn) \\ \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x(n)-A-Bn)n \end{bmatrix} = \begin{bmatrix} \frac{N}{\sigma^2} & \frac{N(N-1)}{\sigma^2} \\ \frac{N(N-1)}{2\sigma^2} & \frac{N(N-1)(2N-1)}{6\sigma^2} \end{bmatrix}\begin{bmatrix} \hat{A}-A \\ \hat{B}-B \end{bmatrix},$$

$$\hat{A} = \frac{2(2N-1)}{N(N+1)}\sum_{n=0}^{N-1} x(n) - \frac{6}{N(N+1)}\sum_{n=0}^{N-1} nx(n),\qquad \hat{B} = -\frac{6}{N(N+1)}\sum_{n=0}^{N-1} x(n) + \frac{12}{N(N^2-1)}\sum_{n=0}^{N-1} nx(n).$$

Hence, the conditions for equality are satisfied and $[\hat{A}\ \hat{B}]^T$ is an efficient and therefore MVU estimator.

32.1.2.8 Vector Transformations CRLB

If $\underline{\alpha} = \underline{g}(\underline{\theta}),\ \underline{g} \equiv r-$ dimensional,

$$C_{\hat{\underline{\alpha}}} - \frac{\partial \underline{g}(\underline{\theta})}{\partial \underline{\theta}} I^{-1}(\underline{\theta}) \frac{\partial \underline{g}(\underline{\theta})^T}{\partial \underline{\theta}} \ge \underline{0};\ \partial \underline{g}(\underline{\theta})/\partial \underline{\theta} \equiv r \times p \text{ Jacobian matrix} = \begin{bmatrix} \frac{\partial g_1(\underline{\theta})}{\partial \theta_1} & \cdots & \frac{\partial g_1(\underline{\theta})}{\partial \theta_p} \\ \vdots & & \vdots \\ \frac{\partial g_r(\underline{\theta})}{\partial \theta_1} & \cdots & \frac{\partial g_r(\underline{\theta})}{\partial \theta_p} \end{bmatrix}$$

32.1.2.9 General Gaussian CRLB Case

$x = N(\underline{\mu}(\underline{\theta}), C(\underline{\theta}))$,

$$I(\underline{\theta})_{ij} = \left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_i}\right]^T C^{-1}(\underline{\theta})\left[\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_j}\right] + \frac{1}{2} tr\left[C^{-1}(\underline{\theta})\frac{\partial C(\underline{\theta})}{\partial \theta_i} C^{-1}(\underline{\theta})\frac{\partial C(\underline{\theta})}{\partial \theta_j}\right],$$

$$\frac{\partial \underline{\mu}(\underline{\theta})}{\partial \theta_i} = \left[\frac{\partial [\underline{\mu}(\underline{\theta})]_1}{\partial \theta_i} \;\; \frac{\partial [\underline{\mu}(\underline{\theta})]_2}{\partial \theta_i} \cdots \frac{\partial [\underline{\mu}(\underline{\theta})]_N}{\partial \theta_i}\right]^T, \quad \frac{\partial C(\underline{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [C(\underline{\theta})]_{11}}{\partial \theta_i} & \cdots & \frac{\partial [C(\underline{\theta})]_{1N}}{\partial \theta_i} \\ \vdots & & \vdots \\ \frac{\partial [C(\underline{\theta})]_{N1}}{\partial \theta_i} & \cdots & \frac{\partial [C(\underline{\theta})]_{NN}}{\partial \theta_i} \end{bmatrix}$$

If the parameter is scalar θ, $\underline{x} = N(\underline{\mu}(\theta), C(\theta))$,

$$I(\theta) = \left[\frac{\partial \underline{\mu}(\theta)}{\partial \theta}\right]^T C^{-1}(\theta)\left[\frac{\partial \underline{\mu}(\theta)}{\partial \theta}\right] + \frac{1}{2} tr\left[\left(C^{-1}(\theta)\frac{\partial C(\theta)}{\partial \theta}\right)^2\right]$$

Example 1

$x(n) = A + v(n)$, $n = 0,1,\cdots,N-1$, $v(n) = \text{WGN}$, A = Gaussian r.v. with $\mu = 0$ and $\text{var}(A) = \sigma_A^2$, A is independent of $v(n)$, σ_A^2 = unknown.

$[C(\sigma_A^2)]_{ij} = E\{x(i-1)x(j-1)\} = E\{(A+v(i-1))(A+v(j-1))\} = \sigma_A^2 + \sigma^2\delta_{ij}$, $C(\sigma_A^2) = \sigma_A^2 \underline{1}\underline{1}^T + \sigma^2 I$, $\underline{1} = [1\ 1\ \cdots 1]^T$,

$$C^{-1}(\sigma_A^2) = \frac{1}{\sigma^2}\left(I - \frac{\sigma_A^2}{\sigma^2 + N\sigma_A^2}\underline{1}\underline{1}^T\right), \quad \frac{\partial C(\sigma_A^2)}{\partial \sigma_A^2} = \underline{1}\underline{1}^T, \quad C^{-1}(\sigma_A^2)\frac{\partial C(\sigma_A^2)}{\partial \sigma_A^2} = \frac{1}{\sigma^2 + N\sigma_A^2}\underline{1}\underline{1}^T,$$

$$I(\theta) = \frac{1}{2} tr\left[\left(\frac{1}{\sigma^2 + N\sigma_A^2}\right)^2 \underline{1}\underline{1}^T\underline{1}\underline{1}^T\right] = \frac{1}{2}\left(\frac{1}{\sigma^2 + N\sigma_A^2}\right)^2, \quad \text{var}(\sigma_A^2) \ge 2\left(\sigma_A^2 + \frac{\sigma^2}{N}\right)^2$$

See (32.1.2.6) and (32.1.2.3).

32.1.3 Linear Models in Estimation

32.1.3.1 MVU Estimation with Gaussian Noise

$\underline{x} = H\underline{\theta} + \underline{v}$, $\underline{x} = N \times 1$ observation vector, $H = N \times p$ known observation matrix and rank p, $\underline{v} = N \times 1$ noise vector with *p.d.f.* $N(\underline{0}, \sigma^2 I)$, $\underline{\theta} = p \times 1$ vector parameter (to be estimated), $\hat{\underline{\theta}} = (H^T H)^{-1} H^T \underline{x} \equiv$ MVU estimator, $C_{\hat{\underline{\theta}}} = \sigma^2 (H^T H)^{-1} \equiv$ covariance matrix of $\hat{\underline{\theta}}$, for the linear model the MVU estimator is efficient (attains the CRLB).

Example 1

$x(t_n) = \theta_1 + \theta_2 t_n + \theta_3 t_n^2 + v(t_n)$, $n = 0,1,\cdots,N-1$, (to fit a second-order curve to data $x(t_n)$), $\underline{x} = [x(t_0)\cdots x(t_{N-1})]^T$, $\underline{\theta} = [\theta_1\ \theta_2\ \theta_3]^T$,

$$H = \begin{bmatrix} 1 & t_0 & t_0^2 \\ \vdots & \vdots & \vdots \\ 1 & t_{N-1} & t_{N-1}^2 \end{bmatrix},$$

$\hat{\underline{\theta}} = (H^T H)^{-1} H^T \underline{x}$, hence the estimated curve is $\hat{s}(t) = \sum_{i=1}^{3} \hat{\theta}_i t^{i-1}$.

32.1.3.2 MVU Estimator of General Model

$\underline{x} = H\underline{\theta} + \underline{s} + \underline{v}$, $\underline{x} = N \times 1$ observation vector, $H = N \times p$ known observation matrix, $\underline{s} = N \times 1$ known signal, $\underline{v} = N \times 1$ noise with p.d.f.

$N(\underline{0}, C)$, C = covariance matrix, $\hat{\theta} =$ MVU estimator $= (H^T C^{-1} H)^{-1} H^T C^{-1}(\underline{x} - \underline{s})$, $C_{\hat{\underline{\theta}}} = (H^T C^{-1} H)^{-1}$.

Example 1

$x(n) = A + v(n)$, $n = 0,1,\cdots,N-1$, $v(n)$ = colored Gaussian noise with $N \times N$ covariance matrix C. H = $\underline{1} = [1\,1 \cdots 1]^T$, then $\hat{A} = (H^T C^{-1} H)^{-1} C^{-1} \underline{x} = \dfrac{\underline{1}^T C^{-1} \underline{x}}{\underline{1}^T C^{-1} \underline{1}}$, $\operatorname{var}(\hat{A}) = (H^T C^{-1} H)^{-1} = 1/[\underline{1}^T C^{-1} \underline{1}]$. If we set C^{-1} = $D^T D$, $D = N \times N$ invertible matrix, then

$$\hat{A} = \frac{\underline{1}^T D^T D \underline{x}}{\underline{1}^T D^T D \underline{1}} = \frac{(D\underline{1})^T \underline{x}'}{\underline{1}^T D^T D \underline{1}} = \sum_{n=0}^{N-1} d_n x'(n),$$

$\underline{x}' = D\underline{x} \equiv$ prewhitened data, $d_n = [D\underline{1}]_n / \underline{1}^T D^T D \underline{1}$.

32.1.4 General MVU Estimation

32.1.4.1 Sufficient Statistic t(x)

If $p(\underline{x};\theta) = g(t(\underline{x}),\theta) h(\underline{x})$, $g(\cdot)$ is a function of $\underline{x}$ through $t(\underline{x})$ only, $h(\underline{x})$ is a function only of $\underline{x}$, then $t(\underline{x})$ is a sufficient statistic.

32.1.4.2 Complete Statistic

A statistic is complete if there is only one function of the statistic that is unbiased.

32.1.4.3 Unbiased Estimator

$\hat{\theta} = E\{\check{\theta} | t(\underline{x})\}$, $\check{\theta} =$ unbiased estimator of θ, $t(\underline{x})$ = sufficient statistic (see 32.1.4.1) for θ, $\hat{\theta} =$ unbiased and has lesser or equal variance than that of $\check{\theta}$ for all θ. $\hat{\theta} =$ MVU estimator if the sufficient statistic is complete (see 32.1.4.2).

32.1.4.4 Unbiased Estimator (vector parameter)

If $p(\underline{x};\underline{\theta}) = g(\underline{t}(x),\underline{\theta}) h(\underline{x})$, then $\underline{t}(\underline{x})$, an $r \times 1$ statistic, is a sufficient statistic for $\underline{\theta}$. $g(\cdot)$ depends only on $\underline{x}$ through $t(\underline{x})$ and on $\underline{\theta}$ and $h(\cdot)$ depends only on $\underline{x}$.

Example

$x(n) = A \cos 2\pi f_0 n + v(n)$, $n = 0,1,\cdots,N-1$, $\underline{\theta} = [A \;\; \sigma^2]^T$,

$$p(\underline{x};\theta) = [1/(2\pi\sigma^2)^{N/2}]\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n) - A\cos 2\pi f_0 n)^2\right]$$

$$= [1/(2\pi\sigma^2)^{N/2}]\exp\left[-\frac{1}{2\sigma^2}\left(\sum_{n=0}^{N-1}x^2(n) - 2A\sum_{n=0}^{N-1}x(n)\cos 2\pi f_0 n + A^2\sum_{n=0}^{N-1}\cos^2 2\pi f_0 n\right)\right]\cdot 1$$

$$= g(\underline{t}(\underline{x}),\underline{\theta})\cdot h(\underline{x}),\ \ h(\underline{x}) = 1,\ \ \underline{t}(\underline{x}) = \left[\sum_{n=0}^{N-1}x(n)\cos 2\pi f_o n \quad \sum_{n=0}^{N-1}x^2(n)\right]^T$$

32.1.5 Maximum Likelihood Estimation (MLE)

32.1.5.1 Maximum Likelihood Estimation (MLE)

The MLE for a scalar parameter is defined to be the value of θ that maximizes $p(\underline{x};\theta)$ for fixed $\underline{x}$. $\frac{\partial \ln p(\underline{x};\theta)}{\partial\theta} = 0.$

Example 1

$x(n) = A + v(n),\ n = 0,1,\cdots,N-1,\ v(n) =$ white Gaussian noise (WGN), A = unknown parameter, $p(\underline{x};A)$

$$= [1/(2\pi\sigma^2)^{N/2}]\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n) - A)^2\right],\quad \frac{\partial \ln p(\underline{x};\theta)}{\partial A} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x(n) - A) = 0,\quad \hat{A} = \frac{1}{N}\sum_{n=0}^{N-1}x(n)$$

32.1.5.2 MLE-vector Parameters

$\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial\underline{\theta}} = \underline{0}$ (see Chapters 33, 33.34.5)

Example 1

$x(n) = A + v(n),\ n = 0,1,\cdots,N-1,\ v(n) = WGN,\ \underline{\theta} = [A\ \sigma^2]^T,\ \frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial A} = \left(\frac{1}{\sigma^2}\right)\sum_{n=0}^{N-1}(x(n) - A) = 0,$ or $\hat{A} = \bar{x} =$ sample mean,

$$\frac{\partial \ln p(\underline{x};\underline{\theta})}{\partial\sigma^2} = -(N/2\sigma^2) + (1/\sigma^4)\sum_{n=0}^{N-1}(x(n) - A)^2 = 0 \text{ or } \sigma^2 = (1/N)\sum_{n=0}^{N-1}(x(n) - \bar{x})^2$$

since $\hat{A} = \bar{x},\ \ \hat{\underline{\theta}} = [\bar{x}\ \ (1/N)\sum_{n=0}^{N-1}(x(n) - \bar{x})^2]^T.$

32.1.5.3 MLE-Linear Model

$\underline{x} = H\underline{\theta} + \underline{v},\ H = N\times p$ matrix with $N > p$ and rank p, $\underline{\theta} = p\times 1$ parameter vector, $\underline{v}$ = noise vector with p.d.f. $N(\underline{0},C),\ \hat{\underline{\theta}} = (H^TC^{-1}H)^{-1}H^TC^{-1}\underline{x},\ \hat{\underline{\theta}} \sim N(\underline{\theta},(H^TC^{-1}H)^{-1}).$

32.1.6 Least Squares (LS)

32.1.6.1 Definition

The Least squares approach attempts to minimize the squared difference between the given data $x(n)$ and the assumed signal (noiseless data). $s(n)$ = deterministic.

32.1.6.2 Error Criterion

$$J(\theta)=\sum_{n=0}^{N-1}[x(n)-s(n)]^2,\ J(\cdot)\ \text{depends on}\ \theta\ \text{via}\ s(n).$$

Example 1

$$s(n)=A,\ J(A)=\sum_{n=0}^{N-1}(x(n)-A)^2,\ \frac{\partial J(A)}{\partial A}=0\ \text{we obtain}\ \hat{A}=\frac{1}{N}\sum_{n=0}^{N-1}x(n)=\bar{x}$$

32.1.6.3 Linear LS

$$s(n)=\theta h(n),\ h(n)=\text{known sequence},\ J(\theta)=\sum_{n=0}^{N-1}[x(n)-\theta h(n)]^2=\ \text{error criterion},$$

$$\hat{\theta}=\sum_{n=0}^{N-1}x(n)h(n)\Big/\sum_{n=0}^{N-1}h^2(n),\ J_{\min}=J(\hat{\theta})=\sum_{n=0}^{N-1}x^2(\theta)-\hat{\theta}\sum_{n=0}^{N-1}x(n)h(n)=\ \text{minimum LS error}$$

32.1.6.4 Linear LS-Vector Parameters

$\underline{s}=H\underline{\theta}$, $\underline{\theta}=p\times 1$ vector parameter, $H=$ observation matrix = $N\times p$ known matrix with rank p, $\underline{s}$ = $[s(0)\ s(1)\cdots s(N-1)]^T=$ signal linear in the unknown parameter,

$$J(\underline{\theta})=\sum_{n=0}^{N-1}(x(n)-s(n))^2=(\underline{x}-H\underline{\theta})^T(\underline{x}-H\underline{\theta})=\underline{x}^T\underline{x}-2\underline{x}^TH\underline{\theta}+\underline{\theta}^TH^TH\underline{\theta},$$

$\dfrac{J(\underline{\theta})}{\partial\underline{\theta}}=-2H^T\underline{x}+2H^TH\underline{\theta}=0$ then $\hat{\underline{\theta}}=(H^TH)^{-1}H^T\underline{x},\ H^TH\underline{\theta}=H^T\underline{x}\equiv$ normal equations, $J_{\min}$ = $J(\hat{\underline{\theta}})$ = $(\underline{x}-H(H^TH)^{-1}H^T\underline{x})^T(\underline{x}-H(H^TH)^{-1}H^T\underline{x})=\underline{x}^T(I-H(H^TH)^{-1}H^T)\underline{x}$

32.1.6.5 Linear LS Weighted-Vector Parameter

$J(\underline{\theta})=(\underline{x}-H\underline{\theta})^TW(\underline{x}-H\underline{\theta})$, $W=N\times N$ positive definite weighting matrix, $\hat{\underline{\theta}}=(H^TWH)^{-1}H^TW\underline{x}$, $J_{\min}=$ minimum LS error $=\underline{x}^T(W-WH(H^TWH)^{-1}H^TW)\underline{x}$

32.1.6.6 Order-Recursive LS

$$J_{\min,k+1}=J_{\min,k}-\frac{(\underline{h}_{k+1}^TP_k^{\perp}\underline{x})^2}{\underline{h}_{k+1}^TP_k^{\perp}\underline{h}_{k+1}},$$

$\hat{\underline{\theta}}=(H_k^TH_k)^{-1}H_k^T\underline{x}$, $H_k=N\times k$ observation matrix, $J_{\min,k}=$ minimum LS error based on $H_k=(\underline{x}-H_k\hat{\underline{\theta}}_k)^T(\underline{x}-H_k\hat{\underline{\theta}}_k)$, $H_{k+1}=[H_k\ \ \underline{h}_{k+1}]\equiv[N\times k\ \ N\times 1]$ (add a column),

$$\hat{\underline{\theta}}_{k+1}=\begin{bmatrix}\hat{\underline{\theta}}_k-\dfrac{(H_k^TH_k)^{-1}H_k^T\underline{h}_{k+1}\underline{h}_{k+1}^TP_k^{\perp}\underline{x}}{\underline{h}_{k+1}^TP_k^{\perp}\underline{h}_{k+1}}\\ \dfrac{\underline{h}_{k+1}^TP_k^{\perp}\underline{x}}{\underline{h}_{k+1}^TP_k^{\perp}\underline{h}_{k+1}},\end{bmatrix}\equiv\begin{bmatrix}k\times 1\\ 1\times 1\end{bmatrix}$$

where $P_k^{\perp}=I-H_k(H_k^TH_k)^{-1}H_k^T=$ projection matrix onto the subspace orthogonal to that spanned by the columns of H_k.

Example

(Line fitting) Since $s_1(n) = A_1$ and $s_2(n) = A_2 + B_2 n$ for $n = 0,1,\cdots,N-1$ we have $H_1 = [1\ 1 \cdots 1]^T = \underline{1}^T$, $H_2 = [H_1\ \ \underline{h}_2]$, $\underline{h}_2 = [0\ 1 \cdots N-1]^T$, $\hat{A}_1 = \hat{\theta}_1 = (H_1^T H_1)^{-1} H_1^T \underline{x} = \bar{x}$,

$$J_{\min 1} = (\underline{x} - H_1\hat{\theta}_1)^T(\underline{x} - H_1\hat{\theta}_1) = \sum_{n=0}^{N-1}(x(n)-\bar{x})^2,\ \bar{x} = \text{ sample mean},$$

$$\hat{\underline{\theta}}_2 = [\hat{A}_2\ \ \hat{B}_2]^T = \left[\theta_1 - \frac{(H_1^T H_1)^{-1} H_1^T \underline{h}_2 \underline{h}_2^T P_1^{\perp}\underline{x}}{\underline{h}_2^T P_1^{\perp}\underline{h}_2}\ \ \frac{\underline{h}_2^T P_1^{\perp}\underline{x}}{\underline{h}_2^T P_1^{\perp}\underline{h}_2}\right]^T,$$

$$(H_1^T H_1)^{-1} = 1/N,\ \ P_1^{\perp} = I - H_1(H_1^T H_1)^{-1} - H_1^T = I - \frac{1}{N}\underline{1}\,\underline{1}^T,$$

$$P_1^{\perp}\underline{x} = \underline{x} - \frac{1}{N}\underline{1}\,\underline{1}^T\underline{x} = \underline{x} - \bar{x}\underline{1},\ \ \underline{h}_2^T P_1^{\perp}\underline{x} = \underline{h}_2^T\underline{x} - \bar{x}\,\underline{h}_2^T\underline{1} = \sum_{n=0}^{N-1} n\,x(n) - \bar{x}\sum_{n=0}^{N-1} n,$$

$$\underline{h}_2^T P_1^{\perp}\underline{h}_2 = \underline{h}_2^T\underline{h}_2 - \tfrac{1}{N}(\underline{h}_2^T\underline{1})^2 = \sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2,$$

$$\hat{\underline{\theta}}_2 = \left[\bar{x} - \frac{\frac{1}{N}\sum_{n=0}^{N-1} n\left[\sum_{n=0}^{N-1} n\,x(n) - \bar{x}\sum_{n=0}^{N-1} n\right]}{\sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2}\ \ \frac{\sum_{n=0}^{N-1} n\,x(n) - \bar{x}\sum_{n=0}^{N-1} n}{\sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2}\right]^T = \left[\bar{x} - \frac{1}{N}\sum_{n=0}^{N-1} n\hat{B}_2\right],$$

$$\hat{B}_2 = \frac{\sum_{n=0}^{N-1} n\,x(n) - \frac{N(N-1)}{2}\bar{x}}{N(N^2-1)/12} = -\frac{6}{N(N+1)}\sum_{n=0}^{N-1} x(n) + \frac{12}{N(N^2-1)}\sum_{n=0}^{N-1} n\,x(n),$$

$$\hat{A} = \bar{x} - \frac{1}{N}\sum_{n=0}^{N-1} n\hat{B}_2 = \bar{x} - \frac{N-1}{2}\hat{B}_2,\ \ J_{\min 2} = J_{\min 1} - \frac{(\underline{h}_2^T P_1^{\perp}\underline{x})^2}{\underline{h}_2^T P_1^{\perp}\underline{h}_2}$$

32.1.6.7 Sequential Least Squares

$\hat{\theta}(N) = \hat{\theta}(N-1) + \frac{1}{N+1}[x(N) - \hat{\theta}(N-1)]$, $\hat{\theta}(N-1) \equiv$ LSE based on $\{x(0), x(1), \cdots, x(N-1)\}$, the argument of $\hat{\theta}$ denotes the index of the most recent data point observed, $[1/(N+1)][x(N) - \hat{\theta}(N-1)] \equiv$ correction term.

32.1.6.8 Sequential Least Squares Error

$$J_{\min}(N) = J_{\min}(N-1) + \frac{N}{N+1}(x(N) - \hat{\theta}(N-1))^2$$

32.1.6.9 Nonlinear LS by Transformation of Parameters

$\underline{\alpha} = \underline{g}(\underline{\theta})$, $\underline{g} = p-$ dimensional function of $\underline{\theta}$ whose inverse exists. $\underline{s}(\underline{\theta}(\underline{\alpha})) = \underline{s}(\underline{g}^{-1}(\underline{\alpha})) = H\underline{\alpha} \equiv$ linear in $\underline{\alpha}$, $\hat{\underline{\theta}} = \underline{g}^{-1}(\hat{\underline{\alpha}})$ where $\hat{\underline{\alpha}} = (H^T H)^{-1} H^T \underline{x}$, $\hat{\underline{\theta}} = \underline{g}^{-1}(\hat{\underline{\alpha}})$, $J =$ error criterion $= (\underline{x} - \underline{s}(\underline{\theta}))^T (\underline{x} - \underline{s}(\underline{\theta}))$.

Example 1

$s(n) = A\cos(2\pi f_0 n + \varphi)$, $n = 0,1,\cdots,N-1$, A and φ to be estimated, $f_0 =$ known, $J = \sum_{n=0}^{N-1} [x(n) - A\cos(2\pi f_0 n + \varphi)]^2$, $s(n) = \alpha_1 \cos 2\pi f_0 n + \alpha_2 \sin 2\pi f_0 n$ where $\alpha_1 = A\cos\varphi$ and $\alpha_2 = -A\sin\varphi$, $\underline{s} = H\underline{\alpha}$

where $\underline{\alpha}[\alpha_1 \;\; \alpha_2]^T$ and

$$H = \begin{bmatrix} 1 & 0 \\ \cos 2\pi f_0 & \sin 2\pi f_0 \\ \vdots & \vdots \\ \cos 2\pi f_0 (N-1) & \sin 2\pi f_0 (N-1) \end{bmatrix},$$

$\hat{\alpha} = (H^T H)^{-1} H^T \underline{x}$, $\hat{\underline{\theta}}$ is found from the inverse transformation $\underline{g}^{-1}(\underline{\alpha})$. Hence, $A = [\alpha_1^2 + \alpha_2^2]^{1/2}$, $\varphi = \tan^{-1}(-\alpha_2 / \alpha_1)$, and $\hat{\underline{\theta}} = [\hat{A} \;\; \hat{\varphi}]^T = [\sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2} \quad \tan^{-1}(-\hat{\alpha}_1^2 / \hat{\alpha}_2^2)]^T$.

32.1.6.10 Nonlinear LS by Separation

$\underline{s} = H(\underline{\alpha})\underline{\beta} =$ separable, $\underline{\theta} = [\underline{\alpha} \;\; \underline{\beta}]^T = [(p-q)\times 1 \;\; q \times 1]^T$, $H(\underline{\alpha}) = N \times q$ dependent on $\underline{\alpha}$. Model linear in $\underline{\beta}$ but nonlinear in $\underline{\alpha}$ which implies minimization with respect to $\underline{\beta}$ which results in a function of $\underline{\alpha}$. $J(\underline{\alpha},\underline{\beta}) = (\underline{x} - H(\underline{\alpha})\underline{\beta})^T (\underline{x} - H(\underline{\alpha})\underline{\beta})$, $\hat{\underline{\beta}} = (H^T(\underline{\alpha})H(\underline{\alpha}))^{-1} H^T(\underline{\alpha})\underline{x} \equiv$ minimizes J, $J(\underline{\alpha},\hat{\underline{\beta}})$ = LS error = $\underline{x}^T [I - H(\underline{\alpha})(H^T(\underline{\alpha})H(\underline{\alpha}))^{-1} H^T(\underline{\alpha})]\underline{x}$ which reduces to a minimization of $\underline{x}^T H(\underline{\alpha})(H^T(\underline{\alpha})H(\underline{\alpha}))^{-1} H^T(\underline{\alpha})\underline{x}$.

32.1.7 Method of Moments

32.1.7.1 Scalar Parameter

$$\mu'_k = E\{x^k(n)\} = h(\theta),\; \theta = h^{-1}(\mu'_k),\; \hat{\mu}'_k = \frac{1}{N}\sum_{n=0}^{N-1} x^k(n),\; \hat{\theta} = h^{-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} x^k(n),\right),$$

Example

$x(n) = A + v(n)$, $n = 0,1,\cdots,N-1$, $v(n) =$ WGN with variance σ^2, $A \equiv$ to be estimated. We know

$$\mu'_k = E\{x(n)\} = A \equiv h(\theta) \text{ and } \theta = h^{-1}(\mu'_1) = \mu'_1 \text{ and hence } \hat{\theta} \equiv \hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\;,$$

32.1.7.2 Vector Parameter

$$\underline{\mu}' = \underline{h}(\underline{\theta}) \text{ or } [\mu'_1 \; \mu'_2 \cdots \mu'_p]^T = [h_1(\theta_1,\cdots,\theta_p) \quad h_2(\theta_1,\cdots,\theta_p) \;\cdots\; h_p(\theta_1,\cdots,\theta_p)]^T$$

$$\underline{\theta} = \underline{h}^{-1}(\underline{\mu}'),\ \hat{\underline{\theta}} = \underline{h}^{-1}(\hat{\underline{\mu}}')\ \text{where}\ ,\hat{\underline{\mu}}' = \left[\frac{1}{N}\sum_{n=0}^{N-1} x(n) \quad \frac{1}{N}\sum_{n=0}^{N-1} x^2(n)\cdots\frac{1}{N}\sum_{n=0}^{N-1} x^p(n)\right]^T$$

Example

Let

$$p(x(n);\varepsilon) = \frac{1-\varepsilon}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2}\frac{x^2(n)}{\sigma_1^2}\right) + \frac{\varepsilon}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2}\frac{x^2(n)}{\sigma_2^2}\right),$$

ε = mixture parameter, $0 < \varepsilon < 1$, σ_1 and σ_2 are unknown variances of the individual Gaussian p.d.f.'s $p(x(n);\varepsilon)$ is thought of as the p.d.f. of r.v. obtained from $N(0,\sigma_1^2)$ with probability 1 – ε, and from a $N(0,\sigma_2^2)$ p.d.f. with probability ε.

$\mu_2' = E\{x^2(n)\} = (1-\varepsilon)\sigma_1^2 + \varepsilon\sigma_2^2$, $\mu_4' = E\{x^4(n)\} = 3(1-\varepsilon)\sigma_1^4 + 3\varepsilon\sigma_2^4$, $\mu_6' = E\{x^6(n)\} = 15(1-\varepsilon)\sigma_1^6 + 15\varepsilon\sigma_2^6$. Setting $u = \sigma_1^2 + \sigma_2^2$ and $\upsilon = \sigma_1^2\sigma_2^2$ in the above equation we obtain: $u = (\mu_6' - 5\mu_4'\mu_2')/(5\mu_4' - 15\mu_2'^2)$, $\upsilon = \mu_2' u - \frac{\mu_4'}{3}$. We first find u and then υ and, hence, σ_1 and σ_2, which are: $\sigma_1^2 = (u + \sqrt{u^2 - 4\upsilon})/2$, $\sigma_2^2 = \upsilon/\sigma_1^2$. But $E\{x^2(n)\} = \int x^2(n)\,p(x(n);\varepsilon)dx(n) = (1-\varepsilon)\sigma_1^2 + \varepsilon\sigma_1^2 = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n)$ or $\varepsilon = (\mu_2 - \sigma_1^2)/(\sigma_2^2 - \sigma_1^2)$.

32.1.8 Bayesian MSE

32.1.8.1 Definition:

$B_{mse}(\hat{\theta}) = E\{(\theta - \hat{\theta})^2\} = \iint (\theta - \hat{\theta})^2 p(\underline{x},\theta)\,d\underline{x}\,d\theta$, the operator E with respect to the joint p.d.f. $p(\underline{x},\theta)$.

32.1.8.2 Prior Knowledge ($\theta \equiv$ random)

$p(\theta)$ = assigned prior *p.d.f.* of θ, $p(\theta|\underline{x})$ = posterior p.d.f. after data were observed, $\hat{\theta} = E\{\theta|\underline{x}\} = \int \theta p(\theta|\underline{x})\,d\theta$, $p(\theta|\underline{x}) = p(\underline{x}|\theta)p(\theta)/\int p(\underline{x}|\theta)p(\theta)\,d\theta$

Example 1

p.d.f. of θ is $1/2A_0$ for $-A_0 \le \theta \le A_0$ and zero everywhere else, $x(n) = \theta + v(n)$ $n = 0,1,\cdots,N-1$, $v(n)$ = GWN with variance σ^2 and independent of θ. Hence $B_{mse}(\hat{\theta}) = \iint (\theta - \hat{\theta})^2 p(\underline{x},\theta)\,d\underline{x}\,d\theta = \int\left[\int (\theta - \hat{\theta})^2 p(\theta|\underline{x})\,d\theta\right]p(\underline{x})\,d\underline{x}$ since $p(\underline{x},\theta) = p(A|\underline{x})p(\underline{x}) \equiv$ Bayes relationship. Since $p(\underline{x}) \ge 0$ implies that the internal integral can be minimized for each $\underline{x}$ and hence $B_{mse}(\hat{\theta})$ will be minimized. $\frac{\partial}{\partial\theta}\int (\theta - \hat{\theta})^2 p(\theta|\underline{x})\,d\theta = -2\int \theta p(\theta|\underline{x})\,d\theta + 2\hat{\theta}\int p(\theta|\underline{x})\,d\theta = 0$ and hence $\hat{\theta} = \int \theta p(\theta|\underline{x})\,d\theta = E\{\theta|\underline{x}\}$ since the conditional p.d.f. must integrate to one.

$$p_x(x(n)|\theta) = p_v(x(n) - \theta|\theta) = p_v(x(n) - \theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}(x(n) - \theta)^2\right]$$

and hence

$$p_x(\underline{x}|\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x(n)-\theta)^2\right],$$

$$p(\theta|\underline{x}) = p(\underline{x}|\theta)\,p(\theta) \Big/ \int p(\underline{x}|\theta)\,p(\theta)\,d\theta = \frac{\frac{1}{2A_0}p(\underline{x}|\theta)}{\int_{-A_0}^{A_0}(\underline{x}|\theta)\,p(\theta)\,d\theta} \quad |\theta| \le A_0$$

and zero for $|\theta| > A_0$. Next we must find c such that

$$p(\theta|\underline{x}) = \frac{1}{c\sqrt{\frac{2\pi\sigma^2}{N}}}\exp\left[-\frac{1}{2\sigma^2/N}(\theta-\bar{x})^2\right] \quad |\theta| \le A_0$$

must integrate to one. Hence,

$$c = \int_{-A_0}^{A}\left[1 \quad \sqrt{\frac{2\pi\sigma^2}{N}}\right]\exp\left[-\frac{1}{2\sigma^2/N}(\theta-\bar{x})^2\right]d\theta$$

and thus MMSE estimator, which is the mean of $p(\theta|\underline{x})$ is

$$\hat{\theta} = E\{\theta|\underline{x}\} = \int_{-\infty}^{\infty}\theta p(\theta|\underline{x})d\theta$$

$$= \left[\int_{-A_0}^{A_0} A\left(1 \quad \left(\sqrt{2\pi\frac{\sigma^2}{N}}\right)\right)\exp\left[-\frac{1}{2\sigma^2/N}(\theta-\bar{x})^2\right]d\theta \Big/ \left[\int_{-A_0}^{A_0}\left[1 \quad \left(\sqrt{2\pi\frac{\sigma^2}{N}}\right)\right]\exp[-\frac{1}{2\frac{\sigma^2}{N}}(\theta-\bar{x})^2]d\theta\right]\right.$$

32.1.8.3 Vector Form

$$\hat{\underline{\theta}} = E\{\underline{\theta}|\underline{x}\} = \int \underline{\theta}p(\underline{\theta}|\underline{x})\,d\underline{\theta}, \quad p(\underline{\theta}|\underline{x}) = p(\underline{x}|\underline{\theta})\,p(\underline{\theta}) / \int p(\underline{x}|\underline{\theta})\,p(\underline{\theta})\,d\underline{\theta}$$

32.1.8.4 Linear Model (posterior p.d.f. for the general linear model)

$\underline{x} = H\underline{\theta} + \underline{v}$, $\underline{x} = N \times 1$ data vector, $H =$ known $N \times p$ matrix, $\underline{\theta} = p \times 1$ random vector with prior p.d.f. $N(\underline{\mu}_\theta, C_\theta)$ and $\underline{v} = N \times 1$ noise vector with p.d.f. $N(\underline{0}, C_v)$ and independent of $\underline{\theta}$, then the posterior p.d.f. $p(\underline{\theta}|\underline{x})$ is Gaussian with mean ?

$$E\{\underline{\theta}|\underline{x}\} = \underline{\mu}_\theta + C_\theta H^T(H\,C_\theta H^T + C_v)^{-1}(\underline{x} - H\underline{\mu}_\theta)$$

$$C_{\theta|x} = \text{cov}\,ariance = C_\theta - C_\theta H^T(H\,C_\theta H^T + C_v)^{-1}H\,C_\theta$$

32.2 Statistical Hypotheses

32.2.1 Definitions

32.2.1.1 **Statistical Hypothesis** is a conjecture that a parameter, e.g., θ = mean of a Gaussian process, is larger than a specific value ($\theta > 75$).

32.2.1.2 **Alternative Hypothesis** is the value of the parameter in 32.2.1.1 is set less than the specific value of 32.2.1.1 ($\theta < 75$).

32.2.1.3 **Test** is a rule we devise that will tell us what decision to make once the experimental values have been determined. Such a rule is called a test of the statistical hypothesis H_0: $\theta < 75$ against the alternative hypothesis H_1: $\theta > 75$. A test leads to a decision to accept or reject the hypothesis under consideration.

32.2.1.4 **Critical Region** Let C be that subset of the sample space which, in accordance with a prescribed test, leads to the rejection of the hypothesis under consideration. Then C is called the critical region.

32.2.1.5 **Power Function** The power function of a test that yields the probability that the sample point falls in the critical region C of the test; a function that yields the probability of rejecting the hypothesis under consideration.

32.2.1.6 **Power** The value of the power function at a parameter point is called the power of the test at that point.

32.2.1.7 **Significance Level** The significance level of the test (or the *size* of the critical region C) is the maximum value (supremum) of the power function of the test when H_0 is true (H_0 is a hypothesis to be tested against an alternative hypothesis H_1 in accordance with a prescribed test).

Example 1

Let X have p.d.f. $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta},\ 0 < x < \infty$ and $f(x;\theta) = 0$ otherwise. Test H_0:$\theta = 2$ (simple hypothesis) against the alternative hypothesis H_1:$\theta = 4$. $\{\theta;\theta = 2,4\}$ Random sample X_1,X_2 of size n = 2. C = critical region = $\{(x_1,x_2); 9.5 \le x_1 + x_2 < \infty\}$ will determine the power function and the significance level of the test. $f(x;2)$ specified by H_0 and $f(x;4)$ specified by H_1. The power function is defined at two points $\theta = 2$ and $\theta = 4$. Power function of the test is given by $P\{(X_1,X_2)\in C\}$. If H_0 is true, $\theta = 2$, and the joint p.d.f. of X_1 and X_2 is $f(x_1;2)f(x_2;2) = \tfrac{1}{4}e^{-(x_1+x_2)/2}$ for $0 < x_1 < \infty$, $0 < x_2 < \infty$, and zero otherwise. $P\{(X_1,X_2)\in C\} = 1 - P\{(X_1,X_2)\in C^*\} = 1 - \int_0^{9.5}\int_0^{9.5-x_2} \tfrac{1}{4}e^{-(x_1+x_2)/2}dx_1\,dx_2 \cong 0.05$ (C^* is the complement of C). If H_1 is true, $\theta = 4$, then $f(x_1;4)f(x_2;4) = \tfrac{1}{16}e^{-(x_1+x_2)/4},\ 0 < x_1 < \infty,\ 0 < x_2 < \infty$ and zero otherwise. $P\{(X_1,X_2)\in C\} = 1 - \int_0^{9.5}\int_0^{9.5-x_2} \tfrac{1}{16}e^{-(x_1+x_2)/4}dx_1\,dx_2 \cong 0.31$. The power test is 0.05 for $\theta = 2$ and 0.31 for $\theta = 4$. Hence, the probability of rejecting H_0 when H_0 is true is 0.05, and the probability of rejecting H_0

when H_0 is false is 0.31. Since the significance level of this test (size of C) is the power of the test when H_0 is true, the significance level of this test is 0.05.

32.2.2 Neyman-Pearson Theorem

32.2.2.1 Neyman-Pearson Theorem

$X_1,\cdots,X_n$ = random sample from a distribution with p.d.f. $f(x;\theta)$. $L(\theta;x_1,x_2,\cdots x_n)=$ joint *p.d.f.* = $f(x_1;\theta)\,f(x_2;\theta)\cdots f(x_n;\theta)$. Let θ' and θ'' be distinct fixed vales and k = positive numbers. C = subset of the sample space such that

a) $[L(\theta';x_1,\cdots x_n)/L(\theta'';x_1,\cdots x_n)]\le k$ for each point $(x_1,\cdots,x_n)\in C$,
b) $[L(\theta';x_1,\cdots x_n)/L(\theta'';x_1,\cdots x_n)]\ge k$ for each point $(x_1,\cdots,x_n)\in C^*$ (complement of C)
c) $\alpha = P\{(X_1,\cdots,X_n)\in C;\ H_0\}$.

Then C is a best critical region of size α for testing the simple hypothesis $H_0:\theta=\theta'$ against the alternative simple hypothesis $H_1:\theta=\theta''$.

Example 1

$X_1,\cdots,X_n$ = random sample from a distribution with p.d.f. $f(x;\theta)=\frac{1}{\sqrt{2\pi}}\exp(-\frac{(x-\theta)^2}{2}),\ -\infty<x<\infty.$ Test $H_0:\theta=\theta'=0$ against the alternative hypothesis $H_1:\theta=\theta''=1$. Now $L(\theta';x_1,\cdots x_n)/L(\theta'';x_1,\cdots x_n)$ $=\left[\left(\frac{1}{\sqrt{2\pi}}\right)^n\exp\left[-\left(\sum_1^n x_i^2\right)\Big/2\right]\right]\Big/\left[(1/\sqrt{2\pi})^n\exp\left[-\left(\sum_1^n(x_i-1)^2\right)\Big/2\right]\right]=\exp\left[-\sum_1^n x_i+\frac{n}{2}\right]$. If $k>0$ the set of all points $(x_1,x_2,\cdots,x_n)$ such that $\exp\left[-\sum_1^n x_i+\frac{n}{2}\right]\le k$ is a best critical region. The inequality holds if and only if $-\sum_1^n x_i+\frac{n}{2}\le\ln k$ or $\sum_1^n x_i\ge\frac{n}{2}-\ln k=c$. Hence, the set $C=\{(x_1,x_2,\cdots,x_n);\ \sum_1^n x_i\ge c\}$ where c can be determined so that the size of the critical region is a desired number α. The event $\sum_1^n x_i\ge c$ is equivalent to the event $\bar{X}=\sum x_i|n\ge c/n=c_1$, say, so the test may be based upon the statistic $\bar{X}$. If H_0 is true $(\theta=\theta'=0)$, then $\bar{X}$ has a distribution that is $N(0,1/n)$. If $(x_1,x_2,\cdots,x_n)$ are the experimental values, then $\bar{x}=\sum_{i=1}^n x_i|n$. If $\bar{x}\ge c_1$, the simple hypothesis $H_0:\theta=\theta'=0$ would be rejected at the significant level α; if $\bar{x}<c_1$, H_0 would be accepted. The probability of rejecting H_0, when H_0 is true, is α; the probability of rejecting H_0, when H_0 is false, is the value of the power of the test at $\theta=\theta''=1$. That is

$$P\{\bar{X}\ge c_1;H_1\}=\int_{c_1}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1/n}}\exp\left(-\frac{(\bar{x}-1)^2}{2(1/n)}\right)d\bar{x}\,.$$

For example, if n = 25 and if we select $\alpha = 0.05$, then from Table 34.2 in Chapter 34 (interpolate) we find $c_1 = 1.645/\sqrt{25} = 0.329$. Hence the power of this best test of H_0 against H_1 is 0.05, when H_0 is true, and is

$$\int_{0.329}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1/25}} \exp\left(-\frac{(\bar{x}-1)^2}{2(1/25)}\right) d\bar{x} = 0.999$$

when H_1 is true.

32.2.2.2 Likelihood Ratio

$\lambda(x_1, x_2, \cdots, x_n) = \lambda = \dfrac{L(\hat{\omega})}{L(\hat{\Omega})}$, $L(\hat{\omega})$ = maximum of $L(\omega)$, $L(\hat{\Omega})$ = maximum of $L(\Omega)$, $\Omega =$ the set of all parameter points $(\theta_1, \theta_2, \cdots, \theta_m)$, ω subset of Ω, $(X_1, X_2, \cdots, X_n) = n$ mutually stochastically independent r.v.'s having, respectively, the p.d.f. $f_i(x_i; \theta_1, \theta_2, \cdots, \theta_m)$ $i = 1, 2, \cdots, n$; $L(\omega) = \prod_{i=1}^{n} f_i(x_i; \theta_1, \cdots, \theta_m)$, $(\theta_1, \cdots, \theta_m) \in \omega$, $L(\Omega) = \prod_{i=1}^{n} f_i(x_i; \theta_1, \cdots, \theta_m)$, $(\theta_1, \cdots, \theta_m) \in \Omega$.

32.2.2.3 Likelihood Ration Test Principle

$H_0: (\theta_1, \cdots, \theta_m) \in \omega$, is rejected if and only if $\lambda(x_1, x_2, \cdots, x_n) = \lambda \le \lambda_o =$ positive proper function. The function λ defines an r.v. $\lambda(X_1, \cdots, X_n)$ and the significance level of the test is given by $\alpha = P\{\lambda(X_1, \cdots, X_n) \le \lambda_o; H_o\}$

32.2.3 Hypothesis Testing for the Mean of a Normal Distribution: The Two-Tailed t-test

32.2.3.1 $x_1, x_2, \cdots, x_n = n$ normally distributed observations. We wish to test whether or not the sample supports the hypothesis that $\mu_x = \bar{x}$ is $\mu_0 =$ same test value for μ_x which we have specified according to some objectives of our investigation.

Steps

1. Hypothesis specifications: Set μ_0 and $\alpha =$ the size (probability) of the Type I error to be tolerated.

   $H_0: \mu_x = \mu_0$, $H_1: \mu_x \ne \mu_0$; α
2. Test statistic:

$$t_0 = \frac{\bar{x} - \mu_0}{s_{\bar{x}}} = \text{test statistic}, \ \bar{x} = \sum x_i / n, \ \ s_{\bar{x}} = \sqrt{s_x^2 / n}, \ \ s_x^2 = \sum_{1}^{n} (x_i - \bar{x})(x_i - \bar{x}) / (n-1)$$

3. Assumption: $x_i \equiv N(\mu_x, \sigma^2)$. If H_0 is true $t_0 = (\bar{x} - \mu_0)/s_{\bar{x}} = (\bar{x} - \mu_x)/s_{\bar{x}}$ is a member of t-distribution with (n – 1) degrees of freedom.
4. Critical region: Reject H_0 and accept H_1 if $|t_0| = \left|(\bar{x} - \mu_0)/\sqrt{s_x^2/n}\right| > t(n-1; \alpha/2)$.

   Accept H_0 if $|t_0| < t(n-1; \alpha/2)$.
5. P-value: P-value = $2P\{t(n-1) > |t_0|\}$

6. Confidence interval: $\bar{x}\left(1 \mp \frac{t_{\alpha/2}}{t_0}\right)$

Example 1

If $x_1 = 17,\ x_2 = 16,\ x_3 = 18,\ x_4 = 21;\ n = 4.$ Test whether or not $\mu = 10$ and take $\alpha = 0.05.$

Steps

1. $H_0\!: \mu = 10,\ \ H_1\!: \mu \neq 10;\ \ \alpha = 0.05$
2. $\bar{x} = 74/4 = 18,\ \sum_1^4 (x_i - \bar{x})(x_i - \bar{x}) = 14,\ s_x^2 = 14/3,\ s_{\bar{x}} = \sqrt{14/12},\ \ t_0 = \frac{18-10}{\sqrt{14/12}} = 7.4$
3. Assume $x_i = N(\mu = 10; \sigma^2),\ i = 1,\cdots,4,$ the test statistic is a member of the t-distribution with $n-1 = 3$ degrees of freedom.
4. In the 3-degree of freedom row (see Table 34.3, Chapter 34) and column $1-(0.05/2) = 0.975$ we find $t(3;0.05/2) = 3.182$. The test statistic value $7.4 > 3.182$ has fallen in the critical region. The test hypothesis is false and prefers the alternative hypothesis.
5. The test statistic value 7.4 is even greater than 5.841 found in the column $1-0.005 = 0.995$ in the 3-degree of freedom row. Hence $P < 2(0.005) = 0.01$ which implies that the error is very small.
6. 95% confidence interval for μ_x is $18 \mp (3.182)(\sqrt{14/12}) = 14.56,\ 21.44 \cong 14.5,\ 21.5.$

32.2.4 Hypothesis Testing for the Variance of a Normal Distribution: The Two-Tailed χ^2-Test

32.2.4.1 Steps

1. Hypothesis specification: $H_0\!:\sigma_x^2 = \sigma_0^2,\ \ H_1\!:\sigma_x^2 \neq \sigma_0^2;\ \alpha.$ Investigator specifies the numerical values of σ_0^2 and α.
2. Test statistic: Given $x_1, x_2, \cdots, x_n$ calculate s_x^2 (see 32.2.3.1) and the test statistic

$$\chi_0^2 = \frac{(n-1)s_x^2}{\sigma_0^2} = \frac{\sum_1^n (x_i - \bar{x})(x_i - \bar{x})}{\sigma_0^2}$$

3. Distribution: If $x_i \equiv N(\mu, \sigma_x^2),\ i = 1,\cdots,n,$ then $\sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})/\sigma_x^2 \equiv \chi^2(n-1).$ Hence if H_0 is true and $\sigma_x^2 = \sigma_0^2,$ the test statistic is also a $\chi^2(n-1)$-variate; that is $\sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})/\sigma_0^2$ $\equiv \chi^2(n-1)$ so that, when H_0 is true tabulated $\chi^2(n-1)$ probabilities can be used as probabilities for $\sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})/\sigma_0^2$

4. Critical region: $n-1 \equiv$ degrees of freedom, $\chi^2(n-1;1-\alpha/2)$ = lower ritical value, $\chi^2(n-1;\alpha/2)$ = upper critical value, $P\{\chi^2 < \chi^2(n-1;1-\alpha/2)\} = \frac{\alpha}{2} = P\{\chi^2 > \chi^2(n-1;\alpha/2)\}$, the two-tail region from 0 to $\chi^2_{1-\alpha/2}$ and from $\chi^2_{\alpha/2}$ to ∞ comprise the critical region. We reject H_0 for H_1 if the test statistic $\chi_0^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 / \sigma_0^2$ fall in the region; we accept H_0 if $\chi^2_{1-\alpha/2} < \chi_0^2 < \chi^2_{\alpha/2}$
5. P-value: $2P\{\chi^2(n-1) > \chi_0^2\}$ or $2P\{\chi^2(n-1) < \chi_0^2\}$, $\chi_0^2 =$ right or left tail of the $\chi^2(n-1)$-distribution.
6. Confidence intervals: A $100(1-\alpha)\%$ confidence interval for the population variance is from $\sum_{i=1}^{n}(x_i - \bar{x})^2 / \chi^2_{\alpha/2}$ to $\sum_{i=1}^{n}(x_i - \bar{x})^2 / \chi^2_{1-\alpha/2}$

Example 1

If $n = 16$ and $\sum_{i=1}^{n}(x_i - \bar{x}) = 135$, then $s_x^2 = 135/15 = 19$ with 15 degrees of freedom. At $\alpha = 0.05$, is this result statistically compatible with the hypothesis that $\sigma_x^2 = 20$?

1. $H_0 : \sigma_x^2 = 20$, $H_1 : \sigma_x^2 \neq 20$, $\alpha = 0.05$
2. $\chi_0^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 / \sigma_0^2 = 135/20 = 6.75 \equiv$ test statistic
3. If $x_i = N(\mu, \sigma_x^2)$, $i = 1, \cdots, 16$, the test statistic is a χ_0^2 (15)-variate
4. From Table 34.4 in Chapter 34 and 15 degrees of freedom we obtain $\chi^2(15;1-0.05/2)$ = $\chi^2(15;\, 0.975) = 6.26$ and $\chi^2(15;0.025) = 27.5$ (you should read on $F = 1 - 0.975$ and $F = 1 -$ 0.025). The test statistic value, $\chi_0^2 = 6.75$, lies between the two critical values and we accept H_0 to the unknown with possible small probability of having made a type II error.

32.2.5 One-Sided Alternative Hypothesis for Means

32.2.5.1 One-tailed t-test for the population mean of a normal distribution.

Steps

1. Hypothesis specification: $H_0 : \mu_x \geq \mu_0$, $H_1 : \mu_x < \mu_0$, α
2. Test statistic: n observations, $\bar{x} = \sum_i x_i / n, s_{\bar{x}} = \sqrt{s_x^2 / n}$, $t_0 =$ test statistic $= (\bar{x} - \mu_0)/s_{\bar{x}}$ = $(\bar{x} - \mu_0)/\sqrt{s_x^2 / n}$
3. Assumptions $x_i \equiv N(\mu_x, \sigma_x^2)$
4. Critical region: $-\infty$ to $-t_\alpha = -t(n-1;\alpha)$. If $t_0 < -t_\alpha$ we reject H_0 in favor of H_1. If $\bar{x} > \mu_0$, H_0 is accepted.

5. P-value: $P = P\{t(n-1) < t_0\} = P\{t(n-1) > -t_0\}$
6. Confidence interval: The $100(1-\alpha)\%$ confidence interval for μ_x is $\bar{x} \mp t_{\alpha/2}\sqrt{s_x^2/n}$

Example 1

Given $\bar{x} = 12.5$, $s_x^2 = 11/3$, $n-1 =$ degrees of freedom = 3, $x_i \equiv$ normally distributed.

Steps

1. $H_0: \mu_x \le 10$, $H_1: \mu_x > 10$, $\alpha = 0.05$
2. Test statistic: $\dfrac{\bar{x}-\mu_0}{s_{\bar{x}}} = \dfrac{12.5-10}{\sqrt{(11/3)/4}} = 2.61$
3. Assuming H_0 is true and $x_i \equiv N(10, \sigma^2)$, the test statistic is a random $t(3)$-variate
4. Since H_1 is right-sided, the critical region comprises those $t(3)$-values exceeding $t_{0.05} = t(3; 0.05)$ = 2.353 (see Table 34.3, Chapter 34). Since $t_0 = 2.611 > 2.353$ we accept H_1
5. $P\{t(3) > 2.611\} = 0.05 - \dfrac{2.611-2.353}{3.182-2.353}(0.05-0.025) \cong 0.04$ (by interpolation of the same table)
6. The 95% confidence interval $12.5 \mp 3.182\sqrt{11/12} = 9.4, 15.5$

32.2.6 One-Sided Tests for the Population Variance of a Normal Distribution

32.2.6.1 Steps

1. Hypothesis specifications: $H_0: \sigma_x^2 = \sigma_0^2$, $H_1: \sigma_x^2 < \sigma_0^2$, α

   $H_0: \sigma_x^2 = \sigma_0^2$, $H_1: \sigma_x^2 > \sigma_0^2$, α
2. Test statistic (both cases): $\chi_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2} = \dfrac{\sum_i (x_i - \bar{x})^2}{\sigma_0^2}$
3. If $x_i \equiv N(\mu_x, \sigma_x^2 = \sigma_0^2)$, the test statistic is a $\chi^2(n-1)$-variate
4. Critical region: (a) For the first set of 1) the critical region is from 0 to $\chi_{1-\alpha}^2 = \chi^2(n-1; 1-\alpha)$ (b) For the second set of 1) the critical region is from $\chi_\alpha^2 = \chi^2(n-1; \alpha)$ to ∞.
5. (a) For the first set of 1) P-value is $P\{\chi^2(n-1) < \chi_0^2\}$ see 2, (b) for the second set of 1) P-value is $P\{\chi^2(n-1) > \chi_0^2\}$

Example 1

Given n = 16, x_i normally distributed, $s_x^2 = 9$. Was $\sigma_x^2 > 5$?

1. Specifications: $H_0: \sigma_x^2 \le 5$, $H_1: \sigma_x^2 > 5$, $\alpha = 0.05$
2. Test statistic: $\chi_0^2 = (n-1)s_x^2/\sigma_0^2 = (15)(9)/5 = 27$

3,4. Given normality, since $\chi_0^2 = 27 > 24.50 = \chi^2(15; 0.05)$

(see Table 34.4, Chapter 34; note that the table gives 1-tail values), the test statistic lies in the critical region and, hence, we accept H_1.

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33

Matrices

33.1 Notation and Some General Properties

33.1.1 Notation

Capital letters will denote matrices, e.g., A, B, N; the lower case of the first few letters of the alphabet will denote constants, i.e., a, b, c, d; and the lower case of letters in the last part of the alphabet will denote vectors, e.g., x, y, z. Lo wer case Greek letters will denote vectors, except for λ which will denote eigenvalues.

33.1.2 Notation

$A = [a_{ij}]$ = matrix where i refers to ith row and j refers to jth column. A_{nxm} = matrix with n rows and m columns. $a_{ij} \equiv a(i, j)$ the element on the ith row and jth column.

33.1.3 Identity I

Matrix with ones along the diagonal from left to right and zeros for all the other elements.

Example

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{a } 2 \times 2 \text{ identity matrix}$$

33.1.4 Diagonal D

Matrix with elements along the diagonal and zeros for all the other elements.

Example

$$D = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \text{ a } 2 \times 2 \text{ diagonal matrix}$$

33.1.5 Transpose

$$A^T = [a_{ji}]$$

Example

If

$$A_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{31} \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \text{ a } 2 \times 3 \text{ matrix}$$

33.1.6 Properties of Transposition

1. $(aA)^T = (Aa)^T = A^T a = aA^T$
2. $(aA + bB)^T = aA^T + bB^T$
3. $(A^T)^T = A$
4. $A^T = B^T$ if $A = B$
5. $(AB)^T = B^T A^T$
6. $D = D^T$ if D is diagonal
7. If $A = A^T$, A is *symmetric*
8. If $A = -A^T$, A is *skew-symmetric*
9. $A^T A$ and AA^T are symmetric
10. If A is nonsingular (its determinant is not zero), then A^T and A^{-1} are nonsingular and $(A^T)^{-1} = (A^{-1})^T$. A^{-1} means the inverse of matrix A.

33.1.7 Vector-Vector Multiplication

$z = x^T y$.

Example

$$a = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \times 1 + 2 \times 4 = 11$$

33.1.8 Matrix-Vector Multiplication

$z =' Ax$; $z_i \sum_{j=1}^{n} a(i, j) x(j)$,

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

***Note*:**: A is 2 × 2, x is a 2 × 1 and the result is 2 × 1. In general, if A is $m \times n$ the vector must be $n \times 1$ and the result will be $m \times 1$ vector.

33.1.9 Matrix-Matrix Multiplication

$C = AB$ or $C_{m \times n} = A_{m \times n} B_{n \times q}$. $c(i, j) = \sum_{k=1}^{n} a(i, k) b(k, j)$.

Example

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 4 \times 4 + 1 \times 1 & 1 \times 2 + 4 \times 0 + 1 \times 2 & 1 \times 3 + 4 \times 0 + 1 \times 1 \\ 0 \times 1 + 2 \times 4 + 1 \times 1 & 0 \times 2 + 2 \times 0 + 1 \times 2 & 0 \times 3 + 2 \times 0 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 4 \\ 9 & 2 & 1 \end{bmatrix}$$

$C_{2 \times 3} = A_{2 \times 3} B_{2 \times 3}$

33.1.10 Hermitian

$A^H = (A^T)^* =$ complex conjugate of the transpose of A.

Example

If

$$A = \begin{bmatrix} 1+j2 & 0 \\ 3-j & j4 \end{bmatrix}$$

then

$$A^H = \begin{bmatrix} 1-j2 & 3+j \\ 0 & -j4 \end{bmatrix}$$

Properties

1. $(A+B)^H = A^H + B^H$
2. $(AB)^H = B^H A^H$

33.1.10.1 Inverse of a Hermitian

$(A^H)^{-1} = (A^{-1})^H$

33.1.11 Block Matrix

$$\mathbf{A} = [A(ij)] = \begin{bmatrix} A(1,1) & \cdots & A(1,q) \\ \vdots & \vdots & \vdots \\ A(p.1) & \cdots & A(p,q) \end{bmatrix}$$

Each Matrix A (i,j) has the same dimension $m \times n$. $\mathbf{A}$ has dimensions $p \times q$. The *scalar dimension* of $\mathbf{A}$ $pm \times qn$.

33.1.12 Reflection (or exchange) Matrix J

$$J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

reverses the rows or columns of a matrix.

Example

$$J_{2\times2}A_{2\times3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \end{bmatrix} = \begin{bmatrix} a(2,1) & a(2,2) & a(2,3) \\ a(1,1) & a(1,2) & a(2,3) \end{bmatrix} = \text{reversed the rows}$$

$$J_{3\times3}A_{2\times3} = \begin{bmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a(3,1) & a(1,2) & a(1,1) \\ a(2,3) & a(2,2) & a(2,1) \end{bmatrix} = \text{reversed the columns}$$

33.1.13 Persymmetric P

An $n \times n$ square matrix that is symmetric about its cross diagonal (from right to left). $a(i, j) = a(n - j + 1, n - i + 1)$

Example

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{13} \\ a_{31} & a_{32} & a_{22} & a_{12} \\ a_{41} & a_{31} & a_{21} & a_{11} \end{bmatrix}$$

33.1.13.1 Inverse of Persymmetric

If P is persymmetric, P^{-1} is also persymmetric.

33.1.14 Centrosymmetric

An $n \times n$ square matrix with the property $r(i, j) = r^*(n - i + 1, n - j + 1)$.

33.1.15 Doubly Symmetric

A square matrix that is Hermitian about the principal diagonal and persymmetric about the cross diagonal:

$$r(i, j) = r^*(j, i) = r(n - j + 1, n - i + 1) = r^*(n - i + 1, n - j + 1)$$

Example

$$R = \begin{bmatrix} r(1,1) & r^*(2,1) & r^*(3,1) & r^*(4,1) \\ r(2,1) & r(2,2) & r^*(3,2) & r^*(3,1) \\ r(3,1) & r(3,2) & r(2,2) & r^*(2,1) \\ r(4,1) & r(3,1) & r(2,1) & r(1,1) \end{bmatrix}$$

33.1.15.1 Symmetric

$A = A^T$; $A^{-1} = (A^{-1})^T$ (inverse is also symmetric).

33.1.16 Toeplitz

A matrix A with equal elements along any diagonal, $a(i, j) = a(i - j)$.

33.1.17 Square Toeplitz

A square Toeplitz is Hermitian and a special case of a persymmetric matrix: $a^*(k) = a(-k)$ and $A = J A^* J$. A Hermitian Toeplitz matrix is *centrosymmetric*.

33.1.18 Hankel

A matrix A with equal elements along any cross diagonal, $a(i, j) = a(i + j - n - 1)$. A square Hankel matrix A has the property: $A^H = A$.

33.1.19 Right-Circulant

The relationship of its elements of an $n \times n$ right-circulant is matrix given by

$$a(i,j) = \begin{cases} a(j-1) & \text{for } j-i \ge 0 \\ a(n-j+1) & \text{for } j-i < 0 \end{cases}$$

Example

$$A = \begin{bmatrix} a(0) & a(1) & a(2) & a(3) \\ a(3) & a(0) & a(1) & a(2) \\ a(2) & a(3) & a(0) & a(1) \\ a(1) & a(2) & a(3) & a(0) \end{bmatrix}$$

33.1.20 Left-Circulant (n × n matrix)

The relationship of its elements is given by

$$a(i,j) = \begin{cases} a(n+1-i-j) & \text{for } j+i \le n+1 \\ a(2n+1-i-j) & \text{for } j+i > n+1 \end{cases}$$

Example

$$A = \begin{bmatrix} a(3) & a(2) & a(1) & a(0) \\ a(2) & a(1) & a(0) & a(3) \\ a(1) & a(0) & a(3) & a(2) \\ a(0) & a(3) & a(2) & a(1) \end{bmatrix}$$

33.1.21 Vanderonde (m × n)

$a(i,j) = x_j^{i-1}$ for $1 \le i < m, 1 \le j \le n$

Example

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & \vdots & \\ x_1^{m-1} & x_2^{m-1} & \cdots & x_n^{m-1} \end{bmatrix}$$

33.1.22 Upper Triangular (n × m)

$a(i,j) = 0$ for $j < i$

Example

$$\begin{bmatrix} a(1,1) & a(1,2) \\ 0 & a(2,2) \end{bmatrix}$$

33.1.23 Lower Triangular

$a(i, j) = 0$ for $j > i$

Example

$$\begin{bmatrix} a(1,1) & 0 \\ a(2,1) & a(2,2) \end{bmatrix}$$

33.1.24 Circulant Permutation (n × n)

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

If A is circulant then $A = \sum_{k=0}^{n-1} a_{k+1} C^k$ ($C^0 = I = C^n$ and a's are the entries of the first row of A).

33.1.25 Upper Hessenberg

$a(i, j) = 0$ for $i > j + 1$

33.1.26 Lower Hessenberg:

A is lower Hessenberg if A^T is upper Hessenberg.

33.2 Determinants (of square matrices)

33.2.1 Definition

$\det(A) \equiv |A| = \sum_{j=1}^{n} a_{ij} |A_{ij}|$ where A_{ij} is the cofactor of A for any i .

33.2.2 Cofactor

$A_{ij} = (-1)^{i+j} m_{ij}$ where m_{ij} is called the minor of a_{ij} .

Example

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}|A_{11}| + a_{12}|A_{12}| + a_{13}|A_{13}|$$

$$= a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{23} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

33.2.3 Product of Matricies

$\det(AB) = \det(A)\det(B)$

33.2.4 Transpose

$\det(A) = \det(A^T)$

33.2.5 Change of Sign

If two rows (columns) of a matrix are interchanged, the determinant of the matrix changes sign.

33.2.6 If two rows of a matrix A are identical $\det(A) = 0$

33.2.7 The determinant is not changed if the elements of the i[th] row are multiplied by a scalar k and the results are added to the corresponding elements of the h[th] row, $h \neq i$.

33.2.8 If $\det(A) = 0$, A is a singular matrix.

33.3 Rank of Matrices

33.3.1 Elementary Matrix

E_{rs} is an elementary matrix produced by exchanging two rows (or two columns) of an identity matrix I (ones along the diagonal and zeros everywhere else).

33.3.2 Row Exchange of A

$E_{rs}^{©}$ A exchanges two rows of A if E_{rs} has been constructed from an identity matrix by exchanging rows r and s.

33.3.3 Column Exchange of A

A E_{rs} exchanges two columns of A if E_{rs} has been constructed from an identity matrix by exchanging columns r and s.

33.3.4 Rank

rank $r \equiv r_A \equiv r(A)$ is the number of linearly independent rows and columns in the matrix.

1. r is a positive integer
2. r is equal to or less than the smaller of its number of rows or columns
3. When multiplying by elementary matrices, the rank does not change.
4. When $r \neq o$ of A, there exists at least one nonsingular square submatrix of A.

33.3.5 Calculating Rank

Elementary row operation until the matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & -1 & 2 & -2 \\ 5 & -4 & 0 & -7 \end{bmatrix}, \text{ row } 2 - 3\times \text{ (row 1) and row } 3 - 5\times \text{ (row 1)}$$

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -7 & -10 & -11 \\ 0 & -14 & -20 & -22 \end{bmatrix}$$, row 3–2× (row 2), $$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -7 & -10 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ hence the rank r =2.

33.4 Additional Rank Properties

33.4.1 If $r(A_{n\times n}) = n$, A is nonsingular and full rank.

33.4.2 If $r(A_{n\times n}) < n$, then A is singular.

33.4.3 If $r(A_{p\times q}) = p < q$, A has full row rank.

33.4.4 If $r(A_{p\times q}) = q < p$, A has full column rank.

33.4.5 AC, CB , and ACB have the same rank if A and B are nonsingular and C is any matrix.

33.4.6 $r(AB)$ cannot exceed the rank of either A or B.

33.5 Vectors

33.5.1 Vector Space

Let V_n be a set of n-component vectors such that for every two vectors in V_n the sum of the two vectors is also in V_n and for each vector in V_n and each scalar, the product is in V_n. This set V_n is called a vector space.

33.5.2 Subspace

Let S_n be a subset of vectors in V_n . If S_n is a vector space, then it is called a (vector) subspace.

Example

The set of vectors $x^T = [a_1\ a_2\ 0]$ for all real numbers a_1 and a_2 is a subspace of the three-dimensional space R_3.

33.5.3 Linear Dependent and Independent

The n-dimensional of m vectors $\{x_1, x_2, \cdots, x_m\}$ are linear dependent if we find even one c_1 different from zero such that $\sum_{i=0}^{m} c_i x_i = 0$. If this relation is satisfied only when all c_i 's are zeros, the vectors are independent.

33.5.4 If the rank r of the matrix $V = [v_1, v_2, \cdots, v_m]$, where v_1 's are $n \times 1$ vectors, is less than m, then the vectors are dependent.

33.5.5 Basis

If $\{v_1, v_2, \cdots, v_m\}$ is a basis of a space, then any vector v has a unique linear combination of the given basis:

$$v = \sum_{i=1}^{m} c_i v_i$$

33.5.6 Orthogonal

If $x^T y = 0$, then the vectors x and y are orthogonal, $x^T y = \sum_{i=1}^{n} x_i y_i$

33.5.7 Orthogonal Basis

If $\{v_1, v_2, \cdots, v_m\}$ is a basis and $v_i^T v_j = 0$ for all $i \neq j = 1, 2, \cdots, m,$ the basis is orthogonal.

33.5.8 Orthonormalization

If $\{v_1, v_2, \cdots, v_m\}$ is a basis, then $\{z_1, z_2, \cdots, z_m\}$ is an orthonormal basis

$$y_1 = v_1 \qquad z_1 = \frac{y_1}{\sqrt{y_1^T y_1}}$$

$$y_2 = v_2 - \frac{y_1^T v_2}{y_1^T y_1} y_1 \qquad z_2 = \frac{y_2}{\sqrt{y_2^T y_2}}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$y_m = v_m - \frac{y_1^T v_m}{y_1^T y_1} y_1 - \frac{y_2^T v_m}{y_2^T y_2} y_2 - \cdots - \frac{y_{m-1}^T v_m}{y_{m-1}^T y_{m-1}} y_{m-1} \qquad z_m = \frac{y_m}{\sqrt{y_m^T y_m}}$$

33.6 Quadratic Form

33.6.1 Definition

$$f(x) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j = x^T A x \quad (A = n \times n \text{ matrix})$$

33.6.2 Congruent Matrices

$$x^T A x = y^T C^T A C y = y^T B y \quad (B = C^T A C)$$

A and B are congruent. C= $n \times n$ nonsingular matrix and A and B have the same rank.

33.6.3 Positive Definite

When $A_{n \times n}$ in 33.4.1 has r = n

33.6.4 Positive Semidefinite

When $A_{n \times n}$ in 33.4.1 has r < n

33.7 Orthogonal Matrices

33.7.1 Definition

$P_{n \times n}$ matrix is orthogonal if and only if $P^{-1} = P^T$

33.7.2 Partitioned

$P = [p_1, p_2, \cdots, p_n]$, where p_i ‘s are $n \times 1$ matrices (vectors) of the column of P, then $p_i^T p_i = 1$ for $i = 1, 2, \cdots, n$, and $p_i^T p_j = 0$ for $i \neq j$

33.7.3 $P^T P = I$ if P is orthogonal.

33.7.4 det (P) is +1 or -1

33.7.5 If A is any $n \times n$ mstrix and P an $n \times n$ orthogonal, then $\det(A) = \det(P^T AP)$.

33.7.6 $P^T AP = D$ where A is an $n \times n$ matrix, P is an $n \times n$ orthogonal matrix, and D is a diagonal matrix.

33.8 Inverse Matrices

33.8.1 Uniqueness

If A^{-1} (inverse of A) is such that $A^{-1}A = AA^{-1} = I$ then A^{-1} is unique for a given A.

33.8.2 Adjoint or (adjugate)

If the elements of a matrix are replaced by their cofactors and then transposed, it is called adjoint.

33.8.3 Inverse $(n \times n)$

$$A^{-1} = \frac{1}{|A|} adj\ A, |A| \neq 0$$

Example

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}, \quad adj\ A = \begin{bmatrix} 6 & -4 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ -4 & 1 \end{bmatrix}, |A| = (6-12) = -6,$$

$$A^{-1} = \frac{1}{-6}\begin{bmatrix} 6 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & +\frac{1}{2} \\ +\frac{2}{3} & -\frac{1}{6} \end{bmatrix}, A^{-1}A = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{6} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

33.8.4 Properties

1. $A^{-1}A = AA^{-1} = I$

2. A is unique

2. $|A^{-1}| = 1/|A|$

4. $(A^{-1})^{-1} = A$

5. $(A^T)^{-1} = (A^{-1})^T$

6. $(A^{-1})^T = A^{-T}$ if $A^T = A$

7. $(AB)^{-1} = B^{-1}A^{-1}$

8. $(D[x_i])^{-1} = D[1/x_i]$ for $x_i \neq 0$

9. $(I + AB)^{-1} = I - A(I + BA)^{-1}B$

10. $(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$,
$A = n \times n, B = n \times m, C = m \times m, D = m \times n$

11. $(A + xy^H)^{-1} = A^{-1} - \dfrac{(A^{-1}x)(y^H A^{-1})}{1 + y^H A^{-1}x}$, x and y vectors

12. a) If $Y = \begin{bmatrix} A & D \\ C & B \end{bmatrix}$ then

$$Y^{-1} = \begin{bmatrix} A^{-1} + A^{-1}D\Delta^{-1}CA^{-1} & -A^{-1}D\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix}, \Delta = B - CA^{-1}D$$

$$Y^{-1} = \begin{bmatrix} \Lambda^{-1} & -\Lambda^{-1}DB^{-1} \\ -B^{-1}C\Lambda^{-1} & B^{-1} + B^{-1}C\Lambda^{-1}DB^{-1} \end{bmatrix}, \Lambda = A - DB^{-1}C$$

b) If $D = x, C = y^H$, and $B = a$ then

$$Y^{-1} = \begin{bmatrix} A^{-1} + bA^{-1}xy^H A^{-1} & -\beta A^{-1}x \\ -\beta y^H A^{-1} & \beta \end{bmatrix}, \beta = (a - y^H A^{-1}x)^{-1}$$

13. $B = aA + b(I - A),\ B^{-1} = \frac{1}{a}A + \frac{1}{b}(I - A), \quad A_{k\times k} =$ independent, a, b = constants

14. $A = B^{-1} + CD^{-1}C^H$, $\quad A^{-1} = B - BC(D + C^H BC)^{-1} C^H B$, H stands for Hermitian (conjuga transpose)

33.9 Linear Transformations, Characteristic Roots (eigenvalues)

33.9.1 Definitions

$y = Ax$

If $y_1 = Ax_1$ and $y_2 = Ax_2$ then $y = y_1 + y_2 = A(x_1 + x_2)$
If $y = Ax$ and $z = By$, then $z = BAx$

33.9.2 Characteristic Roots (eigenvalues)

$Ax = \lambda x$ then there exists n complex roots (or real), given by $|A - \lambda I| = 0$ where $|\cdot|$ indicates determinant of the matrix.

33.9.3 If *Ax*=lx, *x* is the eigenvector (characteristic vector).

33.9.4 Properties

1. If A is singular, at least one root is zero.
2. If $C_{n\times n}$ is nonsingular, then $A_{n\times n}, C^{-1}AC, CAC^{-1}$ have the same eigenvalues.
3. $A_{n\times n}$ has the same eigenvalues of A^T but not necessarily the same eigenvectors.
4. If λ_i is the eigenvalue of $A_{n\times n}$ and x_i its eigenvector, then λ_i^k is the eigenvalue of A^k and x_i is the eigenvector of A^k : $A^k x_i = \lambda_i^k x_i$
5. If λ_i is the eigenvalue of $A_{n\times n}$ then $1/\lambda_i$ is an eigenvalue of A^{-1} (if it exists).
6. If λ_i is the eigenvalue of $A_{n\times n}$ then x_i is real.

Example

$$A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}, \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 36 = 0, \lambda_1 = -5, \lambda_2 = 7, Ax_1 = \lambda_1 x_1,$$

$$A\begin{bmatrix} 2 \\ -3 \end{bmatrix} = -5\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad Ax_2 = \lambda_2 x_2, A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad x_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

7. If A has eigenvalue λ then $f(A)$ has eigenvalue $f(\lambda)$.

Example

$f(A) = A^3 + 7A^2 + A + 5I$, then $(A^3 + 7A^2 + A + 5I)x = A^3x + 7A^2x + 5Ix = \lambda^3 x + 7\lambda^2 x + \lambda x + 5x$ $= (\lambda^3 + 7\lambda^2 + \lambda + 5)x$. Hence, $f(\lambda) = \lambda^3 + 7\lambda^2 + \lambda + 5$ is an eigenvalue

Example

$$f(A) = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \text{ has eigenvalue } e^{\lambda}.$$

8. $\sum_{i=1}^{n} \lambda_i = tr\,(A)$. (tr stands for trace.)

9. $\prod_{i=1}^{n} \lambda_i = |A|$.

10. The nonzero eigenvectors $x_1, x_2, \cdots, x_n$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are linearly independent.
11. The eigenvalues of a Hermitian matrix are real.
12. A Hermitian matrix is positive definite if and only if the eigenvalues of A are positive $\lambda_k > 0$.
13. The eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal, i.e., $\lambda_i \neq \lambda_j$, then $v_i^T v_j = 0$.
14. Eigenvalue decomposition: $A = V\Lambda V^{-1}$, $A = n \times n$ has n linear independent vectors, V = contain the eigenvectors of A, Λ = diagonal matrix containing the eigenvalues.
15. Any Hermitian matrix A may be decomposed as $A = V\Lambda V^H = \lambda_1 v_1 v_1^H + \lambda_2 v_2 v_2^H + \cdots + \lambda_n v_n v_n^H$ where λ_i are the eigenvalues of A and v_i are a set of orthogonal eigenvectors, Λ = diagonal matrix containing the eigenvalues.
16. $B = n \times n$ matrix with eigenvalues λ_i, $A = B + aI$, A and B have the same eigenvectors, and the eigenvalues of A are $\lambda_i + a$.
17. $A = B + aI,\ B = n \times n$ with rank one (u_1 eigenvector of B and λ its eigenvalue),

$$A^{-1} = \frac{1}{a+\lambda} u_1 u_1^H + \frac{1}{a} I - \frac{1}{a} u_1 u_1^H = \frac{1}{a} I - \frac{\lambda}{a(a+\lambda)} u_1 u_1^H$$

18. A = symmetric positive definite matrix, $x^T A x = 1$ differs an ellipse in n dimensions whose axes are in the direction of the eigenvectors vi of A with the half-length of these axes equal to $1/\sqrt{\lambda_i}$
19. The largest eigenvalue of an $n \times n$ matrix $A = \{a_{ij}\}$ is bounded by $\lambda_{\max} \leq \max_i \sum_{j=1}^{n} a_{ij}$ (bounded by the maximum row sum of the matrix).
20. $A = n \times n$ Hermitian with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, y = arbitrary complex vector, a = arbitrary complex number, if $\tilde{A} = (n+1) \times (n+1)$ Hermitian = $\begin{bmatrix} A & y \\ y^H & a \end{bmatrix}$ then $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_{n+1}$ are interlaced with those of A: $\tilde{\lambda}_1 \leq \lambda_1 \leq \tilde{\lambda}_2 \leq \lambda_2 \leq \cdots \lambda_n \leq \tilde{\lambda}_{n+1}$
21. Eigenvalues are bounded by the maximum and minimum values of the associated power spectral density of the data: $S_{\min} \leq \lambda_i \leq S_{\max}$, $i = 1, \cdots, m$.
22. Eigenvalue spread: $x(R) = \dfrac{\lambda_{\max}}{\lambda_{\min}}$, $R \equiv$ correlation matrix of a discrete-time stochastic process.

23. Karhunen-Loeve expansion: $x(n) = \sum_{i=1}^{M} c_i(n) v_i$, $x(n) = [x(n)\ x(n-1) \cdots x(n-M)]^T$, v_i = eigenvector of the matrix R corresponding to eigenvalue λ_i, R = correlation matrix of the wide-sense stationary process *x(n)*, $c_i(n)$ = constants = coefficients of expansion, $c_i(n) = v_i^H x(n)$, $i = 1, 2, \cdots, M$, $E\{c_i(n)\} = 0$, $E\{c_i(n) c_j^*(n)\} = \lambda_i$ if $i = j$ and zero if $i \neq j$, $E\{|c_i(n)|^2\} = \lambda_i$ $\sum_{i=1}^{M} |c_i(n)|^2 = \|x(n)\|^2$ where $\|\cdot\|$ is the norm $(\|x(n)\|^2 = x^H X)$ and H stands for Hermitian (conjugate transpose).

33.9.5 Finding Eigenvectors

$A - \lambda_k I = \begin{bmatrix} R_k & C_k \\ D & E \end{bmatrix}$, $r(R_k) = r(A - \lambda_k I)$, $x_k = \begin{bmatrix} -R_k C_k y \\ y \end{bmatrix}$, y arbitrary of order $n - r(A - \lambda_k I)$, R_k same rank as $A - \lambda_k I$.

Example

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}, r(A - \lambda_1 I) = 2, \lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -4, n - r(A - \lambda_1 I) = 3 - 2 = 1,$$

$$y = a,\ x_1 = \begin{bmatrix} \frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ \frac{1}{4}a \\ a \end{bmatrix} \text{ and for } a = 4,\ x_1 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

In similar procedures, we find

$x_2 = [-a \quad -\frac{1}{2}a \quad a]^T$ and for $a = 2$ $x_2 = [-2 \quad -1 \quad 2]^T$. $x_3 = [a/13 - 3a/13\ a]^T$ and for $a = 13$ $x_3 = [-1 - 3\ \ 13]^T$.

Example

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \lambda_1 = 1 \text{ with multiplicity } 2(m_1 = 2), \lambda_2 = -1 \text{ with } m_2 = 1.$$

For $\lambda_1 = 1, (A - \lambda_1 I) = \begin{bmatrix} -2 & -2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, r(A - \lambda_1 I) = 1, n - r(A - \lambda_1 I) = 3 - 1 = 2$,

$$x_1 = \begin{bmatrix} -(-\frac{1}{2})(-2 - 2)y \\ y \end{bmatrix} = \begin{bmatrix} -a - b \\ a \\ b \end{bmatrix}.$$

Similarly for $\lambda_2 = -1,\ x_2 = [2a - a\ a]^T$. For $a = -b,\ x_1 = [0\ 1\ -1]^T$ and for $a = b = 1\ x_1^* = [-2\ 1\ 1]^T$. For $a = 1,\ x_2 = [2\ -1\ 1]^T$ and all three eigenvectors are independent.

33.9.6 Similar Matrices

A and B are similar if there exists Q such that $B = Q^{-1}AQ$.

33.9.7 Properties of Similar Matrices

1. $|B| = |A|$
2. If the eigenvalues of λ_i of A are distinct, then there exists Q such that $B = Q^{-1}AQ$.
3. If X is a matrix of eigenvectors x_i corresponding to eigenvalues λ_i' then $X^{-1}AX=$ diag $\{\lambda_1, \lambda_2, \cdots \lambda_n\} \equiv D$.
4. If $Q^{-1}AQ = T_u$ (T_u = upper triangular) the diagonal elements of T_u are the eigenvalues of A.
5. Similar matrices have the same set of eigenvalues.
6. Power: $A = XDX^{-1}$ (D=diagonal) implies $A^2 = XDX^{-1}XDX^{-1} = XD^2X^{-1}$ and hence $A^k = XD^kX$.
7. Inverse: $A^{-1} = XD^{-1}X^{-1}$
8. Difference Equations: If A is diagonizable then $x(n) = Ax(n-1) = y$ or

$$x(n) = A(Ax(n-2) + y = A^2x(n-2) + (A + I)y = \cdots$$

$$= A^n x(0) + (A^{n-1} + A^{n-2} + \cdots A = I)y$$

If $A^k \to 0$ as $k \to \infty$ and $(I - A)$ exists then

$$x(n) = A^n x(0) + (I - A^n)(I - A)y$$

33.10 Symmetric Matrices

33.10.1 Eigenvalues (characteristic roots) of a real symmetric matrix are real.

33.10.2 Real symmetric matrices are diagonizable with eigenvalues along the diagonal $P^TAP = D$, P = orthogonal matrix.

33.10.3 Eigenvectors are orthogonal.

33.10.4 If λ_k is an eigenvalue of $A_{n\times n}$ of multiplicity m_k, then $A - \lambda_k I$ has rank $n - m$, and is singular.

33.10.5 If there are m zero eigenvalues of an $n \times n$ real symmetric matrix, then its rank is n - m.

Example

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, (\lambda + 1)^2(\lambda - 5) = 0,\ \lambda_1 = 5 \text{ with } m_1 = 1,\ \lambda_2 = -1 \text{ with } m_2 = 2.$$

From (33.9.5)

$$A - \lambda_1 I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix},\ x_1 = \begin{bmatrix} \frac{1}{12}\begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix};$$

for λ_2

$$A-\lambda_2 I=\begin{bmatrix}2 & 2 & 2\\ 2 & 2 & 2\\ 2 & 2 & 2\end{bmatrix}, x_2=\begin{bmatrix}-\frac{1}{2}[2\ 2]y\\ y=\begin{bmatrix}a_1\\ a_2\end{bmatrix}\end{bmatrix}=\begin{bmatrix}-(a_1+a_2)\\ a_1\\ a_2\end{bmatrix}$$

if we set $a_1=a_2=1$, then $x_2^{(1)}=[-2\ 1\ 1]^T$. But $x_2^T x_2^{(1)}=0$ or $2(a_1+a_2)+a_1+a_2=0$ or $a_1=-1, a_2$ =1 which gives the second vector $x_2^{(2)}=[0\ -1\ 1]$, with $a=1, x_1=[1\ 1\ 1]^T$ and the normalized eigenvector matrix is

$$Q=\frac{1}{6}\begin{bmatrix}\sqrt{2} & -2 & 0\\ \sqrt{2} & 1 & -\sqrt{3}\\ \sqrt{2} & 1 & -\sqrt{3}\end{bmatrix}$$

and gives $Q^T AQ=D=diag\{-5,-1,-1\}$ and $QQ^T=I$.

33.11 Geometric Interpretation

33.11.1 $x=[a_1\ a_2\ \cdots a_n]^T=\sum_{i=0}^{n} a_i\varepsilon_i$ where $\varepsilon_i=[0\ 0\cdots 1\cdots 0]^T$ (one at the i^{th} place).

33.11.2 Distance in Euclidean E_n space

$$d=\left[\sum_{i=1}^{n}(a_i-b_i)^2\right]^{1/2}=[(x-y)^T(x-y)]^{1/2} \text{ where } x=[a_1\ a_2\ \cdots a_n]^T \text{ and } y=[b_1\ b_2\ \cdots b_n]$$

33.11.3 Projection

$p_k^n \equiv$ plane through the origin with $B=[y_1\ y_2\cdots y_k]$ on $n\times k$ matrix of rank k. Then the projection of x on P_k^n is $z=B(B^TB)^{-1}B^Tx$.

Example

$$B=\begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix} \text{ of } y_1=[1\ 1\ 0]^T$$

and $y_2=[1\ 0\ 1]$ in P_2^3 plane through the origin. Hence,

$$z=\begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}^{-1}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}4/3\\ 2/3\\ 2/3\end{bmatrix}$$

33.11.4 Distance from Projection

$$d=[(x-B(B^TB)^{-1}B^Tx)^T(x-B(B^TB)^{-1}B^Tx)]^{1/2}$$

33.11.5 Quadratic Form Matrix

$G = I - B(B^T B)^{-1} B$ has the properties $G = G^T$ and $G = G^2$ (G is a symmetric idepotent matrix).

33.12 Relationships of Vector Spaces

33.12.1 Sum of Vector Subspaces

$S = S_1 \oplus S_2 = \{y = x_1 + x_2 ; x_1 \in S_1 \text{ and } x_2 \in S_2\}$

33.12.2 Sum and Intersection

If S_1 and S_2 belong to subspace V_n of E_n then $S_1 \otimes S_2$ and $S_1 \cap S_2$ are subspaces of V_n

33.12.3 Orthogonal Subspaces

$S_1 \perp S_2$ if for each f_i vector in S_1 and for each vector y_i in S_2 $x_i^T y = 0$.

33.12.4 Orthogonal Complement

Let $S_1 \in E_n$ then $S_1^\perp \in E_n$ is the orthogonal complement of S_1 if $S_1 \perp S_1^\perp$ and $S_1 \otimes S_2 = E_n$. For each subspace S_1 in E_n the orthogonal complement $S_1^\perp$ always exists and it is unique.

Example

In E_4 let the subspace S be spanned by the linearly independent vectors $x_1 = [1\ 0\ -1\ 1]^T$ and $x_2 = [1\ 1\ 0\ 1]^T$. From (33.5.8) we obtain the orthogonal basis for $S: y_1 = x_1 = [1\ 0\ -1\ 1]^T$ and $y_2 = x_1 = \frac{x_1^T x_2}{x_1^T x_1} y_1 = \frac{1}{3}[1\ 3\ 2\ 1]^T$ with $y_1^T y_2 = 0$ (orthogonal). If we find z_1 and z_2 such that $\{y_1, y_2, z_1, z_2\}$ is an orthogonal basis for E_4 then $\{z_1, z_2\}$ is the basis for $S^\perp$. By inspection $z_1 = [1\ 0\ 0\ -1]^T$ and $z_2 = [1\ -2\ 2\ 1]^T$.

33.12.5 Column Space

If we write $A_{n \times m}$ matrix in the form $A = [x_1 x_2 \ldots x_m]$ and consider the columns x_i 's as vectors in E_n then the space spanned by the x_i 's is called the column space of A. ***Note:*** If A is $n \times n$ and nonsingular, the column space of A is E_n.

33.13 Functions of Matrices

33.13.1 Polynomial of Matrices

$p_q(A) = a_q A^q + \cdots + a_1 A + a_0 I \quad (A = n \times n \text{ matrix})$

33.13.2 Characteristic Polynomial

If $p_n(x) = \det(A - \lambda I) = \sum_{i=0}^{n} c_i \lambda^i$ then $p_n(A) = 0$.

33.13.3 Norm

$\|A\| \equiv$ norm is a real-valued (non-negative) function of A (if the elements a_{ij} of A) that satisfies the following:

1. $\|A\| \ge 0$ and $\|A\| = 0$ if and only if A = 0;
2. $\|cA\| = |c|\|A\|, c =$ any scalar;
3. $\|A + B\| \le \|A\| + \|B\|$;
4. $\|AB\| \le \|A\|\,\|B\|$

Example

Let $f(A) = \Sigma\Sigma|a_{ij}|$

1. $f(A) \ge 0$ and $f(A) = 0$ if $A = 0$
2. $f(cA) = \Sigma\Sigma|ca_{ij}| = |c|\Sigma\Sigma a_{ij} = |c|f(A)$
3. $f(A+B) = \Sigma\Sigma\left(|a_{ij} + b_{ij}|\right) \le \Sigma\Sigma\left(|a_{ij}| + |b_{ij}|\right) = f(A) + f(B)$
4. $f(AB) = f(C) = \sum_i \sum_j |c_{ij}| = \sum_i \sum_j \left|\sum_t a_{it} b_{tj}\right| \le \sum_i \sum_j \sum_t |a_{it}||b_{tj}| \le \sum_i \sum_j \left(\sum_t |a_{it}| \sum_s |b_{sj}| \right) = f(A) f(B)$

Hence, $f(A) = \Sigma\Sigma|a_{ij}|$ is a norm.

33.13.4 SpecialNorms

1. $\|A\|_1 = \max_j \left(\sum_{i=1}^{n} |a_{ij}| \right) =$ maximum of sums of absolute values of column elements.
2. $\|A\|_2 =$ [maximum eigenvalue root of $A^T A]^{1/2}$ called the *spectrum* norm.
3. $\|A\|_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right) =$ maximum of sums of absolute values of row elements.
4. $\|A\|_E = [\sum_i \sum_j \left(|a_{ij}|\right)^2]^{1/2} =$ Euclidean norm.

33.13.5 Properties of Norms (all matrices are n × n; refer also to 33.13.4)

1. $\|A^q\| \le (\|A\|)^q$ for norms in (33.13.4)
2. $0 \le |(\|A\| - \|B\|)| \le \|A - B\|$ for norms in(33.13.4)
3. $\|A\|_1 = \|A^T\|_\infty$ for norms in (33.13.4)
4. $\|AB\|_E \le \|A\|_E \|B\|_2$
5. $\|A\|_E^2 =$ trace $(A^T A) = \|A^T\|_E^2$
6. $\|A\|_E = \|A^T\|_E$
7. $\|A\|_2 \le \|A\|_E \le \sqrt{n}\|A\|_2$

8. $\|D\|_m = \max|d_{ii}|$ for $m = 1, 2, \infty$ and $D = [d_{ij}]$

9. $\|PA\|_m = \|AP\|_m = \|P^T AP\|_m$, $m = E$ or 2 and P is orthogonal.

10. $\|A\|_2^2 = \max_{x \in S}[x^T A^T Ax / x^T x]$, $S \equiv$ set of all $n \times 1$ real vectors except the 0 vector.

11. $\|A\|_2 =$ max root of A,A is non-negative

33.14 Generalized Inverse

33.14.1 Generalized Inverse (g-inverse) of $A_{m \times n}$

A^- is generalized inverse of A if:

1. AA^- is symmetric,
2. $A^- A$ is symmetric,
3. $AA^- A = A$
4. $A^- AA^- = A^-$

33.14.2 Generalized Inverse

If A=0 hen we may write $A_{m \times n} = B_{m \times r} C_{r \times n}$ (A has rank > 0, B and C have rank r), and

$$A^- = C^T (CC^T)^{-1} (B^T B)^{-1} B^T$$

33.14.3 Properties of g-inverse

1. A^- is unique
2. $(A^T)^- = (A^-)^T$
3. $(A^-)^- = A$
4. $r(A) = r(A^-)$
5. If rank A is r, then $A^-, AA^-, A^- A, AA^- A, A^- AA^-$ have rank r
6. $(A^T A)^- = A^- A^{T-}$
7. $(AA^-)^- = AA^-$, $(A^- A)^- = A^- A$
8. $(PAQ)^- = Q^T A^- P^T$, $P_{m \times m} =$ orthogonal, $Q_{n \times n}$ = orthogonal, $A_{m \times n}$
9. If $A = A^T \equiv$ symmetric then $A^- = (A^- 0 =$ symmetric
10. If $A = A^T$ then $AA^- = A^- A$
11. If $A =$ nonsingular then $A^- = A^{-1}$
12. If $A = A^T$ and $A = A^2$ then $A^- = A$
13. If $D_{n \times n}[d_{ii}]$ is diagonal, then $D^- = [d_{ii}^{-1}]$ if $d_{ii} \neq 0$ and zero if $d_{ii} = 0$
14. If $A_{m \times n}$ has rank *m*, then $A^- = A^T (AA^T)^{-1}$ and $AA^- = I$
15. If $A_{m \times n}$ has rank *n*, then $A^- = (A^T A)^{-1} A^T$ and $A^- A = I$

16. If $A^- = A^T(AA^T)^- = (A^TA)^- A^T$ whatever the rank of A

17. AA^-, A^-A, $I - A^-A$ and $I - A^-A$ are all symmetric idempotent

18. $(BC)^- = C^-B^-$ if $B_{m\times r}$ has rank >0 and $C_{r\times m}$ has rank r

19. $(cA)^- = \frac{1}{c}A^-$, $c =$ nonzero scalar

20. If $A = A_1 + A_2 + \cdots + A_n$ and $A_iA_j^T = 0$ and $A_i^TA_j = 0$ for all $i, j = 1,\cdots,n$

$i \neq j$ then $A^- = A_1^- + \cdots + A_n^-$

21. If $A_{n\times m}$ is any matrix, $K_{m\times m}$ is nonsingular matrix and $B = AK$, then $BB^- = AA^-$

22. If $A^TA = AA^T$ then $A^-A = AA^-$ and $(A^n)^- = (A^-)^n$

Note: In general $(A^n)^- \neq (A^-)^n$ for $m \times m$ matrix.

23. If $A = \begin{bmatrix} B \\ C \end{bmatrix}$ and $BC^T = 0$ then $A^- = [B^-, C^-]$, $A^-A = B^-B + C^-C$,

$$AA^- = \begin{bmatrix} BB^- & 0 \\ 0 & CC^- \end{bmatrix}$$

24. If $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ then $A^- = \begin{bmatrix} B^- & 0 \\ 0 & C^- \end{bmatrix}$, $A^-A = \begin{bmatrix} B^-B & 0 \\ 0 & C^-C \end{bmatrix}$,

$$AA^- = \begin{bmatrix} BB^- & 0 \\ 0 & CC^- \end{bmatrix}$$

25. $x^- = (x^Tx)^{-1}x^T$, $x =$ vector

26. $A^- = A^T$ if and only if A^TA is independent

27. The column spaces of A and AA^- are the same.

The column spaces of A^- and A^-A are the same.

The column space of $I - AA^-$ is the orthogonal complement of the column space of A.

The column space of $I - A^-A$ is the orthogonal complement of the column space of A^T

33.15 Generalized Matrices Computation

33.15.1 Computation (theorem)

Let X be an $m \times t$ matrix and x_t be the t^{th} column vector of X. Then we write $X = [X_{t-1}, x_t]$, and the g-inverse of X is equal to Y where

$$Y = \begin{bmatrix} X_{t-1}^- - X_{t-1}^- x_t y_t^- \\ y_t^- \end{bmatrix}, \quad y_t^- \equiv \text{ g-inverse of } y_t$$

$$y_t = \begin{cases} (1 - X_{t-1}X_{t-1}^- & \text{if } x_t \neq X_{t-1}X_{t-1}^- x_t \quad \text{(case I)} \\ \dfrac{[1 + x_t^T(X_{t-1}X_{t-1}^T)^- x_t](X_{t-1}X_{t-1}^T)^- x_t}{x_t^T(X_{t-1}X_{t-1}^T)^-(X_{t-1}X_{t-1}^T)^- x_t}, & \text{if } x_t = X_{t-1}X_{t-1}^- x_t \quad \text{(case II)} \end{cases}$$

33.15.2 Algorithm for 33.15.1

1. Set $Z_2 = [X_1\ x_2] \equiv [x_1\ x_2]$,
2. Compute $X_1^- \equiv x_1^-$ (see Section 33.14.3.25)
3. Compute $X_1^- x_2$
4. Compute $X_1 X_1^- x_2$ (if $x_2 \neq X_1 X_1^- x_2$ use case I, otherwise use case II)
5. Compute y_2^-
6. Compute $Z_2^- = X_2^- = \begin{bmatrix} X_1^- - X_1^- x_2 y_2^- \\ y_2^- \end{bmatrix}$
7. $Z_3^- = X_3^- = \begin{bmatrix} Z_2^- - Z_2^- x_3 y_2^- \\ y_3^- \end{bmatrix}$, $Z_3 = [Z_3 x_3]$ etc.

Example

$$X = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix},$$

1. $Z_2 = [X_1, x_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -0 \end{bmatrix},$
2. $X_1^- \equiv x_1^- = \frac{1}{6}[1 \quad 0 \quad -1 \quad 2]$
3. $X_1^- x_2 = -\frac{3}{6}$
4. $X_1 X_1^- x_2 = \begin{bmatrix} -\frac{3}{6} & 0 & \frac{3}{6} & -\frac{6}{6} \end{bmatrix}^T$
5. since $x_2 \neq X_1 X_1^- x_2$ case I applies and $y_2 = x_2 - X_1 X_1^- x_2 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 \end{bmatrix}^T$ and from

   (14.3.25) $y_2^- = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$
6. $Z_2^- = X_2^- = \begin{bmatrix} X_1^- - X_1^- x_2 y_2^- \\ y_2^- \end{bmatrix} = \begin{bmatrix} \frac{2}{6} & \frac{2}{6} & 0 & \frac{2}{6} \\ \frac{2}{6} & \frac{4}{6} & \frac{2}{6} & 0 \end{bmatrix}$
7. $Z_3^- = X_3^- = \begin{bmatrix} Z_2^- - Z_2^- x_3 y_2^- \\ y_3^- \end{bmatrix}$, $Z_3 = [Z_2\ x_3] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

8. $Z_2^- x_3 = X_2^- x_3 = \begin{bmatrix} -\frac{2}{6} & -\frac{6}{6} \end{bmatrix}^T$

9. $X_2 X_2^- x_3 = \frac{1}{6}[-2 \quad -6 \quad -4 \quad 2]^T$

10. $x_3 \neq X_2 X_2^- x_3$ and implies case I, $y_3^- = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$

$$Z_3^- = X_3^- \equiv X^- = \frac{1}{15}\begin{bmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{bmatrix}$$

33.15.3 Block Form

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is an $r \times r$ matrix of rank r, then

$$A^- = \begin{bmatrix} A_{11}^T B A_{11}^T & A_{11}^T B A_{21}^T \\ A_{12}^T B A_{11}^T & A_{12}^T B A_{21}^T \end{bmatrix}$$

where $B = (A_{11}A_{11}^T + A_{12}A_{12}^T)^{-1} A_{11} (A_{11}^T A_{11} + A_{21}^T A_{21})^{-1}$

33.15.4

Steps to find A^- of an $A_{m\times n}$ matrix of rank r

1. Compute $B = A^T A$
2. Let $C_1 = I$
3. Compute $C_{i+1} = I(1/i)\mathrm{tr}(\mathrm{C_i} B) - \mathrm{C_i} B$ for $i = 1, 2, \cdots r-1$.
4. Compute $rC_r A^T / tr(C_r B) = A^- (C_{r+1} B = 0$ and $\mathrm{tr}(C_r B) \neq 0)$

Example

$A \equiv X$ of example in (33.15.2)

1. $B = AA^T = \begin{bmatrix} 6 & -3 & 1 \\ -3 & 3 & -2 \\ 1 & -2 & 10 \end{bmatrix}$

2. $C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $C_2 = Itr(C_1 B) - C_1 B = \begin{bmatrix} 13 & 3 & -1 \\ 3 & 16 & 2 \\ -1 & 2 & 9 \end{bmatrix}$

$$C_3 = I\frac{1}{2}\operatorname{tr}(C_2B) - C_2B = \begin{bmatrix} 26 & 28 & 3 \\ 28 & 59 & 9 \\ 3 & 9 & 9 \end{bmatrix}, \quad r \le 3 \text{ and } C_3B \ne 0 \text{ which implies r=3.}$$

4. $$A^- = \frac{3C_3A^T}{\operatorname{tr}(C_3B)} = \frac{3}{225}\begin{bmatrix} 20 & 25 & 5 & 30 \\ 10 & 50 & 40 & 15 \\ -15 & 0 & 15 & 15 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{bmatrix} \text{ (see 33.15.2)}$$

33.16 Conditional Inverse (or c-inverse)

33.16.1 $A_{m\times n}$ is conditional inverse A^c if $AA^cA = A$. **Note**: A g-inverse is conditional, but the reverse is not always true.

33.16.2 Properties

1. Conditional inverse always exists but may not be unique.
2. If A is $m\times n$ then A^c is $n\times m$.

33.16.3 Hermite Form H

An $n\times n$ matrix H is of Hermite form if: l) H is upper triangular, 2) only zeros and ones are on the diagonal ; 3) if a row has a zero on the diagonal, the whole row is zeros; 4) if a row has a one on the diagonal, then every off-diagonal element is zero in the column in which the one appears.

33.16.4 Properties of Hermite Form

1. $H = H^2$
2. There exists nonsingular B such that BA=H where A is any $n\times n$ matrix.
3. B of (2) is conditional inverse of A.
4. If c-inverse of $A_{m\times n}$ is A^c then A^cA and AA^c are idempotent.
5. $\operatorname{rank}(A) = \operatorname{rank}(A^cA) = \operatorname{rank}(AA^c) \le \operatorname{rank}(A^c)$
6. $\operatorname{rank}(A) = \operatorname{rank}(H)$.
7. $A_{n\times n} =$ nonsingular, its Hermite form is $I_{n\times n}$.
8. A^TA, A have the same Hermite form.
9. A^-A and A have the same Hermite form.
10. Rank of A is equal to the number of diagonal elements of H that are equal to one.
11. If $ABA = kA$ then $(1/k)B$ is c-inverse of A.

33.16.5 Steps to find $A^c = B$

$A_{m\times n}$ and $m > n$, $A_0 = [\text{A } 0]$, $0 \equiv m\times m - n$ zero matrix, $B_0 =$ nonsingular such that $B_0A_0 = H$ partition $B_0 = \begin{bmatrix} B \\ B_1 \end{bmatrix}$ with $B = n\times m$ matrix, then $A^c = B$.

Example

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} B \\ B_1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix} = A^c$$

33.17 System of Linear Equation

33.17.1 Definition

$A_x = y$

33.17.2 Existence of Solution of 33.17.1

1) A is an $n \times n$ matrix and nonsingular; 2) solution exists if and only if rank(A) = rank$([A\ y])$ $([A\ y]$ $\equiv$ augmented matrix); 3) if y is in the column space of A; 4) if there is A^c (c-inverse) of A such that $AA^c y = y$; 5) if there is A^- such that $AA^- y = y$; and 6) the $m \times n$ matrix A has rank m.

33.17.3 Solution of Homogeneous Equation (Ax = 0)

If rank (A) < n, there is a solution different than x = 0.

33.17.4 Solutions of Ax = y

1. If a solution exists, $x_0 = A^c y + (I - A^c A)h$ is a solution where h is any $n \times 1$ vector. A is an $m \times n$ matrix.
2. As in (1) $x_0 = A^- y + (1 - A^- A)h$.
3. If $Ax = y$ is consistent (a solution exists) then $x_0 = A^- y$ and $x_1 = A^c y$ are solutions. A is an $m \times n$ matrix.
4. $x_0 = A^- y$ in (3) is unique if and only if $A^- A = I$.
5. The system in (3) has a unique solution if rank $(A) = n$.
6. If a unique solution exists, then it is $A^- y$ and $A^- y = A^c y$ for any A^c.
7. If the system in (3) has rank r>0 and $y \neq 0$, then there are exactly $n - r + 1$ linearly independent vectors that satisfy the system.
8. The set of solutions of $Ax = 0$ is the orthogonal complement of the column space of A^T.

33.18 Approximate Solutions of Linear Equations

33.18.1 Definition of Approximate System

$Ax - y = \varepsilon(x), \quad \varepsilon(x) \equiv$ error depending on vector x.

33.18.2 Best Approximate Solution

$x_0 = A^- y$ (always exists and is unique).

33.18.3 Minimization of Sum of Squares of Deviations

$\Sigma \varepsilon_i^2 = \varepsilon^T(x)\varepsilon(x) = (Ax - y)^T (Ax - y)$.

33.18.4 Minimum of $\varepsilon^T(X)\varepsilon(X)$

For $A_{m\times n}$ any $y_{n\times 1}$ the minimum of $\varepsilon^T(x)\varepsilon(x)$ as x varies over E_n is $y^T(I - AA^-)y$.

33.18.5 Linear Model (in statistics)

$z = X\beta + \varepsilon$, $z_{n\times 1} =$ observed quantities, $X_{n\times p} =$ known (correlation matrix), $\beta_{p\times 1}$ = unknown vector, $\varepsilon =$ error vector (deviation of observations from the expected value).

33.18.6 Normal Equation

$X^T X\hat{\beta} = X^T z$ or $Ax = y$ $(A_{p\times p} = X^T X\hat{\beta}, = x$ and $X^T z = y)$

33.18.7 Properties of Normal Equation

1. The system in (33.18.6) is consistent.
2. $\hat{\beta} = (X^T X)^- X^T z + [I - (X^T X)^-(X^T X)]h = X^- z + [I - X^- X]h =$ general solution, h is any p × 1 vector.
3. Any vector $\hat{\beta}$ that satisfies (33.18.6) leaves $\hat{\beta}^T X^T z$ invariant.

33.19 Least Squares Solution of Linear Systems

33.19.1 Definition

If x_0 is the least square solution (LSS) to (33.18.1) $(Ax - y = \varepsilon(x)),$, then for any other vector x in E_n the following relation holds: $(Ax - y)^T(Ax - y) \ge (Ax_0 - y)^T(Ax_0 - y)$. ***Note***: Best approximate solution (BAS) contains LSS, but the reverse is not always true.

33.19.2 Solution

If $x_0 = By \equiv$ least square solution to $Ax - y = \varepsilon(x)$ then a) $ABA = A$, and b) $AB \equiv$ symmetric.

33.19.3 Properties of Least Square Solution

1. If $A_{m\times n}$ and B is such that ABA=A and AB is symmetric then $AB = AA^-$
2. $(Ax_0 - y)^T(Ax_0 - y) = y^T(I - AA^-)y$
3. If x_0 satisfies $Ax = AA^- y$ then x_0 is LSS
4. If x_0 satisfies $A^T Ax = A^T y$ (normal equation) then x_0 is LSS
5. if $A_{m\times n}$ has rank n then the LSS unique

33.19.4 Least Square Inverse

A^L is least square inverse if and only if a) $AA^L A = A$, and b) $AA^L = (AA^L)^T$. ***Note:*** $A^L \equiv$ c-inverse of A, $A^- \equiv$ c-inverse ad L-inverse of A.

33.19.5 Solution with L-Inverse

$x_0 = A^L y + (I - A^L A)h,$ h is any $n\times 1$ vector.

33.19.6 Properties of L-Inverse

1. $(A^T A)^c \equiv$ c-inverse of $A^T A$, $A_{m\times n}$ then $B = (A^T A)^c A^T$ is L-inverse of A
2. $AA^L = AA^- \equiv$ symmetric idempotent
3. If $AA^L y = y$ the system is consistent

33.19.7 Solution with Constraint

a) If the system $Ax - \alpha = \varepsilon(x)$, $A_{m\times n}$ of rank p (see 33.18.1) is subject to condition $\beta x = \beta$ which is a consistent set of equations, its solution is:

$$x_0 = A^- \alpha - (A^T A)^{-1} B^T [B(A^T A)^{-1} B^T]^{-1} (BA^- \alpha - \beta)$$

b) If $Ax = \alpha$ is not consistent then

$$x_0 = B^- \beta + [A(I - B^- B)]^- (\alpha - (AB^- \beta) + (I - B^- B), \quad (I - [A(I - B^- B)]^- [A(I - B^- B)]h)$$

h=any $p\times 1$ vector, $A \equiv n\times p$ matrix and $B \equiv q\times p$ matrix

33.19.8 Solution with Weighted Least Squares

For the system $y = X\beta + \varepsilon$, if the covariance matrix of ε is not $\sigma^2 I$ but a positive definite matrix V, then the vector β which minimizes $\varepsilon^T V \varepsilon$ is $\beta = (X^T V^{-1} X)^c X^T V^{-1} y$.

33.20 Partitioned Matrices

33.20.1 The Inverse of a Partitioned Matrix

$$B_{n\times n} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad B^{-1} = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad |B| \neq 0, \quad |B_{11}| \neq 0, |B_{22}| \neq 0$$

1. A_{11}^{-1}, A_{22}^{-1} exist
2. $[B_{11} - B_{12} B_{22}^{-1} B_{21}]^{-1}$ and $[B_{22} - B_{21} B_{11}^{-1} B_{12}]^{-1}$ exist
3. $B^{-1} = \begin{bmatrix} [B_{11} - B_{12} B_{22}^{-1} B_{21}]^{-1} & -B_{11}^{-1} B_{12} [B_{22} - B_{21} B_{11}^{-1} B_{12}]^{-1} \\ -B_{22}^{-1} B_{21} [B_{11} - B_{12} B_{22}^{-1} B_{21}]^{-1} & [B_{22} - B_{21} B_{11}^{-1} B_{12}]^{-1} \end{bmatrix}$
4. $A_{11} = [B_{11} - B_{12} B_{22}^{-1} B_{21}]^{-1} = B_{11}^{-1} + B_{11}^{-1} B_{12} A_{22} B_{21} B_{11}^{-1}$
5. $A_{12} = -B_{11}^{-1} B_{12} [B_{22} - B_{21} B_{11}^{-1} B_{12}]^{-1} = -B_{11}^{-1} B_{12} A_{22}$
6. $A_{22} = [B_{22} - B_{21} B_{11}^{-1} B_{12}]^{-1} = B_{22}^{-1} B_{22}^{-1} B_{21} A_{22} B_{21} B_{11}^{-1}$
7. $A_{21} = -B_{22}^{-1} B_{21} [B_{11} - B_{12} B_{22}^{-1} B_{21}]^{-1} = -B_{22}^{-1} B_{21} A_{11}$
8. $|B| = \dfrac{|B_{11}|}{|A_{22}|} = \dfrac{|B_{22}|}{|A_{11}|}, \quad |B_{11} A_{11}| = |B_{22} A_{22}|$
9. $|B| = |B_{22}| \, |B_{11} - B_{12} B_{22}^{-1} B_{21}|, \quad B_{22} =$ nonsingular
10. $|B| = |B_{11}| \, |B_{22} - B_{21} B_{11}^{-1} B_{12}|$

33.21 Inverse Patterned Matrices

33.21.1 k × k Lower Triangular

$$C = \begin{bmatrix} a_1b_1 & 0 & \cdots & 0 \\ a_2b_1 & a_2b_2 & \cdots & 0 \\ \vdots & & & \\ a_kb_1 & a_kb_2 & \cdots & a_kb_k \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} (a_1b_1)^{-1} & 0 & \cdots & \cdots & 0 & 0 \\ -(b_2a_1)^{-1} & (a_2b_2)^{-1} & 0 & \cdots & 0 & 0 \\ 0 & -(b_3a_2)^{-1} & (a_3b_3)^{-1} & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & & & (b_{k-1}a_{k-1})^{-1} & \\ 0 & 0 & \cdots & \cdots & -(b_ka_{k-1})^{-1} & (a_kb_k)^{-1} \end{bmatrix}$$

33.21.2 Diagonal Plus Matrix

$C = D + qab^T, C^{-1} = D^{-1} + pa^*b^{*T},\quad D =$ nonsingular, $q \neq -[\Sigma a_ib_i / d_{ii}]^{-1}, p = -q\left(1 + q\sum_{i=1}^{k} a_ib_id_{ii}^{-1}\right.$

$a_i^* = a_i / d_{ii}, b_i^* = b_i / d_{ii}, d_{ii} \equiv i^{th}$

Example

$V \equiv$ variance-covariance of k-dimensional random variable $\left(p_i > 0, \sum_{i=1}^{k} p_i < 1\right)$

$$V = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & \cdots & -p_1p_k \\ -p_1p_2 & p_2(1-p_2) & -p_2p_3 & & -p_2p_k \\ \vdots & & & & \\ -p_1p_k & -p_2p_k & -p_3p_k & \cdots & p_k(1-p_k) \end{bmatrix} = D + qab^T$$

$$= \begin{bmatrix} p_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_3 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & p_k \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} [p_1\ p_2 \cdots p_k],$$

$$V^{-1} = D^{-1} + ba^*b^{*T},\ a_i^* = b_i^* = 1,\ a^*b^{*T} = J,\ q = \left[1 - \sum_{i=1}^{k} p_i\right]^{-1}$$

33.21.3 Equal Diagonal and All Other Elements

$$C_{3\times 3} = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \quad C_{k\times k} = (a-b)I + bJ,\ C^{-1} = \frac{1}{a-b}[I - \frac{b}{a+(k-1)b} J],$$

$$a \neq b,\ a \neq -(k-1)b,\ J = 11^T = \text{ matrix of ones } (1 = [1 \quad 1 \ldots 1]^T)$$

33.22 Determinants of Patterned Matrices

33.22.1 Lower Triangular

This determinant of 33.21.1 is $|C| = a_1 a_2 \cdots a_k b_1 b_2 \cdots b_k$

33.22.2 Diagonal Plus Matrix

The determinant of 33.21.2 is $\det(C) = \left[1 + q\sum_j \frac{a_j b_j}{d_{jj}}\right]\prod_i d_{ii}$ 33.22.3

$C_{k\times k} = (a-b)I + bJ \quad (J \equiv \text{ matrix of ones }): \quad \det(C) = (a-b)^{k-1}[a + (k-1)/b]$

33.23 Characteristic Equations of Patterned Matrices

33.23.1 Triangular $(k \times k)$: $\prod_{i=1}^{k}(a_{ii} - \lambda) = 0$, eigenvalues $a_{11}, a_{22}, \cdots, a_{kk}$

33.23.2 $C = D + qab^T$ (see 21.2): $\left(1 + q\sum_{i=1}^{k}\frac{a_i b_i}{d_{ii} - \lambda}\right)\prod_{i=1}^{k}(d_{ii} - \lambda) = 0$

33.23.3 $C = dI + qab^T$: $\left(d + q\sum_{i=1}^{k} a_i b_i - \lambda\right)(d-\lambda)^{k-1} = 0, \quad k-1$ eigenvalues are equal to d and one is equal to $d + q\sum_{i=1}^{k} a_i b_i$

33.24 Triangular Matrices

33.24.1 Decomposition to Triangular Matrices

A=RT, R=lower real triangular, T = upper real triangular, $A_{k\times k} \equiv$ real with every leading principal minor is nonzero.

33.24.2 Upper Triangular

$A = T^T T, \quad T =$ upper triangular, $A_{k\times k} =$ positive definite.

Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 17 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

33.24.3 Upper and Diagonal

$A = T^T DT$, $T =$ upper triangular (real and unique) with $t_{ii} = 1$, D= real diagonal, $A_{n\times n} =$ positive definite.

33.24.4 Product of Lower Triangular

$R = R_1 R_2 \cdots R_n$, $R_i =$ lower triangular, R =lower triangular.

33.24.5 Product of Upper Triangular

$T = T_1 T_2 \cdots T_n$, T_i upper triangular, T=upper triangular.

33.24.6 Inverse

R^{-1} = lower, T^{-1} = upper, R = lower, T = upper

33.24.7 Determinant

$$\det(R \text{ or } T) = \left(\prod_{i=1}^{k} R_{ii} \text{ or } \prod_{i=1}^{k} T_{ii} \right), \quad R_{k\times k} \text{ and } T_{k\times k}$$

33.24.8 Eigenvalues

Eigenvalues (R or T)= $(r_{11}, r_{22}, \cdots r_{kk}$ or $t_{11}, t_{22}, \cdots t_{kk})$.

33.24.9 Orthogonal Decomposition

$PA = T, P =$ real orthogonal matrix, $A_{k\times k} =$ real, T = upper real triangular, $t_{ii} > 0$ for $i = 1, 2, \cdots k$.

33.24.10 Eigenvalues

$T = P^{-1}AP$, A = real and its roots are real, P = nonsingular, T = upper triangular, eigenvalues = diagonal elements of T.

33.24.11 Product

i^{th} diagonal element of T^n is t_{ii}^n, $T_{k\times k} =$ upper (lower) triangular.

33.24.12 Inverse

If we set $T^{-1} = B$ then $t_{ii} b_{ii=1}$ for $i = 1, 2, \cdots k$, $T_{k\times k} =$ nonsingular triangular

33.24.13 Orthogonal Decomposition

$P^T AP = T$, $A_{k\times k} =$ real eigenvalues, P=orthogonal, T =upper triangular, eigenvalues of A =diagonal of T.

33.25 Correlation Matrix

33.25.1 Correlation Matrix

$R = [\rho_{ij}],\ \rho_{ij} = \dfrac{\upsilon_{ij}}{\sqrt{\upsilon_{ii}\upsilon_{ij}}}$, V = positive definite covariance matrix $= [\upsilon_{ij}],\ \upsilon_{ij} =$ covariance between y_i and y_j components of the random vectors y, υ_{ii} = variance between y_i and y_i .

33.25.2 Correlation Matrix

$R = D_\upsilon^{-1/2} V D_\upsilon^{-1/2}$, R = correlation matrix of random vector y, V = positive definite covariance matrix, D_υ = diagonal matrix with i^{th} diagonal element υ_{ii} .

33.25.3 Properties of Correlation Matrix

a) $\rho_{ii} = 1, i = 1, 2, \cdots n$; ; b) $-1 < \rho_{ij} < 1$ all $i \neq j$; c) largest eigenvalue is less than n; d) $0 < |R| \leq 1$; e) positive definite.

33.26 Direct Product and Sum of Matrices

33.26.1 Direct Product

$$C_{m_1 m_2 \times n_1 n_2} = A_{m_2 \times n_2} \otimes B_{m_1 \times n_1} = \begin{bmatrix} Ab_{11} & Ab_{12} & \cdots & Ab_{1n_1} \\ Ab_{21} & Ab_{22} & \cdots & Ab_{2n_1} \\ \vdots & \vdots & & \vdots \\ Ab_{m_1 1} & Ab_{m_1 2} & & Ab_{m_1 n_1} \end{bmatrix}$$

$$= [C_{ij}] = [Ab_{ij}] \quad i = 1, 2, \cdots m_1, \quad j = 1, 2, \cdots n_1$$

Example

$$A \otimes B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} Ab_{11} & Ab_{12} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 8 & 4 \\ 0 & -0 & 0 & -4 \end{bmatrix}$$

$$B \otimes A = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} Ba_{11} & Ba_{12} \\ Ba_{21} & Ba_{22} \end{bmatrix} = \begin{bmatrix} 2 & 8 & 1 & 4 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

33.26.2 Properties of Direct Products

1. $A \otimes B \neq B \otimes A$ in general
2. $A_{m_2 \times m_2} \otimes I_{m_1 \times m_1} = \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix}_{m_1 m_2 \times m_1 m_2} = diag(A)$
3. $(aA) \otimes B = A \otimes (aB) = a(A \otimes B), \quad a =$ any scalar
4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
5. $(A \otimes B)^T = A^T \otimes B^T$

Example

$$A^T \otimes B^T = \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 0 \\ 12 & -8 \\ 8 & 0 \end{bmatrix} = (A \otimes B)^T = \begin{bmatrix} 3 & 2 & 12 & 8 \\ -2 & 0 & -8 & 0 \end{bmatrix}^T$$

6. $tr(A \otimes B) = tr(A) tr(B)$, A and B square matrices
7. $(A \otimes B)(F \otimes G) = (AF) \otimes (BG),\ A_{m_1 \times n_1},\ B_{m_2 \times n_2},\ F_{n_1 \times k_1},\ G_{n_2 \times k_2}$
8. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, A and B nonsingular
9. $P \otimes Q =$ orthogonal if P and Q are orthogonal
10. $A \otimes I = diag(A)$
11. $\det(A \otimes B) = \det(B \otimes A) = |A|^n |B|^m,\ A_{m \times m},\ B_{n \times n}$
12. $\det(A \otimes A^{-1}) = \det(A^{-1} \otimes A) = 1,$ A = nonsingular
13. $(I \otimes P)(I \otimes A) = I \otimes T = C,\ A_{m \times m},\ P_{m \times m},\ PA = T,$ T = upper (lower), C = upper (lower)
14. $A_{m \times m} \equiv \{\lambda_1, \lambda_2, \cdots \lambda_m\}$ = characteristic roots (eigenvalues), the eigenvalues of $I \otimes A$ and $A \otimes I$ are of multiplicity of n of the eigenvalues of $A, I_{n \times n}$.
15. $(A + B) \otimes C = A \otimes C + B \otimes C,\ A_{m \times m},\ B_{m \times m},\ C_{n \times n}$
16. $D_1 \otimes D_2 \equiv$ diagonal, $D_1 =$ diagonal, $D_2 =$ diagonal

    $T_1 \otimes T_2 \equiv$ upper (lower) tiangular, $T_1 \equiv$ upper (lower) tiangular

    $T_2 \equiv$ upper (lower) tiangular
17. $A \otimes B =$ positive (semi) definite, A and B = positive (semi) definite
18. $rank\ (A \otimes B) = rank(A)\ rank(B)$

33.27 Direct Sum of Matricies

33.27.1 Definition

$$A = A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},\ A_{1 n_1 \times n_2},\ A_{2 n_2 \times n_2},\ A_{(n_1+n_2) \times (n_1+n_2)}$$

33.27.2 Properties

1. $A_1 \oplus A_2 \neq A_2 \oplus A_1$ in general
2. $(A_1 \oplus A_2) \oplus A_3 = A_1 \oplus (A_2 \oplus A_3)$
3. $A_1 \oplus \cdots \oplus A_k = A_1^T \oplus \cdots \oplus A_k^T = A^T = A$ if and only if A_i is symmetric
4. $\det(A_1 \oplus \cdots \oplus A_k) = \prod_{i=1}^{k} \det(A_i)$

5. $tr(A_1 \oplus \cdots \oplus A_k) = \sum_{i=1}^{k} tr(A_i)$
6. $A_1 \oplus \cdots \oplus A_k$ is upper (lower) triangular if each A_i is upper (lower) triangular
7. $A_1 \oplus \cdots \oplus A_k = A =$ orthogonal, $A^T A = AA^T = I$ if each A_i is orthogonal
8. If $rank(A_i) = r_i$ then $rank(A_1 \oplus \cdots \oplus A_k) = \sum_{i=1}^{k} r_i$
9. $(A_1 \oplus A_2)(A_3 \oplus A_4) = (A_1A_3) \oplus (A_2A_4)$ assuming the operations are defined
10. $A \otimes I = A \oplus A \oplus \cdots \oplus A, \quad A_{n\times n}, \quad I_{m\times m}$

33.28 Circulant Matrices

33.28.1 Definitions

$(j-i)/k \equiv (j-i)$ modulo k, if remainder is positive it is kept, but if it is negative we must add k (Example: 16/5 = 1, -16/5=-1 hence -1+5=4).

$$(j-i)/k = k + j - i \text{ when } i > j \quad 1 \le i \text{ and } j \le k$$

$$(j-i)/k = j - i \text{ when } i \le j \quad 1 \le i \text{ and } j \le k$$

33.28.2 Regular Circulant

$(j-i)/k \equiv (q-p)/k$ which implies that $a_{ij} = a_{pq}, A_{k\times k}$

$$A = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{bmatrix}, \quad a_{ij} = a_{j-i} \text{ for } j \ge i \text{ (elements sbovr disagonal)}$$

$$a_{ij} = a_{k+j-i} \text{ for } j < i \text{ (elements below the diagonal)}$$

33.28.3 Regular Circulants

1. The zero matrix 0,
2. The identity one I,
3. Matrix with all elements equal to a constant,
4. $aA + bB, \quad A_{k\times k}, \quad B_{k\times k}$ regular circulants

33.28.4 Properties

1. $A^T \equiv$ regular circulant if A is also
2. Diagonal elements of A are equal
3. The diagonals parallel to the main one have their elements equal
4. $P^T AP = A,\ A_{k\times k} =$ regular singular, $P = [x_1 x_2 \cdots x_k x_1]$ where $x_i = [0\ 0 \cdots 1 \cdots 0]$ (i^{th} element 1)
5. $AB \equiv$ regular circulant if $A_{k\times k}$ and $B_{k\times k}$ are regular circulants
6. $C^{-1} \equiv$ regular circulant if C is a nonsingular regular circulant
7. AB=BA if both are $k \times k$ regular circulants
8. $C_{k\times k} =$ regular circulant, $[c_0\ c_1 \cdots c_{k-1}] \equiv$ first row,

$\lambda_i =$ eigrnvalues of $C = c_0\omega_i^0 + c_i\omega_i + c_2\omega_i^2 + \cdots + c_{k-1}\omega_i^{k-1}$, $\omega_k \equiv k^{th}$ roots of unity, $x_i = [\omega_i^0 \omega_i \omega_i^2 \cdots \omega_i^{k-1}]^T$ are the eigenvalues

Example

$$C = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 0 & 3 & -1 & 2 \\ 2 & 0 & 3 & -1 \\ -1 & 2 & 0 & 3 \end{bmatrix}, \quad e^{j\frac{2\pi}{4}k} \Rightarrow 1, -1, j, -j;$$

$$\lambda_1 = 3\cdot 1 - 1\cdot 1 + 2\cdot 1^2 + 0\cdot 1^3 = 4 \qquad x_1 = [1\,1\,1\,1]^T$$

$$\lambda_2 = 3\cdot 1 - 1(-1) + 2\cdot(-1)^2 + 0(-1)^3 = 6 \qquad x_2 = [1\,-1\,1\,-1]^T \text{ etc.}$$

9. $C_{k\times k}$ with first row $[c_0\ c_1 \cdots c_{k-1}]$ its

$$\det(C) = \prod_{i=1}^{k} \lambda_i = \prod_{i=1}^{k} (c_0 + c_1\omega_i + c_2\omega_i^2 + \cdots + c_{k-1}\omega_i^{k-1})$$

10. Symmetric regular: $A = \begin{bmatrix} a_0 & a_1 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 \\ a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_1 & a_0 \end{bmatrix} \equiv$ any regular circulant that is also symmetric

11. Properties of symmetric regular: A and B are symmetric regular.
    1. $aA + bB \equiv$ symmetric regular, a and b any real number
    2. $A^T, B^T \equiv$ symmetric regular
    3. AB=BA and AB is symmetric regular circulant
    4. $A^{-1} \equiv$ symmetric regular

12. Symmetric circulant: $A = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{bmatrix}$ $(a_{ij} = a_{(i+j-2)/k})$

13. Properties of symmetric circulant: $A_{k\times k}$ and $B_{k\times k}$ are symmetric circulants
    1. aA and bB $\equiv$ a and b are any real numbers
    2. I $\equiv$ not symmetric circulant
    3. $A^{-I} \equiv$ symmetric circulant
    4. AB $\equiv$ regular circulant (AB $\neq$ BA in general)
    5. If $a_0, a_1, \cdots, a_{k-1}$ are the first row elements of A then $s = a_0 + a_1 + \cdots + a_{k-1}$ is an eigenvalue of A
    6. A^- is the g-inverse of B and is symmetric circulant

33.29 Vandermonde and Fourier Matrices

33.29.1 Definition of Vandermonde Matrix (see 33.1.12)

33.29.2 Properties of Vandermonde

1. $\det(A) = \prod_{t=2}^{k} \prod_{i=2}^{t-1} (a_t - a_i)$
2. rank(A) = r, r= number of distinct a_i values
3. $A^T =$ also a Vandermonde matrix if A is one

Example

$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_{k-1} x_i^{p-1} + \varepsilon_i, \quad i = 1, 2, \cdots n, n \ge p$ Hence, $y = X\beta + \varepsilon$ or

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \cdots & x_1^{p-1} \\ 1 & x_2 & x_2^2 \cdots & x_2^{p-1} \\ \vdots & & & \\ 1 & x_n & x_n^2 \cdots & x_n^{p-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

33.29.3 Fourier Matrix

$$F = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{k-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2k-2} \\ \vdots & & & & \\ 1 & \omega^{k-1} & \omega^{2k-2} & \cdots & \omega^{(k-1)(k-1)} \end{bmatrix}, \quad F_{k\times k}, \omega = e^{j\frac{2\pi}{k}}$$

33.29.4 Properties of the Fourier Matrix

1. $F = F^T, \quad F_{k\times k}$
2. $F^{-1} = \overline{F}, \overline{F} =$ conjugate of F
3. $F^2 = P, P =$ permutation matrix $= [\varepsilon_1 \ \varepsilon_k \ \varepsilon_{k-1} \cdots \varepsilon_2], \quad \varepsilon_j \ j^{th}$ column of the k × k identity matrix.
4. $F^4 = I$
5. $\sqrt{k}F = C + jS,$ C and S are real matrices,

$$c_{pq} = \cos\frac{2\pi}{k}(p-1)(q-1), S_{pq} = \sin\frac{2\pi}{k}(p-1)(q-1)$$

6. $CS = SC, \quad C^2 + S^2 = I$

33.30 Permutation Matrices

33.30.1 Definition

a) Column permutation = n × n matrix resulting from permuting columns of n × n identity matrix.
b) Row permutation = n × n matrix resulting from permuting the rows of n × n identity matrix.

Example

$I = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_4]$, then if $P = [\varepsilon_2\ \varepsilon_4\ \varepsilon_3\ \varepsilon_1]$ we obtain AP=B that moves column 1 of A to column 4 of B, column of A to column I of B, column 3 of A to column 3 of B, and column 4 of A to column 2 of B. If we look at P, we observe that row-wise from I row 1 moved to row 2, row 4 moved to row 1, row 3 remains the same, and row 2 moved to row 4. Hence, PA=C moves row I of A to row 2 of C, row 2 of A to row 4 of C, row 3 of A to row 3 of C, and row 4 of A to row I of C.

33.30.2 Properties

1. $P^T P = PP^T = I,\ P^T = P^{-1}$ which implies P is orthogonal.
2. $P^T AP$ has the same diagonal elements (rearranged) as the diagonal elements of A

33.31 Hadamard Matrices

33.31.1 Definitions

a) H consists of +1 and -1 elements, $H_{n\times n}$
b) $H^T H = nI$

Example

$$H_{2\times 2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ H^T H = 2I$$

33.31.2 Properties

1. $n^{-1/2}H$ is orthogonal, $H_{n\times n}$
2. $HH^T = nI$
3. $H^{-1} = n^{-1}H^T$
4. H^T and nH^{-1} are Hadamard matrices
5. $H_1 \otimes H_2 =$ Hadamard matrix, H_1 and H_2 are Hadamard matrices

Example

$$H \otimes H = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

6. $H_{n\times n},\ n = 1$ or 2 or multiple of 4
7. $\det(H) = n^{n/2}$

33.32 Toeplitz Matrices (T-matrix)

33.32.1 Definition

All elements on each superdiagonal are equal, and all elements on each subdiagonal are equal.

Example

$$A_1 = \begin{bmatrix} 6 & 2 & 0 \\ 5 & 6 & 2 \\ 0 & 5 & 6 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

33.32.2 Properties

1. A regular circulant is a T-matrix but a T-matrix is not necessarily a circulant.
2. A linear combination of T-matrices is a T-matrix.
3. If a_{ij} is definded $a_{ij} = a_{|i-j|}$ then A is a symmetric T-matrix.
4. A^T is a T-matrix.
5. A is symmetric about its secondary diagonals.
6. $A_{n\times n} = \begin{bmatrix} a_0 & a_1 & 0\cdots & 0 & 0 \\ a_2 & a_0 & a_1\cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0\cdots & a_2 & a_0 \end{bmatrix}$, eigenvalues: $\lambda_m = a_0 + 2\sqrt{a_1 a_2}\cos\left(\frac{m\pi}{n+1}\right)$ for $m = 1, 2, \cdots, n$
7. If $a_{ij} = a_{|i-j|}$ for $|i-j| \le 1$ and $a_{|i-j|} = 0$ for $|i-j| > 1$ (symmetric matrix) the eigenvalues are:

   $\lambda_m = a_0 + 2a_1 \cos\frac{m\pi}{n+1}$ for $m = 1, 2, \cdots, n$.
8. A of (7) is positive definite if and only if $a_0 + 2a_1 \cos\frac{m\pi}{n+1} > 0$ for $m = 1, 2, \cdots, n$
9. A of (7) is positive for all n if $a_0 > 0$ and $|a_1 / a_0| \le 1/2$
10. The determinant of $A_{n\times n}$ of (6) is $\det(A) = \prod_{m=1}^{n}\left[a_0 + 2\sqrt{a_1 a_2}\cos\left(\frac{m\pi}{n+1}\right)\right]$

33.32.3 Centrosymmetric matrix (or cross-symmetric): $a_{ij} = a_{n+1-i,n+1-j}$ for all i and j

Example

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 2 & 1 & 3 & 6 \\ 6 & 3 & 1 & 2 \\ 0 & 4 & 2 & 1 \end{bmatrix}$$

33.32.4 Properties of Centrosymmetric Matrix (c-matrix)

1. 0 ; I ; J (all ones); aI + bJ are c-matrices.
2. A^T is a c-matrix if A is.
3. $A = \sum_{i=1}^{K} a_i A_i$ is a c-matrix if A_i 's are c-matrices.
4. AB is a c-matrix if $A_{n\times n}$ and $B_{n\times n}$ are c-matrices.
5. A^{-1} is c-matrix if A is a c-matrix.

33.33 Trace

33.33.1 Definitions

$$tr(A) = \sum_{i=1}^{n} a_{ii},\ A_{n\times n}$$

33.33.2 Properties

1. $tr(AB) = tr(BA), \quad A_{n\times n}, \quad B_{n\times n}$
2. $tr(A) = \sum_{i=1}^{n} \lambda_i, \lambda_i =$ eigenvalues.
3. $tr(A) = tr(P^{-1}AP), \quad A_{n\times n}, \quad P_{n\times n}$
4. $tr(A) = tr(P^{-1}AP), \quad P$ is orthogonal
5. $tr(aA + bB) = a\,tr(A) + b\,tr(B)$
6. $tr(A) = m\,rank(A), A^2 = mA$
7. $tr(A^T) = tr(A)$
8. $tr[I - A(A^TA)^- A^T] = m - r, \; A_{m\times m}, \; rank(A) = r$
9. $tr(AA^T) = tr(A^TA) = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}^2, A_{n\times m}$
10. $tr(A^k) = \sum_{i=1}^{n} \lambda_i^k, \quad A_{n\times n} \quad \lambda_1, \lambda_2 \cdots \lambda_r$
11. $tr(A \otimes B) = tr(A)tr(B)$
12. $tr(A^-) = \sum_{i=1}^{r} \lambda_i^{-1}, A_{n\times n}\; \lambda_1, \lambda_2 \cdots \lambda_r$
13. $tr(I + S) = n, \quad tr(S) = 0, \; S_{n\times n} =$ skew symmetric.
14. $tr(A) = 0$ if $A^k = 0$ for some positive k.
15. $x^TAx = tr(Axx^T)$
16. $tr(A) > 0$ implies A is positive definite.
17. $tr(A) \geq 0$ implies A is positive semidefinite, or non-negative.
18. $tr(AB) \geq 0, A_{n\times n}$ and $B_{n\times n}$ are non-negative.
19. $tr(A^TB) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ij}$
20. $y^Tx = tr(xy^T)$

33.34 Derivatives

33.34.1 Derivative of a Function with Respect to a Vector

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_k} \end{bmatrix}, \quad x = [x_1, x_2, \cdots x_k]^T, \; x_i = \text{ real veriables}, \quad f(x_1, x_2, \cdots x_k)$$

33.34.2 $f(X)=\alpha^T X\beta,\ \frac{\partial f}{\partial X}=\alpha\beta^T,\ X_{m\times n}, \alpha_{m\times 1},\ \beta_{n\times 1},\ \alpha$ and β are vectors of constants.

33.34.3 $f(X)=\alpha^T X\alpha,\ \frac{\partial f}{\partial X}=2\alpha\alpha^T - D_{\alpha\alpha^T},\ \alpha_{k\times 1} =$ vectors of constatns $X_{k\times k} =$ symmetric matrix of independent real variables (except that $x_{ij}=x_{ji}$), $D_{\alpha\alpha^T} = k\times k$ diagonal matrix (i^{th} diagonal = i^{th} diagonal of $\alpha\alpha^T$)

33.34.4 Derivative of a Matrix with Respect to a Scalar

$\frac{\partial Y}{\partial x_{ij}}=\left[\frac{\partial f_{pq}(x_{11},\cdots,x_{mn})}{\partial x_{ij}}\right],\quad Y_{k\times k},\quad y_{pq}=$ elements of $Y=f_{pq}(x_{11},x_{12}\cdots,x_{mn})$,each y_{pq} is a function of an $m\times n$ matrix of independent real variables x_{ij}.

33.34.5 $\frac{\partial f(x;\alpha)}{\partial\alpha}=\begin{bmatrix}\frac{\partial f(x;\alpha)}{\partial\alpha_1}\\ \vdots\\ \frac{\partial f(x;\alpha)}{\partial\alpha_n}\end{bmatrix},\quad x=[x_1,x_2,\cdots,x_r]^T,\quad \alpha=[\alpha_1,\alpha_1,\cdots\alpha_n]^T\ \frac{\partial x(\alpha)}{\partial\alpha}$

33.34.6 $\frac{\partial x(\alpha)}{\partial\alpha}=\begin{bmatrix}\frac{\partial x_1(\alpha)}{\partial\alpha_1} & \cdots & \frac{\partial x_1(\alpha)}{\partial\alpha_n}\\ \vdots & & \\ \frac{\partial x_r(\alpha)}{\partial\alpha_i} & \cdots & \frac{\partial x_r(\alpha)}{\partial\alpha_n}\end{bmatrix},\quad x=[x_1(\alpha),x_2(\alpha),\cdots x_r(\alpha)]^T,\ \alpha=[\alpha_1,\alpha_2,\cdots\alpha_n]^T$

33.34.7 $\frac{\partial x(\alpha)}{\partial\alpha_i}\begin{bmatrix}\frac{\partial x_i(\alpha)}{\partial\alpha_i}\\ \vdots\\ \frac{\partial x_r(\alpha)}{\partial\alpha_i}\end{bmatrix}$

33.34.8 $\nabla_z(\alpha^H z)=\alpha^*,\quad \nabla_z(z^H\alpha)=0,\quad \nabla_z(z^H Rz)=(Rz)^*$

33.34.9 $\nabla_{z^*}(\alpha^H z)=0,\quad \nabla_{z^*}(z^H\alpha)=\alpha,\quad \nabla_{z^*}(z^H Rz)=Rz$

33.34.10 $w_k=x_k+jy_k,\ \frac{\partial}{\partial w_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}-j\frac{\partial}{\partial y_k}\right),\quad \frac{\partial}{\partial w_{k^*}}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}+j\frac{\partial}{\partial y_k}\right),$

$$\frac{\partial w_k}{\partial w_k} = 1, \ \frac{\partial w_k}{\partial w_k^*} = \frac{\partial w_k^*}{\partial w_k} = 0, \ \frac{\partial}{\partial w} = \frac{1}{2}\left[\frac{\partial}{\partial x_0} - j\frac{\partial}{\partial y_0} \ \frac{\partial}{\partial x_1} - j\frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial x_{m-1}} - j\frac{\partial}{\partial y_{m-1}}\right]^T,$$

$$\frac{\partial}{\partial w^*} = \frac{1}{2}\left[\frac{\partial}{\partial x_0} + j\frac{\partial}{\partial y_0} \ \frac{\partial}{\partial x_1} + j\frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial x_{m-1}} + j\frac{\partial}{\partial y_{m-1}}\right]^T$$

Example

$$\frac{\partial x^H w}{\partial w^*} = 0, \quad \frac{\partial w^H x}{\partial w^*} = x, \quad \frac{\partial w^H R w}{\partial w^*} = Rw, \ \text{if} \ \ J = \sigma_d^2 - w^H x - x^H w + w^H R w$$

then $\dfrac{\partial J}{\partial w^*} = -x + Rw$

33.35 Positive Definite Matrices

33.35.1 Positive Definite

$A = CC^T$, $C = n \times n$ full rank or the principal minors are all positive. (The i^{th} principal minor is the determinant if the submatrix formed by deleting all rows and columns with an index greater than I.)

$$A^{-1} = (C^{-1})^T (C^{-1})$$

33.35.2 A = positive definite, $B = m \times n$ full rank with $m \le n$ then $BAB^T =$ positive definite

33.35.3 A = positive definite a) the diagonal elements are positive (non-negative), and b) det A = positive (non-negative)

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